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**CONTRIBUTION À L'ÉTUDE DE L'ÉQUATION DE
BOLTZMANN HOMOGENÈNE**

co-dirigée par Nicolas FOURNIER et Stéphane SEURET

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# CONTRIBUTION À L'ÉTUDE DE L'ÉQUATION DE BOLTZMANN HOMOGENÈ

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Résumé

Dans cette thèse, on étudie principalement l'équation de Boltzmann homogène 3D pour les potentiels durs et les potentiels modérément mous et l'équivalence entre une EDS à sauts et l'EDP correspondante. En particulier, on calcule le spectre multifractal de certains processus stochastiques, on étudie le caractère bien-posé et la propagation du chaos pour l'équation de Boltzmann.

Dans le premier chapitre, on étudie les propriétés trajectorielle pathologiques du processus stochastique $(V_t)_{t \geq 0}$ représentant l'évolution de la vitesse d'une particule typique dans un gaz modélisé par l'équation de Boltzmann pour les potentiels durs ou modérément mous. Nous montrons que ce processus est multifractal et qu'il a un spectre déterministe. Pour les potentiels durs, nous donnons aussi le spectre multifractal du processus $X_t = \int_0^t V_s ds$, représentant l'évolution de la position de la particule typique.

Dans le deuxième chapitre, nous étudions l'unicité de la solution faible à l'équation de Boltzmann dans la classe de toutes les solutions mesures, pour les potentiels modérément mous. Ceci nous permet aussi d'obtenir un taux quantitatif de propagation du chaos pour le système de particules de Nanbu.

Enfin, dans le troisième chapitre, nous étendons le travail de Figalli [22] pour étudier la relation entre une EDS à sauts et l'équation de Fokker-Planck correspondante. On montre que pour toute solution faible $(f_t)_{t \in [0, T]}$ de l'EDP, il existe une solution faible de l'EDS dont les marginales temporelles sont données par la famille $(f_t)_{t \in [0, T]}$.

Mots-clefs: Théorie cinétique, Équation de Boltzmann, Analyse multifractale, Dimension de Hausdorff, Systèmes de particules, Propagation du chaos, Distance de Wasserstein, Existence et unicité, Solutions faibles, EDS à sauts, EDP non-locale.


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# CONTRIBUTION TO THE STUDY OF THE HOMOGENEOUS BOLTZMANN EQUATION

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Abstract

This thesis mainly studies the 3D homogeneous Boltzmann equation for hard potentials and moderately soft potentials and the equivalence between some jumping SDE and the corresponding PDE. In particular, we compute the multifractal spectrum of some stochastic processes, study the well-posedness and the propagation of chaos for the Boltzmann equation.

The purpose of the first chapter is to study the pathwise properties of the stochastic process $(V_t)_{t \geq 0}$, representing the time-evolution of the velocity of a typical particle in a gas modeled by the Boltzmann equation for hard or moderately potentials. We show that this process is multifractal and has a deterministic spectrum. For hard potentials, we also give the multifractal spectrum of the process $X_t = \int_0^t V_s ds$, representing the time-evolution of the position of the typical particle.

The second chapter is devoted to study the uniqueness of the weak solution to the Boltzmann equation in the class of all measure solutions, in the case of moderately soft potentials. This allows us to obtain a quantitative rate of propagation of chaos for Nanbu particle system for this singular interaction.

Finally in the third chapter, we extend Figalli's work [22] to study the relation between some jumping SDE and the corresponding Fokker-Planck equation. We prove that for any weak solution $(f_t)_{t \in [0, T]}$ of the PDE, there exists a weak solution to the SDE of which the time-marginals are given by the family $(f_t)_{t \in [0, T]}$.

Keywords: Kinetic theory, Boltzmann equation, Multifractal analysis, Hausdorff dimension, Particle systems, Propagation of Chaos, Wasserstein distance, Existence et uniqueness, Weak solutions, Jumping SDEs, non-local PDEs.

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Chapter 0

Introduction

0.1 Presentation of the thesis

In this thesis, we study mainly the multifractal nature of the Boltzmann process in Chapter 1, the uniqueness and propagation of chaos of the equation for singular interactions in Chapter 2, and the relationship between some SDE and some PDE in Chapter 3.

0.1.1 Multifractal Analysis

Tanaka [57] associated a Markov process $(V_t)_{t \geq 0}$, solution to a Poisson-driven stochastic differential equation, to the weak solution $(f_t)_{t \geq 0}$ of the Boltzmann equation for the case of Maxwellian molecules. Such a process, called the Boltzmann process, represents the time-evolution of the velocity of a typical particle. Then Fournier and Méléard extended Tanaka's probabilistic interpretation to non-Maxwellian molecules in [30]. In particular, Fournier recently built the stochastic processes related to Boltzmann's equation in [24, Section 5] for $\gamma \in (-1, 1)$ with the usual notation, see below (the Maxwellian case is $\gamma = 0$). Roughly speaking, he proved that for any weak solution $(f_t)_{t \geq 0}$ to the Boltzmann equation, one can associate a Boltzmann process for hard potentials ($\gamma \in (0, 1)$), and for moderately soft potentials, one can also construct such a process for some weak solution to the Boltzmann equation. From both theoretical and physical standpoints, we study the fluctuation of regularity of the sample paths of this velocity process when $\gamma \in (-1, 1)$ in Chapter 1 and we prove it is the same as that of a well-chosen Lévy process, studied by Jaffard [41], though it is absolutely not a Lévy process. Besides, we also considered the position process when $\gamma \in (0, 1)$ which enables us to well understand how the particle behaves.

The main tool to investigate the regularity of the sample paths of stochastic processes is *multifractal analysis*, which was initiated by Orey and Taylor [49] and Perkins [50] to study

fast and slow points of Brownian motion. The Hölder regularity of Brownian motion is almost surely everywhere $1/2$, while the situation is totally different for Lévy processes since its Hölder regularity depends on the point under consideration. Indeed, there is a continuum of possible values for the Hölder exponent of a general Lévy process. Jaffard [41] showed that the sample paths of most Lévy processes are multifractal functions and they have almost surely deterministic spectrum of singularities. Then Barral, Fournier, Jaffard and Seuret [6] studied a very specific ad-hoc Markov process, defined by a Poisson-driven stochastic differential equation, showing that some quite simple processes may display a random non-homogeneous spectrum. Following this, Yang [64] extended their methods to a much more general class of Markov processes, namely, rather general diffusions with jumps. The objects we investigate in Chapter 1 are other important physical examples. The main difficulty is the loss of independence and stationarity. To overcome this, we chose some good jumps of the process, estimated the increment of the process by a special process, analysed the jump times and distinguished the oscillating singularities, by using stochastic analysis and wavelet methods.

0.1.2 Uniqueness for the Boltzmann equation with moderately soft potentials

The existence and uniqueness of a weak solution, i.e. solution in the sense of distributions, to the Boltzmann equation for different potentials has been widely studied in recent decades. The global existence of the weak solution for all potentials was concluded by the seminal work of Villani [59], with very few assumptions on the initial data (finite energy and entropy), using some compactness methods. Uniqueness was studied for different potentials with quite different assumptions, by, among others, Desvillettes, Fournier, Mouhot, Mischler, Wennberg, etc. In Chapter 2, we prove a better *uniqueness* result for all *measure solutions* for a collision kernel without angular cutoff and for *moderately soft potentials* (singular, $\gamma \in (-1, 0)$). This is also very important when studying particle systems. In particular, the convergence (without rate) can be derived almost directly from this uniqueness result, since the tightness is very easy. Previous uniqueness results in this case were assuming a few regularity of the solution, which we completely remove. The main difficulty is singularity. We borrowed some ideas from [27], while relies on regularization, tightness, martingale problems and coupling methods. We obtained a *stability* estimate, combining a truncation technique. There is no doubt that the situation is more complicated because we dealt with jump processes.

0.1.3 Propagation of chaos for the Boltzmann equation with moderately soft potentials

From both physical and numerical standpoints, we also considered *propagation of chaos*, which refers to the convergence of the empirical measure of a particle system to the solution to a non-linear equation, initiated in Kac's work [43] in 1956. He considered the convergence of a *toy* particle system to the solution to the Boltzmann equation. Kac's particle system is similar to the one studied in the present thesis (named Nanbu's system), but each collision modifies the velocities of the two involved particles, while in Nanbu's system, only one of the two particles is deviated. Hence, Kac's system is physically more meaningful. Concerning propagation of chaos for these two particle systems for non-singular interaction, there are many references, see [16, 23, 32, 35, 36, 45, 46, 51, 56]. Concerning the Boltzmann equation, after some early seminal works by Sznitman [56], Graham-Méléard [35] and a very recent breakthrough by Mischler-Mouhot [46], Fournier-Mischler [32] recently proved the propagation of chaos at rate $N^{-1/4}$ for the Nanbu system in the case of hard potentials without cutoff. Concerning singular interaction, there are only very few results, see [37] for the Vlasov equation, [28] for the 2D Navier-Stokes equation, [34] for the 2D subcritical Keller-Segel equation and [27] for the Landau equation. In this thesis, we consider the propagation of chaos with singular interaction for the *Nanbu particle system* in Chapter 2. We make use of the Wasserstein distance with quadratic cost. Following Tanaka's methods in [57], we construct some processes solving some non linear stochastic differential equations driven by Poisson measure and then couple them with the particle system. To our knowledge, this is the first chaos result (with rate) for soft potentials, but we cannot study Kac's system since we haven't found a suitable coupling.

0.1.4 Equivalence between jumping SDEs and non-local PDEs

Probabilistic representations of partial differential equations are powerful tools to study the analytic properties of the equation (well-posedness, regularity,...) since it allows us to use a lot of probabilistic tools. One of them is relying on nonlinear stochastic differential equation in the sense of McKean. In the remarkable work [22], Figalli established the equivalence between continuous SDEs with rough coefficients and related Fokker-Planck equations by martingale problem theory. Concerning the homogeneous Boltzmann equations, the first (partial) result for uniqueness was obtained by Tanaka for Maxwell molecules, and afterwards it was extended to more general cases and also to the Landau equation. It is then natural to ask for a general relationship between *jumping* SDEs and PDEs. In Chapter 3, we extended Figalli's result to jump processes, by proving the equivalence between some jumping SDEs with rough coefficients and *non-local* PDEs (Fokker-Planck or Kolmogorov forward). Roughly speaking, we prove that given any weak solution $(f_t)_{t \in [0, T]}$ to the PDE, there exists a weak solution to the SDE, whose family of time marginals is given by $(f_t)_{t \in [0, T]}$. As a corollary, we deduce: 1)

existence for the PDE is equivalent to (weak) existence for the SDE; 2) uniqueness in law for the SDE implies uniqueness for the PDE. The proof is much more technically involved, though we followed closely the global strategy of [22].

Some results of Ethier and Kurtz's work [20] (extended later by Horowitz and Karandikar [38] and by Bhatt and Karandikar [10]) explained in spirit that if some SDE has a unique solution (in law) for any deterministic initial condition, then the corresponding PDE has a unique weak solution for any reasonable initial condition. Our result is much stronger since *no* uniqueness is required for the SDE.

Our main motivation for this chapter is the uniqueness for some *nonlinear* PDEs. For example, if we study the Boltzmann equation, it directly implies that, for any solution f to the nonlinear equation, we can associate a solution X to the corresponding linear SDE with additionally $X_t \sim f_t$ for all t . In other words, X solves the nonlinear SDE. This was crucial when studying more singular nonlinear equations, such as the Landau or Boltzmann equations for moderately soft potentials, see [27] and [63].

0.2 The Boltzmann equation

In this subsection, we introduce the main mathematical objects we consider in the following two chapters.

The Boltzmann equation. The Boltzmann equation is the main model of kinetic theory. It describes the time evolution of the density $f_t(x, v)$ of particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, in a gas of particles interacting through binary collisions. We consider a 3-dimensional spatially homogeneous case, that is, the gas is initially spatially homogeneous. This property propagates with time, and $f_t(x, v)$ does not depend on x . Then $f_t(v)$ solves

$$\partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v - v_*|, \theta) [f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)], \quad (0.1)$$

where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad (0.2)$$

and θ is the *deviation angle* defined by $\cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$. The cross section $B(|v - v_*|, \theta) \geq 0$ depends on the type of interaction between particles. In this thesis, we assume that the interaction is the important physical inverse power laws interactions: two particles located at a distance r collide due to a repulsive force proportional to $1/r^s$ for some $s > 2$. Then the cross

section $B(|v - v_*|, \theta)$ can be written by (see Cercignani [15]),

$$B(|v - v_*|, \theta) \sin \theta = |v - v_*|^\gamma \beta(\theta), \quad \gamma = \frac{s - 5}{s - 1},$$

where $\beta : (0, \pi] \rightarrow \mathbb{R}_+$ is a measurable function satisfying, near 0,

$$\beta(\theta) \sim \theta^{-1-\nu}, \quad \text{with } \nu = \frac{2}{s-1} \quad \text{satisfying } \gamma + \nu > 0.$$

According to [3], we may assume that $\beta = 0$ on $[\pi/2, \pi]$.

One usually calls *hard* potentials when $s > 5$ (i.e. $\gamma > 0$), *Maxwellian* potentials when $s = 5$ (i.e. $\gamma = 0$), *soft* potentials when $2 < s < 5$ (i.e. $-3 < \gamma < 0$) and *Coulomb* when $s = 2$ (i.e. $\gamma = -3$). For many details on the physical and mathematical theory of the Boltzmann equation, one can see [2, 17, 58, 60].

Parameterization. We now introduce a suitable spherical parameterization of (0.2) as in [31]. For each $x \in \mathbb{R}^3 \setminus \{0\}$, we consider a vector $I(x) \in \mathbb{R}^3$ such that $|I(x)| = |x|$ and $I(x) \perp x$. We also set $J(x) = \frac{x}{|x|} \wedge I(x)$, where \wedge is the vector product. Then the triplet $(\frac{x}{|x|}, \frac{I(x)}{|x|}, \frac{J(x)}{|x|})$ is an orthonormal basis of \mathbb{R}^3 . Then for $x, v, v_* \in \mathbb{R}^3$, $\theta \in (0, \pi]$, $\varphi \in [0, 2\pi)$, we set

$$\begin{cases} \Gamma(x, \varphi) := (\cos \varphi)I(x) + (\sin \varphi)J(x), \\ v'(v, v_*, \theta, \varphi) := v - \frac{1 - \cos \theta}{2}(v - v_*) + \frac{\sin \theta}{2}\Gamma(v - v_*, \varphi), \\ a(v, v_*, \theta, \varphi) := v'(v, v_*, \theta, \varphi) - v, \end{cases} \quad (0.3)$$

then we write $\sigma \in \mathbb{S}^2$ as $\sigma = \frac{v - v_*}{|v - v_*|} \cos \theta + \frac{I(v - v_*)}{|v - v_*|} \sin \theta \cos \varphi + \frac{J(v - v_*)}{|v - v_*|} \sin \theta \sin \varphi$. We observe at once that $\Gamma(x, \varphi)$ is orthogonal to x and has the same norm as x , from which it is easy to check that

$$|a(v, v_*, \theta, \varphi)| = \sqrt{\frac{1 - \cos \theta}{2}} |v - v_*|. \quad (0.4)$$

Weak solutions. We denote by $\mathcal{P}(\mathbb{R}^3)$ the set of probability measures on \mathbb{R}^3 and for $q > 0$, we set

$$\mathcal{P}_q(\mathbb{R}^3) = \{\mu \in \mathcal{P}(\mathbb{R}^3) : m_q(\mu) < \infty\} \quad \text{with } m_q(\mu) := \int_{\mathbb{R}^3} |v|^q \mu(dv).$$

Definition 0.2.1. A measurable family of probability measures $(f_t)_{t \geq 0}$ on \mathbb{R}^3 is called a *weak solution* to (0.1) if it satisfies the following two conditions:

- For all $t \geq 0$,

$$\int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv) \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^2 f_t(dv) = \int_{\mathbb{R}^3} |v|^2 f_0(dv) < \infty. \quad (0.5)$$

- For any bounded globally Lipschitz function $\phi \in Lip(\mathbb{R}^3)$, any $t \in [0, T]$,

$$\int_{\mathbb{R}^3} \phi(v) f_t(dv) = \int_{\mathbb{R}^3} \phi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{A}\phi(v, v_*) f_s(dv_*) f_s(dv) ds \quad (0.6)$$

where

$$\mathcal{A}\phi(v, v_*) = |v - v_*|^\gamma \int_0^{\pi/2} \beta(\theta) d\theta \int_0^{2\pi} [\phi(v + a(v, v_*, \theta, \varphi)) - \phi(v)] d\varphi.$$

Noting that $|a(v, v_*, \theta, \varphi)| \leq C\theta|v - v_*|$ and that $\int_0^{\pi/2} \theta\beta(\theta)d\theta < \infty$, we easily check that $|\mathcal{A}\phi(v, v_*)| \leq C_\phi|v - v_*|^{1+\gamma} \leq C_\phi(1 + |v - v_*|^2)$, so that everything is well-defined in (0.6).

0.3 The Multifractal Nature of Boltzmann Processes

0.3.1 The Boltzmann process

In the first chapter of the thesis, the main objects we deal with are a solution to some SDE associated to (0.1), called the Boltzmann process, and the position process. The Boltzmann process represents the time evolution of the velocity of a typical particle in 3-dimension. It is defined on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, and is a solution to the following SDE

$$V_t = V_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty a(V_{s-}, v, \theta, \varphi) \mathbf{1}_{\{u \leq |V_{s-} - v|^\gamma\}} N(ds, dv, d\theta, d\varphi, du), \quad (0.7)$$

where $N(ds, dv, d\theta, d\varphi, du)$ is a Poisson measure on $[0, \infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0, \infty)$ with intensity $ds f_s(dv) \beta(\theta) d\theta d\varphi du$, where $(f_t)_{t \geq 0}$ is a weak solution to (0.1) and where V_0 is a \mathcal{F}_0 -measurable random variable with law f_0 . Here a is the increment of velocity defined in (0.3). Of course, the associated position process $(X_t)_{t \in [0, 1]}$ is defined by $X_t = \int_0^t V_s ds$.

The Boltzmann process is well-defined thanks to [24, Proposition 5.1] which we recall now.

Proposition 0.3.1. *Let f_0 be a probability measure with $m_2(f_0) < \infty$.*

- *If $\gamma \in (0, 1)$, for any weak solution $(f_t)_{t \geq 0}$ to (0.1) starting from f_0 and satisfying for all $p \geq 2$, all $t_0 > 0$, $\sup_{t \geq t_0} m_p(f_t) < \infty$, there exist a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, a $(\mathcal{F}_t)_{t \geq 0}$ -Poisson measure $N(ds, dv, d\theta, d\varphi, du)$ on $[0, \infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0, \infty)$ with intensity $ds f_s(dv) \beta(\theta) d\theta d\varphi du$ and a càdlàg $(\mathcal{F}_t)_{t \geq 0}$ -adapted process $(V_t)_{t \geq 0}$ satisfying $\mathcal{L}(V_t) = f_t$ for all $t \geq 0$ and solving (0.7).*
- *If $\gamma \in (-1, 0]$, assume additionally that f_0 with $m_p(f_0) < \infty$ for some $p > 2$. There exist a probability space, a Poisson measure N and a càdlàg adapted process $(V_t)_{t \geq 0}$ as in the previous case, satisfying $\mathcal{L}(V_t) = f_t$ for all $t \geq 0$ and solving (0.7).*

Sample path properties of stochastic processes are widely studied since 1970s, there are variety of tools to measure the regularity, among which pointwise and local Hölder exponents are the most recurrent tools used in the literature. Here, we adopt the former one which is defined below.

Definition 0.3.2. A locally bounded function $g : [0, 1] \rightarrow \mathbb{R}^3$ is said to belong to the pointwise Hölder space $C^\alpha(t_0)$ with $t_0 \in [0, 1]$ and $\alpha \notin \mathbb{N}$, if there exist $C > 0$ and a polynomial P_{t_0} of degree less than $\lfloor \alpha \rfloor$, such that for some neighborhood I_{t_0} of t_0 ,

$$|g(t) - P_{t_0}(t)| \leq C|t - t_0|^\alpha, \quad \forall t \in I_{t_0}.$$

The pointwise Hölder exponent of g at point t_0 is given by

$$h_g(t_0) = \sup\{\alpha > 0 : g \in C^\alpha(t_0)\},$$

where by convention $\sup \emptyset = 0$.

In order to describe the size of the set of singularities of a function or a process, we introduce the level sets of the Hölder exponent, called the *iso-Hölder sets* of a function or a process. For example, we consider a function g , the *iso-Hölder sets* of g are denoted, for any $h \geq 0$, by

$$E_g(h) = \{t \geq 0 : h_g(t) = h\}.$$

As we know, the Hölder exponent lacks of stability, and therefore do not completely characterize the local regularity of a function or a stochastic process at a given point. We thus need the notion of Hausdorff dimension.

Definition 0.3.3. Let $A \subset \mathbb{R}^d$ and $0 \leq s \leq d$. The s -dimensional Hausdorff measure of A is defined by

$$\mathcal{H}^s(A) = \lim_{\epsilon \rightarrow 0} \mathcal{H}_\epsilon^s(A) = \lim_{\epsilon \rightarrow 0} \inf \left\{ \sum_{i=1}^{+\infty} |A_i|^s : A \subset \bigcup_{i=1}^{+\infty} A_i \quad \text{and} \quad |A_i| \leq \epsilon \right\}.$$

The limit exists since $\mathcal{H}_\epsilon^s(A)$ is increasing. Finally the Hausdorff dimension of A is defined by

$$\dim_H(A) := \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(A) = +\infty\},$$

and by convention $\dim_H \emptyset = -\infty$.

Brownian motion, the most important stochastic process, was studied by Orey and Taylor [49] and Perkins [50], and it turns out that almost surely for all $t \in [0, 1]$, $h_B(t) = 1/2$. It is universal in probability, but it proved to be too restrictive to model a number of phenomena.

Hence general classes of processes with wider range of behaviors rapidly appeared in the probability literature, such as fractional Brownian motion, multifractional Brownian motion and Lévy processes. Recently, the multifractal behaviour of some rather general (jumping) Markov processes has been studied by Yang [64]. Here, we study the Boltzmann process $(V_t)_{t \geq 0}$ defined by (0.7).

Let us now recall Jaffard's work [41], which provides us the main ideas for studying the multifractal spectrum of the Boltzmann process $(V_t)_{t \geq 0}$ in Chapter 1. Let X be a Lévy process and α be the upper index of Blumenthal-Gettoor [11] of X . Then Jaffard proved that the multifractal spectrum of X is almost surely,

$$D_X(h) = \begin{cases} \alpha h & \text{if } h \in [0, 1/\alpha], \\ -\infty & \text{if } h > 1/\alpha. \end{cases}$$

Glancing at the shape of the jumping SDE (0.7) satisfied by $(V_t)_{t \geq 0}$, one can easily get convinced that it should behave like a Lévy process, although it of course lacks of the independence and stationarity properties. We now write the multifractal spectrum of $(V_t)_{t \geq 0}$.

Theorem 0.3.4. *We consider some $\gamma \in (-1, 1)$, some $\nu \in (0, 1)$ with $\gamma + \nu > 0$. We consider some initial condition f_0 with $m_2(f_0) < \infty$ and assume that it is not a Dirac mass. If $\gamma \in (-1, 0]$, we moreover assume that f_0 with $m_p(f_0) < \infty$ for some $p > 2$. Almost surely, for all $h \geq 0$,*

$$D_V(h) = \begin{cases} \nu h & \text{if } 0 \leq h \leq 1/\nu, \\ -\infty & \text{if } h > 1/\nu. \end{cases} \quad (0.8)$$

We obtain the same spectrum as Lévy process though $(V_t)_{t \geq 0}$ is not Lévy process. We now state the main strategies for getting this spectrum. We first bound the law of $(V_t)_{t \geq 0}$ from below which enables us to choose some independent jump points from all the jumps. These well-chosen jump points constitute the Poisson random measure which allows us to use Shepp's lemma to get a random cover of time interval $[0, 1]$. This random cover implies that the Hölder exponent is bounded by $1/\nu$. The main difficulty is to get the lower bound for Hölder exponent. Compared to Lévy processes, we have two main problems: (1) the Markovian dynamic of V is no longer homogenous since $f_t(dv)$ appears in the intensity of Poisson measure, (2) the nearby future of V at each instant depends on the current state of V . We thus need to handle a delicate study of the small jumps. We make use of a number of ideas found in the recent work of Balança [5].

0.3.2 The position process

In Chapter 1, we also study the multifractal spectrum of the position process $X_t = \int_0^t V_s ds$ for $t \in [0, 1]$ for hard potentials. We first give a definition.

Definition 0.3.5. Let $g : [0, 1] \rightarrow \mathbb{R}^3$ be a locally bounded function and let $G(t) = \int_0^t g(s) ds$. For all $h \geq 0$, we introduce the sets

$$E_g^{cusp}(h) = \{t \in E_g(h) : h_G(t) = 1 + h_g(t)\} \text{ and } E_g^{osc}(h) = \{t \in E_g(h) : h_G(t) > 1 + h_g(t)\}.$$

The times $t \in E_g^{cusp}(h)$ are referred to as cusp singularities, while the times $t \in E_g^{osc}(h)$ are called oscillating singularities. Observe that $E_g(h) = E_g^{cusp}(h) \cup E_g^{osc}(h)$, the union being disjoint: this follows from the fact that obviously, for all $t \in [0, 1]$, $h_G(t) \geq h_g(t) + 1$.

We now exhibit the multifractal spectrum of the position process.

Theorem 0.3.6. Let $\gamma \in (0, 1)$ and $\nu \in (0, 1)$. We consider some initial condition f_0 with $m_2(f_0) < \infty$ and assume that it is not a Dirac mass. We consider a Boltzmann process $(V_t)_{t \in [0, 1]}$ defined by (0.7) and introduce the associated position process $(X_t)_{t \in [0, 1]}$ defined by $X_t = \int_0^t V_s ds$. Almost surely, for all $h \geq 0$,

$$D_X(h) = \begin{cases} \nu(h - 1) & \text{if } 1 \leq h \leq \frac{1}{\nu} + 1, \\ -\infty & \text{if } h > \frac{1}{\nu} + 1 \text{ or } 0 \leq h < 1. \end{cases}$$

During the proof, we also get the following.

Theorem 0.3.7. Under the assumptions of Theorem 0.3.6, we have almost surely:

- for all $h \in [1/(2\nu), 1/\nu)$, $\dim_H(E_V^{osc}(h)) \leq 2h\nu - 1$,
- for all $h \in [0, 1/(2\nu)) \cup (1/\nu, +\infty]$, $E_V^{osc}(h) = \emptyset$,
- for all $h \in [0, 1/\nu]$, $\dim_H(E_V^{cusp}(h)) = h\nu$.

Here again, this work is strongly inspired by the work of Balança [5].

0.4 Uniqueness and propagation of chaos for the Boltzmann equation with moderately soft potentials

In this chapter, we establish a stability principle for 3D homogeneous Boltzmann equation (0.1) in the case of moderately soft potentials ($\gamma \in (-1, 0)$). We also study the Nanbu stochastic particle system which approximates the weak solution.

The Boltzmann equation was devised by Boltzmann [12] in 1872 to depict the behaviour of a dilute gas. We consider 3D homogeneous case, which describes the time evolution of the

density $f_t(v)$ of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The particles interact via binary collisions. These collisions are supposed to be elastic, i.e. mass, momentum and kinetic energy are preserved in a collision process. Here the collision kernel $B(|v - v_*|, \theta)$ is supposed to be in the following form:

$$\begin{cases} B(|v - v_*|, \theta) \sin \theta = |v - v_*|^\gamma \beta(\theta), \\ \exists 0 < c_0 < c_1, \forall \theta \in (0, \pi/2), c_0 \theta^{-1-\nu} \leq \beta(\theta) \leq c_1 \theta^{-1-\nu}, \\ \forall \theta \in [\pi/2, \pi], \beta(\theta) = 0, \end{cases} \quad (0.9)$$

for some $\nu \in (0, 1)$, and $\gamma \in (-1, 0)$ satisfying $\gamma + \nu > 0$. We now introduce, for $\theta \in (0, \pi/2)$ and $z \in [0, \infty)$,

$$H(\theta) = \int_\theta^{\pi/2} \beta(x) dx \quad \text{and} \quad G(z) = H^{-1}(z). \quad (0.10)$$

Under (0.9), it is clear that H is a continuous decreasing function valued in $[0, \infty)$, so it has an inverse function $G : [0, \infty) \mapsto (0, \pi/2)$ defined by $G(H(\theta)) = \theta$ and $H(G(z)) = z$. For $x, v, v_* \in \mathbb{R}^3$, $\theta \in (0, \pi]$, $\varphi \in [0, 2\pi)$, recalling $a(v, v_*, \theta, \varphi)$ introduced in parameterization (0.3), we define

$$c(v, v_*, z, \varphi) = a[v, v_*, G(z/|v - v_*|^\gamma), \varphi] \quad \text{and} \quad c_K(v, v_*, z, \varphi) := c(v, v_*, z, \varphi) \mathbf{1}_{\{z \leq K\}}.$$

Here, we use a substitution that $\theta = G(z/|v - v_*|^\gamma)$ in order to remove the velocity-dependence $|v - v_*|^\gamma$ in the rate. Next, we introduce the definition of Wasserstein distance.

Definition 0.4.1. For $g, \tilde{g} \in \mathcal{P}_2(\mathbb{R}^3)$, let $\mathcal{H}(g, \tilde{g})$ be the set of probability measures on $\mathbb{R}^3 \times \mathbb{R}^3$ with first marginal g and second marginal \tilde{g} . We then set

$$\mathcal{W}_2(g, \tilde{g}) = \inf \left\{ \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - \tilde{v}|^2 R(dv, d\tilde{v}) \right)^{1/2}, \quad R \in \mathcal{H}(g, \tilde{g}) \right\}.$$

We can also define the Wasserstein distance in an equivalent form:

$$\mathcal{W}_2(g, \tilde{g}) = \inf \{ \mathbb{E}[|X - Y|^2]^{1/2}, X \sim g, Y \sim \tilde{g} \}.$$

This is the Wasserstein distance with quadratic cost. It is well-known that the infimum is reached. And more precisely, if g has a density, there is a unique $R \in \mathcal{H}(g, \tilde{g})$ such that $\mathcal{W}_2^2(g, \tilde{g}) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - \tilde{v}|^2 R(dv, d\tilde{v})$ (see Villani [61, Theorem 2.12]).

0.4.1 The stability principle

The purpose of the second work of this thesis is to establish a strong/weak stability estimate for the Boltzmann equation for $\gamma \in (-1, 0)$ in $L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$, which implies a uniqueness

result. This is solved by a probability method, introduced by Tanaka [57]. Let us first recall the well-posedness result of [33, Corollary 2.4] (more general existence results can be found in [59]).

Theorem 0.4.2. *Assume (0.9) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$. Let $q \geq 2$ such that $q > \gamma^2/(\gamma + \nu)$. Let $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} f_0(v) |\log f_0(v)| dv < \infty$ and let $p \in (3/(3 + \gamma), p_0(\gamma, \nu, q))$, where*

$$p_0(\gamma, \nu, q) = \frac{q - \gamma}{q(3 - \nu)/3 - \gamma} \in (3/(3 + \gamma), 3/(3 - \nu)). \quad (0.11)$$

Then (0.1) has a unique weak solution $f \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$.

We extend the above uniqueness result to the all measure solutions in $L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$.

Theorem 0.4.3. *Assume (0.9) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ satisfying $\gamma + \nu > 0$. Let $q \geq 2$ such that $q > \gamma^2/(\gamma + \nu)$. Assume that $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ has a finite entropy, more precisely that $\int_{\mathbb{R}^3} f_0(v) |\log f_0(v)| dv < \infty$. Let $p \in (3/(3 + \gamma), p_0(\gamma, \nu, q))$, recall (0.11), and $(f_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ be the unique weak solution to (0.1) given by Theorem 0.4.2. Then for any other weak solution $(\tilde{f}_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ to (0.1), we have, for any $t \geq 0$,*

$$\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq \mathcal{W}_2^2(f_0, \tilde{f}_0) \exp \left(C_{\gamma, p} \int_0^t (1 + \|f_s\|_{L^p}) ds \right).$$

In particular, we have uniqueness for (0.1) when starting from f_0 in the space of all weak solutions.

Our uniqueness result is thus much better. The major difficulty comes from the singular interaction and the absence of regularity of the weak solution, that cannot compensate the singularity of the coefficients. To overcome this, we adopt some ideas of Fournier-Hauray in [27], which concerns the simpler case of the Landau equation with moderately soft potentials. Let us recall that the Landau equation was derived by Landau in 1936. It has some links with the Boltzmann equation. Indeed, when $\gamma \in (-3, 1]$, the Landau equation can be seen as an approximation of the corresponding Boltzmann equation in the asymptotics of grazing collisions. Villani [59] proves the convergence of the Boltzmann equation to the Landau equation, together with the existence of solutions to the Landau equation in the whole range. When $\gamma = -3$ (Coulomb interaction), it replaces the Boltzmann equation.

The main idea to prove the theorem is to construct a suitable coupling between two weak solutions to (0.1). Let $(f_t)_{t \geq 0}$ be the *strong* solution to (0.1) (i.e. the one of Theorem 0.4.2,

which is slightly regular) and let $(\tilde{f}_t)_{t \geq 0}$ be a weak solution in $L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$. We first build $(X_t)_{t \geq 0}$ with $\mathcal{L}(X_t) = \tilde{f}_t$ solving

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c(X_{s-}, X_s^*(\alpha), z, \varphi) M(ds, d\alpha, dz, d\varphi),$$

where $(X_t^*)_{t \geq 0}$ is a measurable α -process with law \tilde{f}_t , and $M(ds, d\alpha, dz, d\varphi)$ is a Poisson measure.

The existence of the process $(X_t)_{t \geq 0}$ is not easy and we only build a *weak* solution. The difficulty is mainly due to the singularity of the interaction, which cannot be compensated by some regularity of \tilde{f}_t , because \tilde{f}_t is any weak solution. We thus use the strategy of [22] (which deals with continuous diffusion processes). We introduce $\tilde{f}_t^\epsilon = \tilde{f}_t * \phi_\epsilon$, where ϕ_ϵ is the centered Gaussian density with covariance matrix ϵI_3 . We write the PDE satisfied by \tilde{f}_t^ϵ and associate, for each $\epsilon \in (0, 1)$, a solution $(X_t^\epsilon)_{t \geq 0}$ to some SDE. Since both the SDE and the PDE (with $\epsilon \in (0, 1)$ fixed) are well-posed (because the coefficients are regular enough), we conclude that $\mathcal{L}(X_t^\epsilon) = \tilde{f}_t^\epsilon$. Next, we prove that the family $\{(X_t^\epsilon)_{t \geq 0}, \epsilon \in (0, 1)\}$ is tight using the Aldous criterion [1]. Finally, we consider a limit point $(X_t)_{t \geq 0}$, as $\epsilon \rightarrow 0$, of $\{(X_t^\epsilon)_{t \geq 0}, \epsilon \in (0, 1)\}$. Since $\mathcal{L}(X_t^\epsilon) = \tilde{f}_t^\epsilon$, we deduce that $\mathcal{L}(X_t) = f_t$ for each $t \geq 0$. Then, we classically make use of martingale problems to show that $(X_t)_{t \geq 0}$ is indeed a solution of the desired SDE.

On the other hand, we plan to build a f_t -distributed process which couples with the above process $(X_t)_{t \geq 0}$ with the same Poisson measure $M(ds, d\alpha, dz, d\varphi)$. More precisely, we intend to associate to $(f_t)_{t \geq 0}$ the *strong* solution $(W_t)_{t \geq 0}$ to the SDE, driven by $M(ds, d\alpha, dz, d\varphi)$, with f_t -distributed α -process $(W_t^*)_{t \geq 0}$ coupled with $(X_t^*)_{t \geq 0}$. This should be possible, using that $(f_t)_{t \geq 0}$ is slightly regular. But unfortunately, we fail in proving the strong existence of such a process, because there is a problem of parameterization of the sphere, already encountered by Tanaka [57]. We thus introduce a truncated SDE (with a finite number of jumps per unit of time), namely,

$$W_t^K = W_0 + \int_0^t \int_0^1 \int_0^K \int_0^{2\pi} c(W_{s-}^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K}) M(ds, d\alpha, dz, d\varphi).$$

Here $\varphi_{s,\alpha,K}$ is some well-chosen angle, that allows us to overcome the problem of the sphere parametrization, see Lemma 2.2.2, due to Tanaka, in Chapter 3. This equation of course has a unique strong solution $(W_t^K)_{t \geq 0}$, because it is a discrete equation (with finitely many jumps per unit of time).

Finally, we prove that W_t^K goes in law to f_t for each $t \geq 0$, we thus have

$$\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq \limsup_{K \rightarrow \infty} \mathbb{E}[|W_t^K - X_t|^2].$$

Then using the Itô formula, some results in Chapter 3 and some technical and very precise computations, we conclude that

$$\limsup_{K \rightarrow \infty} \mathbb{E}[|W_t^K - X_t|^2] \leq \mathcal{W}_2^2(f_0, \tilde{f}_0) \exp\left(C_{\gamma,p} \int_0^t (1 + \|f_s\|_{L^p}) ds\right),$$

which completes the proof.

0.4.2 The Nanbu particle system

In Chapter 2, we also consider the problem of propagation of chaos for some finite stochastic particle system, which means that the empirical measure of the particle system converges to the unique solution of the Boltzmann equation. Precisely, we consider the simple particle system introduced by Nanbu [48] in 1983. It is a non-symmetric particle system in the sense that at each collision event, only one of the two involved particles is deviated. Since we deal with a non cutoff cross section, which means that there are infinitely many jumps with a very small deviation angle, we study a truncated version of Nanbu's particle system as in [32].

In [32], Fournier and Mischler give an almost optimal explicit rate of convergence for the Boltzmann equation with $\gamma \in [0, 1]$ for Nanbu's system using non-linear stochastic differential equations driven by Poisson measure. Their approach is very technical and the coupling that they built is extremely meaningful for our case.

We now describe the main strategy to get the propagation of chaos rate. Let $f_0 \in \mathcal{P}(\mathbb{R}^3)$, $K \geq 1$ and $N \geq 1$. We consider the unique strong solution $(f_t)_{t \geq 0}$ to (0.1), a family of random variables $(V_0^i)_{i=1, \dots, N}$ with common law f_0 and a family of i.i.d. Poisson measures $(M_i(ds, d\alpha, dz, d\varphi))_{i=1, \dots, N}$. Then we build the family of i.i.d. f_t -distributed Boltzmann processes $(W_t^1, \dots, W_t^N)_{t \geq 0}$ solving, for $i = 1, \dots, N$,

$$W_t^i = V_0^i + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c(W_{s-}^i, W_s^*(\alpha), z, \varphi) M_i(ds, d\alpha, dz, d\varphi). \quad (0.12)$$

We then couple the family $(W_t^1, \dots, W_t^N)_{t \geq 0}$ with the particle system $(V_t^1, \dots, V_t^N)_{t \geq 0}$, which is a strong solution to

$$V_t^i = V_0^i + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c_K(V_{s-}^i, V_{s-}^{j(s,\alpha)}, z, \varphi) M_i(ds, d\alpha, dz, d\varphi), \quad (0.13)$$

$i = 1, \dots, N$, the indice K indicating the level of truncation. Of course, $j(s, \alpha)$ has to be uniform in $\{1, \dots, N\}$ (if α is uniformly distributed on $[0, 1]$), and we couple $j(s, \alpha)$ and $W_s^*(\alpha)$ in such a way that $V_{s-}^{j(s,\alpha)}$ and $W_s^*(\alpha)$ are as close as possible, as in [32, Lemma 4.3]. Actually,

we need to introduce some intermediate couplings, and there are also some problems of sphere parameterization as previously mentioned.

Once we built the suitable coupling, we then compute $\mathcal{W}_2^2(\mu_t^{N,K}, \mu_{\mathbf{W}_t}^N)$, where $\mu_t^{N,K} = \frac{1}{N} \sum_{i=1}^N \delta_{V_t^i}$, $\mu_{\mathbf{W}_t}^N = \frac{1}{N} \sum_{i=1}^N \delta_{W_t^i}$. However, we observe from the stability principle that a regularized empirical measure (i.e. $\bar{\mu}_{\mathbf{W}_t}^N = \mu_{\mathbf{W}_t}^N * \psi_{\epsilon_N}$) is necessary, with a small parameter ϵ_N . Here $\psi_\epsilon = (3/(4\pi\epsilon^3))\mathbf{1}_{\{|x| \leq \epsilon\}}$. Hence, a new difficulty appears: we have to bound the L^p -norm of this regularized empirical measure. While, the statistics knowledge tells us that it should be bounded by $\|f_t\|_{L^p}$ with high probability, for each t fixed, but we need something uniform (locally) in time, so we have to use some continuity arguments, which is not so easy since the processes are of jump type. At the end, we establish the following result.

Theorem 0.4.4. *Consider the assumption (0.9) for $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$ and $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ for some $q > 8$ with a finite entropy. Let $(f_t)_{t \geq 0}$ be the unique weak solution to (0.1) given by Theorem 0.4.2. For each $N \geq 1$, $K \in [1, \infty)$, let $(V_t^{i,K})_{i=1, \dots, N}$ be the unique solution to (0.13). We denote the associated empirical measure by $\mu_{\mathbf{V}_t}^{N,K} = N^{-1} \sum_{i=1}^N \delta_{V_t^{i,K}}$. Then for all $T > 0$,*

$$\sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^{N,K}, f_t)] \leq C_{T,q} \left(1 + N^{-(1-6/q)(2+2\gamma)/3} + K^{1-2/\nu} + N^{-1/2} \right).$$

To our knowledge, the obtained quantitative rate of chaos is the first result for a singular Boltzmann equation (i.e. with $\gamma < 0$). However, it is not sharp and deals with the Nanbu system, which is simpler than Kac's system.

0.5 On the equivalence between some jumping SDEs with rough coefficients and some non-local PDEs

In [22], Figalli study the main relations between the (continuous) SDE

$$dX = b(t, X)dt + \sigma(t, X)dB_t,$$

and the corresponding (local) Fokker-Planck equation

$$\partial_t f_t = - \sum_i \partial_i (b_i f_t) + \frac{1}{2} \sum_{ij} \partial_{ij} (a_{ij} f_t) = 0,$$

where the coefficients $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \mapsto \mathcal{S}_d^+$ (the set of symmetric nonnegative $d \times d$ real matrices) are measurable and bounded. Also, $a(t, x) := \sigma(t, x)\sigma^*(t, x)$

and B is an d -dimensional Brownian motion. He proved that whenever we have existence of a solution $(f_t)_{t \in [0, T]}$ to the PDE, there exists at least one martingale solution $(X_t)_{t \in [0, T]}$ of the SDE such that $X_t \sim f_t$ for all $t \in [0, T]$.

The purpose of the third chapter is to extend such a result to jumping SDEs and their corresponding (non-local) Fokker-Planck equations.

Let $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \mapsto \mathcal{S}_d^+$ and $h : [0, T] \times E \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be measurable functions. The space E is endowed with a σ -field \mathcal{E} and with a σ -finite measure μ . Let $\mathcal{P}(\mathbb{R}^d)$ be the set of probability measures on \mathbb{R}^d and

$$\mathcal{P}_1(\mathbb{R}^d) = \{f \in \mathcal{P}(\mathbb{R}^d) : m_1(f) < \infty\} \quad \text{with} \quad m_1(f) := \int_{\mathbb{R}^d} |x| f(dx).$$

We define $L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$ as the set of all measurable families $(f_t)_{t \in [0, T]}$ of probability measures on \mathbb{R}^d such that $\sup_{[0, T]} m_1(f_t) < \infty$. We assume

Assumption 0.5.1. *There is a constant C such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$,*

$$|\sigma(t, x)| + |b(t, x)| + \int_E |h(t, z, x)| \mu(dz) \leq C(1 + |x|).$$

We consider the d -dimensional stochastic differential equation on the time interval $[0, T]$

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s + \int_0^t \int_E h(s, z, X_{s-}) N(ds, dz), \quad (0.14)$$

where $(B_t)_{t \in [0, T]}$ is a d -dimensional Brownian motion and $N(ds, dz)$ is a Poisson measure on $[0, T] \times E$ with intensity measure $ds \mu(dz)$. The Fokker-Planck (or Kolmogorov forward) equation associated to (0.14) is

$$\partial_t f_t + \operatorname{div}(b(t, \cdot) f_t) = \frac{1}{2} \sum_{i, j=1}^d \partial_{ij}([\sigma(t, \cdot) \sigma^*(t, \cdot)]_{i, j} f_t) + \mathcal{L}_t f_t, \quad (0.15)$$

where $\mathcal{L}_t f_t : \mathbb{R}^d \mapsto \mathbb{R}$ is defined by duality as

$$\int_{\mathbb{R}^d} (\mathcal{L}_t f_t)(x) \varphi(x) dx = \int_{\mathbb{R}^d} \int_E [\varphi(x + h(t, z, x)) - \varphi(x)] f_t(x) dx$$

for any reasonable $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$. We use the notation $\nabla = \nabla_x$, $\operatorname{div} = \operatorname{div}_x$ and $\partial_{ij} = \partial_{x_i x_j}^2$.

We are not able, at the moment, to study a more general jumping SDE with infinite variation jump part, i.e. an SDE driven by a compensated Poisson measure. Here is the main result of Chapter 3.

Theorem 0.5.2. *Suppose Assumption 0.5.1 and consider any weak solution $(f_t)_{t \in [0, T]}$ to (0.15) such that $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$. There exist, on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, a d -dimensional $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion $(B_t)_{t \in [0, T]}$, a $(\mathcal{F}_t)_{t \in [0, T]}$ -Poisson measure $N(dt, dz)$ on $[0, T] \times E$ with intensity measure $dt \mu(dz)$, these two objects being independent, as well as a càdlàg $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted process $(X_t)_{t \in [0, T]}$ solving (0.14) and such that $\mathcal{L}(X_t) = f_t$ for all $t \in [0, T]$.*

If $(X_t)_{t \in [0, T]}$ is a solution to (0.14) with $f_t = \mathcal{L}(X_t)$, a simple application of the Itô formula implies that the family $(f_t)_{t \in [0, T]}$ is a weak solution to (0.15). Hence, we can deduce that

- The existence of a (weak) solution $(X_t)_{t \in [0, T]}$ to (0.14) such that $\mathcal{L}(X_0) = f_0$ is equivalent to the existence of a weak solution $(f_t)_{t \in [0, T]}$ to (0.15) starting from f_0 .
- The uniqueness (in law) of the solution $(X_t)_{t \in [0, T]}$ to (0.14) with $\mathcal{L}(X_0) = f_0$ implies the uniqueness of the weak solution $(f_t)_{t \in [0, T]}$ to (0.15) starting from f_0 .

Our proof uses a smoothing procedure introduced in [22]. Roughly speaking, we first introduce $f_t^\epsilon = f_t * \phi_\epsilon$, where ϕ_ϵ is the centered Gaussian density with covariance matrix ϵI_d . We write the PDE satisfied by f_t^ϵ . In some sense, this PDE is rather complicated because its coefficients depend on f_t itself. However, these coefficients seen as fixed functions, we can associate to this PDE a solution $(X_t^\epsilon)_{t \geq 0}$ to some SDE. Since both the SDE and the PDE (with $\epsilon \in (0, 1)$ fixed) are well-posed (because the coefficients are regular enough), we conclude that $\mathcal{L}(X_t^\epsilon) = f_t^\epsilon$. Next, we prove that the family $\{(X_t^\epsilon)_{t \geq 0}, \epsilon \in (0, 1)\}$ is tight using the Aldous criterion [1]. Finally, we consider a limit point $(X_t)_{t \geq 0}$, as $\epsilon \rightarrow 0$, of $\{(X_t^\epsilon)_{t \geq 0}, \epsilon \in (0, 1)\}$. Since $\mathcal{L}(X_t^\epsilon) = f_t^\epsilon$, we deduce that $\mathcal{L}(X_t) = f_t$ for each $t \geq 0$. Then, we classically make use of martingale problems to show that $(X_t)_{t \geq 0}$ is indeed a solution of the desired SDE.

Chapter 1

The Multifractal Nature of Boltzmann Processes

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We consider the spatially homogeneous Boltzmann equation for (true) hard and moderately soft potentials. We study the pathwise properties of the stochastic process $(V_t)_{t \geq 0}$, which describes the time evolution of the velocity of a typical particle. We show that this process is almost surely multifractal and compute its spectrum of singularities. For hard potentials, we also compute the multifractal spectrum of the position process $(X_t)_{t \geq 0}$.

1.1 Introduction

The Boltzmann equation is the main model of kinetic theory. It describes the time evolution of the density $f_t(x, v)$ of particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, in a gas of particles interacting through binary collisions. In the special case where the gas is initially spatially homogeneous, this property propagates with time, and $f_t(x, v)$ does not depend on x . We refer to the books by Cercignani [15] and Villani [60] for many details on the physical and mathematical theory of this equation, see also the review paper by Alexandre [2].

Tanaka gave in [57] a probabilistic interpretation of the case of Maxwellian molecules: he constructed a Markov process $(V_t)_{t \geq 0}$, solution to a Poisson-driven stochastic differential equation, and such that the law of V_t is f_t for all $t \geq 0$. Such a process $(V_t)_{t \geq 0}$ has a richer structure than the Boltzmann equation, since it contains some information on the history of particles. Physically, $(V_t)_{t \geq 0}$ is interpreted as the time-evolution of the velocity of a typical particle. Fournier and Méléard [30] extended Tanaka's work to non-Maxwellian molecules, see the last part of paper by Fournier [24] for up-to-date results.

In the case of long-range interactions, that is when particles interact through a repulsive force in $1/r^s$ (for some $s > 2$), the Boltzmann equation presents a singular integral (case without cutoff). The reason is that the corresponding process $(V_t)_{t \geq 0}$ jumps infinitely often, i.e. the particle is subjected to infinitely many collisions, on each time interval. In some sense, it behaves, roughly, like a Lévy process.

The Hölder regularity of the sample paths of stochastic processes was first studied by Orey and Taylor [49] and Perkins [50], who showed that the fast and slow points of Brownian motion are located on random sets of times, and they showed that the sets of points with a given pointwise regularity have a fractal nature. Jaffard [41] showed that the sample paths of most Lévy processes are multifractal functions and he obtained their spectrum of singularities. This spectrum is almost surely deterministic: of course, the sets with a given pointwise regularity are extremely complicated, but their Hausdorff dimension is deterministic. Let us also mention the article by Balança [5], in which he extended the results (and simplified some proofs) of Jaffard [41].

What we expect here is that $(V_t)_{t \geq 0}$ should have the same spectrum as a well-chosen Lévy process. This is of course very natural (having a look at the shape of the jumping SDE satisfied by $(V_t)_{t \geq 0}$). There are however many complications, compared to the case of Lévy processes, since we lose all the independence and stationarity properties that simplify many computations and arguments. We will also compute the multifractal spectrum of the position process $(X_t)_{t \geq 0}$, defined by $X_t = \int_0^t V_s ds$, which appears to have multifractal sample paths as well.

By the way, let us mention that, though there are many papers computing the multifractal spectrum of some quite complicated objects, we are not aware of any work concerning general Markov processes, that is, roughly, solutions to jumping (or even non jumping) SDEs. In this paper, we study the important case of the Boltzmann process, as a physical example of jumping SDE. Of course, a number of difficulties have to be surmounted, since the model is rather complicated. However, we follow, adapting everywhere to our situation, the main ideas of Jaffard [41] and Balança [5].

Let us finally mention that Barral, Fournier, Jaffard and Seuret [6] studied a very specific ad-hoc Markov process, showing that quite simple processes may have a random spectrum that depends heavily on the values taken by the process.

1.1.1 The Boltzmann equation

We consider a 3-dimensional spatially homogeneous Boltzmann equation, which depicts the density $f_t(v)$ of particles in a gas, moving with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The density

$f_t(v)$ solves

$$\partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v - v_*|, \cos \theta) [f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)], \quad (1.1)$$

where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \text{and} \quad \cos \theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle. \quad (1.2)$$

The cross section $B(|v - v_*|, \cos \theta) \geq 0$ depends on the type of interaction between particles. It only depends on $|v - v_*|$ and on the cosine of the deviation angle θ . Conservations of mass, momentum and kinetic energy hold for reasonable solutions and we may assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v) dv = 1$. We will assume that there is a measurable function $\beta : (0, \pi] \rightarrow \mathbb{R}_+$ such that

$$\begin{cases} B(|v - v_*|, \cos \theta) \sin \theta = |v - v_*|^\gamma \beta(\theta), \\ \exists 0 < c_0 < C_0, \forall \theta \in (0, \pi/2], c_0 \theta^{-1-\nu} \leq \beta(\theta) \leq C_0 \theta^{-1-\nu}, \\ \forall \theta \in (\pi/2, \pi), \beta(\theta) = 0, \end{cases} \quad (1.3)$$

for some $\nu \in (0, 1)$, and $\gamma \in (-1, 1)$ satisfying $\gamma + \nu > 0$. The last assumption on the function β is not a restriction and can be obtained by symmetry as noted in the introduction of [3]. Note that, when particles collide by pairs due to a repulsive force proportional to $1/r^s$ for some $s > 2$, assumption (1.3) holds with $\gamma = (s - 5)/(s - 1)$ and $\nu = 2/(s - 1)$. Here we will be focused on the cases of hard potentials ($s > 5$), Maxwell molecules ($s = 5$) and moderately soft potentials ($3 < s < 5$).

Next, we give the definition of weak solutions of (1.1). We introduce a notation beforehand. Let f be any probability measure on \mathbb{R}^3 , and we denote

$$m_p(f) = \int_{\mathbb{R}^3} |v|^p f(dv).$$

Definition 1.1.1. Assume (1.3) is true for some $\nu \in (0, 1)$, $\gamma \in (-1, 1)$. A measurable family of probability measures $(f_t)_{t \geq 0}$ on \mathbb{R}^3 is called a weak solution to (1.1) if it satisfies the following two conditions.

