Pluripotential theory and equidistribution problems

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Théorie du pluripotentiel et problèmes d'équidistribution

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Résumé

Cette thèse porte sur la théorie du pluripotentiel et des problèmes d'équidistribution. Elle consiste en 4 chapitres.

Le premier chapitre se consarce à l'étude de la régularité de la solution de l'équation de Monge-Ampère complexe sur une variété kählérienne compacte X. Plus précisement, à l'aide des outils de la géométrie Cauchy-Riemann, on montre que la dernière équation possède une (unique) solution höldérienne pour une large classe géométrique de mesures de probabilités supportées par des sous-variétés réelles de X. Dans le chapitre 2, on étudie l'intersection des courants positifs fermés de grand bidegré. On y prouve que le produit extérieur de deux courants positifs fermés dont l'un possède un superpotentiel continu est positif fermé. Ceci généralise un résultat classique pour les courants de bidegré $(1,1)$.

Les deux chapitres suivants sont des applications de la théorie du pluripotentiel à des problèmes d'équidistribution. Dans le chapitre 3, on donne une vitesse explicite de convergence pour l'équidistribution des points de Fekete dans un compact K de $\mathbb{R}^n$ (ou de la sphère $\mathbb{S}^n$) à bord lisse par morceaux vers la mesure d'équilibre de K. Ici, les points de Fekete sont des bons points dans le problème d'interpolation d'une fonction continue sur K par des polynômes. Un tel contrôle de vitesse est crucial en pratique qu'on utilise les points de Fekete. La thèse se termine par le chapitre 4 où on prouve un analogue de la loi de Weyl pour les résonances d'un opérateur de Schrödinger générique sur $\mathbb{R}^n$ avec n impair. Les résonances sont des objets centraux dans l'étude des opérateurs de Schrödinger. Elles jouent un rôle similaire à celui des valeurs propres dans le cadre compact.

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Introduction

Ma thèse porte sur la théorie du pluripotentiel et des problèmes d'équidistribution. Étant fondé par Lelong et Oka dans des années 1940, voir, e.g., [49], la théorie du pluripotentiel est devenue une branche très active de Mathématiques ayant des influences importantes dans d'autres domaines comme Géométrie Complexe et Dynamique Complexe. Les objets centraux de la théorie sont les fonctions plurisousharmoniques et les courants positifs fermés. Rappelons qu'une fonction plurisousharmonique (psh) dans un ouvert de $\mathbb{C}^n$ est une fonction fortement semi-continue supérieurement f telle que f est locallement intégrable et $dd^c f := \frac{i}{\pi}\partial\bar{\partial} f \geq 0$ au sens des courants.

Cette thèse se compose de 4 chapitres qui correspondent à mes 4 articles dans l'ordre [84, 83, 82, 35]. Dans ce qui suit, j'y présenterai les résultats principaux et expliquerai leurs motivations.

Le premier problème auquel nous nous intéressons concerne l'intersection des courants positifs fermés. Soit X une variété compacte complexe de dimension n et T_1, T_2 deux courants positifs fermés sur X. Contrairement au cas des formes différentielles, en général l'intersection (le produit extérieur) $T_1 \wedge T_2$ n'est pas toujours bien définie. Une question centrale dans la théorie du pluripotentiel est de déterminer, dans quelle circonstance, one peut étendre le dernier produit à des courants. On remarque que si T_1, T_2 sont respectivement les courants d'intégration des sous-variétés complexes compactes V_1, V_2 de X, alors $T_1 \wedge T_2$ (au cas où on peut le définir) devrait être un candidat naturel pour le courant défini par l'intersection $V_1 \cap V_2$.

Dans le cas de bidegré $(1,1)$, le problème est bien compris. Supposons que T_1 soit un courant de bidegré $(1,1)$ dont les potentiels locaux sont bornés. Autrement dit, $T_1 = dd^c u + \eta$, où η est une forme différentielle fermée lisse sur X et u est une fonction quasi-psh bornée sur X (une fonction est dite quasi-psh si elle est localement la somme d'une fonction psh et une fonction lisse). Alors, d'après [2, 10], on définit

$$dd^c u \wedge T_2 := dd^c(uT_2), \quad T_1 \wedge T_2 := dd^c u \wedge T_2 + \eta \wedge T_2. \tag{0.0.1}$$

Nous renvoyons à [17, 37] pour un traitement détaillé. Grâce à (0.0.1), on peut définir l'auto-intersection $T_1^n := T_1 \wedge \cdots \wedge T_1$ (n termes) qui est un courant positif fermé de bidegré maximal sur X. Il en résulte que T_1^n est une mesure positive dont la masse égale $\int_X \eta^n$ pour une raison cohomologique. Maitenant supposons que X admette une forme de Kähler ω qui est normalisée telle que $\int_X \omega^n = 1$. Comme ci-dessus, pour une fonction quasi-psh u bornée avec $dd^c u + \omega \geq 0$, l'intersection $(dd^c u + \omega)^n$ est une mesure de probabilité bien définie et applelée celle de Monge-Ampère.

Soit $\mathcal{P}(X)$ l'espace des mesures de probabilité de X. On veut comprendre la classe $\mathcal{M}$ suivante de mesures de Monge-Ampère :

$$\mathcal{M} := \{\mu \in \mathcal{P}(X) : \mu = (dd^c u + \omega)^n \text{ pour } u \text{ quasi-psh höldérienne et } dd^c u \geq -\omega\}.$$

Dans la théorie du pluripotentiel et ses applications, les fonctions (et plus généralement les courants) ont souvent une faible régularité, e.g., elles sont höldériennes. C'est la raison pour laquelle la dernière classe de mesures est très utile. Déterminer si une mesure appartient à $\mathcal{M}$ est lié à la résolution de *l'équation de Monge-Ampère*

$$(dd^c u + \omega)^n = \mu. \tag{0.0.2}$$

La dernière équation joue un rôle crucial dans la géométrie complexe et la théorie du pluripotentiel. Yau [85] a prouvé que si μ est une forme volume lisse de X, (0.0.2) admet une solution lisse, donc une telle μ est un élément de $\mathcal{M}$. Kołodziej [46, 45] a introduit des outils de la théorie du pluripotentiel pour étudier (0.0.2). Depuis, de nombreux auteurs [47, 20, 22, 63, 24] contributent à la compréhension de la classe $\mathcal{M}$. Une conséquence du résultat principal du chapitre 1 est le théorème suivant.

Théorème A. *Soit K une sous-variété $\mathcal{C}^3$ réelle immergée générique de X. Toutes les mesures de probabilité à support compact dans K avec densités L^p pour un certain $p > 1$ appartiennent à $\mathcal{M}$.*

Ici, une sous-variété réelle K de X est dite *(Cauchy-Riemann) générique* si l'espace tangent d'un point arbitraire de K n'est pas inclus dans un hyperplan complexe de l'espace tangent de X en ce point.

L'intersection $T_1 \wedge T_2$ est moins bien comprise lorsque les bidegrés de T_1, T_2 sont supérieurs à $(1,1)$. Un analogue de (0.0.1) est la théorie des *super-potentiels* due à Dinh et Sibony [30, 32], voir aussi [34] pour un autre avancement. Ils ont introduit la notion de super-potentiel d'un courant positif fermé généralisant le potentiel dans le cas de bidegré $(1,1)$. Lorsque T_1 possède un super-potentiel continu, alors ils ont défini $T_1 \wedge T_2$ qui est un courant fermé. Cependant, il n'est pas clair de leur définition que l'intersection $T_1 \wedge T_2$ soit encore positif. En établissant une autre définition équivalente, dans le chapitre 2, on prouve le résultat suivant.

Théorème B. *Supposons que T_1 possède un super-potentiel continu. Alors, $T_1 \wedge T_2$ au sens de Dinh-Sibony est un courant positif fermé.*

On s'intéresse maitenant à des problèmes d'équidistribution qui seront étudiés à l'aide de la théorie du pluripotentiel. En général, une propriété d'équidistribution pour une famille de certains objets dépendant d'un paramètre réel signifie que la famille est assymptotiquement équidistribuée quand le paramètre tend vers l'infini. Dans le chapitre 3, on considère la distribution des *points de Fekete*. Soit K un sous-ensemble de la sphère $\mathbb{S}^n$ de $\mathbb{R}^{n+1}$. Les points de Fekete sont des bons choix pour le problème d'interpolation des fonctions continues sur K par des polynômes à $(n+1)$ variables, voir [52]. Précisément, on les définit comme suit.

Pour $m \in \mathbb{N}$, soit $\mathcal{P}_m$ l'espace vectoriel des restrictions des polynômes de $(n+1)$ variables de degré au plus m à K. Désignons par N_m la dimension de $\mathcal{P}_m$. Fixons une base $\{p_j\}_{1 \leq j \leq N_m}$

de $\mathcal{P}_m$. Soit $(x_j)_{1\leq j\leq N_m} \subset K$ avec $\det[p_j(x_l)]_{1\leq j,l\leq N_m} \neq 0$. Alors, pour une fonction continue quelconque f définie sur K, il existe un unique polynôme $p_f \in \mathcal{P}_m$ tel que $p_f(x_j) = f(x_j)$ pour $1 \leq j \leq N_m$. Il est prouvé que si $\{x_j\}$ maximise la valeur absolue de $\det[p_j(x_l)]$ sur K^{N_m}, alors p_f est bien proche de f dans le sens que

$$\|p_f - f\| \leq \big(1 + N_m\big) \inf_{p\in\mathcal{P}_m} \|p - f\|,$$

où $\|\cdot\|$ est la norme sup sur K, voir [8]. Une famille $\{x_j\}_{1\leq j\leq N_m}$ vérifiant la dernière condition est appelée un *point de Fekete d'ordre* m. Comme il est impossible de calculer explicitement les points de Fekete pour $n \geq 1$ ou $m > 1$, on essaie de comprendre leur comportement asymptotique lorsque leur ordre tend vers l'infini. Posons

$$\mu_m := \frac{1}{N_m} \sum_{j=1}^{N_m} \delta_{x_j},$$

où δ_x est la masse de Dirac en x. Une conséquence directe d'un résultat de Berman, Boucksom and Witt Nyström dans [4, 23] dit que μ_m converge faiblement vers la mesure d'équilibre de K. De façon géométrique, on dit que les points de Fekete sont asymptotiquement équidistribués par rapport à la mesure d'équilibre. Un taux explicite de la convergence a été obtenu par Dinh, Nguyên et Ma dans [23] sous des hypothèses abstraites sur le compact K et son poid, voir [51] pour un autre résultat. Dans le chapitre 3, on montre le résultat suivant.

Théorème C. *Soit K l'adhérence d'un ouvert de $\mathbb{S}^n$ dont le bord est lisse non-dégénéré par morceaux. Pour tout $\epsilon \in (0, 1)$, il existe une constante c_ϵ indépendent de $k \in \mathbb{N}$ telle que*

$$\mathrm{dist}_1(\mu_k, \mu_{eq}) \leq c_\epsilon k^{-1/72+\epsilon}, \tag{0.0.3}$$

où dist_1 est une distance équivalente à celle de Kantorovich-Wasserstein entre les mesures positives sur X, voir Section 3.1 pour la définition de dist_1.

Ce dernier théorème est en fait un cas spécial d'un résultat plus général dans le contexte d'une variété complexe X munie d'un fibré en droite positif. De façon similaire, on peut définir les points de Fekete pour un sous-ensemble de $\mathbb{R}^n$ et obtenir un analogue du (0.0.3) dans ce cas.

Dans le chapitre 4, on considère un autre problème d'équidistribution. Soit (M, g) une variété compacte (réelle) de dimension n. Soit V une fonction lisse et Δ l'opérateur de Laplace-Beltrami sur M. L'opérateur $P := -\Delta + V$ agissant sur $\mathcal{C}^2(M)$ est appelé un opérateur de Schrödinger. On peut considérer P en tant qu'un auto-opérateur non-borné de $L^2(M)$, où le dernier espace est celui de fonctions de carré intégrable sur M. C'est un fait fondamental que P est diagonalisable, c'est-à-dire il existe une base orthogonale $\{\varphi_j\}_{j\in\mathbb{N}}$ de $L^2(M)$ telle que pout tout $j \in \mathbb{N}$, φ_j est lisse et

$$P\varphi_j = \lambda_j\varphi_j,$$

pour certains nombres réels λ_j tels que $\lambda_j \to +\infty$ quand $j \to \infty$. Étant donné un réel r, on note par $N_V(r)$ le nombre de valeurs propres λ_j avec $|\lambda_j| \leq r^2$. La loi de Weyl classique nous dit que $N_V(r) = c_V r^n + o(r^n)$ lorsque $r \to \infty$, où c_V est une constante indépendente de r, voir [91]. En comparant à des estimés du type (0.0.3), la dernière égalité donne moins

information mais elle nous fournit une perspective importante sur la distribution des valeurs propres dans $\mathbb{R}$.

Dans le cadre non-compact, P n'est plus diagonalisable, ici le potentiel V est une fonction bornée réelle ou complexe à support compact. On se restreint au cas où M *est l'espace euclidien de dimension impaire* n (le cas de dimension paire est de caractère différent). Alors P a au plus un nombre fini de valeurs propres. Par conséquent, ces valeurs propres ne sont pas suffisantes pour décrire P. Dans ce cas, *les résonances,* qui sont les pôles d'une fonction méromorphe sur $\mathbb{C}$ à valeurs dans un espace d'opérateurs bornés agissant sur un certain espace de fonctions sur M, peuvent être vues comme des remplaçants des valeurs propres, voir [92]. Soulignons ici que pour une variété compacte, les résonances sont exactement les valeurs propres. Comme dans le cadre compact, on s'intéresse à la distribution des résonances dans $\mathbb{C}$. Une conséquence du résultat principal du chapitre 4 est la loi de Weyl suivante pour des résonances.

Théorème D. *Suposons que V soit à support compact dans la boule de rayon $a > 0$ de $\mathbb{R}^n$ et générique dans un sens de la théorie du pluripotentiel sur cette boule. Alors, le nombre $n_V(r)$ de résonances de $-\Delta + V$ de module $\leq r$ vérifie*

$$n_V(r) = c_d a^d r^d + O(r^{d-\frac{3}{16}+\epsilon}) \quad \textit{lorsque } r \to \infty$$

pour toute constante $\epsilon > 0$, où c_d est une constante dimensionnelle.

Nous allons donner dans les chapitres suivants les détails sur les résultats décrits ci-dessus. Le lecteur trouvera une brève description des preuves dans l'introduction de chaque chapitre.

Chapter 1

Équation de Monge-Ampère avec solutions höldériennes

Let (X, ω) be a compact n-dimensional Kähler manifold on which the integral of ω^n is 1. Let K be an immersed real $\mathcal{C}^3$ submanifold of X such that the tangent space at any point of K is not contained in any complex hyperplane of the (real) tangent space at that point of X. Let μ be a probability measure compactly supported on K with L^p density for some $p > 1$. We prove that the complex Monge-Ampère equation $(dd^c\varphi + \omega)^n = \mu$ has a Hölder continuous solution. This chapter is based on the article [84].

1.1 Introduction

Let X be a compact Kähler manifold of dimension n and let ω be a fixed Kähler form on X so normalized that $\int_X \omega^n = 1$. The aim of this paper is to give a useful explicit class of measures for which the complex Monge-Ampère equation has a Hölder continuous solution. Recall that a real $\mathcal{C}^1$ manifold K is said to be *immersed* in X if there is an injective $\mathcal{C}^1$ immersion from K to X. In this case we say that K is an immersed $\mathcal{C}^1$ submanifold of X. An immersed *real* $\mathcal{C}^1$ submanifold K of X is said to be *generic CR* (or *generic* for simplicity) in the sense of the Cauchy-Riemann geometry if the tangent space at any point of K is not contained in a complex hyperplane of the tangent space at that point of X. Such a submanifold has the real dimension at least n. A function $\varphi : X \to [-\infty, \infty)$ is *quasi-p.s.h.* if it is locally the sum of a p.s.h. function and a smooth one. A quasi-p.s.h. function is said to be *ω-p.s.h.* if we have $dd^c\varphi + \omega \geq 0$ in the sense of currents. The following is our main result.

Theorem 1.1.1. *Let K be a generic immersed $\mathcal{C}^3$ submanifold of X of real codimension $d > 0$. Let μ be a probability measure compactly supported on K with L^p density for some $p > 1$. Then the Monge-Ampère equation $(dd^c\varphi + \omega)^n = \mu$ has an ω-p.s.h. solution φ which is Hölder continuous with Hölder exponent α, for any positive number $\alpha < \frac{2(p-1)}{3d(n+1)p}$.*

Note that our proof still holds if K is $\mathcal{C}^{2,\beta}$ for some $\beta \in (0, 1)$. In this case one just needs to replace the $\mathcal{C}^{2,1/2}$ regularity in Section 1.3 by $\mathcal{C}^{2,\beta'}$ one for $\beta' \in (0, \beta)$. For simplicity, we only consider the $\mathcal{C}^3$ regularity as in Theorem 1.1.1. Secondly, if the Monge-Ampère equation has

a Hölder continuous solution, then that solution is unique up to an additive constant. This is a direct consequence of results in [45, 21].

For a probability measure μ on X, the associated complex Monge-Ampère equation

$$(dd^c\varphi + \omega)^n = \mu \tag{1.1.1}$$

has been extensively studied since the fundamental paper [85] of Yau in which he proved that (1.1.1) has a unique smooth solution if μ is a (smooth) Riemannian volume form vol_X of X. Later Kołodziej showed that the Monge-Ampère equation admits a unique continuous solution for a larger class of measures μ which contains $\mu = fvol_X$ with $f \in L^p(X)$ for $p > 1$, see [46, 45]. For the last measures, he also obtained Hölder regularity of the solution in [47]. The Hölder exponent of that solution is then made precise by Demailly, Dinew, Guedj, Hiep, Kolodziej and Zeriahi in [20] using the regularization method in [19] and the stability theorem in [22]. Moreover, in [63] Hiep obtains the Hölder regularity for $\mu = fvol_Y$, where vol_Y is the volume form of a compact real hypersurface Y of X and $f \in L^p(Y)$ for $p > 1$.

Recently, Dinh and Nguyên in [24] show that the class of probability measures μ, for which (1.1.1) admits a Hölder continuous solution, is exactly the class of probability measures whose super-potentials are Hölder continuous, see Definition 1.1.3 below. They then recover the aforementioned results in [47, 63, 20]. By [24], we know that if a probability measure μ having a Hölder continuous super-potential of order $\beta \in (0, 1]$, then the solution of (1.1.1) is Hölder continuous of order β' for any $0 < \beta' < 2\beta/(n + 1)$. For more information on the complex Monge-Ampère equation, the readers may consult the survey [64].

Theorem 1.1.1 above combined with [24, Pro. 4.4] yields the following nice exponential estimate, see also [74, 26, 43].

Corollary 1.1.2. *Let K be a generic immersed $\mathcal{C}^3$ submanifold of X. Let $\tilde{K}$ be a compact subset of K. Then the restriction of the Lebesgue measure on K to $\tilde{K}$ is moderate, that is, there exist two positive constants α and c such that for any ω-p.s.h. function φ on X with $\sup_X \varphi = 0$ we have*

$$\int_{\tilde{K}} e^{-\alpha\varphi} d\, vol_K \leq c.$$

Before presenting the idea of the proof of Theorem 1.1.1, we need to recall some definitions. Let μ be a probability measure on X. Let $\mathscr{C}$ be the set of ω-p.s.h. functions φ on X such that $\int_X \varphi\, \omega^n = 0$. We define the distance dist_{L^1} on $\mathscr{C}$ by putting

$$\mathrm{dist}_{L^1}(\varphi_1, \varphi_2) := \int_X |\varphi_1 - \varphi_2|\, \omega^n,$$

for every $\varphi_1, \varphi_2 \in \mathscr{C}$.

Definition 1.1.3. *The super-potential of μ (of mean 0) is the function $\mathscr{U} : \mathscr{C} \to \mathbb{R}$ given by $\mathscr{U}(\varphi) := \int_X \varphi d\mu$. We say that $\mathscr{U}$ is Hölder continuous with Hölder exponent $\alpha \in (0, 1]$ if it is so with respect to the distance dist_{L^1}.*

By [24, The. 1.3, Cor. 4.5], Theorem 1.1.1 is a direct consequence of the following result.

Theorem 1.1.4. *Let K be a generic immersed $\mathcal{C}^3$ submanifold of X of real codimension $d > 0$. Let $\tilde{K}$ be a compact subset of K and $\mathbf{1}_{\tilde{K}}$ the characteristic function of $\tilde{K}$. Let vol_K be an arbitrary $\mathcal{C}^3$ Riemannian volume form of K. Then the super-potential of $\mathbf{1}_{\tilde{K}} vol_K$ is Hölder continuous with Hölder exponent α for any positive number $\alpha < 1/(3d)$.*

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. A $\mathcal{C}^1$ *analytic disc* in X is a $\mathcal{C}^1$ map from $\overline{\mathbb{D}}$ to X which is holomorphic on $\mathbb{D}$. For a nonempty arc $I \subset \partial\mathbb{D}$, an analytic disc f is said to be *I-attached to a subset K* of X if $f(I)$ belongs to K. When we do not want to mention I, we simply say an analytic disc partly attached to K. Throughout this paper, for every parameter τ, we will systematically use the notation $\lesssim_\tau$ or $\lesssim$ which means $\leq$ up to a constant depending only on (τ, X, K, ω) or on (X, K, ω) respectively. A similar convention is applied to $\gtrsim_\tau$ and $\gtrsim$.

The idea of the proof of Theorem 1.1.4 is as follows. Observe that the codimension d of K is at most equal to n. We consider below the case where $d = n$. The other cases can be deduced from it. Let $\varphi_1, \varphi_2 \in \mathscr{C}$ and $\varphi := \varphi_1 - \varphi_2$. To show the Hölder regularity of the super-potential of vol_K, by definition we need to bound the L^1-norm of φ with respect to vol_K by a power of the L^1-norm of φ on X. Since one can approximate any ω-p.s.h. function on X by a decreasing smooth ones (see [7]), it is enough to prove the desired property for smooth φ_1, φ_2 with $\varphi_1 \geq \varphi_2$, see Proposition 1.5.1 and Lemma 1.5.2. In this case, φ is *smooth and nonnegative.* This reduction is crucial in our proof. Observe that by compactness of $\tilde{K}$, it suffices to estimate

$$\int_{\tilde{K}'} \varphi \, dvol_K,$$

for small open subsets $\tilde{K}'$ of $\tilde{K}$. For each point $a \in K$, we will construct a $\mathcal{C}^{2,1/2}$-differentiable family $\tilde{F}_{\{\boldsymbol{\tau} \in Z\}}$ of analytic discs partly attached to K parameterized by $\boldsymbol{\tau}$ in a compact manifold Z of real dimension $(2n-2)$ which roughly satisfies the following two properties:

(i) the restriction of $\tilde{F}$ to $\partial\mathbb{D} \times Z$ is a submersion onto an open neighborhood K' of a in K, where we consider $\tilde{F}$ as a map from $\overline{\mathbb{D}} \times Z$ to X.

(ii) the restriction of $\tilde{F}$ to $\mathbb{D} \times Z$ is a diffeomorphism onto an open subset of X.

Put $\tilde{K}' := \tilde{K} \cap K'$ for $a \in \tilde{K}$. These $\tilde{K}'$ covers $\tilde{K}$. By the change of variables theorem and Property (i), we have

$$\int_{\tilde{K}'} \varphi \, dvol_K \leq \int_{K'} \varphi \, dvol_K \lesssim \int_{\partial\mathbb{D}\times Z} \varphi \circ \tilde{F}. \tag{1.1.2}$$

Since $\tilde{F}$ is holomorphic on $\mathbb{D}$ and $\mathcal{C}^2$ on $\overline{\mathbb{D}}$, observe that $\varphi \circ \tilde{F}$ is the difference of two $\mathcal{C}^2$ subharmonic functions on $\overline{\mathbb{D}}$.

Our second step is to bound $\int_{\partial\mathbb{D}\times Z} \varphi \circ \tilde{F}$ by a quantity involving $\int_{\mathbb{D}\times Z} \varphi \circ \tilde{F}$. For this purpose, we will establish a crucial inequality in dimension one which shows that L^1-norm on $\partial\mathbb{D}$ of a nonnegative $\mathcal{C}^2$ function on $\overline{\mathbb{D}}$ is bounded by a function of its L^1-norm on $\mathbb{D}$ and some Hölder norm of its Laplacian on $\mathbb{D}$. The ingredients for the proof of the last inequality are Riesz's representation formula and a general interpolation inequality for currents on manifolds with boundary. Note that a version of that interpolation inequality for manifolds without boundary was firstly used by Dinh and Sibony in [30].

The problem will be solved if one is able to bound $\int_{\mathbb{D}\times Z} \varphi \circ \tilde{F}$ by a constant times $\|\varphi\|_{L^1(X)}$. Taking into account Property (ii), one is tempted to use the change of variables by $\tilde{F}$. However, the Jacobian of $\tilde{F}$ is small near the boundary $\partial\mathbb{D} \times Z$. This is due to a general fact that any family of analytic discs satisfying Property (i) should degenerate at $\partial\mathbb{D}$ because of its attachment to K. So we need a precise control of the Jacobian of $\tilde{F}$ from below and prove some estimates on the integrals of p.s.h. functions and their dd^c on a tubular neighborhood

of $\tilde{K}$. These estimates are of independent interest. Consequently, we will get

$$\int_{\mathbb{D}\times Z} \varphi \circ \tilde{F} \lesssim_{\alpha_2} \left(\int_X \varphi \, dvol_X \right)^{\alpha_2}, \tag{1.1.3}$$

for any $\alpha_2 \in (0, 1/n)$. Combining these above inequalities gives the Hölder regularity of the super-potential of vol_K.

The paper is organized as follows. Section 1.2 is devoted to proving the above mentioned interpolation inequality for currents. In Section 1.3, we construct the desired family of analytic discs $\tilde{F}$. In Section 1.4, we present (1.1.2) and (1.1.3). Finally, we prove Theorem 1.1.4 in Section 1.5. At the beginning of Section 1.3, we will fix some notations which will be used for the rest of the paper.

1.2 Interpolation theory

Let M be a compact smooth manifold of dimension m. Fix a partition of unity subordinated to a finite covering of local charts of M. For $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, let $\mathcal{C}^{k,\alpha}(M)$ be the space of $\mathcal{C}^k$ functions on M whose partial derivatives of order k are Hölder continuous of order α. We endow the last space with the usual norm. For $t \in [0, \infty)$, denote by $\mathcal{C}^t(M)$ the space $\mathcal{C}^{[t],t-[t]}(M)$ where $[t]$ is the integer part of t. Let $\Lambda^l T^*M$ be the l^{th}-exterior power of the cotangent vector bundle T^*M for $1 \leq l \leq m$. Let $\mathcal{C}^t(M, \Lambda^l T^*M)$ be the set of l-differential forms with $\mathcal{C}^l$ coefficients. Using the above fixed partition of unity, we can equip $\mathcal{C}^t(M, \Lambda^l T^*M)$ with the norm $\| \cdot \|_{\mathcal{C}^t}$ which is the maximum of the $\mathcal{C}^t$ norms of its coefficients.

Let T be an l-current of order 0, *i.e.*, there is a constant C such that $|\langle T, \Phi \rangle| \leq C \|\Phi\|_{\mathcal{C}^0}$ for every smooth $(m-l)$-form Φ. For $t \in [0, \infty)$, define

$$\|T\|_{\mathcal{C}^{-t}} := \sup_{\Phi \text{ smooth }, \|\Phi\|_{\mathcal{C}^t}=1} |\langle T, \Phi \rangle|. \tag{1.2.1}$$

We will write $\|T\|$ instead of $\|T\|_{\mathcal{C}^{-0}}$ which is the usual mass norm of T. Dinh and Sibony in [30] proved that for any $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$, we have

$$\|T\|_{\mathcal{C}^{-t_2}} \leq \|T\|_{\mathcal{C}^{-t_1}} \leq c \|T\|^{1-t_1/t_2} \|T\|^{t_1/t_2}_{\mathcal{C}^{-t_2}}, \tag{1.2.2}$$

for some constant c independent of T. This inequality is very useful when dealing with continuous functionals on differential forms because one can reduce the problem to the smooth case. In this section, we will establish a generalization of (1.2.2) for compact smooth manifolds with boundary.

Let M be a compact smooth manifold of dimension m with boundary. Cover M by a finite number of local charts U_j. Take a partition of unity ϕ_j subordinated to this covering. By the aid of these ϕ_j, as above we can define the Banach spaces $\mathcal{C}^t(M)$ with the usual norms for $t \in [0, \infty)$. Denote by $\mathrm{Int}M$ the interior of M. Let $\mathcal{C}^t_c(\mathrm{Int}M)$ be the subspace of $\mathcal{C}^t(M)$ of $f \in \mathcal{C}^t(M)$ with compact support in $\mathrm{Int}M$. Let $\tilde{\mathcal{C}}^t(M)$ be the subspace of $\mathcal{C}^t(M)$ consisting of f with $f|_{\partial M} \equiv 0$. We can also define $\tilde{\mathcal{C}}^t(M, \Lambda^l T^*M)$ and $\mathcal{C}^t_c(M, \Lambda^l T^*M)$ in the same way as above.

Let T be an l-current of order 0 on $\mathrm{Int}M$. Assume that its mass is finite, that is,

$$\|T\| := \sup_{\Phi \text{ smooth }, \|\Phi\|_{\mathcal{C}^0_c(\mathrm{Int}M)}=1} |\langle T, \Phi\rangle| < \infty. \tag{1.2.3}$$

In our application, M will be $\overline{\mathbb{D}}$ and T will be the restriction of a continuous form on $\mathbb{C}$ to $\mathbb{D}$. By Riesz's representation theorem, T is a differential form whose coefficients are Radon measures on M with finite total variations. Hence, for any continuous differential form Φ on $\mathrm{Int}M$ with $\|\Phi\|_{\mathcal{C}^0} < \infty$, the value of T at Φ is well-defined. Then the current T can be extended to be a continuous linear functional on $\tilde{\mathcal{C}}^t(M, \Lambda^l T^*M)$. Let $\|T\|_{\tilde{\mathcal{C}}^{-t}(M)}$ be the norm of T as a continuous linear functional on $\tilde{\mathcal{C}}^t(M, \Lambda^l T^*M)$. As mentioned at the beginning of the section, we will prove the following analogue of (1.2.2).

Proposition 1.2.1. *Let T be a l-current of order 0 on $\mathrm{Int}M$. Assume that T has finite mass. Let $t_0, t_1, t_2 \in [0, \infty)$ with $t_0 < t_1 < t_2$. Let t_* be the unique real number for which $t_1 = t_* t_0 + (1 - t_*)t_2$. Then we have*

$$\|T\|_{\tilde{\mathcal{C}}^{-t_2}(M)} \leq \|T\|_{\tilde{\mathcal{C}}^{-t_1}(M)} \leq C\|T\|^{t_*}_{\tilde{\mathcal{C}}^{-t_0}(M)} \|T\|^{1-t_*}_{\tilde{\mathcal{C}}^{-t_2}(M)}, \tag{1.2.4}$$

for some constant C independent of T.

The remaining part of this section is devoted to prove the last proposition. Using a partition of unity as above, that proposition is a direct consequence of Corollary 1.2.9 at the end of this section. We first recall some notations and results from the interpolation theory of Banach spaces and refer to [53, 78] for a general treatment of the theory. Then we compute some interpolation spaces of $\tilde{\mathcal{C}}^t(M)$, see Corollary 1.2.8 below.

Let A_0 and A_1 be two Banach spaces which are continuously embedded to a Hausdorff topological vector space $\mathcal{A}$. Let B_0 and B_1 be two Banach spaces which are continuously embedded to a Hausdorff topological vector space $\mathcal{B}$. Let T be a linear operator from $\mathcal{A}$ to $\mathcal{B}$. Assume that $T|_{A_j} : A_j \to B_j$ are bounded for $j = 0, 1$. The interpolation theory of Banach spaces is to search for Banach subspaces $A \subset \mathcal{A}$ and $B \subset \mathcal{B}$ such that the restriction $T|_A : A \to B$ is a bounded linear operator. The spaces A and B are called interpolation spaces. We will recall below a classical construction of such spaces.

For $0 < t < \infty$ and $a \in A_0 + A_1$, define

$$K(t, a; A_0, A_1) := \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \tag{1.2.5}$$

where $a_0 \in A_0, a_1 \in A_1$. Let α be a constant in $(0, 1)$. The following class of Banach spaces is of great importance in the interpolation theory.

Definition 1.2.2. *Let $(A_0, A_1)_{\alpha,\infty}$ be the subspace of $A_0 + A_1$ consisting of $a \in A_0 + A_1$ for which the following quantity*

$$\|a\|_{(A_0,A_1)_{\alpha,\infty}} := \sup_{t>0} t^{-\alpha} K(t, a; A_0, A_1) \tag{1.2.6}$$

is finite. The last formula defines a norm on $(A_0, A_1)_{\alpha,\infty}$ which make it to be a Banach space.

The following fundamental theorem explains the role of the space $(A_0, A_1)_{\alpha,\infty}$.

Theorem 1.2.3. *[53, Th. 1.1.6] Let A_0, A_1, B_0, B_1 and T be as above. Let $\alpha \in (0,1)$. Then the restriction $T_{(A_0,A_1)_{\alpha,\infty}}$ of T to $(A_0,A_1)_{\alpha,\infty}$ is a bounded linear operator from $(A_0,A_1)_{\alpha,\infty}$ to $(B_0,B_1)_{\alpha,\infty}$ and*

$$\|T|_{(A_0,A_1)_{\alpha,\infty}}\| \leq \|T|_{A_0}\|^{1-\alpha}\|T|_{A_0}\|^{\alpha},$$

where $\|\cdot\|$ is the norm of bounded linear operators.

Let $m \in \mathbb{N}^*$ and $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Let $\mathcal{C}^k(\mathbb{R}^m)$ (respectively $\mathcal{C}^{k,\alpha}(\mathbb{R}^m)$) be the set of $\mathcal{C}^k$ functions (respectively $\mathcal{C}^{k,\alpha}$) on $\mathbb{R}^m$. For $t \in \mathbb{R}^+$, define $\mathcal{C}^t(\mathbb{R}^m) := \mathcal{C}^{[t],t-[t]}(\mathbb{R}^m)$. Let $\mathcal{C}^t_b(\mathbb{R}^m)$ be the subset of $\mathcal{C}^t(\mathbb{R}^m)$ consisting of elements whose $\mathcal{C}^t$ norms are bounded.

Let Ω be a bounded open subset of $\mathbb{R}^m$ with smooth boundary. Let $\partial\Omega$ be its boundary. Then $\overline{\Omega}$ is a smooth compact manifold with boundary which is itself a global chart. We have the Banach spaces $\mathcal{C}^t(\overline{\Omega})$ and $\tilde{\mathcal{C}}^t(\overline{\Omega})$ as above. In what follows, we will give a description of the interpolation space

$$\big(\tilde{\mathcal{C}}^{t_0}(\overline{\Omega}), \tilde{\mathcal{C}}^{t_2}(\overline{\Omega})\big)_{\alpha,\infty} \tag{1.2.7}$$

for $0 \leq t_0 < t_2 < \infty$. The corresponding interpolation spaces for $\mathcal{C}^t(\overline{\Omega})$ and $\mathcal{C}^t_b(\mathbb{R}^m)$ are already known, see Theorems 2.7.2 and 4.5.2 in [78].

It should be noted that the spaces $\big(\mathcal{C}^{t_0}(\overline{\Omega}), \mathcal{C}^{t_2}(\overline{\Omega})\big)_{\alpha,\infty}$ are easily determined by using the result mentioned above for $\mathcal{C}^t_b(\mathbb{R}^m)$ and the fact that the restriction from $\mathcal{C}^t_b(\mathbb{R}^m)$ to $\mathcal{C}^t(\overline{\Omega})$ is a retraction, see [78, Th. 4.5.1]. Nevertheless, this property is no longer true if we replace $\mathcal{C}^t(\overline{\Omega})$ by $\tilde{\mathcal{C}}^t(\overline{\Omega})$ because even the restriction map from $\mathcal{C}^t_b(\mathbb{R}^m)$ to $\tilde{\mathcal{C}}^t(\overline{\Omega})$ is not well-defined. In order to compute (1.2.7), we will follow the original strategy for $\mathcal{C}^t_b(\mathbb{R}^m)$ in [78], see also [53]. Although, in essence, our below results can be implicitly deduced from [78], we will present them in a simplified and detailed way which is therefore accessible for a wider audience.

The following lemma is well-known but for the reader's convenience, a complete proof will be given.

Lemma 1.2.4. *For every $t \in [0,\infty)$, every $f \in \mathcal{C}^t(\overline{\Omega})$ can be extended to be a function $Ef \in \mathcal{C}^t(\mathbb{R}^m)$ such that $\|Ef\|_{\mathcal{C}^t(\mathbb{R}^m)} \leq C\|f\|_{\mathcal{C}^t(\overline{\Omega})}$, where C is a constant independent of f.*

Proof. We will use a reflexion argument. By using a partition of unity subordinated to a suitable finite covering of $\overline{\Omega}$, we can suppose that $\overline{\Omega} = \mathbb{R}^{m-1} \times \mathbb{R}^+$. Let $f \in \mathcal{C}^t(\mathbb{R}^{m-1} \times \mathbb{R}^+)$. Let $[t]$ be the integer part of t. Let $a_1, \cdots, a_{[t]+1}$ be real numbers which are chosen later. Define $Ef := f$ on $\mathbb{R}^{m-1} \times \mathbb{R}^+$ and

$$Ef(x_1, \cdots, x_n) := \sum_{k=1}^{[t]+1} a_k f(x_1, \cdots, x_{n-1}, -kx_n)$$

otherwise. It is easy to see that Ef is continuous on $\mathbb{R}^m$. Now we will choose a_k such that $Ef \in \mathcal{C}^{[t]}$. If we can do so, we also get $Ef \in \mathcal{C}^t$ because $D^{[t]}Ef$ is $\mathcal{C}^{t-[t]}$ on $\mathbb{R}^{m-1} \times \mathbb{R}^+$, hence, on the whole $\mathbb{R}^m$ by its defining formula. One only needs to be concerned with the x_n-direction. Direct computations show that

$$\partial^l_{x_n} Ef(x_1, \cdots, x_{n-1}, 0) = \Big(\sum_{k=1}^{[t]+1} (-k)^l a_k \Big) \partial^l_{x_n} f(x_1, \cdots, x_{n-1}, 0),$$

for $0 \leq l \leq [t]$. The regularity condition on Ef is equivalent to the linear system $\sum_{k=1}^{[t]+1}(-k)^l a_k = 1$ for $0 \leq l \leq [t]$. Its determinant is a Vandermonde one. Hence the system has a unique solution $(a_1, \cdots, a_{[t]+1})$. When $f|_{\partial\Omega} = 0$, it is clear from the defining formula of Ef that $Ef|_{\partial\Omega} = 0$. The proof is finished. □

Proposition 1.2.5. *For every $\alpha \in (0,1)$ and every $k \in \mathbb{N}^*$, we have*

$$\big(\tilde{\mathcal{C}}^0(\overline{\Omega}), \tilde{\mathcal{C}}^k(\overline{\Omega})\big)_{\alpha,\infty} \supset \tilde{\mathcal{C}}^{\alpha k}(\overline{\Omega}), \tag{1.2.8}$$

where the last inclusion means a continuous inclusion between Banach spaces.

Proof. Let $f \in \tilde{\mathcal{C}}^{\alpha k}(\overline{\Omega})$. Put $t := \alpha k$. We write below $\lesssim$ to indicate $\leq$ up to a constant independent of (f, ϵ). By Lemma 1.2.4, we can extend f to be a function F in $\mathcal{C}^t(\mathbb{R}^m)$ with

$$\|F\|_{\mathcal{C}^t(\mathbb{R}^m)} \leq C\|f\|_{\tilde{\mathcal{C}}^t(\overline{\Omega})},$$

for some constant C independent of f and $F|_{\partial\Omega} = 0$. Let $\mathbb{B}_r$ denotes the ball of radius $r > 0$ centered at 0 in $\mathbb{R}^m$ and $\mathbb{B}_r^+$ denotes the subset of $\mathbb{B}_r$ consisting of $\mathbf{x} = (x_1, \cdots, x_n)$ with $x_n \geq 0$. Since $\overline{\Omega}$ is compact, we can cover $\partial\Omega$ by a finite number of small open subsets $\{U_j\}_{1\leq j\leq N}$ of $\mathbb{R}^m$ such that in each U_j, by a suitable change of coordinates Ψ_j, we have

$$\Psi_j\big(\overline{\Omega} \cap U_j\big) = \mathbb{B}_2^+$$

and $\Psi_j\big(\partial\Omega \cap U_j\big) = \mathbb{B}_2^+ \cap \{x_n \geq 0\}$. Without loss of generality, we can suppose that $\Psi_j^{-1}(\mathbb{B}_1^+)$ also covers $\partial\Omega$. Put

$$U_0 := \Omega \backslash \cup_{1\leq j\leq N} \Psi_j^{-1}(\overline{\mathbb{B}}_1^+).$$

The family $\{U_j\}_{0\leq j\leq N}$ covers $\overline{\Omega}$. Let $\{\chi_j\}_{0\leq j\leq N}$ be smooth functions of $\mathbb{R}^m$ such that $0 \leq \chi_j \leq 1$ for $0 \leq j \leq N$, and $\mathrm{supp}\chi_j \Subset \Psi_j^{-1}(\mathbb{B}_{5/4})$ for $1 \leq j \leq N$ and $\mathrm{supp}\chi_0 \Subset U_0$, and $\sum_{0\leq j\leq N} \chi_j = 1$ on $\overline{\Omega}$.

Define $F_j := (\chi_j F)\circ\Psi_j^{-1}$. By the properties of (Ψ_j, F) mentioned above, we have $F_j|_{x_n=0} = 0$. Let χ be a nonnegative smooth function on $\mathbb{R}^m$ which is compactly supported on $\mathbb{B}_1$ such that $\int_{\mathbb{R}^m} \chi d\mathbf{x} = 1$. Taylor's expansion for F_j gives

$$F_j(\mathbf{x}) = F_j(\mathbf{x}-\mathbf{y}) + DF_j(\mathbf{x}-\mathbf{y})\mathbf{y} + \cdots + \frac{1}{[t]!}D^{[t]}F_j(\mathbf{x}-\mathbf{y})\mathbf{y}^{[t]} + R_j(\mathbf{x},\mathbf{y})\mathbf{y}^{[t]}, \tag{1.2.9}$$

where $R_j(\mathbf{x},\mathbf{y})$ is, for $\mathbf{x}$ fixed, a $\mathcal{C}^{t-[t]}$ linear functional on $(\mathbb{R}^m)^{[t]}$ and we have

$$R_j(\mathbf{x},0) = 0, \quad \|R_j\|_{\mathcal{C}^{t-[t]}} \lesssim \|F_j\|_{\mathcal{C}^t} \leq C\|f\|_{\mathcal{C}^t}.$$

Hence, one gets

$$|R_j(\mathbf{x},\mathbf{y})| \lesssim |\mathbf{y}|^{t-[t]}\|f\|_{\mathcal{C}^t}. \tag{1.2.10}$$

Put

$$\epsilon_0 := \min\{1/4, \mathrm{dist}(U_0, \partial\Omega)\}.$$

Let $\epsilon \in (0, \epsilon_0)$. For $0 \leq j \leq N$, we define

$$F_{j,\epsilon}(\mathbf{x}) := \int_{\mathbb{R}^m} \big[F_j(\mathbf{x}-\epsilon\mathbf{y}) + DF_j(\mathbf{x}-\epsilon\mathbf{y})(\epsilon\mathbf{y}) + \cdots + \frac{1}{[t]!}D^{[t]}F_j(\mathbf{x}-\epsilon\mathbf{y})(\epsilon\mathbf{y})^{[t]}\big]\chi(\mathbf{y})d\mathbf{y}. \tag{1.2.11}$$

Observe that $F_{0,\epsilon}$ is a smooth function in $\tilde{\mathcal{C}}^\infty(\overline{\Omega})$ by the choice of ϵ and $F_{j,\epsilon}$ is smooth on $\mathbb{R}^m$ and compactly supported on $\mathbb{B}_{3/2}$ for $1 \leq j \leq N$. A property of the convolution implies that $F_{j,\epsilon}$ converges to F_j in $\mathcal{C}^0$-topology. Precisely, using (1.2.9), (1.2.11) and (1.2.10) yields that

$$|F_{j,\epsilon}(\mathbf{x}) - F_j(\mathbf{x})| \leq \epsilon^{[t]} \int_{\mathbb{R}^m} |R_j(\mathbf{x}, \epsilon\mathbf{y})|\chi(\mathbf{y})d\mathbf{y} \leq C\epsilon^t \|f\|_{\mathcal{C}^t}, \tag{1.2.12}$$

for every $\mathbf{x}$. Let τ be a smooth function on $\mathbb{R}$ compactly supported on $[-2, 2]$ such that $\tau \equiv 1$ on $[-3/2, 3/2]$. Define

$$F'_{j,\epsilon}(x_1, \cdots, x_n) := F_{j,\epsilon}(x_1, \cdots, x_{n-1}, x_n) - \tau(x_n)F_{j,\epsilon}(x_1, \cdots, x_{n-1}, 0),$$

for $1 \leq j \leq N$ and we put $F'_{0,\epsilon} := F_{0,\epsilon}$ for consistence. We immediately see that $F'_{j,\epsilon} = 0$ on $\{x_n = 0\}$ and $\mathrm{supp}F'_{j,\epsilon} \subset \mathbb{B}_2$. As a consequence, $F'_{j,\epsilon} \circ \Psi_j$ is smooth on $\mathbb{R}^m$ and vanishes on $\partial\Omega$. We deduce from (1.2.12) and the fact that $F_j|_{\{x_n=0\}} \equiv 0$ that

$$\begin{aligned} |F'_{j,\epsilon}(\mathbf{x}) - F_j(\mathbf{x})| \leq |F_{j,\epsilon}(\mathbf{x}) - F_j(\mathbf{x})| + \\ |F_{j,\epsilon}(x_1, \cdots, x_{n-1}, 0) - F_j(x_1, \cdots, x_{n-1}, 0)| \leq 2C\epsilon^t \|f\|_{\mathcal{C}^t}. \end{aligned} \tag{1.2.13}$$

Define

$$g_{1,\epsilon} := \sum_{0 \leq j \leq N} F'_{j,\epsilon} \circ \Psi_j|_{\overline{\Omega}} \in \tilde{\mathcal{C}}^\infty(\overline{\Omega})$$

and $g_{0,\epsilon} := f - g_{1,\epsilon} \in \tilde{\mathcal{C}}^0(\overline{\Omega})$. We have $f = g_{0,\epsilon} + g_{1,\epsilon}$. In view of (1.2.6), we have to estimate $\|g_{0,\epsilon}\|_{\tilde{\mathcal{C}}^0(\overline{\Omega})}$ and $\|g_{1,\epsilon}\|_{\tilde{\mathcal{C}}^k(\overline{\Omega})}$. Since $f = \sum_{0 \leq j \leq N} F_j \circ \Psi_j$, we have

$$g_{0,\epsilon} = \sum_{0 \leq j \leq N} (F_j \circ \Psi_j - F'_{j,\epsilon} \circ \Psi_j).$$

Taking into account (1.2.13), one gets

$$\|g_{0,\epsilon}\|_{\tilde{\mathcal{C}}^0(\overline{\Omega})} \lesssim \epsilon^t \|f\|_{\tilde{\mathcal{C}}^t(\overline{\Omega})}. \tag{1.2.14}$$

For $0 \leq l \leq [t]$, we define

$$G_{j,l}(\mathbf{x}, \mathbf{y}) := D^l F_j(\mathbf{y}) + D^{l+1}F_j(\mathbf{y})(\mathbf{x}-\mathbf{y}) + \cdots + \frac{1}{([t]-l)!}D^{[t]}F_j(\mathbf{y})(\mathbf{x}-\mathbf{y})^{[t]-l}$$

which is the Taylor expansion up to the $([t]-l)$ order of $D^l F_j(\mathbf{x})$ at $\mathbf{y}$. Thus arguing as in (1.2.10), we get

$$|G_{j,l}(\mathbf{x}, \mathbf{y}) - D^l F_j(\mathbf{x})| \lesssim \|f\|_{\mathcal{C}^t}|\mathbf{x}-\mathbf{y}|^{t-l}. \tag{1.2.15}$$

The equality (1.2.11) can be rewritten as

$$F_{j,\epsilon}(\mathbf{x}) = \epsilon^{-m}\int_{\mathbb{R}^m}\big[F_j(\mathbf{y}') + DF_j(\mathbf{y}')(\mathbf{x}-\mathbf{y}') + \cdots + \frac{1}{[t]!}D^{[t]}F_j(\mathbf{y}')(\mathbf{x}-\mathbf{y}')^{[t]}\big]\chi(\frac{\mathbf{x}-\mathbf{y}'}{\epsilon})d\mathbf{y}'.$$

Differentiating the last equality in $\mathbf{x}$ for k' times gives

$$\begin{aligned} D^{k'}_{\mathbf{x}}F_{j,\epsilon}(\mathbf{x}) &= \epsilon^{-m-k'+l}\sum_{0\le l\le \min\{k',[t]\}}\int_{\mathbb{R}^m} G_{j,l}(\mathbf{x},\mathbf{y}')\otimes D^{k'-l}\chi(\frac{\mathbf{x}-\mathbf{y}'}{\epsilon})d\mathbf{y}' \\ &= \epsilon^{-k'+l}\sum_{0\le l\le \min\{k',[t]\}}\int_{\mathbb{R}^m} G_{j,l}(\mathbf{x},\mathbf{x}-\epsilon\mathbf{y})\otimes D^{k'-l}\chi(\mathbf{y})d\mathbf{y} \end{aligned} \tag{1.2.16}$$

by a suitable change of coordinates. Since $\int_{\mathbb{R}^m} D^l_{\mathbf{x}}\chi(\mathbf{y})d\mathbf{y} = 0$ for any $l \ge 1$, we obtain

$$\int_{\mathbb{R}^m} G_{j,l}(\mathbf{x},\mathbf{x}-\epsilon\mathbf{y})\otimes D^{k'-l}\chi(\mathbf{y})d\mathbf{y} = \int_{\mathbb{R}^m}\big(G_{j,l}(\mathbf{x},\mathbf{x}-\epsilon\mathbf{y}) - D^lF_j(\mathbf{x})\big)\otimes D^{k'-l}\chi(\mathbf{y})d\mathbf{y} \tag{1.2.17}$$

which is of absolute value $\lesssim \epsilon^{t-l}\|f\|_{\mathcal{C}^t}$ by using (1.2.15) and the fact that $\mathrm{supp}\chi \subset \mathbb{B}_1$. Combining (1.2.16) with (1.2.17) gives

$$|D^{k'}_{\mathbf{x}}F_{j,\epsilon}(\mathbf{x})| \lesssim \epsilon^{-k'+t}\|f\|_{\mathcal{C}^t}$$

which implies that

$$\|g_{1,\epsilon}\|_{\tilde{\mathcal{C}}^k(\overline{\Omega})} \lesssim \epsilon^{-k+t}\|f\|_{\mathcal{C}^t} \tag{1.2.18}$$

by choosing $k' = k$. Taking into account (1.2.14), (1.2.18) and (1.2.5), one deduces that

$$\epsilon^{-\alpha k}K\big(\epsilon^k, f; \tilde{\mathcal{C}}^0(\overline{\Omega}), \tilde{\mathcal{C}}^k(\overline{\Omega})\big) \le \epsilon^t\big(\|g_{0,\epsilon}\|_{\tilde{\mathcal{C}}^0} + \epsilon^k\|g_{1,\epsilon}\|_{\tilde{\mathcal{C}}^k}\big) \lesssim \|f\|_{\tilde{\mathcal{C}}^t(\overline{\Omega})},$$

for every $\epsilon \in (0,\epsilon_0)$. When $\epsilon \ge \epsilon_0$, since

$$f = f + 0 \in \tilde{\mathcal{C}}^0(\overline{\Omega}) + \tilde{\mathcal{C}}^1(\overline{\Omega}),$$

we have

$$\epsilon^{-\alpha k}K\big(\epsilon^k, f; \tilde{\mathcal{C}}^0(\overline{\Omega}), \tilde{\mathcal{C}}^k(\overline{\Omega})\big) \le \epsilon_0^{-\alpha k}\|f\|_{\tilde{\mathcal{C}}^0(\overline{\Omega})} \le \epsilon_0^{-\alpha k}\|f\|_{\tilde{\mathcal{C}}^{\alpha k}(\overline{\Omega})}.$$

Hence, $f \in \big(\tilde{\mathcal{C}}^0(\overline{\Omega}), \tilde{\mathcal{C}}^k(\overline{\Omega})\big)_{\alpha,\infty}$. The proof is finished. □

For every $h \in \mathbb{R}^m$ and every a function g on $\mathbb{R}^m$, define the operator

$$\Delta_h g(x) := g(x+h) - g(x)$$

for every $x \in \mathbb{R}^m$. The following property is crucial for the next proposition.

Lemma 1.2.6. *Let $\alpha \in (0,1)$ and l be an integer ≥ 1. For $g \in \mathcal{C}^\alpha_b(\mathbb{R}^m)$, we put*

$$\|g\|_{\alpha,\Delta,l} := \|g\|_{\mathcal{C}^0} + \sup_{x,h\in\mathbb{R}^m, h\neq 0}\frac{|\Delta^l_h g|}{|h|^\alpha}.$$

Then the last formula defines a norm on $\mathcal{C}^\alpha_b(\mathbb{R}^m)$ which is equivalent to its usual $\mathcal{C}^\alpha$ norm. More precisely, there exists a positive constant $C_{l,\alpha}$ depending only on (l,α) such that for every g, we have

$$C^{-1}_{l,\alpha}\|g\|_{\mathcal{C}^\alpha} \le \|g\|_{\alpha,\Delta,l} \le C_{l,\alpha}\|g\|_{\mathcal{C}^\alpha}.$$

Proof. This is a simplification of Lemma 1.13.4 in [78]. When $l = 1$, the two norms are identical. Consider $l \geq 2$. Observe that it is enough to prove the desired result for $l = 2$ because the general case can easily follow by induction. It is clear that $\|g\|_{\alpha,\Delta,2} \leq 2\|g\|_{\mathcal{C}^\alpha}$. We now prove the converse inequality. The key argument is the following formula:

$$g(x+h) - g(x) = \frac{1}{2}\big(g(x+2h) - g(x)\big) - \frac{g(x+2h) - 2g(x+h) + g(x)}{2}.$$

Dividing the last equality by $|h|^\alpha$ gives

$$\frac{|g(x+h) - g(x)|}{|h|^\alpha} \leq 2^{\alpha-1}\frac{|g(x+2h) - g(x)|}{|2h|^\alpha} + \frac{|g(x+2h) - 2g(x+h) + g(x)|}{2|h|^\alpha}.$$

By taking the supremum over $\{(x,h) \in \mathbb{R}^{2m}, h \neq 0\}$ in the last inequality, we deduce that

$$\|g\|_{\mathcal{C}^\alpha} \leq 2^{\alpha-1}\|g\|_{\mathcal{C}^\alpha} + \|g\|_{\alpha,\Delta,2}.$$

Since $2^{\alpha-1} < 1$ we get the desired conclusion. The proof is finished. □

Proposition 1.2.7. *Let k be a positive integer and let α be a real number in $(0,1)$. Assume that $\alpha k \in (0,1)$. Then we have*

$$\big(\tilde{\mathcal{C}}^0(\overline{\Omega}), \tilde{\mathcal{C}}^k(\overline{\Omega})\big)_{\alpha,\infty} \subset \tilde{\mathcal{C}}^{\alpha k}(\overline{\Omega}). \tag{1.2.19}$$

Proof. Let take an element $f \in \big(\tilde{\mathcal{C}}^0(\overline{\Omega}), \tilde{\mathcal{C}}^k(\overline{\Omega})\big)_{\alpha,\infty}$. Suppose that $f = g_0 + g_1$ with $g_0 \in \tilde{\mathcal{C}}^0(\overline{\Omega})$ and $g_1 \in \tilde{\mathcal{C}}^k(\overline{\Omega})$. We have $\Delta_h^k f = \Delta_h^k g_0 + \Delta_h^k g_1$. By using Taylor's expansion of g_1, observe that $|\Delta_h^k g_1| \leq C|h|^k\|g_1\|_{\mathcal{C}^k}$ for some constant C independent of (g_1, h). On the other hand, $|\Delta_h^k g_0| \leq 2^l\|g_0\|_{\mathcal{C}^0}$. Combining these inequalities gives

$$|\Delta_h^k f| \leq 2^l\|g\|_{\mathcal{C}^0} + C|h|^k\|g_1\|_{\mathcal{C}^k} \lesssim \|g\|_{\mathcal{C}^0} + |h|^k\|g_1\|_{\mathcal{C}^k},$$

for every (g_0, g_1) with $f = g_0 + g_1$. Taking the infimum in the last inequality in (g_0, g_1), we obtain

$$|\Delta_h^k f| \lesssim K\big(h^k, f; \tilde{\mathcal{C}}^0(\overline{\Omega}), \tilde{\mathcal{C}}^1(\overline{\Omega})\big) \leq |h|^{\alpha k}\|f\|_{\big(\tilde{\mathcal{C}}^0(\overline{\Omega}), \tilde{\mathcal{C}}^k(\overline{\Omega})\big)_{\alpha,\infty}}.$$

As a consequence, one gets

$$\|f\|_{\alpha k,\Delta,k} \lesssim \|f\|_{\big(\tilde{\mathcal{C}}^0(\overline{\Omega}), \tilde{\mathcal{C}}^k(\overline{\Omega})\big)_{\alpha,\infty}}.$$

By Lemma 1.2.6 and the hypothesis that $\alpha k < 1$, we obtain the desired result. The proof is finished. □

Corollary 1.2.8. *For every $\alpha \in (0,1)$, every real nonnegative numbers t_1 and t_2, we have*

$$\big(\tilde{\mathcal{C}}^{t_1}(\overline{\Omega}), \tilde{\mathcal{C}}^{t_2}(\overline{\Omega})\big)_{\alpha,\infty} \supset \tilde{\mathcal{C}}^{\alpha t_2 + (1-\alpha)t_1}(\overline{\Omega}). \tag{1.2.20}$$

Proof. For simplicity, we give a proof for $t_2 = k \in \mathbb{N}^*$ and $t_1 = \beta \in [0,1)$. The general case can be deduced by using similar arguments. By a consequence of the reiteration theorem (see [53, Re. 1.3.7]), we have the following general formula:

$$\big((A_0, A_1)_{\theta,\infty}, A_1\big)_{\alpha,\infty} = (A_0, A_1)_{(1-\alpha)\theta+\alpha,\infty}.$$

Applying the last equality to

$$A_0 = \tilde{\mathcal{C}}^0(\overline{\Omega}), A_1 = \tilde{\mathcal{C}}^k(\overline{\Omega}) \quad \text{and} \quad \theta = \beta/k$$

and using the fact that $(A_0, A_1)_{\theta,\infty} = \tilde{\mathcal{C}}^\beta(\overline{\Omega})$ (by Proposition 1.2.7 and 1.2.5), we obtain the desired inclusion. The proof is finished. □

Since $(\mathbb{R}, \mathbb{R})_{\alpha,\infty} = \mathbb{R}$ for any $\alpha \in (0,1)$, applying Theorem 1.2.3 to

$$A_0 = \tilde{\mathcal{C}}^{t_0}(\overline{\Omega}), \quad A_1 = \tilde{\mathcal{C}}^{t_2}(\overline{\Omega}), \quad B_0 = B_1 = \mathbb{R},$$

and then using Corollary 1.2.8, we obtain the following result.

