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Renyu Xu

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Renyu Xu. Some coloring problems of graphs. Discrete Mathematics [cs.DM]. Université Paris Saclay (COmUE); Shandong University (Jinan, Chine), 2017. English. NNT: 2017SACLS104 . tel-01561631

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THÈSE DE DOCTORAT  
DE  
SHANDONG UNIVERSITY  
ET DE  
L'UNIVERSITÉ PARIS-SACLAY  
PRÉPARÉE À  
L'UNIVERSITÉ PARIS-SUD

ÉCOLE DOCTORALE N°580  
Sciences et technologies de l'information et de la communication  
GRADUATE SCHOOL of Mathematics

Spécialité Informatique

Par

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Quelques problèmes de coloration du graphe

**Thèse présentée et soutenue à Jinan, le 27 Mai 2017 :**

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# Some Coloring Problems of Graph

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## ABSTRACT

The study of graph theory started two hundred years ago. The earliest known paper was written by Euler(1736)to solve the Konigsberg seven-bridge problem. Graph coloring has been one of the most important directions of graph theory since the arose of the well-known Four Color Problem. Graph coloring has real-life applications in optimization, computer science and network design. Here, we study the total coloring, list coloring, neighbor sum distinguishing total coloring and linear  $L$ -choosable arboricity.

All graphs in this thesis are simple, undirected and finite graphs. Let  $G = (V, E)$  be a graph. For a vertex  $v \in V(G)$ , let  $N_G(v)$  be the set of neighbors of  $v$  in  $G$  and let  $d_G(v) = |N_G(v)|$  be the degree of  $v$  in  $G$ . The maximum degree and minimum degree of  $G$  is denoted by  $\Delta(G)$  and  $\delta(G)$ , respectively. For convenience, throughout this thesis, we set  $\Delta = \Delta(G)$  and  $\delta = \delta(G)$ .

A  $k$ -total-coloring of a graph  $G$  is a coloring of  $V(G) \cup E(G)$  using  $(1, 2, \dots, k)$  colors such that no two adjacent or incident elements receive the same color. The total chromatic number  $\chi''(G)$  is the smallest integer  $k$  such that  $G$  has a  $k$ -total-coloring. As early as 1960s, Vizing and Behzad independently conjectured that for any graph  $G$ ,  $\Delta \leq \chi''(G) \leq \Delta + 2$ . This

conjecture was known as Total Coloring Conjecture. This conjecture has been confirmed for general graphs with  $\Delta \leq 5$ . For planar graphs, the only open case is  $\Delta = 6$ . It is interesting to notice that many planar graphs are proved to be  $\chi''(G) = \Delta + 1$ , i.e., the exact result has been obtained. Up to date, for each planar graph with  $\Delta \geq 9$ ,  $\chi''(G) = \Delta + 1$ . However, for planar graphs with  $4 \leq \Delta \leq 8$ , no one has found counterexamples that are not  $(\Delta + 1)$ -total-colorable. So, Wang Yingqian et al. conjectured that planar graphs with  $\Delta \geq 4$  are  $(\Delta + 1)$ -totally-colorable. In chapter 2, we study total coloring of planar graphs and obtain three results: (1) Let  $G$  be a planar graph with maximum degree  $\Delta \geq 8$ . If every two chordal 6-cycles are not adjacent in  $G$  or any 6-cycle of  $G$  contains at most one chord, then  $\chi''(G) = \Delta + 1$ . (2) Let  $G$  be a planar graph  $G$  with maximum degree  $\Delta \geq 8$ . If any 7-cycle of  $G$  contains at most two chords, then  $\chi''(G) = \Delta + 1$ . (3) Let  $G$  be a planar graph without intersecting chordal 5-cycles, that is, every vertex is incident with at most one chordal 5-cycle. If  $\Delta \geq 7$ , then  $\chi''(G) = \Delta + 1$ .

A mapping  $L$  is said to be an assignment for a graph  $G$  if it assigns a list  $L(v)$  of colors to each vertex  $v \in V(G)$ . If it is possible to color  $G$  so that every vertex gets a color from its list and no two adjacent vertices receive the same color, then we say that  $G$  is  $L$ -colorable. A graph  $G$  is  $k$ -choosable if  $G$  is an  $L$ -colorable for any assignment  $L$  for  $G$  satisfying  $|L(v)| \geq k$  for every vertex  $v \in V(G)$ . We prove that if every 5-cycle of  $G$  is not simultaneously adjacent to 3-cycles and 4-cycles, then  $G$  is 4-choosable. A mapping  $L$  is said to be a *total assignment* for a graph  $G$  if it assigns a list  $L(x)$  of colors to each element  $x \in V(G) \cup E(G)$ . Given a total assignment  $L$  of  $G$ , an  *$L$ -total coloring* of  $G$  is a proper total coloring such that each element receives a color from its own list. A graph  $G$  is  *$k$ -total-choosable* if  $G$  has a proper

$L$ -total-coloring for every preassigned total assignment  $L$  with  $|L(x)| \geq k$  for every  $x \in V \cup E$ . The *list total chromatic number* or *total choosability* of  $G$ , denoted  $\chi''_l(G)$ , is the smallest integer  $k$  such that  $G$  is  $k$ -total-choosable. The *list edge chromatic number* (or *edge choosability*)  $\chi'_l(G)$  are defined similarly in terms of coloring only edges. In chapter 3, if every 5-cycles of  $G$  is not adjacent to 4-cycles, we prove that  $\chi'_l(G) = \Delta$ ,  $\chi''_l(G) = \Delta + 1$  if  $\Delta(G) \geq 8$ , and  $\chi'_l(G) \leq \Delta + 1$ ,  $\chi''_l(G) \leq \Delta + 2$  if  $\Delta(G) \geq 6$ .

Recently, magic and antimagic labellings and the irregularity strength and other colorings and labellings related to “sum” of the colors have received much attention. Among them there are the famous 1 – 2 – 3 Conjecture and 1 – 2 Conjecture. In chapter 4, we will give the definition of neighbor sum distinguishing total coloring. We also list the research progress and the corresponding conjectures of neighbor sum distinguishing total coloring. Let  $f(v)$  denote the sum of the colors of a vertex  $v$  and the colors of all incident edges of  $v$ . A total  $k$ -neighbor sum distinguishing-coloring of  $G$  is a total  $k$ -coloring of  $G$  such that for each edge  $uv \in E(G)$ ,  $f(u) \neq f(v)$ . The smallest number  $k$  is called the *neighbor sum distinguishing total chromatic number*, denoted by  $\chi''_{\Sigma}(G)$ . Pilśniak and Woźniak conjectured that for any graph  $G$  with maximum degree  $\Delta(G)$  holds that  $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$ . This conjecture has been proved for complete graphs, cycles, bipartite graphs, subcubic graphs, sparse graphs, series parallel graphs and planar graphs with  $\Delta \geq 14$ . We prove for a graph  $G$  with maximum degree  $\Delta(G)$  which can be embedded in a surface  $\Sigma$  of Euler characteristic  $\chi(\Sigma) \geq 0$ , then  $\chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 2, 16\}$ .

Lastly, we study the linear  $L$ -choosable arboricity of graph. A linear forest is a graph in which each component is a path. The *linear arboricity*  $la(G)$  of a graph  $G$  as defined by Harary is the minimum number of linear forests in  $G$ , whose union is the set of all edges of  $G$ . Akiyama et al. posed the

following conjecture: For any regular graph  $G$ ,  $la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ . Clearly,  $la(G) \geq \lceil \frac{\Delta(G)}{2} \rceil$ . So for every regular graph  $G$ , we have  $la(G) \geq \lceil \frac{\Delta(G)+1}{2} \rceil$ . Hence, the conjecture above is equivalent to the linear arboricity conjecture: For any simple graph  $G$ ,  $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ . A list assignment  $L$  to the edges of  $G$  is the assignment of a set  $L(e) \subseteq N$  of colors to every edge  $e$  of  $G$ , where  $N$  is the set of positive integers. If  $G$  has a coloring  $\varphi(e)$  such that  $\varphi(e) \in L(e)$  for every edge  $e$  and  $(V(G), \varphi^{-1}(i))$  is a linear forest for any  $i \in C_\varphi$ , where  $C_\varphi = \{\varphi(e) | e \in E(G)\}$ , then we say that  $G$  is linear  $L$ -colorable and  $\varphi$  is a linear  $L$ -coloring of  $G$ . We say that  $G$  is linear  $k$ -choosable if it is linear  $L$ -colorable for every list assignment  $L$  satisfying  $|L(e)| \geq k$  for all edges  $e$ . The list linear arboricity  $la_{list}(G)$  of a graph  $G$  is the minimum number  $k$  for which  $G$  is linear  $k$ -list colorable. It is obvious that  $la(G) \leq la_{list}(G)$ . In chapter 5, we prove that if  $G$  is a planar graph such that every 7-cycle of  $G$  contains at most two chords, then  $G$  is linear  $\lceil \frac{\Delta+1}{2} \rceil$ -choosable if  $\Delta(G) \geq 6$ , and  $G$  is linear  $\lceil \frac{\Delta}{2} \rceil$ -choosable if  $\Delta(G) \geq 11$ .

Chapter 6 is the conclusion of the thesis. We give some graphs that can be studied in the future and we show some graph coloring problems for future works.

**Keywords:** Total coloring; List coloring; Neighbor sum distinguishing total coloring, linear  $L$ -choosable arboricity.

Quelques problèmes de coloration du graphe

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### Résumé

La théorie des graphes est un domaine de recherche actif depuis 200 ans. Le plus ancien article de théorie des graphes connu a été rédigé par Euler en 1736, pour résoudre le problème dit des ponts de Königsberg. La coloration de graphe est l'une des branches les plus importantes de la théorie des graphes, depuis l'émergence du fameux problème des 4 couleurs. La coloration de graphe a des applications pratiques dans l'optimisation, l'informatique et la conception de réseau. Dans la présente thèse nous allons étudier le coloriage total, le coloriage par liste, le coloriage total somme-des-voisins-distinguant et l'arboricité linéaire  $L$ -sélectionnable.

Tous les graphes abordés dans la thèse sont simples, non-orientés et finis. Soit  $G = (V, E)$  un graphe. Pour une sommet  $v \in V(G)$ , soit  $N_G(v)$  l'ensemble des voisins de  $v$  dans  $G$  and soit  $d_G(v) = |N_G(v)|$  le degré de  $v$  dans  $G$ . Le degré maximum et le degré minimum de  $G$  sont notés respectivement  $\Delta(G)$  et  $\delta(G)$ . On simplifiera par la suite  $\Delta = \Delta(G)$  et  $\delta = \delta(G)$ .

Un  $k$ -coloriage total d'un graphe  $G$  est un coloriage de  $V(G) \cup E(G)$  utilisant  $(1, 2, \dots, k)$  couleurs tel qu'aucune paire d'éléments adjacents ou incidents ne reçoivent la même couleur. Le nombre chromatique total  $\chi''(G)$  est le plus petit entier  $k$  tel que  $G$  admette un  $k$ -coloriage total. Dès les

années 1960, Vizing et Behzad ont conjecturé indépendamment que pour tout graphe  $G$ ,  $\Delta \leq \chi''(G) \leq \Delta + 2$ . Cette conjecture est connue sous le nom de Total Coloring Conjecture. Elle a été confirmée pour les graphes quelconques tels que  $\Delta \leq 5$ . Pour les graphes planaires, le seul cas encore ouvert est quand  $\Delta = 6$ . Il est intéressant de remarquer que pour de nombreux graphes planaires,  $\chi''(G) = \Delta + 1$ , c.-à-d que la relation exacte est connue. Par exemple, pour tout graphe  $\Delta \geq 9$ ,  $\chi''(G) = \Delta + 1$ . Pour les graphes planaires tels que  $4 \leq \Delta \leq 8$ , personne n'a réussi à trouver de graphe qui ne soient pas  $(\Delta + 1)$ -totally-colorable. Wang Yingqian et al. ont donc conjecturé que les graphes planaires avec  $\Delta \geq 4$  sont donc  $(\Delta + 1)$ -totalement-coloriable. Dans le chapitre 2, nous étudions la coloration totale de graphe planaires et obtenons 3 résultats : (1) Soit  $G$  un graphe planaire avec pour degré maximum  $\Delta \geq 8$ . Si toutes les paires de 6-cycles cordaux ne sont pas adjacentes dans  $G$ , alors  $\chi''(G) = \Delta + 1$ . (2) Soit  $G$  un graphe planaire avec pour degré maximum  $\Delta \geq 8$ . Si tout 7-cycle de  $G$  contient au plus deux cordes, alors  $\chi''(G) = \Delta + 1$ . (3) Soit  $G$  un graphe planaire sans 5-cycles cordaux qui s'intersectent, c'est à dire tel que tout sommet ne soit incident qu'à au plus un seul 5-cycle cordal. Si  $\Delta \geq 7$ , alors  $\chi''(G) = \Delta + 1$ .

Une relation  $L$  est appelé assignation pour un graphe  $G$  s'il met en relation chaque sommet  $v \in V(G)$  à une liste de couleur. S'il est possible de colorier  $G$  tel que la couleur de chaque sommet soit présente dans la liste qu'il lui a été assignée, et qu'aucune paire de sommets adjacents n'aient la même couleur, alors on dit que  $G$  est  $L$ -coloriable. Un graphe  $G$  est  $k$ -selectionnable si  $G$  est  $L$ -coloriable pour toute assignation  $L$  de  $G$  qui satisfie  $|L(v)| \geq k$  pour tout sommet  $v \in V(G)$ . Nous démontrons que si chaque 5-cycle de  $G$  n'est pas simultanément adjacent à des 3-cycles et des 4-cycles, alors  $G$  est 4-sélectionnable. Une relation  $L$  est appelée une *assignation totale* d'un graphe

$G$  si elle assigne une liste  $L(x)$  de couleurs à chaque élément  $x \in V(G) \cup E(G)$ . Étant donné une assignation totale  $L$  pour  $G$ , une coloration  $L$ -totale de  $G$  est une coloration totale propre tel que la couleur de chaque élément soit présente dans la liste qui lui a été assignée. Un graphe  $G$  est  $k$ -total-sélectionnable si  $G$  a une coloration propre  $L$ -totale pour toute assignation totale  $G$  telle que  $|L(x)| \geq k$  pour tout  $x \in V \cup E$ . La sélectionnabilité totale de  $G$ , ou nombre chromatique total de liste de  $G$ , noté  $\chi_l''(G)$ , est le plus petit nombre entier  $k$  tel que  $G$  soit  $k$ -totalement sélectionnable. Le nombre arête-chromatique  $\chi'(G)$  et le nombre arête-chromatique de liste (ou arête-sélectionnabilité)  $\chi_l'(G)$  sont défini de manière similaire en ne coloriant que les arêtes. Dans le chapitre 3, nous prouvons que si aucun des 5-cycles de  $G$  n'est adjacent à un 4-cycles, alors  $\chi_l'(G) = \Delta$  et  $\chi_l''(G) = \Delta + 1$  si  $\Delta(G) \geq 8$ , et  $\chi_l'(G) \leq \Delta + 1$  et  $\chi_l''(G) \leq \Delta + 2$  si  $\Delta(G) \geq 6$ .

Récemment, les colorations avec étiquetage magique et antimagique, avec poids non uniforme, etc, et les colorations liés à la “somme” des couleurs ont reçu beaucoup d'attention. Parmi ces recherches ont trouvé la Conjecture 1 – 2 – 3 et Conjecture 1 – 2 . Dans le chapitre 4, nous allons fournir une définition du coloriage total somme-des-voisins-distinguant, et passer en revue les progrès et conjecture concernant ce type de coloriage. Soit  $f(v)$  la somme des couleurs d'un sommet  $v$  et des toutes les arêtes incidentes à  $v$ . Un  $k$ -coloriage total somme-des-voisins-distinguant de  $G$  est un  $k$  coloriage total de  $G$  tel que pour chaque arête  $uv \in E(G)$ ,  $f(u) \neq f(v)$ . Le plus petit  $k$  tel qu'on ait un tel coloriage sur  $G$  est appelé le *nombre chromatique total somme-des-voisins-distinguant*, noté  $\chi_{\Sigma}''(G)$ . Pilśniak et Woźniak ont conjecturé que pour tout graphe  $G$  avec degré maximum  $\Delta(G)$ , on a  $\chi_{\Sigma}''(G) \leq \Delta(G) + 3$ . Cette conjecture a été prouvée pour les graphes complets, , les graphes cycles, les graphes bipartis, les graphes subcubiques, les graphes

creux (sparse graphs), les graphes séries parallèles et les graphes planaires avec  $\Delta \geq 13$ . Nous avons démontré que si un graphe  $G$  avec degré maximum  $\Delta(G)$  peut être embedded dans une surface  $\Sigma$  de caractéristique eulérienne  $\chi(\Sigma) \geq 0$ , alors  $\chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 2, 16\}$ .

Pour finir, nous étudions l'arborescence  $L$ -sélectable linéaire d'un graphe. Une forêt linéaire est un graphe pour lequel chaque composante connexe est une chemin. L'arboricité linéaire  $la(G)$  d'un graphe  $G$  tel que définie par Harary est le nombre minimum de forêts linéaires dans  $G$ , dont l'union est égale à  $V(G)$ . Akiyama et al. ont proposé la conjecture suivante : Pour tout graphe régulier  $G$ ,  $la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ . Clairement,  $la(G) \geq \lceil \frac{\Delta(G)}{2} \rceil$ . Donc pour tout graphe régulier  $G$ , on a que  $la(G) \geq \lceil \frac{\Delta(G)+1}{2} \rceil$ . Ainsi, la conjecture précédente est équivalente à la conjecture de linéarité arborescente, qui s'énonce ainsi : Pour tout graphe simple  $G$ ,  $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ . Une assignation par liste  $L$  pour les arrêtes de  $G$  est l'assignation d'un ensemble  $L(e) \subseteq \mathbb{N}$  de couleurs à chaque arête  $e$  de  $G$ . Si  $G$  admet une coloration  $\varphi(e)$  tel que  $\varphi(e) \in L(e)$  pour toute arête  $e$  et  $(V(G), \varphi^{-1}(i))$  est une forêt linéaire pour tout  $i \in C_{\varphi}$ , où  $C_{\varphi} = \{\varphi(e) | e \in E(G)\}$ , alors on dit que  $G$  est linéairement  $L$ -colorable et  $\varphi$  est une  $L$ -coloration linéaire de  $G$ . On dit que  $G$  est linéairement  $k$ -sélectionnable si il est linéairement  $L$ -colorable poru toute assignation par liste  $L$  satisfiant  $|L(e)| \geq k$  pour toutes les arêtes  $e$  de  $G$ . L'arborescence linéaire de liste  $la_{list}(G)$  d'un graphe  $G$  est le nombre minimum  $k$  tel que  $G$  soit linéairement  $k$ -liste colorable. Il est évident que  $la(G) \leq la_{list}(G)$ . Dans le chapitre 5, nous prouvons que si  $G$  est une graphe planaire tel que tout 7-cycle de  $G$  contienne au plus deux cordes, alors  $G$  est linéairement  $\lceil \frac{\Delta+1}{2} \rceil$ -sélectionnable si  $\Delta(G) \geq 6$ , et  $G$  est linéairement  $\lceil \frac{\Delta}{2} \rceil$ -sélectionnable si  $\Delta(G) \geq 11$ .

Le chapitre 6 est la conclusion de cette thèse. Nous fournissons quelques

problèmes de coloration de graphe pour des travaux futurs.

**Mots clefs:** Coloration totale; Coloration par liste; Coloriage total somme-des-voisins-distinguant; Arboricité linéaire  $L$ -déterminable.



Symbol Description

$G$	Graph
$K_n$	Complete graph with $n$ vertices
$K_{m,n}$	Complete bipartite graph with partitions of size $ V_1  = m$ and $ V_2  = n$
$P_n$	Path with $n$ vertices
$C_n$	Cycle with $n$ vertices
$V(G)$	The vertex set of graph $G$
$E(G)$	The edge set of graph $G$
$F(G)$	The face set of graph $G$
$N_G(v)$	The neighbor vertices of $v$ in $G$
$d_G(v)$	Degree of vertex $v$ in $G$
$d_G(f)$	Degree of face $f$ in $G$
$\delta(G)$	Minimum degree of $G$
$\Delta(G)$	Maximum degree of $G$
$n_d$	The number of vertices which have degree $d$
$n_d(v)$	The number of vertices which have degree $d$ and adjacent to $v$
$n_d(f)$	The number of vertices which have degree $d$ and incident with $f$
$f_d(v)$	The number of faces which have degree $d$ and incident with $v$
$ X $	The number of $X$
$\lfloor x \rfloor$	Largest integer not larger than $x$
$\lceil x \rceil$	Least integer not less than $x$
$g(G)$	The girth of $G$
$mad(G)$	Maximum average degree of $G$
$\mu(G)$	Multiplicity of $G$

$\chi(G)$	Vertex chromatic number of $G$
$\chi'(G)$	Edge chromatic number of $G$
$\chi''(G)$	Total chromatic number of $G$
$\chi_l(G)$	List vertex chromatic number of $G$
$\chi'_l(G)$	List edge chromatic number of $G$
$\chi''_l(G)$	List total chromatic number of $G$
$\chi''_a(G)$	Adjacent vertex distinguishing total chromatic number of $G$
$\chi''_{\Sigma}(G)$	Neighbor sum distinguish total chromatic number of $G$
$la(G)$	Linear arboricity of $G$
$la_{list}(G)$	List linear arboricity of $G$

# Chapter 1 Introduction

Graph is a powerful tool to establish mathematical model. Graph theory is a branch of the research object of graph in mathematics. Four color problem, Chinese postman problem, Hamilton problem are the classic problems in graph theory. Graph theory is penetrated into other natural science and social science in various aspects, including computer science, physics, chemistry, biology, genetics, psychology, economics, management science and so on. At the same time, the graph theory itself has made great progress. It is closely connected with other mathematical disciplines. In many areas graph theory formed a cross discipline which plays a role in promoting each other.

The study of graph theory started over two hundreds years ago. The earliest known paper is due to Euler (1736) about the seven bridges of Königsberg. Frederick Guthrie(1833-1886) proposed four color map problem: A planar map is a set of pairwise disjoint subsets of the plane, called regions. A simple map is one whose regions are connected open sets. Two regions of a map are adjacent if their respective closures have a common point that is not a corner of the map. A point is a corner of a map if and only if it belongs to the closures of at least three regions. Then we get the theorem: The regions of any simple planar map can be colored with only four colors, in such a way that any two adjacent regions have different colors. The four color theorem was proved in 1976 by Kenneth Appel and Wolfgang Haken. It was the first major theorem to be proved using a computer. Appel and Haken's approach started by showing that there is a particular set of 1,936 maps, each of which cannot be part of a smallest-sized counterexample to the four color theorem. (If they did appear, you could make a smaller counter-example.) Appel and Haken used a special-purpose computer pro-

gram to confirm that each of these maps had this property. Additionally, any map that could potentially be a counterexample must have a portion that looks like one of these 1,936 maps. Showing this required hundreds of pages of hand analysis. Appel and Haken concluded that no smallest counterexamples exist because any must contain, yet do not contain, one of these 1,936 maps. This contradiction means there are no counterexamples at all and that the theorem is therefore true. Initially, their proof was not accepted by all mathematicians because the computer-assisted proof was infeasible for a human to check by hand. Since then the proof has gained wider acceptance, although doubts remain. Since then more and more scholars began to pay attention to the coloring problems, graph coloring theory has developed rapidly and become a classic content of graph theory. The four color problem is essentially a graph coloring problem, which indicates the direction of graph coloring theory. The graph coloring theory is an important branch of graph theory. Moreover, it has important applications in the field of optimization, computer science, network design, including the schedule problem, storage problem, task arrangement and so on. At the same time, the data transmission in the network, the problem of Hessians matrix calculation, can be directly or indirectly into graph coloring problems.

Since 1960s, graph theory has developed very fast and numerous results on graph theory sprung forth. There are many nice and celebrated problems in graph theory. As a subfield in discrete mathematics, graph theory has attracted much attention from all sides. In this thesis, we study the total coloring, list coloring, neighbor sum distinguish total coloring and list linear coloring.

In this chapter, we give a short but relatively complete introduction. It is divided into four sections. Some basic definition and notations are given

in Section 1.1. We describe some categories of graph in Section 1.2. We introduce some progress of above four graph coloring in Section 1.3.

### §1.1 Basic Definitions and Notations

A *graph*  $G$  is an ordered tuple  $(V(G), E(G))$  and an incidence function  $\psi_G$  that associates with each edge of  $G$  an unordered pair of (not necessarily distinct) vertices of  $G$ , where  $V(G)$  is a nonempty set,  $E(G)$  is a set of disjoint from  $V(G)$ . If  $e$  is an edge,  $u$  and  $v$  are vertices such that  $\psi(e) = uv$ , then  $e$  is said to *join*  $u$  and  $v$ , and the vertices  $u$  and  $v$  are called the *ends* of  $e$ . A *loop* is an edge with identical ends. Two edges (which are not loops) are said to be *parallel* if they have the same pair of ends. A graph is *simple* if it has neither loops nor parallel edges. If the vertex set and edge set of a graph  $G$  are finite set, then it is called a *finite graph*. If the vertex set and edge set of a graph  $G$  are empty set, then it is called an *empty graph*.

All graphs considered in this thesis are simple, finite and undirected. If  $S$  is a set, we shall denote by  $|S|$  the cardinality of  $S$ . For a vertex  $v \in V$ , let  $N_G(v)$  denote the set of vertices adjacent to  $v$  and let  $d_G(v) = |N(v)|$  denote the *degree* of  $v$ . Set  $\delta(G) = \min\{d_G(v) : v \in V(G)\}$ , the minimum degree of  $G$ , and  $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$ , the maximum degree of  $G$ . A  $k$ -*vertex*, a  $k^-$ -*vertex* or a  $k^+$ -*vertex* is a vertex of degree  $k$ , at most  $k$  or at least  $k$ , respectively. If there is no confusion, we use  $V$ ,  $E$ ,  $d(v)$ ,  $\delta$ ,  $\Delta$ ,  $N(v)$  instead of  $V(G)$ ,  $E(G)$ ,  $d_G(v)$ ,  $\delta(G)$ ,  $\Delta(G)$ ,  $N_G(v)$ , respectively.

A graph is *connected* when there is a path between every pair of vertices. A *subgraph*  $H = (V(H), E(H))$  of a graph  $G$  is a graph with  $V(H) \subseteq V(G)$ ,  $E(H) \subseteq E(G)$ . We write  $H \subseteq G$  is a subgraph of  $G$ . Given a nonempty subset  $V'$  of  $V(G)$ , the subgraph with vertex set  $V'$  and edge set  $\{uv \in E(G) | u, v \in V'\}$  is called the subgraph of  $G$  *induced* by  $V'$ , denoted  $G[V']$ .

We say that  $G[V']$  is an *induced subgraph* of  $G$ . A *spanning subgraph* of  $G$  is a subgraph  $H$  with  $V(H) = V(G)$ .

## §1.2 Some Special graphs

A *walk* in  $G$  is a finite non-null sequence  $W := v_0e_1v_1e_2v_2 \cdots e_kv_k$ , whose terms are alternately vertices and edges of  $G$  (not necessarily distinct), such that the ends of  $e_i$  ( $1 \leq i \leq k$ ) are  $v_{i-1}$  and  $v_i$ . We say that  $v_0$  and  $v_k$  are connected by  $W$ . If the edge  $e_1, e_2, \dots, e_k$  of a walk  $W$  are distinct,  $W$  is called a *trail*. In addition, if the vertices  $v_1, v_2, \dots, v_k$  are distinct,  $W$  is called a *path*. If  $v_0 = v_k$  ( $k \geq 2$ ) and  $v_1, v_2, \dots, v_{k-1}$  are distinct, then  $W$  is called a *cycle*. The length of a path or a cycle is the number of its edges. A path or a cycle of length  $k$  is called a  $k$ -path or  $k$ -cycle, respectively; the path or cycle is *odd* or *even* according to the parity of its length. The *girth* of a graph  $G$  is the length of a shortest cycle contained in the graph, denoted by  $g(G)$ . Let  $C = (v_1, v_2, \dots, v_k)$  ( $k \geq 4$ ) be a cycle. If there is an edge  $v_iv_j$  with  $j \not\equiv i \pm 1 \pmod{k}$ , then the edge  $v_iv_j$  is called to be a *chord* of  $C$ . We say that two cycles are *adjacent* (or *intersecting*) if they share at least one edge (or one vertex, respectively).

A *tree* is an undirected graph in which any two vertices are connected by exactly one path; that is to say, a tree has no cycles. A *forest* is an undirected graph, all of whose connected components are trees; in other words, the graph consists of a disjoint union of trees. A *linear forest* is a graph in which each component is a path.

A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. If a complete graph is of order  $n$ , we denote it by  $K_n$ .

A graph is *bipartite* if there exists a partition  $(X, Y)$  of  $V(G)$  such that

each edge of  $G$  has one end in  $X$  and one end in  $Y$ , such a partition  $(X, Y)$  is called a bipartition of graph  $G$ . Equivalently, a graph is bipartite if it does not contain any odd-cycle. A *complete bipartite* graph is a bipartite graph with bipartition  $(X, Y)$  in which each vertex of  $X$  is joined to each vertex of  $Y$ . Let  $K_{m,n}$  denote the complete bipartite graph such that  $|X| = m$  and  $|Y| = n$ .

*Surfaces* in this paper are compact, connected 2-dimensional manifolds without boundary. All embedded graphs considered in this paper are *2-cell-embeddings*.

The *Euler characteristic* is a topological invariant, a number that describes a topological space's shape. It is commonly denoted by  $\chi$ . The Euler characteristic  $\chi$  was classically defined for the surfaces of polyhedra, according to the formula  $\chi = |V(G)| - |E(G)| + |F(G)|$ , where  $|V(G)|$  is the number of vertices,  $|E(G)|$  is the number of edges,  $|F(G)|$  is the number of faces. If  $G$  be a graph which can be embedded in a surface of nonnegative Euler characteristic, then  $|V(G)| - |E(G)| + |F(G)| \geq 0$ .

A *planar* graph is a graph which can be embedded in a plane in such a way that no two edges intersect geometrically except at a vertex to which they are both incident. It can be drawn on the plan such that its edges intersect only at their ends. Such a drawing is called a *planar embedding* of the graph. Give a planar embedding of a planar graph, it divided the plan into a set of connected regions, including an outer unbounded connected region. A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of  $K_5$  or  $K_{3,3}$ . For a face  $f$  of  $G$ , the *degree*  $d(f)$  is the number of edges incident with it, where each cut-edge is counted twice. A *k-face*, a *k<sup>-</sup>-face* or a *k<sup>+</sup>-face* is a face of degree  $k$ , at most  $k$  or at least  $k$ , respectively. We use  $(v_1, v_2, \dots, v_d)$  to denote a face (or a cycle) whose

boundary vertices are  $v_1, v_2, \dots, v_d$  in the clockwise order. For convenience, we denote by  $n_d(v)$  the number of  $d$ -vertices adjacent to the vertex  $v$ ,  $n_d(f)$  the number of  $d$ -vertices incident with the face  $f$ , and  $f_d(v)$  the number of  $d$ -faces incident with  $v$ .

Euler's formula states that if a finite, connected, planar graph is drawn in the plane without any edge intersections), then

**Theorem 1.2.1.** (*Euler's formula*)

$$|V| - |E| + |F| = 2$$

The “discharging method” is used to prove that every graph in a certain class contains some subgraph from a specified list. The presence of the desired subgraph is then often used to prove a coloring result. Most commonly, discharging is applied to planar graphs. Initially, a charge is assigned to each face and each vertex of  $V(G) \cup F(G)$ . The charges are assigned so that they sum to a small positive number. However, each discharging rule maintains the sum of the charges. During the discharging the charge at each face or vertex may be redistributed to nearby faces and vertices, as required by a set of discharging rules. The rules are designed so that after the discharging phase each face or vertex with positive charge lies in one of the desired subgraphs. Since the sum of the charges is positive, some face or vertex must have a positive charge. Many discharging arguments use one of a few standard initial charge functions (these are listed below). Successful application of the discharging method requires creative design of discharging rules.

In all figures of the thesis, vertices marked  $\bullet$  have no edges of  $G$  incident with them other than those shown and vertices marked  $\circ$  are  $3^+$ -vertices.

The terminology and notation used but undefined in this paper can be found in [14].

### §1.3 Some coloring problems of graphs

#### 1. Vertex coloring

A  $k$ -vertex-coloring of a graph  $G$  is a mapping  $c: V(G) \rightarrow \{1, 2, \dots, k\}$ , if no two adjacent vertices are assigned a same color. The minimum  $k$  for which a graph  $G$  is  $k$ -vertex-colorable is called its *chromatic number*, denoted by  $\chi := \chi(G)$ . Obviously, thinking about the vertex coloring we just need to consider when  $G$  is a simple graph. Vertex coloring of infinite graphs with a finite number of colors, can always be reduced to finite instances. If the chromatic number of a simple graph is at most 2, whether the simple graph a bipartite graph is in polynomial time. When  $k \geq 3$ , Karp [71] proved whether a simple graph is  $k$ -colorable is NP-hard. Brooks observed that every graph  $G$  may be colored by  $\Delta(G) + 1$  colors, where  $\Delta(G)$  is the maximum degree of  $G$ , and he characterized the graphs for which  $\Delta(G)$  colors are not enough.

**Theorem 1.3.1.** (*Brooks Theorem*) [23]

$\chi(G) \leq \Delta(G) + 1$  holds for every graph  $G$ . Moreover,  $\chi(G) = \Delta(G) + 1$  if and only if either  $\Delta(G) \neq 2$  and  $G$  has a complete  $(\Delta(G) + 1)$ -graph  $K_{\Delta(G)+1}$  as a connected component, or  $\Delta(G) = 2$  and  $G$  has an odd cycle as a connected component.

#### 2. Edge coloring

A  $k$ -edge-coloring of a graph  $G$  is a mapping  $c: E(G) \rightarrow \{1, 2, \dots, k\}$ , if two adjacent edges are assigned distinct color. The minimum  $k$  for which a graph  $G$  is  $k$ -edge-colorable is called its *chromatic number*, denoted by  $\chi' := \chi'(G)$ . Ensure the exact edge chromatic number of a graph is a very difficult problem. Holyer [56] proved this problem is a NP-complete problem for the graph with  $\Delta(G) \geq 3$ . Obviously,  $\chi' \geq \Delta$ . The breakthrough was the theorem of Vizing [113] obtained independently by Gupta [51].

**Theorem 1.3.2.** (*Vizing Theorem*, [113], [51])

Let  $G$  be a multigraph of multiplicity  $\mu(G)$ , then  $\chi' \leq \Delta + \mu$ .

From this result one can prove the theorem of Shannon,

**Theorem 1.3.3.** (*Shannon Theorem*, [100])

Let  $G$  be a graph, then  $\chi' \leq \frac{3\Delta}{2}$ .

It follows immediately from Vizing's theorem that  $\chi'(G)$  is equal to either  $\Delta(G)$  or  $\Delta(G) + 1$  when  $G$  is simple.

### 3. Total coloring

Vizing and Behazd posed independently the definition of total coloring.

If a mapping  $\phi : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$  satisfied these three condition below:

- (i) If two vertices  $v_1, v_2 \in V(G)$  are adjacent, and  $\phi(v_1) \neq \phi(v_2)$ ;
- (ii) If two edges  $e_1, e_2 \in E(G)$  are adjacent, and  $\phi(e_1) \neq \phi(e_2)$ ;
- (iii) If vertex  $v$  and edge  $e$  are incident in  $G$ , and  $\phi(v) \neq \phi(e)$

we see  $\phi$  is a proper *total-k-coloring* of a graph  $G$ .

A proper *total-k-coloring* of a graph  $G$  is a coloring of  $V(G) \cup E(G)$  using  $k$  colors such that no two adjacent or incident elements receive the same color. The *total chromatic number*  $\chi''(G)$  is the smallest integer  $k$  such that  $G$  has a total-k-coloring.

Obvious,  $\chi''(G) \geq \Delta + 1$ .

Behzad [9] in 1965 and Vizing [114] in 1968 posed independently the famous conjecture, known as the Total Coloring Conjecture(TCC):

**Conjecture 1.3.1.** For any graph  $G$ ,  $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$ .

(Actually, Vizing posed is a general total coloring conjecture: Let  $G$  be a multigraph of multiplicity  $\mu(G)$ , then  $\chi''(G) \leq \Delta(G) + \mu(G) + 1$ )

Clearly, the lower bound is trivial. The upper bound has been unsolved completely.

1971, Rosenfeld [95] and Vijayaditya [112] use different method to prove TCC is true for the graph of  $\Delta(G) \leq 3$ . If graph  $G$  satisfy  $\chi''(G) = \Delta + 2$ , we say  $G$  is class one. If graph  $G$  satisfy  $\chi''(G) = \Delta + 1$ , we say  $G$  is class two. Yap [158] proved that if  $n$  is odd, then  $\chi''(K_n) = n$ , or  $\chi''(K_n) = n + 1$ . Chew and Yap [38], Hoffman and Rodger [55] proved independently that for any complete  $r$ - partite  $G$ , where  $r$  is odd, is class one. As research continues, TCC is proved true for interval graphs [11], series-parallel graphs [156], Halin graphs [163]. For graph of maximum degree  $\Delta \leq 2$  graph,  $\chi''(G) \leq \Delta + 2$  is obvious. 1977, Kostochka [72] proved if  $G$  is a simple graph of maximum degree  $\Delta = 4$ , then  $\chi''(G) \leq 6$ , that is to say TCC is true. Kostochka [73] in 1996 proved for any simple graph  $G$  with  $\Delta = 5$ ,  $\chi''(G) \leq \Delta + 2$ . Michael and Bruce [86] in 1998 proved if the maximum degree  $\Delta$  of  $G$  is large enough, then  $\chi''(G) \leq \Delta + C$ ,  $C$  is a larger constant. This conjecture has not been solved for graphs  $G$  when  $\Delta(G) \geq 6$ . But for maximum degree  $\Delta \geq 6$  of planar graphs, it has obtained several related results. Borodin [16] in 1989 proved TCC is true for maximum degree  $\Delta(G) \geq 9$  of planar graphs, then Yap [158] reduce the bound of maximum degree to 8, Sanders and Zhao [98] in 1999 proved that TCC is true for maximum degree  $\Delta(G) = 7$  of planar graphs. For the planar graph with  $\Delta(G) = 6$ , (a)if there does not contain  $k$ -cycle with chords, where  $k \in \{4, 5, 6\}$ , then TCC holds [57]; (b) if  $5^-$ -cycles are not adjacent, then TCC holds; (c) if  $v_5^4 + 2(v_5^{5^+} + v_6^4) + 3v_6^5 + 4v_6^{6^+} < 24$ , where  $v_n^k$  represents the number of vertices of degree  $n$  which lie on  $k$  distinct triangle [76].

Through the study constantly in recent years, the researchers found that a lot of classes of  $G$  not only meet Total Coloring Conjecture, but

also their total chromatic number could get the lower bound, that is to say  $\chi''(G) = \Delta + 1$ . In the process of total coloring, it has been found the examples of non  $(\Delta + 1)$ -total colorable when  $\Delta \leq 3$ , such as  $K_2$ ,  $(3k + 2)$ -cycle (where  $k \geq 1$ ),  $K_4$ , but it has not been found the examples of non  $(\Delta + 1)$ -total colorable when  $4 \leq \Delta \leq 8$ . Arroyo [97] proved to determine whether a graph satisfies  $\chi''(G) = \Delta + 1$  is a NP-C problem. McDiarmid and Arroyo [85] further point out for any fixed  $k \geq 3$ , determine whether a  $k$ -regular bipartite  $G$  satisfies  $\chi''(G) = \Delta + 1$  is a NP-C problem. Li et al. [79] proved that for Halin graphs of maximum degree is 4, the total chromatic number is 5. Jianliang Wu [156] proved that for series parallel graphs of minimum degree at least 3, the total chromatic number is  $\Delta + 1$ . In 1987, Borodin [15] proved that for graph  $G$  with maximum degree  $\Delta(G) \geq 16$  holds  $\chi''(G) = \Delta + 1$ . In 1989, he himself improved the result to  $\Delta(G) \geq 14$  holds  $\chi''(G) = \Delta + 1$  [16]. In 1997, he improved to  $\Delta(G) \geq 12$  [18], moreover,  $\Delta(G) = 11$  [20]. In 2007, Weifan Wang [138] improved the result to  $\Delta(G) = 10$  holds  $\chi''(G) = \Delta + 1$ . In 2008, Kowalik et al. [74] improved the conclusion to  $\Delta(G) = 9$ . As so far, planar graphs with  $\Delta = 4, 5, 6, 7, 8$  have not been proved  $\chi''(G) = \Delta + 1$  completely. Yingqian Wang and Lan Shen [101] posed the Total Coloring Conjecture for planar graphs, i.e. PTCC:

**Conjecture 1.3.2.** *For any planar graph  $G$  with  $4 \leq \Delta \leq 8$ , then  $\chi''(G) = \Delta + 1$ .*

For the total coloring of planar graphs with  $\Delta = 4, 5, 6, 7, 8$ , there are some conclusions as follows in restriction conditions:

**Theorem 1.3.4.** *Let  $G$  be a planar graph, the girth of  $G$  is  $g$ , maximum degree is  $\Delta$ , if  $G$  satisfy one of the conditions below, then  $\chi''(G) = \Delta + 1$ .*

- (1)  $\Delta \geq 8$  and for every vertex  $v \in V(G)$ , there is an integer  $k \in \{3, 4, 5, 6, 7, 8\}$

- such that  $v$  is incident with at most one cycle of length  $k$  [132] ;*
- (2)  $\Delta \geq 8$  and for each vertex  $v \in V(G)$ , there are two integers  $i, j \in \{3, 4, 5\}$  such that any two cycles of length  $i$  and  $j$ , which contain  $v$ , are not adjacent [119] ;
- (3)  $\Delta \geq 8$  is an  $F_5$ -free [26] ;
- (4)  $\Delta \geq 8$  contains no 5-cycles with two chords [29] ;
- (5)  $\Delta \geq 8$  contains no adjacent chordal 5-cycles [111] ;
- (6)  $\Delta \geq 7$  and for every vertex  $v \in V(G)$ , there is an integer  $k \in \{3, 4, 5, 6, 7, 8\}$  such that  $v$  is incident with no cycles of length  $k$  [25] ;
- (7)  $\Delta \geq 7$  and no 3-cycle is adjacent to a cycle of length less than 6 [116] ;
- (8)  $\Delta \geq 7$  and  $G$  contains no intersecting 3-cycles [117] ;
- (9)  $\Delta \geq 7$  and  $G$  contains no adjacent 4-cycles [124] ;
- (10)  $\Delta \geq 7$  and  $G$  contains no intersecting 5-cycles [118] ;
- (11)  $\Delta \geq 7$  and  $G$  contains no chordal 5-cycles [145] ;
- (12)  $\Delta \geq 7$  and  $G$  contains no chordal 6-cycles [119] ;
- (13)  $\Delta \geq 7$  and  $G$  contains no chordal 7-cycles [24] ;
- (14)  $\Delta \geq 6$  and  $G$  contains no adjacent  $5^-$ -cycles [159] ;
- (15)  $\Delta \geq 6$  and  $G$  contains no 4-cycles [101] ;
- (16)  $\Delta \geq 6$  and  $G$  contains no chordal 5-cycles and 6-cycles [157] ;
- (17)  $\Delta \geq 6$  and  $G$  contains no intersecting 4-cycles and  $G$  contains no intersecting 3-cycles, or 5-cycles, or 6-cycles [109] ;
- (18)  $\Delta \geq 5$  and  $G$  contains no 4-cycles and 6-cycles [57] ;
- (19)  $(\Delta, g) \in \{(7, 4), (5, 5), (4, 6)\}$  [21] ;
- (20)  $(\Delta, k) \in \{(6, 5), (5, 7), (4, 14)\}$ , where  $G$  has no cycle of length from 4 to  $k$  [135] ;
- (21)  $(\Delta, k) \in \{(5, 5), (4, 11)\}$ , where  $G$  contains no intersecting 3-cycles and  $G$  has no cycles of length from 4 to  $k$  [107] ;

That is to say, the graphs satisfying the above conditions is the second class.

We will give the relative results of total coloring of planar graphs in Chapter 2.

#### 4. List coloring

A mapping  $L$  is said to be an *assignment* for a graph  $G$  if it assigns a list  $L(v)$  of colors to each vertex  $v \in V(G)$ . If it is possible to color  $G$  so that every vertex gets a color from its list and no two adjacent vertices receive the same color, then we say that  $G$  is  *$L$ -colorable*. A graph  $G$  is  *$k$ -choosable* if  $G$  is  $L$ -colorable for any assignment  $L$  with  $|L(v)| \geq k$  for every vertex  $v \in V(G)$ .

Choosability of planar graphs has been extensively studied. Thomassen [110] proved that every planar graph is 5-choosable. Voigt [115] and Mirzakhani [87] presented examples of non-4-choosable planar graphs, respectively. Moreover, Gutner [52] investigated that determining a planar graph whether 4-choosable is NP-hard. So, finding nice sufficient conditions for a planar graph to be 4-choosable is of interest. It is shown that  $G$  is 4-choosable if it is a planar graph without 4-cycles [75], 5-cycles [141], 6-cycles [48], 7-cycles [47], intersecting triangles [137] and 4-cycles adjacent to 3-cycles [17].

Let  $G$  be a planar graph, we prove that if every 5-cycle of  $G$  is not simultaneously adjacent to 3-cycles and 4-cycles, then  $G$  is 4-choosable in chapter 3.

The mapping  $L$  is said to be an *edge assignment* for the graph  $G$  if it assigns a list  $L(x)$  of possible colors to each element  $x \in E(G)$ . If  $G$  has a proper edge coloring  $\varphi$  such that  $\varphi(e) \in L(e)$  for all  $e \in E(G)$ , then we say that  $G$  is an *edge colorable*. We say that  $G$  is *edge- $L$ -choosable* if it is edge-

$L$ -colorable for every edge assignment  $L$  satisfying  $|L(e)| \geq k$  for all elements  $x \in E(G)$ . The *list edge chromatic number*  $\chi'_l(G)$  of  $G$  is the smallest integer  $k$  such that  $G$  is edge- $L$ -choosable when  $|L(e)| \geq k$  for all elements  $e \in E(G)$ . We can obtain:  $\chi'_l(G) \geq \chi'(G) \geq \Delta$ .