- For all $t \geq 0$,

$$\int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv) \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^2 f_t(dv) = \int_{\mathbb{R}^3} |v|^2 f_0(dv) < \infty. \quad (1.4)$$

- For any bounded globally Lipschitz-continuous function $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$, any $t \geq 0$,

$$\int_{\mathbb{R}^3} \phi(v) f_t(dv) = \int_{\mathbb{R}^3} \phi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L_B \phi(v, v_*) f_s(dv_*) f_s(dv) ds, \quad (1.5)$$

where v' and θ are defined by (1.2), and

$$L_B\phi(v, v_*) := \int_{\mathbb{S}^2} B(|v - v_*|, \cos\theta)(\phi(v') - \phi(v))d\sigma.$$

The existence of a weak solution to (1.1) is now well established (see [59] and [44]). In particular, when $\gamma \in (0, 1)$, it is shown in [44] that for any f_0 with $m_2(f_0) < \infty$, there exists a weak solution $(f_t)_{t \geq 0}$ to (1.1) satisfying $\sup_{t \geq t_0} m_p(f_t) < \infty$ for all $p \geq 2$, all $t_0 > 0$. Some uniqueness results can be found in [33].

1.1.2 The Boltzmann process

We first parameterize (1.2) as in [31]. For each $x \in \mathbb{R}^3 \setminus \{0\}$, we consider the vector $I(x) \in \mathbb{R}^3$ such that $|I(x)| = |x|$ and $I(x) \perp x$. We also set $J(x) = \frac{x}{|x|} \wedge I(x)$, where \wedge is the vector product. The triplet $(\frac{x}{|x|}, \frac{I(x)}{|x|}, \frac{J(x)}{|x|})$ is an orthonormal basis of \mathbb{R}^3 . Then for $x, v, v_* \in \mathbb{R}^3$, $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$, we set

$$\begin{cases} \Gamma(x, \varphi) := (\cos\varphi)I(x) + (\sin\varphi)J(x), \\ v'(v, v_*, \theta, \varphi) := v - \frac{1 - \cos\theta}{2}(v - v_*) + \frac{\sin\theta}{2}\Gamma(v - v_*, \varphi), \\ \mathbf{a}(v, v_*, \theta, \varphi) := v'(v, v_*, \theta, \varphi) - v. \end{cases} \quad (1.6)$$

Let us observe at once that $\Gamma(x, \varphi)$ is orthogonal to x and has the same norm as x , from which it is easy to check that

$$|\mathbf{a}(v, v_*, \theta, \varphi)| = \sqrt{\frac{1 - \cos\theta}{2}}|v - v_*|. \quad (1.7)$$

Definition 1.1.2. Let $(f_t)_{t \geq 0}$ be a weak solution to the Boltzmann equation (1.1). On some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, we consider a \mathcal{F}_0 -measurable random variable V_0 with law f_0 , a Poisson measure $N(ds, dv, d\theta, d\varphi, du)$ on $[0, \infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0, \infty)$ with intensity $ds f_s(dv) \beta(\theta) d\theta d\varphi du$. A càdlàg $(\mathcal{F}_t)_{t \geq 0}$ -adapted process $(V_t)_{t \geq 0}$ with values in \mathbb{R}^3 is then called a Boltzmann process if it solves

$$V_t = V_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \mathbf{a}(V_{s-}, v, \theta, \varphi) \mathbf{1}_{\{u \leq |V_{s-} - v|^\gamma\}} N(ds, dv, d\theta, d\varphi, du). \quad (1.8)$$

From Proposition 5.1 in [24], we have slightly different results for different potentials: when $\gamma \in (0, 1)$, i.e. hard potentials, we can associate a Boltzmann process to any weak solution to (1.1), but when $\gamma \in (-1, 0)$, i.e. moderately soft potentials, we can only prove existence of a weak solution to (1.1) to which it is possible to associate a Boltzmann process.

Proposition 1.1.3. Let f_0 be a probability measure with $m_2(f_0) < \infty$. Assume (1.3) for some $\gamma \in (-1, 1)$, $\nu \in (0, 1)$.

- If $\gamma \in (0, 1)$, for any weak solution $(f_t)_{t \geq 0}$ to (1.1) starting from f_0 and satisfying

$$\text{for all } p \geq 2, \text{ all } t_0 > 0, \quad \sup_{t \geq t_0} m_p(f_t) < \infty,$$

there exist a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, a $(\mathcal{F}_t)_{t > 0}$ -Poisson random measure $N(ds, dv, d\theta, d\varphi, du)$ on the space $[0, \infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0, \infty)$ with intensity $ds f_s(dv) \beta(\theta) d\theta d\varphi du$ and a càdlàg $(\mathcal{F}_t)_{t \geq 0}$ -adapted process $(V_t)_{t \geq 0}$ satisfying $\mathcal{L}(V_t) = f_t$ for all $t \geq 0$ and solving (1.8).

- If $\gamma \in (-1, 0]$, assume additionally that f_0 with $m_p(f_0) < \infty$ for some $p > 2$. There exist a probability space, a Poisson measure N and a càdlàg adapted process $(V_t)_{t \geq 0}$ as in the previous case, satisfying $\mathcal{L}(V_t) = f_t$ for all $t \geq 0$ and solving (1.8).

The Boltzmann equation depicts the velocity distribution of a dilute gas which is made up of a large number of molecules. So, the corresponding Boltzmann process $(V_t)_{t \geq 0}$ represents the time evolution of the velocity of a typical particle. When this particle collides with another one, its velocity changes suddenly. It is thus a jump process.

1.1.3 Recalls on multifractal analysis

In this part, we recall the definition of the main objects in multifractal analysis.

Definition 1.1.4. A locally bounded function $g : [0, 1] \rightarrow \mathbb{R}^3$ is said to belong to the pointwise Hölder space $C^\alpha(t_0)$ with $t_0 \in [0, 1]$ and $\alpha \notin \mathbb{N}$, if there exist $C > 0$ and a polynomial P_{t_0} of degree less than $\lfloor \alpha \rfloor$, such that for some neighborhood I_{t_0} of t_0 ,

$$|g(t) - P_{t_0}(t)| \leq C|t - t_0|^\alpha, \quad \forall t \in I_{t_0}.$$

The pointwise Hölder exponent of g at point t_0 is given by

$$h_g(t_0) = \sup\{\alpha > 0 : g \in C^\alpha(t_0)\},$$

where by convention $\sup \emptyset = 0$. The level sets of the pointwise Hölder exponent of the function g are called the iso-Hölder sets of g , and are denoted, for any $h \geq 0$, by

$$E_g(h) = \{t \geq 0 : h_g(t) = h\}.$$

We now recall the definition of the Hausdorff measures and dimension, see [21] for details.

Definition 1.1.5. Given a subset A of \mathbb{R} , given $s > 0$ and $\epsilon > 0$, the s -Hausdorff pre-measure \mathcal{H}_ϵ^s using balls of radius less than ϵ is given by

$$\mathcal{H}_\epsilon^s(A) = \inf \left\{ \sum_{i \in J} |I_i|^s : (I_i)_{i \in J} \in \mathcal{P}_\epsilon(A) \right\},$$

where $\mathcal{P}_\epsilon(A)$ is the set of all countable coverings of A by intervals with length at most ϵ . The s -Hausdorff measure of A is defined by

$$\mathcal{H}^s(A) = \lim_{\epsilon \rightarrow 0} \mathcal{H}_\epsilon^s(A).$$

Finally the Hausdorff dimension of A is defined by

$$\dim_H(A) := \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(A) = +\infty\},$$

and by convention $\dim_H \emptyset = -\infty$.

We use the concept of spectrum of singularities to describe the distribution of the singularities of a function g .

Definition 1.1.6. Let $g : [0, 1] \rightarrow \mathbb{R}^3$ be a locally bounded function. The spectrum of singularities (or multifractal spectrum) of g is the function $D_g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{-\infty\}$ defined by

$$D_g(h) = \dim_H(E_g(h)).$$

The iso-Hölder sets $E_g(h)$ are random for most studied stochastic processes, but stochastic processes such as Lévy processes [41], Lévy processes in multifractal time [7] and fractional Brownian motion have a deterministic multifractal spectrum. In the case of a Lévy process, it is easy to see that because of the Blumenthal 0-1 law and the Markov property, these Hausdorff dimensions are deterministic.

1.1.4 Main Results

Now, we give the main results of this paper.

Theorem 1.1.7. We assume (1.3) for some $\gamma \in (-1, 1)$, some $\nu \in (0, 1)$ with $\gamma + \nu > 0$. We consider some initial condition f_0 with $m_2(f_0) < \infty$ and assume that it is not a Dirac mass. If $\gamma \in (-1, 0]$, we moreover assume that f_0 with $m_p(f_0) < \infty$ for some $p > 2$. We consider a Boltzmann process $(V_t)_{t \in [0, 1]}$ as introduced in Proposition 1.1.3. Almost surely, for all $h \geq 0$,

$$D_V(h) = \begin{cases} \nu h & \text{if } 0 \leq h \leq 1/\nu, \\ -\infty & \text{if } h > 1/\nu. \end{cases} \quad (1.9)$$

The condition that f_0 is not a Dirac mass is important: if $V_0 = v_0$ a.s. for some deterministic $v_0 \in \mathbb{R}^3$, then $V_t = v_0$ for all $t \geq 0$ a.s. (which is a.s. a C^∞ function on $[0, \infty)$).

It is obvious from the proof that the spectrum of singularities is *homogeneous*: we could prove similarly that a.s., for any $0 \leq t_0 < t_1 < \infty$, all $h \geq 0$, $\dim_H(E_V(h) \cap [t_0, t_1]) = D_V(h)$.

Finally, it is likely that the same result holds true for very soft potentials. However, there are several technical difficulties, and the proof would be much more intricate.

Now we exhibit the multifractal spectrum of the position process. For simplicity, we only consider the case of hard potentials.

Theorem 1.1.8. *We assume (1.3) for some $\gamma \in (0, 1)$ and some $\nu \in (0, 1)$. We consider some initial condition f_0 with $m_2(f_0) < \infty$ and assume that it is not a Dirac mass. We consider a Boltzmann process $(V_t)_{t \in [0,1]}$ as introduced in Proposition 1.1.3 and introduce the associated position process $(X_t)_{t \in [0,1]}$ defined by $X_t = \int_0^t V_s ds$. Almost surely, for all $h \geq 0$,*

$$D_X(h) = \begin{cases} \nu(h-1) & \text{if } 1 \leq h \leq \frac{1}{\nu} + 1, \\ -\infty & \text{if } h > \frac{1}{\nu} + 1 \text{ or } 0 \leq h < 1. \end{cases} \quad (1.10)$$

This result is very natural once Theorem 1.1.7 is checked: we expect that at some given time t , the pointwise exponent of X is the one of V plus 1. However, this is not always true. Balança [5] has shown that such an oscillatory phenomenon do occur for Lévy processes, but on a very small set of points.

Definition 1.1.9. *Let $g : [0, 1] \rightarrow \mathbb{R}^3$ be a locally bounded function and let $G(t) = \int_0^t g(s) ds$. For all $h \geq 0$, we introduce the sets*

$$E_g^{cusp}(h) = \{t \in E_g(h) : h_G(t) = 1 + h_g(t)\} \text{ and } E_g^{osc}(h) = \{t \in E_g(h) : h_G(t) > 1 + h_g(t)\}. \quad (1.11)$$

The times $t \in E_g^{cusp}(h)$ are referred to as cusp singularities, while the times $t \in E_g^{osc}(h)$ are called oscillating singularities. Observe that $E_g(h) = E_g^{cusp}(h) \cup E_g^{osc}(h)$, the union being disjoint: this follows from the fact that obviously, for all $t \in [0, 1]$, $h_G(t) \geq h_g(t) + 1$. We will prove the following.

Theorem 1.1.10. *Under the assumptions of Theorem 1.1.8, we have almost surely:*

- for all $h \in [1/(2\nu), 1/\nu)$, $\dim_H(E_V^{osc}(h)) \leq 2h\nu - 1$,
- for all $h \in [0, 1/(2\nu)) \cup (1/\nu, +\infty]$, $E_V^{osc}(h) = \emptyset$,
- for all $h \in [0, 1/\nu]$, $\dim_H(E_V^{cusp}(h)) = h\nu$.

Actually, we will first prove Theorem 1.1.10 which, together with Theorem 1.1.7, implies Theorem 1.1.8.

1.2 Localization of the problem

In the following sections, we consider a Boltzmann process $(V_t)_{t \in [0,1]}$ associated to a weak solution $(f_t)_{t \in [0,1]}$ to (1.1), and driven by a Poisson measure $N(ds, dv, d\theta, d\varphi, du)$ on $[0, 1] \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0, \infty)$ with intensity $ds f_s(dv) \beta(\theta) d\theta d\varphi du$.

For $B \geq 1$, setting $H_B(v) = \frac{|v| \wedge B}{|v|} v$, we define, for $t \in [0, 1]$,

$$V_t^B := V_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \mathbf{a}(H_B(V_{s-}), v, \theta, \varphi) \mathbf{1}_{\{u \leq |H_B(V_{s-}) - v|^\gamma\}} N(ds, dv, d\theta, d\varphi, du), \quad (1.12)$$

where \mathbf{a} is defined in (1.6). We define the corresponding position process, for $t \in [0, 1]$, as

$$X_t^B = \int_0^t V_s^B ds. \quad (1.13)$$

In the rest of the paper, we will check the following two localized claims.

Proposition 1.2.1. *Let $B \geq 1$ be fixed. We assume (1.3) for some $\gamma \in (-1, 1)$, some $\nu \in (0, 1)$ with $\gamma + \nu > 0$. We consider the localized process introduced in (1.12). Almost surely, for all $h \geq 0$,*

$$D_{V^B}(h) = \begin{cases} \nu h & \text{if } 0 \leq h \leq 1/\nu, \\ -\infty & \text{if } h > 1/\nu. \end{cases}$$

Proposition 1.2.2. *Let $B \geq 1$ be fixed. We assume (1.3) for some $\gamma \in (0, 1)$, some $\nu \in (0, 1)$. We consider the localized process $(V_t^B)_{t \geq 0}$ defined in (1.12). Then almost surely,*

- for all $h \in [1/(2\nu), 1/\nu)$, $\dim_H(E_{V^B}^{osc}(h)) \leq 2h\nu - 1$,
- for all $h \in [0, 1/(2\nu)) \cup (1/\nu, +\infty]$, $E_{V^B}^{osc}(h) = \emptyset$,
- for all $h \in [0, 1/\nu]$, $\dim_H(E_{V^B}^{cusp}(h)) = h\nu$.

Once these propositions are verified, Theorems 1.1.7 and 1.1.10 are immediately deduced.

Proof of Theorems 1.1.7 and 1.1.10. Since $\sup_{[0,1]} |V_t| < +\infty$ a.s. (because V is a càdlàg process), the event $\Omega_B = \{\sup_{[0,1]} |V_t| \leq B\}$ a.s. increases to Ω as B increases to infinity. But on Ω_B , we obviously have that $(V_t^B)_{t \in [0,1]} = (V_t)_{t \in [0,1]}$. Hence on Ω_B , it holds that for all $h \in [0, +\infty]$, $D_V(h) = D_{V^B}(h)$, $\dim_H(E_V^{osc}(h)) = \dim_H(E_{V^B}^{osc}(h))$ and $\dim_H(E_V^{cusp}(h)) = \dim_H(E_{V^B}^{cusp}(h))$. The conclusion then follows from the above two propositions. \square

We thus fix $B \geq 1$ for the rest of the paper.

1.3 Study of the velocity process

1.3.1 Preliminaries

First, we need to bound f_t from below. The following lemma is purely deterministic.

Lemma 1.3.1. *There exist $a, b, c > 0$, such that for any $w \in \mathbb{R}^3$, any $t \in [0, 1]$,*

$$f_t(\mathcal{H}_w) \geq b, \quad (1.14)$$

where $\mathcal{H}_w = \{v \in \mathbb{R}^3 : |v - w| \geq a, |v| \leq c\}$.

Proof. As f_0 is not a Dirac mass, there exist $v_1 \neq v_2$ such that $v_1, v_2 \in \text{Supp} f_0$. We set $a = \frac{|v_1 - v_2|}{6}$.

Step 1. We first show that there exists $b > 0$, such that for all $w \in \mathbb{R}^3, t \in [0, 1]$, $f_t(\{v : |v - w| \geq a\}) \geq 2b$. First, if $|w| \geq \sqrt{2m_2(f_0)} + a =: M$, recalling that $m_2(f_t) = m_2(f_0)$ for all $t \geq 0$,

$$\begin{aligned} f_t(\{v : |v - w| \geq a\}) &\geq f_t(\{v : |v| \leq |w| - a\}) = 1 - f_t(\{v : |v| > |w| - a\}) \\ &\geq 1 - \frac{m_2(f_0)}{(|w| - a)^2} \geq 1 - \frac{m_2(f_0)}{2m_2(f_0)} = \frac{1}{2}. \end{aligned}$$

Next, we consider a bounded nonnegative globally Lipschitz-continuous function $\phi : \mathbb{R}_+ \rightarrow [0, 1]$, such that for all $v > 0$, $\mathbf{1}_{B(0,a)^c}(v) \geq \phi(|v|) \geq \mathbf{1}_{B(0,2a)^c}(v)$, and define $F(t, w) = \int_{\mathbb{R}^3} \phi(|w - v|) f_t(dv)$. We know that $t \mapsto F(t, w)$ is continuous for each $w \in \mathbb{R}^3$ by Lemma 3.3 in [24]. Moreover, $F(t, w)$ is (uniformly in t) continuous in w by the Lipschitz-continuity of ϕ . So $F(t, w)$ is continuous on $[0, 1] \times \mathbb{R}^3$. Since for all $t > 0$, $\text{Supp} f_t = \mathbb{R}^3$ by Theorem 1.2 in [24], we get $F(t, w) \geq f_t(B(w, 2a)^c) > 0, \forall (t, w) \in (0, 1] \times \overline{B(0, M)}$. When $t = 0$, recalling that $v_1, v_2 \in \text{Supp} f_0$ and $a = \frac{|v_1 - v_2|}{6}$, we easily see that for all $w \in \mathbb{R}^3$, either $B(v_1, a) \subset B(w, 2a)^c$ or $B(v_2, a) \subset B(w, 2a)^c$, whence $F(0, w) \geq \min\{f_0(B(v_1, a)), f_0(B(v_2, a))\} > 0$. Since $[0, 1] \times \overline{B(0, M)}$ is compact and $F(t, w)$ is continuous, there exists $b_1 > 0$, such that $f_t(B(w, a)^c) \geq F(t, w) \geq b_1$ for all $(t, w) \in [0, 1] \times \overline{B(0, M)}$. So we conclude by choosing $b = \min(\frac{1}{2}, b_1)/2$.

Step 2. We now conclude. Using Step 1,

$$f_t(\{v : |v - w| \geq a, |v| \leq c\}) \geq f_t(\{v : |v - w| \geq a\}) - f_t(\{v : |v| > c\}) \geq 2b - \frac{m_2(f_0)}{c^2}.$$

So, we complete the proof by taking $c = \sqrt{\frac{m_2(f_0)}{b}}$. \square

1.3.2 Random fractal sets associated with the Poisson process

First, we introduce some notations. Recall that h_{V^B} , E_{V^B} , D_{V^B} respectively the Hölder exponent, iso-Hölder set and spectrum of singularities of the Boltzmann process $(V_t^B)_{t \in [0,1]}$. The notation \mathcal{L} represents the Lebesgue measure. \mathcal{J} designates the set of the jump times of the process V^B , that is,

$$\mathcal{J} := \{s \in [0, 1] : |\Delta V_s^B| \neq 0\}.$$

For $m \geq 1$, we also introduce

$$\mathcal{J}_m := \{s \in \mathcal{J} : |\Delta V_s^B| \leq 2^{-m}\}, \quad \tilde{\mathcal{J}}_m := \{s \in \mathcal{J} : 2^{-m-1} < |\Delta V_s^B| \leq 2^{-m}\}.$$

For $\delta > 0$ and $m \geq 1$, we define the sets

$$A_\delta^m := \bigcup_{s \in \mathcal{J}_m} [s - |\Delta V_s^B|^\delta, s + |\Delta V_s^B|^\delta], \quad \tilde{A}_\delta^m := \bigcup_{s \in \tilde{\mathcal{J}}_m} [s - |\Delta V_s^B|^\delta, s + |\Delta V_s^B|^\delta].$$

Finally, for $\delta > 0$, we define

$$A_\delta = \limsup_{m \rightarrow +\infty} A_\delta^m = \limsup_{m \rightarrow +\infty} \tilde{A}_\delta^m. \quad (1.15)$$

The main result of this subsection states that

Proposition 1.3.2. *We have a.s. the following properties:*

- (1) for all $\delta \in (0, \nu)$, $A_\delta \supset [0, 1]$,
- (2) there exists a (random) positive sequence $(\epsilon_m)_{m \geq 1}$ decreasing to 0, such that

$$\mathcal{L} \left(A_\nu^* \cap [0, 1] \right) = 1,$$

where we use the notation $A_\delta^* = \limsup_{m \rightarrow +\infty} \tilde{A}_{\delta(1-\epsilon_m)}^m$, for all $\delta \in (0, \infty)$.

Remark 1.3.3. *We observe at once that for any $\delta > \delta' > 0$, $A_\delta \subset A_\delta^* \subset A_{\delta'}$.*

We study A_δ because of the following heuristics: if $t \in A_\delta$ with δ large, then t is rather close to many large jump times of V^B , so that V^B will not be very regular at t . On the contrary, if t only belongs to those A_δ 's where δ is small, then this means that t is rather far away from the jumps of V^B , so that V^B will be rather regular at t .

We introduce A_δ^* (which resembles very much A_δ) for technical reasons, mainly because at the *critical* value $\delta = \nu$, we cannot prove (and it may be false) that A_ν has full Lebesgue measure.

The rest of this subsection is devoted to proving this proposition. We first recall the Shepp lemma, first discovered in [53], in the version used in [41].

Lemma 1.3.4. *We consider a Poisson measure $\pi(ds, dy) = \sum_{s \in \mathcal{D}} \delta_{(s, y_s)}$ on $[0, 1] \times (0, 1)$ with intensity $ds\mu(dy)$, where μ is a measure on $(0, 1)$. We consider the set $U = \cup_{s \in \mathcal{D}} (s - y_s, s + y_s)$. If*

$$\int_0^1 \exp\left(2 \int_t^1 \mu((y, 1)) dy\right) dt = +\infty,$$

then almost surely, $[0, 1] \subset U$.

We write $N = \sum_{s \in \mathcal{D}} \delta_{(s, v_s, \theta_s, \varphi_s, u_s)}$, where $v_s, \theta_s, \varphi_s, u_s$ are the quanta corresponding to the jump time $s \in \mathcal{D}$. For convenience, we consider this Poisson measure by adding a family of independent variables $(x_s)_{s \in \mathcal{D}}$, which are uniformly distributed in $[0, 1]$ and independent of $v_s, \theta_s, \varphi_s, u_s$, so that $O := \sum_{s \in \mathcal{D}} \delta_{(s, v_s, \theta_s, \varphi_s, u_s, x_s)}$ is a Poisson measure on $[0, 1] \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi] \times [0, \infty) \times [0, 1]$ with intensity $ds f_s(dv) \beta(\theta) d\theta d\varphi du dx$. According to Lemma 1.3.1, we know that $f_s(\mathcal{H}_w) \geq b$ for all $s \in [0, 1]$ and all $w \in \mathbb{R}^3$. Then we can get the following lemma.

Lemma 1.3.5. *For $m \geq 1$, we introduce*

$$\mathcal{J}'_m := \left\{ s \in \mathcal{D} : u_s \leq d^\gamma, v_s \in \mathcal{H}_{H_B(V_{s-})}, \theta_s \leq K2^{-m}, x_s \leq \frac{b}{f_s(\mathcal{H}_{H_B(V_{s-})})} \right\},$$

where $K = 1/(B + c)$ and where $d = a$ (if $\gamma \in (0, 1)$) or $d = B + c$ (if $\gamma \in (-1, 0]$). Then we have

$$\mathcal{J}'_m \subset \mathcal{J}_m \quad \text{and} \quad \bigcup_{s \in \mathcal{J}'_m} \left[s - \left(\frac{a\theta_s}{4}\right)^\delta, s + \left(\frac{a\theta_s}{4}\right)^\delta \right] \subset A_\delta^m. \quad (1.16)$$

Proof. We recall that, for all $s \in [0, 1]$, $|H_B(V_{s-})| = \left| \frac{|V_{s-}| \wedge B}{|V_{s-}|} V_{s-} \right| \leq B$ and that $v_s \in \mathcal{H}_{H_B(V_{s-})}$ implies that $|H_B(V_{s-}) - v_s| \geq a$ and $|v_s| \leq c$. Then for all $m \geq 1$, for all $s \in \mathcal{J}'_m$, we have (recall (1.7))

$$\begin{aligned} |\Delta V_s^B| &= \sqrt{\frac{1 - \cos \theta_s}{2}} |H_B(V_{s-}) - v_s| \mathbf{1}_{\{u_s \leq |H_B(V_{s-}) - v_s|^\gamma\}} \\ &\leq \theta_s |H_B(V_{s-}) - v_s| \leq K2^{-m}(B + c) = 2^{-m}, \end{aligned}$$

i.e. $\mathcal{J}'_m \subset \mathcal{J}_m$.

In addition, for all $s \in \mathcal{J}'_m$, using that $|H_B(V_{s-}) - v_s| \geq a$, that $1 - \cos \theta \geq \theta^2/8$ on $(0, \pi/2]$, and that the indicator equals 1, we have

$$|\Delta V_s^B| = \sqrt{\frac{1 - \cos \theta_s}{2}} |H_B(V_{s-}) - v_s| \mathbf{1}_{\{u_s \leq |H_B(V_{s-}) - v_s|^\gamma\}} \geq \frac{a\theta_s}{4}.$$

Indeed, the indicator equals 1 because we always have $u_s \leq d^\gamma \leq |H_B(V_{s-}) - v_s|^\gamma$ (if $\gamma \in (0, 1)$, then $|H_B(V_{s-}) - v_s| \geq a$ and $d = a$, while if $\gamma \in (-1, 0]$, then $|H_B(V_{s-}) - v_s| \leq B + c$ and $d = B + c$). Since $|\Delta V_s^B| \geq a\theta_s/4$ and $\mathcal{J}'_m \subset \mathcal{J}_m$, the lemma follows. \square

Lemma 1.3.6. *Let $m \geq 1$ and $\delta > 0$ be fixed. The random measure*

$$\mu_m^\delta(ds, dy) = \sum_{s \in \mathcal{J}'_m} \delta_{(s, (a\theta_s/4)^\delta)}$$

is a Poisson measure on $[0, 1] \times (0, \infty)$ with intensity $ds h_m^\delta(y) dy$, where

$$h_m^\delta(y) = \frac{8\pi d^\gamma b}{a\delta} \beta\left(\frac{4}{a} y^{1/\delta}\right) y^{\frac{1}{\delta}-1} \mathbf{1}_{\{y \leq (aK2^{-(m+2)})^\delta \wedge (a\pi/8)^\delta\}}.$$

Moreover, we have

$$c_1 y^{-1-\frac{\nu}{\delta}} \mathbf{1}_{\{y \leq (aK2^{-(m+2)})^\delta \wedge (a\pi/8)^\delta\}} \leq h_m^\delta(y) \leq C_1 y^{-1-\frac{\nu}{\delta}} \mathbf{1}_{\{y \leq (aK2^{-(m+2)})^\delta \wedge (a\pi/8)^\delta\}},$$

for some constants $0 < c_1 < C_1$ (depending on B, δ).

Proof. By Jacod-Shiryaev [40] [Chapter 2, Theorem 1.8], it suffices to check that the compensator of the random measure $\mu_m^\delta(ds, dy)$ is $ds h_m^\delta(y) dy$, i.e., for any predictable process $W(s, y)$,

$$\begin{aligned} & \int_0^t \int_0^\infty W(s, y) (\mu_m^\delta(ds, dy) - ds h_m^\delta(y) dy) \\ &= \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \int_0^1 W(s, (a\theta/4)^\delta) \times \mathbf{1}_{\{v \in \mathcal{H}_{H_B}(V_{s-}), \theta \leq K2^{-m}, u \leq d^\gamma, x \leq b/f_s(\mathcal{H}_{H_B}(V_{s-}))\}} \\ & \quad \times O(ds, dv, d\theta, d\varphi, du, dx) - \int_0^t \int_0^\infty W(s, y) h_m^\delta(y) ds dy \end{aligned}$$

is a local martingale. Recalling that $O(ds, dv, d\theta, d\varphi, du, dx)$ is a Poisson measure with intensity $ds f_s(dv) \beta(\theta) d\theta d\varphi du dx$, we know that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \int_0^1 W(s, (a\theta/4)^\delta) \mathbf{1}_{\{v \in \mathcal{H}_{H_B}(V_{s-}), \theta \leq K2^{-m}, u \leq d^\gamma, x \leq b/f_s(\mathcal{H}_{H_B}(V_{s-}))\}} \\ & \quad \times (O(ds, dv, d\theta, d\varphi, du, dx) - ds f_s(dv) \beta(\theta) d\theta d\varphi du dx) \end{aligned}$$

is a local martingale. Thus, we only need to prove that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \int_0^1 W(s, (a\theta/4)^\delta) \mathbf{1}_{\{v \in \mathcal{H}_{H_B}(V_{s-}), \theta \leq K2^{-m}, u \leq d^\gamma, x \leq b/f_s(\mathcal{H}_{H_B}(V_{s-}))\}} \\ & \quad \times ds f_s(dv) \beta(\theta) d\theta d\varphi du dx \\ &= \int_0^t \int_0^\infty W(s, y) h_m^\delta(y) ds dy. \end{aligned}$$

Actually,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \int_0^1 W(s, (a\theta/4)^\delta) \mathbf{1}_{\{v \in \mathcal{H}_{HB}(V_{s-}), \theta \leq K2^{-m}, u \leq d^\gamma, x \leq b/f_s(\mathcal{H}_{HB}(V_{s-}))\}} \\ & \quad \times ds f_s(dv) \beta(\theta) d\theta d\varphi dudx \\ & = 2\pi d^\gamma b \int_0^t \int_0^{\pi/2} W(s, (a\theta/4)^\delta) \mathbf{1}_{\{\theta \leq K2^{-m}\}} ds \beta(\theta) d\theta. \end{aligned}$$

Using the substitution $y = (a\theta/4)^\delta$, we conclude that the intensity of μ_m^δ is indeed $dsh_m^\delta(y)dy$. From (1.3), we can easily get the bounds for $h_m^\delta(y)$. \square

Now, we give the

Proof of Proposition 1.3.2. We start with (1) and thus fix $\delta \in (0, \nu)$. By Lemma 1.3.6, we know that the random measure $\mu_m^\delta = \sum_{s \in \mathcal{J}'_m} \delta_{(s, (a\theta_s/4)^\delta)}$ is a Poisson measure on $[0, 1] \times (0, 1)$ with intensity $ds h_m^\delta(y)dy$, where

$$h_m^\delta(y) \geq c_1 y^{-1-\frac{\nu}{\delta}} \mathbf{1}_{\{y \leq (aK2^{-(m+2)})^\delta \wedge (a\pi/8)^\delta\}}.$$

Clearly, for all $m \geq 1$, $\delta \in (0, \nu)$,

$$\int_0^1 \exp\left(2 \int_t^1 \int_y^1 h_m^\delta(z) dz dy\right) dt = \infty,$$

since $2 \int_t^1 (\int_y^1 h_m^\delta(z) dz) dy \geq 2c_1 \frac{\delta^2}{(\nu-\delta)^\nu} t^{1-\frac{\nu}{\delta}}$. Applying Lemma 1.3.4, we deduce that almost surely, for all $m \geq 1$,

$$[0, 1] \subset \bigcup_{s \in \mathcal{J}'_m} \left[s - \left(\frac{a\theta_s}{4}\right)^\delta, s + \left(\frac{a\theta_s}{4}\right)^\delta \right].$$

Consequently, almost surely,

$$[0, 1] \subset \limsup_{m \rightarrow +\infty} \bigcup_{s \in \mathcal{J}'_m} \left[s - \left(\frac{a\theta_s}{4}\right)^\delta, s + \left(\frac{a\theta_s}{4}\right)^\delta \right].$$

Recalling (1.15) and (1.16), we deduce that $[0, 1] \subset A_\delta$ almost surely.

We next prove (2). We set $m_1 = 1$. By (1), we have a.s.

$$[0, 1] \subset A_{\nu(1-1/2)} \subset \bigcup_{m \geq m_1} \tilde{A}_{\nu(1-1/2)}^m.$$

Hence we can find $m_2 > m_1$ such that

$$\mathcal{L} \left(\bigcup_{m_1 \leq m < m_2} \tilde{A}_{\nu/2}^m \cap [0, 1] \right) \geq 1 - \frac{1}{2}.$$

Similarly, we have almost surely, $[0, 1] \subset A_{\nu(1-1/3)} \subset \bigcup_{m \geq m_2} \tilde{A}_{\nu(1-1/3)}^m$, therefore we can find $m_3 > m_2$ such that

$$\mathcal{L} \left(\bigcup_{m_2 \leq m < m_3} \tilde{A}_{\nu(1-1/3)}^m \cap [0, 1] \right) \geq 1 - \frac{1}{2^2}.$$

By induction, we can find an increasing sequence $(m_j)_{j \geq 1}$ such that, for all $j \geq 2$,

$$\mathcal{L} \left(\bigcup_{m_{j-1} \leq m < m_j} \tilde{A}_{\nu(1-1/j)}^m \cap [0, 1] \right) \geq 1 - \frac{1}{2^{j-1}}.$$

So, from the Fatou lemma, we have

$$\begin{aligned} & \mathcal{L} \left(\limsup_{j \rightarrow +\infty} \bigcup_{m_{j-1} \leq m < m_j} \tilde{A}_{\nu(1-1/j)}^m \cap [0, 1] \right) \\ & \geq \limsup_{j \rightarrow +\infty} \mathcal{L} \left(\bigcup_{m_{j-1} \leq m < m_j} \tilde{A}_{\nu(1-1/j)}^m \cap [0, 1] \right) \geq 1. \end{aligned}$$

We now put $\epsilon_m = \frac{1}{j}$ for $m \in [m_{j-1}, m_j)$ and note that

$$\limsup_{j \rightarrow +\infty} \bigcup_{m \in [m_{j-1}, m_j)} \tilde{A}_{\nu(1-\epsilon_m)}^m = \limsup_{m \rightarrow +\infty} \tilde{A}_{\nu(1-\epsilon_m)}^m.$$

The conclusion follows. □

1.3.3 Study of the Hölder exponent of V^B

We now study the pointwise Hölder exponent of the localized Boltzmann process V^B .

Definition 1.3.7. For all $t \in [0, 1]$, the index of approximation of t is defined by

$$\delta_t := \sup\{\delta > 0 : t \in A_\delta\}.$$

For all $t \in [0, 1]$, the index of approximation of t reflects directly the relation between t and jump times of V^B . If δ_t is large, then t is close to many large jumps of V^B .

Remark 1.3.8. Recalling Remark 1.3.3 and Proposition 1.3.2, we see that almost surely, for all $t \in [0, 1]$, $\delta_t = \sup\{\delta > 0 : t \in A_\delta^*\}$ and $\delta_t \geq \nu$.

If $t \in \mathcal{J}$, we know that $h_{V^B}(t) = 0$. Then for $t \in [0, 1] \setminus \mathcal{J}$, we claim that the Hölder exponent is the inverse of the index of approximation.

Proposition 1.3.9. *Almost surely, for all $t \in [0, 1] \setminus \mathcal{J}$, $h_{V^B}(t) = 1/\delta_t$.*

To prove this claim, we need the following two lemmas. The first lemma, that will give the upper bound for $h_{V^B}(t)$, can be found in [41] and is as follows.

Lemma 1.3.10. *Let $f : \mathbb{R} \rightarrow \mathbb{R}^3$ be a function discontinuous on a dense set of points and let $t \in \mathbb{R}$. Let $(t_n)_{n \geq 1}$ be a real sequence converging to t and such that f has left and right limits at each t_n . Then*

$$h_f(t) \leq \liminf_{n \rightarrow \infty} \frac{\log |f(t_n+) - f(t_n-)|}{\log |t_n - t|}.$$

For the lower bound of $h_{V^B}(t)$, we will use Lemma 1.3.11 below, that relies on some ideas of [5]. We first introduce, for $m > 0$, the following two processes:

$$\begin{aligned} V_t^{B,m} &:= V_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \mathbf{a}(H_B(V_{s-}), v, \theta, \varphi) \mathbf{1}_{\{u \leq |H_B(V_{s-}) - v|^\gamma\}} \\ &\quad \times \mathbf{1}_{\{|\mathbf{a}(H_B(V_{s-}), v, \theta, \varphi)| \leq 2^{-m}\}} N(ds, dv, d\theta, d\varphi, du), \\ Z_t^{B,m} &:= \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \theta |H_B(V_{s-}) - v| \mathbf{1}_{\{u \leq |H_B(V_{s-}) - v|^\gamma\}} \\ &\quad \times \mathbf{1}_{\{|\theta |H_B(V_{s-}) - v|/4 \leq 2^{-m}\}} N(ds, dv, d\theta, d\varphi, du). \end{aligned}$$

We can immediately observe that the process $Z_t^{B,m}$ is almost surely increasing as a function of t . We also notice that a.s., for all $x, y \in [0, 1]$,

$$|V_x^{B,m} - V_y^{B,m}| \leq |Z_x^{B,m} - Z_y^{B,m}|. \quad (1.17)$$

This comes from the inequality $\theta |H_B(V_{s-}) - v|/4 \leq |\mathbf{a}(H_B(V_{s-}), v, \theta, \varphi)| \leq \theta |H_B(V_{s-}) - v|$, which follows from (1.7).

Lemma 1.3.11. *There exists some constant $C_B > 0$, such that*

(1) *for all $\delta > \nu$, all $m \geq 1$,*

$$\mathbb{P} \left[\sup_{x, y \in [0, 1], |x-y| \leq 2^{-m}} \left| V_x^{B, \frac{m}{\delta}} - V_y^{B, \frac{m}{\delta}} \right| \geq m 2^{-m/\delta} \right] \leq C_B e^{-m/4}, \quad (1.18)$$

(2) *for all $m \geq 1$, all $\lambda \in [0, 2^m]$,*

$$\mathbb{E} \left[e^{\lambda Z_1^{B,m}} \right] \leq e^{C_B \lambda 2^{-m(1-\nu)}}. \quad (1.19)$$

Proof. We first prove (1.18). Setting $\lambda = 3 \times 2^{m/\delta}$, recalling (1.17) and that $Z_t^{B, \frac{m}{\delta}}$ is almost surely increasing in t , we get

$$\begin{aligned}
& \mathbb{P} \left[\sup_{x, y \in [0, 1], |x-y| \leq 2^{-m}} \left| V_x^{B, \frac{m}{\delta}} - V_y^{B, \frac{m}{\delta}} \right| \geq m 2^{-m/\delta} \right] \\
& \leq \mathbb{P} \left[\sup_{x, y \in [0, 1], |x-y| \leq 2^{-m}} \left| Z_x^{B, \frac{m}{\delta}} - Z_y^{B, \frac{m}{\delta}} \right| \geq m 2^{-m/\delta} \right] \\
& \leq \sum_{k=0}^{2^m-1} \mathbb{P} \left[\left(Z_{(k+1)2^{-m}}^{B, \frac{m}{\delta}} - Z_{k2^{-m}}^{B, \frac{m}{\delta}} \right) \geq \frac{m 2^{-m/\delta}}{3} \right] \\
& \leq \sum_{k=0}^{2^m-1} e^{-m} \mathbb{E} \left[\exp \left\{ \lambda \left(Z_{(k+1)2^{-m}}^{B, \frac{m}{\delta}} - Z_{k2^{-m}}^{B, \frac{m}{\delta}} \right) \right\} \right] \\
& =: \sum_{k=0}^{2^m-1} e^{-m} I_k.
\end{aligned}$$

We then set

$$J_k(t) := \mathbb{E} \left[\exp \left\{ \lambda \left(Z_{t+k2^{-m}}^{B, \frac{m}{\delta}} - Z_{k2^{-m}}^{B, \frac{m}{\delta}} \right) \right\} \right].$$

Observe that $I_k = J_k(2^{-m})$. For all $t \geq 0$, we have, by the Itô formula,

$$\begin{aligned}
J_k(t) = 1 + 2\pi \mathbb{E} \left[\int_{k2^{-m}}^{t+k2^{-m}} \int_{\mathbb{R}^3} \int_0^{\pi/2} \exp \left\{ \lambda \left(Z_s^{B, \frac{m}{\delta}} - Z_{k2^{-m}}^{B, \frac{m}{\delta}} \right) \right\} (e^{\lambda \theta |H_B(V_s) - v|} - 1) \right. \\
\left. \times |H_B(V_s) - v|^{\gamma} \mathbf{1}_{\{\theta |H_B(V_s) - v|/4 \leq 2^{-m/\delta}\}} \beta(\theta) d\theta f_s(dv) ds \right].
\end{aligned}$$

From $\lambda \theta |H_B(V_s) - v| \leq 4\lambda 2^{-m/\delta} = 12$, we have $e^{\lambda \theta |H_B(V_s) - v|} - 1 \leq C \lambda \theta |H_B(V_s) - v|$ for some positive constant C . Using this estimate and recalling (1.3), we get

$$\begin{aligned}
J_k(t) \leq 1 + C \lambda \mathbb{E} \left[\int_{k2^{-m}}^{t+k2^{-m}} \int_{\mathbb{R}^3} \int_0^{\pi/2} \exp \left\{ \lambda \left(Z_s^{B, \frac{m}{\delta}} - Z_{k2^{-m}}^{B, \frac{m}{\delta}} \right) \right\} \right. \\
\left. \times \theta^{-\nu} |H_B(V_s) - v|^{\gamma+1} \mathbf{1}_{\{\theta |H_B(V_s) - v|/4 \leq 2^{-m/\delta}\}} d\theta f_s(dv) ds \right].
\end{aligned}$$

Moreover,

$$\begin{aligned}
& |H_B(V_s) - v|^{\gamma+1} \int_0^{\pi/2} \theta^{-\nu} \mathbf{1}_{\{\theta |H_B(V_s) - v|/4 \leq 2^{-m/\delta}\}} d\theta \\
& \leq C |H_B(V_s) - v|^{\gamma+1} (|H_B(V_s) - v| 2^{m/\delta})^{\nu-1} \\
& \leq C |H_B(V_s) - v|^{\gamma+\nu} 2^{m(\nu-1)/\delta}.
\end{aligned}$$

Since $\gamma + \nu \in (0, 2)$ by assumption, we have $|H_B(V_s) - v|^{\gamma+\nu} \leq C(1 + |v|^2 + |H_B(V_s)|^2)$, whence

$$J_k(t) \leq 1 + C\lambda 2^{m(\nu-1)/\delta} \mathbb{E} \left[\int_{k2^{-m}}^{t+k2^{-m}} \int_{\mathbb{R}^3} \exp \left\{ \lambda \left(Z_s^{B, \frac{m}{\delta}} - Z_{k2^{-m}}^{B, \frac{m}{\delta}} \right) \right\} \right. \\ \left. \times (1 + |H_B(V_s)|^2 + |v|^2) f_s(dv) ds \right].$$

Since $|H_B(V_s)| \leq B$, and by (1.4), we have a.s.

$$\int_{\mathbb{R}^3} (1 + |H_B(V_s)|^2 + |v|^2) f_s(dv) \leq 1 + B^2 + m_2(f_0).$$

Using finally that $\lambda 2^{m(\nu-1)/\delta} = 3 \times 2^{m\nu/\delta}$, we find that for all t , a.s.

$$J_k(t) \leq 1 + C_B 2^{m\nu/\delta} \int_0^t J_k(s) ds \quad .$$

Hence $J_k(t) \leq \exp(C_B 2^{m\nu/\delta} t)$ by the Gronwall inequality, so that $I_k = J_k(2^{-m}) \leq \exp(C_B 2^{-m(1-\frac{\nu}{\delta})}) \leq C_B$ because $\delta \geq \nu$. Finally,

$$\mathbb{P} \left[\sup_{x, y \in [0, 1], |x-y| \leq 2^{-m}} \left| V_x^{B, \frac{m}{\delta}} - V_y^{B, \frac{m}{\delta}} \right| \geq m 2^{-m/\delta} \right] \leq \sum_{k=0}^{2^m-1} e^{-m} I_k \leq C_B e^{-m} 2^m \leq C_B e^{-m/4}.$$

This completes the proof of (1.18). We only sketch the proof of (1.19), since it is very similar. First, by Itô Formula,

$$\mathbb{E} \left[e^{\lambda Z_t^{B, m}} \right] \\ = 1 + 2\pi \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} e^{\lambda Z_s^{B, m}} (e^{\lambda \theta |H_B(V_s) - v|} - 1) |H_B(V_s) - v|^\gamma \right. \\ \left. \times \mathbf{1}_{\{\theta |H_B(V_s) - v|/4 \leq 2^{-m}\}} \beta(\theta) d\theta f_s(dv) ds \right].$$

Since $\lambda \theta |H_B(V_s) - v| < 4$ (because $\lambda \leq 2^m$), a similar computation as previously shows that

$$\mathbb{E} \left[e^{\lambda Z_t^{B, m}} \right] \leq 1 + C_B \lambda 2^{m(\nu-1)} \mathbb{E} \left[\int_0^t e^{\lambda Z_s^{B, m}} ds \right] \leq 1 + C_B \lambda 2^{m(\nu-1)} \int_0^t \mathbb{E} [e^{\lambda Z_s^{B, m}}] ds.$$

Owing to the Grönwall inequality, we deduce that $\mathbb{E}[e^{\lambda Z_t^{B, m}}] \leq e^{C_B \lambda 2^{m(\nu-1)} t}$. Taking $t = 1$, we obtain the conclusion. \square

Now, we can proceed to the

Proof of Proposition 1.3.9. Upper Bound. Here we prove that for all $t \in [0, 1]$, it holds that $h_{VB}(t) \leq 1/\delta_t$. To this end, we check that for all $\delta > 0$, all $t \in A_\delta$, $h_{VB}(t) \leq 1/\delta$. Let thus $\delta > 0$ and $t \in A_\delta$. By definition of A_δ , for all $m \geq 1$, there exists $t_m \in \mathcal{J}$, such that $|t_m - t| \leq |\Delta V_{t_m}^B|^\delta$ and $|\Delta V_{t_m}^B| \leq 2^{-m}$. From Lemma 1.3.10, we directly deduce that

$$h_{VB}(t) \leq \liminf_{m \rightarrow \infty} \frac{\log |\Delta V_{t_m}^B|}{\log |t_m - t|} \leq \liminf_{m \rightarrow \infty} \frac{\log |\Delta V_{t_m}^B|}{\log |\Delta V_{t_m}^B|^\delta} = \frac{1}{\delta}.$$

Lower Bound. In this part we show that almost surely, for all $t \in [0, 1] \setminus \mathcal{J}$, $h_{VB}(t) \geq 1/\delta_t$. To get this, we need to check that for all $\delta > \nu$, if $t \notin A_\delta$, then $h_{VB}(t) \geq 1/\delta$. Let thus $\delta > \nu$ and $t \notin A_\delta$.

By Lemma 1.3.11-(1) and Borel-Cantelli's lemma, there almost surely exists $m_0 \geq 1$ such that for all $m > m_0$, for all $x, y \in [0, 1]$ satisfying $|x - y| \leq 2^{-m}$,

$$|V_x^{B, m/\delta} - V_y^{B, m/\delta}| \leq m2^{-m/\delta}. \quad (1.20)$$

Since $t \notin A_\delta$, there exists $m_1 > m_0$, such that for all $s \in \mathcal{J}$ satisfying $|\Delta V_s^B| \leq 2^{-m_1}$, we have

$$|s - t| > |\Delta V_s^B|^\delta. \quad (1.21)$$

For all $r \in [0, 1]$, we define

$$U_{t,r}^{m_1} := \sum_{s \in [t \wedge r, t \vee r] \cap \mathcal{J}} |\Delta V_s^B| \mathbf{1}_{\{|\Delta V_s^B| > 2^{-m_1}\}},$$

and we observe that

$$|V_t^B - V_r^B| \leq |V_t^{B, m_1} - V_r^{B, m_1}| + U_{t,r}^{B, m_1}.$$

Since $t \notin \mathcal{J}$ and since the process V^B has almost surely a finite number of jump greater than 2^{-m_1} , we can almost surely find $\epsilon_1 > 0$ such that, for all $r \in (t - \epsilon_1, t + \epsilon_1)$, $U_{t,r}^{m_1} = 0$.

Next, we put $\epsilon_2 = 2^{-m_1-1}$. Then for each $r \in (t - \epsilon_2, t + \epsilon_2)$, we set $m_r = \lfloor \log_2 \frac{1}{|t-r|} \rfloor > m_1$, for which $2^{-m_r-1} < |t-r| \leq 2^{-m_r}$. Then for all $r \in (t - \epsilon_2, t + \epsilon_2)$, we write

$$|V_t^{B, m_1} - V_r^{B, m_1}| \leq |V_t^{B, m_r/\delta} - V_r^{B, m_r/\delta}| + \sum_{s \in [t \wedge r, t \vee r] \cap \mathcal{J}} |\Delta V_s^B| \mathbf{1}_{\{2^{-m_r/\delta} < |\Delta V_s^B| \leq 2^{-m_1}\}}.$$

According to (1.21), for $s \in [t \wedge r, t \vee r] \cap \mathcal{J}$, $|\Delta V_s^B| \leq 2^{-m_1}$ implies that $|\Delta V_s^B| < |s-t|^{1/\delta} \leq |r-t|^{1/\delta} \leq 2^{-m_r/\delta}$, whence the second term $\sum_{s \in [t \wedge r, t \vee r] \cap \mathcal{J}} |\Delta V_s^B| \mathbf{1}_{\{2^{-m_r/\delta} < |\Delta V_s^B| \leq 2^{-m_1}\}}$ vanishes.

To summarize, we have checked that for all $r \in \left(t - (\epsilon_1 \wedge \epsilon_2), t + (\epsilon_1 \wedge \epsilon_2)\right)$,

$$|V_t^B - V_r^B| \leq \left|V_t^{B, m_r/\delta} - V_r^{B, m_r/\delta}\right|.$$

Furthermore, since $m_r > m_0$, we conclude from (1.20) that, still for $r \in \left(t - (\epsilon_1 \wedge \epsilon_2), t + (\epsilon_1 \wedge \epsilon_2)\right)$,

$$|V_t^B - V_r^B| \leq m_r 2^{-m_r/\delta} \leq \frac{2^{1/\delta}}{\log 2} \log\left(\frac{1}{|t-r|}\right) |t-r|^{1/\delta}.$$

This implies that $h_{V^B}(t) \geq 1/\delta$ and ends the proof. \square

1.3.4 Hausdorff dimension of the sets A_δ^*

Now, we compute the Hausdorff dimension of A_δ^* , which will be used for giving the spectrum of singularities and the proof of Proposition 1.2.1 in the next subsection.

Proposition 1.3.12. *Almost surely, for all $\delta > \nu$,*

$$\dim_H(A_\delta^*) = \nu/\delta \text{ and } \mathcal{H}^{\nu/\delta}(A_\delta^*) = +\infty.$$

To check this proposition, we need the *mass transference principle*, proved in [9], Theorem 2 (applied in dimension $k = 1$ and with the function $f(x) = x^\alpha$).

Lemma 1.3.13. *Let $\alpha \in (0, 1)$ be fixed. Let $\{F_i = [x_i - r_i, x_i + r_i]\}_{i \in \mathbb{N}}$ be a sequence of intervals in \mathbb{R} with radius $r_i \rightarrow 0$ as $i \rightarrow +\infty$. Suppose that*

$$\mathcal{L}(\limsup_{i \rightarrow +\infty} F_i^\alpha \cap [0, 1]) = 1,$$

where $F_i^\alpha := [x_i - r_i^\alpha, x_i + r_i^\alpha]$. Then,

$$\mathcal{H}^\alpha(\limsup_{i \rightarrow +\infty} F_i \cap [0, 1]) = \mathcal{H}^\alpha([0, 1]) = +\infty.$$

Proof of Proposition 1.3.12. Lower Bound. We fix $\delta > \nu$. For all $m \geq 1$, we set

$$N_m := \#\tilde{\mathcal{J}}_m = \#\{s \in \mathcal{J} : 2^{-m-1} < |\Delta V_s^B| \leq 2^{-m}\}.$$

We can write $\tilde{\mathcal{J}}_m = \{T_1^m, \dots, T_{N_m}^m\}$, ordered chronologically. Then we define a sequence $(F_{\delta, j})_{j \geq 1}$ of intervals as follows. For $j \geq 1$, there is a unique $m \geq 1$ and $i \in \{1, 2, \dots, N_m\}$ such that $j = \sum_{k=0}^{m-1} N_k + i$ and write

$$F_{\delta, j} := \left[T_i^m - |\Delta V_{T_i^m}^B|^{\delta(1-\epsilon_m)}, T_i^m + |\Delta V_{T_i^m}^B|^{\delta(1-\epsilon_m)}\right],$$

where ϵ_m is defined in Proposition 1.3.2. By this way, we get a sequence of intervals $(F_{\delta,j})_{j \geq 1}$ of radius tending to 0 and such that, for all $\alpha > 0$, $\limsup_{j \rightarrow +\infty} F_{\delta,j}^\alpha = A_{\alpha\delta}^*$ (this is obvious by definition of A_δ^* , see Remark 1.3.3). Particularly, taking $\alpha = \frac{\nu}{\delta} \in (0, 1)$, we get

$$\limsup_{j \rightarrow +\infty} F_{\delta,j}^{\nu/\delta} = A_\nu^*.$$

Thus by Proposition 1.3.2-(2),

$$\mathcal{L}\left(\limsup_{j \rightarrow +\infty} F_{\delta,j}^{\nu/\delta} \cap [0, 1]\right) = 1.$$

Consequently, by Lemma 1.3.13, we have

$$\mathcal{H}^{\nu/\delta}\left(\limsup_{j \rightarrow +\infty} F_{\delta,j} \cap [0, 1]\right) = +\infty,$$

that is,

$$\mathcal{H}^{\nu/\delta}\left(A_\delta^* \cap [0, 1]\right) = +\infty.$$

Then $\mathcal{H}^{\nu/\delta}(A_\delta^*) = +\infty$ and $\dim_H(A_\delta^*) \geq \nu/\delta$.

