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Zhenjian Wang

► **To cite this version:**

Zhenjian Wang. Projective groups and line arrangements. General Mathematics [math.GM]. Université Côte d'Azur, 2017. English. NNT : 2017AZUR4034 . tel-01590471

**HAL Id: tel-01590471**

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Submitted on 19 Sep 2017

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École doctorale en Science Fondamentales et Appliquées EDSFA  
Unité de recherche: Laboratoire Jean-Alexandre Dieudonné

**Thèse de doctorat**  
Présentée en vue de l'obtention du  
grade de docteur en Mathématiques  
de  
UNIVERSITÉ CÔTE D'AZUR

par  
**Zhenjian WANG**

**Groupes Projectifs et Arrangements de Droites**  
**Projective Groups and Line Arrangements**

Dirigée par Alexandru DIMCA  
et codirigée par Sorin DUMITRESCU

Soutenue le 19 Juin 2017  
Devant de jury composé de:

M. Nero BUDUR	Professeur, KU Leuven	Rapporteur
M. Alexandru DIMCA	Professeur, Université Côte d'Azur	Directeur de thèse
M. Sorin DUMITRESCU	Professeur, Université Côte d'Azur	Co-directeur de thèse
M. Vincent KOZIARZ	Professeur, Université de Bordeaux	Examineur
M. Xavier ROULLEAU	Professeur, Université d'Aix-Marseille	Rapporteur



# Groupes Projectifs et Arrangements de Droites

**Résumé :** Le but de cette thèse est de considérer différentes questions sur les groupes projectifs et sur les arrangements de droites dans le plan projectif  $\mathbb{P}^2$ . Un groupe projectif est un groupe qui est isomorphe au groupe fondamental d'une variété projective lisse complexe. Pour étudier les groupes projectifs, des techniques sophistiquées de topologie algébrique et de géométrie algébrique ont été développées pendant les dernières décennies, par exemple la théorie des variétés caractéristiques combinée avec la théorie de Hodge s'est montrée être un outil puissant. Les arrangements de droites dans le plan projectif  $\mathbb{P}^2$  ont une place centrale dans l'étude des groupes projectifs. En effet, il y a beaucoup de questions ouvertes sur les groupes projectifs, et la théorie des arrangements d'hyperplans, en particulier celle des arrangements de droites, qui est un domaine très actif de recherche, peut suggérer des solutions à ces problèmes. En outre, les problèmes sur les groupes fondamentaux de complémentaires des arrangements d'hyperplans peuvent être réduits au cas des arrangements de droites, en utilisant le bien connu Théorème de Zariski du type de Lefschetz. Assez souvent, pour étudier les groupes projectifs ou quasi-projectifs, on considère d'abord les arrangements de droites pour obtenir des idées intuitives.

Dans le Chapitre 1 nous donnons une présentation du contenu de cette thèse, énumérant les résultats principaux suivant les chapitres.

Dans le Chapitre 2 nous donnons un résumé des techniques utilisées dans cette thèse. Les outils principaux sont la théorie de Hodge classique et la théorie de Hodge mixte, la théorie des variétés caractéristiques et certaines propriétés des arrangements d'hyperplans.

Nous étudions en détail les propriétés des variétés caractéristiques dans les Chapitres 3 et 4. Dans le Chapitre 3, les relations principales entre les variétés caractéristiques, les variétés de résonance et les premières variétés logarithmiques sont établies en utilisant le modèle de Gysin construit par J. Morgan. Dans le Chapitre 4 nous démontrons plusieurs théorèmes sur les sous-espaces isotropes. Nous montrons que ces théorèmes sur les sous-espaces isotropes restent valides même dans le cas des surfaces non-kähleriennes. De plus, nous obtenons des inégalités sur les nombres de Hodges qui font intervenir les variétés caractéristiques.

Dans le Chapitre 5 nous discutons en principal la géométrie de la fibre de Milnor d'une courbe plane projective, en la reliant aux propriétés des morphismes construits à partir de multineux, une classe spéciale de pinceaux de courbes. Nous montrons que la fibre générale d'un tel morphisme est connexe, donc on obtient des pinceaux irrationnels ; des généralisations sont proposées, au cas où les polynômes ne sont pas de produits de facteurs linéaires. Nous donnons après cela un critère pour décider la non-1-formalité de fibres de Milnor, ainsi que de nouveaux exemples des arrangements de droites dont la fibre de Milnor associé n'est pas 1-formelle.

Dans le Chapitre 6 nous discutons certaines surfaces algébrique associées aux arrangements de droites. Le but principal est d'obtenir des formules compactes pour les nombres de Chern des surfaces projectives lisses obtenues par compactifications des fibres de Milnor. Nous montrons ensuite que ces surfaces ne sont jamais de quotients de boules.

Dans le dernier Chapitre, nous montrons que les surfaces ainsi obtenues à partir d'un arrangement ayant seulement des points doubles et triples sont de surfaces de type général dans la classification de Kodaira. La question ouverte principale sur la topologie de la fibre de Milnor est si la monodromie associée est déterminée par la combinatoire. Nous expliquons la relation étroite entre cette question ouverte majeure et l'irrégularité des surfaces correspondantes. En outre, nous explorons les surfaces obtenues à partir d'arrangements de courbes, en compactifiant leurs fibres de Milnor dans des espaces projectifs pondérés.

Dans cette thèse nous obtenons aussi des résultats d'un intérêt indépendant, par exemple sur les morphismes définis sur un produit d'espaces projectifs dans le Chapitre 4, sur la fibre générale de certains morphismes dans le Chapitre 5 et les critères sur les surfaces de type générales au Chapitre 7.

**Mots-clés :** arrangements de droites, fibre de Milnor, théorie de Hodge, groupe fondamental, variété caractéristique, 1-formalité, surfaces projectives de type général.

# Projective Groups and Line Arrangements

**Abstract:** The objective of this thesis is to investigate various questions about projective groups and line arrangements in the projective plane  $\mathbb{P}^2$ . A projective group is a group which is isomorphic to the fundamental group of a smooth complex projective variety. To study projective groups, sophisticated techniques in algebraic topology and algebraic geometry have been developed in the passed decades, for instance, the theory of cohomology jump loci, together with Hodge theory, has been proven a powerful tool. Line arrangements in the projective plane  $\mathbb{P}^2$  are of special interest in the study of projective groups. Indeed, there are many open questions related to projective groups, and the theory of hyperplane arrangements, and in particular that of line arrangements, which is quite an active area of research, may provide insights for these problems. Furthermore, problems concerning the fundamental groups of the complements of hyperplane arrangements can be reduced to the case of line arrangements, due to the celebrated Zariski theorem of Lefschetz type. Very often, in the study of projective groups or quasi-projective groups, one usually considers line arrangements first to get some intuitive ideas.

In Chapter 1, we give an outline of the content of this thesis, containing a brief exposition of the main results in the following chapters.

In Chapter 2, we give a review of fundamental materials that we used throughout the thesis. The crucial tools are classical Hodge theory and mixed Hodge theory, theory of cohomology jump loci and the special properties of hyperplane arrangements.

We explore in detail properties of cohomology jump loci in Chapter 3 and Chapter 4. In Chapter 3, the main relations between characteristic varieties, resonance varieties and first logarithmic varieties are established via the Gysin model constructed by J. Morgan. In Chapter 4, various isotropic subspace theorems are proved. We extend the well-known isotropic subspace theorems to the case of non-Kählerian surfaces. In addition, some inequalities about Hodge numbers are given in terms of the cohomology jump loci.

In Chapter 5, we discuss mainly geometry of the Milnor fiber of a projective plane curve by showing properties of morphisms associated to multinefs, a special class of pencils of curves. We prove that the morphisms have connected generic fibers, thus they give irrational pencils; some extensions are also given, for instance, morphisms concerning homogeneous polynomials which are not necessarily linear forms. After that, we give a criterion for non-1-formality of the Milnor fiber and some new examples of line arrangements whose Milnor fibers are not 1-formal.

In Chapter 6, we discuss some algebraic surfaces associated to line arrangements. The main part is to give compact formulae for the Chern numbers of the associated projective surfaces. Then with the help of the developed surface theory, we show that the associated surfaces can never be ball quotients.

In the last Chapter, we show the associated surfaces are of general type if the line arrangements have only nodes or triple points as singularities. The open question about the topology of the Milnor fiber being combinatorially determined is also discussed. Moreover, we also make an exploration of surfaces associated to curves arrangements and compactifications of the Milnor fibers in weighted projective spaces.

In this thesis, we also prove some theorems that are of independent interest and can be used elsewhere, for instance, we prove properties concerning morphisms from products of projective spaces in Chapter 4, we show that some morphisms have generic connected fibers in Chapter 5 and we give criteria for a projective surface to be of general type in Chapter 7.

**Keywords:** line arrangements, Milnor fiber, Hodge theory, fundamental group, coho-

mology jump loci, 1-formal, projective surfaces of general type.

# Acknowledgements

First of all, I would like to thank Professor Alexandru Dimca and Professor Sorin Dumitrescu, under whose supervision I completed this thesis. Many progresses achieved in this thesis go back to the unique insights from them and my discussions with them. Furthermore, I would like to thank Professor Nero Budur and Professor Xavier Roulleau for reviewing the thesis, and Professor Vincent Koziarz for his attendance in the jury.

I would also like to show my special gratitude to Professor Alexandru Dimca and Professor Xiuxiong Chen who was my former advisor in China, because they gave the indispensable guidance to me concerning my long-term future. In 2013, I managed to come to France to pursue the Master degree although there was a huge number of difficulties. Without their help, I could have never tackled the obstacles about languages, application letters and even financial difficulties. What is more, during the last four years, Professor Dimca provided me with crucial aids not only in my studies but also in my everyday life, for instance, bank account, trains or flights and supermarkets, etc, and Professor Chen is always caring a lot about my PhD career and my future.

During my stay in France, I enjoyed quite different sorts of culture compared to Chinese culture. I met many people from different countries, with different hobbies and having different characteristics. I also attended many conferences meeting a numbers of mathematicians from different communities. They study different topics and give different ideas about mathematics and even the world. In addition, I visited many countries, like the US, Italy, Spain, Germany, Poland, etc. Having experienced so many things, I have my horizons broadened; although coming from one of the least developed regions in China, I have already enlarged my former narrow sights. It was unimaginable four years ago that I could achieve all these progresses, and these progresses could happen only due to the help I received from Professor Chen and Professor Dimca. It is said that there is hard time during our life, and then there is unexpected help for us. They are such helpers for me.

This thesis was done in Laboratoire J.A. Dieudonné, which provided me with financial support and stimulating research atmosphere. Here I would like to thank the everybody in laboratory and the colleagues in the department, especially the secretaries, like Isabelle De Angelis, Angélique Guitard and Clara Salaun. As a foreigner in France, I do not know much about local affairs, and they help me quite a lot to manage to have a good life and excellent study conditions here. Moreover, I would like to thank my officemates Hacem Zelaci and Rinel Foguen Tchuendom for their help with my French.

Last but not the least, I would like to thank my parents for their support. I am not an obedient son, thus I also thank them for their understanding and tolerance.



# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Motivations and Contexts . . . . .	1
1.2	Conventions and Notations . . . . .	2
1.3	Contents of Thesis . . . . .	6
<b>2</b>	<b>Preliminaries</b>	<b>9</b>
2.1	Hodge theory and mixed Hodge theory . . . . .	9
2.2	Basics on cohomology jump loci . . . . .	15
2.3	Hyperplane arrangements . . . . .	19
<b>3</b>	<b>First Logarithmic Resonance Varieties</b>	<b>27</b>
3.1	History . . . . .	27
3.2	Case 1: $H^1$ having pure weight 2 . . . . .	30
3.3	Case 2: $X$ being compact . . . . .	30
3.4	Case 3: $X$ being a punctured curve . . . . .	31
3.5	Case 4: general setting . . . . .	33
<b>4</b>	<b>Isotropic Subspace Theorems and Inequalities of Hodge Numbers</b>	<b>36</b>
4.1	Castelnuovo-De Franchis theorems . . . . .	36
4.2	Isotropic subspace theorems . . . . .	40
4.3	Morphisms from products of projective spaces . . . . .	43
4.4	Inequalities of Hodge numbers . . . . .	46
<b>5</b>	<b>Geometry of Milnor Fibers of Line Arrangements</b>	<b>50</b>
5.1	Topological properties of Milnor fibers . . . . .	51
5.2	Morphisms associated to multinets . . . . .	56
5.3	1-formality of Milnor fiber . . . . .	59
5.4	Cohomology jump loci of Milnor fiber . . . . .	60
<b>6</b>	<b>Surfaces associated to line arrangements and Chern numbers</b>	<b>63</b>
6.1	Basic topology . . . . .	63
6.2	Intersection theory on surfaces . . . . .	64
6.3	Projective surfaces and the Miyaoka–Yau inequality . . . . .	70
6.4	Surfaces associated to line arrangements . . . . .	75

6.5	Resolution of singularities . . . . .	80
6.6	Examples of resolutions of surfaces . . . . .	84
6.7	Numerical invariants for minimal resolutions . . . . .	89
6.8	The minimal resolution is not a ball quotient . . . . .	97
<b>7</b>	<b>Surfaces of general type associated to line arrangements</b>	<b>99</b>
7.1	General type criteria . . . . .	99
7.2	line arrangements with nodes or triple points . . . . .	102
7.3	Hodge numbers and Chern numbers . . . . .	105
7.4	Extensions of irrational pencils of the Milnor fiber . . . . .	111
7.5	Curve arrangements . . . . .	112
	<b>Bibliography</b>	<b>120</b>

## 1.1 Motivations and Contexts

The study of geometry and topology of Kähler manifolds has quite a long history. On the one hand, the cohomology theory of Kähler manifolds, which arises from the classical Hodge theory, is much richer than that of the more general differential or Riemannian manifolds. The Hodge decomposition and Lefschetz decomposition give much restrictive conditions on the geometry and topology for a manifold being compact Kähler, for an excellent exposition of Hodge theory on compact Kähler manifolds, we refer to [Voi1]; more generally, for quasi-projective manifolds, mixed Hodge theory can be established and similar results can be obtained, see [PeS],[Del71],[Del74] and also Appendix C in [Dim92] for a short review. On the other hand, the class of Kähler manifolds contains lots of differential and topological spaces we encounter frequently, for example, projective manifolds and complements of line arrangements in the complex projective plane  $\mathbb{P}^2$ . These manifolds provide a number of interesting examples in geometry and topology.