Corollary 1.2.9. *Let Ω be a bounded open subset of $\mathbb{R}^m$ with smooth boundary. Let t_0, t_1 and t_2 be three real numbers such that $0 \leq t_0 < t_1 < t_2$. Let S be a bounded linear map from $\tilde{\mathcal{C}}^{t_0}(\overline{\Omega})$ to $\mathbb{R}$. Then the restriction $S|_{\tilde{\mathcal{C}}^{t_j}(\overline{\Omega})}$ of S to $\tilde{\mathcal{C}}^{t_j}(\overline{\Omega})$ for $j = 1$ or 2 is also a bounded linear map from $\tilde{\mathcal{C}}^{t_j}(\overline{\Omega})$ to $\mathbb{R}$ and*

$$\|S|_{\tilde{\mathcal{C}}^{t_1}(\overline{\Omega})}\| \leq c\|S|_{\tilde{\mathcal{C}}^{t_0}(\overline{\Omega})}\|^{t_*}\|S|_{\tilde{\mathcal{C}}^{t_2}(\overline{\Omega})}\|^{1-t_*},$$

where c is a constant independent of S and t_* is the unique real number for which $t_1 = t_* t_0 + (1 - t_*)t_2$.

1.3 Analytic discs partly attached to a generic submanifold

Firstly we fix some notations which will be valid throughout the rest of paper. For every Riemannian smooth manifold Y, any $a \in Y$ and $r \in \mathbb{R}^+$, we denote by $\mathbb{B}_Y(a, r)$ the ball of radius r centered at a of Y and by vol_Y the Riemannian volume form of Y. When $Y = \mathbb{R}^m$ for some $m \in \mathbb{N}$ with the Euclidean metric, we write $\mathbb{B}_m(a, r)$ instead of $\mathbb{B}_Y(a, r)$ and $\mathbb{B}_m$ instead of $\mathbb{B}_m(0, 1)$. In particular, when $Y = \mathbb{C} \simeq \mathbb{R}^2$ and $a = 0$, we put $\mathbb{D}_r := \mathbb{B}_2(0, r)$ and $\mathbb{D} := \mathbb{B}_2(0, 1)$. For every $m \in \mathbb{N}^*$, we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ via the formula $\mathbb{C}^m = \mathbb{R}^m + i\mathbb{R}^m$.

Let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$ and $\partial^+\mathbb{D} := \{\xi \in \mathbb{D} : \mathrm{Re}\,\xi \geq 0\}$. We sometimes identify $\xi \in \mathbb{D}$ with $\theta \in (-\pi, \pi]$ by letting $\xi = e^{i\theta}$. *An analytic disc f in X* is a holomorphic mapping from $\mathbb{D}$ to X which is continuous up to the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. For an interval $I \subset \partial\mathbb{D}$, *$f$ is said to be I-attached to a subset $E \subset X$* if $f(I) \subset E$. When $I = \partial^+\mathbb{D}$, an analytic disc I-attached to E is said to be *half-attached to E*.

Let K be a generic immersed $\mathcal{C}^3$ submanifold of X. Observe that the dimension of K is at least n. *Throughout the paper, we only consider the case where $\dim K = n$, hence its codimension d equals n.* This is in fact the most interesting case and the general case will be easily deduced from it. In Section 1.5, we will explain the necessary modifications to get Theorem 1.1.4 when $\dim K > n$.

Our goal is to for each $a \in K$ construct a $\mathcal{C}^{2,1/2}$-differentiable family of analytic discs partly attached to K which covers an open neighborhood of a in X. It should be noted that any family of discs partly attached to K degenerates near K due to its attachment to K. Controlling such behaviour around K is actually the key point in this section. We also need that the part of this family lying in K must cover an open neighborhood of 0 in K. Constructing analytic discs is an important tool in Cauchy-Riemann geometry. Generally, one uses a suitable Bishop-type equation together with a choice of initial data depending on situations to

obtain the desired result. The reader may also consult [1, 58, 59] and references therein for more information. In what follows, we will apply the same strategy combining with the ideas from [82].

The following local coordinates are frequently used in the Cauchy-Riemann geometry.

Lemma 1.3.1. *Through every point a of K, there exist local holomorphic coordinates $(W, \mathbf{z})$ of X around a such that in that local coordinates, the point a is the origin and $K \cap W$ is the graph over $\mathbb{B}_n$ of a $\mathcal{C}^3$ map h from $\overline{\mathbb{B}}_n$ to $\mathbb{R}^n$ which satisfies $D^j h(0) = 0$ with $j = 0, 1, 2$, where Dh denotes the differential of h. Moreover, $\|h\|_{\mathcal{C}^3}$ is bounded uniformly in $a \in \tilde{K}$.*

Proof. The existence of such h with $h(0) = Dh(0) = 0$ is well-known, see [1] for example. In order to obtain the additional property $D^2h(0) = 0$, one will need to perform a change of coordinates, we refer to [58, Sec. 6.10] for details. The proof is finished. □

From now on, fix an arbitrary point $a \in K$ and we confine ourselves to the local chart described in Lemma 1.3.1. In other words, we will work on $\mathbb{C}^n$ and

$$K' := \{\mathbf{z} = \mathbf{x} + ih(\mathbf{x}) \in \mathbb{C}^n : \mathbf{x} \in \mathbb{B}_n\},$$

where we have $h(0) = Dh(0) = 0$. For most of the time, the last condition is enough for our purposes, we will only need $D^2h(0) = 0$ in the proof of Proposition 1.4.5. The property of h yields that there is a constant c_0 for which

$$|h(\mathbf{x})| \leq c_0|\mathbf{x}|^2, \quad |Dh(\mathbf{x})| \leq c_0|\mathbf{x}|, \tag{1.3.1}$$

for every $\mathbf{x} \in \overline{\mathbb{B}}_n$.

In this paragraph, we prepare some useful facts about harmonic functions on the unit disc which will be indispensable for studying Bishop-type equations later. Denote by $z = x+iy$ the complex variable on $\mathbb{C}$ and by $\xi = e^{i\theta}$ the variable on $\partial\mathbb{D}$. Let $u_0(\xi)$ be an arbitrary continuous function on $\partial\mathbb{D}$. Recall that u_0 can be extended uniquely to be a harmonic function on $\mathbb{D}$ which is continuous on $\overline{\mathbb{D}}$. Since this correspondence is bijective, without stating explicitly, we will freely identify u_0 with its harmonic extension on $\mathbb{D}$. We will write $u_0(z) = u_0(x + iy)$ to indicate the harmonic extension of $u_0(e^{i\theta})$. It is well-known that the Cauchy transform of u_0, given by

$$\mathcal{C}u_0(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(e^{i\theta})\frac{e^{i\theta} + z}{e^{i\theta} - z}d\theta,$$

is a holomorphic function on $\mathbb{D}$ whose real part is u_0. Let $\mathcal{T}u_0$ be the imaginary part of $\mathcal{C}u_0$. Decomposing the last formula into the real and imaginary parts, we obtain

$$u_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1 - |z|^2)}{|e^{i\theta} - z|^2}u_0(e^{i\theta})d\theta. \tag{1.3.2}$$

and

$$\mathcal{T}u_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(ze^{-i\theta} - \bar{z}e^{i\theta})}{i|e^{i\theta} - z|^2}u_0(e^{i\theta})d\theta.$$

The function $\mathcal{T}u_0$ is harmonic on $\mathbb{D}$ but is not always continuous up to the boundary of $\mathbb{D}$. Let k be an arbitrary natural number and let β be an arbitrary number in $(0, 1)$. A result of

Privalov (see [58, Th. 4.12]) implies that if u_0 belongs to $\mathcal{C}^{k,\beta}(\partial\mathbb{D})$, then $\mathcal{T}u_0$ is continuous up to $\partial\mathbb{D}$ and $\|\mathcal{T}u_0\|_{\mathcal{C}^{k,\beta}(\partial\mathbb{D})}$ is bounded by $\|u_0\|_{\mathcal{C}^{k,\beta}(\partial\mathbb{D})}$ times a constant independent of u_0. Hence, the linear self-operator of $\mathcal{C}^{k,\beta}(\partial\mathbb{D})$ defined by sending u_0 to the restriction of $\mathcal{T}u_0$ onto $\partial\mathbb{D}$ is bounded and called *the Hilbert transform*. For simplicity, we also denote it by $\mathcal{T}$. For our later purposes, it is convenient to use a modified version $\mathcal{T}_1$ of $\mathcal{T}$ defined by

$$\mathcal{T}_1 u_0 := \mathcal{T}u_0 - \mathcal{T}u_0(1).$$

Hence we always have $\mathcal{T}_1 u_0(1) = 0$ and

$$\partial_\theta \mathcal{T}_1 u_0 = \partial_\theta \mathcal{T} u_0 = \mathcal{T}\partial_\theta u_0, \tag{1.3.3}$$

provided that $u_0 \in \mathcal{C}^{1,\beta}(\partial\mathbb{D})$ with $\beta \in (0,1)$, see [58, p.121] for a proof. The boundedness of $\mathcal{T}$ on $\mathcal{C}^{k,\beta}(\partial\mathbb{D})$ implies that there is a constant $C_{k,\beta} > 1$ such that for any $v \in \mathcal{C}^{k,\beta}(\partial\mathbb{D})$ we have

$$\|\mathcal{T}_1 v\|_{\mathcal{C}^{k,\beta}(\partial\mathbb{D})} \leq C_{k,\beta}\|v\|_{\mathcal{C}^{k,\beta}(\partial\mathbb{D})}. \tag{1.3.4}$$

Extending $u_0, \mathcal{T}_1 u_0$ harmonically to $\mathbb{D}$. By construction, the function $f(z) := -\mathcal{T}_1 u_0(z) + i u_0(z)$ is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ provided that u_0 is in $\mathcal{C}^{\beta}(\partial\mathbb{D})$ with $0 < \beta < 1$. By [59, Th. 4.2], $\|f\|_{\mathcal{C}^{k,\beta}(\overline{\mathbb{D}})}$ is bounded by $\|f\|_{\mathcal{C}^{k,\beta}(\partial\mathbb{D})}$ times a constant depending only on (k,β). Since $\|u_0\|_{\mathcal{C}^{k,\beta}(\overline{\mathbb{D}})} \leq \|f\|_{\mathcal{C}^{k,\beta}(\overline{\mathbb{D}})}$ and $\|f\|_{\mathcal{C}^{k,\beta}(\partial\mathbb{D})} \leq (1 + C_{k,\beta})\|u_0\|_{\mathcal{C}^{k,\beta}(\partial\mathbb{D})}$ by (1.3.4), we have

$$\|u_0\|_{\mathcal{C}^{k,\beta}(\overline{\mathbb{D}})} \leq C'_{k,\beta}\|u_0\|_{\mathcal{C}^{k,\beta}(\partial\mathbb{D})}, \tag{1.3.5}$$

for some constant $C'_{k,\beta}$ depending only on (k,β). A direct consequence of the above inequalities is that when u_0 is smooth on $\partial\mathbb{D}$, the associated holomorphic function f is also smooth on $\overline{\mathbb{D}}$.

Lemma 1.3.2. *There exist a function $u_0 \in \mathcal{C}^\infty(\partial\mathbb{D})$ and two positive constants (θ_{u_0}, c_{u_0}) such that $u_0(e^{i\theta}) = 0$ for $\theta \in [-\theta_{u_0}, \theta_{u_0}] \subset [-\pi/2, \pi/2]$ and $\partial_x u_0(1) = -1$ and $u_0(z) > c_{u_0}(1 - |z|)$ for every $z \in \mathbb{D}$.*

Proof. Let u be a smooth function on $\partial\mathbb{D}$ vanishing on $\partial^+\mathbb{D}$. By Poisson's formula, we have

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-|z|^2)}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta. \tag{1.3.6}$$

Differentiating (1.3.6) gives

$$\partial_x u(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(e^{i\theta})}{\cos\theta - 1} d\theta.$$

Note that the last integral is well-defined because u vanishes on $\partial^+\mathbb{D}$. It is easy to choose a smooth u so that the above integral is equal to -1 and $u \equiv 0$ on $\partial^+\mathbb{D}$ and $u(e^{i\theta}) > 0$ for $|\theta| \geq 3\pi/2$. The last property and (1.3.6) show that $u(z) > 0$ for every $z \in \mathbb{D}$.

We have chosen u with the property that $\partial_x u(1) = -1$ and $u(z) > 0$ for $z \in \mathbb{D}$. This implies that $\partial_x u(e^{i\theta}) \leq -1/2$ for every $\theta \in [-\theta_0, \theta_0] \subset (-\pi/2, \pi/2)$ for θ_0 close enough to 1. Since u vanishes on $\partial^+\mathbb{D}$, we have

$$0 = \partial_\theta u(e^{i\theta}) = -\partial_x u(e^{i\theta})\sin\theta + \partial_y u(e^{i\theta})\cos\theta$$

which yields

$$\partial_y u(e^{i\theta}) = \partial_x u(e^{i\theta}) \tan\theta \tag{1.3.7}$$

for $\theta \in [-\theta_0, \theta_0]$. Let $z = |z|e^{i\theta} \in \mathbb{D}$ such that $\theta \in [-\theta_0, \theta_0]$. Taylor's expansion for u at $e^{i\theta}$ gives

$$u(|z|e^{i\theta}) = u(e^{i\theta}) + (|z|\cos\theta - \cos\theta)\partial_x u(e^{i\theta}) + (|z|\sin\theta - \sin\theta)\partial_y u(e^{i\theta}) + O\big((1-|z|)^2\big) \tag{1.3.8}$$

$$= \frac{(|z|-1)\partial_x u(e^{i\theta})}{\cos\theta} + O\big((1-|z|)^2\big) \quad \text{(by (1.3.7))}.$$

By our choice of θ_0, the last equality gives

$$u(|z|e^{i\theta}) \geq (1-|z|)/2 - \|u\|_{\mathcal{C}^2(\mathbb{D})}(1-|z|)^2 \geq (1-|z|)/4, \tag{1.3.9}$$

for $|z| \geq 1 - 1/4\|u\|^{-1}_{\mathcal{C}^2(\mathbb{D})}$. When $|z| \leq 1 - 1/4\|u\|^{-1}_{\mathcal{C}^2(\mathbb{D})}$, we have $u(z) \geq c$ for some constant c independent of z. This combined with the fact that $(1-|z|) \leq 1$ implies that there is a positive constant c' for which $u(z) \geq c'(1-|z|)$ for $|z| \leq 1 - 1/4\|u\|^{-1}_{\mathcal{C}^2(\mathbb{D})}$. In summary, we can find a positive constant c' for which

$$u(z) \geq c'(1-|z|),$$

for $z = |z|e^{i\theta} \in \mathbb{D}$ with $\theta \in [-\theta_0, \theta_0]$.

Now let Ω be a simply connected subdomain of $\mathbb{D}$ with smooth boundary such that Ω is strictly convex and $\overline{\Omega} \cap \overline{\mathbb{D}} = [e^{-i\theta_0/2}, e^{i\theta_0/2}]$. By Painvelé's theorem (see, for example, [3, Th. 3.1] or [48, Th. 5.3.8]), there is a smooth diffeomorphism Φ from $\overline{\mathbb{D}}$ to $\overline{\Omega}$ which is a biholomorphism from $\mathbb{D}$ to Ω and $\Phi(1) = 1$. Define $u_0' := u \circ \Phi$ which is a smooth function on $\overline{\mathbb{D}}$ and harmonic on $\mathbb{D}$. We immediately have $u_0'(z) > 0$ on $\mathbb{D}$.

Since $\Phi(1) = 1$ and Φ sends $\partial\mathbb{D}$ to $\partial\Omega$, there is a small positive constant θ' such that $\Phi([e^{-i\theta_0'}, e^{i\theta_0'}])$ is contained in $[e^{-i\theta_0/2}, e^{i\theta_0/2}]$. This yields $u_0'(e^{i\theta}) = 0$ for $|\theta| \leq \theta_0'$ and $\mathrm{Re}^2\,\Phi(e^{i\theta}) + \mathrm{Im}^2\,\Phi(e^{i\theta}) = 1$ on $[e^{-i\theta_0'}, e^{i\theta_0'}]$. Differentiating the last inequality at $\theta' = 0$ gives

$$\mathrm{Re}\,\Phi(1)\partial_y \mathrm{Re}\,\Phi(1) + \mathrm{Im}\,\Phi(1)\partial_y \mathrm{Im}\,\Phi(1) = 0$$

which combined with $\Phi(1) = 1$ implies that $\partial_y \mathrm{Re}\,\Phi(1) = 0$. The last equality coupled with the fact that Φ is holomorphic implies

$$\det D_{(x,y)}\Phi(1) = \big(\partial_x \mathrm{Re}\,\Phi(1)\big)^2 + \big(\partial_y \mathrm{Re}\,\Phi(1)\big)^2 = \big(\partial_x \mathrm{Re}\,\Phi(1)\big)^2.$$

As a result, we have $\partial_x \mathrm{Re}\,\Phi(1) \neq 0$. On the other hand, since

$$|\Phi(1)|^2 = 1 = \max_{x\in[0,1]} |\Phi(x)|^2,$$

we have

$$0 \leq \partial_x |\Phi(x)|^2|_{x=1} = \mathrm{Re}\,\Phi(1)\partial_x \mathrm{Re}\,\Phi(1) + \mathrm{Im}\,\Phi(1)\partial_x \mathrm{Im}\,\Phi(1) = \partial_x \mathrm{Re}\,\Phi(1).$$

Hence, one gets $\partial_x\Phi(1) > 0$. Direct computations gives

$$\partial_x u_0'(1) = \partial_x u(1)\partial_x \mathrm{Re}\,\Phi(1) + \partial_y u(1)\partial_x \mathrm{Im}\,\Phi(1) = -\partial_x \mathrm{Re}\,\Phi(1) < 0.$$

Define $u_0 := u_0'/\partial_x \operatorname{Re}\Phi(1)$. We obtain $\partial_x u_0(1) = -1$ and $u_0(e^{i\theta}) = 0$ for $|\theta| \leq \theta_0'$. It remains to check that

$$u_0(z) \geq c''(1 - |z|), \tag{1.3.10}$$

for some constant $c'' > 0$. Since $u_0(z) > 0$ and $u(z) > 0$ on $\mathbb{D}$ and $\partial\Omega \cap \overline{\mathbb{D}} = [e^{-i\theta_0/2}, e^{i\theta_0/2}]$, it is enough to check (1.3.10) for z so that $w = \Phi(z)$ is close to $[e^{-i\theta_0/2}, e^{i\theta_0/2}]$. Let $w = \Phi(z) \in \Omega$ close to $[e^{-i\theta_0/2}, e^{i\theta_0/2}]$. By our choice of Ω, the axe Ow is transverse to $\partial\Omega$ at a unique point $w' = \Phi(z')$ for $z' \in \partial\mathbb{D}$. The $\mathcal{C}^1$- boundedness of Φ^{-1} imply that $|w - w'| \geq c_1|z - z'|$ for some constant c_1 independent of (z, z'). On the other hand, since $\Omega \subset \mathbb{D}$, we have $|w - w'| \leq 1 - |w|$. Hence,

$$1 - |w| \geq c_1|z - z'| \geq c_1(1 - |z|),$$

because $z' \in \partial\mathbb{D}$. Write $w = |w|e^{i\theta_w}$. Note that $\theta_w \in (\theta_0, \theta_0)$ if w is close enough to $[e^{-i\theta_0/2}, e^{i\theta_0/2}]$. We deduce that

$$u_0'(z) = u(\Phi(z)) = u(w) \geq c'(1 - |w|) \geq c'c_1(1 - |z|).$$

Hence, one gets (1.3.10). The proof is finished. □

We are now ready to introduce the Bishop equation which allows us to construct the promised family of analytic discs. Let u_0 be a function described in Lemma 1.3.2 and θ_{u_0} be the constant there. Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \overline{\mathbb{B}}_{n-1} \subset \mathbb{R}^{n-1}$. Define $\boldsymbol{\tau}_1^* := (1, \boldsymbol{\tau}_1) \in \mathbb{R}^n$ and $\boldsymbol{\tau}_2^* := (0, \boldsymbol{\tau}_1) \in \mathbb{R}^n$ and $\boldsymbol{\tau} := (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$. Let t be a positive number in $(0, 1)$ which plays a role as a scaling parameter in the equation (1.3.11) below.

In order to construct an analytic disc partly attached to K, it suffices to find a map

$$U : \partial\mathbb{D} \to \mathbb{B}_n \subset \mathbb{R}^n,$$

which is Hölder continuous, satisfying the following Bishop-type equation

$$U_{\boldsymbol{\tau},t}(\xi) = t\boldsymbol{\tau}_2^* - \mathcal{T}_1\big(h(U_{\boldsymbol{\tau},t})\big)(\xi) - t\mathcal{T}_1 u_0(\xi)\,\boldsymbol{\tau}_1^*, \tag{1.3.11}$$

Indeed, suppose that (1.3.11) has a solution. For simplicity, we use the same notation $U_{\boldsymbol{\tau},t}(z)$ to denote the harmonic extension of $U_{\boldsymbol{\tau},t}(\xi)$ to $\mathbb{D}$. Let $P_{\boldsymbol{\tau},t}(z)$ be the harmonic extension of $h\big(U_{\boldsymbol{\tau},t}(\xi)\big)$ to $\mathbb{D}$. Define

$$F(z, \boldsymbol{\tau}, t) := U_{\boldsymbol{\tau},t}(z) + iP_{\boldsymbol{\tau},t}(z) + it\,u_0(z)\,\boldsymbol{\tau}_1^*$$

which is a family of analytic discs parametrized by $(\boldsymbol{\tau}, t)$. For any $\xi \in [e^{-i\theta_{u_0}}, e^{i\theta_{u_0}}]$, the defining formula of F and the fact that $u_0 \equiv 0$ on $[e^{-i\theta_{u_0}}, e^{i\theta_{u_0}}]$ imply that

$$F(\xi, \boldsymbol{\tau}, t) = U_{\boldsymbol{\tau},t}(\xi) + iP_{\boldsymbol{\tau},t}(\xi) = U_{\boldsymbol{\tau},t}(\xi) + ih\big(U_{\boldsymbol{\tau},t}(\xi)\big) \in K.$$

In other words, F is $[e^{-i\theta_{u_0}}, e^{i\theta_{u_0}}]$-attached to K. In what follows, it is convenient to regard $U_{\boldsymbol{\tau},t}(z)$ as a function of the variable $(z, \boldsymbol{\tau})$.

Proposition 1.3.3. *There are a positive number $t_1 \in (0,1)$ and a real number $c_1 > 0$ satisfying the following property. For any $t \in (0,t_1]$ and any $\boldsymbol{\tau} \in \overline{\mathbb{B}}_{n-1}^2$, the equation (1.3.11) has a unique solution $U_{\boldsymbol{\tau},t}$ which is $\mathcal{C}^{2,\frac{1}{2}}$ in $(\xi,\boldsymbol{\tau})$ and such that*

$$\|D^j_{(\xi,\boldsymbol{\tau})} U_{\boldsymbol{\tau},t}\|_{\mathcal{C}^{\frac{1}{2}}(\partial\mathbb{D})} \leq c_1 t, \tag{1.3.12}$$

for any $\boldsymbol{\tau} \in \overline{\mathbb{B}}_{n-1}^2$ and $j = 0, 1$ or 2, where $D_{(\xi,\boldsymbol{\tau})}$ is the differential with respect to both $(\xi,\boldsymbol{\tau})$ and $D^2_{(\xi,\boldsymbol{\tau})} := D_{(\xi,\boldsymbol{\tau})} \circ D_{(\xi,\boldsymbol{\tau})}$.

Proof. This is a direct consequence of a general result due to Tumanov, see [59, Th. 4.19] or see [82, Pro. 4.2] for a more simple proof adapted to our present situation. □

Let $U_{\boldsymbol{\tau},t}$ be the unique solution of (1.3.11). As above we also use $U_{\boldsymbol{\tau},t}(z)$ to denote its harmonic extension to $\mathbb{D}$. Let $F(z,\boldsymbol{\tau},t)$ and $P_{\boldsymbol{\tau},t}$ be as above. Our goal is to study the behaviour of the image of the family $F(\cdot,\boldsymbol{\tau},t)$ near K, or in other words when z is close to $[e^{-i\theta_{u_0}}, e^{i\theta_{u_0}}] \subset \partial\mathbb{D}$.

Lemma 1.3.4. *There exists a constant c_2 so that for every $t \in (0,t_1]$ and every $(z,\boldsymbol{\tau}) \in \overline{\mathbb{D}} \times \overline{\mathbb{B}}_{n-1}^2$, we have*

$$\|D^j_{(z,\boldsymbol{\tau})} U_{\boldsymbol{\tau},t}(z)\| \leq c_2 t \quad \textit{and} \quad \|D^j_{(z,\boldsymbol{\tau})} P_{\boldsymbol{\tau},t}(z)\| \leq c_2 t^2, \tag{1.3.13}$$

for $j = 0, 1, 2$.

Proof. In view of (1.3.5) and (1.3.12), the first inequality of (1.3.13) is obvious and for the second one, it is enough to estimate the $\mathcal{C}^{1/2}(\partial\mathbb{D})$-norms of $D^j_{(\xi,\boldsymbol{\tau})} P_{\boldsymbol{\tau},t}(\xi)$ for $j = 0, 1, 2$. Since $P_{\boldsymbol{\tau},t}(\xi) = h\big(U_{\boldsymbol{\tau},t}(\xi)\big)$ on $\partial\mathbb{D}$, we have

$$\partial_\xi P_{\boldsymbol{\tau},t}(\xi) = Dh\big(U_{\boldsymbol{\tau},t}(\xi)\big)\partial_\xi U_{\boldsymbol{\tau},t}(\xi).$$

This combined with (1.3.1) and (1.3.12) yields that

$$\|\partial_\xi P'_{\mathbf{z},t,\boldsymbol{\tau}}\|_{\mathcal{C}^{1/2}(\partial\mathbb{D})} \leq c_0 \|U_{\boldsymbol{\tau},t}\|_{\mathcal{C}^{1/2}(\partial\mathbb{D})} \|\partial_\xi U_{\boldsymbol{\tau},t}\|_{\mathcal{C}^{1/2}(\partial\mathbb{D})} \leq c_0 c_1 t^2.$$

By similar arguments, we also have $|\partial^j_\xi P_{\boldsymbol{\tau}}(\xi)| \lesssim t^2$ with $j = 0, 2$. To deal with the other partial derivatives, observe that for $0 \leq j \leq 2$, $D^j_{\boldsymbol{\tau}} P_{\boldsymbol{\tau},t}$ is the harmonic extension of $D^j_{\boldsymbol{\tau}} h\big(U_{\boldsymbol{\tau},t}(\cdot)\big)$ to $\mathbb{D}$. Hence, in order to estimate $D^k_z D^j_{\boldsymbol{\tau}} P_{\boldsymbol{\tau},t}$ for $0 \leq k, j \leq 2$, we can apply the same reasoning as above. Thus the proof is finished. □

Proposition 1.3.5. *There are three constants $t_2 \in (0,t_1]$, $\theta_0 \in (0,\theta_{u_0})$ and $\epsilon_0 > 0$ such that for any $\boldsymbol{\tau}_1 \in \overline{\mathbb{B}}_{n-1}$ and $t \in (0,t_2]$ the map $F(\cdot,\boldsymbol{\tau}_1,t) : [e^{-i\theta_0}, e^{i\theta_0}] \times \overline{\mathbb{B}}_{n-1} \to K$ is a diffeomorphism onto its image which contains the graph of h over $\mathbb{B}_n(0,t\epsilon_0)$.*

Proof. By Cauchy-Riemann equations, we have

$$\partial_y U_{\boldsymbol{\tau},t}(1) = -t\partial_x u_0(1)\boldsymbol{\tau}_1^* - \partial_x P_{\boldsymbol{\tau},t}(1) = t\boldsymbol{\tau}_1^* - \partial_x P_{\boldsymbol{\tau},t}(1).$$

The last term is $O(t^2)$ by Lemma 1.3.4. Thus the first component of $\partial_y U_{\boldsymbol{\tau},t}(1)$ is greater than $t/2$ provided that $t \leq t_2$ small enough. A direct computation gives $\partial_y U_{\boldsymbol{\tau},t}(1) = \partial_\theta U_{\boldsymbol{\tau},t}(1)$. Consequently, the first component of $\partial_\theta U_{\boldsymbol{\tau},t}(1)$ is greater than $t/2$ for $t \leq t_2$.

On the other hand, by (1.3.11), we have $U_{\boldsymbol{\tau},t}(1) = t\boldsymbol{\tau}_2^*$ which implies $\partial_{\boldsymbol{\tau}_2} U_{\boldsymbol{\tau},t}(1)$ is a $(n, n-1)$ matrix whose the fist row is 0 and the other rows form the identity matrix. Combining with the above argument shows that $D_{\boldsymbol{\tau}_2,\theta} U_{\boldsymbol{\tau},t}(1)$ is a nondegenerate matrix. This coupled with the fact that $F(e^{i\theta}, \boldsymbol{\tau}_1, t) = U_{\boldsymbol{\tau},t}(e^{i\theta})$ for $\theta \in [-\theta_0, \theta_0]$ implies the desired result. The existence of ϵ_0 is obvious. The proof is finished. □

For $a \in \mathbb{C}^n$ and $A \subset \mathbb{C}^n$, $\operatorname{dist}(a, A)$ denotes the distance from a to A.

Proposition 1.3.6. *There are two constants $t_3 \in (0, t_2]$, $r_0 > 0$ such that for every $t \in (0, t_3)$, the restriction F_1 of F to $\big(\mathbb{B}_2(1, r_0) \cap \mathbb{D}\big) \times \overline{\mathbb{B}}_{n-1}^2$ is a diffeomorphism onto its image and for any $(z, \boldsymbol{\tau})$, we have*

$$\big|\det DF_1(z, \boldsymbol{\tau}, t)\big| \gtrsim t^{2n}\big[1 - |z|\big]^{n-1} \tag{1.3.14}$$

and

$$t(1 - |z|) \lesssim \operatorname{dist}\big(F_1(z, \boldsymbol{\tau}, t), K'\big) \lesssim t(1 - |z|). \tag{1.3.15}$$

Proof. Let r_0, t_3 be two positive small constants to be chosen later. For the moment, we take r_0 to be small enough so that if $z = |z|e^{i\theta} \in \mathbb{B}_2(1, r_0) \cap \mathbb{D}$, then $\theta \in (\theta_0, \theta_0)$, thus we have $u_0(e^{i\theta}) = 0$. Fix a constant $t \in (0, t_3]$. Provided that t_3 and r_0 are small enough we will prove in the order (1.3.15), (1.3.14) and finally that F_1 is a diffeomorphism. Extend h to be a $\mathcal{C}^3$ function on $\mathbb{R}^n$. Let $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\Psi(\mathbf{x} + i\mathbf{y}) := \mathbf{x} + i\mathbf{y} - ih(\mathbf{x}).$$

One can see without difficulty that Ψ is a diffeomorphism sending K' to $\mathbb{B}_n$, where we embed

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n.$$

Let $F_1' := \Psi \circ F_1$. We have

$$\operatorname{Im} F_1'(z, \boldsymbol{\tau}, t) = P_{\boldsymbol{\tau},t}(z) - h\big(U_{\boldsymbol{\tau},t}(z)\big) + tu_0(z)\boldsymbol{\tau}_1^* \quad \text{and} \quad \operatorname{Re} F_1'(z, \boldsymbol{\tau}, t)(z) = U_{\boldsymbol{\tau},t}(z). \tag{1.3.16}$$

By the above property of Ψ, it suffices to prove the required property for $(F_1', \mathbb{B}_n)$ in place of (F_1, K'). Note that $P_{\boldsymbol{\tau},t}(z)$ and $h\big(U_{\boldsymbol{\tau},t}(z)\big)$ are identical on $\partial\mathbb{D}$. This together with (1.3.13) yields

$$P_{\boldsymbol{\tau},t}(z) - h\big(U_{\boldsymbol{\tau},t}(z)\big) = t^2(1 - |z|)R_0(z, \boldsymbol{\tau}, t), \tag{1.3.17}$$

where $R_0(z, \boldsymbol{\tau}, t)$ is $\mathcal{C}^1$ in $(z, \boldsymbol{\tau})$ so that $\|R_0(\cdot, t)\|_{\mathcal{C}^1} \lesssim 1$. Remember that t is fixed, so we do not consider it as a variable when taking the $\mathcal{C}^1$ norm. On the other hand, by our choice of u_0 and Lemma 1.3.2, one has $u_0(z) \gtrsim (1 - |z|)$. By this and (1.3.17) and (1.3.16), we obtain

$$\begin{aligned}\operatorname{dist}\big(F_1(z, \boldsymbol{\tau}, t), K'\big) &\gtrsim \operatorname{dist}\big(F_1'(z, \boldsymbol{\tau}, t), \mathbb{R}^n\big) \\ &= |\operatorname{Im} F_1'(z, \boldsymbol{\tau}, t)| \gtrsim t(1 - |z|)|\boldsymbol{\tau}_1^*| - t^2(1 - |z|).\end{aligned}$$

Thus if t is sufficiently small, the first inequality of (1.3.15) follows.

For t_3 small enough, $U_{\boldsymbol{\tau},t}(z) \in \mathbb{B}_n$. Hence, we get

$$\operatorname{dist}\big(F_1(z,\boldsymbol{\tau},t), K'\big) \lesssim \operatorname{dist}\big(F_1'(z,\boldsymbol{\tau},t), \mathbb{B}_n\big) \lesssim |\operatorname{Im} F_1'(z,\boldsymbol{\tau},t)|.$$

Write $z = |z|e^{i\theta} \in \mathbb{B}_2(1,r_0) \cap \mathbb{D}$. Hence $\theta \in [-2r_0, 2r_0] \subset (\theta_0, \theta_0)$ if r_0 is small enough. Since $u_0(e^{i\theta}) = 0$, we deduce from (1.3.8) that

$$u_0(z) = (1-|z|) + \theta(1-|z|)R_1(z) + (1-|z|)^2 R_2(z), \tag{1.3.18}$$

where R_j is smooth function with $\|R_j\|_{\mathcal{C}^1} \lesssim 1$ for $j = 1, 2$. Put $\epsilon := \max\{2r_0, t\}$. We choose (t, r_0) to be so small that $\epsilon < 1/2$. Put

$$T_0(z,\boldsymbol{\tau},t) := tR_0(z,\boldsymbol{\tau},t) + \big(\theta R_1(z) + (1-|z|)R_2(z)\big)\boldsymbol{\tau}_1^* \tag{1.3.19}$$

which satisfies

$$\|T_0\|_{\mathcal{C}^0} \lesssim \epsilon, \quad \|D_{\boldsymbol{\tau}} T_0\|_{\mathcal{C}^0} \lesssim \epsilon \tag{1.3.20}$$

because $|\theta| \leq 2r_0$ and $1 - |z| \leq r_0$. Combining (1.3.18), (1.3.17) and (1.3.16) gives

$$\operatorname{Im} F_1'(z,\boldsymbol{\tau},t) = t(1-|z|)\big[\boldsymbol{\tau}_1^* + T_0(z,\boldsymbol{\tau},t)\big]. \tag{1.3.21}$$

Consequently, using (1.3.20) we obtain

$$|\operatorname{Im} F_1'(z,\boldsymbol{\tau},t)| \lesssim t(1-|z|)$$

which proves the second inequality of (1.3.15).

By (1.3.11) and the Cauchy-Riemann equations, we have $U_{\boldsymbol{\tau},t}(1) = t\boldsymbol{\tau}_2^*$ and

$$\partial_y U_{\boldsymbol{\tau},t}(z) = -\partial_x P_{\boldsymbol{\tau},t}(z) - t\partial_x u_0(z)\boldsymbol{\tau}_1^*$$

and

$$\partial_x U_{\boldsymbol{\tau},t}(z) = \partial_y P_{\boldsymbol{\tau},t}(z) + t\partial_y u_0(z)\boldsymbol{\tau}_1^*.$$

Observe that

$$\partial_\theta U_{\boldsymbol{\tau},t}(e^{i\theta}) = -\partial_x U_{\boldsymbol{\tau},t}(e^{i\theta})\sin\theta + \partial_y U_{\boldsymbol{\tau},t}(e^{i\theta})\cos\theta.$$

These above equalities combined with (1.3.13) and Taylor's expansion to $U_{\boldsymbol{\tau},t}(e^{i\theta})$ at $\theta = 0$ gives

$$U_{\boldsymbol{\tau},t}(e^{i\theta}) = t\boldsymbol{\tau}_2^* + t^2 R_3(\theta,\boldsymbol{\tau},t) + t\theta\boldsymbol{\tau}_1^* + t\theta^2 R_4(\theta)\boldsymbol{\tau}_1^*, \tag{1.3.22}$$

where

$$R_3(\theta,\boldsymbol{\tau},t) := \int_0^\theta \big[\partial_y P_{\boldsymbol{\tau},t}(e^{is})\cos s - \partial_x P_{\boldsymbol{\tau},t}(e^{is})\sin s\big]ds$$

which is of $\mathcal{C}^1$ norm $\lesssim 1$, and $R_4(\theta)$ is a $\mathcal{C}^1$ function satisfying $\|R_4\|_{\mathcal{C}^1} \lesssim 1$. Remark that in (1.3.22), we used the $\mathcal{C}^3$ differentiability of u_0 and R_4 comes from the remainder of the Taylor expansion of u_0 at 1 up to the order 2.

Using (1.3.22), Taylor's expansion for $\operatorname{Re} F_1'(z,\boldsymbol{\tau},t)$ at $\tilde{z} = e^{i\theta}$ implies

$$\operatorname{Re} F_1'(z,\boldsymbol{\tau},t) = t\boldsymbol{\tau}_2^* + t\theta\boldsymbol{\tau}_1^* + t^2 R_3(\theta,\boldsymbol{\tau},t) + t\theta^2 R_4(\theta)\boldsymbol{\tau}_1^* + t(1-|z|)R_5(z,\boldsymbol{\tau},t), \tag{1.3.23}$$

for some $\mathcal{C}^1$ function $R_5(z,\boldsymbol{\tau},t)$ with $\|R_j(\cdot,t)\|_{\mathcal{C}^1}\lesssim 1$. Define

$$T_1(z,\boldsymbol{\tau},t):=tR_3(\theta,\boldsymbol{\tau},t)+\theta^2R_4(\theta)\boldsymbol{\tau}_1^*+(1-|z|)R_5(z,\boldsymbol{\tau},t), \tag{1.3.24}$$

which satisfies

$$\|D_{\boldsymbol{\tau},\theta}T_1\|_{\mathcal{C}^0}\lesssim\epsilon, \tag{1.3.25}$$

where we use the polar coordinate $(|z|,\theta)$ for z. Combining (1.3.23), (1.3.25), (1.3.21) and (1.3.20) gives (1.3.14).

Let $\boldsymbol{\rho}=t\boldsymbol{\rho}_2+i\,t\boldsymbol{\rho}_1$ be an arbitrary point in the image of F'. This means that

$$\boldsymbol{\rho}=F_1'(z^0,\boldsymbol{\tau}^0,t), \tag{1.3.26}$$

for some $(z^0,\boldsymbol{\tau}^0)$. Let $\theta^0\in(-\pi/2,\pi/2)$ be the argument of z^0. Then $z^0=|z^0|e^{i\theta^0}$. We will prove that the equation

$$F_1'(z,\boldsymbol{\tau},t)=\boldsymbol{\rho} \tag{1.3.27}$$

has a unique solution, *i.e* F_1' is injective. The equation (1.3.27) is equivalent to the system of the two following equations

$$\operatorname{Re}F_1'(z,\boldsymbol{\tau},t)=t\boldsymbol{\rho}_2 \tag{1.3.28}$$

and

$$\operatorname{Im}F_1'(z,\boldsymbol{\tau},t)=t\boldsymbol{\rho}_1. \tag{1.3.29}$$

Write $T_j=(T_{j1},\cdots,T_{jn})$ for $j=0$ or 1 and $\boldsymbol{\rho}_j=(\rho_{j1},\cdots,\rho_{jn})$ for $j=1,2$. Define

$$\tilde{\boldsymbol{\rho}}_1:=\frac{\boldsymbol{\rho}_1}{1-|z|}.$$

We also write $\tilde{\boldsymbol{\rho}}_1=(\tilde\rho_{11},\cdots,\tilde\rho_{1n})$. Recall that $\boldsymbol{\tau}_j^*=(1,\boldsymbol{\tau}_j)$ for $j=1$ or 2 and $\boldsymbol{\tau}_j=(\tau_{j1},\cdots,\tau_{j(n-1)})$. We have

$$\tilde\rho_{1k}-\tilde\rho_{11}\,\frac{\rho_{1k}}{\rho_{11}}=0, \tag{1.3.30}$$

for $2\le k\le n$. The variable $(\tilde\rho_{11},\theta)$ will be used as a substitute for z. If $(z,\boldsymbol{\tau},t)$ is a solution of (1.3.27), identifying the first component of (1.3.21) and (1.3.29) imply

$$1+T_{01}(z,\boldsymbol{\tau},t)=\tilde\rho_{11}$$

which in turn yields $|\tilde\rho_{11}-1|\lesssim\epsilon$ by (1.3.20). Hence if $(z,\boldsymbol{\tau},t)$ is a solution of (1.3.27), we get

$$1/2\le\tilde\rho_{11}\le 3/2. \tag{1.3.31}$$

By (1.3.29) again and the fact that $\boldsymbol{\tau}_1\in\mathbb{B}_{n-1}$, one also gets

$$\Big|\frac{\rho_{1k}}{\rho_{11}}\Big|\approx|\tau_{1(k-1)}|\le 3/2, \tag{1.3.32}$$

for $2 \leq k \leq n$. Since $z = |z|e^{i\theta}$, we have

$$z = \big(1 - \frac{\rho_{11}}{\tilde{\rho}_{11}}\big)e^{i\theta}.$$

From now on, we will consider T_0, T_1 as functions of $(\tilde{\rho}_{11}, \theta, \boldsymbol{\tau})$. Define

$$G = (G_1, G_2, G_3) : \overline{\mathbb{B}}^2_{n-1} \times [\frac{1}{2}, \frac{3}{2}] \times \mathbb{R}^{n-1} \times [-2r_0, 2r_0] \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n-1}$$

by putting

$$G_1(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta) := \boldsymbol{\tau}_1^* + T_0(\theta, \tilde{\rho}_{11}, \boldsymbol{\tau}, t) - \tilde{\boldsymbol{\rho}}_1, \quad G_2(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta) := \boldsymbol{\tau}_2^* + \theta\boldsymbol{\tau}_1^* + T_1(\theta, \tilde{\rho}_{11}, \boldsymbol{\tau}, t) - \boldsymbol{\rho}_2$$

and

$$G_3(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta) := \big(\tilde{\rho}_{12} - \tilde{\rho}_{11}\,\frac{\rho_{1k}}{\rho_{11}}, \cdots, \tilde{\rho}_{1n} - \tilde{\rho}_{11}\,\frac{\rho_{1k}}{\rho_{11}}\big).$$

By (1.3.30), (1.3.23) and (1.3.21), resolving the system (1.3.28)-(1.3.29) is equivalent to finding $(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta)$ for which

$$G(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta) = 0. \tag{1.3.33}$$

By (1.3.26), we know that $\mathbf{a}^0 := (\boldsymbol{\tau}^0, \tilde{\boldsymbol{\rho}}_1^0, \theta^0)$ is a solution of (1.3.33), where

$$\tilde{\boldsymbol{\rho}}_1^0 := \frac{\boldsymbol{\rho}_1}{1 - |z^0|}.$$

Suppose that there is an another solution $\mathbf{a} = (\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta)$ of (1.3.33). By a direct computation, one gets

$$\partial_{\tilde{\rho}_{11}}(1 - |z|) = -\frac{\rho_{11}}{\tilde{\rho}_{11}^2} = -(1 - |z|)\tilde{\rho}_{11}^{-1} = O(1 - |z|) \lesssim \epsilon$$

by (1.3.31). This coupled with (1.3.19) and (1.3.24) yields

$$|T_0(\mathbf{a}, t) - T_0(\mathbf{a}^0, t)| \lesssim \epsilon|\mathbf{a} - \mathbf{a}^0| + |\theta - \theta'|. \tag{1.3.34}$$

and

$$|T_1(\mathbf{a}, t) - T_1(\mathbf{a}^0, t)| \lesssim \epsilon|\mathbf{a} - \mathbf{a}^0|. \tag{1.3.35}$$

Using (1.3.35) and identifying the first component of the equation $G_2(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta) = 0$ imply

$$|\theta - \theta^0| \leq |T_1(\mathbf{a}, t) - T_1(\mathbf{a}^0, t)| \lesssim \epsilon|\mathbf{a} - \mathbf{a}^0|. \tag{1.3.36}$$

By doing the same thing for $G_1(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta) = 0$ and using (1.3.36), we also obtain

$$|\tilde{\rho}_{11} - \tilde{\rho}_{11}^0| \leq |T_0(\mathbf{a}, t) - T_0(\mathbf{a}^0, t)| \lesssim \epsilon|\mathbf{a} - \mathbf{a}^0|.$$

Using the last inequality, the equality $G_3(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta) = 0$ and (1.3.32), one infers

$$|\tilde{\boldsymbol{\rho}}_1 - \tilde{\boldsymbol{\rho}}_1^0| \lesssim |\tilde{\rho}_{11} - \tilde{\rho}_{11}^0| \lesssim \epsilon|\mathbf{a} - \mathbf{a}^0|. \tag{1.3.37}$$

Similarly, using $G_1(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta) = 0$ gives

$$|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_1^0| \leq |T_0(\mathbf{a}, t) - T_0(\mathbf{a}^0, t)| + |\tilde{\boldsymbol{\rho}}_1 - \tilde{\boldsymbol{\rho}}_1^0| \lesssim \epsilon |\mathbf{a} - \mathbf{a}^0|. \tag{1.3.38}$$

Finally, using $G_2(\boldsymbol{\tau}, \tilde{\boldsymbol{\rho}}_1, \theta) = 0$ gives

$$|\boldsymbol{\tau}_2 - \boldsymbol{\tau}_2^0| \lesssim \epsilon |\mathbf{a} - \mathbf{a}^0|. \tag{1.3.39}$$

Summing the inequalities from (1.3.36) to (1.3.39) and taking into account that

$$|\mathbf{a} - \mathbf{a}^0| \leq |\boldsymbol{\tau}_2 - \boldsymbol{\tau}_2^0| + |\boldsymbol{\tau}_1 - \boldsymbol{\tau}_1^0| + |\tilde{\boldsymbol{\rho}}_1 - \tilde{\boldsymbol{\rho}}_1^0| + |\theta - \theta^0|$$

show that $\mathbf{a} = \mathbf{a}^0$. This means that (1.3.33) has a unique solution, or equivalently, so does (1.3.27) if r_0 and t are small enough. The proof is finished. □

Let Ω be a simply connected subdomain of $\mathbb{D}$ with smooth boundary such that Ω is strictly convex and $\overline{\Omega} \cap \overline{\mathbb{D}} = [e^{-i\theta_1}, e^{i\theta_1}]$ for some $\theta_1 \in (0, \theta_0)$ and $\overline{\Omega} \subset \mathbb{B}_2(1, r_0)$. By Painvelé's theorem as in the proof of Lemma 1.3.2, there is a smooth diffeomorphism Φ from $\overline{\mathbb{D}}$ to $\overline{\Omega}$ which is a biholomorphism from $\mathbb{D}$ to Ω and $\Phi(1) = 1$.

Define $\tilde{F}(z, \boldsymbol{\tau}, t) := F\big(\Phi(z), \boldsymbol{\tau}, t\big)$ which is again a $\mathcal{C}^{2,1/2}$ family of analytic discs partly attached to K.

Proposition 1.3.7. (i) *There are positive constants $\tilde{\theta}_0$ and $\tilde{\epsilon}_0$ so that for every $\boldsymbol{\tau}_1 \in \overline{\mathbb{B}}_{n-1}$ and $t \in (0, t_3]$, the restriction map $\tilde{F}(\cdot, \boldsymbol{\tau}_1, t) : [e^{-i\tilde{\theta}_0}, e^{i\tilde{\theta}_0}] \times \overline{\mathbb{B}}_{n-1} \to K'$ is a diffeomorphism onto its image which contains the graph of h over $\mathbb{B}_n(0, t\tilde{\epsilon}_0)$.*

(ii) *Let t_3 be the constant in Proposition 1.3.6. Then for any $t \in (0, t_3]$, the map $\tilde{F}(\cdot, t)$ is a diffeomorphism from $\mathbb{D} \times \overline{\mathbb{B}}_{n-1}^2$ onto its image in $\mathbb{D}^n \subset \mathbb{C}^n$, and for any $(z, \boldsymbol{\tau})$ we have*

$$\big|\det D\tilde{F}(z, \boldsymbol{\tau}, t)\big| \gtrsim t^{n+1} \operatorname{dist}^{n-1}\big(\tilde{F}(z, \boldsymbol{\tau}, t), K'\big) \tag{1.3.40}$$

and

$$t(1 - |z|) \lesssim \operatorname{dist}\big(\tilde{F}(z, \boldsymbol{\tau}, t), K'\big). \tag{1.3.41}$$

Proof. Property (i) is a direct consequence of Propositions 1.3.5. By the differentiability of Φ^{-1} on $\overline{\Omega}$, we have $(1 - |\Phi(z)|) \gtrsim 1 - |z|$ for every $z \in \mathbb{D}$. Hence, by (1.3.15), we get (1.3.41). The inequality (1.3.40) follows immediately from the fact that

$$\big|\det DF_1(z, \boldsymbol{\tau}, t)\big| \gtrsim t^{n+1} \operatorname{dist}^{n-1}\big(F_1(z, \boldsymbol{\tau}, t), K'\big)$$

which is in turn implied by (1.3.15) and (1.3.14). The proof is finished. □

Using the local coordinates of K at the beginning of this section and choosing $t = t_3$, the last proposition can be rephrased as follows.

Proposition 1.3.8. *Let a be an arbitrary point of K. Then there exist positive constants $\tilde{\epsilon}_a, \tilde{\theta}_a$ and a $\mathcal{C}^{2,1/2}$ diffeomorphism $\tilde{F}_a : \mathbb{D} \times \overline{\mathbb{B}}_{n-1}^2 \to X$ onto its image such that the two following properties hold:*

(i) *for every $\boldsymbol{\tau}_1 \in \overline{\mathbb{B}}_{n-1}$, the restriction map $\tilde{F}_a(\cdot, \boldsymbol{\tau}_1) : [e^{-i\tilde{\theta}_a}, e^{i\tilde{\theta}_a}] \times \mathbb{B}_{n-1} \to K$ is a diffeomorphism onto its image which contains the graph of h over $\mathbb{B}_K(a, \tilde{\epsilon}_a)$.*

(ii) there is an open relatively compact neighborhood K'_a of a in K such that for any $(z, \boldsymbol{\tau})$, we have

$$\big|\det D\tilde{F}_a(z,\boldsymbol{\tau})\big| \gtrsim \operatorname{dist}^{n-1}\big(\tilde{F}_a(z,\boldsymbol{\tau}), K'_a\big) \tag{1.3.42}$$

and

$$(1-|z|) \lesssim \operatorname{dist}\big(\tilde{F}_a(z,\boldsymbol{\tau}), K'_a\big). \tag{1.3.43}$$

1.4 Some estimates for p.s.h. functions

In this section, we will prove some key estimates for p.s.h functions and their dd^c on $\mathbb{C}^n$. For a Borel subset A of $\mathbb{R}^m$ with $m \in \mathbb{N}$, denote by $|A|$ the volume of A with respect to the canonical volume form $vol_{\mathbb{R}^m}$. In what follows, for simplicity, we will write $\int_A f$ instead of $\int_A f dvol_{\mathbb{R}^m}$ for every Borel set $A \subset \mathbb{R}^m$ and every integrable function f on A. In particular, this convention is applied to $\mathbb{C}^n = \mathbb{R}^{2n}$.

Lemma 1.4.1. *Let V be an open subset of $\mathbb{C}^n$ and let V_1 be a compact subset of V. Let φ be a p.s.h. function on V. Then there exists a constant c independent of φ such that for any Borel set $V_2 \subset V_1$, we have*

$$\int_{V_2} |\varphi| \le c|V_2| \max\{1, -\log|V_2|\} \int_V |\varphi|. \tag{1.4.1}$$

Proof. If $\varphi \equiv 0$ or $\int_V |\varphi| = \infty$, then there is nothing to prove. Now suppose $\varphi \neq 0$ and $\int_V |\varphi| < \infty$. Let $\varphi_1 = \varphi / \int_V |\varphi|$. We have $\int_V |\varphi_1| = 1$. As a result, there exist two positive constants (c_1, α_1) independent of φ_1 for which

$$\int_{V_1} e^{\alpha_1|\varphi_1|} \le c_1. \tag{1.4.2}$$

Let $\mathbf{1}_{V_2}$ be the characteristic function of V_2. Let $\mu := |V_2|^{-1}\mathbf{1}_{V_2} vol_{\mathbb{C}^n}$ which is a probability measure supported in V_2. We have

$$\int_{V_2} |\varphi_1| = \alpha_1^{-1} \int_{V_2} \log e^{\alpha_1|\varphi_1|} = \alpha_1^{-1}|V_2| \int_{V_2} \log e^{\alpha_1|\varphi_1|}\, d\mu.$$

This together with the concavity of the $\log$ function implies

$$\int_{V_2} |\varphi_1| \le \alpha_1^{-1}|V_2| \log \int_{V_2} e^{\alpha_1|\varphi_1|}\, d\mu$$

which, by (1.4.2), is less than or equal to

$$\alpha_1^{-1}|V_2|\big(\log c_1 - \log|V_2|\big).$$

Hence (1.4.1) follows. The proof is finished. □

Now let h, K' be as in Section 1.3. Let ϵ be a real positive number and K'_ϵ the compact subset of $\mathbb{C}^n$ consisting of points of distance $\le \epsilon$ to K'. Obviously, the volume of K'_ϵ is $\lesssim \epsilon^n$. Using Lemma 1.4.1 for $V_2 = K'_\epsilon$, we get the following.

Corollary 1.4.2. *Let V be an open subset of $\mathbb{C}^n$ containing H_1. Let φ be a p.s.h. function on V. Then there is a constant c independent of φ for which*

$$\int_{K'_\epsilon} |\varphi| \leq c\epsilon^n |\log \epsilon| \int_V |\varphi| \tag{1.4.3}$$

for every $\epsilon \leq 1/2$.

Now we will give a similar estimate for the mass of $dd^c\varphi$ on K'_ϵ. We begin with a general result.