Observing that the family of intervals $F_{\delta,j}^{\nu/\delta}$ does not depend on δ , we can clearly apply Lemma 1.3.13 simultaneously for all $\delta > \nu$ and we conclude that a.s., for all $\delta > \nu$, $\mathcal{H}^{\nu/\delta}(A_\delta^*) = +\infty$ and $\dim_H(A_\delta^*) \geq \nu/\delta$.

Upper Bound. Let $\delta > \nu$ be fixed. To get the upper bound for $\dim_H(A_\delta^*)$, we show first that a.s., $\dim_H(A_\delta) \leq \frac{\nu}{\delta}$. For all $m \geq 1$,

$$N_m = \sum_{s \in \mathcal{J}} \mathbf{1}_{\{2^{-m-1} < |\Delta V_s^B| \leq 2^{-m}\}} \leq \sum_{s \in \mathcal{J}} 2^{m+1} |\Delta V_s^B| \mathbf{1}_{\{|\Delta V_s^B| \leq 2^{-m}\}} \leq 2^{m+1} Z_1^{B,m}.$$

This estimate is obtained by using (1.17). Then

$$\mathbb{P}[N_m \geq m2^{m\nu}] \leq \mathbb{P}[Z_1^{B,m} \geq \frac{1}{2}m2^{m(\nu-1)}].$$

Setting $\lambda = 2^{m(1-\nu)}$, we get

$$\mathbb{P}[Z_1^{B,m} \geq \frac{1}{2}m2^{m(\nu-1)}] = \mathbb{P}[\lambda Z_1^{B,m} \geq m/2] \leq e^{-\frac{m}{2}} \mathbb{E}[e^{\lambda Z_1^{B,m}}].$$

Since $\lambda = 2^{m(1-\nu)} \leq 2^m$, we infer from Lemma 1.3.11-(2) that

$$\mathbb{E}[e^{\lambda Z_1^{B,m}}] \leq C_B.$$

Hence we obtain

$$\mathbb{P}[N_m \geq m2^{m\nu}] \leq C_B e^{-m/2}.$$

According to the Borel-Cantelli lemma, we know that, almost surely there exists $M > 0$ such that, for all $m > M$, $N_m < m2^{m\nu}$.

Next, by definition of \tilde{A}_δ^k ,

$$\bigcup_{k \geq m} \tilde{A}_\delta^k \subset \bigcup_{k \geq m} \bigcup_{s \in \tilde{\mathcal{J}}_k} [s - 2^{-k\delta}, s + 2^{-k\delta}],$$

so, recalling Definition 1.5, for all $\alpha > 0$, and all $m > M$, a.s.,

$$\mathcal{H}_{2^{-m\delta+1}}^\alpha \left(\bigcup_{k \geq m} \tilde{A}_\delta^k \right) \leq 2^\alpha \sum_{k \geq m} N_k 2^{-k\delta\alpha} \leq 2^\alpha \sum_{k \geq m} k 2^{k(\nu-\delta\alpha)}.$$

But recalling (1.15), $A_\delta \subset \bigcup_{k \geq m} \tilde{A}_\delta^k$, whence, for all $\alpha > 0$, and all $m > M$, a.s.,

$$\mathcal{H}_{2^{-m\delta+1}}^\alpha(A_\delta) \leq 2^\alpha \sum_{k \geq m} k 2^{k(\nu-\delta\alpha)}.$$

Consequently,

$$\mathcal{H}^\alpha(A_\delta) = \lim_{m \rightarrow +\infty} \mathcal{H}_{2^{-m\delta+1}}^\alpha(A_\delta) \leq 2^\alpha \lim_{m \rightarrow +\infty} \sum_{k \geq m} k 2^{k(\nu-\delta\alpha)}.$$

It follows that $\mathcal{H}^\alpha(A_\delta) = 0$ for all $\alpha > \nu/\delta$. Thus, $\dim_H(A_\delta) \leq \nu/\delta$ by Definition 1.1.5. Since $A_\delta^* \subset A_{\delta'}$ for any $\delta' \in (0, \delta)$, we easily conclude that, a.s.,

$$\dim_H(A_\delta^*) \leq \nu/\delta.$$

We have shown that for all $\delta > \nu$, a.s., $\dim_H(A_\delta^*) \leq \nu/\delta$. Using the a.s. monotonicity of $\delta \mapsto A_\delta^*$, it is not hard to conclude that a.s., for all $\delta > \nu$, $\dim_H(A_\delta^*) \leq \nu/\delta$. \square

1.3.5 Spectrum of singularity of V^B

Using Proposition 1.3.9, we can easily get the following relationship between $E_{V^B}(h)$ and A_δ^* .

Proposition 1.3.14. *Almost surely, for all $h > 0$,*

$$E_{V^B}(h) = \left(\bigcap_{\delta \in (0, 1/h)} A_\delta^* \right) \setminus \left(\bigcup_{\delta > 1/h} A_\delta^* \right).$$

and

$$E_{V^B}(0) = \left(\bigcap_{\delta \in (0, \infty)} A_\delta^* \right).$$

Remark 1.3.15. Due to Remark 1.3.3, Proposition 1.3.14 also holds when replacing everywhere A_δ^* by A_δ .

We now can finally give the

Proof of Proposition 1.2.1. We first deal with the case where $h \in (0, 1/\nu]$. By Propositions 1.3.14 and 1.3.12,

$$D_{VB}(h) = \dim_H \left(E_{VB}(h) \right) \leq \dim_H \left(\bigcap_{\delta \in (0, 1/h)} A_\delta^* \right) \leq \inf_{\delta \in (0, 1/h)} \dim_H(A_\delta^*) = h\nu.$$

On the other hand, we observe that (recall that $\delta \mapsto A_\delta^*$ is decreasing)

$$D_{VB}(h) = \dim_H \left(E_{VB}(h) \right) \geq \dim_H \left(A_{1/h}^* \setminus \left(\bigcup_{\delta > 1/h} A_\delta^* \right) \right).$$

But

$$\mathcal{H}^{h\nu} \left(A_{1/h}^* \setminus \left(\bigcup_{\delta > 1/h} A_\delta^* \right) \right) = \mathcal{H}^{h\nu}(A_{1/h}^*) - \mathcal{H}^{h\nu} \left(\bigcup_{\delta > 1/h} A_\delta^* \right).$$

For all $\delta > 1/h$, $\dim_H(A_\delta^*) = \frac{\nu}{\delta} < h\nu$, thus $\mathcal{H}^{h\nu}(A_\delta^*) = 0$. Moreover, recalling that A_δ^* is decreasing when $\delta > \nu$, hence

$$\mathcal{H}^{h\nu} \left(\bigcup_{\delta > 1/h} A_\delta^* \right) = 0.$$

Next, Proposition 1.3.12 (if $h\nu < 1$) and Proposition 1.3.2 (if $h\nu = 1$) imply that

$$\mathcal{H}^{h\nu}(A_{1/h}^*) > 0.$$

Consequently, $\dim_H \left(A_{1/h}^* \setminus \left(\bigcup_{\delta > 1/h} A_\delta^* \right) \right) \geq h\nu$, whence finally, $D_{VB}(h) \geq h\nu$. We have checked that for $h \in (0, 1/\nu]$, it holds that $D_{VB}(h) = h\nu$.

When $h = 0$, we immediately get, using Proposition 1.3.12, that

$$\dim_H \left(E_{VB}(0) \right) = \dim_H \left(\bigcap_{\delta \in (0, \infty)} A_\delta^* \right) \leq \inf_{\delta \in (0, \infty)} \frac{\nu}{\delta} = 0.$$

Since furthermore $E_{VB}(0) \supset \mathcal{J}$ is a.s. not empty, we conclude that $\dim_H \left(E_{VB}(0) \right) = 0$.

Finally, when $h > \frac{1}{\nu}$, we want to show that $\dim_H \left(E_{VB}(h) \right) = -\infty$, i.e. that $E_{VB}(h) = \emptyset$. This claim immediately follows from Remark 1.3.8 and Proposition 1.3.9, since for all $t \in [0, 1] \setminus \mathcal{J}$, $h_{VB}(t) = \frac{1}{\delta_t} \leq \frac{1}{\nu}$, and for $t \in \mathcal{J}$, $h_{VB}(t) = 0$. \square

1.4 Study of the position process

The goal of this last section is to prove Proposition 1.2.2. We thus only consider the case of hard potentials $\gamma \in (0, 1)$. Since $X_t^B = \int_0^t V_s^B ds$, we obviously have a.s., for all $t \in [0, 1]$,

$$h_{XB}(t) \geq 1 + h_{VB}(t). \quad (1.22)$$

Recall that by Definition, $t \in E_{VB}^{osc}(h)$ if $h_{XB}(t) > 1 + h_{VB}(t)$ and $t \in E_{VB}^{cusp}(h)$ if $h_{XB}(t) = 1 + h_{VB}(t)$. Inspired by the ideas of Balança [5], we will prove several technical lemmas to get Proposition 1.2.2.

1.4.1 Preliminaries

For any $m > 0$ and any interval $[r, t] \subset [0, 1]$, we set

$$H_{[r,t]}^m := \#\{s \in [r, t] \cap \mathcal{J} : |\Delta V_s^B| \geq 2^{-m}\}. \quad (1.23)$$

Lemma 1.4.1. *For any $m \geq 1$ and any interval $[r, t] \subset [0, 1]$,*

(1) *we have*

$$H_{[r,t]}^m \leq R_{[r,t]}^m,$$

where $R_{[r,t]}^m = \int_r^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \mathbf{1}_{\{\theta(B+|v|) \geq 2^{-m}\}} \mathbf{1}_{\{u \leq (B+|v|)^\gamma\}} N(ds, dv, d\theta, d\varphi, du)$;

(2) *and, with $a > 0$ introduced in Lemma 1.3.1 (this actually holds true for any value of $a > 0$),*

$$H_{[r,t]}^m \geq S_{[r,t]}^m,$$

where $S_{[r,t]}^m = \int_r^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \mathbf{1}_{\{|v - H_B(V_{s-})| \geq a\}} \mathbf{1}_{\{\theta \geq 2^{-m+2/a}\}} \mathbf{1}_{\{u \leq a^\gamma\}} N(ds, dv, d\theta, d\varphi, du)$.

Proof. By definition of V^B , see (1.12), we have

$$H_{[r,t]}^m = \int_r^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \mathbf{1}_{\{|\mathbf{a}(H_B(V_{s-}), v, \theta, \varphi)| \geq 2^{-m}\}} \mathbf{1}_{\{u \leq |H_B(V_{s-}) - v|^\gamma\}} N(ds, dv, d\theta, d\varphi, du).$$

Then the claims immediately follow from $\frac{\theta}{4} |H_B(V) - v| \leq |\mathbf{a}(H_B(V), v, \theta, \varphi)| \leq \theta(B + |v|)$, see (1.7), and $|H_B(V) - v|^\gamma \leq (B + |v|)^\gamma$. \square

Remark 1.4.2. *It follows from their definitions that $S_{[r,t]}^m$ and $R_{[r,t]}^m$ are \mathcal{F}_t -measurable, that $R_{[r,t]}^m$ is independent of \mathcal{F}_r and is a Poisson variable with parameter $\lambda_{[r,t]}^m$, where*

$$\lambda_{[r,t]}^m = \int_r^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \mathbf{1}_{\{\theta(B+|v|) \geq 2^{-m}\}} \mathbf{1}_{\{u \leq (B+|v|)^\gamma\}} ds f_s(dv) \beta(\theta) d\theta d\varphi du. \quad (1.24)$$

Using (1.3) and that $m_2(f_s) = m_2(f_0)$ for all $s \in [0, 1]$, one easily checks that there exists a constant $C_B > 0$ such that $\lambda_{[r,t]}^m \leq C_B 2^{m\nu} |t - r|$ for all $m > 0$ and all $0 \leq r \leq t \leq 1$.

Indeed,

$$\begin{aligned}
\lambda_{[r,t]}^m &\leq 2\pi C_0 \int_r^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \mathbf{1}_{\{\theta(B+|v|) \geq 2^{-m}\}} (B+|v|)^\gamma \theta^{-1-\nu} ds f_s(dv) d\theta \\
&\leq C \int_r^t \int_{\mathbb{R}^3} 2^{m\nu} (B+|v|)^{\gamma+\nu} ds f_s(dv) \\
&\leq 2^{m\nu} C \int_r^t \int_{\mathbb{R}^3} (1+B^2+|v|^2) ds f_s(dv) \\
&\leq C_B 2^{m\nu} |t-r|.
\end{aligned}$$

1.4.2 Refined study of the jumps

The goal of this part is to prove the following crucial fact.

Proposition 1.4.3. *Fix $\epsilon > 0$ and set $\alpha = \nu(1 - 2\epsilon)$ and $\beta = \nu(1 + 4\epsilon)$. Almost surely, there exists $M \geq 1$, such that for all $m \geq M$, for all $t \in [0, 1]$, there exists $t_m \in B(t, 2^{-m\alpha})$ such that $|\Delta V_{t_m}^B| \geq 2^{-m}$ and there is no other jump of size greater than $2^{-m(1+\epsilon)}$ in $B(t_m, 2^{-m\beta}/3)$.*

We start with an intermediate result.

Lemma 1.4.4. *Fix $\epsilon > 0$, $\alpha = \nu(1 - 2\epsilon)$ and $\beta = \nu(1 + 4\epsilon)$. For any interval $I = [t_0, t_3] \subset [0, 1]$ with length $2^{-m\beta}$, divide $I = [t_0, t_1] \cup [t_1, t_2] \cup [t_2, t_3]$ into three consecutive intervals with length $2^{-m\beta}/3$. Consider the event*

$$A_I^{m,\epsilon} = \{H_{[t_0,t_1]}^{m(1+\epsilon)} = 0\} \cap \{H_{[t_1,t_2]}^{m(1+\epsilon)} = H_{[t_1,t_2]}^m = 1\} \cap \{H_{[t_2,t_3]}^{m(1+\epsilon)} = 0\}.$$

There exist some constants $c_B > 0$ and $m_\epsilon > 0$ such that, for all $m \geq m_\epsilon$, all intervals $I \subset [0, 1]$ with length $2^{-m\beta}$,

$$\mathbb{P}[A_I^{m,\epsilon} | \mathcal{F}_{t_0}] \geq c_B 2^{-4m\nu\epsilon}. \quad (1.25)$$

Proof. We introduce $A_1 = \{H_{[t_0,t_1]}^{m(1+\epsilon)} = 0\}$, $A_2 = \{H_{[t_1,t_2]}^{m(1+\epsilon)} = H_{[t_1,t_2]}^m = 1\}$ and $A_3 = \{H_{[t_2,t_3]}^{m(1+\epsilon)} = 0\}$, so that $A_I^{m,\epsilon} = A_1 \cap A_2 \cap A_3$.

Step 1. First we write, since $A_1 \cap A_2 \in \mathcal{F}_{t_2}$,

$$\mathbb{P}[A_I^{m,\epsilon} | \mathcal{F}_{t_0}] = \mathbb{E} [\mathbf{1}_{A_1 \cap A_2} \mathbb{P}[A_3 | \mathcal{F}_{t_2}] | \mathcal{F}_{t_0}].$$

But using Lemma 1.4.1 and Remark 1.4.2,

$$\mathbb{P}[A_3 | \mathcal{F}_{t_2}] = \mathbb{P}[H_{[t_2,t_3]}^{m(1+\epsilon)} = 0 | \mathcal{F}_{t_2}] \geq \mathbb{P}[R_{[t_2,t_3]}^{m(1+\epsilon)} = 0 | \mathcal{F}_{t_2}] = \exp(-\lambda_{[t_2,t_3]}^{m(1+\epsilon)}) \geq \frac{1}{2}$$

for all m large enough (depending only on ϵ), since $\lambda_{[t_2, t_3]}^{m(1+\epsilon)} \leq C_B 2^{m\nu} 2^{-m\beta} / 3 \leq C_B 2^{-3m\epsilon} / 3$. Consequently, for all m large enough (depending only on $\epsilon > 0$), we a.s. have

$$\mathbb{P}[A_I^{m, \epsilon} | \mathcal{F}_{t_0}] \geq \frac{1}{2} \mathbb{P}[A_1 \cap A_2 | \mathcal{F}_{t_0}]. \quad (1.26)$$

Step 2. We next write

$$\mathbb{P}[A_1 \cap A_2 | \mathcal{F}_{t_0}] = \mathbb{E} \left[\mathbf{1}_{A_1} \mathbb{P}[A_2 | \mathcal{F}_{t_1}] \middle| \mathcal{F}_{t_0} \right].$$

But using again Lemma 1.4.1,

$$A_2 = \{H_{[t_1, t_2]}^m \geq 1\} \setminus \{H_{[t_1, t_2]}^{m(1+\epsilon)} \geq 2\} \supset \{S_{[t_1, t_2]}^m \geq 1\} \setminus \{R_{[t_1, t_2]}^{m(1+\epsilon)} \geq 2\}.$$

Thus,

$$\mathbb{P}[A_2 | \mathcal{F}_{t_1}] \geq \mathbb{P}[S_{[t_1, t_2]}^m \geq 1 | \mathcal{F}_{t_1}] - \mathbb{P}[R_{[t_1, t_2]}^{m(1+\epsilon)} \geq 2 | \mathcal{F}_{t_1}].$$

First, by Remark 1.4.2,

$$\mathbb{P}[R_{[t_1, t_2]}^{m(1+\epsilon)} \geq 2 | \mathcal{F}_{t_1}] = 1 - \left(1 + \lambda_{[t_1, t_2]}^{m(1+\epsilon)}\right) \exp\left(-\lambda_{[t_1, t_2]}^{m(1+\epsilon)}\right) \leq \left(\lambda_{[t_1, t_2]}^{m(1+\epsilon)}\right)^2 \leq C_B 2^{-6m\nu\epsilon}.$$

Next, we put $Y_t := S_{[t_1, t]}^m$ for $t \geq t_1$ and observe, according to Itô's Formula, that

$$\begin{aligned} \mathbf{1}_{\{Y_t=0\}} &= 1 + \int_{t_1}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \mathbf{1}_{\{|v-H_B(V_{s-})| \geq a\}} \mathbf{1}_{\{u \leq a^\gamma\}} \mathbf{1}_{\{\theta \geq 2^{-m+2}/a\}} \\ &\quad \times \left(\mathbf{1}_{\{Y_{s-} + \Delta Y_s = 0\}} - \mathbf{1}_{\{Y_{s-} = 0\}}\right) N(ds, dv, d\theta, d\varphi, du) \\ &= 1 - \int_{t_1}^t \int_{\mathbb{R}^3} \int_{2^{-m+2}/a}^{\pi/2} \int_0^{2\pi} \int_0^{a^\gamma} \mathbf{1}_{\{|v-H_B(V_{s-})| \geq a\}} \mathbf{1}_{\{Y_{s-} = 0\}} N(ds, dv, d\theta, d\varphi, du). \end{aligned}$$

Hence, for all $t \geq t_1$,

$$\begin{aligned} &\frac{d}{dt} \mathbb{E}[\mathbf{1}_{\{Y_t=0\}} | \mathcal{F}_{t_1}] \\ &= -\mathbb{E} \left[\int_{\mathbb{R}^3} \int_{2^{-m+2}/a}^{\pi/2} \int_0^{2\pi} \int_0^{a^\gamma} \mathbf{1}_{\{|v-H_B(V_t)| \geq a\}} \mathbf{1}_{\{Y_t=0\}} f_t(dv) \beta(\theta) d\theta d\varphi du \middle| \mathcal{F}_{t_1} \right]. \end{aligned}$$

Using (1.3) and Lemma 1.3.1 (which implies that $f_s(\{v \in \mathbb{R}^3 : |v - H_B(V_s)| \geq a\}) \geq b > 0$ a.s. for all $s \in [0, 1]$), we easily deduce that

$$\frac{d}{dt} \mathbb{E}[\mathbf{1}_{\{Y_t=0\}} | \mathcal{F}_{t_1}] \leq -\kappa 2^{m\nu} \mathbb{E}[\mathbf{1}_{\{Y_t=0\}} | \mathcal{F}_{t_1}],$$

for some positive constant κ . Integrating this inequality, we deduce that a.s., for all $t \geq t_1$,

$$\mathbb{E}(\mathbf{1}_{\{Y_t=0\}} | \mathcal{F}_{t_1}) \leq \exp\{-\kappa 2^{m\nu}(t - t_1)\}.$$

Consequently,

$$\begin{aligned} \mathbb{P}[S_{[t_1, t_2]}^m \geq 1 | \mathcal{F}_{t_1}] &= 1 - \mathbb{E}(\mathbf{1}_{\{Y_{t_2}=0\}} | \mathcal{F}_{t_1}) \\ &\geq 1 - \exp\{-\kappa 2^{m\nu}(t_2 - t_1)\} = 1 - \exp\{-\kappa 2^{-4m\nu\epsilon}/3\}. \end{aligned}$$

Finally, for all m large enough (depending only on ϵ), we a.s. have

$$\mathbb{P}[A_2 | \mathcal{F}_{t_1}] \geq 1 - \exp\{-\kappa 2^{-4m\nu\epsilon}/3\} - C_B 2^{-6m\nu\epsilon} \geq c_B 2^{-4m\nu\epsilon}.$$

Step 3. Finally, exactly as Step 1, we obtain that for all m large enough,

$$\mathbb{P}[A_1 | \mathcal{F}_{t_0}] \geq \frac{1}{2}.$$

Step 4. It suffices to gather Steps 1, 2 and 3 to conclude the proof. \square

Proof of Proposition 1.4.3. We thus fix $\epsilon > 0$ and consider α and β as in the statement. For $m > 0$, we introduce the notation $r_m = 2^{-m\beta}/3$. We also introduce the number $q_m^2 := \lfloor 2^{m(\beta-\alpha)} \rfloor$, the length $\ell_m := q_m^2 2^{-m\beta}$ (we have $\ell_m \leq 2^{-m\alpha}$ and $\ell_m \simeq 2^{-m\alpha}$) and the number $q_m^1 := \lfloor 1/\ell_m \rfloor + 1$ (we have $q_m^1 \simeq 2^{m\alpha}$). We consider a covering of $[0, 1]$ by q_m^1 consecutive intervals $I_1^m, \dots, I_{q_m^1}^m$ with length ℓ_m . Next, we divide each I_i^m into q_m^2 consecutive intervals $I_{i,1}^m, \dots, I_{i,q_m^2}^m$ with length $2^{-m\beta}$. Finally, we divide each $I_{i,j}^m$ into three consecutive intervals with length r_m , writing $I_{i,j}^m = [t_{i,j}^m, t_{i,j}^m + r_m) \cup [t_{i,j}^m + r_m, t_{i,j}^m + 2r_m) \cup [t_{i,j}^m + 2r_m, t_{i,j+1}^m)$. We consider the event

$$\begin{aligned} A_{i,j}^m &= \{H_{[t_{i,j}^m, t_{i,j}^m + r_m)}^{m(1+\epsilon)} = 0\} \cap \{H_{[t_{i,j}^m + r_m, t_{i,j}^m + 2r_m)}^{m(1+\epsilon)} = H_{[t_{i,j}^m + r_m, t_{i,j}^m + 2r_m)}^m = 1\} \\ &\quad \cap \{H_{[t_{i,j}^m + 2r_m, t_{i,j+1}^m)}^{m(1+\epsilon)} = 0\}. \end{aligned}$$

According to Lemma 1.4.4, we know that if m is large enough (depending only on ϵ), a.s., for all i, j

$$\mathbb{P}[A_{i,j}^m | \mathcal{F}_{t_{i,j}^m}] \geq c_B 2^{-4m\nu\epsilon}. \quad (1.27)$$

We now consider, for each i , the event

$$K_{m,i} = \bigcap_{j=1}^{q_m^2} (A_{i,j}^m)^c.$$

Then, we easily deduce from (1.27), together with the fact that $A_{i,1}^m, \dots, A_{i,j-1}^m \in \mathcal{F}_{t_{i,j}^m}$ for all $j = 1, \dots, q_m^2 - 1$, that

$$\mathbb{P}(K_{m,i}) \leq (1 - c_B 2^{-4m\nu\epsilon})^{q_m^2} \leq (1 - c_B 2^{-4m\nu\epsilon})^{2^{m(\beta-\alpha)} - 1}.$$

Thus for m large enough (depending only on ϵ), we conclude that

$$\mathbb{P}(K_{m,i}) \leq \exp(-c_B 2^{-4m\nu\epsilon} 2^{m(\beta-\alpha)}) = \exp(-c_B 2^{2m\nu\epsilon}).$$

Next, we introduce the event $K_m = \bigcup_{i=1}^{q_m^1} K_{m,i}$. Clearly, for m large enough, (allowing the value of the constant $c_B > 0$ to change)

$$\mathbb{P}(K_m) \leq q_m^1 \exp(-c_B 2^{2m\nu\epsilon}) \leq \exp(-c_B 2^{2m\nu\epsilon}).$$

Finally, using the Borel-Cantelli lemma, we conclude that there a.s. exists $M > 0$ such that for all $m \geq M$, the event K_m^c is realized (whence for all $i = 1, \dots, q_m^1$, there is $j \in \{1, \dots, q_m^2\}$ such that $A_{i,j}^m$ is realized). This implies that a.s., for all $m \geq M$, for all $t \in [0, 1]$, considering $i \in \{1, \dots, q_m^1\}$ such that $t \in I_i^m$ and $j \in \{1, \dots, q_m^2\}$ such that $A_{i,j}^m$ is realized, V^B has exactly one jump greater than $2^{-m(1+\epsilon)}$ in the time interval $I_{i,j}^m$, this jump is greater than 2^{-m} and happens at some time t_m located in the middle of $I_{i,j}^m$ (more precisely, the distance between t_m and the extremities of $I_{i,j}^m$ is at least r_m). We clearly have $|t_m - t| \leq \ell_m \leq 2^{-m\alpha}$, $|\Delta V_{t_m}^B| \geq 2^{-m}$, and V^B has no other jump of size greater than $2^{-m(1+\epsilon)}$ in $B(t_m, r_m) \subset I_{i,j}^m$. The proof is complete. \square

1.4.3 Uniform bound for the Hölder exponent of X^B

We show here that $D_{X^B}(h) = -\infty$ for all $h > 1 + 1/\nu$. We use a general result for primitives of discontinuous functions. It based on Proposition 1 in [4], recalled in the following lemma.

Lemma 1.4.5. *Let $\eta > 0$ and let $N > \eta$ be an integer. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function and let ψ be a C^∞ compactly supported function with its N first moments vanishing, i.e. $\int_{\mathbb{R}} x^k \psi(x) dx = 0$ for $k = 0, \dots, N - 1$. The wavelet transform of g is defined by*

$$W_\psi(g, a, b) = \frac{1}{a} \int_{\mathbb{R}} g(t) \psi\left(\frac{t-b}{a}\right) dt. \quad (1.28)$$

If $g \in C^\eta(t_0)$, then there exists a constant $C > 0$ such that for all $a > 0$, all $b \in [t_0 - 1, t_0 + 1]$,

$$|W_\psi(g, a, b)| \leq C (a^\eta + |t_0 - b|^\eta). \quad (1.29)$$

Now, we give the following general result. For any function $g : \mathbb{R} \rightarrow \mathbb{R}$, and any interval $I \subset \mathbb{R}$, we set

$$\text{osc}_I(g) = \sup_{x \in I} g(x) - \inf_{x \in I} g(x).$$

Lemma 1.4.6. *Let $g : [0, \infty) \rightarrow \mathbb{R}$ be a càdlàg function, discontinuous on a dense set of points, let $G(t) = \int_0^t g(s)ds$. Let $t > 0$ and let $(t_m)_{m \geq 1}$ be a sequence of discontinuities of the function g converging to t . For all $s \in \mathbb{R}$, all $m \geq 1$, we define*

$$g_m(s) = g(s) - J_m \mathbf{1}_{\{s \geq t_m\}}, \quad (1.30)$$

where $J_m = g(t_m+) - g(t_m-)$. Assume that for all $m \geq 1$, there exist $r_m > 0$ and $\delta_m > 0$ such that

$$\text{osc}_{[t_m-r_m, t_m+r_m]}(g_m) \leq \delta_m \text{ and } \lim_{m \rightarrow +\infty} \frac{\delta_m}{|J_m|} = 0. \quad (1.31)$$

Then

$$h_G(t) \leq \liminf_{m \rightarrow +\infty} \frac{\log(r_m |J_m|)}{\log(|t_m - t| + r_m)}. \quad (1.32)$$

Proof. Let φ be a positive C^∞ function, supported on $[0, 1]$ satisfying $\int_{\mathbb{R}} \varphi(x)dx = 1$.

For $k \geq 1$, let $\psi_k(t) = \varphi^{(k)}(t)$, it is clear that ψ_k is C^∞ , supported on $[0, 1]$ and that its k first moments vanish, so it is a *wavelet*.

We now pick an integer N such that $N-2$ is larger than the right hand side of (1.32), and we denote by $c_N(a, b) := W_{\psi_N}(g, a, b)$ and $C_{N+1}(a, b) := W_{\psi_{N+1}}(G, a, b)$ the wavelet transforms of g and G using the wavelet ψ_N and ψ_{N+1} , respectively. An integration by parts shows that

$$c_N(a, b) = -\frac{1}{a} C_{N+1}(a, b). \quad (1.33)$$

We fix $\theta \in (0, 1)$ such that $\psi_{N-1}(\theta) > 0$. It follows from (1.30) that $c_N(r_m, t_m - \theta r_m) = P_m + Q_m$, where

$$\begin{aligned} P_m &= \frac{1}{r_m} \int_{-\infty}^{+\infty} J_m \mathbf{1}_{\{s \geq t_m\}} \psi_N \left(\frac{s - t_m + \theta r_m}{r_m} \right) ds \\ &= \frac{J_m}{r_m} \int_{t_m}^{+\infty} \psi_N \left(\frac{s - t_m + \theta r_m}{r_m} \right) ds = -J_m \psi_{N-1}(\theta) \end{aligned}$$

and

$$\begin{aligned} Q_m &= \frac{1}{r_m} \int_{-\infty}^{+\infty} g_m(s) \psi_N \left(\frac{s - t_m + \theta r_m}{r_m} \right) ds \\ &= \frac{1}{r_m} \int_{-\infty}^{+\infty} (g_m(s) - g_m(t_m)) \psi_N \left(\frac{s - t_m + \theta r_m}{r_m} \right) ds, \end{aligned}$$

where we used that ψ_N has a vanishing integral. Observing that

$$\text{supp} \left(\psi_N \left(\frac{\cdot - t_m + \theta r_m}{r_m} \right) \right) \subset [t_m - r_m, t_m + r_m]$$

and recalling (1.31), we deduce that $|Q_m| \leq 2\|\psi_N\|_\infty \delta_m$. As a conclusion,

$$|c_N(r_m, t_m - \theta r_m)| \geq |P_m| - |Q_m| \geq \psi_{N-1}(\theta) |J_m| - 2\|\psi_N\|_\infty \delta_m \geq c |J_m|$$

for all m large enough, since $\lim_{m \rightarrow +\infty} \frac{\delta_m}{|J_m|} = 0$ by assumption. Then we obtain according to (1.33),

$$|C_{N+1}(r_m, t_m - \theta r_m)| \geq cr_m |J_m|. \quad (1.34)$$

Assume that $G \in C^\eta(t)$ for some $\eta > \liminf_{m \rightarrow +\infty} [\log(r_m |J_m|) / [\log(|t_m - t| + r_m)]]$. We apply Lemma 1.4.5 with $g = G$, $\psi = \psi_{N+1}$, $a = r_m$, $b = t_m - \theta r_m$. Hence, there is a constant C such that for all m ,

$$|C_{N+1}(r_m, t_m - \theta r_m)| \leq C (r_m^\eta + |t - t_m + \theta r_m|^\eta) \leq C (r_m + |t - t_m|)^\eta.$$

This contradicts (1.34), so necessarily (1.32) hold true. \square

We next apply this lemma to our position process to get a uniform upper bound for all pointwise Hölder exponents of X^B .

Proposition 1.4.7. *Almost surely, for all $t \in [0, 1]$, the Hölder exponent of X^B satisfies*

$$h_{X^B}(t) \leq 1 + \frac{1}{\nu}. \quad (1.35)$$

Proof. We fix $\epsilon > 0$ and set $\alpha = \nu(1 - 2\epsilon)$ and $\beta = \nu(1 + 4\epsilon)$. We show that a.s., $h_{X^B}(t) \leq (1 + \beta)/\alpha$ for all $t \in [0, 1]$. This clearly suffices since $\epsilon > 0$ can be chosen arbitrarily small.

Proposition 1.4.3 shows that there a.s. exists $M > 0$, such that for all $m \geq M$, for all $t \in [0, 1]$, there exists $t_m \in B(t, 2^{-m\alpha})$ such that $|\Delta V_{t_m}^B| \geq 2^{-m}$ and such that there is no other jump of size greater than $2^{-m(1+\epsilon)}$ in $B(t_m, r_m)$, with $r_m := 2^{-m\beta}/3$.

We now fix $t \in [0, 1]$. Up to extraction, one can assume that the first coordinate \tilde{V}_s^B of the three-dimensional vector V_s^B satisfies $|\Delta \tilde{V}_{t_m}^B| \geq 2^{-m}/3$. We now apply Lemma 1.4.6 with $g = \tilde{V}^B$ and $r_m = 2^{-m\beta}/3$. We thus introduce $g_m(s) = g(s) - \Delta \tilde{V}_{t_m}^B \mathbf{1}_{\{s \geq t_m\}}$. Since V^B (and so \tilde{V}_s^B) has no jump with size greater than $2^{-m(1+\epsilon)}$ within the interval $B(t_n, r_n) = (t_m - r_m, t_m + r_m)$, we observe that

$$\text{osc}_{B(t_n, r_n)}(g_m) \leq 2 \times \sup_{x, y \in [0, 1], |x-y| \leq 2^{-m\beta}} |V_x^{B, m(1+\epsilon)} - V_y^{B, m(1+\epsilon)}|.$$

Next, using Lemma 1.3.11-(1) (with $\delta = \beta/(1 + \epsilon) > \nu$) and the Borel-Cantelli Lemma, we deduce that there is a.s. $M' > 0$ such that, for all $m \geq M'$, all $0 < x < y < 1$ with $|x - y| < 2^{-m\beta}$, $|V_x^{B,m(1+\epsilon)} - V_y^{B,m(1+\epsilon)}| \leq m\beta 2^{-m(1+\epsilon)}$. That is,

$$\text{osc}_{B(t_n, r_n)}(g_m) \leq 2m\beta 2^{-m(1+\epsilon)}.$$

Since furthermore $\lim_{m \rightarrow +\infty} \frac{2m\beta 2^{-m(1+\epsilon)}}{|\Delta \tilde{V}_{t_m}^B|} \leq \lim_{m \rightarrow +\infty} \frac{2m\beta 2^{-m(1+\epsilon)}}{2^{-m}/3} = 0$, we can apply Lemma 1.4.6 with $\delta_m = 2m\beta 2^{-m(1+\epsilon)}$:

$$h_{X^B}(t) \leq \liminf_{m \rightarrow +\infty} \frac{\log(r_m |\Delta \tilde{V}_{t_m}^B|)}{\log(|t_m - t| + r_m)} \leq \liminf_{m \rightarrow +\infty} \frac{\log(2^{-m(1+\beta)}/9)}{\log(2 \cdot 2^{-m\alpha})} = \frac{1 + \beta}{\alpha}.$$

We used that $r_m |\Delta \tilde{V}_{t_m}^B| \geq (2^{-m}/3)(2^{-m\beta}/3)$ and that $|t_m - t| + r_m \leq 2^{-m\alpha} + 2^{-m\beta}/3 \leq 2 \cdot 2^{-m\alpha}$. This ends the proof. \square

1.4.4 Study of the oscillating singularities of X^B

To characterize more precisely the set of oscillating times, we first give the following lemma.

Lemma 1.4.8. *Let $\delta > \nu$, $\epsilon > 0$ and $k \in \mathbb{N}$ satisfy $\delta > \nu(1 + \epsilon)(k + 1)/k$. For all $m \in \mathbb{N}$, let $(I_j^m)_{j=1, \dots, \lfloor 2^{m\delta} \rfloor + 1}$ be the covering of $[0, 1]$ composed of successive intervals of length $2^{-m\delta}$. Almost surely, there exists $M \geq 1$ such that for all $m \geq M$, for all $j = 1, \dots, \lfloor 2^{m\delta} \rfloor$, recalling (1.23),*

$$H_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)} \leq k, \quad (1.36)$$

Proof. Using Lemma 1.4.1 and Remark 1.4.2,

$$\mathbb{P}\left(H_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)} > k\right) \leq \mathbb{P}\left(R_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)} > k\right) \leq \sum_{\ell=k+1}^{+\infty} \frac{(\lambda_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)})^\ell}{\ell!} e^{-\lambda_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)}} \leq (\lambda_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)})^{k+1},$$

where the value of $\lambda_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)}$ is given by equation (1.24). But, since the length of $I_j^m \cup I_{j+1}^m$ is $2 \cdot 2^{-m\delta}$, we apply the upper bound found for $\lambda_{[r,s]}^m$ in Remark 1.4.2 in order to get $\lambda_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)} \leq 2C_B 2^{m\nu(1+\epsilon)-m\delta}$, so that

$$\mathbb{P}\left(H_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)} > k\right) \leq 2C_B 2^{1+m(k+1)(\nu(1+\epsilon)-\delta)}.$$

Consequently,

$$\mathbb{P}\left(\bigcup_{j=1}^{\lfloor 2^{m\delta} \rfloor + 1} \left\{H_{I_j^m \cup I_{j+1}^m}^{m(1+\epsilon)} > k\right\}\right) \leq 2C_B 2^{m\delta} 2^{m(k+1)(\nu(1+\epsilon)-\delta)} = 2C_B 2^{-mk(\delta-\nu(1+\epsilon)(k+1)/k)}.$$

By assumption, this is the general term of a convergent series. We conclude thanks to the Borel-Cantelli lemma. \square

We first study the case where $h \in [0, 1/(2\nu))$.

Lemma 1.4.9. *Almost surely, for all $h \in [0, 1/(2\nu))$, $E_{V^B}^{osc}(h) = \emptyset$.*

Proof. According to (1.22), it is sufficient to check that for $h \in [0, 1/(2\nu)]$, for all $t \in E_{V^B}(h)$, $h_{XB}(t) \leq 1 + h$. We fix $\epsilon > 0$ so small that there exists $\delta \in (\max\{2\nu(1 + \epsilon), 1/(h + \epsilon)\}, 1/h)$. Next, we fix $t \in E_{V^B}(h)$. By Remark 1.3.15, we know that $t \in A_{1/(h+\epsilon)}$. Hence for all $n \geq 1$, we can find $m_n \geq n$ and $t_n \in \tilde{\mathcal{J}}_{m_n}$ (that is $|\Delta V_{t_n}^B| \in (2^{-m_n-1}, 2^{-m_n})$) such that $|t_n - t| \leq |\Delta V_{t_n}^B|^{1/(h+\epsilon)} \leq 2^{-m_n/(h+\epsilon)}$. Applying Lemma 1.4.8 with $k = 1$ (since $\delta > 2\nu(1 + \epsilon)$), we deduce that V^B has no other jump of size greater than $2^{-m_n(1+\epsilon)}$ in $B(t_n, 2^{-m_n\delta})$.

As we did before, up to extraction, we can e.g. assume that the first coordinate \tilde{V}^B of V^B satisfies $|\Delta \tilde{V}_{t_n}^B| \geq 2^{-m_n}/3$ for all $n \geq 1$.

We then apply Lemma 1.4.6 with $g(s) = \tilde{V}_s^B$ and $g_n(s) = g(s) - \Delta \tilde{V}_{t_n}^B \mathbf{1}_{\{s \geq t_n\}}$, with the choices $r_n = 2^{-m_n\delta}$ and $\delta_n = m_n \delta 2^{-m_n(1+\epsilon)}$. It indeed holds true that $\lim_{n \rightarrow +\infty} \delta_n / |\Delta \tilde{V}_{t_n}^B| = 0$ and, thanks to Lemma 1.3.11-(1) (which is licit because $\delta/(1 + \epsilon) > \nu$) and the Borel-Cantelli Lemma, we deduce that a.s., for all n sufficiently large,

$$\text{osc}_{B(t_n, r_n)}(\tilde{V}_s^B) \leq \sup_{x, y \in [0, 1], |x-y| \leq 2^{-m_n\delta}} |V_x^{B, m_n(1+\epsilon)} - V_y^{B, m_n(1+\epsilon)}| \leq m_n \delta 2^{-m_n(1+\epsilon)}.$$

We conclude from Lemma 1.4.6 that

$$h_{XB}(t) \leq \liminf_n \frac{\log(r_n |\Delta \tilde{V}_{t_n}^B|)}{\log(|t_n - t| + r_n)} \leq \liminf_n \frac{\log(2^{-m_n(1+\delta)}/3)}{\log(2 \cdot 2^{-m_n/(h+\epsilon)})} = (1 + \delta)(h + \epsilon).$$

We used that $r_n |\Delta \tilde{V}_{t_n}^B| \geq (2^{-m_n}/3) 2^{-m_n\delta}$ while $|t_n - t| + r_n \leq 2^{-m_n/(h+\epsilon)} + 2^{-m_n\delta} \leq 2 \cdot 2^{-m_n/(h+\epsilon)}$. Letting $\epsilon \rightarrow 0$ (whence $\delta \rightarrow 1/h$), we conclude that $h_{XB}(t) \leq 1 + h$ as desired. \square

Before computing the dimension of $E_{V^B}^{osc}(h)$ when $h \in [1/(2\nu), 1/\nu)$, we need to count those jump times that are very close to each other.

Lemma 1.4.10. *For $\epsilon > 0$ and $m > 0$, denote by $0 < T_1^{\epsilon, m} < \dots < T_{K_{\epsilon, m}}^{\epsilon, m} < 1$ the successive instants of jumps of V^B with size greater than $2^{-m(1+\epsilon)}$. For $\delta > 0$, we introduce*

$$N_m^{\delta, \epsilon} = \sum_{i=1}^{K_{\epsilon, m}} \mathbf{1}_{\{T_i^{\epsilon, m} - T_{i-1}^{\epsilon, m} \leq 2^{-m\delta}\}}$$

with the convention that $T_0^{\epsilon,m} = 0$. For any fixed $\epsilon > 0$ and $\delta > 0$, there a.s. exists $M > 0$ such that for all $m > M$,

$$N_m^{\delta,\epsilon} \leq 2^{-m\delta+2m\nu(1+2\epsilon)}.$$

Proof. Recalling Lemma 1.4.1, we see that $\{T_1^{\epsilon,m}, \dots, T_{K_{\epsilon,m}}^{\epsilon,m}\} \subset \{S_1^{\epsilon,m}, \dots, S_{L_{\epsilon,m}}^{\epsilon,m}\}$, where $0 < S_1^{\epsilon,m} < \dots < S_{L_{\epsilon,m}}^{\epsilon,m}$ are the successive instants of jump of the counting process $R_{[0,t]}^{m(1+\epsilon)}$. Consequently,

$$N_m^{\delta,\epsilon} \leq \tilde{N}_m^{\delta,\epsilon} := \sum_{i=1}^{L_{\epsilon,m}} \mathbf{1}_{\{S_i^{\epsilon,m} - S_{i-1}^{\epsilon,m} \leq 2^{-m\delta}\}}.$$

By Remark 1.4.2, we know that $R_{[0,t]}^{m(1+\epsilon)}$ is an inhomogeneous Poisson process with intensity bounded by $C_B 2^{m(1+\epsilon)\nu}$. Consequently,

$$\mathbb{P}\left[L_{\epsilon,m} \geq 2^{m\nu(1+2\epsilon)}\right] \leq 2^{-m\nu(1+2\epsilon)} C_B 2^{m(1+\epsilon)\nu} \leq C_B 2^{-m\nu\epsilon}.$$

Hence, applying the Borel-Cantelli lemma, we know that almost surely, there exists $M' \geq 1$ such that for all $m \geq M'$,

$$L_{\epsilon,m} \leq 2^{m\nu(1+2\epsilon)} \quad \text{and thus} \quad N_m^{\delta,\epsilon} \leq \sum_{i=1}^{2^{m\nu(1+2\epsilon)}} \mathbf{1}_{\{S_i^{\epsilon,m} - S_{i-1}^{\epsilon,m} \leq 2^{-m\delta}\}}.$$

But for all $i \geq 1$, $S_i^{\epsilon,m} - S_{i-1}^{\epsilon,m}$ is bounded from above by an exponential random variable with parameter $C_B 2^{m(1+\epsilon)\nu}$, so that $\mathbb{P}(S_i^{\epsilon,m} - S_{i-1}^{\epsilon,m} \leq 2^{-m\delta}) \leq 1 - \exp(-C_B 2^{m(1+\epsilon)\nu} 2^{-m\delta}) \leq C_B 2^{m(1+\epsilon)\nu - m\delta}$ and thus

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{2^{m\nu(1+2\epsilon)}} \mathbf{1}_{\{S_i^{\epsilon,m} - S_{i-1}^{\epsilon,m} \leq 2^{-m\delta}\}} \geq 2^{-m\delta+2m\nu(1+2\epsilon)}\right) &\leq 2^{m\delta-2m\nu(1+2\epsilon)} 2^{m\nu(1+2\epsilon)} C_B 2^{m(1+\epsilon)\nu - m\delta} \\ &= C_B 2^{-m\nu\epsilon}. \end{aligned}$$

By the Borel-Cantelli lemma again, there exists a.s. a constant $M'' > 0$ such that for all $m \geq M''$,

$$\sum_{i=1}^{2^{m\nu(1+2\epsilon)}} \mathbf{1}_{\{S_i^{\epsilon,m} - S_{i-1}^{\epsilon,m} \leq 2^{-m\delta}\}} \leq 2^{-m\delta+2m\nu(1+2\epsilon)}.$$

As a conclusion, a.s. we have $N_m^{\delta,\epsilon} \leq 2^{-m\delta+2m\nu(1+2\epsilon)}$ for all $m \geq M' \vee M''$. Choosing $M = M' \vee M''$ completes the proof. \square

Now we treat the case where $h \in [1/(2\nu), 1/\nu)$.

Proposition 1.4.11. *Almost surely, for $h \in [1/(2\nu), 1/\nu)$, $\dim_H(E_{\nu B}^{\text{osc}}(h)) \leq 2h\nu - 1$.*

Proof. We divide the proof into several steps.

Step 1. For any fixed $\epsilon > 0$, $\delta \in (\nu, 2\nu]$ and $m \geq 1$, we consider the sets

$$F_m(\delta, \epsilon) = \bigcup_{\{i: T_i^{\epsilon, m} - T_{i-1}^{\epsilon, m} \leq 2^{-m\delta}\}} ([T_{i-1}^{\epsilon, m} - 2^{-m\delta}, T_{i-1}^{\epsilon, m} + 2^{-m\delta}] \cup [T_i^{\epsilon, m} - 2^{-m\delta}, T_i^{\epsilon, m} + 2^{-m\delta}]),$$

where the family $T_i^{\epsilon, m}$ has been introduced in Lemma 1.4.10, and the associated limsup set

$$G(\delta, \epsilon) = \limsup_{m \rightarrow +\infty} F_m(\delta, \epsilon).$$

For every $n \geq 1$, $\bigcup_{m \geq n} F_m(\delta, \epsilon)$ forms a covering of $G(\delta, \epsilon)$ by sets of diameter less than $2^{-n\delta+2}$, and Lemma 1.4.10 allows to bound by above the cardinality of such sets. Hence, choosing $s > \frac{2\nu(1+2\epsilon)}{\delta} - 1$, a.s. for every n large enough one has

$$\mathcal{H}_{2^{-n\delta+2}}^s(G(\delta, \epsilon)) \leq \sum_{m \geq n} 2^{-m\delta s + 2s} N_m^{\delta, \epsilon} \leq \sum_{m \geq n} 2^{2s} 2^{-m(s+1)\delta + 2m\nu(1+2\epsilon)}.$$

We deduce that $\lim_{n \rightarrow +\infty} \mathcal{H}_{2^{-n\delta+2}}^s(G(\delta, \epsilon)) = 0$, hence $\mathcal{H}^s(G(\delta, \epsilon)) = 0$. Therefore,

$$\dim_H(G(\delta, \epsilon)) \leq \frac{2\nu(1+2\epsilon)}{\delta} - 1.$$

Step 2. Here we fix $h \in [1/(2\nu), 1/\nu]$, we consider $\epsilon > 0$ such that $1/[(h+\epsilon)(1+\epsilon)] > \nu$, we set $\delta_\epsilon = 1/(h+\epsilon)$ and we prove that $E_{V^B}^{\text{osc}}(h) \subset G(\delta_\epsilon, \epsilon)$.

We consider $t \in E_{V^B}(h) \setminus G(\delta_\epsilon, \epsilon)$ and we show that $h_{X^B}(t) = 1+h$, which will imply indeed that $t \in E_{V^B}^{\text{cusp}}(h)$.

Since $t \notin G(\delta_\epsilon, \epsilon)$, there exists $N \geq 1$ such that for all $m \geq N$, $t \notin F_m(\delta_\epsilon, \epsilon)$. Moreover, for any $0 < \eta \leq \epsilon$, since $t \in E_{V^B}(h)$, by Remark 1.3.15, we know that $t \in A_{\delta_\eta}$ (because $\delta_\eta = 1/(h+\eta) < 1/h$), so that for all $n \geq 1$, there exist $m_n \geq n$ and $t_n \in B(t, 2^{-m_n\delta_\eta})$ such that $|\Delta V_{t_n}^B| \geq 2^{-m_n}$. Observing that $F_{m_n}(\delta_\eta, \eta) \subset F_{m_n}(\delta_\epsilon, \epsilon)$ since $0 < \eta \leq \epsilon$ and $\delta_\eta \geq \delta_\epsilon$. Hence $t \notin F_{m_n}(\delta_\eta, \eta)$ (for all n large enough), whence, there is also no other jump in $B(t, 2^{-m_n\delta_\eta})$ with size greater than $2^{-m_n(1+\eta)}$.

As in the previous proofs, up to extraction, we deduce that $|\Delta \tilde{V}_{t_n}^B| \geq 2^{-m_n}/3$ for all n , where \tilde{V}^B is one of the three coordinates of V^B . Since V^B (and so \tilde{V}^B) has no jump with size greater than $2^{-m_n(1+\eta)}$ in $B(t_n, 2^{-m_n\delta_\eta})$, we may use Lemma 1.3.11-(1) (because $\delta_\eta/(1+\eta) = \frac{1}{(h+\eta)(1+\eta)} \geq \frac{1}{(h+\epsilon)(1+\epsilon)} > \nu$) and the Borel-Cantelli Lemma, we deduce that a.s. for all n sufficiently large, setting $r_n = 2^{-m_n\delta_\eta}$,

$$\text{osc}_{B(t_n, r_n)}(\tilde{V}^B) \leq 2 \times \sup_{x, y \in [0, 1], |x-y| \leq 2^{-m_n\delta_\eta}} |V_x^{B, m_n(1+\eta)} - V_y^{B, m_n(1+\eta)}| \leq 2m_n\delta_\eta 2^{-m_n(1+\eta)}.$$

Moreover,

$$\lim_{n \rightarrow +\infty} \frac{2m_n \delta_\eta 2^{-m_n(1+\eta)}}{|\Delta \tilde{V}_{t_n}^B|} \leq \lim_{n \rightarrow +\infty} \frac{2m_n \delta_\eta 2^{-m_n(1+\eta)}}{2^{-m_n}/3} = 0.$$

Applying Lemma 1.4.6 with $g = \tilde{V}^B$, $r_n = 2^{-m_n \delta_\eta}$ and $\delta_n = 2m_n \delta_\eta 2^{-m_n(1+\eta)}$, we obtain

$$h_{X^B}(t) \leq \liminf_{n \rightarrow +\infty} \frac{\log(r_n |\Delta \tilde{V}_{t_n}^B|)}{\log(r_n + |t_n - t|)} \leq \liminf_{n \rightarrow +\infty} \frac{\log(2^{-m_n(1+\delta_\eta)}/3)}{\log(2 \cdot 2^{-m_n \delta_\eta})} = \frac{1 + \delta_\eta}{\delta_\eta} = 1 + h + \eta \quad (1.37)$$

because $r_n |\Delta \tilde{V}_{t_n}^B| \geq 2^{-m_n(1+\delta_\eta)}/3$ and $r_n + |t_n - t| \leq 2 \cdot 2^{-m_n \delta_\eta}$. Since (1.37) is satisfied for any $0 < \eta \leq \epsilon$, then a.s. $h_{X^B}(t) \leq 1 + h$. That is, $E_{V^B}^{osc}(h) \subset G(\delta_\epsilon, \epsilon)$.