As a generalization of the classical coloring of graphs, list coloring has been extensively studied, and one of the famous conjectures is the list coloring conjecture.

**Conjecture 1.3.3.** *If  $G$  is a multigraph, then  $\chi'_l(G) = \chi'(G)$ .*

The list edge coloring conjecture was formulated by Vizing, Gupta, Abertson and Collins [67], Bollobás and Harris [12], and it is well known as the List Coloring Conjecture. Vizing's theorem give the upper bound  $\Delta(G) + 1$  of the edge chromatic number  $\chi'(G)$ , so if the list edge coloring conjecture holds, then  $\chi'_l(G) \leq \Delta(G) + 1$  directly. The list edge coloring has been extensively studied and a large number of results have been obtained in the planar graph.

**Theorem 1.3.5.** *Let  $G$  be a planar graph, the maximum degree of  $G$  is  $\Delta$ , if  $G$  satisfies one of the conditions below, then  $\chi'_l(G) = \Delta + 1$ .*

- (1)  $\Delta(G) \geq 9$  [22];
- (2)  $\Delta(G) = 8$  [13];
- (3)  $\Delta(G) \geq 7$  and  $G$  contains no chordal 7-cycles [27];
- (4)  $\Delta(G) \geq 7$  and  $G$  contains no chordal 6-cycles [50];
- (5)  $\Delta(G) \geq 6$  and  $G$  contains no adjacent 3-cycles [59];
- (6)  $\Delta(G) \geq 6$  and the 3-cycle of  $G$  is not adjacent to 5-cycle [84];
- (7)  $\Delta(G) \geq 6$  and  $G$  contains no chordal 5-cycles [59];
- (8)  $\Delta(G) \geq 6$  and  $G$  contains no chordal 6-cycles [42];
- (9)  $\Delta(G) \geq 5$  and  $G$  contains no 3-cycle [160];

(10)  $\Delta(G) \geq 5$  and  $G$  contains no 4-cycles [104];

(11)  $\Delta(G) \geq 5$  and  $G$  contains no 5-cycles [142].

Borodin et al. proved that if  $G$  is a graph which can be embedded in a surface of Euler characteristic and  $\Delta \geq 12$ , then  $\chi'_i(G) = \Delta$  [19]. We have similar conclusions in planar graphs.

**Theorem 1.3.6.** *Let  $G$  be a planar graph, the maximum degree of  $G$  is  $\Delta$ , the girth is  $g$ , if  $G$  satisfies one of the conditions below, then  $\chi'_i(G) = \Delta$ .*

(1)  $\Delta(G) \geq 8$  and the 3-cycle of  $G$  is not adjacent to 4-cycle [80];

(2)  $\Delta(G) \geq 8$  and the 3-cycle of  $G$  is not adjacent to 5-cycle [84];

(3)  $\Delta(G) \geq 8$  and  $G$  contains no adjacent 4-cycles [134];

(4)  $\Delta(G) \geq 8$  and  $G$  contains no chordal 5-cycles [130];

(5)  $\Delta(G) \geq 7$  and the 4-cycle of  $G$  is not adjacent to  $4^-$ -cycle [125];

(6)  $\Delta(G) \geq 7$  and  $G$  contains no  $4^-$ -cycles [84];

(7)  $\Delta(G) \geq 7$  and  $G$  contains no 5-cycles and 6-cycles [82];

(8)  $\Delta(G) \geq 6$  and  $G$  contains no 4-cycles and 6-cycles [59];

(9)  $(\Delta(G), g) \in \{(7, 4), (6, 5), (5, 8), (4, 14)\}$ , where  $g$  is the girth of  $G$  [19];

(10)  $(\Delta(G), k) \in \{(7, 4), (5, 5), (4, 6), (3, 10)\}$ , where  $k$  satisfies that  $G$  has no cycle of length from 4 to  $k$  [58].

Similarly, we can give the definition of the list total chromatic number  $\chi''_i(G)$  of  $G$ , identically, the formula below holds:  $\chi''_i(G) \geq \chi'(G) \geq \Delta + 1$ . For list total coloring, we have a well known conjecture, too.

**Conjecture 1.3.4.** *For any graph  $G$ , we have  $\chi''_i(G) = \chi''(G)$ .*

The list total coloring conjecture was formulated independently by Borodin, Kostochka and Woodall [19], Juvan, Mohar and Šekovski [68], and it is well known as the List Total Coloring Conjecture. TCC conjecture that the up-

per bound of total chromatic number  $\chi''(G)$  is  $\Delta(G) + 2$ , so if the list total coloring conjecture holds, we have  $\chi_l''(G) \leq \Delta(G) + 2$ .

**Theorem 1.3.7.** *Let  $G$  be a planar graph, the maximum degree is  $\Delta$ , if  $G$  satisfies one of the condition below, then  $\chi_l''(G) \leq \Delta + 2$ .*

- (1)  $\Delta(G) \geq 9$  [60];
- (2)  $\Delta(G) \geq 7$  and  $G$  contains no chordal 7-cycles [43];
- (3)  $\Delta(G) \geq 7$  and  $G$  contains no adjacent 3-cycles [81];
- (4)  $\Delta(G) \geq 6$  and the 3-cycle of  $G$  is not adjacent to 4-cycles [80];
- (5)  $\Delta(G) \geq 6$  and the 3-cycle of  $G$  is not adjacent to 5-cycles [84];
- (6)  $\Delta(G) \geq 6$  and  $G$  contains no chordal 6-cycles [42];
- (7)  $\Delta(G) \geq 5$  and  $G$  contains no 3-cycles [160];
- (8)  $\Delta(G) \geq 5$  and  $G$  contains no 4-cycles [104];
- (9)  $\Delta(G) \geq 5$  and  $G$  contains no 5-cycles [142].

Similarly, Borodin et al. proved that if  $G$  is a graph which can be embedded in a surface of Euler characteristic and  $\Delta \geq 12$ , then  $\chi_l''(G) = \Delta + 1$  [19]. We have similar conclusions in planar graphs.

**Theorem 1.3.8.** *Let  $G$  be a planar graph, the maximum degree is  $\Delta$ , if  $G$  satisfies one of the condition below, then  $\chi_l''(G) = \Delta + 1$ .*

- (1)  $\Delta(G) \geq 8$  and the 3-cycle of  $G$  is not adjacent to 4-cycles [80];
- (2)  $\Delta(G) \geq 8$  and the 3-cycle of  $G$  is not adjacent to 5-cycles [84];
- (3)  $\Delta(G) \geq 8$  and  $G$  contains no adjacent 4-cycles [134];
- (4)  $\Delta(G) \geq 8$  and  $G$  contains no chordal 5-cycles [130];
- (5)  $\Delta(G) \geq 7$  and the 4-cycle of  $G$  is not adjacent to  $4^-$ -cycle [125];
- (6)  $\Delta(G) \geq 7$  and  $G$  contains no  $4^-$ -cycles [84];
- (7)  $\Delta(G) \geq 7$  and  $G$  contains no 5-cycles and 6-cycles [82];
- (8)  $\Delta(G) \geq 6$  and  $G$  contains no 4-cycles and 6-cycles [59];
- (9)  $(\Delta(G), g) \in \{(7, 4), (6, 5), (5, 8), (4, 14)\}$ , where  $g$  is the girth of  $G$  [19].

Let  $G$  be a planar graph with maximum degree  $\Delta$ , if every 5-cycles of  $G$  is not adjacent to 3-cycles or is not intersecting to 4-cycles, we prove that  $\chi'_l(G) = \Delta$  and  $\chi''_l(G) = \Delta + 1$  if  $\Delta(G) \geq 8$ , and  $\chi'_l(G) \leq \Delta + 1$  and  $\chi''_l(G) \leq \Delta + 2$  if  $\Delta(G) \geq 6$ , where  $\chi'_l(G)$  and  $\chi''_l(G)$  denote the list edge chromatic number and list total chromatic number of  $G$ , respectively. We will illustrate in Chapter 3.

### 5. Neighbor sum distinguishing total coloring

In the study of irregular networks, usually assigned each of the edges of graph  $G$  a positive integer, making for each vertices of graph  $G$ , its associated edge weights are different. This method effectively promoted the development of the theory of graph coloring, and produced a lot of branches, including “vertices distinguishing edge coloring, adjacent vertex distinguishing edge coloring, adjacent vertex distinguishing total coloring”. Recently, colorings and labellings concerning the sums of the colors have received much attention. The family of such problems includes, e.g. vertex-coloring  $k$ -edge-weighting (Kalkowski et al. [69]), total weight choosability (Przybyło and Woźniak [93]; Wong and Zhu [146]), magic and antimagic labellings (Huang et al. ([65]; Wong and Zhu [147]) and the irregularity strength (Przybyło [90] [91]).

Given a total- $k$ -coloring  $\phi$  of  $G$ , let  $C_\phi(v)$  denote the set of colors of the edges incident to  $v$  and the color of  $v$ . This total  $k$ -coloring is called adjacent vertex distinguishing, or it is a total- $k$ -avd-coloring for short, if for each edge  $uv$ ,  $C_\phi(u)$  is different from  $C_\phi(v)$ . The smallest  $k$  is called the adjacent vertex distinguishing total chromatic number, denoted by  $\chi''_a(G)$ . Zhang et al. [164] proposed that the following conjecture.

**Conjecture 1.3.5.** (*Adjacent vertex distinguishing total coloring conjecture*)  
 For any graph  $G$  with at least two vertices,  $\chi''_a(G) \leq \Delta(G) + 3$ .

Conjecture 1.3.5 was confirmed for graphs with maximum degree at most three independently by Chen [35], Wang [122] and Hulgan [66] Wang and Wang [144] proved that this conjecture holds for outerplanar graphs and Wang and Wang [143] proved that  $K_4$ -minor free graphs satisfied this conjecture. Recently Huang and Wang proved that Conjecture 1.3.5 holds for planar graphs with maximum degree at least 11 [63] and they also proved that  $\chi''_a(G) \leq \Delta(G) + 2$  for planar graphs with maximum degree at least 14 [140]. And Huang et al. [64] also proved for graphs with maximum degree  $\Delta \geq 3$ , we have  $\chi''_a(G) \leq 2\Delta$ .

An *edge  $k$ -weighting* of  $G$  is a function  $w : E(G) \rightarrow [k] := \{1, 2, \dots, k\}$ . An edge  $k$ -weighting  $w$  is a *proper vertex coloring by sums* if  $\sum_{e \ni u} w(e) \neq \sum_{e \ni v} w(e)$  for every  $uv \in E(G)$ . Denote by  $\chi^e_\Sigma := \chi^e_\Sigma(G)$  the smallest value of  $k$  such that a graph  $G$  has a edge  $k$ -weighting which is a proper vertex coloring by sums. In 2004, Karoński, Łuczak and Thomason [70] introduced the following conjecture:

**Conjecture 1.3.6.** (1-2-3 Conjecture) *If graph  $G$  has no connected component isomorphic to  $K_2$ , then  $\chi^e_\Sigma \leq 3$ .*

Addario-Berry, Dalal, McMiarmid, Reed and Thomason in 2007 [2] showed that every graph without isolated edges admits a vertex-coloring 30-edge-weighting, equivalent,  $\chi^e_\Sigma \leq 30$ . This bound was improved to 16 by Addario-Berry, Dalal and Reed [3], equivalent,  $\chi^e_\Sigma \leq 16$ . And later improved to 13 by Tao Wang and Qinglin Yu [136], equivalent,  $\chi^e_\Sigma \leq 13$ . Recently, Kalkowski, Karoński and Pfender in 2010 [69] showed that every graph without isolated edges admits a vertex-coloring 5-edge-weighting, equivalent,  $\chi^e_\Sigma \leq 5$ .

Researchers consider those where addition of edge weights as the coloring method is replaced by another operation. In particular, consider variations

where colors are obtained by taking a product, set, multiset of weights from edges incident to  $v$  for each  $v \in V(G)$ . If such a coloring is proper, then the edge  $k$ -weighting of  $G$  is a proper vertex coloring by products, sets, multisets, respectively. Denoted by  $\chi_{\Pi}^e, \chi_s^e$  and  $\chi_m^e$ . For a graph  $G$ ,  $\chi_{\Pi}^e \leq 4$ , see [105];  $\chi_s^e = \lceil \log_2 \chi \rceil + 1$ , see [53];  $\chi_m^e \leq 4$ , see [1]. More information in [10].

A total weighting of a graph  $G$  is an assignment of a real number weight to each  $e \in E(G)$  and each  $v \in V(G)$ . In 2010, Przybyło and Woźniak [92] proposed the following conjecture:

**Conjecture 1.3.7.** (1-2 Conjecture) For any graph  $G$ ,  $\chi_{\Sigma}^t(G) \leq 2$

Given a total  $k$ -weighting of  $G$ , we consider vertex coloring obtained by taking either the product, set, multiset of weights taken from the edges incident to  $v$  and from  $v$  itself for each  $v \in V(G)$ . If such a coloring is proper, then the total  $k$ -weighting of  $G$  is a *proper vertex coloring* by products, sets, multisets, respectively. The smallest values of  $k$  for which a proper coloring of each type exists for a graph  $G$  are denoted  $\chi_{\Pi}^t, \chi_s^t$  and  $\chi_m^t$ .

In a total  $k$ -coloring of  $G$ , let  $f(v)$  denote the total sum of colors of the edges incident to  $v$  and the color of  $v$ . If for each edge  $uv$ ,  $f(u) \neq f(v)$ , we call such total  $k$ -coloring a *total  $k$  neighbor sum distinguishing coloring*. The smallest number  $k$  is called the *neighbor sum distinguishing total chromatic number*, denoted by  $\chi_{\Sigma}''(G)$ .

For neighbor sum distinguishing total colorings, we have the following conjecture due to Przybyło and Woźniak [89].

**Conjecture 1.3.8.** (*Neighbor sum distinguishing total coloring conjecture*) For any graph  $G$  with at least two vertices,  $\chi_{\Sigma}''(G) \leq \Delta(G) + 3$ .

We could see Conjecture 1.3.8 is proposed according to Conjecture 1.3.5, for a graph  $G$ , if it has a  $k$ -total coloring  $\phi$ . For any adjacent vertices

$u, v \in V(G)$ , if  $C_\phi(u) = C_\phi(v)$  holds, then  $f(u) = f(v)$ . Hence, if the  $k$ -total coloring of  $G$  is neighbor sum distinguishing, then it must be adjacent distinguishing. So if Conjecture 1.3.8 can be proved, then Conjecture 1.3.5 will be a corollary. Przybyło and Woźniak [89] have already proved Conjecture 1.3.8 is true in complete graph, cycles and bipartite graphs. Dong and Wang [41] proved Conjecture 1.3.8 is true for graphs with bounded maximum average degree  $mad \leq \frac{2}{5}$ . Li et al. [78] proved Conjecture 1.3.8 holds for  $K_4$ -minor free graphs  $G$ , moreover, if  $\Delta(G) \geq 4$ , then,  $\chi''_\Sigma(G) \leq \Delta(G) + 2$ . By using the famous Combinatorial Nullstellensatz, Ding et al. [40] proved that  $\chi''_\Sigma(G) \leq 2\Delta(G) + col(G) - 1$ , where  $col(G)$  is the coloring number of  $G$ . Later Ding et al. improved this bound to  $\Delta(G) + 2col(G) - 2$ . Moreover they proved this assertion in its list version. Cheng et al. [37] proved that  $\chi''_\Sigma(G) \leq \Delta(G) + 2$  for planar graph  $G$  with  $\Delta(G) \geq 14$ .

In Chapter 4, it is proved that the total neighbor sum distinguishing chromatic number of  $G$  is  $\Delta(G) + 2$  if  $\Delta(G) \geq 14$ , where  $\Delta(G)$  is the maximum degree of  $G$ .

## 6. List linear arboricity

A *linear forest* is a graph in which each component is a path. A map  $\varphi$  from  $E(G)$  to  $\{1, 2, \dots, t\}$  is called a *t-linear coloring* if the induced subgraph of edges having the same color  $\alpha$  is a linear forest for  $1 \leq \alpha \leq t$ . The linear arboricity  $la(G)$  of a graph  $G$  defined by Harary [54] is the minimum number  $t$  for which  $G$  has a  $t$ -linear coloring. For a real number  $x$ ,  $\lceil x \rceil$  is the least integer not less than  $x$  and  $\lfloor x \rfloor$  is the largest integer not larger than  $x$ . Akiyama et al. [4] conjectured that  $la(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$  for any simple regular graph  $G$ . The conjecture is equivalent to the following conjecture.

**Conjecture 1.3.9.** (*Linear arboricity conjecture*) For any graph  $G$ ,  $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ .

The linear arboricity has been determined for complete bipartite graphs [4], complete regular multipartite graphs [154], Halin graphs [148], series-parallel graphs [153] and regular graphs with  $\Delta = 3, 4$  [5] and  $5, 6, 8$  [46].

For planar graphs, more results about the linear arboricity are obtained. Conjecture 1.3.10 has already been proved to be true for all planar graphs (see [149] and [152]). Wu [149] proved that for a planar graph  $G$  with girth  $g$  and maximum degree  $\Delta$ ,  $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$  if  $\Delta(G) \geq 13$ , or  $\Delta(G) \geq 7$  and  $g \geq 4$ , or  $\Delta(G) \geq 5$  and  $g \geq 5$ ,  $\Delta(G) \geq 3$  and  $g \geq 6$ . Recently, M. Cygan et al. [39] proved that if  $G$  is a planar graph with  $\Delta \geq 9$ , then  $la(G) = \lceil \frac{\Delta}{2} \rceil$ , and then they posed the following conjecture (the conjecture has also been posed in [150]):

**Conjecture 1.3.10.** (*Linear arboricity conjecture for planar graph*) For any planar graph  $G$  of maximum degree  $\Delta \geq 5$ ,  $la(G) = \lceil \frac{\Delta}{2} \rceil$ .

There are more partial results to support the conjecture 1.3.10. The linear arboricity of a planar graph  $G$  is  $\lceil \frac{\Delta}{2} \rceil$  if it satisfies one of the following conditions: (1)  $\Delta(G) \geq 7$  and  $G$  contains no 5-cycles with two chords [34]; (2)  $\Delta(G) \geq 7$  and  $G$  contains no chordal  $i$ -cycles for some  $i \in \{4, 5, 6, 7\}$  ([32, 33, 123]); (3)  $\Delta \geq 7$  and for each vertex  $v \in V(G)$ , there exist two integers  $i_v, j_v \in \{3, 4, 5, 6, 7, 8\}$  such that any two  $i_v, j_v$ -cycles incident with  $v$  are not adjacent ([31, 133]); (4)  $\Delta \geq 5$  and  $G$  contains no 4-cycles ([151]); (5)  $\Delta \geq 5$  and  $G$  has no intersecting 4-cycles and intersecting 5-cycles ([30]); (6)  $\Delta \geq 5$  and  $G$  has no 5-, 6-cycles with chords ([33]); (7)  $\Delta \geq 5$  and any 4-cycle is not adjacent to an  $i$ -cycle for any  $i \in \{3, 4, 5\}$  or  $G$  has no intersecting 4-cycles and intersecting  $i$ -cycles for either  $i = 3$  or  $i = 6$  ([108]); (8)  $\Delta \geq 5$  and any two 4-cycles are not adjacent, and any 3-cycle is not adjacent to a 5-cycle ([128]).

A list assignment  $L$  to the edges of  $G$  is the assignment of a set  $L(e) \subseteq N$

of colors to every edge  $e$  of  $G$ , where  $N$  is the set of positive integers. If  $G$  has a coloring  $\varphi(e)$  such that  $\varphi(e) \in L(e)$  for every edge  $e$  and  $(V(G), \varphi^{-1}(i))$  is a linear forest for any  $i \in C_\varphi$ , where  $C_\varphi = \{\varphi(e) | e \in E(G)\}$ , then we say that  $G$  is linear  $L$ -colorable and  $\varphi$  is a linear  $L$ -coloring of  $G$ . We say that  $G$  is linear  $k$ -choosable if it is linear  $L$ -colorable for every list assignment  $L$  satisfying  $|L(e)| = k$  for all edges  $e$ . The list linear arboricity  $la_{list}(G)$  of a graph  $G$  is the minimum number  $k$  for which  $G$  is linear  $k$ -list colorable. It is obvious that  $la(G) \leq la_{list}(G)$ . In [7] and [150], the following conjecture is posed independently.

**Conjecture 1.3.11.** (*List linear arboricity conjecture*) For any graph  $G$ ,  $la(G) = la_{list}(G)$ .

Very few results are known about the conjecture 1.3.11. An and Wu [7] proved by using the results of [149] that  $la_{list}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$  for any planar graph having  $\Delta(G) \geq 9$ , and for a planar graph  $G$ ,  $la_{list}(G) = \lceil \frac{\Delta(G)}{2} \rceil$  if  $\Delta(G) \geq 13$ , or  $\Delta(G) \geq 7$  and  $G$  contains no  $i$ -cycles for some  $i \in \{3, 4, 5\}$ . In [162],  $la_{list}(G) \leq \max\{4, \lceil \frac{\Delta(G)+1}{2} \rceil\}$  and  $la_{list}(G) = \lceil \frac{\Delta}{2} \rceil$  if  $\Delta(G) \geq 11$  for a  $F_5$ -free planar graph  $G$ .

In Chapter 5, we prove that if  $G$  is a planar graph such that every 7-cycle of  $G$  contains at most two chords, then  $G$  is linear  $\lceil \frac{\Delta+1}{2} \rceil$ -choosable if  $\Delta(G) \geq 6$ , and  $G$  is linear  $\lceil \frac{\Delta}{2} \rceil$ -choosable if  $\Delta(G) \geq 11$ .

## §1.4 Main results

In Chapter 2, we use the formula of Euler and discharging method to study the total coloring problems of planar graphs, and prove the following conclusions:

**Conclusion 1** Let  $G$  be a planar graph with maximum degree  $\Delta \geq 8$ . If every 6-cycle of  $G$  contains at most one chord in  $G$ , then  $\chi''(G) = \Delta + 1$ .

**Conclusion 2** Let  $G$  be a planar graph with maximum degree  $\Delta \geq 8$ . If chordal 6-cycles are not adjacent in  $G$ , then  $\chi''(G) = \Delta + 1$ .

**Conclusion 3** Let  $G$  be a planar graph with maximum degree  $\Delta \geq 8$ . If every 7-cycle of  $G$  contains at most two chords, then  $\chi''(G) = \Delta + 1$ .

**Conclusion 4** Let  $G$  be a planar graph without intersecting chordal 5-cycles, that is, every vertex is incident with at most one chordal cycle of length 5. If  $\Delta \geq 7$ , then  $\chi''(G) = \Delta + 1$ .

Conclusion 1, Conclusion 2, Conclusion 3 and Conclusion 4 are all about total coloring of planar graphs, these conclusions solve the total coloring of planar graph conjecture (PTCC) partly. Conclusion 1, Conclusion 2 and Conclusion 3 are aimed at planar graphs with  $\Delta \geq 8$ . Conclusion 1 covers the planar graphs with  $\Delta \geq 8$  and contain no 6-cycles [62]; Conclusion 2 covers the planar graphs with  $\Delta \geq 8$  and contain no chordal 6-cycles [102], or contain no intersecting 6-cycles [132], or contain no adjacent 6-cycles [119], or contain no intersecting chordal 6-cycles [131]; Conclusion 3 covers the planar graphs with  $\Delta \geq 8$  and contain no 7-cycles [96] or contain no chordal 7-cycle [129], this conclusion prove every 7-cycle of  $G$  contains at most two chords directly, covered every 7-cycle of  $G$  contains at most one chords naturally. Conclusion 4 is directed against planar graphs with  $\Delta \geq 7$ . It covers the planar graphs with  $\Delta \geq 7$  and contain no 5-cycles [103], or contain no chordal 5-cycles [145] or contain no intersecting 5-cycles [118]. Equivalently promote planar graphs with  $\Delta \geq 8$  and contain no intersecting chordal 5-cycles.

In Chapter 3, according the properties of planar graphs, we study the list vertex coloring, list edge coloring and list total coloring and get the following conclusions:

**Conclusion 5** Let  $G$  be a planar graph. If every 5-cycles of  $G$  is not adjacent simultaneously to 3-cycles and 4-cycles, then  $G$  is 4-choosable.

Conclusion 5 is in regard to list vertex coloring. It prove that the planar graph is 4-choosable either if every 3-cycles of  $G$  is not adjacent to 5-cycles or if every 4-cycles of  $G$  is not adjacent to 5-cycles. The conclusion improved the 4-choosable planar graph if contain no 4-cycles [75], or 5-cycles [141].

**Conclusion 6** Let  $G$  be a planar graph with  $\Delta \geq 8$ , if every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $\chi'_l(G) = \Delta$ .

**Conclusion 7** Let  $G$  be a planar graph with  $\Delta \geq 8$ , if every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $\chi''_l(G) = \Delta + 1$ .

**Conclusion 8** Let  $G$  be a planar graph with  $\Delta \geq 6$ , if every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $\chi'_l(G) \leq \Delta + 1$ .

**Conclusion 9** Let  $G$  be a planar graph with  $\Delta \geq 6$ , if every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $\chi''_l(G) \leq \Delta + 2$ .

Conclusion 6 and Conclusion 8 are aimed at list edge coloring. They solve the list edge coloring conjecture of planar graphs to some extent. Conclusion 7 and Conclusion 9 concern about list total coloring. They prove the list total coloring conjecture of planar graphs to a certain degree.

In Chapter 4, we consider the total neighbor sum distinguishing chromatic number of embedded in a surface  $\Sigma$  of Euler characteristic, and obtain two corollaries:

**Conclusion 10** Let  $G$  be a graph with maximum degree  $\Delta(G)$  which can be embedded in a surface  $\Sigma$  of Euler characteristic  $\chi(\Sigma) \geq 0$ , then  $\chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 2, 16\}$ .

**Conclusion 11** Let  $G$  be a graph which can be embedded in a surface  $\Sigma$  of Euler characteristic  $\chi(\Sigma) \geq 0$ . If  $\Delta(G) \geq 14$ , then  $\chi''_{\Sigma}(G) \leq \Delta(G) + 2$ .

**Conclusion 12** Let  $G$  be a graph which can be embedded in a surface

$\Sigma$  of Euler characteristic  $\chi(\Sigma) \geq 0$ . If  $\Delta(G) \geq 14$ , then  $\chi_a''(G) \leq \Delta(G) + 2$ .

Conclusion 10 is the first conclusion about total neighbor sum distinguishing chromatic number of embedded in a surface non negative  $\Sigma$  of Euler characteristic. It is the best result of this kind of graph at present. Meanwhile, this conclusion include some results of planar graphs.

In Chapter 5, we investigate the list linear arboricity of planar graph, and have the following conclusions:

**Conclusion 13** Let  $G$  be a planar graph. If every 7-cycles of  $G$  contains at most two chords, then  $\lceil \frac{\Delta(G)}{2} \rceil \leq la_{list}(G) \leq \max\{4, \lceil \frac{\Delta(G)+1}{2} \rceil\}$ .

**Conclusion 14** Let  $G$  be a planar graph. If  $\Delta(G) \geq 6$  and every 7-cycles of  $G$  contains at most two chords, then  $\lceil \frac{\Delta(G)}{2} \rceil \leq la_{list}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ .

**Conclusion 15** Let  $G$  be a planar graph, and every 7-cycles of  $G$  contains at most two chords. Then  $G$  is linear  $k$ -choosable, where  $k \geq \max\{6, \lceil \frac{\Delta(G)}{2} \rceil\}$ .

**Conclusion 16** Let  $G$  be a planar graph. If  $\Delta(G) \geq 11$  and every 7-cycles of  $G$  contains at most two chords, then  $la_{list}(G) = \lceil \frac{\Delta(G)}{2} \rceil$ .

Conclusion 13 and Conclusion 15 are with respect to list linear arboricity of planar graph. The corollaries of Conclusion 14 and Conclusion 16 solve the list linear arboricity conjecture of planar graphs partly.

## Chapter 2 Total Coloring

### §2.1 Basic definitions and properties

A  $k$ -total-coloring of a graph  $G = (V, E)$  is a coloring of  $V \cup E$  using  $k$  colors such that no two adjacent or incident elements receive the same color. A graph  $G$  is *total- $k$ -colorable* if it admits a  $k$ -total-coloring. The *total chromatic number*  $\chi''(G)$  of  $G$  is the smallest integer  $k$  such that  $G$  has a  $k$ -total-coloring. Clearly,  $\chi''(G) \geq \Delta + 1$ . Since more and more planar graphs have been determined the total chromatic number  $\chi''(G) = \Delta + 1$ . In this chapter, we mainly study the total coloring of planar graphs. In the first subsection we discuss the total coloring problems of planar graph  $G$  with  $\Delta \geq 8$ , and in the second subsection we discuss the total coloring problems of planar graph  $G$  with  $\Delta \geq 7$ .

### §2.2 Planar graph $G$ with maximum degree $\Delta \geq 8$

Hou et al. [62] proved:

**Lemma 2.2.1.** *For planar graph  $G$  with  $\Delta \geq 8$ , if it contains no 6-cycles, then  $\chi''(G) = \Delta + 1$ .*

Shen and Wang [102] extended this result:

**Lemma 2.2.2.** *For planar graph  $G$  with  $\Delta \geq 8$ , if it contains no 6-cycles with chords, then  $\chi''(G) = \Delta + 1$ .*

Roussel and Zhu [96] proved

**Lemma 2.2.3.** *For planar graph  $G$  with  $\Delta \geq 8$ , and for each vertex  $x$ , there is an integer  $k_x \in \{3, 4, 5, 6, 7, 8\}$  such that there is no  $k_x$ -cycle which contains  $x$ , then  $\chi''(G) = \Delta + 1$ .*

Wang et al. [131] extended this result:

**Lemma 2.2.4.** *Let  $G$  be a planar graph with  $\Delta \geq 8$  and without adjacent cycles of size  $i$  and  $j$ , for some  $3 \leq i \leq j \leq 5$ ,  $\chi''(G) = \Delta + 1$ .*

These results are all about planar graph with ( $\Delta \geq 8$ ) in order to prove their total chromatic number is ( $\Delta + 1$ ). We extend these result and get the following results: 1. Let  $G$  be a planar graph with maximum degree  $\Delta \geq 8$ . If every 6-cycle of  $G$  contains at most one chord or chordal 6-cycles are not adjacent in  $G$ , then  $\chi''(G) = \Delta + 1$ . 2. Let  $G$  be a planar graph with maximum degree  $\Delta \geq 8$ . If every 7-cycle of  $G$  contains at most two chords, then  $\chi''(G) = \Delta + 1$ .

**§2.2.1 If every 6-cycle of  $G$  contains at most one chord or chordal 6-cycles are not adjacent in  $G$**

**Theorem 2.2.1.** *Let  $G$  be a planar graph with maximum degree  $\Delta \geq 8$ . If every 6-cycle of  $G$  contains at most one chord or chordal 6-cycles are not adjacent in  $G$ , then  $\chi''(G) = \Delta + 1$ .*

According to [74], the theorem 2.2.1 is true for  $\Delta \geq 9$ . So we assume in the following that  $\Delta = 8$ . Let  $G = (V, E)$  be a minimal counterexample to the planar graph  $G$  with maximum degree  $\Delta = 8$ , such that  $|V| + |E|$  is minimal and  $G$  has been embedded in the plane. Then every proper subgraph of  $G$  is total-9-colorable. First we give some lemmas for  $G$ .

**Lemma 2.2.5.** [16] (a)  $G$  is 2-connected.

(b) If  $uv$  is an edge of  $G$  with  $d(u) \leq 4$ , then  $d(u) + d(v) \geq \Delta + 2 = 10$ .

By Lemma 2.2.5(a): There is no 1-vertex in  $G$ . By Lemma 2.2.5(b): any two neighbors of a 2-vertex are 8-vertices. Any three neighbors of a 3-vertex are 7<sup>+</sup>-vertices. Any four neighbors of a 4-vertex are 6<sup>+</sup>-vertices.

**Lemma 2.2.6.**  *$G$  has no configurations depicted in Figure 2.1, where  $v$  denotes the vertex of degree of 7.*

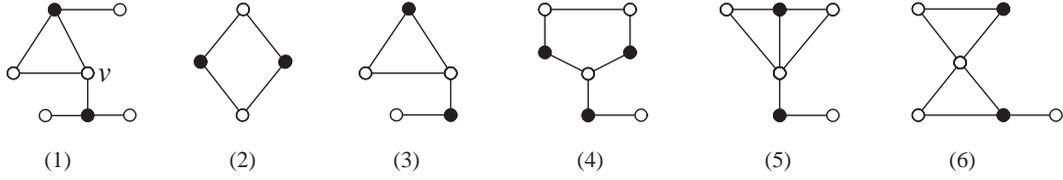


Figure 2.1

*Proof.* The proof of (1), (2), (4) and (6) can be found in [139], (3) can be found in [102], (5) can be found in [74]. □

**Lemma 2.2.7.** *Suppose  $v$  is a  $d$ -vertex of  $G$  with  $d \geq 5$ . Let  $v_1, \dots, v_d$  be the neighbor of  $v$  and  $f_1, f_2, \dots, f_d$  be faces incident with  $v$ , such that  $f_i$  is incident with  $v_i$  and  $v_{i+1}$ , for  $i \in \{1, 2, \dots, d\}$ . Let  $d(v_1) = 2$  and  $\{v, u_1\} = N(v_1)$ . Then  $G$  does not satisfy one of the following conditions (see Figure 2.2).*

- (1) there exists an integer  $k$  ( $2 \leq k \leq d-1$ ) such that  $d(v_{k+1}) = 2$ ,  $d(v_i) = 3$  ( $2 \leq i \leq k$ ) and  $d(f_j) = 4$  ( $1 \leq j \leq k$ ).
- (2) there exist two integers  $k$  and  $t$  ( $2 \leq k < t \leq d-1$ ) such that  $d(v_k) = 2$ ,  $d(v_i) = 3$  ( $k+1 \leq i \leq t$ ),  $d(f_t) = 3$  and  $d(f_j) = 4$  ( $k \leq j \leq t-1$ ).
- (3) there exist two integers  $k$  and  $t$  ( $3 \leq k \leq t \leq d-1$ ) such that  $d(v_i) = 3$  ( $k \leq i \leq t$ ),  $d(f_{k-1}) = d(f_t) = 3$  and  $d(f_j) = 4$  ( $k \leq j \leq t-1$ ).

*Proof.* Suppose  $G$  satisfies all of the conditions (1)-(3). If  $d(f_i) = 4$ , then let  $u_i$  be adjacent to  $v_i$  and  $v_{i+1}$ . By the minimality of  $G$ ,  $G' = G - vv_1$  has a  $(\Delta + 1)$ -total-coloring  $\phi$ . Let  $C(x) = \{\phi(xy) : y \in N(x)\} \cup \{\phi(x)\}$ . First we erase the colors on all  $3^-$ -vertices adjacent to  $v$ . We have  $\phi(v_1u_1) \notin$

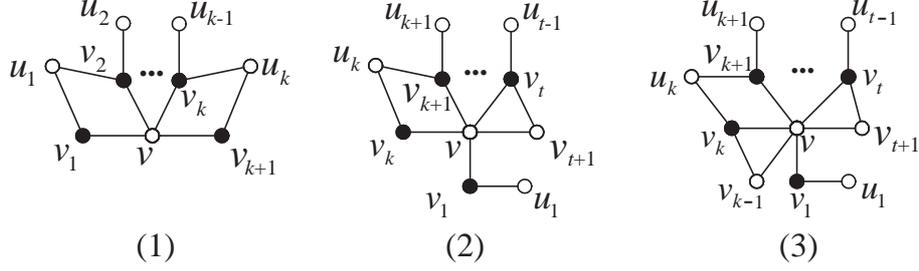


Figure 2.2

$C(v)$ , for otherwise, the number of the forbidden colors for  $vv_1$  is at most  $\Delta$ , so  $vv_1$  can be properly colored and by properly recoloring the erased vertices, we get a  $(\Delta + 1)$ -total-coloring of  $G$ , a contradiction. Without loss of generality, assume that  $C(v) = \{1, 2, \dots, d\}$  with  $\phi(vv_i) = i$  ( $i \in \{2, 3, \dots, d\}$ ),  $\phi(v_1u_1) = d + 1$  and  $\phi(v) = 1$ . Thus we have  $d + 1 \in C(v_i)$  for all  $i \in \{2, 3, \dots, d\}$ , for otherwise, we can recolor  $vv_i$  with  $d + 1$  and color  $vv_1$  with  $i$ , and by properly recoloring the erased vertices, we get a  $(\Delta + 1)$ -total-coloring of  $G$ , a contradiction, too. In the following we consider (1)-(3) one by one.

(1) Since  $d + 1 \in C(v_i)$  for all  $i \in \{2, 3, \dots, d\}$ , there is a vertex  $u_s$  ( $1 \leq s \leq k$ ) such that  $d + 1$  appears at least twice on  $u_s$ , a contradiction to  $\phi$ .

(2) Since  $d + 1 \in C(v_i)$  for all  $i \in \{2, 3, \dots, d\}$ ,  $\phi(v_ku_k) = \phi(v_{k+1}u_{k+1}) = \dots = \phi(v_{t-1}u_{t-1}) = \phi(v_tu_{t+1}) = d + 1$ . We also have  $\phi(v_tu_{t-1}) = t + 1$ . For otherwise, we can recolor  $v_tv_{t+1}$  with  $t + 1$ ,  $vv_{t+1}$  with  $d + 1$  and color  $vv_1$  with  $t + 1$ . By properly recoloring the erased vertices, we get a  $(\Delta + 1)$ -total-coloring of  $G$ , a contradiction. Similarly,  $\phi(v_{t-1}u_{t-2}) = \phi(v_{t-2}u_{t-3}) = \dots = \phi(v_{k+1}u_k) = t + 1$ . So we can recolor  $vv_{t+1}$  with  $d + 1$ ,  $v_tv_{t+1}$  with  $t + 1$ ,  $v_tu_{t-1}$  with  $d + 1$ ,  $v_{t-1}u_{t-1}$  with  $t + 1, \dots, v_{k+1}u_{k+1}$  with  $t + 1$ ,  $v_{k+1}u_k$  with  $d + 1$ ,  $v_ku_k$  with  $t + 1$  and color  $vv_1$  with  $t + 1$ . By properly recoloring the erased

vertices, we get a  $(\Delta + 1)$ -total-coloring of  $G$ , also a contradiction.

(3) If  $d + 1 \notin \{\phi(v_{k-1}v_k) \cup \phi(v_tv_{t+1})\}$ , then there is a vertex  $u_s$  ( $k \leq s \leq t - 1$ ) such that  $d + 1$  appears at least twice on  $u_s$ , a contradiction to  $\phi$ . So without loss of generality, assume  $\phi(v_{k-1}v_k) = d + 1$ . If  $\phi(v_{k+1}u_k) = d + 1$ , then  $\phi(v_{k+2}u_{k+1}) = \phi(v_{k+3}u_{k+2}) = \dots = \phi(v_tu_{t-1}) = d + 1$ . By the discussion of (2), we also have  $\phi(v_ku_k) = \phi(v_{k+1}u_{k+1}) = \dots = \phi(v_{t-1}u_{t-1}) = \phi(v_tv_{t+1}) = k - 1$ . Then we can recolor  $vv_{k-1}$  with  $d + 1$ ,  $v_{k-1}v_k$  with  $k - 1$ ,  $v_ku_k$  with  $d + 1$ ,  $v_{k+1}u_k$  with  $k - 1$ ,  $\dots$ ,  $v_{t-1}u_{t-1}$  with  $d + 1$ ,  $v_tu_{t-1}$  with  $k - 1$ ,  $v_tv_{t+1}$  with  $t + 1$ ,  $vv_{t+1}$  with  $k - 1$  and color  $vv_1$  with  $t + 1$ . By properly recoloring the erased vertices, we get a  $(\Delta + 1)$ -total-coloring of  $G$ , a contradiction. If  $\phi(v_{k+1}u_{k+1}) = d + 1$ , then  $\phi(v_{k+2}u_{k+2}) = \phi(v_{k+3}u_{k+3}) = \dots = \phi(v_{t-1}u_{t-1}) = \phi(v_tv_{t+1}) = d + 1$ . Similarly, we have  $\phi(v_tu_{t-1}) = \phi(v_{t-1}u_{t-2}) = \dots = \phi(v_{k+1}u_k) = t + 1$ . Let  $\phi(v_ku_k) = s$ . Then we can recolor  $vv_{t+1}$  with  $d + 1$ ,  $v_tv_{t+1}$  with  $t + 1$ ,  $v_tu_{t-1}$  with  $d + 1$ ,  $v_{t-1}u_{t-1}$  with  $t + 1$ ,  $\dots$ ,  $v_{k+1}u_{k+1}$  with  $t + 1$ ,  $v_{k+1}u_k$  with  $s$ ,  $v_ku_k$  with  $t + 1$ , and color  $vv_1$  with  $t + 1$ . By properly recoloring the erased vertices, we get a  $(\Delta + 1)$ -total-coloring of  $G$ , a contradiction, too. □

We will use the ‘‘Discharging method’’ to complete the proof of Theorem 2.2.1.

We know, by  $\sum_{v \in V} d(v) = 2|E|$ ,  $\sum_{f \in F} d(f) = 2|E|$  the Euler’s formula  $|V| - |E| + |F| = 2$ , we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

We define  $ch$  the initial charge that  $ch(x) = 2d(x) - 6$  for each  $x \in V$  and  $ch(x) = d(x) - 6$  for each  $x \in F$ . So  $\sum_{x \in V \cup F} ch(x) = -12 < 0$ . In the

following, we will reassign a new charge denoted by  $ch'(x)$  to each  $x \in V \cup F$  according to the discharging rules. If we can show that  $ch'(x) \geq 0$  for each  $x \in V \cup F$ , then we get an obvious contradiction to  $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -12$ , which completes our proof. Now we define the discharging rules as follows.

**R1.** Each 2-vertex receives 1 from each of its neighbors.

**R2.** Let  $f$  be a 3-face. If  $f$  is incident with a  $3^-$ -vertex, then it receives  $\frac{3}{2}$  from each of its two incident  $7^+$ -vertices. If  $f$  is incident with a 4-vertex, then it receives  $\frac{5}{4}$  from each of its two incident  $6^+$ -vertices. If  $f$  is not incident with any  $4^-$ -vertex, then it receives 1 from each of its two incident  $5^+$ -vertices.

**R3.** Let  $f$  be a 4-face. If  $f$  is incident with two  $3^-$ -vertices, then it receives 1 from each of its two incident  $7^+$ -vertices. If  $f$  is incident with only one  $3^-$ -vertex, then it receives  $\frac{3}{4}$  from each of its two incident  $7^+$ -vertices; and  $\frac{1}{2}$  from the left incident  $4^+$ -vertex. If  $f$  is not incident with any  $3^-$ -vertex, then it receives  $\frac{1}{2}$  from each of its incident  $4^+$ -vertices.

**R4.** Each 5-face receives  $\frac{1}{3}$  from each of its incident  $4^+$ -vertices.

Next, we show that  $ch'(x) \geq 0$  for all  $x \in V \cup F$ . It is easy to check that  $ch'(f) \geq 0$  for all  $f \in F$  and  $ch'(v) \geq 0$  for all 2-vertices  $v \in V$  by the above discharging rules. If  $d(v) = 3$ , then  $ch'(v) = ch(v) = 0$ . If  $d(v) = 4$ , then  $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$  by R2 and R3. For  $d(v) \geq 5$ , we need the following structural lemma.

**Lemma 2.2.8.** (1) *Suppose that every 6-cycle of  $G$  contains at most one chord. Then we have the following results.*

(a)  *$G$  has no configurations depicted in Figure 2.3(1), Figure 2.3(2) and Figure 2.3(3);*

(b) *Suppose  $G$  has a subgraph isomorphic to Figure 2.3(4). Then  $d(f_1) \geq 4$  and  $d(f_2) \neq 4$ . More over if  $d(f_1) = 4$ , then  $d(f_2) \geq 5$ ;*

(c) If  $G$  has a subgraph isomorphic to Figure 2.3(5), then  $d(f_1) \geq 5$  and  $d(f_2) \geq 5$ .

(2) Suppose that all chordal 6-cycles are not adjacent. Then we have the following results.

(d) If  $G$  has a subgraph isomorphic to Figure 2.3(5), then  $\max\{d(f_1), d(f_2)\} \geq 4$ ;

(e)  $G$  has no configurations depicted in Figure 2.3(6) and Figure 2.3(7).

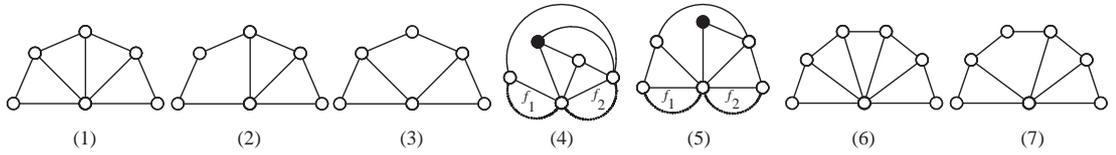


Figure 2.3

Suppose  $d(v) = 5$ . Then  $f_3(v) \leq 4$  by Lemma 2.2.8. If  $f_3(v) = 4$ , then  $f_{6+}(v) \geq 1$ , so  $ch'(v) \geq ch(v) - 1 \times 4 = 0$ . If  $f_3(v) \leq 3$ , then  $ch'(v) \geq ch(v) - 1 \times f_3(v) - \frac{1}{2} \times (5 - f_3(v)) = \frac{3-f_3(v)}{2} \geq 0$ . Suppose  $d(v) = 6$ . Then  $f_3(v) \leq 4$  and  $ch'(v) \geq ch(v) - \frac{5}{4} \times f_3(v) - \frac{1}{2} \times (6 - f_3(v)) = \frac{3(4-f_3(v))}{4} \geq 0$ . Suppose  $d(v) = 7$ . Then  $f_3(v) \leq 5$ . By Lemma 2.2.6(1),  $v$  is incident with at most two 3-faces are incident with a  $3^-$ -vertex, that is,  $v$  sends  $\frac{3}{2}$  to each of the two 3-faces and at most  $\frac{5}{4}$  to each other 3-face. If  $f_3(v) = 5$ , then  $f_{5+}(v) \geq 1$ , and  $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - \frac{5}{4} \times 3 - \frac{3}{4} \times 1 - \frac{1}{3} \times 1 = \frac{1}{6} > 0$ . If  $2 \leq f_3(v) \leq 4$ , then  $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - \frac{5}{4} \times (f_3(v) - 2) - 1 \times (5 - f_3(v)) - \frac{3}{4} \times 2 = \frac{4-f_3(v)}{4} \geq 0$ . If  $f_3(v) \leq 2$ , then  $ch'(v) \geq ch(v) - \frac{3}{2} \times f_3(v) - 1 \times (7 - f_3(v)) = \frac{2-f_3(v)}{2} > 0$ .