As is well-known, every finitely presented group is the fundamental group of a compact differential manifold of dimension greater than or equal to four, see [Man]. However, not every group can be the fundamental group of a quasi-projective manifold. A finitely presented group is said to be *projective* (resp. *quasi-projective*) if it can be realized as the fundamental group of a projective (resp. quasi-projective) manifold. The study of cohomology jump loci gives specific conditions for a group to be quasi-projective. Together with the mixed Hodge theory, this study has proven to be a powerful tool to study quasi-projective groups. In such a study of quasi-projective manifolds, hyperplane arrangements stand out.

In the theory of hyperplane arrangements, one of the most remarkable feature is to deeply understand the geometry and topology of the Milnor fiber  $F$ , for example, its Betti numbers or the characteristic polynomial of the monodromy, and study whether they are determined by the combinatorics of the arrangement. As a matter of fact, investigations on the monodromy action from different points of view often give various results on other aspects, e.g., the Betti numbers. Moreover, if the study concerns only the fundamental group of the complement of the arrangement, then the discussions can often be further reduced to the case of line arrangements in  $\mathbb{P}^2$ . In dimension 2, the Milnor fiber is an algebraic surface, so surface theory may also be applied.

One can construct interesting algebraic surfaces from line arrangements. A typical example in this direction is the Hirzebruch surface constructed from a line arrangement  $\mathcal{A}$ , from study of which, some new and very useful inequalities about the combinatorics of  $\mathcal{A}$  are obtained, see [Hi]. The Hirzebruch surface does not seem to be closely related to the Milnor fiber,

however, we can construct surfaces directly from the Milnor fiber. The hope is that using the very developed surface theory, more properties of the Milnor fiber will be revealed.

Our basic philosophy in this thesis is to apply various tools from different branches of mathematics to study the Milnor fiber of an arrangement. For a line arrangement  $\mathcal{A}$  in  $\mathbb{P}^2$ , its Milnor fiber  $F$  is on the one hand is a quasi-projective manifold, hence we may use mixed Hodge theory to study its cohomology jump loci; on the other hand,  $F$  is also a surface, for which the well-known surface theory may help. Moreover, a line arrangement is a quite special object in topology and much has been well understood about the complement  $M = M(\mathcal{A})$  for the arrangement  $\mathcal{A}$ , see [OT]. Since  $F$  is clearly a Galois cover of  $M$ , it is a natural idea to study  $F$  using the known results on  $M$  and studying the monodromy associated to the Galois cover. In addition, although  $F$  is a smooth manifold, it has a compactification  $\overline{F}$  in  $\mathbb{P}^3$  which has isolated singularities; some properties of  $\overline{F}$  reflect the properties of  $F$ , thus singularity theory can also be applied to study  $F$ . Of course, tools from differential geometry, differential topology, algebraic geometry and analytic geometry can be applied.

Therefore, our study of the Milnor fiber is via a mixture of different theories. Naturally, during such a study of the Milnor fiber  $F$  using various tools, we can hope not only to understand better its topology and geometry, but to obtain some results which are true in more general settings and which can be applied elsewhere to study quite different objects. For example, we prove in Chapter 5, Section 1, that some morphisms have connected generic fibers, and these results can be applied to study deformations of hypersurfaces, see also [Wang].

## 1.2 Conventions and Notations

In this section, we give an outlined description of the conventions and notations that will be applied throughout the whole thesis.

A real  $n$ -dimensional manifold will be a connected  $C^\infty$  manifold that appears in differential geometry, namely it is a Hausdorff second countable topological space  $X$  equipped with an open cover

$$X = \bigcup_{i \in I} U_i,$$

and a system of homeomorphisms  $\phi_i : U_i \rightarrow V_i \subseteq \mathbb{R}^n, i \in I$  where  $V_i$  is an open set in  $\mathbb{R}^n$  such that whenever  $U_i \cap U_j \neq \emptyset$ ,

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$$

is  $C^\infty$ . A complex manifold is defined similarly, with the main point being that  $\mathbb{R}^n$  is replaced by  $\mathbb{C}^n$  and  $\phi_i \circ \phi_j^{-1}$  is biholomorphic map. Typical examples of complex manifolds are: the complex Euclidean space  $\mathbb{C}^n$  and the complex projective space  $\mathbb{P}^n$ .

Let  $X$  be a real manifold, the sheaf of complex valued differential forms of degree  $p$  will be denoted by  $\mathcal{A}_X^p$ , while the sheaf of real valued differential forms of degree  $p$  is denoted by  $\mathcal{A}_{X,\mathbb{R}}^p$ . Also denote  $\Gamma(X, -) : \mathcal{F} \mapsto \Gamma(X, \mathcal{F})$  the global section functor acting on sheaves over  $X$ , which to each sheaf  $\mathcal{F}$ , associates the set of global sections  $\Gamma(X, \mathcal{F})$ . Denote

$$A_X^p = \Gamma(X, \mathcal{A}_X^p).$$

Denote by  $\Omega_{X,\mathbb{R}}^p$  the vector bundle of real  $p$ -forms.

When  $X$  is a complex manifold, we have the sheaf of holomorphic functions on  $X$ , denoted by  $\mathcal{O}_X$ . The sheaf of meromorphic functions on  $X$  will be denoted by  $\mathcal{M}_X$ . We also have the sheaf of holomorphic forms of degree  $p$ , denoted by  $\Omega_X^p$ , which is also, up to an identification,

the holomorphic bundle of holomorphic  $p$ -forms. In addition, we denote  $\mathcal{A}_X^{p,q}$  the sheaf of differential forms of type  $(p, q)$  on  $X$ , while we denote  $\Omega_X^{p,q}$  the corresponding vector bundles.

For cohomology of  $X$ , we use  $H^k(X)$  to denote the  $k$ -th De Rham cohomology or singular cohomology of  $X$  with complex coefficients. More generally, let  $A$  be a ring, for instance  $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ , denote by  $H^k(X, A)$  be the  $k$ -th singular cohomology with coefficients in  $A$ .

For any sheaf  $\mathcal{F}$  over  $X$ , denote  $H^k(X, \mathcal{F})$  the  $k$ -th cohomology of  $X$  with coefficients in  $\mathcal{F}$ . More generally, let  $A$  be any ring and  $\mathcal{F}^\bullet$  be a bounded from below complex of sheaves of  $A$ -modules on  $X$ , denote by  $\mathbb{H}^k(X, \mathcal{F}^\bullet)$  be the  $k$ -th hypercohomology; this cohomology is given as follows: let  $I^\bullet$  be an injective resolution of  $\mathcal{F}^\bullet$ , then

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) = H^k(\Gamma(X, I^\bullet)).$$

A complex manifold  $X$  is said to be a quasi-projective (resp. projective) manifold if it is also a quasi-projective (resp. projective) variety. Let  $X$  be a quasi-projective manifold. A projective manifold  $\bar{X}$  is said to be a good compactification of  $X$ , if there is an injective morphism  $j : X \hookrightarrow \bar{X}$  such that  $j(X)$  is a Zariski open subset of  $\bar{X}$  and in addition,  $D = \bar{X} - j(X)$  is a simple normal crossing divisor in  $\bar{X}$ , i.e.,  $D$  admits an irreducible component decomposition  $D = \bigcup_i D_i$  such that each  $D_i$  is a smooth hypersurface in  $\bar{X}$  and for any  $p \in D$ , there exists a holomorphic coordinate neighbourhood  $(U, (z_i))$  centered at  $p$  for  $\bar{X}$  in which  $D \cap U$  is a union of coordinate hyperplanes:

$$D \cap U = \{z_1 z_2 \cdots z_k = 0\}.$$

Very often, we will not explicitly write out the morphism  $j$ , and simply identify  $X$  with  $j(X)$ .

Now let  $\bar{X}$  be a good compactification of a quasi-projective manifold  $X$  and let  $D = \bar{X} - X$ . The sheaf  $\Omega_{\bar{X}}^p(*D)$  is the sheaf of meromorphic  $p$ -forms on  $\bar{X}$  with poles along  $D$ . Alternatively, since  $D$  is a simple normal crossing divisor, we may choose a local coordinate neighbourhood  $(U, (z_i))$  as above in  $\bar{X}$  such that

$$D \cap U = \{f_D = 0\}$$

with  $f_D = z_1 z_2 \cdots z_k$ . Then

$$\Omega_{\bar{X}}^p(*D)(U) = \left\{ \frac{\omega}{f_D^\ell} \quad : \quad \omega \in \Omega_{\bar{X}}^p(U), \ell \geq 0 \right\} = \varinjlim_{\ell} \Omega_{\bar{X}}^p(\ell D).$$

Clearly,  $(\Omega_{\bar{X}}^\bullet(*D), d)$  is a complex of sheaves. The sheaf of logarithmic poles of degree  $p$  along  $D$ , denoted by  $\Omega_{\bar{X}}^p(\log D)$ , is defined as follows:

$$\Omega_{\bar{X}}^p(\log D)(U) = \left\{ \eta \in \Omega_{\bar{X}}^p(*D)(U) \quad : \quad f_D \eta, f_D d\eta \in \Omega_{\bar{X}}^p(U) \right\};$$

in view of the fact that  $D \cap U = \{z_1 z_2 \cdots z_k = 0\}$ , we obtain

$$\Omega_{\bar{X}}^1(\log D)(U) = \mathcal{O}_{\bar{X}}(U) \left\langle \frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}, dz_{k+1}, \dots, dz_n \right\rangle$$

and

$$\Omega_{\bar{X}}^p(\log D)(U) = \bigwedge^p \Omega_{\bar{X}}^1(\log D)(U).$$

In particular,  $\Omega_{\bar{X}}^p(\log D)(U)$  is locally free for every  $p$ .  $(\Omega_{\bar{X}}^\bullet(\log D), d)$  is also a complex of sheaves, and in fact it computes the cohomology of  $X$ , namely, we have

$$H^k(X) = \mathbb{H}^k(\bar{X}, \Omega_{\bar{X}}^\bullet(\log D)).$$

The complex  $\Omega_{\mathbb{X}}^{\bullet}(\log D)$  admits two natural filtrations, which induces two filtrations on  $H^k(X)$  and gives the mixed Hodge structure on  $H^k(X)$ . For more details, see [Voi1] and Chapter 2 of this thesis.

Now we turn to the most important object in this thesis: hyperplane arrangement. A hyperplane arrangement  $\mathcal{A}$  in  $\mathbb{P}^n$  is a finite union of hyperplanes:

$$\mathcal{A} = \{H_1, \dots, H_d\}$$

where  $H_i : L_i = 0$  is a hyperplane in  $\mathbb{P}^n$  defined by a linear form  $L_i$ . Denote  $|\mathcal{A}| = d$  the number of hyperplanes in  $\mathcal{A}$ . The defining polynomial of  $\mathcal{A}$  is

$$Q = Q(\mathcal{A}) = \prod_{j=1}^d L_j.$$

Let  $V(Q)$  be the projective variety in  $\mathbb{P}^n$  defined by  $Q = 0$ , which is also the union of hyperplanes in  $\mathcal{A}$ . The complement of  $\mathcal{A}$  is defined as

$$M = M(\mathcal{A}) = \mathbb{P}^n \setminus V(Q) = \mathbb{P}^n \setminus \bigcup_{j=1}^d H_j,$$

while the Milnor fiber of  $\mathcal{A}$  is

$$F = F(\mathcal{A}) : Q = 1$$

in  $\mathbb{C}^{n+1}$ . It is clear that we have a natural map

$$p : F \rightarrow M$$

induced by the projection  $\mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ . Clearly,  $p$  is a Galois cover of degree  $d$ .

Hyperplane arrangements have quite special properties compared to other algebraic or geometric objects. There has been an excellent book on hyperplane arrangements by Orlik and Terao [OT], in which many combinatorial properties of  $\mathcal{A}$ , properties of topology and geometry of  $M$  and their relations are given.

Here is a list of symbols we will use in the sequel.

A manifold is always connected unless otherwise stated.

$\mathbb{P}^n$	complex projective space of dimension $n$
$H^k(X, A)$	singular cohomology of $X$ with coefficients in a ring $A$
$H^k(X)$	simpler notation for $H^k(X, \mathbb{C})$
$H^k(X, \mathcal{F})$	cohomology of $X$ with coefficients in sheaf $\mathcal{F}$
$\mathbb{H}^k(X, \mathcal{F}^\bullet)$	hypercohomology of $X$ for a complex of sheaves $\mathcal{F}^\bullet$
$F^p H^k(X)$	Hodge filtration for $H^k(X)$
$W_m H^k(X)$	Weight filtration for $H^k(X)$
$Gr_F^p$	$p$ -th graded piece for the filtration $F^\bullet$
$Gr_m^W$	$m$ -th graded piece for the filtration $W_\bullet$
$j : X \hookrightarrow \bar{X}$	$\bar{X}$ is a good compactification of $X$
$\chi(X)$	the topological Euler characteristic of $X$
$f : X \rightarrow C$	an irrational pencil where $C$ is a smooth curve with $\chi(C) < 0$

In Chapters 2-4, the following symbols are used:

$\mathcal{V}_1(X)$	the first characteristic variety of $X$
$\mathcal{R}_1(X)$	the first resonance variety of $X$
$TC_1(\mathcal{V}_1(X))$	the tangent cone at the identity 1 of $\mathcal{V}_1(X)$
$L\mathcal{R}_1(X)$	the first logarithmic resonance variety of $X$
$h^{1,1}(X)$	$\dim H^1(X, \Omega_X^1)$ when $X$ is compact
$h^{2,0}(X)$	$\dim H^0(X, \Omega_X^2)$ when $X$ is compact
$F^k(X)$	simpler notation for $H^0(\bar{X}, \Omega_{\bar{X}}^k(\log X))$ when $X$ is quasi-projective

In Chapters 5-7, the following symbols are fixed:

$\mathcal{A}$	a line arrangement in $\mathbb{P}^2$ or hyperplane arrangement in $\mathbb{P}^n$
$Q$	defining equation of $\mathcal{A}$
$M$	complement of $\mathcal{A}$
$F$	Milnor fiber of $\mathcal{A}$
$p : F \rightarrow M$	the natural Galois cover
$h : F \rightarrow F$	the monodromy map $h(x) = \exp(2\pi\sqrt{-1}/d)x$
$h^*$	the monodromy action in cohomology
$H^k(F)_a$	eigenspace of $h^*$ in $H^k(F)$
$H^k(F)_1$	invariant subspace for $h^*$
$H^k(F)_{\neq 1}$	direct sum of eigenspaces of eigenvalues $\neq 1$

In Chapters 6-7, the following symbols are fixed:

$\bar{F}$	compactification of $F$ in $\mathbb{P}^3$
$\tilde{F}$	minimal desingularization of $\bar{F}$
$t_r$	number of singular points of multiplicity $r$
$K_X$	the canonical divisor
$c_1^2(X), c_2(X)$	Chern numbers
$MY(X)$	Miyaoka-Yau number, equal to $3\chi(X) - K_X^2$
$DCI, DCII$	differences of Chern numbers
$DCI_{r,d}, DCII_{r,d}$	differences of Chern numbers for a singular point of multiplicity $r$

## 1.3 Contents of Thesis

The content of this thesis is closely related to the study of the above aspects, namely, cohomology jump loci, Hodge theory and theory of line arrangements. We shall recall and extend some natural constructions of projective manifolds from line arrangements in  $\mathbb{P}^2$ .