Lemma 1.4.3. *Let V, V_1, V_2 be open subsets of $\mathbb{C}^n$ such that $V_2 \Subset V_1 \Subset V$. Let T be a closed positive current of bidimension (p,p) on V and λ a real number > 1. Let φ and ρ be two bounded p.s.h. functions on V. Let A be a subset of V_2 and $a_{\epsilon,\varphi}$ be an upper bound for $|\varphi|$ on $V_1 \cap \{\rho \leq \epsilon\}$ for $\epsilon > 0$. Assume that ρ is bounded by 1 on V. Then there is a constant c independent of T, A, ρ, φ such that*

$$\int_{A\cap\{\rho\leq\epsilon\}} T \wedge (dd^c\varphi)^p \leq c[\epsilon^{-1}a_{\lambda\epsilon,\varphi}]^p \|T\|_{V_1}, \tag{1.4.4}$$

for every $\epsilon \in (0,1)$.

Proof. We prove (1.4.4) by induction on p. When $p = 0$, the conclusion is obvious. Suppose (1.4.4) holds for $p-1$. We need to prove its validity for p. Let χ be a smooth function compactly supported in some $V'_1 \Subset V_1$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on V_2. Let ϵ be a positive constant. Choose a constant $\lambda' \in (1, \lambda)$. Define

$$\rho_\epsilon := \max\{\rho, \lambda'\epsilon\} - \max\{\rho, \epsilon\}$$

which is the difference of two bounded p.s.h. functions on V. Clearly, we have $0 \leq \rho_\epsilon \leq (\lambda' - 1)\epsilon$ and $\rho_\epsilon = (\lambda' - 1)\epsilon$ on $\{\rho \leq \epsilon\}$ and $\rho_\epsilon = 0$ on $\{\rho \geq \lambda'\epsilon\}$. This yields

$$\int_{A\cap\{\rho\leq\epsilon\}} T \wedge (dd^c\varphi)^p \leq \frac{\epsilon^{-1}}{\lambda' - 1} \int_V \chi\rho_\epsilon T \wedge (dd^c\varphi)^p \lesssim \epsilon^{-1} \int_V \chi\rho_\epsilon T \wedge (dd^c\varphi)^p \tag{1.4.5}$$

which is, by integration by parts, equal to

$$\epsilon^{-1} \int_V \rho_\epsilon\varphi dd^c\chi \wedge T \wedge (dd^c\varphi)^{p-1} + \epsilon^{-1} \int_V \varphi\chi dd^c\rho_\epsilon \wedge T \wedge (dd^c\varphi)^{p-1} + R, \tag{1.4.6}$$

where

$$R = 2\epsilon^{-1} \int_V \varphi d\chi \wedge d^c\rho_\epsilon \wedge T \wedge (dd^c\varphi)^{p-1}.$$

Denote by R_1 and R_2 respectively the first and second terms in (1.4.6). We are now going to estimate R_1, R and finally R_2. Let ω be the canonical Kähler form on $\mathbb{C}^n$. Since $dd^c\chi \lesssim \omega$ and $|\varphi| \leq a_{2\epsilon,\varphi}$ on $\mathrm{supp}\rho_\epsilon$, we get

$$R_1 \leq \epsilon^{-1} a_{\lambda'\epsilon,\varphi} \int_{V'_1\cap\{\rho\leq\lambda'\epsilon\}} \omega \wedge T \wedge (dd^c\varphi)^{p-1}.$$

Applying the induction hypothesis to $\omega \wedge T, \lambda'\epsilon$ in place of T, ϵ implies

$$R_1 \leq \epsilon^{-1} a_{\lambda'\epsilon,\varphi} \int_{V_1' \cap \{\rho \leq \lambda'\epsilon\}} \omega \wedge T \wedge (dd^c\varphi)^{p-1} \lesssim [\epsilon^{-1} a_{\lambda\epsilon,\varphi}]^p. \tag{1.4.7}$$

As to R, the Cauchy-Schwarz inequality applied to a suitable scalar product gives

$$\begin{aligned} |R|^2 &\leq \epsilon^{-2} \int_{V_1'} |\varphi| \mathbf{1}_{\{\rho \leq \lambda'\epsilon\}} d\chi \wedge d^c\chi \wedge T \wedge (dd^c\varphi)^{p-1} \int_{V_1'} |\varphi| d\rho_\epsilon \wedge d^c\rho_\epsilon \wedge T \wedge (dd^c\varphi)^{p-1} \\ &\lesssim [\epsilon^{-1} a_{\lambda'\epsilon,\varphi}]^{p+1} \int_{V_1'} d\rho_\epsilon \wedge d^c\rho_\epsilon \wedge T \wedge (dd^c\varphi)^{p-1} \end{aligned} \tag{1.4.8}$$

by induction hypothesis and the fact that $d\rho_\epsilon \wedge d^c\rho_\epsilon$ is positive and supported on $\{\rho \leq \lambda'\epsilon\}$. Denote by R' the last integral. Since ρ_ϵ is the difference of two bounded p.s.h. functions on V, so is ρ_ϵ^2. More precisely, since $|\rho| \leq 1$ on V we can find four p.s.h function ψ_j with $1 \leq j \leq 4$ so that they are bounded independent of ϵ and

$$\rho_\epsilon^2 = \psi_1 - \psi_2 \quad \text{and} \quad \rho_\epsilon = \psi_3 - \psi_4. \tag{1.4.9}$$

We also have

$$dd^c\rho_\epsilon^2 = 2d\rho_\epsilon \wedge d^c\rho_\epsilon + 2\rho_\epsilon dd^c\rho_\epsilon.$$

Note that each side of the last equality is well-defined. Substituting this to the defining formula of R', then using (1.4.9), one gets

$$R' \leq \sum_{j=1}^4 \int_{V_1' \cap \{\rho \leq \lambda'\epsilon\}} dd^c\psi_j \wedge T \wedge (dd^c\varphi)^{p-1}$$

which, by induction hypothesis, is $\lesssim$

$$[\epsilon^{-1} a_{\lambda\epsilon,\varphi}]^{p-1} \sum_{j=1}^4 \|dd^c\psi_j \wedge T\|_{V_1''},$$

where V_1'' be a relatively compact subset of V_1 which is open and contains $\overline{V}_1'$. By the Chern-Levine-Nirenberg inequality, the last sum is $\lesssim \|T\|_{V_1}$. Combining with (1.4.8), we obtain

$$R \leq [\epsilon^{-1} a_{\lambda\epsilon,\varphi}]^p \|T\|_{V_1}. \tag{1.4.10}$$

Bounding R_2 is done similarly. The proof is finished. □

Lemma 1.4.4. *Let f be a real $\mathcal{C}^2$ function on an open set $V \subset \mathbb{C}^n$. Let $g(t) := |t| \log(|t| + 2)$ for $t \in \mathbb{R}$. Let ω be the canonical Kähler form on $\mathbb{C}^n$. Then we have*

$$12\, dd^c(g \circ f) \geq df \wedge d^c f - 2n \|D^2 f\|_{L^\infty(V)}\, \omega$$

as currents on V.

Proof. By direct computations, one obtains for $t > 0$,

$$g'(t) = 1 - \frac{2}{2+t} + \log(2+t), \quad g''(t) = \frac{2}{(2+t)^2} + \frac{1}{2+t}$$

and for $t < 0$,

$$g'(t) = -1 + \frac{2}{2-t} - \log(2-t), \quad g''(t) = \frac{2}{(2-t)^2} + \frac{1}{2-t}\cdot$$

For $k \geq 3$, we are going to construct a sequence of $\mathcal{C}^2$ convex function g_k of uniformly bounded L^∞ norm converging pointwise to g. To this end, we define

$$q_k(t) := \frac{2}{(2+|t|)^2} + \frac{1}{2+|t|} \quad \text{for } t \geq 1/k$$

and on $[-1/k, 1/k]$, let $q_k(t)$ be the piece-wise affine function satisfying the two following properties:

(i) q_k is affine on $[-1/k, 0]$ and on $[0, 1/k]$, $q_k(0) = 2kg'(1/k) - q_k(1/k) \geq 1$,

(ii) q_k is continuous on $\mathbb{R}$.

The value of $q(0)$ is in fact chosen such that

$$\int_{-1/k}^{1/k} q_k(t)dt = g'(1/k) - g'(-1/k).$$

This property ensures the existence of a unique $\mathcal{C}^2$ convex function $g_k(t)$ on $\mathbb{R}$ satisfying $g_k(t) \equiv g(t)$ for $|t| \geq 1/k$ and $g_k''(t) = q_k(t)$. One can check that g_k is uniformly bounded and g_k converges to g. Hence $g_k(f)$ converges weakly to $g(f)$ as currents. On the other hand, direct computations give

$$g_k''(f) \geq \min\{1/3, 2k\log 2 - 1\} = 1/3, \quad |g_k'(t)| \leq \log 3 + 2 \leq 4$$

for $|t| \leq 1$ and

$$dd^c g_k(f) = g_k''(f)df \wedge d^c f + g_k'(f)dd^c f \geq 12^{-1}\big(df \wedge d^c f - 2n\|D^2 f\|_{L^\infty}\,\omega\big).$$

The proof is finished. □

Proposition 1.4.5. *Let φ be a p.s.h. function on an open set $V \subset \mathbb{C}^n$. Let A be a generic $\mathcal{C}^3$ submanifold of dimension n of V. Let A_1 be a compact subset of A and for $\epsilon > 0$, let $A_{1,\epsilon}$ be the set of points in $\mathbb{C}^n$ of distance $\leq \epsilon$ to A_1. Then there is a constant c independent of φ, ϵ for which we have*

$$\int_{A_{1,\epsilon}} dd^c\varphi \wedge \omega^{n-1} \leq c\epsilon^{n-1} \int_V |\varphi|, \tag{1.4.11}$$

where ω is the canonical Kähler form of $\mathbb{C}^n$.

Proof. Let δ be a small positive number which will be chosen later. Observe that the question is local. By using a partition of unity and Lemma 1.3.1, it is enough to prove the desired result for the case where A is the graph of a $\mathcal{C}^3$ map h over $\mathbb{B}_n(0,3\delta)$ such $h(0) = Dh(0) = D^2h(0) = 0$ and $\|h\|_{\mathcal{C}^3}$ is bounded independently of chosen local charts (hence, in particular, independent of δ); and A_1 is the part of the graph over $\overline{\mathbb{B}}_n(0,\delta)$. We can assume that

$$A_{1,\epsilon} = \{\mathbf{x} + i\mathbf{y} : \mathbf{x} \in \overline{\mathbb{B}}_n(0,\delta), |\mathbf{y} - h(\mathbf{x})| \leq \epsilon\}$$

and $V = \mathbb{B}_n(0,3\delta) + i\mathbb{B}_n$.

Let g be the function defined in Lemma 1.4.4. We write $\mathbf{z} = (z_1, \cdots, z_n)$, $\mathbf{y} = (y_1, \cdots, y_n)$ and $h = (h_1, \cdots, h_n)$. Since $|D^2h| \lesssim \delta$ on $\mathbb{B}_n(0,3\delta)$, one has

$$|D^2\big(y_j - h_j(\mathbf{x})\big)\| \lesssim \delta$$

for $1 \leq j \leq n$. Using this and Lemma 1.4.4, we see that the function

$$\rho(\mathbf{z}) := \sum_{j=1}^{n} g\big(y_j - h_j(\mathbf{x})\big)$$

satisfies

$$dd^c\rho \geq \sum_{j=1}^{n} \big(\frac{i}{4\pi} dz_j \wedge d\bar{z}_j - \delta M dz_j \wedge d\bar{z}_j\big),$$

for some constant M independent of δ. Thus if δ is small enough independently of ϵ, the function ρ is p.s.h. on V. It is clear that $A_{1,\epsilon} \subset A_1 \cap \{\rho \leq 2n\epsilon\}$. Let $\varphi_1(\mathbf{z}) := |\mathbf{y} - h(\mathbf{x})|^2$. A direct computation shows that φ_1 is also p.s.h. on V. Note that $|\varphi_1| \lesssim \epsilon^2$ on $\{\rho \leq 2\epsilon.\}$ Now applying Lemma 1.4.3 to (ρ, φ_1) and to $T := dd^c\varphi$, we obtain

$$\int_{A_{1,\epsilon}} dd^c\varphi \wedge (dd^c\varphi_1)^{n-1} \lesssim \epsilon^{n-1}\|dd^c\varphi\|_{\mathbb{B}_n(0,2\delta)+i\mathbb{B}_n(0,1/2)} \lesssim \epsilon^{n-1}\int_V |\varphi|.$$

The last inequality together with the fact that $dd^c\varphi_1 \gtrsim \omega$ gives the desired result. The proof is finished. □

Note that a similar technique was used by Sibony in [69] when dealing with the extension of positive closed currents (or more generally pluripositive currents) through a CR submanifold. For $\epsilon \in (0,1]$, let K'_ϵ be as above. The following is just a direct consequence of Proposition 1.4.5.

Corollary 1.4.6. *Let V be an open subset of $\mathbb{C}^n$ containing K'_1. Let φ be a p.s.h. function on V. Then there is a constant c independent of φ, ϵ for which we have*

$$\int_{K'_\epsilon} dd^c\varphi \wedge \omega^{n-1} \leq c\epsilon^{n-1}\int_V |\varphi|, \tag{1.4.12}$$

where ω is the canonical Kähler form of $\mathbb{C}^n$.

Now we are going to give some applications of these above estimates to our present problem. Firstly, we prove some auxiliary lemmas. Let $t_3, \tilde{\epsilon}_0$ and $\tilde{\theta}_0$ be the constants in Proposition 1.3.7. Let $\tilde{F}$ be the family of analytic discs defined there. For simplicity, from now on, we denote $\tilde{F}(z, \boldsymbol{\tau}, t_3)$ by $\tilde{F}(z, \boldsymbol{\tau})$. Recall that the image of $\tilde{F}$ is contained in $\mathbb{D}^n$. Put $\tilde{\epsilon}'_0 := t_3\tilde{\epsilon}_0$.

Lemma 1.4.7. *There exists a positive constant c_0 such that for any Borel function g on $\mathbb{D}^n$, we have*

$$\int_{\mathbb{B}_n(0,\tilde{\epsilon}_0')} |g\big(\mathbf{x}, h(\mathbf{x})\big)| \leq c_0 \int_{[e^{-i\tilde{\theta}_0}, e^{i\tilde{\theta}_0}]\times\mathbb{B}_{n-1}^2} |g \circ \tilde{F}(e^{i\theta}, \boldsymbol{\tau})|. \tag{1.4.13}$$

Proof. This is a direct consequence of Property (i) of Proposition 1.3.7 and the change of variables theorem. The proof is finished. □

Lemma 1.4.8. *Let g be a Borel function on $\mathbb{D}^n$.*

(i) If $n = 1$, then

$$\int_{\mathbb{D}\times\mathbb{B}_{n-1}(0,1)^2} |g \circ \tilde{F}(z, \boldsymbol{\tau})| \leq c_1 \int_{\mathbb{D}^n} |g(\mathbf{z})|,$$

for some constant c_1 independent of g.

(ii) Assume $n > 1$. Let t_0 and δ_0 be two positive real numbers such that $t_0 + \delta_0 > n - 1 > \delta_0$. Let

$$M_g := \sup_{\epsilon\in(0,1)} \epsilon^{-t_0} \int_{K_\epsilon'} |g(\mathbf{z})|$$

and $\lambda_0 := t_0 + \delta_0 - n + 1$. Assume $M_g < \infty$. Then we have

$$\int_{\mathbb{D}\times\mathbb{B}_{n-1}^2} (1 - |z|)^{\delta_0} |g \circ \tilde{F}(z, \boldsymbol{\tau})| \leq \frac{2^{t_0} c_1 M_g}{\lambda_0} \left[\int_{\mathbb{D}^n} |g(\mathbf{z})| \right]^{\frac{\lambda_0}{t_0}},$$

for some constant c_1 independent of g, t_0, δ_0.

Proof. When $n = 1$, the desired inequality is a direct application of the change of variables theorem and (1.3.40). Consider now $n > 1$. Put $Y := \mathbb{D} \times \mathbb{B}_{n-1}^2$. Let ϵ be a small positive number which will be chosen later. Set

$$Y_{\epsilon,0} := \{(z, \boldsymbol{\tau}) \in Y : \operatorname{dist}\big(\tilde{F}(z, \boldsymbol{\tau}), K'\big) \geq \epsilon\}$$

and

$$Y_{\epsilon,k} := \{(z, \boldsymbol{\tau}) \in Y : 2^{-k}\epsilon \leq \operatorname{dist}\big(\tilde{F}(z, \boldsymbol{\tau}), K'\big) \leq 2^{-k+1}\epsilon\},$$

for $k \in \mathbb{N}$. It is clear that $Y = \cup_{k=0}^{\infty} Y_{\epsilon,k}$. By definition of K_ϵ', we have

$$\tilde{F}(Y_{\epsilon,k}) \subset K'_{2^{-k+1}\epsilon}. \tag{1.4.14}$$

Denote by vol_Y the canonical volume form on Y. Write

$$\begin{aligned}
\int_Y (1 - |z|)^{\delta_0} |g \circ \tilde{F}|\, dvol_Y &= \sum_{k=0}^{\infty} \int_{Y_{\epsilon,k}} (1 - |z|)^{\delta_0} |g \circ \tilde{F}|\, dvol_Y \qquad (1.4.15)\\
&\lesssim \sum_{k=0}^{\infty} \int_{Y_{\epsilon,k}} (1 - |z|)^{\delta_0} |g \circ \tilde{F}(z, \boldsymbol{\tau})| \frac{|\det D\tilde{F}(z, \boldsymbol{\tau})|}{\operatorname{dist}^{n-1}\big(\tilde{F}(z, \boldsymbol{\tau}), K'\big)}\, dvol_Y\\
&\quad \text{(by (1.3.40))}\\
&\lesssim \sum_{k=0}^{\infty} (2^{-k}\epsilon)^{-n+1+\delta_0} \int_{Y_{\epsilon,k}} |g \circ \tilde{F}|\, |\det D\tilde{F}|\, dvol_Y,
\end{aligned}$$

by definition of $Y_{\epsilon,k}$, (1.3.41) and the fact that $-n+1+\delta_0<0$. By change of variables, the last integral equals

$$\int_{\tilde{F}(Y_{\epsilon,k})}|g|$$

which is, for $k\geq 1$, less than or equal to

$$\int_{H_{2^{-k+1}\epsilon}}|g|\leq (2^{-k+1}\epsilon)^{t_0}M_g$$

by definition of M_g and (1.4.14). This coupled with (1.4.15) yields that

$$\begin{aligned}\int_Y(1-|z|)^{\delta_0}|g\circ\tilde{F}| &\lesssim \epsilon^{-n+1+\delta_0}\int_{\mathbb{D}^n}|g|+2^{t_0}\epsilon^{\lambda_0}M_g\sum_{k=1}^{\infty}2^{-k\lambda_0} \\ &\lesssim \epsilon^{-n+1+\delta_0}\int_{\mathbb{D}^n}|g|+\frac{2^{t_0}\epsilon^{\lambda_0}M_g}{2^{\lambda_0}-1}.\end{aligned} \tag{1.4.16}$$

Choose $\epsilon=\|g\|_{L^1(\mathbb{D}^n)}^{1/t_0}$. Using (1.4.16) and the fact that $2^{\lambda_0}\geq 1+\lambda_0$, we get the desired inequality. The proof is finished. □

The following will be crucial for our later purpose.

Corollary 1.4.9. *Let V be an open subset of $\mathbb{C}^n$ containing $K_1'\cup\overline{\mathbb{D}}^n$. Let φ be a p.s.h. function on V. Let $\delta\in(0,1)$. Define $\gamma:=\delta/(n-1)$ if $n>1$ and $\gamma=1$ otherwise. Then we have*

$$\int_{\mathbb{D}\times\mathbb{B}_{n-1}^2}(1-|z|)^{\delta}dd^c(\varphi\circ\tilde{F})(z,\boldsymbol{\tau})\lesssim_\delta\|\varphi\|_{L^1(V)}^{\gamma}. \tag{1.4.17}$$

Furthermore, we have

$$\int_{\{1-2\epsilon\leq|z|\leq 1\}\times\mathbb{B}_{n-1}^2}(1-|z|)dd^c(\varphi\circ\tilde{F})(z,\boldsymbol{\tau})\lesssim_\delta\epsilon^{1-\frac{\delta(n-1)}{\delta+n-1}}\max\{\|\varphi\|_{L^1(V)}^{\gamma},\|\varphi\|_{L^1(V)}\}, \tag{1.4.18}$$

for every $\epsilon\in(0,1)$.

Proof. Firstly we prove (1.4.17). The case where $n=1$ is clear. Consider $n>1$. Let $V_1\Subset V$ be an open subset of V. Fix a decreasing sequence of smooth p.s.h. functions φ_l converging pointwise to φ on V_1 and $\|\varphi_l\|_{L^1(V_1)}\leq 2\|\varphi\|_{L^1(V)}$. Let $\delta\in(0,1)$. Since

$$dd^c\varphi_l=\frac{i}{\pi}\sum_{1\leq j,k\leq n}\frac{\partial^2\varphi_l}{\partial z_j\partial\bar{z}_k}dz_j\wedge d\bar{z}_k\geq 0,$$

using Corollary 1.4.6, there is a positive constant c independent of φ such that for every j,k,l we have

$$\int_{K_\epsilon'}\left|\frac{\partial^2\varphi_l}{\partial z_j\partial\bar{z}_k}\right|\leq c\epsilon^{n-1}\int_{V_1}|\varphi_l|\leq c\epsilon^{n-1}\|\varphi\|_{L^1(V)} \tag{1.4.19}$$

which infers that the constant M_g, defined in Lemma 1.4.8 for

$$g:=\frac{\partial^2\varphi_l}{\partial z_j\partial\bar{z}_k},\quad t_0=n-1,\quad \delta_0=\delta,$$

is finite. Hence applying that lemma to the above mentioned (g, t_0, δ_0) gives

$$\int_{\mathbb{D}\times\mathbb{B}^2_{n-1}} (1-|z|)^\delta dd^c(\varphi_l \circ \tilde{F})(z,\boldsymbol{\tau}) \lesssim_\delta \|\varphi\|^{\frac{\delta}{n-1}}_{L^1(V)}. \tag{1.4.20}$$

On the other hand, since $dd^c\varphi_l \circ \tilde{F}$ converges weakly to $dd^c\varphi_l \circ \tilde{F}$ on $\mathbb{D}$, we have

$$\liminf_{l\to\infty} \big\langle dd^c\big(\varphi_l \circ \tilde{F}(\cdot,\boldsymbol{\tau})\big), f\big\rangle \geq \big\langle dd^c\big(\varphi \circ \tilde{F}(\cdot,\boldsymbol{\tau})\big), f\big\rangle, \tag{1.4.21}$$

for every positive continuous function f on $\mathbb{D}$. Letting $l \to \infty$ in (1.4.20) and then using (1.4.21) and Fatou's lemma, we get the desired result.

Now we prove (1.4.18). As above, it is enough to prove it for φ smooth. Set $W := \{1-2\epsilon \leq |z| \leq 1\} \times \mathbb{B}^2_{n-1}$. Let r be a positive constant. Denote by W_1 the subset of W containing $(z,\boldsymbol{\tau})$ with $\operatorname{dist}\big(\tilde{F}(z,\boldsymbol{\tau}), K'\big) \geq r$ and by W_2 the complement of W_1 in W. Let ϵ be a positive constant in $(0,1)$. Using (1.3.40) and the change of variables by $\tilde{F}$ on W_1 gives

$$\int_{W_1} (1-|z|)dd^c(\varphi\circ\tilde{F}) \lesssim \epsilon \int_{W_1} dd^c(\varphi\circ\tilde{F}) \lesssim \epsilon r^{-n+1} \int_{\tilde{F}(W_1)} dd^c\varphi \wedge \omega^{n-1} \lesssim \epsilon r^{-n+1}\|\varphi\|_{L^1(V)}.$$

By the proof of Lemma 1.4.8 applied to $g = \frac{\partial^2\varphi_l}{\partial z_j \partial \bar{z}_k}$, $t_0 = n-1$ and $\delta_0 = \delta$, we have

$$\begin{aligned}\int_{W_2} (1-|z|)dd^c(\varphi\circ\tilde{F}) &\leq \epsilon^{1-\delta} \int_{W_2} (1-|z|)^\delta dd^c(\varphi\circ\tilde{F}) \\ &\lesssim_\delta \epsilon^{1-\delta}\Big[\int_{\tilde{F}(W_2)} dd^c\varphi\wedge\omega^{n-1}\Big]^{\frac{\delta}{n-1}} \leq \epsilon^{1-\delta} r^\delta \|\varphi\|^\gamma_{L^1(V)}\end{aligned}$$

by (1.4.12) and the fact that $\tilde{F}(W_2)$ is contained in $K'_r \times \mathbb{B}^2_{n-1}$. Choose $r := \epsilon^{\frac{\delta}{\delta+n-1}}$. Taking the sume of the last two inequalities gives (1.4.18). The proof is finished. □

1.5 Hölder continuity for super-potentials

Recall that $\mathscr{C}$ defined at Introduction is a compact subset of the set of ω-p.s.h. functions on X with respect to L^1-topology. Hence there is a positive number r_0 such that

$$\|\varphi_0\|_{L^1(X)} \leq r_0 \quad \text{and} \quad \|\max\{\varphi_1,\varphi_2\}\|_{L^1(X)} \leq r_0,$$

for every $\varphi_0, \varphi_1, \varphi_2 \in \mathscr{C}$. Let $\mathscr{C}'$ be the set of ω-p.s.h. functions φ on X such that $\|\varphi\|_{L^1(X)} \leq 2r_0$. In this section, we will finish the proof of Theorem 1.1.4. In order to do so, we will prove the following which is actually equivalent to Theorem 1.1.4 by Lemma 1.5.2 below. Remember that we are still assuming that $\dim K = n$. Let $\tilde{K}$ be the compact subset of K as in Theorem 1.1.4.

Proposition 1.5.1. *Let α be a positive number strictly less than $1/(3n)$. Then for any $\varphi_1, \varphi_2 \in \mathscr{C}'$ such that $\varphi_1 \geq \varphi_2$, we have*

$$\int_{\tilde{K}} (\varphi_1-\varphi_2)dvol_K \leq c\int_X (\varphi_1-\varphi_2)dvol_X + c\bigg(\int_X (\varphi_1-\varphi_2)dvol_X\bigg)^\alpha, \tag{1.5.1}$$

where c is a constant independent of φ_1, φ_2.

Lemma 1.5.2. *Proposition 1.5.1 implies Theorem 1.1.4.*

Proof. Take $\varphi_1, \varphi_2 \in \mathscr{C}$. Observe that $\max\{\varphi_1, \varphi_2\}, \varphi_1, \varphi_2 \in \mathscr{C}'$ and $\max\{\varphi_1, \varphi_2\} \geq \varphi_j$ for $j = 1, 2$. Hence, we can apply (1.5.1) to $\max\{\varphi_1, \varphi_2\}, \varphi_j$ with $j = 1, 2$. Using these two inequalities and the fact that

$$|\varphi_1 - \varphi_2| = 2\max\{\varphi_1, \varphi_2\} - \varphi_1 - \varphi_2$$

gives

$$\|\varphi_1 - \varphi_2\|_{L^1(\mathbf{1}_{\tilde{K}} vol_K)} \lesssim \max\{\|\varphi_1 - \varphi_2\|_{L^1(X)}, \|\varphi_1 - \varphi_2\|^{\alpha}_{L^1(X)}\}$$

which means that $\mathbf{1}_{\tilde{K}} vol_K$ has Hölder continuous super-potential with Hölder exponent α. The proof is finished. □

The remaining of this section is devoted to prove Proposition 1.5.1. By [7], it is enough to prove (1.5.1) for φ_1, φ_2 smooth. We will firstly show that for any nonnegative $\mathcal{C}^2$ function v on $\overline{\mathbb{D}}$, the integral of v over $\partial\mathbb{D}$ can be bounded by a quantity of the L^1-norm of v over $\mathbb{D}$ and some Hölder norm of its Laplacian. This together with the exponent estimates in the last section are the key ingredients in the proof of Proposition 1.5.1. We will reuse the notations from Section 2 for $M = \overline{\mathbb{D}}$.

Lemma 1.5.3. *Let v be a nonnegative $\mathcal{C}^2$ functions on $\overline{\mathbb{D}}$. Let $\beta \in (1, 2)$. Then we have*

$$\int_{\partial\mathbb{D}} v d\xi \lesssim_\beta \|dd^c v\|_{\tilde{\mathcal{C}}^{-\beta}(\overline{\mathbb{D}})} + \int_{\mathbb{D}} v. \tag{1.5.2}$$

Proof. By Riesz's representation formula, we have

$$v(z) = \int_{-\pi}^{\pi} P(e^{i\theta}, z) v(e^{i\theta}) d\theta + \int_{\{|\eta|<1\}} \log \frac{|z-\eta|}{|1 - z\bar{\eta}|} dd^c v, \tag{1.5.3}$$

for $z \in \mathbb{D}$, where $P(\xi, z)$ is the Poisson kernel given by

$$P(\xi, z) = (2\pi)^{-1}(|\xi|^2 - |z|^2)|\xi - z|^{-2}.$$

This implies that

$$\begin{aligned}\int_{\mathbb{D}_{1/2}} v(z) = \int_{-\pi}^{\pi} v(e^{i\theta}) d\theta \int_{z\in\mathbb{D}_{1/2}} P(e^{i\theta}, z) + \\ + \int_{\mathbb{D}} dd^c v(\eta) \int_{z\in\mathbb{D}_{1/2}} \log \frac{|z-\eta|}{|1 - z\bar{\eta}|}.\end{aligned} \tag{1.5.4}$$

Set

$$f(\eta) := \int_{\{|z|<1/2\}} \log \frac{|z-\eta|}{|1 - z\bar{\eta}|} = \int_{\{|z|<1/2\}} \log|z - \eta| - \int_{\{|z|<1/2\}} \log|1 - z\bar{\eta}|.$$

Observe that $f(e^{i\theta}) = 0$ because

$$\log \frac{|z - e^{i\theta}|}{|1 - ze^{-i\theta}|} = 0$$

for any $z \in \mathbb{D}$. This means that $f|_{\partial\mathbb{D}} \equiv 0$. We claim that f is indeed in $\tilde{\mathcal{C}}^\beta(\overline{\mathbb{D}})$ for every $\beta \in (1,2)$. Since $z \in \mathbb{D}_{1/2}$ and $\eta \in \mathbb{D}$, the function

$$\int_{\mathbb{D}_{1/2}} \log|1 - z\bar{\eta}| dxdy$$

is smooth in $\eta \in \mathbb{D}$. Hence, we only need to take care of $\int_{z\in\mathbb{D}_{1/2}} \log|z-\eta|$. It is clear that

$$\partial_\eta \int_{z\in\mathbb{D}_{1/2}} \log|z-\eta| = -\frac{1}{2}\int_{z\in\mathbb{D}_{1/2}} \frac{\bar{z}-\bar{\eta}}{|z-\eta|^2} = -\frac{1}{2}\int_{z\in\mathbb{D}_{1/2}} \frac{1}{z-\eta}. \tag{1.5.5}$$

Let g be the right-hand side of the last equation. We will show that $g \in \mathcal{C}^\alpha(\overline{\mathbb{D}})$ for every $\alpha \in (0,1)$. If we can do so, then $\partial_\eta f \in \mathcal{C}^\alpha(\overline{\mathbb{D}})$, using similar argument we also gets $\partial_{\bar{\eta}} f \in \mathcal{C}^\alpha(\overline{\mathbb{D}})$, hence $f \in \tilde{\mathcal{C}}^\beta(\overline{\mathbb{D}})$ for $\beta \in (1,2)$. Let $\alpha \in (0,1)$. For $(\eta,\eta') \in \mathbb{D}^2$, consider the difference

$$\begin{aligned} \Big|\frac{1}{z-\eta} - \frac{1}{z-\eta'}\Big| = \Big|\frac{\eta-\eta'}{(z-\eta)(z-\eta')}\Big| &\leq \frac{|\eta-\eta'|^\alpha}{|(z-\eta)(z-\eta')|^\alpha}\Big|\frac{1}{z-\eta} - \frac{1}{z-\eta'}\Big|^{1-\alpha} \\ &\leq |\eta-\eta'|^\alpha\Big[\frac{1}{|z-\eta|\,|z-\eta')|^\alpha} + \frac{1}{|z-\eta)|^\alpha|z-\eta|}\Big]. \end{aligned} \tag{1.5.6}$$

It is not difficult to see that the integration of the right-hand side of (1.5.6) over $z \in \mathbb{D}_{1/2}$ is bounded by $|\eta-\eta'|^\alpha$ times a constant depending only on α. Thus one gets $g \in \mathcal{C}^\alpha(\overline{\mathbb{D}})$. As explained above, this yields $f \in \tilde{\mathcal{C}}^\beta(\overline{\mathbb{D}})$. The last property combined with (1.5.4) gives

$$\Big|\int_{-\pi}^{\pi} v(e^{i\theta})d\theta \int_{z\in\mathbb{D}_{1/2}} P(e^{i\theta},z)\Big| \leq \|v\|_{L^1(\mathbb{D}_{1/2})} + \|dd^c v\|_{\tilde{\mathcal{C}}^{-\beta}(\overline{\mathbb{D}})}\|f\|_{\tilde{\mathcal{C}}^\beta(\overline{\mathbb{D}})}. \tag{1.5.7}$$

By our hypothesis that $v \geq 0$ and the fact that $P(e^{i\theta},z) \gtrsim 1$ for $z \in \mathbb{D}_{1/2}$, using (1.5.7), one obtains that

$$\int_{\partial\mathbb{D}} v d\xi \lesssim_\beta \|v\|_{L^1(\mathbb{D}_{1/2})} + \|dd^c v\|_{\tilde{\mathcal{C}}^{-\beta}(\overline{\mathbb{D}})}. \tag{1.5.8}$$

The proof is finished. □

Proposition 1.5.4. *Let v be a nonnegative $\mathcal{C}^2$ function on $\overline{\mathbb{D}}$. Let $\epsilon, \beta_0 \in (0,1)$ and $\beta \in (1,2)$. Let γ be the unique real number for which $\beta = \gamma\beta_0 + (1-\gamma)2$. Then we have*

$$\begin{aligned} \int_{\partial\mathbb{D}} v d\xi \lesssim_{(\beta_0,\beta)} \|v\|_{L^1(\mathbb{D})} + \epsilon^{-2(1-\gamma)}\|dd^c v\|^\gamma_{\tilde{\mathcal{C}}^{-\beta_0}(\overline{\mathbb{D}})}\|v\|^{1-\gamma}_{L^1(\mathbb{D})} + \\ + \|dd^c v\|^\gamma_{\tilde{\mathcal{C}}^{-\beta_0}(\overline{\mathbb{D}})}\Big[\int_{1-2\epsilon\leq|z|\leq 1}(1-|z|)|dd^c v|\Big]^{1-\gamma}. \end{aligned} \tag{1.5.9}$$

Proof. Firstly we will estimate $\|dd^c v\|_{\tilde{\mathcal{C}}^{-2}(\overline{\mathbb{D}})}$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi \equiv 0$ on $[-1,1]$ and $\chi \equiv 1$ outside $[-2,2]$. For $\epsilon \in (0,1)$, put $\chi_\epsilon(z) := \chi(\frac{1-|z|}{\epsilon})$ for $z \in \mathbb{D}$. We have $\mathrm{supp}\chi_\epsilon \subset \{z : |z| \leq 1-\epsilon\}$ and $\chi_\epsilon(z) = 1$ for z with $|z| \leq 1-2\epsilon$. Let Φ be a function in $\tilde{\mathcal{C}}^2(\overline{\mathbb{D}})$ with $\|\Phi\|_{\mathcal{C}^2} \leq 1$. Since $\Phi \equiv 0$ on $\partial\mathbb{D}$ we have $|\Phi(z)| \leq 1-|z|$. Decompose

$$\langle dd^c v, \Phi\rangle = \langle dd^c v, \chi_\epsilon\Phi\rangle + \langle dd^c v, (1-\chi_\epsilon)\Phi\rangle.$$

Denote by I_1, I_2 respectively the first and second terms in the right-hand side of the last equality. By properties of Φ and χ_ϵ, one gets

$$|I_2| \leq 2 \int_{1-2\epsilon \leq |z| \leq 1} (1-|z|)|dd^c v|.$$

On the other hand, performing an integration by parts gives

$$|I_1| \leq \int_{\mathbb{D}} |v dd^c(\chi_\epsilon \Phi)| \lesssim \epsilon^{-2} \int_{\mathbb{D}} |v|.$$

Hence, we obtain

$$\|dd^c v\|_{\tilde{\mathcal{C}}^{-2}(\overline{\mathbb{D}})} = \sup_{\{\Phi \in \tilde{\mathcal{C}}^2(\mathbb{D}) : \|\Phi\|_{\mathcal{C}^2} \leq 1\}} \big|\langle dd^c v, \Phi\rangle\big| \lesssim \epsilon^{-2} \int_{\mathbb{D}} |v| + \int_{1-2\epsilon \leq |z| \leq 1} (1-|z|)|dd^c v|. \tag{1.5.10}$$

Now applying Proposition 1.2.1 to $dd^c v$ and $M = \overline{\mathbb{D}}$, one gets

$$\|dd^c v\|_{\tilde{\mathcal{C}}^{-\beta}(\overline{\mathbb{D}})} \lesssim \|dd^c v\|^{\gamma}_{\tilde{\mathcal{C}}^{-\beta_0}(\overline{\mathbb{D}})} \|dd^c v\|^{1-\gamma}_{\tilde{\mathcal{C}}^2(\overline{\mathbb{D}})}.$$

The last inequality combined with (1.5.2) and (1.5.10) gives (1.5.9). The proof is finished. □

We are now about to prove the local version of Proposition 1.5.1. Given a point $a \in K$, a small open neighborhood K' of a in K can be described as in Section 1.3. Namely, there are a $\mathcal{C}^3$ map h from $\overline{\mathbb{B}}_n$ to $\mathbb{R}^n$ with $h(0) = Dh(0) = 0$ and local holomorphic coordinates in X such that

$$K' := \{\mathbf{x} + ih(\mathbf{x}) : \mathbf{x} \in \mathbb{B}_n\} \subset \mathbb{D}_2^n.$$

Let $\tilde{F}(z, \boldsymbol{\tau})$, $t_3, \tilde{\epsilon}_0'$ and $\tilde{\theta}_0$ be as in Section 1.4. The couple $(K', \mathbb{D}_2^n)$ is considered as the local counterpart of (K, X). One can replace $\mathbb{D}_2^n$ by any polydisks $\mathbb{D}_r^n$ with $r > 1$ without making any differences in what follows.

Let $\beta_0 \in (0,1)$. For every positive continuous $(1,1)$-current T on an open neighborhood of $\overline{\mathbb{D}}$, we have

$$\|T\|_{\tilde{\mathcal{C}}^{-\beta_0}(\overline{\mathbb{D}})} \leq \int_{\mathbb{D}} (1-|z|)^{\beta_0} T. \tag{1.5.11}$$

Let φ_1 and φ_2 be two $\mathcal{C}^2$ p.s.h. functions on $\mathbb{D}_2^n$ such that $\varphi_1 \geq \varphi_2$ and $\|\varphi_j\|_{L^1(\mathbb{D}_2^n)} \leq 1$ for $j = 1, 2$. Put $\varphi := \varphi_1 - \varphi_2$ which is $\mathcal{C}^2$ and nonnegative. Define

$$g_1(\boldsymbol{\tau}) := \|dd^c\big(\varphi \circ \tilde{F}(\cdot, \boldsymbol{\tau})\big)\|_{\tilde{\mathcal{C}}^{-\beta_0}(\overline{\mathbb{D}})}$$

which is less than or equal to

$$\|dd^c\big(\varphi_1 \circ \tilde{F}(\cdot, \boldsymbol{\tau})\big)\|_{\tilde{\mathcal{C}}^{-\beta_0}(\overline{\mathbb{D}})} + \|dd^c\big(\varphi_2 \circ \tilde{F}(\cdot, \boldsymbol{\tau})\big)\|_{\tilde{\mathcal{C}}^{-\beta_0}(\overline{\mathbb{D}})}. \tag{1.5.12}$$

Since $\tilde{F}$ is $\mathcal{C}^2$, so is $\varphi_j \circ \tilde{F}$ for $j = 1, 2$. Using (1.5.11) for $T = dd^c\big(\varphi_j \circ \tilde{F}(\cdot, \boldsymbol{\tau})\big)$ and (1.4.17), we deduce that the integral of the sum (1.5.12) with respect to $\boldsymbol{\tau} \in \mathbb{B}^2_{n-1}$ is $\lesssim_{\beta_0} 1$. This implies

$$\int_{\mathbb{B}^2_{n-1}} g_1(\boldsymbol{\tau}) d\boldsymbol{\tau} \lesssim_{\beta_0} 1. \tag{1.5.13}$$

Put

$$g_2(\boldsymbol{\tau}) := \|\varphi \circ \tilde{F}(\cdot, \boldsymbol{\tau})\|_{L^1(\mathbb{D})}.$$

By Corollary 1.4.2, the function φ satisfy the hypothesis of Lemma 1.4.8 for $\delta_0 = 0$ and $t_0 = n - 1 + \epsilon$ with $\epsilon \in (0, 1)$. As a result, we get

$$\int_{\mathbb{B}^2_{n-1}} g_2(\boldsymbol{\tau}) d\boldsymbol{\tau} \lesssim_{\epsilon} \left[\int_{\mathbb{D}^n_2} |\varphi| \right]^{\frac{\epsilon}{n-1+\epsilon}}. \tag{1.5.14}$$

For $\epsilon' \in (0, 1)$, we define

$$g_3(\boldsymbol{\tau}, \epsilon') := \int_{1-2\epsilon' \le |z| \le 1} (1 - |z|) dd^c\big(\varphi_1 \circ \tilde{F}(\cdot, \boldsymbol{\tau})\big) + \int_{1-2\epsilon' \le |z| \le 1} (1 - |z|) dd^c\big(\varphi_2 \circ \tilde{F}(\cdot, \boldsymbol{\tau})\big).$$

By (1.4.18), we have

$$\int_{\mathbb{B}^2_{n-1}} g_3(\boldsymbol{\tau}, \epsilon') d\boldsymbol{\tau} \lesssim_{\delta} (\epsilon')^{1 - \frac{\delta(n-1)}{n-1+\delta}}, \tag{1.5.15}$$

for any $\delta \in (0, 1)$.

Proposition 1.5.5. *Let φ_1 and φ_2 be two $\mathcal{C}^2$ p.s.h. functions on $\mathbb{D}^n_2$ such that $\varphi_1 \ge \varphi_2$ and $\|\varphi_j\|_{L^1(\mathbb{D}^n_2)} \le 1$ for $j = 1, 2$. Let $\varphi := \varphi_1 - \varphi_2$. Then we have*

$$\int_{\mathbb{B}_n(0, \epsilon_0'^*)} \varphi\big(\mathbf{x}, h(\mathbf{x})\big) d\mathbf{x} \lesssim_{\delta} \|\varphi\|^{\frac{1}{3n} - \delta}_{L^1(\mathbb{D}^n_2)}, \tag{1.5.16}$$

for any $\delta \in (0, \frac{1}{3n})$.

Proof. Let $\epsilon, \epsilon', \beta_0 \in (0, 1)$ and $\beta \in (1, 2)$. Let g_1, g_2, g_3 be as above. Applying Lemma 1.4.7 to $g = \varphi$ gives

$$\int_{\mathbb{B}_n(0, \epsilon_0'^*)} |\varphi\big(\mathbf{x}, h(\mathbf{x})\big)| d\mathbf{x} \lesssim \int_{\mathbb{B}^2_{n-1}} d\boldsymbol{\tau} \int_{\partial\mathbb{D}} |\varphi \circ \tilde{F}(\cdot, \boldsymbol{\tau})| d\xi.$$

Put $\gamma := \frac{2-\beta}{2-\beta_0}$. Applying Proposition 1.5.4 to $v = \varphi \circ \tilde{F}(\cdot, \boldsymbol{\tau}) \in \mathcal{C}^2$ shows that the right-hand side of the last inequality is

$$\lesssim_{(\beta_0, \beta)} \int_{\mathbb{B}^2_{n-1}} g_2 d\boldsymbol{\tau} + (\epsilon')^{-2(1-\gamma)} \int_{\mathbb{B}^2_{n-1}} g_1^{\gamma} g_2^{1-\gamma} d\boldsymbol{\tau} + \int_{\mathbb{B}^2_{n-1}} g_1^{\gamma} g_3^{1-\gamma}(\cdot, \epsilon') d\boldsymbol{\tau}.$$

The first term of the last sum is

$$\lesssim_{\epsilon} \|\varphi\|^{\frac{\epsilon}{n-1+\epsilon}}_{L^1(\mathbb{D}^n_2)}$$

by (1.5.14). On the other hand, by the Hölder inequality, the second one is $\leq$

$$(\epsilon')^{-2(1-\gamma)}\|g_1\|_{L^1}^{\gamma}\|g_2\|_{L^1}^{1-\gamma}$$

and the third one is $\leq$

$$\|g_1\|_{L^1}^{\gamma}\|g_3(\cdot,\epsilon')\|_{L^1}^{1-\gamma},$$

where the L^1-norm is taken over $\mathbb{B}_{n-1}^2$. Taking into account (1.5.13) and (1.5.14), one obtains

$$(\epsilon')^{-2(1-\gamma)}\|g_1\|_{L^1}^{\gamma}\|g_2\|_{L^1}^{1-\gamma} \lesssim_{\beta_0,\epsilon} (\epsilon')^{-2(1-\gamma)}\|\varphi\|_{L^1(\mathbb{D}_2^n)}^{\frac{\epsilon(1-\gamma)}{n-1+\epsilon}}.$$

By (1.5.13) and (1.5.15), we have

$$\|g_1\|_{L^1}^{\gamma}\|g_3(\cdot,\epsilon')\|_{L^1}^{1-\gamma} \lesssim_{\beta_0,\delta} (\epsilon')^{(1-\frac{\delta(n-1)}{n-1+\delta})(1-\gamma)},$$

for every $\epsilon' \in (0,1)$. Put

$$a_1 := \frac{\epsilon}{(n-1+\epsilon)(3-\frac{\delta(n-1)}{n-1+\delta})}, \quad a_2 := \frac{\epsilon(1-\frac{\delta(n-1)}{n-1+\delta})(1-\gamma)}{(n-1+\epsilon)(3-\frac{\delta(n-1)}{n-1+\delta})}.$$

Choose $\epsilon' := \|\varphi\|_{L^1(\mathbb{D}_2^n)}^{a_1}$. Combining all these above inequalities, we get

$$\int_{\mathbb{B}_n(0,\epsilon_0'^*)} |\varphi\big(\mathbf{x},h(\mathbf{x})\big)|d\mathbf{x} \lesssim_{(\beta_0,\beta,\delta,\epsilon)} \|\varphi\|_{L^1(\mathbb{D}_2^n)}^{a_2}.$$

Observe that $a_2 \to \frac{1}{3n}$ as $\epsilon \to 1$, $\beta \to 2$, $\beta_0 \to 0$, $\delta \to 0$. Thus, the proof is finished. □

End of the proof of Proposition 1.5.1 in the case where $\dim K = n$. Given any $a \in K$, let $\tilde{F}_a$ and $\tilde{\epsilon}_a$ be as in Proposition 1.3.8. Since $\tilde{K}$ is compact, we can cover it by a finite number of ball $B_K(a,\tilde{\epsilon}_a)$. Hence, in order to prove (1.5.1), it is enough to restrict ourselves to local charts. In other words, we are now being in the situation with the model $(K',\mathbb{D}_2^n)$ described above. Moreover, by subtracting a suitable common smooth function, we can assume that φ_1,φ_2 in (1.5.1) are $\mathcal{C}^2$ p.s.h. functions on $\mathbb{D}_2^n$. Hence, the desired result follows directly from Proposition 1.5.5. The proof is finished. □

We now deal with the case where the dimension of K is greater than n. Let $n_K := \dim K > n$. Since K is generic, we have $T_aK + JT_aK = T_aX$, where $a \in K$ and J is the complex structure of X. We then deduce that $T_aK \cap JT_aK$ is of even dimension which equals $2n_K - 2n$. The codimension d of K equals $2n - n_K$.

Proposition 1.5.6. *Let a be a point in K. There exist local $\mathcal{C}^2$ coordinates (W,Ψ) of X around a such that the following properties hold:*

(i) $\Psi : W \to \mathbb{C}^d \times \mathbb{C}^{n_K-n}$ is a $\mathcal{C}^2$ diffeomorphism onto its image which equals

$$\big(\mathbb{B}_d + i\mathbb{B}_d(0,2)\big) \times \mathbb{D}^{n_K-n}$$

and $\Psi(p) = 0$ and $\Psi^{-1}(\mathbf{z}_1,\mathbf{z}_2)$ is holomorphic in $\mathbf{z}_1$ for every fixed $\mathbf{z}_2 \in \mathbb{D}^{n_K-n}$,

(ii) there is a $\mathcal{C}^2$ map $h(\operatorname{Re}\mathbf{z}_1, \mathbf{z}_2)$ from $\overline{\mathbb{B}}_d \times \mathbb{D}^{n_K-n}$ to $\mathbb{R}^d$ so that for every $\mathbf{z}_2$ fixed, $h(\cdot, \mathbf{z}_2) \in \mathcal{C}^3$ and

$$D^j_{\operatorname{Re}\mathbf{z}_1} h(0, \mathbf{z}_2) = 0$$

for $j = 0$ or 1 and

$$\Psi(K \cap W) = \big\{(\mathbf{z}_1, \mathbf{z}_2) \in \big(\mathbb{B}_d + i\mathbb{R}^d\big) \times \mathbb{D}^{n_K-n} : \operatorname{Im}\mathbf{z}_1 = h(\operatorname{Re}\mathbf{z}_1, \mathbf{z}_2)\big\}.$$

Proof. It is well-known that in suitable holomorphic local coordinates, K is given by

$$K = \big\{(\mathbf{z}_1, \mathbf{z}_2) \in \big(\mathbb{B}_d + i\mathbb{R}^d\big) \times \mathbb{D}^{n_K-n} : \operatorname{Im}\mathbf{z}_1 = \tilde{h}(\operatorname{Re}\mathbf{z}_1, \operatorname{Re}\mathbf{z}_2, \operatorname{Im}\mathbf{z}_2)\big\}$$

where $\tilde{h}$ is a $\mathcal{C}^3$ map of uniformly bounded $\mathcal{C}^3$ norm in p and $\tilde{h}(0) = D\tilde{h}(0) = 0$, see [1]. For $\mathbf{z}_2$ fixed, we choose the tangent space of the graph of $\tilde{h}(\cdot, \mathbf{z}_2)$ at 0 and its orthogonal subspace as new holomorphic coordinates of $\mathbb{C}^d$. These new coordinates depend $\mathcal{C}^2$ on (but in general not holomorphically) on the parameter $\mathbf{z}_2$. In these new coordinates, one easily see that K is given by the formula given in the asssertion (ii) for some $\mathcal{C}^2$ map h with the desired properties. The proof is finished. □

Remark 1.5.7. *As in Lemma 1.3.1 we can obtain furthermore that $D^2_{\operatorname{Re}\mathbf{z}_1} h(0, \mathbf{z}_2) = 0$ and $\|h(\cdot, \mathbf{z}_2)\|_{\mathcal{C}^3}$ is bounded uniformly in $a = (\mathbf{z}_1, \mathbf{z}_2) \in \tilde{K}$ but in this case we will lose a unit for the regularity in $\mathbf{z}_2$,* i.e *Ψ and h are only $\mathcal{C}^1$ in $\mathbf{z}_2$.*

Thanks to Proposition 1.5.6, we can consider K locally as a family of generic submanifolds of $\mathbb{C}^d$ of dimension d parameterized by $\mathbf{z}_2 \in \mathbb{D}^{n_K-n}$. This allows us to reduce the question to the previous case where we already dealt with generic submanifolds of minimal dimension. By compactness of $\tilde{K}$, we can cover it by local charts W as in Proposition 1.5.6. From now on, we work exclusively on a such local chart. Hence, we can identify K with $\Psi(K \cap W)$. Let h and Ψ be as in that proposition. The map h will be seen as a family of maps of $\mathbf{z}_1$ parameterized by $\mathbf{z}_2$. For $\mathbf{z}_2 \in \mathbb{D}^{n_K-n}$, define

$$K'_{\mathbf{z}_2} := \{\mathbf{z}_1 \in \big(\mathbb{B}_d + i\mathbb{R}^d\big) : \operatorname{Im}\mathbf{z}_1 = h(\operatorname{Re}\mathbf{z}_1, \mathbf{z}_2)\big\}$$

which is identified with $K'_{\mathbf{z}_2} \times \{\mathbf{z}_2\} \subset \mathbb{C}^n$. Then K is foliated by $K'_{\mathbf{z}_2}$.

We are now going to construct a family of analytic discs partly attached to K. The strategy will be almost identical with what we did. Let u_0 be a function described in Lemma 1.3.2 and θ_{u_0} be the constant there. Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \overline{\mathbb{B}}_{d-1} \subset \mathbb{R}^{d-1}$. Define $\boldsymbol{\tau}^*_1 := (1, \boldsymbol{\tau}_1) \in \mathbb{R}^d$ and $\boldsymbol{\tau}^*_2 := (0, \boldsymbol{\tau}_1) \in \mathbb{R}^d$ and $\boldsymbol{\tau} := (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$. Let t be a positive number in $(0, 1)$. Consider the following modified version of the equation 1.3.11:

$$U_{\boldsymbol{\tau},\mathbf{z}_2,t}(\xi) = t\boldsymbol{\tau}^*_2 - \mathcal{T}_1\big(h(U_{\boldsymbol{\tau},\mathbf{z}_2,t}; \mathbf{z}_2)\big)(\xi) - t\mathcal{T}_1 u_0(\xi)\, \boldsymbol{\tau}^*_1, \tag{1.5.17}$$

where $U : \partial\mathbb{D} \to \mathbb{B}_d$ is Hölder continuous.

Since $h(0, \mathbf{z}_2) = D_{\operatorname{Re}\mathbf{z}_1} h(0, \mathbf{z}_2) = 0$ for every $\mathbf{z}_2$, we can use the same reason mentioned in the proof of Proposition 1.3.3 to show that if t is small enough, the equation (1.5.17) has a unique solution $U_{\boldsymbol{\tau},\mathbf{z}_2,t}$ in $\mathcal{C}^{2,1/2}(\partial\mathbb{D} \times \mathbb{B}^2_{d-1})$ for $\mathbf{z}_2$ fixed so that $U_{\boldsymbol{\tau},\mathbf{z}_2,t} \in \mathcal{C}^1$ as a function of

$(z, \boldsymbol{\tau}, \mathbf{z}_2)$. We use the same notation $U_{\boldsymbol{\tau},\mathbf{z}_2,t}$ to denote the harmonic extension of $U_{\boldsymbol{\tau},\mathbf{z}_2,t}$ to $\mathbb{D}$. Let $P_{\boldsymbol{\tau},\mathbf{z}_2,t}(z)$ be the harmonic extension of $h\big(U_{\boldsymbol{\tau},\mathbf{z}_2,t}(\xi), \mathbf{z}_2\big)$ to $\mathbb{D}$. Define

$$F(z, \boldsymbol{\tau}, \mathbf{z}_2, t) := U_{\boldsymbol{\tau},\mathbf{z}_2,t}(z) + iP_{\boldsymbol{\tau},\mathbf{z}_2,t}(z) + it\, u_0(z)\, \boldsymbol{\tau}_1^*$$

which is a family of analytic discs to $\mathbb{C}^d$ parametrized by $(\boldsymbol{\tau}, \mathbf{z}_2, t)$. By our choice of u_0, we have $F(\xi, \boldsymbol{\tau}, \mathbf{z}_2, t) \in K_{\mathbf{z}_2}$ for $\xi \in [e^{-i\theta_{u_0}}, e^{i\theta_{u_0}}]$. Now define

$$F'(z, \boldsymbol{\tau}, \mathbf{z}_2, t) := \big(F_{\boldsymbol{\tau},\mathbf{z}_2,t}(z), \mathbf{z}_2\big) \in \mathbb{C}^n$$

which is a family of analytic discs to X partly attached to K. Here we used an essential fact that the $\mathcal{C}^2$ coordinates $(\mathbf{z}_1, \mathbf{z}_2)$ are holomorphic in $\mathbf{z}_1$. Proposition 1.3.6 with n replaced by d implies that for two positive constants (t, r_0) small enough, F' is a diffeomorphism on

$$\big(\mathbb{B}_2(1, r_0) \cap \mathbb{D}\big) \times \overline{\mathbb{B}}_{d-1}^2 \times \mathbb{D}^{n_K - n}$$

and its differential satisfies

$$\big|\det DF'(z, \boldsymbol{\tau}, \mathbf{z}_2, t)\big| \gtrsim t^{d+1} \operatorname{dist}^{d-1}\big(F'(z, \boldsymbol{\tau}, \mathbf{z}_2, t), K'_{\mathbf{z}_2}\big) \gtrsim t^{2d}(1 - |z|)^{d-1}. \tag{1.5.18}$$

Now applying the same arguments right before Proposition 1.3.7, one gets the following.