Suppose  $d(v) = 8$ . Then  $ch(v) = 10$ . Let  $v_1, \dots, v_8$  be neighbors of  $v$  in the clockwise order and  $f_1, f_2, \dots, f_8$  be faces incident with  $v$ , such that  $f_i$  is incident with  $v_i$  and  $v_{i+1}$ , for  $i \in \{1, 2, \dots, 8\}$ , and  $f_9 = f_1$ .

Suppose  $n_2(v) = 0$ . Then  $f_3(v) \leq 6$ . If  $f_3(v) = 6$ , then  $f_{5+}(v) \geq 2$ , so  $ch'(v) \geq 10 - \frac{3}{2} \times 6 - \frac{1}{3} \times 2 = \frac{1}{3} > 0$ . If  $f_3(v) = 5$ , then  $f_{5+}(v) \geq 1$ ,

so  $ch'(v) \geq 10 - \frac{3}{2} \times 5 - 1 \times 2 - \frac{1}{3} \times 1 = \frac{1}{6} > 0$ . If  $f_3(v) \leq 4$ , then  $ch'(v) \geq 10 - \frac{3}{2} \times f_3(v) - 1 \times (8 - f_3(v)) \geq 0$ .

Suppose  $n_2(v) = 1$ . Without loss of generality, assume  $d(v_1) = 2$ .

Suppose  $v_1$  is incident with a 3-cycle  $f_1$ .

By Lemma 2.2.8,  $f_3(v) \leq 6$  and all 3-faces incident with no  $3^-$ -vertex except  $f_1$  by Lemma 2.2.6(6). If  $f_3(v) = 6$ , then  $f_{5^+}(v) \geq 2$ , so  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - \frac{5}{4} \times 5 - \frac{1}{3} \times 2 = \frac{7}{12} > 0$ . If  $4 \leq f_3(v) \leq 5$ , then  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - \frac{5}{4} \times (f_3(v) - 1) - 1 \times (6 - f_3(v)) - \frac{3}{4} \times 2 = \frac{5 - f_3(v)}{4} \geq 0$ . If  $1 \leq f_3(v) \leq 3$ , then  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - \frac{5}{4} \times (f_3(v) - 1) - 1 \times (8 - f_3(v)) = \frac{3 - f_3(v)}{4} \geq 0$ .

Suppose  $v_1$  is not incident with a 3-cycle.

Suppose every 6-cycle of  $G$  contains at most one chord. Then  $f_3(v) \leq 5$  by Lemma 2.2.6(2)-(4). If  $4 \leq f_3(v) \leq 5$ , then  $f_{5^+}(v) \geq 2$ , so  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times (f_3(v) - 1) - 1 \times 1 - 1 \times (6 - f_3(v)) - \frac{1}{3} \times 2 = \frac{17 - 3f_3(v)}{6} > 0$ . If  $f_3(v) = 3$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 3 - 1 \times 4 - \frac{1}{3} \times 1 = \frac{1}{6} > 0$ . If  $f_3(v) = 2$ , then  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 2 - 1 \times 6 = 0$ . If  $f_3(v) = 1$ , then without loss of generality,  $d(f_2) = 3$ , i.e.  $d(v_3) = 3$  and  $d(v_2) \geq 7$ , so  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - 1 \times 6 - \frac{3}{4} \times 1 = \frac{3}{4} > 0$ . If  $f_3(v) = 0$ , then  $ch'(v) \geq 10 - 1 - 1 \times 8 = 1 > 0$ .

Suppose any two chordal 6-cycles are not adjacent. Then  $f_3(v) \leq 5$  by Lemma 2.2.6(2)-(4). If  $f_3(v) \geq 4$ , then  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 2 - \frac{5}{4} \times (f_3(v)) - \frac{3}{4} \times (8 - f_3(v)) = \frac{5 - f_3(v)}{2} \geq 0$ . If  $f_3(v) = 3$ , then  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 3 - \frac{3}{4} \times 5 = \frac{3}{4} > 0$ . If  $1 \leq f_3(v) \leq 2$ , then  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times f_3(v) - 1 \times (6 - 2f_3(v)) - \frac{3}{4} \times (2 + f_3(v)) = \frac{6 - f_3(v)}{4} > 0$ . If  $f_3(v) = 0$ , then  $ch'(v) \geq 10 - 1 - 1 \times 8 = 1 > 0$ .

Note that next Lemma 2.2.9 is also true for general planar graphs if we just use the above discharging rules.

**Lemma 2.2.9.** *Suppose  $d(v) = 8$  and  $2 \leq n_2(v) \leq 8$ . Then  $ch'(v) \geq 0$ .*

*Proof.* Since  $d(v) = 8$ , then  $ch(v) = 10$ . First we give a Claim for conve-

nience.

**Claim 2.3.** *Suppose that  $d(v_i) = d(v_{i+k+1}) = 2$  and  $d(v_j) \geq 3$  for  $i + 1 \leq j \leq i + k$ . Then  $v$  sends at most  $\phi$  (in total) to  $f_i$  and  $f_{i+1}, f_{i+2}, \dots, f_{i+k}$ , where  $\phi = \frac{5k+1}{4}$  ( $k = 1, 2, 3, 4, 5$ ), see Figure 2.4.*

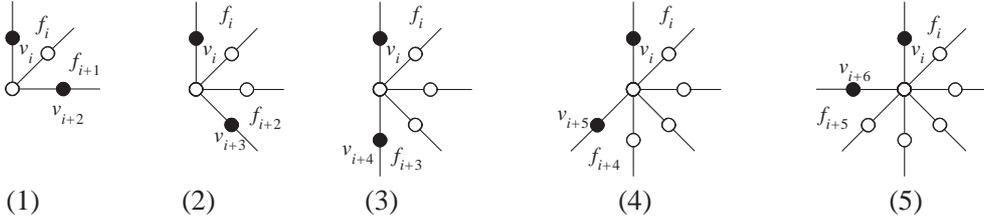


Figure 2.4

By Lemma 2.2.6,  $d(f_i) \geq 4$  and  $d(f_{i+k}) \geq 4$ .

**Case (a).**  $k = 1$  By Lemma 2.2.7(1), we have  $d(v_{i+1}) \geq 4$  or  $\max\{d(f_i), d(f_{i+1})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{4} \times 2, 1 + \frac{1}{3}\} = \frac{3}{2}$  by R3.

**Case (b).**  $k = 2$  If  $d(f_{i+1}) = 3$ , then  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+2})\} \geq 5$  by Lemma 2.2.7(2), and it follows that  $\phi \leq \max\{\frac{3}{4} + \frac{5}{4} + \frac{3}{4}, \frac{1}{3} + \frac{3}{2} + \frac{3}{4}\} = \frac{11}{4}$ . Otherwise,  $d(f_{i+1}) \geq 4$ , then  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2})\} \geq 5$  by Lemma 2.2.7(1), and it follows that  $\phi \leq \max\{1 + \frac{3}{4} \times 2, 1 \times 2 + \frac{1}{3}\} = \frac{5}{2} < \frac{11}{4}$ .

**Case (c).**  $k = 3$  Suppose  $d(f_{i+1}) = d(f_{i+2}) = 3$ . Then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(f_i) \geq 5$  and  $d(f_{i+3}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{1}{3} \times 2 = \frac{11}{3}$ . If  $\min\{d(v_{i+1}), d(v_{i+3})\} \geq 4$ , then  $\phi \leq \frac{5}{4} \times 2 + \frac{3}{4} \times 2 = 4$ . Suppose  $d(f_{i+1}) = 3$  and  $d(f_{i+2}) \geq 4$ . If  $d(v_{i+1}) = 3$ , then  $d(v_{i+2}) \geq 7$  and  $d(f_i) \geq 5$ , so  $\phi \leq \frac{1}{3} + \frac{3}{2} + \frac{3}{4} + 1 = \frac{43}{12}$ . If  $d(v_{i+2}) = 3$ , then  $d(v_{i+1}) \geq 7$  and  $d(v_{i+3}) \geq 4$ , so  $\phi \leq \frac{3}{4} + \frac{3}{2} + \frac{3}{4} + \frac{3}{4} = \frac{15}{4}$ . If  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$ ,  $\phi \leq \frac{3}{4} + \frac{5}{4} + \frac{3}{4} + 1 = \frac{15}{4}$ . It is similar with  $d(f_{i+2}) = 3$  and  $d(f_{i+1}) \geq 4$ .

Suppose  $\min\{d(f_{i+1}), d(f_{i+2})\} \geq 4$ . Then  $\max\{d(v_{i+1}), d(v_{i+2}), d(v_{i+3})\}$

$\geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2}), d(f_{i+3})\} \geq 5$ , so  $\phi \leq \max\{1 \times 2 + \frac{3}{4} \times 2, 1 \times 3 + \frac{1}{3}\} = \frac{7}{2}$ . So  $\phi \leq \max\{\frac{11}{3}, 4, \frac{43}{12}, \frac{15}{4}, \frac{7}{2}\} = 4$ .

**Case (d).**  $k = 4$  Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = 3$ . Then  $\min\{d(v_{i+2}), d(v_{i+3})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$  and  $d(f_{i+4}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + 1 \times 1 + \frac{1}{3} \times 2 = \frac{14}{3}$ . If  $\min\{d(v_{i+1}), d(v_{i+4})\} \geq 4$ , then  $\phi \leq \frac{5}{4} \times 3 + \frac{3}{4} \times 2 = \frac{21}{4}$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = 3, d(f_{i+3}) \geq 4$ . Then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(v_{i+4}) \geq 4$  and  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{3}{4} \times 2 + \frac{1}{3} \times 1 = \frac{29}{6}$ . If  $\min\{d(v_{i+1}), d(v_{i+3})\} \geq 4$ , then  $\phi \leq \frac{5}{4} \times 2 + 1 \times 1 + \frac{3}{4} \times 2 = 5$ . Similar with  $d(f_{i+2}) = d(f_{i+3}) = 3, d(f_{i+1}) \geq 4$ . Suppose  $d(f_{i+1}) = d(f_{i+3}) = 3, d(f_{i+2}) \geq 4$ . Then  $\max\{d(v_{i+2}), d(v_{i+3})\} \geq 4$  by Lemma 2.2.7(3), so  $\phi \leq \frac{3}{2} \times 1 + \frac{5}{4} \times 1 + \frac{3}{4} \times 3 = 5$ .

Suppose  $d(f_{i+1}) = 3, d(f_{i+2}) \geq 4$  and  $d(f_{i+3}) \geq 4$ . If  $d(v_{i+1}) = 3$ , then  $d(v_{i+2}) \geq 7$  and  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + 1 \times 2 + \frac{3}{4} \times 1 + \frac{1}{3} \times 1 = \frac{55}{12}$ . If  $d(v_{i+2}) = 3$ , then  $d(v_{i+1}) \geq 7$  and  $\max\{d(v_{i+3}), d(v_{i+4})\} \geq 4$ , so  $\phi \leq \frac{3}{2} \times 1 + 1 \times 1 + \frac{3}{4} \times 3 = \frac{19}{4}$ . Otherwise,  $\phi \leq \frac{5}{4} \times 1 + 1 \times 2 + \frac{3}{4} \times 2 = \frac{19}{4}$ . It is similar with  $d(f_{i+3}) = 3, d(f_{i+1}) \geq 4$  and  $d(f_{i+2}) \geq 4$ . Suppose  $d(f_{i+2}) = 3, d(f_{i+1}) \geq 4$  and  $d(f_{i+3}) \geq 4$ . If  $d(v_{i+2}) = 3$  or  $d(v_{i+3}) = 3$ , then  $\phi \leq \frac{3}{2} \times 1 + 1 \times 1 + \frac{3}{4} \times 3 = \frac{19}{4}$ . Otherwise,  $\phi \leq \frac{5}{4} \times 1 + 1 \times 2 + \frac{3}{4} \times 2 = \frac{19}{4}$ . Suppose  $\min\{d(f_{i+1}), d(f_{i+2}), d(f_{i+3})\} \geq 4$ . Then  $\max\{d(v_{i+1}), d(v_{i+2}), d(v_{i+3}), d(v_{i+4})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2}), d(f_{i+3}), d(f_{i+4})\} \geq 5$ , so  $\phi \leq \max\{1 \times 3 + \frac{3}{4} \times 2, 1 \times 4 + \frac{1}{3}\} = \frac{9}{2}$ . So  $\phi \leq \max\{\frac{14}{3}, \frac{21}{4}, \frac{29}{6}, 5, \frac{55}{12}, \frac{19}{4}, \frac{9}{2}\} = \frac{21}{4}$ .

**Case (e).**  $k = 5$  If  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = d(f_{i+4}) = 3$ , then  $\min\{d(v_{i+2}), d(v_{i+3}), d(v_{i+4})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+5}) = 3$ , then  $d(f_i) \geq 5$  and  $d(f_{i+5}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} \times 2 + \frac{1}{3} \times 2 = \frac{37}{6}$ . If  $\min\{d(v_{i+1}), d(v_{i+5})\} \geq 4$ , then  $\phi \leq \frac{5}{4} \times 4 + \frac{3}{4} \times 2 = \frac{13}{2}$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = 3$ ,  $d(f_{i+4}) \geq 4$ , then  $\min\{d(v_{i+2}), d(v_{i+3})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$  and  $d(v_{i+5}) \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 + 1 + \frac{1}{3} \times 2\} = \frac{35}{6}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+4}) \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 2 + 1 + \frac{3}{4} + \frac{1}{3} = \frac{73}{12}$ . If  $d(v_{i+4}) = 3$  and  $d(v_{i+1}) \geq 4$ , then  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 2 + \frac{3}{4} \times 3 = \frac{25}{4}$ . Otherwise,  $\phi \leq \frac{5}{4} \times 3 + 1 + \frac{3}{4} \times 2 = \frac{25}{4}$ . It is similar with  $d(f_{i+2}) = d(f_{i+3}) = d(f_{i+4}) = 3$ ,  $d(f_{i+1}) \geq 4$ . Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+4}) = 3$ ,  $d(f_{i+3}) \geq 4$ , then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$  and  $d(f_{i+4}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 3 + \frac{3}{4} \times 1 + \frac{1}{3} \times 2 = \frac{71}{12}$ . It is similar with  $d(v_{i+1}) = d(v_{i+3}) = d(v_{i+5}) = 3$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$  and  $d(v_{i+3}) \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} \times 1 + \frac{3}{4} \times 2 + \frac{1}{3} \times 1 = \frac{73}{12}$ . Otherwise,  $\phi \leq \frac{3}{2} \times 1 + \frac{5}{4} \times 2 + \frac{3}{4} \times 3 = \frac{25}{4}$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = 3$ ,  $d(f_{i+3}) \geq 4$  and  $d(f_{i+4}) \geq 4$ , then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(f_i) \geq 5$  and  $\max\{d(v_{i+4}), d(v_{i+5})\} \geq 4$  or  $\max\{d(f_{i+3}), d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 \times 2 + \frac{1}{3} \times 2\} = \frac{35}{6}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+3}) \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} + 1 \times 2 + \frac{3}{4} + \frac{1}{3} = \frac{35}{6}$ . If  $d(v_{i+3}) = 3$  and  $d(v_{i+1}) \geq 4$ , then  $\max\{d(v_{i+4}), d(v_{i+5}) \geq 4\}$  or  $\max\{d(f_{i+3}), d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} + 1 + \frac{3}{4} \times 3, \frac{3}{2} + \frac{5}{4} + 1 \times 2 + \frac{3}{4} + \frac{1}{3}\} = 6$ . Otherwise,  $\phi \leq \frac{5}{4} \times 2 + 1 \times 2 + \frac{3}{4} \times 2 = 6$ . It is similar with  $d(f_{i+3}) = d(f_{i+4}) = 3$ ,  $d(f_{i+1}) \geq 4$  and  $d(f_{i+2}) \geq 4$ . Suppose  $d(f_{i+1}) = d(f_{i+3}) = 3$ ,  $d(f_{i+2}) \geq 4$  and  $d(f_{i+4}) \geq 4$ , then  $\phi \leq \frac{3}{2} \times 2 + \frac{3}{4} \times 4 = 6$ . It is similar with  $d(f_{i+2}) = d(f_{i+4}) = 3$ ,  $d(f_{i+1}) \geq 4$  and  $d(f_{i+3}) \geq 4$ . Suppose  $d(f_{i+1}) = d(f_{i+4}) = 3$ ,  $d(f_{i+2}) \geq 4$  and  $d(f_{i+3}) \geq 4$ , then  $\phi \leq \frac{3}{2} \times 2 + \frac{3}{4} \times 4 = 6$ . Suppose  $d(f_{i+2}) = d(f_{i+3}) = 3$ ,  $d(f_{i+1}) \geq 4$  and  $d(f_{i+4}) \geq 4$ , then  $\phi \leq \frac{3}{2} \times 2 + \frac{3}{4} \times 4 = 6$ .

Suppose  $f_{i+1}, f_{i+2}, f_{i+3}$  and  $f_{i+4}$  has at most one 3-face contains  $3^-$ -vertex, then  $\phi \leq \frac{3}{2} + 1 \times 3 + \frac{3}{4} \times 2 = 6$ .

Suppose  $\min\{d(f_{i+1}), d(f_{i+2}), d(f_{i+3}), d(f_{i+4})\} \geq 4$ , then  $\max\{d(v_{i+1}), d(v_{i+2}), d(v_{i+3}), d(v_{i+4}), d(v_{i+5})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2}), d(f_{i+3}), d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{1 \times 4 + \frac{3}{4} \times 2, 1 \times 5 + \frac{1}{3}\} = \frac{11}{2}$ . In the end,  $\phi \leq \max\{\frac{37}{6}, \frac{13}{2}, \frac{35}{6}, \frac{73}{12}, \frac{25}{4}, \frac{71}{12}, 6, \frac{11}{2}\} = \frac{13}{2}$ .

Next, we prove the Lemma 2.2.9.

If  $n_2(v) = 8$ , then all faces incident with  $v$  are  $6^+$ -faces by Lemma 2.2.6(2)-(4), that is,  $f_{6^+}(v) = 8$ , so  $ch'(v) = 10 - 1 \times 8 = 2 > 0$ . If  $n_2(v) = 7$ , then  $f_{6^+}(v) \geq 6$  and  $f_3(v) = 0$ , so  $ch'(v) \geq 10 - 1 \times 7 - \frac{3}{2} = \frac{3}{2} > 0$  by Claim (a).

Suppose  $n_2(v) \leq 6$ . The possible distributions of 2-vertices adjacent to  $v$  are illustrated in Figure 2.5. For Figure 2.5(1), we have  $f_{6^+}(v) \geq 5$  and

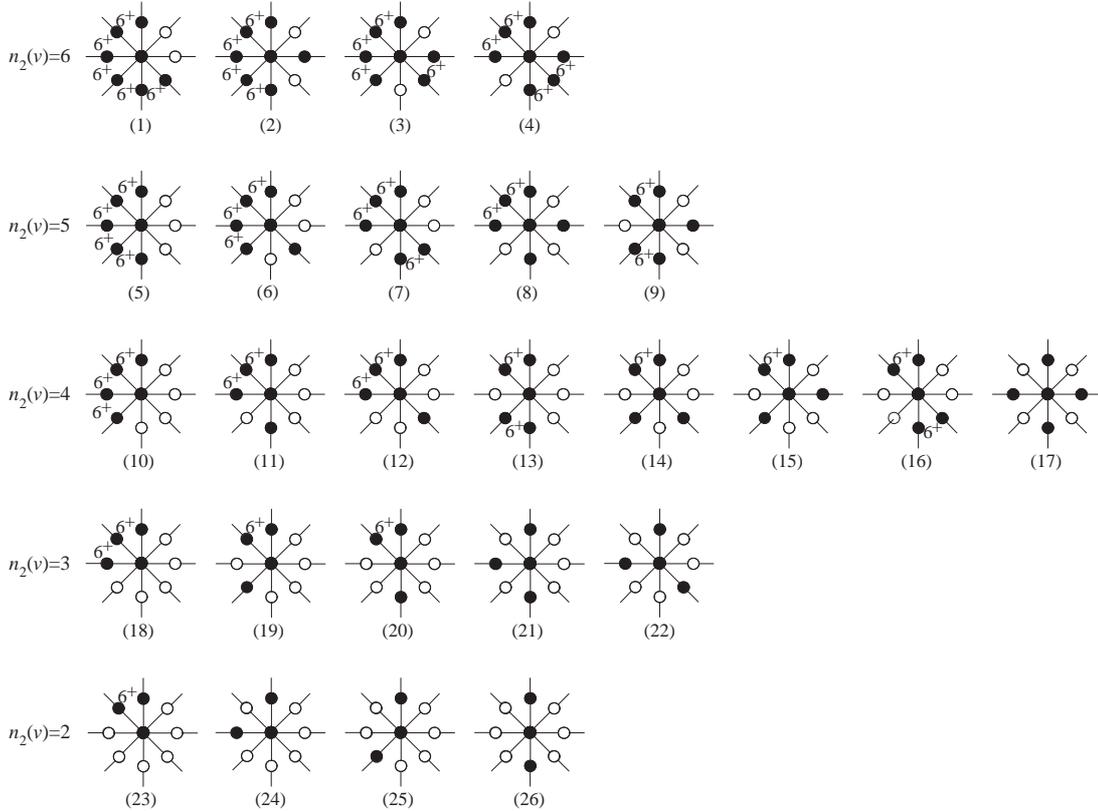


Figure 2.5

$ch'(v) \geq 10 - 1 \times 6 - \frac{11}{4} = \frac{5}{4} > 0$  by Claim (b). For Figure 2.5(2)-(4), we have  $f_{6^+}(v) \geq 4$  and  $ch'(v) \geq 10 - 1 \times 6 - \frac{3}{2} \times 2 = 1 > 0$ . For Figure 2.5(5), we have  $f_{6^+}(v) \geq 4$  and  $ch'(v) \geq 10 - 1 \times 5 - 4 = 1 > 0$  by Claim (c). For Figure 2.5(6)-(7), we have  $f_{6^+}(v) \geq 3$  and  $ch'(v) \geq 10 - 1 \times 5 - \frac{3}{2} - \frac{11}{4} = \frac{3}{4} > 0$ . For Figure 2.5(8)-(9), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 10 - 1 \times 5 - \frac{3}{2} \times 3 = \frac{1}{2} > 0$ . For Figure 2.5(10), we have  $f_{6^+}(v) \geq 3$  and  $ch'(v) \geq 10 - 1 \times 4 - \frac{21}{4} = \frac{3}{4} > 0$  by Claim (d). For Figure 2.5(11) and 2.5(13), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 10 - 1 \times 4 - \frac{3}{2} - 4 = \frac{1}{2} > 0$ . For Figure 2.5(12) and 2.5(16), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 10 - 1 \times 4 - \frac{11}{4} \times 2 = \frac{1}{2} > 0$ . For Figure 2.5(14) and 2.5(15), we have  $f_{6^+}(v) \geq 1$  and  $ch'(v) \geq 10 - 1 \times 4 - \frac{3}{2} \times 2 - \frac{11}{4} = \frac{1}{4} > 0$ . For Figure 2.5(17), we have  $ch'(v) \geq 10 - 1 \times 4 - \frac{3}{2} \times 4 = 0$ . For Figure 2.5(18), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 10 - 1 \times 3 - \frac{13}{2} = \frac{1}{2} > 0$  by Claim (e). For Figure 2.5(19), we have  $f_{6^+}(v) \geq 1$  and  $ch'(v) \geq 10 - 1 \times 3 - \frac{3}{2} - \frac{21}{4} = \frac{1}{4} > 0$ . For Figure 2.5(20), we have  $f_{6^+}(v) \geq 1$  and  $ch'(v) \geq 10 - 1 \times 3 - \frac{11}{4} - 4 = \frac{1}{4} > 0$ . For Figure 2.5(21), we have  $ch'(v) \geq 10 - 1 \times 3 - \frac{3}{2} \times 2 - 4 = 0$ . For Figure 2.5(22), we have  $ch'(v) \geq 10 - 1 \times 3 - \frac{3}{2} - \frac{11}{4} \times 2 = 0$ . For Figure 2.5(23), we have  $f_{6^+}(v) \geq 1$ . Suppose  $d(f_2) = d(f_3) = d(f_4) = d(f_5) = d(f_6) = 3$ . Then  $\min\{d(v_3), d(v_4), d(v_5), d(v_6)\} \geq 4$ . If  $d(v_2) = d(v_6) = 3$ , then  $d(f_1) \geq 5$  and  $d(f_7) \geq 5$  by Lemma 2.2.7, so  $ch'(v) \geq 10 - 1 \times 2 - \frac{3}{2} \times 2 - \frac{5}{4} \times 2 - 1 \times 1 - \frac{1}{3} \times 2 = \frac{5}{6} > 0$ . If  $f_2, f_3, f_4, f_5$  and  $f_6$  are incident with no  $3^-$ -vertex, then  $ch'(v) \geq 10 - 1 \times 2 - \frac{5}{4} \times 5 - \frac{3}{4} \times 2 = \frac{1}{4} > 0$ . For Figure 2.5(24), we have  $ch'(v) \geq 10 - 1 \times 2 - \frac{3}{2} - \frac{13}{2} = 0$ . For Figure 2.5(25), we have  $ch'(v) \geq 10 - 1 \times 2 - \frac{11}{4} - \frac{21}{4} = 0$ . For Figure 2.5(26), we have  $ch'(v) \geq 10 - 1 \times 2 - 4 \times 2 = 0$ . □

Hence we complete the proof of the theorem 2.2.1: Let  $G$  be a planar graph with maximum degree  $\Delta \geq 8$ . If every 6-cycle of  $G$  contains at most one chord or chordal 6-cycles are not adjacent in  $G$ , then  $\chi''(G) = \Delta + 1$ .

**§2.3.1 7-cycles containing at most two chords**

**Theorem 2.3.1.** *Let  $G$  be a planar graph with maximum degree  $\Delta \geq 8$ . If every 7-cycle of  $G$  contains at most two chords, then  $\chi''(G) = \Delta + 1$ .*

*Proof.* According to [74], Theorem 2.3.1 holds for planar graphs with  $\Delta \geq 9$ . So we have  $\Delta = 8$ . Arguing by contradiction, let  $G = (V, E)$  be a minimal counterexample to the planar graph  $G$  with maximum degree  $\Delta = 8$ , such that  $|V| + |E|$  is minimal and  $G$  has been embedded in the plane. Then every proper subgraph of  $G$  is total-9-colorable. First we give some lemmas for  $G$ .

**Lemma 2.3.1.** [16] (a)  $G$  is 2-connected.

(b) If  $uv$  is an edge of  $G$  with  $d(u) \leq 4$ , then  $d(u) + d(v) \geq \Delta + 2 = 10$ .

(c) The subgraph induced by all edges joining 2-vertices to 8-vertices in  $G$  is a forest.

By Lemma 2.3.1(a): There is no 1-vertex in  $G$ . By Lemma 2.3.1(b): any two neighbors of a 2-vertex are 8-vertices. Any three neighbors of a 3-vertex are  $7^+$ -vertices. Any four neighbors of a 4-vertex are  $6^+$ -vertices.

**Lemma 2.3.2.**  $G$  has no configurations depicted in Figure 2.6, where  $v$  denotes the vertex of degree of 7.

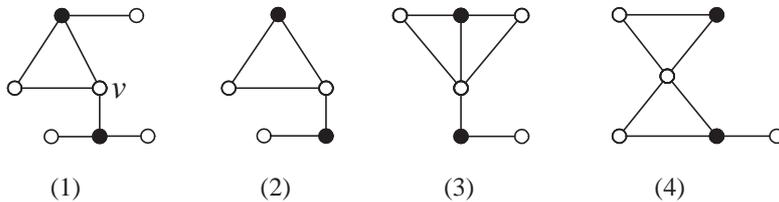


Figure 2.6

*Proof.* The proof showing that (1) and (5) cannot be configurations contained in  $G$  can be found in [139]. Those for (2), (3) and (4) can be found in [74].

□

**Lemma 2.3.3.** *Suppose  $v$  is a  $d$ -vertex of  $G$  with  $d \geq 5$ . Let  $v_1, \dots, v_d$  be the neighbor of  $v$  and  $f_1, f_2, \dots, f_d$  be faces incident with  $v$ , such that  $f_i$  is incident with  $v_i$  and  $v_{i+1}$ , for  $i \in \{1, 2, \dots, d\}$ . Let  $d(v_1) = 2$  and  $\{v, u_1\} = N(v_1)$ . Then  $G$  does not satisfy one of the following conditions (see Figure 2.2).*

- (1) *there exists an integer  $k$  ( $2 \leq k \leq d - 1$ ) such that  $d(v_{k+1}) = 2$ ,  $d(v_i) = 3$  ( $2 \leq i \leq k$ ) and  $d(f_j) = 4$  ( $1 \leq j \leq k$ ).*
- (2) *there exist two integers  $k$  and  $t$  ( $2 \leq k < t \leq d - 1$ ) such that  $d(v_k) = 2$ ,  $d(v_i) = 3$  ( $k + 1 \leq i \leq t$ ),  $d(f_t) = 3$  and  $d(f_j) = 4$  ( $k \leq j \leq t - 1$ ).*
- (3) *there exist two integers  $k$  and  $t$  ( $3 \leq k \leq t \leq d - 1$ ) such that  $d(v_i) = 3$  ( $k \leq i \leq t$ ),  $d(f_{k-1}) = d(f_t) = 3$  and  $d(f_j) = 4$  ( $k \leq j \leq t - 1$ ).*

See the proof of Lemma 2.3.3 in Lemma 2.2.7.

Now we will use “Discharging method” to complete the proof of Theorem 2.3.1.

By the Euler’s formula  $|V| - |E| + |F| = 2$  and  $\sum_{v \in V} d(v) = 2|E|$ ,  $\sum_{f \in F} d(f) = 2|E|$ , we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

For each  $x \in V \cup F$ , we define the initial charge

$$ch(x) = \begin{cases} 2d(x) - 6 & \text{if } x \in V \\ d(x) - 6 & \text{if } x \in F \end{cases}$$

So  $\sum_{x \in V \cup F} ch(x) = -12 < 0$ . In the following, we will reassign a new charge

denoted by  $ch'(x)$  to each  $x \in V \cup F$  according to the discharging rules. If we can show that  $ch'(x) \geq 0$  for each  $x \in V \cup F$ , then we get an obvious contradiction to  $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -12$ , which completes our proof.

Let  $ch(v \rightarrow f)$  be the amount that a vertex  $v$  sends a face  $f$ . We define the discharging rules as follows.

**R1.** Each 2-vertex receives 1 from each of its neighbors.

**R2.** Let  $f$  be a 3-face.

**R2.1** If  $f$  is incident with a  $3^-$ -vertex, then it receives  $\frac{3}{2}$  from each of its two incident  $7^+$ -vertices.

**R2.2** If  $f$  is incident with a 4-vertex, then it receives  $\frac{1}{2}$  from the 4-vertex and  $\frac{5}{4}$  from each of its two incident  $6^+$ -vertices.

**R2.3** Suppose that all vertices incident with  $f$  are  $5^+$ -vertices, and  $v$  is a vertex incident with  $f$ . If  $d(v) = 5$ , then  $ch(v \rightarrow f) = (4 - \frac{f_4(v)}{2} - \frac{f_5(v)}{3})/f_3(v)$ . If  $d(v) \geq 6$ , then  $ch(v \rightarrow f) = \frac{5}{4}$ .

**R3.** Let  $f$  be a 4-face and incident with a  $4^+$ -vertex  $v$ . If  $4 \leq d(v) \leq 5$ , then  $f$  receives  $\frac{1}{2}$  from  $v$ ; Otherwise

$$ch(v \rightarrow f) = \begin{cases} 1 & \text{if } n_{3^-}(f) = 2, \\ \frac{3}{4} & \text{if } n_{3^-}(f) = 1 \text{ and } n_{5^-}(f) = 2, \\ \frac{2}{3} & \text{if } n_{3^-}(f) = 1 \text{ and } n_{6^+}(f) = 3, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

**R4.** Each 5-face receives  $\frac{1}{3}$  from each of its incident  $4^+$ -vertices.

Since every 7-cycle contains at most two chords, we have the following lemma.

**Lemma 2.3.4.**  *$G$  has no configurations depicted in Figure 2.7, where all the*

vertices showing in Figure 2.7 are different.

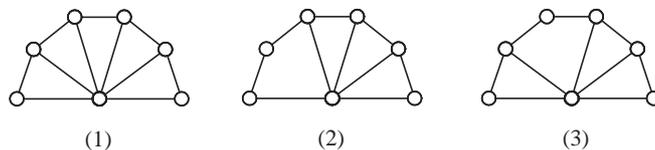


Figure 2.7

Next, it suffices to show that  $ch'(x) \geq 0$  for all  $x \in V \cup F$ . Let  $f$  be a face of  $G$ . If  $d(f) \geq 6$ , then  $f$  does not send out any charge and hence  $ch'(f) = ch(f) \geq 0$ . If  $d(f) = 5$ , then  $f$  is incident with at least three  $4^+$ -vertices by Lemma 2.3.1(b), and it follows that  $ch'(f) \geq ch(f) + \frac{1}{3} \times 3 = 0$  by R4. If  $d(f) = 4$ , then  $ch'(f) \geq ch(f) + \min\{\frac{1}{2} \times 4, \frac{2}{3} \times 3, \frac{3}{4} \times 2 + \frac{1}{2}, 1 \times 2\} = 0$  by R3. Suppose that  $d(f) = 3$ . Let  $f = v_1v_2v_3v_1$  and assume that  $d(v_1) \leq d(v_2) \leq d(v_3)$ . If  $d(v_1) \leq 3$ , then  $v_2$  and  $v_3$  are  $7^+$ -vertices by Lemma 2.3.1 and it follows that  $ch'(f) = 3 - 6 + \frac{3}{2} \times 2 = 0$ . If  $d(v_1) = 4$ , then  $v_2$  and  $v_3$  are  $6^+$ -vertices and it follows that  $ch'(f) = 3 - 6 + \frac{1}{2} + \frac{5}{4} \times 2 = 0$ . Suppose that  $d(v_1) \geq 5$ . Note that if a 5-vertex  $x$  is incident with four 3-faces, then any  $5^+$ -vertex adjacent to  $x$  must be incident with two  $5^+$ -faces. So we have  $ch'(f) = 3 - 6 + \min\{\frac{4}{5} + 2 \times \frac{10}{9}, 3 \times \frac{19}{18}\} = 0$  by R2.3. Hence we prove that  $ch'(f) \geq 0$  for all faces  $f$ .

Let  $v$  be a vertex of  $G$ . If  $d(v) = 2$ , then  $ch'(v) = -2 + 1 \times 2 = 0$  by R1. If  $d(v) = 3$ , then  $ch'(v) = ch(v) = 0$ . If  $d(v) = 4$ , then  $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$ . If  $d(v) = 5$ , then  $ch'(v) \geq 0$  by R2-R4. Suppose that  $d(v) = 6$ . Then  $f_3(v) \leq 4$ . If  $f_3(v) = 4$ , then either  $f_4(v) = 1$  and  $f_{6^+}(v) \geq 1$ , or  $f_{5^+}(v) \geq 2$ . So  $ch'(v) \geq ch(v) - \frac{5}{4} \times 4 - \max\{1 + 0, \frac{1}{3} \times 2\} = 0$  by Lemma 2.3.4. If  $f_3(v) \leq 3$ , then  $ch'(v) \geq ch(v) - \frac{5}{4} \times f_3(v) - \frac{2}{3} \times (6 - f_3(v)) = \frac{24 - 7f_3(v)}{12} \geq 0$ . Suppose  $d(v) = 7$ . Then  $f_3(v) \leq 5$ . By Lemma 2.3.2(1),  $v$  is incident with at most two 3-faces are incident with a  $3^-$ -vertex, that is,  $v$  sends  $\frac{3}{2}$  to each of the two

3-faces and at most  $\frac{5}{4}$  to each other 3-face. If  $f_3(v) = 5$ , then  $f_{5^+}(v) \geq 1$ , and  $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - \frac{5}{4} \times 3 - \frac{3}{4} \times 1 - \frac{1}{3} \times 1 = \frac{1}{6} > 0$ . If  $2 \leq f_3(v) \leq 4$ , then  $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - \frac{5}{4} \times (f_3(v) - 2) - 1 \times (5 - f_3(v)) - \frac{3}{4} \times 2 = \frac{4 - f_3(v)}{4} \geq 0$ . If  $f_3(v) \leq 1$ , then  $ch'(v) \geq ch(v) - \frac{3}{2} \times f_3(v) - 1 \times (7 - f_3(v)) = \frac{2 - f_3(v)}{2} > 0$ .

Finally, we assume that  $d(v) = 8$ . Then  $ch(v) = 10$ . Let  $v_1, \dots, v_8$  be neighbors of  $v$  in the clockwise order and  $f_1, f_2, \dots, f_8$  be faces incident with  $v$ , such that  $f_i$  is incident with  $v_i$  and  $v_{i+1}$ , for  $i \in \{1, 2, \dots, 8\}$ , and  $f_9 = f_1$ .

Suppose  $n_2(v) = 0$ . Then  $f_3(v) \leq 6$ . If  $f_3(v) = 6$ , then  $f_{5^+}(v) \geq 2$ , so  $ch'(v) \geq 10 - \frac{3}{2} \times 6 - \frac{1}{3} \times 2 = \frac{1}{3} > 0$ . If  $f_3(v) = 5$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq 10 - \frac{3}{2} \times 5 - 1 \times 2 - \frac{1}{3} \times 1 = \frac{1}{6} > 0$ . If  $f_3(v) \leq 4$ , then  $ch'(v) \geq 10 - \frac{3}{2} \times f_3(v) - 1 \times (8 - f_3(v)) \geq 0$ .

Suppose  $n_2(v) = 1$ . Without loss of generality, assume  $d(v_1) = 2$ . Suppose that  $v_1$  is incident with a 3-cycle  $f_1$ . By Lemma 2.3.4,  $f_3(v) \leq 5$  and all 3-faces incident with no  $3^-$ -vertex except  $f_1$  by Lemma 2.3.2(4). If  $f_3(v) = 5$ , then  $f_{5^+}(v) \geq 2$ , so  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - \frac{5}{4} \times 4 - 1 \times 1 - \frac{1}{3} \times 2 = \frac{5}{6} > 0$ . If  $f_3(v) = 4$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - \frac{5}{4} \times 3 - 1 \times 3 - \frac{1}{3} \times 1 = \frac{5}{12} > 0$ . If  $1 \leq f_3(v) \leq 3$ , then  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 1 - \frac{5}{4} \times (f_3(v) - 1) - 1 \times (8 - f_3(v)) = \frac{3 - f_3(v)}{4} \geq 0$ . Suppose  $v_1$  is not incident with a 3-cycle. By Lemma 2.3.4,  $f_3(v) \leq 5$ . If  $f_3(v) = 5$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times 2 - \frac{5}{4} \times 3 - 1 \times 1 - \frac{3}{4} \times 1 - \frac{1}{3} \times 1 = \frac{1}{6} > 0$  by Lemma 2.3.2(2)-(4). If  $f_3(v) \leq 4$ , then  $ch'(v) \geq 10 - 1 - \frac{3}{2} \times f_3(v) - 1 \times (8 - f_3(v)) \geq 0$ .

For  $2 \leq n_2(v) \leq 8$ , we need to prove the following claim firstly.

**Claim 2.4.** *Suppose that  $d(v_i) = d(v_{i+k+1}) = 2$  and  $d(v_j) \geq 3$  for  $i + 1 \leq j \leq i + k$  ( see Figure 2.8). Then  $v$  sends at most  $\phi$  (in total) to  $f_i, f_{i+1}, \dots, f_{i+k}$ , where*

$$\phi = \begin{cases} \frac{5k+1}{4} & \text{if } k = 1, 3, 4, 5, 6 \\ \frac{8}{3} & \text{if } k = 2. \end{cases}$$

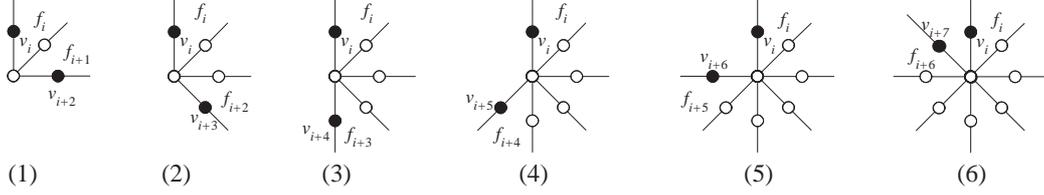


Figure 2.8

*Proof.* By Lemma 2.3.2,  $d(f_i) \geq 4$  and  $d(f_{i+k}) \geq 4$ .

(a)  $k = 1$  By Lemma 2.3.3(1), we have  $d(v_{i+1}) \geq 4$  or  $\max\{d(f_i), d(f_{i+1})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{4} \times 2, 1 + \frac{1}{3}\} = \frac{3}{2}$  by R3.

(b)  $k = 2$  If  $d(f_{i+1}) = 3$ , then  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+2})\} \geq 5$  by Lemma 2.3.3(2), and it follows that  $\phi \leq \max\{\frac{3}{4} + \frac{5}{4} + \frac{2}{3}, \frac{1}{3} + \frac{3}{2} + \frac{2}{3}\} = \frac{8}{3}$ . Otherwise,  $d(f_{i+1}) \geq 4$ , then  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2})\} \geq 5$  by Lemma 2.3.3(1), and it follows that  $\phi \leq \max\{1 + \frac{3}{4} \times 2, 1 \times 2 + \frac{1}{3}\} = \frac{5}{2} < \frac{8}{3}$ .

(c)  $k = 3$  Suppose  $d(f_{i+1}) = d(f_{i+2}) = 3$ . Then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(f_i) \geq 5$  and  $d(f_{i+3}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{1}{3} \times 2 = \frac{11}{3}$ . If  $\min\{d(v_{i+1}), d(v_{i+3})\} \geq 4$ , then  $\phi \leq \frac{5}{4} \times 2 + \frac{3}{4} \times 2 = 4$ . Suppose  $d(f_{i+1}) = 3$  and  $d(f_{i+2}) \geq 4$ . If  $d(v_{i+1}) = 3$ , then  $d(v_{i+2}) \geq 7$  and  $d(f_i) \geq 5$ , so  $\phi \leq \frac{1}{3} + \frac{3}{2} + \frac{2}{3} + 1 = \frac{7}{2}$ . If  $d(v_{i+2}) = 3$ , then  $d(v_{i+1}) \geq 7$  and  $d(v_{i+3}) \geq 4$  or  $\max\{d(f_{i+2}), d(f_{i+3})\} \geq 5$ , so  $\phi \leq \max\{\frac{2}{3} + \frac{3}{2} + \frac{3}{4} + \frac{3}{4}, \frac{2}{3} + \frac{3}{2} + 1 + \frac{1}{3}\} = \frac{11}{3}$ . If  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$ , then  $\phi \leq \frac{3}{4} + \frac{5}{4} + \frac{2}{3} + 1 = \frac{11}{3}$ . It is similar with  $d(f_{i+2}) = 3$  and  $d(f_{i+1}) \geq 4$ .

Suppose  $\min\{d(f_{i+1}), d(f_{i+2})\} \geq 4$ . Then  $\max\{d(v_{i+1}), d(v_{i+2}), d(v_{i+3})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2}), d(f_{i+3})\} \geq 5$ , so  $\phi \leq \max\{1 \times 2 + \frac{3}{4} \times 2, 1 \times 3 + \frac{1}{3}\} = \frac{7}{2}$ . So  $\phi \leq \max\{\frac{11}{3}, 4, \frac{7}{2}\} = 4$ .

(d)  $k = 4$  Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = 3$ . Then  $\min\{d(v_{i+2}), d(v_{i+3})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$  and  $d(f_{i+4}) \geq 5$ ,

so  $\phi \leq \frac{3}{2} \times 2 + 1 + \frac{1}{3} \times 2 = \frac{14}{3}$ . If  $\min\{d(v_{i+1}), d(v_{i+4})\} \geq 4$ , then  $\phi \leq \max\{\frac{5}{4} \times 3 + \frac{3}{4} + \frac{2}{3}, \frac{5}{4} \times 3 + \frac{3}{4} \times 2\} = \frac{21}{4}$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = 3$ ,  $d(f_{i+3}) \geq 4$ . Then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(v_{i+4}) \geq 4$  and  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{3}{4} \times 2 + \frac{1}{3} = \frac{29}{6}$ . If  $\min\{d(v_{i+1}), d(v_{i+3})\} \geq 4$ , then  $\phi \leq \frac{5}{4} \times 2 + 1 + \frac{3}{4} \times 2 = 5$ . It is similar with  $d(f_{i+2}) = d(f_{i+3}) = 3$ ,  $d(f_{i+1}) \geq 4$ . Suppose  $d(f_{i+1}) = d(f_{i+3}) = 3$ ,  $d(f_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $d(f_{i+3}) \geq 5$  and  $d(f_{i+4}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{2}{3} + \frac{1}{3} \times 2 = \frac{13}{3}$ . If  $d(v_{i+2}) = d(v_{i+3}) = 3$ , then  $d(f_{i+3}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{2}{3} \times 2 + \frac{1}{3} = \frac{14}{3}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+4}) \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{2}{3} \times 2 + \frac{1}{3} = \frac{14}{3}$ . It is similar with  $d(v_{i+3}) = 3$  and  $d(v_{i+1}) \geq 4$ . If  $d(v_{i+2}) = 3$  and  $\min\{d(v_{i+3}), d(v_{i+4})\} \geq 4$ , then  $d(v_{i+1}) \geq 7$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} + \frac{3}{4} + \frac{2}{3} \times 2 = \frac{29}{6}$ . It is similar with  $d(v_{i+3}) = 3$  and  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$ . Otherwise,  $\phi \leq \frac{5}{4} \times 2 + \frac{3}{4} \times 2 + \frac{2}{3} = \frac{14}{3}$ .