For the topology of quasi-projective manifolds, we achieve some inequalities on Hodge numbers by considering characteristic varieties, resonance varieties and their extensions. After that, we focus our attention to some special quasi-projective manifolds, e.g., the Milnor fibers of line arrangements.

### 1.3.1 Chapter 2

In Chapter 2, we mainly review some materials of fundamental importance for our study, namely, mixed Hodge theory and cohomology jump loci; we also give several examples of interesting objects in our study, for instance, formal spaces and hyperplane arrangements. For hyperplane arrangements, we concentrate on the affine Milnor fiber and the monodromy action. These are classical results and are in fact well-known. We state some of the basic results and list some references for interested readers to consult for the proofs.

### 1.3.2 Chapter 3

In Chapter 3, we present a new point of view on the first logarithmic resonance varieties for smooth quasi-projective varieties, which are defined by A. Dimca [Dim10]. With the help of the Gysin model of a quasi-projective variety, we discuss the relations between the first logarithmic variety and the tangent cone at the identity of the first characteristic variety.

The main point is to prove these results without using the relations between characteristic varieties and irrational pencils described by Arapura [Ara] and Bauer [Bau]. The only ingredient we use is the basic structure theorem about the characteristic variety  $\mathcal{V}_1(X)$  established in [DP14],[DPS09], namely, the tangent cone  $TC_1(\mathcal{V}_1(X))$  is a mixed Hodge substructure of  $H^1(X)$ . Then using computations via the Gysin model, we establish the relations between the first logarithmic variety  $LR_1(X)$  and the tangent cone  $TC_1(\mathcal{V}_1(X))$ .

The key tool to give such relations is Proposition 3.1.2. Then we first consider several special interesting cases separately:  $H^1(X)$  of pure weight 2, see Proposition 3.2.1;  $X$  being projective, see Corollary 3.3.2;  $X$  being a punctured curve, see Corollary 3.4.1. Finally we consider the general case, see Proposition 3.5.1 and Corollary 3.5.2.

### 1.3.3 Chapter 4

In Chapter 4, we aim at giving some inequalities on Hodge numbers.

We first prove some basic theorems, especially isotropic subspace theorems, that are very useful in the study of cohomology jump loci. These theorems, e.g., Theorem 4.1.1, can be found in some books and papers, see [BHPV],[Ara]; however, we give much more detailed proofs. Moreover, we provide several theorems for compact smooth surfaces that are not necessarily Kähler, therefore the corresponding results are not contained in the classical forms which are stated only for Kähler manifolds. See Theorem 4.1.6 and Theorem 4.1.11, Theorem 4.2.2.

Furthermore, we prove some properties for morphisms with sources products of projective spaces, see Lemma 4.3.1 and Lemma 4.3.2; these properties will be used, together with

isotropic subspace theorems, to deduce inequalities about Hodge numbers; they are also interesting in their own right.

Finally, we prove some inequalities on Hodge numbers, achieving our aim. We first prove Theorem 4.4.3 and Corollary 4.4.9, which extend the case known before of compact Kähler manifolds without any irrational pencils described in Corollary 4.4.5. Furthermore, we show that similar results hold for smooth quasi-projective varieties, see Theorem 4.4.13 and Corollary 4.4.16, where the role of  $\mathcal{V}_1(X)$  is replaced by  $L\mathcal{V}_1(X)$ .

### 1.3.4 Chapter 5

The main theme in Chapter 5 is to understand the geometry and topology of the Milnor fiber  $F$  of a line arrangement  $\mathcal{A}$  in  $\mathbb{P}^2$ . However, we prove much more general results that hopefully have in their own interest.

We first investigate the connectivity of the generic fiber in a pencil of projective hypersurfaces, see Lemma 5.1.1. Then we prove a number of technical results, see Lemmata 5.1.4, 5.1.9, 5.1.10.

Next we specialize to the case of a line arrangement  $\mathcal{A}$  which carries a multinet structure. Let  $M = M(\mathcal{A})$  be the complement of the line arrangement  $\mathcal{A}$  in  $\mathbb{P}^2$ . We prove that the associated morphism  $f : M \rightarrow C$  (where  $C$  is obtained from  $\mathbb{P}^1$  by deleting  $\geq 3$  points) has a generic connected fiber, see Proposition 5.2.1.

Moreover, starting from  $f : M \rightarrow C$ , there is natural way to construct a morphism  $g : F \rightarrow H$ , where  $H$  is a smooth curve of negative Euler characteristic; such a construction was introduced by Dimca and Papadima [DP11].

Our first main result here is Proposition 5.2.3, which gives sufficient conditions for  $g$  to be itself an irrational pencil, i.e., its generic fiber is connected. Using this result, we show that the Milnor fiber of any monomial arrangement

$$\mathcal{A}(m, m, 3) : (x^m - y^m)(y^m - z^m)(z^m - x^m) = 0, \quad m \geq 3$$

is not 1-formal, generalizing the case  $m = 3$  treated by Zuber in [Zub]. This is done by showing that in these cases  $g^*W_1H^1(H) \neq W_1H^1(F)$ . On the other hand, when

$$g^*W_1H^1(H) = W_1H^1(F)$$

holds, then we prove that the multinet structure on  $\mathcal{A}$  is unique, see Theorem 5.4.3 and Corollary 5.4.4. Recall that the Ceva arrangement  $\mathcal{A}(3, 3, 3)$  has 4 distinct multinet structures, see Section 2.3.10 (4), hence the above criterion is quite sharp.

### 1.3.5 Chapter 6

One of the main open questions in hyperplane arrangement theory is whether the topology of the Milnor fiber  $F$ , e.g., its Betti numbers or the characteristic polynomial of the monodromy action on each  $H^m(F)$ , are determined by the combinatorics. In Chapter 6, we consider the compactification  $\overline{F}$  of  $F$  in  $\mathbb{P}^3$  in the case of a line arrangement  $\mathcal{A}$  in  $\mathbb{P}^2$ . If  $\widetilde{F}$  is the minimal desingularization of  $\overline{F}$ , then we compute the Chern numbers of  $\widetilde{F}$  in terms of the combinatorics. The hope is that using the classification theory of smooth projective surfaces, we can better understand the topology of the Milnor fiber  $F$  in this case.

In order to compute the Chern numbers

$$c_1^2(\widetilde{F}) = K_{\widetilde{F}}^2, \quad c_2(\widetilde{F}) = \chi(\widetilde{F}),$$

we first determine the corresponding values for the singular surface  $\bar{F}$  in (6.3.2) and (6.3.3). Then we express the differences

$$DCI = K_{\bar{F}}^2 - K_{\tilde{F}}^2, \quad DCII = \chi(\tilde{F}) - \chi(\bar{F})$$

in terms of contributions coming from the singularities of  $\bar{F}$ , which are in bijection with the multiple points of the line arrangement  $\mathcal{A}$ . Hence, one writes

$$DCI = \sum_r t_r DCI_{r,d}, \quad DCII = \sum_r t_r DCII_{r,d}$$

where  $t_r$  is the number of singular points of multiplicity  $r \geq 2$  in  $\mathcal{A}$  and  $d = |\mathcal{A}|$  is the number of lines in  $\mathcal{A}$ .

The hard part, done in Section 6.7, is to find compact formulae for the invariants  $DCI_{r,d}$  and  $DCII_{r,d}$ , without the implicit use of the associated continued fractions coming from the resolution of weighted homogeneous surface singularities recalled in Theorem 6.5.2.

A by-product of these computations is the fact that  $\tilde{F}$  is not a ball quotient, see Theorem 6.8.1 and Remark 6.8.3.

### 1.3.6 Chapter 7

In Chapter 7, we use the results in Chapter 6 to show that many of the surfaces  $\tilde{F}$  associated to line arrangements  $\mathcal{A}$  are of general type. We also explicitly compute the Hodge numbers of several such surfaces.

Using two simple criteria, Proposition 7.1.1 and Proposition 7.1.2, which may be of independent interest, we show that some of the surfaces  $\tilde{F}$  are of general type. A result of this type is Theorem 7.2.5, where line arrangements with only nodes and triple points are discussed.

Then we explain in Section 7.3 that the open question about the topology of  $F$  being determined by the combinatorics mentioned above is partially equivalently to the question whether the Hodge numbers of the smooth surface  $\tilde{F}$  are determined by the combinatorics. Example 7.2.2 shows that the numbers of multiple points are enough to determine the Chern numbers of  $\tilde{F}$ , but are not enough to determine the Betti numbers of  $\tilde{F}$ .

A similar phenomenon is that the numbers  $t_r$  of multiple points of  $\mathcal{A}$  determine the spectrum of the line arrangement  $\mathcal{A}$ , see [BS], but not the Poincaré-Deligne polynomial, see [Dim10P].

Next, we discuss whether an irrational pencil  $g : F \rightarrow H$  considered in Chapter 5 can be extended to a morphism  $\bar{g} : \tilde{F} \rightarrow \bar{H}$ , where  $\bar{H}$  is a smooth compactification of  $H$ . This is done in Section 7.4

Finally, our computations of Chern numbers apply to a minimal resolution of singularities of a cyclic cover of  $\mathbb{P}^2$  ramified over any curve having only strict ordinary  $r$ -multiple points. In particular, we can apply this technique to

- (i) the curves considered by Bailet [Bai16] and get in this way surfaces with larger  $b_1(\tilde{F})$ .
- (ii) arrangements of nonrational curves in  $\mathbb{P}^2$  and hopefully obtain some ball quotients.

An attempt in this direction is done in Section 7.5.

The study of geometry and topology of (connected) Kähler manifolds has quite a long history. On the one hand, the cohomology theory of a Kähler manifold, which arises from the classical Hodge theory, is much more fruitful than that of a more general differential or Riemannian manifold. The Hodge decomposition and Lefschetz decomposition of the cohomology give much restrictive conditions on the geometry and topology for a manifold being compact Kähler; more generally, for a quasi-projective manifold, mixed Hodge theory can be established and similar restrictions can be obtained. On the other hand, the class of Kähler manifolds contains lots of differential and topological spaces we encounter frequently, for example, projective manifolds and complements of line arrangements in the projective plane. These manifolds provide a number of interesting resources in geometry and topology.

As is well-known, every finitely presented group can be realized as the fundamental group of a compact differential manifold of dimension greater than or equal to 4, see for instance [Man]. However, not every group can be the fundamental group of a compact Kähler manifold, a fortiori, a projective manifold. As is shown by Serre [Ser], any finite group can be realized as the fundamental group of a projective manifold; and in the same paper, the author proposes the problem about characterizing or finding properties of quasi-projective groups, that is, finitely presented groups that can be obtained as the fundamental groups of quasi-projective manifolds.

The study of cohomology jump loci provides specific conditions for a group to be quasi-projective. Together with the Hodge theory for Kähler manifolds, this study has been proven to be a powerful tool to study the fundamental group of a Kähler manifold. Historically, one usually studies its linear representations in order to study a group. And a representation of the fundamental group corresponds naturally to a local system of vector spaces, and quite often, the local systems can be equipped with more restrictive structures, for example, variation of Hodge structures. When the set of all representations of the fundamental group are taken into consideration, these special structures put restrictions on the cohomology jump loci.

In this chapter, we shall make an overview of the fundamental materials we shall use in the sequel, principally divided into two different kinds: differential analysis and algebraic geometry. Of course, one point of view is influenced by and influences the other.

## 2.1 Hodge theory and mixed Hodge theory

In this section, we make a survey of some Hodge theory and mixed Hodge theory that we will need. For some basic definitions and properties of Hodge structures and mixed Hodge

structures, we refer to the book [PeS] and to the Appendix C in [Dim92].

### 2.1.1 Hodge theory for compact Kähler manifolds

By a Kähler manifold, we refer to a complex manifold  $X$  which is equipped with a Hermitian metric  $g$  such that the associated fundamental form (called *Kähler form*)  $\omega = \omega_g$  is closed. Locally,  $g$  can be written as

$$g = \sum_{j,k} g_{j\bar{k}} dz^j d\bar{z}^k$$

while

$$\omega = \frac{1}{2} \sum_{j,k} g_{j\bar{k}} dz^j \wedge d\bar{z}^k.$$

An equivalent characterization of Kähler manifold is the existence of holomorphic normal coordinates, namely, holomorphic coordinates  $(z^1, \dots, z^n)$  centered at any given point  $x \in X$  in which

$$g_{j\bar{k}} = \delta_{jk} + O(|z|^2), \quad j, k = 1, \dots, n.$$

Typical examples of Kähler manifolds are the Euclidean space  $\mathbb{C}^n$ , the complex tori  $\mathbb{C}^n/\mathbb{Z}^{2n}$ , Riemann surfaces, projective manifolds and blowups of any given Kähler manifold.

There are topological restrictions for a compact manifold  $X$  to be Kähler, for instance, its first Betti number must be even, which follows immediately from the Hodge decomposition of  $H^1(X, \mathbb{C})$ , see Theorem 2.1.2 below. As a consequence, if a finitely presented group  $G$  is isomorphic to the fundamental group of a compact Kähler manifold, then the rank of  $G$  must necessarily be even. To get more delicate restrictions for a group being Kähler, namely, being the fundamental group of a compact Kähler manifold, a more precise study of the structure of the cohomology ring  $H^\bullet(X, \mathbb{C})$  is indispensable.

Briefly summarized, there are Hodge decompositions, Lefschetz decompositions and polarizations on the cohomology ring  $H^\bullet(X, \mathbb{C})$  for a compact Kähler manifold  $X$ .

#### Hodge decomposition

Given a compact Riemannian manifold, one can represent its cohomology by harmonic forms. This is the classical Hodge theorem. There, one obtains two naturally defined differential operators, namely, the exterior differential  $d$  and its formal adjoint  $d^*$ , via which one defines the Laplace-Beltrami operator  $\Delta = dd^* + d^*d$ . A harmonic form is a differential form which is annihilated by  $\Delta$ , or equivalently, by both  $d$  and  $d^*$ .