Step 3. From step 2 we deduce that $E_{V^B}^{osc}(h) \subset \bigcap_{\epsilon \downarrow 0} G(\delta_\epsilon, \epsilon)$. Hence,

$$\dim_H(E_{V^B}^{osc}(h)) \leq \dim_H\left(\bigcap_{\epsilon \downarrow 0} G(\delta_\epsilon, \epsilon)\right) = \inf_{\epsilon \downarrow 0} (2\nu(1 + 2\epsilon)(h + \epsilon) - 1) = 2h\nu - 1.$$

This ends the proof. \square

1.4.5 Conclusion

Proof of Proposition 1.2.2. First, we now from Proposition 1.2.1 that $E_{V^B}(h) = \emptyset$ for $h > 1/\nu$, so that obviously $E_{V^B}^{osc}(h) = \emptyset$. If now $h = 1/\nu$, then we deduce from Proposition 1.4.7 that $E_{V^B}^{osc}(h) = \emptyset$, simply because a.s., for all $t \in [0, 1]$, $h_{X^B}(t) \leq 1 + 1/\nu$.

As shown in Lemma 1.4.9, we also know that $E_{V^B}^{osc}(h) = \emptyset$ for all $h \in [0, 1/(2\nu))$ and as seen in Proposition 1.4.11, $\dim_H(E_{V^B}^{osc}(h)) \leq 2h\nu - 1$ for all $h \in [1/(2\nu), 1/\nu)$.

It remains to verify that for all $h \in [0, 1/\nu]$, $\dim_H(E_{V^B}^{cusp}(h)) = h\nu$. If $h \in [0, 1/(2\nu))$ or $h = 1/\nu$, it is obvious because $E_{V^B}^{osc}(h) = \emptyset$ and by Proposition 1.2.1. If next $h \in [1/(2\nu), 1/\nu)$, it follows from the fact that $E_{V^B}^{cusp}(h) = E_{V^B}(h) \setminus E_{V^B}^{osc}(h)$ with $\dim_H(E_{V^B}(h)) = h\nu$ (by Proposition 1.2.1) and $\dim_H(E_{V^B}^{osc}(h)) \leq 2h\nu - 1 < h\nu$. \square

Finally, we verify that Theorems 1.1.7 and 1.1.10 imply Theorem 1.1.8.

Proof of Theorem 1.1.8. For any $h \in [1, 1 + 1/\nu]$, we have $E_X(h) \supset E_V^{cusp}(h - 1)$, whence $\dim_H(E_X(h)) \geq \dim_H(E_V^{cusp}(h - 1)) = (h - 1)\nu$ by Theorem 1.1.10.

Next we obviously have a.s., for all $t \in [0, 1]$,

$$h_X(t) \geq h_V(t) + 1, \quad (1.38)$$

whence $E_X(h) \subset \bigcup_{h' \leq h-1} E_V(h')$. We thus infer from Theorem 1.1.7 that $E_X(h) = \emptyset$ when $h < 1$. But when $h \in [1, 1 + 1/\nu]$, recalling Proposition 1.3.14 and the fact that A_δ^* is decreasing with δ , we deduce that $\bigcup_{h' \leq h-1} E_V(h') \subset \bigcup_{h' \leq h-1} \bigcap_{\delta \in (0, 1/h')} A_\delta^* \subset \bigcap_{\delta < h-1} A_\delta^*$. Whence we derive $\dim_H(E_X(h)) \leq (h-1)\nu$ from Proposition 1.3.12.

It only remains to verify that $E_X(h) = \emptyset$ when $h > 1 + 1/\nu$. But in such a case, we know from Proposition 1.4.7 that $E_{X^B}(h) = \emptyset$, whence $E_X(h) = \bigcup_{B \geq 1}^{+\infty} E_{X^B}(h) = \emptyset$. \square

Chapter 2

Uniqueness and propagation of chaos for the Boltzmann equation with moderately soft potentials

This work is re-submitted after revision required by *Ann. Appl. Probab.*

We prove a strong/weak stability estimate for the 3D homogeneous Boltzmann equation with moderately soft potentials ($\gamma \in (-1, 0)$) using the Wasserstein distance with quadratic cost. This in particular implies the uniqueness in the class of all weak solutions, assuming only that the initial condition has a finite entropy and a finite moment of sufficiently high order. We also consider the Nanbu N -stochastic particle system which approximates the weak solution. We use a probabilistic coupling method and give, under suitable assumptions on the initial condition, a rate of convergence of the empirical measure of the particle system to the solution of the Boltzmann equation for this singular interaction.

2.1 Introduction

2.1.1 The Boltzmann equation

We consider a 3-dimensional spatially homogeneous Boltzmann equation, which depicts the density $f_t(v)$ of particles in a gas, moving with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The density $f_t(v)$ solves

$$\partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v - v_*|, \theta) [f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)], \quad (2.1)$$

where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \quad (2.2)$$

and θ is the *deviation angle* defined by $\cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$. The *collision Kernel* $B(|v - v_*|, \theta) \geq 0$ depends on the type of interaction between particles. It only depends on $|v - v_*|$ and on the cosine of the deviation angle θ . Conservations of mass, momentum and kinetic energy hold for reasonable solutions and we may assume without loss of generality that $\int_{\mathbb{R}^3} f_t(v) dv = 1$ for all $t \geq 0$.

2.1.2 Assumptions

We will assume that there is a measurable function $\beta : (0, \pi] \rightarrow \mathbb{R}_+$ such that

$$\begin{cases} B(|v - v_*|, \theta) \sin \theta = |v - v_*|^\gamma \beta(\theta), \\ \exists 0 < c_0 < c_1, \forall \theta \in (0, \pi/2), c_0 \theta^{-1-\nu} \leq \beta(\theta) \leq c_1 \theta^{-1-\nu}, \\ \forall \theta \in [\pi/2, \pi], \beta(\theta) = 0, \end{cases} \quad (2.3)$$

for some $\nu \in (0, 1)$, and $\gamma \in (-1, 0)$ satisfying $\gamma + \nu > 0$.

The last assumption $\beta = 0$ on $[\pi/2, \pi]$ is not a restriction and can be obtained by symmetry as noted in the introduction of [3]. This assumption corresponds to a classical physical example, inverse power laws interactions: when particles collide by pairs due to a repulsive force proportional to $1/r^s$ for some $s > 2$, assumption (2.3) holds with $\gamma = (s - 5)/(s - 1)$ and $\nu = 2/(s - 1)$. Here we will focus on the case of moderately soft potentials, i.e. $s \in (3, 5)$.

2.1.3 Some notations

Let us denote by $\mathcal{P}(\mathbb{R}^3)$ the set of probability measures on \mathbb{R}^3 and by $Lip(\mathbb{R}^3)$ the set of bounded globally Lipschitz functions $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$. When $f \in \mathcal{P}(\mathbb{R}^3)$ has a density, we also denote this density by f . For $q > 0$, we set

$$\mathcal{P}_q(\mathbb{R}^3) = \{f \in \mathcal{P}(\mathbb{R}^3) : m_q(f) < \infty\} \quad \text{with} \quad m_q(f) := \int_{\mathbb{R}^3} |v|^q f(dv).$$

We now introduce, for $\theta \in (0, \pi/2)$ and $z \in [0, \infty)$,

$$H(\theta) = \int_{\theta}^{\pi/2} \beta(x) dx \quad \text{and} \quad G(z) = H^{-1}(z). \quad (2.4)$$

Under (2.3), it is clear that H is a continuous decreasing function valued in $[0, \infty)$, so it has an inverse function $G : [0, \infty) \mapsto (0, \pi/2)$ defined by $G(H(\theta)) = \theta$ and $H(G(z)) = z$.

Furthermore, it is easy to verify that there exist some constants $0 < c_2 < c_3$ such that for all $z > 0$,

$$c_2(1+z)^{-1/\nu} \leq G(z) \leq c_3(1+z)^{-1/\nu}, \quad (2.5)$$

and we know from [25] that there exists a constant $c_4 > 0$ such that for all $x, y \in \mathbb{R}_+$,

$$\int_0^\infty (G(z/x) - G(z/y))^2 dz \leq c_4 \frac{(x-y)^2}{x+y}. \quad (2.6)$$

Let us now introduce the Wasserstein distance with quadratic cost on $\mathcal{P}_2(\mathbb{R}^3)$. For $g, \tilde{g} \in \mathcal{P}_2(\mathbb{R}^3)$, let $\mathcal{H}(g, \tilde{g})$ be the set of probability measures on $\mathbb{R}^3 \times \mathbb{R}^3$ with first marginal g and second marginal \tilde{g} . We then set

$$\mathcal{W}_2(g, \tilde{g}) = \inf \left\{ \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - \tilde{v}|^2 R(dv, d\tilde{v}) \right)^{1/2}, R \in \mathcal{H}(g, \tilde{g}) \right\}.$$

Here the infimum is actually a minimum, for more details on this distance, one can see [61, Chapter 2].

2.1.4 Weak solutions

We now introduce a suitable spherical parameterization of (2.2) as in [31]. For each $x \in \mathbb{R}^3 \setminus \{0\}$, we consider a vector $I(x) \in \mathbb{R}^3$ such that $|I(x)| = |x|$ and $I(x) \perp x$. We also set $J(x) = \frac{x}{|x|} \wedge I(x)$, where \wedge is the vector product. Then the triplet $(\frac{x}{|x|}, \frac{I(x)}{|x|}, \frac{J(x)}{|x|})$ is an orthonormal basis of \mathbb{R}^3 . Then for $x, v, v_* \in \mathbb{R}^3$, $\theta \in (0, \pi]$, $\varphi \in [0, 2\pi)$, we set

$$\begin{cases} \Gamma(x, \varphi) := (\cos \varphi)I(x) + (\sin \varphi)J(x), \\ v'(v, v_*, \theta, \varphi) := v - \frac{1-\cos \theta}{2}(v - v_*) + \frac{\sin \theta}{2}\Gamma(v - v_*, \varphi), \\ a(v, v_*, \theta, \varphi) := v'(v, v_*, \theta, \varphi) - v, \end{cases} \quad (2.7)$$

then we write $\sigma \in \mathbb{S}^2$ as $\sigma = \frac{v-v_*}{|v-v_*|} \cos \theta + \frac{I(v-v_*)}{|v-v_*|} \sin \theta \cos \varphi + \frac{J(v-v_*)}{|v-v_*|} \sin \theta \sin \varphi$, and observe at once that $\Gamma(x, \varphi)$ is orthogonal to x and has the same norm as x , from which it is easy to check that

$$|a(v, v_*, \theta, \varphi)| = \sqrt{\frac{1 - \cos \theta}{2}} |v - v_*|. \quad (2.8)$$

Let us now give the definition of weak and strong solutions to (2.1).

Definition 2.1.1. Assume (2.3) is true for some $\nu \in (0, 1)$, $\gamma \in (-1, 0)$ with $\gamma + \nu > 0$. A measurable family of probability measures $(f_t)_{t \geq 0}$ is called a weak solution to (2.1) if it satisfies the following two conditions:

- For all $t \geq 0$,

$$\int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv) \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^2 f_t(dv) = \int_{\mathbb{R}^3} |v|^2 f_0(dv) < \infty. \quad (2.9)$$

- For any bounded globally Lipschitz function $\phi \in Lip(\mathbb{R}^3)$, any $t \in [0, T]$,

$$\int_{\mathbb{R}^3} \phi(v) f_t(dv) = \int_{\mathbb{R}^3} \phi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{A}\phi(v, v_*) f_s(dv_*) f_s(dv) ds \quad (2.10)$$

where

$$\mathcal{A}\phi(v, v_*) = |v - v_*|^\gamma \int_0^{\pi/2} \beta(\theta) d\theta \int_0^{2\pi} [\phi(v + a(v, v_*, \theta, \varphi)) - \phi(v)] d\varphi.$$

We observe that $|\mathcal{A}\phi(v, v_*)| \leq C_\phi |v - v_*|^{1+\gamma} \leq C_\phi (1 + |v - v_*|^2)$ from $|a(v, v_*, \theta, \varphi)| \leq C\theta|v - v_*|$ and $\int_0^{\pi/2} \theta \beta(\theta) d\theta < \infty$, (2.10) is thus well-defined.

Definition 2.1.2. Assume (2.3) is true for some $\nu \in (0, 1)$, $\gamma \in (-1, 0)$ with $\gamma + \nu > 0$. A measurable family of probability measures $(f_t)_{t \geq 0}$ is called a strong solution to (2.1) if $(f_t)_{t \geq 0} \in L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$.

Let us now recall the well-posedness result of (2.1) in [33, Corollary 2.4] (more general existence results can be found in [59]).

Theorem 2.1.3. Assume (2.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$. Let $q \geq 2$ such that $q > \gamma^2/(\gamma + \nu)$. Let $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} f_0(v) |\log f_0(v)| dv < \infty$ and let $p \in (3/(3 + \gamma), p_0(\gamma, \nu, q))$, where

$$p_0(\gamma, \nu, q) = \frac{q - \gamma}{q(3 - \nu)/3 - \gamma} \in (3/(3 + \gamma), 3/(3 - \nu)). \quad (2.11)$$

Then (2.1) has a unique weak solution $f \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$.

The explicit value of $p_0(\gamma, \nu, q)$ are not properly stated in [33, Corollary 2.4]. However, following its proof (see the end of Step 3), we see that $f \in L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ as soon as $1 < p < 3/(3 - \nu)$ and $-\gamma(p - 1)/(1 - p(3 - \nu)/3) < q$. This precisely rewrites as $p \in (1, p_0(\gamma, \nu, q))$.

2.1.5 The particle system

Let us now recall the Nanbu particle system introduced by [48]. It is the $(\mathbb{R}^3)^N$ -valued Markov process with infinitesimal generator \mathcal{L}_N defined as follows: for any bounded Lipschitz function $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ and $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$,

$$\mathcal{L}_N \phi(\mathbf{v}) = \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{S}^2} [\phi(\mathbf{v} + (v'(v_i, v_j, \sigma) - v_i)\mathbf{e}_i) - \phi(\mathbf{v})] B(|v_i - v_j|, \theta) d\sigma,$$

where $v\mathbf{e}_i = (0, \dots, v, \dots, 0) \in (\mathbb{R}^3)^N$ with v at the i -th place for $v \in \mathbb{R}^3$.

In other words, the system contains N particles with velocities $\mathbf{v} = (v_1, \dots, v_N)$. Each pair of particles (with velocities (v_i, v_j)), interact, for each $\sigma \in \mathbb{S}^2$, at rate $B(|v_i - v_j|, \theta)/N$. Then one changes the velocity v_i to $v'(v_i, v_j, \sigma)$ given by (2.2) but v_j remains unchanged. That is, only one particle is changed at each collision.

The fact that $\int_0^\pi \beta(\theta) d\theta = \infty$ (i.e. β is non cutoff) means that there are infinitely many jumps with a very small deviation angle. It is thus impossible to simulate it directly. For this reason, we will study a truncated version of Nanbu's particle system applying a cutoff procedure as [32], who were studying the Nanbu system for *hard potentials* and *Maxwell molecules*, and [16], who were dealing with the Kac system for *Maxwell molecules*. Our particle system with cutoff corresponds to the generator $\mathcal{L}_{N,K}$ defined, for any bounded Lipschitz function $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ and $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, by

$$\begin{aligned} \mathcal{L}_{N,K} \phi(\mathbf{v}) = \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{S}^2} [\phi(\mathbf{v} + (v'(v_i, v_j, \sigma) - v_i)\mathbf{e}_i) - \phi(\mathbf{v})] B(|v_i - v_j|, \theta) \\ \times \mathbf{1}_{\{\theta \geq G(K/|v_i - v_j|^\gamma)\}} d\sigma, \end{aligned} \quad (2.12)$$

with G defined by (2.4).

The generator $\mathcal{L}_{N,K}$ uniquely defines a strong Markov process with values in $(\mathbb{R}^3)^N$. This comes from the fact that the corresponding jump rate is finite and constant: for any configuration $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, it holds that $N^{-1} \sum_{i \neq j} \int_{\mathbb{S}^2} B(|v_i - v_j|, \theta) \mathbf{1}_{\{\theta \geq G(K/|v_i - v_j|^\gamma)\}} d\sigma = 2\pi(N-1)K$. Indeed, for any $z \in [0, \infty)$, we have $\int_{\mathbb{S}^2} B(x, \theta) \mathbf{1}_{\{\theta \geq G(K/x^\gamma)\}} d\sigma = 2\pi K$, which is easily checked recalling that $B(x, \theta) = x^\gamma \beta(\theta)$ and the definition of G .

2.1.6 Main results

Now, we give our uniqueness result for the Boltzmann equation.

Theorem 2.1.4. *Assume (2.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ satisfying $\gamma + \nu > 0$. Let $q \geq 2$ such that $q > \gamma^2/(\gamma + \nu)$. Assume that $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ with a finite entropy, i.e.*

$\int_{\mathbb{R}^3} f_0(v) |\log f_0(v)| dv < \infty$. Let $p \in (3/(3 + \gamma), p_0(\gamma, \nu, q))$, recall (2.11), and $(f_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ be the unique weak solution to (2.1) given by Theorem 2.1.3. Then for any other weak solution $(\tilde{f}_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ to (2.1), we have, for any $t \geq 0$,

$$\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq \mathcal{W}_2^2(f_0, \tilde{f}_0) \exp\left(C_{\gamma,p} \int_0^t (1 + \|f_s\|_{L^p}) ds\right).$$

In particular, we have uniqueness for (2.1) when starting from f_0 in the space of all weak solutions in the sense of Definition 2.1.1.

The novelty of Theorem 2.1.4 is that no regularity at all is assumed concerning \tilde{f} . In particular, we have uniqueness among all weak solutions, while in [33], uniqueness is proved only in the class of weak solutions lying in $L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ for some $p > 3/(3 + \gamma)$.

Next, we write the following conclusion concerning the particle system.

Theorem 2.1.5. Assume (2.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$. Let $q > 6$ such that $q > \gamma^2/(\gamma + \nu)$ and let $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ with a finite entropy. Let $(f_t)_{t \geq 0}$ be the unique weak solution to (2.1) given by Theorem 2.1.3. For each $N \geq 1$, $K \in [1, \infty)$, let $(V_t^i)_{i=1, \dots, N}$ be the Markov process with generator $\mathcal{L}_{N,K}$ (see (2.12)) starting from an i.i.d. family $(V_0^i)_{i=1, \dots, N}$ of f_0 -distributed random variables. We denote the associated empirical measure by $\mu_t^{N,K} = N^{-1} \sum_{i=1}^N \delta_{V_t^i}$. Then for all $T > 0$,

$$\sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq C_{T,q} \left(N^{-(1-6/q)(2+2\gamma)/3} + K^{1-2/\nu} + N^{-1/2} \right).$$

We thus obtain a quantitative rate of chaos for the Nanbu's system with a singular interaction. To our knowledge, this is the first result in this direction. However, there is no doubt this rate is not the hoped optimal rate $N^{-1/2}$ like in the hard potential case [32].

2.1.7 Known results, strategies and main difficulties

Let us give a non-exhaustive overview of the known results on the well-posedness of (2.1) for different potentials. First, the global existence of weak solution for the Boltzmann equation concerning all potentials was concluded by Villani in [59], with rather few assumptions on the initial data (finite energy and entropy), using some compactness methods. However, the uniqueness results are less well-understood. For *hard potentials* ($\gamma \in (0, 1)$) with *angular cutoff* ($\int_0^\pi \beta(\theta) d\theta < \infty$), there are some optimal results obtained by Mischler-Wennberg [47], where they gave the existence of a unique weak L^1 solution to (2.1) with the minimal assumption that $\int_{\mathbb{R}^3} (1 + |v|^2) f_0(v) dv < \infty$. This was extended to weak measure solutions

by Lu-Mouhot [44]. For the difficult case *without* angular cutoff, the first uniqueness result was obtained by Tanaka [57] concerning *Maxwell molecules* ($\gamma = 0$). See also Toscani-Villani [58], who proved uniqueness for Maxwell molecules imposing that $\int_0^\pi \theta \beta(\theta) d\theta < \infty$ and that $\int_{\mathbb{R}^3} (1 + |v|^2) f_0(dv) < \infty$. Subsequently, Desvillettes-Mouhot [18] (relying on a weighted W_1^1 space) and Fournier-Mouhot [33] (using the Wasserstein distance \mathcal{W}_1) successively gave the uniqueness and stability for both *hard potentials* ($\gamma \in (0, 1]$) and *moderately soft potentials* ($\gamma \in (-1, 0)$ and $\nu \in (0, 1)$) under different assumptions on initial data. For *moderately soft potentials*, the result in [33] is much better since they use less assumptions on the initial condition than [18]. Finally, let us mention another work [25], where Fournier-Guérin proved a local (in time) uniqueness result with $f_0 \in L^p(\mathbb{R}^3)$ for some $p > 3/(3 + \gamma)$ for the *very soft potentials* ($\gamma \in (-3, 0)$ and $\nu \in (0, 2)$).

In this paper (Theorem 2.1.4), we obtain a better uniqueness result in the case of a collision kernel without angular cutoff when $\gamma \in (-1, 0)$ and $\nu \in (0, 1 - \gamma)$, that is, the uniqueness holds in the class of all measure solutions in $L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$. This is very important when studying particle systems. For example, a convergence result without rate would be almost immediate from our uniqueness: the tightness of the empirical measure of the particle system is not very difficult, as well as the fact that any limit point is a weak solution to (2.1). Since such a weak solution is unique by Theorem 2.1.4, the convergence follows. Such a conclusion would be very difficult to obtain when using the uniqueness proved in [33], because one would need to check that any limit point of the empirical measure belongs to $L_{loc}^1([0, \infty), L^p(\mathbb{R}^3))$ for some $p > 3/(3 + \gamma)$, which seems very difficult.

In order to extend the uniqueness result for all measure solutions, extra difficulty is inevitable and the methods of [25, 33] will *not* work. However, Fournier-Hauray [27] provide some ideas to overcome this, in the simpler case of the Landau equation for moderately soft potentials. Here we follow these ideas, which rely on coupling methods. Consider two weak solutions f and \tilde{f} in $L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ to (2.1), with possibly two different initial conditions and assume that f is *strong*, in the sense that it belongs to $L_{loc}^1([0, \infty), L^p(\mathbb{R}^3))$. First, we associate to the weak solution \tilde{f} a *weak* solution $(X_t)_{t \geq 0}$ to some Poisson-driven SDE. This uses a smoothing procedure as in [22, 27], but the situation is consequently more complicated because we deal with jump processes. Next, we try to associate to the *strong* solution f a *strong* solution $(W_t)_{t \geq 0}$ to another SDE (driven by the same Poisson measure as $(X_t)_{t \geq 0}$), as [27] did. But we did not manage to do this properly and we had to use a truncation procedure which though complicates our computation. Then, roughly, we estimate $\mathcal{W}_2^2(f_t, \tilde{f}_t)$ by computing $\mathbb{E}[|X_t - W_t|^2]$ as precisely as possible.

The terminology *propagation of chaos*, which is equivalent to the convergence of the empirical measure of a particle system to the solution to a nonlinear equation, was first formulated by Kac [43]. He was studying the convergence of a *toy* particle system as a step to the rigor-

ous derivation of the Boltzmann equation. Kac's particle system is similar to the one studied in the present paper, but each collision modifies the velocities of the two involved particles, while in Nanbu's system, only one of the two particles is deviated. Hence, Kac's system is physically more meaningful. Afterwards, McKean [45] and Grünbaum [36] extended Kac's ideas to study the chaos property for different models with bounded collision kernels. Sznitman [56] then showed the chaos property (for Kac's system without rate) for the *hard spheres* ($\gamma = 1$ and $\nu = 0$). Following Tanaka's probabilistic interpretation for the Boltzmann equation with *Maxwell molecules*, Graham-Méléard [35] were the first to give a rate of chaos for (2.1), concerning both Kac and Nanbu models, for Maxwell molecules with cutoff ($\gamma = 0$ and $\int_0^\pi \beta(\theta)d\theta < \infty$), using the total variation distance. Fontbona-Guérin-Méléard [23] first gave explicit rates for Nanbu type diffusive approximations of the Landau equation with *Maxwell molecules* by coupling arguments, using the \mathcal{W}_2 distance. Recently, some important progresses have been made. First, Mischler-Mouhot [46] obtained a uniform (in time) rate of convergence of Kac's particle system of order $N^{-\epsilon}$ (for Maxwell molecules without cutoff) and $(\log N)^{-\epsilon}$ (for hard spheres, i.e. $\gamma = 1$ and $\nu = 0$), with some small $\epsilon > 0$, in \mathcal{W}_1 distance between the joint law of the particle system and $f_t^{\otimes N}$. This result, entirely relying on analytic methods, is noticeable, although the rates are clearly not sharp. Then, Fournier-Mischler [32] proved the propagation of chaos at rate $N^{-1/4}$ for the Nanbu system and for hard potentials without cutoff ($\gamma \in [0, 1]$ and $\nu \in (0, 1)$) using the \mathcal{W}_2 distance. Finally, as mentioned in Section 1.5, Cortez-Fontbona [16] used two coupling techniques and the \mathcal{W}_2 distance for Kac's system and obtained a uniform in time estimate for the Boltzmann equation with *Maxwell molecules* ($\gamma = 0$) under some suitable moments assumptions on the initial datum. Let us mention that the time-uniformity uses the recent nice results of Rousset [51].

In this paper (Theorem 2.1.5), we obtain, to our knowledge, the first chaos result (with rate) for soft potentials (which are, of course, more difficult), but it is a bit unsatisfying: (1) we cannot study Kac's system (which is physically more reasonable than Nanbu's system) because it is not readily to exhibit a suitable coupling; (2) our consideration is merely for $\gamma \in (-1, 0)$, since some basic estimates in Section 2 do not hold any more if $\gamma \leq -1$; (3) our rate is not sharp. However, since the interaction is singular, it seems hopeless to get a perfect result.

In terms of the propagation of chaos with a singular interaction, there are only very few results. In one dimension, Bossy-Talay [13] and Jourdain [42] concerned the viscous Burgers equation and a viscous scalar conservation law by a family of stochastic particles with a discontinuous interaction kernel (i.e. particles interact through the Heaviside function). Let us also mention the work of Cépa-Lépingle [14] which considered the very singular Brownian motion model introduced by Dyson [19]. For high dimensions, Hauray-Jabin [37] considered a deterministic system of particles interacting through a force of the type $1/|x|^\alpha$ with $\alpha < 1$, in

dimension $d \geq 3$, and proved the mean field limit and the propagation of chaos to the Vlasov equation. Also, Fournier-Hauray-Mischler [28] proved the convergence of the vortex model to the 2D Navier-Stokes equation with a singular Biot-Savart kernel using some entropy dissipation technique. Following the method of [28], Godinho-Quñinao [34] proved the propagation of chaos of some particle system to the 2D subcritical Keller-Segel equation. For the very subcritical case for this equation, Fournier-Jourdain [29] proved the existence for the particle system and that its flow of empirical measures converges to a weak solution of the Keller-Segel equation. Recently, Fournier-Hauray [27] proved propagation of chaos for the 3D Landau equation with a singular interaction ($\gamma \in (-2, 0)$) for the Nanbu diffusive particle system using the \mathcal{W}_2 distance. Actually, they gave a quantitative rate of chaos when $\gamma \in (-1, 0)$, while the convergence without rate was checked when $\gamma \in (-2, 0)$ by the entropy dissipation technique.

Roughly speaking, to prove our propagation of chaos result, we consider an approximate version of our stability principle, with a discrete L^p norm as in [27]. Here, we list the main difficulties: The trajectory of a typical particle related to the Boltzmann equation is a jump process so that all the continuity arguments used in [27] have to be changed. In particular, a detailed study of small and large jumps is required. Also, the solution to the Landau equation lies in $L^1_{loc}([0, \infty), L^2(\mathbb{R}^3))$, while the one of the Boltzmann equation lies in $L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ for some p smaller than 2. This causes a few difficulties in Section 2.5, because working in L^p is slightly more complicated.

2.1.8 Arrangement of the paper and final notations

In Section 2, we give some basic estimates. In Section 3, we establish the strong/weak stability principle for (2.1). In Section 4, we construct the suitable coupling. In Section 5, we bound the L^p norm of a blob approximation of an empirical measure in terms of the L^p norm of the weak solution. Finally, in Section 6, we prove the convergence of the particle system.

In the sequel, C stands for a positive constant whose value may change from line to line. When necessary, we will indicate in subscript the parameters it depends on.

In the whole paper, we consider two probability spaces by Tanaka's idea for the probabilistic interpretation of the Boltzmann equation in Maxwell molecules case: the first space is the abstract space $(\Omega, \mathcal{F}, \mathbb{P})$ and the second is $([0, 1], \mathcal{B}([0, 1]), d\alpha)$.

2.2 Preliminaries

Above all, let us recall that for $\gamma \in (-1, 0)$, $p > 3/(3 + \gamma)$ and $f \in \mathcal{P}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, it holds that

$$\begin{aligned} \sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\gamma f(dv_*) &\leq \sup_{v \in \mathbb{R}^3} \int_{|v - v_*| \leq 1} |v - v_*|^\gamma f(dv_*) + \sup_{v \in \mathbb{R}^3} \int_{|v - v_*| \geq 1} |v - v_*|^\gamma f(dv_*) \\ &\leq 1 + C_{\gamma,p} \|f\|_{L^p(\mathbb{R}^3)}, \end{aligned} \quad (2.13)$$

where $C_{\gamma,p} = \sup_{v \in \mathbb{R}^3} [\int_{|v - v_*| \leq 1} |v - v_*|^{p\gamma/(p-1)} dv_*]^{(p-1)/p} = [\int_{|v_*| \leq 1} |v_*|^{p\gamma/(p-1)} dv_*]^{(p-1)/p} < \infty$, since $p > 3/(3 + \gamma)$ by assumption.

Let us now classically rewrite the collision operator by making disappear the velocity-dependence $|v - v_*|^\gamma$ in the *rate* using a substitution.

Lemma 2.2.1. *We assume (2.3) and recall (2.4) and (2.7). For $z \in [0, \infty)$, $\varphi \in [0, 2\pi)$, $v, v_* \in \mathbb{R}^3$ and $K \in [1, \infty)$, we define*

$$c(v, v_*, z, \varphi) := a[v, v_*, G(z/|v - v_*|^\gamma), \varphi] \quad \text{and} \quad c_K(v, v_*, z, \varphi) := c(v, v_*, z, \varphi) \mathbf{1}_{\{z \leq K\}}. \quad (2.14)$$

For any $\phi \in Lip(\mathbb{R}^3)$, any $v, v_* \in \mathbb{R}^3$,

$$\mathcal{A}\phi(v, v_*) = \int_0^\infty dz \int_0^{2\pi} d\varphi [\phi(v + c(v, v_*, z, \varphi)) - \phi(v)]. \quad (2.15)$$

For any $N \geq 1$, $K \in [1, \infty)$, $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, any bounded measurable $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$,

$$\mathcal{L}_{N,K}\phi(\mathbf{v}) = \frac{1}{N} \sum_{i \neq j} \int_0^\infty dz \int_0^{2\pi} d\varphi [\phi(\mathbf{v} + c_K(v_i, v_j, z, \varphi) \mathbf{e}_i) - \phi(\mathbf{v})]. \quad (2.16)$$

This lemma is stated in [32, Lemma 2.2] when $\gamma \in [0, 1]$, but the proof does not use this fact: it actually holds true for any $\gamma \in \mathbb{R}$. Next, let us recall Lemma 2.3 in [32] which is an accurate version of Tanaka's trick in [57]. Here, we adopt the notation (2.7).

Lemma 2.2.2. *There exists some measurable function $\varphi_0 : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto [0, 2\pi)$ such that for all $X, Y \in \mathbb{R}^3$, all $\varphi \in [0, 2\pi)$,*

$$|\Gamma(X, \varphi) - \Gamma(Y, \varphi + \varphi_0(X, Y))| \leq |X - Y|.$$

The rest of the section is an adaption of [32, Section 3], which assumes that $\gamma \in [0, 1]$, to the case where $\gamma \in (-1, 0)$. When compared with [25], what is new is that in the inequalities (2.17) and (2.18) below, only $|v - v_*|^\gamma$ appears (while in [25], there is $|v - v_*|^\gamma + |\tilde{v} - \tilde{v}_*|^\gamma$). This is very useful to get a strong/weak stability estimate: we will be able to use the regularity of only one of the two solutions to be compared. Let us mention that it seems impossible to extend our ideas to the more singular case where $\gamma \leq -1$.

Lemma 2.2.3. *There is a constant C such that for any $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$, any $K \geq 1$,*

$$\int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi) - c(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*))|^2 d\varphi dz \quad (2.17)$$

$$\leq C(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2)|v - v_*|^\gamma$$

$$\int_0^\infty \int_0^{2\pi} \left(|v + c(v, v_*, z, \varphi) - \tilde{v} - c_K(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*))|^2 - |v - \tilde{v}|^2 \right) d\varphi dz \quad (2.18)$$

$$\leq C(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2)|v - v_*|^\gamma + C|v - v_*|^{2+2\gamma/\nu} K^{1-2/\nu}$$

$$\int_0^\infty \int_0^{2\pi} |c_K(v, v_*, z, \varphi)|^2 d\varphi dz \leq C|v - v_*|^{\gamma+2}, \int_0^\infty \left| \int_0^{2\pi} c_K(v, v_*, z, \varphi) d\varphi \right| dz \leq C|v - v_*|^{\gamma+1} \quad (2.19)$$

$$\int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)|^2 d\varphi dz \leq C|v - v_*|^{\gamma+2}, \int_0^\infty \left| \int_0^{2\pi} c(v, v_*, z, \varphi) d\varphi \right| dz \leq C|v - v_*|^{\gamma+1} \quad (2.20)$$

Proof. For $x > 0$, we set $\Phi_K(x) = \pi \int_0^K (1 - \cos G(z/x^\gamma)) dz$ and $\Psi_K(x) = \pi \int_K^\infty (1 - \cos G(z/x^\gamma)) dz$. We introduce the shortened notation $x = |v - v_*|$, $\tilde{x} = |\tilde{v} - \tilde{v}_*|$, $\varphi_0 = \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*)$, $c = c(v, v_*, z, \varphi)$, $c_K = c_K(v, v_*, z, \varphi) = c\mathbf{1}_{\{z \leq K\}}$, $\tilde{c} = c(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0)$ and $\tilde{c}_K = c_K(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0) = \tilde{c}\mathbf{1}_{\{z \leq K\}}$.

Step 1. We first verify that $\Phi_K(x) \leq Cx^\gamma$ and that $|\Phi_K(x) - \Phi_K(\tilde{x})| \leq C|x^\gamma - \tilde{x}^\gamma|$. First, we immediately see that $\Phi_K(x) \leq \pi \int_0^\infty G^2(z/x^\gamma) dz = x^\gamma \pi \int_0^\infty G^2(z) dz$ which implies the first point (recall (2.5)). To check the second point, it suffices to verify that $F_K(x) = \int_0^K (1 - \cos G(z/x)) dz$ has a bounded derivative (uniformly in $K \geq 1$). But we have $F_K(x) = x \int_0^{K/x} (1 - \cos G(z)) dz$ so that

$$|F'_K(x)| \leq \int_0^\infty (1 - \cos G(z)) dz + x(K/x^2)(1 - \cos G(K/x)) \leq C + (K/x)G^2(K/x),$$

which is uniformly bounded by (2.5).

Step 2. Proceeding as in the proof of [32, Lemma 3.1], we see that $\int_0^\infty \int_0^{2\pi} |c_K|^2 d\varphi dz = x^2 \Phi_K(x)$, which is bounded by $Cx^{\gamma+2}$ by Step 1. Also, recalling (2.7) and (2.14), using that $\int_0^{2\pi} \Gamma(X, \varphi) d\varphi = 0$, we see that we have $\int_0^{2\pi} c_K d\varphi = -\pi(v - v_*)(1 - \cos G(z/x^\gamma))$, whence $\int_0^\infty \left| \int_0^{2\pi} c_K d\varphi \right| dz = x \Phi_K(x) \leq Cx^{\gamma+1}$ by Step 1. All this proves (2.19), from which (2.20) follows by letting K increase to infinity.

Step 3. Let us denote by $I_K = \int_0^K \int_0^{2\pi} |c - \tilde{c}|^2 d\varphi dz$, by $J_K = \int_0^K \int_0^{2\pi} (|v + c - \tilde{v} - \tilde{c}|^2 - |v - \tilde{v}|^2) d\varphi dz$ and by $L_K = \int_K^\infty \int_0^{2\pi} (|v + c - \tilde{v}|^2 - |v - \tilde{v}|^2) d\varphi dz$. Proceeding exactly as in

the proof of [32, Lemma 3.1], we see that $J_K \leq A_1^K + A_2^K$ and $L_K \leq A_3^K$, where

$$\begin{aligned} A_1^K &= 2x\tilde{x} \int_0^K \left(G(z/x^\gamma) - G(z/\tilde{x}^\gamma) \right)^2 dz, \\ A_2^K &= [|v - \tilde{v}| + |v_* - \tilde{v}_*|] | (v - v_*)\Phi_K(x) - (\tilde{v} - \tilde{v}_*)\Phi_K(\tilde{x}) |, \\ A_3^K &= (x^2 + 2|v - \tilde{v}|x)\Psi_K(x). \end{aligned}$$

Also, $I_K = J_K - 2(v - \tilde{v}) \cdot \int_0^K \int_0^{2\pi} (c - \tilde{c}) d\varphi dz$ and, as seen in the proof of [32, Lemma 3.1], $\int_0^K \int_0^{2\pi} c d\varphi dz = -(v - v_*)\Phi_K(x)$, so that $I_K \leq J_K + A_4^K$ with

$$A_4^K = 2|v - \tilde{v}| | (v - v_*)\Phi_K(x) - (\tilde{v} - \tilde{v}_*)\Phi_K(\tilde{x}) |.$$

First, we immediately deduce from (2.6) that

$$A_1^K \leq 2c_4 x\tilde{x} \frac{(x^\gamma - \tilde{x}^\gamma)^2}{x^\gamma + \tilde{x}^\gamma} \leq 2c_4 (x - \tilde{x})^2 \min(x^\gamma, \tilde{x}^\gamma) \leq C(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) |v - v_*|^\gamma.$$

For the second inequality, we used that $|x^\gamma - \tilde{x}^\gamma| \leq |x^{-1} - \tilde{x}^{-1}| (x \wedge \tilde{x})^{1+\gamma}$ (because $\gamma \in (-1, 0)$) so that

$$x\tilde{x} \frac{|x^\gamma - \tilde{x}^\gamma|^2}{x^\gamma + \tilde{x}^\gamma} \leq (x\tilde{x})^{1+|\gamma|} \frac{|x^{-1} - \tilde{x}^{-1}|^2 (x \wedge \tilde{x})^{2\gamma+2}}{x^{|\gamma|} + \tilde{x}^{|\gamma|}} \leq (x\tilde{x})^{|\gamma|-1} \frac{|x - \tilde{x}|^2 (x\tilde{x})^{1+\gamma}}{x^{|\gamma|} + \tilde{x}^{|\gamma|}} = \frac{|x - \tilde{x}|^2}{x^{|\gamma|} + \tilde{x}^{|\gamma|}},$$

which is indeed bounded by $(x - \tilde{x})^2 \min(x^\gamma, \tilde{x}^\gamma)$.

We now verify that $A_2^K \leq C(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) |v - v_*|^\gamma$. By Step 1, for any $X, Y \in \mathbb{R}^3$,

$$\begin{aligned} |X\Phi_K(|X|) - Y\Phi_K(|Y|)| &\leq |Y| |\Phi_K(|X|) - \Phi_K(|Y|)| + |X - Y| \Phi_K(|X|) \\ &\leq C|Y| \left| |X|^\gamma - |Y|^\gamma \right| + C|X - Y| |X|^\gamma. \end{aligned}$$

Since again $|x^\gamma - \tilde{x}^\gamma| \leq |x^{-1} - \tilde{x}^{-1}| (x \wedge \tilde{x})^{1+\gamma}$, we conclude that $|X\Phi_K(|X|) - Y\Phi_K(|Y|)| \leq C|X - Y| |X|^\gamma$, whence

$$A_2^K \leq C [|v - \tilde{v}| + |v_* - \tilde{v}_*|] | (v - v_*) - (\tilde{v} - \tilde{v}_*) | \min\{x^\gamma, \tilde{x}^\gamma\}$$

as desired.

We next observe that $A_4^K \leq 2A_2^K$.

Finally, we see that $\Psi_K(x) \leq C \int_K^\infty G^2(z/x^\gamma) dz \leq C \int_K^\infty (z/x^\gamma)^{-2/\nu} dz = Cx^{2\gamma/\nu} K^{1-2/\nu}$ and that $\Psi_K(x) \leq C \int_0^\infty G^2(z/x^\gamma) dz \leq C \int_0^\infty (1 + z/x^\gamma)^{-2/\nu} dz = Cx^\gamma$ according to (2.5), which imply $\Psi_K(x) \leq C \min\{x^\gamma, x^{2\gamma/\nu} K^{1-2/\nu}\}$. Hence,

$$A_3^K = (x^2 + 2|v - \tilde{v}|x)\Psi_K(x) \leq C|v - \tilde{v}|^2 |v - v_*|^\gamma + C|v - v_*|^{2+2\gamma/\nu} K^{1-2/\nu},$$

because $2|v - \tilde{v}|x \leq |v - \tilde{v}|^2 + x^2$ and $x^2\Psi_K(x) \leq Cx^{2+2\gamma/\nu}K^{1-2/\nu}$.

The left hand side of (2.18) is nothing but $J_K + L_K$, which is bounded by $A_1^K + A_2^K + A_3^K$: (2.18) is proved. Finally, the left hand side of (2.17) equals $\lim_{K \rightarrow \infty} I_K$ and we know that $I_K \leq A_1^K + A_2^K + A_4^K$, which is (uniformly in K) bounded by $(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2)|v - v_*|^\gamma$ as desired. \square

2.3 Stability

In this section, we first introduce an auxiliary space $([0, 1], \mathcal{B}([0, 1]), d\alpha)$, and a stochastic process defined on the latter space is called an α -processes and we denote the expectation on $[0, 1]$ by \mathbb{E}_α and the laws by \mathcal{L}_α . Our goal of this section is to prove Theorem 2.1.4.

Let us first give the outline of the proof. Let $(f_t)_{t \geq 0}$ be the *strong* solution to (2.1) and let $(\tilde{f}_t)_{t \geq 0}$ be a weak solution. We first build $(X_t)_{t \geq 0}$ with $\mathcal{L}(X_t) = \tilde{f}_t$ solving

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c(X_{s-}, X_s^*(\alpha), z, \varphi) M(ds, d\alpha, dz, d\varphi),$$

where $(X_t^*)_{t \geq 0}$ is a measurable α -process with law \tilde{f}_t , and $M(ds, d\alpha, dz, d\varphi)$ is a Poisson measure. This process $(X_t)_{t \geq 0}$ can be interpreted as the velocity of a typical particle. Each time it has a jump, say at some time t , it means that the typical particle has collided with another particle, of which the velocity is independent and represented by X_t^* . Of course, X_t^* has to be \tilde{f}_t -distributed.

The existence of the process $(X_t)_{t \geq 0}$ is not easy and we only build a weak solution. The difficulty is mainly due to the singularity of the interaction, which cannot be compensated by some regularity of \tilde{f}_t , because \tilde{f}_t is any weak solution. We thus use the strategy of [22] (which deals with continuous diffusion processes). We introduce $\tilde{f}_t^\epsilon = \tilde{f}_t * \phi_\epsilon$, where ϕ_ϵ is the centered Gaussian density with covariance matrix ϵI_3 . We write the PDE satisfied by \tilde{f}_t^ϵ and associate, for each $\epsilon \in (0, 1)$, a solution $(X_t^\epsilon)_{t \geq 0}$ to some SDE. Since both the SDE and the PDE (with $\epsilon \in (0, 1)$ fixed) are well-posed (because the coefficients are regular enough, see Lemma 2.3.4), we conclude that $\mathcal{L}(X_t^\epsilon) = \tilde{f}_t^\epsilon$. Next, we prove that the family $\{(X_t^\epsilon)_{t \geq 0}, \epsilon \in (0, 1)\}$ is tight using the Aldous criterion [1]. Finally, we consider a limit point $(X_t)_{t \geq 0}$, as $\epsilon \rightarrow 0$, of $\{(X_t^\epsilon)_{t \geq 0}, \epsilon \in (0, 1)\}$. Since $\mathcal{L}(X_t^\epsilon) = \tilde{f}_t^\epsilon$, we deduce that $\mathcal{L}(X_t) = \tilde{f}_t$ for each $t \geq 0$. Then, we classically make use of martingale problems to show that $(X_t)_{t \geq 0}$ is indeed a solution of the desired SDE.

Next, we would like to associate to $(f_t)_{t \geq 0}$ a solution $(W_t)_{t \geq 0}$ to the SDE, driven by the same Poisson measure M , with f_t -distributed α -process $(W_t^*)_{t \geq 0}$ coupled with $(X_t^*)_{t \geq 0}$, that

is,

$$W_t = W_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c(W_{s-}, W_s^*(\alpha), z, \varphi + \varphi_0(X_{s-} - X_s^*(\alpha), W_{s-} - W_s^*(\alpha))) \times M(ds, d\alpha, dz, d\varphi),$$

where the f_t -distributed W_t^* is optimally coupled with X_t^* for each $t \geq 0$. Unfortunately, we cannot prove that such a process exists, because of the term $\varphi + \varphi_0(X_{s-} - X_s^*(\alpha), W_{s-} - W_s^*(\alpha))$. Such a problem was already encountered by Tanaka [57], and we more or less solve it as he did, by introducing, for all $K \geq 1$,

$$W_t^K = W_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c_K(W_{s-}^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K}) M(ds, d\alpha, dz, d\varphi)$$

with $\varphi_{s,\alpha,K} = \varphi_0(X_{s-} - X_s^*(\alpha), W_{s-}^K - W_s^*(\alpha))$ as a coupling SDE. This equation of course has a unique strong solution $(W_t^K)_{t \geq 0}$, but the computation becomes more complicated.

Finally, we observe that

$$\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq \limsup_{K \rightarrow \infty} \mathbb{E}[|W_t^K - X_t|^2],$$

because W_t^K goes in law to f_t for each $t \geq 0$.

Using the Itô formula, we find

$$\mathbb{E}[|W_t^K - X_t|^2] = \mathbb{E}[|W_0 - X_0|^2] + \mathbb{E} \left[\int_0^t \int_0^1 \Delta_s^K(\alpha) d\alpha ds \right],$$

where

$$\Delta_s^K(\alpha) := \int_0^\infty \int_0^{2\pi} \left(|W_s^K - X_s + c_{K,W}(s) - c_X(s)|^2 - |W_s^K - X_s|^2 \right) d\varphi dz$$

with the shortened notation

$$c_{K,W}(s) := c_K(W_s^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K}) \quad \text{and} \quad c_X(s) := c(X_s, X_s^*(\alpha), z, \varphi).$$

Then we deduce from Section 2 that

$$\begin{aligned} \Delta_s^K(\alpha) &\leq C(|W_s^K - X_s|^2 + |W_s^*(\alpha) - X_s^*(\alpha)|^2) |W_s^K - W_s^*(\alpha)|^\gamma \\ &\quad + C|W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu} K^{1-2/\nu}. \end{aligned}$$

It is then not too hard to conclude, using technical computations, that

$$\limsup_{K \rightarrow \infty} \mathbb{E}[|W_t^K - X_t|^2] \leq \mathcal{W}_2^2(f_0, \tilde{f}_0) \exp \left(C_{\gamma,p} \int_0^t (1 + \|f_s\|_{L^p}) ds \right),$$

which completes the proof.

We first state the following result, of which the proof lies at the end of the section.

Proposition 2.3.1. *Assume (2.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$. Consider any weak solution $(\tilde{f}_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ to (2.1). Then there exists, on some probability space, a random variable X_0 with law \tilde{f}_0 , independent of a Poisson measure $M(ds, d\alpha, dz, d\varphi)$ on $[0, \infty) \times [0, 1] \times [0, \infty) \times [0, 2\pi)$ with intensity $dsd\alpha dzd\varphi$, a measurable family $(X_t^*)_{t \geq 0}$ of α -random variables such that $\mathcal{L}_\alpha(X_t^*) = \tilde{f}_t$ and a càdlàg adapted process $(X_t)_{t \geq 0}$ solving*

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c(X_{s-}, X_s^*(\alpha), z, \varphi) M(ds, d\alpha, dz, d\varphi) \quad (2.21)$$

and such that for all $t \geq 0$, $\mathcal{L}(X_t) = \tilde{f}_t$.

We are unfortunately not able to say anything about uniqueness (in law) for this SDE, except if \tilde{f} is a *strong* solution, and this is precisely the reason why things are complicated. We really need to use the ideas of [22] to produce, for $(\tilde{f}_t)_{t \geq 0}$ given, a solution $(X_t)_{t \geq 0}$ of which the time marginals are $(\tilde{f}_t)_{t \geq 0}$.

Proposition 2.3.2. *Assume (2.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$, that $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ for some $q \geq 2$ such that $q > \gamma^2/(\gamma + \nu)$ and that f_0 has a finite entropy. Fix $p \in (3/(3 + \gamma), p_0(\gamma, \nu, q))$. Let $(f_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ be the corresponding unique weak solution to (2.1) given by Theorem 2.1.3. Consider also the Poisson measure M , the process $(X_t)_{t \geq 0}$ and the family $(X_t^*)_{t \geq 0}$ built in Proposition 2.3.1 (associated to another weak solution $(\tilde{f}_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$). Let $W_0 \sim f_0$ (independent of M) be such that $\mathbb{E}[|W_0 - X_0|^2] = \mathcal{W}_2^2(f_0, \tilde{f}_0)$ and, for each $t \geq 0$, an α -random variable W_t^* such that $\mathcal{L}_\alpha(W_t^*) = f_t$ and $\mathbb{E}_\alpha[|W_t^* - X_t^*|^2] = \mathcal{W}_2^2(f_t, \tilde{f}_t)$. Then for $K \geq 1$, the equation*

$$W_t^K = W_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c_K(W_{s-}^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K}) M(ds, d\alpha, dz, d\varphi), \quad (2.22)$$

with $\varphi_{s,\alpha,K} = \varphi_0(X_{s-} - X_s^*(\alpha), W_{s-}^K - W_s^*(\alpha))$, has a unique solution. Moreover, setting $f_t^K = \mathcal{L}(W_t^K)$ for each $t \geq 0$, it holds that for all $T > 0$,

$$\lim_{K \rightarrow \infty} \sup_{[0, T]} \mathcal{W}_2^2(f_t^K, f_t) = 0. \quad (2.23)$$

Remark 2.3.3. *As recalled in the previous section, the infimum in the definition of Wasserstein distance is actually a minimum. Since the strong solution $f_t \in \mathcal{P}_2(\mathbb{R}^3)$ has a density for all $t \geq 0$, there is a unique $R_t \in \mathcal{H}(f_t, \tilde{f}_t)$ such that $\mathcal{W}_2^2(f_t, \tilde{f}_t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - \tilde{v}|^2 R_t(dv, d\tilde{v})$ (see Villani [61, Theorem 2.12]). We then know that $(t, \alpha) \mapsto (W_t^*(\alpha), X_t^*(\alpha))$ can be chosen measurable from Fontbona-Guérin-Méléard [23, Theorem 1.3].*

Proof. For any $K \geq 1$, the Poisson measure involved in (2.22) is actually finite (because $c_K = c\mathbf{1}_{\{z \leq K\}}$), so the existence and uniqueness for this equation is obvious. It only remains to prove (2.23), which has already been done in [25, Lemma 4.2], where the formulation of the equation is slightly different. But one easily checks that $(W_t^K)_{t \geq 0}$ is a (time-inhomogeneous) Markov process with the same generator as the one defined by [25, Eq. (4.1)], because for all bounded measurable function $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$ and all $t \geq 0$, a.s.,

$$\begin{aligned} & \int_0^1 \int_0^\infty \int_0^{2\pi} \left[\phi(w + c_K(w, W_t^*(\alpha), z, \varphi + \varphi_0(X_{t-} - X_t^*(\alpha), w - W_t^*(\alpha))) - \phi(w) \right] d\varphi dz d\alpha \\ &= \int_0^1 \int_0^\infty \int_0^{2\pi} \left[\phi(w + c_K(w, v, z, \varphi)) - \phi(w) \right] d\varphi dz f_t(dv) \end{aligned}$$

by the 2π -periodicity of c_K (in φ) and since $\mathcal{L}_\alpha(W_t^*) = f_t$. \square

Now, we use these coupled processes to conclude the

Proof of Theorem 2.1.4. We consider a weak solution $(\tilde{f}_t)_{t \geq 0}$ to (2.1), with which we associate the objects M , $(X_t)_{t \geq 0}$, $(X_t^*)_{t \geq 0}$ as in Proposition 2.3.1. We then consider f_0 satisfying the assumptions of Theorem 2.1.3 and the corresponding unique weak solution $(f_t)_{t \geq 0}$ belonging to $L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ (with $p \in (3/(3+\gamma), p_0(\gamma, \nu, q))$) and we consider $(W_t^K)_{t \geq 0}$, $(W_t^*)_{t \geq 0}$ built in Proposition 2.3.2 for any $K \geq 1$. We know that $\mathcal{W}_2^2(f_0, \tilde{f}_0) = \mathbb{E}[|W_0 - X_0|^2]$ and that $\mathcal{W}_2^2(f_t, \tilde{f}_t) = \mathbb{E}_\alpha[|W_t^* - X_t^*|^2]$ for all $t \geq 0$. Using that $W_t^K \sim f_t^K$ and $X_t \sim \tilde{f}_t$ for each $t \geq 0$, we deduce from (2.23) that for all $t \geq 0$,

$$\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq \limsup_{K \rightarrow \infty} \mathbb{E}[|W_t^K - X_t|^2] =: J_t. \quad (2.24)$$

Next, we focus on the time interval $[0, T]$ for any fixed $T > 0$, and split the proof into several steps.