Suppose  $d(f_{i+1}) = 3$ ,  $d(f_{i+2}) \geq 4$  and  $d(f_{i+3}) \geq 4$ . If  $d(v_{i+1}) = 3$ , then  $d(v_{i+2}) \geq 7$  and  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + 1 \times 2 + \frac{2}{3} + \frac{1}{3} = \frac{9}{2}$ . If  $d(v_{i+2}) = 3$ , then  $d(v_{i+1}) \geq 7$  and  $\max\{d(v_{i+3}), d(v_{i+4})\} \geq 4$  or  $\max\{d(f_{i+2}), d(f_{i+3}), d(f_{i+4})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + 1 + \frac{3}{4} \times 2 + \frac{2}{3}, \frac{3}{2} + 1 \times 2 + \frac{2}{3} + \frac{1}{3}\} = \frac{9}{2}$ . Otherwise,  $\phi \leq \frac{5}{4} + 1 \times 2 + \frac{3}{4} + \frac{2}{3} = \frac{14}{3}$ . It is similar with  $d(f_{i+3}) = 3$ ,  $d(f_{i+1}) \geq 4$  and  $d(f_{i+2}) \geq 4$ . Suppose  $d(f_{i+2}) = 3$ ,  $d(f_{i+1}) \geq 4$  and  $d(f_{i+3}) \geq 4$ . If  $d(v_{i+2}) = 3$ , then  $d(v_{i+1}) \geq 4$  or  $\max\{d(f_i), d(f_{i+1}) \geq 5\}$ , so  $\phi \leq \max\{\frac{3}{2} + 1 + \frac{3}{4} \times 2 + \frac{2}{3}, \frac{3}{2} + 1 \times 2 + \frac{2}{3} + \frac{1}{3}\} = \frac{14}{3}$ . Otherwise,  $\phi \leq \frac{5}{4} + 1 \times 2 + \frac{3}{4} + \frac{2}{3} = \frac{14}{3}$ . It is similar with  $d(v_{i+3}) = 3$ .

Suppose  $\min\{d(f_{i+1}), d(f_{i+2}), d(f_{i+3})\} \geq 4$ . Then  $\max\{d(v_{i+1}), d(v_{i+2}), d(v_{i+3}), d(v_{i+4})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2}), d(f_{i+3}), d(f_{i+4})\} \geq 5$ , so  $\phi \leq \max\{1 \times 3 + \frac{3}{4} \times 2, 1 \times 4 + \frac{1}{3}\} = \frac{9}{2}$ . So  $\phi \leq \max\{\frac{14}{3}, \frac{21}{4}, \frac{29}{6}, 5, \frac{13}{3}, \frac{19}{4}, \frac{9}{2}\} = \frac{31}{6}$ .

(e)  $k = 5$  Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = d(f_{i+4}) = 3$ , then

$\min\{d(v_{i+2}), d(v_{i+3}), d(v_{i+4})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+5}) = 3$ , then  $d(f_i) \geq 5$  and  $d(f_{i+5}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} \times 2 + \frac{1}{3} \times 2 = \frac{37}{6}$ . If  $\min\{d(v_{i+1}), d(v_{i+5})\} \geq 4$ , then  $\phi \leq \frac{5}{4} \times 4 + \frac{3}{4} \times 2 = \frac{13}{2}$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = 3$ ,  $d(f_{i+4}) \geq 4$ , then  $\min\{d(v_{i+2}), d(v_{i+3})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$  and  $d(v_{i+5}) \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 + 1 + \frac{1}{3} \times 2\} = \frac{35}{6}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+4}) = 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} + 1 + 1 + \frac{3}{4} + \frac{1}{3} = \frac{35}{6}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+4}) \geq 5$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 2 + 1 + \frac{2}{3} + \frac{1}{3} = 6$ . If  $d(v_{i+4}) = 3$  and  $d(v_{i+1}) = 4$ , then  $d(v_{i+5}) \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} + 1 + \frac{3}{4} \times 3, \frac{3}{2} + \frac{5}{4} + 1 + 1 + \frac{3}{4} + \frac{1}{3}\} = 6$ . If  $d(v_{i+4}) = 3$  and  $d(v_{i+1}) \geq 5$ , then  $d(v_{i+5}) \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} \times 2 + \frac{3}{4} \times 2 + \frac{2}{3}, \frac{3}{2} + \frac{5}{4} \times 2 + 1 + \frac{2}{3} + \frac{1}{3}\} = \frac{37}{6}$ . Otherwise,  $\phi \leq \frac{5}{4} \times 3 + 1 + \frac{3}{4} + \frac{2}{3} = \frac{37}{6}$ . Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+4}) = 3$ ,  $d(f_{i+3}) \geq 4$ , then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$  and  $d(f_{i+3}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 3 + \frac{2}{3} + \frac{1}{3} \times 2 = \frac{35}{6}$ . It is similar with  $d(v_{i+1}) = d(v_{i+3}) = d(v_{i+5}) = 3$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$  and  $\min\{d(v_{i+4}), d(v_{i+5})\} \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} + \frac{3}{4} + \frac{2}{3} + \frac{1}{3} = 6$ . It is similar with  $d(v_{i+1}) = d(v_{i+4}) = 3$  and  $d(v_{i+3}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+5}) = 3$  and  $d(v_{i+3}) \geq 4$ , then  $d(f_i) \geq 5$  and  $d(f_{i+5}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} + \frac{1}{2} + \frac{1}{3} \times 2 = \frac{65}{12}$ . Otherwise,  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 2 + \frac{3}{4} \times 2 + \frac{2}{3} = \frac{37}{6}$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = 3$ ,  $d(f_{i+3}) \geq 4$  and  $d(f_{i+4}) \geq 4$ , then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(f_i) \geq 5$  and  $\max\{d(v_{i+4}), d(v_{i+5})\} \geq 4$  or  $\max\{d(f_{i+3}), d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 \times 2 + \frac{1}{3} \times 2\} = \frac{35}{6}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+3}) \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} + 1 \times 2 + \frac{3}{4} + \frac{1}{3} = \frac{35}{6}$ . If  $d(v_{i+3}) = 3$  and  $d(v_{i+1}) \geq 4$ , then  $\max\{d(v_{i+4}), d(v_{i+5})\} \geq 4$  or  $\max\{d(f_{i+3}), d(f_{i+4}), d(f_{i+5})\} \geq 5$ ,

so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} + 1 + \frac{3}{4} \times 3, \frac{3}{2} + \frac{5}{4} + 1 \times 2 + \frac{3}{4} + \frac{1}{3}\} = 6$ . Otherwise,  $\phi \leq \frac{5}{4} \times 2 + 1 \times 2 + \frac{3}{4} \times 2 = 6$ . Suppose  $d(f_{i+1}) = d(f_{i+3}) = 3$ ,  $d(f_{i+2}) \geq 4$  and  $d(f_{i+4}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + 1 + \frac{2}{3} \times 2 + \frac{1}{3} = \frac{17}{3}$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$  and  $d(v_{i+5}) \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + \frac{3}{4} \times 2 + \frac{2}{3} + \frac{1}{3}, \frac{3}{2} \times 2 + 1 + \frac{2}{3} + \frac{1}{3}\} = \frac{11}{2}$ . If  $d(v_{i+2}) = d(v_{i+3}) = 3$ , then  $d(f_{i+2}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + 1 + \frac{2}{3} \times 2 + \frac{1}{3} = \frac{17}{3}$ . If  $d(v_{i+2}) = d(v_{i+4}) = 3$ , then  $d(v_{i+5}) \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + \frac{3}{4} \times 2 + \frac{2}{3} \times 2, \frac{3}{2} \times 2 + 1 + \frac{2}{3} \times 2 + \frac{1}{3}\} = \frac{35}{6}$ . It is similar with  $d(f_{i+1}) = d(f_{i+4}) = 3$ ,  $d(f_{i+2}) \geq 4$  and  $d(f_{i+3}) \geq 4$ . Suppose  $d(f_{i+2}) = d(f_{i+3}) = 3$ ,  $d(f_{i+1}) \geq 4$  and  $d(f_{i+4}) \geq 4$ , then  $d(v_{i+3}) \geq 4$ . If  $d(v_{i+2}) = d(v_{i+4}) = 3$ , then  $\max\{d(v_{i+1}), d(v_{i+5})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + \frac{3}{4} \times 4, \frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 1 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 \times 2 + \frac{1}{3} \times 2\} = 6$ . If  $d(v_{i+2}) = 3$  and  $d(v_{i+4}) \geq 4$ , then  $d(v_{i+1}) \geq 4$  or  $\max\{d(f_i), d(f_{i+1})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} + 1 + \frac{3}{4} \times 3, \frac{3}{2} + \frac{5}{4} + 1 \times 2 + \frac{3}{4} + \frac{1}{3}\} = 6$ . It is similar with  $d(v_{i+4}) = 3$  and  $d(v_{i+2}) \geq 4$ . Otherwise,  $\phi \leq \frac{5}{4} \times 2 + 1 \times 2 + \frac{3}{4} \times 2 = 6$ .

Suppose only one of  $f_{i+1}$ ,  $f_{i+2}$ ,  $f_{i+3}$  and  $f_{i+4}$  is 3-face, assume that  $d(f_{i+1}) = 3$ . If  $d(v_{i+1}) = 3$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + 1 \times 3 + \frac{2}{3} + \frac{1}{3} = \frac{11}{2}$ . If  $d(v_{i+2}) = 3$ , then  $\max\{d(v_{i+3}), d(v_{i+4}), d(v_{i+5})\} \geq 4$  or  $\max\{d(f_{i+2}), d(f_{i+3}), d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + 1 \times 2 + \frac{3}{4} \times 2 + \frac{2}{3}, \frac{3}{2} + 1 \times 3 + \frac{2}{3} + \frac{1}{3}\} = \frac{17}{3}$ . Otherwise,  $\phi \leq \frac{5}{4} + 1 \times 3 + \frac{3}{4} + \frac{2}{3} = \frac{17}{3}$ . Suppose  $\min\{d(f_{i+1}), d(f_{i+2}), d(f_{i+3}), d(f_{i+4})\} \geq 4$ , then  $\max\{d(v_{i+1}), d(v_{i+2}), d(v_{i+3}), d(v_{i+4}), d(v_{i+5})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2}), d(f_{i+3}), d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{1 \times 4 + \frac{3}{4} \times 2, 1 \times 5 + \frac{1}{3}\} = \frac{11}{2}$ .

At last,  $\phi \leq \max\{\frac{37}{6}, \frac{13}{2}, \frac{35}{6}, 6, \frac{65}{12}, \frac{17}{3}, \frac{11}{2}\} = \frac{13}{2}$ .

(f)  $k = 6$  Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = d(f_{i+4}) = d(f_{i+5}) = 3$ , then  $\min\{d(v_{i+2}), d(v_{i+3}), d(v_{i+4}), d(v_{i+5})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+6}) = 3$ ,

then  $d(f_i) \geq 5$  and  $d(f_{i+6}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} \times 2 + 1 + \frac{1}{3} \times 2 = \frac{43}{6}$ . If  $\min\{d(v_{i+1}), d(v_{i+6})\} \geq 4$ , then  $\phi \leq \frac{5}{4} \times 5 + \frac{3}{4} + \frac{2}{3} = \frac{23}{3}$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = d(f_{i+4}) = 3$  and  $d(f_{i+5}) \geq 4$ , then  $\min\{d(v_{i+2}), d(v_{i+3}), d(v_{i+4})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+5}) = 3$ , then  $d(f_i) \geq 5$  and  $d(v_{i+6}) \geq 4$  or  $\max\{d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + \frac{5}{4} \times 2 + \frac{3}{4} \times 2 + \frac{1}{3}, \frac{3}{2} \times 2 + \frac{5}{4} \times 2 + 1 + \frac{1}{3} \times 2\} = \frac{22}{3}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+5}) \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 3 + 1 + \frac{3}{4} + \frac{1}{3} = \frac{19}{3}$ . If  $d(v_{i+5}) = 3$  and  $d(v_{i+1}) \geq 4$ , then  $d(v_{i+6}) \geq 4$  or  $\max\{d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} \times 3 + \frac{3}{4} \times 3, \frac{3}{2} + \frac{5}{4} \times 3 + 1 + \frac{3}{4} + \frac{1}{3}\} = \frac{15}{2}$ . If  $\min\{d(v_{i+1}), d(v_{i+5})\} \geq 4$ , then  $\phi \leq \frac{5}{4} \times 4 + 1 + \frac{3}{4} \times 2 = \frac{15}{2}$ . Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = d(f_{i+5}) = 3$  and  $d(f_{i+4}) \geq 4$ , then  $\min\{d(v_{i+2}), d(v_{i+3})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+4}) = d(v_{i+5}) = 3$ , then  $d(f_i) \geq 5$  and  $d(f_{i+4}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 3 + 1 + \frac{2}{3} + \frac{1}{3} \times 2 = \frac{41}{6}$ . It is similar with  $d(v_{i+1}) = d(v_{i+4}) = d(v_{i+6}) = 3$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$  and  $\min\{d(v_{i+5}), d(v_{i+6})\} \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} + 1 + \frac{3}{4} + \frac{2}{3} + \frac{1}{3} = 7$ . If  $d(v_{i+1}) = d(v_{i+5}) = 3$  and  $d(v_{i+4}) \geq 5$ , then  $d(f_i) \geq 5$ , so  $\frac{3}{2} \times 2 + \frac{5}{4} \times 2 + \frac{2}{3} \times 2 + \frac{1}{3} = \frac{43}{6}$ . It is similar with  $d(v_{i+4}) = d(v_{i+5}) = 3$  and  $d(v_{i+1}) \geq 5$  or  $d(v_{i+4}) = d(v_{i+6}) = 3$  and  $d(v_{i+1}) \geq 5$ . If  $d(v_{i+1}) = d(v_{i+6}) = 3$  and  $d(v_{i+4}) \geq 4$ , then  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} \times 2 + \frac{2}{3} + \frac{1}{3} \times 2 = \frac{41}{6}$ . If  $d(v_{i+1}) = d(v_{i+5}) = 3$  and  $d(v_{i+4}) = 4$ , then  $d(f_i) \geq 5$ , then  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} + 1 + \frac{3}{4} + \frac{2}{3} + \frac{1}{3} = 7$ . It is similar with  $d(v_{i+4}) = d(v_{i+5}) = 3$  and  $d(v_{i+1}) = 4$  or  $d(v_{i+4}) = d(v_{i+6}) = 3$  and  $d(v_{i+1}) = 4$ . If  $d(v_{i+1}) = 3$  and  $\min\{d(v_{i+4}), d(v_{i+5}), d(v_{i+6})\} \geq 4$ , then  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 3 + \frac{3}{4} + \frac{2}{3} + \frac{1}{3} = 7$ . It is similar with  $d(v_{i+6}) = 3$  and  $\min\{d(v_{i+1}), d(v_{i+4}), d(v_{i+5})\} \geq 4$ . Otherwise,  $\phi \leq \frac{5}{4} \times 4 + \frac{3}{4} \times 2 + \frac{2}{3} = \frac{43}{6}$ . Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+4}) = d(f_{i+5}) = 3$  and  $d(f_{i+3}) \geq 4$ , then  $\min\{d(v_{i+2}), d(v_{i+5})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+4}) = d(v_{i+5}) = d(v_{i+6}) = 3$ , then  $\min\{d(f_i), d(f_{i+3}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \frac{3}{2} \times 4 + \frac{1}{3} \times 3 = 7$ . If  $d(v_{i+1}) = d(v_{i+6}) = 3$ , then  $\max\{d(v_{i+3}), d(v_{i+4})\} \geq 4$  or  $d(f_{i+3}) \geq 5$ , so

$\phi \leq \max\{\frac{3}{2} \times 3 + \frac{5}{4} + \frac{3}{4} + \frac{1}{3} \times 2, \frac{3}{2} \times 4 + \frac{1}{3} \times 3\} = 7$ . if  $\min\{d(v_{i+1}), d(v_{i+6})\} \geq 4$  and  $\max\{d(v_{i+3}), d(v_{i+4})\} \geq 4$ , then  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 3 + \frac{3}{4} \times 3 = \frac{15}{2}$ . Otherwise,  $\phi \leq \frac{5}{4} \times 4 + \frac{3}{4} \times 2 + \frac{1}{2} = 7$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = 3$ , and  $\min\{d(f_{i+4}), d(f_{i+5})\} \geq 4$ , then  $\min\{d(v_{i+2}), d(v_{i+3})\} \geq 4$ . if  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$  and  $\max\{d(v_{i+5}), d(v_{i+6})\} \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + 1 + \frac{3}{4} \times 2 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 + 1 \times 2 + \frac{1}{3} \times 2\} = \frac{41}{6}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+4}) = 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} + 1 + 1 \times 2 + \frac{3}{4} + \frac{1}{3} = \frac{41}{6}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+4}) \geq 5$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 2 + 1 \times 2 + \frac{2}{3} + \frac{1}{3} = 7$ . If  $d(v_{i+4}) = 3$  and  $d(v_{i+1}) = 4$ , then  $\max\{d(v_{i+5}), d(v_{i+6})\} \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} + 1 + 1 + \frac{3}{4} \times 3, \frac{3}{2} + \frac{5}{4} + 1 + 1 \times 2 + \frac{3}{4} + \frac{1}{3}\} = 7$ . If  $d(v_{i+4}) = 3$  and  $d(v_{i+1}) \geq 5$ , then  $\max\{d(v_{i+5}), d(v_{i+6})\} \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{2}{3}, \frac{3}{2} + \frac{5}{4} \times 2 + 1 \times 2 + \frac{2}{3} + \frac{1}{3}\} = \frac{43}{6}$ . Otherwise,  $\phi \leq \frac{5}{4} \times 3 + 1 \times 2 + \frac{3}{4} + \frac{2}{3} = \frac{43}{6}$ . Suppose  $d(f_{i+2}) = d(f_{i+3}) = d(f_{i+4}) = 3$  and  $\min\{d(f_{i+1}), d(f_{i+5})\} \geq 4$ , then  $\min\{d(v_{i+3}), d(v_{i+4})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+2}) = d(v_{i+5}) = d(v_{i+6}) = 3$ , then  $\max\{d(f_i), d(f_{i+1})\} \geq 5$  and  $\max\{d(v_{i+5}), d(v_{i+6})\} \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + 1 + 1 \times 2 + \frac{1}{3} \times 2 = \frac{20}{3}$ . If  $d(v_{i+1}) = d(v_{i+2}) = d(v_{i+5}) = 3$  and  $d(v_{i+6}) \geq 4$ , then  $\max\{d(f_i), d(f_{i+1})\} \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + 1 + 1 + \frac{3}{4} \times 2 + \frac{1}{3} = \frac{41}{6}$ . It is similar with  $d(v_{i+2}) = d(v_{i+5}) = d(v_{i+6}) = 3$  and  $d(v_{i+1}) \geq 4$ . If  $d(v_{i+2}) = 3$  and  $d(v_{i+5}) = 4$ , then  $d(v_{i+1}) \geq 4$  or  $\max\{d(f_i), d(f_{i+1})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} + 1 + 1 + \frac{3}{4} \times 3, \frac{3}{2} + \frac{5}{4} + 1 + 1 \times 2 + \frac{3}{4} + \frac{1}{3}\} = 7$ . It is similar with  $d(v_{i+5}) = 3$  and  $d(v_{i+2}) = 4$ . If  $d(v_{i+2}) = 3$  and  $d(v_{i+5}) \geq 5$ , then  $d(v_{i+1}) \geq 4$  or  $\max\{d(f_i), d(f_{i+1})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{2}{3}, \frac{3}{2} + \frac{5}{4} \times 2 + 1 \times 2 + \frac{2}{3} + \frac{1}{3}\} = \frac{43}{6}$ . It is similar with  $d(v_{i+5}) = 3$  and  $d(v_{i+2}) \geq 5$ . If  $d(v_{i+2}) = d(v_{i+5}) = 4$ , then  $\phi \leq \frac{5}{4} \times 2 + 1 + \frac{3}{4} \times 4 = \frac{13}{2}$ . If  $d(v_{i+2}) = 4$  and  $d(v_{i+5}) \geq 5$ , then  $\phi \leq \frac{5}{4} \times 2 + 1 \times 2 + \frac{3}{4} + \frac{2}{3} = \frac{43}{6}$ . It is similar

with  $d(v_{i+5}) = 3$  and  $d(v_{i+2}) \geq 5$ . Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+4}) = 3$  and  $\min\{d(f_{i+3}), d(f_{i+5})\} \geq 4$ , then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = d(v_{i+4}) = 3$ , then  $\min\{d(f_i), d(f_{i+3})\} \geq 5$ , so  $\phi \leq \frac{3}{2} \times 3 + 1 + \frac{2}{3} + \frac{1}{3} \times 2 = \frac{41}{6}$ . If  $d(v_{i+1}) = d(v_{i+3}) = d(v_{i+5}) = 3$ , then  $d(f_i) \geq 5$  and  $d(v_{i+6}) \geq 4$  or  $\max\{d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 3 + \frac{3}{4} \times 2 + \frac{2}{3} + \frac{1}{3}, \frac{3}{2} \times 3 + 1 + \frac{2}{3} + \frac{1}{3} \times 2\} = 7$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$  and  $\min\{d(v_{i+4}), d(v_{i+5})\} \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} + 1 + \frac{3}{4} + \frac{2}{3} + \frac{1}{3} = 7$ . It is similar with  $d(v_{i+1}) = d(v_{i+4}) = 3$  and  $d(v_{i+3}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+5}) = 3$  and  $d(v_{i+3}) \geq 4$ , then  $d(f_i) \geq 5$  and  $d(v_{i+6}) \geq 4$  or  $\max\{d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + \frac{5}{4} + \frac{3}{4} \times 2 + \frac{1}{2} + \frac{1}{3}, \frac{3}{2} \times 2 + \frac{5}{4} + \frac{1}{2} + \frac{1}{3} \times 2\} = \frac{79}{12}$ . If  $d(v_{i+1}) = d(v_{i+5}) = 3$  and  $d(v_{i+3}) \geq 5$ , then  $d(f_i) \geq 5$  and  $d(v_{i+6}) \geq 4$  or  $\max\{d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{2}{3} + \frac{1}{3}, \frac{3}{2} \times 2 + 1 + 1 + \frac{1}{2} + \frac{1}{3} \times 2\} = \frac{13}{2}$ . If  $d(v_{i+3}) = d(v_{i+4}) = 3$  and  $d(v_{i+1}) \geq 4$ , then  $d(f_{i+3}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + \frac{5}{4} + 1 + \frac{3}{4} + \frac{2}{3} + \frac{1}{3} = 7$ . If  $d(v_{i+3}) = d(v_{i+5}) = 3$  and  $d(v_{i+1}) \geq 4$ , then  $d(v_{i+6}) \geq 4$  or  $\max\{d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + \frac{5}{4} + \frac{3}{4} \times 3 + \frac{2}{3}, \frac{3}{2} \times 2 + \frac{5}{4} + 1 + \frac{3}{4} + \frac{2}{3} + \frac{1}{3}\} = \frac{43}{6}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+3}) = d(v_{i+5}) = 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 2 + 1 + \frac{3}{4} + \frac{1}{2} + \frac{1}{3} = \frac{79}{12}$ . If  $d(v_{i+1}) = 3$ ,  $d(v_{i+5}) = 4$  and  $d(v_{i+3}) \geq 5$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} + 1 + 1 + \frac{3}{4} + \frac{2}{3} + \frac{1}{3} = \frac{13}{2}$ . If  $d(v_{i+3}) = 3$ ,  $d(v_{i+1}) = 4$  and  $\min\{d(v_{i+4}), d(v_{i+5})\} \geq 4$ , then  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{2}{3} = \frac{43}{6}$ . If  $d(v_{i+4}) = 3$  and  $\min\{d(v_{i+1}), d(v_{i+3})\} \geq 4$ , then  $\phi \leq \frac{3}{2} + \frac{5}{4} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{2}{3} = \frac{43}{6}$ . If  $d(v_{i+5}) = 3$ ,  $d(v_{i+1}) \geq 4$  and  $d(v_{i+3}) = 4$ , then  $d(v_{i+6}) \geq 4$  or  $\max\{d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} \times 2 + \frac{3}{4} \times 2 + \frac{1}{2}, \frac{3}{2} + \frac{5}{4} \times 2 + 1 + \frac{3}{4} + \frac{1}{2} + \frac{1}{3}\} = \frac{27}{4}$ . If  $d(v_{i+5}) = 3$ ,  $d(v_{i+1}) \geq 4$  and  $d(v_{i+3}) \geq 5$ , then  $d(v_{i+6}) \geq 4$  or  $\max\{d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} + 1 + \frac{3}{4} \times 3 + \frac{2}{3}, \frac{3}{2} + \frac{5}{4} + 1 + 1 + \frac{3}{4} + \frac{2}{3} + \frac{1}{3}\} = \frac{40}{3}$ . Otherwise,  $\phi \leq \frac{5}{4} \times 3 + \frac{3}{4} \times 3 + \frac{1}{2} = \frac{13}{2}$ . It is similar with  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+5}) = 3$  or  $d(f_{i+2}) = d(f_{i+3}) = d(f_{i+5}) = 3$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = 3$  and  $\min\{d(f_{i+3}), d(f_{i+4}), d(f_{i+5})\} \geq 4$ , then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(f_i) \geq 5$  and  $\max\{d(v_{i+4}), d(v_{i+5}), d(v_{i+6})\} \geq 4$  or  $\max\{d(f_{i+3}), d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 \times 2 + \frac{3}{4} \times 2 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 \times 3 + \frac{1}{3} \times 2\} = \frac{41}{6}$ . If  $d(v_{i+1}) = 3$  and  $d(v_{i+3}) \geq 4$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{5}{4} + 1 \times 3 + \frac{3}{4} + \frac{1}{3} = \frac{41}{6}$ . If  $d(v_{i+3}) = 3$  and  $d(v_{i+1}) \geq 4$ , then  $\max\{d(v_{i+4}), d(v_{i+5}), d(v_{i+6})\} \geq 4$  or  $\max\{d(f_{i+3}), d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} + 1 \times 2 + \frac{3}{4} \times 3, \frac{3}{2} + \frac{5}{4} + 1 \times 3 + \frac{3}{4} + \frac{1}{3}\} = 7$ . Otherwise,  $\phi \leq \frac{5}{4} \times 2 + 1 \times 3 + \frac{3}{4} \times 2 = 7$ . Suppose  $d(f_{i+2}) = d(f_{i+3}) = 3$  and  $\min\{d(f_{i+1}), d(f_{i+4}), d(f_{i+5})\} \geq 4$ , then  $d(v_{i+3}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+2}) = d(v_{i+4}) = d(v_{i+5}) = 3$ , then  $\max\{d(f_i), d(f_{i+1})\} \geq 5$  and  $\max\{d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + 1 \times 3 + \frac{1}{3} \times 2 = \frac{20}{3}$ . If  $d(v_{i+2}) = d(v_{i+4}) = 3$  and  $d(v_{i+1}) \geq 4$ , then  $\max\{d(v_{i+5}), d(v_{i+6})\} \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 4, \frac{3}{2} \times 2 + 1 \times 2 + \frac{3}{4} \times 1 + \frac{1}{3}\} = 7$ . It is similar with  $d(v_{i+2}) = d(v_{i+4}) = 3$  and  $\max\{d(v_{i+5}), d(v_{i+6})\} \geq 4$ . If  $d(v_{i+2}) = 3$  and  $d(v_{i+4}) \geq 4$ , then  $d(v_{i+1}) \geq 4$  or  $\max\{d(f_i), d(f_{i+1})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + \frac{5}{4} + 1 \times 2 + \frac{3}{4} \times 3, \frac{3}{2} + \frac{5}{4} + 1 \times 3 + \frac{3}{4} + \frac{1}{3}\} = 7$ . It is similar with  $d(v_{i+4}) = 3$  and  $d(v_{i+2}) \geq 4$ . Otherwise,  $\phi \leq \frac{5}{4} \times 2 + 1 \times 3 + \frac{3}{4} \times 2 = 7$ . Suppose  $d(f_{i+1}) = d(f_{i+3}) = 3$  and  $\min\{d(f_{i+2}), d(f_{i+4}), d(f_{i+5})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + 1 \times 2 + \frac{2}{3} \times 2 + \frac{1}{3} = \frac{20}{3}$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$  and  $\max\{d(v_{i+5}), d(v_{i+6})\} \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{2}{3} + \frac{1}{3}, \frac{3}{2} \times 2 + 1 \times 2 + \frac{2}{3} + \frac{1}{3}\} = \frac{13}{2}$ . If  $d(v_{i+2}) = d(v_{i+3}) = 3$ , then  $d(f_{i+2}) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + 1 \times 2 + \frac{2}{3} \times 2 + \frac{1}{3} = \frac{20}{3}$ . If  $d(v_{i+2}) = d(v_{i+4}) = 3$ , then  $\max\{d(v_{i+5}), d(v_{i+6})\} \geq 4$  or  $\max\{d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{2}{3} \times 2, \frac{3}{2} \times 2 + 1 \times 2 + \frac{2}{3} \times 2 + \frac{1}{3}\} = \frac{41}{6}$ . If  $d(v_{i+2}) = 3$  and  $\min\{d(v_{i+3}), d(v_{i+4})\} \geq 4$ , then  $\phi \leq \frac{3}{2} + \frac{5}{4} + 1 \times 2 + \frac{3}{4} + \frac{2}{3} \times 2 = \frac{41}{6}$ . It is similar with  $d(f_{i+1}) = d(f_{i+5}) = 3$  and  $\min\{d(f_{i+2}), d(f_{i+3})\} \geq 4$ .

Suppose  $d(f_{i+1}) = d(f_{i+4}) = 3$  and  $\min\{d(f_{i+2}), d(f_{i+3}), d(f_{i+5})\} \geq 4$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} \times 2 + 1 \times 2 + \frac{2}{3} \times 2 + \frac{1}{3} = \frac{20}{3}$ . If  $d(v_{i+2}) = d(v_{i+4}) = 3$ , then  $d(v_{i+3}) \geq 4$  or  $\max\{d(f_{i+2}), d(f_{i+3})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} \times 2 + 1 + \frac{3}{4} \times 2 + \frac{2}{3} \times 2, \frac{3}{2} \times 2 + 1 \times 2 + \frac{2}{3} \times 2 + \frac{1}{3}\} = \frac{41}{6}$ . It is similar with  $d(v_{i+2}) = d(v_{i+5}) = 3$ . If  $d(v_{i+1}) = d(v_{i+5}) = 3$ , then  $d(v_{i+6}) \geq 4$  or  $\max\{d(f_{i+5}), d(f_{i+6})\} \geq 5$ , then  $\phi \leq \max\{\frac{3}{2} \times 2 + \frac{3}{4} \times 2 + \frac{2}{3} \times 2 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2\} = \frac{37}{6}$ . It is similar with  $d(f_{i+2}) = d(f_{i+4}) = 3$  and  $\min\{d(f_{i+1}), d(f_{i+3}), d(f_{i+5})\} \geq 4$ .

Suppose only one of  $f_{i+1}, f_{i+2}, f_{i+3}$  and  $f_{i+4}$  is 3-face, assume  $d(f_{i+1}) = 3$ . If  $d(v_{i+1}) = 3$ , then  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + 1 \times 4 + \frac{2}{3} + \frac{1}{3} = \frac{13}{2}$ . If  $d(v_{i+2}) = 3$ , then  $\max\{d(v_{i+3}), d(v_{i+4}), d(v_{i+5}), d(v_{i+6})\} \geq 4$  or  $\max\{d(f_{i+2}), d(f_{i+3}), d(f_{i+4}), d(f_{i+5}), d(f_{i+6})\} \geq 5$ , so  $\phi \leq \max\{\frac{3}{2} + 1 \times 3 + \frac{3}{4} \times 2 + \frac{2}{3}, \frac{3}{2} + 1 \times 4 + \frac{2}{3} + \frac{1}{3}\} = \frac{20}{3}$ . Otherwise,  $\phi \leq \frac{5}{4} + 1 \times 4 + \frac{3}{4} + \frac{2}{3} = \frac{20}{3}$ . Suppose  $\min\{d(f_{i+1}), d(f_{i+2}), d(f_{i+3}), d(f_{i+4})\} \geq 4$ , then  $\max\{d(v_{i+1}), d(v_{i+2}), d(v_{i+3}), d(v_{i+4}), d(v_{i+5})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2}), d(f_{i+3}), d(f_{i+4}), d(f_{i+5})\} \geq 5$ , so  $\phi \leq \max\{1 \times 5 + \frac{3}{4} \times 2, 1 \times 6 + \frac{1}{3}\} = \frac{13}{2}$ .

At last,  $\phi \leq \max\{\frac{43}{6}, \frac{23}{3}, \frac{22}{3}, \frac{19}{3}, \frac{15}{2}, \frac{41}{6}, 7, \frac{20}{3}, \frac{13}{2}, \frac{79}{12}, \frac{27}{4}, \frac{40}{3}, \frac{37}{6}\} = \frac{23}{3}$ .

□

**Proof of Theorem 2.3.1(continued).** If  $n_2(v) = 8$ , then all faces incident with  $v$  are 6<sup>+</sup>-faces by Lemma 2.3.2(2)-(4), that is,  $f_{6^+}(v) = 8$ , so  $ch'(v) = 10 - 1 \times 8 = 2 > 0$ . If  $n_2(v) = 7$ , then  $f_{6^+}(v) \geq 6$  and  $f_3(v) = 0$ , so  $ch'(v) \geq 10 - 1 \times 7 - \frac{3}{2} = \frac{3}{2} > 0$  by claim (a).

Suppose  $n_2(v) \leq 6$ . The possible distributions of 2-vertices adjacent to  $v$  are illustrated in Figure 2.5.

For Figure 2.5(1), we have  $f_{6^+}(v) \geq 5$  and  $ch'(v) \geq 10 - 1 \times 6 - \frac{8}{3} = \frac{4}{3} > 0$  by claim (b). For Figure 2.5(2)-(4), we have  $f_{6^+}(v) \geq 4$  and  $ch'(v) \geq 10 - 1 \times 6 - \frac{3}{2} \times 2 = 1 > 0$ . For Figure 2.5(5), we have  $f_{6^+}(v) \geq 4$  and

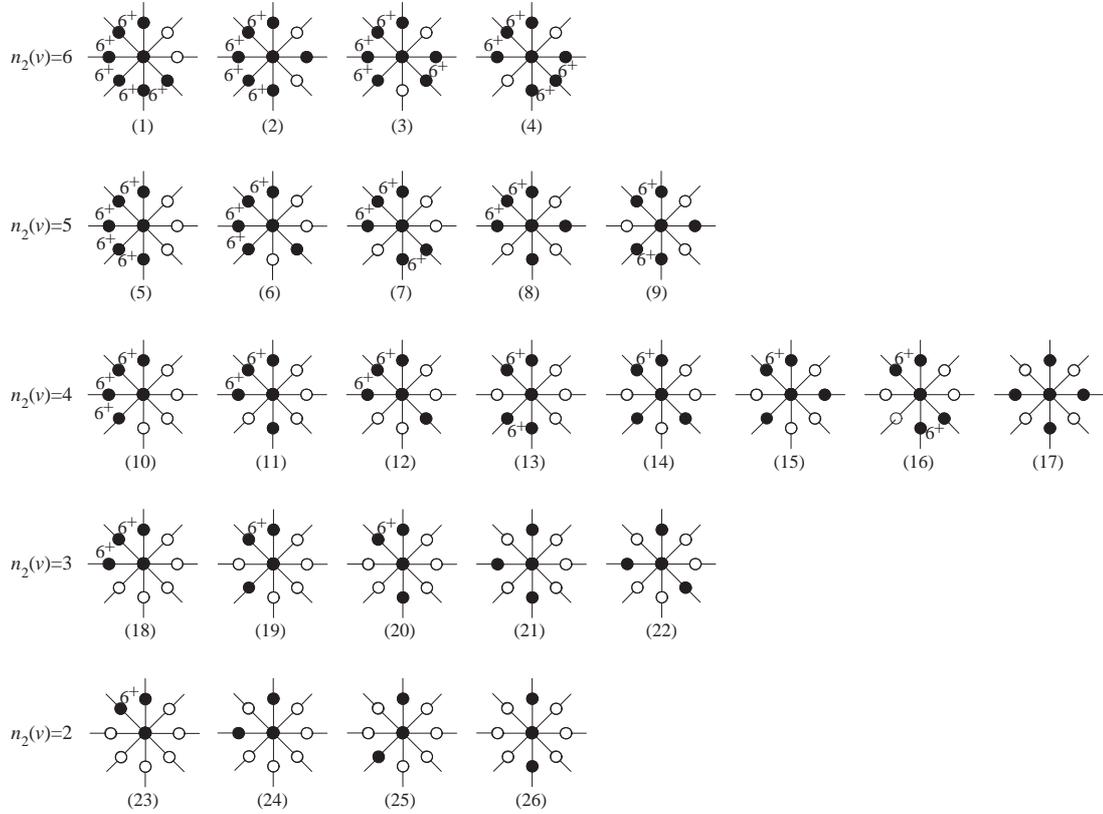


Figure 2.5

$ch'(v) \geq 10 - 1 \times 5 - 4 = 1 > 0$  by claim (c). For Figure 2.5(6)-(7), we have  $f_{6^+}(v) \geq 3$  and  $ch'(v) \geq 10 - 1 \times 5 - \frac{3}{2} - \frac{8}{3} = \frac{5}{6} > 0$ . For Figure 2.5(8)-(9), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 10 - 1 \times 5 - \frac{3}{2} \times 3 = \frac{1}{2} > 0$ . For Figure 2.5(10), we have  $f_{6^+}(v) \geq 3$  and  $ch'(v) \geq 10 - 1 \times 4 - \frac{31}{6} = \frac{5}{6} > 0$  by claim (d). For Figure 2.5(11) and Figure 2.5(13), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 10 - 1 \times 4 - \frac{3}{2} - 4 = \frac{1}{2} > 0$ . For Figure 2.5(12) and Figure 2.5(16), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 10 - 1 \times 4 - \frac{8}{3} \times 2 = \frac{2}{3} > 0$ . For Figure 2.5(14) and Figure 2.5(15), we have  $f_{6^+}(v) \geq 1$  and  $ch'(v) \geq 10 - 1 \times 4 - \frac{3}{2} \times 2 - \frac{8}{3} = \frac{1}{3} > 0$ . For Figure 2.5(17), we have  $ch'(v) \geq 10 - 1 \times 4 - \frac{3}{2} \times 4 = 0$ . For Figure 2.5(18), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 10 - 1 \times 3 - \frac{13}{2} = \frac{1}{2} > 0$  by claim (e). For Figure 2.5(19), we have  $f_{6^+}(v) \geq 1$  and  $ch'(v) \geq 10 - 1 \times 3 - \frac{3}{2} - \frac{21}{4} = \frac{1}{4} > 0$ .

For Figure 2.5(20), we have  $f_{6^+}(v) \geq 1$  and  $ch'(v) \geq 10 - 1 \times 3 - \frac{8}{3} - 4 = \frac{1}{3} > 0$ . For Figure 2.5(21), we have  $ch'(v) \geq 10 - 1 \times 3 - \frac{3}{2} \times 2 - 4 = 0$ . For Figure 2.5(22), we have  $ch'(v) \geq 10 - 1 \times 3 - \frac{3}{2} - \frac{8}{3} \times 2 = \frac{1}{6}$ . For Figure 2.5(23), we have  $ch'(v) \geq 10 - 1 \times 2 - \frac{31}{4} = \frac{1}{4} > 0$ . For Figure 2.5(24), we have  $ch'(v) \geq 10 - 1 \times 2 - \frac{3}{2} - \frac{13}{2} = 0$ . For Figure 2.5(25), we have  $ch'(v) \geq 10 - 1 \times 2 - \frac{8}{3} - \frac{21}{4} = \frac{1}{12} > 0$ . For Figure 2.5(26), we have  $ch'(v) \geq 10 - 1 \times 2 - 4 \times 2 = 0$ .

Hence we complete the proof of the theorem 2.3.1.

## §2.5 Planar graph with $\Delta \geq 7$

Shen and Wang [103] proved

**Lemma 2.5.1.** *Let  $G$  be a planar graph without 5-cycles. If  $\Delta \geq 7$ , then  $\chi''(G) = \Delta + 1$ .*

Wang et al. [145] proved

**Lemma 2.5.2.** *Let  $G$  be a planar graph without chordal 5-cycles. If  $\Delta \geq 7$ , then  $\chi''(G) = \Delta + 1$ .*

Wang and Wu [118] proved

**Lemma 2.5.3.** *Let  $G$  be a planar graph without intersecting 5-cycles. If  $\Delta \geq 7$ , then  $\chi''(G) = \Delta + 1$ .*

These results are all about planar graph with ( $\Delta \geq 7$ ) in order to prove their total chromatic number is ( $\Delta + 1$ ). We extend these result and get the following result: Let  $G$  be a planar graph without intersecting chordal 5-cycles. If  $\Delta \geq 7$ , then  $\chi''(G) = \Delta + 1$ .

**Theorem 2.5.1.** *Let  $G$  be a planar graph without intersecting chordal 5-cycles, that is, every vertex is incident with at most one chordal cycle of length 5. If  $\Delta \geq 7$ , then  $\chi''(G) = \Delta + 1$ .*

According to [127], the theorem 2.5.1 is true for planar graph with  $\Delta \geq 8$  and satisfy that every vertex is incident with at most one chordal cycle of length 5. So we assume in the following that  $\Delta = 8$ .

Let  $G = (V, E)$  be a minimal counterexample to the planar graph  $G$  with maximum degree  $\Delta = 7$ , such that  $|V| + |E|$  is minimal and  $G$  has been embedded in the plane. Then every proper subgraph of  $G$  is total-8-colorable. First we give some lemmas for  $G$ .

**Lemma 2.5.4.** [16] (a)  $G$  is 2-connected.

(b) If  $uv$  is an edge of  $G$  with  $d(u) \leq 4$ , then  $d(u) + d(v) \geq \Delta + 2$ .

(c) The subgraph induced by all edges joining 2-vertices to 7-vertices in  $G$  is a forest.

By Lemma 2.5.4(a): There is no 1-vertex in  $G$ . By Lemma 2.5.4(b): any two neighbors of a 2-vertex are 7-vertices. Any three neighbors of a 3-vertex are 6<sup>+</sup>-vertices. Any four neighbors of a 4-vertex are 5<sup>+</sup>-vertices.

**Lemma 2.5.5.**  $G$  has no configurations depicted in Figure 2.9, where  $w$  denotes the vertex of degree of 6.

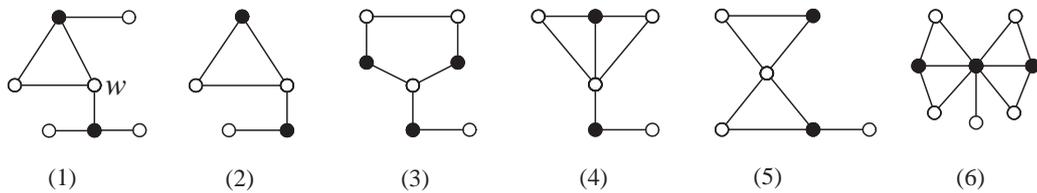


Figure 2.9

*Proof.* The proof of (1), (3) and (5) can be found in [139], (2) can be found in [102], (4) and (6) can be found in [74]. □

**Lemma 2.5.6.** [83]  *$G$  contains no 3-face incident with more than one 4-vertex.*

**Lemma 2.5.7.** *Let  $v$  be a vertex of  $G$ , and  $d(v) = d \geq 5$ . Let  $v_1, \dots, v_d$  be the neighbor of  $v$  and  $f_1, f_2, \dots, f_d$  be faces incident with  $v$ , such that  $f_i$  is incident with  $v_i$  and  $v_{i+1}$ , for  $i \in \{1, 2, \dots, d\}$ . Let  $d(v_1) = 2$  and  $\{v, u_1\} = N(v_1)$ . Then  $G$  does not satisfy one of the following conditions (see Figure 2.2).*

- (1) *there exists an integer  $k$  ( $2 \leq k \leq d-1$ ) such that  $d(v_{k+1}) = 2$ ,  $d(v_i) = 3$  ( $2 \leq i \leq k$ ) and  $d(f_j) = 4$  ( $1 \leq j \leq k$ ).*
- (2) *there exist two integers  $k$  and  $t$  ( $2 \leq k < t \leq d-1$ ) such that  $d(v_k) = 2$ ,  $d(v_i) = 3$  ( $k+1 \leq i \leq t$ ),  $d(f_t) = 3$  and  $d(f_j) = 4$  ( $k \leq j \leq t-1$ ).*
- (3) *there exist two integers  $k$  and  $t$  ( $3 \leq k \leq t \leq d-1$ ) such that  $d(v_i) = 3$  ( $k \leq i \leq t$ ),  $d(f_{k-1}) = d(f_t) = 3$  and  $d(f_j) = 4$  ( $k \leq j \leq t-1$ ).*

See the proof of Lemma 2.5.7 in Lemma 2.2.7.

Now we will use “Discharging method” to complete the proof of Theorem 2.5.1.

By the Euler’s formula  $|V| - |E| + |F| = 2$  and  $\sum_{v \in V} d(v) = 2|E|$ ,  $\sum_{f \in F} d(f) = 2|E|$ , we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

For each  $x \in V \cup F$ , we define the initial charge

$$ch(x) = \begin{cases} 2d(x) - 6 & \text{if } x \in V \\ d(x) - 6 & \text{if } x \in F \end{cases}$$

So  $\sum_{x \in V \cup F} ch(x) = -12 < 0$ . In the following, we will reassign a new charge denoted by  $ch'(x)$  to each  $x \in V \cup F$  according to the discharging rules. If we can show that  $ch'(x) \geq 0$  for each  $x \in V \cup F$ , then we get an obvious contradiction to  $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -12$ , which completes our proof.