Proposition 1.5.8. *There exists a map $\tilde{F} : \mathbb{D} \times \mathbb{B}_{d-1}^2 \times \mathbb{D}^{n_K - n} \to X$ which is a diffeomorphism onto its image such that the following three properties hold:*

(i) there are positive constants $\tilde{\theta}_0$ and $\tilde{\epsilon}_0$ so that for every $\boldsymbol{\tau}_1 \in \overline{\mathbb{B}}_{d-1}$ the restriction map $\tilde{F}(\cdot, \boldsymbol{\tau}_1) : [e^{-i\tilde{\theta}_0}, e^{i\tilde{\theta}_0}] \times \overline{\mathbb{B}}_{d-1} \times \mathbb{D}^{n_K - n} \to K$ is a diffeomorphism onto its image which contains the graph of h over $\mathbb{B}_d(0, \tilde{\epsilon}_0) \times \mathbb{D}^{n_K - n}$,

(ii) $\tilde{F}(\cdot, \boldsymbol{\tau}, \mathbf{z}_2)$ is an analytic disc to X and

$$\big|\det D\tilde{F}(z, \boldsymbol{\tau}, \mathbf{z}_2)\big| \gtrsim \operatorname{dist}^{d-1}\big(\tilde{F}(z, \boldsymbol{\tau}, \mathbf{z}_2, t), K'_{\mathbf{z}_2}\big) \gtrsim (1 - |z|)^{d-1}. \tag{1.5.19}$$

Proposition 1.5.8 and Remark 1.5.7 allow us to repeat all of arguments in the proof of Theorem 1.1.4 in the case where $n_K = n$ for our present situation. Hence, this finishes the proof of Theorem 1.1.4.

Chapter 2

Intersection de courants de grand bidegré

Let X be a compact Kähler manifold of dimension n. Let T and S be two positive closed currents on X of bidegree (p,p) and (q,q) respectively with $p+q \leq n$. Assume that T has a continuous super-potential. We prove that the wedge product $T \wedge S$, defined by Dinh and Sibony, is a positive closed current. This chapter is based on the article [83].

2.1 Introduction

Let X be a compact Kähler manifold of dimension n. Let T and S be two positive closed currents on X of bidegree (p,p) and (q,q) respectively with $p+q \leq n$. In [18], Demailly asked the question to define the intersection $T \wedge S$. The theory of intersections of currents of bidegree $(1,1)$ is well developed, see, e.g., [2, 10, 17, 38]. So the question of Demailly concerns currents of higher degree.

The problem was recently solved by Dinh and Sibony in [33] using their theory of super-potentials (see also [28]). Assume that T has continuous super-potentials (see [33] or Section 2.2 for the terminology). Then the wedge product $T \wedge S$ is well-defined. It is known that this product is the difference of two positive closed currents. The operator satisfies basic properties like the commutativity and the associativity when intersect several currents. The Hodge cohomology class of $T \wedge S$ is the cup product of the ones of T and S. Moreover, $T \wedge S$ depends continuously on S. Therefore, it is positive when S can be approximated by smooth positive closed forms. The last property of approximation is satisfied when X is a homogeneous manifold and also in the case of some dynamical Green currents. The purpose of this work is to prove the positivity of $T \wedge S$ in the general setting. We have the following theorem.

Theorem 2.1.1. *Let X be a compact Kähler manifold of dimension n. Let T and S be two positive closed currents on X of bidegree (p,p) and (q,q) respectively with $p+q \leq n$. Assume that T has a continuous super-potential. Then the intersection current $T \wedge S$ is a positive closed current of bidegree $(p+q,p+q)$.*

In Section 2.2, we will recall some basic properties of positive closed currents and their

super-potentials. In Section 2.3, we will introduce an alternative definition of $T \wedge S$ which is a positive closed current. We then show that this definition is equivalent to the one by Dinh and Sibony. The above result will follow immediately. We will present now the main idea.

Suppose first that T and S are positive closed smooth forms of X. Let π_j $(j = 1, 2)$ be the projections from $X \times X$ to the first and second components respectively. We have $T \otimes S = \pi_1^*(T) \wedge \pi_2^*(S)$. This is a positive closed smooth form on $X \times X$. Then one can compute $T \wedge S$ via the formula

$$T \wedge S = (\pi_j)_*(T \otimes S \wedge [\Delta]) \text{ for } j = 1, 2, \tag{2.1.1}$$

where $[\Delta]$ is the current of integration on the diagonal Δ of $X \times X$.

Observe that because of $[\Delta]$, the formula (2.1.1) can not be extended to general singular currents T and S. We can however use the theory of intersection with $(1, 1)$-currents if in the place of Δ we have a hypersurface. This is the reason why we consider the blow-up $\widehat{X \times X}$ of $X \times X$ along Δ. Let Π be the natural projection from $\widehat{X \times X}$ to $X \times X$ and $\widehat{\Delta} = \Pi^{-1}(\Delta)$ be the exceptional hypersurface. Recall from [6, 81] that the blow-up of a compact Kähler manifold along a submanifold is also Kähler. Let $\widehat{\omega}$ be a Kähler form of $\widehat{X \times X}$. Observe that $\Pi_*(\widehat{\omega}^{n-1} \wedge [\widehat{\Delta}])$ is a non-zero positive closed current of $X \times X$ supported on Δ and has the same dimension as Δ. Therefore, it equals a constant times $[\Delta]$, see, e.g., [17]. By normalizing $\widehat{\omega}$, we can suppose that

$$\Pi_*(\widehat{\omega}^{n-1} \wedge [\widehat{\Delta}]) = [\Delta]. \tag{2.1.2}$$

Put $\widehat{T \otimes S} = \Pi^*(T \otimes S)$ and $\Pi_j = \pi_j \circ \Pi$ $(j = 1, 2)$. Then (2.1.1) can be rewritten as

$$T \wedge S = (\Pi_j)_*(\widehat{T \otimes S} \wedge \widehat{\omega}^{n-1} \wedge [\widehat{\Delta}]). \tag{2.1.3}$$

In general, when T and S are only positive closed currents, one still can define $\widehat{T \otimes S}$ as a positive closed current outside $\widehat{\Delta}$ and extend it by 0 through $\widehat{\Delta}$. We can show that $\widehat{T \otimes S} \wedge \widehat{\omega}^{n-1} \wedge [\widehat{\Delta}]$ is well-defined provided that T has a continuous super-potential. In this case, we can use (2.1.3) as an alternative definition of $T \wedge S$ which gives a positive closed current, see Corollary 2.3.5. Proposition 2.3.7 below shows that this definition is equivalent to the one of Dinh and Sibony.

2.2 Super-potential of positive closed currents

We will recall now some basic facts and refer to [33] for details. Let X be a compact Kähler manifold of dimension n and ω be a Kähler form on X. It is well-known that the de Rham cohomology of currents and smooth forms are canonically equal (see [41, Chap. 3]). Denote them by $H^r(X, \mathbb{C})$ with $0 \leq r \leq n$. For any closed current T of degree r, denoted by $\{T\}$ its cohomology class in $H^r(X, \mathbb{C})$. Let $H^{p,p}(X, \mathbb{R})$ be the vector subspace of $H^{p,p}(X, \mathbb{C})$ spanned by the classes of closed real $2p$-forms. Since a closed positive (p, p)-current is real, its class belongs to $H^{p,p}(X, \mathbb{R})$. If V is an analytic subset of X of dimension $n - p$, it defines a positive closed current $[V]$ of bidegree (p, p) by integration over V. Its class will be denoted by $\{V\}$ for simplicity.

Let $\mathcal{C}_p$ be the convex cone of positive closed (p,p)-currents on X and $\mathcal{D}_p$ be the real vector space generated by $\mathcal{C}_p$. Since the Kähler form ω is strictly positive, the set $\mathcal{D}_p$ contains all real closed smooth (p,p)-forms. Let $\mathcal{D}_p^0$ be the subspace of $\mathcal{D}_p$ of currents belonging to the class 0 in $H^{p,p}(X,\mathbb{R})$. We recall the notion of $*$-norm on $\mathcal{D}_p$. Consider first a positive closed current S in $\mathcal{D}_p$. Define its $*$-norm by

$$\|S\|_* = |\langle S, \omega^{n-p}\rangle|$$

which is equal to the mass of S. In general, since any $S \in \mathcal{D}_p$ can be written as the difference of two positive closed currents, define

$$\|S\|_* = \inf(\|S^+\|_* + \|S^-\|_*),$$

where the infimum is taken over all $S^+, S^- \in \mathcal{C}_p$ such that $S = S^+ - S^-$. By compactness property of positive closed currents, the above infimum is attained for some S^+ and S^-. We say that S_k converges to S in $\mathcal{D}_p$ for the $*$-topology if S_k converges to S weakly as currents and $\|S_k\|_*$ is bounded independently of k. The following result is due to Dinh and Sibony, see [33, Th. 2.4.4] and also [27, Th. 1.1].

Proposition 2.2.1. *There is a positive constant c such that for all $S \in \mathcal{D}_p$, there exist smooth forms $S_k \in \mathcal{D}_p$ with $k \in \mathbb{N}$ such that S_n converges weakly to S and $\|S_k\|_* \leq c\|S\|_*$ for all k.*

Let T be in $\mathcal{D}_p$ and R be in $\mathcal{D}_q^0$. By dd^c-lemma for currents (see [40, Th. 1.2.1]), there is a real $(q-1,q-1)$-current U_R such that $dd^c U_R = R$. We call U_R a potential of R. Consider the following important example of R. Let V be a hypersurface of X and β_0 be a smooth form of the same cohomology class with $[V]$. Then $R = [V] - \beta_0$ is in $\mathcal{D}_1^0$. One can construct an explicit potential U_R as follows. Consider the holomorphic line bundle of X associated with V and σ a holomorphic section whose divisor is V. Take a smooth Hermitian metric on this line bundle and denote by $|\cdot|$ the norm induced by this metric. By Poincaré-Lelong formula, there is a smooth form β_1 such that

$$dd^c \log|\sigma| = [V] - \beta_1.$$

Since $\{\beta_0\} = \{V\} = \{\beta_1\}$, there is a smooth function f on X such that $dd^c f = \beta_0 - \beta_1$. The function $U_R := \log|\sigma| - f$ is a potential of R. Note that U_R is smooth outside V and if σ' is a holomorphic function on an open neighborhood W of a point of V such that its divisor is $V \cap W$, then

$$U_R(x) - \log|\sigma'| \text{ is smooth on } W. \tag{2.2.1}$$

Consider now a current $R \in \mathcal{D}_{n-p+1}^0$ and an $(n-p,n-p)$-current U_R which is a potential of R. Let $\alpha = (\alpha_1, \cdots, \alpha_h)$ with $h = \dim H^{p,p}(X,\mathbb{R})$ be a fixed family of real smooth closed (p,p)-forms such that the family of classes $\{\alpha\} = (\{\alpha_1\}, \cdots, \{\alpha_h\})$ is a basis of $H^{p,p}(X,\mathbb{R})$. By adding to U_R a suitable closed smooth form, we can assume that $\langle U_R, \alpha_i\rangle = 0$ for $i = 1, \cdots, h$. We say that U_R is α-normalized.

Definition 2.2.2. *([33, Def. 3.2.2]) Let T be a current in $\mathcal{D}_p$ as above. The α-normalized super-potential $\mathcal{U}_T$ of T is the function defined on smooth forms $R \in \mathcal{D}_{n-p+1}^0$ and given by*

$$\mathcal{U}_T(R) = \langle T, U_R\rangle,$$

where U_R is an α-normalized smooth potential of R. We say that T has a continuous super-potential if $\mathcal{U}_T$ can be extended to a function on $\mathcal{D}^0_{n-p+1}$ which is continuous with respect to the $*$-topology. In this case, the extension is also denoted by $\mathcal{U}_T$.

By [33, Lem. 3.2.1], $\mathcal{U}_T(R)$ does not depend on the choice of an α-normalized U_R. And the continuity of $\mathcal{U}_T$ does not depend on α. Observe that when $\{T\} = 0$, the α-normalized super-potential of T does not depend on α. Indeed, in this case, it is the restriction of any potential U_T of T to the set of smooth forms in $\mathcal{D}^0_{n-p+1}$. Assume that T has a continuous super-potential. Take any current $S \in \mathcal{D}_q$. Let $(a_1, \cdots, a_h)$ be the coefficients of $\{T\}$ in the basis $\{\alpha\}$. Define $T \wedge S$ to be the real $(p+q, p+q)$-current satisfying

$$\langle T \wedge S, \Phi\rangle := \mathcal{U}_T\big(dd^c\Phi \wedge S\big) + \sum_{1\leq j\leq h} a_j\langle \alpha_j, \Phi \wedge S\rangle, \tag{2.2.2}$$

for any real smooth $(n-p-q, n-p-q)$-form Φ.

2.3 Alternative definition for the intersection of currents

Let $X, \widehat{X\times X}, \omega, \widehat{\omega}, \Pi, \Pi_j, \pi_j, \Delta, \widehat{\Delta}$ be as in the previous sections. Consider two currents $T \in \mathcal{D}_p$ and $S \in \mathcal{D}_q$ as above with $p+q \leq n$. Let h, a_j and α_j with $1 \leq j \leq h$ be as in the last section. From now on, assume that T is positive and has a continuous super-potential. Note that $\Pi_j = \pi_j \circ \Pi$ are submersions, for a proof see [33] or the proof of Lemma 2.3.2 below. Define $\widehat{T} = \Pi_1^*(T)$ and $\widehat{S} = \Pi_2^*(S)$. They are positive closed currents on $\widehat{X\times X}$. Put $\widehat{\alpha}_j = \Pi_1^*(\alpha_j)$ for $1 \leq j \leq h$.

Lemma 2.3.1. *The current $\widehat{T}$ has a continuous super-potential.*

Proof. Suppose that the classes $\{\widehat{\alpha}_j\}$ are linearly dependent. Then there exist real numbers b_j with $1 \leq j \leq h$ which are not simultaneously equal to zero and a smooth form $\widehat{\gamma}$ such that $\sum_{j=1}^h b_j\widehat{\alpha}_j = d(\widehat{\gamma})$. Taking the wedge product with $\widehat{\omega}^n$ in the last equality and then using the push-forward by $(\Pi_1)_*$ give

$$\sum_{j=1}^h b_j\alpha_j \wedge (\Pi_1)_*(\widehat{\omega}^n) = d\big((\Pi_1)_*(\widehat{\gamma}\wedge\widehat{\omega}^n)\big). \tag{2.3.1}$$

Note that $(\Pi_1)_*\widehat{\omega}^n$ is actually a nonzero constant since $\widehat{\omega}^n$ is closed and positive. We deduce that the left-hand side of (2.3.1) is a non-trivial linear combination of α_j, $1 \leq j \leq h$. However this contradicts the fact that $\{\alpha_j\}$ are linearly independent. Hence, the classes $\{\widehat{\alpha}_j\}$ are linearly independent. Complete them to be basis $\widehat{\alpha}'$ of $H^{p,p}(\widehat{X\times X}, \mathbb{R})$. Let $\mathcal{U}_{\widehat{T}}$ be the $\widehat{\alpha}'$-normalized super-potential of $\widehat{T}$.

Put $\alpha_T = \sum_{j=1}^h a_j\alpha_j$ and $\widehat{\alpha}_T = \Pi_1^*\alpha_T$. Remark that α_T and $\widehat{\alpha}_T$ are in the same cohomology classes with T and $\widehat{T}$ respectively. Let $U_{T-\alpha_T}$ be a potential of $T - \alpha_T$. Then $U_{\widehat{T}-\widehat{\alpha}_T} := \Pi_1^*U_{T-\alpha_T}$ is a potential of $\widehat{T} - \widehat{\alpha}_T$. By definition, for any smooth form $\tilde{R} \in \mathcal{D}^0_{2n-p+1}(\widehat{X\times X})$, we have

$$\mathcal{U}_{\widehat{T}}(\tilde{R}) = \langle \widehat{T}, U_{\tilde{R}}\rangle = \langle \widehat{T} - \widehat{\alpha}_T, U_{\tilde{R}}\rangle = \langle U_{\widehat{T}-\widehat{\alpha}_T}, \tilde{R}\rangle$$

By our choice of potentials, the last quantity equals

$$\langle U_{T-\alpha_T}, (\Pi_1)_* \tilde{R} \rangle = \mathcal{U}_T((\Pi_1)_* \tilde{R}).$$

The continuity of $\mathcal{U}_T$ now implies immediately the same property for $\mathcal{U}_{\widehat{T}}$. The proof is finished. □

Thanks to Lemma 2.3.1, one can define $\widehat{T} \wedge \widehat{S}$ as in (2.2.2). Recall that $T \otimes S$ is a positive closed $(p+q, p+q)$-current on $X \times X$ depending continuously on T and S. Its action on smooth forms can be described as follows. Let x be local coordinates of X. They induce naturally local coordinates (x, y) on $X \times X$. For a smooth form $\Phi(x, y)$ of $X \times X$, we have

$$\langle T \otimes S, \Phi \rangle = \big\langle T, S\big(\Phi(x, \cdot)\big)\big\rangle = \big\langle S, T\big(\Phi(\cdot, y)\big)\big\rangle. \tag{2.3.2}$$

Let Π' be the restriction of Π to $\widehat{X \times X} \backslash \widehat{\Delta}$. The current

$$\widehat{T \otimes S} = \Pi'^*(T \otimes S)$$

is well-defined and positive closed on $\widehat{X \times X} \backslash \widehat{\Delta}$ because Π' is biholomorphic. By Proposition 5.1 of [29], the mass of $\widehat{T \otimes S}$ is bounded. Hence, it can be extended by zero to be a positive closed current of $\widehat{X \otimes X}$ through $\widehat{\Delta}$, see [17, 70, 75]. We still denote by $\widehat{T \otimes S}$ the extended current. Take a smooth closed $(1,1)$-form $\widehat{\beta}$ with $\{\widehat{\beta}\} = \{\widehat{\Delta}\}$. Since $\widehat{\Delta}$ is a hypersurface, choose a potential

$$\hat{u} = U_{[\widehat{\Delta}] - \widehat{\beta}} \tag{2.3.3}$$

of $[\widehat{\Delta}] - \widehat{\beta}$ as in Section 2.2. It is smooth outside $\widehat{\Delta}$ and its behaviour near $\widehat{\Delta}$ is described by (2.2.1). By adding a constant to $\hat{u}$ if necessary, we can assume that $\hat{u} \leq -1$.

Lemma 2.3.2. *The current $\hat{u}\widehat{S}$ is well-defined. Moreover, if smooth forms $S_k \in \mathcal{D}_q$ converge to S in the $*$-topology, then $\hat{u}\widehat{S}_k$ converge weakly to $\hat{u}\widehat{S}$.*

Proof. We prove the first assertion. For any smooth $(2n-q, 2n-q)$-form $\hat{\eta}$ on $\widehat{X \times X}$, we will show that $(\Pi_2)_*(\hat{u}\hat{\eta})$ is a smooth form on X. This allows us to define

$$\langle \hat{u}\widehat{S}, \hat{\eta} \rangle = \langle S, (\Pi_2)_*(\hat{u}\hat{\eta}) \rangle. \tag{2.3.4}$$

To see that $(\Pi_2)_*(\hat{u}\hat{\eta})$ is smooth, we just need to work locally. Consider local coordinates $(W, x = (x_1, \cdots, x_n))$ on a chart W of X. Without loss of generality, we can suppose W is diffeomorphic to the unit ball $\mathbb{B}_1$ in $\mathbb{C}^n$. Consider induced local coordinates (x, y) on $W \times W$. We have $\Delta \cap (W \times W) = \{x = y\}$. Define new local coordinates (x', y) on $W \times W$ by putting $x' := x - y$. Hence Δ is given by the equation $x' = 0$. The set $\Pi^{-1}(W \times W)$ is biholomorphic to the manifold M in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{P}^{n-1}$ defined by

$$M = \big\{(x', y, [v]) : y \in \mathbb{B}_1,\, x' + y \in \mathbb{B}_1,\, [v] \in \mathbb{P}^{n-1} \text{ and } x' \in [v]\big\},$$

where $[v] = [v_1 : v_2 : \cdots : v_n]$ denotes the homogeneous coordinates of $\mathbb{P}^{n-1}$. Let M_j $(1 \leq j \leq n)$ be the open subset of M containing all points $(x', y, [v]) \in M$ with $v_j \neq 0$. They form an open covering of M. For $(x', y, [v]) \in M_1$, we have $x'_1 v_j = x'_j v_1$. Choose $v_1 = 1$, then

$x'_j = x'_1 v_j$. We deduce that $\big(x'_1, v_2, \cdots, v_n, y\big)$ are coordinates on M_1 and $\widehat{\Delta} \cap M_1 = \{x'_1 = 0\}$. Since $\Pi_2(x'_1, v_2, \cdots, v_n, y) = y$, we see that

$$\begin{aligned}(\Pi_2)_*(\hat{u}\hat{\eta}) &= \int_{x'_1, v_2, \cdots, v_n} \hat{u}(x'_1, v_2, \cdots, v_n, y)\hat{\eta}(x'_1, v_2, \cdots, v_n, y) \\ &= \int_{x'_1, v_2, \cdots, v_n} \log|x'_1|\hat{\eta}(x'_1, v_2, \cdots, v_n, y) \\ &+ \quad \int_{x'_1, v_2, \cdots, v_n} \hat{u}'(x'_1, v_2, \cdots, v_n, y)\hat{\eta}(x'_1, v_2, \cdots, v_n, y),\end{aligned}$$

where $\hat{u}'(x'_1, v_2, \cdots, v_n, y)$ is a smooth function, see (2.2.1). This implies that the last integral defines a smooth form in y. It is also clear that the integral involving $\log|x'_1|$ depends smoothly in y. The proof of the first assertion is finished. The second assertion is a direct consequence of the identity (2.3.4). The proof is finished. □

Proposition 2.3.3. *We have* $\widehat{T} \wedge \widehat{S} = \widehat{T \otimes S}$.

Proof. Consider first the case where S is smooth. So $\widehat{T} \wedge \widehat{S}$ is the usual wedge product of a current with a smooth form. We then see that $\widehat{T} \wedge \widehat{S} = \Pi^*(T \otimes S) = \widehat{T \otimes S}$ outside $\widehat{\Delta}$. Observe that the fibers of the submersion Π_1 are transverse to $\widehat{\Delta}$. Therefore, $\widehat{T}$ has no mass on $\widehat{\Delta}$. Hence, $\widehat{T} \wedge \widehat{S}$ has no mass on $\widehat{\Delta}$. We deduce that $\widehat{T} \wedge \widehat{S} = \widehat{T \otimes S}$ in this case because $\widehat{T \otimes S}$ has no mass on $\widehat{\Delta}$ by definition.

In general, by Proposition 2.2.1, there is a sequence of smooth forms $S_k \in \mathcal{D}_q$ converging to S in the $*$-topology. The first case and the continuity on S imply that $\widehat{T} \wedge \widehat{S} = \widehat{T \otimes S}$ outside $\widehat{\Delta}$. It remains to show that the restriction $\mathbf{1}_{\widehat{\Delta}}(\widehat{T} \wedge \widehat{S})$ of $\widehat{T} \wedge \widehat{S}$ vanishes. This is equivalent to say that

$$\int_{\widehat{\Delta}} \widehat{T} \wedge \widehat{S} \wedge \widehat{\Phi} = 0, \tag{2.3.5}$$

for any smooth form $\widehat{\Phi}$ of bidegree $2n-p-q$. By Proposition 2.2.1, we can write $S = S^+ - S^-$ where S^+ and S^- are approximable by smooth positive closed forms. Since $\widehat{T} \wedge \widehat{S} = \widehat{T} \wedge \widehat{S}^+ - \widehat{T} \wedge \widehat{S}^-$, we only need to verify that $\mathbf{1}_{\widehat{\Delta}}(\widehat{T} \wedge \widehat{S}^{\pm}) = 0$. Therefore, without loss of generality, assume that $\widehat{T} \wedge \widehat{S}$ is positive. Consequently, it suffices to prove (2.3.5) for $\widehat{\Phi} = \widehat{\omega}^{2n-p-q}$.

Let χ be a convex increasing smooth function on $\mathbb{R}$ such that $\chi(t) = 0$ if $t \leq -1/4$, $\chi(t) = t$ for $t \geq 1/4$ and $0 \leq \chi' \leq 1$. For each positive integer k, put

$$\hat{u}_k = \chi(\hat{u} + k) - k.$$

This is a smooth negative quasi-p.s.h. function since $\hat{u} \leq -1$. The functions $\hat{u}_k$ decrease to $\hat{u}$ and $-\hat{u}_k/k$ decrease to the characteristic function $\mathbf{1}_{\widehat{\Delta}}$ of $\widehat{\Delta}$ as $k \to \infty$. The first property implies that $\widehat{S} \wedge dd^c\hat{u}_k$ converges weakly to $\widehat{S} \wedge dd^c\hat{u}$, see Lemma 2.3.2. We also have

$$dd^c\hat{u}_k = [\chi''(\hat{u} + k)]d\hat{u} \wedge d^c\hat{u} + \chi'(\hat{u} + k)dd^c\hat{u} \geq \chi'(\hat{u} + k)dd^c\hat{u} \geq -c\widehat{\omega},$$

for some positive constant c. This yields that $dd^c\hat{u}_k = (dd^c\hat{u}_k + c\widehat{\omega}) - c\widehat{\omega}$ which is the difference of two positive closed currents in the same cohomology class $c\{\widehat{\omega}\}$. We deduce that $dd^c\hat{u}_k$ is $*$-bounded uniformly in k and then so is $\widehat{S} \wedge dd^c\hat{u}_k \wedge \widehat{\omega}^{2n-p-q}$ because we have

$$\|\widehat{S} \wedge dd^c\hat{u}_k \wedge \widehat{\omega}^{2n-p-q}\|_* \leq c\|S\|_*\|dd^c\hat{u}_k\|_*, \tag{2.3.6}$$

for a positive constant c depending only on (X, ω). It follows that

$$\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q} \to \widehat{S} \wedge dd^c \hat{u} \wedge \widehat{\omega}^{2n-p-q}$$

in the $*$-topology. The equality (2.3.5) with $\widehat{\Phi} = \widehat{\omega}^{2n-p-q}$ is equivalent to

$$\big\langle \widehat{T} \wedge \widehat{S}, -\frac{\hat{u}_k}{k} \cdot \widehat{\omega}^{2n-p-q} \big\rangle \to 0 \text{ as } k \to \infty. \tag{2.3.7}$$

Applying the formula (2.2.2) to $\widehat{T} \wedge \widehat{S}$ gives

$$\big\langle \widehat{T} \wedge \widehat{S}, -\frac{\hat{u}_k}{k} \cdot \widehat{\omega}^{2n-p-q} \big\rangle = -\frac{1}{k} \mathcal{U}_{\widehat{T}}\big(\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}\big) - \frac{1}{k} \big\langle \widehat{\alpha}_T, \hat{u}_k \widehat{S} \wedge \widehat{\omega}^{2n-p-q} \big\rangle,$$

where $\widehat{\alpha}_T = \sum_{j=1}^h a_j \widehat{\alpha}_j$. The last quantity converges to 0 as $k \to \infty$ for the mass norm of $\hat{u}_k \widehat{S}$ is bounded independently of k by Lemma 2.3.2. On the other hand, the continuity of $\mathcal{U}_{\widehat{T}}$ gives

$$\mathcal{U}_{\widehat{T}}\big(\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}\big) \to \mathcal{U}_{\widehat{T}}\big(\widehat{S} \wedge dd^c \hat{u} \wedge \widehat{\omega}^{2n-p-q}\big)$$

which is finite, as $k \to \infty$. Hence we get (2.3.7). The proof is finished. □

Lemma 2.3.4. *The current $\hat{u}(\widehat{T} \wedge \widehat{S})$ is well-defined. Denote it by $\hat{u}\widehat{T} \wedge \widehat{S}$ for simplicity. For any closed real smooth form $\widehat{\Phi}$ of $\widehat{X \times X}$ of the right bidegree, we have*

$$\langle \hat{u}\widehat{T} \wedge \widehat{S}, \widehat{\Phi} \rangle = \mathcal{U}_{\widehat{T}}\big(dd^c(\hat{u}\widehat{S} \wedge \widehat{\Phi})\big) + \sum_{j=1}^h a_j \langle \widehat{S}, \hat{u}\widehat{\alpha}_j \wedge \widehat{\Phi} \rangle. \tag{2.3.8}$$

In particular, $\langle \hat{u}\widehat{T} \wedge \widehat{S}, \widehat{\Phi} \rangle$ depends continuously on S.

Proof. Using the computation in the proof of Proposition 2.3.3, we have

$$\langle \widehat{T} \wedge \widehat{S}, \hat{u} \cdot \widehat{\omega}^{2n-p-q} \rangle = \lim_{k \to \infty} \mathcal{U}_{\widehat{T}}\big(\widehat{S} \wedge dd^c \hat{u}_k \wedge \widehat{\omega}^{2n-p-q}\big) + \big\langle \widehat{\alpha}_T, \hat{u}_k \widehat{S} \wedge \widehat{\omega}^{2n-p-q} \big\rangle,$$

where $\hat{u}_k$ is defined as in Proposition 2.3.3. The same arguments at the end of the above proposition show that the last limit is finite. The first assertion follows. Note that each smooth closed form Φ can be written as the difference of two positive closed forms. Hence it is enough to prove (2.3.8) for positive closed forms Φ. The computations in Proposition 2.3.3 still hold for Φ in place of $\widehat{\omega}^{2n-p-q}$. Hence (2.3.8) follows.

In order to prove the last assertion, it is enough to prove it for positive closed forms Φ by the same reason as above. Let $\{S_l\}_{l \in \mathbb{N}}$ be a sequence of currents in $\mathcal{D}_q$ which converges to S in the $*$-topology. Put $\widehat{S}_l = \Pi_2^*(S_l)$. It is clear that $\widehat{S}_l$ converges to $\widehat{S}$ in the $*$-topology. Lemma 2.3.2 implies that $dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi})$ converges weakly to $dd^c(\hat{u}\widehat{S} \wedge \widehat{\Phi})$ and

$$\lim_{k \to \infty} dd^c(\hat{u}_k \widehat{S}_l \wedge \widehat{\Phi}) = dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi}), \tag{2.3.9}$$

for any $l \in \mathbb{N}$. Applying (2.3.6) to S_k in place of S, we see that the mass of $dd^c(\hat{u}_k \widehat{S}_l \wedge \widehat{\Phi})$ is bounded independently of k and l. This combined with (2.3.9) yields that the $*$-norm of $dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi})$ is bounded independently of l. We deduces that $dd^c(\hat{u}\widehat{S}_l \wedge \widehat{\Phi})$ converges to $dd^c(\hat{u}\widehat{S} \wedge \widehat{\Phi})$ in the $*$-topology. The continuity of $\mathcal{U}_{\widehat{T}}$ now implies that the right-hand side of (2.3.8) depends continuously on S. The proof is finished.

□

Corollary 2.3.5. *Define the intersection $\widehat{T\otimes S}\wedge[\widehat{\Delta}]$ by putting*

$$\widehat{T\otimes S}\wedge[\widehat{\Delta}]=dd^c\big(\hat{u}\widehat{T\otimes S}\big)+\widehat{T\otimes S}\wedge\widehat{\beta},\tag{2.3.10}$$

see (2.3.3) for the definition of $\widehat{\beta}$. Then $\widehat{T\otimes S}\wedge[\widehat{\Delta}]$ is positive when S is positive.

Proof. We only need to prove the positivity. This property is classical since the current $[\widehat{\Delta}]$ is of bidegree $(1,1)$. We give here a proof for the sake of the reader. Fix a small open subset $\widehat{W}$ of $\widehat{X\times X}$ biholomorphic to a ball. We can find a smooth function $\hat{v}$ on $\widehat{W}$ such that $dd^c\hat{v}=\widehat{\beta}$. Hence the function $\hat{u}'=\hat{u}+\hat{v}$ satisfies $dd^c\hat{u}'=[\widehat{\Delta}]\geq 0$. So $\hat{u}'$ is p.s.h. on $\widehat{W}$. We then have $\widehat{T\otimes S}\wedge[\widehat{\Delta}]=dd^c\big(\hat{u}'\widehat{T\otimes S}\big)$ on $\widehat{W}$. If $\hat{u}'_k$ is a sequence of smooth p.s.h. functions on $\widehat{W}$ decreasing to $\hat{u}'$, then the last current is the limit of $dd^c\big(\hat{u}'_k\widehat{T\otimes S}\big)$ which is clearly positive since it equals $dd^c\hat{u}'_k\wedge\widehat{T\otimes S}$. The proof is finished. □

Lemma 2.3.6. *Let Y be a closed subset of X. Let R be a positive (p,p)-current of X and let R_k be a sequence of positive (p,p)-currents of X converging weakly to R as currents in $X\backslash Y$. Assume that R has no mass on Y and the masses of R_k converge to the one of R. Then R_k converges weakly to R in X.*

Proof. For each $\epsilon>0$, let Y_ϵ be the set of points in X of distance less than ϵ to Y. Let χ_ϵ be a continuous function on X such that $0\leq\chi_\epsilon\leq 1$ and $\chi_\epsilon=1$ on $X\backslash Y_{2\epsilon}$ and $\chi_\epsilon=0$ on $\overline{Y}_\epsilon$. Take any continuous real form Φ on X of bidegree $n-p$. We need to prove that

$$R_k(\Phi)\to R(\Phi)\quad\text{ as }\quad k\to\infty.\tag{2.3.11}$$

Since a continuous form can be written as the difference of two continuous positive forms, we can assume that Φ is positive. The hypothesis on R_k implies that $R_k(\chi_\epsilon\Phi)$ converges to $R(\chi_\epsilon\Phi)$. Hence in order to prove (2.3.11), it is sufficient to show that

$$\lim_{\epsilon\to 0}\delta_\epsilon=0,\tag{2.3.12}$$

where

$$\delta_\epsilon=\limsup_{k\to\infty}\int_{\overline{Y}_{2\epsilon}}R_k(\Phi).$$

Let $\mu_k=R_k\wedge\omega^{n-p}$ and $\mu=R\wedge\omega^{n-p}$ be the trace measures of R_k and R respectively. Observe that δ_ϵ is less than a constant times

$$\limsup_{k\to\infty}\mu_k(\overline{Y}_{2\epsilon})=\|R\|-\liminf_{k\to\infty}\mu_k(X\backslash\overline{Y}_{2\epsilon}).$$

Since the set $X\backslash\overline{Y}_{2\epsilon}$ is an open subset of $X\backslash Y$, the last limit is greater than $\mu(X\backslash\overline{Y}_{2\epsilon})$. Hence we get

$$\limsup_{k\to\infty}\int_{\overline{Y}_{2\epsilon}}R_k(\Phi)\lesssim\|R\|-\mu(X\backslash\overline{Y}_{2\epsilon})=\mu(\overline{Y}_{2\epsilon}).$$

The last quantity converges to zero as $\epsilon\to 0$ because μ has no mass on Y. The proof is finished. □

Proposition 2.3.7. *For $j = 1$ or 2, we have*

$$T \wedge S = (\Pi_j)_*\big(\widehat{T \otimes S} \wedge [\widehat{\Delta}] \wedge \widehat{\omega}^{n-1}\big), \tag{2.3.13}$$

where $T \wedge S$ is defined as in (2.2.2).

Proof. As explained in Introduction, the formula (2.3.13) holds for smooth forms T and S. We consider now the general case. We already know that $T \wedge S$ depends continuously on S for the $*$-topology. Let $\{S_k\}_{k\in\mathbb{N}}$ be a sequence of smooth forms in $\mathcal{D}_q$ which converges to S in the $*$-topology. Put $\widehat{S}_k = \Pi_2^*(S_k)$ and $R_k = \hat{u}\widehat{T} \wedge \widehat{S}_k$. It follows from Lemma 2.3.4 that the masses of R_k converge to the mass of $R = \hat{u}\widehat{T} \wedge \widehat{S}$. Moreover, R_k converges to R in $\widehat{X \times X}\backslash\widehat{\Delta}$. Applying Lemma 2.3.6 to $\widehat{X \times X}$ in the place of X, R_k and R, we see that the right-hand sides of (2.3.13), which is defined in Corollary 2.3.5, also depends continuously on S for the $*$-topology. Hence approximating S by smooth forms allows us to assume that S is smooth. Now Lemma 2.3.2 applied to $\widehat{T}$ in place of $\widehat{S}$ implies that the right-hand side of (2.3.13) is continuous in T. When S is smooth, it is clear that $T \wedge S$ depends continuously on T. Therefore, (2.3.13) holds since we can approximate T by closed smooth forms. The proof is finished. □

Chapter 3

Distribution des points de Fekete

Let L be a positive line bundle over a compact complex projective manifold X and $K \subset X$ be a compact set which is regular in a sense of pluripotential theory. A Fekete configuration of order k is a finite subset of K maximizing a Vandermonde type determinant associated with the power L^k of L. Berman, Boucksom and Witt Nyström proved that the empirical measure associated with a Fekete configuration converges to the equilibrium measure of K as $k \to \infty$. Dinh, Ma and Nguyen obtained an estimate for the rate of convergence. Using techniques from Cauchy-Riemann geometry, we show that the last result holds when K is a real nondegenerate $\mathcal{C}^5$-piecewise submanifold of X such that its tangent space at any regular point is not contained in a complex hyperplane of the tangent space of X at that point. In particular, the estimate holds for Fekete points on some compact sets in $\mathbb{R}^n$ or the unit sphere in $\mathbb{R}^{n+1}$. This chapter is based on the article [82].

3.1 Introduction

The aim of this paper is to obtain an estimate on the rate of convergence of Fekete points on some compact sets toward the equilibrium state. In the view of possible applications, the class of compact sets that we consider is large enough and the required conditions for our compact sets are simple to check. Before introducing the general complex setting, let us discuss a simple but already important case of Fekete points for a compact K of the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$.

For each $k \in \mathbb{N}$, let $\mathcal{P}_k(K)$ be the real vector space of all real polynomials of degree at most k in $(n+1)$ variables restricted to K. Let N_k be the dimension of $\mathcal{P}_k(K)$. Given a basis $S = \{s_1, \cdots, s_{N_k}\}$ of $\mathcal{P}_k(K)$, consider the Vandermonde determinant $\det S$ of S defined by assigning each point $x = (x_1, \cdots, x_{N_k}) \in (\mathbb{S}^n)^{N_k}$ to

$$\det S(x) := \det[s_j(x_l)]_{1 \le j,l \le N_k}.$$

A Fekete point of order k for K is a point $x \in K^{N_k}$ maximizing the absolute function $|\det S|$ on K^{N_k}. It is easy to see that this definition does not depend on the choice of the basis S. The study of Fekete points is motivated by the fact that they are good choices of points for the interpolation problem of functions by polynomials, see, e.g., [8, 76] and references therein for more information. For any Fekete point $x = (x_1, \cdots, x_{N_k})$ of order k, the probability measure

δ_x on K, defined by

$$\delta_x := \frac{1}{N_k}\sum_{j=1}^{N_k}\delta_{x_j},$$

is called the Fekete measure of order k associated with x. We are interested in the distribution of Fekete points of order k as $k \to \infty$. A natural way to formulate this question is to study the limit points of Fekete measures in the space of probability measures on K.

Let μ_{eq} be the equilibrium measure of $(K, 0)$ which is defined in Section 3.2. When $K = \mathbb{S}^n$, the measure μ_{eq} is simply the normalized volume form on $\mathbb{S}^n$ induced by the Euclidean metric on $\mathbb{R}^{n+1}$. In this case, J. Marzo and J. Ortega-Cerdà in [54] prove that Fekete measures of order k converge weakly to μ_{eq} as $k \to \infty$. In general, a recent result of R. Berman, S. Boucksom and D. Witt Nyström in [4] shows that the weak convergence also holds for any compact K of $\mathbb{S}^n$ which is non-pluripolar in the natural complexification $\mathbb{S}^n_{\mathbb{C}}$ of $\mathbb{S}^n$. In fact, this result holds in a general setting of Fekete points associated with a big line bundle over a compact complex manifold. Also in this general setting, Dinh, Ma and Nguyen [23] introduced a notion of $(\mathcal{C}^\alpha, \mathcal{C}^{\alpha'})$-regularity and obtained a precise estimate on the speed of convergence of Fekete points when the compact K satisfies this property. We will show that such a property holds for the closures K of open subsets of $\mathbb{S}^n$ with nondegenerate piecewise smooth boundary (see Definition 3.1.3). As a consequence, we will have the following.

Theorem 3.1.1. *Let K be the closure of an open subset of $\mathbb{S}^n$ with nondegenerate piecewise smooth boundary. For every $\epsilon \in (0, 1)$, there is a positive constant c_ϵ independent of $k \in \mathbb{N}$ such that for any Fekete measure μ_k of order k of K, we have*

$$\mathrm{dist}_1(\mu_k, \mu_{eq}) \leq c_\epsilon k^{-1/72+\epsilon}. \tag{3.1.1}$$

Recall that for any two probability measures μ, μ' on a compact differentiable manifold X and a real number $\gamma > 0$, define

$$\mathrm{dist}_\gamma(\mu, \mu') := \sup_{\|v\|_{\mathcal{C}^\gamma} \leq 1} \big|\langle \mu - \mu', v\rangle\big|,$$

where v is a smooth real-valued function. This distance induces the weak topology on the space of probability measures on X. For two positive numbers γ, γ' with $\gamma < \gamma'$, the distances dist_γ and $\mathrm{dist}_{\gamma'}$ are related by the inequalities

$$\mathrm{dist}_{\gamma'} \leq \mathrm{dist}_\gamma \leq c\,\mathrm{dist}_{\gamma'}^{\gamma/\gamma'},$$

for some positive constant c, see [30, 78]. Note that $\mathrm{dist}_1(\mu, \mu')$ is equivalent to the Kantorovich-Wasserstein distance. We have a better estimate when $K = \mathbb{S}^n$.

Theorem 3.1.2. *For every $\epsilon \in (0, 1)$, there is a positive constant c_ϵ depending only on (n, ϵ) such that for any Fekete measure μ_k of order k of $\mathbb{S}^n$, we have*

$$\mathrm{dist}_1(\mu_k, \mu_{eq}) \leq c_\epsilon k^{-1/36+\epsilon}. \tag{3.1.2}$$

It is worth mentioning also that when K is the closure of an open subset of $\mathbb{R}^n$ with nondegenerate piecewise smooth boundary, one can define the Fekete points in K and Fekete

measures in the same way as above. The analogue of the inequality $(3.1.1)$ also holds for this case. This could be deduced from our general result by using the compact complexification $\mathbb{P}^n$ of $\mathbb{R}^n$, where $\mathbb{P}^n$ is the complex projective space of dimension n.

In order to prove Theorem 3.1.1, we will work in the following context of complex geometry. Let X be a compact n-dimensional complex projective manifold admitting an ample line bundle L. Fix a smooth Hermitian metric h_0 on L with positive curvature. Let ω_0 be the first Chern form of (L, h_0) which is a Kähler form on X. For $k \in \mathbb{N}$, let $H^0(X, L^k)$ be the complex vector space of global holomorphic sections of L^k. We also use N_k to denote the dimension of $H^0(X, L^k)$. This will cause no ambiguity because we discuss essentially the general case from now on. Consider a basis

$$S = (s_1, s_2, \cdots, s_{N_k})$$

of $H^0(X, L^k)$ which can be seen to be a section of the vector bundle $L^k \times \cdots \times L^k$ of rank N_k over X^{N_k}. The determinant

$$\det S(p) := \det[s_i(p_j)]_{1 \le i,j \le N_k}$$

with $p = (p_1, \cdots, p_{N_k}) \in X^{N_k}$ defines a section of the determinant line bundle $L^{\boxtimes N_k}$ of the last bundle. The metric h_0 induces naturally a metric on $L^{\boxtimes N_k}$. Denote by $|\det S|$ the norm of $\det S$ measured by this natural metric on $L^{\boxtimes N_k}$.

Let K be a compact subset of X. Let ϕ be a continuous function on K. *The weighted Vandermonde determinant* $|\det S|_\phi$ at a point $p = (p_1, \cdots, p_{N_k}) \in K^{N_k}$, by definition, is

$$|\det S|_\phi(p) := |\det S(p)| e^{-k\phi(p_1) - \cdots - k\phi(p_{N_k})}.$$

A *Fekete configuration of order* k associated with (K, ϕ) and (L, h_0) is a point in K^{N_k} maximizing the above weighted Vandermonde determinant on K^{N_k}. The associated probability measure

$$\frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{p_j}$$

on K is called a *Fekete measure of order* k.

Recall that a convex polyhedron K'' in $\mathbb{R}^M$ with $M \in \mathbb{N}^*$ is the intersection of a finite number of closed half-spaces in $\mathbb{R}^M$. Its dimension is defined to be the one of the smallest vector subspace of $\mathbb{R}^M$ containing it. Such subspace is said to support K''. We define the complementary polyhedron of K'' to be the complement of K'' in the vector subspace supporting it. That complementary polyhedron is clearly a finite union of convex ones.

Definition 3.1.3. *A subset K of a real M-dimensional smooth manifold Y is called a nondegenerate $\mathcal{C}^5$-piecewise submanifold of dimension m if for every point $p \in K$, there exists a local chart (W_p, Ψ) of Y such that Ψ is a $\mathcal{C}^5$-diffeomorphism from W_p to the unit ball of $\mathbb{R}^M$ and $\Psi(K \cap W_p)$ is the intersection with the unit ball of a finite union of convex polyhedra of the same dimension m. In particular, when $\Psi(K \cap W_p)$ is the intersection with the unit ball of a convex polyhedron of dimension m or of the complementary polyhedron of a convex one of dimension m, we say that K is a $\mathcal{C}^5$ submanifold of dimension m with nondegenerate piecewise boundary.*

Let K be a nondegenerate $\mathcal{C}^5$-piecewise submanifold of dimension m of some smooth manifold Y. *A regular point* of K is a point of K such that there exists an open neighborhood

of it diffeomorphic to an open subset of $\mathbb{R}^m$. The regular part of K is the set of regular points of K. *The singularity of* K is the complement of the regular part of K in K.

Now let K be a nondegenerate $\mathcal{C}^5$-piecewise submanifold of X. Since X is a complex manifold, its real tangent spaces have a natural complex structure. We say that K is *generic* in the sense of Cauchy-Riemann geometry if the tangent space at any regular point p of K is not contained in a complex hyperplane of the (real) tangent space at p of X. One can see without difficulty that the dimension of a generic K is at least n. Here is our second main result.

Theorem 3.1.4. *Let α be a real number in $(0,1)$. Let K be a compact generic nondegenerate $\mathcal{C}^5$-piecewise submanifold of X. Let ϕ be a function of Hölder class $\mathcal{C}^\alpha$ on K. Then for every $0<\gamma<2$, there is a constant $c>0$ such that for every integer $k\geq 1$ and for every Fekete measure μ_k of order k associated with (K,ϕ), we have*

$$\operatorname{dist}_\gamma\left(\mu_k, \mu_{eq}(K,\phi)\right) \leq ck^{-\beta\gamma}(\log k)^{3\beta\gamma}, \tag{3.1.3}$$

where $\mu_{eq}(K,\phi)$ is the equilibrium measure of (K,ϕ) (see Definition 3.2.1) and $\beta=\alpha/(48+24\alpha)$. When K has no singularity, the constant β can be chosen to be $\alpha/(24+12\alpha)$.

In general, when K is an arbitrary non-pluripolar compact subset of X and $\phi\in\mathcal{C}^0(K)$, Boucksom, Berman and Witt Nyström in [4] proved that μ_k converges weakly to μ_{eq}. Using a different technique, Lev and Ortega-Cerdà in [51] obtained an optimal speed for the dist_1 provided that $K=X$ and ϕ is smooth ω_0-strictly p.s.h. and the metric $e^{-2\phi}h_0$ is strictly positive. Very recently, Dinh, Ma and Nguyen in [23] introduced the notion of $(\mathcal{C}^\alpha,\mathcal{C}^{\alpha'})$-regularity and proved an estimate for the rate of convergence for every compact K satisfying this property, see Theorem 3.2.2. In particular, they showed that the closure of an open subset of X with smooth boundary enjoys such regularity. Finally, we would like to remark that the case of Fekete's points can be regarded as the extreme case of some random point processes called β-ensembles as $\beta\to\infty$. Similar equidistribution results are also obtained for the latter. We refer to [5, 25, 9] and references therein for physical interpretations of β-ensembles and recent results on them.

In this work, we will prove that the compact K in Theorem 3.1.4 satisfies the regularity mentioned above. Hence Theorem 3.1.4 will follow immediately. For the proof, we develop ideas from [23, Th. 2.7]. The key point is to construct families of analytic discs partly attached to K in X with useful properties. These families will allow us to reduce the question to the case of dimension one. The reader may consult [58, 59, 60] for applications of analytics discs. Although there are plenty of works concerning families of analytics discs, it seems that there is no result which can be used directly for our problem. We will establish a Bishop-type equation and prove that it has a (unique) solution which suits our purposes. For the reader's convenience, a self-contained proof will be given. The construction is inspired by the work of Merker and Porten in [58, 59]. We also underline that the case where the singularity of K is nonempty requires much more sophisticated technical arguments than the case without singularity although the ideas used in the both situations are similar.

In Section 3.2, we prove the aforementioned regularity property of K by admitting the existence of special families of analytic discs whose proof is postponed until Section 3.3 and 3.4. In Section 3.3, we prove the existence of the above families of analytic discs in the simplest case by constructing special analytic discs partly attached to $\mathbb{R}^n$ or $(\mathbb{R}^+)^n$ in $\mathbb{C}^n$. In

Section 3.4, we show that the required families can be obtained as small deformations of the previous ones constructed in Section 3.3.

3.2 $(\mathcal{C}^\alpha, \mathcal{C}^{\alpha'})$-regularity of generic submanifolds

We first recall some definitions. Let $d^c := \frac{i}{2\pi}(\bar\partial - \partial)$, hence $dd^c = \frac{i}{\pi}\partial\bar\partial$. A function $\psi : X \to \mathbb{R} \cup \{-\infty\}$ is called *quasi-plurisubharmonic* (quasi-p.s.h. for short) if it is locally the sum of a plurisubharmonic function and a smooth one. A quasi-p.s.h. function is called ω_0-*p.s.h.* if $dd^c\psi + \omega_0 \geq 0$. Let K be a compact subset of X and ϕ be a real-valued continuous function on K. The pair (K, ϕ) is called *a weighted compact subset* of X and ϕ is called a *weight* on K. The *equilibrium weight* associated with (K, ϕ) is the upper semi-continuous regularization ϕ_K^* of the function

$$\phi_K := \sup\{\psi(z) : \psi\ \omega_0\text{-p.s.h.}, \psi \leq \phi \text{ on } K\}.$$

Since the constant function $-\max_K |\phi|$ is a ω_0-p.s.h. and bounded above by ϕ on K, we have $\phi_K \geq -\max_K |\phi|$. It follows that ϕ_K is bounded from below. Recall that K is said to be pluripolar if it is locally contained in $\{\psi = -\infty\}$ for some (local) p.s.h. function ψ, otherwise we say that K is non-pluripolar. It is well-known that ϕ_K is bounded from above if and only if K is non-pluripolar. In this case, ϕ_K^* is a bounded ω_0-p.s.h. function. The Monge-Ampère measure $(\omega_0 + dd^c\phi_K^*)^n$ is hence well-defined. Its mass on X equals $\int_X \omega_0^n$ by Stokes' theorem. The *equilibrium measure* of (K, ϕ) is the normalized Monge-Ampère measure defined by

$$\mu_{eq}(K, \phi) := \frac{\big(dd^c(\phi_K^*) + \omega_0\big)^n}{\int_X \omega_0^n}. \tag{3.2.1}$$

Recall that $\mu_{eq}(K, \phi)$ is a probability measure supported on K. It is a direct consequence of Theorem 3.2.3 below that any compact generic nondegenerate $\mathcal{C}^5$-piecewise submanifold of X is non-pluripolar.

Fix a Riemannian metric on X. For $p \in X$ and $r > 0$, let $B_X(p, r)$ be the ball centered at p of radius r of X. Put $B_X^*(p, r) := B_X(p, r)\backslash\{p\}$. Recall that for $0 < \alpha < 1$, $\mathcal{C}^\alpha(X)$ is the space of real functions of Hölder class $\mathcal{C}^\alpha$ on X with the norm defined by

$$\|\phi\|_{\mathcal{C}^\alpha(X)} := \sup_{p\in X} |\phi(p)| + \sup_{p\neq p', p, p'\in X} \frac{|\phi(p) - \phi(p')|}{\operatorname{dist}(p, p')^\alpha},$$

where dist denotes the distance on X. The space $\mathcal{C}^\alpha(K)$ is defined similarly.

Definition 3.2.1. *For $\alpha \in (0, 1)$ and $\alpha' \in (0, 1)$, a non-pluripolar compact K is said to be $(\mathcal{C}^\alpha, \mathcal{C}^{\alpha'})$-regular if for any positive constant C, the set $\{\phi_K : \phi \in \mathcal{C}^\alpha(K)$ and $\|\phi\|_{\mathcal{C}^\alpha(K)} \leq C\}$ is a bounded subset of $\mathcal{C}^{\alpha'}(X)$.*

By definition, if K is $(\mathcal{C}^\alpha, \mathcal{C}^{\alpha'})$-regular, for any Hölder continuous function ϕ of order $\alpha > 0$ on K, we have $\phi_K^* = \phi_K$ because the latter is also Hölder continuous. The notion of $(\mathcal{C}^\alpha, \mathcal{C}^{\alpha'})$-regularity is essential in our work. The reason is the following result.

Theorem 3.2.2. *([23, Th. 1.5]) Let K be a non-pluripolar compact subset of X. Let $\alpha \in (0,1)$, $\alpha' \in (0,1)$ and $\gamma \in (0,2]$. Assume that K is $(\mathcal{C}^\alpha, \mathcal{C}^{\alpha'})$-regular. Let ϕ be a $\mathcal{C}^\alpha$ real-valued function on K. Then, there is $c > 0$ such that for every $k > 1$, we have*

$$\mathrm{dist}_\gamma\left(\mu_k, \mu_{eq}(K,\phi)\right) \leq ck^{-\beta\gamma}(\log k)^{3\beta\gamma},$$

with $\beta = \alpha'/(24 + 12\alpha')$.

Theorem 3.1.4 is a direct consequence of Theorem 3.2.2 and Theorem 3.2.3 below.

Theorem 3.2.3. *Let α be an arbitrary number in $(0,1)$. Then any compact generic nondegenerate $\mathcal{C}^5$-piecewise submanifold K of X is $(\mathcal{C}^\alpha, \mathcal{C}^{\alpha/2})$-regular. Particularly, when K has no singularity, K is $(\mathcal{C}^\alpha, \mathcal{C}^\alpha)$-regular.*

When K is of maximal real dimension, the regularity of K can be improved, see Remark 3.2.7 for more details.

Remark 3.2.4. *Consider the case where $\dim_{\mathbb{R}} K = n$. This is the case of our great interest. Then, the regularity of K obtained in Theorem 3.2.3 is optimal. For simplicity, let take $n = 1$ and $X = \mathbb{P}^1 = \{[z_0 : z_1] : (z_0, z_1) \in \mathbb{C}^2 \backslash \{0\}\}$. Let ω_0 be the Fubini-Study form on $\mathbb{P}^1$. Using the local coordinates $[z : 1]$, $\mathbb{P}^1$ can be seen as the compactification of $\mathbb{C}$ with a point at infinity. An ω_0-p.s.h. functions is equal to $\psi(z) - \frac{1}{2}\log(1 + |z|^2)$ on $\mathbb{C}$, where ψ is a subharmonic function on $\mathbb{C}$, such that the last difference is bounded above.*

Let $K = [-1,1]$. Choose $\phi \equiv 0$ on K. Using [44, Cor. 5.4.5], we get $\phi_K(z) = \log|z + \sqrt{z^2-1}|$ on $\mathbb{C}$, where the square root is chosen such that

$$|z + \sqrt{z^2 - 1}| \geq 1.$$

Comparing $\phi_K(z)$ with 0 when z is close to 1, one sees that $\phi_K \in \mathcal{C}^{1/2}(X) \backslash \mathcal{C}^{1/2+\epsilon}(X)$ for any $\epsilon > 0$. In higher dimension, the same arguments also work for $X = \mathbb{P}^n$, $K = [-1,1]^n \subset \mathbb{C}^n$ and $\phi \equiv 0$.

Before giving the proof of Theorem 3.2.3, we need to recall the definition of analytic discs partly attached to a subset of X. Throughout the paper, let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. *An analytic disc f* in X is a holomorphic mapping from $\mathbb{D}$ to X which is continuous up to the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. For an interval $I \subset \partial\mathbb{D}$, *$f$ is said to be I-attached to a subset $E \subset X$* if $f(I) \subset E$. In particular, we say that f is *half-attached to E* if $f(\partial^+\mathbb{D}) \subset E$, where $\partial^+\mathbb{D} = \{\xi \in \partial\mathbb{D} : \mathrm{Re}\,\xi \geq 0\}$. The crucial ingredient in the proof of Theorem 3.2.3 is Proposition 3.2.5 below which shows the existence of special families of analytic discs partly attached to K in X. Its proof will be given in Section 3.4.