Step 1. By the Itô formula, we know that

$$\mathbb{E}[|W_t^K - X_t|^2] = \mathbb{E}[|W_0 - X_0|^2] + \mathbb{E} \left[\int_0^t \int_0^1 \Delta_s^K(\alpha) d\alpha ds \right],$$

where

$$\Delta_s^K(\alpha) := \int_0^\infty \int_0^{2\pi} \left(|W_s^K - X_s + c_{K,W}(s) - c_X(s)|^2 - |W_s^K - X_s|^2 \right) d\varphi dz$$

with the shortened notation

$$c_{K,W}(s) := c_K(W_s^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K}), \quad c_X(s) := c(X_s, X_s^*(\alpha), z, \varphi).$$

We then show that

$$\begin{aligned} \Delta_s^K(\alpha) &\leq C(|W_s^K - X_s|^2 + |W_s^*(\alpha) - X_s^*(\alpha)|^2)|W_s^K - W_s^*(\alpha)|^\gamma \\ &\quad + C|W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu} K^{1-2/\nu}, \end{aligned} \quad (2.25)$$

and

$$\begin{aligned} \Delta_s^K(\alpha) &\leq C|W_s^K - W_s^*(\alpha)|^{\gamma+2} + C|X_s - X_s^*(\alpha)|^{\gamma+2} \\ &\quad + C|W_s^K - X_s|(|W_s^K - W_s^*(\alpha)|^{\gamma+1} + |X_s - X_s^*(\alpha)|^{\gamma+1}). \end{aligned} \quad (2.26)$$

First, Lemma 2.2.3 (inequality (2.18)) precisely tells us that (2.25) holds true. Next, we observe that

$$\begin{aligned} \Delta_s^K(\alpha) &\leq 2 \int_0^\infty \int_0^{2\pi} (|c_{K,W}(s)|^2 + |c_X(s)|^2) d\varphi dz \\ &\quad + 2|W_s^K - X_s| \left| \int_0^\infty \int_0^{2\pi} (c_{K,W}(s) - c_X(s)) d\varphi dz \right|. \end{aligned}$$

Hence, using (2.19) and (2.20), the proof of (2.26) is concluded.

Step 2. Set $\kappa(\gamma) = \min((\gamma + 1)/|\gamma|, |\gamma|/2) > 0$. We verify that there exists a constant $C(T, f_0, \tilde{f}_0, f) > 0$ (depending on $T, m_2(f_0), m_2(\tilde{f}_0), \int_0^t \|f_s\|_{L^p} ds$), such that for all $\ell \geq 1$ (and all $K \geq 1$),

$$I_t^{i,\ell} \leq C(T, f_0, \tilde{f}_0, f) \ell^{-\kappa(\gamma)}, \quad i = 1, 2, 3, 4,$$

where

$$\begin{aligned} I_t^{1,\ell} &:= \mathbb{E} \left[\int_0^t \int_0^1 |W_s^K - W_s^*(\alpha)|^{\gamma+2} \mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \geq \ell\}} d\alpha ds \right], \\ I_t^{2,\ell} &:= \mathbb{E} \left[\int_0^t \int_0^1 |X_s - X_s^*(\alpha)|^{\gamma+2} \mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \geq \ell\}} d\alpha ds \right], \\ I_t^{3,\ell} &:= \mathbb{E} \left[\int_0^t \int_0^1 |W_s^K - X_s| |W_s^K - W_s^*(\alpha)|^{\gamma+1} \mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \geq \ell\}} d\alpha ds \right], \\ I_t^{4,\ell} &:= \mathbb{E} \left[\int_0^t \int_0^1 |W_s^K - X_s| |X_s - X_s^*(\alpha)|^{\gamma+1} \mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \geq \ell\}} d\alpha ds \right]. \end{aligned}$$

Since $\gamma \in (-1, 0)$ and $\kappa(\gamma) \leq (\gamma + 2)/|\gamma|$, we have

$$I_t^{1,\ell} \leq \ell^{-(\gamma+2)/|\gamma|} T \leq \ell^{-\kappa(\gamma)} T.$$

Similarly,

$$I_t^{3,\ell} \leq \ell^{-(\gamma+1)/|\gamma|} \int_0^t \mathbb{E} \left[|W_s^K - X_s| \right] ds.$$

Using (2.9) for $(f_t)_{t \geq 0}$ and $(\tilde{f}_t)_{t \geq 0}$, (2.23), and that $m_2(f_s^K) \leq 2m_2(f_s) + 2\mathcal{W}_2^2(f_s, f_s^K)$, we know that $\mathbb{E}[|W_s^K - X_s|] \leq C(1 + m_2(f_s^K) + m_2(\tilde{f}_s)) \leq C(T, f_0, \tilde{f}_0)$. Hence,

$$I_t^{3,\ell} \leq C(T, f_0, \tilde{f}_0)\ell^{-\kappa(\gamma)}.$$

Since $\gamma + 2 \in (1, 2)$, it follows from the Hölder inequality that

$$\begin{aligned} I_t^{2,\ell} &\leq \mathbb{E} \left[\left(\int_0^t \int_0^1 |X_s - X_s^*(\alpha)|^2 d\alpha ds \right)^{\frac{\gamma+2}{2}} \left(\int_0^t \int_0^1 \mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \geq \ell\}} d\alpha ds \right)^{\frac{|\gamma|}{2}} \right] \\ &\leq C\mathbb{E} \left[\left(\int_0^t (|X_s|^2 + m_2(\tilde{f}_s)) ds \right)^{\frac{\gamma+2}{2}} \left(\int_0^t \int_0^1 \frac{|W_s^K - W_s^*(\alpha)|^\gamma}{\ell} d\alpha ds \right)^{\frac{|\gamma|}{2}} \right] \end{aligned}$$

Since $\mathcal{L}_\alpha(W_s^*) = f_s$, we have $\int_0^1 |W_s^K - W_s^*(\alpha)|^\gamma d\alpha = \int_{\mathbb{R}^3} |W_s^K - v|^\gamma f_s(dv) \leq 1 + C_{\gamma,p} \|f_s\|_{L^p}$ by (2.13), so that

$$\begin{aligned} I_t^{2,\ell} &\leq \ell^{\gamma/2} \left(1 + \int_0^t (\mathbb{E}[|X_s|^2] + m_2(\tilde{f}_s)) ds \right) \left(\int_0^t (1 + C_{\gamma,p} \|f_s\|_{L^p}) ds \right)^{\frac{|\gamma|}{2}} \\ &\leq \ell^{\gamma/2} \left(1 + 2m_2(\tilde{f}_0)T \right) \left(1 + \int_0^t (1 + C_{\gamma,p} \|f_s\|_{L^p}) ds \right) \leq C(T, \tilde{f}_0, f)\ell^{-\kappa(\gamma)}. \end{aligned}$$

For $I_t^{4,\ell}$, we use the triple Hölder inequality to write

$$\begin{aligned} I_t^{4,\ell} &\leq \mathbb{E} \left[\int_0^t |W_s^K - X_s|^2 ds \right]^{\frac{1}{2}} \times \mathbb{E} \left[\int_0^t \int_0^1 |X_s - X_s^*(\alpha)|^2 d\alpha ds \right]^{\frac{1+\gamma}{2}} \\ &\quad \times \mathbb{E} \left[\int_0^t \int_0^1 \mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \geq \ell\}} d\alpha ds \right]^{\frac{|\gamma|}{2}}. \end{aligned}$$

Thus $I_t^{4,\ell} \leq C(T, f_0, \tilde{f}_0, f)\ell^{-\kappa(\gamma)}$: use that $\mathbb{E}[|X_s|^2] = \mathbb{E}_\alpha[|X_s^*|^2] = m_2(\tilde{f}_0)$, that $m_2(f_s^K) \leq 2m_2(f_s) + 2\mathcal{W}_2^2(f_s, f_s^K)$ as before and treat the last term of the product the same as we study $I_t^{2,\ell}$.

Step 3. According to Step 1, we now bound $\Delta_s^K(\alpha)$ by (2.25) when $|W_s^K - W_s^*(\alpha)|^\gamma \leq \ell$ and by (2.26) when $|W_s^K - W_s^*(\alpha)|^\gamma \geq \ell$:

$$\begin{aligned} &\mathbb{E}[|W_t^K - X_t|^2] \\ &\leq \mathbb{E}[|W_0 - X_0|^2] + C \sum_{i=1}^4 I_t^{i,\ell} + CK^{1-2/\nu} \mathbb{E} \left[\int_0^t \int_0^1 |W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu} d\alpha ds \right] \\ &\quad + C\mathbb{E} \left[\int_0^t \int_0^1 (|W_s^K - X_s|^2 + |W_s^*(\alpha) - X_s^*(\alpha)|^2) \min(|W_s^K - W_s^*(\alpha)|^\gamma, \ell) d\alpha ds \right]. \end{aligned}$$

It then follows from Step 2 that for all $\ell \geq 1$, all $K \geq 1$,

$$\begin{aligned} \mathbb{E}[|W_t^K - X_t|^2] &\leq \mathcal{W}_2^2(f_0, \tilde{f}_0) + C(T, f_0, \tilde{f}_0, f)\ell^{-\kappa(\gamma)} \\ &\quad + CK^{1-2/\nu}\mathbb{E}\left[\int_0^t \int_0^1 |W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu} d\alpha ds\right] \\ &\quad + C\mathbb{E}\left[\int_0^t \int_0^1 |W_s^K - X_s|^2 |W_s^K - W_s^*(\alpha)|^\gamma d\alpha ds\right] \\ &\quad + C\mathbb{E}\left[\int_0^t \int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2 \min(|W_s^K - W_s^*(\alpha)|^\gamma, \ell) d\alpha ds\right]. \end{aligned} \quad (2.27)$$

Since $\gamma + \nu > 0$, it holds that $2 + 2\gamma/\nu > 0$. As a consequence, like in Step 2,

$$\mathbb{E}\left[\int_0^t \int_0^1 |W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu} d\alpha ds\right] \leq C_T[1 + \mathbb{E}[|W_s^K|^2] + m_2(f_0)] \leq C(T, f_0, \tilde{f}_0),$$

which gives

$$\lim_{K \rightarrow \infty} K^{1-2/\nu}\mathbb{E}\left[\int_0^t \int_0^1 |W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu} d\alpha ds\right] = 0.$$

Moreover, we recall that a.s. $\int_0^1 |W_s^K - W_s^*(\alpha)|^\gamma d\alpha \leq 1 + C_{\gamma,p}\|f_s\|_{L^p}$ as in Step 2, whence

$$\mathbb{E}\left[\int_0^t \int_0^1 |W_s^K - X_s|^2 |W_s^K - W_s^*(\alpha)|^\gamma d\alpha ds\right] \leq \int_0^t \mathbb{E}[|W_s^K - X_s|^2](1 + C_{\gamma,p}\|f_s\|_{L^p}) ds.$$

Letting $K \rightarrow \infty$, by dominated convergence, we find (recall (2.24))

$$\limsup_K \mathbb{E}\left[\int_0^t \int_0^1 |W_s^K - X_s|^2 |W_s^K - W_s^*(\alpha)|^\gamma d\alpha ds\right] \leq \int_0^t J_s(1 + C_{\gamma,p}\|f_s\|_{L^p}) ds.$$

Next, it is obvious that for each $\ell \geq 1$ fixed, for all $s \in [0, T]$, all $\alpha \in [0, 1]$, the function $v \mapsto \min(|v - W_s^*(\alpha)|^\gamma, \ell)$ is bounded and continuous. By (2.23), we conclude that $\lim_{K \rightarrow \infty} \mathbb{E}[\min(|W_s^K - W_s^*(\alpha)|^\gamma, \ell)] = \mathbb{E}[\min(|W_s - W_s^*(\alpha)|^\gamma, \ell)]$ and, by dominated convergence, that, still for $\ell \geq 1$ fixed,

$$\begin{aligned} &\lim_{K \rightarrow \infty} \mathbb{E}\left[\int_0^t \int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2 \min(|W_s^K - W_s^*(\alpha)|^\gamma, \ell) d\alpha ds\right] \\ &= \int_0^t \int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2 \mathbb{E}[\min(|W_s - W_s^*(\alpha)|^\gamma, \ell)] d\alpha ds. \end{aligned}$$

But since $W_s \sim f_s$, we have, for each α fixed, $\mathbb{E}[\min(|W_s - W_s^*(\alpha)|^\gamma, \ell)] \leq \int_{\mathbb{R}^3} |W_s^*(\alpha) - v|^\gamma f_s(dv) \leq 1 + C_{\gamma,p}\|f_s\|_{L^p}$ by (2.13). Furthermore, we have $\int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2 d\alpha =$

$\mathbb{E}_\alpha[|W_s^* - X_s^*|^2] = \mathcal{W}_2^2(f_s, \tilde{f}_s) \leq J_s$. All in all, we have checked that

$$\begin{aligned} & \lim_{K \rightarrow \infty} \mathbb{E} \left[\int_0^t \int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2 \min(|W_s^K - W_s^*(\alpha)|^\gamma, \ell) d\alpha ds \right] \\ & \leq C \int_0^t J_s (1 + \|f_s\|_{L^p}) ds. \end{aligned}$$

Gathering all the previous estimates to let $K \rightarrow \infty$ in (2.27): for each $\ell \geq 1$ fixed,

$$J_t \leq \mathcal{W}_2^2(f_0, \tilde{f}_0) + C(T, f_0, \tilde{f}_0, f) \ell^{-\kappa(\gamma)} + C \int_0^t J_s (1 + \|f_s\|_{L^p}) ds.$$

Letting now $\ell \rightarrow \infty$ and using the Grönwall lemma, we find

$$J_t \leq \mathcal{W}_2^2(f_0, \tilde{f}_0) \exp \left(C_{\gamma,p} \int_0^t (1 + \|f_s\|_{L^p}) ds \right).$$

Since $\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq J_t$, this completes the proof. \square

It remains to prove Proposition 2.3.1. We start with a technical result.

Lemma 2.3.4. *Assume (2.3) for some $\gamma \in (-1, 0)$, some $\nu \in (0, 1)$ with $\gamma + \nu > 0$ and recall that the deviation function c was defined by (2.14). Consider $f \in \mathcal{P}_2(\mathbb{R}^3)$ and $\phi_\epsilon(x) = (2\pi\epsilon)^{-3/2} e^{-|x|^2/(2\epsilon)}$. Set $f^\epsilon(w) = (f * \phi_\epsilon)(w)$.*

(i) *There exists a constant $C > 0$ such that for all $x \in \mathbb{R}^3$, all $\epsilon \in (0, 1)$,*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| \frac{\phi_\epsilon(v-x)}{f^\epsilon(x)} d\varphi dz f(dv) f(dv_*) \leq C \left(1 + \sqrt{m_2(f)} + |x| \right),$$

(ii) *For all $\epsilon \in (0, 1)$, all $R > 0$, there is a constant $C_{R,\epsilon} > 0$ (depending only on $m_2(f)$) such that for all $x, y \in B(0, R)$,*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| \left| \frac{\phi_\epsilon(v-x)}{f^\epsilon(x)} - \frac{\phi_\epsilon(v-y)}{f^\epsilon(y)} \right| d\varphi dz f(dv) f(dv_*) \leq C_{R,\epsilon} |x-y|.$$

Proof. We start with (i) and set $I_\epsilon(x) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| \frac{\phi_\epsilon(v-x)}{f^\epsilon(x)} d\varphi dz f(dv) f(dv_*)$. Using (2.8) and (2.5), we see that $|c(v, v_*, z, \varphi)| \leq G(z/|v-v_*|^\gamma) |v-v_*| \leq C(1+z/|v-v_*|^\gamma)^{-1/\nu} |v-v_*|$. Hence

$$\begin{aligned} I_\epsilon(x) & \leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty (1+z/|v-v_*|^\gamma)^{-1/\nu} |v-v_*| \frac{\phi_\epsilon(v-x)}{f^\epsilon(x)} dz f(dv) f(dv_*) \\ & = C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^{1+\gamma} \frac{\phi_\epsilon(v-x)}{f^\epsilon(x)} f(dv) f(dv_*). \end{aligned}$$

Using now that $|v - v_*|^{1+\gamma} \leq 1 + |v| + |v_*|$, we find

$$\begin{aligned} I_\epsilon(x) &\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v| + |v_*|) \frac{\phi_\epsilon(v-x)}{f^\epsilon(x)} f(dv) f(dv_*) \\ &\leq C \left(1 + \sqrt{m_2(f)} + \frac{\int_{\mathbb{R}^3} |v| \phi_\epsilon(v-x) f(dv)}{f^\epsilon(x)} \right). \end{aligned}$$

To conclude the proof of (i), it remains to study $J_\epsilon(x) = (f^\epsilon(x))^{-1} \int_{\mathbb{R}^3} |v| \phi_\epsilon(v-x) f(dv)$. We introduce $L := \sqrt{2m_2(f)}$, for which $f(B(0, L)) \geq 1/2$ (because $f(B(0, L)^c) \leq m_2(f)/L^2$). Using that $\{v \in \mathbb{R}^3 : |v| \leq 2|x| + L\} \cup \{v \in \mathbb{R}^3 : |v-x| \geq |x| + L\} = \mathbb{R}^3$, we write

$$J_\epsilon(x) = \frac{\int_{\mathbb{R}^3} |v| \phi_\epsilon(v-x) f(dv)}{\int_{\mathbb{R}^3} \phi_\epsilon(v-x) f(dv)} \leq 2|x| + L + \frac{\int_{|v-x| \geq |x|+L} |v| \phi_\epsilon(v-x) f(dv)}{\int_{|v-x| \leq |x|+L} \phi_\epsilon(v-x) f(dv)}.$$

Since ϕ_ϵ is radial and decreasing,

$$\int_{|v-x| \geq |x|+L} |v| \phi_\epsilon(v-x) f(dv) \leq \phi_\epsilon(|x| + L) \sqrt{m_2(f)}$$

and

$$\int_{|v-x| \leq |x|+L} \phi_\epsilon(v-x) f(dv) \geq \phi_\epsilon(|x| + L) f(B(x, |x| + L)) \geq \phi_\epsilon(|x| + L)/2$$

owing to the fact that $B(0, L) \subset B(x, |x| + L)$. Hence,

$$J_\epsilon(x) \leq 2|x| + L + 2\sqrt{m_2(f)} \leq 2|x| + 4\sqrt{m_2(f)}$$

and this completes the proof of (i).

For point (ii), we set

$$\Delta_\epsilon(x, y) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| |F_\epsilon(x, v) - F_\epsilon(y, v)| d\varphi dz f(dv) f(dv_*),$$

where $F_\epsilon(v, x) := (f^\epsilon(x))^{-1} \phi_\epsilon(v-x)$. Exactly as in point (i), we start with

$$\begin{aligned} \Delta_\epsilon(x, y) &\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{1+\gamma} |F_\epsilon(v, x) - F_\epsilon(v, y)| f(dv) f(dv_*) \\ &\leq C \int_{\mathbb{R}^3} (1 + \sqrt{m_2(f)} + |v|) |F_\epsilon(v, x) - F_\epsilon(v, y)| f(dv) \\ &\leq C|x-y| \int_{\mathbb{R}^3} (1 + \sqrt{m_2(f)} + |v|) \left(\sup_{a \in B(0, R)} |\nabla_x F_\epsilon(v, a)| \right) f(dv) \end{aligned}$$

for all $x, y \in B(0, R)$. But we have

$$\nabla_x F_\epsilon(v, a) = \frac{1}{\epsilon} \frac{\phi_\epsilon(v-a) \int_{\mathbb{R}^3} (v-u) \phi_\epsilon(u-a) f(du)}{(f^\epsilon(a))^2}. \quad (2.28)$$

Indeed, recalling that $\phi_\epsilon(x) = (2\pi\epsilon)^{-3/2} e^{-|x|^2/(2\epsilon)}$, we observe that

$$\nabla_x \phi_\epsilon(v-x) = \frac{1}{\epsilon} (v-x) \phi_\epsilon(v-x) \text{ and } \nabla_x f^\epsilon(x) = \frac{1}{\epsilon} \int_{\mathbb{R}^3} \phi_\epsilon(u-x) (u-x) f(du).$$

Since $F_\epsilon(v, a) := (f^\epsilon(a))^{-1} \phi_\epsilon(v-a)$, we have

$$\begin{aligned} \nabla_x F_\epsilon(v, a) &= \frac{\nabla_x \phi_\epsilon(v-a) f^\epsilon(a) - \phi_\epsilon(v-a) \nabla_x f^\epsilon(a)}{(f^\epsilon(a))^2} \\ &= \frac{\phi_\epsilon(v-a)}{\epsilon} \frac{(v-a) f^\epsilon(a) - \int_{\mathbb{R}^3} \phi_\epsilon(u-a) (u-a) f(du)}{(f^\epsilon(a))^2} \\ &= \frac{\phi_\epsilon(v-a)}{\epsilon} \frac{\int_{\mathbb{R}^3} \phi_\epsilon(u-a) (v-a) f(du) - \int_{\mathbb{R}^3} \phi_\epsilon(u-a) (u-a) f(du)}{(f^\epsilon(a))^2}, \end{aligned}$$

whence (2.28). Using now that $J_\epsilon(a) = (f^\epsilon(a))^{-1} \int_{\mathbb{R}^3} |u| \phi_\epsilon(u-a) f(du) \leq 2|a| + 4\sqrt{m_2(f)}$ as proved in (i),

$$\begin{aligned} |\nabla_x F_\epsilon(v, a)| &\leq \frac{1}{\epsilon} \frac{\phi_\epsilon(v-a)}{f^\epsilon(a)} \frac{\int_{\mathbb{R}^3} (|v| + |u|) \phi_\epsilon(u-a) f(du)}{f^\epsilon(a)} \\ &\leq \frac{1}{\epsilon} \frac{\phi_\epsilon(v-a)}{f^\epsilon(a)} \left(|v| + 2|a| + 4\sqrt{m_2(f)} \right). \end{aligned}$$

But we know that $\phi_\epsilon(x) \leq (2\pi\epsilon)^{-3/2}$ and that

$$f^\epsilon(a) \geq \int_{|v-a| \leq |a|+L} \phi_\epsilon(v-a) f(dv) \geq \phi_\epsilon(|a|+L) f(B(a, |a|+L)) \geq \phi_\epsilon(|a|+L)/2$$

since $B(0, L) \subset B(a, |a|+L)$. Hence,

$$\sup_{a \in B(0, R)} |\nabla_x F_\epsilon(v, a)| \leq \frac{2}{\epsilon} e^{(R+L)^2/(2\epsilon)} \left(|v| + 2R + 4\sqrt{m_2(f)} \right).$$

Consequently, for all $x, y \in B(0, R)$,

$$\begin{aligned} \Delta_\epsilon(x, y) &\leq \frac{2C}{\epsilon} e^{(R+L)^2/(2\epsilon)} |x-y| \int_{\mathbb{R}^3} \left(1 + \sqrt{m_2(f)} + |v| \right) \left(|v| + 2R + 4\sqrt{m_2(f)} \right) f(dv) \\ &\leq C_{R,\epsilon} |x-y|, \end{aligned}$$

where $C_{R,\epsilon}$ depends only on R, ϵ and $m_2(f)$ (recall that $L := \sqrt{2m_2(f)}$). \square

Finally, we end the section with the

Proof of Proposition 2.3.1. We consider any given weak solution $(\tilde{f}_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ to (2.1) and we write the proof in several steps.

Step 1. We introduce $\phi_\epsilon(x) = (2\pi\epsilon)^{-3/2}e^{-|x|^2/(2\epsilon)}$ and $\tilde{f}_t^\epsilon(w) = (\tilde{f}_t * \phi_\epsilon)(w)$. For each $t \geq 0$, we see that \tilde{f}_t^ϵ is a positive smooth function. We claim that for any $\psi \in Lip(\mathbb{R}^3)$,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \psi(w) \tilde{f}_t^\epsilon(dw) = \int_{\mathbb{R}^3} \tilde{f}_t^\epsilon(dw) \tilde{\mathcal{A}}_{t,\epsilon} \psi(w),$$

where

$$\tilde{\mathcal{A}}_{t,\epsilon} \psi(w) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} [\psi(w + c(v, v_*, z, \varphi)) - \psi(w)] \frac{\phi_\epsilon(v - w)}{\tilde{f}_t^\epsilon(w)} d\varphi dz \tilde{f}_t(dv_*) \tilde{f}_t(dv). \quad (2.29)$$

Indeed, $\tilde{f}_t^\epsilon(w) = \int_{\mathbb{R}^3} \phi_\epsilon(v - w) \tilde{f}_t(dv)$ since $\phi_\epsilon(x)$ is even. According to (2.10) and (2.15), we have

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{f}_t^\epsilon(w) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} [\phi_\epsilon(v - w + c(v, v_*, z, \varphi)) - \phi_\epsilon(v - w)] d\varphi dz \tilde{f}_t(dv_*) \tilde{f}_t(dv) \\ &= \int_{\mathbb{R}^3} \int_0^K \int_0^{2\pi} \left[\int_{\mathbb{R}^3} \phi_\epsilon(v - w + c(v, v_*, z, \varphi)) \tilde{f}_t(dv) - \tilde{f}_t^\epsilon(w) \right] d\varphi dz \tilde{f}_t(dv_*) \\ &\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_K^\infty \int_0^{2\pi} [\phi_\epsilon(v - w + c(v, v_*, z, \varphi)) - \phi_\epsilon(v - w)] d\varphi dz \tilde{f}_t(dv_*) \tilde{f}_t(dv) \end{aligned}$$

for any $K \geq 1$. We thus have, for any $\psi \in Lip(\mathbb{R}^3)$,

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \psi(w) \tilde{f}_t^\epsilon(dw) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^K \int_0^{2\pi} \int_{\mathbb{R}^3} \phi_\epsilon(v - w + c(v, v_*, z, \varphi)) \psi(w) \tilde{f}_t(dv) d\varphi dz \tilde{f}_t(dv_*) dw \\ &\quad - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^K \int_0^{2\pi} \psi(w) \tilde{f}_t^\epsilon(w) d\varphi dz \tilde{f}_t(dv_*) dw \\ &\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_K^\infty \int_0^{2\pi} [\phi_\epsilon(v - w + c(v, v_*, z, \varphi)) - \phi_\epsilon(v - w)] \psi(w) d\varphi dz \tilde{f}_t(dv_*) \tilde{f}_t(dv) dw. \end{aligned}$$

Using the change of variables $w - c(v, v_*, z, \varphi) \mapsto w$, we see that the first integral of the RHS equals

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^K \int_0^{2\pi} \int_{\mathbb{R}^3} \phi_\epsilon(v - w) \psi(w + c(v, v_*, z, \varphi)) \tilde{f}_t(dv) d\varphi dz \tilde{f}_t(dv_*) dw.$$

Consequently,

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \psi(w) \tilde{f}_t^\epsilon(dw) \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^K \int_0^{2\pi} \left[\int_{\mathbb{R}^3} \psi(w + c(v, v_*, z, \varphi)) \frac{\phi_\epsilon(v-w)}{\tilde{f}_t^\epsilon(w)} \tilde{f}_t(dw) - \psi(w) \right] \tilde{f}_t^\epsilon(w) d\varphi dz \tilde{f}_t(dv_*) dw \\
&\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_K^\infty \int_0^{2\pi} [\phi_\epsilon(v-w + c(v, v_*, z, \varphi)) - \phi_\epsilon(v-w)] \psi(w) d\varphi dz \tilde{f}_t(dv_*) \tilde{f}_t(dw) dw \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^K \int_0^{2\pi} \int_{\mathbb{R}^3} [\psi(w + c(v, v_*, z, \varphi)) - \psi(w)] \frac{\phi_\epsilon(v-w)}{\tilde{f}_t^\epsilon(w)} \tilde{f}_t(dw) d\varphi dz \tilde{f}_t(dv_*) \tilde{f}_t^\epsilon(dw) \\
&\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_K^\infty \int_0^{2\pi} [\phi_\epsilon(v-w + c(v, v_*, z, \varphi)) - \phi_\epsilon(v-w)] \psi(w) d\varphi dz \tilde{f}_t(dv_*) \tilde{f}_t(dw) dw.
\end{aligned}$$

Letting K increase to infinity, one easily ends the step.

Step 2. We set $F_{t,\epsilon}(v, x) = (\tilde{f}_t^\epsilon(x))^{-1} \phi_\epsilon(v-x)$. For a given X_0^ϵ with law \tilde{f}_0^ϵ , and a given independent Poisson measure $N(ds, dv, dv_*, dz, d\varphi, du)$ on $[0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \times [0, \infty) \times [0, 2\pi) \times [0, \infty)$ with intensity $ds \tilde{f}_s(dw) \tilde{f}_s(dv_*) dz d\varphi du$, there exists a pathwise unique solution to

$$X_t^\epsilon = X_0^\epsilon + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} \int_0^\infty c(v, v_*, z, \varphi) \mathbf{1}_{\{u \leq F_{s,\epsilon}(v, X_{s-}^\epsilon)\}} N(ds, dv, dv_*, dz, d\varphi, du). \quad (2.30)$$

This classically follows from Lemma 2.3.4, which precisely tells us that the coefficients of this equation satisfy some *at most linear growth condition* (point (i)) and some *local Lipschitz condition* (point (ii)).

Step 3. We now prove that $\mathcal{L}(X_t^\epsilon) = \tilde{f}_t^\epsilon$ for each $t \geq 0$. We thus introduce $g_t^\epsilon = \mathcal{L}(X_t^\epsilon)$. By the Itô formula, we see that for all $\psi \in Lip(\mathbb{R}^3)$,

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \psi(w) g_t^\epsilon(dw) \\
&= \int_{\mathbb{R}^3} g_t^\epsilon(dw) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} \left(\psi(w + c(v, v_*, z, \varphi)) - \psi(w) \right) F_{t,\epsilon}(v, w) d\varphi dz \tilde{f}_t(dv_*) \tilde{f}_t(dw) \\
&= \int_{\mathbb{R}^3} g_t^\epsilon(dw) \tilde{\mathcal{A}}_{t,\epsilon} \psi(w).
\end{aligned}$$

Thus $(\tilde{f}_t^\epsilon)_{t \geq 0}$ and $(g_t^\epsilon)_{t \geq 0}$ satisfy the same equation and we of course have $g_0^\epsilon = \tilde{f}_0^\epsilon$ by construction. The following uniqueness result allows us to conclude the step: for any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^3)$, there exists at most one family $(\mu_t) \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ such that for any $\psi \in Lip(\mathbb{R}^3)$, any $t \geq 0$,

$$\int_{\mathbb{R}^3} \psi(w) \mu_t(dw) = \int_{\mathbb{R}^3} \psi(w) \mu_0(dw) + \int_0^t ds \int_{\mathbb{R}^3} \mu_s(dw) \tilde{\mathcal{A}}_{s,\epsilon} \psi(w). \quad (2.31)$$

This must be classical (as well as Step 2 is), but we find no precise reference and thus make use of martingale problems. A càdlàg adapted \mathbb{R}^3 -valued process $(Z_t)_{t \geq 0}$ on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is said to solve the martingale problem $MP(\tilde{\mathcal{A}}_{t,\epsilon}, \mu_0, Lip(\mathbb{R}^3))$ if $\mathbb{P} \circ Z_0 = \mu_0$ and if for all $\psi \in Lip(\mathbb{R}^3)$, $(M_t^{\psi,\epsilon})_{t \geq 0}$ is a $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ -martingale, where

$$M_t^{\psi,\epsilon} = \psi(Z_t) - \int_0^t \tilde{\mathcal{A}}_{s,\epsilon} \psi(Z_s) ds.$$

According to [10, Theorem 5.2] (see also [10, Remark 3.1, Theorem 5.1] and [38, Theorem B.1]), it suffices to check the following points to conclude the uniqueness for (2.31).

(i) there exists a countable family $(\psi_k)_{k \geq 1} \subset Lip(\mathbb{R}^3)$ such that for all $t \geq 0$, the closure (for the bounded pointwise convergence) of $\{(\psi_k, \tilde{\mathcal{A}}_{t,\epsilon} \psi_k), k \geq 1\}$ contains $\{(\psi, \tilde{\mathcal{A}}_{t,\epsilon} \psi), \psi \in Lip(\mathbb{R}^3)\}$,

(ii) for each $w_0 \in \mathbb{R}^3$, there exists a solution to $MP(\tilde{\mathcal{A}}_{t,\epsilon}, \delta_{w_0}, Lip(\mathbb{R}^3))$,

(iii) for each $w_0 \in \mathbb{R}^3$, uniqueness (in law) holds for $MP(\tilde{\mathcal{A}}_{t,\epsilon}, \delta_{w_0}, Lip(\mathbb{R}^3))$.

The fact that (2.30) has a pathwise unique solution proved in Step 2 (there we can of course replace X_0^ϵ by any deterministic point $w_0 \in \mathbb{R}^3$) immediately implies (ii) and (iii). Point (i) is very easy (recall that $\epsilon > 0$ is fixed here).

Step 4. In this step, we check that the family $((X_t^\epsilon)_{t \geq 0})_{\epsilon > 0}$ is tight in $\mathbb{D}([0, \infty), \mathbb{R}^3)$. To do this, we use the Aldous criterion [1], see also [40, p 321], i.e. it suffices to prove that for all $T > 0$,

$$\sup_{\epsilon \in (0,1)} \mathbb{E} \left[\sup_{[0,T]} |X_t^\epsilon| \right] < \infty, \quad \lim_{\delta \rightarrow 0} \sup_{\epsilon \in (0,1)} \sup_{S, S' \in \mathcal{S}_T(\delta)} \mathbb{E} [|X_{S'}^\epsilon - X_S^\epsilon|] = 0, \quad (2.32)$$

where $\mathcal{S}_T(\delta)$ is the set containing all pairs of stopping times (S, S') satisfying $0 \leq S \leq S' \leq S + \delta \leq T$.

First, since $X_t^\epsilon \sim \tilde{f}_t^\epsilon = \tilde{f}_t \star \phi_\epsilon$, we have $\mathbb{E}[|X_t^\epsilon|^2] \leq 2(m_2(\tilde{f}_t) + 3\epsilon) \leq 2m_2(\tilde{f}_0) + 6$. Thus for any $T > 0$, using Lemma 2.3.4-(i),

$$\begin{aligned} & \mathbb{E} \left[\sup_{[0,T]} |X_t^\epsilon| \right] \\ & \leq \mathbb{E}[|X_0^\epsilon|] + \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| \frac{\phi_\epsilon(v - X_s^\epsilon)}{\tilde{f}_s^\epsilon(X_s^\epsilon)} d\varphi dz \tilde{f}_s(dv) \tilde{f}_s(dv_*) ds \right] \\ & \leq \mathbb{E}[|X_0^\epsilon|] + C \mathbb{E} \left[\int_0^T (1 + |X_s^\epsilon|) ds \right] \leq C_T. \end{aligned}$$

Furthermore, for any $T > 0$, $\delta > 0$ and $(S, S') \in \mathcal{S}_T(\delta)$, using again Lemma 2.3.4-(i),

$$\begin{aligned} & \mathbb{E}[|X_{S'}^\epsilon - X_S^\epsilon|] \\ & \leq \mathbb{E} \left[\int_S^{S+\delta} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| \frac{\phi_\epsilon(v - X_s^\epsilon)}{\tilde{f}_s^\epsilon(X_s^\epsilon)} d\varphi dz \tilde{f}_s^\epsilon(dv) \tilde{f}_s^\epsilon(dv_*) ds \right] \\ & \leq C \mathbb{E} \left[\int_S^{S+\delta} (1 + |X_s^\epsilon|) ds \right] \leq C \mathbb{E} \left[\delta \sup_{[0, T]} (1 + |X_s^\epsilon|) \right] \leq C_T \delta. \end{aligned}$$

Hence (2.32) holds true and this completes the step.

Step 5. We thus can find some $(X_t)_{t \geq 0}$ which is the limit in law (for the Skorokhod topology) of a sequence $(X_t^{\epsilon_n})_{t \geq 0}$ with $\epsilon_n \searrow 0$. Since $\mathcal{L}(X_t^{\epsilon_n}) = \tilde{f}_t^{\epsilon_n}$ by Step 3 and since $\tilde{f}_t^{\epsilon_n} \rightarrow \tilde{f}_t$ by definition, we have $\mathcal{L}(X_t) = \tilde{f}_t$ for each $t \geq 0$. It only remains to show that $(X_t)_{t \geq 0}$ is a (weak) solution to (2.21). Using the theory of martingale problems, see Jacod [39, Theorem 13.55], it classically suffices to prove that for any $\psi \in C_b^1(\mathbb{R}^3)$, the process $\psi(X_t) - \psi(X_0) - \int_0^t \mathcal{B}_s \psi(X_s) ds$ is a martingale, where

$$\mathcal{B}_t \psi(x) = \int_0^1 \int_0^\infty \int_0^{2\pi} \left(\psi(x + c(x, X_t^*(\alpha), z, \varphi)) - \psi(x) \right) d\varphi dz d\alpha.$$

But since $\mathcal{L}_\alpha(X_t^*) = \tilde{f}_t$, this rewrites (recall (2.15))

$$\mathcal{B}_t \psi(x) = \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} \left(\psi(x + c(x, v_*, z, \varphi)) - \psi(x) \right) d\varphi dz \tilde{f}_t(dv_*) = \int_{\mathbb{R}^3} \mathcal{A} \psi(x, v_*) \tilde{f}_t(dv_*).$$

We thus have to prove that for any $0 \leq s_1 \leq \dots \leq s_k \leq s \leq t \leq T$, any $\psi_1, \dots, \psi_k \in C_b^1(\mathbb{R}^3)$, and any $\psi \in C_b^1(\mathbb{R}^3)$,

$$\mathbb{E}[\mathcal{F}(X)] = 0,$$

where $\mathcal{F} : \mathbb{D}([0, \infty), \mathbb{R}^3) \mapsto \mathbb{R}$ is defined by

$$\mathcal{F}(\lambda) = \left(\prod_{i=1}^k \psi_i(\lambda_{s_i}) \right) \left(\psi(\lambda_t) - \psi(\lambda_s) - \int_s^t \mathcal{B}_r \psi(\lambda_r) dr \right).$$

We of course start from $\mathbb{E}[\mathcal{F}_{\epsilon_n}(X^{\epsilon_n})] = 0$, where, recalling (2.29),

$$\mathcal{F}_\epsilon(\lambda) = \left(\prod_{i=1}^k \psi_i(\lambda_{s_i}) \right) \left(\psi(\lambda_t) - \psi(\lambda_s) - \int_s^t \tilde{\mathcal{A}}_{r, \epsilon} \psi(\lambda_r) dr \right).$$

We then write

$$\left| \mathbb{E}[\mathcal{F}(X)] \right| \leq \left| \mathbb{E}[\mathcal{F}(X)] - \mathbb{E}[\mathcal{F}(X^{\epsilon_n})] \right| + \left| \mathbb{E}[\mathcal{F}(X^{\epsilon_n})] - \mathbb{E}[\mathcal{F}_{\epsilon_n}(X^{\epsilon_n})] \right|.$$

On the one hand, we know from [24, Lemma 3.3] that $(x, v_*) \mapsto \mathcal{A}\psi(x, v_*)$ is continuous on $\mathbb{R}^3 \times \mathbb{R}^3$ and bounded by $C|x - v_*|^{\gamma+1}$. We thus easily deduce that \mathcal{F} is continuous at each $\lambda \in \mathbb{D}([0, \infty), \mathbb{R}^3)$ which does not jump at s_1, \dots, s_k, s, t (this is a.s. the case of $X \in \mathbb{D}([0, \infty), \mathbb{R}^3)$ because it has no deterministic time jump by the Aldous criterion). We also deduce that $|\mathcal{F}(\lambda)| \leq C(1 + \int_0^t \int_{\mathbb{R}^3} |\lambda_r - v_*|^{\gamma+1} \tilde{f}_r(dv_*) dr)$. Using that $0 < \gamma + 1 < 1$, that $\sup_{\epsilon \in (0,1)} \mathbb{E}[\sup_{[0,T]} |X_t^\epsilon|] < \infty$ by Step 4 and recalling that X^{ϵ_n} goes in law to X , we easily conclude that $|\mathbb{E}[\mathcal{F}(X)] - \mathbb{E}[\mathcal{F}(X^{\epsilon_n})]|$ tends to 0 as $n \rightarrow \infty$.

On the other hand, since $|\mathcal{F}(\lambda) - \mathcal{F}_\epsilon(\lambda)| \leq C|\int_s^t (\mathcal{B}_r\psi(\lambda_r) - \tilde{\mathcal{A}}_{r,\epsilon}\psi(\lambda_r))dr|$ and $X_r^\epsilon \sim \tilde{f}_r^\epsilon$,

$$\begin{aligned} & \left| \mathbb{E}[\mathcal{F}(X^{\epsilon_n})] - \mathbb{E}[\mathcal{F}_{\epsilon_n}(X^{\epsilon_n})] \right| \\ & \leq C \int_s^t \mathbb{E} \left[\left| \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^3} \psi(X_r^{\epsilon_n} + c(v, v_*, z, \varphi)) \right. \right. \\ & \quad \left. \left. \left[\frac{\phi_{\epsilon_n}(v - X_r^{\epsilon_n})}{\tilde{f}_r^{\epsilon_n}(X_r^{\epsilon_n})} \tilde{f}_r(dv) - \delta_{X_r^{\epsilon_n}}(dv) \right] d\varphi dz \tilde{f}_r(dv_*) \right| \right] dr \\ & = C \int_s^t \left| \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(w + c(v, v_*, z, \varphi)) \right. \\ & \quad \left. \left[\phi_{\epsilon_n}(v - w) \tilde{f}_r(dv) - \tilde{f}_r^{\epsilon_n}(w) \delta_w(dv) \right] dw d\varphi dz \tilde{f}_r(dv_*) \right| dr. \end{aligned}$$

But we can write

$$\begin{aligned} & \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(w + c(v, v_*, z, \varphi)) \tilde{f}_r^{\epsilon_n}(w) \delta_w(dv) dw \\ & = \int_{\mathbb{R}^3} \psi(w + c(w, v_*, z, \varphi)) \tilde{f}_r^{\epsilon_n}(w) dw \\ & = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(w + c(w, v_*, z, \varphi)) \phi_{\epsilon_n}(v - w) \tilde{f}_r(dv) dw, \end{aligned}$$

so that

$$\begin{aligned} & \left| \mathbb{E}[\mathcal{F}(X^{\epsilon_n})] - \mathbb{E}[\mathcal{F}_{\epsilon_n}(X^{\epsilon_n})] \right| \\ & \leq C \int_s^t \left| \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[\psi(w + c(v, v_*, z, \varphi)) - \psi(w + c(w, v_*, z, \varphi)) \right] \right. \\ & \quad \left. \phi_{\epsilon_n}(v - w) \tilde{f}_r(dv) dw d\varphi dz \tilde{f}_r(dv_*) \right| dr \\ & = C \int_s^t \left| \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[\psi(w + c(v, v_*, z, \varphi + \varphi_0(v - v_*, w - v_*))) \right. \right. \\ & \quad \left. \left. - \psi(w + c(w, v_*, z, \varphi)) \right] \phi_{\epsilon_n}(v - w) \tilde{f}_r(dv) dw d\varphi dz \tilde{f}_r(dv_*) \right| dr. \end{aligned}$$

The last equality uses the 2π -periodicity of c . We now put

$$R_n(v, v_*, z, \varphi) := \int_{\mathbb{R}^3} \left[\psi(w + c(v, v_*, z, \varphi + \varphi_0(v - v_*, w - v_*))) - \psi(w + c(w, v_*, z, \varphi)) \right] \phi_{\epsilon_n}(v - w) dw,$$

and show the following two things:

(a) for all $v, v_* \in \mathbb{R}^3$, all $z \in [0, \infty)$ and $\varphi \in [0, 2\pi)$, $\lim_{n \rightarrow \infty} R_n(v, v_*, z, \varphi) = 0$;

(b) there is a constant $C > 0$ such that for all $n \geq 1$, all $v, v_* \in \mathbb{R}^3$, all $z \in [0, \infty)$ and $\varphi \in [0, 2\pi)$,

$$|R_n(v, v_*, z, \varphi)| \leq C(1 + |v - v_*|)(1 + z)^{-1/\nu},$$

which belongs to $L^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times [0, \infty) \times [0, 2\pi))$, $dr \tilde{f}_r(dv_*) \tilde{f}_r(dv) dz d\varphi$ because $(\tilde{f}_t)_{t \geq 0} \in L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$ by assumption.

By dominated convergence, we will deduce that $\lim_{n \rightarrow \infty} \left| \mathbb{E}[\mathcal{F}(X^{\epsilon_n})] - \mathbb{E}[\mathcal{F}_{\epsilon_n}(X^{\epsilon_n})] \right| = 0$ and this will conclude the proof.

We first study (a). Since $\psi \in C_b^1(\mathbb{R}^3)$, we immediately observe that

$$\begin{aligned} & \left| \psi(w + c(v, v_*, z, \varphi + \varphi_0(v - v_*, w - v_*))) - \psi(w + c(w, v_*, z, \varphi)) \right| \\ & \leq C_\psi \left| c(v, v_*, z, \varphi + \varphi_0(v - v_*, w - v_*)) - c(w, v_*, z, \varphi) \right|. \end{aligned} \quad (2.33)$$

Recalling that

$$c(v, v_*, z, \varphi) = -\frac{1 - \cos G(z/|v - v_*|^\gamma)}{2}(v - v_*) + \frac{\sin G(z/|v - v_*|^\gamma)}{2}\Gamma(v - v_*, \varphi),$$

we have

$$\begin{aligned} & \left| c(v, v_*, z, \varphi + \varphi_0(v - v_*, w - v_*)) - c(w, v_*, z, \varphi) \right| \\ & \leq \frac{|\cos G(z/|v - v_*|^\gamma) - \cos G(z/|w - v_*|^\gamma)|}{2} |v - v_*| + \frac{|1 - \cos G(z/|w - v_*|^\gamma)|}{2} |v - w| \\ & \quad + \frac{|\sin G(z/|v - v_*|^\gamma) - \sin G(z/|w - v_*|^\gamma)|}{2} |\Gamma(v - v_*, \varphi + \varphi_0)| \\ & \quad + \frac{|\sin G(z/|w - v_*|^\gamma)|}{2} |\Gamma(v - v_*, \varphi + \varphi_0) - \Gamma(w - v_*, \varphi)|. \end{aligned}$$

Using that $|\Gamma(v - v_*, \varphi + \varphi_0)| = |v - v_*|$ and Lemma 2.2.2, we obtain

$$\begin{aligned} & \left| c(v, v_*, z, \varphi + \varphi_0(v - v_*, w - v_*)) - c(w, v_*, z, \varphi) \right| \\ & \leq C |G(z/|v - v_*|^\gamma) - G(z/|w - v_*|^\gamma)| |v - v_*| + C |v - w|. \end{aligned}$$

We deduce from (2.4) that $|G'(z)| = 1/\beta(G(z)) \leq C$ by (2.3), whence

$$\begin{aligned} & \left| c(v, v_*, z, \varphi + \varphi_0(v - v_*, w - v_*)) - c(w, v_*, z, \varphi) \right| \\ & \leq Cz \left| |v - v_*|^{|\gamma|} - |w - v_*|^{|\gamma|} \right| |v - v_*| + C|v - w|. \end{aligned}$$

Using again the inequality $|x^\alpha - y^\alpha| \leq |x - y|(x \vee y)^{\alpha-1}$ for $\alpha \in (0, 1)$, and $x, y \geq 0$, we have

$$\left| |v - v_*|^{|\gamma|} - |w - v_*|^{|\gamma|} \right| \leq |v - w| |v - v_*|^{|\gamma|-1}.$$

We thus get

$$\left| c(v, v_*, z, \varphi + \varphi_0(v - v_*, w - v_*)) - c(w, v_*, z, \varphi) \right| \leq C(z|v - v_*|^{|\gamma|} + 1)|v - w|.$$

Consequently,

$$R_n(v, v_*, z, \varphi) \leq C_\psi (z|v - v_*|^{|\gamma|} + 1) \int_{\mathbb{R}^3} |v - w| \phi_{\epsilon_n}(v - w) dw,$$

which clearly tends to 0 as $n \rightarrow \infty$. This ends the proof of (a).

For (b), start again from (2.33) to write

$$\begin{aligned} & \left| \psi(w + c(v, v_*, z, \varphi + \varphi_0(v - v_*, w - v_*))) - \psi(w + c(w, v_*, z, \varphi)) \right| \\ & \leq \left| \psi(w + c(v, v_*, z, \varphi)) - \psi(w) \right| + \left| \psi(w) - \psi(w + c(w, v_*, z, \varphi)) \right| \\ & \leq C_\psi (|c(v, v_*, z, \varphi)| + |c(w, v_*, z, \varphi)|). \end{aligned}$$

Moreover, since $|c(v, v_*, z, \varphi)| \leq G(z/|v - v_*|^\gamma) |v - v_*| \leq C|v - v_*|(1 + |v - v_*|^{|\gamma|} z)^{-1/\nu}$ by (2.8) and (2.5), we observe that

$$\begin{aligned} R_n(v, v_*, z, \varphi) & \leq C|v - v_*|(1 + |v - v_*|^{|\gamma|} z)^{-1/\nu} \\ & \quad + C \int_{\mathbb{R}^3} |w - v_*|(1 + |w - v_*|^{|\gamma|} z)^{-1/\nu} \phi_{\epsilon_n}(v - w) dw. \end{aligned}$$

Since $1 + |v - v_*|^{|\gamma|} z \geq (1 \wedge |v - v_*|^{|\gamma|})(1 + z)$ for $z \in [0, \infty)$,

$$|v - v_*|(1 + |v - v_*|^{|\gamma|} z)^{-1/\nu} \leq |v - v_*|(1 + z)^{-1/\nu} (1 \wedge |v - v_*|^{|\gamma|})^{-1/\nu}.$$

Using that $|\gamma|/\nu < 1$, we deduce that

$$|v - v_*|(1 + |v - v_*|^{|\gamma|} z)^{-1/\nu} \leq (1 + |v - v_*|)(1 + z)^{-1/\nu}.$$

As a conclusion,

$$R_n(v, v_*, z, \varphi) \leq C \left(1 + |v - v_*| + \int_{\mathbb{R}^3} |w - v_*| \phi_{\epsilon_n}(v - w) dw \right) (1 + z)^{-1/\nu},$$

which is easily bounded (recall that $\epsilon_n \in (0, 1)$) by $C(1 + |v| + |v_*|)(1 + z)^{-1/\nu}$ as desired. \square

2.4 The coupling

2.4.1 Main ideas of the proof of Theorem 2.1.5

The proof of Theorem 2.1.5 is very technical, so let us exhibit the main ideas. We consider the unique *strong* solution $(f_t)_{t \geq 0}$ to (2.1) given in Theorem 2.1.3. We first couple $(W_t^1, \dots, W_t^N)_{t \geq 0}$ (i.i.d copies of $(W_t)_{t \geq 0}$ solution to the SDE associated to $(f_t)_{t \geq 0}$) and the Nanbu particle system $(V_t^1, \dots, V_t^N)_{t \geq 0}$ in such a way that, roughly, as soon as possible, each time W_t^i has a jump $c(W_{t-}^i, W_t^*(\alpha), z, \varphi)$, V_t^i also has a jump $c_K(V_{t-}^i, V_t^j, z, \varphi)$ with V_t^j as close as possible to $W_t^*(\alpha)$. So, we construct a coupling between $W_t^*(\alpha)$ (with law f_t) and V_t^j (with law $\mu_t^{N,K}$) in Lemma 2.4.2 as Fournier-Mischler [32], see also [23]. Unfortunately, there are many problems: we have to use in a complicated way the function φ_0 of Lemma 2.2.2, and to use an intermediate coupling between the empirical measure of the V_t^i 's and the W_t^i 's.

To get the convergence rate, we roughly apply the stability principle in Theorem 2.1.4, and find that $\mathcal{W}_2^2(\mu_t^{N,K}, \mu_{\mathbf{W}_t}^N)$ should be bounded by (some natural error terms) $\times \exp\left(C_{\gamma,p} \int_0^t (1 + \|\mu_{\mathbf{W}_t}^N\|_{L^p}) ds\right)$, but it is not correct since the empirical measure does not have a finite L^p norm. We thus consider a regularized version (i.e. $\bar{\mu}_{\mathbf{W}_t}^N = \mu_{\mathbf{W}_t}^N * \psi_{\epsilon_N}$), with a small parameter ϵ_N . Here $\psi_\epsilon = (3/(4\pi\epsilon^3)) \mathbf{1}_{\{|x| \leq \epsilon\}}$. This introduces some additional error terms, but we are able to bound, uniformly in N , the L^p -norm of $\bar{\mu}_{\mathbf{W}_t}^N$. This is difficult, but not surprising. Indeed, it is well-known from statistics that, if (X_1, \dots, X_N) are i.i.d with density $g \in L^p$, then $\|\frac{1}{N} \sum_{i=1}^N \delta_{X_i} * \psi_{\epsilon_N}\|_{L^p} \leq 2\|g\|_{L^p}$ with high probability if ϵ_N is well-chosen. So for each fixed $t \geq 0$, we apply such a principle, but we need to get something similar (locally) uniformly in time. For this, we use some continuity properties of the W_t^i 's, and again this is complicated since they are only càdlàg.

Now we have all this in mind, we realize that we also need to take into account the regularization (by convolution with ψ_{ϵ_N}) when introducing the coupling between the V_t^i 's and the W_t^i 's.

2.4.2 The coupling

To get the convergence of the particle system, we construct a suitable coupling between the particle system with generator $\mathcal{L}_{N,K}$ defined by (2.16) and the realization of the weak solution to (2.1), following the ideas of [32].