On a compact Kähler manifold  $X$ , one can further define more operators. First, since  $X$  is a complex manifold, the differential forms on  $X$  can be decomposed according to types

$$\mathcal{A}_X^k = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q}$$

and thus the exterior differential  $d$  is also decomposed into two first-order differential operators  $\partial$  and  $\bar{\partial}$ , with  $d = \partial + \bar{\partial}$ . More precisely, for a differential form  $\alpha$  of type  $(p, q)$ ,

$$\partial\alpha = (p+1, q)\text{-component of } d\alpha$$

and

$$\bar{\partial}\alpha = (p, q+1)\text{-component of } d\alpha.$$

One also gets their formal adjoints  $\partial^*$  and  $\bar{\partial}^*$  such that  $d^* = \partial^* + \bar{\partial}^*$ . Furthermore, we also obtain the associated Laplace operators

$$\Delta_{\partial} = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Second, wedge product with the Kähler form  $\omega$  on  $X$  provides us with another operator, called Lefschetz operator  $L$ :

$$L = \omega \wedge .$$

It is a zero-order operator, with formal adjoint denoted by  $\Lambda$ .

The most important relations between these operators are the *Kähler identities*, one of which is

$$[\Lambda, \bar{\partial}] = -\sqrt{-1}\partial^*.$$

The other identities can be obtained by complex conjugation and taking adjoint. An important consequence of these identities is

$$\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}.$$

In particular,  $\Delta$  preserves the type of forms, or more briefly,  $\Delta$  is *homogeneous*. Therefore, a  $k$ -form  $\alpha = \sum_{p+q=k} \alpha^{p,q}$  is harmonic if and only if each component  $\alpha^{p,q}$  is. Thus, we obtain the following theorem.

**Theorem 2.1.2** (Hodge Decomposition Theorem, see [Voil]). *Suppose  $X$  is a compact Kähler manifold of complex dimension  $n$ . Let  $H^{p,q}$  be the cohomology of  $X$  that can be represented by closed  $(p, q)$ -forms.*

*Then we have the following decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}, \quad 0 \leq k \leq 2n.$$

Moreover,

- (i) *this decomposition does not depend on the specific choices of Kähler metrics on  $X$ ;*
- (ii) *complex conjugation gives an isomorphism  $\overline{H^{q,p}} \cong H^{p,q}$ ;*
- (iii) *any cohomology belonging to  $H^{p,q}$  can be represented by a unique  $\Delta_{\bar{\partial}}$ -harmonic form of type  $(p, q)$ , hence by Dolbeault theorem, we have an isomorphism*

$$H^{p,q} \cong \mathcal{H}^{p,q} \cong H^q(X, \Omega_X^p),$$

*where  $\mathcal{H}^{p,q}$  is the set of  $\Delta_{\bar{\partial}}$ -harmonic forms of type  $(p, q)$ ;*

- (iv) *Serre duality:  $H^{p,q} \cong H^{n-p, n-q}$ .*

### Lefschetz decomposition

A delicate discussion about the Lefschetz operator  $L$  and its adjoint  $\Lambda$  gives the following hard Lefschetz isomorphism theorem.

**Theorem 2.1.3** (Hard Lefschetz Theorem, see [Voi1]). *Let  $X$  be a compact Kähler manifold with Kähler form  $\omega$ .*

*Then the morphism of vector bundles*

$$L^{n-k} = \omega^{n-k} \wedge : \Omega_{X,\mathbb{R}}^k \rightarrow \Omega_{X,\mathbb{R}}^{2n-k}$$

*or equivalently, the operator of order zero  $L^{n-k} : \mathcal{A}_{X,\mathbb{R}}^k \rightarrow \mathcal{A}_{X,\mathbb{R}}^{2n-k}$  with  $k \leq n$ , is an isomorphism.*

*As a consequence,*

$$L^{n-k} : H^k(X, \mathbb{R}) \rightarrow H^{2n-k}(X, \mathbb{R}), \quad k \leq n$$

*is an isomorphism.*

**Definition 2.1.4.** We say that a form  $\alpha$  on  $X$  of degree  $k \leq n$  is *primitive* if  $L^{n-k+1}\alpha = 0$ , or equivalently  $\Lambda\alpha = 0$ .

We have

**Theorem 2.1.5** (Lefschetz Decomposition Theorem, see [Voi1]). *Let  $X$  be a compact Kähler manifold.*

*Then every cohomology class  $\alpha \in H^k(X, \mathbb{R})$  admits a unique decomposition*

$$\alpha = \sum_r L^r \alpha_r$$

*where the  $\alpha_r$  are of degree  $k - 2r \leq \min(k, 2n - k)$  and are primitive in the sense that  $L^{n-k+2r+1}\alpha_r = 0$  in  $H^{2n-k+2r+2}(X, \mathbb{R})$ .*

*As a consequence, let*

$$H^k(X, \mathbb{C})_{\text{prim}} = \ker(L^{n-k+1} : H^k(X, \mathbb{C}) \rightarrow H^{2n-k+2}(X, \mathbb{C})), \quad k \leq n$$

*then*

$$H^k(X, \mathbb{C}) = \bigoplus_{0 \leq k-2r \leq \min(k, 2n-k)} L^r H^{k-2r}(X, \mathbb{C})_{\text{prim}}.$$

*Moreover, this decomposition is compatible with the Hodge decomposition.*

## Polarization

Now let  $X$  be a compact Kähler manifold with Kähler metric  $g$  and Kähler form  $\omega$ . Define the following intersection form on  $H^k(X, \mathbb{R})$ ,  $k \leq n$ :

$$Q(\alpha, \beta) = \int_X L^{n-k} \alpha \wedge \beta = \int_X \omega^{n-k} \wedge \alpha \wedge \beta,$$

then  $Q$  is nondegenerate by Poincaré duality and hard Lefschetz theorem 2.1.3. Moreover,  $Q$  is symmetric when  $k$  is even, alternating when  $k$  is odd, and thus

$$H_k(\alpha, \beta) = (\sqrt{-1})^k Q(\alpha, \bar{\beta})$$

defines a Hermitian form on  $H^k(X, \mathbb{C})$ .

**Theorem 2.1.6.** (see [Voi1]) *Let  $X$  be a compact Kähler manifold, then we have the following:*

(i) the Hodge decomposition in Theorem 2.1.2

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}, \quad 0 \leq k \leq n$$

is an orthogonal decomposition for  $H_k$ ;

(ii) the Lefschetz decomposition in Theorem 2.1.5

$$H^k(X, \mathbb{C}) = \bigoplus_{\substack{r \\ 2r \leq k}} L^r H^{k-2r}(X, \mathbb{C})_{\text{prim}}, \quad k \leq n$$

is also an orthogonal decomposition for  $H_k$ ;

(iii) on each primitive component  $L^r H^{k-2r}(X, \mathbb{C})_{\text{prim}}$ ,  $H_k$  induces the form  $(-1)^r H_{k-2r}$ ;

(iv)  $(-1)^{k(k-1)/2} (\sqrt{-1})^{p-q-k} H_k$  is positive definite on the complex subspace  $H_{\text{prim}}^{p,q} = H^{p,q} \cap H^k(X, \mathbb{C})_{\text{prim}}$ .

### 2.1.7 Mixed Hodge theory

#### Mixed Hodge theory for quasi-projective manifolds

Let  $X$  be a quasi-projective manifold. By compactification and then desingularization, we may assume  $X = \bar{X} \setminus D$ , where  $\bar{X}$  is a projective manifold,  $D$  is a divisor in  $\bar{X}$  with normal crossings; we may assume furthermore that each irreducible component of  $D$  is a smooth hypersurface in  $\bar{X}$ . Denote by  $j : X \hookrightarrow \bar{X}$  and  $i : D \hookrightarrow \bar{X}$  the natural inclusions.

The sheaves of logarithmic forms  $\Omega_{\bar{X}}^{\bullet}(\log D)$  and  $Rj_* \mathbb{C}_X$  are isomorphic, and hence computes the cohomology  $H^{\bullet}(X, \mathbb{C})$ :

$$H^k(X, \mathbb{C}) = \mathbb{H}^k(\bar{X}, \Omega_{\bar{X}}^{\bullet}(\log D)).$$

We may define two filtrations on  $\Omega_{\bar{X}}^{\bullet}(\log D)$ , namely the weight filtration  $W$  defined over  $\mathbb{Q}$ , which is increasing and the Hodge filtration  $F$  defined over  $\mathbb{C}$ , which is decreasing, and these two filtrations induce naturally two filtrations on the cohomology  $H^{\bullet}(X, \mathbb{C})$ . In fact, we have the following.

**Theorem 2.1.8** (Mixed Hodge Theory for quasi-projective manifolds, see [Voi1]). *Let  $X$  be a quasi-projective manifold with  $X = \bar{X} \setminus D$  where  $\bar{X}$  is a smooth compactification of  $X$  and  $D$  a normal crossing divisor in  $\bar{X}$ ; assume that moreover  $D = \cup_{i \in I} D_i$  is the irreducible component decomposition of  $D$  with  $D_i$  a smooth hypersurface in  $\bar{X}$ .*

*Then  $H^{\bullet}(X, \mathbb{C})$  admits a naturally defined mixed Hodge structure (MHS) defined via the weight filtration  $W$  and the Hodge filtration  $F$  on  $\Omega_{\bar{X}}^{\bullet}(\log D)$ , such that*

(i) the spectral sequence  ${}_W E_r^{\bullet, \bullet}$  associated to the weight filtration  $W$  begins with

$${}_W E_1^{p,q} \cong H^{2p+q}(D^{(-p)}, \mathbb{C}), \quad p \leq 0, q \geq 0$$

and degenerates at  $E_2$ , where

$$D^{(-p)} = \bigsqcup_{\substack{i_1 < \dots < i_{-p} \\ i_1, \dots, i_{-p} \in I}} D_{i_1} \cap \dots \cap D_{i_{-p}};$$

by convention,  $D^{(0)} = \bar{X}$ .

(ii) the morphism

$$d_1 : {}_W E_1^{p,q} \cong H^{2p+q}(D^{(-p)}, \mathbb{C}) \rightarrow {}_W E_1^{p+1,q} \cong H^{2p+q+2}(D^{(-p-1)}, \mathbb{C})$$

restricted on each component  $H^{2p+q}(D_{i_1} \cap \cdots \cap D_{i_{-p}}, \mathbb{C})$  can be identified, up to sign, with the Gysin map induced by the various inclusions

$$D_{i_1} \cap \cdots \cap D_{i_{-p}} \cap D_j \hookrightarrow D_{i_1} \cap \cdots \cap D_{i_{-p}}, \quad j \in I \setminus \{i_1, \dots, i_{-p}\}.$$

(iii) the spectral sequence  ${}_F E_r^{\bullet, \bullet}$  associated to the Hodge filtration  $F$  starts with

$${}_F E_1^{p,q} = H^q(\bar{X}, \Omega_{\bar{X}}^p(\log D))$$

and degenerates at  $E_1$ .

### Mixed Hodge theory for arbitrary algebraic varieties

Given an arbitrary complex algebraic variety  $X$ . If  $X$  is not smooth, then apply a resolution of singularities; and if it is not compact, then use a smooth compactification. Eventually, we can equip the cohomology  $H^\bullet(X, \mathbb{C})$  with a MHS. More precisely, we have the following.

**Theorem 2.1.9** (Mixed Hodge Theory for algebraic varieties, see [Dim92], Appendix C). *Let  $X$  be a complex algebraic variety.*

*Then there is a MHS on  $H^\bullet(X, \mathbb{Q})$  such that the following properties hold for all  $k \geq 0$ :*

(i) *the weight filtration  $W$  on  $H^k(X, \mathbb{Q})$  satisfies*

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2k} = H^k(X, \mathbb{Q})$$

*namely,  $Gr_i^W H^k(X, \mathbb{Q}) \neq 0$  only for  $i \in [0, 2k]$ ;*

(ii) *the Hodge filtration  $F$  on  $H^k(X, \mathbb{C})$  satisfies*

$$H^k(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m \supseteq F^{m+1} = 0,$$

*namely,  $Gr_F^p H^k(X, \mathbb{C}) \neq 0$  only for  $p \in [0, k]$ ;*

(iii) *if  $X$  is smooth, then  $W_{k-1} H^k(X, \mathbb{Q}) = 0$ , namely all the weights on  $H^k(X, \mathbb{Q})$  are  $\geq k$ , and moreover,*

$$W_k H^k(X, \mathbb{Q}) = j^* H^k(\bar{X}, \mathbb{Q})$$

*for any good compactification  $j : X \hookrightarrow \bar{X}$ ;*

(iv) *if  $X$  is compact, then  $W_k H^k(X, \mathbb{Q}) = H^k(X, \mathbb{Q})$ , namely all the weights on  $H^k(X, \mathbb{Q})$  are  $\leq k$ , and moreover,*

$$W_{k-1} H^k(X, \mathbb{Q}) = \ker(p^* : H^k(X, \mathbb{Q}) \rightarrow H^k(\tilde{X}, \mathbb{Q}))$$

*for any surjective map  $p : \tilde{X} \rightarrow X$  with  $\tilde{X}$  compact and smooth;*

(v) *the assignment  $X \mapsto H^k(X, \mathbb{Q})$  is functorial, i.e., any morphism  $f : X \rightarrow X'$  of algebraic varieties induces a MHS morphism*

$$f^* : H^k(X', \mathbb{Q}) \rightarrow H^k(X, \mathbb{Q}).$$

Unless otherwise explicitly mentioned in the sequel, we shall write  $H^\bullet(X)$  for  $H^\bullet(X, \mathbb{C})$  for an algebraic variety  $X$ , equipped with the MHS above.

## 2.2 Basics on cohomology jump loci

In order to study the fundamental group of a manifold, one considers its representations; to study the properties of the linear representations, one considers the cohomology jump loci: mainly characteristic varieties and resonance varieties.

### 2.2.1 Definitions

We shall focus on the case of 1-dimensional representations of the fundamental group, which is already very interesting.

Let  $X$  be a quasi-projective manifold, or more generally a connected finite CW complex. Let

$$\mathbb{T}(X) = \text{Hom}(\pi_1(X), \mathbb{C}^*)$$

be the parameter space of 1-dimensional representations of  $\pi_1(X)$ , which is also called *the character variety* of  $X$ . It is an algebraic group whose irreducible component containing the identity is an algebraic torus  $\mathbb{T}^0(X) = (\mathbb{C}^*)^{b_1(X)}$ ; and when  $H_1(X, \mathbb{Z})$  is torsion free,  $\mathbb{T}(X) = \mathbb{T}^0(X)$ . For  $\rho \in \mathbb{T}(X)$ , we shall denote  $L_\rho$  for the corresponding rank 1 local system on  $X$ .