Proposition 3.2.5. *There are positive constants c_0, r_0 and $\theta_0 \in (0, \pi/2)$ such that for any $p_0 \in K$ and any $p \in B_X^*(p_0, r_0)$, there exist an open neighborhood W_{p_0} of p_0 in X independent of p which is biholomorphic to the unit ball of $\mathbb{C}^n$ and a $\mathcal{C}^1$ analytic disc $f : \overline{\mathbb{D}} \to W_{p_0}$ such that f is $[e^{-i\theta_0}, e^{i\theta_0}]$-attached to K, $\mathrm{dist}(f(1), p_0) \leq c_0\delta$ with $\delta = \mathrm{dist}(p, p_0)$, $\|f\|_{\mathcal{C}^1} \leq c_0$ and there is $z^* \in \mathbb{D}$ so that $|1 - z^*| \leq \sqrt{c_0\delta}$ and $f(z^*) = p$. When K has no singularity, z^* can be chosen so that $|1 - z^*| \leq c_0\delta$.*

We will also need the following lemma in complex dimension one.

Lemma 3.2.6. *Let $\theta_0 \in (0, \pi/2), \beta \in (0,1)$ and let $c > 0$. Let ψ be a subharmonic function on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Assume that*

$$\psi(e^{i\theta}) \leq c|\theta|^\beta \quad \textit{for} \quad \theta \in (-\theta_0, \theta_0) \quad \textit{and} \quad \sup_{\theta\in[-\pi,\pi]} \psi(e^{i\theta}) \leq c. \tag{3.2.2}$$

Then, there exists a constant C depending only on (θ_0, β, c) so that for any $z \in \mathbb{D}$, we have

$$\psi(z) \leq C|1-z|^\beta.$$

Proof. Observe that the function $|\theta|^\beta$ is Hölder continuous of order β in θ. By using this fact and suitable cut-off functions in $\mathcal{C}^\infty(\partial\mathbb{D})$, we see that there exists a function ψ_1 in $\mathcal{C}^\beta(\partial\mathbb{D})$ so that

$$\psi_1(e^{i\theta}) = c|\theta|^\beta \quad \text{for} \quad \theta \in (-\theta_0/2, \theta_0/2)$$

and

$$\psi_1(e^{i\theta}) \geq c \quad \text{for} \quad \theta \in [-\pi, -3\theta_0/4] \cup [3\theta_0/4, \pi].$$

By (3.2.2), we have $\psi(e^{i\theta}) \leq \psi_1(e^{i\theta})$ on $\partial\mathbb{D}$. Extend ψ_1 harmonically to $\mathbb{D}$. Denote also by ψ_1 its harmonic extension. It is classical that $\psi_1 \in \mathcal{C}^\beta(\overline{\mathbb{D}})$, see (3.3.4) for details. Since ψ is subharmonic on $\mathbb{D}$ and $\psi \leq \psi_1$ on $\partial\mathbb{D}$, we have $\psi \leq \psi_1$ on $\mathbb{D}$. As a result,

$$\psi(z) \leq \psi_1(z) \leq \psi_1(1) + \|\psi_1\|_{\mathcal{C}^\beta}|1-z|^\beta,$$

for any $z \in \mathbb{D}$. The desired inequality follows because $\psi_1(1) = 0$. The proof is finished. □

Proof of Theorem 3.2.3. By [23, Th. 2.7], the compact set X is itself $(\mathcal{C}^\beta, \mathcal{C}^\beta)$-regular for any $\beta \in (0,1)$. Let $\tilde{\phi}$ be the function on X defined by

$$\tilde{\phi}(p) := \min_{p'\in K}[\phi(p') + A\,\mathrm{dist}(p,p')^{\alpha/2}]$$

for $p \in X$ and $A \gg \|\phi\|_{\mathcal{C}^\alpha}$ is a fixed big constant. It is not difficult to see that $\tilde{\phi}$ is $\mathcal{C}^{\alpha/2}$ and $\tilde{\phi} = \phi$ on K. Namely, we have

$$|\tilde{\phi}(p) - \tilde{\phi}(p')| \leq A\,\mathrm{dist}(p,p')^{\alpha/2}$$

for all $p, p' \in X$. By $(\mathcal{C}^{\alpha/2}, \mathcal{C}^{\alpha/2})$-regularity of X, we have $\tilde{\phi}_X = \tilde{\phi}^*_X$ which is ω_0-p.s.h. and of Hölder class $\mathcal{C}^{\alpha/2}$. Hence, the proof of Theorem 3.2.3 is finished if we can show that

$$\phi^*_K = \tilde{\phi}_X. \tag{3.2.3}$$

Clearly, by definition of ϕ^*_K and $\tilde{\phi}_X$, we have $\phi^*_K \geq \tilde{\phi}_X$. Thus, to prove (3.2.3), it is enough to prove that

$$\phi_K \leq \tilde{\phi}$$

on X because this implies $\phi^*_K \leq \tilde{\phi}$ thanks to the continuity of the last function. Since A is big enough and ϕ_K is bounded on X, we only need to check that

$$\phi_K(p) \leq \tilde{\phi}(p), \tag{3.2.4}$$

for p close to K. This inequality is clear for $p \in K$.

Fix a $p \notin K$ close to K. Let p_0 be a point in K such that

$$\operatorname{dist}(p, p_0) = \min_{p' \in K} \operatorname{dist}(p, p').$$

Put $\delta = \operatorname{dist}(p, p_0)$. It is shown in the proof of [23, Th. 2.7] that $\tilde{\phi}(p) \geq \tilde{\phi}(p_0) + A'\delta^{\alpha/2}$ for some big constant A' independent of p ($A' \to \infty$ if $A \to \infty$). Hence, in order to prove (3.2.4), it suffices to prove that

$$\phi_K(p) \leq \tilde{\phi}(p_0) + A'\delta^{\alpha/2} = \phi(p_0) + A'\delta^{\alpha/2}. \tag{3.2.5}$$

As already said above, we only need to consider δ small. Precisely, we will suppose that $\delta < r_0$, where r_0 is the constant in Proposition 3.2.5. Let f, W_{p_0},c_0, θ_0 be the analytic disc corresponding to (p_0, p), the open neighborhood of p_0 and the constants respectively in that proposition. There is $z^* \in \mathbb{D}$ with $|1 - z^*| \leq \sqrt{c_0\delta}$ so that $f(z^*) = p$.

Let ψ be an ω_0-p.s.h. function on X so that $\psi \leq \phi$ on K. Since W_{p_0} is biholomorphic to the unit ball of $\mathbb{C}^n$, there exists a smooth potential ψ_{ω_0} of ω_0 on $W_{p_0}, i.e,$ we have

$$dd^c\psi_{\omega_0} = \omega_0 \quad \text{on} \quad W_{p_0}.$$

Hence, $\psi_0 := \psi + \psi_{\omega_0}$ is a p.s.h. function on W_{p_0} and $\psi_0 \leq \phi_0 := \phi + \psi_{\omega_0}$ on $W_{p_0} \cap K$. By the smoothness of ψ_{ω_0}, the function ϕ_0 is also Hölder continuous of order α on any compact subset of W_{p_0}, hence on $\overline{f(\mathbb{D})}$. Define $\psi_1 := \psi_0 \circ f$, and $\phi_1 := \phi_0 \circ f$. Observe that ψ_1 is a p.s.h. function on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. We also have

$$\psi_1(z^*) = \psi_0(p) \quad \text{and} \quad \phi_1(1) = \phi_0\big(f(1)\big). \tag{3.2.6}$$

Since $\|f\|_{\mathcal{C}^1} \leq c_0$, the function ϕ_1 is Hölder continuous of order α with a Hölder constant independent of p, p_0 and f. On the other hand, since the disc f is $[e^{-i\theta_0}, e^{i\theta_0}]$-attached to K, we have $\psi_1(e^{i\theta}) \leq \phi_1(e^{i\theta})$ for $\theta \in [-\theta_0, \theta_0]$. This together with the Hölder continuity of ϕ_1 yield that

$$\psi_1(e^{i\theta}) \leq \phi_1(1) + c|\theta|^\alpha,$$

for $\theta \in [-\theta_0, \theta_0]$ and for some positive constant c. Applying Lemma 3.2.6 to the subharmonic function $\big(\psi_1 - \phi_1(1)\big)$ gives

$$\psi_1(z^*) \leq \phi_1(1) + C|1 - z^*|^\alpha \quad \text{for some positive constant } C. \tag{3.2.7}$$

Combining (3.2.6), (3.2.7) and the definitions of ψ_0, ϕ_0, one obtains

$$\psi(p) \leq \phi_0\big(f(1)\big) + C|1 - z^*|^\alpha,$$

for every ω_0-p.s.h. ψ on X with $\psi \leq \phi$ on K. Taking the supremum over all such ψ in the last inequality and using the definition of ϕ_K give

$$\phi_K(p) \leq \phi_0\big(f(1)\big) + C|1 - z^*|^\alpha \leq \phi(p_0) + \|\phi\|_{\mathcal{C}^\alpha}|1 - z^*|^\alpha + C|1 - z^*|^\alpha \leq \phi(p_0) + A'\delta^{\alpha/2}$$

because $\phi \in \mathcal{C}^\alpha$ and $|1 - z^*| \leq \sqrt{c_0\delta}$. Now consider the case where K has no singularity. Define

$$\tilde{\phi}'(p) := \min_{p' \in K}[\phi(p') + A \operatorname{dist}(p, p')^\alpha]$$

for $p \in X$ and some fixed big constant $A \gg \|\phi\|_{\mathcal{C}^\alpha}$. By using the same argument as above with $\tilde{\phi}'$ in place of $\tilde{\phi}$ and the fact that $|1 - z^*| \leq c_0\delta$, we get the desired conclusion. The proof is finished. □

Proof of Theorems 3.1.1 and 3.1.2. We first prove Theorem 3.1.1. Recall that we have canonical inclusions: $\mathbb{R}^{n+1} \subset \mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$. Let $\mathbb{S}^n_{\mathbb{C}}$ be the complexification of $\mathbb{S}^n$ in $\mathbb{P}^{n+1}$ defined by the equation

$$z_0^2 + \cdots + z_n^2 = z_{n+1}^2,$$

where $[z_0 : \cdots : z_{n+1}]$ are the homogeneous coordinates on $\mathbb{P}^{n+1}$. We see that $\mathbb{S}^n$ is a compact generic submanifold of $\mathbb{S}^n_{\mathbb{C}}$. Choose $X = \mathbb{S}^n_{\mathbb{C}}$, $K = \mathbb{S}^n$, $\phi = 0$ and $L = \mathcal{O}(1)|_X$ is the restriction of the hyperplane line bundle of $\mathbb{P}^{n+1}$ to X. Observe that the restriction $H^0(X, L^k)|_{\mathbb{S}^n}$ of $H^0(X, L^k)$ to $\mathbb{S}^n$ is a complex vector space of (complex) dimension $\dim_{\mathbb{R}} \mathcal{P}_k(\mathbb{S}^n)$. As K is non-pluripolar in $\mathbb{S}^n_{\mathbb{C}}$, any nonzero holomorphic function on an open subset of $\mathbb{S}^n_{\mathbb{C}}$ can not annihilate on the whole K. As a result, we have $\dim_{\mathbb{R}} \mathcal{P}_k(\mathbb{S}^n) = \dim_{\mathbb{R}} \mathcal{P}_k(K)$. This allows one to choose a common basis for the two vector spaces $H^0(X, L^k)|_{\mathbb{S}^n}$ and $\mathcal{P}_k(K)$ when defining Fekete points. Therefore, Fekete points in the complex case are those defined on K as in Introduction. Theorem 3.1.1 is now a direct corollary of Theorem 3.1.4 with the choice of (X, L, K, ϕ) as above, $\gamma = 1$ and $\alpha = 1 - \epsilon$, for $\epsilon > 0$.

Consider the case where $K = \mathbb{S}^n$, the equilibrium measure $\mu_{eq}(K, 0)$ coincides with the normalized volume form on $\mathbb{S}^n$ induced by the Euclidean metric on $\mathbb{R}^{n+1}$ because $\mu_{eq}(K, \phi)$ is preserved by the actions of the orthogonal matrix group on $\mathbb{S}^n$. Theorem 3.1.2 is hence obtained in a similar way by using the fact that $\mathbb{S}^n$ has no boundary. The proof is finished. □

Remark 3.2.7. *We discuss here very briefly the case where $\dim_{\mathbb{R}} K = 2n$ in which the regularity of K can be improved. For simplicity, we consider the following simple model in the complex dimension 1. Let $X = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ as in Remark 3.2.4. Let K be the compact convex polygon in $\mathbb{C}$. Denote by $S_1, S_2, \cdots, S_m$ the consecutive vertices of K. Let $\pi < \gamma_j < 2\pi$ be the exterior angle at S_j of K, for $1 \leq j \leq m$. Put $\gamma = \max_{1 \leq j \leq m} \gamma_j$. Then, K is $(\mathcal{C}^\alpha, \mathcal{C}^{\alpha\pi/\gamma})$-regular. When $\gamma_j = \pi$ for all j, we re-obtain [23, Th. 2.7]. The idea for the proof is as follows. Let $\phi \in \mathcal{C}^\alpha(K)$. In order to get the above regularity of K, it is enough to show that given any p.s.h. function ψ on $\mathbb{C}$ so that $\psi \leq \phi$ on K, then for every j, we have $\psi(z) \leq \phi(S_j) + A|z - S_j|^{\alpha\pi/\gamma}$ for every z close to S_j and for some fixed big constant A. Let L_1 be the open domain of $\mathbb{C}$ limited by the two rays S_1S_m and S_1S_2 which does not contain K. Using an affine change of coordinates, we can suppose that $S_1 = 0$, the ray S_1S_2 is $\{z = (x, 0) : x \geq 0\}$ and $L_1 \supset \mathbb{H} := \{\operatorname{Im} z > 0\}$. Using the map $(z - i)/(z + i)$ sending $\mathbb{H}$ biholomorphically to $\mathbb{D}$, one easily sees that the map $\Psi(z) := (z^{\pi/\gamma} - i)(z^{\pi/\gamma} + i)$ is a biholomorphism from L_1 to $\mathbb{D}$. Clearly, Ψ is Hölder continuous of order π/γ on an open neighborhood of S_1 in $\overline{L}_1$. An application of Lemma 3.2.6 to $\psi \circ \Psi^{-1}$ gives the desired result.*

3.3 Two special families of analytic discs

3.3.1 Hilbert transform

Denote by $z = x + iy$ the complex variable on $\mathbb{C}$ and by $\xi = e^{i\theta}$ the variable on $\partial\mathbb{D}$. For any $m \in \mathbb{N}$ and $r > 0$, let $B_m(0, r)$ be the Euclidean ball centered at 0 of radius r of $\mathbb{R}^m$ and

let $B_m^*(0,r) = B_m(0,r)\backslash\{0\}$. Denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^m$. The same notations will be used for $\mathbb{C}^n$ that we sometimes identify with $\mathbb{R}^{2n}$. Let Z be a submanifold of $\mathbb{R}^m$. The Euclidean metric on $\mathbb{R}^m$ induces a metric on Z. For $\beta \in (0,1)$ and $k \in \mathbb{N}$, let $\mathcal{C}^{k,\beta}(Z)$ be the space of real-valued functions on Z which are differentiable up to the order k and whose k^{th} derivatives are Hölder continuous of order β. This is a Banach space with the $\mathcal{C}^{k,\beta}$-norm given by

$$\|v\|_{k,\beta,Z} := \|v\|_{k,Z} + \sup_{\xi\neq\xi',\xi,\xi'\in Z} \frac{\|D^k v(\xi) - D^k v(\xi')\|}{|\xi-\xi'|^\beta},$$

where $\|\cdot\|_{k,Z} := \|\cdot\|_{\mathcal{C}^k(Z)}$ and $D^k v$ denotes the k^{th}-differential of v. In the proof, we will only use this norm for $Z = \overline{\mathbb{D}}$ or $\partial\mathbb{D}$. When Z is clear from the context, we will remove the subscript Z from the above notation of norm. For any tuple $v = (v_0, \cdots, v_m)$ consisting of functions in $\mathcal{C}^{k,\beta}(Z)$, we define its $\mathcal{C}^{k,\beta}$-norm to be the maximum of the ones of its components.

Recall that an arbitrary continuous function $u_0(\xi)$ on $\partial\mathbb{D}$ can be extended uniquely to be a harmonic function on $\mathbb{D}$ which is continuous on $\overline{\mathbb{D}}$. Since this correspondence is bijective, without stating explicitly, we will freely identify u_0 with its harmonic extension on $\mathbb{D}$. We will write $u_0(z) = u_0(x+iy)$ to indicate the harmonic extension of $u_0(e^{i\theta})$. It is well-known that the Cauchy transform of u_0, given by

$$\mathcal{C}u_0(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(e^{i\theta})\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta,$$

is a holomorphic function on $\mathbb{D}$ whose real part is u_0. Decomposing the last formula into the real and imaginary parts, we obtain that

$$u_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-|z|^2)}{|e^{i\theta}-z|^2} u_0(e^{i\theta})d\theta. \tag{3.3.1}$$

and

$$\mathcal{T}u_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(ze^{-i\theta}-\bar{z}e^{i\theta})}{i|e^{i\theta}-z|^2} u_0(e^{i\theta})d\theta.$$

The function $\mathcal{T}u_0$ is harmonic on $\mathbb{D}$ but is not always continuous up to the boundary of $\mathbb{D}$. Let k be an arbitrary natural number and let β be an arbitrary number in $(0,1)$. A result of Privalov (see [58, Th. 4.12] or [1, Sec. 6.1]) implies that if u_0 belongs to $\mathcal{C}^{k,\beta}(\partial\mathbb{D})$, then $\mathcal{T}u_0$ is continuous up to $\partial\mathbb{D}$ and $\|\mathcal{T}u_0\|_{k,\beta,\partial\mathbb{D}}$ is bounded by $\|u_0\|_{k,\beta,\partial\mathbb{D}}$ times a constant independent of u_0. Hence, the linear self-operator of $\mathcal{C}^{k,\beta}(\partial\mathbb{D})$ defined by sending u_0 to the restriction of $\mathcal{T}u_0$ onto $\partial\mathbb{D}$ is bounded and called *the Hilbert transform*. For simplicity, we also denote it by $\mathcal{T}$. In the method of analytic discs, it is convenient to use a modified version $\mathcal{T}_1$ of $\mathcal{T}$ defined by

$$\mathcal{T}_1 u_0 := \mathcal{T}u_0 - \mathcal{T}u_0(1).$$

Hence we always have $\mathcal{T}_1 u_0(1) = 0$ and

$$\partial_\theta \mathcal{T}_1 u_0 = \partial_\theta \mathcal{T} u_0 = \mathcal{T}\partial_\theta u_0, \tag{3.3.2}$$

see [58, p.121] for a proof. The boundedness of $\mathcal{T}$ on $\mathcal{C}^{k,\beta}(\partial\mathbb{D})$ implies that there is a constant $C_{k,\beta} > 1$ such that for any $v \in \mathcal{C}^{k,\beta}(\partial\mathbb{D})$ we have

$$\|\mathcal{T}_1 v\|_{k,\beta,\partial\mathbb{D}} \leq C_{k,\beta}\|v\|_{k,\beta,\partial\mathbb{D}}. \tag{3.3.3}$$

Extending $u_0, \mathcal{T}_1 u_0$ harmonically to $\mathbb{D}$. By construction, the function $f(z) := -\mathcal{T}_1 u_0(z) + i u_0(z)$ is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ provided that u_0 is in $\mathcal{C}^\beta(\partial\mathbb{D})$ with $0 < \beta < 1$. By [59, Th. 4.2], $\|f\|_{k,\beta,\overline{\mathbb{D}}}$ is bounded by $\|f\|_{k,\beta,\partial\mathbb{D}}$ times a constant depending only on (k, β). Since $\|u_0\|_{k,\beta,\overline{\mathbb{D}}} \leq \|f\|_{k,\beta,\overline{\mathbb{D}}}$ and $\|f\|_{k,\beta,\partial\mathbb{D}} \leq (1 + C_{k,\beta})\|u_0\|_{k,\beta,\partial\mathbb{D}}$ by (3.3.3), we have

$$\|u_0\|_{k,\beta,\overline{\mathbb{D}}} \leq C'_{k,\beta} \|u_0\|_{k,\beta,\partial\mathbb{D}}, \tag{3.3.4}$$

for some constant $C'_{k,\beta}$ depending only on (k, β). A direct consequence of the above inequalities is that when u_0 is smooth on $\partial\mathbb{D}$, the associated holomorphic function f is also smooth on $\overline{\mathbb{D}}$.

3.3.2 Analytic discs half-attached to $\mathbb{R}^n$ in $\mathbb{C}^n$

The goal of this subsection is to construct a special family of analytic discs half-attached to $\mathbb{R}^n$ in $\mathbb{C}^n$. The main result is Proposition 3.3.4 presented at the end of the subsection. The reader should keep in mind that the idea that we use below will be constantly applied later.

In what follows, we identify $\mathbb{C}^n$ with $\mathbb{R}^n + i\mathbb{R}^n$. Let $\mathbf{z} \in B^*_{2n}(0, 1)$. Let $u = (u_1, \cdots, u_n)$ be a vector with components $u_j \in \mathcal{C}^{k,\beta}(\partial\mathbb{D})$ for $1 \leq j \leq n$ such that $u \equiv 0$ on $\partial^+\mathbb{D}$. Then, $\mathcal{T}_1 u := (\mathcal{T}_1 u_1, \cdots, \mathcal{T}_1 u_n)$ is a vector in $\big(\mathcal{C}^{k,\beta}(\partial\mathbb{D})\big)^n$. As above, extend u and $\mathcal{T}_1 u$ harmonically to $\mathbb{D}$. By the last subsection, u and $\mathcal{T}_1 u$ belong to $\big(\mathcal{C}^{k,\beta}(\overline{\mathbb{D}})\big)^n$. It follows that the map

$$f := -\mathcal{T}_1 u + iu$$

is a $\mathcal{C}^{k,\beta}$ mapping from $\overline{\mathbb{D}}$ to $\mathbb{C}^n$ which is holomorphic on $\mathbb{D}$ and $f|_{\partial^+\mathbb{D}} \subset \mathbb{R}^n$. In other words, f is a $\mathcal{C}^{k,\beta}$ analytic disc half-attached to $\mathbb{R}^n$ in $\mathbb{C}^n$ with $f(1) = 0$. We are going to choose u depending on the parameter $\mathbf{z}$ such that there exist $z^* \in \mathbb{D}$ (depending on $\mathbf{z}$) and a constant c_0 independent of $\mathbf{z}$ for which

$$\|f\|_{3,\overline{\mathbb{D}}} \leq c_0, \tag{3.3.5}$$

and

$$f(z^*) = \mathbf{z} \quad \text{and} \quad |1 - z^*| \leq c_0 |\mathbf{z}|. \tag{3.3.6}$$

Recall that we will systematically identify continuous functions on $\partial\mathbb{D}$ with their harmonic extension to $\mathbb{D}$. Hence, for any continuous function u on $\partial\mathbb{D}$, we can speak of its derivatives in (x, y) as the ones of its harmonic extension, where $z = x + iy \in \mathbb{D}$.

Lemma 3.3.1. *There exists a function $u \in \mathcal{C}^\infty(\partial\mathbb{D})$ vanishing on $\partial^+\mathbb{D}$ so that $\partial_x u(1) = -1$.*

Proof. Differentiating (3.3.1) gives

$$\partial_x u(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{u(e^{i\theta})}{\cos\theta - 1} d\theta.$$

Note that the last integral is well-defined because u vanishes on $\partial^+\mathbb{D}$. It is easy to choose a smooth u so that the above integral is equal to -1 and $u \equiv 0$ on $\partial^+\mathbb{D}$. The proof is finished. □

Lemma 3.3.2. *Let u be a functions as in Lemma 3.3.1. Then there exist two smooth functions g_1, g_2 defined on $[0,1]$ so that $u(1-s+is) = s + s^2 g_1(s)$ and $-\mathcal{T}_1 u(1-s+is) = s + s^2 g_2(s)$, for every $s \in [0,1]$.*

Proof. By hypothesis, we have $\partial_x u(1) = -1$ and $\partial_y u(1) = 0$ because u vanishes on $\partial^+\mathbb{D}$. On the other hand, the Cauchy-Riemann equations imply $-\partial_x \mathcal{T}_1 u(1) = \partial_y u(1) = 0$ and $-\partial_y \mathcal{T}_1 u(1) = -\partial_x u(1) = 1$. Now using Taylor's expansions with integral remainders to $u(1-s+is)$ and $-\mathcal{T}_1 u(1-s+is)$ at $s=0$, there exist two functions $g_1(s)$ and $g_2(s)$ satisfying the desired property. The proof is finished. □

We denote by $|\cdot|_{\ln}$ the usual norm of linear endomorphisms of $\mathbb{R}^m$ for $m \in \mathbb{N}$. Since any real matrix of order m can be canonically associated with a such linear endomorphism, one can use the norm $|\cdot|_{\ln}$ for the real matrices. We will repeatedly use the following known version of the inverse function theorem.

Lemma 3.3.3. *Let $m \in \mathbb{N}^*$. Let Φ_0 be a function from $B_m(0,1)$ to $\mathbb{R}^m$. Assume that there are a nondegenerate square matrix A of order m and a M-Lipschitz function g on $B_m(0,1)$ for some constant $M > 0$ such that*

$$\Phi_0(\mathbf{z}) = A\mathbf{z} + g(\mathbf{z}),$$

for every $\mathbf{z} \in B_m(0,1)$ and $g(0) = 0$, $|A^{-1}|_{\ln} M < 1$. Then, for every $0 < r < 1$ and every $\tilde{\mathbf{z}} \in B_m\big(0, \frac{1-|A^{-1}|_{\ln} M}{|A^{-1}|_{\ln}} r\big)$, there exists a unique point $\mathbf{z}^* \in B_m(0,r)$ such that $\Phi_0(\mathbf{z}^*) = \tilde{\mathbf{z}}$.

Proof. Since g is M-Lipschitz on $B_m(0,1)$, we have

$$|g(\mathbf{z}) - g(\mathbf{z}')| \le M|\mathbf{z} - \mathbf{z}'|,$$

for all $\mathbf{z}, \mathbf{z}' \in B_m(0,1)$. In particular, we have $|g(\mathbf{z})| \le M|\mathbf{z}|$ because $g(0) = 0$. Let $\tilde{\mathbf{z}}$ be a point in $B_m\big(0, \frac{1-|A^{-1}|_{\ln} M}{|A^{-1}|_{\ln}} r\big)$. The equation $\Phi_0(\mathbf{z}) = \tilde{\mathbf{z}}$ is equivalent to $\mathbf{z} = A^{-1}\big(\tilde{\mathbf{z}} - g(\mathbf{z})\big)$. Let $r \in (0,1)$. Define

$$R(\mathbf{z}) := A^{-1}\big(\tilde{\mathbf{z}} - g(\mathbf{z})\big),$$

for $\mathbf{z} \in \overline{B}_m(0,r)$. Then R is a self-map of $\overline{B}_m(0,r)$. Indeed, we have

$$|R(\mathbf{z})| \le |A^{-1}|_{\ln} |\tilde{\mathbf{z}} - g(\mathbf{z})| \le |A^{-1}|_{\ln} \big(|\tilde{\mathbf{z}}| + M|\mathbf{z}|\big) \le |A^{-1}|_{\ln} \big(\frac{1 - |A^{-1}|_{\ln} M}{|A^{-1}|_{\ln}} r + Mr\big) = r,$$

for any $\mathbf{z} \in \overline{B}_m(0,r)$. Additionally, similar estimates also gives

$$|R(\mathbf{z}) - R(\mathbf{z}')| \le |A^{-1}|_{\ln} M |\mathbf{z} - \mathbf{z}'|.$$

Since $|A^{-1}|_{\ln} M < 1$, R is a contraction of $\overline{B}_m(0,r)$. Since the last metric space is compact, the fixed point theorem applied to R implies that R has a unique fixed point $\mathbf{z}^* \in \overline{B}_m(0,r)$. Equivalently, there is a unique point $\mathbf{z}^* \in \overline{B}_m(0,r)$ for which $\Phi_0(\mathbf{z}^*) = \tilde{\mathbf{z}}$. □

For any two vectors $\mathbf{v}^j = (v_1^j, \cdots, v_n^j) \in \mathbb{R}^n$ with $j = 1$ or 2, we denote by $\mathbf{v^1} \cdot \mathbf{v}^2$ the vector in $\mathbb{R}^n$ whose l^{th} component is $v_l^1 v_l^2$ for $1 \le l \le n$. Let u be a function as in Lemma

3.3.1. By abuse of notation, denote also by u the vector of $\mathcal{C}^\infty(\partial\mathbb{D})^n$ whose components are all equal to u. We define

$$u_{\mathbf{z},t}(e^{i\theta}) := tu(e^{i\theta}) \cdot \frac{\operatorname{Im}\mathbf{z}}{|\mathbf{z}|}, \tag{3.3.7}$$

for any $\mathbf{z} \in B_{2n}^*(0,1)$ and $t \in (0,1]$. Extend $u_{\mathbf{z},t}$ harmonically to $\mathbb{D}$. Define

$$F(z,\mathbf{z},t) := t(\operatorname{Re}\mathbf{z} - \operatorname{Im}\mathbf{z}) - \mathcal{T}_1(u_{\mathbf{z},t})(z) + iu_{\mathbf{z},t}(e^{i\theta}), \tag{3.3.8}$$

for any $z \in \overline{\mathbb{D}}$, $\mathbf{z} \in B_{2n}^*(0,1)$ and $t \in (0,1]$. We have following properties of F.

Proposition 3.3.4. *The map $F : \overline{\mathbb{D}} \times B_{2n}^*(0,1) \times (0,1] \to \mathbb{C}^n$ is smooth and the three following properties hold:*

(i) for any $\mathbf{z} \in B_{2n}^*(0,1)$ and $t \in (0,1]$, the mapping $F(\cdot,\mathbf{z},t)$ is a smooth analytic disc half-attached to $\mathbb{R}^n$ in $\mathbb{C}^n$, and $F(1,\mathbf{z},t) = t(\operatorname{Re}\mathbf{z} - \operatorname{Im}\mathbf{z}) \in B_n(0,2t) \subset \mathbb{R}^n$,

(ii) there exists a constant $r_0 > 0$ so that for any $\mathbf{z} \in B_{2n}^*(0,r_0)$ and $t \in (0,1]$, there exists $\mathbf{z}^* \in B_{2n}^*(0,1)$ for which

$$F(1 - |\mathbf{z}^*| + i|\mathbf{z}^*|, \mathbf{z}^*, t) = t\mathbf{z}$$

and $|\mathbf{z}^*| \leq 2|\mathbf{z}|$,

(iii) there exists a constant $c_0 > 1$ so that for any $\mathbf{z} \in B_{2n}^*(0,1)$ and $t \in (0,1]$, we have

$$\|F(\cdot,\mathbf{z},t)\|_3 \leq tc_0 \text{ and } \quad \|D_{\mathbf{z}}F(\cdot,\mathbf{z},t)\|_2 \leq tc_0|\mathbf{z}|^{-1}, \tag{3.3.9}$$

where $D_{\mathbf{z}}$ is the differential with respect to $\mathbf{z}$.

Proof. The properties (i) and (iii) automatically hold by our construction. It remains to prove (ii). Fix $t \in (0,1]$. For every $\mathbf{z} \in B_{2n}^*(0,1)$, define $\Phi(\mathbf{z}) := F(1 - |\mathbf{z}| + i|\mathbf{z}|, \mathbf{z}, t)$ and $\Phi(0) := 0$. Applying Lemma 3.3.2 to each component of $u_{\mathbf{z},t}$ and $s = |\mathbf{z}|$, using (3.3.7) and (3.3.8), there exists a smooth map $g_0 : [0,1] \to \mathbb{R}^n$ such that

$$\Phi(\mathbf{z}) = t(\operatorname{Re}\mathbf{z} - \operatorname{Im}\mathbf{z}) + t\operatorname{Im}\mathbf{z} + t|\mathbf{z}|^2 g_0(|\mathbf{z}|) \cdot \frac{\operatorname{Im}T(\mathbf{z})}{|\mathbf{z}|} = t\mathbf{z} + t|\mathbf{z}|g_0(|\mathbf{z}|) \cdot \operatorname{Im}T(\mathbf{z}).$$

Put

$$g(\mathbf{z}) := t|\mathbf{z}|g_0(|\mathbf{z}|) \cdot \operatorname{Im}T(\mathbf{z}).$$

Let $r_0 < 1/16 \min\{\|g_0\|_1^{-1}, 1\}$. Observe that $g(0) = 0$ and

$$\|g\|_{1, B_{2n}^*(0,2r_0)} \leq t/4 < t/2$$

Thus, g is $t/2$-Lipschitz on $\overline{B}_{2n}(0,2r_0)$. Applying Lemma 3.3.3 to Φ in place of Φ_0, $A = t\mathrm{Id}$ and g as above shows that for any $\mathbf{z} \in B_{2n}^*(0,r_0)$, there exists $\mathbf{z}^* \in B_{2n}^*(0,2r_0)$ for which $\Phi(\mathbf{z}^*) = t\mathbf{z}$. Moreover, the last equation implies that

$$t|\mathbf{z}| \geq t|\mathbf{z}^*| - |g(\mathbf{z}^*)| \geq t|\mathbf{z}^*| - t/2|\mathbf{z}^*|.$$

Hence, $|\mathbf{z}^*| \leq 2|\mathbf{z}|$. The proof is finished. □

For each $\mathbf{z} \in B_{2n}^*(0,r_0)$, define $f(z) := F(z,\mathbf{z}^*,1)$ and $z^* := 1 - |\mathbf{z}^*| + i|\mathbf{z}^*|$. It is clear that f and z^* satisfy the two conditions (3.3.5) and (3.3.6).

3.3.3 Analytic discs partly attached to $(\mathbb{R}^+)^n$ in $\mathbb{C}^n$

The goal of this subsection is to construct a family F' of analytic discs which somewhat resembles the one in Proposition 3.3.4 and partly attached to $(\mathbb{R}^+)^n$ in $\mathbb{C}^n$, where $\mathbb{R}^+$ is the set of nonnegative real numbers. The arguments used in the last subsection do not permit us to control the position of the part of the boundary of the disc in $\mathbb{R}^n$. The idea is to construct discs which look like the image of F under the map $(z_1, \cdots, z_n) \longmapsto (z_1^2, \cdots, z_n^2)$, this image is half-attached to $(\mathbb{R}^+)^n$, where F is the family in the last subsection.

At the end of this subsection, we also introduce an another family $F'_{\boldsymbol{\tau}}$ of discs half-attached to $\mathbb{R}^n$ parametrized by $\boldsymbol{\tau} \in B_n(0,2)$ which contains F' as a subfamily. Let us explain why we need such $F'_{\boldsymbol{\tau}}$. In the general case considered in Section 3.4, the required analytic discs in Proposition 3.2.5 can be obtained as a small perturbation of F'. Due to the nonsmoothness of $(\mathbb{R}^+)^n$ (or due to the singularity of K in the general case), any family of discs partly attached to $(\mathbb{R}^+)^n$ is generally no longer so when being perturbed. Hence, in order to control the perturbed family, one should embed F' in the bigger family $F'_{\boldsymbol{\tau}}$ which is more stable under perturbation.

Define

$$\rho_1(\theta) := \frac{1}{2\pi(\cos\theta - 1)} \quad \text{and} \quad \rho_2(\theta) := -\frac{\sin\theta}{2\pi(\cos\theta - 1)^2}, \tag{3.3.10}$$

for $\theta \in [-\pi, \pi]$.

Lemma 3.3.5. *Let u be harmonic function on $\mathbb{D}$ and smooth on $\overline{\mathbb{D}}$. Assume that u vanishes on $\partial^+\mathbb{D}$. Then, we have $\partial_y u(1) = 0$, $\partial_y^2 u(1) = \partial_x u(1)$, $\partial_x^2 u(1) = -\partial_x u(1)$ and*

$$\partial_x u(1) = \int_{-\pi}^{\pi} u(e^{i\theta})\rho_1(\theta)d\theta \quad \textit{and} \quad \partial_x\partial_y u(1) = \int_{-\pi}^{\pi} u(e^{i\theta})\rho_2(\theta)d\theta.$$

Proof. Firstly, observe that for an arbitrary $\mathcal{C}^2$ function $u(x+iy)$ on $\overline{\mathbb{D}}$, we have

$$\partial_\theta u(e^{i\theta})|_{\theta=0} = \partial_y u(1) \quad \text{and} \quad \partial_\theta^2 u(e^{i\theta})|_{\theta=0} = \partial_y^2 u(1) - \partial_x u(1). \tag{3.3.11}$$

Now let u be the function in the statement. The last two equalities combined with the fact that $u|_{\partial^+\mathbb{D}} \equiv 0$ imply that

$$\partial_y u(1) = 0, \quad \partial_y^2 u(1) = \partial_\theta^2 u(1) + \partial_x u(1) = \partial_x u(1). \tag{3.3.12}$$

Since $\Delta u(z) = 0$, we get $\partial_x^2 u(1) = -\partial_y^2 u(1) = -\partial_x u(1)$. On the other hand, it is computed in the proof of Lemma 3.3.1 that $\partial_x u(1) = \int_{-\pi}^{\pi} u(e^{i\theta})\rho_1(\theta)d\theta$. Differentiating the Poisson kernel at $(1,0)$ gives

$$\begin{aligned}
\partial_x|_{x=1}\partial_y|_{y=0}\frac{1-|z|^2}{2\pi|e^{i\theta}-z|^2} &= 2\partial_x|_{x=1}\frac{-y|e^{i\theta}-z|^2 + (1-x^2-y^2)\sin\theta - y(1-|\mathbf{z}|^2)}{2\pi|e^{i\theta}-(x+iy)|^4}\bigg|_{y=0} \\
&= 2\sin\theta\,\partial_x|_{x=1}\frac{1-x^2}{2\pi(x^2-2x\cos\theta+1)^2} \\
&= 4\sin\theta\,\frac{-x}{2\pi(x^2-2x\cos\theta+1)^2}\bigg|_{x=1} = -\frac{\sin\theta}{2\pi(\cos\theta-1)^2} = \rho_2(\theta).
\end{aligned}$$

Combining this with (3.3.1) shows that $\partial_x\partial_y u(1) = \int_{-\pi}^{\pi} u(e^{i\theta})\rho_2(\theta)d\theta$. The proof is finished. □

Corollary 3.3.6. *Let u be a function as in Lemma 3.3.5. There are smooth functions g_1, g_2 defined on $[0,1]$ and g_3 defined on $[-\pi/2,\pi/2]$ so that for any $s\in[0,1]$, we have*

$$u(1-s)=-s\partial_x u(1)-s^2\frac{\partial_x u(1)}{2}+s^3g_1(s)\quad\text{and}\quad -\mathcal{T}_1u(1-s)=s^2\frac{\partial_x\partial_y u(1)}{2}+s^3g_2(s),$$

and

$$-\mathcal{T}_1u(e^{i\theta})=-\partial_x u(1)\theta-\frac{\partial_x\partial_y u(1)}{2}\theta^2+\theta^3g_3(\theta),$$

for any $\theta\in[-\pi/2,\pi/2]$. Moreover, there is a constant c independent of u for which $\|g_j\|_0\leq c\|u\|_{4,\partial\mathbb{D}}$ for $j=1,2,3$.

Proof. This is an analogue of Lemma 3.3.2. Recall that we have

$$-\partial_x\mathcal{T}_1u(1)=\partial_y u(1)=0\quad\text{and}\quad -\partial_x\mathcal{T}_1u(z)=\partial_y u(z).$$

Thus, $-\partial_x^2\mathcal{T}_1u(z)=\partial_x\partial_y u(z)$. Letting $z=1$ in the last equality gives $-\partial_x^2\mathcal{T}_1u(1)=\partial_x\partial_y u(1)$. Using the last equalities and (3.3.12) and Taylor's expansions at $s=0$ for $u(1-s)$ and for $-\mathcal{T}_1u(1-s)$, we get g_1,g_2 and the first two equalities. By (3.3.11), we have

$$-\partial_\theta\mathcal{T}_1u(e^{i\theta})=-\partial_y\mathcal{T}_1u(1)=-\partial_x u(1)$$

and

$$-\partial_\theta^2\mathcal{T}_1u(e^{i\theta})=-\partial_y^2\mathcal{T}_1u(1)-\partial_x\mathcal{T}_1u(1)=-\partial_y^2\mathcal{T}_1u(1)=-\partial_x\partial_y u(1).$$

This combined with Taylor's expansion at $\theta=0$ of $-\mathcal{T}_1u(e^{i\theta})$ gives g_3 and the third equality. Since g_j are the remainders in Taylor's expansions up to the order 2, we also see that there is a constant c independent of u so that for $1\leq j\leq 3$,

$$\|g_j\|_0\leq\max\{\|u\|_{3,\overline{\mathbb{D}}},\|\mathcal{T}_1u\|_{3,\overline{\mathbb{D}}}\}\leq c\|u\|_{4,\partial\mathbb{D}},$$

by (3.3.4) and (3.3.3). The proof is finished. □

Let $\tilde{z}\in\mathbb{D}$, $\delta\in(0,1]$ and $\gamma\in(0,1]$. We want to construct a function u on $\partial\mathbb{D}$ which is differentiable enough such that $\partial_x u(1)$ and $\partial_x\partial_y u(1)$ equal to prescribed values. Note that we always identify u with its harmonic extension on $\mathbb{D}$. Precisely, we want to choose u so that

$$-\delta\partial_x u(1)-\delta^2\frac{\partial_x u(1)}{2}=\operatorname{Im}\tilde{z}\quad\text{and}\quad\delta^2\frac{\partial_x\partial_y u(1)}{2}=\operatorname{Re}\tilde{z}-\gamma.\tag{3.3.13}$$

The last system is equivalent to

$$\partial_x u(1)=-\frac{2\operatorname{Im}\tilde{z}}{\delta(2+\delta)}\quad\text{and}\quad\partial_x\partial_y u(1)=-\frac{2(\gamma-\operatorname{Re}\tilde{z})}{\delta^2}.\tag{3.3.14}$$

In order to construct a such u satisfying the last property, we will need the following lemma.

Lemma 3.3.7. *Let m be a positive integer. Let $\{a_j\}_{1\leq j\leq m}$ be real smooth functions on $\partial\mathbb{D}$ such that they are linearly independent in $\mathcal{C}^\infty(\partial\mathbb{D})$. Then there exist $b_j\in\mathcal{C}^\infty(\partial\mathbb{D})$ with $1\leq j\leq m$ so that*

$$\int_{-\pi}^{\pi}b_j(e^{i\theta})a_{j'}(e^{i\theta})d\theta=\delta^j_{j'},$$

for all $1\leq j,j'\leq m$, where $\delta^j_{j'}$ is the Kronecker delta.

Proof. Let $L_j : \mathcal{C}^\infty(\partial\mathbb{D}) \to \mathbb{R}$ be the linear functional defined by

$$L_j(v) = \int_{-\pi}^{\pi} a_j(e^{i\theta})v(e^{i\theta})d\theta,$$

for $v \in \mathcal{C}^\infty(\partial\mathbb{D})$ and $1 \leq j \leq m$. The linear independence of a_j and the density of smooth functions in $L^2(\partial\mathbb{D})$ imply that $\{L_j\}_{1\leq j\leq m}$ are linearly independent. A basic result of linear algebra says that for any $1 \leq j \leq m$,

$$\big(\bigcap_{j'\neq j} \operatorname{Ker} L_{j'}\big)\setminus \operatorname{Ker} L_j \neq \varnothing.$$

In the other words, there is $b_j \in \mathcal{C}^\infty(\partial\mathbb{D})$ satisfying $L_{j'}(b_j) = \delta^j_{j'}$. The proof is finished. □

Lemma 3.3.8. *There exist two functions $u_1(e^{i\theta}), u_2(e^{i\theta}) \in \mathcal{C}^\infty(\mathbb{D})$ vanishing on $\partial^+\mathbb{D}$ such that*

$$\partial_x u_1(1) = \partial_x\partial_y u_2(1) = 1 \quad \textit{and} \quad \partial_x u_2(1) = \partial_x\partial_y u_1(1) = 0. \tag{3.3.15}$$

Proof. By Lemma 3.3.5, the condition (3.3.15) is equivalent to

$$\int_{-\pi}^{\pi} u_1\rho_1 d\theta = \int_{-\pi}^{\pi} u_2\rho_2 d\theta = 1 \quad \text{and} \quad \int_{-\pi}^{\pi} u_1\rho_2 d\theta = \int_{-\pi}^{\pi} u_2\rho_1 d\theta = 0.$$

Put $\partial^-\mathbb{D} = \partial\mathbb{D}\setminus\partial^+\mathbb{D}$. Let $\chi \in \mathcal{C}^\infty(\partial\mathbb{D})$ with $\operatorname{supp}\chi \subset \partial^-\mathbb{D}$ and $\chi \not\equiv 0$. Let $a_1 = \chi\rho_1(\theta)$, $a_2 = \chi\rho_2(\theta)$. Observe that these functions are linearly independent in $\mathcal{C}^\infty(\partial\mathbb{D})$. This allows us to apply Lemma 3.3.7 to a_1, a_2. Hence, we obtain $b_1, b_2 \in \mathcal{C}^\infty(\partial\mathbb{D})$ with

$$\int_{-\pi}^{\pi} b_j a_{j'} d\theta = \delta^j_{j'}.$$

Let $u_1 := \chi b_1$ and $u_2 := \chi b_2$. One easily checks that u_1 and u_2 satisfies the desired property. The proof is finished. □

Define

$$u_{\tilde z,\delta,\gamma}(e^{i\theta}) = -u_1(e^{i\theta})\frac{2\operatorname{Im}\tilde z}{\delta(2+\delta)} - u_2(e^{i\theta})\frac{2(\gamma - \operatorname{Re}\tilde z)}{\delta^2}. \tag{3.3.16}$$

We deduce from Lemma 3.3.8 and (3.3.14) that $u_{\tilde z,\delta,\gamma}$ enjoys the property (3.3.13). The following explains our choice of $u_{\tilde z,\delta,\gamma}$.

Lemma 3.3.9. *Let $(\tilde z, \delta, \gamma) \in \mathbb{D} \times (0,1]^2$ so that*

$$\gamma \geq 2|\tilde z| \quad \textit{and} \quad 2\sqrt{\gamma} \geq \delta \geq \frac{\sqrt{\gamma}}{2}. \tag{3.3.17}$$

Then, there are a positive constant θ_0 independent of $(\tilde z, \delta, \gamma)$ and a smooth function $g_{\tilde z,\delta,\gamma}(s)$ defined on $[0,1]$ depending smoothly on the parameter $(\tilde z, \delta, \gamma)$ such that $\|g_{\tilde z,\delta,\gamma}\|_1$ is bounded independently of $(\tilde z, \delta, \gamma)$ and the analytic disc

$$f_{\tilde z,\delta,\gamma} := \gamma - \mathcal{T}_1 u_{\tilde z,\delta,\gamma} + iu_{\tilde z,\delta,\gamma}$$

is $[e^{-i\theta_0}, e^{i\theta_0}]$-attached to $\mathbb{R}^+$ in $\mathbb{C}$ and $f_{\tilde z,\delta,\gamma}(1-\delta) = \tilde z + \delta^3 g_{\tilde z,\delta,\gamma}(\delta)$. Moreover, the quantities

$$\delta\|D_\delta\, g_\delta(\cdot)\|_0 \quad \textit{and} \quad \|D_{(\tilde z,\gamma)} g_{\tilde z,\delta,\gamma}(\cdot)\|_0$$

are bounded independently of $(\tilde z, \delta, \gamma)$, where g is considered as a function of $(s, \tilde z, \delta, \gamma)$ and $D_{(\tilde z,\delta,\gamma)}$ is the differential with respect to $(\tilde z, \delta, \gamma)$.

Proof. Corollary (3.3.6), (3.3.16) and (3.3.13) show that there exist smooth functions g_1, g_2 defined on $[0,1]$ depending smoothly on $(\tilde{z},\delta,\gamma)$ for which

$$f(1-\delta)=\gamma+\operatorname{Re}\tilde{z}-\gamma+\delta^3 g_1(\delta)+i\big(\operatorname{Im}\tilde{z}+\delta^3 g_2(\delta)\big)=\tilde{z}+\delta^3\big(g_1(\delta)+ig_2(\delta)\big).$$

Hence, it is immediate to see that the function

$$g_{\tilde{z},\delta,\gamma}(\delta):=g_1(\delta)+ig_2(\delta)$$

satisfies $f(1-\delta)=\tilde{z}+\delta^3 g(\delta)$. By the hypothesis on $(\tilde{z},\delta,\gamma)$, we have

$$\Big|\frac{2\operatorname{Im}\tilde{z}}{\delta(2+\delta)}\Big|\le 2,\quad \frac{1}{4}\le\frac{2(\gamma-\operatorname{Re}\tilde{z})}{\delta^2}\le 12. \tag{3.3.18}$$

This yields that $\|g_1\|_1$ and $\|g_2\|_1$ are bounded independently of $(\tilde{z},\delta,\gamma)$, hence, so is $\|g\|_1$. Estimating $\delta\|D_\delta\, g_{\tilde{z},\delta,\gamma}(\cdot)\|_0$ and $\|D_{(\tilde{z},\gamma)}g_{\tilde{z},\delta,\gamma}(\cdot)\|_0$ is done similarly.

Now we prove that $f_{\tilde{z},\delta,\gamma}$ is partly attached to $\mathbb{R}^+$. To this end, it suffices to check the sign of the real part of $f_{\tilde{z},\delta,\gamma}$. Using again Corollary (3.3.6), (3.3.16) and (3.3.13) implies that for $\theta\in[-\pi/2,\pi/2]$, we have

$$\operatorname{Re} f_{\tilde{z},\delta,\gamma}(e^{i\theta})=\gamma+\frac{2\operatorname{Im}\tilde{z}}{\delta(2+\delta)}\theta+\frac{2(\gamma-\operatorname{Re}\tilde{z})}{\delta^2}\theta^2+\theta^3 g_3(\theta),$$

where $g_3(\theta)$ is a smooth function on $[-\pi/2,\pi/2]$ whose supnorm is bounded independently of $(\tilde{z},\delta,\gamma)$. Let c be a such upper bound of $\|g_3\|_0$. Put

$$\tilde{f}_1(\theta):=\gamma+\frac{2\operatorname{Im}\tilde{z}}{\delta(2+\delta)}\theta+\frac{\gamma-\operatorname{Re}\tilde{z}}{\delta^2}\theta^2$$

and

$$\tilde{f}_2(\theta):=\frac{\gamma-\operatorname{Re}\tilde{z}}{\delta^2}\theta^2+\theta^3 g_3(\theta).$$

We have $\operatorname{Re} f_{\tilde{z},\delta,\gamma}(e^{i\theta})=\tilde{f}_1(\theta)+\tilde{f}_2(\theta)$. By the second inequality of (3.3.18), one sees that

$$\tilde{f}_2(\theta)\ge\theta^2/8+\theta^3 g_3(\theta)\ge\theta^2\big(1/8-|\theta|\|g_3\|_0\big)\ge 0$$

provided that $|\theta|\le\min\{\pi/2,1/(8c)\}$. Observe that $\tilde{f}_1(\theta)$ is a quadratic polynomial in θ. Its discriminant equals to

$$\frac{1}{\delta^2}\left[\frac{\operatorname{Im}^2\tilde{z}}{(2+\delta)^2}-\gamma(\gamma-\operatorname{Re}\tilde{z})\right]\le\frac{1}{\delta^2}\big[\operatorname{Im}^2\tilde{z}-\gamma^2/2\big]\le 0,$$

because $\delta\ge 0$ and $\gamma\ge 2|\tilde{z}|$. This means that $\tilde{f}_1(\theta)\ge 0$ for all θ. Hence, $\operatorname{Re} f_{\tilde{z},\delta,\gamma}(e^{i\theta})\ge 0$ for $|\theta|\le\theta_0:=\min\{\pi/2,1/(8c)\}$. The proof is finished. □

Now let $\mathbf{z}=(z_1,\cdots,z_n)\in B^*_{2n}(0,\frac{1}{2n})$ and let $t\in(0,1]$. In the formula (3.3.16), let

$$\gamma=2|\mathbf{z}|,\quad \delta=\sqrt{|\mathbf{z}|}\quad\text{and}\quad \tilde{z}=z_j,$$

where z_j is the j^{th} component of $\mathbf{z}$, denote by $u'_{\mathbf{z};j}$ the function $u_{\tilde{z},\delta,\gamma}$ with the above choice of $(\tilde{z},\delta,\gamma)$.

Define $u'_{\mathbf{z},t}$ to be the vector of $\mathcal{C}^\infty(\mathbb{D})^n$ whose j^{th} component is equal $tu'_{\mathbf{z};j}$, for $1 \leq j \leq n$. Extend $u'_{\mathbf{z},t}$ harmonically to $\mathbb{D}$. Define

$$F'(z,\mathbf{z},t) := 2t(|\mathbf{z}|,\cdots,|\mathbf{z}|) - \mathcal{T}_1(u'_{\mathbf{z},t})(z) + iu'_{\mathbf{z},t}, \tag{3.3.19}$$

for $z \in \overline{\mathbb{D}}$, where $(|\mathbf{z}|,\cdots,|\mathbf{z}|) \in \mathbb{R}^n$. Then, F' is a family of analytic discs half-attached to $\mathbb{R}^n$ and $F'(1,\mathbf{z},t) = t(|\mathbf{z}|,\cdots,|\mathbf{z}|) \in (\mathbb{R}^+)^n$. By (3.3.14), we have

$$\partial_x u'_{\mathbf{z},t}(1) = -\frac{2t\,\mathrm{Im}\,\mathbf{z}}{\sqrt{|\mathbf{z}|}(2+\sqrt{|\mathbf{z}|})} \quad \text{and} \quad \partial_x\partial_y u'_{\mathbf{z},t}(1) = -\frac{2t(2|\mathbf{z}| - \mathrm{Re}\,\mathbf{z})}{|\mathbf{z}|}, \tag{3.3.20}$$

where, for simplicity, we wrote $|\mathbf{z}|$ for $(|\mathbf{z}|,\cdots,|\mathbf{z}|)$ in the last equality.

Proposition 3.3.10. *The map $F' : \overline{\mathbb{D}} \times B^*_{2n}(0,\frac{1}{2n}) \times (0,1] \to \mathbb{C}^n$ is smooth and there exists two constants $r'_0 \in (0,1/4)$ and $c_0 > 1$ such that the three following conditions hold:*

(i) for any $\mathbf{z} \in B^*_{2n}(0,\frac{1}{2n})$ and $t \in (0,1]$, the mapping $F'(\cdot,\mathbf{z},t)$ is a smooth analytic disc $[e^{ic_0^{-1}}, e^{ic_0^{-1}}]$-attached to $(\mathbb{R}^+)^n$ in $\mathbb{C}^n$, and

$$F'(1,\mathbf{z},t) = 2t(|\mathbf{z}|,\cdots,|\mathbf{z}|) \in B_n(0,1) \cap (\mathbb{R}^+)^n,$$

(ii) for any $\mathbf{z} \in B^*_{2n}(0,r'_0)$ and $t \in (0,1]$, there exists an $\mathbf{z}^* \in B^*_{2n}(0,2r'_0)$ for which

$$F'(1-\sqrt{|\mathbf{z}^*|},\mathbf{z}^*,t) = t\mathbf{z}$$

and $|\mathbf{z}^*| \leq 2|\mathbf{z}|$,

(iii) for any $\mathbf{z} \in B^*_{2n}(0,\frac{1}{2n})$ and $t \in (0,1]$, we have

$$\|F'(\cdot,\mathbf{z},t)\|_5 \leq tc_0 \text{ and } \quad \|D_{\mathbf{z}}F'(\cdot,\mathbf{z},t)\|_4 \leq tc_0|\mathbf{z}|^{-1}, \tag{3.3.21}$$

where $D_{\mathbf{z}}$ is the differential with respect to $\mathbf{z}$.

Proof. Since $F'(z,\mathbf{z},t) = tF'(z,\mathbf{z},1)$, it is enough to verify the three above conditions for $t=1$. It is clear that $(\gamma,\delta,\tilde{z}) = (2|\mathbf{z}|,\sqrt{|\mathbf{z}|},z_j)$ satisfies the condition (3.3.17) for $1 \leq j \leq n$. Hence, direct consequences of Lemma 3.3.9 and (3.3.19) show that there exists a constant $c_0 > 1$ for which the property (i) and (iii) hold. It remains to verify (ii). We will use the same idea as in the proof of Proposition 3.3.4.