Lemma 2.4.1. *Assume (2.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu \in (0, 1)$. Let $N \geq 1$ be fixed. Let $q \geq 2$ such that $q > \gamma^2/(\gamma + \nu)$. Let $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ with a finite entropy and let $(f_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ (with $p \in (3/(3+\gamma), p_0(\gamma, \nu, q))$) be the unique weak solution to (2.1) given by Theorem 2.1.3. Then there exists, on some probability*

space, a family of i.i.d. random variables $(V_0^i)_{i=1,\dots,N}$ with common law f_0 , independent of a family of i.i.d. Poisson measures $(M_i(ds, d\alpha, dz, d\varphi))_{i=1,\dots,N}$ on $[0, \infty) \times [0, 1] \times [0, \infty) \times [0, 2\pi)$, with intensity $dsd\alpha dzd\varphi$, a measurable family $(W_t^*)_{t \geq 0}$ of α -random variables with α -law $(f_t)_{t \geq 0}$ and N i.i.d. càdlàg adapted processes $(W_t^i)_{t \geq 0}$ solving, for each $i = 1, \dots, N$,

$$W_t^i = V_0^i + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c(W_{s-}^i, W_s^*(\alpha), z, \varphi) M_i(ds, d\alpha, dz, d\varphi). \quad (2.34)$$

Moreover, $W_t^i \sim f_t$ for all $t \geq 0$, all $i = 1, \dots, N$. Also, for all $T > 0$,

$$\mathbb{E} \left[\sup_{[0, T]} |W_t^1|^q \right] \leq C_{T, q}. \quad (2.35)$$

Proof. Except for the moment estimate (2.35), it suffices to apply Proposition 2.3.1. A simpler proof could be handled here because we deal with the *strong* solution $f \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L_{loc}^1([0, \infty), L^p(\mathbb{R}^3))$. We now prove (2.35), which is more or less classical. We thus fix $q \geq 2$. It is clear that

$$|v + c(v, v_*, z, \varphi)|^q - |v|^q \leq C_q (|v|^{q-1} + |c(v, v_*, z, \varphi)|^{q-1}) |c(v, v_*, z, \varphi)|.$$

Due to (2.8) and (2.5), $|c(v, v_*, z, \varphi)| \leq |v - v_*|$, $|c(v, v_*, z, \varphi)| \leq (1 + z/|v - v_*|^\gamma)^{-1/\nu} |v - v_*|$, whence

$$\begin{aligned} & \int_0^\infty \int_0^{2\pi} \left| |v + c(v, v_*, z, \varphi)|^q - |v|^q \right| d\varphi dz \\ & \leq C_q \int_0^\infty \int_0^{2\pi} (1 + |v|^{q-1} + |v_*|^{q-1}) (1 + z/|v - v_*|^\gamma)^{-1/\nu} |v - v_*| d\varphi dz \\ & = C_q (1 + |v|^{q-1} + |v_*|^{q-1}) |v - v_*|^{1+\gamma} \\ & \leq C_q (1 + |v|^q + |v_*|^q), \end{aligned} \quad (2.36)$$

because $0 < 1 + \gamma < 1$. It then easily follows from the Itô formula and $\mathcal{L}_\alpha(W_t^*) = f_t = \mathcal{L}(W_t^1)$ that

$$\begin{aligned} \mathbb{E} \left[\sup_{[0, t]} |W_s^1|^q \right] & \leq \mathbb{E}[|V_0^1|^q] + C_q \int_0^t \int_0^1 \mathbb{E} \left[1 + |W_s^1|^q + |W_s^*(\alpha)|^q \right] d\alpha ds \\ & \leq \mathbb{E}[|V_0^1|^q] + C_q \int_0^t \left(1 + \mathbb{E} \left[\sup_{[0, s]} |W_u^1|^q \right] \right) ds. \end{aligned}$$

We thus conclude (2.35) by the Grönwall lemma. \square

Next, let us recall [32, Lemma 4.3] below in order to construct our coupling.

Lemma 2.4.2. Consider $(f_t)_{t \geq 0}$ and $(W_t^*)_{t \geq 0}$ introduced in Lemma 2.4.1 and fix $N \geq 1$. For $\mathbf{v} = (v_1, v_2, \dots, v_N) \in (\mathbb{R}^3)^N$, we introduce the empirical measure $\mu_{\mathbf{v}}^N := N^{-1} \sum_{i=1}^N \delta_{v_i}$. Then for all $t \geq 0$, all $\mathbf{v} \in (\mathbb{R}^3)^N$ and all $\mathbf{w} \in (\mathbb{R}^3)_{\bullet}^N$, with $(\mathbb{R}^3)_{\bullet}^N := \{\mathbf{w} \in (\mathbb{R}^3)^N : w_i \neq w_j \ \forall i \neq j\}$, there are α -random variables $Z_t^*(\mathbf{w}, \alpha)$ and $V_t^*(\mathbf{v}, \mathbf{w}, \alpha)$ such that the α -law of $(Z_t^*(\mathbf{w}, \cdot), V_t^*(\mathbf{v}, \mathbf{w}, \cdot))$ is $N^{-1} \sum_{i=1}^N \delta_{(w_i, v_i)}$ and $\int_0^1 |W_t^*(\alpha) - Z_t^*(\mathbf{w}, \alpha)|^2 d\alpha = \mathcal{W}_2^2(f_t, \mu_{\mathbf{w}}^N)$.

Remark 2.4.3. We know from [23] and the fact that f_t has a density for each $t \geq 0$ that the map $(t, \mathbf{v}, \mathbf{w}, \alpha) \mapsto (Z_t^*(\mathbf{w}, \alpha), V_t^*(\mathbf{v}, \mathbf{w}, \alpha))$ can be chosen measurable.

Observe that $\mathcal{L}_{\alpha}(Z_t^*(\mathbf{w}, \cdot)) = \mu_{\mathbf{w}}^N$ and $\mathcal{L}_{\alpha}(V_t^*(\mathbf{v}, \mathbf{w}, \cdot)) = \mu_{\mathbf{v}}^N$ for all fixed $t \geq 0$, $\mathbf{v} \in (\mathbb{R}^3)^N$ and $\mathbf{w} \in (\mathbb{R}^3)_{\bullet}^N$. No regularity of $Z_t^*(\mathbf{w}, \alpha)$ or $V_t^*(\mathbf{v}, \mathbf{w}, \alpha)$ is required in any of their variables.

Owing to technical reasons, we need to introduce some more notations.

Notation 2.4.4. We consider an α -random variable Y with uniform distribution on $B(0, 1)$ (independent of everything else) and, for $\epsilon \in (0, 1)$, $t \geq 0$, $\alpha \in [0, 1]$, $\mathbf{v} \in (\mathbb{R}^3)^N$ and $\mathbf{w} \in (\mathbb{R}^3)_{\bullet}^N$, we set $W_t^{*,\epsilon}(\alpha) = W_t^*(\alpha) + \epsilon Y(\alpha)$ and $V_t^{*,\epsilon}(\mathbf{v}, \mathbf{w}, \alpha) = V_t^*(\mathbf{v}, \mathbf{w}, \alpha) + \epsilon Y(\alpha)$. It holds that $\mathcal{L}_{\alpha}(W_t^{*,\epsilon}) = f_t * \psi_{\epsilon}$ and $\mathcal{L}_{\alpha}(V_t^{*,\epsilon}(\mathbf{v}, \mathbf{w}, \cdot)) = \mu_{\mathbf{v}}^N * \psi_{\epsilon}$, where $\psi_{\epsilon}(x) = (3/(4\pi\epsilon^3)) \mathbf{1}_{\{|x| \leq \epsilon\}}$.

At last, we built a suitable realisation for the particle system.

Lemma 2.4.5. Consider all the objects introduced in Lemmas 2.4.1-2.4.2 and Notation 2.4.4. Set $\mathbf{W}_s = (W_s^1, \dots, W_s^N)$, which a.s. belongs to $(\mathbb{R}^3)_{\bullet}^N$ (because f_s has a density for all $s \geq 0$). Fix $K \geq 1$ and $\epsilon \in (0, 1)$. There is a unique strong solution $(\mathbf{V}_t)_{t \geq 0} = (V_t^1, \dots, V_t^N)_{t \geq 0}$ to, for $i = 1, \dots, N$,

$$V_t^i = V_0^i + \int_0^t \int_0^1 \int_0^{\infty} \int_0^{2\pi} c_K(V_{s-}^i, V_s^*(\mathbf{V}_{s-}, \mathbf{W}_{s-}, \alpha), z, \varphi + \varphi_{i,\alpha,s}) M_i(ds, d\alpha, dz, d\varphi), \quad (2.37)$$

where $\varphi_{i,\alpha,s} := \varphi_{i,\alpha,s}^1 + \varphi_{i,\alpha,s}^2 + \varphi_{i,\alpha,s}^3$ with

$$\begin{aligned} \varphi_{i,\alpha,s}^1 &= \varphi_0(W_{s-}^i - W_s^*(\alpha), W_{s-}^i - W_s^{*,\epsilon}(\alpha)), \\ \varphi_{i,\alpha,s}^2 &= \varphi_0(W_{s-}^i - W_s^{*,\epsilon}(\alpha), V_{s-}^i - V_s^{*,\epsilon}(\mathbf{V}_{s-}, \mathbf{W}_{s-}, \alpha)), \\ \varphi_{i,\alpha,s}^3 &= \varphi_0(V_{s-}^i - V_s^{*,\epsilon}(\mathbf{V}_{s-}, \mathbf{W}_{s-}, \alpha), V_{s-}^i - V_s^*(\mathbf{V}_{s-}, \mathbf{W}_{s-}, \alpha)). \end{aligned}$$

Moreover, $(\mathbf{V}_t)_{t \geq 0}$ is a Markov process with generator $\mathcal{L}_{N,K}$. If $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ for some $q \geq 2$, then $\mathbb{E}[\sup_{[0,T]} |V_t^1|^q] \leq C_{T,q}$ (this last constant not depending on N, K nor $\epsilon \in (0, 1)$).

Proof. Since $c_K = c \mathbf{1}_{\{z \leq K\}}$, the Poisson measures involved in (2.37) are finite. Hence the existence and uniqueness results hold for (2.37). Next, we check that $(\mathbf{V}_t)_{t \geq 0}$ is a Markov

process with generator $\mathcal{L}_{N,K}$: for all bounded measurable function $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$, all $t \geq 0$, a.s.,

$$\begin{aligned}
 & \sum_{i=1}^N \int_0^1 \int_0^\infty \int_0^{2\pi} \left[\phi(\mathbf{v} + c_K(v_i, V_t^*(\mathbf{v}, \mathbf{w}, \alpha), z, \varphi + \varphi_{i,\alpha,t})\mathbf{e}_i) - \phi(\mathbf{v}) \right] d\varphi dz d\alpha \\
 &= \sum_{i=1}^N \int_0^1 \int_0^\infty \int_0^{2\pi} \left[\phi(\mathbf{v} + c_K(v_i, V_t^*(\mathbf{v}, \mathbf{w}, \alpha), z, \varphi)\mathbf{e}_i) - \phi(\mathbf{v}) \right] d\varphi dz d\alpha \\
 &= \sum_{i=1}^N N^{-1} \sum_{j=1}^N \int_0^\infty \int_0^{2\pi} \left[\phi(\mathbf{v} + c_K(v_i, v_j, z, \varphi)\mathbf{e}_i) - \phi(\mathbf{v}) \right] d\varphi dz \\
 &= N^{-1} \sum_{i \neq j} \int_0^\infty \int_0^{2\pi} \left[\phi(\mathbf{v} + c_K(v_i, v_j, z, \varphi)\mathbf{e}_i) - \phi(\mathbf{v}) \right] d\varphi dz,
 \end{aligned}$$

This is nothing but $\mathcal{L}_{N,K}\phi(\mathbf{v})$, recall Lemma 2.2.1. We used the 2π -periodicity of c_K in φ for the first equality, that $\mathcal{L}_\alpha(V_t^*(\mathbf{v}, \mathbf{w}, \cdot)) = \mu_{\mathbf{v}}^N$ for the second one, and that $c_K(v_i, v_i, z, \varphi) = 0$ for the last one.

Finally, we verify that $\sup_{[0,T]} \mathbb{E}[|V_t^1|^q] \leq C_{T,q}$ if $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ for some $q \geq 2$: it immediately follows from the Itô formula, (2.36) and exchangeability that

$$\begin{aligned}
 \mathbb{E}[|V_t^1|^q] &\leq \mathbb{E}[|V_0^1|^q] + C_q \int_0^t \int_0^1 \mathbb{E}\left[1 + |V_s^1|^q + |V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^q\right] d\alpha ds \\
 &\leq \mathbb{E}[|V_0^1|^q] + C_q N^{-1} \sum_{i=1}^N \int_0^t \mathbb{E}\left[1 + |V_s^1|^q + |V_s^i|^q\right] ds \\
 &\leq \mathbb{E}[|V_0^1|^q] + C_q \int_0^t \mathbb{E}\left[1 + |V_s^1|^q\right] ds,
 \end{aligned}$$

The Grönwall lemma allows us to complete the proof. \square

Remark 2.4.6. *The exchangeability holds for the family $\{(W_t^i, V_t^i)_{t \geq 0}, i = 1, \dots, N\}$. Indeed, the family $\{(W_t^i)_{t \geq 0}, i = 1, \dots, N\}$ is i.i.d. by construction, so that the exchangeability follows from the symmetry and pathwise uniqueness for (2.37).*

2.5 Bound in L^p of a blob approximation of an empirical measure

An empirical measure cannot be in some L^p space with $p > 1$, so we will consider a blob approximation, inspired by [27, Proposition 5.5] and [37]. But we deal with a jump process, so we need to overcome a few additional difficulties.

First, the following fact can be checked as Lemma 5.3 in [27] (the norm and the step of the subdivision are different, but this obviously changes nothing).

Lemma 2.5.1. *Let $p \in (1, 2)$ and $(f_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ such that $m_2(f_t) = m_2(f_0)$ for all $t \geq 0$.*

(i) *There is a constant $M_p > 0$, such that for all $t \geq 0$, $\|f_t\|_{L^p} \geq M_p$.*

(ii) *For any $T > 0$, we can find a subdivision $(t_\ell^N)_{\ell=0}^{K_N+1}$ satisfying $0 = t_0^N < t_1^N < \dots < t_{K_N}^N \leq T \leq t_{K_N+1}^N$, such that $\sup_{\ell=0, \dots, K_N} (t_{\ell+1}^N - t_\ell^N) \leq N^{-2}$ with $K_N \leq 2TN^2$ and*

$$\int_0^T h_N(t) dt \leq 2 \int_0^T \|f_t\|_{L^p} dt,$$

with $h_N(t) = \sum_{\ell=1}^{K_N+1} \|f_{t_\ell^N}\|_{L^p} \mathbf{1}_{\{t \in (t_{\ell-1}^N, t_\ell^N]\}}$.

The goal of the section is to prove the following crucial fact.

Proposition 2.5.2. *Assume (2.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$. Let $q \geq 2$ such that $q > \gamma^2/(\gamma + \nu)$ and let $p \in (3/(3 + \gamma), p_0(\gamma, \nu, q)) \subset (1, 3/2)$. Consider $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ with a finite entropy and $(f_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{loc}([0, \infty), L^p(\mathbb{R}^3))$ the corresponding unique solution to (2.1) given by Theorem 2.1.3. Consider $(W_t^i)_{i=1, \dots, N, t \geq 0}$ the solution to (2.34) and set $\mu_{\mathbf{W}_t}^N = N^{-1} \sum_{i=1}^N \delta_{W_t^i}$. Fix $\delta \in (0, 1)$, set $\epsilon_N = N^{-(1-\delta)/3}$ and define $\bar{\mu}_{\mathbf{W}_t}^N = \mu_{\mathbf{W}_t}^N * \psi_{\epsilon_N}$, where ψ_ϵ was defined in Notation 2.4.4. Finally, fix $T > 0$ and consider h_N built in Lemma 2.5.1. We have*

$$\mathbb{P}\left(\forall t \in [0, T], \|\bar{\mu}_{\mathbf{W}_t}^N\|_{L^p} \leq 13500(1 + h_N(t))\right) \geq 1 - C_{T, q, \delta} N^{1-\delta q/3}.$$

Throughout the section, we fix $N \geq 1$, $\delta \in (0, 1)$, and $\epsilon_N = N^{-(1-\delta)/3}$ and adopt the assumptions and notations of Proposition 2.5.2. We also put $r = p/(p - 1)$.

In order to extend [27, Proposition 5.5], it is necessary to study some properties of the paths of the processes defined by (2.34). Following Lemma 3.11 in [62], we introduce, for each $i = 1, \dots, N$,

$$\begin{aligned} \widetilde{W}_t^i &= V_0^i + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c(W_{s-}^i, W_s^*(\alpha), z, \varphi) \mathbf{1}_{\{|c(W_{s-}^i, W_s^*(\alpha), z, \varphi)| \leq N^{-1/3}\}} \\ &\quad \times M_i(ds, d\alpha, dz, d\varphi). \end{aligned} \quad (2.38)$$

Lemma 2.5.3. *For all $T > 0$,*

$$\mathbb{P}\left[\sup_{[0, T]} |W_t^1| \leq N^{\delta/3}, \sup_{s, t \in [0, T], |s-t| \leq N^{-2}} |\widetilde{W}_t^1 - \widetilde{W}_s^1| \geq \epsilon_N\right] \leq C_T N^2 e^{-N^{\delta/3}}.$$

Proof. Let us denote by \tilde{p} the probability we want to bound.

Step 1. We introduce

$$Z_t^1 = \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} G(z/|W_{s-}^1 - W_s^*(\alpha)|^\gamma) |W_{s-}^1 - W_s^*(\alpha)| \\ \times \mathbf{1}_{\left\{G(z/|W_{s-}^1 - W_s^*(\alpha)|^\gamma) |W_{s-}^1 - W_s^*(\alpha)|/4 \leq N^{-1/3}\right\}} M_1(ds, d\alpha, dz, d\varphi).$$

It is clear that Z_t^1 is almost surely increasing in t , and that a.s., for all $s, t \in [0, T]$,

$$|\widetilde{W}_t^1 - \widetilde{W}_s^1| \leq |Z_t^1 - Z_s^1|, \quad (2.39)$$

since for any $v, v_* \in \mathbb{R}^3$ (recall (2.8))

$$G(z/|v - v_*|^\gamma) |v - v_*|/4 \leq |c(v, v_*, z, \varphi)| \leq G(z/|v - v_*|^\gamma) |v - v_*|.$$

We now consider the stopping time $\tau_N = \inf \{t \geq 0 : |W_t^1| > N^{\delta/3}\}$ and deduce from (2.39) and the Markov inequality that

$$\tilde{p} \leq \mathbb{P} \left[\sup_{[0, T]} |W_t^1| \leq N^{\delta/3}, \quad \sup_{s, t \in [0, T], |s-t| \leq N^{-2}} |Z_t^1 - Z_s^1| \geq \epsilon_N \right] \\ \leq \mathbb{P} \left[\sup_{s, t \in [0, T], |s-t| \leq N^{-2}} |Z_{t \wedge \tau_N}^1 - Z_{s \wedge \tau_N}^1| \geq \epsilon_N \right].$$

Since $[0, T] \subset \bigcup_{k=0}^{\lfloor N^2 T \rfloor} [k/N^2, (k+1)/N^2)$ and Z_t^1 is almost surely increasing in t , we deduce that on $\{\sup_{s, t \in [0, T], |s-t| \leq N^{-2}} |Z_{t \wedge \tau_N}^1 - Z_{s \wedge \tau_N}^1| \geq \epsilon_N\}$, there exists $k \in \{0, 1, \dots, \lfloor N^2 T \rfloor\}$ for which there holds $(Z_{((k+1)N^{-2}) \wedge \tau_N}^1 - Z_{(kN^{-2}) \wedge \tau_N}^1) \geq \epsilon_N/3$. Hence,

$$\tilde{p} \leq \sum_{k=0}^{\lfloor N^2 T \rfloor} \mathbb{P} \left[\left(Z_{((k+1)N^{-2}) \wedge \tau_N}^1 - Z_{(kN^{-2}) \wedge \tau_N}^1 \right) \geq \frac{\epsilon_N}{3} \right] \\ \leq \sum_{k=0}^{\lfloor N^2 T \rfloor} e^{-N^{\delta/3}} \mathbb{E} \left[\exp \left\{ 3N^{1/3} \left(Z_{((k+1)N^{-2}) \wedge \tau_N}^1 - Z_{(kN^{-2}) \wedge \tau_N}^1 \right) \right\} \right] \\ =: \sum_{k=0}^{\lfloor N^2 T \rfloor} e^{-N^{\delta/3}} I_k.$$

Step 2. We now prove that I_k is (uniformly) bounded, which will complete the proof. We put

$$J_k(t) =: \mathbb{E} \left[\exp \left\{ 3N^{1/3} \left(Z_{(t+kN^{-2}) \wedge \tau_N}^1 - Z_{(kN^{-2}) \wedge \tau_N}^1 \right) \right\} \right].$$

It is obvious that $I_k = J_k(N^{-2})$. Applying the Itô formula, we find

$$\begin{aligned} J_k(t) &= 1 + 2\pi \mathbb{E} \left[\int_{(kN^{-2}) \wedge \tau_N}^{(t+kN^{-2}) \wedge \tau_N} \int_0^1 \int_0^\infty \exp \left\{ 3N^{1/3} \left(Z_s^1 - Z_{(kN^{-2}) \wedge \tau_N}^1 \right) \right\} \times \right. \\ &\quad \left. \left(e^{3N^{1/3} G(z/|W_s^1 - W_s^*(\alpha)|^\gamma) |W_s^1 - W_s^*(\alpha)|} - 1 \right) \mathbf{1}_{\left\{ G(z/|W_s^1 - W_s^*(\alpha)|^\gamma) |W_s^1 - W_s^*(\alpha)|/4 \leq N^{-1/3} \right\}} dz d\alpha ds \right]. \end{aligned}$$

Since $3N^{1/3} G(z/|W_s^1 - W_s^*(\alpha)|^\gamma) |W_s^1 - W_s^*(\alpha)| \leq 12$ (thanks to the indicator function), we have

$$e^{3N^{1/3} G(z/|W_s^1 - W_s^*(\alpha)|^\gamma) |W_s^1 - W_s^*(\alpha)|} - 1 \leq CN^{1/3} G(z/|W_s^1 - W_s^*(\alpha)|^\gamma) |W_s^1 - W_s^*(\alpha)|$$

for a positive constant C . Then using (2.5), we see that

$$\mathbf{1}_{\left\{ G(z/|W_s^1 - W_s^*(\alpha)|^\gamma) |W_s^1 - W_s^*(\alpha)|/4 \leq N^{-1/3} \right\}} \leq \mathbf{1}_{\left\{ z \geq CN^{\nu/3} |W_s^1 - W_s^*(\alpha)|^{\gamma+\nu} - |W_s^1 - W_s^*(\alpha)|^\gamma \right\}}.$$

Hence,

$$\begin{aligned} J_k(t) &\leq 1 + CN^{1/3} \mathbb{E} \left[\int_{(kN^{-2}) \wedge \tau_N}^{(t+kN^{-2}) \wedge \tau_N} \int_0^1 \int_0^\infty \exp \left\{ 3N^{1/3} \left(Z_s^1 - Z_{(kN^{-2}) \wedge \tau_N}^1 \right) \right\} \right. \\ &\quad \left. \times \left(1 + z/|W_s^1 - W_s^*(\alpha)|^\gamma \right)^{-1/\nu} |W_s^1 - W_s^*(\alpha)| \mathbf{1}_{\left\{ z \geq CN^{\nu/3} |W_s^1 - W_s^*(\alpha)|^{\gamma+\nu} - |W_s^1 - W_s^*(\alpha)|^\gamma \right\}} dz d\alpha ds \right]. \end{aligned}$$

But, we have

$$\begin{aligned} &|W_s^1 - W_s^*(\alpha)| \int_0^\infty \left(1 + z/|W_s^1 - W_s^*(\alpha)|^\gamma \right)^{-1/\nu} \mathbf{1}_{\left\{ z \geq CN^{\nu/3} |W_s^1 - W_s^*(\alpha)|^{\gamma+\nu} - |W_s^1 - W_s^*(\alpha)|^\gamma \right\}} dz \\ &= CN^{-(1-\nu)/3} |W_s^1 - W_s^*(\alpha)|^{\nu+\gamma} \\ &\leq CN^{-(1-\nu)/3} (1 + |W_s^1|^2 + |W_s^*(\alpha)|^2) \end{aligned}$$

since $\gamma + \nu \in (0, 1)$. Using now that $\int_0^1 |W_s^*(\alpha)|^2 d\alpha = m_2(f_0)$ and that $|W_s^1| \leq N^{\delta/3}$ for all $s \leq \tau_N$, we conclude that

$$\begin{aligned} J_k(t) &\leq 1 + CN^{\nu/3} (1 + m_2(f_0) + N^{2\delta/3}) \int_0^t J_k(s) ds \\ &\leq 1 + CN^{(\nu+2\delta)/3} \int_0^t J_k(s) ds. \end{aligned}$$

It follows from the Grönwall lemma that $J_k(t) \leq \exp(CN^{(\nu+2\delta)/3}t)$, and thus that $I_k = J_k(N^{-2})$ is uniformly bounded, because $(\nu + 2\delta)/3 < 2$ (recall that $\nu \in (0, 1)$ and $\delta \in (0, 1)$). \square

Next, we study the large jumps of $(W_t^1)_{t \geq 0}$.

Lemma 2.5.4. *There exists $C > 0$ such that for any $\ell \in \{1, \dots, K_N + 1\}$,*

$$\mathbb{P}\left[\exists t \in (t_{\ell-1}^N, t_\ell^N] : |\Delta W_t^1| > N^{-1/3}\right] \leq CN^{-(2-\nu/3)}.$$

Proof. Let us fix ℓ and set $A = \{\exists t \in (t_{\ell-1}^N, t_\ell^N] : |\Delta W_t^1| > N^{-1/3}\}$. After noting that

$$A = \left\{ \int_{t_{\ell-1}^N}^{t_\ell^N} \int_0^1 \int_0^\infty \int_0^{2\pi} \mathbf{1}_{\{|c(W_{s-}^i, W_s^*(\alpha), z, \varphi)| > N^{-1/3}\}} M_1(ds, d\alpha, dz, d\varphi) \geq 1 \right\},$$

we have

$$\mathbb{P}(A) \leq \mathbb{E} \left[\int_{t_{\ell-1}^N}^{t_\ell^N} \int_0^1 \int_0^\infty \int_0^{2\pi} \mathbf{1}_{\{|c(W_{s-}^1, W_s^*(\alpha), z, \varphi)| > N^{-1/3}\}} M_1(ds, d\alpha, dz, d\varphi) \right]$$

by the Markov inequality. But, (2.8) and (2.5) tell us that $|c(v, v_*, z, \varphi)| \leq C(1 + z/|v - v_*|^\gamma)^{-1/\nu} |v - v_*|$. Hence,

$$\begin{aligned} \mathbb{P}(A) &\leq 2\pi \mathbb{E} \left[\int_{t_{\ell-1}^N}^{t_\ell^N} \int_0^1 \int_0^\infty \mathbf{1}_{\{C(1+z/|W_s^1 - W_s^*(\alpha)|^\gamma)^{-1/\nu} |W_s^1 - W_s^*(\alpha)| > N^{-1/3}\}} dz d\alpha ds \right] \\ &\leq 2\pi \mathbb{E} \left[\int_{t_{\ell-1}^N}^{t_\ell^N} \int_0^1 \int_0^\infty \mathbf{1}_{\{z < CN^{\nu/3} |W_s^1 - W_s^*(\alpha)|^{\gamma+\nu}\}} dz d\alpha ds \right] \\ &= CN^{\nu/3} \mathbb{E} \left[\int_{t_{\ell-1}^N}^{t_\ell^N} \int_0^1 |W_s^1 - W_s^*(\alpha)|^{\gamma+\nu} d\alpha ds \right]. \end{aligned}$$

Finally, using that $|W_s^1 - W_s^*(\alpha)|^{\gamma+\nu} \leq 1 + |W_s^1|^2 + |W_s^*(\alpha)|^2$ and that $\int_0^1 |W_s^*(\alpha)|^2 d\alpha = \mathbb{E}[|W_s^1|^2] < \infty$, we conclude that $\mathbb{P}(A) \leq CN^{\nu/3}(t_{\ell+1}^N - t_\ell^N) \leq CN^{\nu/3-2}$ as desired. \square

Lemma 2.5.5. *For $\ell = 1, \dots, K_N + 1$, we introduce*

$$I_\ell = \{i \in \{1, \dots, N\} : \exists t \in (t_{\ell-1}^N, t_\ell^N] \text{ such that } |\Delta W_t^i| > N^{-1/3}\}, \quad (2.40)$$

and the event

$$\begin{aligned} \Omega_{T,N}^1 &= \left\{ \forall i \in \{1, \dots, N\}, \sup_{[0,T]} |W_t^i| \leq N^{\delta/3} \text{ and } \sup_{s,t \in [0,T], |s-t| \leq N^{-2}} |\widetilde{W}_t^i - \widetilde{W}_s^i| \leq \epsilon_N \right\} \\ &\cap \left\{ \forall \ell = 1, \dots, K_N + 1, \#(I_\ell) \leq N\epsilon_N^{3/r} \right\}. \end{aligned}$$

Then we have

$$\mathbb{P}[\Omega_{T,N}^1] \geq 1 - C_{T,q,\delta} N^{1-q\delta/3}.$$

Proof. We write $\Omega_{T,N}^1 = \Omega_{T,N}^{1,1} \cap \Omega_{T,N}^{1,2}$, where

$$\Omega_{T,N}^{1,1} := \left\{ \forall i \in \{1, \dots, N\}, \sup_{[0,T]} |W_t^i| \leq N^{\delta/3} \text{ and } \sup_{s,t \in [0,T], |s-t| \leq N^{-2}} |\widetilde{W}_t^i - \widetilde{W}_s^i| \leq \epsilon_N \right\},$$

$$\Omega_{T,N}^{1,2} := \left\{ \forall \ell = 1, \dots, K_N + 1, \#(I_\ell) \leq N\epsilon_N^{3/r} \right\},$$

Step 1. Here we estimate $\mathbb{P}[(\Omega_{T,N}^{1,1})^c]$. Using the Markov inequality, (2.35) and Lemma 2.5.3, we get

$$\begin{aligned} & \mathbb{P}[(\Omega_{T,N}^{1,1})^c] \\ & \leq N \mathbb{P} \left[\left\{ \sup_{[0,T]} |W_t^1| \leq N^{\delta/3} \text{ and } \sup_{|s-t| \leq N^{-2}} |\widetilde{W}_t^1 - \widetilde{W}_s^1| \leq \epsilon_N \right\}^c \right] \\ & = N \mathbb{P} \left[\sup_{[0,T]} |W_t^1| \geq N^{\delta/3} \right] + N \mathbb{P} \left[\sup_{[0,T]} |W_t^1| \leq N^{\delta/3} \text{ and } \sup_{|s-t| \leq N^{-2}} |\widetilde{W}_t^1 - \widetilde{W}_s^1| \geq \epsilon_N \right] \\ & \leq N \mathbb{E} \left[\sup_{[0,T]} |W_t^1|^q \right] N^{-q\delta/3} + C_T N^3 e^{-N^{\delta/3}} \leq C_{T,q} N^{1-q\delta/3}. \end{aligned}$$

Step 2. We now prove that $\mathbb{P}[(\Omega_{T,N}^{1,2})^c] \leq C_T \exp(-N^\delta)$. For any fixed $\ell \in \{1, \dots, K_N + 1\}$, we introduce $A_N^\ell = \{\exists t \in (t_{\ell-1}^N, t_\ell^N] : |\Delta W_t^1| > N^{-1/3}\}$. Then we observe that $\#(I_\ell)$ follows a Binomial distribution with parameters N and $\mathbb{P}(A_N^\ell)$. Using again the Markov inequality, we observe that

$$\mathbb{P}[(\Omega_{T,N}^{1,2})^c] \leq \sum_{\ell=1}^{K_N+1} \mathbb{P}[\#(I_\ell) \geq N\epsilon_N^{3/r}] \leq \sum_{\ell=1}^{K_N+1} \mathbb{E}[\exp(\#(I_\ell))] \exp(-N\epsilon_N^{3/r}).$$

But,

$$\mathbb{E}[\exp(\#(I_\ell))] = \exp(N \log(1 + (e-1)\mathbb{P}(A_N^\ell))) \leq \exp(N(e-1)\mathbb{P}(A_N^\ell)).$$

Hence,

$$\mathbb{P}[(\Omega_{T,N}^{1,2})^c] \leq \sum_{\ell=1}^{K_N+1} \exp(N(e-1)\mathbb{P}(A_N^\ell)) \exp(-N\epsilon_N^{3/r}).$$

We know from Lemma 2.5.4 that $\mathbb{P}(A_N^\ell) \leq CN^{-(2-\nu/3)}$, hence $N\mathbb{P}(A_N^\ell) \leq CN^{-1+\nu/3} \leq C$. We thus deduce that

$$\mathbb{P}[(\Omega_{T,N}^{1,2})^c] \leq C(K_N + 1) \exp(-N\epsilon_N^{3/r}) \leq C(2TN^2 + 1) \exp(-N\epsilon_N^{3/r}) \leq C_T \exp(-N^\delta),$$

since $N\epsilon_N^{3/r} = N^{1/p+\delta/r}$ and since $1/p + \delta/r > \delta$. This ends the proof. \square

2.5. BOUND IN L^p OF A BLOB APPROXIMATION OF AN EMPIRICAL MEASURE 91

We now give the

Proof of Proposition 2.5.2. Consider the partition \mathcal{P}_N of \mathbb{R}^3 in cubes with side length ϵ_N and its subset \mathcal{P}_N^δ consisting of cubes that have non-empty intersection with $B(0, N^{\delta/3})$. Then we observe that $\#(\mathcal{P}_N^\delta) \leq (2(N^{\delta/3} + \epsilon_N)\epsilon_N^{-1})^3 \leq 64N^\delta\epsilon_N^{-3} = 64N$. We split the proof into several steps.

Step 1. For $(x_1, \dots, x_N) \in (B(0, N^{\delta/3}))^N$ and $(y_1, \dots, y_N) \in (B(0, N^{\delta/3}))^N$, we set

$$I = \{i \in \{1, \dots, N\} : |x_i - y_i| > \epsilon_N\},$$

and denote the empirical measure of $\mathbf{y} = (y_1, \dots, y_N) \in (\mathbb{R}^3)^N$ by $\mu_{\mathbf{y}}^N = N^{-1} \sum_{i=1}^N \delta_{y_i}$. The goal of this step is to show that

$$\|\mu_{\mathbf{y}}^N * \psi_{\epsilon_N}\|_{L^p} \leq \left(\frac{3}{4\pi}\right)^{1/r} \frac{\#(I)}{N\epsilon_N^{3/r}} + 3375 \left(N^{-p}\epsilon_N^{-3(p-1)} \sum_{D \in \mathcal{P}_N^\delta} (\#\{i \in \{1, \dots, N\} : x_i \in D\})^p\right)^{1/p}.$$

Indeed, recalling that $\psi_\epsilon(x) = (3/(4\pi\epsilon^3))\mathbf{1}_{\{|x| \leq \epsilon\}}$, we observe that

$$\begin{aligned} & \mu_{\mathbf{y}}^N * \psi_{\epsilon_N}(v) \\ &= \frac{1}{N} \sum_{i=1}^N \psi_{\epsilon_N}(v - y_i) \mathbf{1}_{\{|x_i - y_i| > \epsilon_N\}} + N^{-1} \sum_{i=1}^N \psi_{\epsilon_N}(v - y_i) \mathbf{1}_{\{|x_i - y_i| \leq \epsilon_N\}} \\ &= \frac{1}{N} \sum_{i \in I} \psi_{\epsilon_N}(v - y_i) + \frac{3}{4\pi N \epsilon_N^3} \#\{i \in \{1, \dots, N\} : y_i \in B(v, \epsilon_N), |y_i - x_i| \leq \epsilon_N\} \\ &\leq \frac{1}{N} \sum_{i \in I} \psi_{\epsilon_N}(v - y_i) + \frac{3}{4\pi N \epsilon_N^3} \#\{i \in \{1, \dots, N\} : x_i \in B(v, 2\epsilon_N)\}. \end{aligned}$$

Hence,

$$\mu_{\mathbf{y}}^N * \psi_{\epsilon_N}(v) \leq \frac{1}{N} \sum_{i \in I} \psi_{\epsilon_N}(v - y_i) + \frac{3}{4\pi N \epsilon_N^3} \sum_{D \in \mathcal{P}_N^\delta} \#\{i \in \{1, \dots, N\} : x_i \in D\} \mathbf{1}_{\{D \cap B(v, 2\epsilon_N) \neq \emptyset\}}.$$

We then deduce that

$$\begin{aligned} & \|\mu_{\mathbf{y}}^N * \psi_{\epsilon_N}\|_{L^p} \\ &\leq \frac{1}{N} \left\| \sum_{i \in I} \psi_{\epsilon_N}(\cdot - y_i) \right\|_{L^p} + \frac{3}{4\pi N \epsilon_N^3} \left\| \sum_{D \in \mathcal{P}_N^\delta} \#\{i \in \{1, \dots, N\} : x_i \in D\} \mathbf{1}_{\{D \cap B(\cdot, 2\epsilon_N) \neq \emptyset\}} \right\|_{L^p}. \end{aligned}$$

Since $\|\psi_{\epsilon_N}(\cdot - y_i)\|_{L^p} = \left(\frac{3}{4\pi}\right)^{1/r} \epsilon_N^{-3/r}$, we have

$$\frac{1}{N} \left\| \sum_{i \in I} \psi_{\epsilon_N}(\cdot - y_i) \right\|_{L^p} \leq \frac{1}{N} \sum_{i \in I} \|\psi_{\epsilon_N}(\cdot - y_i)\|_{L^p} \leq \left(\frac{3}{4\pi}\right)^{1/r} \frac{\#(I)}{N\epsilon_N^{3/r}}.$$

On the other hand, let $A := \left\| \sum_{D \in \mathcal{D}_N^\delta} \#\{i \in \{1, \dots, N\} : x_i \in D\} \mathbf{1}_{\{D \cap B(\cdot, 2\epsilon_N) \neq \emptyset\}} \right\|_{L^p}$, then

$$\begin{aligned} A^p &= \int_{\mathbb{R}^3} \left(\sum_{D \in \mathcal{D}_N^\delta} \#\{i : x_i \in D\} \mathbf{1}_{\{D \cap B(v, 2\epsilon_N) \neq \emptyset\}} \right)^p dv \\ &= \int_{\mathbb{R}^3} \left(\sum_{D, D' \in \mathcal{D}_N^\delta} \#\{i : x_i \in D\} \#\{i : x_i \in D'\} \mathbf{1}_{\{D \cap B(v, 2\epsilon_N) \neq \emptyset, D' \cap B(v, 2\epsilon_N) \neq \emptyset\}} \right)^{p/2} dv \\ &\leq \int_{\mathbb{R}^3} \sum_{D, D' \in \mathcal{D}_N^\delta} (\#\{i : x_i \in D\})^{p/2} (\#\{i : x_i \in D'\})^{p/2} \mathbf{1}_{\{D \cap B(v, 2\epsilon_N) \neq \emptyset, D' \cap B(v, 2\epsilon_N) \neq \emptyset\}} dv \end{aligned}$$

because $p \in (1, 2)$. From $x^2 + y^2 \geq 2xy$ and a symmetry argument, we see that

$$A^p \leq \sum_{D \in \mathcal{D}_N^\delta} (\#\{i : x_i \in D\})^p \int_{\mathbb{R}^3} \mathbf{1}_{\{D \cap B(v, 2\epsilon_N) \neq \emptyset\}} \sum_{D' \in \mathcal{D}_N^\delta} \mathbf{1}_{\{D' \cap B(v, 2\epsilon_N) \neq \emptyset\}} dv.$$

But, for each $v \in \mathbb{R}^3$, $\sum_{D' \in \mathcal{D}_N^\delta} \mathbf{1}_{\{D' \cap B(v, 2\epsilon_N) \neq \emptyset\}} = \#\{D' \in \mathcal{D}_N^\delta : D' \cap B(v, 2\epsilon_N) \neq \emptyset\} \leq 5^3$. And for each $D \in \mathcal{D}_N^\delta$, $\{v \in \mathbb{R}^3 : D \cap B(v, 2\epsilon_N) \neq \emptyset\}$ is included by a ball of radius $3\epsilon_N$. Therefore, $\int_{\mathbb{R}^3} \mathbf{1}_{\{D \cap B(v, 2\epsilon_N) \neq \emptyset\}} dv \leq 4\pi(3\epsilon_N)^3/3$. Hence,

$$A^p \leq \frac{5^3 4\pi (3\epsilon_N)^3}{3} \sum_{D \in \mathcal{D}_N^\delta} (\#\{i : x_i \in D\})^p.$$

Consequently,

$$\begin{aligned} &\|\mu_{\mathbf{y}}^N * \psi_{\epsilon_N}(v)\|_{L^p} \\ &\leq \left(\frac{3}{4\pi}\right)^{1/r} \frac{\#(I)}{N\epsilon_N^{3/r}} + \frac{3}{4\pi N\epsilon_N^3} A \\ &\leq \left(\frac{3}{4\pi}\right)^{1/r} \frac{\#(I)}{N\epsilon_N^{3/r}} + \left(\frac{3}{4\pi}\right)^{1/r} (15)^{3/p} \left(N^{-p}\epsilon_N^{-3(p-1)} \sum_{D \in \mathcal{D}_N^\delta} (\#\{i : x_i \in D\})^p\right)^{1/p}. \end{aligned}$$

Since $(15)^{3/p} \leq 15^3 = 3375$, this ends the step.

Step 2. In this step, we extend the proof of [27, Step 3-Proposition 5.5] to show that there are some constants $C > 0$ and $c > 0$ (depending on δ and M_p of Lemma 2.5.1) such that for all fixed $t \in [0, T + 1]$,

$$\mathbb{P}[(\Omega_{t,N}^2)^c] \leq C \exp(-cN^{\delta/r}),$$

where

$$\Omega_{t,N}^2 = \left\{ N^{-p}\epsilon_N^{-3(p-1)} \sum_{D \in \mathcal{D}_N^\delta} \left(\#\{i \in \{1, \dots, N\} : W_t^i \in D\} \right)^p \leq 2^{p+1} \|f_t\|_{L^p}^p \right\}.$$

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To this end, we introduce, for $D \in \mathcal{P}_N^\delta$, $A_D = \#\{i : W_t^i \in D\}$. Then $A_D \sim B(N, f_t(D))$ and we have

$$\mathbb{P}(A_D \geq x) \leq \exp(-x/8) \quad \text{for all } x \geq 2Nf_t(D). \quad (2.41)$$

Indeed, $\mathbb{P}(A_D \geq x) \leq e^{-x} \mathbb{E}[\exp(A_D)] = e^{-x} \exp[N \log(1 + f_t(D)(e-1))] \leq e^{-x} \exp[N(e-1)f_t(D)]$. If $x \geq 2Nf_t(D)$, we thus have

$$\mathbb{P}(A_D \geq x) \leq \exp[-x + x(e-1)/2] \leq \exp(-x/8).$$

Next, it follows from the Hölder inequality that

$$\|f_t\|_{L^p}^p \geq \sum_{D \in \mathcal{P}_N^\delta} \int_D |f_t(v)|^p dv \geq \epsilon_N^{-3p/r} \sum_{D \in \mathcal{P}_N^\delta} (f_t(D))^p.$$

On the other hand, we observe from $\#(\mathcal{P}_N^\delta) \leq 64N^\delta \epsilon_N^{-3}$ that

$$\|f_t\|_{L^p}^p \geq 64^{-1} N^{-\delta} \epsilon_N^3 \sum_{D \in \mathcal{P}_N^\delta} \|f_t\|_{L^p}^p.$$

Using the two previous inequalities, we find that

$$2^{p+1} \|f_t\|_{L^p}^p \geq \sum_{D \in \mathcal{P}_N^\delta} (2^p \epsilon_N^{-3p/r} (f_t(D))^p + 2^p 64^{-1} N^{-\delta} \epsilon_N^3 \|f_t\|_{L^p}^p).$$

Consequently, on $(\Omega_{t,N}^2)^c$, we have

$$\begin{aligned} \sum_{D \in \mathcal{P}_N^\delta} A_D^p &> N^p \epsilon_N^{3(p-1)} 2^{p+1} \|f_t\|_{L^p}^p \\ &\geq N^p \epsilon_N^{3(p-1)} \sum_{D \in \mathcal{P}_N^\delta} (2^p \epsilon_N^{-3p/r} (f_t(D))^p + 2^p 64^{-1} N^{-\delta} \epsilon_N^3 \|f_t\|_{L^p}^p), \end{aligned}$$

so that there is at least one $D \in \mathcal{P}_N^\delta$ with

$$A_D^p \geq N^p \epsilon_N^{3(p-1)} [2^p \epsilon_N^{-3p/r} (f_t(D))^p + 2^p 64^{-1} N^{-\delta} \epsilon_N^3 \|f_t\|_{L^p}^p].$$

Hence,

$$\mathbb{P}[(\Omega_{t,N}^2)^c] \leq \sum_{D \in \mathcal{P}_N^\delta} \mathbb{P}\left(A_D \geq N \epsilon_N^{3/r} [2^p \epsilon_N^{-3p/r} (f_t(D))^p + 2^p 64^{-1} N^{-\delta} \epsilon_N^3 \|f_t\|_{L^p}^p]^{1/p}\right).$$

But we can apply (2.41), because

$$x_N := N \epsilon_N^{3/r} [2^p \epsilon_N^{-3p/r} (f_t(D))^p + 2^p 64^{-1} N^{-\delta} \epsilon_N^3 \|f_t\|_{L^p}^p]^{1/p}$$

enjoys the property that $x_N \geq N\epsilon_N^{3/r} [2^p \epsilon_N^{-3p/r} (f_t(D))^p]^{1/p} = 2N f_t(D)$:

$$\mathbb{P}[(\Omega_{t,N}^2)^c] \leq \sum_{D \in \mathcal{D}_N^\delta} \exp(-x_N/8).$$

Using that $x_N \geq N\epsilon_N^{3/r} (2^p 64^{-1} N^{-\delta} \epsilon_N^3 \|f_t\|_{L^p}^p)^{1/p} = cN^{\delta/r} \|f_t\|_{L^p}$, that $\#(\mathcal{D}_N^\delta) \leq 64N$ and that $\|f_t\|_{L^p} \geq M_p$, we deduce that

$$\begin{aligned} \mathbb{P}[(\Omega_{t,N}^2)^c] &\leq \sum_{D \in \mathcal{D}_N^\delta} \exp(-cN^{\delta/r} \|f_t\|_{L^p}/8) \\ &\leq 64N \exp(-cM_p N^{\delta/r}/8) \leq C \exp(-cM_p N^{\delta/r}/10). \end{aligned}$$

This ends the step.

Step 3. We finally consider the event

$$\Omega_{T,N} = \Omega_{T,N}^1 \cap \left(\bigcap_{\ell=1}^{K_N+1} \Omega_{t_\ell^N, N}^2 \right),$$

where $\Omega_{T,N}^1$ is defined in Lemma 2.5.5, and the sequence $(t_\ell^N)_{\ell=0}^{K_N+1}$ satisfying $0 = t_0^N < t_1^N < \dots < t_{K_N}^N \leq T \leq T_{K_N+1}^N$, with $K_N \leq 2TN^2$ and $\sup_{i=0, \dots, K_N} (t_{\ell+1}^N - t_\ell^N) \leq N^{-2}$ is built in Lemma 2.5.1. We also recall that $h_N(t) = \sum_{\ell=1}^{K_N+1} \|f_{t_\ell^N}\|_{L^p} \mathbf{1}_{\{t \in (t_{\ell-1}^N, t_\ell^N]\}}$.

According to Lemma 2.5.5 and Step 2, we see that

$$\begin{aligned} \mathbb{P}[\Omega_{T,N}^c] &\leq \mathbb{P}[(\Omega_{T,N}^1)^c] + \sum_{\ell=1}^{K_N+1} \mathbb{P}[(\Omega_{t_\ell^N, N}^2)^c] \\ &\leq C_{T,q,\delta} N^{1-q\delta/3} + C(K_N+1) \exp(-cN^{\delta/r}) \leq C_{T,q,\delta} N^{1-q\delta/3}. \end{aligned}$$

Finally, we show that on $\Omega_{T,N}$, for all $t \in [0, T]$, $\|\bar{\mu}_{\mathbf{W}_t}^N\|_{L^p} \leq 13500(1 + h_N(t))$. Recall that \widetilde{W}_t^i is defined by (2.38) and that I_ℓ is given by (2.40), we have

- (i) for all $i = 1, \dots, N$, and for all $t \in [0, T+1]$, $W_t^i \in B(0, N^{\delta/3})$ (according to $\Omega_{T,N}^1$);
- (ii) for all $\ell = 1, \dots, K_N+1$, all $t \in (t_{\ell-1}^N, t_\ell^N]$, all $i \in \{1, \dots, N\} \setminus I_\ell$, $|W_t^i - W_{t_\ell^N}^i| = |\widetilde{W}_t^i - \widetilde{W}_{t_\ell^N}^i| \leq \epsilon_N$, and $\#(I_\ell) \leq N\epsilon_N^{3/r}$ (by definition of \widetilde{W}^i and I_ℓ and thanks to $\Omega_{T,N}^1$);
- (iii) For all $\ell = 1, \dots, K_N+1$, $N^{-p} \epsilon_N^{-3(p-1)} \sum_{D \in \mathcal{D}_N^\delta} \left(\#\{i \in \{1, \dots, N\} : W_{t_\ell^N}^i \in D\} \right)^p \leq 2^{p+1} \|f_{t_\ell^N}\|_{L^p}^p$ (according to $\bigcap_{\ell=1}^{K_N+1} \Omega_{t_\ell^N, N}^2$).

Using Step 1 with $\bar{\mu}_{\mathbf{W}_t}^N = \mu_{\mathbf{W}_t}^N * \psi_{\epsilon_N}$, we deduce that on $\Omega_{T,N}$, for all $t \in [0, T]$, choosing ℓ such that $t \in (t_{\ell-1}^N, t_\ell^N]$, we have

$$\begin{aligned} \|\bar{\mu}_{\mathbf{W}_t}^N\|_{L^p} &\leq \left(\frac{3}{4\pi}\right)^{1/r} \frac{\#(I_\ell)}{N\epsilon_N^{3/r}} + 3375 \left(N^{-p} \epsilon_N^{-3(p-1)} \sum_{D \in \mathcal{D}_N^\delta} (\#\{i \in \{1, \dots, N\} : W_{t_\ell}^i \in D\})^p \right)^{1/p} \\ &\leq 1 + 3375.2^{(p+1)/p} \|f_{t_\ell}^N\|_{L^p} \\ &= 1 + 3375.2^{(p+1)/p} h_N(t). \end{aligned}$$

This completes the proof, since $3375.2^{(p+1)/p} \leq 3375.4 = 13500$. \square

2.6 Estimate of the Wasserstein distance

This last section is devoted to the proof of Theorem 2.1.5. In the whole section, we assume (2.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$. We consider $q > 6$ such that $q > \gamma^2/(\gamma + \nu)$, $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ with a finite entropy, and $(f_t)_{t \geq 0}$ the unique weak solution to (2.1) given by Theorem 2.1.3. We fix $p \in (3/(3 + \gamma), p_0(\gamma, \nu, q))$ and know that $(f_t)_{t \geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L_{loc}^1([0, \infty), L^p(\mathbb{R}^3))$.

We fix $N \geq 1$, $K \geq 1$ and put $\epsilon_N = N^{-(1-\delta)/3}$ with $\delta = 6/q$. Consider $(V_t^i)_{t \geq 0}$ for $i = 1, \dots, N$, defined by (2.37) with the choice $\epsilon = \epsilon_N$. We know by Lemma 2.4.5 that $(V_t^i)_{i=1, \dots, N, t \geq 0}$ is a Markov process with generator $\mathcal{L}_{N,K}$, see (2.12), starting from $(V_0^i)_{i=1, \dots, N}$, which is an i.i.d. family of f_0 -distributed random variables. We set $\mu_{\mathbf{V}_t}^N = N^{-1} \sum_1^N \delta_{V_t^i}$. So the goal of the section is to prove that

$$\sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, f_t)] \leq C_{T,q} \left(N^{-(1-6/q)(2+2\gamma)/3} + K^{1-2/\nu} + N^{-1/2} \right). \quad (2.42)$$

We consider $(W_t^i)_{t \geq 0}$, for $i = 1, \dots, N$ defined by (2.34) and recall that for all $t \geq 0$, the family $(W_t^i)_{i=1, \dots, N}$ is i.i.d. and f_t -distributed.

First, we introduce the following shortened notations:

$$\begin{aligned} c_W(s) &:= c(W_s^1, W_s^*(\alpha), z, \varphi), \\ c_W^N(s) &:= c(W_s^1, W_s^{*, \epsilon_N}(\alpha), z, \varphi + \varphi_{1, \alpha, s}^1), \\ c_V^N(s) &:= c(V_s^1, V_s^{*, \epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha), z, \varphi + \varphi_{1, \alpha, s}^1 + \varphi_{1, \alpha, s}^2), \\ c_{K,V}^N(s) &:= c_K(V_s^1, V_s^{*, \epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha), z, \varphi + \varphi_{1, \alpha, s}^1 + \varphi_{1, \alpha, s}^2), \\ c_{K,V}(s) &:= c_K(V_s^1, V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha), z, \varphi + \varphi_{1, \alpha, s}), \end{aligned}$$

with the notations of Section 4. Let us now prove an intermediate result.