We use  $(d(v_1), d(v_2), \dots, d(v_k)) \rightarrow (c_1, c_2, \dots, c_k)$  to denote that the vertex  $v_i$  sends the face the amount of charge  $c_i$  for  $i = 1, 2, \dots, k$ . Now we define the discharging rules as follows:

**R1.** Each 2-vertex receives 1 from each of its neighbors.

**R2.** For a 3-face  $(v_1, v_2, v_3)$ , let

$$(3^-, 6^+, 6^+) \rightarrow (0, \frac{3}{2}, \frac{3}{2}),$$

$$(4, 5^+, 5^+) \rightarrow (\frac{1}{2}, \frac{5}{4}, \frac{5}{4}),$$

$$(5^+, 5^+, 5^+) \rightarrow (1, 1, 1).$$

**R3.** For a 4-face  $(v_1, v_2, v_3, v_4)$ , let

$$(3^-, 6^+, 3^-, 6^+) \rightarrow (0, 1, 0, 1),$$

$$(3^-, 6^+, 4, 6^+) \rightarrow (0, \frac{3}{4}, \frac{1}{2}, \frac{3}{4}),$$

$$(3^-, 6^+, 5, 6^+) \rightarrow (0, \frac{11}{16}, \frac{5}{8}, \frac{11}{16}),$$

$$(3^-, 6^+, 6^+, 6^+) \rightarrow (0, \frac{1}{2}, 1, \frac{1}{2}),$$

$$(4^+, 4^+, 4^+, 4^+) \rightarrow (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}).$$

**R4.** For a 5-face  $(v_1, v_2, v_3, v_4, v_5)$ ,

**R4.1** let

$$(3^-, 6^+, 3^-, 6^+, 6^+) \rightarrow (0, 0, 0, \frac{1}{2}, \frac{1}{2}),$$

$$(3^-, 6^+, 4, 4, 6^+) \rightarrow (0, 0, \frac{1}{2}, \frac{1}{2}, 0),$$

$$\begin{aligned}
 (3^-, 6^+, 4, 5, 6^+) &\rightarrow (0, \frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4}), \\
 (3^-, 6^+, 5, 5^+, 6^+) &\rightarrow (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), \\
 (3^-, 6, 6, 6, 6) &\rightarrow (0, \frac{1}{2}, 0, 0, \frac{1}{2}), \\
 (3^-, 6^+, 6, 7^+, 6^+) &\rightarrow (0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}), \\
 (3^-, 7^+, 7^+, 7^+, 7^+) &\rightarrow (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}),
 \end{aligned}$$

**R4.2** If 5-face  $f$  is not incident with any  $3^-$ -vertices, then it receives  $x$  from each of its incident 4-vertices; and  $y$  from each of the other  $5^+$ -vertices indicates  $u$ . Let the number of 4-vertices is  $p$  ( $p$  is an integer and  $p \geq 0$ ).

**R4.2(a)** If  $p = 0$ , then  $x = 0$  and  $y = \frac{1}{5}$ ;

**R4.2(b)** If  $p = 1$ , then  $x = \frac{1}{2}$  and  $y = 0$  while  $u$  is adjacent with the 4-vertex, or  $y = \frac{1}{4}$  while  $u$  is not adjacent with the 4-vertex;

**R4.2(c)** If  $2 \leq p \leq 5$ , then  $x = \frac{1}{p}$ ,  $y = 0$ ;

Next, we show that  $ch'(x) \geq 0$  for all  $x \in V \cup F$ . It is easy to check that  $ch'(f) \geq 0$  for all  $f \in F$  and  $ch'(v) \geq 0$  for all 2-vertices  $v \in V$  by the above discharging rules. If  $d(v) = 3$ , then  $ch'(v) = ch(v) = 0$ . If  $d(v) = 4$ , then  $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$ . For  $d(v) \geq 5$ , we need the following structural lemma. Let  $v_1, \dots, v_d$  be the neighbor of  $v$  and  $f_1, f_2, \dots, f_d$  be faces incident with  $v$ , such that  $f_i$  is incident with  $v_i$  and  $v_{i+1}$ , for  $i \in \{1, 2, \dots, d\}$ .

**Lemma 2.5.8.** *Since  $G$  contains no intersecting chordal 5-cycles, we have the following results. Suppose  $G$  has a subgraph isomorphic to Figure 2.10.*

*Then*

- (1) *If  $d(f_i) \geq 4$  and  $d(f_{i+2}) \geq 4$ , then  $\max\{d(f_i), d(f_{i+2})\} \geq 5$ ;*
- (2) *If  $d(f_i) \geq 4$ , then  $d(f_i) \geq 5$ .*
- (3) *If  $d(f_i) \geq 4$  and  $d(f_{i+3}) \geq 4$ , then  $\max\{d(f_i), d(f_{i+3})\} \geq 5$ ;*
- (4) *If  $d(f_{i+1}) \geq 4$ , then  $d(f_{i+1}) \geq 5$ ;*
- (5) *If  $d(f_{i+1}) \geq 4$ , then  $d(f_{i+1}) \geq 5$ ;*
- (6) *If  $d(f_i) \geq 4$  and  $d(f_{i+4}) \geq 4$ , then  $d(f_i) \geq 5$  and  $d(f_{i+4}) \geq 5$ .*

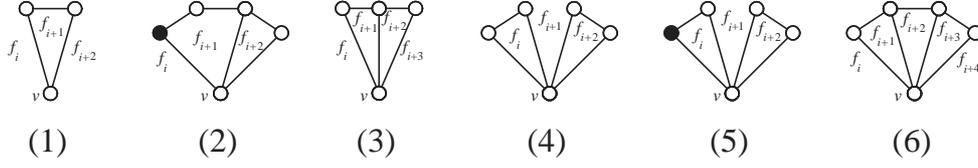


Figure 2.10

We also use  $(v_1, v_2, \dots, v_d)$  to denote a face whose boundary vertices are  $v_1, v_2, \dots, v_d$  in the clockwise order.

Suppose  $d(v) = 5$ . Then  $f_3(v) \leq 3$ . Suppose  $f_3(v) = 3$ . Then  $f_{5^+}(v) \geq 2$  by Lemma 2.5.8. If all the 3-faces are  $(5, 5^+, 4)$ -faces, then  $v$  is incident with a 4-vertex of  $5^+$ -face, so  $ch'(v) \geq ch(v) - 3 \times \frac{5}{4} - 1 \times \frac{1}{4} = 0$ . If at least one of the 3-faces are  $(5, 5^+, 5^+)$ -faces, then  $ch'(v) \geq ch(v) - 2 \times \frac{5}{4} - 1 \times 1 - 2 \times \frac{1}{4} = 0$ . If  $1 \leq f_3(v) \leq 2$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq ch(v) - f_3(v) \times \frac{5}{4} - (4 - f_3(v)) \times \frac{5}{8} - 1 \times \frac{1}{4} = \frac{6-3f_3(v)}{4} \geq 0$ . If  $f_3(v) = 0$ , then  $ch'(v) \geq ch(v) - 5 \times \frac{5}{8} = \frac{7}{8} > 0$ .

Suppose  $d(v) = 6$ . Then  $f_3(v) \leq 4$ . By Lemma 2.5.5(1),  $v$  is incident with at most two 3-faces are incident with a  $3^-$ -vertex, that is,  $v$  sends  $\frac{3}{2}$  to each of the two 3-faces and at most  $\frac{5}{4}$  to each other 3-face. If  $3 \leq f_3(v) \leq 4$ , then  $f_{5^+}(v) \geq 2$ , so  $ch'(v) \geq ch(v) - 2 \times \frac{3}{2} - (f_3(v) - 2) \times \frac{5}{4} - (4 - f_3(v)) \times 1 - 2 \times \frac{1}{4} = \frac{4-f_3(v)}{4} \geq 0$ . If  $1 \leq f_3(v) \leq 2$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq ch(v) - f_3(v) \times \frac{3}{2} - (5 - f_3(v)) \times 1 - (2 - f_3(v)) \times \frac{1}{2} = 0$ . If  $f_3(v) = 0$ , then  $ch'(v) \geq ch(v) - 6 \times 1 = 0$ .

Suppose  $d(v) = 7$ . Then  $ch(v) = 8$ . Let  $v_1, \dots, v_7$  be neighbors of  $v$  in the clockwise order and  $f_1, f_2, \dots, f_7$  be faces incident with  $v$ , such that  $f_i$  is incident with  $v_i$  and  $v_{i+1}$ , for  $i \in \{1, 2, \dots, 7\}$ , and  $f_8 = f_1$ . First we give Lemma 2.5.9 for convenience.

**Lemma 2.5.9.** *Suppose that  $d(v_i) = d(v_{i+k+1}) = 2$  and  $d(v_j) \geq 3$  for  $i+1 \leq j \leq i+k$ . Then  $v$  sends at most  $\phi$  (in total) to  $f_i$  and  $f_{i+1}, f_{i+2}, \dots, f_{i+k}$ , where  $\phi = \frac{2k+1}{2}$  ( $k = 1, 2, 3, 4$ ), (see Figure 2.11).*

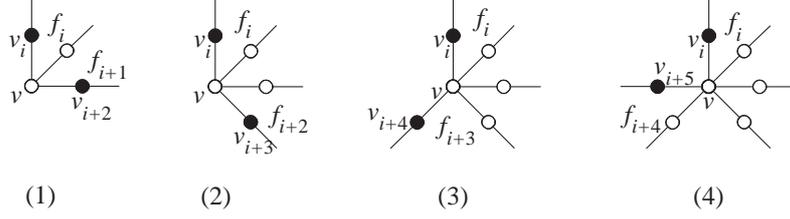


Figure 2.11

By Lemma 2.5.7,  $d(f_i) \geq 4$  and  $d(f_{i+k}) \geq 4$ .

**Case 1.**  $k = 1$ . By Lemma 2.5.7(1), we have  $d(v_{i+1}) \geq 4$  or  $\max\{d(f_i), d(f_{i+1})\} \geq 5$ , so  $\phi \leq \max\{2 \times \frac{3}{4}, 1 + 0\} = \frac{3}{2}$ .

**Case 2.**  $k = 2$ . If  $d(f_{i+1}) = 3$ , then  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+2})\} \geq 5$  by Lemma 2.5.7(2), and it follows that  $\phi \leq \max\{\frac{3}{4} + \frac{5}{4} + \frac{1}{2}, \frac{1}{2} + \frac{3}{2} + 0\} = \frac{5}{2}$ . Otherwise,  $d(f_{i+1}) \geq 4$ , then  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2})\} \geq 5$ , and it follows that  $\phi \leq \max\{1 + 2 \times \frac{3}{4}, 2 \times 1 + 0\} = \frac{5}{2}$ .

**Case 3.**  $k = 3$ . Suppose  $d(f_{i+1}) = d(f_{i+2}) = 3$ . Then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(f_i) \geq 5$ ,  $d(f_{i+3}) \geq 5$  and  $v$  sends 0 to  $f_i$  and  $f_{i+3}$ , so  $\phi \leq \frac{3}{2} \times 2 = 3$ . If  $\min\{d(v_{i+1}), d(v_{i+3})\} \geq 4$ , then  $\max\{d(f_i), d(f_{i+3})\} \geq 5$ , so  $\phi \leq 2 \times \frac{5}{4} + \frac{3}{4} + \frac{1}{4} = \frac{7}{2}$ .

Suppose  $d(f_{i+1}) = 3$  and  $d(f_{i+2}) \geq 4$ . If  $d(v_{i+1}) = 3$ , then  $d(v_{i+2}) \geq 6$  and  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + \frac{1}{2} + 1 = 3$ . If  $d(v_{i+2}) = 3$ , then  $d(v_{i+1}) \geq 6$ ,  $d(v_{i+3}) \geq 4$  and  $\max\{d(f_i), d(f_{i+2})\} \geq 5$ , so  $\phi \leq \frac{1}{2} + \frac{3}{2} + 2 \times \frac{3}{4} = \frac{7}{2}$ . If  $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$ , then  $\max\{d(f_i), d(f_{i+2})\} \geq 5$ , so  $\phi \leq \frac{1}{2} + \frac{5}{4} + \frac{3}{4} + 1 = \frac{7}{2}$ . It is similar with  $d(f_{i+2}) = 3$  and  $d(f_{i+1}) \geq 4$ .

Suppose  $\min\{d(f_{i+1}), d(f_{i+2})\} \geq 4$ . Then  $\max\{d(v_{i+1}), d(v_{i+2}), d(v_{i+3})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2}), d(f_{i+3})\} \geq 5$ , so  $\phi \leq \max\{2 \times 1 + 2 \times \frac{3}{4}, 3 \times 1 + 0\} = \frac{7}{2}$ . So  $\phi \leq \max\{3, \frac{7}{2}\} = \frac{7}{2}$ .

**Case 4.**  $k = 4$ . Suppose  $d(f_{i+1}) = d(f_{i+2}) = d(f_{i+3}) = 3$ . Then

$\min\{d(v_{i+2}), d(v_{i+3})\} \geq 4$ ,  $d(f_i) \geq 5$  and  $d(f_{i+4}) \geq 5$ . If  $d(v_{i+1}) = d(v_{i+4}) = 3$ , then  $v$  sends 0 to  $f_i$  and  $f_{i+4}$ , so  $\phi \leq 2 \times \frac{3}{2} + 1 = 4$ . If  $\min\{d(v_{i+1}), d(v_{i+4})\} \geq 4$ , then  $\phi \leq 3 \times \frac{5}{4} + \frac{1}{2} + \frac{1}{4} = \frac{9}{2}$ .

Suppose  $d(f_{i+1}) = d(f_{i+2}) = 3$  and  $\max\{d(f_i), d(f_{i+3})\} \geq 5$ . Then  $d(v_{i+2}) \geq 4$ . If  $d(v_{i+1}) = d(v_{i+3}) = 3$ , then  $d(v_{i+4}) \geq 4$  and  $v$  sends 0 to  $5^+$ -faces, so  $\phi \leq 2 \times \frac{3}{2} + 2 \times \frac{3}{4} = \frac{9}{2}$ . If  $\min\{d(v_{i+1}), d(v_{i+3})\} \geq 4$ , then  $\phi \leq 2 \times \frac{5}{4} + 1 + \frac{3}{4} + \frac{1}{4} = \frac{9}{2}$ . Similar with  $d(f_{i+2}) = d(f_{i+3}) = 3$  and  $\max\{d(f_{i+1}), d(f_{i+4})\} \geq 5$ . Suppose  $d(f_{i+1}) = d(f_{i+3}) = 3$  and  $\max\{d(f_i), d(f_{i+4})\} \geq 5$ . Then  $d(f_{i+2}) \geq 5$  and  $\max\{d(v_{i+2}), d(v_{i+3})\} \geq 4$  by Lemma 2.5.7(3), so  $\phi \leq 2 \times \frac{3}{2} + \frac{3}{4} + \frac{1}{2} = \frac{17}{4}$ .

Suppose  $d(f_{i+1}) = 3$  and  $\max\{d(f_i), d(f_{i+2})\} \geq 5$ . If  $d(v_{i+1}) = 3$ , then  $d(v_{i+2}) \geq 6$  and  $d(f_i) \geq 5$ , so  $\phi \leq \frac{3}{2} + 2 \times 1 + \frac{3}{4} = \frac{17}{4}$ . If  $d(v_{i+2}) = 3$ , then  $d(v_{i+1}) \geq 6$  and  $\max\{d(v_{i+3}), d(v_{i+4})\} \geq 4$ , so  $\phi \leq \frac{3}{2} + 1 + 2 \times \frac{3}{4} + \frac{1}{2} = \frac{9}{2}$ . Otherwise,  $\phi \leq \frac{5}{4} + 2 \times 1 + \frac{3}{4} + \frac{1}{2} = \frac{9}{2}$ . It is similar with  $d(f_{i+3}) = 3$  and  $\max\{d(f_{i+2}), d(f_{i+4})\} \geq 5$ . Suppose  $d(f_{i+2}) = 3$  and  $\max\{d(f_{i+1}), d(f_{i+3})\} \geq 5$ . If  $d(v_{i+2}) = 3$  or  $d(v_{i+3}) = 3$ , then  $\phi \leq \frac{3}{2} + 1 + 2 \times \frac{3}{4} + \frac{1}{2} = \frac{9}{2}$ . Otherwise,  $\phi \leq \frac{5}{4} + 2 \times 1 + \frac{3}{4} + \frac{1}{2} = \frac{9}{2}$ .

Suppose  $\min\{d(f_{i+1}), d(f_{i+2}), d(f_{i+3})\} \geq 4$ . Then  $\max\{d(v_{i+1}), d(v_{i+2}), d(v_{i+3}), d(v_{i+4})\} \geq 4$  or  $\max\{d(f_i), d(f_{i+1}), d(f_{i+2}), d(f_{i+3}), d(f_{i+4})\} \geq 5$ , so  $\phi \leq \max\{3 \times 1 + 2 \times \frac{3}{4}, 4 \times 1 + 0\} = \frac{9}{2}$ . So  $\phi \leq \max\{4, \frac{9}{2}, \frac{17}{4}\} = \frac{9}{2}$ .

□

Next, we prove  $d(v) = 7$ .

If  $n_2(v) = 7$ , then all faces incident with  $v$  are  $6^+$ -faces by Lemma 2.5.4-2.5.5, that is,  $f_{6^+}(v) = 7$ , so  $ch'(v) = 8 - 7 \times 1 = 1 > 0$ . If  $n_2(v) = 6$ , then  $f_{6^+}(v) \geq 5$  and  $f_3(v) = 0$ , so  $ch'(v) \geq 8 - 6 \times 1 - \frac{3}{2} = \frac{1}{2} > 0$  by Lemma 2.5.9.

Suppose  $2 \leq n_2(v) \leq 5$ . The possible distributions of 2-vertices adjacent to  $v$  are illustrated in Figure 2.12.

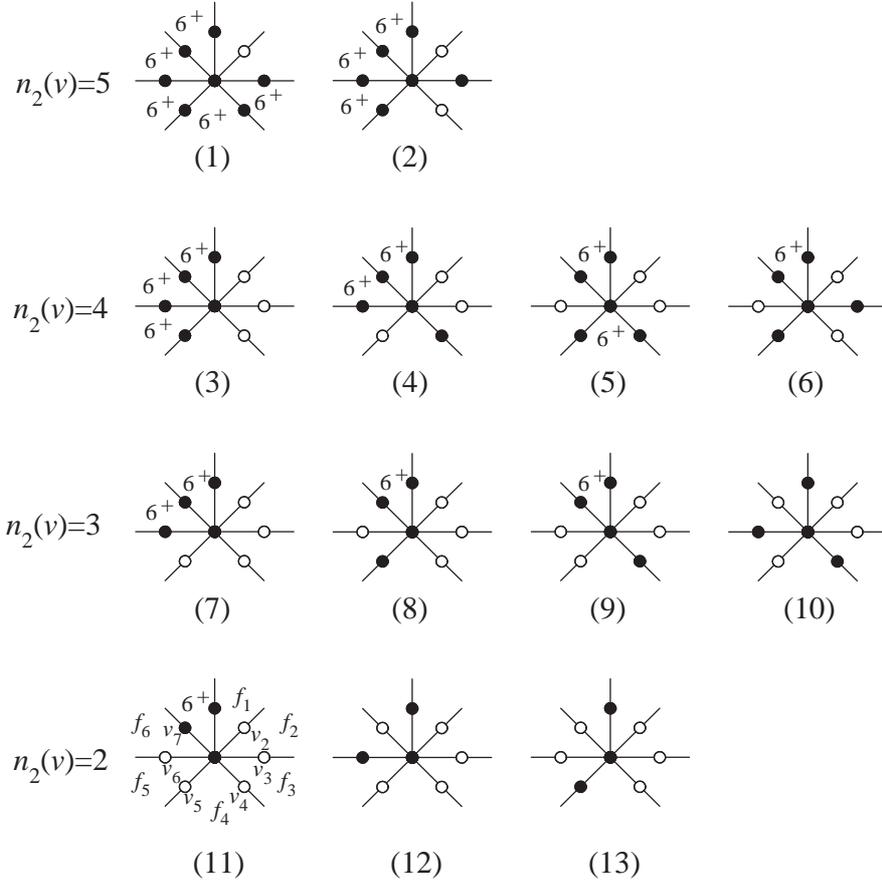


Figure 2.12

For Figure 2.12(1), we have  $f_{6^+}(v) \geq 4$  and  $ch'(v) \geq 8 - 5 \times 1 - \frac{5}{2} = \frac{1}{2} > 0$  by Lemma 2.5.9. For Figure 2.12(2), we have  $f_{6^+}(v) \geq 3$  and  $ch'(v) \geq 8 - 5 \times 1 - 2 \times \frac{3}{2} = 0$ . For Figure 2.12(3), we have  $f_{6^+}(v) \geq 3$  and  $ch'(v) \geq 8 - 4 \times 1 - \frac{7}{2} = \frac{1}{2} > 0$ . For Figure 2.12(4), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 8 - 4 \times 1 - \frac{3}{2} - \frac{5}{2} = 0$ . For Figure 2.12(5), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 8 - 4 \times 1 - \frac{3}{2} - \frac{5}{2} = 0$ . For Figure 2.12(6), as  $v$  is incident at least one  $6^+$ -vertex [145], then  $ch'(v) \geq 8 - 4 \times 1 - 2 \times \frac{3}{2} - 2 \times \frac{1}{2} = 0$ . For Figure 2.12(7), we have  $f_{6^+}(v) \geq 2$  and  $ch'(v) \geq 8 - 3 \times 1 - \frac{9}{2} = \frac{1}{2} > 0$ . For Figure 2.12(8), we have  $f_{6^+}(v) \geq 1$  and  $ch'(v) \geq 8 - 3 \times 1 - \frac{3}{2} - \frac{7}{2} = 0$ . For Figure 2.12(9), we

have  $f_{6^+}(v) \geq 1$  and  $ch'(v) \geq 8 - 3 \times 1 - \frac{5}{2} \times 2 = 0$ . For Figure 2.12(10), as  $v$  is incident at least one  $6^+$ -vertex [145], then  $ch'(v) \geq 8 - 3 \times 1 - \frac{5}{2} - \frac{3}{2} - 2 \times \frac{1}{2} = 0$ . For Figure 2.12(11), we have  $f_{6^+}(v) \geq 1$ . Suppose  $d(f_2) = d(f_3) = d(f_4) = 3$ . Then  $\min\{d(v_3), d(v_4)\} \geq 4$ ,  $d(f_1) \geq 5$  and  $d(f_5) \geq 5$ . If  $d(v_2) = d(v_5) = 3$ , then  $v$  sends 0 to  $f_1$  and  $f_5$ , so  $ch'(v) \geq 8 - 2 \times 1 - 2 \times \frac{3}{2} - 1 - 1 = 1 > 0$ . If  $\min\{d(v_2), d(v_5)\} \geq 4$ , then  $ch'(v) \geq 8 - 2 \times 1 - 3 \times \frac{5}{4} - 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{2} > 0$ . Suppose  $d(f_2) = d(f_3) = d(f_5) = 3$ . Then  $d(v_3) \geq 4$ ,  $\max\{d(v_4), d(v_4)\} \geq 4$ ,  $d(f_4) \geq 5$  and  $\max\{d(f_1), d(f_6)\} \geq 5$ . If  $d(v_2) = d(v_4) = 3$ , then  $d(f_1) \geq 5$  and  $v$  sends 0 to  $f_1$  and  $f_4$ , so  $ch'(v) \geq 8 - 2 \times 1 - 3 \times \frac{3}{2} - \frac{1}{2} = 1 > 0$ . If  $\min\{d(v_2), d(v_4)\} \geq 4$ , then  $ch'(v) \geq 8 - 2 \times 1 - \frac{3}{2} - 2 \times \frac{5}{4} - \frac{3}{4} - \frac{1}{2} - \frac{1}{4} = \frac{1}{2} > 0$ . If  $1 \leq f_3(v) \leq 2$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq 8 - 2 \times 1 - f_3(v) \times \frac{5}{4} - (4 - f_3(v)) \times 1 - \frac{3}{4} - \frac{1}{2} = \frac{3 - f_3(v)}{4} > 0$ . If  $f_3(v) = 0$ , then  $ch'(v) \geq 8 - 2 \times 1 - 6 \times 1 = 0$ . For Figure 2.12(12), we have  $ch'(v) \geq 8 - 2 \times 1 - \frac{3}{2} - \frac{9}{2} = 0$ . For Figure 2.12(13), we have  $ch'(v) \geq 8 - 2 \times 1 - \frac{5}{2} - \frac{7}{2} = 0$ .

Suppose  $n_2(v) = 1$ . Without loss of generality, assume  $d(v_1) = 2$ .

Suppose  $v_1$  is incident with a 3-cycle  $f_1$ . Then  $f_3(v) \leq 5$  and all 3-faces incident with no  $3^-$ -vertex except  $f_1$  by Lemma 2.5.5(5). If  $f_3(v) = 5$ , then  $f_{6^+}(v) \geq 1$ , so  $ch'(v) \geq 8 - 1 - \frac{3}{2} - 4 \times \frac{5}{4} - \frac{1}{2} = 0$ . If  $3 \leq f_3(v) \leq 4$ , then  $f_{5^+}(v) \geq 2$ , so  $ch'(v) \geq 8 - 1 - \frac{3}{2} - (f_3(v) - 1) \times \frac{5}{4} - 1 - (4 - f_3(v)) \times \frac{3}{4} - \frac{1}{2} - \frac{1}{4} = \frac{4 - f_3(v)}{4} \geq 0$ . If  $1 \leq f_3(v) \leq 2$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq 8 - 1 - \frac{3}{2} - (f_3(v) - 1) \times \frac{5}{4} - (6 - f_3(v)) \times 1 = \frac{3 - f_3(v)}{4} > 0$ .

Suppose  $v_1$  is not incident with a 3-cycle. Then  $f_3(v) \leq 4$ . If  $f_3(v) = 4$ , then  $f_{5^+}(v) \geq 3$ , so  $ch'(v) \geq 8 - 1 - \frac{3}{2} - 3 \times \frac{5}{4} - 3 \times \frac{1}{2} = \frac{1}{4} > 0$ . If  $f_3(v) = 3$ , then  $f_{5^+}(v) \geq 2$ , so  $ch'(v) \geq 8 - 1 - 3 \times \frac{5}{4} - 2 \times 1 - 2 \times \frac{1}{2} = \frac{1}{4}$ . If  $1 \leq f_3(v) \leq 2$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq 8 - 1 - f_3(v) \times \frac{5}{4} - (5 - f_3(v)) \times 1 - \frac{3}{4} - \frac{1}{2} = \frac{3 - f_3(v)}{4} > 0$ . If  $f_3(v) = 0$ , then  $ch'(v) \geq 8 - 1 - 7 \times 1 = 0$ .

Suppose  $n_2(v) = 0$ . Then  $f_3(v) \leq 5$ . If  $f_3(v) = 5$ , then  $f_{5^+}(v) \geq 2$ , so

$ch'(v) \geq 8 - 5 \times \frac{2}{3} - \frac{1}{2} = 0$ . If  $3 \leq f_3(v) \leq 4$ , then  $f_{5^+}(v) \geq 2$ , so  $ch'(v) \geq 8 - f_3(v) \times \frac{3}{2} - (5 - f_3(v)) \times 1 - (f_3(v) - 2) \times \frac{1}{2} = f_3(v) - 4 \geq 0$ . If  $1 \leq f_3(v) \leq 2$ , then  $f_{5^+}(v) \geq 1$ , so  $ch'(v) \geq 8 - f_3(v) \times \frac{3}{2} - (6 - f_3(v)) \times 1 - 1 \times \frac{1}{2} = \frac{3 - f_3(v)}{2} > 0$ . If  $f_3(v) = 0$ , then  $ch'(v) \geq 8 - 7 \times 1 = 1 > 0$ .

Hence we complete the proof of the theorem 2.5.1



## Chapter 3 List Coloring

### §3.1 List vertex coloring

In this section, we mainly proved the theorem below:

**Theorem 3.1.1.** *Let  $G$  be a planar graph. If every 5-cycles of  $G$  is not adjacent simultaneously to 3-cycles and 4-cycles, then  $G$  is 4-choosable.*

**Proof** Arguing by contradiction, we assume that  $G = (V, E)$  is a counterexample to Theorem 3.1.1 having the fewest vertices. Embed  $G$  into the plane, then

(1)  $\delta(G) \geq 4$  (see [75]).

(2)  $G$  does not contain a 5-cycle  $(v_1, v_2, \dots, v_5)$  adjacent to a 3-cycle  $(v_1, v_2, u)$  such that  $d(u) = 4$  and  $d(v_i) = 4$  for every  $i \in \{1, 2, \dots, 5\}$  (see [75]).

(3)  $G$  does not contain a 5-cycle  $(v_1, v_2, \dots, v_5)$  such that  $v_i v_j \in E(G)$  ( $1 \leq i < j - 1 \leq 4$ ), that is, any 5-cycle has no chord.

(4) If two 3-faces are adjacent, then each of the other faces adjacent to one of the two 3-faces is a  $6^+$ -face.

By Euler's formula  $|V| - |E| + |F| = 2$ , we have

$$\sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (d(f) - 4) = -4(|V| - |E| + |F|) = -8 < 0$$

For each  $x \in V \cup F$ , we define the initial charge

$$ch(x) = \begin{cases} d(x) - 4 & \text{if } x \in V \\ d(x) - 4 & \text{if } x \in F \end{cases}$$

So  $\sum_{x \in V \cup F} ch(x) = -8 < 0$ . If we can define suitable discharging rules such that, for every  $x \in V \cup F$ , the *final* charge of  $x$ , denoted  $ch'(x)$ , is non-negative, then we get an obvious contradiction to  $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -8$ , which completes our proof.

Let  $w(x \rightarrow y)$  be the charge transferred from  $x$  to  $y$  for all  $x, y \in V \cup F$ .

We define the discharging rules as follows:

**R1.** Let  $f$  be a 3-face  $(u, v, w)$  of  $G$ .

**R1.1.** If  $f$  is not adjacent to a 3-face, then  $w(f' \rightarrow f) = \frac{1}{3}$  for any  $5^+$ -face  $f'$  adjacent to  $f$ .

**R1.2.** Suppose that  $uv$  is incident with two 3-faces and  $d(u) \leq d(v)$ . If  $d(u) = d(v) = 4$ , then  $w(f' \rightarrow f) = \frac{1}{2}$  for each  $6^+$ -face  $f'$  adjacent to  $f$ ; Otherwise,  $w(v \rightarrow f) = \frac{1}{3}$  and  $w(f' \rightarrow f) = \frac{1}{3}$  for each  $6^+$ -face  $f'$  adjacent to  $f$ .

**R2.** Let  $f$  be a 5-face  $(v_1, v_2, \dots, v_5)$  of  $G$  and  $f_i$  be the other face incident with  $v_i v_{i+1}$  for  $i \in \{1, 2, \dots, 5\}$ , where all the subscripts are taken modulo 5.

**R2.1.** Suppose that all  $f_i$  ( $1 \leq i \leq 5$ ) are 3-faces  $(v_i, v_{i+1}, u_i)$ . If  $n_4(f) = 5$ , that is,  $f$  is a  $(4, 4, 4, 4, 4)$ -face, then  $w(u_i \rightarrow f) = \frac{1}{6}$  ( $1 \leq i \leq 5$ ); Otherwise,  $f$  receives  $2/(3n_{5^+}(f))$  from each of  $5^+$ -vertices incident with  $f$ .

**R2.2.** Suppose that  $f$  is adjacent to four 3-faces, without loss of generality,  $f_i$  is a 3-face  $(v_i, v_{i+1}, u_i)$  of  $G$ , where  $i = 1, 2, 3, 4$ . If  $n_4(f) = 5$ , then  $w(u_i \rightarrow f) = \frac{1}{6}$  ( $2 \leq i \leq 4$ ); Otherwise,  $f$  receives  $1/(3n_{5^+}(f))$  from each of  $5^+$ -vertices incident with  $f$ .

In the following, we will check that  $ch'(x) \geq 0$  for each  $x \in V \cup F$ . Let  $f \in F$ . If  $d(f) = 3$ , then  $ch'(f) \geq ch(f) + \max\{\frac{1}{2} \times 2, \frac{1}{3} \times 3\} = 0$  by (3), (4) and R1. If  $d(f) = 4$ , then  $ch'(f) = ch(f) = 0$ . Suppose  $d(f) = 5$ . Note that if  $f$  is adjacent to a 3-face  $f'$ , then  $f$  is not adjacent to any 4-cycle and it follows that all faces incident with  $f'$  must be  $5^+$ -faces. If  $f$  is adjacent to at most three 3-faces, then  $ch'(f) \geq ch(f) - \frac{1}{3} \times 3 = 0$  by R1; Otherwise,  $ch'(f) \geq ch(f) + \min\{\frac{1}{6} \times 5 - \frac{1}{3} \times 5, \frac{2}{3n_{5^+}(f)} \times n_{5^+}(f) - \frac{1}{3} \times 5, \frac{1}{6} \times 3 - \frac{1}{3} \times 4, \frac{1}{3n_{5^+}(f)} \times n_{5^+}(f) - \frac{1}{3} \times 4\} = 0$  by R2. Suppose that  $f$  is a  $k$ -face  $(v_1, v_2, \dots, v_k)$ , where  $k \geq 6$ . We denote by  $f_i$  the face adjacent to  $f$  and incident with  $v_i v_{i+1}$  where all the subscripts are taken modulo  $k$ . If  $w(f \rightarrow f_i) = \frac{1}{2}$ , then  $d(v_i) = 4$  or  $d(v_{i+1}) = 4$  and  $f_{i-1}$  ( or  $f_{i+1}$ ) must be a  $6^+$ -face since every 5-cycle of  $G$  is not simultaneously adjacent to 3-cycles or 4-cycles, and this can be equivalent to say that  $f$  sends  $\frac{1}{3}$  to  $f_i$  and  $\frac{1}{6}$  to  $f_{i-1}$  (or  $f_{i+1}$ , respectively). According to this averaging, every  $f_i$  receives at most  $\frac{1}{3}$  from  $f$ . So  $ch'(f) \geq ch(f) - \frac{1}{3} \times d(f) \geq 0$ .

Let  $v \in V(G)$ . If  $d(v) = 4$ , then  $ch'(v) = ch(v) = 0$  by R1 and R2. Suppose  $d(v) = k \geq 5$ . Let  $N(v) = \{v_1, \dots, v_k\}$  and  $f_1, f_2, \dots, f_k$  be faces incident with  $v$  such that  $f_i$  is incident with  $v_i$  and  $v_{i+1}$ ,  $f_{i(i+1)}$  be the face adjacent to  $(v, v_i, v_{i+1})$  for  $i \in \{1, 2, \dots, k\}$  where the subscripts are taken modulo  $k$ .

Suppose that  $k = 5$ . Then  $f_3(v) \leq 3$ , that is,  $v$  is incident with at most three 3-faces. If  $f_3(v) = 3$ , then  $v$  is incident with two  $6^+$ -faces, and it follows from R1 and R2 that  $ch'(v) \geq ch(v) - \frac{1}{3} \times 2 - \frac{1}{6} > 0$ . If  $f_3(v) \leq 1$ , then we also have  $ch'(v) \geq ch(v) - \frac{1}{3} \times 2 - \frac{1}{6} > 0$  by R1 and R2.2. So we assume that  $f_3(v) = 2$ . If  $f_i$  and  $f_{i+1}$  are two 3-faces for some  $i \in \{1, 2, \dots, 5\}$ , then  $ch'(v) \geq ch(v) - \frac{1}{3} \times 2 > 0$  by R1; Otherwise, without loss of generality, assume that  $f_1$  and  $f_3$  are the two 3-faces. We denote a 5-face  $f$  by  $5^t$ -face

if  $f$  is a  $(5, 4, 4, 4, 4)$ -face and adjacent to  $t$  3-faces, where  $t \geq 4$ . If  $f_2$  is a  $5^5$ -face or  $f_5$  is a  $5^4$ -face, then the  $5^+$ -face  $f_{12}$  incident with  $v_1v_2$  can not be a  $(4, 4, 4, 4, 4)$ -face for  $k \in \{4, 5\}$  by statement (2). This means that if  $w(v \rightarrow f_2) = \frac{2}{3}$  or  $w(v \rightarrow f_5) = \frac{1}{3}$ , then  $w(v \rightarrow f_{12}) = 0$ . Similarly, if  $f_2$  is a  $5^5$ -face or  $f_4$  is a  $5^4$ -face, then the  $5^+$ -face incident with  $v_3v_4$  must not be a  $(4, 4, 4, 4, 4)$ -face. Meanwhile, at most one in  $\{f_4, f_5\}$  is a  $5^4$ -face. So  $ch'(v) \geq ch(v) - \max\{\frac{2}{3} + \frac{1}{3}, \frac{1}{3} \times 2 + \frac{1}{6} \times 2\} = 0$ .

Suppose  $k \geq 6$ . By R2.1, if  $w(v \rightarrow f_i) = \frac{2}{3}$  for some  $i(1 \leq i \leq k)$ , then  $f_{i-1}, f_{i+1}$  are 3-faces and  $w(v \rightarrow f_{i-1}) = w(v \rightarrow f_{i+1}) = 0$ , that can be equivalent to say that  $v$  sends  $\frac{1}{3}$  to  $f_i$ ,  $\frac{1}{6}$  to  $f_{i-1}$  and  $\frac{1}{6}$  to  $f_{i+1}$ . Every charge of  $\frac{1}{6}$  by  $v$  to a  $5^+$ -face incident with  $v_i v_{i+1}$  can be looked at as giving  $\frac{1}{6}$  to  $f_i$ . According to this averaging, every face receive at most  $\frac{1}{3}$  from  $v$ . So  $ch'(v) \geq ch(v) - d(v) \times \frac{1}{3} \geq 0$ .

Hence we complete the proof of Theorem 3.1.1.

### §3.2 List edge coloring and List total coloring

In the section above, we discuss the 4-choosable of a planar graph  $G$ , that is, vertex list coloring. In this section, we mainly discuss the list edge coloring and list total coloring of planar graph  $G$ . Let  $G$  be a planar graph with maximum degree  $\Delta$ , if every 5-cycles of  $G$  is not adjacent to 4-cycles, we prove that  $\chi'_l(G) = \Delta$  and  $\chi''_l(G) = \Delta + 1$  if  $\Delta(G) \geq 8$ , and  $\chi'_l(G) \leq \Delta + 1$  and  $\chi''_l(G) \leq \Delta + 2$  if  $\Delta(G) \geq 6$ , where  $\chi'_l(G)$  and  $\chi''_l(G)$  denote the list edge chromatic number and list total chromatic number of  $G$ , respectively.

First, we introduce three lemmas used in our proofs.

A critical edge  $k$ -choosable graph  $G$  is that  $G$  is not edge  $k$ -choosable and  $G - x$  is edge  $k$ -choosable for any element  $x \in V \cup E$ , where  $k$  is a positive integer. Similarly, one can define critical total  $k$ -choosable graphs.

A critical total  $k$ -choosable graph  $G$  is that  $G$  is not total  $k$ -choosable and  $G - x$  is total  $k$ -choosable for any element  $x \in V \cup E$ , where  $k$  is a positive integer.

**Lemma 3.2.1.** [58] *The following hold for any critical edge  $k$ -choosable graph  $G$  with maximum degree  $\Delta \leq k$ .*

- (a)  $G$  is connected.
- (b) If  $e = uv$  is an edge in  $G$ , then  $d(u) + d(v) \geq k + 2$ .
- (c)  $G$  has no even cycle  $(v_1, v_2, \dots, v_{2t})$  with  $d(v_1) = d(v_3) = \dots = d(v_{2t-1}) = 2$ .

**Lemma 3.2.2.** [28] *The following hold for any critical total  $k$ -choosable graph  $G$  with maximum degree  $\Delta \leq k - 1$ .*

- (a)  $G$  is connected.
- (b) If  $e = uv$  is an edge in  $G$  with  $d(u) \leq \frac{k-1}{2}$ , then  $d(u) + d(v) \geq k + 1$ .  
In particular,  $\delta(G) \geq k + 1 - \Delta$  and so  $G$  has no 1-vertex.
- (c)  $G$  has no even cycle  $(v_1, v_2, \dots, v_{2t})$  with  $d(v_i) \leq \min\{\frac{k-1}{2}, k+1-\Delta\}$  for each odd  $i$ .

**Lemma 3.2.3.** [155] *Let  $G$  be a critical edge  $\Delta$ -choosable graph or a critical total  $\Delta + 1$ -choosable graph. For any integer  $2 \leq k \leq \lfloor \frac{\Delta}{2} \rfloor$ , let  $X_k = \{x \in V(G) | d(x) \leq k\}$  and  $Y_k = \cup_{x \in X_k} N(x)$ . If  $X_k \neq \emptyset$ , then there exists a bipartite subgraph  $M_k$  of  $G$  with partite sets  $X_k$  and  $Y_k$  such that  $d_{M_k}(x) = 1$  for each  $x \in X_k$  and  $d_{M_k}(y) \leq k - 1$  for each  $y \in Y_k$ .*

### §3.2.1 Planar graph $G$ with maximum degree $\Delta \geq 8$

**Theorem 3.2.1.** *Let  $G$  be a planar graph with  $\Delta \geq 8$ , if every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $\chi'_l(G) = \Delta$  and  $\chi''_l(G) = \Delta + 1$ .*

**Proof** Arguing by contradiction, we assume that  $G = (V, E, F)$  is a counterexample to Theorem 3.2.1 having the fewest vertices. If  $G$  is not edge-

$\Delta$ -choosable, we suppose that  $L$  is an edge assignment of  $G$  with  $|L(e)| = \Delta$  for every edge  $e \in E$  such that  $G$  is not edge- $L$ -colorable. If  $G$  is not total- $(\Delta + 1)$ -choosable, we suppose that  $L$  is a total assignment of  $G$  with  $|L(x)| = \Delta + 1$  for every  $x \in V \cup E$  such that  $G$  is not total- $L$ -colorable.

From Lemmas 3.2.1(b) and 3.2.2(b), we claim that  $\delta(G) \geq 2$  and every 2-vertex is adjacent to two  $\Delta$ -vertices.

Let  $G_2$  be the subgraph induced by the edges incident with the 2-vertices of  $G$ . It follows from Lemma 3.2.3 that  $G_2$  contains a matching  $M$  that saturates all 2-vertices. If  $uv \in M$  and  $d(u) = 2$ , then  $v$  is called the 2-master of  $u$  and  $u$  is called the dependent of  $v$ . It is easy to see that each 2-vertex has a 2-master and each vertex of maximum degree can be the 2-master of at most one 2-vertex. Let  $X$  be the set of vertices of degree at most 3 and  $Y = \cup_{x \in X} N(x)$ . By Lemma 3.2.3,  $G$  contains a bipartite subgraph  $M = (X, Y)$  such that  $d_M(x) = 1$  and  $d_M(y) = 2$  for all  $x \in X$  and  $y \in Y$ . We call  $y$  the 3-master of  $x$  if  $xy \in M$  and  $x \in X$ . Therefore, each vertex of degree at most 3 has a 3-master, and each vertex of degree at least  $\Delta - 1$  can be the 3-master of at most two vertices. Now, we consider two cases depending upon the maximum degree  $\Delta$  of  $G$ .

In the following, we will consider two cases of different maximum degree in order to prove the Theorem 3.2.1.

**Case 3.2.1.**  $\Delta = 8$

By the Euler's formula  $|V| - |E| + |F| = 2$ , we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

For each  $x \in V \cup F$ , we define  $c(x)$  the initial charge. Let  $c(v) = 2d(v) - 6$

for each  $v \in V$  and  $c(f) = d(f) - 6$  for each  $f \in F$ . So  $\sum_{x \in V \cup F} c(x) = -12 < 0$ . In the following, we will reassign a new charge denoted by  $c'(x)$  to each  $x \in V \cup F$  according to the discharging rules. If we can show that  $c'(x) \geq 0$  for each  $x \in V \cup F$ , then we get an obvious contradiction to  $0 \leq \sum_{x \in V \cup F} c'(x) = \sum_{x \in V \cup F} c(x) = -12 \leq 0$ , which completes our proof.

Let  $w(v \rightarrow f)$  be the amount that a vertex  $v$  sends to its incident face  $f$ . We define the discharging rules as follows.

**R1.** Each  $7^+$ -face  $f$  sends 1 to each incident 2-vertex  $v$ , if  $f$  is adjacent to a 4-face and  $v$  is incident with a 3-face.

**R2.** Each 2-vertex  $v$  receives 1 from its 2-master, moreover, if  $v$  is incident with  $7^+$ -face  $f$  and 3-face, where  $f$  is adjacent to a 4-face, then  $v$  sends  $\frac{1}{2}$  to its neighbor.