The *characteristic varieties* of  $X$  are the jump loci for the cohomology of  $X$ , with coefficients in rank 1 local systems, i.e.,

$$\mathcal{V}_k^j(X) = \{\rho \in \mathbb{T}(X) \quad : \quad \dim H^j(X, \mathcal{L}_\rho) \geq k\}.$$

When  $j = 1$ , we use the simpler notation  $\mathcal{V}_k(X) = \mathcal{V}_k^1(X)$ .

For higher dimensional representations, one can similarly define the characteristic varieties and discuss their properties, see [DP14]; however, it turns out that a lot of valuable information for the higher dimensional representations can be deduced by just considering some 1-dimensional sub-representations, see [MPPS].

To define resonance varieties, we consider a commutative differential graded algebra (CDGA)  $K$ . By definition, a  $\mathbb{k}$ -CDGA  $K = (K^\bullet, d)$  (where  $\mathbb{k} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ ) is equipped with a multiplication under which  $K = \bigoplus_{p \geq 0} K^p$  is a commutative graded algebra, and a differential  $d$  which satisfies the graded Leibniz rule, i.e.,  $d(a \cdot b) = da \cdot b + (-1)^p a \cdot db$  for  $a \in K^p$ . Unless otherwise specifically stated in the sequel, any CDGA is also finitely generated with unity,  $K^0 = \mathbb{k}$ , and  $\mathbb{k} = \mathbb{C}$ .

The *resonance varieties* of a CDGA  $K$  are the cohomology jump loci for the cohomology of the twisted differentials (also known as *Aomoto complexes* of  $K$ ), namely,

$$\mathcal{R}_k^j(K) = \{\alpha \in H^1(K) \quad : \quad \dim H^j(K, d_\alpha) \geq k\}.$$

where we have written  $d_\alpha = d + \alpha \wedge$ . When  $j = 1$ , we use the simpler notation  $\mathcal{R}_k(K) = \mathcal{R}_k^1(K)$ .

We have the following well-known basic results.

**Proposition 2.2.2** (See for instance [Ara]). *The following properties hold:*

- (i) *for any connected finite CW complex, the characteristic variety  $\mathcal{V}_k^j(X)$  is a closed algebraic subset of  $\mathbb{T}(X)$  for any  $k, j$ .*
- (ii) *for any finitely generated CDGA  $K$ ,  $\mathcal{R}_k^j(K)$  is a closed algebraic subset of  $H^1(K)$  for all  $k, j$ .*

Let  $X$  be a connected finite CW complex, a fortiori a quasi-projective manifold, the resonance variety of  $X$  is defined as

$$\mathcal{R}_k^j(X) = \mathcal{R}_k^j(H^\bullet(X)),$$

where  $H^\bullet(X)$  is regarded as a CDGA with zero differential.

### 2.2.3 Formality

Formality of a topological space plays a central role in the study of the cohomology jump loci. To survey some results on this aspect, we first recall several equivalence relations in the category of CDGAs. These results are quite fundamental and classical, and for a more detailed exposition, we refer to [DP14] for instance.

A morphism of CDGA  $\phi : K \rightarrow K'$  is called *quasi-isomorphism* if the induced map on cohomology  $\phi^* : H^i(K) \rightarrow H^i(K')$  is an isomorphism for all  $i \geq 0$ ; it is called a *q-isomorphism* ( $1 \leq q < \infty$ ) if the induced morphism  $\phi^* : H^i(K) \rightarrow H^i(K')$  is an isomorphism for  $i \leq q$  and a monomorphism for  $i = q + 1$ . Two CDGAs  $K$  and  $K'$  are called *q-equivalent* (respectively, *weakly equivalent*) if there is a zig-zag sequence consisting of  $q$ -isomorphisms of CDGAs (respectively, quasi-isomorphisms) connecting them. In the sequel, we shall apply an abuse of notation to denote  $\infty$ -isomorphisms for quasi-isomorphisms,  $\infty$ -equivalences for weak equivalences. Notation:  $K \simeq_q K'$  ( $1 \leq q \leq \infty$ ) if  $K$  and  $K'$  are  $q$ -equivalent.

Let  $1 \leq q \leq \infty$ . A CDGA  $K$  is called *q-formal* (for  $q = \infty$ , *formal*) if it is  $q$ -equivalent to its cohomology algebra  $H^\bullet(K)$  (with the zero differential).

By [Sul], a connected finite CW-complex has Sullivan's model of polynomial forms,  $\Omega^\bullet(X)$  which is a CDGA defined over  $\mathbb{Q}$ .  $X$  is called *q-formal* (for  $q = \infty$ , *formal*) if so is  $\Omega^\bullet(X)$ . If a CDGA  $K \simeq_q \Omega^\bullet(X)$ , then  $K$  is called a *q-model* (for  $q = \infty$ , *model*) of  $X$ . Note that in general  $\Omega^\bullet(X)$  is not finitely generated, but it can still be  $q$ -equivalent to a finitely generated CDGA.

Many interesting spaces we encounter have some formalities.

**Example 2.2.4.** (1) Compact Kähler manifolds are formal, as is shown in [DGMS].

In particular, smooth projective varieties are formal.

(2) Let  $X$  be a quasi-projective manifold. Then  $X$  is 1-formal if  $H^1(X)$  is pure of weight 2, see [Mor].

For example, hypersurface complements in  $\mathbb{P}^n$  are 1-formal for  $n \geq 2$ . Note also that there are varieties with  $H^1$  pure of weight 1, but not 1-formal.

(3) Hyperplane arrangement complements are formal, see [Bri].

(4) Spheres are formal, see [He].

(5) Products of formal spaces are formal and wedges of formal spaces are formal, see [He].

(6) Given a compact, connected Lie group  $G$ . Let  $K$  denote the connected component of its neutral element  $e$ , in the subgroup of elements fixed by a given involution. The quotient  $G/K$ , which is a symmetric space, is then a formal space. This is proved in [Cartan].

### 2.2.5 Properties of cohomology jump loci

Here we summarize some properties about cohomology jump loci that will be needed in the sequel.

### Characteristic varieties and irrational pencils

Let  $X$  be a quasi-Kähler  $X$ , namely,  $X = \overline{X} \setminus D$  with  $\overline{X}$  compact Kähler and  $D$  a normal crossing divisor. The characteristic variety  $\mathcal{V}_1(X)$  has close relations with irrational pencils of  $X$ , as first proved in [Ara]. By definition, an irrational pencil of  $X$  is an equivalence class of surjective morphisms  $f : X \rightarrow C$  with  $C$  being a smooth curve, satisfying the following two conditions: (i)  $\chi(C) < 0$ ; (ii) a generic fiber of  $f$  is connected. Here the equivalence relation is defined as follows: two morphisms  $f : X \rightarrow C$  and  $f' : X \rightarrow C'$  are equivalent if there exists an isomorphism  $\eta : C \rightarrow C'$  such that  $f' = f \circ \eta$ .

We denote  $\mathcal{E}(X)$  the set of all irrational pencils of  $X$ . To any  $f \in \mathcal{E}(X)$ , we associate  $W_f = f^*\mathbb{T}(f(X))$ .

Let  $IC_1(X)$  be the set of irreducible components of positive dimension (i.e. dimension  $> 0$ ) passing through 1 of the characteristic variety  $\mathcal{V}_1(X)$ . Then we have the following (see also [Ara],[Dim08],[DPS09]).

**Theorem 2.2.6.** *Suppose  $X$  is a (connected) quasi-Kähler manifold. Then the following hold.*

- (i) for any  $f \in \mathcal{E}(X)$ ,  $W_f$  is an irreducible component of  $\mathcal{V}_1(X)$ ;
- (ii) the map  $f \mapsto W_f$  gives a one-to-one correspondence between  $\mathcal{E}(X)$  and  $IC_1(X)$ .
- (iii) for any irreducible components  $V_1$  and  $V_2$  of  $\mathcal{V}_1(X)$ ,  $V_1 \cap V_2$  is a finite set.

In particular,  $\mathcal{E}(X)$  is finite.

So the irreducible components of tangent cone  $TC_1(\mathcal{V}_1(X))$  at 1 of the characteristic variety  $\mathcal{V}_1(X)$  have close relations with irrational pencils of  $X$ . They are also closely related to maximal isotropic subspaces of  $H^1(X)$ , see [Bau],[Cat],[Dim08].

### Relations between characteristic varieties and resonance varieties —without formality assumptions

Let  $X$  be a connected finite CW complex. The basic relations between characteristic varieties of  $X$  and resonance varieties of a  $q$ -models of  $X$  are given as follows, see also [DP14], Theorem B.

**Theorem 2.2.7.** *Let  $X$  is a connected finite CW complex and  $K$  a CDGA with  $\dim K^i < \infty$  for  $i \leq q$  which is a  $q$ -model for  $X$  ( $q \geq 1$ ). Then for all  $j \leq q$  and  $k \geq 0$ , there is a local analytic isomorphism  $(\mathcal{R}_k^j(K), 0) \cong (\mathcal{V}_k^j(X), 1)$ , induced by the exponential map  $\exp : H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathbb{C}^*) = \mathbb{T}^0(X)$ .*

We say that a CDGA  $K$  has *positive weights* if there exist decompositions

$$H^i = \bigoplus_{j \in \mathbb{Z}} K_j^i$$

for all  $i$  such that  $K_j^\bullet \cdot K_k^\bullet \subseteq K_{j+k}^\bullet$  and  $K_j^1 = 0$  for  $j \leq 0$ . A subspace  $E \subseteq K^1$  is called *weighted homogeneous* if  $E = \bigoplus_j (E \cap K_j^1)$ . Note that a decomposition as above gives a  $\mathbb{C}^*$ -action on  $K^1$ , and  $E$  is weighted homogeneous means exactly that  $E$  is invariant under this  $\mathbb{C}^*$ -action.

A typical example of a CDGA having positive weights is given by the Gysin model of a quasi-projective manifold.

**Example 2.2.8** (Gysin model of a quasi-projective manifold). Let  $X$  be a quasi-projective manifold. Pick any smooth compactification,  $X = \overline{X} - D$ , where  $D = \cup_{i \in I} D_i$  is a union of smooth divisors with normal crossings.

There is an associated CDGA defined over  $\mathbb{Q}$ ,  $A = (A^\bullet, d) = (A^\bullet(\overline{X}, D), d)$ , called the Gysin model of the compactification, constructed as follows. As a vector space,

$$A^k = \bigoplus_{p+l=k} A^{p,l}$$

where

$$A^{p,l} = \bigoplus_{|S|=l} H^p\left(\bigcap_{i \in S} D_i, \mathbb{C}\right)(-l)$$

where  $S$  runs through the  $l$ -element subsets of  $I$  and  $(-l)$  denotes the Tate twist. The multiplication is induced by cup product, and has the property that  $A^{p,l} \cdot A^{p',l'} \subseteq A^{p+p',l+l'}$ . The differential,  $d : A^{p,l} \rightarrow A^{p+2,l-1}$ , is defined by using the various Gysin maps coming from intersections of divisors. See [FM] for full details.

Morgan proved in [Mor] that  $\Omega^\bullet(X) \simeq_\infty A$ , and moreover, the associated cohomology isomorphism preserves  $\mathbb{Q}$ -structures. By definition the weight of  $A^{p,l}$  is  $p + 2l$ , and we clearly obtain a positive weight decomposition of  $(A^\bullet, d)$ , inducing a splitting of Deligne's weight filtration on  $H^\bullet(X)$ .

**Theorem 2.2.9** (See [DP14], Theorem C, D). *Let  $X$  be a connected finite CW complex and  $K$  a CDGA with  $\dim K^i < \infty$  for  $i \leq q$  which is a  $q$ -model for  $X$  ( $q \geq 1$ ). If  $K$  is defined over  $\mathbb{Q}$  and has positive weights, and the isomorphism induced by the zig-zag of  $q$ -equivalences,  $H^1(X, \mathbb{C}) \cong H^1(K)$ , preserves  $\mathbb{Q}$ -structures. Then the following hold:*

- (i) *for all  $j \leq q$  and  $k \geq 0$ ,  $\mathcal{R}_k^j(K)$  is a finite union of linear subspaces of  $H^1(X, \mathbb{C}) \cong H^1(K)$  that are defined over  $\mathbb{Q}$  and weighted homogeneous;*
- (ii) *for all  $j \leq q$  and  $k \geq 0$ , every irreducible component of  $\mathcal{V}_k^j(X)$  passing through 1 is a subtorus of the character torus  $\mathbb{T}^0(X)$ .*

*In particular, when  $X$  is a quasi-projective manifold and  $A = (A^\bullet, d)$  is the Gysin model of  $X$ , then  $\mathcal{R}_k^j(A)$  is a union of linear subspaces of  $H^1(X, \mathbb{C})$ .*

In fact, when  $X$  is a quasi-projective manifold, every irreducible component of  $\mathcal{V}_k^j(X)$  of positive dimension is a translated subtorus of the character variety  $\mathbb{T}(X)$ , see [BWens]. In addition, let  $A$  be its Gysin model, then each irreducible component of  $\mathcal{R}_k^j(A)$  is a mixed Hodge substructure of  $H^1(X, \mathbb{Q})$  for all  $j, k$ , see Theorem D in [DP14].

**Theorem 2.2.10** (See [MPPS]). *Let  $X$  be a finite CW complex and  $K$  a CDGA with  $\dim K^i < \infty$  for  $i \leq q$  which is a  $q$ -model for  $X$  ( $q \geq 1$ ). Then the following hold:*

- (i) *for all  $j \leq q$  and  $k \geq 0$ , the tangent cone  $TC_0(\mathcal{R}_k^j(K))$  at 0 to the resonance variety  $\mathcal{R}_k^j(K)$  is contained in the resonance variety  $\mathcal{R}_k^j(X)$  of  $X$ , namely,  $TC_0(\mathcal{R}_k^j(K)) \subseteq \mathcal{R}_k^j(X)$ .*
- (ii) *suppose that, moreover,  $K$  is defined over  $\mathbb{Q}$  and has positive weights, and the isomorphism induced by the zig-zag of  $q$ -equivalences,  $H^1(X, \mathbb{C}) \cong H^1(K)$ , preserves  $\mathbb{Q}$ -structures. Then for all  $j \leq q$  and  $k \geq 0$ ,  $\mathcal{R}_k^j(K) \subseteq \mathcal{R}_k^j(X)$ .*

### Relations between characteristic varieties and resonance varieties —with formality assumptions

For a space with some formality, the characteristic varieties and resonance varieties admit special properties. We just make a brief summary as follows.