Fix $t \in (0,1]$. Let $\Phi'(\mathbf{z}) := F'(1-\sqrt{|\mathbf{z}|},\mathbf{z},t)$ for $\mathbf{z} \in B^*_{2n}(0,\frac{1}{2n})$ and $\Phi'(0) := 0$. By the above reason and Lemma 3.3.9, there exists a smooth map $g_{\mathbf{z}}(s) : [0,1] \to \mathbb{R}^n$ depending smoothly on $\mathbf{z} \in B^*_{2n}(0,\frac{1}{2n})$ such that

$$\Phi'(\mathbf{z}) = t\mathbf{z} + t|\mathbf{z}|^{3/2} g_{\mathbf{z}}(\sqrt{|\mathbf{z}|}).$$

Note that the homogeneity of F' in t implies that $g_{\mathbf{z}}$ is independent of t. Put

$$g'(\mathbf{z}) := t|\mathbf{z}|^{3/2} g_{\mathbf{z}}(\sqrt{|\mathbf{z}|}).$$

Observe that $g'(0) = 0$ and

$$D_{\mathbf{z}}g'(\mathbf{z}) = \frac{3t|\mathbf{z}|^{1/2}}{2} g_{\mathbf{z}}(|\sqrt{|\mathbf{z}|}) + t|\mathbf{z}|^{3/2}\big\{D_{\mathbf{z}}g_{\mathbf{z}}(\sqrt{|\mathbf{z}|}) + D_s g_{\mathbf{z}}(\sqrt{|\mathbf{z}|})D_{\mathbf{z}}\sqrt{|\mathbf{z}|}\big\}.$$

Lemma 3.3.9 for $(\gamma, \delta, \tilde{z}) = (2|\mathbf{z}|, \sqrt{|\mathbf{z}|}, z_j)$ implies that $\sqrt{|\mathbf{z}|}D_{\mathbf{z}}g_{\mathbf{z}}(\sqrt{|\mathbf{z}|})$ and $D_s g_{\mathbf{z}}(\sqrt{|\mathbf{z}|})$ are bounded independently of $\mathbf{z}$. As a consequence, we have

$$|D_{\mathbf{z}}g'(\mathbf{z})| \leq c|\mathbf{z}|^{1/2}t,$$

for some constant c independent of $(\mathbf{z}, t)$. Let $r_0' := (2c)^{-2}/3$. The last inequality yields that $|D_{\mathbf{z}}g'(\mathbf{z})| \leq t/2$ for $\mathbf{z} \in B_{2n}^*(0, 3r_0')$. Thus, g' is $t/2$-Lipschitz on $\overline{B}_{2n}(0, 2r_0')$. Applying Lemma 3.3.3 to Φ' in place of Φ_0, $A = t\mathrm{Id}$ and g' as above shows that for any $\mathbf{z} \in B_{2n}^*(0, r_0')$, there exists $\mathbf{z}^* \in B_{2n}^*(0, 2r_0')$ for which $\Phi'(\mathbf{z}^*) = t\mathbf{z}$. Moreover, the last equation implies that

$$t|\mathbf{z}| \geq t|\mathbf{z}^*| - |g(\mathbf{z}^*)| \geq t|\mathbf{z}^*| - t/2|\mathbf{z}^*|.$$

Hence, $|\mathbf{z}^*| \leq 2|\mathbf{z}|$. The proof is finished. □

As explained at the beginning, let us now introduce a new parameter $\boldsymbol{\tau} \in B_n(0, 2)$ and a family $F_{\boldsymbol{\tau}}'$ of analytic discs half-attached to $\mathbb{R}^n$ contains F' as a subfamily.

Lemma 3.3.11. *Let u_1 be the function in Lemma 3.3.8. Then, the function $\tilde{u} := 10u_1$ is smooth on $\partial\mathbb{D}$ and vanishes on $\partial^+\mathbb{D}$ and*

$$\partial_x\tilde{u}(1) = 10, \quad \partial_x\partial_y\tilde{u}(1) = 0. \tag{3.3.22}$$

Proof. This is obvious by the properties of u_1. The proof is finished. □

Let $\mathbf{z}$ and t be as above. We define

$$u'_{\mathbf{z},t,\boldsymbol{\tau}}(z) := u'_{\mathbf{z},t}(z) + t\boldsymbol{\tau} \cdot \tilde{u}(z), \tag{3.3.23}$$

where $\boldsymbol{\tau} \in B_n(0, 2)$, see the paragraph right after Lemma 3.3.3 for the notation. The parameter $\boldsymbol{\tau}$ will play a role as a control parameter. Define

$$F'_{\boldsymbol{\tau}}(z, \mathbf{z}, t) := 2t(|\mathbf{z}|, \cdots, |\mathbf{z}|) - \mathcal{T}_1 u'_{\mathbf{z},t,\boldsymbol{\tau}}(z) + iu'_{\mathbf{z},t,\boldsymbol{\tau}}(z), \tag{3.3.24}$$

for $\mathbf{z} \in B_{2n}^*(0, \frac{1}{2n}), t \in (0, 1]$ and $\boldsymbol{\tau} \in B_n(0, 2)$. By construction, $F'_{\boldsymbol{\tau}}$ is a family of discs half-attached to $\mathbb{R}^n$ and when $\boldsymbol{\tau} = 0$ we have $F'_0 \equiv F'$ which is the family constructed earlier.

By choosing the constant c_0 in Proposition 3.3.10 big enough, for any $\mathbf{z} \in B_{2n}^*(0, \frac{1}{2n}), t \in (0, 1]$ and $\boldsymbol{\tau} \in B_n(0, 2)$, we have

$$\|D_{\boldsymbol{\tau}}^j F'_{\boldsymbol{\tau}}(\cdot, \mathbf{z}, t)\|_4 \leq tc_0 \text{ and } \|D_{\boldsymbol{\tau}}^j D_{\mathbf{z}} F'_{\boldsymbol{\tau}}(\cdot, \mathbf{z}, t)\|_3 \leq tc_0|\mathbf{z}|^{-1}, \tag{3.3.25}$$

for $j = 0, 1$ and

$$D_{\boldsymbol{\tau}}^2 F'_{\boldsymbol{\tau}}(1, \mathbf{z}, t) \equiv 0, \quad D_{\boldsymbol{\tau}} u'_{\mathbf{z},t,\boldsymbol{\tau}}(1) = 10t, \tag{3.3.26}$$

where the right-hand side of the last equality denotes the diagonal matrix of order n whose coefficients on the diagonal are all equal to $10t$.

3.4 Analytic discs partly attached to K

Fix a smooth Riemannian metric on X. Let p_0 be an arbitrary point of K. Our goal is to construct special families of analytic discs partly attached to K in a small neighborhood of p_0 in X. Since K is a generic submanifold, its dimension is at least n. We first study the case where $\dim_{\mathbb{R}} K = n$. Then we deduce the case of higher dimension by considering (local) generic submanifolds of K. In what follows, the notations $\gtrsim$ and $\lesssim$ respectively mean $\geq$ and $\leq$ up to a positive constant depending only on the geometry of (K, X).

3.4.1 The case where K has no singularity

In this subsection, we consider the case where K has no singularity and $\dim_{\mathbb{R}} K = n$. The $\mathcal{C}^3$-differentiability of K is enough for our proof. The local coordinates described in Lemma 3.4.1 below are used widely in the Cauchy-Riemann geometry. Since we need to use concrete estimates uniform in p_0, a complete proof will be presented. We refer to the beginning of Subsection 3.3.1 for the notation of the norms of maps below.

Lemma 3.4.1. *There exist constants $c_1, r_K > 1$ depending only on (K, X) and a local chart (W_{p_0}, Ψ) around p_0, where $\Psi : W_{p_0} \to B_{2n}(0, r_K)$ is biholomorphic with $\Psi(p_0) = 0$ such that the two following conditions hold:*

(i) we have

$$\|\Psi\|_1 \leq c_1, \quad \|\Psi^{-1}\|_1 \leq c_1,$$

(ii) there is a $\mathcal{C}^3$ map h from $\overline{B}_n(0,1)$ to $\mathbb{R}^n$ so that $h(0) = Dh(0) = 0$, where Dh denotes the differential of h, and

$$\Psi(K \cap W_{p_0}) \supset \{(\mathbf{x}, h(\mathbf{x})) : \mathbf{x} \in \overline{B}_n(0,1)\},$$

where the canonical coordinates on $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ are denoted by $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, and

$$\|h\|_3 \leq c_1. \tag{3.4.1}$$

Proof. We cover X by a finite family of charts (W_j, Ψ_j), where W_j is an open subset of X and Ψ_j is a biholomorphic map from W_j to the ball $B_{2n}(0,2)$. We choose these charts so that $\Psi_j^{-1}\big(B_{2n}(0,1)\big)$ also cover X. This choice is independent of p_0. Consider a chart (W_{j_0}, Ψ_{j_0}) such that p_0 belongs to $\Psi_{j_0}^{-1}\big(B_{2n}(0,1)\big)$. Define $W_{p_0} := W_{j_0}$ and $\Psi := \Psi_{j_0} - \Psi_{j_0}(p_0)$. Let $\mathbf{z} = (z_1, \cdots, z_n)$ be the coordinates on $\mathbb{C}^n$. Identify $K \cap W_{p_0}$ with $\Psi(K \cap W_{p_0})$ for convenience. By the hypothesis on K, we have

$$T^{\mathbb{R}}_{\mathbf{z}} K + iT^{\mathbb{R}}_{\mathbf{z}} K = \mathbb{C}^n.$$

This implies that there are a positive constant c_1 independent of p_0 and a linear change of coordinates $\Psi' = (\Psi'_1, \cdots, \Psi'_n)$ of $\mathbb{C}^n$ such that

$$\|\Psi'\|_1 \leq c_1, \quad \|\Psi'^{-1}\|_1 \leq c_1 \tag{3.4.2}$$

and

$$\Psi'(p_0) = 0 \quad \text{and} \quad \Psi'(T^{\mathbb{R}}_{p_0} K) = \{\operatorname{Im} \Psi'_k = 0, 1 \leq k \leq n\},$$

where $T^{\mathbb{R}}_{\mathbf{z}}K$ is considered naturally as an affine subspace of $\mathbb{C}^n$. Replacing Ψ by $\Psi' \circ \Psi$, we can suppose that $\Psi(p_0) = 0$ and

$$\Psi_*(T^{\mathbb{R}}_{p_0}K) = \{\operatorname{Im} z_k = 0, 1 \le k \le n\} = \{\mathbf{z} = \mathbf{x} + i0\} \equiv \mathbb{R}^n.$$

By rescaling Ψ (by a constant independently of p_0) if necessary, the submanifold

$$K \cap W_{p_0} \cap \{\mathbf{z} \in \mathbb{C}^n : \operatorname{Re}\mathbf{z} \in \overline{B}_n(0,1)\}$$

is the graph of a $\mathcal{C}^3$ map $h = (h_1, \cdots, h_n)$ over $\overline{B}_n(0,1)$ of $\mathbb{R}^n$. By construction, we have $h(0) = Dh(0) = 0$. The compactness of K and (3.4.2) insure that there is a positive constant c_1 independent of p_0 such that $\|h\|_3 \le c_1$. The proof is finished. □

From now on, we only use the local coordinates introduced in Lemma 3.4.1 and identify points in W_{p_0} with those in $B_{2n}(0, r_K)$ via Ψ. Property (i) of Lemma 3.4.1 implies that the distance on X is uniformly comparable with the Euclidean distance measured by the local coordinates given in Lemma 3.4.1. Hence, in what follows, we make no distinction between these two distances. The estimate (3.4.1) implies that

$$|h(\mathbf{x})| \le c_1|\mathbf{x}|^2, \quad |Dh(\mathbf{x})| \le c_1|\mathbf{x}| \quad \text{for } |\mathbf{x}| \le 1. \tag{3.4.3}$$

For each $\mathbf{z} \in B^*_{2n}(0,1)$ and $t \in (0,1]$, let $u_{\mathbf{z},t}$ be the map defined in (3.3.7). Let F and c_0 be the family of analytic discs and the constant respectively in Proposition 3.3.4. In order to construct an analytic disc half-attached to K, it suffices to find a map

$$U : \partial\mathbb{D} \to B_n(0,1) \subset \mathbb{R}^n,$$

which is Hölder continuous, satisfying the following Bishop-type equation

$$U_{\mathbf{z},t}(\xi) = t(\operatorname{Re}\mathbf{z} - \operatorname{Im}\mathbf{z}) - \mathcal{T}_1\big(h(U_{\mathbf{z},t})\big)(\xi) - \mathcal{T}_1 u_{\mathbf{z},t}(\xi), \tag{3.4.4}$$

where $\mathbf{z}$ and t are parameters in $B^*_{2n}(0,1)$ and $(0,1)$ respectively. Indeed, suppose that (1.3.11) has a solution. For simplicity, we use the same notation $U_{\mathbf{z},t}(z)$ to denote the harmonic extension of $U_{\mathbf{z},t}(\xi)$ to $\mathbb{D}$. Let $P_{\mathbf{z},t}(z)$ be the harmonic extension of $h\big(U_{\mathbf{z},t}(\xi)\big)$ to $\mathbb{D}$. Define

$$F^h(z, \mathbf{z}, t) := U_{\mathbf{z},t}(z) + iP_{\mathbf{z},t}(z) + iu_{\mathbf{z},t}(z)$$

which is a family of analytic discs parametrized by $(\mathbf{z}, t)$. For any $\xi \in \partial^+\mathbb{D}$, the defining formula of F^h and the fact that $u_{\mathbf{z},t} \equiv 0$ on $\partial^+\mathbb{D}$ imply that

$$F^h(\xi, \mathbf{z}, t) = U_{\mathbf{z},t}(\xi) + iP_{\mathbf{z},t}(\xi) = U_{\mathbf{z},t}(\xi) + ih\big(U_{\mathbf{z},t}(\xi)\big) \in K$$

by Property (ii) of Lemma 3.4.1. In other words, F^h is half-attached to K. Moreover we have

$$F^h(1, \mathbf{z}, t) = t(\operatorname{Re}\mathbf{z} - \operatorname{Im}\mathbf{z}) + ih(t\operatorname{Re}\mathbf{z} - t\operatorname{Im}\mathbf{z}).$$

In what follows, it is convenient to regard $U_{\mathbf{z},t}(z)$ as a function of three variables $(z, \mathbf{z}, t)$.

Proposition 3.4.2. *There is a small positive number $t_1 \in (0,1)$ independent of $(\mathbf{z}, p_0)$ so that for any $t \in (0, t_1]$ and any $\mathbf{z} \in B^*_{2n}(0,1)$, the equation (1.3.11) has a unique solution $U_{\mathbf{z},t}$ such that $U_{\mathbf{z},t}(\xi)$ is $\mathcal{C}^{2,\frac{1}{2}}$ in ξ, the partial derivative $D_{\mathbf{z}}U_{\mathbf{z},t}$ exists and is $\mathcal{C}^{1,\frac{1}{2}}$ in $\xi \in \partial\mathbb{D}$. Moreover, the two following estimates hold:*

$$\|U_{\mathbf{z},t}(\cdot)\|_{2,\frac{1}{2}} \leq 4c_0 t, \quad \|D_{\mathbf{z}}U_{\mathbf{z},t}(\cdot)\|_{1,\frac{1}{2}} \leq 4c_0 t|\mathbf{z}|^{-1}. \tag{3.4.5}$$

Proposition 3.4.2 except (3.4.5) is a direct corollary of a more general result due to Tumanov, see [59, Th. 4.19]. Since we do not need the optimal regularity for $U_{\mathbf{z},t}$ (whereas it is the case for Tumanov's result), the proof is simpler. We will follow the presentation in [59]. Firstly, we need the following preparatory lemma on the norms of the Hölder spaces.

Lemma 3.4.3. *Let g_1 and g_2 be functions defined on $\partial\mathbb{D}$ with suitable differentiability. Then we have*

$$\|g_1 g_2\|_{\frac{1}{2}} \leq \|g_1\|_{\frac{1}{2}} \|g_2\|_{\frac{1}{2}} \tag{3.4.6}$$

and

$$\|g_1 g_2\|_{1,\frac{1}{2}} \leq 4\|g_1\|_{1,\frac{1}{2}} \|g_2\|_{1,\frac{1}{2}}. \tag{3.4.7}$$

Moreover, there exists a positive constant $c(n)$ such that for any maps g_1, g_2 from $\partial\mathbb{D}$ to $B_n(0,1)$ and any function f on $B_n(0,1)$, we have

$$\|f \circ g_1 - f \circ g_2\|_{1,\frac{1}{2}} \leq c(n)\{1 + \|g_2\|_{1,\frac{1}{2}}\}\|f\|_2 \|g_1 - g_2\|_{1,\frac{1}{2}} \tag{3.4.8}$$

and

$$\|f \circ g_1\|_{1,\frac{1}{2}} \leq c(n)\big\{\|Df \circ g_1\|_0 \|g_1\|_{1,\frac{1}{2}} + \|f\|_2 \|g_1\|^2_{1,\frac{1}{2}}\big\}. \tag{3.4.9}$$

and

$$\|f \circ g_1\|_{1,\frac{1}{2}} \leq c(n)\{1 + \|g_1\|_{1,\frac{1}{2}}\}\|f\|_2 \|g_1\|_{1,\frac{1}{2}}. \tag{3.4.10}$$

Proof. Write

$$g_1(\xi)g_2(\xi) - g_1(\xi')g_2(\xi') = g_1(\xi)\big(g_2(\xi) - g_2(\xi')\big) + \big(g_1(\xi) - g_2(\xi')\big)g_2(\xi'),$$

for any $\xi, \xi' \in \partial\mathbb{D}$. Using the last equality and the definition of the $\mathcal{C}^{\frac{1}{2}}$, one easily gets (3.4.6). Since $D(g_1 g_2) = (Dg_1)g_2 + g_1 Dg_2$, using (3.4.6) and the definition of $\mathcal{C}^{1,\frac{1}{2}}$ gives

$$\begin{aligned}\|g_1 g_2\|_{1,\frac{1}{2}} &\leq \|g_1 g_2\|_1 + \|D(g_1 g_2)\|_{\frac{1}{2}} \\ &\leq \|g_1\|_1 \|g_2\|_1 + \|Dg_1\|_{\frac{1}{2}} \|g_2\|_1 + \|g_1\|_1 \|Dg_2\|_{\frac{1}{2}} \leq 4\|g_1\|_{1,\frac{1}{2}} \|g_2\|_{1,\frac{1}{2}}.\end{aligned}$$

Hence, (3.4.7) follows.

Now we prove (3.4.8). Let g_1, g_2, f be as in the hypothesis of (3.4.8). We have

$$D\big(f(g_1) - f(g_2)\big) = Df(g_1)Dg_1 - Df(g_2)Dg_2 = Df(g_1)(Dg_1 - Dg_2) + \big(Df(g_1) - Df(g_2)\big)Dg_2.$$

Applying (3.4.6) to the last sum shows that there exists a positive constant $c'(n)$ depending only on n so that

$$\begin{aligned}\|f(g_1)-f(g_2)\|_{1,\frac{1}{2}} &\le \|f(g_1)-f(g_2)\|_1 + \|D\big(f(g_1)-f(g_2)\big)\|_{\frac{1}{2}}\\ &\le c'(n)\Big\{\|f\|_1\|g_1-g_2\|_1 + \|f\|_2\|g_1-g_2\|_0\|g_2\|_1+\\ &\quad + \|Df(g_1)\|_{\frac{1}{2}}\|Dg_1-Dg_2\|_{\frac{1}{2}} + \|f\|_2\|g_1-g_2\|_{\frac{1}{2}}\|g_2\|_{1,\frac{1}{2}}\Big\}\\ &\le c'(n)\{1+3\|g_2\|_{1,\frac{1}{2}}\}\|f\|_2\|g_1-g_2\|_{1,\frac{1}{2}}.\end{aligned}$$

Hence, (3.4.8) follows by choosing $c(n)=3c'(n)$. The inequality (3.4.9) is deduced by using the same method and (3.4.10) is a direct consequence of (3.4.9). The proof is finished. □

Proof of Proposition 3.4.2. Let $C_{1,1/2}$ and $C_{2,1/2}$ be the constants $C_{k,\beta}$ appearing in (3.3.3) with $k=1,2$ and $\beta=1/2$. Let $c(n)$ be the constant in Lemma 3.4.3. Define

$$t_1 := \big(40c_0c_1c(n)\max\{C_{1,1/2},C_{2,1/2}\}+4c_0\big)^{-2}.$$

Fix $t\in(0,t_1)$ and $\mathbf{z}\in B^*_{2n}(0,1)$. Let $\mathcal{A}$ be the set of $\mathcal{C}^{1,\frac{1}{2}}$ maps

$$U:\partial\mathbb{D}\to\overline{B}_n(0,4c_0t)$$

such that $\|U\|_{1,\frac{1}{2}}\le 4c_0t$. Endow $\mathcal{A}$ with the $\mathcal{C}^{1,\frac{1}{2}}$-norm making it become a closed subset of a suitable Banach space. Define

$$G(U) := t(\operatorname{Re}\mathbf{z}-\operatorname{Im}\mathbf{z}) - \mathcal{T}_1\big(h(U)\big) - \mathcal{T}_1u_{\mathbf{z},t}. \tag{3.4.11}$$

We will show that G is a well-defined self-map of $\mathcal{A}$ and is a contraction.

Let $U\in\mathcal{A}$. Note that since $\mathbf{z}\in B^*_{2n}(0,1)$, we have $|\operatorname{Re}\mathbf{z}-\operatorname{Im}\mathbf{z}|\le 2$. By (3.3.3) and Proposition 3.3.4, we get

$$\|G(U)\|_{1,\frac{1}{2}} \le 2t + \|\mathcal{T}_1\|_{1,\frac{1}{2}}\|h(U)\|_{1,\frac{1}{2}} + \|F(\cdot,\mathbf{z},t)\|_2 \le C_{1,1/2}\|h(U)\|_{1,\frac{1}{2}} + 3c_0t \tag{3.4.12}$$

because we chose $c_0>1$. The inequality (3.4.9) for $f=h$ and $g_1=U$ combined with (3.4.3) yields that

$$\|h(U)\|_{1,\frac{1}{2}} \le 2c(n)c_1\|U\|^2_{1,\frac{1}{2}}. \tag{3.4.13}$$

We deduce from (3.4.13) and (3.4.12) that

$$\|G(U)\|_{1,\frac{1}{2}} \le 2c_1c(n)C_{1,1/2}\|U\|^2_{1,\frac{1}{2}} + 3c_0t \le 4c_0t \tag{3.4.14}$$

and similarly using (3.4.11) gives

$$\begin{aligned}\|G(U)-G(U')\|_{1,\frac{1}{2}} &\le 2c_1c(n)C_{1,1/2}\|U-U'\|^2_{1,\frac{1}{2}}\\ &\le 16c_0c_1c(n)C_{1,1/2}t\|U-U'\|_{1,\frac{1}{2}} \le t^{1/2}\|U-U'\|_{1,\frac{1}{2}},\end{aligned} \tag{3.4.15}$$

for any $U, U' \in \mathcal{A}$. The inequality (3.4.14) shows that G is well-defined. And the contractivity of G follows from (3.4.15). By the fixed point theorem, G has a unique fixed point $U_{\mathbf{z},t} \in \mathcal{A}$. In other words, the equation (1.3.11) has a unique solution $U_{\mathbf{z},t} \in \mathcal{C}^{1,\frac{1}{2}}(\mathbb{D})$ and $\|U_{\mathbf{z},t}\|_{1,\frac{1}{2}} \leq 4c_0 t$.

Now we explain why $U_{\mathbf{z},t} \in \mathcal{C}^{2,\frac{1}{2}}(\partial\mathbb{D})$. Define $\mathcal{A}'$ to be the subset of $\mathcal{A}$ consisting of U with $\|U\|_{2,\frac{1}{2}} \leq 4c_0 t$. Since $\mathcal{A}'$ is a closed subset with respect to the $\mathcal{C}^{2,\frac{1}{2}}$-norm in a suitable Banach space and $h, F \in \mathcal{C}^3$, similar arguments as above applied to the $\mathcal{C}^{2,\frac{1}{2}}$-norm show that for t small enough, precisely $t \in (0, t_1)$, G is a self-contraction of $\mathcal{A}'$ and $U_{\mathbf{z},t}$ is the unique fixed point of G. Note that in this argument, we need to use the constant $C_{2,1/2}$ which explains its presence in the definition of t_1. Therefore, $U_{\mathbf{z},t} \in \mathcal{C}^{2,\frac{1}{2}}$ and it satisfies

$$\|U_{\mathbf{z},t}\|_{2,\frac{1}{2}} \leq 4c_0 t.$$

Now we investigate the dependence of $U_{\mathbf{z},t}$ on the parameter $\mathbf{z}$. Observe that if $U_{\mathbf{z},t}$ is differentiable in $\mathbf{z}$ and $D_{\mathbf{z}}U_{\mathbf{z},t}$ is at least $\mathcal{C}^\beta$ in ξ for some $\beta \in (0,1)$, then by (1.3.11) and $\mathcal{C}^\beta$-boundedness of $\mathcal{T}_1$ we must have

$$D_{\mathbf{z}}U_{\mathbf{z},t} = -\mathcal{T}_1\big(Dh(U_{\mathbf{z},t})D_{\mathbf{z}}U_{\mathbf{z},t}\big) - \mathcal{T}_1 D_{\mathbf{z}}u_{\mathbf{z},t}. \tag{3.4.16}$$

This leads us to study the equation

$$V = -\mathcal{T}_1\big(H \cdot V\big) - \mathcal{T}_1 D_{\mathbf{z}}u_{\mathbf{z},t}, \tag{3.4.17}$$

where $H(\xi) = Dh(U_{\mathbf{z},t}(\xi))$ is a $\mathcal{C}^2$ matrix function in $\xi \in \partial\mathbb{D}$. This equation is of the same type as (1.3.11). Since

$$\|D_{\mathbf{z}}u_{\mathbf{z},t}(\cdot)\|_2 \leq tc_0|\mathbf{z}|^{-1} \quad \text{(see (3.3.9))}, \tag{3.4.18}$$

and $H \in \mathcal{C}^2$, the same arguments as above show that the equation (3.4.17) has a unique solution $V_{\mathbf{z},t}$ in $\mathcal{C}^{1,\frac{1}{2}}(\partial\mathbb{D})$ with

$$\|V_{\mathbf{z},t}\|_{1,\frac{1}{2}} \leq 4c_0|\mathbf{z}|^{-1}t. \tag{3.4.19}$$

Furthermore, if we define $V'^0_{\mathbf{z},t} = D_{\mathbf{z}}u_{\mathbf{z},t}$ and

$$V'^{k+1}_{\mathbf{z},t} = -\mathcal{T}_1\big(H \cdot V'^k_{\mathbf{z},t}\big) - \mathcal{T}_1 D_{\mathbf{z}}u_{\mathbf{z},t}$$

for $k \in \mathbb{N}^*$, then

$$\|V'^k_{\mathbf{z},t} - V_{\mathbf{z},t}\|_{1,\frac{1}{2}} \leq t^{k/2}|\mathbf{z}|^{-1} \tag{3.4.20}$$

thanks to the $t^{1/2}$-contractivity of the self-map defining the recurrence relation of $V'^k_{\mathbf{z},t}$.

We now relate $V_{\mathbf{z},t}$ to $U_{\mathbf{z},t}$. Note that by (3.3.9), $u_{\mathbf{z},t} \in \mathcal{A}$ for $t \in (0, t_1)$. Let $\{U^k_{\mathbf{z},t}\}_{k\in\mathbb{N}}$ be the sequence in $\mathcal{A}$ defined by

$$U^0_{\mathbf{z},t} = u_{\mathbf{z},t}, \quad U^k_{\mathbf{z},t} = G(U^{k-1}_{\mathbf{z},t}) \quad \text{for} \quad k \geq 1.$$

Since $u_{\mathbf{z},t}$ is $\mathcal{C}^4$ in $(z, \mathbf{z})$ and $h \in \mathcal{C}^3$ and $\mathcal{T}_1$ is a linear $\mathcal{C}^{2,\frac{1}{2}}$-bounded operator, the functions $U^k_{\mathbf{z},t}$ are $\mathcal{C}^{2,\frac{1}{2}}$ in $(z, \mathbf{z})$ for all $k \geq 0$. Define

$$V^k_{\mathbf{z},t} := D_{\mathbf{z}}U^k_{\mathbf{z},t} \in \mathcal{C}^{1,\frac{1}{2}} \quad \text{in} \quad (z, \mathbf{z}),$$

for $k \in \mathbb{N}$. By definition of $U^k_{\mathbf{z},t}$, the sequence $V^k_{\mathbf{z},t}$ is defined by the induction relation

$$V^{k+1}_{\mathbf{z},t} = -\mathcal{T}_1\big(Dh(U^k_{\mathbf{z},t})V^k_{\mathbf{z},t}\big) - \mathcal{T}_1 D_{\mathbf{z}} u_{\mathbf{z},t},$$

for $k \geq 0$. Using (3.4.18), the induction on k and the above technique in the proof of (3.4.19), we obtain that

$$\|V^k_{\mathbf{z},t}\|_{1,\frac{1}{2}} \leq 4c_0 t|\mathbf{z}|^{-1}. \tag{3.4.21}$$

Since G is $t^{1/2}$-contraction, we have

$$\|U^k_{\mathbf{z},t} - U_{\mathbf{z},t}\|_{1,\frac{1}{2}} \leq t^{k/2}. \tag{3.4.22}$$

We now compare $V'^k_{\mathbf{z},t}$ and $V^k_{\mathbf{z},t}$. Their difference is

$$V^{k+1}_{\mathbf{z},t} - V'^{k+1}_{\mathbf{z},t} = -\mathcal{T}_1\big[\big(Dh(U^k_{\mathbf{z},t}) - Dh(U_{\mathbf{z},t})\big)V^k_{\mathbf{z},t}\big] - \mathcal{T}_1\big[Dh(U_{\mathbf{z},t})(V^k_{\mathbf{z},t} - V'^k_{\mathbf{z},t})\big]. \tag{3.4.23}$$

Applying (3.4.7) and (3.3.3) to each term in the right-hand side of (3.4.23) gives

$$\begin{aligned}\|V^{k+1}_{\mathbf{z},t} - V'^{k+1}_{\mathbf{z},t}\|_{1,\frac{1}{2}} \leq 4C_{1,1/2}\|Dh(U^k_{\mathbf{z},t}) - Dh(U_{\mathbf{z},t})\|_{1,\frac{1}{2}}\|V^k_{\mathbf{z},t}\|_{1,\frac{1}{2}} + \\ + 4C_{1,1/2}\|Dh(U_{\mathbf{z},t})\|_{1,\frac{1}{2}}\|V^k_{\mathbf{z},t} - V'^k_{\mathbf{z},t}\|_{1,\frac{1}{2}}.\end{aligned} \tag{3.4.24}$$

The second term of the right-hand side of (3.4.24) is less than or equal to

$$8c_1c(n)C_{1,1/2}\|U_{\mathbf{z},t}\|_{1,\frac{1}{2}}\|V^k_{\mathbf{z},t} - V'^k_{\mathbf{z},t}\|_{1,\frac{1}{2}}$$

thanks to (3.4.10) and (3.4.1). By the first inequality of (3.4.5) and (3.4.21), the last quantity is less than or equal to

$$32c_0c_1c(n)C_{1,1/2}t|\mathbf{z}|^{-1}\|V^k_{\mathbf{z},t} - V'^k_{\mathbf{z},t}\|_{1,\frac{1}{2}}.$$

In a similar way, the first term of the right-hand side of (3.4.24) is less than or equal to

$$8c_1C_{1,1/2}\|U^k_{\mathbf{z},t} - U_{\mathbf{z},t}\|_{1,\frac{1}{2}}\|V^k_{\mathbf{z},t}\|_{\frac{1}{2}}$$

thanks to (3.4.8) and (3.4.1). By (3.4.21) and (3.4.22), the last quantity is also less than or equal to

$$32c_0c_1c(n)C_{1,1/2}t^{(k+2)/2}|\mathbf{z}|^{-1}.$$

Hence, we just proved that

$$\|V^{k+1}_{\mathbf{z},t} - V'^{k+1}_{\mathbf{z},t}\|_{1,\frac{1}{2}} \leq 32c_0c_1c(n)C_{1,1/2}\Big[t^{(k+2)/2}|\mathbf{z}|^{-1} + t\|V^k_{\mathbf{z},t} - V'^k_{\mathbf{z},t}\|_{1,\frac{1}{2}}\Big].$$

By induction on k and the last inequality, one easily deduces that

$$\|V^{k+1}_{\mathbf{z},t} - V'^{k+1}_{\mathbf{z},t}\|_{1,\frac{1}{2}} \leq t^{k/2}|\mathbf{z}|^{-1}$$

for all $k \in \mathbb{N}$. Combining with the fact that $V'^k_{\mathbf{z},t} \to V_{\mathbf{z},t}$, we get $V^k_{\mathbf{z},t} \to V_{\mathbf{z},t}$. Integrating the last limit with respect to $\mathbf{z}$, one sees that $U_{\mathbf{z},t}$ is differentiable on $\mathbf{z}$ and $D_{\mathbf{z}}U_{\mathbf{z},t} = V_{\mathbf{z},t}$. In particular, $D_{\mathbf{z}}U_{\mathbf{z},t}$ belongs to $\mathcal{C}^{1,\frac{1}{2}}(\partial\mathbb{D})$. The proof is finished. □

Let t be a real number in $(0, t_1)$ as in Proposition 3.4.2. Define the map

$$\Phi^h : B_{2n}(0,1) \to \mathbb{C}^n$$

by putting $\Phi^h(0) = 0$ and $\Phi^h(\mathbf{z}) = F^h(1 - |\mathbf{z}| + i|\mathbf{z}|, \mathbf{z}, t)$ for $\mathbf{z} \neq 0$. Let Φ and g be the maps defined in the proof of Proposition 3.3.4. Recall that $\Phi(\mathbf{z}) = F(1 - |\mathbf{z}| + i|\mathbf{z}|, \mathbf{z}, t)$ and $\Phi(\mathbf{z}) = t\mathbf{z} + g(\mathbf{z})$ and

$$\|g\|_{1, B^*_{2n}(0, 2r_0)} \leq t/4, \quad g(0) = 0.$$

We want to prove that $\Phi^h\big(B_{2n}(0,1)\big)$ contains an open neighborhood of 0 just as what we did for Φ. To this end, we compare below these two maps and their derivatives.

Lemma 3.4.4. *There is a positive constant c_2 independent of $\mathbf{z}, p_0$ and of t such that for all $\mathbf{z} \in B^*_{2n}(0,1)$, we have*

$$\big|\Phi^h(\mathbf{z}) - \Phi(\mathbf{z})\big| \leq c_2 t^2 |\mathbf{z}|, \tag{3.4.25}$$

and

$$\big|D_{\mathbf{z}}\Phi^h(\mathbf{z}) - D_{\mathbf{z}}\Phi(\mathbf{z})\big| \leq c_2 t^2. \tag{3.4.26}$$

Proof. Let $\mathbf{z} \in B^*_{2n}(0,1)$. Let $z^* := 1 - |\mathbf{z}| + i|\mathbf{z}|$. We deduce from (3.4.4) and the definition of F^h that

$$E(z, \mathbf{z}, t) := F^h(z, \mathbf{z}, t) - F(z, \mathbf{z}, t) = -\mathcal{T}_1(P_{\mathbf{z},t})(z) + iP_{\mathbf{z},t}(z)$$

is a holomorphic map in z. Substituting z by z^* in the last equality gives

$$\Phi^h(\mathbf{z}) - \Phi(\mathbf{z}) = E(z^*, \mathbf{z}, t). \tag{3.4.27}$$

Recall that $P_{\mathbf{z},t}(\xi) = h\big(U_{\mathbf{z},t}(\xi)\big)$, for $\xi \in \mathbb{D}$. By (3.4.13), (3.4.5) and (3.3.4), we have

$$\|P_{\mathbf{z},t}(\cdot)\|_{1, \frac{1}{2}, \overline{\mathbb{D}}} \lesssim \|P_{\mathbf{z},t}(\cdot)\|_{1, \frac{1}{2}, \partial\mathbb{D}} \lesssim t^2.$$

This yields that

$$\|E(\cdot, \mathbf{z}, t)\|_{1, \frac{1}{2}, \overline{\mathbb{D}}} \lesssim \|\mathcal{T}_1(P_{\mathbf{z},t})\|_{1, \frac{1}{2}, \overline{\mathbb{D}}} + \|P_{\mathbf{z},t}\|_{1, \frac{1}{2}, \overline{\mathbb{D}}} \lesssim \|P_{\mathbf{z},t}\|_{1, \frac{1}{2}, \overline{\mathbb{D}}} \lesssim t^2. \tag{3.4.28}$$

We have

$$E(1, \mathbf{z}, t) = ih\big(U_{\mathbf{z},t}(1)\big) = ih(t \operatorname{Re}\mathbf{z} - t \operatorname{Im}\mathbf{z})$$

which is of modulus less than or equal to $t^2|\operatorname{Re}\mathbf{z} - \operatorname{Im}\mathbf{z}|^2 \leq 2t^2|\mathbf{z}|^2$ by (3.4.3). Using the last inequality and (3.4.28), one has

$$|E(z^*, \mathbf{z}, t)| \leq |E(z^*, \mathbf{z}, t) - E(1, \mathbf{z}, t)| + |E(1, \mathbf{z}, t)| \leq \|E(\cdot, \mathbf{z}, t)\|_{1, \overline{\mathbb{D}}} |1 - z^*| + 2t^2|\mathbf{z}|^2 \lesssim t^2|\mathbf{z}|.$$

Using this and (3.4.27), one gets (3.4.25).

Differentiating (3.4.27) gives

$$D_{\mathbf{z}}\Phi^h(\mathbf{z}) - D_{\mathbf{z}}\Phi(\mathbf{z}) = D_{\mathbf{z}}E(z^*, \mathbf{z}, t) + D_z E(z^*, \mathbf{z}, t) \begin{bmatrix} -D_{\mathbf{z}}|\mathbf{z}| \\ D_{\mathbf{z}}|\mathbf{z}| \end{bmatrix}. \tag{3.4.29}$$

By (3.4.28), we have

$$\big|D_z E(z^*, \mathbf{z}, t) D_{\mathbf{z}}|\mathbf{z}|\big| \lesssim t^2 \big|D_{\mathbf{z}}|\mathbf{z}|\big| \lesssim t^2.$$

Hence it remains to estimate the first term in the right-hand side of (3.4.29). Observe that

$$D_{\mathbf{z}} E(z, \mathbf{z}, t) = -\mathcal{T}_1\big(D_{\mathbf{z}} P_{\mathbf{z},t}\big)(z) + i D_{\mathbf{z}} P_{\mathbf{z},t}(z),$$

for all $z \in \overline{\mathbb{D}}$. Let $H(\xi) = Dh(U_{\mathbf{z},t}(\xi))$ the function defined in the proof of Proposition 3.4.2. By definition of $P_{\mathbf{z},t}$, we have

$$D_{\mathbf{z}} P_{\mathbf{z},t}(\xi) = H(\xi) D_{\mathbf{z}} U_{\mathbf{z},t}(\xi).$$

Using (3.4.7) together with (3.4.5) gives

$$\|D_{\mathbf{z}} P_{\mathbf{z},t}\|_{1,\frac{1}{2},\overline{\mathbb{D}}} \leq \|D_{\mathbf{z}} P_{\mathbf{z},t}\|_{1,\frac{1}{2},\partial\mathbb{D}} \leq 4\|H\|_{1,\frac{1}{2},\partial\mathbb{D}} \|D_{\mathbf{z}} U_{\mathbf{z},t}\|_{1,\frac{1}{2},\partial\mathbb{D}} \lesssim t^2 |\mathbf{z}|^{-1}.$$

We have $D_{\mathbf{z}} E(1, \mathbf{z}, t) = i D_{\mathbf{z}} h(t \operatorname{Re}\mathbf{z} - t \operatorname{Im}\mathbf{z})$ which is clearly of absolute value $\lesssim t^2$ by (3.4.3). Combining this with (3.3.3) yields that

$$\begin{aligned} |D_{\mathbf{z}} E(z^*, \mathbf{z}, t)| &= |D_{\mathbf{z}} E(z^*, \mathbf{z}, t) - D_{\mathbf{z}} E(1, \mathbf{z}, t)| + |D_{\mathbf{z}} E(1, \mathbf{z}, t)| \\ &\leq |1 - z^*| \|D_{\mathbf{z}} E(\cdot, \mathbf{z}, t)\|_{1,\overline{\mathbb{D}}} + t^2 \lesssim |1 - z^*| \|D_{\mathbf{z}} P_{\mathbf{z},t}\|_{1,\overline{\mathbb{D}}} + t^2 \lesssim t^2. \end{aligned}$$

The proof is finished. □

Lemma 3.4.5. *Let r_0 be the constant in Proposition 3.3.4. There is a positive number $t_2 < t_1$ independent of p_0 and of $\mathbf{z}$ such that for any $t \in (0, t_2]$, the set $\Phi^h\big(B_{2n}(0, r_0)\big)$ contains the ball $B_{2n}(0, r_0 t/2)$.*

Proof. Define $g^h(\mathbf{z}) := \Phi^h(\mathbf{z}) - t\mathbf{z}$. We have $g^h = \Phi^h - \Phi + g$. By Lemma 3.4.4, $\Phi^h - \Phi$ is $t/4$-Lipschitz for $t \leq t_2 := \min\{t_1, (4c_2)^{-1}\}$. Combining with the fact that g is $t/4$-Lipschitz on $\overline{B}_{2n}(0, 2r_0)$ implies that g^h is $t/2$-Lipschitz on $\overline{B}_{2n}(0, 2r_0)$ for $t \in (0, t_2)$. Now, an application of Lemma 3.3.3 to $\Phi^h = t\mathrm{Id} + g^h$ gives the desired result. The proof is finished. □

Proof of Proposition 3.2.5 for the case without singularity. In our chosen local coordinates around p_0, we have $p_0 = 0$ and $p = \mathbf{z}$. Let $t \in (0, t_2)$ be as in Lemma 3.4.5. For any $\mathbf{z} \in B^*_{2n}(0, r_0 t/2)$, there is $\mathbf{z}^* \in B_{2n}(0, r_0)$ for which $\Phi^h(\mathbf{z}^*) = \mathbf{z}$. We deduce from (3.4.25) that

$$|\mathbf{z} - \Phi(\mathbf{z}^*)| \leq c_2 t^2 |\mathbf{z}^*| \leq \frac{t|\mathbf{z}^*|}{4}.$$

At the end of the proof of Proposition 3.3.4, we proved that $|\Phi(\mathbf{z}^*)| \geq t|\mathbf{z}^*|/2$. This implies that $|\mathbf{z}^*| \leq 4|\mathbf{z}|/t$. Let $f(z) := F^h(z, \mathbf{z}^*, t_2)$ and $z^* = 1 - |\mathbf{z}| + i|\mathbf{z}|$. The last inequality implies that

$$|1 - z^*| \leq 2|\mathbf{z}^*| \leq 8|\mathbf{z}|/t.$$

The analytic disc f clearly satisfies all requirements in Proposition 3.2.5.

Now, we explain how to obtain the desired analytic discs when $\dim_{\mathbb{R}} K > n$. Since we only consider small discs near K, it is enough to work in a small chart and identify K with a submanifold of $B_{2n}(0, 1)$ with $0 \in K$. Choose a real linear space A through 0 such that A intersects $K \cap B_{2n}(0, 2r)$ transversally at a generic manifold of dimension n, where $r > 0$

is a positive number. We can choose r small enough such that this property also holds for any linear subspace A' parallel to A which intersects $K \cap B_{2n}(0, 2r)$. Let $p_0 \in K \cap B_{2n}(0, r)$ and $p \in B_{2n}(0, 1)$ close to p_0. Let K' be the intersection of $K \cap B_{2n}(0, 2r)$ with the linear space A' through p_0 and parallel to A. The construction in the last subsections can be applied to (K', X, p_0, p) without changes. We obtain analytic discs half-attached to K', hence half-attached to K, with the properties described in Proposition 3.2.5. The proof is finished. □

3.4.2 The case where K has singularity

We treat the case where K is a compact generic nondegenerate $\mathcal{C}^5$-piecewise submanifold of X. Actually, $\mathcal{C}^4$-differentiability is enough for our proof but in order to avoid some involvedly technical points, we will use $\mathcal{C}^5$-differentiability.

The case of higher dimension will be treated at the end of this subsection also by considering generic submanifolds of K. The following is an analogue of Lemma 3.4.1.

Lemma 3.4.6. *There exist constants $c_1, r_K > 1$ depending only on (K, X) and a local chart (W_{p_0}, Ψ) around p_0, where $\Psi : W_{p_0} \to B_{2n}(0, r_K)$ is biholomorphic with $\Psi(p_0) = 0$ such that the two following conditions hold:*

(i) we have

$$\|\Psi\|_1 \leq c_1, \quad \|\Psi^{-1}\|_1 \leq c_1,$$

(ii) there is a $\mathcal{C}^5$ map h defined on $\overline{B}_n(0, 1)$ with $h(0) = Dh(0) = 0$, so that

$$\Psi(K \cap W_{p_0}) \supset \big\{(\mathbf{x}, h(\mathbf{x})) : \mathbf{x} \in (\mathbb{R}^+)^n \cap \overline{B}_n(0, 1)\big\}$$

and

$$\|h\|_5 \leq c_1. \tag{3.4.30}$$

Proof. Firstly, observe that by definition of K, through every point p on the singularity of K, there exist a local chart W of p in X so that $K \cap W$ is the intersection of W with a finite union of convex polyhedra of dimension n in $\mathbb{R}^{2n}$. Here we identified W and $K \cap W$ with their images in $\mathbb{R}^{2n}$. Let K'' be one of these convex polyhedra containing p. Let K' be the intersection of W with the linear subspace of $\mathbb{R}^{2n}$ supporting K''. Since $K'' \cap W$ is a generic submanifold of W (because $K \cap W$ is so), K' is a generic smooth submanifold without boundary of W by shrinking W if necessary. Note that $p \in K'' \subset K \cap W \cap K'$ and $\dim K'' = \dim K' = n$.

The above observation shows that we can cover K by a finite number of holomorphic charts (W_j, Ψ_j) of X such that there are generic n-dimensional submanifolds K'_j without boundary of W_j and subsets K''_j of $K'_j \cap K \cap W_j$ diffeomorphic to the intersection of W_j with a convex polyheron of dimension n via suitable local charts of X. Without loss of generality, we can suppose that Ψ_j are biholomorphisms from W_j to $B_{2n}(0, 2)$ and the open sets $\Psi_j^{-1}\big(B_{2n}(0, 1)\big)$ also cover K.

Consider a chart (W_{j_0}, Ψ_{j_0}) such that $p_0 \in \Psi_{j_0}^{-1}\big(B_{2n}(0, 1)\big)$. As above, we can suppose that $p_0 \in K''_{j_0}$. Put $W_{p_0} := W_{j_0}$. Using the fact that K'_{j_0} is a generic n-dimensional smooth submanifold of W_{j_0} and arguing as in Lemma 3.4.1, we see that by replacing Ψ_{j_0} by the

composition of Ψ_{j_0} with a suitable affine linear map of $\mathbb{C}^n$, one obtain $\Psi_{j_0}(p_0) = 0$ and K'_{j_0} contains the graph of a $\mathcal{C}^5$ map $h(\mathbf{x})$ over $\overline{B}_n(0,1)$ and $h(0) = Dh(0) = 0$.

By the choice of K''_{j_0} and rescaling Ψ_{j_0} if necessary, there exist $\mathcal{C}^5$ functions $\tau_j(\mathbf{x})$ defined on an open neighborhood of $\overline{B}_n(0,1)$ with $1 \leq j \leq n$ such that

$$K \cap W_{p_0} \supset K''_{j_0} \supset \{(\mathbf{x}, h(\mathbf{x})) : \tau_j(\mathbf{x}) \geq 0 \text{ for all } 1 \leq j \leq n\}$$

and the Jacobian $D(\tau_1, \cdots, \tau_n)/D\mathbf{x}$ is of maximal rank in $B_n(0,1)$. Write $\mathbf{x} = (x_1, \cdots, x_n)$. Since every linear change of coordinates in $\mathbb{R}^n$ can be extended naturally to be a complex linear change of $\mathbb{C}^n$, using a suitable complex linear change of coordinates in $\mathbb{C}^n$ allows one to assume that tangent space of $\{\tau_j = 0\}$ at 0 is $\{x_j = 0\}$ for $1 \leq j \leq n$. Notice that the distortion caused by the change of coordinates is bounded independently of p_0. For $\mathbf{x} \in \overline{B}_n(0,1)$, write

$$\tau_j(\mathbf{x}) = \tau_j(0) + \sum_{l=1}^{n} \partial_{x_l}\tau_j(0)x_l + O(|\mathbf{x}|^2) \geq x_j - \|\tau_j\|_2 \sum_{l=1}^{n} |x_l|^2. \tag{3.4.31}$$

Put $C = \sup_{1\leq j\leq n} \|\tau_j\|_2$. Define

$$Q_n = \big\{\mathbf{x} \in \mathbb{R}^n : x_j \geq \frac{1}{3n}\sum_{l=1}^{n} x_l \quad \forall\, 1 \leq j \leq n\big\} \subset (\mathbb{R}^+)^n.$$

For $\mathbf{x} \in Q_n$ with $|\mathbf{x}| \leq \frac{1}{3nC}$, the inequality (3.4.31) yields that

$$\tau_j(\mathbf{x}) \geq x_j - C\frac{1}{3nC}\sum_{l=1}^{n} x_l \geq x_j - \frac{1}{3n}\sum_{l=1}^{n} x_l \geq 0,$$

for all $1 \leq j \leq n$. We deduce that

$$K \cap W_{p_0} \supset \big\{(\mathbf{x}, h(\mathbf{x})) : \mathbf{x} \in Q_n \cap \overline{B}_n(0, \frac{1}{3nC})\big\}.$$

The composition of a suitable linear change of coordinates in $\mathbb{R}^n$ with a dilation in $\mathbb{R}^n$ will map Q_n onto $(\mathbb{R}^+)^n$ and map $\overline{B}_n(0, \frac{1}{3nC})$ onto a neighborhood of $\overline{B}_n(0,1)$. This map can be extended to be a holomorphic change of coordinates Ψ' in $\mathbb{C}^n$. Composing Ψ_{j_0} with Ψ', we get the desired change of coordinates and the property (ii). The proof is finished. □

Let $K_h := \big\{(\mathbf{x}, h(\mathbf{x})) : \mathbf{x} \in \overline{B}_n(0,1)\big\}$ which is a $\mathcal{C}^5$ submanifold of $B_{2n}(0,1)$. Property (ii) of Lemma 3.4.6 implies that

$$|h(\mathbf{x})| \leq c_1|\mathbf{x}|^2, \quad |Dh(\mathbf{x})| \leq c_1|\mathbf{x}|, \quad \text{for } |\mathbf{x}| \leq 1. \tag{3.4.32}$$

To establish the desired family of analytic discs in this context, we follow the same strategy as in the previous case. Let $F'_{\boldsymbol{\tau}}, u'_{\mathbf{z},t,\boldsymbol{\tau}}, c_0$ be the maps and the constant defined in (3.3.24), (3.3.23) and (3.3.25) respectively. As in the last subsection, consider the following Bishop-type equation

$$U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi) = 2t(|\mathbf{z}|, \cdots, |\mathbf{z}|) - \mathcal{T}_1\big(h(U'_{\mathbf{z},t,\boldsymbol{\tau}})\big)(\xi) - \mathcal{T}_1 u'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi), \tag{3.4.33}$$

for $\mathbf{z} \in B^*_{2n}(0, \frac{1}{2n})$, $t \in (0,1]$ and $\boldsymbol{\tau} \in B_n(0,2)$. For simplicity, we use the same notation $U'_{\mathbf{z},t,\boldsymbol{\tau}}(z)$ to denote the harmonic extension of $U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi)$ to $\mathbb{D}$. Let $P'_{\mathbf{z},t,\boldsymbol{\tau}}(z)$ be the harmonic extension of $h\big(U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi)\big)$ to $\mathbb{D}$. If $U'_{\mathbf{z},t,\boldsymbol{\tau}}$ is a solution of (3.4.33) which is at least Hölder continuous, then

$$F'^h_{\boldsymbol{\tau}}(z,\mathbf{z},t) := U'_{\mathbf{z},t,\boldsymbol{\tau}}(z) + iP'_{\mathbf{z},t,\boldsymbol{\tau}}(z) + iu'_{\mathbf{z},t,\boldsymbol{\tau}}(z)$$

is clearly a family of analytic discs half-attached to K_h and

$$F'^h_{\boldsymbol{\tau}}(1,\mathbf{z},t) = 2t(|\mathbf{z}|,\cdots,|\mathbf{z}|) + ih(2t|\mathbf{z}|,\cdots,2t|\mathbf{z}|) \in (\mathbb{R}^+)^n.$$

Our goal is to obtain a stronger property that $F'^h_{\boldsymbol{\tau}}$ is I-attached to $K \subset K_h$, for some interval $I \subset \partial\mathbb{D}$ containing 1. In view of (ii) of Lemma 3.4.6, it suffices to prove that

$$U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi) \geq 0 \quad \text{ for } \xi \in I.$$

Here for any $r \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$, we write $\mathbf{v} \geq r$ to indicate that each component of $\mathbf{v}$ is greater than or equal to r. A similar convention is applied to $\mathbf{v} \leq r$.

Proposition 3.4.7. *There is a positive number $t_1 \in (0,1)$ independent of $(p_0,\mathbf{z},\boldsymbol{\tau})$ so that for any $t \in (0,t_1)$ and any $\mathbf{z} \in B^*_{2n}(0,\frac{1}{2n})$, the equation (3.4.33) has a unique solution $U'_{\mathbf{z},t,\boldsymbol{\tau}}$ such that $U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi)$ is $\mathcal{C}^{4,\frac{1}{2}}$ in ξ, for $1 \leq j \leq 4$ the differential $D^j_{(\mathbf{z},\boldsymbol{\tau})}U'_{\mathbf{z},t,\boldsymbol{\tau}}$ exists and is $\mathcal{C}^{4-j,\frac{1}{2}}$ in $\xi \in \partial\mathbb{D}$. Moreover, the following estimates hold:*

$$\|U'_{\mathbf{z},t,\boldsymbol{\tau}}(\cdot)\|_{4,\frac{1}{2}} \leq 4c_0 t, \quad \|D^{j'}_{\boldsymbol{\tau}}U'_{\mathbf{z},t,\boldsymbol{\tau}}(\cdot)\|_{4-j,\frac{1}{2}} \leq 4c_0 t, \quad \|D^j_{\boldsymbol{\tau}}D_{\mathbf{z}}U'_{\mathbf{z},t,\boldsymbol{\tau}}(\cdot)\|_{3-j,\frac{1}{2}} \leq 4c_0|\mathbf{z}|^{-1}t, \tag{3.4.34}$$

for $j = 0,1,2,3$ and $j' = 1,2,3,4$.

Proof. By (3.3.23)-(3.3.26), we see that the arguments in the proof of Proposition 3.4.2 still work for this case. Hence, the proof is finished. □

From now on, let t_1 be the constant in Proposition 3.4.7 and let $t \in (0,t_1)$. Let $U'_{\mathbf{z},t,\boldsymbol{\tau}}$ be the solution of (3.4.33) described in Proposition 3.4.7. For $\xi \in \partial\mathbb{D}$, write $\xi = e^{i\theta}$ with $\theta \in [-\pi,\pi)$.

Lemma 3.4.8. *There exists a constant c_2 independent of $(p_0,\mathbf{z},t,\boldsymbol{\tau})$ so that for any $(\mathbf{z},t,\boldsymbol{\tau})$, we have*

$$\|P'_{\mathbf{z},t,\boldsymbol{\tau}}(\cdot)\|_{4,\frac{1}{2},\overline{\mathbb{D}}} \leq c_2 t^2, \quad \|D^j_{\boldsymbol{\tau}}P'_{\mathbf{z},t,\boldsymbol{\tau}}(\cdot)\|_{4-j,\frac{1}{2},\overline{\mathbb{D}}} \leq c_2 t^2, \tag{3.4.35}$$

for $j = 1,2,3,4$ and

$$\|D^j_{\boldsymbol{\tau}}D_{\mathbf{z}}P'_{\mathbf{z},t,\boldsymbol{\tau}}(\cdot)\|_{3-j,\frac{1}{2},\overline{\mathbb{D}}} \leq c_2 t^2|\mathbf{z}|^{-1}, \tag{3.4.36}$$

for $j = 0,1,2,3$.