Lemma 2.6.1. *There is $C > 0$ such that a.s.,*

$$\begin{aligned} & I_0^N(s) + I_1^N(s) + I_2^N(s) + I_3^N(s) \\ & \leq C\epsilon_N^{2+2\gamma} + C|W_s^1 - V_s^1|^2 + CK^{1-2/\nu} \int_0^1 |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^{2+2\gamma/\nu} d\alpha \\ & \quad + C \int_0^1 \left(|W_s^1 - V_s^1|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 \right) |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma d\alpha. \end{aligned}$$

where

$$I_0^N(s) := \int_0^1 \int_0^\infty \int_0^{2\pi} \left(2(W_s^1 - V_s^1) \cdot (c_W^N(s) - c_{K,V}^N(s)) + |c_W^N(s) - c_{K,V}^N(s)|^2 \right) d\varphi dz d\alpha,$$

$$I_1^N(s) := \int_0^1 \int_0^\infty \int_0^{2\pi} 2(W_s^1 - V_s^1) \cdot (c_W(s) - c_W^N(s) + c_{K,V}^N(s) - c_{K,V}(s)) d\varphi dz d\alpha,$$

$$I_2^N(s) := \int_0^1 \int_0^\infty \int_0^{2\pi} |c_W(s) - c_W^N(s) + c_{K,V}^N(s) - c_{K,V}(s)|^2 d\varphi dz d\alpha,$$

$$I_3^N(s) := \int_0^1 \int_0^\infty \int_0^{2\pi} 2(c_W^N(s) - c_{K,V}^N(s)) \cdot (c_W(s) - c_W^N(s) + c_{K,V}^N(s) - c_{K,V}(s)) d\varphi dz d\alpha.$$

Proof. First recall that $|W_s^{*,\epsilon_N}(\alpha) - V_s^{*,\epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 = |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2$, see Notation 2.4.4. It thus follows from (2.18) (with $v = W_s^1$, $v_* = W_s^{*,\epsilon_N}(\alpha)$, $\tilde{v} = V_s^1$ and $\tilde{v}_* = V_s^{*,\epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha)$) that

$$\begin{aligned} I_0^N(s) & \leq C \int_0^1 \left(|W_s^1 - V_s^1|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 \right) |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma d\alpha \\ & \quad + CK^{1-2/\nu} \int_0^1 |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^{2+2\gamma/\nu} d\alpha. \end{aligned}$$

Next, we study $I_1^N(s)$. As seen in the proof of Lemma 2.2.3,

$$\int_0^\infty \int_0^{2\pi} c(v, v_*, z, \varphi) d\varphi dz = -(v - v_*)\Phi(|v - v_*|),$$

and

$$\int_0^\infty \int_0^{2\pi} c_K(v, v_*, z, \varphi) d\varphi dz = -(v - v_*)\Phi_K(|v - v_*|),$$

where $\Phi(x) = \pi \int_0^\infty (1 - \cos G(z/x^\gamma)) dz$ and $\Phi_K(x) = \pi \int_0^K (1 - \cos G(z/x^\gamma)) dz$. Then,

$$\begin{aligned} I_1^N(s) & = 2(W_s^1 - V_s^1) \cdot \int_0^1 \left[- (W_s^1 - W_s^*(\alpha))\Phi(|W_s^1 - W_s^*(\alpha)|) \right. \\ & \quad + (W_s^1 - W_s^{*,\epsilon_N}(\alpha))\Phi(|W_s^1 - W_s^{*,\epsilon_N}(\alpha)|) \\ & \quad - (V_s^1 - V_s^{*,\epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha))\Phi_K(|V_s^1 - V_s^{*,\epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha)|) \\ & \quad \left. + (V_s^1 - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha))\Phi_K(|V_s^1 - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|) \right] d\alpha. \end{aligned}$$

But we have checked that $|X\Phi_K(|X|) - Y\Phi_K(|Y|)| \leq C|X - Y||X|^\gamma$ for any $X, Y \in \mathbb{R}^3$ in the proof of Lemma 2.2.3, and it of course also holds true that

$$|X\Phi(|X|) - Y\Phi(|Y|)| \leq C|X - Y||X|^\gamma.$$

Thus

$$\begin{aligned} & I_1^N(s) \\ & \leq C|W_s^1 - V_s^1| \int_0^1 \left[|W_s^*(\alpha) - W_s^{*,\epsilon_N}(\alpha)| |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma \right. \\ & \quad \left. + |V_s^{*,\epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)| |V_s^1 - V_s^{*,\epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^\gamma \right] d\alpha \\ & = C|W_s^1 - V_s^1| \int_0^1 |\epsilon_N Y(\alpha)| \left[|W_s^1 - W_s^*(\alpha) - \epsilon_N Y(\alpha)|^\gamma \right. \\ & \quad \left. + |V_s^1 - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha) - \epsilon_N Y(\alpha)|^\gamma \right] d\alpha \\ & \leq C|W_s^1 - V_s^1|^2 + C\epsilon_N^2 \int_0^1 |Y(\alpha)|^2 \left[|W_s^1 - W_s^*(\alpha) - \epsilon_N Y(\alpha)|^{2\gamma} \right. \\ & \quad \left. + |V_s^1 - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha) - \epsilon_N Y(\alpha)|^{2\gamma} \right] d\alpha. \end{aligned}$$

But Y is independent of $(W_s^*, V_s^*(\mathbf{V}_s, \mathbf{W}_s, \cdot))$ and it holds that

$$\sup_{x \in \mathbb{R}^3} \int_0^1 |x - \epsilon_N Y(\alpha)|^{2\gamma} |Y(\alpha)|^2 d\alpha \leq \int_0^1 |\epsilon_N Y(\alpha)|^{2\gamma} |Y(\alpha)|^2 d\alpha = C\epsilon_N^{2\gamma}$$

by recalling that $\gamma \in (-1, 0)$ and that Y is uniformly distributed on $B(0, 1)$, so that finally,

$$I_1^N(s) \leq C|W_s^1 - V_s^1|^2 + C\epsilon_N^{2+2\gamma}.$$

For $I_2^N(s)$, we first write $I_2^N(s) \leq A + B$, where

$$A = 2 \int_0^1 \int_0^\infty \int_0^{2\pi} |c_W(s) - c_W^N(s)|^2 d\varphi dz d\alpha$$

and

$$B = 2 \int_0^1 \int_0^\infty \int_0^{2\pi} |c_{K,V}^N(s) - c_{K,V}(s)|^2 d\varphi dz d\alpha.$$

We first apply (2.17) with $v = W_s^1$, $v_* = W_s^{*,\epsilon_N}(\alpha)$, $\tilde{v} = W_s^1$ and $\tilde{v}_* = W_s^*(\alpha)$:

$$\begin{aligned} A & \leq C \int_0^1 |W_s^*(\alpha) - W_s^{*,\epsilon_N}(\alpha)|^2 |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma d\alpha \\ & = C\epsilon_N^2 \int_0^1 |Y(\alpha)|^2 |W_s^1 - W_s^*(\alpha) - \epsilon_N Y(\alpha)|^\gamma d\alpha. \end{aligned}$$

Using that $\sup_{x \in \mathbb{R}^3} \int_0^1 |x - \epsilon_N Y(\alpha)|^\gamma |Y(\alpha)|^2 d\alpha \leq \int_0^1 |\epsilon_N Y(\alpha)|^\gamma |Y(\alpha)|^2 d\alpha = C\epsilon_N^\gamma$ and arguing as in the study of $I_1^N(s)$, we conclude that $A \leq C\epsilon_N^{2+\gamma} \leq C\epsilon_N^{2+2\gamma}$. The other term B is treated in the same way (observe that (2.17) obviously also holds when replacing c by $c_K = c\mathbf{1}_{\{z \leq K\}}$).

We finally treat $I_3^N(s)$. It is obvious that

$$I_3^N(s) \leq \int_0^1 \int_0^\infty \int_0^{2\pi} |c_W^N(s) - c_{K,V}^N(s)|^2 d\varphi dz d\alpha + I_2^N(s).$$

But

$$\begin{aligned} & \int_0^\infty \int_0^{2\pi} |c_W^N(s) - c_{K,V}^N(s)|^2 d\varphi dz \\ &= \int_0^K \int_0^{2\pi} |c_W^N(s) - c_V^N(s)|^2 d\varphi dz + \int_K^\infty \int_0^{2\pi} |c_W^N(s)|^2 d\varphi dz. \end{aligned}$$

Applying first (2.17) with $v = W_s^1$, $v_* = W_s^{*,\epsilon_N}(\alpha)$, $\tilde{v} = V_s^1$ and $\tilde{v}_* = V_s^{*,\epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha)$, we find that

$$\begin{aligned} & \int_0^K \int_0^{2\pi} |c_W^N(s) - c_V^N(s)|^2 d\varphi dz \\ & \leq C(|W_s^1 - V_s^1|^2 + |W_s^{*,\epsilon_N}(\alpha) - V_s^{*,\epsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2) |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma \\ & = C(|W_s^1 - V_s^1|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2) |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma. \end{aligned}$$

Moreover, as seen in the proof of Lemma 2.2.3,

$$\int_K^\infty \int_0^{2\pi} |c_W^N(s)|^2 d\varphi dz = |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^2 \Psi_K(|W_s^1 - W_s^{*,\epsilon_N}(\alpha)|),$$

where $\Psi_K(x) = \Phi(x) - \Phi_K(x) \leq C \int_K^\infty G^2(z/x^\gamma) dz \leq Cx^{2\gamma/\nu} K^{1-2/\nu}$. Hence,

$$\int_K^\infty \int_0^{2\pi} |c_W^N(s)|^2 d\varphi dz \leq C|W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^{2+2\gamma/\nu} K^{1-2/\nu}.$$

All this shows that

$$\begin{aligned} I_3^N(s) & \leq I_2^N(s) + C \int_0^1 (|W_s^1 - V_s^1|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2) |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma d\alpha \\ & \quad + CK^{1-2/\nu} \int_0^1 |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^{2+2\gamma/\nu} d\alpha \end{aligned}$$

and this ends the proof. \square

To prove our main result, we need the following estimate which can be found in [26, Theorem 1].

Lemma 2.6.2. *Fix $A > 0$ and $q > 4$. There is a constant $C_{A,q}$ such that for all $f \in \mathcal{P}_q(\mathbb{R}^3)$ verifying $\int_{\mathbb{R}^3} |v|^q f(dv) \leq A$, all i.i.d. family $(X_i)_{i=1,\dots,N}$ of f -distributed random variables,*

$$\mathbb{E} \left[\mathcal{W}_2^2 \left(f, N^{-1} \sum_{i=1}^N \delta_{X_i} \right) \right] \leq C_{A,q} N^{-1/2}.$$

Proposition 2.6.3. *Fix $T > 0$ and recall that h_N was defined in Lemma 2.5.1. Consider the stopping time*

$$\sigma_N = \inf \{ t \geq 0 : \|\bar{\mu}_{\mathbf{W}_t}^N\|_{L^p} \geq 13500(1 + h_N(t)) \},$$

where $\bar{\mu}_{\mathbf{W}_t}^N = \mu_{\mathbf{W}_t}^N * \psi_{\epsilon_N}$ with $\psi_{\epsilon_N}(x) = (3/(4\pi\epsilon_N^3)) \mathbf{1}_{\{|x| \leq \epsilon_N\}}$ and $\mu_{\mathbf{W}_t}^N = N^{-1} \sum_{i=1}^N \delta_{W_t^i}$. We have for all $T > 0$,

$$\sup_{[0,T]} \mathbb{E}[|W_{t \wedge \sigma_N}^1 - V_{t \wedge \sigma_N}^1|^2] \leq C_T(\epsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2}).$$

Proof. We fix $T > 0$ and set $u_t^N = \mathbb{E}[|W_{t \wedge \sigma_N}^1 - V_{t \wedge \sigma_N}^1|^2]$ for all $t \in [0, T]$. By the Itô formula, we have

$$\begin{aligned} u_t^N &= \mathbb{E} \left[\int_0^{t \wedge \sigma_N} \int_0^1 \int_0^\infty \int_0^{2\pi} \left(|W_s^1 - V_s^1 + c_W(s) - c_{K,V}(s)|^2 - |W_s^1 - V_s^1|^2 \right) d\varphi dz d\alpha \right] \\ &= \mathbb{E} \left[\int_0^{t \wedge \sigma_N} \int_0^1 \int_0^\infty \int_0^{2\pi} \left(2(W_s^1 - V_s^1) \cdot (c_W(s) - c_{K,V}(s)) \right. \right. \\ &\quad \left. \left. + |c_W(s) - c_{K,V}(s)|^2 \right) d\varphi dz d\alpha \right] \\ &= \mathbb{E} \left[\int_0^{t \wedge \sigma_N} (I_0^N(s) + I_1^N(s) + I_2^N(s) + I_3^N(s)) ds \right], \end{aligned}$$

where $I_i^N(s)$ was introduced in Lemma 2.6.1 for $i = 0, 1, 2, 3$. We know from Lemma 2.6.1 that

$$u_t^N \leq C t \epsilon_N^{2+2\gamma} + C \int_0^t u_s^N ds + C(J_1^N(t) + J_2^N(t) + J_3^N(t)),$$

where

$$\begin{aligned} J_1^N(t) &= \mathbb{E} \left[\int_0^{t \wedge \sigma_N} \int_0^1 |W_s^1 - V_s^1|^2 |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma d\alpha ds \right], \\ J_2^N(t) &= \mathbb{E} \left[\int_0^{t \wedge \sigma_N} \int_0^1 |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma d\alpha ds \right], \\ J_3^N(t) &= K^{1-2/\nu} \mathbb{E} \left[\int_0^{t \wedge \sigma_N} \int_0^1 |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^{2+2\gamma/\nu} d\alpha ds \right]. \end{aligned}$$

First, we have

$$J_3^N(t) \leq CK^{1-2/\nu}t.$$

Indeed, it suffices to use that $|W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^{2+2\gamma/\nu} \leq C(1 + |W_s^1|^2 + |W_s^{*,\epsilon_N}(\alpha)|^2)$ (because $2 + 2\gamma/\nu \in (0, 2)$), that $|W_s^{*,\epsilon_N}(\alpha)|^2 \leq 2 + 2|W_s^*(\alpha)|^2$ (because $\epsilon_N \in (0, 1)$ and Y takes its values in $B(0, 1)$) and finally that $\mathbb{E}[|W_s^1|^2] = \int_0^1 |W_s^*(\alpha)|^2 d\alpha = m_2(f_0)$.

Next, $\mathcal{L}_\alpha(W_s^{*,\epsilon_N}) = f_s * \psi_{\epsilon_N}$, so that $\int_0^1 |W_s^1 - W_s^{*,\epsilon_N}(\alpha)|^\gamma d\alpha \leq 1 + C_{\gamma,p} \|f_s * \psi_{\epsilon_N}\|_{L^p}$ by (2.13) (recall that $p > 3/(3 + \gamma)$ is fixed since the beginning of the section). Of course, $\|f_s * \psi_{\epsilon_N}\|_{L^p} \leq \|f_s\|_{L^p}$, and we conclude that

$$J_1^N(t) \leq C_{\gamma,p} \int_0^t (1 + \|f_s\|_{L^p}) u_s^N ds.$$

On the other hand, using the exchangeability and that $W_s^{*,\epsilon_N}(\alpha) = W_s^*(\alpha) + \epsilon_N Y(\alpha)$, with $Y(\alpha)$ independent of $W_s^*(\alpha)$ and $V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)$ introduced in Notation 2.4.4, we have

$$\begin{aligned} J_2^N(t) &= \mathbb{E} \left[\int_0^{t \wedge \sigma_N} \int_0^1 |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 N^{-1} \sum_{i=1}^N |W_s^i - \epsilon_N Y(\alpha) - W_s^*(\alpha)|^\gamma d\alpha ds \right] \\ &= \mathbb{E} \left[\int_0^{t \wedge \sigma_N} \int_0^1 |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 \right. \\ &\quad \left. \times \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |w - x - W_s^*(\alpha)|^\gamma \psi_{\epsilon_N}(x) \mu_{\mathbf{W}_s}^N(dw) dx \right) d\alpha ds \right] \\ &= \mathbb{E} \left[\int_0^{t \wedge \sigma_N} \int_0^1 |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 \left(\int_{\mathbb{R}^3} |w - W_s^*(\alpha)|^\gamma \bar{\mu}_{\mathbf{W}_s}^N(dw) \right) d\alpha ds \right]. \end{aligned}$$

But $\int_{\mathbb{R}^3} |W_s^*(\alpha) - w|^\gamma \bar{\mu}_{\mathbf{W}_s}^N(dw) \leq C_{\gamma,p} (1 + \|\bar{\mu}_{\mathbf{W}_s}^N\|_{L^p})$ by (2.13), so that

$$J_2^N(t) \leq C_{\gamma,p} \mathbb{E} \left[\int_0^{t \wedge \sigma_N} \int_0^1 (1 + \|\bar{\mu}_{\mathbf{W}_s}^N\|_{L^p}) |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 d\alpha ds \right].$$

We now deduce from Lemma 2.4.2 that

$$\begin{aligned} &\int_0^1 |W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 d\alpha \\ &\leq 2 \int_0^1 \left(|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2 + |Z_s^*(\mathbf{W}_s, \alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^2 \right) d\alpha \\ &= 2\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N) + 2 \frac{1}{N} \sum_{i=1}^N |W_s^i - V_s^i|^2. \end{aligned}$$

Using the exchangeability and that $\|\bar{\mu}_{\mathbf{W}_s}^N\|_{L^p} \leq 13500(1 + h_N(s))$ for all $s \leq \tau_N$, it holds that

$$J_2^N(t) \leq C \int_0^t (1 + h_N(s)) \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)] ds + C \int_0^t (1 + h_N(s)) u_s^N ds.$$

We thus have checked that

$$\begin{aligned} u_t^N &\leq Ct(\epsilon_N^{2+2\gamma} + K^{1-2/\nu}) + C \int_0^t (1 + h_N(s)) \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)] ds \\ &\quad + C \int_0^t (1 + \|f_s\|_{L^p} + h_N(s)) u_s^N ds. \end{aligned}$$

But for each $t \geq 0$, the family $(W_t^i)_{i=1, \dots, N}$ is i.i.d. and f_t -distributed. Furthermore, we have $\sup_{[0, T]} \mathbb{E}[|W_t^1|^q] < \infty$ ($q > 6$) by (2.35). Hence Lemma 2.6.2 tells us that

$$\sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)] \leq C_T N^{-1/2}. \quad (2.43)$$

Using the Grönwall lemma, we deduce that

$$\begin{aligned} \sup_{[0, T]} u_t^N &\leq C_T \left(\epsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2} \int_0^T (1 + h_N(s)) ds \right) \\ &\quad \times \exp \left(C \int_0^T (1 + \|f_s\|_{L^p} + h_N(s)) ds \right). \end{aligned}$$

But $\int_0^T h_N(s) ds \leq 2 \int_0^T \|f_s\|_{L^p} ds$ by Lemma 2.5.1-(ii). And $f \in L_{loc}^1([0, \infty), L^p(\mathbb{R}^3))$. We thus conclude that

$$\sup_{[0, T]} u_t^N \leq C_T (\epsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2})$$

as desired. \square

Now, we give the

Proof of Theorem 2.1.5. As explained at the beginning of the section, we only have to prove (2.42). Recall that $\sigma_N = \inf\{t \geq 0 : \|\bar{\mu}_{\mathbf{W}_t}^N\|_{L^p} \geq 13500(1 + h_N(t))\}$, that $q > 6$ and that $\delta = 6/q$. It is clear that $\mathbb{P}[\sigma_N \leq T] \leq C_{T, q, \delta} N^{1-q\delta/3} = C_{T, q} N^{-1}$ from Proposition 2.5.2. Then for $t \in [0, T]$, we write

$$\begin{aligned} \sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, f_t)] &\leq 2 \sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, \mu_{\mathbf{W}_t}^N) + \mathcal{W}_2^2(\mu_{\mathbf{W}_t}^N, f_t)] \\ &\leq 2 \sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, \mu_{\mathbf{W}_t}^N)] + C_T N^{-1/2} \end{aligned}$$

by (2.43). But, by exchangeability, we have

$$\mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, \mu_{\mathbf{W}_t}^N)] \leq \mathbb{E}\left[N^{-1} \sum_{i=1}^N |W_t^i - V_t^i|^2\right] = \mathbb{E}[|W_t^1 - V_t^1|^2].$$

Moreover,

$$\begin{aligned} \mathbb{E}[|W_t^1 - V_t^1|^2] &\leq \mathbb{E}[|W_{t \wedge \sigma_N}^1 - V_{t \wedge \sigma_N}^1|^2] + \mathbb{E}[|W_t^1 - V_t^1|^2 \mathbf{1}_{\{\sigma_N \leq T\}}] \\ &\leq C_T(\epsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2}) + C\mathbb{E}[|W_t^1|^4 + |V_t^1|^4]^{1/2}(\mathbb{P}(\sigma_N \leq T))^{1/2}, \end{aligned}$$

by Proposition 2.6.3, and the Cauchy-Schwarz inequality. Noting that $\mathbb{E}[|W_t^1|^4] \leq C_T$ by (2.35), and that $\mathbb{E}[|V_t^1|^4] \leq C_T \mathbb{E}[|V_0^1|^4]$ by Lemma 2.4.5, we deduce that

$$\mathbb{E}[|W_t^1 - V_t^1|^2] \leq C_{T,q}(\epsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2}).$$

All in all, we have proved that

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, f_t)] \leq C_{T,q}(\epsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2}).$$

This is precisely (2.42), since $\epsilon_N^{2+2\gamma} = N^{-(1-6/q)(2+2\gamma)/3}$, with $\epsilon_N = N^{-(1-\delta)/3}$ and $\delta = 6/q$. \square

Chapter 3

On the equivalence between some jumping SDEs with rough coefficients and some non-local PDEs

This work was written in collaboration with Nicolas Fournier.

We study some jumping SDE and the corresponding Fokker-Planck (or Kolmogorov forward) equation, which is a non-local PDE. We assume only some measurability and growth conditions on the coefficients. We prove that for any weak solution $(f_t)_{t \in [0, T]}$ of the PDE, there exists a weak solution to the SDE of which the time marginals are given by $(f_t)_{t \in [0, T]}$. As a corollary, we deduce that for any given initial condition, existence for the PDE is equivalent to weak existence for the SDE and uniqueness in law for the SDE implies uniqueness for the PDE. This extends some ideas of Figalli [22] concerning continuous SDEs and local PDEs.

3.1 Introduction

We consider the d -dimensional stochastic differential equation posed on some time interval $[0, T]$

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s + \int_0^t \int_E h(s, z, X_{s-}) N(ds, dz), \quad (3.1)$$

where $(B_t)_{t \in [0, T]}$ is a d -dimensional Brownian motion and $N(ds, dz)$ is a Poisson measure on $[0, T] \times E$ with intensity measure $ds \mu(dz)$. The coefficients $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \mapsto \mathcal{S}_d^+$ and $h : [0, T] \times E \times \mathbb{R}^d \mapsto \mathbb{R}^d$ are supposed to be at least measurable. The space E is endowed with a σ -field \mathcal{E} and with a σ -finite measure μ and \mathcal{S}_d^+ is the set

of nonnegative symmetric $d \times d$ real matrices. The Fokker-Planck (or Kolmogorov forward) equation associated to (3.1) is

$$\partial_t f_t + \operatorname{div}(b(t, \cdot) f_t) = \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}([\sigma(t, \cdot) \sigma^*(t, \cdot)]_{i,j} f_t) + \mathcal{L}_t f_t, \quad (3.2)$$

where $\mathcal{L}_t f_t : \mathbb{R}^d \mapsto \mathbb{R}$ is defined by

$$\int_{\mathbb{R}^d} (\mathcal{L}_t f_t)(x) \varphi(x) dx = \int_{\mathbb{R}^d} \int_E [\varphi(x + h(t, z, x)) - \varphi(x)] f_t(x) dz$$

for any reasonable $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$. We use the notation $\nabla = \nabla_x$, $\operatorname{div} = \operatorname{div}_x$ and $\partial_{ij} = \partial_{x_i x_j}^2$.

Let $\mathcal{P}(\mathbb{R}^d)$ be the set of probability measures on \mathbb{R}^d and

$$\mathcal{P}_1(\mathbb{R}^d) = \{f \in \mathcal{P}(\mathbb{R}^d) : m_1(f) < \infty\} \quad \text{with} \quad m_1(f) := \int_{\mathbb{R}^d} |x| f(dx).$$

We define $L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$ as the set of all measurable families $(f_t)_{t \in [0, T]}$ of probability measures on \mathbb{R}^d such that $\sup_{t \in [0, T]} m_1(f_t) < \infty$.

3.1.1 Main result

We will suppose the following conditions.

Assumption 3.1.1. *The functions $\sigma : [0, T] \times \mathbb{R}^d \mapsto \mathcal{S}_d^+$, $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $h : [0, T] \times E \times \mathbb{R}^d \mapsto \mathbb{R}^d$ are measurable and there is a constant C such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$,*

$$|\sigma(t, x)| + |b(t, x)| + \int_E |h(t, z, x)| \mu(dz) \leq C(1 + |x|).$$

We set $a(t, x) = \sigma(t, x) \sigma^*(t, x)$, which satisfies $|a(t, x)| \leq C(1 + |x|^2)$.

Definition 3.1.2. *Suppose Assumption 3.1.1. A measurable family $(f_t)_{t \in [0, T]}$ of probability measures on \mathbb{R}^d is called a weak solution to (3.2) if for all $\varphi \in C_c^2(\mathbb{R}^d)$, all $t \in [0, T]$,*

$$\int_{\mathbb{R}^d} \varphi(x) f_t(dx) = \int_{\mathbb{R}^d} \varphi(x) f_0(dx) + \int_0^t \int_{\mathbb{R}^d} [\mathcal{A}_s \varphi(x) + \mathcal{B}_s \varphi(x)] f_s(dx) ds, \quad (3.3)$$

with the diffusion operator $\mathcal{A}_s \varphi(x) := b(s, x) \cdot \nabla \varphi(x) + \frac{1}{2} \sum_{i,j=1}^d a_{ij}(s, x) \partial_{ij} \varphi(x)$ and the jump operator $\mathcal{B}_s \varphi(x) := \int_E [\varphi(x + h(s, z, x)) - \varphi(x)] \mu(dz)$.

We will check the following facts in the appendix, implying in particular that (3.3) makes sense.

Remark 3.1.3. *Suppose Assumption 3.1.1.*

(i) For $\varphi \in C_c^2(\mathbb{R}^d)$, $\sup_{[0,T] \times \mathbb{R}^d} (|\mathcal{A}_s \varphi(x)| + |\mathcal{B}_s \varphi(x)|) < \infty$.

(ii) Any weak solution $(f_t)_{t \in [0,T]}$ to the equation (3.2) starting from $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$ belongs to $L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$.

(iii) If $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$, the weak formulation (3.3) automatically extends to all functions $\varphi \in C^2(\mathbb{R}^d)$ such that $(1 + |x|)[|\varphi(x)| + |\nabla \varphi(x)| + |D^2 \varphi(x)|]$ is bounded.

Point (iii) is far from optimal, but sufficient for our purpose. Our main result reads as follows.

Theorem 3.1.4. *Suppose Assumption 3.1.1 and consider any weak solution $(f_t)_{t \in [0,T]}$ to (3.2) such that $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$. There exist, on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, a d -dimensional $(\mathcal{F}_t)_{t \in [0,T]}$ -Brownian motion $(B_t)_{t \in [0,T]}$, a $(\mathcal{F}_t)_{t \in [0,T]}$ -Poisson measure $N(dt, dz)$ on $[0, T] \times E$ with intensity measure $dt \mu(dz)$, these two objects being independent, as well as a càdlàg $(\mathcal{F}_t)_{t \in [0,T]}$ -adapted process $(X_t)_{t \in [0,T]}$ solving (3.1) and such that $\mathcal{L}(X_t) = f_t$ for all $t \in [0, T]$.*

For $(X_t)_{t \in [0,T]}$ a solution to (3.1) and for $f_t = \mathcal{L}(X_t)$, a simple application of the Itô formula (to compute $\int_{\mathbb{R}^d} \varphi(x) f_t(dx) = \mathbb{E}[\varphi(X_t)]$ with $\varphi \in C_c^2(\mathbb{R}^d)$) shows that the family $(f_t)_{t \in [0,T]}$ is a weak solution to (3.2). The following corollary is thus immediately deduced from Theorem 3.1.4.

Corollary 3.1.5. *Suppose Assumption 3.1.1 and fix $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$.*

(i) *The existence of a (weak) solution $(X_t)_{t \in [0,T]}$ to (3.1) such that $\mathcal{L}(X_0) = f_0$ is equivalent to the existence of a weak solution $(f_t)_{t \in [0,T]}$ to (3.2) starting from f_0 .*

(ii) *The uniqueness (in law) of the solution $(X_t)_{t \in [0,T]}$ to (3.1) with $\mathcal{L}(X_0) = f_0$ implies the uniqueness of the weak solution $(f_t)_{t \in [0,T]}$ to (3.2) starting from f_0 .*

In almost all models arising from applied sciences, the jump operator is given under the form $\mathcal{B}_s \varphi(x) = \int_F [\varphi(x + g(s, y, x)) - \varphi(x)] \kappa(s, y, x) \nu(dy)$, meaning that when in the position x at time s , the process jumps to $x + g(s, y, x)$ at rate $\kappa(s, y, x) \nu(dy)$. Here F is a measurable space endowed with a σ -finite measure ν and we have two measurable functions $g : [0, T] \times F \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\kappa : [0, T] \times F \times \mathbb{R}^d \mapsto \mathbb{R}_+$. Introducing $E = F \times \mathbb{R}_+$, $\mu(dy, du) = \nu(dy) du$ and $h(s, (y, u), x) = g(s, y, x) \mathbf{1}_{\{u \leq \kappa(s, y, x)\}}$, one easily verifies that $\mathcal{B}_s \varphi(x) = \int_E [\varphi(x + h(s, (y, u), x)) - \varphi(x)] \mu(dy, du)$. Our results thus apply if $\int_F |g(s, y, x)| \kappa(s, y, x) \nu(dy) \leq C(1 + |x|)$.

3.1.2 Motivation

Stochastic differential equations with jumps are now playing an important role in modeling and applied sciences. We refer to the book of Situ [54] for all basic results and a lot of possible applications. The book of Jacod [39] contains many important results about weak and strong existence and uniqueness, relations between SDEs and martingale problems, etc. See also the survey paper of Bass [8].

Existence for PDEs is often more developed than for SDEs, so Theorem 3.1.4 might be useful to derive some new weak existence results for the SDE (3.1).

Our main motivation is the uniqueness for some nonlinear PDEs, for which the use of nonlinear (in the sense of McKean) SDEs has proved to be a powerful tool. For example, the first (partial) uniqueness result concerning the homogeneous Boltzmann for long range interactions was derived by Tanaka [57]. He was studying the simplest case of Maxwell molecules. Unfortunately, he was only able to prove the uniqueness in law of the nonlinear SDE associated to the Boltzmann equation. Horowitz and Karandikar [38] were able to deduce the uniqueness for the (same) Boltzmann equation proceeding as follows. Let us recall that the original equation writes $\partial_t f_t = Q(f_t, f_t)$, for some quadratic nonlocal operator Q . For f a solution, they consider the linear PDE $\partial_t g_t = Q(g_t, f_t)$, with unknown g satisfying $g_0 = f_0$. They prove uniqueness in law for the (linear) SDE associated to this PDE (for any initial condition). They deduce, extending some results of Ethier and Kurtz [20, Chap.4, Propositions 9.18 and 9.19], the uniqueness for the linear PDE (for any initial condition). So the unique solution (with $g_0 = f_0$) to $\partial_t g_t = Q(g_t, f_t)$ is f itself. Consequently, the time marginals of the solution X to the linear SDE (when $X_0 \sim f_0$), which solve $\partial_t g_t = Q(g_t, f_t)$ are necessarily $(f_t)_{t \in [0, T]}$. Thus X actually solves the nonlinear SDE. Since uniqueness in law holds for the nonlinear SDE by Tanaka [57], they deduce that there is at most one solution to the Boltzmann equation $\partial_t f_t = Q(f_t, f_t)$, for some given reasonable initial condition f_0 .

Let us recall that the above mentioned results of Ethier and Kurtz (extended by Horowitz and Karandikar [38, Theorem B1] and by Bhatt and Karandikar [10, e.g. Theorems 4.1 and 5.2]) state in spirit that if some SDE has a unique solution (in law) for any deterministic initial condition, then the corresponding PDE has a unique weak solution for any reasonable initial condition.

Our result is much stronger, since it does not require at all uniqueness for (3.1). If, for example, studying the Boltzmann equation, it directly implies that, to any solution f to the nonlinear equation (seen here as a solution to the linear equation $\partial_t g_t = Q(g_t, f_t)$), we can associate a solution X to the corresponding linear SDE with additionally $X_t \sim f_t$ for all t . In other words, X solves the nonlinear SDE. This might look anodyne, but this was crucial when studying more singular nonlinear equations, such as the Landau or Boltzmann equations for

moderately soft potentials, see [27] and [63]. Indeed, in such cases, we really need to use some physical symmetries to prove uniqueness : it is absolutely not clear that uniqueness holds for the linear PDE $\partial_t g_t = Q(g_t, f_t)$, since one really uses that the two arguments of Q are the same.

We hope the above discussion shows that Theorem 3.1.4 is an interesting variation of the mentioned results of Ethier and Kurtz [20]. As already said, the method we use was initiated by Figalli [22] for continuous SDEs ($h = 0$) with bounded coefficients. The boundedness assumption was relaxed in [27, Appendix B]. A special jumping SDE (with $a = b = 0$ and a special jump operator) was considered in [63] to study a singular homogeneous Boltzmann equation. We decided to write down the general case in the present paper. We did not want to assume some boundedness of the coefficients, although it complicates the proofs without introducing new deep ideas, because it is very useful for practical purposes.

Finally, as explained in the next subsection, we are not able to prove a general result when the jump part of the SDE has infinite variations, and this is a rather important limitation.

3.1.3 Strategy of the proof and plan of the paper

At many places, the situation is technically more involved, but the global strategy is exactly the same as that introduced by Figalli [22, Theorem 2.6]. Let $(f_t)_{t \in [0, T]}$ be a given weak solution to (3.2).

I. In Section 3.2, we introduce $f_t^\epsilon = f_t \star \phi_\epsilon$, where ϕ_ϵ is the centered Gaussian density with covariance matrix ϵI_d . We compute the PDE satisfied by f_t^ϵ : we find that $\partial_t f_t^\epsilon + \operatorname{div}(b^\epsilon(t, \cdot) f_t^\epsilon) = \frac{1}{2} \sum_{i,j} \partial_{i,j} (a_{i,j}^\epsilon(t, \cdot) f_t^\epsilon) + \mathcal{L}_t^\epsilon f_t^\epsilon$, for some coefficients a^ϵ , b^ϵ and some jump operator \mathcal{L}_t^ϵ . Let us mention that $a^\epsilon(t, \cdot)$, $b^\epsilon(t, \cdot)$ and \mathcal{L}_t^ϵ of course depend on f_t .

II. Still in Section 3.2, we prove that a^ϵ , b^ϵ and the coefficient of the jump operator \mathcal{L}^ϵ satisfy
 (i) the same linear growth conditions as a , b , \mathcal{L} , uniformly in $\epsilon \in (0, 1)$,
 (ii) some (non-uniform) local Lipschitz conditions.

III. In Section 3.3, we use II to build, for each $\epsilon \in (0, 1)$, a solution $(X_t^\epsilon)_{t \in [0, T]}$ to some SDE of which the Fokker-Planck equation is the PDE satisfied by $(f_t^\epsilon)_{t \in [0, T]}$. Since both the SDE and the PDE (with $\epsilon \in (0, 1)$ fixed) are well-posed (because the coefficients are regular enough), we conclude that $\mathcal{L}(X_t^\epsilon) = f_t^\epsilon$. Indeed, the time marginals of $(X_t^\epsilon)_{t \in [0, T]}$ satisfy the same PDE as $(f_t^\epsilon)_{t \in [0, T]}$.

IV. Still in Section 3.3, we prove that the family $\{(X_t^\epsilon)_{t \in [0, T]}, \epsilon \in (0, 1)\}$ is tight. This is rather easy from the Aldous criterion [1], using only II-(ii).

V. In Section 3.4, we finally consider a limit point $(X_t)_{t \in [0, T]}$, as $\epsilon \rightarrow 0$, of $\{(X_t^\epsilon)_{t \in [0, T]}, \epsilon \in (0, 1)\}$. Since $\mathcal{L}(X_t^\epsilon) = f_t^\epsilon$ by III, we deduce that $\mathcal{L}(X_t) = f_t$ for each $t \in [0, T]$. It then

remains to show that $(X_t)_{t \in [0, T]}$ is a weak solution to (3.1) and we classically make use of martingale problems. Since the coefficients a, b, h are possibly rough, we have to approximate them by some continuous (in x) coefficients $\tilde{a}, \tilde{b}, \tilde{h}$. We use that we already know the time marginals of $(X_t)_{t \in [0, T]}$: we can take $\tilde{a}(t, \cdot)$, $\tilde{b}(t, \cdot)$ and $\tilde{h}(t, \cdot, z)$ close to $a(t, \cdot)$, $b(t, \cdot)$ and $h(t, \cdot, z)$ in $L^1(f_t)$.

The proof of Remark 3.1.3 is written in an appendix.

To conclude this paragraph, let us mention a few difficulties. The regularized jump operator, in its weak form writes $\int_{\mathbb{R}^d} \mathcal{L}_t^\epsilon f_t^\epsilon(y) \varphi(y) dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_E [\varphi(y + h(t, z, x)) - \varphi(y)] \phi_\epsilon(x - y) f_t^\epsilon(dx) dy$. We found no *regular* Poisson representation of the associated SDE. We use an indicator function, see (3.4). This is why we are not able to treat the case of an infinite variation jump term: we do not know how to prove that a SDE like (3.4), with a compensated Poisson measure and some weaker condition on h (something like $\int_E |h(s, z, x)|^2 \mu(dz) \leq C(1 + |x|^2)$), is well-posed.

Although this should be classical since the coefficients are rather regular for $\epsilon \in (0, 1)$ fixed, we found no reference about the uniqueness for the PDE satisfied by $(f_t^\epsilon)_{t \in [0, T]}$ (see Lemma 3.2.1). We have not been able to write down a deterministic proof. We thus use that the corresponding SDE is well-posed (for any deterministic initial condition) and we apply a result of Horowitz and Karandikar [38].

3.1.4 Convention

During the whole paper, we always suppose Assumption 3.1.1 and that $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$. We use the generic notation C for a positive finite constant, of which the value may change from line to line. It is allowed to depend only on the dimension d , on the parameters a, b, h, E, μ, T of our equations, and on the weak solution $(f_t)_{t \in [0, T]}$ to (3.2) under study. When a constant depends on another parameter, we indicate it in subscript. For example, C_ϵ is a constant allowed to depend only on $a, b, h, E, \mu, T, (f_t)_{t \in [0, T]}$ and on ϵ .

3.2 Regularization

We introduce the following regularization procedure, as Figalli in [22], see also [63].

Lemma 3.2.1. *For $(f_t)_{t \in [0, T]} \in L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$ a weak solution to (3.2) and $\epsilon \in (0, 1)$, we set*

$$f_t^\epsilon(y) := \int_{\mathbb{R}^d} \phi_\epsilon(x - y) f_t(dx) = (f_t \star \phi_\epsilon)(y) \quad \text{with} \quad \phi_\epsilon(x) = (2\pi\epsilon)^{-d/2} e^{-|x|^2/(2\epsilon)}.$$

Then for any test function $\psi \in C_c^2(\mathbb{R}^d)$, any $t \in [0, T]$,

$$\int_{\mathbb{R}^d} \psi(y) f_t^\epsilon(y) dy = \int_{\mathbb{R}^d} \psi(y) f_0^\epsilon(y) dy + \int_0^t \int_{\mathbb{R}^d} [\mathcal{A}_{s,\epsilon} \psi(y) + \mathcal{B}_{s,\epsilon} \psi(y)] f_s^\epsilon(y) dy ds,$$

with

$$\begin{aligned} \mathcal{A}_{t,\epsilon} \psi(y) &= b^\epsilon(t, y) \cdot \nabla \psi(y) + \frac{1}{2} \sum_{i,j=1}^d a_{ij}^\epsilon(t, y) \partial_{ij} \psi(y), \\ \mathcal{B}_{t,\epsilon} \psi(y) &= \int_E \int_{\mathbb{R}^d} [\psi(y + h(t, z, x)) - \psi(y)] F_t^\epsilon(x, y) f_t(dx) \mu(dz), \end{aligned}$$

where

$$\begin{aligned} a^\epsilon(t, y) &:= \frac{\int_{\mathbb{R}^d} \phi_\epsilon(x - y) a(t, x) f_t(dx)}{f_t^\epsilon(y)}, \\ b^\epsilon(t, y) &:= \frac{\int_{\mathbb{R}^d} \phi_\epsilon(x - y) b(t, x) f_t(dx)}{f_t^\epsilon(y)}, \\ F_t^\epsilon(x, y) &:= \frac{\phi_\epsilon(x - y)}{f_t^\epsilon(y)}. \end{aligned}$$

Proof. It is obvious that $f_t^\epsilon(y) > 0$ for each $(t, y) \in [0, T] \times \mathbb{R}^d$. We first apply (3.3) with the choice $\varphi(x) = \phi_\epsilon(x - y)$ (with some fixed $y \in \mathbb{R}^d$), which is licit by Remark 3.1.3-(iii). We then integrate the obtained equality against $\psi \in C_c^2(\mathbb{R}^d)$. This gives

$$\int_{\mathbb{R}^d} \psi(y) f_t^\epsilon(y) dy = \int_{\mathbb{R}^d} \psi(y) f_0^\epsilon(y) dy + \int_0^t (I_s + J_s) ds,$$

where

$$I_t := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(y) \mathcal{A}_t \phi_\epsilon(x - y) f_t(dx) dy \quad \text{and} \quad J_t := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(y) \mathcal{B}_t \phi_\epsilon(x - y) f_t(dx) dy.$$

First,

$$I_t = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(y) b(t, x) \cdot \nabla \phi_\epsilon(x - y) f_t(dx) dy + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{i,j=1}^d \psi(y) a_{ij}(t, x) \partial_{ij} \phi_\epsilon(x - y) f_t(dx) dy.$$

But we have $\int_{\mathbb{R}^d} \psi(y) \nabla \phi_\epsilon(x - y) dy = \int_{\mathbb{R}^d} \phi_\epsilon(x - y) \nabla \psi(y) dy$ as well as $\int_{\mathbb{R}^d} \psi(y) \partial_{ij} \phi_\epsilon(x -$

$y)dy = \int_{\mathbb{R}^d} \phi_\epsilon(x-y) \partial_{ij} \psi(y) dy$, so that

$$\begin{aligned} I_t &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_\epsilon(x-y) b(t,x) \cdot \nabla \psi(y) f_t(dx) dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(t,x) \phi_\epsilon(x-y) \partial_{ij} \psi(y) f_t(dx) dy \\ &= \int_{\mathbb{R}^d} b^\epsilon(t,y) \cdot \nabla \psi(y) f_t^\epsilon(y) dy + \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}^\epsilon(t,y) \partial_{ij} \psi(y) f_t^\epsilon(y) dy \\ &= \int_{\mathbb{R}^d} \mathcal{A}_{t,\epsilon} \psi(y) f_t^\epsilon(y) dy \end{aligned}$$

as desired. For the jump term, we use a similar computation as in [63, Proposition 3.1]. Since μ is σ -finite, there exists a non-decreasing sequence $(E_n)_{n \geq 1} \subset E$ such that $\bigcup_{n=1}^\infty E_n = E$ and $\mu(E_n) < \infty$ for each $n \geq 1$. We fix n and write

$$\begin{aligned} J_t &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{E_n} \psi(y) \phi_\epsilon(x-y+h(t,z,x)) \mu(dz) f_t(dx) dy \\ &\quad - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{E_n} \psi(y) \phi_\epsilon(x-y) \mu(dz) f_t(dx) dy \\ &\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{E \setminus E_n} \psi(y) [\phi_\epsilon(x-y+h(t,z,x)) - \phi_\epsilon(x-y)] \mu(dz) f_t(dx) dy. \end{aligned}$$

Using the change of variables $y - h(t, z, x) \mapsto y$, we see that

$$\int_{\mathbb{R}^d} \psi(y) \phi_\epsilon(x-y+h(t,z,x)) dy = \int_{\mathbb{R}^d} \psi(y+h(t,z,x)) \phi_\epsilon(x-y) dy,$$

and consequently,

$$\begin{aligned} J_t &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{E_n} [\psi(y+h(t,z,x)) - \psi(y)] \phi_\epsilon(x-y) \mu(dz) f_t(dx) dy \\ &\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{E \setminus E_n} \psi(y) [\phi_\epsilon(x-y+h(t,z,x)) - \phi_\epsilon(x-y)] \mu(dz) f_t(dx) dy. \end{aligned}$$

Observe now that

$$|\psi(y+h(t,z,x)) - \psi(y)| \phi_\epsilon(x-y) \leq C |h(t,z,x)| \phi_\epsilon(x-y) \in L^1(\mu(dz) f_t(dx) dy)$$

and $|\psi(y) [\phi_\epsilon(x-y+h(t,z,x)) - \phi_\epsilon(x-y)]| \leq C_\epsilon |\psi(y)| |h(t,z,x)| \in L^1(\mu(dz) f_t(dx) dy)$: this uses that $\psi \in C_c^2(\mathbb{R}^d)$, Assumption 3.1.1 and that $f_t \in \mathcal{P}_1(\mathbb{R}^d)$. We thus can let $n \rightarrow \infty$:

$$J_t = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_E [\psi(y+h(t,z,x)) - \psi(y)] \phi_\epsilon(x-y) \mu(dz) f_t(dx) dy = \int_{\mathbb{R}^d} \mathcal{B}_{t,\epsilon} \psi(y) f_t^\epsilon(y) dy,$$

which completes the proof. \square

Let us now give some growth and regularity estimates on the regularized coefficients.

Lemma 3.2.2. *Let $(f_t)_{t \in [0, T]} \in L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$ be a weak solution to (3.2) and recall that $a^\epsilon, b^\epsilon, F^\epsilon$ were introduced in Lemma 3.2.1.*

(i) *There exists a constant $C > 0$ such that for all $\epsilon \in (0, 1)$, all $y \in \mathbb{R}^d$, all $t \in [0, T]$,*

$$|b^\epsilon(t, y)| + |a^\epsilon(t, y)|^{1/2} + \int_{\mathbb{R}^d} \int_E |h(t, z, x)| F_t^\epsilon(x, y) \mu(dz) f_t(dx) \leq C(1 + |y|).$$

(ii) *For all $\epsilon \in (0, 1)$ and $R > 0$, there is $C_{R, \epsilon} > 0$ such that for all $y_1, y_2 \in B(0, R)$, all $t \in [0, T]$,*

$$\begin{aligned} |b^\epsilon(t, y_1) - b^\epsilon(t, y_2)| + |a^\epsilon(t, y_1) - a^\epsilon(t, y_2)| + |[a^\epsilon(t, y_1)]^{1/2} - [a^\epsilon(t, y_2)]^{1/2}| \\ + \int_{\mathbb{R}^d} \int_E |h(t, z, x)| |F_t^\epsilon(x, y_1) - F_t^\epsilon(x, y_2)| \mu(dz) f_t(dx) \leq C_{R, \epsilon} |y_1 - y_2|. \end{aligned}$$

Proof. We start with (i). By Assumption 3.1.1,

$$\begin{aligned} & |b^\epsilon(t, y)| + |a^\epsilon(t, y)|^{1/2} + \int_{\mathbb{R}^d} \int_E |h(t, z, x)| F_t^\epsilon(x, y) \mu(dz) f_t(dx) \\ & \leq C \frac{\int_{\mathbb{R}^d} \phi_\epsilon(x - y)(1 + |x|) f_t(dx)}{f_t^\epsilon(y)} + C \left[\frac{\int_{\mathbb{R}^d} \phi_\epsilon(x - y)(1 + |x|)^2 f_t(dx)}{f_t^\epsilon(y)} \right]^{1/2} \\ & =: CI_\epsilon(t, y) + CJ_\epsilon(t, y). \end{aligned}$$

Since for y fixed, $[f_t^\epsilon(y)]^{-1} \phi_\epsilon(x - y) f_t(dx)$ is a probability measure, we infer from Cauchy-Schwarz that $I_\epsilon(t, y) \leq J_\epsilon(t, y)$. We thus only have to prove that $[J_\epsilon(t, y)]^2 \leq C(1 + |y|^2)$. Let $L := 2 \sup_{[0, T]} m_1(f_t) + 2$. We use that

$$1 + |x| \leq 1 + |y| + |x - y| \leq 1 + 2|y| + L + |x - y| \mathbf{1}_{\{|x-y| > |y|+L\}}$$

to write

$$\begin{aligned} [J_\epsilon(t, y)]^2 & \leq 2 \frac{\int_{\mathbb{R}^d} (1 + 2|y| + L)^2 \phi_\epsilon(x - y) f_t(dx)}{f_t^\epsilon(y)} + 2 \frac{\int_{|x-y| \geq |y|+L} |x - y|^2 \phi_\epsilon(x - y) f_t(dx)}{f_t^\epsilon(y)} \\ & \leq 2(1 + 2|y| + L)^2 + 2 \frac{(|y| + L)^2 \phi_\epsilon(|y| + L)}{f_t^\epsilon(y)}. \end{aligned}$$

For the second term, we used that $|y| + L \geq 2 \geq \sqrt{2\epsilon}$ and that $z \mapsto |z|^2 \phi_\epsilon(z)$ is radially symmetric and decreasing on $\{|z| \geq \sqrt{2\epsilon}\}$. To conclude the proof of (i), it suffices to note that

$$f_t^\epsilon(y) \geq \int_{|x-y| \leq |y|+L} \phi_\epsilon(x - y) f_t(dx) \geq \phi_\epsilon(|y| + L) f_t(B(y, |y| + L)) \geq \phi_\epsilon(|y| + L)/2$$

because $z \mapsto \phi_\epsilon(z)$ is radially symmetric decreasing and because

$$f_t(B(y, |y| + L)) \geq f_t(B(0, L)) \geq 1/2,$$

since $f_t(B(0, L)^c) \leq m_1(f_t)/L \leq 1/2$.

For point (ii), it suffices to prove that $\nabla_y b^\epsilon(t, y)$, $\nabla_y a^\epsilon(t, y)$, $D_y^2 a^\epsilon(t, y)$ are locally bounded on $[0, T] \times \mathbb{R}^d$, as well as $G^\epsilon(t, y) := \int_{\mathbb{R}^d} \int_E |h(t, z, x)| |\nabla_y F_t^\epsilon(x, y)| \mu(dz) f_t(dx)$. No uniformity in ϵ is required here. By Stroock and Varadhan [55, Theorem 5.2.3], the local boundedness of $D_y^2 a^\epsilon(t, y)$ implies that of $\nabla_y ([a^\epsilon(t, y)]^{1/2})$.

First, one easily checks that $y \mapsto (f_t^\epsilon(y))^{-1}$ is of class C^∞ for each $t \in [0, T]$ and that it is locally bounded, as well as its derivatives of order 1 and 2, on $[0, T] \times \mathbb{R}^d$. This uses in particular the lower bound $f_t^\epsilon(y) \geq \phi_\epsilon(|y| + L)/2$ proved a few lines above.

Recall that by definition, we have $a^\epsilon(t, y) = (f_t^\epsilon(y))^{-1} \int_{\mathbb{R}^d} \phi_\epsilon(x - y) a(t, x) f_t(dx)$ and $b^\epsilon(t, y) = (f_t^\epsilon(y))^{-1} \int_{\mathbb{R}^d} \phi_\epsilon(x - y) b(t, x) f_t(dx)$. Recall finally that $|a(t, x)| + |b(t, x)| \leq C(1 + |x|^2)$. So concerning a^ϵ and b^ϵ , our goal is only to check that

$$K_\epsilon(t, y) := \int_{\mathbb{R}^d} [|\nabla_y \phi_\epsilon(x - y)| + |D_y^2 \phi_\epsilon(x - y)|] (1 + |x|^2) f_t(dx)$$

is locally bounded on $[0, T] \times \mathbb{R}^d$. But using that $(1 + |z|^2)[|\nabla \phi_\epsilon(z)| + |D^2 \phi_\epsilon(z)|]$ is bounded on \mathbb{R}^d , we deduce that $[|\nabla_y \phi_\epsilon(x - y)| + |D_y^2 \phi_\epsilon(x - y)|] (1 + |x|^2) \leq C_\epsilon(1 + |y|^2)$, whence $K_\epsilon(t, y) \leq C_\epsilon(1 + |y|^2)$.

Next, one has $|\nabla_y F_t^\epsilon(x, y)| \leq C_\epsilon (f_t^\epsilon(y))^{-2} [\phi_\epsilon(x - y) |\nabla f_t^\epsilon(y)| + f_t^\epsilon(y) |\nabla \phi_\epsilon(x - y)|]$. Using again that f_t^ϵ is smooth and positive, the goal concerning G^ϵ is to verify that

$$L_\epsilon(t, y) := \int_{\mathbb{R}^d} \int_E |h(t, z, x)| [\phi_\epsilon(x - y) + |\nabla \phi_\epsilon(x - y)|] \mu(dz) f_t(dx)$$

is locally bounded. By Assumption 3.1.1,

$$L_\epsilon(t, y) \leq \int_{\mathbb{R}^d} [\phi_\epsilon(x - y) + |\nabla \phi_\epsilon(x - y)|] (1 + |x|) f_t(dx) \leq C_\epsilon(1 + |y|)$$

as previously, because $(1 + |z|)[\phi_\epsilon(z) + |\nabla \phi_\epsilon(z)|]$ is bounded. \square

3.3 Study of the regularized equations

In this section, we build a realization of the regularized weak solution $(f_t^\epsilon)_{t \in [0, T]}$.