**Corollary 2.2.11.** *Let  $X$  be a finite CW complex which is  $q$ -formal ( $q \geq 1$ ). Then the following hold:*

1. *for all  $j \leq q$  and  $k \geq 0$ , there is a local isomorphism  $(\mathcal{R}_k^j(X), 0) \cong (\mathcal{V}_k^j(X), 1)$ , induced by the exponential map  $\exp : H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathbb{C}^*)$ . In particular,  $TC_0(\mathcal{R}_k^j(X)) \cong TC_1(\mathcal{V}_k^j(X))$ .*
2. *suppose that, moreover,  $X$  admits a  $q$ -model  $K$ ,  $\dim K^i < \infty, i \leq q$ , which is defined over  $\mathbb{Q}$  and has positive weights, and the isomorphism induced by the zig-zag of  $q$ -equivalences,  $H^1(X, \mathbb{C}) \cong H^1(K)$ , preserves  $\mathbb{Q}$ -structures. Then for all  $j \leq q$  and  $k \geq 0$ ,  $TC_1(\mathcal{V}_k^j(X)) = \mathcal{R}_k^j(X)$ .*
3. *In addition, if  $X$  is a quasi-projective manifold and  $q$ -formal, then for all  $j \leq q$  and  $k \geq 0$ , any irreducible component of  $\mathcal{R}_k^j(X)$  is a mixed Hodge substructure of  $H^1(X)$ .*

Indeed, since  $H^\bullet(X)$  is a  $q$ -model for  $X$ , the statements follow directly from Theorem 2.2.9.

**Corollary 2.2.12.** *Let  $X$  be a quasi-Kähler manifold which is 1-formal, then  $TC_1(\mathcal{V}_1(X)) = \mathcal{R}_1(X)$  and moreover the map  $f \mapsto f^*H^1(f(X))$  gives a one-to-one correspondence between  $\mathcal{E}(X)$  and the set of irreducible components of  $\mathcal{R}_1(X)$ . In addition, any two distinct irreducible components of  $\mathcal{R}_1(X)$  intersect only at the origin.*

Indeed, the statements follows directly from Theorem 2.2.6.

In the sequel, we shall mainly concentrate on  $\mathcal{V}_1(X)$ ,  $\mathcal{R}_1(X)$  and 1-formality for a quasi-projective manifold  $X$ . Here the following points are worth to re-emphasize:

1.  $\mathcal{V}_1(X)$ ,  $\mathcal{R}_1(X)$  and the 1-formality depend only on the fundamental group  $\pi_1(X)$  of  $X$ ;
2. Each irreducible component of  $\mathcal{V}_1(X)$  passing through 1 is a subtorus of  $\mathbb{T}^0(X)$ ;
3. Each irreducible component of  $TC_1(\mathcal{V}_1(X))$  is a mixed Hodge substructure of  $H^1(X)$ ;
4.  $TC_1(\mathcal{V}_1(X)) = \mathcal{R}_1(A)$ , where  $A$  is the Gysin model for  $X$ ;
5. Always  $TC_1(\mathcal{V}_1(X)) \subseteq \mathcal{R}_1(X)$ ; equality holds when  $X$  is 1-formal.

## 2.3 Hyperplane arrangements

In this section, we survey some basic notions and properties for hyperplane arrangements. For more details, we refer to the excellent book [OT].

### 2.3.1 Definitions

A *hyperplane arrangement*  $\mathcal{A}$  in  $\mathbb{P}^n$  is a finite set of hyperplanes in  $\mathbb{P}^n$ , i.e.,

$$\mathcal{A} = \{H_1, \dots, H_d\}$$

where  $H_i : L_i = 0$  with  $L_i$  a nonzero linear form (defined up to a scalar multiplication) for  $i = 1, \dots, d$ . The polynomial

$$Q = Q(\mathcal{A}) = \prod_{i=1}^d L_i$$

is called the *defining polynomial* of  $\mathcal{A}$ . By definition,  $Q$  is a *reduced* polynomial of degree  $d = |\mathcal{A}|$ , defined up to a scalar multiplication.

The complement

$$M(\mathcal{A}) = \mathbb{P}^n \setminus V(Q) = \mathbb{P}^n \setminus \bigcup_{i=1}^d H_i$$

is called the *complement* of the arrangement  $\mathcal{A}$ , and moreover,

$$F(\mathcal{A}) = Q^{-1}(1)$$

is called the *Milnor fiber* of the arrangement, where  $Q$  is viewed as a map  $Q : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ .

More generally, given a homogeneous polynomial  $Q \in \mathbb{C}[x_0, \dots, x_n]$  of degree  $d$ , one can similarly define the complement

$$M(Q) = \mathbb{P}^n \setminus V(Q),$$

where  $V(Q) : Q = 0$  is the projective hypersurface in  $\mathbb{P}^n$ , and the (affine) Milnor fiber

$$F(Q) : Q = 1$$

in  $\mathbb{C}^{n+1}$ .

When there is no confusion, we shall denote  $M$  for the complement and  $F$  the Milnor fiber, without mentioning the arbitrarily chosen polynomial  $Q$  or the hyperplane arrangement  $\mathcal{A}$ .

### 2.3.2 Complement, Milnor fiber and monodromy

Here we only make a review of some basic definitions concerning the Milnor fiber and the monodromy action. For a more detailed description, we refer to [Dim92].

With the notations above, let  $\xi = \exp(2\pi\sqrt{-1}/d)$ , Define the *geometric monodromy*  $h : F \rightarrow F$  by

$$h(x_0, \dots, x_n) = (\xi x_0, \dots, \xi x_n),$$

then  $h$  is an algebraic morphism and gives an action on  $F$  of group

$$\mu_d = \langle \xi \rangle = \{z \in \mathbb{C} : z^d = 1\} \cong \langle h \rangle \cong \mathbb{Z}/d\mathbb{Z}$$

with quotient

$$F/\mu_d = M.$$

In fact, the natural projection  $p : F \rightarrow M$  is a regular covering with Deck transformation group  $\mu_d$ .

The map  $h$  induces the *algebraic monodromy* on cohomology

$$h^* : H^i(F, \mathbb{Q}) \rightarrow H^i(F, \mathbb{Q})$$

which is also a MHS morphism for each  $i \geq 0$ .

Since  $h^d = \text{Id}$ , we have  $h^*$  is diagonalizable with possible eigenvalues  $d$ -th roots of unity  $\xi^k, k = 0, 1, \dots, d-1$ . Denote, for  $i \geq 0$ ,

$$H^i(F)_a = \ker(a \cdot \text{Id} - h^* \quad : \quad H^i(F) \rightarrow H^i(F))$$

the eigenspace corresponding to  $a \in \mathbb{C}$ . Clearly,  $H^i(F)_1 = p^*H^i(M)$  for all  $i$ .

A more general form of the following theorem is proved in [CS95].

**Theorem 2.3.3.** *Let  $Q \in \mathbb{C}[x_0, \dots, x_n]$  be a (possibly not square-free) homogeneous polynomial of degree  $d \geq 2$  and  $F : Q = 1$  be its Milnor fiber. If  $a$  and  $b$  generate the same subgroup of  $\mu_d = \mathbb{Z}/d\mathbb{Z}$ , then the eigenspaces  $H^i(F)_a$  and  $H^i(F)_b$  are isomorphic for all  $i \geq 0$ .*

Furthermore, we have

$$H^{n-1}(F, \mathbb{Q}) = H^{n-1}(F, \mathbb{Q})_1 \oplus H^{n-1}(F, \mathbb{Q})_{\neq 1}$$

where  $H^{n-1}(F, \mathbb{Q})_1 = \ker(\text{Id} - h^*)$  and  $H^{n-1}(F, \mathbb{Q})_{\neq 1} = \ker(\text{Id} + h^* + \dots + h^{*(d-1)})$ . In addition, the following hold.

**Theorem 2.3.4** (see [DP11]). *If  $Q \in \mathbb{C}[x_0, \dots, x_n]$  is a square-free homogeneous polynomial of degree  $d$  such that  $V(Q) \subseteq \mathbb{P}^n$  has only isolated singularities, and let  $F$  be its Milnor fiber. Then the mixed Hodge structure on  $H^{n-1}(F, \mathbb{Q})$  splits, i.e., the subspaces  $H^{n-1}(F, \mathbb{Q})_1$  and  $H^{n-1}(F, \mathbb{Q})_{\neq 1}$  inherit pure Hodge structure from  $H^{n-1}(F, \mathbb{Q})$ , such that  $H^{n-1}(F, \mathbb{Q})_1$  (respectively  $H^{n-1}(F, \mathbb{Q})_{\neq 1}$ ) has weight  $n$  (respectively  $n-1$ ).*

### 2.3.5 Cohomology of complements of hyperplane arrangements

Given a hyperplane arrangement  $\mathcal{A} = \{H_1, \dots, H_d\}$  in  $\mathbb{P}^n$ , with  $H_i : L_i = 0$ . We can associate to it a graded algebra  $R(\mathcal{A})$  generated by degree 1 elements

$$\omega_i = \frac{1}{2\pi\sqrt{-1}} \frac{dL_i}{L_i}, \quad i = 1, \dots, d.$$

Then a fundamental theorem in the theory of hyperplane arrangements is that the map

$$\omega_i \mapsto [\omega_i]$$

gives an algebra isomorphism  $R(\mathcal{A}) \cong H^\bullet(M(\mathcal{A}))$ . For more algebras constructed from the combinatorics of  $\mathcal{A}$ , see [OT].

Moreover, the following hold:

1.  $H^\bullet(M(\mathcal{A}))$  is generated by  $H^1(M(\mathcal{A}))$ , and  $\omega_i = \frac{1}{2\pi\sqrt{-1}} \frac{dL_i}{L_i}, \quad i = 1, \dots, d-1$  give a basis for  $H^1(M(\mathcal{A}))$ .
2. For  $0 \leq k \leq n$ ,  $H^k(M(\mathcal{A}))$  is pure of weight  $2k$ , and of type  $(k, k)$ .

Indeed, it suffices to show that  $\omega_i$  is of type  $(1, 1)$  for all  $i$ . Fix  $i$ , and let  $L_i : M(\mathcal{A}) \rightarrow \mathbb{C}^*$  be the map induced by  $L_i$ . Then

$$\omega_i = L_i^* \left( \frac{1}{2\pi\sqrt{-1}} \frac{dz}{z} \right);$$

the desired result immediately follows since  $L_i^*$  is a MHS morphism and  $\frac{1}{2\pi\sqrt{-1}} \frac{dz}{z} \in H^1(\mathbb{C}^*)$  is of type  $(1, 1)$ .

Furthermore,

3.  $H^\bullet(M(\mathcal{A}), \mathbb{Z})$  is torsion free.
4.  $M(\mathcal{A})$  is formal, see [Bri].

### 2.3.6 Complement and Milnor fiber of a line arrangement

Let  $M = M(Q)$  for a homogenous polynomial  $Q \in \mathbb{C}[x_0, \dots, d]$ .

If  $n > 2$ , then by the Zariski theorem of Lefschetz type (see for instance [Dim92]), for a generic 2-dimensional plane  $P \cong \mathbb{P}^2 \subseteq \mathbb{P}^n$ , the natural homomorphism of fundamental groups

$$\pi_1(M \cap P) \rightarrow \pi_1(M),$$

induced by the inclusion  $M \cap P \rightarrow M$ , is an isomorphism. Hence, for our purpose of studying the fundamental group, it suffices to study only the case  $n = 2$ , and hence complements of plane curves and lines arrangements in  $\mathbb{P}^2$ .

Now given a line arrangement  $\mathcal{A} = \{L_1, \dots, L_d\}$  in  $\mathbb{P}^2$ . Let  $Q = \prod_i L_i$  be its defining equation,  $M = M(\mathcal{A})$  its complement and  $F : Q = 1$  its Milnor fiber,  $p : F \rightarrow M$  the natural projection; the geometric monodromy  $h : F \rightarrow F$  defined by  $h(x) = \xi \cdot x$  with  $\xi = \exp(2\pi\sqrt{-1}/d)$ , the algebraic monodromy  $h^i : H^i(F) \rightarrow H^i(F)$  for all  $i \geq 0$ , the eigenspaces  $H^i(F)_a = \ker(a \cdot \text{Id} - h^* : H^i(F) \rightarrow H^i(F))$  and  $H^i(F)_{\neq 1} = \bigoplus_{a \neq 1} H^i(F)_a$ .

Since  $M$  is formal, the irreducible components of  $\mathcal{R}_1(M)$  correspond to irrational pencils. Let  $f : M \rightarrow C$  be an irrational pencil of  $M$ , then  $f^* : H^1(C) \rightarrow H^1(M)$  is injective, hence  $W_1 H^1(C) = 0$  since  $H^1(M)$  has pure weight 2. Denote  $\overline{C}$  a smooth compactification of  $C$ , then from  $W_1 H^1(C) \cong H^1(\overline{C})$ , we have  $H^1(\overline{C}) = 0$  and thus  $\overline{C} \cong \mathbb{P}^1$ , i.e.,  $C$  is  $\mathbb{P}^1$  deleting  $k \geq 3$  points:  $C = \mathbb{P}^1 \setminus \{(-\beta_i, \alpha_i) : i = 1, \dots, k\}$ . Consequently, the morphism  $f$  extends to a rational map  $\overline{f} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$  with  $\overline{f} = (Q_1, Q_2)$  where  $Q_1, Q_2$  are two coprime homogenous polynomials. Note that the indeterminacies  $B$  of  $\overline{f}$  form a finite set, because  $\mathbb{P}^2$  is smooth and  $\mathbb{P}^1$  is proper. Hence  $\overline{f}$  is also defined even on a Zariski open subset of  $V(Q) \subseteq \mathbb{P}^2$ .

Further, we claim that  $\overline{f}(\mathbb{P}^2 - B) = \mathbb{P}^1$ . Indeed, otherwise  $\overline{f}$  gives a rational map  $\overline{f} : \mathbb{P}^2 \dashrightarrow \mathbb{C}$ , which is necessarily a morphism by Hartog's theorem; but then  $\overline{f}$  is constant by the maximal principle, contradiction.

Now for each  $i \leq k$ , we consider the Zariski closure the preimage  $\overline{f}^{-1}((-\beta_i, \alpha_i))$ , namely, the curve  $Q_i := \alpha_i Q_1 + \beta_i Q_2 = 0$  in  $\mathbb{P}^2$ . It can not intersect  $M$  since  $f(M)$  avoids the point  $(-\beta_i, \alpha_i)$ , so  $V(Q_i) \subseteq V(Q)$ , and thus  $Q_i$  has only irreducible factors  $L_i, i = 1, \dots, L_d$ , namely,  $Q_i = L_1^{m_{i,1}} L_2^{m_{i,2}} \dots L_d^{m_{i,d}}$  for some  $m_{i,j} \geq 0, j = 1, \dots, d$ . All these motivate the theory of multinets associated to a line arrangement.