Proof. In view of (3.3.4), it is enough to estimate the norms of $P'_{\mathbf{z},t,\boldsymbol{\tau}}$ and $D^j_{(\mathbf{z},\boldsymbol{\tau})}P'_{\mathbf{z},t,\boldsymbol{\tau}}$ on $\partial\mathbb{D}$, for $j = 1,2$. Since $P'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi) = h\big(U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi)\big)$ on $\partial\mathbb{D}$, we have

$$\partial_\xi P'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi) = Dh\big(U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi)\big)\partial_\xi U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi).$$

This combined with (3.4.32) and (3.4.34) yields that

$$|\partial_\xi P'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi)| \le c_1 |U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi)|\, |\partial_\xi U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi)| \le 4c_0c_1t^2.$$

By similar arguments, we also have $|\partial^j_\xi P'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi)| \lesssim t^2$ with $j = 1, 2$. Hence, we obtain the first inequality in (3.4.35). For the proofs of the remaining inequalities, observe that $D^j_{(\mathbf{z},\boldsymbol{\tau})} P'_{\mathbf{z},t,\boldsymbol{\tau}}$ is the harmonic extension of $D^j_{(\mathbf{z},\boldsymbol{\tau})} h\big(U'_{\mathbf{z},t,\boldsymbol{\tau}}(\cdot)\big)$ to $\mathbb{D}$. Hence, analogous reasoning gives the desired result. The proof is finished. □

Lemma 3.4.9. *We always have*

$$|\partial_x U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)| \lesssim t^2|\mathbf{z}|, \quad |D_{\boldsymbol{\tau}}\partial_x U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)| \lesssim t^2|\mathbf{z}| \quad \textit{and} \quad |D_{\mathbf{z}}\partial_x U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)| \lesssim t^2.$$

Proof. By (3.3.11), one has

$$\partial_y P'_{\mathbf{z},t,\boldsymbol{\tau}}(1) = \partial_\theta h\big(U'_{\mathbf{z},t,\boldsymbol{\tau}}(e^{i\theta})\big)|_{\theta=0} = Dh\big(U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)\big)\partial_\theta U'_{\mathbf{z},t,\boldsymbol{\tau}}(1).$$

Since $U'_{\mathbf{z},t}(1) = 2t(|\mathbf{z}|, \cdots, |\mathbf{z}|)$, using (3.4.32) and (3.4.34), we get

$$|\partial_y P'_{\mathbf{z},t,\boldsymbol{\tau}}(1)| \lesssim t^2|\mathbf{z}|.$$

The Cauchy-Riemann equations for $F'^h_{\boldsymbol{\tau}}$ give

$$\partial_x U'_{\mathbf{z},t,\boldsymbol{\tau}}(z) = \partial_y P'_{\mathbf{z},t,\boldsymbol{\tau}}(z) + \partial_y u'_{\mathbf{z},t,\boldsymbol{\tau}}(z),$$

for all $z \in \overline{\mathbb{D}}$. Substituting z by 1 in the last equation, we obtain

$$\partial_x U'_{\mathbf{z},t,\boldsymbol{\tau}}(1) = \partial_y P'_{\mathbf{z},t,\boldsymbol{\tau}}(1) + \partial_y u'_{\mathbf{z},t,\boldsymbol{\tau}}(1) = \partial_y P'_{\mathbf{z},t,\boldsymbol{\tau}}(1) = O(t^2|\mathbf{z}|),$$

because $u'_{\mathbf{z},t,\boldsymbol{\tau}}$ vanishes on $\partial^+\mathbb{D}$. Hence, the first desired inequality follows. As to the second one, by differentiating the last inequality with respect to $\boldsymbol{\tau}$, we get

$$\begin{aligned} D_{\boldsymbol{\tau}}\partial_x U'_{\mathbf{z},t,\boldsymbol{\tau}}(1) &= D_{\boldsymbol{\tau}}\partial_y P'_{\mathbf{z},t,\boldsymbol{\tau}}(1) \\ &= Dh\big(U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)\big) D_{\boldsymbol{\tau}}\partial_\theta U'_{\mathbf{z},t,\boldsymbol{\tau}}(1) + D^2h\big(U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)\big)\big\{D_{\boldsymbol{\tau}}U'_{\mathbf{z},t,\boldsymbol{\tau}}(1), \partial_\theta U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)\big\}. \end{aligned} \tag{3.4.37}$$

On the other hand, differentiating (3.4.33) with respect to $\boldsymbol{\tau}$ gives

$$D_{\boldsymbol{\tau}}U'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi) = -\mathcal{T}_1\big(D_{\boldsymbol{\tau}}h(U'_{\mathbf{z},t,\boldsymbol{\tau}})\big)(\xi) - \mathcal{T}_1 D_{\boldsymbol{\tau}}u'_{\mathbf{z},t,\boldsymbol{\tau}}(\xi).$$

In particular, this implies that

$$D_{\boldsymbol{\tau}}U'_{\mathbf{z},t,\boldsymbol{\tau}}(1) = 0. \tag{3.4.38}$$

Now using the same reason as above, (3.4.37) and (3.4.38) implies the second desired inequality. The third one is proved in the same way with the remark that in this case, (3.4.38) is replaced by the equation $D_{\mathbf{z}}U'_{\mathbf{z},t,\boldsymbol{\tau}}(1) = O(t)$. The proof is finished. □

Lemma 3.4.10. *There exist a positive constant $t_2 < t_1$ independent of $(p_0, \mathbf{z}, \boldsymbol{\tau}, t)$ and a $\mathcal{C}^1$ function*

$$\boldsymbol{\tau}(\mathbf{z}, t) : B^*_{2n}(0, \frac{1}{2n}) \times (0, t_2) \to B_n(0, 1)$$

so that for any $(\mathbf{z}, t) \in B^*_{2n}(0, \frac{1}{2n}) \times (0, t_2)$, we have

$$\partial_\theta U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(e^{i\theta})|_{\theta=0} = \frac{2t \operatorname{Im} \mathbf{z}}{\sqrt{|\mathbf{z}|}(2 + \sqrt{|\mathbf{z}|})} \tag{3.4.39}$$

and

$$\left| \partial^2_\theta U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(e^{i\theta})|_{\theta=0} - \frac{2t(2|\mathbf{z}| - \operatorname{Re} \mathbf{z})}{|\mathbf{z}|} \right| \leq c_2 t^2. \tag{3.4.40}$$

Proof. The Cauchy-Riemann equations for $F'^h_{\boldsymbol{\tau}}$ give

$$\partial_y U'_{\mathbf{z},t,\boldsymbol{\tau}}(z) = -\partial_x P'_{\mathbf{z},t,\boldsymbol{\tau}}(z) - \partial_x u'_{\mathbf{z},t,\boldsymbol{\tau}}(z). \tag{3.4.41}$$

Combining this with (3.3.11) gives

$$\partial_\theta U'_{\mathbf{z},t,\boldsymbol{\tau}}(1) = \partial_y U'_{\mathbf{z},t}(1) = -\partial_x P'_{\mathbf{z},t,\boldsymbol{\tau}}(1) - \partial_x u'_{\mathbf{z},t,\boldsymbol{\tau}}(1). \tag{3.4.42}$$

Fix $\mathbf{z}$ and t. Define $\Phi_0(\boldsymbol{\tau}) := \partial_\theta U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)$. By definition of $u'_{\mathbf{z},t,\boldsymbol{\tau}}(1)$ and (3.3.20), we have

$$\Phi_0(0) = -\partial_x P'_{\mathbf{z},t,0}(1) + \frac{2t \operatorname{Im} \mathbf{z}}{\sqrt{|\mathbf{z}|}(2 + \sqrt{|\mathbf{z}|})}. \tag{3.4.43}$$

The first inequality of (3.4.35) implies that

$$\left| \Phi_0(0) - \frac{2t \operatorname{Im} \mathbf{z}}{\sqrt{|\mathbf{z}|}(2 + \sqrt{|\mathbf{z}|})} \right| \leq c_2 t^2 \leq t^{3/2}, \tag{3.4.44}$$

for t small enough. Differentiating (3.4.42) with respect to $\boldsymbol{\tau}$ gives

$$D^j_{\boldsymbol{\tau}} \Phi_0(\boldsymbol{\tau}) = -D^j_{\boldsymbol{\tau}} \partial_x P'_{\mathbf{z},t,\boldsymbol{\tau}}(1) - D^j_{\boldsymbol{\tau}} \partial_x u'_{\mathbf{z},t,\boldsymbol{\tau}}(1),$$

for $j = 1$ or 2. By (3.3.26) and (3.4.35), we see that

$$\| D^2_{\boldsymbol{\tau}} \Phi_0 \|_0 \leq c_2 t^2 \leq t^{3/2}, \tag{3.4.45}$$

and

$$\frac{1}{ct} \geq \left| [D_{\boldsymbol{\tau}} \Phi_0(0)]^{-1} \right|_{\ln} \geq c[10t + c_2 t^2]^{-1} \geq \frac{c}{11t}, \tag{3.4.46}$$

for t small enough and some constant $c > 0$ independent of $(\mathbf{z}, t, \boldsymbol{\tau})$, where we recall that the norm $|\cdot|_{\ln}$ of a square matrix is the one of its associated linear map. Taylor's expansion for Φ_0 at $\boldsymbol{\tau} = 0$ gives

$$\Phi_0(\boldsymbol{\tau}) = \Phi_0(0) + D_{\boldsymbol{\tau}} \Phi_0(0) \boldsymbol{\tau} + g_0(\boldsymbol{\tau}), \tag{3.4.47}$$

where $g_0(\boldsymbol{\tau})$ is $t^{3/2}$-Lipschitz by (3.4.45) and $g_0(0)=0$. A direct application of Lemma 3.3.3 to Φ_0 with $A=D_{\boldsymbol{\tau}}\Phi_0(0)$ and $M=t^{3/2}$ implies that for t small enough, Φ_0 is an injection on $B_n(0,1)$ and

$$\Phi\big(B_n(0,1)\big)\supset B_n\big(\Phi_0(0),ct\big)$$

Note that when t is small, we see that

$$\frac{2t\,\mathrm{Im}\,\mathbf{z}}{\sqrt{|\mathbf{z}|}(2+\sqrt{|\mathbf{z}|})}\in B_n\big(\Phi_0(0),ct\big)$$

thanks to (3.4.44). This yields that there exists a unique $\boldsymbol{\tau}(\mathbf{z},t)\in B_n(0,1)$ such that

$$\Phi_0\big(\boldsymbol{\tau}(\mathbf{z},t)\big)=-\frac{2t\,\mathrm{Im}\,\mathbf{z}}{\sqrt{|\mathbf{z}|}(2+\sqrt{|\mathbf{z}|})}\cdot$$

The differentiability of $\boldsymbol{\tau}(\mathbf{z},t)$ is implied directly from the implicit function theorem for $\Phi_0(\boldsymbol{\tau},\mathbf{z},t)$, where we recovered the variable $(\mathbf{z},t)$ to indicate the dependence of Φ_0 on them. By definition of Φ_0, (3.4.39) follows.

Recall that $u'_{\mathbf{z},t,\boldsymbol{\tau}}=u'_{\mathbf{z},t}+t\boldsymbol{\tau}\cdot\tilde{u}$ and $\partial_x\partial_y\tilde{u}(1)=0$. Now differentiating (3.4.41) with respect to y and using (3.3.11) and Lemma 3.4.9 yield

$$\begin{aligned}\partial_\theta^2U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)&=\partial_y^2U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)+O(t^2)=-\partial_y\partial_xP'_{\mathbf{z},t,\boldsymbol{\tau}}(1)-\partial_y\partial_xu'_{\mathbf{z},t,\boldsymbol{\tau}}(1)+O(t^2)\\&=-\partial_y\partial_xP'_{\mathbf{z},t,\boldsymbol{\tau}}(1)+\frac{2t(2|\mathbf{z}|-\mathrm{Re}\,\mathbf{z})}{|\mathbf{z}|}+O(t^2)\quad\text{(by (3.3.20))}.\end{aligned}$$

This combined with (3.4.35) implies (3.4.40). The proof is finished. □

Corollary 3.4.11. *There exists a positive constant $t_3<t_2$ so that for any $t\in(0,t_3)$ we can find a positive number $\theta_t\in(0,\pi/2)$ such that such that for any $\mathbf{z}\in B^*_{2n}(0,\frac{1}{2n})$ the analytic disc $F'^h_{\boldsymbol{\tau}(\mathbf{z},t)}(\cdot,\mathbf{z},t)$ is $[e^{-i\theta_t},e^{i\theta_t}]$-attached to K.*

Proof. Write $U'_{\mathbf{z},t,\boldsymbol{\tau}}=(U'_{\mathbf{z},t,\boldsymbol{\tau};1},\cdots,U'_{\mathbf{z},t,\boldsymbol{\tau};n})$. We only need to prove that $U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t);j}(e^{i\theta})\geq 0$ for $|\theta|$ small enough and $1\leq j\leq n$. Fix $1\leq j\leq n$. Put

$$(\tilde{z},\delta,\gamma):=(z_j,\sqrt{|\mathbf{z}|},2|\mathbf{z}|)$$

which satisfies the condition (3.3.17). We will mimic the proof of Lemma 3.3.9. By Lemma 3.4.10 and Taylor's expansion of $U'_{\mathbf{z},t,\boldsymbol{\tau};j}(e^{i\theta})$ at $\theta=0$, we have

$$t^{-1}U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t);j}\geq\gamma+\frac{2\,\mathrm{Im}\,\tilde{z}}{\delta(2+\delta)}\theta+\frac{2(\gamma-\mathrm{Re}\,\tilde{z})}{\delta^2}\theta^2-c_2t\theta^2+\theta^3g'_{\mathbf{z}}(\theta),$$

where $g'_{\mathbf{z}}(\theta)$ is a function on $[-\pi/2,\pi/2]$ whose supnorm is bounded by $t^{-1}\|U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}\|_3\leq 4c_0$ by (3.4.34). Put

$$\tilde{f}_1(\theta):=\gamma+\frac{2\,\mathrm{Im}\,\tilde{z}}{\delta(2+\delta)}\theta+\frac{\gamma-\mathrm{Re}\,\tilde{z}}{\delta^2}\theta^2$$

and

$$\tilde{f}_2(\theta):=\frac{\gamma-\mathrm{Re}\,\tilde{z}}{\delta^2}\theta^2-c_2t\theta^2+\theta^3g'_{\mathbf{z}}(\theta).$$

We have

$$t^{-1}U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t);j} = \tilde{f}_1(\theta) + \tilde{f}_2(\theta).$$

Arguing as in Lemma 3.3.9 shows that $\tilde{f}_1(\theta) \geq 0$ for all $\theta \in [-\pi/2, \pi/2]$ and $\tilde{f}_2(\theta) \geq 0$ provided that t is small enough and $|\theta| \leq \theta_t$ for some $\theta_t > 0$ independent of $\mathbf{z}$. Hence, $U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t);j}(e^{i\theta}) \geq 0$ for all $1 \leq j \leq n$ and t small enough and $|\theta| \leq \theta_t$. The proof is finished. □

We will need the following estimates on the function $\boldsymbol{\tau}(\mathbf{z}, t)$.

Lemma 3.4.12. *Let $t, \mathbf{z}$ and $\boldsymbol{\tau}(\mathbf{z}, t)$ be as in Lemma 3.4.10. Then, there exist positive constants c_3 and $t_4 < t_3$ which are both independent of $(p_0, \mathbf{z}, t)$ so that for any $(\mathbf{z}, t) \in B^*_{2n}(0, \frac{1}{2n}) \times (0, t_4)$, we have*

$$|\boldsymbol{\tau}(\mathbf{z}, t)| \leq c_3 t \quad \textit{and} \quad |D_{\mathbf{z}}\boldsymbol{\tau}(\mathbf{z}, t)| \leq c_3 t|\mathbf{z}|^{-1}. \tag{3.4.48}$$

Proof. We reuse the notation in the proof of in Lemma 3.4.10. Recall that $\Phi_0(\boldsymbol{\tau}, \mathbf{z}, t) = \partial_\theta U'_{\mathbf{z},t,\boldsymbol{\tau}}(1)$. Thus, (3.4.34) implies that

$$|D_{\mathbf{z}} D_{\boldsymbol{\tau}} \Phi_0(\boldsymbol{\tau}, \mathbf{z}, t)| \leq 4c_0 t|\mathbf{z}|^{-1}. \tag{3.4.49}$$

Since

$$\Phi_0\big(\boldsymbol{\tau}(\mathbf{z}, t), \mathbf{z}, t\big) = \frac{2t \operatorname{Im} \mathbf{z}}{\sqrt{|\mathbf{z}|}(2 + \sqrt{|\mathbf{z}|})},$$

using (3.4.47) and (3.4.43), we have

$$-\partial_x P'_{\mathbf{z},t,0}(1) = D_{\boldsymbol{\tau}}\Phi_0(0, \mathbf{z}, t)\boldsymbol{\tau}(\mathbf{z}, t) + g_0\big(\boldsymbol{\tau}(\mathbf{z}, t)\big). \tag{3.4.50}$$

Since g_0 is $c_2 t^2$-Lipschitz (see (3.4.45)) and $g_0(0) = 0$, we deduce from (3.4.46) that

$$|\partial_x P'_{\mathbf{z},t,0}(1)| \geq ct|\boldsymbol{\tau}(\mathbf{z}, t)| - c_2 t^2 |\boldsymbol{\tau}(\mathbf{z}, t)| \geq t|\boldsymbol{\tau}(\mathbf{z}, t)|(c - c_2 t).$$

This combined with (3.4.35) implies that

$$|\boldsymbol{\tau}(\mathbf{z}, t)| \leq \frac{t}{c - c_2 t} \lesssim t, \tag{3.4.51}$$

for t small enough. Hence, the first inequality of (3.4.48) follows.

We now prove the second one. Differentiating the equality (3.4.50) with respect to $\mathbf{z}$ and using (3.4.36) give

$$\big[D_{\boldsymbol{\tau}}\Phi_0\big(0, \mathbf{z}, t\big) + D_{\boldsymbol{\tau}} g_0\big(\boldsymbol{\tau}(\mathbf{z}, t)\big)\big] D_{\mathbf{z}}\boldsymbol{\tau}(\mathbf{z}, t) + D_{\mathbf{z}} D_{\boldsymbol{\tau}}\Phi_0\big(\boldsymbol{\tau}(\mathbf{z}, t), \mathbf{z}, t\big)\boldsymbol{\tau}(\mathbf{z}, t) = O(t^2|\mathbf{z}|^{-1}).$$

This together with (3.4.51) and (3.4.49) yields that

$$\big[D_{\boldsymbol{\tau}}\Phi_0\big(0, \mathbf{z}, t\big) + D_{\boldsymbol{\tau}} g_0\big(\boldsymbol{\tau}(\mathbf{z}, t)\big)\big] D_{\mathbf{z}}\boldsymbol{\tau}(\mathbf{z}, t) = O(t^2|\mathbf{z}|^{-1}).$$

Multiplying the two sides of the last equality by $D_{\boldsymbol{\tau}}\Phi_0\big(0, \mathbf{z}, t\big)^{-1}$ and using (3.4.46) and

$$|D_{\boldsymbol{\tau}} g_0\big(\boldsymbol{\tau}(\mathbf{z}, t)\big)|_{\ln} = O(t^2) \quad \text{(by (3.4.45))},$$

we get

$$|D_{\mathbf{z}}\boldsymbol{\tau}|_{\ln} \lesssim t^{-1}t^2|\mathbf{z}|^{-1} \lesssim t|\mathbf{z}|^{-1}.$$

The proof is finished. □

Let $t \in (0, t_4)$. Define the map

$$\Phi'^h : B_{2n}(0, \frac{1}{2n}) \to \mathbb{C}^n$$

by putting $\Phi'^h(0) = 0$ and $\Phi'^h(\mathbf{z}) = F'^h_{\boldsymbol{\tau}(\mathbf{z},t)}(z^*, \mathbf{z}, t)$ for $\mathbf{z} \neq 0$, where $z^* := 1 - \sqrt{|\mathbf{z}|}$. Our goal is to obtain similar estimates for Φ'^h as in Lemma 3.4.4. However, due to the presence of $\boldsymbol{\tau}$, direct comparisons between Φ'^h and Φ' do not work efficiently as in the case without singularity. In order to get the expected results, we will use the technique in Corollary 3.3.6.

Lemma 3.4.13. *There is a positive constant c_4 independent of $\mathbf{z}, p_0$ and t such that for all $\mathbf{z} \in B^*_{2n}(0, \frac{1}{2n})$, we have*

$$|\Phi'^h(\mathbf{z}) - t\mathbf{z}| \leq c_4|\mathbf{z}|(t^2 + t\sqrt{|\mathbf{z}|}), \tag{3.4.52}$$

and

$$\left|D_{\mathbf{z}}\Phi'^h(\mathbf{z}) - t\mathrm{Id}\right| \leq c_4(t^2 + t\sqrt{|\mathbf{z}|}). \tag{3.4.53}$$

Proof. We want to study the behavior of $F'_{\boldsymbol{\tau}(\mathbf{z},t)}(z)$ near $z = 1$. By using Taylor's expansions, it is sufficient to estimate its partial derivatives at 1. Put

$$\tilde{F}'(z, \mathbf{z}, t) := F'_{\boldsymbol{\tau}(\mathbf{z},t)}(z, \mathbf{z}, t).$$

Differentiating the last equality and using the second inequality of (3.4.48) and (3.4.34), one has

$$\|D_{\mathbf{z}}\tilde{F}'(\cdot, \mathbf{z}, t)\|_3 \lesssim \|D_{\mathbf{z}}F'_{\boldsymbol{\tau}(\mathbf{z},t)}(\cdot, \mathbf{z}, t)\|_3 + \|D_{\boldsymbol{\tau}}F'_{\boldsymbol{\tau}(\mathbf{z},t)}(z, \mathbf{z}, t)\|_3 \, |D_{\mathbf{z}}\boldsymbol{\tau}(\mathbf{z}, t)| \lesssim t|\mathbf{z}|^{-1}. \tag{3.4.54}$$

Lemma 3.4.10 implies that

$$\partial_x \operatorname{Im} \tilde{F}'(1, \mathbf{z}, t) = -\partial_y U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(1) = -\frac{2t \operatorname{Im} \mathbf{z}}{\sqrt{|\mathbf{z}|}(2 + \sqrt{|\mathbf{z}|})}.$$

On the other hand, we have

$$\operatorname{Im} \tilde{F}'(z, \mathbf{z}, t) = P'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(z) + u'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(z).$$

Hence,

$$\begin{aligned}
\partial_x^2 \operatorname{Im} \tilde{F}'(1, \mathbf{z}, t) &= \partial_x^2 P'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(1) + \partial_x^2 u'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(1) \\
&= \partial_x^2 P'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(1) + \partial_x^2 u'_{\mathbf{z},t}(1) + t\boldsymbol{\tau}(\mathbf{z}, t) \cdot \partial_x^2 \tilde{u}(1) \\
&= \partial_x^2 P'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(1) + \frac{2t \operatorname{Im} \mathbf{z}}{\sqrt{|\mathbf{z}|}(2 + \sqrt{|\mathbf{z}|})} + t\boldsymbol{\tau}(\mathbf{z}, t) \cdot \partial_x^2 \tilde{u}(1),
\end{aligned}$$

by (3.3.20). By Taylor's expansion for $\operatorname{Im} \tilde{F}'(\cdot, \mathbf{z}, t)$ at $z = 1$ up to the order 3 and using (3.4.35) and the first inequality of (3.4.48), there is a function $g''_{\mathbf{z},t;1}$ defined on $[0, 1]$ so that $g''_{\mathbf{z},t;1}(s)$ is $\mathcal{C}^{1,\frac{1}{2}}$ in $(s, \mathbf{z})$ and for any $s \in [0, 1]$, we have

$$\begin{aligned}
\operatorname{Im} \tilde{F}'(1 - s, \mathbf{z}, t) = s\frac{2t \operatorname{Im} \mathbf{z}}{\sqrt{|\mathbf{z}|}(2 + \sqrt{|\mathbf{z}|})} + s^2\frac{2t \operatorname{Im} \mathbf{z}}{2\sqrt{|\mathbf{z}|}(2 + \sqrt{|\mathbf{z}|})} \\
+ ts^2\boldsymbol{\tau}(\mathbf{z}, t) \cdot \partial_x^2 \tilde{u}(1) + s^3 g''_{\mathbf{z},t;1}(s),
\end{aligned} \tag{3.4.55}$$

and

$$\|g''_{\mathbf{z},t;1}\|_1 \lesssim \|F'_{\boldsymbol{\tau}(\mathbf{z},t)}(\cdot,\mathbf{z},t)\|_4 \lesssim t.$$

Additionally, (3.4.54) also imply that

$$\|D_{\mathbf{z}} g''_{\mathbf{z},t;1}\|_0 \lesssim \|D_{\mathbf{z}}\tilde{F}'(\cdot,\mathbf{z},t)\|_3 \lesssim t|\mathbf{z}|^{-1}.$$

Define

$$g'_{\mathbf{z},t;1}(s) := t^{-1}s g''_{\mathbf{z},t;1}(s) + \boldsymbol{\tau}(\mathbf{z},t)\cdot\partial_x^2\tilde{u}(1).$$

Thus,

$$|g'_{\mathbf{z},t;1}|_1 \lesssim s+t, \quad \|D_{\mathbf{z}} g'_{\mathbf{z},t;1}\|_0 \lesssim t|\mathbf{z}|^{-1} + s|\mathbf{z}|^{-1}.$$

Letting $s=\sqrt{|\mathbf{z}|}$ in (3.4.55) and using (3.3.13), (3.3.14), we obtain

$$\operatorname{Im}\tilde{F}'(1-\sqrt{|\mathbf{z}|},\mathbf{z},t) = t\operatorname{Im}\mathbf{z} + t\tilde{g}'_{t;1}(\mathbf{z}), \tag{3.4.56}$$

where $\tilde{g}'_{t;1}(\mathbf{z}) := |\mathbf{z}| g'_{\mathbf{z},t;1}(\sqrt{|\mathbf{z}|})$. Direct computations give

$$|\tilde{g}'_{t;1}(\mathbf{z})| \lesssim (t+\sqrt{|\mathbf{z}|})|\mathbf{z}| \quad \text{and} \quad \|D_{\mathbf{z}}\tilde{g}'_{t;1}\|_0 \lesssim t+\sqrt{|\mathbf{z}|}. \tag{3.4.57}$$

Analogous arguments and Lemma 3.4.9 also show that

$$\partial_x \operatorname{Re}\tilde{F}'(1,\mathbf{z},t) = \partial_x U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(1) = O(t^2|\mathbf{z}|),$$

$$D_{\mathbf{z}}\partial_x \operatorname{Re}\tilde{F}'(1,\mathbf{z},t) = D_{\mathbf{z}}\partial_x U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(1) + D_{\boldsymbol{\tau}}\partial_x U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(1) D_{\mathbf{z}}\boldsymbol{\tau}(\mathbf{z},t) = O(t^2),$$

and

$$\partial_x^2 \operatorname{Re}\tilde{F}'(1,\mathbf{z},t) = -\partial_y^2 \operatorname{Re}\tilde{F}'(1,\mathbf{z},t) = -\partial_\theta^2 U'_{\mathbf{z},t,\boldsymbol{\tau}(\mathbf{z},t)}(1) + O(t^2) = -\frac{2t(2|\mathbf{z}| - \operatorname{Re}\mathbf{z})}{|\mathbf{z}|} + O(t^2)$$

by (3.4.40). Hence, as above there exists a $\mathcal{C}^1$ function $\tilde{g}'_{t;2}(\mathbf{z})$ on $B^*_{2n}(0,\frac{1}{2n})$ such that

$$\operatorname{Re}\tilde{F}'(1-\sqrt{|\mathbf{z}|},\mathbf{z},t) = t\operatorname{Re}\mathbf{z} + t\tilde{g}'_{t;2}(\mathbf{z}) \tag{3.4.58}$$

and

$$|\tilde{g}'_{t;2}(\mathbf{z})| \lesssim (t+\sqrt{|\mathbf{z}|})|\mathbf{z}| \quad \text{and} \quad \|D_{\mathbf{z}}\tilde{g}'_{t;2}\|_0 \lesssim t+\sqrt{|\mathbf{z}|}. \tag{3.4.59}$$

By (3.4.56) and (3.4.58), we get

$$\Phi'^h(\mathbf{z}) = \tilde{F}'(1-\sqrt{|\mathbf{z}|},\mathbf{z},t) = t\mathbf{z} + t(\tilde{g}'_{t;2} + i\tilde{g}'_{t;1}).$$

Using (3.4.59) and (3.4.57), we get the desired results. The proof is finished. □

The proof for the following lemma is similar to Lemma 3.4.5.

Lemma 3.4.14. *There are positive constant $t_5 < t_4$ and $r'_0 < 1/(2n)$ independent of $(p_0,\mathbf{z})$ such that for any $t\in(0,t_5]$ and any $\mathbf{z}\in B^*_{2n}(0,r'_0)$, the set $\Phi'^h\big(B_{2n}(0,r'_0)\big)$ contains the ball $B_{2n}(0,r'_0t/2)$.*

Proof of Proposition 3.2.5 for the case with singularity. In our chosen local coordinates around p_0, we have $p_0 = 0$ and $p = \mathbf{z}$. Let $t \in (0, t_5]$ and $\mathbf{z} \in B^*_{2n}(0, r'_0)$ as in Lemma 3.4.14. Without loss of generality, we can suppose that

$$t_5 + \sqrt{r'_0} \leq 1/(2c_4).$$

For any $\mathbf{z} \in B^*_{2n}(0, r'_0 t/2)$, there exists $\mathbf{z}^* \in B_{2n}(0, r'_0)$ for which $\Phi'^h(\mathbf{z}^*) = \mathbf{z}$. We deduce from (3.4.52) that

$$|\mathbf{z} - t\mathbf{z}^*| \leq c_4|\mathbf{z}^*|(t^2 + t\sqrt{|\mathbf{z}|^*}) \leq \frac{|t\mathbf{z}^*|}{2}.$$

Let $f(z) := F'^h(z, \mathbf{z}^*, t_5)$ and $z^* := 1 - \sqrt{|\mathbf{z}^*|}$. The last inequality implies that

$$|1 - z^*|^2 \leq 2|\mathbf{z}|/t.$$

As in the case without singularity, the analytic disc f satisfies all properties in Proposition 3.2.5.

Now, we explain how to obtain the desired analytic discs when $\dim_{\mathbb{R}} K > n$. In the last subsection, we sliced K by generic n-dimensional submanifolds K' in a uniform way. Then, one just applied the previous result for K' to get discs partly attached to K. In our present case, such slicing does not always work due to the fact that a hypersurface passing an edge of K may only intersect K at that point. Hence, we do not get a such a family K' as above. We will use the same idea with some additional caution. As just mentioned, we only need to take care of the edges of K. Let p_e be an edge of K. By definition of K, there exists a local chart $(\tilde{W}_{p_e}, \tilde{\Psi})$ of p_e in X such that $\tilde{\Psi}$ is a diffeomorphism from $\tilde{W}_{p_e}$ to $B_{2n}(0, 2)$ and $\tilde{\Psi}(K \cap \tilde{W}_{p_e})$ is the intersection of a finite union of convex polyhedra with $B_{2n}(0, 2)$. For simplicity, we identify K with $\tilde{\Psi}(K)$ and suppose that K is just a convex polyhedron. Hence, it is easy to choose a $(3n - \dim K)$-dimensional subspace H_{p_e} of $\mathbb{R}^{2n}$ such that the affine subspace $p_e + H_{p_e}$ intersects K at a n-dimensional convex polyhedron K'_{p_e} which is generic at p_e in the sense of the Cauchy-Riemann geometry: $K'_{p_e} + JK'_{p_e} = \mathbb{R}^{2n}$ where J is the complex structure of X, we identified $T_{p_e}X$ with R^{2n}. Since p_e is an edge, the last property implies that the same thing also holds for any $p_0 \in \mathbb{R}^{2n}$ close enough to $p_e, i.e.$ $(p_0 + H_{p_e}) \cap K = K_{p_0}$ and K_{p_0} generic at p_0. To summarize, we just get a family of generic n-dimensional local submanifolds K'_{p_0} of K uniformly in p_0. Now apply the above result for each K_{p_0}, we get the desired conclusion. The proof is finished. □

Chapter 4

Loi de Weyl pour les résonances d'un opérateur de Schrödinger générique

Let $-\Delta + V$ be the Schrödinger operator acting on $L^2(\mathbb{R}^d, \mathbb{C})$ with $d \geq 3$ odd. Here V is a bounded real or complex function vanishing outside the closed ball of center 0 and of radius a. Let $n_V(r)$ denote the number of resonances of $-\Delta + V$ with modulus $\leq r$. We show that if the potential V is generic in a sense of pluripotential theory then

$$n_V(r) = c_d a^d r^d + O(r^{d-\frac{3}{16}+\epsilon}) \quad \text{as } r \to \infty$$

for any $\epsilon > 0$, where c_d is a dimensional constant. This chapter is based on the article [35].

4.1 Introduction

Let Δ denote the Laplacian on $\mathbb{R}^d$ with d odd. Let V be a bounded complex-valued function with support in the closed ball $\overline{\mathbb{B}}_a$ of center 0 and of radius a in $\mathbb{R}^d$. The purpose of this work is to study the asymptotic number of resonances associated to the Schrödinger operator $-\Delta+V$ acting on $L^2(\mathbb{R}^d, \mathbb{C})$. The problem has a long history and was intensively investigated during the last three decades. We refer to [12, 14, 72, 77, 80, 89, 90, 91] and to the references therein for an introduction to the subject.

Recall that for $\lambda \in \mathbb{C}$ large enough with $\mathrm{Im}(\lambda) > 0$, the operator $R_V(\lambda) := (-\Delta+V-\lambda^2)^{-1}$ on $L^2(\mathbb{R}^d, \mathbb{C})$ is well-defined and is bounded. It depends holomorphically on the parameter λ. If χ is a smooth function with compact support such that $\chi V = V$, one can extend $\chi R_V(\lambda)\chi$ to a family of operators which depends meromorphically on $\lambda \in \mathbb{C}$. The poles of this family, which are called *the resonances* of the operator $-\Delta+V$, and their multiplicities do not depend on the choice of χ. Denote by $n_V(r)$ the number of resonances of modulus $\leq r$ counted with multiplicity.

In dimension $d = 1$, Zworski obtained in [86] that

$$n_V(r) = \frac{4}{\pi} a r + o(r) \quad \text{as} \quad r \to \infty, \tag{4.1.1}$$

where $2a$ is the diameter of the support of V, see also [39, 66, 71, 90]. From now on, we assume that $d \geq 3$.

The upper bound for the number of resonances is now well-understood. Define

$$N_V(r) := \int_0^r \frac{n_V(t) - n_V(0)}{t} dt.$$

Generalizing some results by Melrose [55, 56, 57], Zworski obtained in [87] the following estimate

$$dN_V(r) \leq c_d a^d r^d + o(r^d) \quad \text{as} \quad r \to \infty,$$

where the sharp constant c_d was identified by Stefanov in [77], see Section 4.2 for the definition of c_d and [73, 79] for more general results. Lower bound for the number of resonances is only known in special cases that we will discuss below.

For $0 < \delta \leq 1$, define

$$\mathfrak{M}_a^\delta := \Big\{ V \in L^\infty(\mathbb{B}_a, \mathbb{C}) : \ n_V(r) - c_d a^d r^d = O(r^{d-\delta+\epsilon}) \text{ as } r \to \infty \text{ for every } \epsilon > 0 \Big\}.$$

This is a subset of the following family introduced by Christiansen in [15]

$$\mathfrak{M}_a := \Big\{ V \in L^\infty(\mathbb{B}_a, \mathbb{C}) : \ n_V(r) - c_d a^d r^d = o(r^d) \text{ as } r \to \infty \Big\}.$$

Our first main result is the following theorem.

Theorem 4.1.1. *Let V be a radial real-valued function of class $\mathscr{C}^2$ on $\overline{\mathbb{B}}_a$. Write $V(x) = V(\|x\|)$ and assume that $V(a) \neq 0$. Then V belongs to $\mathfrak{M}_a^{3/4}$.*

This result generalizes a theorem of Zworski in [88] which says that V belongs to $\mathfrak{M}_a$, see also Stefanov [77, Th.3]. The proof will be given in Section 4.4. It follows Zworski's approach and is based on some refinements of his arguments.

Consider now a connected open set Ω in $\mathbb{C}^p$ and a uniformly bounded family V_ϑ of potentials in $L^\infty(\mathbb{B}_a, \mathbb{C})$ depending holomorphically on the parameter $\vartheta \in \Omega$. Our second main result is the following theorem.

Theorem 4.1.2. *Let V_ϑ be a holomorphic family of potentials as above. Suppose there are $\vartheta_0 \in \Omega$ and $0 < \delta \leq 1$ such that V_{ϑ_0} belongs to $\mathfrak{M}_a^\delta$. Then there is a pluripolar set $E \subset \Omega$ such that $V_\vartheta \in \mathfrak{M}_a^{\delta/4}$ for all $\vartheta \in \Omega \setminus E$.*

Note that pluripolar sets in Ω are of Hausdorff dimension at most equal to $2p - 2$ and their intersections with $\mathbb{R}^p$ have zero p-dimensional volume, see e.g. [26, 50, 65] and also Section 4.5 for the definition. Therefore, in the last theorem, most of potentials V_ϑ belong to $\mathfrak{M}_a^{\delta/4}$. If V is a potential as in Theorem 4.1.1 and if V' is an arbitrary potential in $L^\infty(\mathbb{B}_a, \mathbb{C})$, then for almost every $\vartheta \in \mathbb{C}$ and almost every $\vartheta \in \mathbb{R}$ the potential $\vartheta V + (1 - \vartheta)V'$ belongs to $\mathfrak{M}_a^{3/16}$. For such a potential, the number of resonances is asymptotically $c_d a^d r^d$. Therefore, this property holds for "most of the" potentials in $L^\infty(\mathbb{B}_a, \mathbb{C})$ or in $L^\infty(\mathbb{B}_a, \mathbb{R})$ if we only consider real potentials. The notion of pluripolar sets can be extended to $L^\infty(\mathbb{B}_a, \mathbb{C})$ and the property holds for V out of such a set. We don't know if $\mathfrak{M}_a^{3/16}$ contains a dense G_δ set in $L^\infty(\mathbb{B}_a, \mathbb{C})$. This question was communicated to us by Peter D. Hislop, see [16] for a partial answer.

A version of Theorem 4.1.2 has been obtained by Christiansen in [15], where assuming $V_{\vartheta_0} \in \mathfrak{M}_a$, she proved that the counting function $N_{V_\vartheta}(r)$ satisfies

$$\limsup_{r\to\infty} \frac{dN_{V_\vartheta}(r)}{r^d} = c_d a^d$$

for ϑ outside a pluripolar set, see also [12, 16]. In particular, this property holds for generic potentials V in $L^\infty(\mathbb{B}_a, \mathbb{C})$ or in $L^\infty(\mathbb{B}_a, \mathbb{R})$.

The proof of Theorem 4.1.2 will be given in Section 4.5. It partially follows Christiansen's approach. We also prove and use there some property of plurisubharmonic functions (see Lemma 4.5.2 below) and an upper bound for $N_V(r)$ which generalizes the above estimate (4.1.1) by Zworski and Stefanov, see Theorem 4.3.1 in Section 4.3 below.

Note that Christiansen constructed in [13] examples of complex Schrödinger operators without resonances. This shows that the exceptional set E in Theorem 4.1.2 is not always empty. In comparison with similar results from complex dynamics, it is reasonable to believe that E is always a finite or countable union of analytic subsets of Ω, see e.g. [31].

Sá Barreto and Zworski showed in [68] that any Schrödinger operator with compactly supported real potential admits an infinite number of resonances, see also [11, 67]. The sharp asymptotic behavior for the number of resonances in this case is still unknown.

Notation and convention. Denote by $\mathbb{B}_a$ the open ball of center 0 and of radius a in $\mathbb{R}^d$ and $\mathbb{D}(z, r)$ the disc of center z and of radius r in $\mathbb{C}$. Let $\mathbb{S}^{d-1}$ denote the unit sphere in $\mathbb{R}^d$. Define $\mathbb{B} := \mathbb{B}_1$, $\mathbb{D}(r) := \mathbb{D}(0, r)$, $\mathbb{D} := \mathbb{D}(1)$, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, $\mathbb{R}_\pm := \{t \in \mathbb{R} : \ \pm t \geq 0\}$ and $\mathbb{C}_\pm := \{z \in \mathbb{C} : \ \pm \operatorname{Im} z > 0\}$. The functions ρ, ζ, Ai, J_ν, the constant c_d, the sets $\mathbf{\Omega}, \mathbf{K}_+$ and

the space H_l are introduced in Section 4.2; the sets $\mathbf{\Omega}_c^\nu, \mathbf{\Omega}_c^\nu(k), \mathbf{J}_c^\nu$ in Section 4.4. Define $\arg z := \theta$ and $\log z := \log r + i\theta$ for $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi]$. All the constants we will use depend only on $a, d, \|V\|_\infty$ and can be changed from line to line. The notation $\lesssim$ and $\gtrsim$ means inequalities up to a multiplicative constant. An expression likes $f(z) \sim g(z)$ as $z \to \infty$ means $f(z)/g(z) \to 1$ when $|z| \to \infty$. An expression likes

$$f(z) \sim g(z) \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

means

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{N-1} \frac{a_n}{z^n} + O(z^{-N}) \quad \text{as} \quad |z| \to \infty$$

for each $N \geq 0$, see [62, p.16].

4.2 Some properties of Bessel functions

In this section, we give some properties of Bessel functions and of other auxiliary functions that will be used later in the proofs of the main theorems. We refer to Olver [61, 62] for details.

Let ρ be the continuous function on $\overline{\mathbb{C}}_+ \setminus \{0\}$ defined by

$$\rho(z) := \log \frac{1 + \sqrt{1 - z^2}}{z} - \sqrt{1 - z^2} \tag{4.2.1}$$

which extends the real-valued function in $z \in (0,1)$ given by the same formula. Let $\mathbf{\Omega}$ be the following union of a half-plane and a half-strip

$$\mathbf{\Omega} := \big\{z \in \mathbb{C} : \ \mathrm{Re}\, z < 0\big\} \cup \big\{z \in \mathbb{C} : \ -\pi < \mathrm{Im}\, z < 0, \mathrm{Re}\, z \geq 0\big\}.$$

Then, the function ρ defines a bijection between $\overline{\mathbb{C}}_+ \setminus \{0\}$ and $\overline{\mathbf{\Omega}}$. Moreover, it is holomorphic on $\mathbb{C}_+$ and sends the intervals

$$[1,\infty), (0,1], [-1,0), (-\infty,-1]$$

respectively and bijectively to

$$i\mathbb{R}_+, \mathbb{R}_+, \big\{z \in \mathbb{C} : \ \mathrm{Im}\, z = -\pi, \mathrm{Re}\, z \geq 0\big\} \quad \text{and} \quad i(-\infty,-\pi].$$

A direct computation gives

$$\frac{\partial \rho}{\partial z} = -\frac{\sqrt{1-z^2}}{z} \quad \text{and} \quad |\rho| \sim \mathrm{const}\, |1-z|^{3/2} \quad \text{as} \quad z \to 1. \tag{4.2.2}$$

As in [61, 62], one can find an injective continuous function $\zeta : \overline{\mathbb{C}}_+ \setminus \{0\} \to \overline{\mathbb{C}}_-$ which sends bijectively $(0,1]$ to $\mathbb{R}_+$ and satisfies

$$\frac{2}{3}\zeta^{3/2}(z) = \rho(z). \tag{4.2.3}$$

The function ζ is holomorphic on $\mathbb{C}_+$.

Consider the convex domain

$$\mathbf{K}_+ := \big\{z \in \mathbb{C}_+ : \ \mathrm{Re}\, \rho(z) > 0\big\}.$$

Its boundary is the union of the interval $[-1,1]$ and the curve $\rho^{-1}([-i\pi,0])$ joining the two points -1 and 1. This is the upper half of the domain $\mathbf{K}$ considered in [61, 62], [77, p.126] and [88, p.377]. Note that $\mathbf{K}_+$ contains the half-disc $\overline{\mathbb{D}}(\frac{1}{2}) \cap \mathbb{C}_+$.

Recall that the dimensional constant c_d used in Introduction was defined in [77, 88, 87]. It is equal to

$$\begin{aligned} c_d &= \frac{2d}{\pi(d-2)!} \int_{z=x+iy \in \mathbb{C}_+} \frac{\max(-\mathrm{Re}\, \rho(z), 0)}{|z|^{d+2}} dxdy \\ &= \frac{2d}{\pi(d-2)!} \int_{z=x+iy \in \mathbb{C}_+ \setminus \mathbf{K}_+} -\frac{\mathrm{Re}\, \rho(z)}{|z|^{d+2}} dxdy \\ &= \frac{2\mathrm{vol}(\mathbb{B})^2}{(2\pi)^d} + \frac{2}{\pi d(d-2)!} \int_{\partial \mathbf{K}_+} \frac{|1-z^2|^{1/2}}{|z|^{d+1}} |dz|. \end{aligned} \tag{4.2.4}$$

We will need some basic properties of the Airy function $\mathrm{Ai}(\cdot)$, of its derivative $\mathrm{Ai}'(\cdot)$ and of the Bessel function $J_\nu(\cdot)$ with a large positive parameter ν. The functions $\mathrm{Ai}(\cdot)$ and $\mathrm{Ai}'(\cdot)$ are entire. The function $J_\nu(\cdot)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$. For $w \in \mathbb{C} \setminus \mathbb{R}_-$, define $\xi := \frac{2}{3} w^{3/2}$,

where we use the principal branch for the function $w \mapsto w^{3/2}$. There are real numbers u_s and v_s such that

$$\mathrm{Ai}(w) \sim \frac{e^{-\xi}}{2\pi^{1/2} w^{1/4}} \sum_{s=0}^{\infty} \frac{u_s}{(-\xi)^s} \quad \text{and} \quad \mathrm{Ai}'(w) \sim \frac{w^{1/4} e^{-\xi}}{2\pi^{1/2}} \sum_{s=0}^{\infty} \frac{v_s}{(-\xi)^s} \tag{4.2.5}$$

as $|w| \to \infty$ in $|\arg w| \le \pi - \delta$ for every fixed constant $\delta > 0$.

For the values of $\mathrm{Ai}(\cdot)$ and $\mathrm{Ai}'(\cdot)$ on $\mathbb{C} \setminus \mathbb{R}_+$, we need other formulas. With the above notation, there are real numbers a_s, b_s, a'_s, b'_s such that

$$\mathrm{Ai}(-w) \sim \frac{1}{\pi^{1/2} w^{1/4}} \left[\cos(\xi - \frac{\pi}{4}) \big(1 + \sum_{s=1}^{\infty} \frac{a_s}{\xi^{2s}}\big) + \sin(\xi - \frac{\pi}{4}) \sum_{s=0}^{\infty} \frac{b_s}{\xi^{2s+1}} \right] \tag{4.2.6}$$

and

$$\mathrm{Ai}'(-w) \sim \frac{w^{1/4}}{\pi^{1/2}} \left[\sin(\xi - \frac{\pi}{4}) \big(1 + \sum_{s=1}^{\infty} \frac{a'_s}{\xi^{2s}}\big) + \cos(\xi - \frac{\pi}{4}) \sum_{s=0}^{\infty} \frac{b'_s}{\xi^{2s+1}} \right] \tag{4.2.7}$$

as $|w| \to \infty$ with $|\arg w| \le \frac{2\pi}{3} - \delta$ for every fixed constant $\delta > 0$.

For the Bessel function $J_\nu(\cdot)$, when $\nu \to \infty$, the following relation holds uniformly in $0 \le \arg z \le \pi - \delta$ with any fixed constant $\delta > 0$

$$J_\nu(\nu z) \sim \big(\frac{4\zeta}{1-z^2}\big)^{1/4} \left[\frac{\mathrm{Ai}(\nu^{2/3}\zeta)}{\nu^{1/3}} \big(1 + \sum_{s=1}^{\infty} \frac{A_s(\zeta)}{\nu^{2s}}\big) + \frac{\mathrm{Ai}'(\nu^{2/3}\zeta)}{\nu^{5/3}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{\nu^{2s}} \right], \tag{4.2.8}$$

where A_s and B_s are holomorphic functions in ζ, see Olver [61, (4.24)]. Note that a similar property holds for $z \in \mathbb{C} \setminus (-\infty, 1]$ and $0 \le |\arg z| \le \pi - \delta$ by Schwarz reflection principle.

We will need the following estimates.

Lemma 4.2.1. *Let $M > 0$ be a fixed constant large enough. Then, there is a constant $A > 0$ such that for ν large enough and for $z \neq 0$ with $\mathrm{Re}(z) \ge 0$, $\mathrm{Im}(z) \ge 0$ we have*

$$|J_\nu(\nu z)| \le A \max(1, -\log|z|) e^{-\nu \mathrm{Re}\,\rho} \quad \textit{when} \quad \nu^{2/3}|1-z| \ge M$$

and

$$|J_\nu(\nu z)| \le A \quad \textit{when} \quad \nu^{2/3}|1-z| \le M.$$

Proof. Assume that $\nu^{2/3}|1-z| \le M$. Then z is close to 1. We deduce from (4.2.2) and (4.2.3) that $|\zeta| \sim \mathrm{const}\,|1-z|$. So $|\zeta|$, $|\nu^{2/3}\zeta|$ and the first factor in the right-hand side of (4.2.8) are bounded. Therefore, we deduce from (4.2.8) that $|J_\nu(\nu z)|$ is bounded.

Assume now that $\nu^{2/3}|1-z| \ge M$. Then $|\nu^{2/3}\zeta|$ and $|\nu\rho|$ are bounded below by a large positive constant. This allows us to use the identities (4.2.5), (4.2.6) and (4.2.7). We distinguish two cases. Consider first the case where $-\pi \le \arg(\zeta) \le -\frac{\pi}{2}$. In this case, we have $\mathrm{Re}\,\rho \le 0$. Then, we can apply the relations (4.2.8), (4.2.6) and (4.2.7) to $w := -\nu^{2/3}\zeta$ and $\xi := -i\nu\rho$. We have

$$\begin{aligned} |J_\nu(\nu z)| &\lesssim \frac{|w|^{1/4}|\zeta|^{1/4}}{\nu^{1/3}|1-z^2|^{1/4}} \Big(\big|\sin\big(-i\nu\rho - \frac{\pi}{4}\big)\big| + \big|\cos\big(-i\nu\rho - \frac{\pi}{4}\big)\big| \Big) \\ &\lesssim \frac{|\zeta|^{1/2}}{|1-z^2|^{1/4}} e^{-\nu\,\mathrm{Re}\,\rho}. \end{aligned}$$

We obtain the result using that $|\zeta| \lesssim -\log|z|$ as $z \to 0$, $|\zeta| \lesssim |1-z|$ as $z \to 1$ and $|\zeta| \lesssim |z|^{2/3}$ as $z \to \infty$.

It remains to treat the case where $-\frac{\pi}{2} \leq \arg(\zeta) \leq 0$. In this case, we do not need to know the sign of $\operatorname{Re}\rho$. We can apply the relations (4.2.8) and (4.2.5) to $w := \nu^{2/3}\zeta$ and $\xi := \nu\rho$. Similar estimates as above give the result. □

Recall that the zeros of the function $J_\nu(\nu z)$, except 0, are real, simple and larger than 1, see [61, (7.4)] and [62, p.246]. So the corresponding values of ρ belong to $i\mathbb{R}_+$. Fix an integer k_0 large enough. We say that a solution of $J_\nu(\nu z) = 0$ is of first type if the corresponding value of ρ satisfies $|\nu\rho| < k_0\pi$ and of second type otherwise. Let $\widetilde{z}_{\nu,k_0}, \widetilde{z}_{\nu,k_0+1}, \dots$ be the solutions of second type of $J_\nu(\nu z) = 0$ written in increasing order. Define $\widetilde{\rho}_{\nu,k} := \rho(\widetilde{z}_{\nu,k})$. We will need later the following lemma.

Lemma 4.2.2. *For ν large enough the number of solutions of first type of $J_\nu(\nu z) = 0$ is bounded by a constant independent of ν. Moreover, there is a constant $\epsilon_0 > 0$ such that for ν large enough and for $k_0 \leq k \leq \epsilon_0\nu^4$, we have*

$$\Big|\widetilde{\rho}_{\nu,k} - \big(\frac{3\pi i}{4\nu} + \frac{k\pi i}{\nu}\big)\Big| \leq \frac{1}{\nu}.$$

Proof. Assume that ν is large enough. Consider the solutions z of first type. As above, we can apply (4.2.8) and (4.2.6), (4.2.7) to $w := -\nu^{2/3}\zeta$ and $\xi := -i\nu\rho$. We can see using Rouché theorem that w is almost equal to a solution of $\operatorname{Ai}(-w) = 0$ in a bounded interval. So the number of solutions of first type is bounded.

We prove now the second assertion in the lemma. Recall that the function ρ sends bijectively $[1,\infty)$ to $i\mathbb{R}_+$. So the $\widetilde{\rho}_{\nu,k}$ are in $i\mathbb{R}_+$ and the sequence $|\widetilde{\rho}_{\nu,k}|$ is increasing. We will only consider the zeros of $J_\nu(\nu z)$ such that $k_0\pi\nu^{-1} \leq |\rho| < \epsilon\nu^3$ for some fixed small constant $\epsilon > 0$. For such a zero, we have $|\nu^{2/3}\zeta| < 2\epsilon\nu^{8/3}$.

We apply again (4.2.8) and (4.2.6), (4.2.7) to $w := -\nu^{2/3}\zeta$ and $\xi := -i\nu\rho$. Using that $|w^{1/4}| \ll \nu^{2/3}$, we see that ξ is a positive number large enough (because k_0 is a large constant) satisfying an equation of the form

$$\cos\big(\xi - \frac{\pi}{4}\big) = \gamma_\nu(\xi),$$

where $\gamma_\nu(\xi)$ is a holomorphic function on the domain

$$\big\{\xi \in \mathbb{C} :\ (k_0-1)\pi < |\xi| < \epsilon\nu^4, \operatorname{Re}\xi > 0, |\operatorname{Im}\xi| < 1\big\}$$

such that $|\gamma_\nu|$ is bounded by a very small constant independent of ν. We use here the property that $\cos(\xi - \frac{\pi}{4})$ and $\sin(\xi - \frac{\pi}{4})$ are bounded on the considered domain.

Choose a constant $\epsilon_0 \ll \epsilon$. We can now apply Rouché's theorem and deduce that the first $\epsilon_0\nu^4 - k_0 + 1$ zeros of second type of $J_\nu(\nu z)$ satisfy the lemma. □

Let H_l denote the vector space of harmonic homogeneous polynomials of degree l on $\mathbb{R}^d$. These polynomials are used to describe the eigenfunctions of the Dirichlet Laplacian on the unit ball $\mathbb{B}$ of $\mathbb{R}^d$. We recall some basic properties that will be used later. For the following result, see Corollary 2.53 in [36].

Lemma 4.2.3. *We have*

$$\dim H_l = (2l+d-2)\frac{(l+d-3)!}{l!(d-2)!} = \frac{2l^{d-2}}{(d-2)!} + O(l^{d-3}) \quad \textit{as} \quad l \to \infty.$$

Define $\nu := l + \frac{d}{2} - 1$. Let $m_l^+(r)$ denote the number of positive zeros of $J_\nu(\cdot)$ which are smaller or equal to r. It is almost equal to the maximal integer k such that $\widetilde{z}_{\nu,k} \leq \frac{r}{\nu}$. The following result is a consequence of the classical Weyl law.

Proposition 4.2.4. *Assume that r is large enough. Then*

(a) $m_l^+(r) = 0$ *when* $l \geq r$;

(b) $\widetilde{z}_{\nu,k} \notin \overline{\mathbb{D}}(\frac{r}{\nu})$ *if* $k \geq \frac{1}{2}cr$ *for some fixed constant* $c > 0$ *large enough; in particular, we have* $m_l^+(r) \leq cr$;

(c) *We have*

$$\sum_{l\geq 0} m_l^+(r)\dim H_l = \frac{\mathrm{vol}(\mathbb{B})^2}{(2\pi)^d} r^d + O(r^{d-1}) \quad \textit{as} \quad r \to \infty.$$

Proof. (a) If $l \geq r$ then $\nu > r$. In this case, if $z > 0$ is a solution of $J_\nu(\nu z) = 0$, by classical properties of Bessel functions, we have $z \geq 1$. Therefore, $\nu z > r$ and hence, $m_l^+(r) = 0$.

(b) Assume that $l \leq r$ and $k \geq \frac{1}{2}cr$ for $c > 0$ large enough. Then, by Lemma 4.2.2, $\widetilde{\rho}_{\nu,k}$ is large and therefore $\widetilde{z}_{\nu,k}$ is larger than 2. For z in $[2, \infty)$, we have

$$\rho = \Big(\sqrt{z^2-1} + \arccos\frac{1}{z}\Big) i.$$

We deduce that $\nu\widetilde{z}_{\nu,k} \gtrsim \nu|\widetilde{\rho}_{\nu,k}| \gtrsim k$ and hence $\widetilde{z}_{\nu,k} > \frac{r}{\nu}$.

(c) Recall that the eigenvalues of the Dirichlet Laplacian on $\mathbb{B}$ are precisely $(\nu\widetilde{z}_{\nu,k})^2$ with multiplicity $\dim H_l$, see Theorem 2.66 in [36] for details. So the infinite sum in the proposition is the number of eigenvalues $\leq r^2$ of the Dirichlet Laplacian on $\mathbb{B}$ counted with multiplicities. By Weyl law [42, Th. 29.3.3], this number is equal to

$$\frac{\mathrm{vol}(\mathbb{B})^2}{(2\pi)^d} r^d + O(r^{d-1}) \quad \text{as} \quad r \to \infty.$$

This completes the proof of the proposition. □

4.3 Upper bound for the number of resonances

In this section, we obtain an upper bound for the number of resonances which improves a result due to Zworski-Stefanov [77, 86]. Consider a general Schrödinger operator $-\Delta + V$ with a bounded complex potential V vanishing outside the ball $\mathbb{B}_a$. Here is the main result in this section which is a consequence of Proposition 4.3.2 and Theorem 4.3.3 below.

Theorem 4.3.1. *With the notation as in Introduction, there is a constant $A > 0$ depending only on d, a and $\|V\|_\infty$ such that*

$$N_V(r) \leq \frac{c_d a^d r^d}{d} + Ar^{d-1}\log r \quad \textit{as } r \to \infty.$$

We first recall some basic notions and results, see [77, 86] for details. Let R_1, R_2, R_3 be real numbers such that $a < R_1 < R_2 < R_3$. Choose also a smooth cut-off function χ_1 (resp. χ_2) vanishing outside $\mathbb{B}_{R_2}$ (resp. $\mathbb{B}_{R_3}$) and equal to 1 on $\mathbb{B}_{R_1}$ (resp. $\mathbb{B}_{R_2}$). These numbers and functions will be specified later.