**R3.** Each  $k$ -vertex ( $2 \leq k \leq 3$ ) receives 1 from each of its 3-masters.

**R4.** Let  $v$  be a 3-vertex, then  $w(v \rightarrow f) =$

$$\left\{ \begin{array}{l} \frac{1}{3} \quad \text{if } d(f) = 3 \text{ and } v \text{ is incident with three 3-faces;} \\ \frac{1}{2} \quad \text{if } d(f) = 3 \text{ and } v \text{ is incident with two 3-faces;} \\ \frac{3}{4} \quad \text{if } d(f) = 3 \text{ and } v \text{ is incident with one 3-faces;} \\ \frac{1}{3} \quad \text{if } d(f) = 4. \end{array} \right.$$

**R5.** Let  $v$  be a 4-vertex, then  $w(v \rightarrow f) =$

$$\left\{ \begin{array}{l} \frac{3}{4} \quad \text{if } d(f) = 3; \\ \frac{1}{2} \quad \text{if } d(f) = 4; \\ \frac{1}{4} \quad \text{if } d(f) = 5. \end{array} \right.$$

**R6.** Let  $v$  be a 5-vertex, then  $w(v \rightarrow f) =$

$$\left\{ \begin{array}{l} 1 \quad \text{if } d(f) = 3; \\ \frac{1}{2} \quad \text{if } 4 \leq d(f) \leq 5. \end{array} \right.$$

**R7.** Let  $v$  be a 6-vertex, then  $w(v \rightarrow f) =$

$$\left\{ \begin{array}{l} \frac{3}{2} \quad \text{if } d(f) = 3; \\ \frac{1}{2} \quad \text{if } 4 \leq d(f) \leq 5. \end{array} \right.$$

$$\mathbf{R8.} \text{ Let } v \text{ be a 7-vertex, then } w(v \rightarrow f) = \begin{cases} \frac{3}{2} & \text{if } d(f) = 3; \\ \frac{3}{4} & \text{if } d(f) = 4; \\ \frac{1}{3} & \text{if } d(f) = 5. \end{cases}$$

$$\mathbf{R9.} \text{ Let } v \text{ be a } 8^+\text{-vertex, then } w(v \rightarrow f) = \begin{cases} \frac{3}{2} & \text{if } f = (2, 8^+, 8^+); \\ \frac{4}{3} & \text{if } f = (3, 7^+, 8^+) \text{ and 3-vertex is incident with three 3-faces}; \\ \frac{5}{4} & \text{if } f = (3, 7^+, 8^+) \text{ and 3-vertex is incident with two 3-faces}; \\ \frac{9}{8} & \text{if } f = (3, 7^+, 8^+) \text{ and 3-vertex is incident with one 3-faces}; \\ \frac{9}{8} & \text{if } f = (4^+, 6^+, 8^+); \\ \frac{3}{4} & \text{if } d(f) = 4; \\ \frac{1}{3} & \text{if } d(f) = 5. \end{cases}$$

In the following, we will check that  $c'(x) \geq 0$  for each  $x \in V \cup F$ . By Lemma 3.2.1 and 3.2.2, each 2-vertex is adjacent to two 8-vertices, each 3-vertex is adjacent to three  $7^+$ -vertices and each 4-vertex is adjacent to four  $6^+$ -vertices. Suppose  $d(f) = 3$ . Then  $c(f) = -3$ . If  $n_2(f) = 1$ , then  $f = (2, 8^+, 8^+)$ , so  $c'(f) \geq c(f) + 2 \times \frac{3}{2} = 0$  by R9. Suppose  $n_3(f) = 1$ . Then  $f = (3, 7^+, 7^+)$ . If 3-vertex is incident with three 3-faces, then  $c'(f) \geq c(f) + \frac{1}{3} + 2 \times \frac{4}{3} = 0$ . If 3-vertex is incident with two 3-faces, then  $c'(f) \geq c(f) + \frac{1}{2} + 2 \times \frac{5}{4} = 0$ . If 3-vertex is incident with one 3-face, then  $c'(f) \geq c(f) + \frac{3}{4} + 2 \times \frac{9}{8} = 0$ . If  $n_4(f) = 1$ , then  $f = (4, 6^+, 6^+)$ , so  $c'(f) \geq c(f) + \frac{3}{4} + 2 \times \frac{9}{8} = 0$ . Otherwise,  $f = (5^+, 5^+, 5^+)$ , so  $c'(f) \geq c(f) + 3 \times 1 = 0$ . Suppose  $d(f) = 4$ . Then  $c(f) = -2$ . Note that if  $n_2(f) = 2$ , that is  $f = (2, 8, 2, 8)$ , then this contradict to Lemma 3.2.2(b), so  $n_2(f) \leq 1$ . If  $n_2(f) = 1$ , then  $f = (2, 8, 3^+, 8)$ , so  $c'(f) \geq c(f) + \frac{1}{3} + 2 \times \frac{5}{6} = 0$ . If  $n_3(f) = 2$ , then  $f = (3, 7^+, 3, 7^+)$ ,  $c'(f) \geq c(f) + 2 \times \frac{1}{3} + 2 \times \frac{3}{4} = \frac{1}{6} > 0$ . Otherwise,  $n_{3^-}(f) = 0$ , then  $c'(f) \geq c(f) + 4 \times \frac{1}{2} = 0$ . Suppose  $d(f) = 5$ . Then  $c(f) = -1$ . If  $n_{3^-}(f) = 2$ , then  $f = (3^-, 7^+, 3^-, 7^+, 7^+)$ , so  $c'(f) \geq c(f) + 3 \times \frac{1}{3} = 0$ . If  $n_{3^-}(f) = 1$ , then  $f = (3^-, 7^+, 4^+, 5^+, 7^+)$ , so  $c'(f) \geq c(f) + \frac{1}{4} + 3 \times \frac{1}{3} = \frac{1}{4} > 0$ .

Otherwise,  $n_{3^-}(f) = 0$ , then  $c'(f) \geq c(f) + 5 \times \frac{1}{4} = \frac{1}{4} > 0$ . Suppose  $d(f) = 6$ . Then  $c'(f) = c(f) = 0$ . Suppose  $d(f) = 7$ . Then  $f$  is incident with at most two 2-vertices. If all the two 2-vertices is incident with 3-face and  $f$  is adjacent to a 4-face, then there will appear 5-cycle adjacent to 4-cycle, so  $c'(f) \geq c(f) - 1 = 0$ . Suppose  $d(f) \geq 8$ . Then  $f$  is incident with at most  $\lfloor \frac{d(f)}{3} \rfloor$  2-vertices, so  $c'(f) \geq c(f) - \lfloor \frac{d(f)}{3} \rfloor \times 1 \geq 0$ .

Let  $v$  be a vertex of  $G$ . Suppose  $d(v) = 2$ . Then  $c(v) = -2$  and  $v$  receives 1 from its 2-master and receives 1 from its 3-master by R2 and R3 (the 2-master and 3-master of  $v$  may be a single vertex). If  $v$  is incident with  $7^+$ -face  $f$  and 3-face, where  $f$  is adjacent to a 4-face, then  $v$  sends  $\frac{1}{2}$  to its neighbor, so  $c'(v) \geq c(v) + 1 + 1 + 1 - \frac{1}{2} \times 2 = 0$ . Otherwise,  $c'(v) \geq c(v) + 1 + 1 = 0$ .

Suppose  $d(v) = 3$ . Then  $v$  receives 1 from its 3-master by R3. If  $v$  is incident with three 3-faces, then  $c'(v) \geq c(v) + 1 - 3 \times \frac{1}{3} = 0$ . If  $v$  is incident with two 3-faces, then  $f_{6^+}(v) \geq 1$ , so  $c'(v) \geq c(v) + 1 - 2 \times \frac{1}{3} = \frac{1}{3} > 0$ . If 3-vertex is incident with one 3-face, then  $f_4(v) = 0$ , so  $c'(v) \geq c(v) + 1 - \frac{3}{4} = \frac{1}{4} > 0$ .

Suppose  $d(v) = 4$ . Then  $f_3(v) \leq 2$ . If  $f_3(v) = 2$ , since every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $f_{5^+}(v) = 2$ , so  $c'(v) \geq c(v) - 2 \times \frac{3}{4} - 2 \times \frac{1}{4} = 0$  by R5. If  $f_3(v) = 1$ , then  $f_{5^+}(v) \geq 2$ , so  $c'(v) \geq c(v) - \frac{3}{4} - \frac{1}{2} - 2 \times \frac{1}{4} = \frac{1}{4} > 0$ . Otherwise,  $f_3(v) = 0$ , then  $c'(v) \geq c(v) - 4 \times \frac{1}{2} = 0$ .

Suppose  $d(v) = 5$ . Then  $f_3(v) \leq 3$ . If  $f_3(v) = 3$ , then  $f_{6^+}(v) = 2$ , so  $c'(v) \geq c(v) - 3 \times 1 = 1 > 0$  by R5. If  $f_3(v) \leq 2$ , then  $c'(v) \geq c(v) - f_3(v) - (5 - f_3(v)) \times \frac{1}{2} = \frac{3 - f_3(v)}{2} \geq 0$ .

Suppose  $d(v) = 6$ . Then  $f_3(v) \leq 4$ . If  $f_3(v) = 4$ , then  $f_{6^+}(v) = 2$ , so  $c'(v) \geq c(v) - 4 \times \frac{3}{2} = 0$  by R6. If  $f_3(v) \leq 3$ , then  $c'(v) \geq c(v) - f_3(v) \times \frac{3}{2} - (6 - f_3(v)) \times \frac{1}{2} = 3 - f_3(v) \geq 0$ .

Suppose  $d(v) = 7$ . Then  $f_3(v) \leq 4$ . Since each 2-vertex is only adjacent to 8-vertices,  $v$  can be 3-masters of at most two 3-vertices. If  $f_3(v) = 4$ , then  $f_{6^+}(v) = 3$ , so  $c'(v) \geq c(v) - 2 \times 1 - 4 \times \frac{3}{2} = 0$  by R8. If  $f_3(v) = 3$ , then  $f_4(v) \leq 1$ , so  $c'(v) \geq c(v) - 2 \times 1 - 3 \times \frac{3}{2} - \max\{\frac{3}{4}, 4 \times \frac{1}{3}\} = \frac{1}{6} > 0$ . If  $f_3(v) \leq 2$ , then  $c'(v) \geq c(v) - 2 \times 1 - f_3(v) \times \frac{3}{2} - (7 - f_3(v)) \times \frac{3}{4} = \frac{3 - f_3(v)}{4} \geq 0$ .

Suppose  $d(v) = 8$ . Then  $f_3(v) \leq 5$ .  $v$  can be the 2-master of a 2-vertex as well as being the 3-masters of up to two 2- or 3-vertices. Thus,  $v$  sends totally at most 3 to the  $3^-$ -vertices adjacent to it by R2 and R3. Let  $v_1, \dots, v_8$  be the neighbor of  $v$  and  $f_1, f_2, \dots, f_8$  be faces incident with  $v$ , such that  $f_i$  is incident with  $v_i$  and  $v_{i+1}$ , for  $i \in \{1, 2, \dots, 7\}$ .

Suppose  $f_3(v) = 5$ . Then  $f_{6^+}(v) \geq 3$  and all the  $6^+$ -faces are adjacent to 4-face. If there is at least one 3-faces incident with  $v$  is  $(2, 8, 8)$ -face, then  $f_{7^+}(v) \geq 1$ , so  $c'(v) \geq c(v) - 3 \times 1 - 5 \times \frac{3}{2} + \frac{1}{2} \geq 0$ . Otherwise,  $c'(v) \geq c(v) - 3 \times 1 - 5 \times \frac{4}{3} = \frac{1}{3} > 0$ .

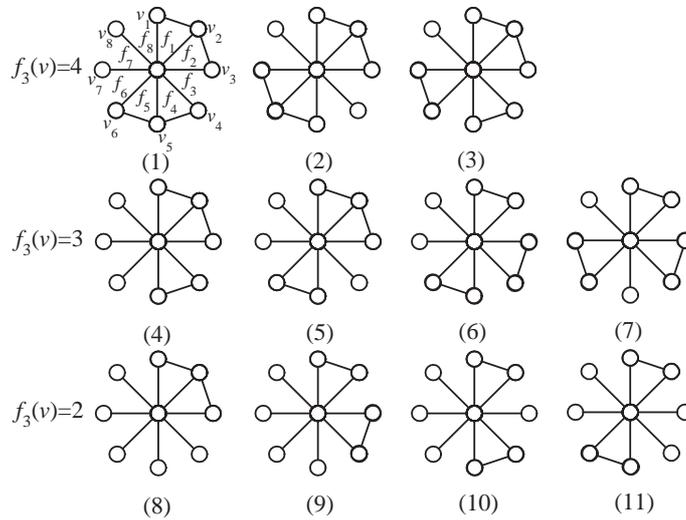


Figure 3.1

Suppose  $f_3(v) = 4$ . Then  $f_4(v) \leq 2$  and  $f_{6^+}(v) \geq 2$ . There are three possibilities in which 3-faces are located. They are shown as configurations

in Figure 3.1(1)-(3).

For Figure 3.1(1), suppose all the four 3-faces incident with  $v$  are  $(2, 8, 8)$ -faces, then  $d(f_3) \geq 8$ , if  $d(f_6) = d(f_8) = 4$ , then  $d(f_7) \geq 6$ , since  $f_3$  is adjacent to 4-face, so  $c'(v) \geq c(v) - 3 \times 1 - 4 \times \frac{3}{2} - 2 \times \frac{3}{4} + \frac{1}{2} = \frac{1}{3} > 0$ . Otherwise,  $d(f_6) \geq 7$  or  $d(f_8) \geq 7$ , then  $c'(v) \geq c(v) - 3 \times 1 - 4 \times \frac{3}{2} - \frac{3}{4} = \frac{3}{4} > 0$ . Suppose there are three 3-faces incident with  $v$  are  $(2, 8, 8)$ -faces. If  $d(v_1) = d(v_3) = d(v_4) = 2$ , then  $d(f_3) \geq 8$  and  $d(f_6) \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 3 \times \frac{3}{2} - \frac{4}{3} - 2 \times \frac{3}{4} + \frac{1}{2} = \frac{1}{6} > 0$ . If  $d(v_1) = d(v_3) = d(v_6) = 2$ , then  $d(f_3) \geq 7$  and  $d(f_7) \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 3 \times \frac{3}{2} - \frac{4}{3} - 2 \times \frac{3}{4} + \frac{1}{2} = \frac{1}{6} > 0$ . Suppose there are two 3-faces incident with  $v$  are  $(2, 8, 8)$ -faces. If  $d(v_1) = d(v_3) = 2$ , then  $d(f_3) \geq 7$  and  $d(f_6) \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 2 \times \frac{3}{2} - 2 \times \frac{4}{3} - 2 \times \frac{3}{4} + \frac{1}{2} = \frac{1}{3} > 0$ . It is similar with  $d(v_1) = d(v_4) = 2$ . Suppose  $d(v_1) = d(v_6) = 2$ , since every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $d(f_3) \geq 6$ . Assume  $d(v_3) = 3$  or  $d(v_4) = 3$ , then  $v_3$  or  $v_4$  is incident with only one 3-face each. If  $d(f_6) = d(f_8) = 4$ , then  $d(f_7) \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 2 \times \frac{3}{2} - 2 \times \frac{9}{8} - 2 \times \frac{3}{4} = \frac{1}{4} > 0$ . Otherwise,  $d(f_6) \geq 7$  or  $d(f_8) \geq 7$ , then  $c'(v) \geq c(v) - 3 \times 1 - 2 \times \frac{3}{2} - 2 \times \frac{9}{8} - \frac{3}{4} + \frac{1}{2} = \frac{3}{2} > 0$ . Suppose there is one 3-faces incident with  $v$  is  $(2, 8, 8)$ -faces. If  $d(v_1) = 2$ , then  $d(f_3) \geq 6$  and  $d(f_6) \geq 6$ . If  $d(f_8) = 4$ , then  $d(f_7) \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - \frac{3}{2} - 3 \times \frac{4}{3} - \frac{3}{4} = \frac{3}{4} > 0$ . Otherwise,  $d(f_8) \geq 7$ , then  $c'(v) \geq c(v) - 3 \times 1 - \frac{3}{2} - 3 \times \frac{4}{3} - \frac{3}{4} + \frac{1}{2} = \frac{5}{4} > 0$ . If  $d(v_3) = 2$ , then  $d(f_3) \geq 7$ ,  $d(f_6) \geq 6$  and  $d(f_8) \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - \frac{3}{2} - 3 \times \frac{4}{3} - \frac{3}{4} + \frac{1}{2} = \frac{5}{4} > 0$ . Suppose there is no 3-faces incident with  $v$  are  $(2, 8, 8)$ -faces. Then  $d(f_3) \geq 6$ ,  $d(f_6) \geq 6$  and  $d(f_8) \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 4 \times \frac{4}{3} - \frac{3}{4} = \frac{11}{12} > 0$ . For Figure 3.1(2), suppose all the four 3-faces incident with  $v$  are  $(2, 8, 8)$ -faces. If  $d(f_3) = d(f_7) = 4$ , then  $d(f_4) \geq 7$  and  $d(f_8) \geq 7$ , since  $f_3$  and  $f_7$  are adjacent to 4-face, so  $c'(v) \geq c(v) - 3 \times 1 - 4 \times \frac{3}{2} - 2 \times \frac{5}{6} + 2 \times \frac{1}{2} = \frac{2}{3} > 0$ . Otherwise,  $d(f_3) \geq 7$  or  $d(f_7) \geq 7$ , then  $c'(v) \geq c(v) - 3 \times 1 - 4 \times \frac{3}{2} - \frac{5}{6} + 2 \times \frac{1}{2} = \frac{3}{2} > 0$ . It is

similar with  $d(f_3) = d(f_8) = 4$ . Suppose there are three 3-faces incident with  $v$  are  $(2, 8, 8)$ -faces. If  $d(v_1) = d(v_3) = d(v_5) = 2$ , then  $d(f_7) \geq 6$ . If  $d(f_3) = d(f_8) = 4$ , then  $d(f_4) \geq 7$ , so  $c'(v) \geq c(v) - 3 \times 1 - 3 \times \frac{3}{2} - \frac{4}{3} - 2 \times \frac{5}{6} + \frac{1}{2} = 0$ . It is similar with  $d(f_4) = d(f_8) = 4$ . If  $d(f_3) = 4$ , then  $d(f_4) \geq 7$  and  $d(f_8) \geq 7$ , then  $c'(v) \geq c(v) - 3 \times 1 - 3 \times \frac{3}{2} - \frac{4}{3} - \frac{5}{6} + 2 \times \frac{1}{2} = \frac{1}{2} > 0$ . Otherwise,  $d(f_3) \geq 7$ ,  $d(f_4) \geq 7$  and  $d(f_8) \geq 7$ , then  $c'(v) \geq c(v) - 3 \times 1 - 3 \times \frac{3}{2} - \frac{4}{3} + 3 \times \frac{1}{2} = \frac{4}{3} > 0$ . It is similar with  $d(v_1) = d(v_3) = d(v_7) = 2$ . Suppose there are two 3-faces incident with  $v$  are  $(2, 8, 8)$ -faces. If  $d(v_1) = d(v_3) = 2$ , then  $d(f_4) \geq 6$  and  $d(f_7) \geq 6$ . Assume  $d(v_5) = 3$  or  $d(v_7) = 3$ , then  $v_5$  or  $v_7$  is incident with only one 3-face each. If  $d(f_3) = d(f_8) = 4$ , then  $c'(v) \geq c(v) - 3 \times 1 - 2 \times \frac{3}{2} - 2 \times \frac{9}{8} - 2 \times \frac{5}{6} = \frac{1}{12} > 0$ . Otherwise,  $d(f_3) \geq 7$  or  $d(f_8) \geq 7$ , then  $c'(v) \geq c(v) - 3 \times 1 - 2 \times \frac{3}{2} - 2 \times \frac{9}{8} - \frac{5}{6} + \frac{1}{2} = \frac{17}{12} > 0$ . It is similar with  $d(v_1) = d(v_5) = 2$ . Suppose there is one 3-faces incident with  $v$  is  $(2, 8, 8)$ -faces. If  $d(v_1) = 2$ , then  $d(f_3) \geq 6$ ,  $d(f_4) \geq 6$  and  $d(f_7) \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - \frac{3}{2} - 3 \times \frac{4}{3} - \frac{5}{6} = \frac{2}{3} > 0$ . Suppose there is no 3-faces incident with  $v$  are  $(2, 8, 8)$ -faces. Then  $d(f_3) \geq 6$ ,  $d(f_4) \geq 6$ ,  $d(f_7) \geq 6$  and  $d(f_8) \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 4 \times \frac{4}{3} = \frac{5}{3} > 0$ .

For Figure 3.1(3), then  $d(f_3) \geq 6$ ,  $d(f_5) \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 4 \times \frac{3}{2} - \frac{3}{4} = \frac{1}{4} > 0$ .

Suppose  $f_3(v) = 3$ . There are four possibilities in which 3-faces are located. They are shown as configurations in Figure 3.1(4)-(7). For Figure 3.1(4), then  $d(f_3) \geq 6$ , if  $\min\{d(f_5), d(f_6)\} = 4$ , then  $\max\{d(f_5), d(f_6)\} \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 3 \times \frac{3}{2} - 2 \times \frac{3}{4} = \frac{11}{12} > 0$ . For Figure 3.1(5), then if  $\min\{d(f_3), d(f_4)\} = 4$ , then  $\max\{d(f_3), d(f_4)\} \geq 6$ , if  $\min\{d(f_6), d(f_7)\} = 4$ , then  $\max\{d(f_6), d(f_7)\} \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , that is, there are at most three 4-faces, so  $c'(v) \geq c(v) - 3 \times 1 - 3 \times \frac{3}{2} - 3 \times \frac{3}{4} = \frac{1}{4} > 0$ .

For Figure 3.1(6), then  $d(f_2) \geq 6$ ,  $d(f_4) \geq 6$ , if  $\min\{d(f_6), d(f_7)\} = 4$ , then  $\max\{d(f_6), d(f_7)\} \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , that is, there are at most two 4-faces, so  $c'(v) \geq c(v) - 3 \times 1 - 3 \times \frac{3}{2} - 2 \times \frac{3}{4} = 1 > 0$ . For Figure 3.1(7), then  $d(f_2) \geq 6$ , if  $\min\{d(f_4), d(f_5)\} = 4$ , then  $\max\{d(f_4), d(f_5)\} \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 3 \times \frac{3}{2} - 2 \times \frac{3}{4} = 1 > 0$ .

Suppose  $f_3(v) = 2$ . There are four possibilities in which 3-faces are located. They are shown as configurations in Figure 3.1(8)-(11). For Figure 3.1(8), then if  $\min\{d(f_3), d(f_4)\} = 4$ , then  $\max\{d(f_3), d(f_4)\} \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 2 \times \frac{3}{2} - 4 \times \frac{3}{4} = 1 > 0$ . For Figure 3.1(9), then  $d(f_2) \geq 6$ , if  $\min\{d(f_4), d(f_5)\} = 4$ , then  $\max\{d(f_4), d(f_5)\} \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 2 \times \frac{3}{2} - 3 \times \frac{3}{4} = \frac{7}{4} > 0$ . For Figure 3.1(10), then if  $\min\{d(f_2), d(f_3)\} = 4$ , then  $\max\{d(f_2), d(f_3)\} \geq 6$ , if  $\min\{d(f_5), d(f_6)\} = 4$ , then  $\max\{d(f_5), d(f_6)\} \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 2 \times \frac{3}{2} - 3 \times \frac{3}{4} = \frac{7}{4} > 0$ . For Figure 3.1(11), then if  $\min\{d(f_2), d(f_3)\} = 4$ , then  $\max\{d(f_2), d(f_3)\} \geq 6$ , if  $\min\{d(f_3), d(f_4)\} = 4$ , then  $\max\{d(f_3), d(f_4)\} \geq 6$ , if  $\min\{d(f_6), d(f_7)\} = 4$ , then  $\max\{d(f_6), d(f_7)\} \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - 2 \times \frac{3}{2} - 4 \times \frac{3}{4} = \frac{7}{6} > 0$ .

Suppose  $f_3(v) = 1$ . Without loss of generality, assume  $d(f_1) = 3$ , then if  $\min\{d(f_2), d(f_3)\} = 4$ , then  $\max\{d(f_2), d(f_3)\} \geq 6$  and if  $\min\{d(f_7), d(f_8)\} = 4$ , then  $\max\{d(f_7), d(f_8)\} \geq 6$ , so  $c'(v) \geq c(v) - 3 \times 1 - \frac{3}{2} - 6 \times \frac{3}{4} = 1 > 0$ . Suppose  $f_3(v) = 0$ . Then  $c'(v) \geq c(v) - 3 \times 1 - 8 \times \frac{3}{4} = 1 > 0$ .

### Case 3.2.2. $\Delta \geq 9$ .

In this case, the initial charge of each element  $x \in V \cup F$  of  $G$  is defined

as  $c(x) = d(x) - 4$ . Euler's formula implies that

$$\sum_{x \in V \cup F} c(x) = \sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (d(f) - 4) = -4(|V| - |E| + |F|) = -8 < 0$$

. We apply the rules below to transfer the charges and get a new charge  $c'(x)$ .

**R1** Each 2-vertex receives 1 from its 2-master.

**R2** Each  $k$ -vertex ( $2 \leq k \leq 3$ ) receives 1 from each of its 3-masters.

**R3** Each 3-face  $f_0$  receives  $\frac{(k-4)l}{k}$  from each of its adjacent  $k$ -faces  $f$  for  $k \geq 5$ , where  $l$  denotes the number of edges shared by  $f_0$  with  $f$ .

**R4** Let  $v$  be a vertex of  $G$ ,  $f$  be a 3-face of  $G$ , then  $w(v \rightarrow f) =$

$$\begin{cases} \frac{d(v)-4}{\lfloor \frac{d(v)}{2} \rfloor + 1} & \text{if } 5 \leq d(v) \leq 7; \\ \frac{2}{5} & \text{if } d(v) = 8; \\ \frac{1}{3} & \text{if } d(v) \geq 9. \end{cases}$$

In the following, we will check that  $c'(x) \geq 0$  for each  $x \in V \cup F$ . By Lemmas above, we know that each 2-vertex is adjacent to two  $\Delta$ -vertices, each 3-vertex is adjacent to three  $(\Delta - 1)^+$ -vertices and each 4-vertex is adjacent to four  $(\Delta - 2)^+$ -vertices. Suppose  $d(f) = 3$ . Then  $c(f) = -1$ . If  $n_2(f) = 1$ , then  $f = (2, \Delta, \Delta)$  and  $f$  is incident with at least one  $6^+$ -face, so  $c'(f) \geq c(f) + 2 \times \frac{1}{3} + \frac{1}{3} = 0$  by R3. Otherwise,  $n_2(f) = 0$ , then  $f$  is adjacent to at most one 3-face and  $f$  is adjacent to no 4-face, so  $c'(f) \geq c(f) + 2 \times \frac{1}{3} + 2 \times \frac{1}{5} = \frac{1}{15} > 0$ . If  $d(f) = 4$ , then  $c'(f) = c(f) = 0$ . If  $d(f) \geq 5$ , then  $c'(f) \geq c(f) - \frac{(d(f)-4)d(f)}{d(f)} = 0$ .

Let  $v$  be a vertex of  $G$ . Note that  $\Delta$ -vertex can be the 2-master of some vertices, and  $(\Delta - 1)$ -vertex can be the 3-master of some vertices. If  $d(v) = 2$ , then  $c(v) = -2$  and  $v$  receives 1 from its 2-master and receives 1 from its 3-master by R1 and R2 (the 2-master and 3-master of  $v$  may be a

single vertex), so  $c'(v) \geq c(v) + 1 + 1 = 0$ . If  $d(v) = 3$ , then  $v$  receives 1 from its 3-master by R2, so  $c'(v) \geq c(v) + 1 = 0$  by R2. If  $d(v) = 4$ , then  $c'(v) = c(v) = 0$ . If  $5 \leq d(v) \leq 7$ , since every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $f_3(v) \leq \lfloor \frac{d(v)}{2} \rfloor + 1$ , so  $c'(v) \geq c(v) - (\lfloor \frac{d(v)}{2} \rfloor + 1) \times \frac{d(v)-4}{\lfloor \frac{d(v)}{2} \rfloor + 1} = 0$ . If  $d(v) = 8$ , then  $v$  can be the 3-master of at most two 3-vertices and  $f_3(v) \leq 5$ , so  $c'(v) \geq c(v) - 2 \times 1 - 5 \times \frac{2}{5} = 0$ . If  $d(v) \geq 9$ , then  $v$  can be the 2-master of a 2-vertex as well as be the 3-master of up to two 2- or 3-vertices, and so  $v$  can send a total of at most 3 to the 2-vertices and 3-vertices adjacent to it, so  $c'(v) \geq c(v) - 3 - (\lfloor \frac{d(v)}{2} \rfloor + 1) \times \frac{1}{3} = \frac{5d(v)-44}{6} > 0$ .

Hence, we complete the proof of Theorem 3.2.1

### §3.2.2 Planar graph $G$ with maximum degree $\Delta \geq 6$

**Theorem 3.2.2.** *Let  $G$  be a planar graph with  $\Delta \geq 6$ , if every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $\chi'_l(G) \leq \Delta + 1$  and  $\chi''_l(G) \leq \Delta + 2$ .*

**Proof** By Theorem 3.2.1, it suffices to consider two cases,  $\Delta = 6$  and  $\Delta = 7$ . Suppose the conclusion does not hold. Again, let  $G = (V, E, F)$  be a plane embedding of a minimal counterexample. If  $G$  is not edge- $\Delta + 1$ -choosable, we suppose that  $L$  is an edge assignment of  $G$  with  $|L(e)| = \Delta + 1$  for every edge  $e \in E$  such that  $G$  is not edge- $L$ -colorable. If  $G$  is not total- $(\Delta + 2)$ -choosable, we suppose that  $L$  is a total assignment of  $G$  with  $|L(x)| = \Delta + 2$  for every  $x \in V \cup E$  such that  $G$  is not total- $L$ -colorable. We prove the edge- $\Delta + 1$ -choosable and total- $(\Delta + 2)$ -choosable of  $G$  separate.

First, we investigate edge- $(\Delta(G) + 1)$ -choosable of planar graphs without adjacent 4-cycles to 5-cycles.

**Claim 3.3.** *If a planar graph  $G$  without adjacent 4-cycles to 5-cycles, and  $\delta(G) \geq 3$ , then there exists an edge  $xy \in E(G)$  such that  $d(x) + d(y) \leq 8$ .*

*Proof.* Suppose that the Claim 3.3 is false for some connected planar graph  $G$  without adjacent 4-cycles to 5-cycles and  $\delta(G) \geq 3$ . Then  $d(x) + d(y) \geq 9$  for every edge  $xy$  of  $G$ .

We rewrite the Euler's formula  $|V| - |E| + |F| = 2$  into the following equivalent form:

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

For each  $x \in V \cup F$ , we define the initial charge

$$c(x) = \begin{cases} 2d(x) - 6 & \text{if } x \in V \\ d(x) - 6 & \text{if } x \in F \end{cases}$$

So  $\sum_{x \in V \cup F} c(x) = -12 < 0$ .

Let  $v$  be a vertex of  $G$ ,  $f$  be a face of  $G$ ,  $W(x \rightarrow y)$  be the charge transferred from  $x$  to  $y$  for all  $x, y \in V \cup F$ . We define the discharging rules as follows.

**R1** If  $d(v) = 4$ , then  $W(v \rightarrow f) = \frac{1}{2}$ .

**R2** If  $d(f) = 3$ , then  $W(v \rightarrow f) = \begin{cases} \frac{5}{4} & \text{if } d(v) = 5; \\ \frac{3}{2} & \text{if } d(v) \geq 6. \end{cases}$

**R3** If  $d(v) \geq 5$ , then  $W(v \rightarrow f) = \begin{cases} 1 & \text{if } d(f) = 4; \\ \frac{1}{2} & \text{if } d(f) = 5. \end{cases}$

In the following, we will check that  $c'(x) \geq 0$  for each  $x \in V \cup F$ .

If  $d(v) = 3$ , then  $c'(v) = c(v) = 0$  by R1-R3. If  $d(v) = 4$ , then  $c'(v) = c(v) - 4 \times \frac{1}{2} = 0$ . If  $d(v) = 5$ , then  $f_3(v) \leq 3$  and it follows that  $c'(v) \geq c(v) - \max\{3 \times \frac{5}{4}, 2 \times \frac{5}{4} + 3 \times \frac{1}{2}, \frac{5}{4} + 4 \times \frac{1}{2}, \frac{5}{4} + 2 \times 1\} = 0$  by R2-R3.

Suppose  $d(v) \geq 6$ . Then  $f_3(v) \leq \lfloor \frac{d(v)}{2} \rfloor + 1$ . If  $f_3(v) = \lfloor \frac{d(v)}{2} \rfloor + 1$ , then  $f_{6^+}(v) = d(v) - f_3(v)$ , so  $c'(v) \geq c(v) - f_3(v) \times \frac{3}{2} = 0$ . Suppose

$f_3(v) = \lfloor \frac{d(v)}{2} \rfloor$ . If  $f_4(v) > 0$ , then  $f_4(v) = f_{6^+}(v)$ , this can be equivalent to say that  $f$  sends  $\frac{1}{2}$  to each 4-face and  $6^+$ -face, otherwise  $f_{5^+}(v) = d(v) - f_3(v)$ , so  $c'(v) \geq c(v) - f_3(v) \times \frac{3}{2} - (d(v) - f_3(v)) \times \frac{1}{2} = 0$ . Otherwise,  $c'(v) \geq c(v) - f_3(v) \times \frac{3}{2} - (d(v) - f_3(v)) \times 1 = 0$ .

Let  $f$  be a face of  $G$ .

Suppose  $d(f) = 3$ , let  $f = (v_1, v_2, v_3)$  and assume that  $d(v_3) \geq d(v_2) \geq d(v_1)$ . If  $d(v_1) = 3$ , then  $d(v_3) \geq d(v_2) \geq 6$ , so  $c'(f) \geq c(f) + 2 \times \frac{3}{2} = 0$  by R2. If  $d(v_1) = 4$ , then  $d(v_3) \geq d(v_2) \geq 5$ , so  $c'(f) \geq c(f) + \frac{1}{2} + 2 \times \frac{5}{4} = 0$  by R3. If  $d(v_1) \geq 5$ , then  $c'(f) \geq c(f) + 3 \times \frac{5}{4} > 0$ .

Suppose  $d(f) = 4$ , let  $f = (v_1, v_2, v_3, v_4)$  and satisfying that  $d(v_1) = \min\{d(v_i) | 1 \leq i \leq 4\}$ . If  $d(v_1) = 3$ , then  $d(v_2) \geq 6$  and  $d(v_4) \geq 6$ , so  $c'(f) \geq c(f) + 2 \times 1 = 0$ . If  $d(v_1) \geq 4$ , then  $d(v_2) \geq 5$  and  $d(v_4) \geq 5$ , so  $c'(f) \geq c(f) + \frac{1}{2} + 2 \times 1 > 0$ .

Suppose  $d(f) = 5$ , let  $f = (v_1, v_2, v_3, v_4, v_5)$  and satisfying that  $d(v_1) = \min\{d(v_i) | 1 \leq i \leq 5\}$ . If  $d(v_1) = 3$ , then  $d(v_2) \geq 6$  and  $d(v_5) \geq 6$ , so  $c'(f) \geq c(f) + 3 \times \frac{1}{2} > 0$ . If  $d(v_1) \geq 4$ , then  $c'(f) \geq c(f) + 5 \times \frac{1}{2} > 0$ .

If  $d(f) \geq 6$ , then  $c'(f) = c(f) = 0$  by R1-R3. □

The proof of edge- $(\Delta(G) + 1)$ -choosable is carried out by induction on  $|V(G)| + |E(G)|$ . It holds trivially when  $|V(G)| + |E(G)| \leq 6$ . Let  $G$  be a planar graph without adjacent 4-cycles to 5-cycles, and  $\Delta(G) \geq 6$ , such that  $|V(G)| + |E(G)| \geq 7$ . By the induction hypothesis,  $G - e$  has an edge- $L$ -coloring  $\phi$ . It is straightforward to extend  $\phi$  to the edge  $e$  because there are at most  $\Delta(G)$  forbidden colors for  $e$ , whereas the number of available colors is at least  $\Delta(G) + 1$ . Suppose that  $\delta(G) \geq 3$ ,  $G$  exists an edge  $xy$  such that  $d(x) + d(y) \leq 8$  by Claim 3.3. The induction hypothesis implies that  $G - xy$  has an edge- $L$ -coloring  $\phi$ . We can color  $xy$  with some color from  $L(xy)$  that was not used by  $\phi$  on the edges adjacent to  $xy$ . Since there exist at most six

such edges and  $L(xy) = k \geq 7$ , the required color is available.

Combining Theorem 3.2.1, let  $G$  be a planar graph with  $\Delta \geq 6$ , if every 5-cycles of  $G$  is not adjacent to 4-cycles, then  $\chi'_l(G) \leq \Delta + 1$ .

Second, we investigate total- $(\Delta(G)+2)$ -choosable of planar graphs without adjacent 4-cycles to 5-cycles.

**Case 3.3.1.**  $\Delta = 6$  First, note that by Lemma 3.2.2(b),  $\delta(G) \geq 3$ , each 3-vertex is adjacent to three 6-vertices, and hence each 4-vertex is adjacent to four  $4^+$ -vertices.

**Claim 3.4.** [81]  $G$  has no  $(4, 4, 5^-)$ -face.

The initial charge  $c(v) = 2d(v) - 6$  for each  $v \in V$  and  $c(f) = d(f) - 6$  for each  $f \in F$ . By the Euler's formula  $|V| - |E| + |F| = 2$ , we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

.

In the following, we will reassign a new charge  $c'(x)$  to each  $x \in V \cup F$  according to the discharging rules.

$$\begin{aligned} \mathbf{R1.} \text{ Let } v \text{ be a 4-vertex, then } w(v \rightarrow f) &= \begin{cases} \frac{3}{4} & \text{if } d(f) = 3; \\ \frac{1}{2} & \text{if } d(f) = 4; \\ \frac{1}{4} & \text{if } d(f) = 5. \end{cases} \\ \mathbf{R2.} \text{ Let } v \text{ be a 5-vertex, then } w(v \rightarrow f) &= \begin{cases} \frac{5}{4} & \text{if } d(f) = 3; \\ \frac{1}{2} & \text{if } d(f) = 4; \\ \frac{1}{4} & \text{if } d(f) = 5. \end{cases} \\ \mathbf{R3.} \text{ Let } v \text{ be a 6-vertex, then } w(v \rightarrow f) &= \begin{cases} \frac{3}{2} & \text{if } d(f) = 3; \\ \frac{3}{4} & \text{if } d(f) = 4; \\ \frac{1}{3} & \text{if } d(f) = 5. \end{cases} \end{aligned}$$

Let  $f$  be a face of  $G$ . Suppose  $d(f) = 3$ . Then  $c(f) = -3$ . If  $n_3(f) = 1$ , then  $f = (3, 6, 6)$ , so  $c'(f) \geq c(f) + 2 \times \frac{3}{2} = 0$  by R3. If  $n_4(f) = 2$ , then  $f = (4, 4, 6)$ , so  $c'(f) \geq c(f) + 2 \times \frac{3}{4} + \frac{3}{2} = 0$  by R1. Otherwise,  $n_4(f) \leq 1$ , then  $f = (4^+, 5^+, 5^+)$ , so  $c'(f) \geq c(f) + \frac{3}{4} + 2 \times \frac{5}{4} = \frac{1}{4} > 0$  by R2. Suppose  $d(f) = 4$ . Then  $c(f) = -2$ . By Lemma 3.2.2(b), if  $n_3(f) = 1$ , then  $f = (3, 6, 4^+, 6)$ , so  $c'(f) \geq c(f) + 2 \times \frac{3}{4} + \frac{1}{2} = 0$ . Otherwise,  $n_3(f) = 0$ , then  $c'(f) \geq c(f) + 4 \times \frac{1}{2} = 0$ . Suppose  $d(f) = 5$ . Then  $c(f) = -1$ . If  $n_3(f) = 2$ , then  $f = (3, 6, 3, 6, 6)$ , so  $c'(f) \geq c(f) + 3 \times \frac{1}{3} = 0$ . Otherwise,  $n_3(f) \leq 1$ , then  $c'(f) \geq c(f) + 4 \times \frac{1}{4} = 0$ . Suppose  $d(f) \geq 6$ . Then  $c'(f) = c(f) \geq 0$ .

Let  $v$  be a vertex of  $G$ . If  $d(v) = 3$ , then  $c'(v) = c(v) = 0$ . Suppose  $d(v) = 4$ . Then  $f_3(v) \leq 2$ . If  $f_3(v) = 2$ , then  $f_{5^+}(v) = 2$ , so  $c'(v) \geq c(v) - 2 \times \frac{3}{4} - 2 \times \frac{1}{4} = 0$ . If  $f_3(v) = 1$ , then  $f_4(v) \leq 1$ , so  $c'(v) \geq c(v) - \frac{3}{4} - \frac{1}{2} - 2 \times \frac{1}{4} = \frac{1}{4} > 0$ . Otherwise,  $c'(v) \geq c(v) - 4 \times \frac{1}{2} = 0$ . Suppose  $d(v) = 5$ . Then  $f_3(v) \leq 3$ . If  $f_3(v) = 3$ , then  $f_{6^+}(v) = 2$ , so  $c'(v) \geq c(v) - 3 \times \frac{5}{4} = \frac{1}{4} > 0$ . Otherwise,  $c'(v) \geq c(v) - 2 \times \frac{5}{4} - 3 \times \frac{1}{2} = 0$ . Suppose  $d(v) = 6$ . Then  $f_3(v) \leq 4$ . If  $f_3(v) = 4$ , then  $f_{6^+}(v) = 2$ , so  $c'(v) \geq c(v) - 4 \times \frac{3}{2} = 0$ . If  $f_3(v) = 3$ , then  $f_4(v) = 0$ , so  $c'(v) \geq c(v) - 3 \times \frac{3}{2} - 3 \times \frac{1}{3} = \frac{1}{2} > 0$ . Otherwise,  $c'(v) \geq c(v) - 2 \times \frac{3}{2} - 4 \times \frac{3}{4} = 0$ .

**Case 3.4.1.**  $\Delta = 7$

By Lemma 3.2.2(b),  $\delta(G) \geq 3$ , each 3-vertex is adjacent to three 7-vertices, each 4-vertex is adjacent to three 6-vertices, and hence each 5-vertex is adjacent to four 5<sup>+</sup>-vertices.

The initial charge  $c(x) = d(x) - 4$  for each  $x \in V \cup F$ . By the Euler's formula, we have

$$\sum_{x \in V \cup F} c(x) = \sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (d(f) - 4) = -8 < 0$$

The discharging rules are as below.

**R1.** Each 3-vertex receives  $\frac{1}{3}$  from each of its adjacent 7-vertices.

**R2.** Each 3-face receives  $\frac{1}{3}$  from each of its incident  $5^+$ -vertices.

**R3** Each 3-face  $f_0$  receives  $\frac{(k-4)l}{k}$  from each of its adjacent  $k$ -faces  $f$  for  $k \geq 5$ , where  $l$  denotes the number of edges shared by  $f_0$  with  $f$ .

Let  $f$  be a face of  $G$ . Suppose  $d(f) = 3$ . Then  $c(f) = -1$ . Note that each  $5^-$ -vertex is adjacent to  $5^+$ -vertex,  $4^-$ -vertex is adjacent to  $6^+$ -vertex, since 3-face is adjacent to at most one 3-face, if 3-face is adjacent to a 3-face, then 3-face is adjacent to two  $6^+$ -faces, otherwise, 3-face is adjacent to three 5-faces, so  $c'(f) \geq c(f) + 2 \times \frac{1}{3} + \min\{2 \times \frac{1}{3}, 3 \times \frac{1}{5}\} = \frac{4}{15} > 0$  by R2 and R3. If  $d(f) = 4$ , then  $c'(f) = c(f) = 0$ . If  $d(f) \geq 5$ , then  $c'(f) \geq c(f) - \frac{(d(f)-4)d(f)}{d(f)} = 0$ .

Let  $v$  be a vertex of  $G$ . If  $d(v) = 3$ , then  $v$  receives  $\frac{1}{3}$  from each of its adjacent  $\Delta$ -vertices, so  $c'(v) \geq c(v) + 3 \times \frac{1}{3} = 0$ . If  $d(v) = 4$ , then  $c'(v) = c(v) = 0$ . If  $d(v) = 5$ , then  $f_3(v) \leq 3$ , so  $c'(v) \geq c(v) - 3 \times \frac{1}{3} = 0$ . If  $d(v) = 6$ , then  $f_3(v) \leq 4$ , so  $c'(v) \geq c(v) - 4 \times \frac{1}{3} = \frac{2}{3} > 0$ . If  $d(v) = 7$ , then  $f_3(v) \leq 4$ , since  $n_3(v) \leq 7 - \lfloor \frac{f_3(v)}{2} \rfloor$ , so  $c'(v) \geq c(v) - f_3(v) \times \frac{1}{3} - n_3(v) \times \frac{1}{3} = \frac{4-f_3(v)}{6} \geq 0$ .

In all cases, we have  $c'(x) \geq 0$  for each  $x \in V \cup F$ , and  $0 \leq \sum_{x \in V \cup F} c'(x) = \sum_{x \in V \cup F} c(x) = -8$ , a contradiction. This completes the proof of Theorem 3.2.2. □

# Chapter 4 Neighbor sum distinguish total coloring

## §4.1 Basic definitions and properties

Recently, colorings and labellings concerning the sums of the colors have received much attention. The family of such problems includes, e.g. vertex-coloring  $k$ -edge-weightings [69], neighbor sum distinguishing edge  $k$ -coloring [44] [45] [49] [120] [121], total weight choosability [93] [146], magic and antimagic labellings [65] [147] and the irregularity strength [90] [91]. Among them there are the 1-2-3 Conjecture [70] and 1-2 Conjecture [92].

A  $k$  total coloring of a graph  $G$  is a mapping  $\phi : V \cup E \rightarrow \{1, 2, \dots, k\}$  such that no two adjacent or incident elements receive the same color. A graph  $G$  is *total  $k$  colorable* if it admits a  $k$  total coloring. The *total chromatic number*  $\chi''(G)$  of  $G$  is the smallest integer  $k$  such that  $G$  has a  $k$ -total-coloring. In a total  $k$ -coloring of  $G$ , let  $f(v)$  denote the total sum of colors of the edges incident to  $v$  and the color of  $v$ . If for each edge  $uv$ ,  $f(u) \neq f(v)$ , we call such total  $k$ -coloring a *total  $k$  neighbor sum distinguishing coloring*. The smallest number  $k$  is called the *neighbor sum distinguishing total chromatic number*, denoted by  $\chi''_{\Sigma}(G)$ .