**Definition 2.3.7** (See [FY]). A  $(k, e)$ -multinet  $\mathcal{M} = (\mathcal{A}; m; B)$  on a line arrangement  $\mathcal{A}$  consists of the following data:

- i) An integer  $k \geq 3$ , and a partition of  $\mathcal{A} = \{L_1, \dots, L_d\}$  into  $k$  subsets, say

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots \cup \mathcal{A}_k.$$

- ii) An assignment of multiplicities on the arrangement, namely,

$$m : \{1, \dots, d\} \rightarrow \mathbb{Z}_{>0}.$$

- iii) A finite set  $B$ , called the base locus.

Moreover, the following conditions are satisfied:

1. There is an integer  $e$  such that

$$\sum_{L_j \in \mathcal{A}_i} m(j) = e$$

for all  $i = 1, \dots, k$ .

2. For any  $L_j \in \mathcal{A}_i$  and  $L_{j'} \in \mathcal{A}_{i'}$  with  $i \neq i'$ ,  $L_j \cap L_{j'}$  belongs to  $B$ .

3. For each  $b \in B$ , the sum

$$n_b = \sum_{L_j \in \mathcal{A}_i : L_j \ni b} m(j)$$

is independent of  $i$ .

4. For each  $i \leq k$ , the space

$$\left( \bigcup_{L_j \in \mathcal{A}_i} L_j \right) \setminus B$$

is connected.

Without loss of generality, we always assume  $\gcd(m(j) : j = 1, \dots, d) = 1$ . If  $m(j) = 1$  for all  $j$ , the multinet is said to be *reduced*. If, in addition, every  $b \in B$  is contained in exactly one line from each  $\mathcal{A}_i$ , the multinet is called a  $(k, e)$ -*net*.

The following theorem summarizes what is known about the non-trivial multinets on arrangements.

**Theorem 2.3.8** (See [Yuz]). *Let  $\mathcal{M} = (\mathcal{A}; m; B)$  be a  $(k, e)$ -multinet on a line arrangement  $\mathcal{A}$ . Then the following hold.*

- (i) *If  $|B| > 1$ , then  $k = 3$  or  $4$ .*
- (ii) *If there is an  $j$  such that  $m(j) > 1$ , then  $k = 3$ .*

Now given a line arrangement  $\mathcal{A} = \{L_1, \dots, L_d\}$  and a  $(k, e)$ -multinet  $\mathcal{M} = (\mathcal{A}; m; B)$  on it, define

$$Q_i = \prod_{L_j \in \mathcal{A}_i} L_j^{m(j)}$$

for each  $i \leq k$ . Then as is shown in [FY],  $Q_1, \dots, Q_k$  belongs to a pencil, namely, the vector space  $\mathbb{C}\langle Q_1, \dots, Q_k \rangle$  has dimension 2. Let  $Q_i = \alpha_i Q_1 + \beta_i Q_2$  for  $i \leq q$  with  $(\alpha_1, \beta_1) = (1, 0)$  and  $(\alpha_2, \beta_2) = (0, 1)$  and  $G(u, v) = \prod_{i=1}^k (\alpha_i u + \beta_i v)$ , then we get a morphism

$$f : M = M(\mathcal{A}) \rightarrow C = \mathbb{P}^1 \setminus V(G), \quad f(x, y, z) = (Q_1(x, y, z), Q_2(x, y, z)).$$

This map  $f$  in fact gives an irrational pencil for  $M$ , which is called *the irrational pencil associated to the multinet  $\mathcal{M}$* . Moreover, we have the following: the following sets coincide

- $\mathcal{E}(M)$  consisting of irrational pencils of  $M$ ;
- $m\mathcal{E}(M)$  consisting of irrational pencils of  $M$  associated to multinets of  $\mathcal{A}$ ;

and, the assignment  $[f] \mapsto f^*\mathbb{T}(f(M))$  gives a one-to-one correspondence between  $\mathcal{E}(M)$  (or  $m\mathcal{E}(X)$ ) and

- $IC_1(M)$  consisting of all irreducible components of  $\mathcal{V}_1(M)$  passing through 1;

in addition, the assignment  $[f] \mapsto f^*H^1(f(M))$  gives a one-to-one correspondence between  $\mathcal{E}(X)$  (or  $m\mathcal{E}(X)$ ) and

- $IC_1(\mathcal{R}_1(M))$  consisting of all irreducible components of  $\mathcal{R}_1(M)$ ;

About the monodromy action, we have (see [LibE])

**Theorem 2.3.9.** *Let  $a \in \mu_d$  and assume that there is a line in  $\mathcal{A}$ , say  $L_1$ , such that the multiplicity  $m_p$  of any multiple points of  $V(Q(\mathcal{A}))$  situating on  $L_1$  satisfies either  $m_p = 2$  or  $a^{m_p} \neq 1$ , then  $H^1(F)_a = 0$ .*

### 2.3.10 Examples

We provide here some examples of arrangements by giving their defining polynomials. We only give brief descriptions of the dimensions of the eigenspaces  $H^1(F)_a$  for the monodromy  $h^* : H^1(F) \rightarrow H^1(F)$ , where  $F$  is the Milnor fiber.

In all the examples, we will use  $(P_1, P_2)$  to denote the pencil determined by  $P_1$  and  $P_2$ . A more standard notation will be given in Chapter 5.

1. (see [CS95]) The  $A_3$ -arrangement is defined by

$$Q = xyz(x-y)(y-z)(z-x)$$

with  $d = 6$  and  $\xi = \exp(2\pi\sqrt{-1}/6)$ . This arrangement is given by three completely reducible fibers of the pencil of conics  $(x(y-z), z(z-x))$ .

$V(Q)$  has 4 triple points and 3 double points, and on any line in the arrangement, there are 2 triple points and 1 double point of  $V(Q)$ . So by Theorem 2.3.9,  $H^1(F)_a \neq 0$  only if  $a = 1, \xi^2, \xi^4$ ; and by Theorem 2.3.3,  $\dim H^1(F)_{\xi^2} = \dim H^1(F)_{\xi^4}$ .

In fact, one has  $\dim H^1(F)_{\xi^2} = \dim H^1(F)_{\xi^4} = 1$ .

2. (see [BDS]) The Hesse arrangement is defined by

$$Q = xyz((x^3 + y^3 + z^3)^3 - 27x^3y^3z^3)$$

with  $d = 12$  and  $\xi = \exp(2\pi\sqrt{-1}/12)$ . This arrangement consists of the 4 completely reducible fibers in the pencil of cubics  $(x^3 + y^3 + z^3, xyz)$ .

$V(Q)$  has 12 double points and 9 points of multiplicity 4. So by Theorem 2.3.9,  $H^1(F)_a \neq 0$  only if  $a = 1, \xi^3, \xi^6, \xi^9$ .

In fact, one has  $\dim H^1(F)_a = 2$  for  $a = \xi^3, \xi^6, \xi^9$ .

3. (see [CS95]) The Pappus configuration  $(9_3)_1$  is given by

$$Q = xyz(x-y)(y-z)(x-y-z)(2x+y+z)(2x+y-z)(-2x+5y-z)$$

consists of three reducible fibers in a cubic pencil.

In this case,  $d = 9$  and  $\xi = \exp(2\pi\sqrt{-1}/9)$ . We have that  $H^1(F)_a \neq 0$  only if  $a = 1, \xi^3, \xi^6$ ; and by Theorem 2.3.3,  $\dim H^1(F)_{\xi^3} = \dim H^1(F)_{\xi^6}$ .

In fact, one has  $\dim H^1(F)_{\xi^3} = \dim H^1(F)_{\xi^6} = 1$ .

4. (see [Bai14],[Dim]) The Ceva arrangement is defined by

$$Q = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3),$$

with  $d = 9$  and  $\xi = \exp(2\pi\sqrt{-1}/9)$ .

This arrangement can be grouped in four different ways such that it contains three completely reducible fibers in a pencil of cubics. More precisely, let

$$\begin{cases} L_1 = x - y, & L_2 = x - \exp(2\pi\sqrt{-1}/3)y, & L_3 = x - \exp(4\pi\sqrt{-1})y, \\ L_4 = y - z, & L_5 = y - \exp(2\pi\sqrt{-1}/3)z, & L_6 = y - \exp(4\pi\sqrt{-1})z, \\ L_7 = z - x, & L_8 = z - \exp(2\pi\sqrt{-1}/3)x, & L_9 = z - \exp(4\pi\sqrt{-1})x, \end{cases}$$

then the arrangement contains three completely reducible fibers belonging to each of the following four pencils:

$$\begin{cases} P_1 = (L_1L_2L_3, L_4L_5L_6), \\ P_2 = (L_1L_4L_7, L_2L_5L_8), \\ P_3 = (L_1L_5L_9, L_2L_6L_8), \\ P_4 = (L_1L_6L_8, L_2L_4L_9). \end{cases}$$

We have  $\dim H^1(F)_a = 2$  for  $a = \xi^3, \xi^6$ , and 0 otherwise with  $a \neq 1$ .

5. (see [Dim]) More generally, consider the arrangement  $\mathcal{A}(m, m, 3)$  defined by

$$Q = (x^m - y^m)(y^m - z^m)(z^m - x^m).$$

Then the eigenvalues of  $h^* : H^1(F) \rightarrow H^1(F)$  are only cubic roots of unity. Set  $\theta = \exp(2\pi\sqrt{-1}/3)$ , then we have two cases:

- (i) If  $m \equiv 0 \pmod{3}$ , then  $\dim H^1(F)_\theta = \dim H^1(F)_{\theta^2} = 2$ ;
- (ii) If  $m \not\equiv 0 \pmod{3}$ , then  $\dim H^1(F)_\theta = \dim H^1(F)_{\theta^2} = 1$ .

Now we list some line arrangements for which  $H^1(F)_{\neq 1} = 0$ .

6. (see [CS95]) The arrangement  $(9_3)_2$  is defined by

$$Q = xyz(x+y)(y+z)(x+3z)(x+2y+z)(x+2y+3z)(4x+6y+6z),$$

with  $d = 9$ . For this arrangement,  $H^1(F)_{\neq 1} = 0$ .

7. (see [Bai14], [CS95]) The  $B_3$ -arrangement is defined by

$$Q = xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$$

for which  $H^1(F)_{\neq 1} = 0$ .

8. (see [Bai14]) The arrangement defined by

$$Q = xy(x^2 - y^2)(x^2 - 4y^2)((2x+y)^2 - z^2)((2x+y)^2 - z^2)((2x+y)^2 - 9z^2)$$

satisfies  $H^1(F)_{\neq 1} = 0$ .

Besides line arrangements, we can also consider curve arrangements. We will consider the singular fibers, which are either line arrangements or nodal cubics, in a pencil of cubics.

9. (see [Bai16]) Consider the pencil of cubics  $C_1 = (Q_1 = 3xyz + y^3 + z^3, Q_2 = 3xyz + x^3 + z^3)$ . Then there are 5 singular fibers in  $C_1$

$$Q_1, \quad Q_2, \quad Q_1 - Q_2, \quad Q_1 - t_1Q_2, \quad Q_1 - t_2Q_2$$

where  $t_1, t_2$  are the two roots of the equation  $t^2 - 3t + 1 = 0$ .

Let  $Q$  be the product of all the singular fibers, namely,

$$Q = Q_1Q_2(Q_1 - Q_2)(Q_1 - t_1Q_2)(Q_1 - t_2Q_2),$$

and  $F : Q = 1$  the Milnor fiber of  $Q$  in  $\mathbb{C}^3$ .

Define the map  $h$  and monodromy action  $h^* : H^1(F) \rightarrow H^1(F)$  as before, we have that the eigenvalues of  $h^*$  are all 5-th roots of unity. Moreover,

$$\dim H^1(F)_a = \begin{cases} 10, & a = 1; \\ 3, & a^5 = 1, a \neq 1. \end{cases}$$

Note also that  $Q$  is a free divisor.

10. (see [Bai16]) Consider the pencil  $C_2 = (Q_1 = y(x - y - z)(2x + y - z), Q_2 = xz(2x - 5y + z))$ , then there are 6 singular fibers in  $C_2$ :

$$Q_1, \quad Q_2, \quad Q_1 - Q_2, \quad Q_1 - t_1Q_2, \quad Q_1 - t_2Q_2, \quad Q_1 - t_3Q_2,$$

where  $t_1, t_2, t_3$  are the three roots of the equation  $125t^3 - 399t^2 + 339t - 1 = 0$ .

Let  $Q = Q_1Q_2(Q_1 - Q_2)(Q_1 - t_1Q_2)(Q_1 - t_2Q_2)(Q_1 - t_3Q_2)$  and  $F$  its Milnor fiber. Then the eigenvalues of the monodromy  $h^* : H^1(F) \rightarrow H^1(F)$  are all 6-th roots of unity. Moreover,

$$\dim H^1(F)_a = \begin{cases} 11, & a = 1; \\ 4, & a^6 = 1, a \neq 1. \end{cases}$$

$Q$  is also a free divisor.

## First Logarithmic Resonance Varieties

### 3.1 History

The study of relations between characteristic varieties and resonance varieties dates back to [CS99], in which the case of hyperplane arrangement complements is discussed. This is the first paper, to my knowledge, concerning such relations between the seemingly quite different cohomology jump loci. After that, A. Libgober in [LibF] discusses these relations in general settings, and claims incorrectly that  $TC_1(\mathcal{V}_1(X)) = \mathcal{R}_1(X)$  for *any* algebraic variety  $X$ . But he proves correctly the inclusion  $TC_1(\mathcal{V}_1(X)) \subseteq \mathcal{R}_1(X)$ . In 2009, the paper [DPS09] came out. In this paper the equality  $TC_1(\mathcal{V}_1(X)) = \mathcal{R}_1(X)$  is established for *1-formal* varieties and many applications are given. In recent papers, such as [DP14] or [BW], the tangent cone is no longer mentioned, but replaced by an isomorphism of germs  $(\mathcal{V}_1(X), 1) \cong (\mathcal{R}_1(X), 0)$ . This is due to the fact that the variety  $R_1(X)$  is in fact a cone, i.e. it is defined by homogeneous equations, and for such varieties, one has  $TC_0(\mathcal{R}_1(X)) = \mathcal{R}_1(X)$ ; what is more, an isomorphism of germs  $(X, 0) = (Y, 0)$  implies an isomorphism of their tangent cones  $TC_0(X) = TC_0(Y)$ .

Now let  $X$  be a quasi-projective manifold such that  $X = \overline{X} \setminus D$ , where  $\overline{X}$  is a smooth compactification of  $X$  and  $D$  is a normal crossing divisor in  $\overline{X}$  and  $D = \cup_{i \in I} D_i$  such that every irreducible component  $D_i$  of  $D$  is a smooth hypersurface.