Proposition 3.3.1. *Let $(f_t)_{t \in [0, T]} \in L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$ be a weak solution to (3.2) and fix $\epsilon \in (0, 1)$. Consider $(f_t^\epsilon)_{t \in [0, T]}$ and $a^\epsilon, b^\epsilon, F^\epsilon$ defined in Lemma 3.2.1 and put $\sigma^\epsilon(t, y) := (a^\epsilon(t, y))^{1/2}$. Consider a random variable X_0^ϵ , a d -dimensional Brownian motion $(B_s)_{s \in [0, T]}$ and a Poisson measure $N(ds, dz, dx, du)$ on $[0, T] \times E \times \mathbb{R}^d \times [0, \infty)$ with intensity measure $ds \mu(dz) f_s(dx) du$, these three objects being independent. We work with the filtration generated by X_0^ϵ, B, N .*

(i) *There is a pathwise unique càdlàg adapted solution $(X_t^\epsilon)_{t \in [0, T]}$ to*

$$\begin{aligned} X_t^\epsilon = & X_0^\epsilon + \int_0^t b^\epsilon(s, X_s^\epsilon) ds + \int_0^t \sigma^\epsilon(s, X_s^\epsilon) dB_s \\ & + \int_0^t \int_E \int_{\mathbb{R}^d} \int_0^\infty h(s, z, x) \mathbf{1}_{\{u \leq F_s^\epsilon(x, X_{s-}^\epsilon)\}} N(ds, dz, dx, du). \end{aligned} \quad (3.4)$$

(ii) *There is a constant C (not depending on ϵ) such that $\mathbb{E}[\sup_{[0, T]} |X_t^\epsilon|] \leq C(1 + \mathbb{E}[|X_0^\epsilon|])$.*

(iii) *If $\mathcal{L}(X_0^\epsilon) = f_0^\epsilon$, then $\mathcal{L}(X_t^\epsilon) = f_t^\epsilon$ for all $t \in [0, T]$.*

Proof. (i) The existence of a pathwise unique solution to (3.4) is more or less standard, because of the linear growth and local Lipschitz properties of the coefficients proved in Lemma 3.2.2. We only prove pathwise uniqueness, the existence being shown similarly, using a localization procedure (to make the coefficients globally Lipschitz continuous) and a Picard iteration. Consider two solutions $(X_t^\epsilon)_{t \in [0, T]}$ and $(\tilde{X}_t^\epsilon)_{t \in [0, T]}$ to (3.4) with $X_0^\epsilon = \tilde{X}_0^\epsilon$ and introduce the stopping time $\tau_R := \inf\{t \in [0, T] : |X_t^\epsilon| \vee |\tilde{X}_t^\epsilon| \geq R\}$, for $R > 0$, with the convention that $\inf \emptyset = T$. Using the Burkholder-Davis-Gundy inequality for the Brownian part, we find

$$\begin{aligned} & \mathbb{E} \left[\sup_{[0, t \wedge \tau_R]} |X_s^\epsilon - \tilde{X}_s^\epsilon| \right] \\ & \leq \mathbb{E} \left[\int_0^{t \wedge \tau_R} |b^\epsilon(s, X_s^\epsilon) - b^\epsilon(s, \tilde{X}_s^\epsilon)| ds + C \left(\int_0^{t \wedge \tau_R} |\sigma^\epsilon(s, X_s^\epsilon) - \sigma^\epsilon(s, \tilde{X}_s^\epsilon)|^2 ds \right)^{1/2} \right. \\ & \quad \left. + \int_0^{t \wedge \tau_R} \int_E \int_{\mathbb{R}^d} |h(s, z, x)| |F_s^\epsilon(x, X_s^\epsilon) - F_s^\epsilon(x, \tilde{X}_s^\epsilon)| f_s(dx) \mu(dz) ds \right]. \end{aligned}$$

By Lemma 3.2.2-(ii), we deduce that

$$\begin{aligned} \mathbb{E} \left[\sup_{[0, t \wedge \tau_R]} |X_s^\epsilon - \tilde{X}_s^\epsilon| \right] & \leq C_{R, \epsilon} \mathbb{E} \left[\int_0^{t \wedge \tau_R} |X_s^\epsilon - \tilde{X}_s^\epsilon| ds + \left(\int_0^{t \wedge \tau_R} |X_s^\epsilon - \tilde{X}_s^\epsilon|^2 ds \right)^{1/2} \right] \\ & \leq C_{R, \epsilon} (t + \sqrt{t}) \mathbb{E} \left[\sup_{[0, t \wedge \tau_R]} |X_s^\epsilon - \tilde{X}_s^\epsilon| \right]. \end{aligned}$$

We deduce that $\mathbb{E}[\sup_{[0, t_R \wedge \tau_R]} |X_s^\epsilon - \tilde{X}_s^\epsilon|] = 0$, where $t_R > 0$ is such that $C_{R, \epsilon}(t_R + \sqrt{t_R}) = 1/2$. But then, the same computation allows us to prove that $\mathbb{E}[\sup_{[t_R \wedge \tau_R, (2t_R) \wedge \tau_R]} |X_s^\epsilon - \tilde{X}_s^\epsilon|] = 0$,

etc, so that we end with $\mathbb{E}[\sup_{[0, T \wedge \tau_R]} |X_s^\epsilon - \tilde{X}_s^\epsilon|] = 0$ for each $R > 0$. Since $\lim_{R \rightarrow \infty} \tau_R = T$ a.s. (because $(X_t^\epsilon)_{t \in [0, T]}$ and $(\tilde{X}_t^\epsilon)_{t \in [0, T]}$ are assumed to be a.s. càdlàg and thus locally bounded on $[0, T]$), we conclude that $\mathbb{E}[\sup_{[0, T]} |X_s^\epsilon - \tilde{X}_s^\epsilon|] = 0$, which was our goal.

(ii) Using the Burkholder-Davis-Gundy inequality for the Brownian part, we find, for $t \in [0, T]$,

$$u_t^\epsilon := \mathbb{E} \left[\sup_{[0, t]} |X_s^\epsilon| \right] \leq \mathbb{E}[|X_0^\epsilon|] + \mathbb{E} \left[\int_0^t |b^\epsilon(s, X_s^\epsilon)| ds \right] + C \mathbb{E} \left[\left(\int_0^t |\sigma^\epsilon(s, X_s^\epsilon)|^2 ds \right)^{1/2} \right] \\ + \mathbb{E} \left[\int_0^t \int_E \int_{\mathbb{R}^d} |h(s, z, x)| F_s^\epsilon(x, X_{s-}^\epsilon) f_s(dx) \mu(dz) ds \right].$$

Inserting the estimates proved in Lemma 3.2.2-(i), we find, for some constant C not depending on $\epsilon \in (0, 1)$ nor on $\mathbb{E}[|X_0^\epsilon|]$,

$$u_t^\epsilon \leq \mathbb{E}[|X_0^\epsilon|] + C \mathbb{E} \left[\int_0^t (1 + |X_s^\epsilon|) ds + \left(\int_0^t (1 + |X_s^\epsilon|^2) ds \right)^{1/2} \right] \leq u_0^\epsilon + C(t + \sqrt{t})(1 + u_t^\epsilon).$$

With $t_0 > 0$ such that $C(t_0 + \sqrt{t_0}) = 1/2$, we conclude that $u_{t_0}^\epsilon \leq 2u_0^\epsilon + 1$. One checks similarly that $u_{2t_0}^\epsilon \leq 2u_{t_0}^\epsilon + 1 \leq 4u_0^\epsilon + 3$. Repeating the argument, we end with $u_T^\epsilon \leq 2^{\lfloor T/t_0 \rfloor + 1} u_0^\epsilon + 2^{\lfloor T/t_0 \rfloor + 1} - 1$.

(iii) We now assume that $\mathcal{L}(X_0^\epsilon) = f_0^\epsilon$ and we set $g_t^\epsilon := \mathcal{L}(X_t^\epsilon)$. A direct application of the Itô formula shows that for all $t \in [0, T]$, recalling the notation of Lemma 3.2.1,

$$\int_{\mathbb{R}^d} \psi(y) g_t^\epsilon(dy) = \int_{\mathbb{R}^d} \psi(y) f_0^\epsilon(dy) + \int_0^t \int_{\mathbb{R}^d} [\mathcal{A}_{s, \epsilon} \psi(y) + \mathcal{B}_{s, \epsilon} \psi(y)] g_s^\epsilon(dy) ds.$$

Recalling Lemma 3.2.1 again, $(f_t^\epsilon)_{t \in [0, T]}$ solves the same equation. The following uniqueness result will thus complete the proof of (iii): for any $\nu_0 \in \mathcal{P}(\mathbb{R}^d)$, there exists at most one measurable family $(\nu_t)_{t \in [0, T]}$ of probability measures such that for all $\psi \in C_c^2(\mathbb{R}^d)$ and all $t \in [0, T]$,

$$\int_{\mathbb{R}^d} \psi(y) \nu_t(dy) = \int_{\mathbb{R}^d} \psi(y) \nu_0(dy) + \int_0^t ds \int_{\mathbb{R}^d} \nu_s(dy) [\mathcal{A}_{s, \epsilon} \psi(y) + \mathcal{B}_{s, \epsilon} \psi(y)]. \quad (3.5)$$

This must be classical (because the coefficients are rather regular), but we found no reference and thus make use of martingale problems. A càdlàg adapted \mathbb{R}^d -valued process $(Y_t)_{t \in [0, T]}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ is said to solve $MP_\epsilon(\nu_0)$ if $\mathcal{L}(Y_0) = \nu_0$ and if

$$\psi(Y_t) - \int_0^t [\mathcal{A}_{s, \epsilon} \psi(Y_s) + \mathcal{B}_{s, \epsilon} \psi(Y_s)] ds$$

is a martingale for all $\psi \in C_c^2(\mathbb{R}^d)$. Due to Horowitz and Karandikar [38, Theorem B1], the following points imply uniqueness for (3.5). Here $C_0(\mathbb{R}^d)$ is the set of continuous functions from \mathbb{R}^d to \mathbb{R} vanishing at infinity.

- (a) $C_c^2(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$ for the uniform convergence topology,
- (b) $(t, y) \mapsto \mathcal{A}_{t,\epsilon}\psi(y) + \mathcal{B}_{t,\epsilon}\psi(y)$ is measurable for all $\psi \in C_c^2(\mathbb{R}^d)$,
- (c) for each $t \in [0, T]$, $\mathcal{A}_{t,\epsilon} + \mathcal{B}_{t,\epsilon}$ satisfies the maximum principle,
- (d) there exists a countable family $(\psi_k)_{k \geq 1} \subset C_c^2(\mathbb{R}^d)$ such that for all $t \in [0, T]$,

$$\overline{\{(\psi_k, \mathcal{A}_{t,\epsilon}\psi_k + \mathcal{B}_{t,\epsilon}\psi_k), k \geq 1\}} \supset \{(\psi, \mathcal{A}_{t,\epsilon}\psi + \mathcal{B}_{t,\epsilon}\psi), \psi \in C_c^2(\mathbb{R}^d)\}$$

where the closure in the left-hand side is under the bounded pointwise convergence,

- (e) for each $y_0 \in \mathbb{R}^d$, there exists a unique (in law) solution to $MP_\epsilon(\delta_{y_0})$.

Points (a) and (b) are obvious. The SDE associated to MP_ϵ is precisely (3.4): $(Y_t)_{t \in [0, T]}$ solves $MP_\epsilon(\nu_0)$ if and only if it is a weak solution to (3.4) and $\mathcal{L}(Y_0) = \nu_0$, see Jacod [39, Theorem 13.55], see also [38, Theorem A1]. Thus (e) follows from (i). For (c), assume that $\psi \in C_c^2(\mathbb{R}^d)$ attains its maximum at y_0 . Then $\mathcal{B}_{t,\epsilon}\psi(y_0) \leq 0$ (this is immediate) and $\mathcal{A}_{t,\epsilon}\psi(y_0) \leq 0$ (because $\nabla\psi(y_0) = 0$ and, since $a(t, y_0)$ is symmetry and nonnegative, $\sum_{i,j} a_{ij}(t, y_0)\partial_{ij}\psi(y_0) \leq 0$). It only remains to prove (d). Consider any countable subset $(\psi_k)_{k \geq 1} \subset C_c^2(\mathbb{R}^d)$ dense in $C_c^2(\mathbb{R}^d)$: for $\psi \in C_c^2(\mathbb{R}^d)$ with $\text{Supp } \psi \subset B(0, M)$, there exists $(\psi_{k_n})_{n \geq 1}$ with $\text{Supp } \psi_{k_n} \subset B(0, 2M)$ such that

$$\lim_{n \rightarrow \infty} (\|\psi - \psi_{k_n}\|_\infty + \|\nabla(\psi - \psi_{k_n})\|_\infty + \|D^2(\psi - \psi_{k_n})\|_\infty) = 0.$$

We will prove more than needed, namely that (i) $\lim_{n \rightarrow \infty} \sup_{[0, T]} \|\mathcal{A}_{t,\epsilon}\psi_{k_n} - \mathcal{A}_{t,\epsilon}\psi\|_\infty = 0$, and (ii) $\lim_{n \rightarrow \infty} \sup_{[0, T]} \|\mathcal{B}_{t,\epsilon}\psi_{k_n} - \mathcal{B}_{t,\epsilon}\psi\|_\infty = 0$.

By Lemma 3.2.2,

$$\begin{aligned} & |\mathcal{A}_{t,\epsilon}(\psi_{k_n} - \psi)(y)| \\ & \leq \|\nabla(\psi_{k_n} - \psi)\|_\infty |b^\epsilon(t, y)| \mathbf{1}_{\{|y| \leq 2M\}} + \frac{1}{2} \|D^2(\psi_{k_n} - \psi)\|_\infty \|a^\epsilon(t, y)\| \mathbf{1}_{\{|y| \leq 2M\}} \\ & \leq C \|\nabla(\psi_{k_n} - \psi)\|_\infty + C \|D^2(\psi_{k_n} - \psi)\|_\infty, \end{aligned}$$

which tends to 0, implying (i). We next write, using that $\text{Supp } (\psi_{k_n} - \psi) \subset B(0, 2M)$,

$$\begin{aligned} |(\psi_{k_n} - \psi)(y + h(t, z, x)) - (\psi_{k_n} - \psi)(y)| & \leq \mathbf{1}_{\{|y| \leq 4M\}} \|\nabla(\psi_{k_n} - \psi)\|_\infty |h(t, z, x)| \\ & \quad + 2\mathbf{1}_{\{|y| \geq 4M\}} \|\psi_{k_n} - \psi\|_\infty \mathbf{1}_{\{|y + h(t, z, x)| \leq 2M\}}. \end{aligned}$$

Observing that

$$\mathbf{1}_{\{|y| \geq 4M, |y + h(t, z, x)| \leq 2M\}} \leq \mathbf{1}_{\{|y| \geq 4M, |h(t, z, x)| \geq |y|/2\}} \leq \mathbf{1}_{\{|y| \geq 4M\}} \frac{2|h(t, z, x)|}{|y|},$$

we deduce that

$$\begin{aligned} |\mathcal{B}_{t,\epsilon}(\psi_{k_n} - \psi)(y)| &\leq \mathbf{1}_{\{|y| \leq 4M\}} \|\nabla(\psi_{k_n} - \psi)\|_\infty \int_E \int_{\mathbb{R}^d} |h(t, z, x)| F_t^\epsilon(x, y) f_t(dx) \mu(dz) \\ &\quad + \mathbf{1}_{\{|y| \geq 4M\}} \|\psi_{k_n} - \psi\|_\infty \int_E \int_{\mathbb{R}^d} \frac{2|h(t, z, x)|}{|y|} F_t^\epsilon(x, y) f_t(dx) \mu(dz). \end{aligned}$$

Recalling that $\int_E \int_{\mathbb{R}^d} |h(t, z, x)| F_t^\epsilon(x, y) f_t(dx) \mu(dz) \leq C(1 + |y|)$ by Lemma 3.2.2, we find

$$\begin{aligned} &|\mathcal{B}_{t,\epsilon}(\psi_{k_n} - \psi)(y)| \\ &\leq \mathbf{1}_{\{|y| \leq 4M\}} C(1 + |y|) \|\nabla(\psi_{k_n} - \psi)\|_\infty + \mathbf{1}_{\{|y| \geq 4M\}} C \frac{1 + |y|}{|y|} \|\psi_{k_n} - \psi\|_\infty \\ &\leq C \|\nabla(\psi_{k_n} - \psi)\|_\infty + C \|\psi_{k_n} - \psi\|_\infty \end{aligned}$$

and the conclusion follows. \square

Lemma 3.3.2. *For $(f_t)_{t \in [0, T]} \in L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$ a weak solution to the equation (3.2) and $\epsilon \in (0, 1)$, consider the process $(X_t^\epsilon)_{t \in [0, T]}$, with $X_0^\epsilon \sim f_0^\epsilon$, introduced in Lemma 3.3.1. The family $\{(X_t^\epsilon)_{t \in [0, T]}, \epsilon > 0\}$ is tight in $\mathbb{D}([0, T], \mathbb{R}^d)$ and any limit point $(X_t)_{t \in [0, T]}$ satisfies $\mathbb{P}(\Delta X_t \neq 0) = 0$ for all $t \in [0, T]$.*

Proof. We use the Aldous criterion [1], see also Jacod and Shiryaev [40, p. 356], which implies tightness and that any limit point $(X_t)_{t \in [0, T]}$ is quasi-left-continuous and thus has no deterministic jump time. It suffices to check that

$$(i) \sup_{\epsilon \in (0, 1)} \mathbb{E}[\sup_{[0, T]} |X_t^\epsilon|] < \infty,$$

(ii) $\lim_{\beta \rightarrow 0} \sup_{\epsilon \in (0, 1)} \sup_{(S, S') \in \mathcal{S}_T(\beta)} \mathbb{E}[|X_{S'}^\epsilon - X_S^\epsilon|] = 0$, where $\mathcal{S}_T(\beta)$ is the set of all pairs of stopping times (S, S') satisfying $0 \leq S \leq S' \leq S + \beta \leq T$ a.s.

Point (i) has already been checked in Lemma 3.3.1-(ii), since $\mathbb{E}[|X_0^\epsilon|] = m_1(f_0^\epsilon) \leq m_1(f_0) + \sqrt{d}\epsilon$. Next, for $S, S' \in \mathcal{S}_T(\beta)$ and $\epsilon \in (0, 1)$, we have

$$\begin{aligned} \mathbb{E}[|X_{S'}^\epsilon - X_S^\epsilon|] &\leq \mathbb{E}\left[\int_S^{S+\beta} |b^\epsilon(s, X_s^\epsilon)| ds\right] + \mathbb{E}\left[\left|\int_S^{S'} \sigma^\epsilon(s, X_s^\epsilon) dB_s\right|\right] \\ &\quad + \mathbb{E}\left[\int_S^{S+\beta} \int_E \int_{\mathbb{R}^d} |h(s, z, x)| F_s^\epsilon(x, X_s^\epsilon) f_s(dx) \mu(dz) ds\right] \\ &\leq C \mathbb{E}\left[\int_S^{S+\beta} (1 + |X_s^\epsilon|) ds\right] + C \mathbb{E}\left[\left(\int_S^{S'} |\sigma^\epsilon(s, X_s^\epsilon)|^2 ds\right)^{1/2}\right], \end{aligned}$$

where the last inequality follows from Lemma 3.2.2-(i) and the Burkholder-Davis-Gundy inequality. But $|\sigma^\epsilon(s, x)|^2 \leq C|a^\epsilon(s, x)| \leq C(1 + |x|^2)$ by Lemma 3.2.2-(i) again, whence

$$\mathbb{E}[|X_{S'}^\epsilon - X_S^\epsilon|] \leq C \mathbb{E}\left[\int_S^{S+\beta} (1 + |X_s^\epsilon|) ds + \left(\int_S^{S+\beta} (1 + |X_s^\epsilon|^2) ds\right)^{1/2}\right].$$

Hence $\mathbb{E}[|X_s^\epsilon - X_s^\xi|] \leq C(\beta + \sqrt{\beta})\mathbb{E}[\sup_{[0,T]}(1 + |X_s^\epsilon|)] \leq C(\beta + \sqrt{\beta})$, which ends the proof. \square

3.4 Conclusion

As Figalli [22], we will need some continuous (in x) approximations of a , b and h .

Lemma 3.4.1. *Let $(f_t)_{t \in [0,T]} \in L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$ be a weak solution to (3.2). For all $\rho > 0$, we can find $\tilde{a} : [0, T] \times \mathbb{R}^d \mapsto \mathcal{S}_d^+$ and $\tilde{b} : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, both continuous and compactly supported, a set $A \in \mathcal{E}$ such that $\mu(A) < \infty$, and a measurable function $\tilde{h} : [0, T] \times E \times \mathbb{R}^d \mapsto \mathbb{R}^d$, continuous on $[0, T] \times \mathbb{R}^d$ for each $z \in E$, such that $\tilde{h}(t, z, x) = 0$ for all $(t, z, x) \in [0, T] \times A^c \times \mathbb{R}^d$ and*

$$\int_0^T \int_{\mathbb{R}^d} \left[\frac{|a(t, x) - \tilde{a}(t, x)|}{1 + |x|} + |b(t, x) - \tilde{b}(t, x)| + \int_E |h(t, z, x) - \tilde{h}(t, z, x)| \mu(dz) \right] f_t(dx) dt < \rho.$$

Proof. For a and b , this follows from the fact, see Rudin [52, Theorem 3.14], that continuous functions with compact support are dense in $L^1([0, T] \times \mathbb{R}^d, dt f_t(dx))$, and that both $a(t, x)/(1 + |x|)$ and $b(t, x)$ belong to this space by Assumption 3.1.1.

Since $h \in L^1([0, T] \times E \times \mathbb{R}^d, dt \mu(dz) f_t(dx))$ by Assumption 3.1.1 and since μ is σ -finite, we can find $A \in \mathcal{E}$ such that $\mu(A) < \infty$ and $\int_0^T \int_{A^c} \int_{\mathbb{R}^d} |h(t, z, x)| f_t(dx) \mu(dz) dt < \rho/3$.

Next, can find a simple function $g = \sum_{n=1}^N \alpha_n \mathbf{1}_{S_n}$, with $\alpha_n \in \mathbb{R}_*$, $S_n \in \mathcal{B}([0, T] \times \mathbb{R}^d) \otimes \mathcal{E}$, such that $\int_0^T \int_A \int_{\mathbb{R}^d} |g(t, z, x) - h(t, z, x)| f_t(dx) \mu(dz) dt < \rho/3$.

But for $S \in \mathcal{B}([0, T] \times \mathbb{R}^d) \otimes \mathcal{E}$ and $\epsilon > 0$, there is $\varphi_{S, \epsilon} : [0, T] \times \mathbb{R}^d \times E \mapsto \mathbb{R}$, measurable, continuous on $[0, T] \times \mathbb{R}^d$ for each $z \in E$ and such that $\int_0^T \int_A \int_{\mathbb{R}^d} |\mathbf{1}_{\{(t, z, x) \in S\}} - \varphi_{S, \epsilon}(t, z, x)| f_t(dx) \mu(dz) dt < \epsilon$. Indeed, when $S = C \times D$ with $C \in \mathcal{B}([0, T] \times \mathbb{R}^d)$ and $D \in \mathcal{E}$, it suffices to consider ψ continuous on $[0, T] \times \mathbb{R}^d$ such that $\int_0^T \int_{\mathbb{R}^d} |\mathbf{1}_{\{(t, x) \in C\}} - \psi(t, x)| f_t(dx) dt < \epsilon/\mu(A)$ and to set $\varphi_{S, \epsilon}(t, z, x) = \psi(t, x) \mathbf{1}_{\{z \in D\}}$. The general case follows from the monotone class theorem.

Finally, $\tilde{h}(t, z, x) = \sum_{n=1}^N \alpha_n \varphi_{S_n, \rho/(3|\alpha_n|2^n)}(t, z, x) \mathbf{1}_{\{z \in A\}}$ is measurable and continuous in (t, x) for each $z \in E$. Writing

$$\begin{aligned} |h(t, z, x) - \tilde{h}(t, z, x)| &\leq |h(t, z, x)| \mathbf{1}_{\{z \in A^c\}} + |g(t, z, x) - h(t, z, x)| \mathbf{1}_{\{z \in A\}} \\ &\quad + \sum_{n=1}^N |\alpha_n| |\varphi_{S_n, \rho/(3|\alpha_n|2^n)}(t, z, x) - \mathbf{1}_{\{(t, z, x) \in S_n\}}| \mathbf{1}_{\{z \in A\}}, \end{aligned}$$

we conclude that $\int_0^T \int_E \int_{\mathbb{R}^d} |h(t, z, x) - \tilde{h}(t, z, x)| f_t(dx) \mu(dz) dt < \rho$ as desired. \square

We now can give the

Proof of Theorem 3.1.4. Let $(f_t)_{t \in [0, T]} \in L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$ be a weak solution to (3.2). For each $\epsilon \in (0, 1)$, consider $(f_t^\epsilon)_{t \in [0, T]}$ introduced in Lemma 3.2.1 and the process $(X_t^\epsilon)_{t \in [0, T]}$ introduced in Lemma 3.3.1-(iii). By Lemma 3.3.2, we can find a sequence $(X_t^{\epsilon_n})_{t \in [0, T]}$ converging in law to some process $(X_t)_{t \in [0, T]}$. Since we know from Lemma 3.3.1 that $\mathcal{L}(X_t^{\epsilon_n}) = f_t^{\epsilon_n}$ for each $t \in [0, T]$, each $n \geq 1$ and since $f_t^{\epsilon_n}$ goes weakly to f_t as $n \rightarrow \infty$ by construction, we deduce that for all $t \in [0, T]$, $\mathcal{L}(X_t) = f_t$. It thus only remains to verify that $X := (X_t)_{t \in [0, T]}$ is a (weak) solution to (3.1). According to the theory of martingale problems, see Jacod [39, Theorem 13.55], it classically suffices to prove that for any $\psi \in C_c^2(\mathbb{R}^d)$, the process

$$\psi(X_t) - \psi(X_0) - \int_0^t [\mathcal{A}_s \psi(X_s) + \mathcal{B}_s \psi(X_s)] ds$$

is a martingale in the filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$. Our goal is thus to check that for any $0 \leq s_1 \leq \dots \leq s_k \leq s \leq t \leq T$, any $\psi_1, \dots, \psi_k \in C_b(\mathbb{R}^d)$ and any $\psi \in C_c^2(\mathbb{R}^d)$, we have $\mathbb{E}[\mathcal{K}(X)] = 0$, where $\mathcal{K} : \mathbb{D}([0, T], \mathbb{R}^d) \mapsto \mathbb{R}$ is defined by

$$\mathcal{K}(\lambda) := \left(\prod_{i=1}^k \psi_i(\lambda_{s_i}) \right) \left(\psi(\lambda_t) - \psi(\lambda_s) - \int_s^t [\mathcal{A}_r \psi(\lambda_r) + \mathcal{B}_r \psi(\lambda_r)] dr \right).$$

We fix $\rho > 0$ and consider \tilde{a} , \tilde{b} and \tilde{h} introduced in Lemma 3.4.1. We introduce $\tilde{\mathcal{A}}_s$ and $\tilde{\mathcal{B}}_s$ exactly as in Definition 3.1.2 with \tilde{a} , \tilde{b} and \tilde{h} instead of a , b and h . We define \tilde{a}^ϵ , \tilde{b}^ϵ , $\tilde{\mathcal{A}}_{s, \epsilon}$ and $\tilde{\mathcal{B}}_{s, \epsilon}$ exactly as in Lemma 3.2.1, with everywhere \tilde{a} , \tilde{b} and \tilde{h} instead of a , b and h . Finally, we define $\tilde{\mathcal{K}}$ (resp. $\tilde{\mathcal{K}}_\epsilon$, resp. \mathcal{K}_ϵ) exactly as \mathcal{K} with \mathcal{A}_r and \mathcal{B}_r replaced by $\tilde{\mathcal{A}}_r$ and $\tilde{\mathcal{B}}_r$ (resp. by $\tilde{\mathcal{A}}_{r, \epsilon}$ and $\tilde{\mathcal{B}}_{r, \epsilon}$, resp. by $\mathcal{A}_{r, \epsilon}$ and $\mathcal{B}_{r, \epsilon}$).

First, $\mathbb{E}[\mathcal{K}_{\epsilon_n}(X^{\epsilon_n})] = 0$. Indeed, since $X^\epsilon = (X_t^\epsilon)_{t \in [0, T]}$ solves (3.4), by the Itô formula,

$$\begin{aligned} & \psi(X_t^\epsilon) - \int_0^t [\mathcal{A}_{r, \epsilon}(X_r^\epsilon) + \mathcal{B}_{r, \epsilon}(X_r^\epsilon)] dr \\ &= \psi(X_t^\epsilon) - \int_0^t b^\epsilon(r, X_r^\epsilon) \cdot \nabla \psi(X_r^\epsilon) dr - \frac{1}{2} \sum_{i, j=1}^d \int_0^t a_{ij}^\epsilon(r, X_r^\epsilon) \partial_{ij} \psi(X_r^\epsilon) dr \\ & \quad - \int_0^t \int_E \int_{\mathbb{R}^d} [\psi(X_r^\epsilon + h(s, z, x)) - \psi(X_r^\epsilon)] F_s^\epsilon(x, X_r^\epsilon) f_r(dx) \mu(dz) dr \end{aligned}$$

is a martingale, which implies the claim. We thus may write, for each $n \geq 1$,

$$\begin{aligned} |\mathbb{E}[\mathcal{K}(X)]| &\leq |\mathbb{E}[\mathcal{K}(X)] - \mathbb{E}[\tilde{\mathcal{K}}(X)]| + |\mathbb{E}[\tilde{\mathcal{K}}(X)] - \mathbb{E}[\tilde{\mathcal{K}}(X^{\epsilon_n})]| \\ & \quad + |\mathbb{E}[\tilde{\mathcal{K}}(X^{\epsilon_n})] - \mathbb{E}[\tilde{\mathcal{K}}_{\epsilon_n}(X^{\epsilon_n})]| + |\mathbb{E}[\tilde{\mathcal{K}}_{\epsilon_n}(X^{\epsilon_n})] - \mathbb{E}[\mathcal{K}_{\epsilon_n}(X^{\epsilon_n})]|. \end{aligned}$$

We now study the four terms. We denote by M a constant such that $\text{Supp } \psi \subset B(0, M)$. We also define $\phi(z) = (2\pi)^{-d/2} e^{-|z|^2/2}$, so that $\phi_\epsilon(z) = \epsilon^{-d/2} \phi(\epsilon^{-1/2}z)$.

Step 1. Here we prove that $\lim_{n \rightarrow \infty} \mathbb{E}[\tilde{\mathcal{K}}(X^{\epsilon_n})] = \mathbb{E}[\tilde{\mathcal{K}}(X)]$. Since X^{ϵ_n} goes in law to X by construction, it suffices to verify that $\tilde{\mathcal{K}}$ is bounded and a.s. continuous at X .

Since \tilde{a} , \tilde{b} and \tilde{h} are continuous in space and time, we easily deduce that $(r, x) \mapsto \tilde{\mathcal{A}}_r \psi(x)$ and $(r, x) \mapsto \tilde{\mathcal{B}}_r \psi(x)$ are continuous and bounded on $[0, T] \times \mathbb{R}^d$. For $\tilde{\mathcal{A}}_r \psi(x) = \tilde{b}(r, x) \cdot \nabla \psi(x) + \frac{1}{2} \sum_{i,j} \tilde{a}_{ij}(r, x) \partial_{ij} \psi(x)$ this is obvious, and for $\tilde{\mathcal{B}}_r \psi(x) = \int_E [\psi(x + \tilde{h}(r, z, x)) - \psi(x)] \mu(dz) = \int_A [\psi(x + \tilde{h}(r, z, x)) - \psi(x)] \mu(dz)$, this follows from the Lebesgue theorem, because ψ is bounded and $\mu(A) < \infty$.

We easily deduce that $\tilde{\mathcal{K}}$ is bounded, and that it is continuous at each $\lambda \in \mathbb{D}([0, T], \mathbb{R}^d)$ which does not jump at s_1, \dots, s_k, s, t . This is a.s. the case of X , see Lemma 3.3.2.

Step 2. Here we check that $\Delta_1 := |\mathbb{E}[\mathcal{K}(X)] - \mathbb{E}[\tilde{\mathcal{K}}(X)]| \leq C\rho$ for some constant C . We have, since $\text{Supp } \psi \subset B(0, M)$,

$$\begin{aligned} |\mathcal{K}(\lambda) - \tilde{\mathcal{K}}(\lambda)| &\leq C \int_0^t [|\mathcal{A}_r \psi(\lambda_r) - \tilde{\mathcal{A}}_r \psi(\lambda_r)| + |\mathcal{B}_r \psi(\lambda_r) - \tilde{\mathcal{B}}_r \psi(\lambda_r)|] dr \\ &\leq C \int_0^t \left(|a(r, \lambda_r) - \tilde{a}(r, \lambda_r)| + |b(r, \lambda_r) - \tilde{b}(r, \lambda_r)| \right) \mathbf{1}_{\{|\lambda_r| < M\}} dr \\ &\quad + C \int_0^t \int_E |h(r, z, \lambda_r) - \tilde{h}(r, z, \lambda_r)| \mu(dz) dr. \end{aligned}$$

Using now that $\mathbf{1}_{\{|x| < M\}} \leq C(1 + |x|)^{-1}$ and that $\mathcal{L}(X_r) = f_r$ for each $r \in [0, T]$, we conclude that

$$\begin{aligned} \Delta_1 &\leq C \int_0^t \int_{\mathbb{R}^d} \left(\frac{|a(r, x) - \tilde{a}(r, x)|}{1 + |x|} + |b(r, x) - \tilde{b}(r, x)| \right) f_r(dx) dr \\ &\quad + C \int_0^t \int_E \int_{\mathbb{R}^d} |h(r, z, x) - \tilde{h}(r, z, x)| f_r(dx) \mu(dz) dr. \end{aligned}$$

This is smaller than $C\rho$ by Lemma 3.4.1.

Step 3. Now we verify that for all $n \geq 1$, $\Delta_2^n = |\mathbb{E}[\tilde{\mathcal{K}}_{\epsilon_n}(X^{\epsilon_n})] - \mathbb{E}[\mathcal{K}_{\epsilon_n}(X^{\epsilon_n})]| \leq C\rho$. As in Step 2,

$$\begin{aligned} \Delta_2^n &\leq C \int_0^t \int_{\mathbb{R}^d} \left(\frac{|a^{\epsilon_n}(r, y) - \tilde{a}^{\epsilon_n}(r, y)|}{1 + |y|} + |b^{\epsilon_n}(r, y) - \tilde{b}^{\epsilon_n}(r, y)| \right) f_r^{\epsilon_n}(y) dy dr \\ &\quad + C \int_0^t \int_E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |h(r, z, x) - \tilde{h}(r, z, x)| \frac{\phi_{\epsilon_n}(x - y)}{f_t^{\epsilon_n}(y)} f_r(dx) f_r^{\epsilon_n}(y) dy \mu(dz) dr. \end{aligned}$$

Recalling (see Lemma 3.2.1) that $a^{\epsilon_n}(r, y)f_r^{\epsilon_n}(y) = \int_{\mathbb{R}^d} \phi_{\epsilon_n}(x - y)a(r, x)f_r(dx)$, and that $\tilde{a}^{\epsilon_n}(r, y)f_r^{\epsilon_n}(y) = \int_{\mathbb{R}^d} \phi_{\epsilon_n}(x - y)\tilde{a}(r, x)f_r(dx)$ and similar formulas for $b^{\epsilon_n}(r, y)f_r^{\epsilon_n}(y)$ and $\tilde{b}^{\epsilon_n}(r, y)f_r^{\epsilon_n}(y)$, we find

$$\begin{aligned} \Delta_2^n &\leq C \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{|a(r, x) - \tilde{a}(r, x)|}{1 + |y|} + |b(r, x) - \tilde{b}(r, x)| \right) \phi_{\epsilon_n}(x - y) f_r(dx) dy dr \\ &\quad + C \int_0^t \int_E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |h(r, z, x) - \tilde{h}(r, z, x)| \phi_{\epsilon_n}(x - y) f_r(dx) dy \mu(dz) dr. \end{aligned}$$

But $\int_{\mathbb{R}^d} \phi_{\epsilon_n}(x - y) dy = 1$ and, since $\frac{1+|x|}{1+|y|} = 1 + \frac{|x|-|y|}{1+|y|} \leq 1 + |x - y| \leq 2 + |x - y|^2$,

$$\int_{\mathbb{R}^d} \frac{(1 + |x|)\phi_{\epsilon_n}(x - y) dy}{1 + |y|} \leq \int_{\mathbb{R}^d} (2 + |x - y|^2)\phi_{\epsilon_n}(x - y) dy = 2 + d\epsilon_n \leq 2 + d.$$

Consequently,

$$\begin{aligned} \Delta_2^n &\leq C \int_0^t \int_{\mathbb{R}^d} \left(\frac{|a(r, x) - \tilde{a}(r, x)|}{1 + |x|} + |b(r, x) - \tilde{b}(r, x)| \right) f_r(dx) dr \\ &\quad + C \int_0^t \int_E \int_{\mathbb{R}^d} |h(r, z, x) - \tilde{h}(r, z, x)| f_r(dx) \mu(dz) dr, \end{aligned}$$

which is smaller than $C\rho$ by Lemma 3.4.1.

Step 4. Finally, we check that $\lim_{n \rightarrow \infty} |\mathbb{E}[\tilde{\mathcal{K}}(X^{\epsilon_n})] - \mathbb{E}[\tilde{\mathcal{K}}_{\epsilon_n}(X^{\epsilon_n})]| = 0$. We first observe that $|\mathbb{E}[\tilde{\mathcal{K}}(X^{\epsilon_n})] - \mathbb{E}[\tilde{\mathcal{K}}_{\epsilon_n}(X^{\epsilon_n})]| \leq C(I_n + J_n)$, where

$$I_n := \mathbb{E} \left[\int_0^t |\tilde{\mathcal{A}}_{r, \epsilon_n} \psi(X_r^{\epsilon_n}) - \tilde{\mathcal{A}}_r \psi(X_r^{\epsilon_n})| dr \right] \text{ and } J_n := \mathbb{E} \left[\int_0^t |\tilde{\mathcal{B}}_{r, \epsilon_n} \psi(X_r^{\epsilon_n}) - \tilde{\mathcal{B}}_r \psi(X_r^{\epsilon_n})| dr \right].$$

Since $\psi \in C_c^2(\mathbb{R}^d)$ and since $\mathcal{L}(X_r^{\epsilon_n}) = f_r^{\epsilon_n}$, we have

$$\begin{aligned} I_n &\leq C \int_0^t \int_{\mathbb{R}^d} (|\tilde{b}^{\epsilon_n}(r, y) - \tilde{b}(r, y)| + |\tilde{a}^{\epsilon_n}(r, y) - \tilde{a}(r, y)|) f_r^{\epsilon_n}(y) dy dr \\ &\leq C \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|\tilde{b}(r, x) - \tilde{b}(r, y)| + |\tilde{a}(r, x) - \tilde{a}(r, y)|) \phi_{\epsilon_n}(x - y) f_r(dx) dy dr. \end{aligned}$$

because $[\tilde{b}^{\epsilon_n}(r, y) - \tilde{b}(r, y)]f_r^{\epsilon_n}(y) = \int_{\mathbb{R}^d} \phi_{\epsilon_n}(x - y)\tilde{b}(r, x)f_r(dx) - \int_{\mathbb{R}^d} \phi_{\epsilon_n}(x - y)\tilde{b}(r, y)f_r(dx)$, with a similar formula concerning \tilde{a} . Using finally the substitution $y = x + \sqrt{\epsilon_n}u$, we find

$$I_n \leq C \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|\tilde{b}(r, x) - \tilde{b}(r, x + \sqrt{\epsilon_n}u)| + |\tilde{a}(r, x) - \tilde{a}(r, x + \sqrt{\epsilon_n}u)|) \phi(u) f_r(dx) dy dr.$$

Hence $\lim_n I_n = 0$ by dominated convergence, since \tilde{a} and \tilde{b} are continuous and bounded.

By the same way, since $f_r^{\epsilon_n}(y) = \int_{\mathbb{R}^d} \phi_{\epsilon_n}(x-y) f_r(dx)$,

$$\begin{aligned} J_n &= \mathbb{E} \left[\int_0^t \left| \int_E \int_{\mathbb{R}^d} \left[\psi(X_r^{\epsilon_n} + \tilde{h}(r, z, x)) - \psi(X_r^{\epsilon_n}) \right] \frac{\phi_{\epsilon_n}(x - X_r^{\epsilon_n})}{f_r^{\epsilon_n}(X_r^{\epsilon_n})} f_r(dx) \mu(dz) \right. \right. \\ &\quad \left. \left. - \int_E \left[\psi(X_r^{\epsilon_n} + \tilde{h}(r, z, X_r^{\epsilon_n})) - \psi(X_r^{\epsilon_n}) \right] \mu(dz) \right| dr \right] \\ &= \mathbb{E} \left[\int_0^t \left| \int_E \int_{\mathbb{R}^d} \left[\psi(X_r^{\epsilon_n} + \tilde{h}(r, z, x)) - \psi(X_r^{\epsilon_n} + \tilde{h}(r, z, X_r^{\epsilon_n})) \right] \right. \right. \\ &\quad \left. \left. \times \frac{\phi_{\epsilon_n}(x - X_r^{\epsilon_n})}{f_r^{\epsilon_n}(X_r^{\epsilon_n})} f_r(dx) \mu(dz) \right| dr \right] \\ &\leq C \mathbb{E} \left[\int_0^t \int_E \int_{\mathbb{R}^d} \left[1 \wedge \left| \tilde{h}(r, z, x) - \tilde{h}(r, z, X_r^{\epsilon_n}) \right| \right] \frac{\phi_{\epsilon_n}(x - X_r^{\epsilon_n})}{f_r^{\epsilon_n}(X_r^{\epsilon_n})} f_r(dx) \mu(dz) dr \right] \end{aligned}$$

because ψ and $\nabla\psi$ are bounded. Using that $\mathcal{L}(X_r^{\epsilon_n}) = f_r^{\epsilon_n}$, the substitution $y = x + \sqrt{\epsilon_n}u$ and the fact that $\tilde{h}(r, z, x) = 0$ if $z \notin A$,

$$\begin{aligned} J_n &\leq C \int_0^t \int_A \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[1 \wedge \left| \tilde{h}(r, z, x) - \tilde{h}(r, z, y) \right| \right] \phi_{\epsilon_n}(x-y) f_r(dx) dy \mu(dz) dr \\ &= C \int_0^t \int_A \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[1 \wedge \left| \tilde{h}(r, z, x) - \tilde{h}(r, z, x + \sqrt{\epsilon_n}u) \right| \right] \phi(u) f_r(dx) dy \mu(dz) dr. \end{aligned}$$

Hence $\lim_n J_n = 0$ by dominated convergence, since h is continuous in x and since $\mu(A) < \infty$.

Conclusion. Gathering Steps 1, 2, 3 and 4, we find that $|\mathbb{E}[\mathcal{K}(X)]| \leq C\rho$. Since ρ can be chosen arbitrarily small, we conclude that $\mathbb{E}[\mathcal{K}(X)] = 0$, which completes the proof. \square

3.5 Appendix

Proof of Remark 3.1.3. First, it is very easy, using only that a and b are locally bounded on $[0, T] \times \mathbb{R}^d$, to show that $\mathcal{A}_t\varphi(x)$ is uniformly bounded as soon as $\varphi \in C_c^2(\mathbb{R}^d)$. The case of $\mathcal{B}_t\varphi$ is more complicated. We consider $\varphi \in C_c^2(\mathbb{R}^d)$ and $M > 0$ such that $\text{Supp } \varphi \subset B(0, M)$ and we write

$$|\mathcal{B}_t\varphi(x)| \leq \mathbf{1}_{\{|x| \leq 2M\}} \|\nabla\varphi\|_\infty \int_E |h(t, z, x)| \mu(dz) + \mathbf{1}_{\{|x| \geq 2M\}} \int_E |\varphi(x + h(t, z, x))| \mu(dz).$$

We observe that $|\varphi(x + h(t, z, x))| \leq \|\varphi\|_\infty \mathbf{1}_{\{|x+h(t,z,x)| \leq M\}}$ and that

$$\mathbf{1}_{\{|x| \geq 2M, |x+h(t,z,x)| \leq M\}} \leq \mathbf{1}_{\{|x| \geq 2M, |h(t,z,x)| \geq |x|/2\}} \leq \mathbf{1}_{\{|x| \geq 2M\}} \frac{2|h(t, z, x)|}{|x|}.$$

Since $\int_E |h(t, z, x)| \mu(dz) \leq C(1 + |x|)$ by assumption, we conclude that

$$|\mathcal{B}_t \varphi(x)| \leq \mathbf{1}_{\{|x| \leq 2M\}} C \|\nabla \varphi\|_\infty (1 + |x|) + \mathbf{1}_{\{|x| \geq 2M\}} \frac{C \|\varphi\|_\infty (1 + |x|)}{|x|},$$

which is bounded. We have proved point (i).

We next prove (ii). We put $\varphi(x) = (1 + |x|^2)^{1/2}$, which satisfies

$$\frac{1 + |x|}{2} \leq \varphi(x) \leq 1 + |x|, \quad |\nabla \varphi| \leq 1 \quad \text{and} \quad |D^2 \varphi| \leq \frac{C}{\varphi}.$$

We also introduce an increasing C^2 function $\chi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\chi(r) = r$ for $r \in [0, 1]$ and $\chi(r) = 2$ for $r \geq 2$. We thus have

$$r \wedge 1 \leq \chi(r) \leq 2(r \wedge 1), \quad |\chi'(r)| \leq C \mathbf{1}_{\{r \leq 2\}} \quad \text{and} \quad |\chi''(r)| \leq C \mathbf{1}_{\{1 \leq r \leq 2\}}.$$

We then set, for $n \geq 1$ and $x \in \mathbb{R}^d$, $\psi_n(x) = n\chi(\varphi(x)/n)$, which satisfies

$$\varphi \wedge n \leq \psi_n \leq 2(\varphi \wedge n), \quad |\nabla \psi_n| \leq C \mathbf{1}_{\{\varphi \leq 2n\}} \quad \text{and} \quad |D^2 \psi_n| \leq \frac{C}{\varphi} \mathbf{1}_{\{\varphi \leq 2n\}}.$$

Consequently, for all $s \in [0, T]$, since $|b(s, \cdot)| \leq C\varphi$ and $|a(s, \cdot)| \leq C\varphi^2$ by Assumption 3.1.1,

$$|\mathcal{A}_s \psi_n| \leq |b(s, \cdot)| |\nabla \psi_n| + |a(s, \cdot)| |D^2 \psi_n| \leq C\varphi \mathbf{1}_{\{\varphi \leq 2n\}} \leq C[\varphi \wedge (2n)] \leq C\psi_n.$$

We next claim that

$$\Delta_n(s, z, x) = |\psi_n(x + h(s, z, x)) - \psi_n(x)| \leq C|h(s, z, x)| \frac{\psi_n(x)}{\varphi(x)}. \quad (3.6)$$

First, if $\varphi(x) \leq 4n$, then we only use that $\nabla \psi_n$ is uniformly bounded to write $\Delta_n(s, z, x) \leq C|h(s, z, x)|$, whence the result because $\psi_n(x) \geq \varphi(x) \wedge n \geq \varphi(x)/4$. Second, if $\varphi(x) \geq 4n$ (whence $|x| \geq 4n - 1 \geq 3n$), since ψ_n is constant (with value $2n$) on $B(0, 2n)^c$ and bounded on \mathbb{R}^d by $2n$, we can write $\Delta_n(s, z, x) \leq 4n \mathbf{1}_{\{|x+h(s,z,x)| \leq 2n\}} \leq 4n \mathbf{1}_{\{|h(s,z,x)| \geq |x|/3\}} \leq 12n|h(s, z, x)|/|x|$. But $12n = 6\psi_n(x)$ and $|x| \geq \varphi(x) - 1 \geq \varphi(x)/2$, whence the result.

We deduce from (3.6), using Assumption 3.1.1, that

$$|\mathcal{B}_s \psi_n(x)| \leq C \frac{\psi_n(x)}{\varphi(x)} \int_E |h(s, z, x)| \mu(dz) \leq C \frac{\psi_n(x)}{\varphi(x)} (1 + |x|) \leq C\psi_n(x).$$

Applying (3.3) with the test function $\psi_n - 2n \in C_c^2(\mathbb{R}^d)$, for which of course $(\mathcal{A}_s + \mathcal{B}_s)(\psi_n - 2n) = (\mathcal{A}_s + \mathcal{B}_s)\psi_n$, and using that f_0 and f_t are probability measures, we find

$$\begin{aligned} \int_{\mathbb{R}^d} \psi_n(x) f_t(dx) &= \int_{\mathbb{R}^d} \psi_n(x) f_0(dx) + \int_0^t \int_{\mathbb{R}^d} (\mathcal{A}_s \psi_n(x) + \mathcal{B}_s \psi_n(x)) f_s(dx) ds \\ &\leq \int_{\mathbb{R}^d} \psi_n(x) f_0(dx) + C \int_0^t \int_{\mathbb{R}^d} \psi_n(x) f_s(dx) ds. \end{aligned}$$

Since $f_0 \in \mathcal{P}_1(\mathbb{R}^d)$ by assumption and since $0 \leq \psi_n(x) \leq 2|x| + 2$, $\sup_{n \geq 1} \int_{\mathbb{R}^d} \psi_n(x) f_0(dx) < \infty$. We thus conclude, by the Gronwall Lemma, that $\sup_{n \geq 1} \sup_{t \in [0, T]} \int_{\mathbb{R}^d} \psi_n(x) f_t(dx) < \infty$, which clearly implies that $(f_t)_{t \in [0, T]} \in L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$, because $\lim_{n \rightarrow 0} \psi_n(x) = \varphi(x) \geq |x|$.

For point (iii), we introduce a family of functions $\chi_n \in C_c^2(\mathbb{R}^d)$, for $n \geq 1$, such that $\mathbf{1}_{\{|x| \leq n\}} \leq \chi_n(x) \leq \mathbf{1}_{\{|x| \leq n+1\}}$ and such that $|D\chi_n(x)| + |D^2\chi_n(x)| \leq C\mathbf{1}_{\{|x| \in [n, n+1]\}}$. We then consider $\varphi \in C^2(\mathbb{R}^d)$ as in the statement, i.e. such that $(1 + |x|)[|\varphi(x)| + |\nabla\varphi(x)| + |D^2\varphi(x)|]$ is bounded. Of course, $\varphi\chi_n \in C_c^2(\mathbb{R}^d)$ for each $n \geq 1$, so that we can apply (3.3). We then let $n \rightarrow \infty$. Since φ is bounded, we obviously have $\lim_n \int_{\mathbb{R}^d} \varphi(x)\chi_n(x)f_t(dx) = \int_{\mathbb{R}^d} \varphi(x)f_t(dx)$. Next, we want to prove that $\lim_n \int_0^t \int_{\mathbb{R}^d} [\mathcal{A}_s(\varphi\chi_n)(x) + \mathcal{B}_s(\varphi\chi_n)(x)] f_s(dx) ds = \int_0^t \int_{\mathbb{R}^d} [\mathcal{A}_s\varphi(x) + \mathcal{B}_s\varphi(x)] f_s(dx) ds$. By dominated convergence and since $(f_t)_{t \in [0, T]} \in L^\infty([0, T], \mathcal{P}_1(\mathbb{R}^d))$ by (ii), it suffices to prove that for all $s \in [0, T]$, $x \in \mathbb{R}^d$,

- (a) $\sup_n |\mathcal{A}_s(\varphi\chi_n)(x)| \leq C(1 + |x|)$, (b) $\lim_n \mathcal{A}_s(\varphi\chi_n)(x) = \mathcal{A}_s\varphi(x)$,
(c) $\sup_n |\mathcal{B}_s(\varphi\chi_n)(x)| \leq C(1 + |x|)$, (d) $\lim_n \mathcal{B}_s(\varphi\chi_n)(x) = \mathcal{B}_s\varphi(x)$.

Point (a) is easy: since $|a(s, x)| + |b(s, x)| \leq C(1 + |x|^2)$ by Assumption 3.1.1 and since $\chi_n, D\chi_n, D^2\chi_n$ are uniformly bounded,

$$\begin{aligned} |\mathcal{A}_s(\varphi\chi_n)(x)| &\leq C(1 + |x|^2)(|D(\varphi\chi_n)(x)| + |D^2(\varphi\chi_n)(x)|) \\ &\leq C(1 + |x|^2)(|\varphi(x)| + |D\varphi(x)| + |D^2\varphi(x)|), \end{aligned}$$

which is bounded by $C(1 + |x|)$ by assumption. Point (b) is not hard, using that

$$\lim_n \nabla(\varphi\chi_n)(x) = \nabla\varphi(x) \quad \text{and} \quad \lim_n \partial_{ij}(\varphi\chi_n)(x) = \partial_{ij}\varphi(x)$$

for each $x \in \mathbb{R}^d$.

Next, $\nabla(\varphi\chi_n)$ is uniformly bounded, so that

$$|(\varphi\chi_n)(x + h(s, z, x)) - (\varphi\chi_n)(x)| \leq C|h(s, z, x)|$$

and thus $|\mathcal{B}_s(\varphi\chi_n)(x)| \leq C \int_E |h(s, z, x)| \mu(dz) \leq C(1 + |x|)$ by Assumption 3.1.1, whence (c). Also, by dominated convergence, since $\lim_n \chi_n(y) = 1$ for all $y \in \mathbb{R}^d$,

$$\begin{aligned} \lim_n \mathcal{B}_s(\varphi\chi_n)(x) &= \lim_n \int_E [(\varphi\chi_n)(x + h(s, z, x)) - (\varphi\chi_n)(x)] \mu(dz) \\ &= \int_E [\varphi(x + h(s, z, x)) - \varphi(x)] \mu(dz), \end{aligned}$$

which is nothing but $\mathcal{B}_s\varphi(x)$ as desired. \square

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