*Surfaces* in this paper are compact, connected 2-dimensional manifolds without boundary. All embedded graphs considered in this paper are 2-cell-embeddings. Let  $G = (V, E, F)$  be an embedded graph. A  $\ell$ -vertex, a  $\ell^-$ -vertex or a  $\ell^+$ -vertex is a vertex of degree  $\ell$ , at most  $\ell$  or at least  $\ell$ , respectively. For a face  $f$  of  $G$ , the *degree*  $d(f)$  is the number of edges incident with it, where each cut-edge is counted twice. A  $l$ -face, a  $l^-$ -face or a  $l^+$ -face is a face of degree  $l$ , at most  $l$  or at least  $l$ , respectively. Suppose that  $H$  is a

subgraph of a given plane graph  $G$ . For  $x \in V(H) \cup F(H)$ , let  $d_H(x)$  denote the degree of  $x$  in  $H$ . We use  $N_\ell^H(x)$  to denote the set of  $\ell$ -vertices adjacent to  $x$  in  $H$ , and  $d_\ell^H(x) = |N_\ell^H(x)|$ . Similarly, we can define  $d_{\ell^+}^H(x)$  and  $d_{\ell^-}^H(x)$ . A vertex  $x$  is small if  $2 \leq d_H(x) \leq 5$ , otherwise it is big. A 4-face is *bad* if it is incident with at least one 2-vertex, and is *special* if it incident with two 2-vertices, the 2-vertex is called *special*. A 3-face is special if it is incident to a 2-vertex. A  $\ell$ -vertex  $u$ , with  $\ell \geq 3$ , is *bad* if each of the faces incident to it is either a 3-face, or a bad 4-face. We use  $d_{\ell b}^H(x)$  to denote the number of bad  $\ell$ -vertices adjacent to  $x$ . If there is no confusion in the context, we usually write  $d_\ell^G(x)$ ,  $d_{\ell^+}^G(x)$ ,  $d_{\ell^-}^G(x)$ ,  $d_{\ell b}^G(x)$  as  $d_\ell(x)$ ,  $d_{\ell^+}(x)$ ,  $d_{\ell^-}(x)$ ,  $d_{\ell b}(x)$ , respectively.

The following lemma will be used in our proof.

**Lemma 4.1.1.** [77] *Suppose  $B_1$  is a set of integers and  $|B_1| = n$ . Let  $B_2 = \left\{ \sum_{i=1}^m x_i \mid x_i \in B_1, m < n, x_i \neq x_j (i \neq j) \right\}$ , then  $|B_2| \geq mn - m^2 + 1$ .*

## §4.2 Neighbor sum distinguishing total coloring

We mainly proved the neighbor sum distinguishing total coloring of a graph which can be embedded in a surface  $\Sigma$  of Euler characteristic.

**Theorem 4.2.1.** *Let  $G$  be a graph with maximum degree  $\Delta(G)$  which can be embedded in a surface  $\Sigma$  of Euler characteristic  $\chi(\Sigma) \geq 0$ , then  $\chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 2, 16\}$ .*

The following is a direct consequence of Theorem 4.2.1.

**Corollary 4.2.2.** *Let  $G$  be a graph which can be embedded in a surface  $\Sigma$  of Euler characteristic  $\chi(\Sigma) \geq 0$ . If  $\Delta(G) \geq 14$ , then  $\chi''_{\Sigma}(G) \leq \Delta(G) + 2$ .*

Obviously, if  $G$  contains two adjacent vertices of maximum degree, then  $\chi''_{\Sigma}(G) \geq \Delta(G) + 2$ . So the bound  $\Delta(G) + 2$  is sharp. Since  $\chi''_a(G) \leq \chi''_{\Sigma}(G)$ , we have the following corollary.

**Corollary 4.2.3.** *Let  $G$  be a graph which can be embedded in a surface  $\Sigma$  of Euler characteristic  $\chi(\Sigma) \geq 0$ . If  $\Delta(G) \geq 14$ , then  $\chi''_a(G) \leq \Delta(G) + 2$ .*

**Proof** Let  $k = \max\{\Delta(G) + 2, 16\}$ . For simplicity, we use “total  $k$ -nsd-coloring” to denote “total  $k$  neighbor sum distinguishing coloring”. Suppose that  $\phi$  is a total  $k$ -nsd-coloring of a graph  $G$  using the color set  $C = \{1, 2, \dots, k\}$ , where  $k \geq 16$ . Assume that  $v \in V(G)$  with  $d(v) \leq 5$ . It is obvious that  $v$  has at most five adjacent vertices and five incident edges, so  $v$  has at most 15 conflicting colors. Since  $|C| \geq 16$ , we may first erase the color of  $v$  and finally recolor it after arguing. In other words, we will omit the coloring for such vertices in the following discussion.

Our proof proceeds by reductio ad absurdum. Assume that  $G = (V, E, F)$  is a minimal counterexample to Theorem 4.2.1 which is embedded in a surface  $\Sigma$  of Euler characteristic  $\chi(\Sigma) \geq 0$ , such that  $|V(G)| + |E(G)|$  is as small as possible. Obviously,  $G$  is connected. Let  $H$  be a proper subgraph of  $G$ . By the minimality of  $G$ ,  $H$  has a total  $k$ -nsd-coloring using the color set  $C = \{1, 2, \dots, k\}$ . We use  $f(v)$  to denote the total sum of colors assigned to a vertex  $v$  and those edges incident to  $v$  in  $G$ . For two adjacent vertices  $u$  and  $v$ , if  $f(u) = f(v)$ , then we call these two vertices conflict on  $\phi$ . Let  $C_{\phi}(v)$  denote the set of colors assigned to a vertex  $v$  and those edges incident to  $v$  in  $H$  and  $m_{\phi}(v)$  denote the sum of colors assigned to a vertex  $v$  and those edges incident to  $v$  in  $H$ . Now we introduce some lemmas first.

**Lemma 4.2.1.** [77] *Let  $v_1, v_2$  be two neighbors of  $v$  in  $G$ ,  $d(v_1) \leq 5$ ,  $d(v_2) \leq 5$ ,  $d(v) + d(v_1) \leq 16$  and  $d(v) + d(v_2) \leq 16$ . Then  $d_{6+}(v) \geq 33 - 2d(v) - d(v_1) - d(v_2)$ . Moreover, if  $d(v_1) \neq d(v_2)$ , then  $d_{6+}(v) \geq 34 - 2d(v) - d(v_1) - d(v_2)$ .*

**Lemma 4.2.2.** [77] *There is no edge  $uv \in E(G)$  such that  $d(v) \leq 7$  and  $d(u) \leq 5$ .*

**Lemma 4.2.3.** [77] *If  $v$  is an 8-vertex of  $G$ , then  $d_1(v) = 0$  and  $d_{5-}(v) \leq 1$ .*

**Lemma 4.2.4.** [77] *Let  $v$  be a 9-vertex of  $G$ .*

- (1) *If  $d_{3-}(v) \geq 1$ , then  $d_{5-}(v) \leq 1$ .*
- (2) *If  $d_{4-}(v) \geq 1$ , then  $d_{5-}(v) \leq 2$ .*

**Lemma 4.2.5.** [77] *Let  $v$  be a 10-vertex of  $G$ .*

- (1) *If  $d_1(v) \neq 0$ , then  $d_{5-}(v) \leq 2$  and  $d_{3-}(v) \leq 1$ .*
- (2) *If  $d_2(v) \neq 0$ , then  $d_{5-}(v) \leq 3$  and  $d_{3-}(v) \leq 1$ .*
- (3) *If  $d_3(v) \neq 0$ , then  $d_{5-}(v) \leq 4$  and  $d_{3-}(v) \leq 2$ .*
- (4) *If  $d_4(v) \neq 0$ , then  $d_{5-}(v) \leq 5$ .*

**Lemma 4.2.6.** *Let  $v$  be an  $\ell$ -vertex of  $G$  with  $\ell \geq 11$ , if  $d_1(v) \geq 1$  and  $d_{2-}(v) \geq 2$ , then  $d_{3-}(v) \leq \lceil \frac{\ell}{2} \rceil - 1$  and  $d_{6+}(v) \geq d_{3-}(v) + 1$ .*

*Proof.* Let  $v_1, v_2, \dots, v_i, i \geq 11$  be the neighbors of  $v$  with  $d(v_1) \leq d(v_2) \leq \dots \leq d(v_i)$ . Then  $d(v_1) = 1$ . Let  $t = d_{3-}(v)$ . Suppose to the contrary that  $t \geq \lceil \frac{\ell}{2} \rceil$  or  $d_{6+}(v) \leq t$ . Note that as  $t \geq \lceil \frac{\ell}{2} \rceil$ , we can obtain  $d_{6+}(v) \leq t$ , so we consider  $d_{6+}(v) \leq t$  following. Let  $H = G - v_1v$ . Then there is a total  $[k]$ -nsd-coloring  $\phi$  of  $H$ . Obviously,  $d(v_2) \leq 2$ , and  $d(v_3) \leq \dots \leq d(v_t) \leq 3$ . We delete all the colors of  $v_i$  and  $vv_i$  for  $1 \leq i \leq t$ , and denote this coloring by  $\phi'$ . Then  $v$  has at most  $t$  conflicting neighbors. Since  $\Delta \geq d(v) = \ell$ ,  $|C \setminus C_{\phi'}(v)| \geq \Delta + 2 - (\ell + 1 - t) \geq t + 1$ . Thus we can choose  $t$  colors from at least  $t + 1$  colors to color  $v_1v, \dots, v_tv$ . Since  $d_{6+}(v) \leq t$ , we can conclude that  $d_{6+}(v) + 1 \leq t + 1$ . By Lemma 4.1.1, we can find at least  $t + 1$  ways to color  $v_1v, \dots, v_tv$  which result at least  $t + 1$  different  $f(v)$ s. Hence, we can find at least one color set  $C' \subseteq C \setminus C_{\phi'}(v)$  to color  $v_1v, \dots, v_tv$ , such that  $v$

does not conflict with its neighbors. Now we color  $v_tv, \dots, v_1v$  one-by-one with colors in  $C'$  as follows. Since  $d(v_i) \leq 3$  for  $3 \leq i \leq t$ , we color  $vv_i$  with color  $a_i$  in  $(C' \setminus \{a_t, \dots, a_{i+1}\}) \setminus C_{\phi'}(v_i)$ . Since  $d(v_2) \leq 2$ , we color  $vv_2$  with color  $a_2$  in  $(C' \setminus \{a_t, \dots, a_3\}) \setminus C_{\phi'}(v_2)$ . Since  $d(v_1) = 1$ , we color  $vv_1$  with color in  $C' \setminus \{a_t, \dots, a_2\}$ . Thus we get a total  $k$ -nsd-coloring of  $G$ , which is a contradiction.  $\square$

**Lemma 4.2.7.** [37] *Let  $v$  be an 11-vertex of  $G$ .*

- (1) *Suppose  $d_1(v) \geq 1$ . Then  $d_{3-}(v) \leq 3$  and  $d_{5-}(v) \leq 5$ , moreover, if  $d_{2-}(v) \geq 2$ , then  $d_{5-}(v) = 2$ .*
- (2) *If  $d_{3-}(v) \geq 1$ , then  $d_{6+}(v) \geq d_{3-}(v) + 1$ .*

**Lemma 4.2.8.** [37] *Let  $v$  be a 12-vertex of  $G$ .*

- (1) *Suppose  $d_1(v) \geq 1$ . If  $d_{3-}(v) \geq 2$ , then  $d_{5-}(v) \leq 6$ . If  $d_1(v) \geq 2$ , then  $d_{3-}(v) \leq 2$  and  $d_{5-}(v) \leq 5$ .*
- (2) *If  $d_{3-}(v) \geq 1$ , then  $d_{6+}(v) \geq d_{3-}(v) + 1$ .*

**Lemma 4.2.9.** [37] *Let  $v$  be a 13-vertex of  $G$ , if  $d_1(v) \geq 4$ , then  $d_{5-}(v) \leq 4$ .*

**Lemma 4.2.10.** [37] *Let  $v$  be an  $\ell (\geq 13)$ -vertex of  $G$  with  $d_2(v) \geq 1$ . Then  $d_{6+}(v) \geq 1$ . Further,*

- (1) *If  $v$  is incident to a special 4-face, then  $d_{3-}(v) \leq \lceil \frac{\ell}{2} \rceil - 1$  and  $d_{6+}(v) \geq d_{3-}(v) + 1$ .*
- (2) *If  $v$  is incident to at least 2 special 3-faces, then  $d_{3-}(v) \leq \lceil \frac{\ell}{2} \rceil - 1$  and  $d_{6+}(v) \geq d_{3-}(v) + 1$ .*

We shall complete the proof of Theorem 4.2.1 by using the "Discharging" method. Let  $G = (V, E, F)$  be a graph which is embedded in a surface of nonnegative Euler characteristic. Let  $H$  be the graph obtained by removing all 1-vertices of  $G$ . Then  $H$  is a connected planar graph with  $\delta(H) \geq 2$ . By

Lemma 4.2.1-4.2.9, we display the relation between  $d(v)$  and  $d_H(v)$  in Table 4.1.

$d(v)$	$\leq 7$	8	9	10	11	12	13	14	15	16	$\dots$
$d_H(v)$	$= d(v)$	8	$\geq 8$	$\geq 9$	$\geq 9$	$\geq 10$	$\geq 9$	$\geq 8$	$\geq 8$	$\geq 9$	$\dots$

**Table 4.1:** The relation between  $d(v)$  and  $d_H(v)$

We list other properties of  $H$  which are collected in the following Claim 4.3.

**Claim 4.3.** *Let  $v$  be a vertex of  $H$ . Then the followings hold.*

- (1) *If  $d_H(v) \leq 7$ , then  $d_H(v) = d(v)$  and  $d_{5-}^H(v) = 0$ .*
- (2) *If  $d_H(v) = 8$ , then  $d_{5-}^H(v) \leq 1$ .*
- (3) *If  $d_H(v) = 9$  with  $d_{4-}^H(v) \geq 1$ , then  $d_{5-}^H(v) \leq 3$ .*
- (4) *If  $d_H(v) = 10$  with  $d_{3-}^H(v) \geq 1$ , then  $d_{6+}^H(v) \geq d_{3-}^H(v) + 2$ .*
- (5) *If  $d_H(v) = 11$  with  $d_{3-}^H(v) \geq 1$ , then  $d_{6+}^H(v) \geq d_{3-}^H(v) + 1$ .*
- (6) *If  $d_H(v) = 12$  with  $d_{2-}^H(v) \geq 1$ , then  $d_{6+}^H(v) \geq d_{3-}^H(v) + 1$ .*
- (7) *If  $d_H(v) \geq 13$  with  $d_{2-}^H(v) \geq 1$ , then  $d_{6+}^H(v) \geq 1$ . Moreover, we have the following:*
  - (a) *If  $v$  is incident to a special 4-face, then  $d_{3-}(v) \leq \lceil \frac{\ell}{2} \rceil - 1$  and  $d_{6+}(v) \geq d_{3-}(v) + 1$ .*
  - (b) *If  $v$  is incident to at least 2 special 3-faces, then  $d_{3-}(v) \leq \lceil \frac{\ell}{2} \rceil - 1$  and  $d_{6+}(v) \geq d_{3-}(v) + 1$ .*

*Proof.* (1) It is trivial by Table 4.1 and Lemma 4.2.2.

(2) By Table 4.1, we only need to prove the cases  $d(v) = 8, 9, 14, 15$ . When  $d(v) = 8$ . It is trivial for  $d(v) = 8$  by Lemma 4.2.3. If  $d(v) = 9$ , then  $d_1(v) = 1$ . Since  $d_{5-}(v) \leq 1$  by Lemma 4.2.4, so  $d_{5-}^H(v) = d_{5-}(v) - d_1(v) = 0$  by (1). If  $d(v) = 14$ , then  $d_1(v) = 6$ . Since  $d_{6+}(v) \geq 7$  by Lemma 4.2.6, so

$d_{5-}^H(v) = d(v) - d_{6+}(v) - d_1(v) \leq 1$ . If  $d(v) = 15$ , then  $d_1(v) = 7$ . Since  $d_{6+}(v) \geq 8$  by Lemma 4.2.6, so  $d_{5-}^H(v) = d(v) - d_{6+}(v) - d_1(v) = 0$ .

(3) By Table 4.1, we only prove the cases  $d(v) = 9, 10, 11$  and  $d(v) \geq 13$ . If  $d(v) = 9$ , then  $d_{5-}(v) \leq 2$  by Lemma 4.2.4. If  $d(v) = 10$ , then  $d_1(v) = 1$ . Since  $d_{5-}(v) \leq 2$  by Lemma 4.2.5, so  $d_{5-}^H(v) = d_{5-}(v) - d_1(v) \leq 1$  by (1). If  $d(v) = 11$ , then  $d_1(v) = 2$ . Since  $d_{5-}(v) = 2$  by Lemma 4.2.7(1), so  $d_{5-}^H(v) = d_{5-}(v) - d_1(v) = 0$ . If  $d(v) = 13$ , then  $d_1(v) = 4$ . Since  $d_{5-}(v) \leq 4$  by Lemma 4.2.9, so  $d_{5-}^H(v) = d_{5-}(v) - d_1(v) = 0$ . If  $d(v) \geq 14$ , then  $d_1(v) \geq 5$ . Since  $d_{6+}(v) \geq 6$  by Lemma 4.2.6, so  $d_{5-}^H(v) = d(v) - d_{6+}(v) - d_1(v) \leq 3$ .

(4) By Table 4.1, we only prove the cases  $d(v) \geq 10$ . If  $d(v) = 10$ , then  $d_{5-}(v) \leq 4$  and  $d_{3-}(v) \leq 2$  by Lemma 4.2.5(3), so  $d_{6+}^H(v) = d(v) - d_{5-}(v) \geq 6 > d_{3-}^H(v) + 2$ . If  $d(v) = 11$ , then  $d_1(v) = 1$ . Since  $d_{3-}(v) \leq 3$  and  $d_{5-}(v) \leq 5$  by Lemma 4.2.7(1), so  $d_{6+}^H(v) = d(v) - d_{5-}(v) \geq 6 > d_{3-}^H(v) + 2$ . If  $d(v) = 12$ , then  $d_1(v) = 2$ . Since  $d_{6+}(v) \geq d_{3-}(v) + 1$  by Lemma 4.2.8(2), so  $d_{6+}^H(v) = d_{6+}(v) \geq d_{3-}(v) + 1 = d_{3-}^H(v) + d_1(v) + 1 > d_{3-}^H(v) + 2$ . If  $d(v) \geq 13$ , then  $d_1(v) \geq 3$ . Since  $d_{6+}(v) \geq d_{3-}(v) + 1$  by Lemma 4.2.6, so  $d_{6+}^H(v) = d_{6+}(v) \geq d_{3-}(v) + 1 = d_{3-}^H(v) + d_1(v) + 1 > d_{3-}^H(v) + 2$ .

(5) By Table 4.1, we only prove the cases  $d(v) \geq 11$ . If  $d(v) = 11$ , it is trivial by Lemma 4.2.7(2). If  $d(v) = 12$ , then  $d_1(v) = 1$ . Since  $d_{6+}(v) \geq d_{3-}(v) + 1$  by Lemma 4.2.8(2), so  $d_{6+}^H(v) = d_{6+}(v) \geq d_{3-}(v) + 1 = d_{3-}^H(v) + d_1(v) + 1 > d_{3-}^H(v) + 1$ . If  $d(v) \geq 13$ , then  $d_1(v) \geq 2$ . Since  $d_{6+}(v) \geq d_{3-}(v) + 1$  by Lemma 4.2.6, so  $d_{6+}^H(v) = d_{6+}(v) \geq d_{3-}(v) + 1 = d_{3-}^H(v) + d_1(v) + 1 > d_{3-}^H(v) + 1$ .

(6) By Table 4.1, we only prove the cases  $d(v) \geq 12$ . If  $d(v) = 12$ , it is trivial by Lemma 4.2.8(2). If  $d(v) \geq 13$ , then  $d_1(v) \geq 1$  and  $d_{2-}(v) \geq 2$ . Since  $d_{6+}(v) \geq d_{3-}(v) + 1$  by Lemma 4.2.6, so  $d_{6+}^H(v) = d_{6+}(v) \geq d_{3-}(v) + 1 = d_{3-}^H(v) + d_1(v) + 1 > d_{3-}^H(v) + 1$ .

(7) If  $d(v) = d_H(v)$ , it is trivial by Lemma 4.2.10. Otherwise  $d_1(v) \geq 1$ , then  $d_{2-}(v) = d_2^H(v) + d_1(v) \geq 2$ . Since  $d_{6+}(v) \geq d_{3-}(v) + 1$  by Lemma 4.2.6, so  $d_{6+}^H(v) = d_{6+}(v) \geq d_{3-}(v) + 1 = d_{3-}^H(v) + d_1(v) + 1 > d_{3-}^H(v) + 1$ .  $\square$

By Euler's formula  $|V| - |E| + |F| = \chi(\Sigma)$ , we have

$$\sum_{v \in V(H)} (d_H(v) - 6) + \sum_{f \in F(H)} (2d_H(f) - 6) = -6\chi(\Sigma) \leq 0.$$

In order to complete the proof, we use the "Discharging method". First, we give an initial charge function  $w(v) = d_H(v) - 6$  for every  $v \in V(H)$ , and  $w(f) = 2d_H(f) - 6$  for every  $f \in F(H)$ . So  $\sum_{x \in V(H) \cup F(H)} w(x) = -6\chi(\Sigma) \leq 0$ . Next, we design a discharging rule and redistribute weights accordingly. Let  $w'$  be the new charge after the discharging. We will show that  $w'(x) \geq 0$  for each  $x \in V(H) \cup F(H)$  and  $\sum_{x \in V(H) \cup F(H)} w'(x) > 0$ . This leads to the following obvious contradiction:

$$0 < \sum_{x \in V(H) \cup F(H)} w'(x) = \sum_{x \in V(H) \cup F(H)} w(x) = -6\chi(\Sigma) \leq 0.$$

Hence demonstrates that no such a counterexample can exist. The discharging rules are defined as follows:

**R1.** Each  $5^+$ -face sends 2 to each of its incident small vertices (counting multiplicity).

**R2.** Let  $f$  be a 4-face of  $H$ . If  $f$  is incident with exactly one 2-vertex, then  $f$  sends 2 to the 2-vertex. Otherwise,  $f$  sends 1 to each of its incident small vertices.

**R3.** Let  $u$  be a  $5^-$ -vertex, denote by  $\beta(u)$  the total sum of charges transferred into  $u$  after (R1) and (R2) were carried out. Each  $8^+$ -vertex sends  $\frac{6-d_H(u)-\beta(u)}{d_H(u)}$  to each of its adjacent small vertices  $u$ .

Let  $\tau(x \rightarrow y)$  be the charge transferred from  $x$  to  $y$ . The following three observations can be easily deduced by Claim 4.3 and R1-R3.

**Observation 4.3.1.** Every face  $f$  is incident to at most  $\lfloor \frac{d_H(f)}{2} \rfloor$   $5^-$ -vertices (counting multiplicity).

**Observation 4.3.2.** Let  $d_H(v) \geq 8$ , and  $u$  be a  $5^-$ -vertex adjacent to  $v$ .

(1) Suppose that  $d_H(u) = 2$ . If  $u$  is a special vertex, then  $\tau(v \rightarrow u) \leq \frac{3}{2}$ . Otherwise,  $\tau(v \rightarrow u) \leq 1$ . Moreover, if  $\tau(v \rightarrow u) = \frac{3}{2}$ , then there exists a 2-vertex  $u'$  such that  $\tau(v \rightarrow u') \leq \frac{1}{2}$ .

(2) Suppose that  $d_H(u) = 3$ . If  $u$  is a bad vertex, then  $\tau(v \rightarrow u) \leq 1$ . If  $u$  is adjacent to only one  $4^+$ -face that is not a bad 4-face, then  $\tau(v \rightarrow u) \leq \frac{2}{3}$ . Otherwise,  $\tau(v \rightarrow u) \leq \frac{1}{3}$ .

(3) Suppose that  $d_H(u) = 4$ . If  $u$  is a bad vertex, then  $\tau(v \rightarrow u) \leq \frac{1}{2}$ . Otherwise,  $\tau(v \rightarrow u) \leq \frac{1}{4}$ .

(4) Suppose that  $d_H(u) = 5$ . Then  $\tau(v \rightarrow u) \leq \frac{1}{5}$ .

*Proof.* (1)  $d_H(u) = 2$ . Suppose that  $u$  is a special vertex, i.e.  $u$  is incident to a special 4-face  $(v, u, x, y)$  and  $d_H(y) = 2$ . If  $u$  is incident to a 3-face, i.e. 3-face  $(v, u, x)$ , then  $\tau(v \rightarrow u) = \frac{6-2-1}{2} = \frac{3}{2}$ . Let  $v_1, \dots, v_\ell$  be neighbors of  $v$  in the clockwise order such that  $u = v_1, y = v_2$  and  $x = v_\ell$ , where  $\ell = d_H(v)$ . By Claim 4.3,  $d_{6^+}^H(v) \geq 2$ . Let  $i$  be the minimum index such that  $d_H(v_i) \geq 3$ . Then  $3 \leq i \leq \ell$  and  $d_H(v_1) = \dots = d_H(v_{i-1}) = 2$ . Let  $f_j$  be the face incident with  $vv_j$  and  $vv_{j+1}$ , for  $j \in \{1, 2, \dots, i-1\}$ . Then  $d_H(f_1) = 4$  and  $d_H(f_j) \geq 4$  for  $j = 2, \dots, i-1$ . Suppose that there exists some  $j$  ( $2 \leq j \leq i-1$ ) such that  $d_H(f_j) \geq 5$ . Let  $p$  be the minimum index such that  $d_H(f_p) \geq 5$  for  $2 \leq p \leq i-1$ . By R1 and R2,  $\tau(f_{p-1} \rightarrow v_p) \geq 1$  and  $\tau(f_p \rightarrow v_p) \geq 2$ . Therefore,  $\tau(v \rightarrow v_p) \leq \frac{1}{2}$ . Suppose that  $d_H(f_2) = \dots = d_H(f_{i-1}) = 4$ . Then  $d_H(f_i) \geq 4$ . By R1 and R2,  $\tau(f_{i-1} \rightarrow v_i) \geq 1$  and  $\tau(f_i \rightarrow v_i) \geq 2$ . Therefore,  $\tau(v \rightarrow v_i) \leq \frac{1}{2}$ .

Suppose that  $u$  is not a special vertex, then  $\tau(v \rightarrow u) \leq \frac{6-2-2}{2} = 1$ .

(2)  $d_H(u) = 3$ . If  $u$  is a bad vertex, then  $\tau(v \rightarrow u) \leq \frac{6-3}{3} = 1$ . If  $u$  is adjacent to only one  $4^+$ -face that is not a bad 4-face, then  $\tau(v \rightarrow u) \leq \frac{6-3-1}{3} = \frac{2}{3}$ . Otherwise,  $\tau(v \rightarrow u) \leq \frac{6-3-2}{3} = \frac{1}{3}$ .

(3) Suppose that  $d_H(u) = 4$ . If  $u$  is a bad vertex, then  $\tau(v \rightarrow u) \leq \frac{6-4}{4} = \frac{1}{2}$ . Otherwise,  $\tau(v \rightarrow u) \leq \frac{6-4-1}{4} = \frac{1}{4}$ .

(4) Suppose that  $d_H(u) = 5$ . Then  $\tau(v \rightarrow u) \leq \frac{6-5}{5} = \frac{1}{5}$ . □

**Observation 4.3.3.** Let  $d_H(v) \geq 8$ , and  $u$  be a 2-vertex adjacent to  $v$ , then

$$\sum_{u \in N_2^H(v)} \tau(v \rightarrow u) \leq d_2^H(v).$$

*Proof.* By observation 4.3.2(1), if  $\tau(v \rightarrow u) = \frac{3}{2}$ , then there exist a neighbor  $u'$  of  $v$  such that  $\tau(v \rightarrow u') \leq \frac{1}{2}$ . Otherwise,  $\tau(v \rightarrow u) \leq 1$ . Hence, we have the following:  $\sum_{u \in N_2^H(v)} \tau(v \rightarrow u) \leq d_2^H(v)$ . □

In the following, we will check that  $w'(x) \geq 0$  for each  $x \in V(H) \cup F(H)$  and  $\sum_{x \in V(H) \cup F(H)} w'(x) > 0$ . Let  $f \in F(H)$ . If  $d_H(f) = 3$ , then  $w'(f) = w(f) = 0$ . If  $d_H(f) = 4$ , then  $f$  is incident with at most two small vertices by Observation 4.3.1. By R2,  $f$  sends at most 2 to all adjacent  $5^-$ -vertices. Thus  $w'(f) \geq w(f) - 2 = 2 \times 4 - 6 - 2 = 0$ . If  $d_H(f) \geq 5$ , then by R1 and Observation 4.3.1,  $w'(f) \geq w(f) - 2 \times \lfloor \frac{d_H(f)}{2} \rfloor = 2d_H(f) - 6 - 2 \times \lfloor \frac{d_H(f)}{2} \rfloor \geq 0$ .

Let  $v \in V(H)$  and  $d_H(v) = \ell$ . Then  $\ell \geq 2$ . Let  $v_1, \dots, v_\ell$  be neighbors of  $v$  in the clockwise order. We use  $f_i$  be the face incident with  $vv_i$  and  $vv_{i+1}$  as boundary edges, for  $1 \leq i \leq \ell$ , where all the subscripts here are taken modulo  $\ell$ . We say that some vertices are *consecutive* if they have consecutive indices on taken modulo  $\ell$ , i.e.  $v_1$  and  $v_2$  are consecutive vertices,  $\dots$ ,  $v_\ell$  and  $v_1$  are consecutive vertices.

If  $2 \leq \ell \leq 5$ , then  $v$  is incident with  $\ell$   $8^+$ -vertices by Claim 4.3(1), so  $w'(v) \geq \ell - 6 + \beta(v) + \ell \times \frac{6-\ell-\beta(v)}{\ell} = 0$  by R1-R3.

If  $\ell = 6$ , then  $w'(v) = w(v) = 6 - 6 = 0$ .

If  $\ell = 7$ , then  $w'(v) = w(v) = 7 - 6 > 0$ .

If  $\ell = 8$ , then  $w(v) = 2$  and  $d_{5^-}^H(v) \leq 1$  by Claim 4.3(2), so  $w'(v) \geq 2 - \frac{3}{2} > 0$  by Observation 4.3.2.

Suppose  $\ell = 9$ . Then  $w(v) = 3$ . If  $d_{4^-}^H(v) \geq 1$ , then  $d_{5^-}^H(v) \leq 3$  by Claim 4.3(3), so  $w'(v) \geq 3 - d_{5^-}^H(v) \geq 0$  by Observation 4.3.2 and 4.3.3. Otherwise,  $w'(v) \geq 3 - \frac{1}{5} \times 9 > 0$ .

Suppose  $\ell = 10$ . Then  $w(v) = 4$ . If  $d_{3^-}^H(v) \geq 1$ , then  $d_{6^+}^H(v) \geq d_{3^-}^H(v) + 2$  by Claim 4.3(4), so  $w'(v) \geq 4 - d_{3^-}^H(v) - \frac{1}{2} \times (10 - d_{3^-}^H(v) - d_{6^+}^H(v)) = 0$  by Observation 4.3.2 and 4.3.3. If  $d_{3^-}^H(v) = 0$  and  $d_{6^+}^H(v) \geq 2$ , then  $w'(v) \geq 4 - \frac{1}{2} \times (10 - 2) = 0$  by Observation 4.3.2. If  $d_{3^-}^H(v) = 0$  and  $d_{6^+}^H(v) \leq 1$ ,  $w'(v) \geq 4 - \frac{1}{2} \times 5 > 0$ .

Suppose  $\ell = 11$ . Then  $w(v) = 5$ . If  $d_{3^-}^H(v) \geq 1$ , then  $d_{6^+}^H(v) \geq d_{3^-}^H(v) + 1$  by Claim 4.3(5), so  $w'(v) \geq 5 - d_{3^-}^H(v) - \frac{1}{2} \times (11 - d_{3^-}^H(v) - d_{6^+}^H(v)) = 0$  by Observation 4.3.2 and 4.3.3. If  $d_{3^-}^H(v) = 0$  and  $d_{6^+}^H(v) \geq 2$ , then  $w'(v) \geq 5 - \frac{1}{2} \times (11 - 2) > 0$  by Observation 4.3.2. If  $d_{3^-}^H(v) = 0$  and  $d_{6^+}^H(v) \leq 1$ ,  $w'(v) \geq 5 - \frac{1}{2} \times 5 > 0$ .

Suppose  $\ell = 12$ . Then  $w(v) = 6$ . If  $d_{3^-}^H(v) = 0$ , then  $w'(v) \geq 6 - \frac{1}{2} \times \ell = 0$  by Observation 4.3.2. If  $d_{2^+}^H(v) \geq 1$ , then  $d_{6^+}^H(v) \geq d_{3^-}^H(v) + 1$  by Claim 4.3(6), so  $w'(v) \geq 6 - d_{3^-}^H(v) - \frac{1}{2} \times (12 - d_{3^-}^H(v) - d_{6^+}^H(v)) > 0$  by Observation 4.3.2 and 4.3.3. Thus we assume  $d_{3^+}^H(v) \geq 1$  and  $d_{2^+}^H(v) = 0$ .

Note that if  $u$  is a bad vertex adjacent to  $v$ , then the faces incident to the edge  $uv$  is 3-faces, so  $d_{3b}^H(v) \leq 6$  by Claim 4.3(1).

(1) If  $d_{3b}^H(v) = 6$ , then  $d_{6^+}^H(v) = 6$ , so  $w'(v) \geq 6 - d_{3b}^H(v) = 0$ .

(2) If  $d_{3b}^H(v) = 5$ , then  $d_{6^+}^H(v) \geq d_{3b}^H(v) + 1$  since that each bad 3-vertex in

$N_{3b}^H(v)$  must be adjacent to two  $6^+$ -vertices in  $N_H(v)$ , so  $w'(v) \geq 6 - d_{3b}^H(v) - \frac{2}{3} \times (12 - d_{3b}^H(v) - d_{6^+}^H(v)) > 0$ .

(3) If  $d_{3b}^H(v) = 4$ , then  $d_{6^+}^H(v) \geq d_{3b}^H(v) + 1$ , so  $w'(v) \geq 6 - d_{3b}^H(v) - \frac{2}{3} \times (12 - d_{3b}^H(v) - d_{6^+}^H(v)) \geq 0$ .

(4) Suppose  $d_{3b}^H(v) = 3$ . Then  $d_{6^+}^H(v) \geq d_{3b}^H(v) + 1$ . If  $d_{6^+}^H(v) \geq 5$ , then  $w'(v) \geq 6 - d_{3b}^H(v) - \frac{2}{3} \times (12 - d_{3b}^H(v) - d_{6^+}^H(v)) > 0$ . Otherwise,  $d_{6^+}^H(v) = 4$ , then there are five consecutive  $5^-$ -vertices  $v_i, v_{i+1}, \dots, v_{i+4}$  in  $N_H(v)$ . So  $f_i, f_{i+1}, f_{i+2}, f_{i+3}$  are  $4^+$ -faces, hence  $\tau(v \rightarrow v_{i+j}) \leq \frac{1}{3}$  for  $j = 1, 2, 3$  by Observation 4.3.2. Therefore,  $w'(v) \geq 6 - d_{3b}^H(v) - \frac{2}{3} \times (12 - d_{3b}^H(v) - d_{6^+}^H(v) - 3) - \frac{1}{3} \times 3 > 0$ .

(5) Suppose  $d_{3b}^H(v) = 2$ . Then  $d_{6^+}^H(v) \geq d_{3b}^H(v) + 1$ . If  $d_{6^+}^H(v) \geq 4$ , then  $w'(v) \geq 6 - d_{3b}^H(v) - \frac{2}{3} \times (12 - d_{3b}^H(v) - d_{6^+}^H(v)) \geq 0$ . Otherwise,  $d_{6^+}^H(v) = 3$ , then there are seven consecutive  $5^-$ -vertices  $v_i, v_{i+1}, \dots, v_{i+6}$  in  $N_H(v)$ . So  $f_i, f_{i+1}, \dots, f_{i+5}$  are  $4^+$ -faces, hence  $\tau(v \rightarrow v_{i+j}) \leq \frac{1}{3}$  for  $j = 1, 2, \dots, 5$  by Observation 4.3.2. Therefore,  $w'(v) \geq 6 - d_{3b}^H(v) - \frac{2}{3} \times (12 - d_{3b}^H(v) - d_{6^+}^H(v) - 5) - \frac{1}{3} \times 5 > 0$ .

(6) Suppose  $d_{3b}^H(v) = 1$ . Then  $d_{6^+}^H(v) \geq d_{3b}^H(v) + 1$ . If  $d_{6^+}^H(v) \geq 4$ , then  $w'(v) \geq 6 - d_{3b}^H(v) - \frac{2}{3} \times (12 - d_{3b}^H(v) - d_{6^+}^H(v)) > 0$ . Otherwise,  $d_{6^+}^H(v) = 2$ , then there are four consecutive  $5^-$ -vertices  $v_i, v_{i+1}, v_{i+2}, v_{i+3}$  in  $N_H(v)$ . So  $f_i, f_{i+1}, f_{i+2}$  are  $4^+$ -faces, hence  $\tau(v \rightarrow v_{i+1}) \leq \frac{1}{3}$  and  $\tau(v \rightarrow v_{i+2}) \leq \frac{1}{3}$  by Observation 4.3.2. Therefore,  $w'(v) \geq 6 - d_{3b}^H(v) - \frac{2}{3} \times (12 - d_{3b}^H(v) - d_{6^+}^H(v) - 2) - \frac{1}{3} \times 2 > 0$ .

(7) Suppose  $d_{3b}^H(v) = 0$ . If  $d_{6^+}^H(v) \geq 3$ , then  $w'(v) \geq 6 - \frac{2}{3} \times (12 - d_{6^+}^H(v)) \geq 0$ . Otherwise,  $d_{6^+}^H(v) \leq 2$ . Since if  $v_{i+1}$  is a  $6^+$ -vertex in  $N_H(v)$ , then the two consecutive  $5^-$ -vertices  $v_i, v_{i+2}$  in  $N_H(v)$  satisfy  $\tau(v \rightarrow v_i) = \frac{2}{3}$  and  $\tau(v \rightarrow v_{i+2}) = \frac{2}{3}$ . So  $w'(v) \geq 6 - \frac{2}{3} \times (2d_{6^+}^H(v)) - \frac{1}{3} \times (12 - d_{6^+}^H(v)) > 0$ .

Suppose  $\ell \geq 13$ . If  $d_{3^-}^H(v) = 0$ , then  $w'(v) \geq 7 - \frac{1}{2} \times \ell > 0$  Observation

4.3.2. Thus we assume  $d_{3^-}^H(v) \geq 1$ . Let  $s$  be the number of special 3-faces incident to  $v$ ,  $N_{3,4,5}^H(v) = N_3^H(v) \cup N_4^H(v) \cup N_5^H(v)$ .

**Case 4.3.1.**  $d_2^H(v) \geq 1$ .

By Claim 4.3(7),  $d_{6^+}^H(v) \geq 1$ .

**Subcase 4.3.1.1.**  $v$  is incident with a special 4-face, or  $s \geq 2$ .

Then  $d_{6^+}^H(v) \geq d_{3^-}^H(v) + 1$  by Claim 4.3(7), so  $w'(v) \geq \ell - 6 - d_{3^-}^H(v) - \frac{1}{2} \times (\ell - d_{3^-}^H(v) - d_{6^+}^H(v)) = \frac{1}{2}\ell - \frac{11}{2} > 0$  by Observation 4.3.2 and 4.3.3.

**Subcase 4.3.1.2.**  $v$  is not incident with any special 4-face, or  $s \leq 1$ .

Then  $d_{6^+}^H(v) + (d_2^H(v) - s) \geq 1$ . Note that if  $v_i$  is incident to a 3-face for  $v_i \in N_2^H(v)$ , then  $\tau(v \rightarrow v_i) \leq 1$ . Otherwise,  $\tau(v \rightarrow v_i) = 0$  by R1 and R2.

Hence, we have the following:

$$\sum_{u \in N_2^H(v)} \tau(v \rightarrow u) \leq s.$$

(1) If  $d_{6^+}^H(v) + (d_2^H(v) - s) \geq 6$ , then  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s)) \geq 0$ .

(2) Suppose  $d_{6^+}^H(v) + (d_2^H(v) - s) = 5$ . If there are three consecutive vertices  $v_i, v_{i+1}, v_{i+2} \in N_{3,4,5}^H(v)$ , then  $f_i$  and  $f_{i+1}$  are 4<sup>+</sup>-faces, hence  $v_i$  and  $v_{i+2}$  can not be bad 3-vertices. So  $\tau(v \rightarrow v_i) \leq \frac{2}{3}$ ,  $\tau(v \rightarrow v_{i+1}) \leq \frac{1}{3}$  and  $\tau(v \rightarrow v_{i+2}) \leq \frac{2}{3}$  by Observation 4.3.2. Thus,  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - \frac{2}{3} \times 2 - \frac{1}{3} - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 3) = \frac{1}{3} > 0$ .

Otherwise, there must be two vertex-disjoint subsets  $\{v_i, v_{i+1}\}, \{v_j, v_{j+1}\} \in N_{3,4,5}^H(v)$ , then  $\tau(v \rightarrow v_i) + \tau(v \rightarrow v_{i+1}) + \tau(v \rightarrow v_j) + \tau(v \rightarrow v_{j+1}) \leq \frac{2}{3} \times 4 = \frac{8}{3}$ , so  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - \frac{8}{3} - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 4) = \frac{1}{3} > 0$ .

(3) Suppose  $d_{6^+}^H(v) + (d_2^H(v) - s) = 4$ . If there are four consecutive vertices  $v_i, v_{i+1}, v_{i+2}, v_{i+3} \in N_{3,4,5}^H(v)$ , then  $\sum_{j=i}^{i+3} \tau(v \rightarrow v_j) \leq \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 2$  by Observation 4.3.2, so  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - 2 - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 4) = 0$ . If there are two vertex-disjoint subsets  $\{v_i, v_{i+1}, v_{i+2}\}, \{v_p, v_{p+1}\} \in N_{3,4,5}^H(v)$ , then  $\sum_{j=i}^{i+2} \tau(v \rightarrow v_j) \leq \frac{2}{3} \times 2 + \frac{1}{3} = \frac{5}{3}$  and  $\tau(v \rightarrow v_p) + \tau(v \rightarrow v_{p+1}) \leq \frac{2}{3} \times 2 = \frac{4}{3}$ , so  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - \frac{5}{3} - \frac{4}{3} - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 5) = 0$ . Otherwise, there must be four vertex-disjoint subsets  $\{v_i, v_{i+1}\}, \{v_j, v_{j+1}\}, \{v_p, v_{p+1}\}, \{v_q, v_{q+1}\} \in N_{3,4,5}^H(v)$ , then  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - \frac{2}{3} \times 8 - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 8) = \frac{2}{3} > 0$ .

(4) Suppose  $d_{6^+}^H(v) + (d_2^H(v) - s) = 3$ . If there are six consecutive vertices  $v_i, v_{i+1}, \dots, v_{i+5} \in N_{3,4,5}^H(v)$ , then  $\sum_{j=i}^{i+5} \tau(v \rightarrow v_j) \leq \frac{2}{3} \times 2 + \frac{1}{3} \times 4 = \frac{8}{3}$  by Observation 4.3.2, so  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - \frac{8}{3} - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 6) = \frac{1}{3} > 0$ . If there are two vertex-disjoint subsets  $\{v_i, v_{i+1}, \dots, v_{i+4}\}, \{v_p, v_{p+1}\} \in N_{3,4,5}^H(v)$  or  $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}, \{v_p, v_{p+1}, v_{p+2}\} \in N_{3,4,5}^H(v)$ , then  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - \frac{2}{3} \times 4 - \frac{1}{3} \times 3 - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 7) = \frac{1}{3} > 0$ . Otherwise, there must be three vertex-disjoint subsets  $\{v_i, v_{i+1}, v_{i+2}\}, \{v_j, v_{j+1}, v_{j+2}\}, \{v_p, v_{p+1}, v_{p+2}\} \in N_{3,4,5}^H(v)$ , then  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - \frac{2}{3} \times 6 - \frac{1}{3} \times 3 - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 9) = 1 > 0$ .

(5) Suppose  $d_{6^+}^H(v) + (d_2^H(v) - s) = 2$ . If there are seven consecutive vertices  $v_i, v_{i+1}, \dots, v_{i+6} \in N_{3,4,5}^H(v)$ , then  $\sum_{j=i}^{i+6} \tau(v \rightarrow v_j) \leq \frac{2}{3} \times 2 + \frac{1}{3} \times 5 = 3$  by Observation 4.3.2, so  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - 3 - (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 7) = 0$ . Otherwise, there are two vertex-disjoint subsets  $\{v_i, v_{i+1}, \dots, v_{i+4}\}, \{v_j, v_{j+1}, v_{j+2}, v_{j+3}\} \in N_{3,4,5}^H(v)$ , then  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - \frac{2}{3} \times 4 - \frac{1}{3} \times 5 - (\ell - d_{6^+}^H(v) -$

$$(d_2^H(v) - s) - 9 = \frac{2}{3} > 0.$$

(6) Suppose  $d_{6^+}^H(v) + (d_2^H(v) - s) = 1$ . Then  $s = 1$ . So  $d_2^H(v) = 1$ ,  $d_{6^+}^H(v) = 1$  and they are consecutive vertices. Therefore,  $w'(v) \geq \ell - 6 - (s + d_3^H(v) + d_4^H(v) + d_5^H(v)) \geq \ell - 6 - 1 - \frac{2}{3} \times 2 - \frac{1}{3} \times (\ell - d_{6^+}^H(v) - (d_2^H(v) - s) - 3) = \frac{2}{3}\ell - 7 > 0$ .

**Case 4.3.2.**  $d_2^H(v) = 0$  and  $d_{3^-}^H(v) \geq 1$ .

Note that if  $u$  is a bad vertex adjacent to  $v$ , then the faces incident to the edge  $uv$  are 3-faces, so  $d_{3b}^H(v) \leq \lfloor \frac{\ell}{2} \rfloor$  by Claim 4.3(1).

(1) If  $d_{3b}^H(v) = \lfloor \frac{\ell}{2} \rfloor$ , then  $d_{6^+}^H(v) = \lceil \frac{\ell}{2} \rceil$ , so  $w'(v) \geq \ell - 6 - d_{3b}^H(v) \geq 0$ .