Define two families of operators $\mathcal{E}_\pm(\lambda) : L^2_{\rm comp}(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ with $\lambda \in \mathbb{C}$ by

$$\mathcal{E}_\pm(\lambda)(f)(w) := \int e^{\pm i\lambda\omega\cdot x} f(x)dx \quad \text{for } f \in L^2_{\rm comp}(\mathbb{R}^d) \text{ and } \omega \in \mathbb{S}^{d-1}.$$

Denote by $\mathcal{E}^*_+(\lambda)$ the transpose operator of $\mathcal{E}_+(\lambda)$ with the same Schwartz kernel. The scattering matrix associated to $-\Delta + V$ is the operator $S_V(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ given by

$$S_V(\lambda) := I - i(2\pi)^{-d} 2^{\frac{1-d}{2}} \lambda^{d-2} \mathcal{E}_-(\lambda)[\Delta, \chi_2] R_V(\lambda)[\Delta, \chi_1]\mathcal{E}^*_+(\lambda),$$

where I denotes the identity operator, see [77, p.120].

The scattering determinant is defined by

$$s_V(\lambda) := \det S_V(\lambda).$$

It satisfies $s_V(\lambda)s_V(-\lambda) = 1$. The poles of s_V are called *the scattering poles*. They are, with a finite number of exceptions, the resonances of $-\Delta + V$ with the same multiplicities. In what follows, we will tend to abuse notation and identify $n_V(r)$, $N_V(r)$ with the similar counting functions for the zeros of $s_V(\lambda)$ on $\mathbb{C}_+$. This does not affect our estimates.

The following result was obtained by Christiansen in [15, (3.2)], see also Stefanov [77, Prop. 2].

Proposition 4.3.2. *We have for r large enough*

$$\left| N_V(r) - \frac{1}{2\pi} \int_0^{2\pi} \log |s_V(re^{i\theta})| d\theta \right| \leq Ar^{d-1},$$

where $A > 0$ is a constant depending only on d, a and $\|V\|_\infty$.

We have the following refinement of [77, Th. 5] where the function h_d is defined by

$$h_d(\theta) := \frac{4}{(d-2)!} \int_0^\infty \frac{\max(-\operatorname{Re}\rho(te^{i\theta}), 0)}{t^{d+1}} dt \quad \text{for} \quad \theta \in [0, \pi].$$

This function is continuous, positive and satisfies $h_d(0) = h_d(\pi) = 0$.

Theorem 4.3.3. *With the above notation, there is a constant $A > 0$ depending only on d, a and $\|V\|_\infty$ such that*

$$\log |s_V(re^{i\theta})| \leq h_d(\theta) a^d r^d + Ar^{d-1} \log r$$

for all r large enough and $\theta \in [0, \pi]$.

In the rest of the section, we give the proof of Theorem 4.3.3. Observe that we can suppose $\theta \leq \pi/2$ since otherwise we can reduce the problem to the first case by replacing V with $\overline{V}$ and λ with $-\overline{\lambda}$. By rescaling, we can also assume that $a = 1$. Choose $R_j := 1 + \frac{j}{r}$ for $j = 1, 2, 3$.

Choose χ_1, χ_2 as above such that $\|\chi_j\|_{\mathscr{C}^2} \leq cr^2$ for some constant $c > 0$ independent of r. Define for $l \geq 1$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$, $\operatorname{Im}\lambda \geq 0$ and $0 < s \leq s'$

$$I_l(\lambda, s, s') := \int_s^{s'} |\lambda|^{2-d} |J_{l+d/2-1}(\lambda t)|^2 t dt$$

and

$$\mu_l^\star(\lambda) := (2\pi)^d I_l(\lambda, R_1, R_2)^{1/2} I_l(\lambda, R_2, R_3)^{1/2} \quad \text{for} \quad \lambda \in \mathbb{C}_+.$$

The following lemma refines an estimate obtained by Stefanov.

Lemma 4.3.4. *There is a constant $A > 0$ such that*

$$\log |s_V(re^{i\theta})| \leq \sum_{l=1}^{\infty} (\dim H_l) \log\left(1 + Ar^{d+4} \mu_l^\star(re^{i\theta})\right) \tag{4.3.1}$$

for r large enough and $0 \leq \theta \leq \frac{\pi}{2}$.

Proof. If $Q : H \to H$ is a bounded linear operator on a Hilbert space such that the spectrum of $(Q^*Q)^{1/2}$ is discrete, denote by $\mu_1(Q), \mu_2(Q), \ldots$ the singular values of Q, i.e. the eigenvalues of $(Q^*Q)^{1/2}$, written in decreasing order and repeated according to their multiplicities. Define

$$\mathbb{H}(\lambda) := -i(2\pi)^{-d} 2^{\frac{1-d}{2}} \lambda^{d-2} [\Delta, \chi_2] R_V(\lambda) [\Delta, \chi_1]$$

and

$$\mathbf{F}(\lambda) := \mathbf{1}_{R_1 < \|x\| < R_2} \mathcal{E}_+^*(\lambda) \mathcal{E}_-(\lambda) \mathbf{1}_{R_2 < \|x\| < R_3}.$$

In the proof of Theorem 5 in [77, p.128], Stefanov obtained that

$$\begin{aligned} \log |s_V(re^{i\theta})| &\leq \sum_{l=1}^{\infty} \log\left(1 + \mu_l(\mathbb{H}(re^{i\theta}) \mathbf{F}(re^{i\theta}))\right) \\ &\leq \sum_{l=1}^{\infty} \log\left(1 + \|\mathbb{H}(re^{i\theta})\|_{L^2 \to L^2} \mu_l(\mathbf{F}(re^{i\theta}))\right). \end{aligned}$$

The last inequality is a consequence of the general inequality $\mu_l(AB) \leq \|A\| \mu_l(B)$.

Stefanov also proved that, up to a permutation of elements, the sequence $\mu_l(\mathbf{F}(re^{i\theta}))$ is constituted by the $\mu_l^\star(re^{i\theta})$'s where each number $\mu_l^\star(re^{i\theta})$ is repeated $(\dim H_l)$ times. Since we only consider sums of positive numbers, this permutation does not affect our computation. So we only have to check that

$$\|\mathbb{H}(re^{i\theta})\|_{L^2 \to L^2} \leq Ar^{d+4}$$

for r large enough and for a large fixed constant $A > 0$.

Choose a smooth function $\rho \leq 1$ with compact support which is equal to 1 on $\mathbb{B}_{R_3}$ and with bounded $\mathscr{C}^1$-norm. Since the operators $[\Delta, \chi_i]$ are of order 1, using the above estimates on χ_i, we only need to check that

$$\|\rho R_V(re^{i\theta}) \rho\|_{H^{-1} \to H^1} = O(r^2).$$

But, this estimate is a consequence of the classical theory of elliptic operators, see Zworski [91] for details. The lemma follows. □

Proof of Theorem 4.3.3. Recall that we only have to consider the case where $0 \le \theta \le \frac{\pi}{2}$ and we have to bound the right-hand side in (4.3.1). Observe that $\log \mu_l^\star(re^{i\theta}) \lesssim r$, see [62, p.59]. Therefore, we only have to consider l larger than any fixed constant.

Let M be the constant in Lemma 4.2.1. Define for $\nu := l + \frac{d}{2} - 1$

$$N := \Big\{ l \in \mathbb{N}^* : \ \nu^{2/3} |1 - \frac{tre^{i\theta}}{\nu}| < M \ \text{ for some } \ t \in (R_1, R_3) \Big\}, \quad N' := \mathbb{N}^* \setminus N$$

and

$$N_1' := \Big\{ l \in N' : \ \frac{rR_3}{\nu} \notin \mathbf{K}_+ \Big\}, \ N_2' := \Big\{ l \in N' : \ \frac{rR_3}{\nu} \le \frac{1}{100} \Big\}, \ N_3' := N' \setminus (N_1' \cup N_2').$$

Denote by $\Sigma, \Sigma', \Sigma_i'$ the sums as in the right-hand side of (4.3.1) but only with l running in N, N' or N_i' respectively. We will bound these sums separately. The theorem is a direct consequence of the estimates given in the 4 cases below.

Case 1. Assume that $l \in N$. Since R_1, R_2, R_3 are close enough to each other and ν is large, we have $\frac{1}{2} < \frac{tr}{\nu} < 2$ for all $t \in (R_1, R_3)$. In particular, we have $\nu < 2rR_3 < 4r$. Moreover, for all $t, t' \in (R_1, R_3)$

$$\nu^{2/3} \Big| 1 - \frac{tre^{i\theta}}{\nu} \Big| - \nu^{2/3} \Big| 1 - \frac{t're^{i\theta}}{\nu} \Big| \le \nu^{2/3} \frac{(R_3 - R_1)r}{\nu} < 1.$$

It follows that

$$\nu^{2/3} \Big| 1 - \frac{tre^{i\theta}}{\nu} \Big| \le M + 1 \quad \text{for every} \quad t \in (R_1, R_3).$$

Applying the second assertion of Lemma 4.2.1 to $\frac{tre^{i\theta}}{\nu}$ and to $M+1$ instead of z and M yields

$$\Sigma \lesssim \sum_{\nu < 4r} (\dim H_l) \log r \lesssim \sum_{l < 4r} l^{d-2} \log r \lesssim r^{d-1} \log r.$$

Case 2. Assume now that $l \in N' = N_1' \cup N_2' \cup N_3'$. Observe that the function $t \mapsto -\operatorname{Re} \rho(tre^{i\theta})$ is increasing since we have by (4.2.2)

$$-\frac{\partial \operatorname{Re} \rho(tz)}{\partial t} = \frac{\operatorname{Re} \sqrt{1 - (tz)^2}}{t} > 0 \text{ for } z = re^{i\theta}.$$

Therefore, by the first assertion in Lemma 4.2.1, we have for some constant $A > 0$

$$\Sigma_i' \le \sum_{l \in N_i'} (\dim H_l) \log \Big(1 + A(\log \nu)^2 r^{d+4} e^{-2\nu \operatorname{Re} \rho\left(\frac{re^{i\theta} R_3}{\nu}\right)} \Big). \tag{4.3.2}$$

Case 2a. Assume that $l \in N_1'$. We have $\nu \lesssim r$ and $-\operatorname{Re} \rho\big(\frac{re^{i\theta}R_3}{\nu}\big) \ge 0$. Hence, by (4.3.2)

$$\begin{aligned} \Sigma_1' &\le \sum_{l \in N_1'} (\dim H_l) \Big[\log \big(A(\log \nu)^2 r^{d+4} \big) - 2\nu \operatorname{Re} \rho\big(\frac{re^{i\theta}R_3}{\nu}\big) \Big] \\ &\le \sum_{l \in N_1'} -\frac{4\nu^{d-1} + O(\nu^{d-2})}{(d-2)!} \operatorname{Re} \rho\big(\frac{re^{i\theta}R_3}{\nu}\big) + O(r^{d-1} \log r). \end{aligned}$$

Since the function $t \mapsto -\operatorname{Re}\rho(te^{i\theta})$ is increasing, we deduce from the last estimates that

$$\begin{aligned}\Sigma_1' &\leq \int_{\frac{re^{i\theta}R_3}{\nu}\notin \mathbf{K}_+} -\frac{4\nu^{d-1}+O(\nu^{d-2})}{(d-2)!}\operatorname{Re}\rho\big(\frac{re^{i\theta}R_3}{\nu}\big)d\nu + O(r^{d-1}\log r)\\ &\leq \frac{4(rR_3)^d}{(d-2)!}\int_{te^{i\theta}\notin \mathbf{K}_+}\frac{-\operatorname{Re}\rho(te^{i\theta})}{t^{d+1}}\,dt + O(r^{d-1}\log r)\\ &\leq \frac{4r^d}{(d-2)!}\int_{te^{i\theta}\notin \mathbf{K}_+}\frac{-\operatorname{Re}\rho(te^{i\theta})}{t^{d+1}}\,dt + O(r^{d-1}\log r).\end{aligned}$$

Case 2b. Assume that $l \in N_2'$. Since r is large, we have $l \geq 90r$. Observe that $\operatorname{Re}\rho(z) \geq -\log|z| - 2$ when $|z| \leq \frac{1}{100}$. Hence, using that $\log(1+t) \leq t$ for $t \geq 0$, we obtain from (4.3.2) that

$$\Sigma_2' \lesssim \sum_{l\geq 90r} l^{d-2}(\log l)^2 r^{d+4}e^{-2l(\log l - \log r - 3)} \lesssim \sum_{l\geq 90r} l^{2d+3}e^{-l}.$$

It follows that Σ_2' is bounded above.

Case 2c. Assume that $l \in N_3'$. We have $l \leq \nu \leq 100rR_3 < 200r$. Since $\operatorname{Re}\rho(\frac{re^{i\theta}R_3}{\nu})$ is positive, we obtain from (4.3.2) that

$$\Sigma_3' \lesssim \sum_{l\leq 200r} l^{d-2}\log r \;=\; O(r^{d-1}\log r).$$

This completes the proof of the theorem. □

4.4 Schrödinger operators with radial potentials

In this section, we give the proof of Theorem 4.1.1. We assume that the potential $V = V(\|x\|)$ satisfies the hypotheses of this theorem. By rescaling, we reduce the problem to the case $a = 1$. Define for $c > 0$

$$\mathbf{\Omega}_c^\nu := \Big\{\rho \in \mathbb{C} :\ \operatorname{Re}\rho < -\frac{\log(c\nu)}{2\nu}\Big\}$$

and

$$\mathbf{J}_c^\nu := \rho^{-1}(\mathbf{\Omega}_c^\nu) = \Big\{z \in \mathbb{C}_+ :\ \operatorname{Re}\rho(z) < -\frac{\log(c\nu)}{2\nu}\Big\}.$$

Following Zworski [88, p.400], the scattering poles are related to the zeros in $\mathbf{J}_c^\nu$ of a family of holomorphic functions of the form (our notation is slightly different from Zworski's one)

$$g_\nu(z) = \frac{e^{-2\nu\rho(z)}}{\nu^2(1-z^2)}(1+\epsilon_\nu(z)) - \sigma(1+\epsilon_\nu'(z)), \tag{4.4.1}$$

where σ is some complex number with $|\sigma|$ bounded below and above by positive constants and $\epsilon_\nu(z)$, $\epsilon_\nu'(z)$ are continuous functions on $\overline{\mathbf{J}}_c^\nu$ which converge uniformly to 0 when $\nu \to \infty$. In comparison with Zworski's notation, for our convenience, we work with variable z in $\mathbb{C}_+$ instead of $\mathbb{C}_-$.

We need to compare g_ν with an auxiliary function h_ν defined by

$$h_\nu(z) := \frac{e^{-2\nu\rho(z)}}{\nu^2(1-z^2)} - \sigma. \tag{4.4.2}$$

In what follows, we often consider g_ν and h_ν as functions on variable $\rho = \rho(z)$.

Let f be the bi-holomorphic map from $\mathbf{\Omega} = \rho(\mathbb{C}_+)$ to $\mathbb{C} \setminus (-\infty, 1]$ defined by

$$f(\rho(z)) := 1 - z^2.$$

We can extend it to a continuous map $f : \overline{\mathbf{\Omega}} \to \mathbb{C} \setminus \{1\}$ which is no more bijective. A direct computations using (4.2.2) gives

$$\frac{\partial f}{\partial \rho} = -\frac{2z^2}{\sqrt{1-z^2}} \quad \text{and} \quad \frac{\partial \log f}{\partial \rho} = -\frac{2z^2}{(1-z^2)^{3/2}}. \tag{4.4.3}$$

We deduce for $z \in \overline{\mathbb{C}}_+ \setminus \{0\}$ outside a neighbourhood of -1 (in particular, for $\operatorname{Im}\rho \geq -\frac{3\pi}{4}$) that

$$\Big|\frac{\partial \log f}{\partial \rho}\Big| \lesssim 1 + \frac{1}{|\rho|} \quad \text{when} \quad \operatorname{Re}\rho < 0. \tag{4.4.4}$$

Define for $k \in \mathbb{Z}$

$$\mathbf{\Omega}_c^\nu(k) := \Big\{\rho \in \mathbf{\Omega}_c^\nu : \ \big|\operatorname{Im}\rho - \big(-\frac{\arg\sigma}{2\nu} + \frac{k\pi}{\nu} + \frac{\pi}{4\nu}\big)\big| < \frac{\pi}{2\nu}\Big\}.$$

These half-strips are disjoint and the union of their closures is equal to $\overline{\mathbf{\Omega}}_c^\nu$.

Lemma 4.4.1. *Assume that ν is large enough and $k \geq -\frac{\nu}{2} - 2$. Then, there is a constant $A > 0$ independent of ν and k such that $|h_\nu(z)| \geq A$ for ρ in the boundary of $\mathbf{\Omega}_c^\nu(k)$ and also for ρ large enough in this domain.*

Proof. For ρ large enough in $\mathbf{\Omega}_c^\nu(k)$, $-\operatorname{Re}\rho$ is a large positive number and $|z| \sim -\operatorname{Re}\rho$. So $|h_\nu(z)|$ is a big number. Consider now the case where ρ belongs to the boundary of $\mathbf{\Omega}_c^\nu(k)$. For $k \geq -\frac{\nu}{2} - 2$ and for ν large enough, $\operatorname{Im}\rho$ is almost larger than $-\frac{\pi}{2}$ and hence $\operatorname{Re} z$ is almost positive. Therefore, $\arg f(\rho)$ belongs to the interval $(-\pi, \delta)$ for some small positive constant δ independent of ν, k.

Assume first that ρ belongs to the horizontal part of $b\mathbf{\Omega}_c^\nu(k)$ which is the union of two half-lines given by

$$\operatorname{Re}\rho \leq -\frac{\log(c\nu)}{2\nu} \quad \text{and} \quad \operatorname{Im}\rho = -\frac{\arg\sigma}{2\nu} + \frac{k\pi}{\nu} + \frac{\pi}{4\nu} \pm \frac{\pi}{2\nu}.$$

The above discussion on $\arg f(\rho)$ implies that $|\arg(\sigma^{-1}h_\nu(z) + 1)|$ is bounded below by $\frac{\pi}{2} - \delta$. It follows that $|h_\nu(z)|$ is bounded below by a positive constant.

It remains to consider the case where ρ belongs to the vertical part of $b\mathbf{\Omega}_c^\nu(k)$. We have

$$\operatorname{Re}\rho = -\frac{\log(c\nu)}{2\nu}.$$

Therefore,

$$|h_\nu(z) + \sigma| = \frac{c}{\nu|1-z^2|}.$$

It is enough to check that the last quantity is small. Since ν is large, this is clear when z is outside a fixed neighbourhood of 1. Otherwise, we deduce from (4.2.2) that

$$|1 - z^2| \gtrsim |\rho|^{2/3} \gtrsim |\operatorname{Re}\rho|^{2/3} \gtrsim \nu^{-2/3}.$$

The result follows. □

Lemma 4.4.2. *Assume that ν is large enough and $k \geq -\frac{\nu}{2} - 2$. Then g_ν and h_ν, as functions on ρ, have the same number of zeros in $\mathbf{\Omega}_c^\nu(k)$ counted with multiplicity.*

Proof. We have for ρ as in Lemma 4.4.1

$$|g_\nu(z) - h_\nu(z)| \leq 2 \max\big(|\epsilon_\nu(z)|, |\epsilon'_\nu(z)|\big) \max\big(|\sigma|, |h_\nu(z) + \sigma|\big).$$

Lemma 4.4.1 implies that the last factor is bounded by a constant times $|h_\nu(z)|$. Since ϵ_ν and ϵ'_ν are small, it is enough to apply Rouché's theorem in order to obtain the result. □

Lemma 4.4.3. *Assume that ν is large enough. Then, for every $k \in \mathbb{Z}$ with $k \geq -\frac{\nu}{2} - 2$, the function g_ν admits a unique zero in $\overline{\mathbf{\Omega}}_c^\nu(k)$ that we denote by $\rho_{\nu,k}$. Moreover, this zero is simple and belongs to $\mathbf{\Omega}_c^\nu(k)$.*

Proof. First, we deduce from Lemma 4.4.1 that h_ν and g_ν have no zero on the boundary of $\mathbf{\Omega}_c^\nu(k)$. By Lemma 4.4.2, we only have to show that h_ν admits a unique zero in $\mathbf{\Omega}_c^\nu(k)$ and this zero is simple.

The zeros of h_ν are exactly the solutions of the following family of equations

$$F_{\nu,k}(\rho) = 0 \quad \text{with} \quad k \in \mathbb{Z}, \tag{4.4.5}$$

where

$$F_{\nu,k}(\rho) := \rho - \Big(-\frac{\log \sigma}{2\nu} - \frac{\log \nu}{\nu} - \frac{\log f(\rho)}{2\nu} + \frac{k\pi i}{\nu} \Big).$$

Therefore, we only have to prove that (4.4.5) admits a unique solution in $\mathbf{\Omega}_c^\nu$ which is simple and belongs to $\mathbf{\Omega}_c^\nu(k)$.

Consider a solution $\rho \in \mathbf{\Omega}_c^\nu$ of (4.4.5). By considering the equation $\operatorname{Im} F_{\nu,k}(\rho) = 0$, we see that $\operatorname{Im} \rho \geq \frac{(k-2)\pi}{\nu}$. So for $k \geq -\frac{\nu}{2} - 2$ and for ν large enough, $\operatorname{Im} \rho$ is almost larger than $-\frac{\pi}{2}$. Therefore, $\arg f(\rho)$ belongs to the interval $(-\pi, \delta)$ for some small positive constant δ. We deduce that

$$\Big| \operatorname{Im} \rho - \big(-\frac{\arg \sigma}{2\nu} + \frac{k\pi}{\nu} + \frac{\pi}{4\nu} \big) \Big| \leq \Big| -\frac{\arg f(\rho)}{2\nu} - \frac{\pi}{4\nu} \Big| < \frac{\pi}{2\nu}.$$

It follows that ρ belongs to $\mathbf{\Omega}_c^\nu(k)$.

We now use the classical argument principle in order to count the number of zeros of $F_{\nu,k}$ in $\mathbf{\Omega}_c^\nu(k)$. Observe that $\operatorname{Im} F_{\nu,k}(\rho)$ is bounded on $\overline{\mathbf{\Omega}}_c^\nu(k)$ and $\operatorname{Re} F_{\nu,k}(\rho) \to -\infty$ when $|\rho| \to \infty$ and $\rho \in \overline{\mathbf{\Omega}}_c^\nu(k)$. We will show in particular that $\operatorname{Re} F_{\nu,k}(\rho)$ changes sign twice on $b\mathbf{\Omega}_c^\nu(k)$.

Consider first the horizontal part of $b\mathbf{\Omega}_c^\nu(k)$ which is the union of two half-lines given by

$$\operatorname{Re} \rho \leq -\frac{\log(c\nu)}{2\nu} \quad \text{and} \quad \operatorname{Im} \rho = -\frac{\arg \sigma}{2\nu} + \frac{k\pi}{\nu} + \frac{\pi}{4\nu} \pm \frac{\pi}{2\nu}.$$

As above, we obtain that $\arg f(\rho)$ belongs to $(-\pi, \delta)$. We then deduce that $\operatorname{Im} F_{\nu,k}(\rho)$ is strictly positive on the upper half-line and strictly negative on the lower one.

Since $|\operatorname{Re} \rho| \gg \frac{1}{\nu}$, the relation (4.4.4) implies that $\operatorname{Re} \rho \mapsto \operatorname{Re} F_{\nu,k}(\rho)$ defines an increasing function on each of the above half-lines. Therefore, in order to obtain the lemma, it suffices to check that $\operatorname{Re} F_{\nu,k}(\rho) > 0$ on the vertical part of $b\mathbf{\Omega}_c^\nu(k)$ which is contained in the line

$$\operatorname{Re} \rho = -\frac{\log(c\nu)}{2\nu}.$$

Assume that ρ satisfies the last identity. When $|\rho| > 1$, $|1 - z|$ is bounded below by a positive constant. Therefore, $\log|f(\rho)|$ is bounded below and $\operatorname{Re} F_{\nu,k}(\rho) > 0$ for ν large enough. Otherwise, we deduce from (4.2.2) that $|f(\rho)| \gtrsim |\rho|^{2/3}$. Therefore, since $|\operatorname{Re}\rho| \gg \frac{1}{\nu}$, we have

$$\log|f(\rho)| \geq \log|\rho|^{2/3} + \text{const} \geq -\frac{2}{3}\log\nu + \text{const}\,.$$

It follows that $\operatorname{Re} F_{\nu,k}(\rho)$ is strictly positive when ν is large enough. This completes the proof of the lemma. □

Denote by $z_{\nu,k}$ the complex number in $\mathbb{C}_+$ such that $\rho(z_{\nu,k}) = \rho_{\nu,k}$. We have the following lemma.

Lemma 4.4.4. *Assume that ν is large enough. Then, for every $k \geq -\frac{\nu}{2} + 2$, we have $\operatorname{Re} z_{\nu,k} > 0$ and $|z_{\nu,k}| > \frac{1}{2}$. Moreover, if $\rho = \rho(z)$ in $\mathbf{\Omega}_c^\nu$ is a zero of $g_\nu(z)$ with $\operatorname{Re} z \geq 0$ and $\operatorname{Im} z \geq 0$, then $z = z_{\nu,k}$ and $\rho = \rho_{\nu,k}$ for some $k \geq -\frac{\nu}{2} - 2$.*

Proof. When $k \geq -\frac{\nu}{2} + 2$, since $\rho_{\nu,k}$ belongs to $\mathbf{\Omega}_c^\nu(k)$, we have $\operatorname{Im}\rho_{\nu,k} > -\frac{\pi}{2}$. Hence, $\operatorname{Re} z_{\nu,k} > 0$. Since $\operatorname{Re}\rho_{\nu,k} < 0$, we have $z_{\nu,k} \notin \mathbf{K}_+$ and hence, $|z_{k,\nu}| > \frac{1}{2}$. If ρ and z are as in the lemma, then $\operatorname{Im}\rho \geq -\frac{\pi}{2}$. Such a point ρ should be in $\overline{\mathbf{\Omega}}_c^\nu(k)$ for some $k \geq -\frac{\nu}{2} - 2$. Lemma 4.4.3 implies the result. □

For $l \in \mathbb{N}$ define $\nu := l + \frac{d}{2} - 1$ (we use here the notation of Stefanov [77] which is slightly different from the one by Zworski [88, (25)]). For $r > 0$, denote by $n_l^+(r)$ (resp. $n_l^-(r)$) the number of points $z_{\nu,k}$ in $\overline{\mathbb{D}}(\frac{r}{\nu})$ with $k > 0$ (resp. $-\frac{\nu}{2} + 2 < k \leq 0$). Theorem 4.1.1 is a consequence of the following two propositions whose proofs will be given at the end of the section.

Proposition 4.4.5. *Assume that ν and r are large enough. Then,*

(a) *$n_l^+(r) = 0$ for $l \geq 2r$;*

(b) *$z_{\nu,k} \notin \overline{\mathbb{D}}(\frac{r}{\nu})$ if $k > cr$ for a fixed constant $c > 0$ large enough; in particular, we have $n_l^+(r) \leq cr$;*

(c) *We have for every constant $\epsilon > 0$*

$$\sum_{0\leq l\leq 2r} n_l^+(r)(\dim H_l) = \frac{\operatorname{vol}(\mathbb{B})^2}{(2\pi)^d} r^d + O(r^{d-3/4+\epsilon}) \quad \text{as} \quad r \to \infty.$$

Proposition 4.4.6. *Assume that ν and r are large enough. Then, $n_l^-(r) = 0$ for $l \geq 2r$. Moreover, we have for every constant $\epsilon > 0$*

$$\sum_{0\leq l\leq 2r} n_l^-(r)(\dim H_l) = \frac{r^d}{\pi d(d-2)!}\int_{\partial\mathbf{K}_+} \frac{|1-z^2|^{1/2}}{|z|^{d+1}}|dz| + O(r^{d-3/4+\epsilon}) \quad \text{as} \quad r \to \infty.$$

End of the proof of Theorem 4.1.1. Using the decomposition of functions into spherical harmonics, Zworski relates scattering poles of $-\Delta + V$ to the zeros of a sequence of functions of the form (4.4.1) with $l \in \mathbb{N}$. More precisely, if $\rho = \rho(z)$ is a zero of $g_\nu(z)$ with $z \in \mathbf{J}_c^\nu$, then

$-\nu z$ is a scattering pole with multiplicity $\dim H_l$, see [86, §2]. If $n_l(r)$ is the number of zeros of $g_\nu(z)$ in $\mathbf{J}_c^\nu \cap \overline{\mathbb{D}}(\frac{r}{\nu})$ with unknown z, Zworski proved that $n_l(r) \lesssim r$ and $n_l(r) = 0$ for $l > 2r$, see [88, p.386]. Therefore, we have

$$n_V(r) = \sum_{l\geq 0} n_l(r)(\dim H_l) = \sum_{0\leq l\leq 2r} n_l(r)(\dim H_l).$$

By Lemma 4.2.3, we only need to consider l large enough.

In our setting with a real potential, the scattering poles are symmetric with respect to the real line $\operatorname{Re} z = 0$. This and Lemma 4.4.4 imply that

$$2n_l^+(r) + 2n_l^-(r) \leq n_l(r) \leq 2n_l^+(r) + 2n_l^-(r) + 4.$$

Now, in order to obtain the result, it suffices to apply Propositions 4.4.5, 4.4.6 and the identity (4.2.4). □

We give now the proofs of the above propositions. For $k \in \mathbb{Z} \setminus \{0\}$ such that $k \geq -\frac{\nu}{2} + 2$, define

$$\rho^\star_{\nu,k} := -\frac{\log \sigma}{2\nu} - \frac{\log \nu}{\nu} - \frac{1}{2\nu} \log f\big(\frac{k\pi i}{\nu}\big) + \frac{k\pi i}{\nu}.$$

These points are easier to count and we will compare them with $\rho_{\nu,k}$.

Lemma 4.4.7. *Assume that ν is large enough, $k \geq -\frac{\nu}{2} + 2$ and $|k| \geq \nu^{1/4}$. Then, we have*

$$|F_{\nu,k}(\rho^\star_{\nu,k})| \ll \nu^{-6/5} \ \text{ and } \ |h_\nu(\rho^\star_{\nu,k})| \ll \nu^{-1/5}.$$

Proof. Observe that

$$|h_\nu(\rho^\star_{\nu,k})| = |\sigma| \Big| e^{-2\nu F_{\nu,k}(\rho^\star_{\nu,k})} - 1 \Big|.$$

So the second inequality in the lemma is a consequence of the first one. We prove now the first inequality. It is enough to check that

$$\Big| \log f(\rho^\star_{\nu,k}) - \log f(\frac{k\pi i}{\nu}) \Big| \ll \nu^{-1/5}.$$

Consider first the case where $|k| \leq \nu$. In this case, $\rho^\star_{\nu,k}$ and $\frac{k\pi i}{\nu}$ are bounded. Since $|k| \geq \nu^{1/4}$, we have $|\rho| \gtrsim \nu^{-3/4}$ for ρ in $\mathbf{\Omega}_c^\nu(k)$. This together with (4.4.4) implies that

$$\Big| \frac{\partial \log f}{\partial \rho} \Big| \lesssim 1 + \frac{1}{|\rho|} \lesssim \nu^{3/4}.$$

Using the last inequality and the estimate

$$\Big| \rho^\star_{\nu,k} - \frac{k\pi i}{\nu} \Big| \lesssim \frac{\log \nu}{\nu},$$

we obtain that

$$\Big| \log f(\rho^\star_{\nu,k}) - \log f(\frac{k\pi i}{\nu}) \Big| \lesssim \nu^{3/4} \frac{\log \nu}{\nu} \ll \nu^{-1/5}.$$

When $k \geq \nu$ the segment joining $\rho^\star_{\nu,k}$ and $\frac{k\pi i}{\nu}$ is contained in the half-plane $\operatorname{Im}\rho > \frac{\pi}{2}$. Therefore, by (4.4.3), we have on this segment

$$\Big|\frac{\partial \log f}{\partial \rho}\Big| \lesssim \frac{1}{|z|} \lesssim \frac{1}{|\rho|} \lesssim \frac{\nu}{k}.$$

Hence, using that

$$\Big|\rho^\star_{\nu,k} - \frac{k\pi i}{\nu}\Big| \lesssim \frac{\log k}{\nu},$$

we obtain again

$$\Big|\log f(\rho^\star_{\nu,k}) - \log f\big(\frac{k\pi i}{\nu}\big)\Big| \lesssim \frac{\nu}{k}\frac{\log k}{\nu} \ll \nu^{-1/5}.$$

This completes the proof of the lemma. □

Lemma 4.4.8. *Assume that ν is large enough, $k \geq -\frac{\nu}{2} + 2$ and $|k| \geq \nu^{1/4}$. Then, we have*

$$|\rho_{\nu,k} - \rho^\star_{\nu,k}| < \frac{2\pi}{\nu}.$$

Proof. Observe that $|\operatorname{Im}\rho^\star_{k,\nu}| \gg \frac{1}{\nu}$. Using (4.2.2), we deduce that $\log\big|f\big(\frac{k\pi i}{\nu}\big)\big| \geq -\frac{2}{3}\log\nu$. Hence,

$$\operatorname{Re}\rho^\star_{\nu,k} \leq -\frac{2\log\nu + \text{const}}{3\nu}.$$

Therefore, the vertical lines of equations

$$\operatorname{Re}\rho - \operatorname{Re}\rho^\star_{\nu,k} = \pm\frac{\pi}{2\nu}$$

are contained in $\mathbf{\Omega}^\nu_c$.

Let Q denote the square of size $\frac{\pi}{\nu} \times \frac{\pi}{\nu}$ limited by the above lines and the horizontal part of $b\mathbf{\Omega}^\nu_c(k)$. It is enough to show that $\rho_{\nu,k}$ belongs to Q. Arguing as in Lemmas 4.4.1, 4.4.2 and 4.4.3, we only have to check that $|h_\nu(z)|$ is bounded below by a positive constant when ρ is on the vertical part of bQ and that $\operatorname{Re}F_{\nu,k}(\rho)$ is positive (resp. negative) on the right (resp. left) vertical part of bQ.

We only consider the case where ρ is on the left vertical part of bQ. The other case can be obtained in the same way. We have

$$\operatorname{Re}\rho - \operatorname{Re}\rho^\star_{\nu,k} = -\frac{\pi}{2\nu}. \tag{4.4.6}$$

So $|\rho - \rho^\star_{\nu,k}| \lesssim \frac{1}{\nu} \ll |\rho^\star_{\nu,k}|$. As in Lemma 4.4.7, we obtain that

$$|\log f(\rho) - \log f(\rho^\star_{\nu,k})| \ll \nu^{-1/5}. \tag{4.4.7}$$

It follows from (4.4.6) and (4.4.7) that

$$\Big|\frac{h_\nu(\rho) + \sigma}{h_\nu(\rho^\star_{\nu,k}) + \sigma}\Big| = e^\pi + O(\nu^{-1/5}).$$

By Lemma 4.4.7, $h_\nu(\rho^\star_{\nu,k})$ is small. Therefore, the last identity implies that $|h_\nu(\rho)|$ is bounded below by a positive constant.

We also deduce from (4.4.6) and the definition of $F_{\nu,k}(\rho)$ that

$$\operatorname{Re} F_{\nu,k}(\rho) - \operatorname{Re} F_{\nu,k}(\rho^\star_{\nu,k}) \leq -\frac{\pi}{2\nu} + \frac{1}{2\nu}|\log f(\rho) - \log f(\rho^\star_{\nu,k})| = -\frac{\pi}{2\nu} + O(\nu^{-6/5}).$$

This and Lemma 4.4.7 imply that $\operatorname{Re} F_{\nu,k}(\rho) < 0$. The result follows. □

Lemma 4.4.9. *Let ϵ_0 be the constant given in Lemma 4.2.2. Let $\epsilon > 0$ be any fixed constant. Assume that ν is large enough and $\nu^{1/4} \leq k \leq \epsilon_0 \nu^4$. Then, we have*

$$|\nu z_{\nu,k} - \nu \widetilde{z}_{\nu,k}| \leq |\nu \widetilde{z}_{\nu,k}|^{1/4+\epsilon}.$$

Proof. By Lemma 4.2.2, we have

$$\Big|\widetilde{\rho}_{\nu,k} - \big(\frac{3\pi i}{4\nu} + \frac{k\pi i}{\nu}\big)\Big| \leq \frac{1}{\nu}.$$

Since k is positive, $f(\frac{k\pi i}{\nu})$ is real and negative. Using Lemma 4.4.8 and the definition of $\rho^\star_{\nu,k}$, we deduce that

$$|\rho_{\nu,k} - \widetilde{\rho}_{\nu,k}| \leq |\rho_{\nu,k} - \rho^\star_{\nu,k}| + |\rho^\star_{\nu,k} - \widetilde{\rho}_{\nu,k}| \lesssim \frac{1}{\nu} + \frac{\log k}{\nu} \lesssim \frac{\log k}{\nu}. \tag{4.4.8}$$

We distinguish two cases.

Assume first that k is bounded below by a fixed small positive constant times ν. Then, $|\rho_{\nu,k}|$ and $|\widetilde{\rho}_{\nu,k}|$ are bounded below by a positive constant. It follows that $\frac{\partial z}{\partial \rho}$ is bounded on the segment joining $\rho_{\nu,k}$ and $\widetilde{\rho}_{\nu,k}$. This and (4.4.8) imply that

$$|\nu z_{\nu,k} - \nu \widetilde{z}_{\nu,k}| \lesssim \nu |\rho_{\nu,k} - \widetilde{\rho}_{\nu,k}| \lesssim \log k \lesssim \log |\nu \widetilde{\rho}_{k,\nu}|.$$

On the other hand, if ρ is in $i\mathbb{R}_+$, we have $z \in [1, +\infty)$ and

$$\rho = \Big(\sqrt{z^2 - 1} + \arccos \frac{1}{z}\Big) i.$$

Therefore, $|\widetilde{\rho}_{\nu,k}| \lesssim |\widetilde{z}_{\nu,k}|$. We conclude that

$$|\nu z_{\nu,k} - \nu \widetilde{z}_{\nu,k}| \lesssim \log |\nu \widetilde{z}_{\nu,k}|.$$

Assume now that k is bounded above by a small constant times ν. Then, $|\rho_{\nu,k}|$, $|\widetilde{\rho}_{\nu,k}|$ are small and $z_{\nu,k}$, $\widetilde{z}_{\nu,k}$ are close to 1 . By (4.2.2), we have $\big|\frac{\partial z}{\partial \rho}\big| \lesssim |\rho|^{-1/3}$ for ρ small. In particular, we have $\big|\frac{\partial z}{\partial \rho}\big| \lesssim \big(\frac{\nu}{k}\big)^{1/3}$ for ρ in the segment joining $\rho_{\nu,k}$ and $\widetilde{\rho}_{\nu,k}$. Hence, since $k \geq \nu^{1/4}$, we obtain

$$|\nu z_{\nu,k} - \nu \widetilde{z}_{\nu,k}| \lesssim \nu |\rho_{\nu,k} - \widetilde{\rho}_{\nu,k}| \big(\frac{\nu}{k}\big)^{1/3} \lesssim (\log k)\big(\frac{\nu}{k}\big)^{1/3} \ll \nu^{1/4+\epsilon}.$$

This completes the proof of the lemma. □

End of the proof of Proposition 4.4.5. (a) Since $\operatorname{Re} \rho_{\nu,k} < 0$, $z_{\nu,k}$ belongs to $\mathbb{C}_+ \setminus \mathbf{K}_+$. In particular, we have $|z_{\nu,k}| > \frac{1}{2}$. Therefore, when $l \geq 2r$, we have $\nu \geq 2r$ and $z_{\nu,k} \notin \overline{\mathbb{D}}(\frac{r}{\nu})$. It follows that $n_l^+(r) = 0$ for $l > 2r$.

(b) Assume that $l \leq 2r$ and $k \geq cr$ for some constant $c > 0$ large enough. Then, $\operatorname{Im} \rho_{\nu,k}$ is bounded below by a large positive constant times $\frac{r}{\nu}$. Using the definition of ρ, we obtain

that $|z_{\nu,k}|$ is almost equal to $|\rho_{\nu,k}|$. In particular, $z_{\nu,k}$ does not belong to $\overline{\mathbb{D}}(\frac{r}{\nu})$. It follows that $n_l^+(r) \leq cr$.

(c) Observe that by (b) and Proposition 4.2.4, if $l \geq L_r := 2(c\epsilon_0^{-1}r)^{1/4}$ and $k \geq \epsilon_0\nu^4$, then $z_{\nu,k}$ and $\widetilde{z}_{\nu,k}$ do not belong to $\overline{\mathbb{D}}(\frac{2r}{\nu})$. Hence, by Lemma 4.4.9, we have for $l \geq L_r$

$$n_l^+(r) \geq m_l^+(r - r^{1/4+\epsilon}) - \nu^{1/4} = m_l^+(r - r^{1/4+\epsilon}) + O(l^{1/4}).$$

This, together with Lemma 4.2.3 and Proposition 4.2.4, implies that

$$\begin{aligned}\sum_{0\leq l\leq 2r} n_l^+(r)(\dim H_l) &\geq \sum_{L_r\leq l\leq 2r} n_l^+(r)(\dim H_l)\\ &\geq \sum_{L_r\leq l\leq 2r} \big[m_l^+(r - r^{1/4+\epsilon}) + O(l^{1/4})\big](\dim H_l)\\ &\geq \frac{\mathrm{vol}(\mathbb{B})^2}{(2\pi)^d} r^d + O(r^{d-3/4+\epsilon}).\end{aligned}$$

For the converse estimate, in the same way, we have for $l \geq L_r$

$$n_l^+(r) \leq m_l^+(r + r^{1/4+\epsilon}) + \nu^{1/4} = m_l^+(r + r^{1/4+\epsilon}) + O(l^{1/4}).$$

Using parts (a) and (b), we obtain

$$\begin{aligned}\sum_{0\leq l\leq 2r} n_l^+(r)(\dim H_l) &= \sum_{L_r\leq l\leq 2r} n_l^+(r)(\dim H_l) + O(r^{d/4+3/4})\\ &\leq \sum_{0\leq l\leq 2r} \big[m_l^+(r + r^{1/4+\epsilon}) + O(l^{1/4})\big](\dim H_l) + O(r^{d/4+3/4})\\ &\leq \frac{\mathrm{vol}(\mathbb{B})^2}{(2\pi)^d} r^d + O(r^{d-3/4+\epsilon}).\end{aligned}$$

This completes the proof of the proposition. □

Define for $-\frac{\nu}{2} - 2 < k \leq 0$

$$\widehat{z}_{\nu,k} := \rho^{-1}\big(\frac{k\pi i}{\nu}\big).$$

This point belongs to $\partial\mathbf{K}_+$ and $\mathrm{Re}\,\widehat{z}_{\nu,k} \geq 0$ if and only if $k \geq -\frac{\nu}{2}$. We have the following lemma.

Lemma 4.4.10. *Let $\epsilon > 0$ be a fixed constant. Assume that ν is large enough and $-\frac{\nu}{2} + 2 < k \leq -\nu^{1/4}$. Then, we have*

$$|\nu z_{\nu,k} - \nu\widehat{z}_{\nu,k}| \leq |\nu\widehat{z}_{\nu,k}|^{1/4+\epsilon}.$$

Proof. Observe that $\big|\frac{k\pi i}{\nu}\big|$ is bounded by $\frac{\pi}{2}$. By Lemma 4.4.8, $|\rho_{\nu,k}|$ is also bounded by $\frac{\pi}{2}$ plus a small constant. Hence, by (4.2.2), we have for ρ in the segment joining $\rho_{\nu,k}$ and $\frac{k\pi i}{\nu}$

$$\Big|\frac{\partial z}{\partial\rho}\Big| \lesssim |\rho|^{-1/3} \lesssim \Big(\frac{\nu}{|k|}\Big)^{1/3}.$$

Moreover,

$$\Big|\rho_{\nu,k} - \frac{k\pi i}{\nu}\Big| \lesssim \frac{\log\nu}{\nu}.$$

It follows that

$$|\nu z_{\nu,k} - \nu\widehat{z}_{\nu,k}| \le (\log\nu)\Big(\frac{\nu}{|k|}\Big)^{1/3} \ll \nu^{1/4+\epsilon} \lesssim (\nu\widehat{z}_{\nu,k})^{1/4+\epsilon}$$

since $\widehat{z}_{\nu,k} \in \partial\mathbf{K}_+$. The lemma follows. □

End of the proof of Proposition 4.4.6. The fact that $n_l^-(r) = 0$ for $l > 2r$ is obtained as in Proposition 4.4.5. Note that by definition, we have $n_l^-(r) \le \nu \lesssim l$. We prove now the second assertion in the proposition.

Denote by $m_l^-(r)$ the number of integers k such that $-\frac{\nu}{2} + 2 \le k \le 0$ and $|\widehat{z}_{\nu,k}| \le \frac{r}{\nu}$. Since $\widehat{z}_{\nu,k} \in \partial\mathbf{K}_+$, we have $m_l^-(r) = 0$ when $l > 2r$. Observe that ρ defines a diffeomorphism between $\partial\mathbf{K}_+$ and $i[0,\pi]$. So if we define

$$\Gamma_{\nu,r} := \Big\{z \in \partial\mathbf{K}_+ :\ |z| \le \frac{r}{\nu}, \operatorname{Re} z > 0, \operatorname{Im} z > 0\Big\}$$

then we have

$$\Big|m_l^-(r) - \frac{\nu}{\pi}\operatorname{length}(\rho(\Gamma_{\nu,r}))\Big| \le 3.$$

Thus,

$$\Big|m_l^-(r) - \frac{\nu}{\pi}\int_{\Gamma_{\nu,r}} id\rho\Big| \le 3. \tag{4.4.9}$$

The last inequality implies that $m_l^-(r) \lesssim \nu \lesssim l$. Observe also that when $l > 2r$ we have $\Gamma_{\nu,r} = \varnothing$. Define $\Gamma := \{z \in \partial\mathbf{K}_+ :\ \operatorname{Re}(z) > 0\}$ and $r' := r - r^{1/4+\epsilon}$. By Lemma 4.4.10, we have

$$n_l^-(r) \ge m_l^-(r') - \nu^{1/4} = m_l^-(r') + O(l^{1/4}).$$

We deduce from the above discussion that

$$\begin{aligned}\sum_{0\le l\le 2r} n_l^-(r)(\dim H_l) &\ge \sum_{0\le l\le 2r}\big(m_l^-(r') + O(l^{1/4})\big)(\dim H_l)\\ &= \sum_{l\ge 0} m_l^-(r')(\dim H_l) + O(r^{d-3/4}).\end{aligned}$$

By (4.4.9), the last sum is equal to

$$\begin{aligned}\sum_{l\ge 0}\frac{2l^{d-1}}{\pi(d-2)!}\int_{\Gamma_{\nu,r'}} id\rho + O(r^{d-3/4}) &= \frac{2}{\pi(d-2)!}\int_{\Gamma} id\rho \sum_{\nu\le\frac{r'}{|z|}} l^{d-1} + O(r^{d-3/4})\\ &= \frac{1}{\pi(d-2)!}\int_{\partial\mathbf{K}_+} id\rho \sum_{\nu\le\frac{r'}{|z|}} l^{d-1} + O(r^{d-3/4})\\ &= \frac{r'^d + O(r'^{d-1})}{\pi d(d-2)!}\int_{\partial\mathbf{K}_+}\frac{id\rho}{|z|^d} + O(r^{d-3/4})\\ &= \frac{r^d}{\pi d(d-2)!}\int_{\partial\mathbf{K}_+}\frac{id\rho}{|z|^d} + O(r^{d-3/4+\epsilon}).\end{aligned}$$

We conclude that

$$\sum_{0\leq l\leq 2r} n_l^-(r)(\dim H_l) \geq \frac{r^d}{\pi d(d-2)!}\int_{\partial \mathbf{K}_+} \frac{|1-z^2|^{1/2}}{|z|^{d+1}}|dz| + O(r^{d-3/4+\epsilon}).$$

We obtain the converse inequality in the same way using that

$$n_l^-(r) \leq m_l^-(r'') + \nu^{1/4} = m_l^-(r'') + O(l^{1/4})$$

with $r'' := r + r^{1/4+\epsilon}$. This completes the proof of the proposition. □

4.5 Generic potentials in a holomorphic family

In this section, we prove Theorem 4.1.2. We need the following result which relates the asymptotic behavior of $n_V(r)$ and of $N_V(r)$. It holds for any bounded complex potential V with compact support.

Proposition 4.5.1. *Let δ and A be strictly positive constants such that $\delta < d$. Then, we have $(a) \Rightarrow (b) \Rightarrow (c)$, where*

(a) $n_V(r) = Ar^d + O(r^{d-\delta})$ *as* $r \to \infty$;

(b) $N_V(r) = \frac{Ar^d}{d} + O(r^{d-\delta})$ *as* $r \to \infty$;

(c) $n_V(r) = Ar^d + O(r^{d-\frac{\delta}{2}})$ *as* $r \to \infty$.

Proof. The proof is similar to Lemma 1 in [77]. The implication $(a) \Rightarrow (b)$ follows from the definition of $N_V(r)$. Assume that the property (b) holds. We show that (c) is also true. We only have to consider r large enough.

Choose a constant $c > 0$ such that $|dN_V(r) - Ar^d| \leq cr^{d-\delta}$. Define $\alpha := cr^{1-\frac{\delta}{2}}$. Since $n_V(r)$ is increasing in r, we have for r large enough

$$\frac{n_V(r)}{r+\alpha} \leq \frac{1}{\alpha}\int_r^{r+\alpha} \frac{n_V(t)}{t}dt \leq \frac{N(r+\alpha)-N(r)}{\alpha} + \frac{n_V(0)}{r}.$$

It follows from the above estimate on $dN_V(r) - Ar^d$ that

$$\begin{aligned}\frac{n_V(r)}{r+\alpha} &\leq \frac{A}{d}\big[(r+\alpha)^d - r^d\big] + \frac{2c(r+\alpha)^{d-\delta}}{d\alpha} + \frac{n_V(0)}{r}\\ &\leq A(r+\alpha)^{d-1} + \frac{2c(r+\alpha)^{d-\delta}}{d\alpha} + \frac{n_V(0)}{r}\\ &\leq A(r+\alpha)^{d-1} + O(r^{d-1-\frac{\delta}{2}}).\end{aligned}$$

Hence,

$$n_V(r) \leq A(r+\alpha)^d + O(r^{d-\frac{\delta}{2}}) \leq Ar^d + O(r^{d-\frac{\delta}{2}}).$$

In the same way, using the inequalities

$$\frac{N(r)-N(r-\alpha)}{\alpha} \leq \frac{1}{\alpha}\int_{r-\alpha}^r \frac{n_V(t)}{t}dt \leq \frac{n_V(r)}{r-\alpha}$$

we obtain

$$n_V(r) \geq Ar^d + O(r^{d-\frac{\delta}{2}}).$$

The proposition follows. □

Following Christiansen [15], we will reduce the problem to the study of a family of plurisubharmonic (p.s.h. for short) functions. The reader will find in Demailly [17] and Lelong-Gruman [50] basic properties of p.s.h. functions.

Recall that a function $\Phi : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *p.s.h.* if it is not identically equal to $-\infty$ and if its restriction to each holomorphic disc is either subharmonic or equal to $-\infty$. A subset of Ω is *pluripolar* if it is contained in the pole set $\{\Phi = -\infty\}$ of a p.s.h. function Φ. The following lemma is crucial for the proof of Theorem 4.1.2.

Lemma 4.5.2. *Let Φ_n, $n = 1, 2, \ldots$, be a sequence of p.s.h. functions on a domain Ω of $\mathbb{C}^p$. Assume there are constants $c > 0$ and $\gamma > 1$ such that $\Phi_n \leq cn^{-\gamma}$ on Ω and $\Phi_n(\vartheta_0) > -cn^{-\gamma}$ for some point $\vartheta_0 \in \Omega$. Then for every $\alpha < \gamma - 1$ there exists a pluripolar set $E \subset \Omega$ such that $\Phi_n(\vartheta) = o(n^{-\alpha})$ for every $\vartheta \in \Omega \setminus E$.*

Proof. Replacing Φ_n by $\Phi_n - cn^{-\gamma}$ allows us to assume that $\Phi_n \leq 0$. Define

$$\Phi_{[m]} := \sum_{n=1}^{m} n^\alpha \Phi_n \quad \text{and} \quad \Phi := \sum_{n=1}^{\infty} n^\alpha \Phi_n.$$

So the sequence $\Phi_{[m]}$ decreases to Φ. It is clear that $\Phi(\vartheta_0) \neq -\infty$. So Φ is a p.s.h. function. Therefore, $E := \Phi^{-1}(-\infty)$ is a pluripolar set. For $\vartheta \notin E$, we have $n^\alpha \Phi_n(\vartheta) \to 0$. The lemma follows. □

For $0 < \delta \leq 1$, define

$$\widetilde{\mathfrak{M}}_a^\delta := \Big\{ V \in L^\infty(\mathbb{B}_a, \mathbb{C}) : \ dN_V(r) - c_d a^d r^d = O(r^{d-\delta+\epsilon}) \text{ as } r \to \infty \text{ for every } \epsilon > 0 \Big\}.$$

Proposition 4.5.1 implies that

$$\mathfrak{M}_a^\delta \subset \widetilde{\mathfrak{M}}_a^\delta \subset \mathfrak{M}_a^{\delta/2}.$$

So Theorem 4.1.2 is a consequence of the following result.

Theorem 4.5.3. *Let Ω be a connected open set of $\mathbb{C}^p$. Let V_ϑ be a uniformly bounded family of potentials in $L^\infty(\mathbb{B}_a, \mathbb{C})$ depending holomorphically on the parameter $\vartheta \in \Omega$. Suppose there are $\vartheta_0 \in \Omega$ and $0 < \delta \leq 1$ such that $V_{\vartheta_0} \in \widetilde{\mathfrak{M}}_a^\delta$. Then there is a pluripolar set $E \subset \Omega$ such that $V_\vartheta \in \widetilde{\mathfrak{M}}_a^{\delta/2}$ for all $\vartheta \in \Omega \setminus E$.*

Proof. Define for r large enough

$$\Psi(r, \vartheta) := \frac{1}{2\pi r^d} \int_0^{2\pi} \log |s_{V_\vartheta}(re^{i\theta})| d\theta - \frac{c_d a^d}{d}.$$

Since V_ϑ depends holomorphically on ϑ, the function s_{V_ϑ} depends also holomorphically on ϑ. Hence, $\Psi(r, \vartheta)$ is p.s.h. on ϑ.

Fix a small constant $\epsilon > 0$. By Theorem 4.3.1 and Proposition 4.3.2, we have

$$\Psi(r,\vartheta) \lesssim \frac{\log r}{r} \quad \text{and} \quad \Psi(r,\vartheta_0) \gtrsim -r^{-\delta+\epsilon}.$$

Define for $k := \frac{2}{\delta}$

$$\Phi_n(\vartheta) := \Psi(n^k,\vartheta).$$

Then

$$\Phi_n(\vartheta) \lesssim (\log n) n^{-k} \quad \text{and} \quad \Phi_n(\vartheta_0) \gtrsim n^{-k(\delta-\epsilon)}.$$

Lemma 4.5.2 implies that for ϑ outside a pluripolar set E_ϵ we have

$$\Phi_n(\vartheta) = o(n^{-k(\delta-\epsilon)+1+\epsilon}) = o(n^{-1+\frac{2\epsilon}{\delta}+\epsilon}).$$

Hence, by Proposition 4.3.2, we have for $r = n^k$ and for $\vartheta \notin E_\epsilon$

$$N_{V_\vartheta}(r) = \frac{c_d a^d r^d}{d} + O(r^{d-\frac{\delta}{2}+\epsilon+\frac{\epsilon\delta}{2}}). \tag{4.5.1}$$

Since $N_{V_\vartheta}(r)$ is increasing in r, we deduce that if $n^k \le r \le (n+1)^k$

$$\begin{aligned}\left|N_{V_\vartheta}(r) - \frac{c_d a^d r^d}{d}\right| &\lesssim (n+1)^{kd} - n^{kd} + O(r^{d-\frac{\delta}{2}+\epsilon+\frac{\epsilon\delta}{2}}) \\ &\lesssim n^{kd-1} + O(r^{d-\frac{\delta}{2}+\epsilon+\frac{\epsilon\delta}{2}}) \\ &\lesssim O(r^{d-\frac{\delta}{2}+\epsilon+\frac{\epsilon\delta}{2}}).\end{aligned}$$

So the property (4.5.1) holds for $r \to \infty$ with $r \in \mathbb{R}_+$.

Finally, define $E := \cup_{n=1}^\infty E_{1/n}$. This is a pluripolar set, see e.g. [50]. We have for all $\vartheta \notin E$ and $\epsilon > 0$

$$N_{V_\vartheta}(r) = \frac{c_d a^d r^d}{d} + O(r^{d-\frac{\delta}{2}+\epsilon}) \quad \text{as} \quad r \to \infty.$$

This completes the proof of the theorem. □

Remark 4.5.4. Let K be a compact subset of Ω. Let μ be the Lebesgue measure on Ω or on a totally real affine space through ϑ_0. Then there are positive constants $c > 0$ and $\alpha > 0$ such that

$$\int_K e^{\alpha|\Phi|} d\mu \le c,$$

see [26]. This allows to bound the size of sets of parameters in K where $N_V(r)$ does not satisfy an estimate likes $|dN_{V_\vartheta}(r) - c_d a^d r^d| \le C r^{d-\delta+\epsilon}$ for a given $\epsilon > 0$ and a large constant $C > 0$.

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