(2) Suppose  $d_{3b}^H(v) \geq 3$ . Since each bad 3-vertex in  $N_{3b}^H(v)$  must be adjacent to two  $6^+$ -vertices in  $N_H(v)$ , then  $d_{6^+}^H(v) \geq d_{3b}^H(v) + 1$ , so  $w'(v) \geq \ell - 6 - d_{3b}^H(v) - \frac{2}{3} \times (\ell - d_{3b}^H(v) - d_{6^+}^H(v)) = \frac{1}{3}\ell - 6 - \frac{1}{3}d_{3b}^H(v) + \frac{2}{3}d_{6^+}^H(v) \geq 0$ .

(3) Suppose  $d_{3b}^H(v) = 2$ . Then  $d_{6^+}^H(v) \geq d_{3b}^H(v) + 1$ . If  $d_{6^+}^H(v) \geq 4$ , then  $w'(v) \geq \ell - 6 - d_{3b}^H(v) - \frac{2}{3} \times (\ell - d_{3b}^H(v) - d_{6^+}^H(v)) = \frac{1}{3}\ell - 6 - \frac{1}{3}d_{3b}^H(v) + \frac{2}{3}d_{6^+}^H(v) \geq 0$ . Otherwise,  $d_{6^+}^H(v) = 3$ , then there are five consecutive  $5^-$ -vertices  $v_i, v_{i+1}, \dots, v_{i+4}$  in  $N_H(v)$ . So  $f_i, f_{i+1}, f_{i+2}, f_{i+3}$  are  $4^+$ -faces, hence  $\tau(v \rightarrow v_{i+j}) \leq \frac{1}{3}$  for  $j = 1, 2, 3$  by Observation 4.3.2. Therefore,  $w'(v) \geq \ell - 6 - d_{3b}^H(v) - \frac{2}{3} \times (\ell - d_{3b}^H(v) - d_{6^+}^H(v) - 3) - \frac{1}{3} \times 3 = \frac{1}{3}(\ell + d_{3b}^H(v) - 13) > 0$ .

(3) Suppose  $d_{3b}^H(v) = 1$ . Then  $d_{6^+}^H(v) \geq d_{3b}^H(v) + 1$ . If  $d_{6^+}^H(v) \geq 3$ , then  $w'(v) \geq \ell - 6 - d_{3b}^H(v) - \frac{2}{3} \times (\ell - d_{3b}^H(v) - d_{6^+}^H(v)) = \frac{1}{3}\ell - 6 - \frac{1}{3}d_{3b}^H(v) + \frac{2}{3}d_{6^+}^H(v) \geq 0$ . Otherwise,  $d_{6^+}^H(v) = 2$ , then there are five consecutive  $5^-$ -vertices  $v_i, v_{i+1}, \dots, v_{i+4}$  in  $N_H(v)$ . So  $w'(v) \geq \ell - 6 - d_{3b}^H(v) - \frac{2}{3} \times (\ell - d_{3b}^H(v) - d_{6^+}^H(v) - 3) - \frac{1}{3} \times 3 = \frac{1}{3}(\ell + d_{3b}^H(v) - 13) > 0$ .

(4) Suppose  $d_{3b}^H(v) = 0$ . If  $d_{6^+}^H(v) \geq 3$ , then  $w'(v) \geq \ell - 6 - d_{3b}^H(v) - \frac{2}{3} \times (\ell - d_{3b}^H(v) - d_{6^+}^H(v)) = \frac{1}{3}\ell - 6 - \frac{1}{3}d_{3b}^H(v) + \frac{2}{3}d_{6^+}^H(v) \geq 0$ . Otherwise,  $d_{6^+}^H(v) \leq 2$ . Since if  $v_{i+1}$  is a  $6^+$ -vertex in  $N_H(v)$ , then the two consecutive

$5^-$ -vertices  $v_i, v_{i+2}$  in  $N_H(v)$  satisfy  $\tau(v \rightarrow v_i) = \frac{2}{3}$  and  $\tau(v \rightarrow v_{i+2}) = \frac{2}{3}$ . So  $w'(v) \geq \ell - 6 - \frac{2}{3} \times (2d_{6^+}^H(v)) - \frac{1}{3} \times (\ell - d_{6^+}^H(v)) = \frac{2}{3}\ell - d_{6^+}^H(v) - 6 > 0$ .

Hence,  $\sum_{x \in V(H) \cup F(H)} w'(x) > 0$ , this contradiction completes the proof of Theorem 4.2.1.

## Chapter 5 List linear arboricity

### §5.1 Basic definitions and properties

A *linear forest* is a graph in which each component is a path. A mapping  $L$  is said to be an *edge assignment* for a graph  $G$  if it assigns a list  $L(e)$  of possible colors to every edge  $e \in G$ . If  $G$  has a coloring  $\varphi(e)$  such that  $\varphi(e) \in L(e)$  for every edge  $e$  and the induced subgraph of edges having the same color  $\alpha$  is a linear forest for any  $\alpha \in \{\varphi(e) | e \in E(G)\}$ , then we say that  $G$  is linear  $L$ -colorable or  $\varphi$  is a linear  $L$ -coloring of  $G$ . We say that  $G$  is linear  $k$ -choosable if it is linear  $L$ -colorable for every list assignment  $L$  satisfying  $|L(e)| \geq k$  for all edges  $e$ . The list linear arboricity  $la_{list}(G)$  of a graph  $G$  is the minimum number  $k$  for which  $G$  is linear  $k$ -list colorable. If  $L(e) = \{1, 2, \dots, t\}$  for any  $e \in E(G)$ , then the linear  $L$ -coloring of  $G$  is called a  $t$ -linear coloring. The linear arboricity  $la(G)$  of a graph  $G$  defined by Harary [54] is the minimum number  $t$  for which  $G$  has a  $t$ -linear coloring. It is obvious that  $la(G) \leq la_{list}(G)$ .

In 1980, Akiyama et al. [4] conjectured that  $la(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$  for any simple regular graph  $G$ . The conjecture is equivalent to the following conjecture 1.3.10.

**Conjecture 1.3.10.** *For any graph  $G$ ,  $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ .*

This has been proved in complete bipartite graphs [4], complete regular multipartite graphs [154], planar graphs [149, 152] and regular graphs with  $\Delta = 3, 4$  [5] and  $\Delta = 5, 6, 8$  [46]. Recently, M. Cygan et al. [39] proved that if  $G$  is a planar graph with  $\Delta \geq 9$ , then  $la(G) = \lceil \frac{\Delta}{2} \rceil$ , and then they posed the following conjecture 1.3.10 (the conjecture has also been posed in [150]).

**Conjecture 1.3.10** *For any planar graph  $G$  of maximum degree  $\Delta \geq 5$ ,*

$$la(G) = \lceil \frac{\Delta}{2} \rceil.$$

Some related results about the conjecture refer to [34]. For the list linear arboricity, the following conjecture is posed in [150] and [7], independently.

**Conjecture 1.3.11** *For any graph  $G$ ,  $la(G) = la_{list}(G)$ .*

The list linear arboricity of a planar graph  $G$  is at most  $\lceil \frac{\Delta(G)+1}{2} \rceil$  if  $\Delta(G) \geq 8$  [7]; or  $\Delta(G) \geq 6$  and  $G$  is  $F_5$ -free [162].  $la_{list}(G) = \lceil \frac{\Delta(G)}{2} \rceil$  if  $\Delta(G) \geq 13$ , or  $\Delta(G) \geq 7$  and  $G$  contains no  $i$ -cycles for some  $i \in \{3, 4, 5\}$  [7]; or  $\Delta(G) \geq 11$  and  $G$  is  $F_5$ -free [162]. Though few results have been reported, the investigation is far from satisfactory. In this chapter, we prove that for a planar graph  $G$  with 7-cycles containing at most two chords,  $la_{list}(G) \leq \max\{4, \lceil \frac{\Delta(G)+1}{2} \rceil\}$  and  $la_{list}(G) = \lceil \frac{\Delta}{2} \rceil$  if  $\Delta(G) \geq 11$ .

## §5.2 List linear arboricity of planar graph

**Lemma 5.2.1.** *Let  $G$  be a planar graph. If every 7-cycles of  $G$  contains at most two chords, then*

- (1)  $G$  has an edge  $uv$  with  $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$ , or
- (2)  $G$  has an even cycle  $c = v_1v_2 \cdots v_{2n}v_1$  with  $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$ .

*Proof.* The proof is carried out by contradiction. Let  $G$  be a minimal counterexample to the lemma with  $|V| + |E|$  minimized. It is obvious that  $G$  is a connected planar graph. By the choice of  $G$ , we have the following observations.

(a) For any edge  $uv$ ,  $d(u) + d(v) \geq \max\{10, \Delta(G) + 3\}$  by (1). Then  $\delta(G) \geq 3$  and all neighbors of an  $i$ -vertex must be  $(10 - i)^+$ -vertices, where  $i = 3, 4$  or  $5$ ;

(b) Let  $V_1$  be the set of 3-vertices of  $G$  and  $G_3$  the subgraph induced by the edges incident with 3-vertices of  $G$ . Then  $G_3$  is a forest.

By (a) or (1), every two 3-vertices are not adjacent, and it follows that  $G_3$  does not contain odd cycles. By (2),  $G_3$  contains no even cycles. So  $G_3$  is a forest and (b) holds.

Thus for any component of  $G_3$ , we select a vertex  $u \notin V_1$  as a root of the tree. Then every 3-vertex has exactly two children. If  $uv \in E(G_3)$ ,  $u \in V_1$  and  $v$  is a child of  $u$ , then  $v$  is called a 3-master of  $u$ . Note that each 3-vertex has exactly two 3-masters and each vertex of degree at least 7 can be the 3-master of at most two 3-vertices.

By the Euler's formula  $|V| - |E| + |F| = 2$ , we have

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -4(|V(G)| - |E(G)| + |F(G)|) = -8 < 0.$$

For each  $x \in V \cup F$ , we define the initial charge

$$ch(x) = \begin{cases} d(x) - 4 & \text{if } x \in V \\ d(x) - 4 & \text{if } x \in F \end{cases}$$

So  $\sum_{x \in V \cup F} ch(x) = -8 < 0$ . In the following, we will reassign a new charge denoted by  $ch'(x)$  to each  $x \in V \cup F$  according to the discharging rules. If we can show that  $ch'(x) \geq 0$  for each  $x \in V \cup F$ , then we get an obvious contradiction to  $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -8$ , which completes our proof.

**R1:** Every 3-vertex receives  $\frac{1}{2}$  from each of its 3-masters.

**R2:** Let  $f$  be a 3-face  $uvw$  with  $d(u) \leq d(v) \leq d(w)$ .

**R2.1:** If  $d(u) = 3$  or  $d(u) = 4$ , then  $f$  receives  $\frac{1}{2}$  from each of  $v$  and  $w$ , respectively.

**R2.2:** If  $d(u) = 5$ , then  $f$  receives  $\frac{1}{f_3(v)}$  from each of  $u$ ,  $v$  and  $w$ , respectively.

**R2.3:** If  $d(u) \geq 6$ , then  $f$  receives  $\frac{1}{3}$  from each of  $u$ ,  $v$  and  $w$ , respectively.

**R3:** Each  $3^+$ -face receives  $\frac{1}{5}$  from each adjacent  $5^+$ -face.

First, we consider the final charge of any face  $f$ . If  $d(f) = 3$ , note that if a 5-vertex  $x$  is incident with four 3-faces, then any  $5^+$ -vertex adjacent to  $x$  must be incident with two  $5^+$ -faces. Then  $ch'(f) = ch(f) + \min\{\frac{1}{2} \times 2, \frac{1}{5} + \frac{1}{3} \times 2 + \frac{1}{5}, \frac{1}{3} \times 3\} = -1 + 1 = 0$  by (a), R2 and R3. If  $d(f) = 4$ , then  $ch'(f) = ch(f) = d(f) - 4 = 0$ . If  $d(f) \geq 5$ , then  $ch'(f) = ch(f) - \frac{1}{5} \times d(f) \geq 0$ .

Second, suppose that  $v$  is any vertex of  $G$ . If  $d(v) = 3$ , then  $v$  has exactly two 3-masters and it follows by R1 that  $ch'(v) \geq ch(v) + \frac{1}{2} \times 2 = 0$ . According to the rules, every 4-vertex retain its initial charge. So  $ch'(v) = ch(v) = 4 - 4 = 0$  if  $d(v) = 4$ . If  $d(v) = 5$ , all neighbors of 5-vertex should be  $5^+$ -vertices, then  $ch'(v) \geq ch(v) + \frac{1}{f_3(v)} \times f_3(v) = 0$ . If  $d(v) = 6$ , then  $f_3(v) \leq 4$ , so  $ch'(v) \geq ch(v) - \max\{\frac{1}{3}, \frac{1}{2}\} \times 4 = 2 - \frac{1}{2} \times 4 = 0$ . If  $d(v) = 7$ , then  $f_3(v) \leq 5$ ,  $v$  can be the 3-master of a 3-vertex, and it follows from R1 and R2 that  $ch'(v) \geq ch(v) - \frac{1}{2} \times 1 - \frac{1}{2} \times 5\} = 0$ . If  $d(v) = 8$ , then  $f_3(v) \leq 6$ , so  $ch'(v) \geq ch(v) - \frac{1}{2} \times 1 - \frac{1}{2} \times 6\} = \frac{1}{2} > 0$ . If  $d(v) \geq 9$ , then  $ch'(v) \geq ch(v) - \frac{1}{2} \times 1 - \frac{1}{2} \times d(v)\} \geq 0$ .

Hence, this completes the proof of Lemma 5.2.1. □

Given a  $t$ -linear coloring  $\varphi$  and  $v \in V(G)$ , we denote by  $C_\varphi^i(v)$  the set of colors appear  $i$  times at  $v$ , where  $i = 0, 1, 2$ . Then  $|C_\varphi^0(v)| + |C_\varphi^1(v)| + |C_\varphi^2(v)| =$

$t$  and  $d(v) = |C_\varphi^1(v)| + 2|C_\varphi^2(v)|$ . Let  $C_\varphi(u, v) = C_\varphi^2(u) \cup C_\varphi^2(v) \cup (C_\varphi^1(u) \cap C_\varphi^1(v))$  for two vertices  $u$  and  $v$ .

**Theorem 5.2.1.** *Let  $G$  be a planar graph. If every 7-cycles of  $G$  contains at most two chords, then  $\lceil \frac{\Delta(G)}{2} \rceil \leq la_{list}(G) \leq \max\{4, \lceil \frac{\Delta(G)+1}{2} \rceil\}$ .*

*Proof.* Let  $G$  be a minimal counterexample to the theorem, that is, there is an edge assignment  $L$  with  $|L(e)| \geq k$  for all  $e \in E(G)$ , where  $k = \max\{4, \lceil \frac{\Delta(G)+1}{2} \rceil\}$ , such that  $G$  is not linear- $L$ -colorable, but all proper subgraphs of  $G$  are linear- $L$ -colorable. By Lemma 5.2.1, we consider two cases as follows.

**Case 5.2.1.**  $G$  contains an edge  $uv$  with  $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$ .

By minimality of  $G$ ,  $G' = G - uv$  has a linear- $L$ -coloring  $\phi$ . Since  $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$ ,  $|C_\phi(u, v)| < k$ , we may extend  $\phi$  to a linear  $L$ -coloring of  $G$  by setting  $\phi(uv) \in L(uv) \setminus C_\phi(u, v)$ , a contradiction.

**Case 5.2.2.**  $G$  has an even cycle  $C = v_0v_1 \cdots v_{2n-1}v_0$  with  $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$ .

Let  $G'$  be the subgraph of  $G$  obtained by deleting the edges of  $C$ . Then  $G'$  has a linear  $L$ -coloring  $\phi$ . In the following, we will construct a linear  $L$ -coloring  $\sigma$ .

Let  $S = L(v_{2n-1}v_0) \setminus C_\phi(v_{2n-1}, v_0)$  and  $N_G(v_{2n-1}) = \{u, v_0, v_{2n-2}\}$ . Since  $|L(v_{2n-1}v_0)| \geq k$ ,  $|S| \geq 1$ . If  $|S| \geq 2$ , then let  $L'(v_{2n-1}v_0) = S$ ; Otherwise we must have  $\phi(v_{2n-1}u) = \alpha \in C_\phi^1(v_0)$ . Let  $L'(v_{2n-1}v_0) = S \cup \{\alpha\}$  and at the same time, if there exists a monochromatic path of color  $\alpha$  between  $v_{2n-1}$  and  $v_0$  passing  $u$ , then  $\alpha \in C_\phi^2(u)$ , and we need to recolor  $v_{2n-1}u$  such that  $\sigma(v_{2n-1}u) \in L(v_{2n-1}u) \setminus C_\phi^2(u)$ . So  $|L'(v_{2n-1}v_0)| \geq 2$ . We define the assignment  $L'$  of other edges of  $C$  such that  $L'(v_i v_{i+1}) = L(v_i v_{i+1}) \setminus C_\phi(v_i, v_{i+1}) (0 \leq i \leq 2n - 2)$ .

If  $L'(v_0v_1) = L'(v_1v_2) = \dots = L'(v_{2n-2}v_{2n-1}) = L'(v_{2n-1}v_0)$ , then we color  $C$  such that  $\sigma(v_{2i}v_{2i+1}) \in L'(v_0v_1)$  and  $\sigma(v_{2i+1}v_{2i+2}) \in L'(v_0v_1) \setminus \sigma(v_{2i}v_{2i+1})$ ,  $i = 0, 1, \dots, n-1$ ; Otherwise, there is an  $i(0 \leq i \leq 2n-1)$  such that  $L'(v_i v_{i+1}) \setminus L'(v_{i+1} v_{i+2}) \neq \emptyset$  or  $L'(v_{i+1} v_{i+2}) \setminus L'(v_i v_{i+1}) \neq \emptyset$ , where the subscripts are taken modulo  $2n$ . Without loss of generality, assume that there is a color  $\beta \in L'(v_0v_1) \setminus L'(v_{2n-1}v_0)$ . First we color  $v_0v_1$  such that  $\sigma(v_0v_1) = \beta$ . Then we assume that  $v_0v_1, v_1v_2, \dots, v_{i-1}v_i (1 \leq i \leq 2n-1)$  has been colored, color  $v_i v_{i+1}$  satisfying

$$\sigma(v_i v_{i+1}) \in \begin{cases} L'(v_i v_{i+1}) & \text{if } |L'(v_i v_{i+1})| = 1, \\ L'(v_i v_{i+1}) \setminus \sigma(v_{i-1} v_i) & \text{otherwise.} \end{cases}$$

Finally, the uncolored edges of  $G$  are colored the same colors as in  $\phi$  of  $G'$ . Thus  $\sigma$  is a linear  $L$ -coloring of  $G$ , a contradiction.

This completes the proof. □

According to the Theorem 5.2.1, it is easy to obtain the following corollary.

**Corollary 5.2.2.** *Let  $G$  be a planar graph. If  $\Delta(G) \geq 6$  and every 7-cycles of  $G$  contains at most two chords, then  $\lceil \frac{\Delta(G)}{2} \rceil \leq la_{list}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ .*

**Theorem 5.2.3.** *Let  $G$  be a planar graph, and every 7-cycles of  $G$  contains at most two chords. Then  $G$  is linear  $k$ -choosable, where  $k \geq \max\{6, \lceil \frac{\Delta(G)}{2} \rceil\}$ .*

*Proof.* Let  $G$  be a minimal counterexample to Theorem 5.2.3. Then there is an edge assignment  $L$  with  $|L(e)| \geq k$  for all  $e \in E(G)$ , where  $k = \max\{6, \lceil \frac{\Delta(G)}{2} \rceil\}$ , such that  $G$  is not linear  $L$ -colorable, but any proper subgraph of  $G$  is linear- $L$ -colorable. By a similar proof as Theorem 2.4 in [7],  $G$  has the following properties.

- (1) For every edge  $uv$  of  $G$ ,  $d(u) + d(v) \geq \max\{14, \Delta(G) + 2\}$ ;

(2)  $G$  has no even cycle  $C = v_1v_2 \cdots v_{2n}v_1$  with  $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$  and  $\max_{1 \leq i \leq n} |n_2(v_{2i-1})| \geq 3$ , where  $n_2(v)$  is the number of 2-vertices adjacent to  $v$ .

By (1),  $\delta(G) \geq 2$  and any two 2-vertices are not adjacent. Let  $G_2$  be the subgraph induced by the edges incident with the 2-vertices of  $G$ . Then  $G_2$  contains no odd cycle. So it follows from (2) that any component of  $G_2$  is either an even cycle or a tree, and then we can find a matching  $M$  in  $G$  saturating all 2-vertices ( $M$  contains alternate edges of every even cycle of  $G_2$ , and if some component of  $G_2$  is a tree  $T$  then we repeatedly add to  $M$  a pendant edge  $e$  of  $T$  and delete the end-vertices of  $e$  from  $T$ ). If  $uv \in M$  and  $d(u) = 2$ ,  $v$  is called a *2-master* of  $u$ . Note that every 2-vertex has a 2-master, which is necessarily a vertex of maximum degree and each vertex of the maximum degree can be the 2-master of at most one 2-vertex.

We define a weight function  $c$  on  $V(G) \cup F(G)$  by letting  $c(v) = 2d(v) - 6$  for each  $v \in V(G)$  and  $c(f) = d(f) - 6$  for each  $f \in F(G)$ . Applying Euler's formula  $|V(G)| - |E(G)| + |F(G)| = 2$ , we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

In the following, we will reassign a new charge denoted by  $c'(x)$  to each  $x \in V \cup F$  according to the discharging rules. If we can show that  $c'(x) \geq 0$  for each  $x \in V \cup F$ , then we get an obvious contradiction to  $0 \leq \sum_{x \in V \cup F} c'(x) = \sum_{x \in V \cup F} c(x) = -12$ , which completes our proof.

Let  $c(v \rightarrow f)$  be the amount that a vertex  $v$  sends a face  $f$ . The rules for redistribution of charge are as follows:

**R1:** Each 2-vertex receives 2 from its 2-master;

**R2:** Let  $f$  be a 3-face. If  $d(v) = 4$ , then  $c(v \rightarrow f) = \frac{1}{2}$ . If  $d(v) = 5$ , then

$c(v \rightarrow f) = \frac{4}{5}$ . If  $6 \leq d(v) \leq 8$ , then  $c(v \rightarrow f) = 1$ . If  $d(v) = 9$ , then  $c(v \rightarrow f) = \frac{11}{10}$ . If  $d(v) \geq 10$ , then  $c(v \rightarrow f) = \frac{3}{2}$ .

**R3:** Let  $f$  be a 4-face. If  $4 \leq d(v) \leq 10$ , then  $c(v \rightarrow f) = \frac{1}{2}$ . If  $d(v) \geq 11$ , then  $c(v \rightarrow f) = 1$ .

**R4:** Let  $f$  be a 5-face. If  $d(v) \geq 4$ , then  $c(v \rightarrow f) = \frac{1}{3}$ .

Let  $c'(x)$  be the resulting charge on  $x$ . In the following, we will check that  $c'(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ .

Let  $f$  be a face of  $G$ . If  $d(f) = 3$ , then  $c'(f) = c(f) + \min\{\frac{3}{2} \times 2, \frac{3}{2} + \frac{1}{2}, \frac{3}{2} + \frac{4}{5}, 1 \times 3\} = 0$  by R2. If  $d(f) = 4$ , then  $c'(f) = c(f) + \min\{\frac{1}{2} \times 4, 1 \times 2\} = 0$  by R3. If  $d(f) = 5$ , then  $c'(f) = c(f) + \min\{\frac{1}{5} \times 5, \frac{1}{3} \times 3\} = 0$  by R4. If  $d(f) \geq 6$ , then  $c'(f) = c(f) \geq 0$ .

Let  $v$  be a vertex of  $G$ . If  $d(v) = 2$ , then  $c'(v) = c(v) + 2 = 0$  by R1. If  $d(v) = 3$ , then  $c'(v) = c(v) = 0$ . If  $d(v) = 4$ , then  $c'(v) = c(v) - \max\{\frac{1}{2} \times 4, \frac{1}{3} \times 4\} = 0$ . If  $d(v) = 5$ , then  $c'(v) = c(v) - \max\{\frac{4}{5} \times 5, \frac{1}{2} \times 5, \frac{1}{3} \times 5\} = 0$ . If  $6 \leq d(v) \leq 8$ , then  $c'(v) = c(v) - \max\{1 \times d(v), \frac{1}{2} \times d(v), \frac{1}{3} \times d(v)\} \geq 0$ . Suppose that  $d(v) \geq 9$ . Since every 7-cycles of  $G$  contains at most two chords,  $f_3(v) \leq d(v) - 2$ . If  $d(v) = 9$ , then  $c'(v) \geq c(v) - \frac{11}{10} \times 7 - 1 \times 2 \geq 0$ . If  $10 \leq d(v) \leq 11$ , then  $c'(v) \geq c(v) - \frac{3}{2} \times (d(v) - 2) - 1 \times 2 \geq 0$ . Suppose  $d(v) \geq 12$ . If  $f_3(v) = d(v) - 2$ , then  $f_{5^+}(v) = 2$ , so  $c'(v) \geq c(v) - 2 - \frac{3}{2} \times (d(v) - 2) - \frac{1}{3} \times 2 \geq 0$ . If  $f_3(v) = d(v) - 3$ , then  $f_{5^+}(v) = 3$ , so  $c'(v) \geq c(v) - 2 - \frac{3}{2} \times (d(v) - 3) - \frac{1}{3} \times 3 \geq 0$ . If  $f_3(v) \leq d(v) - 4$ , then  $c'(v) \geq c(v) - 2 - \frac{3}{2} \times (d(v) - 4) - 1 \times 4 \geq 0$ .

We have checked that  $c'(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ . Hence, this completes the proof of Theorem 5.2.3. □

Based on the above Theorem 5.2.3, we can obtain the following corollary easily.

**Corollary 5.2.4.** *Let  $G$  be a planar graph. If  $\Delta(G) \geq 11$  and every 7-cycles of  $G$  contains at most two chords, then  $la_{list}(G) = \lceil \frac{\Delta(G)}{2} \rceil$ .*



## Chapter 6 Future research

The coloring problem has been the development of hot spots in the study of graph theory, which has a wide application, and according to the different objects and rules, there are lots of coloring problems, such as acyclic vertex coloring, acyclic edge coloring, acyclic total coloring, edge face coloring, vertex edge face coloring, equitable coloring, circular coloring and so on. It has very important theoretical and practical significance to solve these problems.

### §6.1 Some graphs

In this paper, we mainly study the coloring problems in planar graphs. In graph theory, a planar graph is a graph that can be embedded in the plane. It can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. The Polish mathematician Kazimierz · Kuratowski provided a characterization of planar graphs in terms of forbidden graphs, now known as Kuratowski's theorem: A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of  $K_5$  (the complete graph on five vertices) or  $K_{3,3}$ . The Euler characteristic was originally defined for polyhedra and used to prove various theorems about them by using the formula of Euler:  $\chi = |V(G)| - |E(G)| + |F(G)|$ , where  $|V(G)|$  is the number of vertices of  $G$ ,  $|E(G)|$  is the number of edges of  $G$ ,  $|F(G)|$  is the number of faces of  $G$ , including the exterior face. For planar graphs, we have  $|V(G)| - |E(G)| + |F(G)| = 2$ . We can also use discharging method to get some results of  $\chi \geq 0$ .

In the other hand, series-parallel graphs are characterised by having no

subgraph homeomorphic to  $K_4$ . A graph is an series-parallel graph, if it may be turned into  $K_2$  by a sequence of the following operations: Replacement of a pair of parallel edges with a single edge that connects their common endpoints. Replacement of a pair of edges incident to a vertex of degree 2 other than s or t with a single edge. Every series-parallel graph has treewidth at most 2 and branchwidth at most 2. An outerplanar graph is a graph that has a planar drawing for which all vertices belong to the outer face of the drawing. Outerplanar graphs may be characterized by the two forbidden minors  $K_4$  and  $K_{2,3}$ . After studying the structure of the planar graph, we can study these two classes of graphs. We try to extend the results of various kinds of coloring on these graphs, and we can find the relationship between the boundary and the maximum degree.

Moreover, we know for planar graphs, we have  $|E(G)| \leq 3|V(G)| - 6$ . Ringel's [94] motivation was in trying to solve a variation of total coloring for planar graphs, in which one simultaneously colors the vertices and faces of a planar graph, mentioned 1-planar graphs. A 1-planar graph is a graph  $G$  that can be drawn in the Euclidean plane in such a way that each edge has at most one crossing point, where it crosses a single additional edge. Pach and Tóth [88] proved every 1-planar graph with  $n$  vertices has at most  $4n - 8$  edges. More strongly, each 1-planar drawing has at most  $n - 2$  crossings; removing one edge from each crossing pair of edges leaves a planar graph, which can have at most  $3n - 6$  edges, from which the  $4n - 8$  bound on the number of edges in the original 1-planar graph immediately follows. The associated plane graph  $G^\times$  of a 1-plane graph  $G$  is the plane graph that is obtained from  $G$  by turning all crossings of  $G$  into new 4-vertices. A vertex  $v \in V(G^\times) \setminus V(G)$  in  $G^\times$  is called *false* if it is not a vertex of  $G$  and *true* otherwise. For any two false vertices  $u$  and  $v$  in  $G^\times$ ,  $uv \in E(G) \setminus E(G^\times)$ .

For every  $v$  of  $G$ , we have  $d_G(v) = d_{G^\times}(v)$ .

There has been some studies in 1-planar graph.

For a simple graph  $G$ , by Vizing Theorem, the edge chromatic number of  $G$  is  $\Delta$  (i.e.  $G$  is in *class one*) or  $\Delta + 1$  (i.e.  $G$  is in *class two*).

**Conjecture 6.1.1.** [161] *For any 1-planar graph  $G$  with  $\Delta \geq 8$ ,  $G$  is in class one.*

We can conjecture:

**Conjecture 6.1.2.** *For any 1-planar graph  $G$  with  $\Delta \geq 8$ , the total chromatic number  $\chi''(G) = \Delta + 1$ .*

**Conjecture 6.1.3.** *For any 1-planar graph  $G$  with  $\Delta \geq 8$ , the list edge chromatic number  $\chi'_l(G) = \Delta$ , the list total chromatic number  $\chi''_l(G) = \Delta + 1$ .*

**Conjecture 6.1.4.** *For any 1-planar graph  $G$  with  $\Delta \geq 10$ , the neighbor sum distinguish number  $\chi''_{\Sigma}(G) = \Delta + 3$ .*

**Conjecture 6.1.5.** *For any 1-planar graph  $G$  with  $\Delta \geq 13$ , the list linear arboricity  $la_{list}(G) \geq \lceil \frac{\Delta(G)}{2} \rceil$ .*

We can also study other coloring problems in 1-planar graphs.

There are many other graphs which are deserved to study, such as interval graphs, unicyclic graphs, chordal graphs, split graphs, claw-free graphs, hypergraphs, signed graphs and so on.

## §6.2 Future research in planar graphs

### Case 6.2.1. Total coloring

For any planar graph  $G$  with  $\Delta \geq 8$ , we can discuss the following conditions if it is satisfied  $\chi''(G) = \Delta + 1$ .

- (1) 3-cycle is adjacent with two 3-cycles;
- (2) chordal 7-cycles are not adjacent;
- (3) chordal 8-cycles are not adjacent.

For any planar graph  $G$  with  $\Delta \geq 7$ , we can discuss the following conditions if it is satisfied  $\chi''(G) = \Delta + 1$ .

- (1) two 3-cycles are not adjacent;
- (2) two chordal 5-cycles are not adjacent;
- (3) chordal 6-cycles are not intersecting or not adjacent;
- (4) chordal 7-cycles are not intersecting or not adjacent.

For any planar graph  $G$  with  $\Delta \geq 6$ , we can discuss the following conditions if it is satisfied  $\chi''(G) = \Delta + 1$ .

- (1) contains no 5-cycle;
- (2) two 3-cycles are not adjacent;
- (3) two 4-cycles are not adjacent;.

### Case 6.2.2. List coloring

For list vertex coloring, study of 4-choosable proposes the mathematical proof of Four color theorem. We can study the following conditions if it is satisfied 4-choosable.

- (1) contains no 4-cycle;
- (2) contains no adjacent 5-cycles;
- (3) 3-cycle is not adjacent to 6-cycle.

For list edge coloring and list total coloring, we can prove at the same time. We can also use discharge method to prove. For any planar graph  $G$ , we can discuss the following conditions if it is satisfied  $\chi''(G) = \Delta + 1$ .

- (1)  $\Delta(G) \geq 8$  and  $G$  contains no chordal 5-cycles;
- (2)  $\Delta(G) \geq 8$  and 6-cycles of  $G$  contains at most one chord;
- (3)  $\Delta(G) \geq 8$  and 7-cycles of  $G$  contains at most two chords;

- (4)  $\Delta(G) \geq 7$  and 3-cycles of  $G$  is not adjacent to 5-cycles;
- (5)  $\Delta(G) \geq 7$  and  $G$  contains no chordal 6-cycles;
- (6)  $\Delta(G) \geq 6$  and 3-cycles of  $G$  is not adjacent to 5-cycles.

**Case 6.2.3.** Distinguishing total coloring

There are some relative deformation of 1-2-3 conjecture which we have already introduced in Chapter 1. We can use Combinatorial Nullstellensatz [6] and discharging method to prove. We can extend neighbor sum distinguish total coloring to neighbor product distinguish total coloring. For each edge  $uv \in E(G)$ , if the vertex  $u$  is colored by  $\phi(u)$  and the edges incident to  $u$  are colored by  $c, a_1, a_2, \dots, a_n$ , then  $f(u) = c\phi(u)a_1a_2 \cdots a_n$ . Similarly,  $f(v) = c\phi(v)b_1b_2 \cdots b_m$ . Then we call the adjacent vertices  $u$  and  $v$  can be distinguished by products if  $f(u) \neq f(v)$ . The smallest number  $k$  such that  $G$  admits a total neighbor product distinguishing coloring is called the *neighbor product distinguishing total chromatic number*, denoted by  $\chi''_{\Pi}(G)$ .

**Conjecture 6.2.1.** *For any graph  $G$ , the neighbor product distinguishing total chromatic number  $\chi''_{\Pi} \leq \Delta + 3$ .*

We can also study the neighbor set distinguishing total coloring problems and neighbor multiset distinguishing total coloring problems.

**Case 6.2.4.** List linear arboricity

A graph  $G$  is said to be *edge  $k$ -choosable* if, whenever we give lists  $A_e$  of  $k$  colors to each edge  $e \in E(G)$ , there exists a proper edge coloring of  $G$  where each edge is colored with a color from its own list. The *list edge chromatic number*  $\chi'_{list}(G)$  is the smallest integer  $k$  such that  $G$  is edge  $k$ -choosable. For the list edge chromatic number of a planar graph  $G$ , we have the following theorem.

**Theorem 6.2.1.** *Let  $G$  be a planar graph. Then  $\chi'_{list} \leq \Delta(G) + 1$  if one of the following conditions holds.*

- (1)  $\Delta(G) \geq 7$  and  $G$  contains no chordal 7-cycles;
- (2)  $\Delta(G) \geq 7$  and  $G$  contains no chordal 6-cycles;
- (3)  $\Delta(G) \geq 6$  and  $G$  contains no adjacent triangles;
- (4)  $\Delta(G) \geq 6$  and any 3-cycles is not adjacent to 5-cycles;
- (5)  $\Delta(G) \geq 6$  and  $G$  contains no chordal 5-cycles;
- (6)  $\Delta(G) \geq 5$  and  $G$  contains no chordal 4-cycles and chordal 5-cycles;
- (7)  $\Delta(G) \geq 5$  and  $G$  contains no chordal 5-cycles and chordal 6-cycles;
- (8)  $\Delta(G) \geq 5$  and  $G$  contains no  $i$ -cycles for some  $i \in \{3, 4, 5\}$ ;

In fact, to prove these results, it is also to prove a similar structural lemma as Lemma 5.2.1. So according to the proof of Theorem 5.2.1, we can prove the following result.

**Theorem 6.2.2.** *Let  $G$  be a planar graph. Then  $la_{list} \leq \lceil \frac{\Delta(G)+1}{2} \rceil$  if one of the following conditions holds.*

- (1)  $\Delta(G) \geq 6$  and  $G$  contains no chordal 6-cycles, or  $G$  contains no adjacent triangles, or any 3-cycles is not adjacent to 5-cycles, or  $G$  contains no chordal 5-cycles;
- (2)  $\Delta(G) \geq 4$  and  $G$  contains no chordal 4-cycles and chordal 5-cycles, or  $G$  contains no chordal 5-cycles and chordal 6-cycles, or  $G$  contains no  $i$ -cycles for some  $i \in \{3, 4, 5\}$ .

In the proof of Theorem 5.2.3, the mainly properties of  $G$  are  $d(u) + d(v) \geq \Delta(G)+2$  and  $G$  contains no 2-alternating cycle with  $\max_{1 \leq i \leq n} |n_2(v_{2i-1})| \geq 3$ , where  $n_2(v)$  is the number of 2-vertices adjacent to  $v$ . Using  $d(u) + d(v) \geq \Delta(G)+2$  and  $G$  contains no 2-alternating cycle, the list linear arboricity edge chromatic number of planar graph  $G$  is  $\lceil \frac{\Delta(G)}{2} \rceil$  if it satisfies below conditions:

(1)  $(\Delta, k) \in \{(7, 4), (6, 5), (5, 8), (4, 14)\}$ , where  $k$  satisfies that  $G$  has no cycle of length from 4 to  $k$ , where  $k \geq 4$ ;

(2)  $\Delta(G) \geq 8$  and  $G$  is without cycles of length 3 adjacent to cycles of length 5, or  $\Delta(G) \geq 8$  and  $G$  contains no adjacent 4-cycles; or  $\Delta(G) \geq 8$  and  $G$  contains no chordal 5-cycles.

(3)  $\Delta(G) \geq 7$  if any 4-cycle is not adjacent to an  $i$ -cycle for any  $i \in \{3, 4\}$ ; or  $\Delta(G) \geq 7$  and without adjacent cycles of length at most 4; or  $\Delta(G) \geq 7$  and  $g \geq 4$ .



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# Curriculum Vitae

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## Published and submitted papers

1. **Renyu Xu**, Jianliang Wu, Guanghui Wang and Xin Zhang, On the sum of powers of the degrees of graphs, Bulletin of the Australia Mathematical Sciences Society, 88 (2013), 353-358. (SCI)
2. **Renyu Xu** and Jianliang Wu, Total coloring of planar graphs with 7-cycles containing at most two chords, Theoretical Computer Science, 520 (2014), 124-129. (SCI)
3. **Renyu Xu**, Jianliang Wu and Huijuan Wang, Total coloring of planar graphs without some chordal 6-cycles, Bulletin of the Malaysian Mathematical Sciences Society, 38(2015), 561-569.(SCI)
4. **Renyu Xu**, Jianliang Wu and Jin Xu, Neighbor sum distinguishing total colouring of graphs embedded in surfaces of nonnegative Euler characteristic, Journal of Combinatorial Optimization, 31 (2016), 1430-1442.(SCI)
5. Raquel Águeda, Nathann Cohen, Shinya Fujita, Sylvain Legay, Yannis Manoussakis, Yasuko Matsui, Leandro Montero, Reza Naserasr, Yota O-tachi, Tadashi Sakuma, Zsolt Tuza and **Renyu Xu**, Safe Sets in Graphs: Graph Classes and Structural Parameters, 10th Annual International Conference on Combinatorial Optimization and Applications (2016), 241-253.
6. **Renyu Xu** and Jianliang Wu, A sufficient condition for a planar graph to be 4-choosable, Discrete Applied Mathematics, 240(2017),120-122.
7. **Renyu Xu** and Jianliang Wu, The List Linear Arboricity of Planar Graphs with 7-cycles containing at most two chords, Discussiones Mathematicae. (Under review)

8. **Renyu Xu** and Jianliang Wu, Total coloring of planar graphs without intersecting chordal 5-cycles, submitted.
9. **Renyu Xu** and Jianliang Wu, List edge and list total coloring of some planar graphs, submitted.

## Acknowledgements

Time flies. During my graduation study, I have received a lot of knowledge and friendship from several people. These years of accumulated precipitation is coming out of the campus to support, is a warm power once again set sail. The thesis work is nearing completion, I would like to give to all the teachers and students who give me guidance and meticulous care to extend my most sincere thanks.

First of foremost, I would like to thank my supervisor, Prof. Jianliang Wu and Prof. Yannis Manoussakis, For their excellent guidance and patience throughout the course and study of my research. All the progress I have made can not do without the hard work of Prof. Jianliang Wu and Prof. Yannis Manoussakis. Their advices, guidance and insights have been invaluable. I benefit from their profound knowledge and philosophy. They joint training period patient enthusiasm to guide me to carry out academic research, give positive guidance and encouragement of the confusion in my life. Here, I would like to extend my high respect and heartfelt thanks to my teachers.

I would also like to thank Prof. Guizhen Liu. Prof. Liu not only academically, give me a careful guidance, and gave me the kind of care in life. She is a serious working attitude, high work enthusiasm and selfless dedication to everyone who knows her deep impression. I would like to express my sincere gratitude and gratitude to Prof. Liu.

In particular, I would like to thank Professor Guanghui Wang for his help in my life and study, and Prof. Wang's keen insight into the frontiers of the discipline is the direction of my continued academic research on the future.

I am grateful to Shandong University and Shandong University for pro-

viding a good learning atmosphere, rich resources and many opportunities for learning and communication.

I also thank all the faculty in Laboratoire de Recherche en Informatique (Université Paris-Saclay, France), for their hospitality and offering me some convenience.

I would like to express my appreciation to all teachers in school of mathematic at Shandong University.

I wish to thank my family. They have been my most solid backing, thanks to their encouragement and support, so that I continue to progress and successfully complete my studies.

Finally, I would like to express my sincere thanks to the experts and professors review and participate in the my thesis. Because of your hard work, made this paper a successful conclusion. There are many people who helped me in this heartfelt thanks, thank you care about and help you close to my PhD study. Thank you!

**Titre :** Quelques problèmes de coloration du graphe (en français)

**Mots clés :** coloration totale, coloration par liste, coloriage total somme-des-voisins-distinguant et Arboricité linéaire L-déterminable

**Résumé :** La théorie des graphes est un domaine de recherche actif depuis 200 ans. Le plus ancien article de théorie des graphes connu a été rédigé par Euler en 1736, pour résoudre le problème dit des ponts de Königsberg. La coloration de graphe est l'une des branches les plus importantes de la théorie des graphes, depuis l'émergence du fameux problème des 4 couleurs. La coloration de graphe a des applications pratiques dans l'optimisation, l'informatique et la conception de réseau. Dans la présente thèse nous allons étudier le coloriage total, le coloriage par liste, le coloriage total somme-des-voisins-distinguant et l'arboricité linéaire L-sélectionnable.

Un  $k$ -coloriage total d'un graphe  $G$  est un coloriage de  $V(G) \cup E(G)$  utilisant  $(1, 2, \dots, k)$  couleurs tel qu'aucune paire d'éléments adjacents ou incidents

ne reçoivent la même couleur. Le nombre chromatique total  $\chi''(G)$  est le plus petit entier  $k$  tel que  $G$  admette un  $k$ -coloriage total. Un graphe  $G$  est  $k$ -sélectionnable si  $G$  est  $L$ -coloriable pour toute assignation  $L$  de  $G$  qui satisfie  $|L(v)| \geq k$  pour tout sommet  $v \in V(G)$ . Une relation  $L$  est appelée une assignation totale d'un graphe  $G$  si elle assigne une liste  $L(x)$  de couleurs à chaque élément  $x \in V(G) \cup E(G)$ . Soit  $f(v)$  la somme des couleurs d'un sommet  $v$  et des toutes les arêtes incidentes à  $v$ . Un  $k$ -coloriage total somme-des-voisins-distinguant de  $G$  est un  $k$  coloriage total de  $G$  tel que pour chaque arête  $uv \in E(G)$ ,  $f(u) \neq f(v)$ . Une forêt linéaire est un graphe pour lequel chaque composante connexe est une chemin. L'arboricité linéaire  $la(G)$  d'un graphe  $G$  est le nombre minimum de forêts linéaires dans  $G$ , dont l'union est égale à  $V(G)$ .

**Title :** Some coloring problems of graphs

**Keywords :** Total coloring, List coloring, Neighbor sum distinguishing total coloring, linear L-choosable arboricity.

**Abstract :** The study of graph theory started two hundred years ago. The earliest known paper was written by Euler (1736) to solve the Königsberg seven-bridge problem. Graph coloring has been one of the most important directions of graph theory since the arose of the well-known Four Color Problem. Graph coloring has real-life applications in optimization, computer science and network design. Here, we study the total coloring, list coloring, neighbor sum distinguishing total coloring and linear L-choosable arboricity.

A  $k$ -total-coloring of a graph  $G$  is a coloring of  $V(G) \cup E(G)$  using  $(1, 2, \dots, k)$  colors such that no two adjacent or incident elements receive the same color. The total chromatic number  $\chi''(G)$  is the smallest integer  $k$  such that  $G$  has a  $k$ -total-coloring. A mapping  $L$  is said to be an assignment for a graph  $G$  if it assigns a list  $L(v)$  of colors to each vertex  $v \in V(G)$ .

If it is possible to color  $G$  so that every vertex gets a color from its list and no two adjacent vertices receive the same color, then we say that  $G$  is  $L$ -colorable. A graph  $G$  is  $k$ -choosable if  $G$  is an  $L$ -colorable for any assignment  $L$  for  $G$  satisfying  $|L(v)| \geq k$  for every vertex  $v \in V(G)$ . A graph  $G$  is  $k$ -total-choosable if  $G$  has a proper  $L$ -total-coloring for every preassigned total assignment  $L$  with  $|L(x)| \geq k$  for every  $x \in V \cup E$ . Let  $f(v)$  denote the sum of the colors of a vertex  $v$  and the colors of all incident edges of  $v$ . A total  $k$ -neighbor sum distinguishing-coloring of  $G$  is a total  $k$ -coloring of  $G$  such that for each edge  $uv \in E(G)$ ,  $f(u) \neq f(v)$ . A linear forest is a graph in which each component is a path. The linear arboricity  $la(G)$  of a graph  $G$  is the minimum number of linear forests in  $G$ , whose union is the set of all edges of  $G$ .