As in the previous chapter, the cohomology  $H^\bullet(X)$  admits a MHS defined via the logarithmic complex  $\Omega_{\overline{X}}^\bullet(\log D)$ . More precisely, a decreasing Hodge filtration  $F$  and an increasing weight filtration  $W$  can be given on the complex  $\Omega_{\overline{X}}^\bullet(\log D)$ ; and we have the spectral sequences

$${}_F E_1^{p,q} = H^q(\overline{X}, \Omega_{\overline{X}}^p(\log D)) \Rightarrow H^{p+q}(X)$$

degenerating at  $E_1$ , and

$${}_W E_1^{p,q} = H^{2p+q}(D^{(-p)}, \mathbb{C})(p) \Rightarrow H^{p+q}(X)$$

(where  $(p)$  denotes the Tate twist) degenerating at  $E_2$ . In particular, the MHS of  $H^1(X)$  is given by

$$H^1(X) = H^{1,0}(X) \oplus H^{0,1}(X) \oplus H^{1,1}(X)$$

with

$$W_1 H^1(X) = H^{0,1}(X) \oplus H^{1,0}(X), \quad F^1 H^1(X) = H^{1,0}(X) \oplus H^{1,1}(X)$$

and

$$H^{1,1}(X) = F^1 H^1(X) \cap \overline{F^1 H^1(X)}.$$

As a consequence,  $Gr_2^W H^1(X)$  is pure of type  $(1, 1)$ .

In addition,  $X$  admits a model, the Gysin model  $A$ , as in Example 2.2.8, which is given by

$$A^k = \bigoplus_{p+l=k} A^{p,l}$$

where

$$A^{p,l} = \bigoplus_{|S|=l} H^p \left( \bigcap_{i \in S} D_i \right) (-l)$$

where  $S$  runs through all the  $l$ -element subsets of  $I$  and  $(-l)$  denotes the Tate twist. The multiplication is induced by the cup-product and satisfies

$$A^{p,l} \cdot A^{p',l'} \subseteq A^{p+p',l+l'}$$

and the differentials  $d : A^k \rightarrow A^{k+1}$  are induced by the  $d'_1 s : {}_w E_1^{\bullet,\bullet} \rightarrow {}_w E_1^{\bullet,\bullet}$ , see also [Mor] and [DPS15]. In particular, the differential  $d : A^0 \rightarrow A^1$  is the zero map, and the differential  $d : A^1 \rightarrow A^2$  is given by

$$d(\eta, (b_j)_{j \in J}) = \left( \sum_{j \in J} \iota_{j!}(b_j), 0, 0 \right) \quad (3.1.1)$$

for  $\eta \in H^1(\overline{X})$  and  $b_j \in H^0(D_j)$ . Here  $\iota_j : D_j \rightarrow \overline{X}$  denotes the inclusion and  $\iota_{j!} : H^0(D_j) \rightarrow H^2(\overline{X})$  the corresponding Gysin map.

A fundamental result in [DP14] says that the tangent cone  $TC_1(\mathcal{V}_k^j(X))$  can be identified with the resonance variety  $\mathcal{R}_k^j(A)$ . Moreover, it is a finite union of linear subspaces of  $H^1(X)$  and in addition, any irreducible component of  $TC_1(\mathcal{V}_k^j(X)) \cong \mathcal{R}_k^j(X)$  is in fact a rational mixed Hodge substructure of  $H^1(X)$ . A fortiori,  $\mathcal{R}_1(X)$  is a finite union of mixed Hodge substructures of  $H^1(X)$ . On the other hand, unless  $X$  is 1-formal, the resonance variety  $\mathcal{R}_1(X)$  is in general not a union of linear subspaces of  $H^1(X)$ .

To extend the linearity of  $\mathcal{R}_1(X)$  from a 1-formal variety to a general quasi-projective manifold, A. Dimca in [Dim10] introduces the logarithmic cohomology complex

$$F^k(X) = F^k H^k(X) = H^0(\overline{X}, \Omega_{\overline{X}}^k(\log D)),$$

without any formality assumption on  $X$  and defines the first logarithmic resonance variety

$$LR_1(X) = \{\alpha \in F^1(X) \quad : \quad H^1(F^\bullet(X), \alpha \wedge) \neq 0\}.$$

An illustrative theorem about the first logarithmic variety  $LR_1(X)$  in [Dim10] states that  $LR_1(X)$  is a finite union of linear subspaces of  $F^1(X)$ , and for any irreducible component  $I$  of  $LR_1(X)$ ,  $I + \overline{I}$  is an irreducible component of  $TC_1(\mathcal{V}_1(X)) = \mathcal{R}_1(A)$ , where  $A$  is the Gysin model.

In fact, by considering the relations between isotropic subspaces and irrational pencils, A. Dimca proves the following.

**Theorem 3.1.1** (See [Dim10], Proposition 4.5). *For any quasi-projective manifold  $X$ , the following hold.*

- (i) *The (strictly) positive dimensional irreducible components of  $LR_1(X)$  are exactly the maximal isotropic subspaces  $I \subseteq F^1(X)$ , namely  $\bigwedge^2 I = 0$  in  $F^2(X)$ , satisfying  $\dim I \geq 2$ .*

*As a consequence,  $LR_1(X)$  is a union of linear subspaces of  $F^1(X)$ .*

(ii) If  $I$  and  $I'$  are distinct irreducible components of  $LR_1(X)$ , then  $I \cap I' = 0$ .

(iii) The mapping

$$[f] \mapsto I_f = f^*F^1(C) = f^*H^0(\bar{C}, \Omega_{\bar{C}}^1(\log B));$$

induces a bijection between the set  $\mathcal{E}_0(X)$  of equivalence classes of irrational pencils  $f : X \rightarrow C$  with  $g^*(C) \geq 2$  and the set of positive dimensional irreducible components of the first logarithmic resonance variety  $LR_1(X)$ .

Here  $\bar{C}$  is a smooth compactification for  $C$ ,  $B = \bar{C} - C$  is a finite set and  $g^*(C) = b_1(C) - g(\bar{C}) = \dim H^0(\bar{C}, \Omega_{\bar{C}}^1(\log B))$ .

(iv) Moreover, if  $I \neq 0$  is an irreducible component of  $LR_1(X)$ , then  $I + \bar{I}$  is an irreducible component of  $TC_1(\mathcal{V}_1(X))$ .

Conversely, any irreducible component  $E = E_f \neq 0$  of  $TC_1(\mathcal{V}_1(X))$ , not coming from an irrational pencil  $f : X \rightarrow C$  onto a once-punctured elliptic curve  $C$ , is of this form, with  $I = F^1E$ .

In particular,  $\alpha \in LR_1(X)$  if and only if both Hodge type components  $\alpha^{1,0}$  and  $\alpha^{1,1}$  of  $\alpha$  are in the same irreducible component of  $LR_1(X)$ .

In this chapter, we establish the basic relations between the tangent cone  $TC_1(\mathcal{V}_1(X))$  and the first logarithmic variety  $LR_1(X)$  without using the results in Arapura [Ara] and Bauer[Bau] on irrational pencils, see also the next chapter of this thesis; instead, our discussions rely only on computations via the Gysin model above.

We first propose a quite useful proposition for discussing the relations between  $TC_1(\mathcal{V}_1(X))$  and  $LR_1(X)$ .

**Proposition 3.1.2.** *Let  $X$  be a quasi-projective manifold and  $E$  an irreducible component of  $TC_1(\mathcal{V}_1(X))$ , equipped with the induced MHS of  $H^1(X)$ . Then the following hold:*

- (i) if  $I = F^1E \subseteq LR_1(X)$  and has positive dimension, then  $F^1E$  is an irreducible component of  $LR_1(X)$ ;
- (ii) suppose that, moreover, for any irreducible component  $E$  of  $TC_1(\mathcal{V}_1(X))$  with  $F^1E$  of positive dimension,  $F^1E \subseteq LR_1(X)$ , then for any irreducible component  $I \subseteq LR_1(X)$  of positive dimension,  $I + \bar{I}$  is an irreducible component of  $TC_1(\mathcal{V}_1(X))$ .

*Proof.* Given an irreducible component  $E$  of  $TC_1(\mathcal{V}_1(X))$ . Assume  $F^1E \subseteq LR_1(X)$  has positive dimension.

Since  $E$  is a linear subspace of  $H^1(X)$ ,  $I = F^1E$  is an irreducible subvariety of  $F^1(X)$  and therefore there exists an irreducible component  $I'$  of  $LR_1(X)$  such that  $I' \supseteq I$ .

Now consider the Gysin model  $A$  of  $X$ . Since  $d : A^0 \rightarrow A^1$  is the zero map, the first cohomology group  $H^1(X)$  and hence  $F^1(X)$  can be naturally seen as a subspace of  $A^1$ . By the formula (3.1.1) and noting that  $d|_{F^1(X)} = 0$ , one obtains immediately that  $LR_1(X) \subseteq \mathcal{R}_1(A) = TC_1(\mathcal{V}_1(X))$ . In particular,  $I' \subseteq TC_1(\mathcal{V}_1(X))$  and thus, there exists an irreducible component  $E' \subseteq TC_1(\mathcal{V}_1(X))$  satisfying  $I' \subseteq E'$ .

From the choice of  $I'$ , we have  $F^1E \subseteq I' \subseteq E'$ , and hence  $E \subseteq E'$  since  $E'$  is a mixed Hodge substructure of  $H^1(X)$ . Therefore,  $E = E'$  since  $E$  and  $E'$  are both irreducible components of  $TC_1(\mathcal{V}_1(X))$ .

Consequently,  $I' \subseteq E' = E$ , and thus  $I' \subseteq E \cap F^1(X) = F^1E = I$ , namely,  $I = I'$  is an irreducible component of  $LR_1(X)$  and  $E = I + \bar{I}$ . This proves (i).

For (ii), given an irreducible component  $I$  of  $LR_1(X)$ , then it is an irreducible closed subset of  $\mathcal{R}_1(A) = TC_1(\mathcal{V}_1(X))$  and hence  $I \subseteq E$  for some irreducible component  $E$  of  $TC_1(\mathcal{V}_1(X))$ . By assumption,  $F^1E \subseteq LR_1(X)$  and  $I \subseteq E \cap F^1(X) = F^1E$ . Since  $I$  is an irreducible component, one has  $I = F^1E$  and thus  $E = I + \bar{I}$ .  $\square$

## 3.2 Case 1: $H^1$ having pure weight 2

When  $H^1(X)$  is of pure weight 2,  $LR_1(X)$  has special properties.

**Proposition 3.2.1.** *Let  $X$  be a quasi-quasi-projective manifold. If  $H^1(X)$  is pure of weight 2, then  $LR_1(X) = TC_1(\mathcal{V}_1(X))$ .*

*In particular,  $LR_1(X)$  is a union of linear subspaces of  $F^1(X)$ .*

*Proof.* If  $H^1(X)$  is pure of weight 2, then  $X$  is 1-formal (see [Mor]),  $\mathcal{R}_1(X) = TC_1(\mathcal{V}_1(X))$ . Moreover,  $F^1(X) = H^1(X)$ , so by the very definitions of  $\mathcal{R}_1(X)$  and  $LR_1(X)$ , we have  $LR_1(X) = \mathcal{R}_1(X)$ .  $\square$

## 3.3 Case 2: $X$ being compact

We first verify the condition in Proposition 3.1.2 when  $X$  is itself compact, and hence the Gysin model  $A = (H^\bullet(X), d = 0)$  with  $F^k(X) = H^{k,0}(X)$ . Note also that  $X$  is formal in this case.

**Claim 3.3.1.** *Let  $X$  be a projective manifold and  $E \neq 0$  an irreducible component of  $TC_1(\mathcal{V}_1(X))$ . Then  $F^1E \subseteq LR_1(X)$ .*

*Proof.* We use the decomposition

$$H^1(X) = H^{0,1}(X) \oplus H^{1,0}(X) = \overline{F^1(X)} \oplus F^1(X),$$

so any element  $\alpha \in H^1(X)$  can be written as  $\alpha = \bar{\beta} + \gamma$  with  $\beta, \gamma \in F^1(X)$ . Here we have represented the cohomology classes in  $F^1(X)$  by holomorphic (or anti-holomorphic) forms.

Given any  $0 \neq \alpha_1 \in F^1E$ , one has  $\bar{\alpha}_1 + \alpha_1 \in E$ . By definition, there exists  $\bar{\beta}_2 + \gamma_2 \in H^1(X) \setminus \mathbb{C}(\bar{\alpha}_1 + \alpha_1)$  such that

$$(\bar{\alpha}_1 + \alpha_1) \wedge (\bar{\beta}_2 + \gamma_2) = 0,$$

that is,

$$\alpha_1 \wedge \beta_2 = 0, \quad \alpha_1 \wedge \gamma_2 = 0, \quad \bar{\alpha}_1 \wedge \gamma_2 + \alpha_1 \wedge \bar{\beta}_2 = 0.$$

If  $\beta_2 \notin \mathbb{C}\alpha_1$  or  $\gamma_2 \notin \mathbb{C}\alpha_1$ , we are done. Otherwise, suppose

$$\beta_2 = A\alpha_1, \quad \gamma_2 = B\alpha_1,$$

for some  $A, B \in \mathbb{C}$ , then

$$0 = \bar{\alpha}_1 \wedge \gamma_2 + \alpha_1 \wedge \bar{\beta}_2 = (\bar{A} - B)\alpha_1 \wedge \bar{\alpha}_1,$$

meaning that  $(\bar{A} - B)\alpha_1 \wedge \bar{\alpha}_1$  is an exact differential form on  $X$ .

Let  $\omega$  be any Kähler form on  $X$ , then

$$0 = (\bar{A} - B)\alpha_1 \wedge \bar{\alpha}_1 \wedge \omega^{\dim X - 1},$$

meaning that the right hand side is an exact form, so integration over  $X$  gives

$$(\bar{A} - B) \int_X |\alpha_1|^2 = 0.$$

Since  $\alpha_1 \neq 0$ ,  $\bar{A} = B$  and thus

$$\bar{\beta}_2 + \gamma_2 = B(\bar{\alpha}_1 + \alpha) \in \mathbb{C}(\bar{\alpha}_1 + \alpha).$$

Contradiction. □

By applying Proposition 3.1.2, we obtain the following.

**Corollary 3.3.2.** *Let  $X$  be a projective manifold. Then the natural mapping  $E \mapsto F^1 E$  gives a one-to-one correspondence between the set of irreducible components of  $TC_1(\mathcal{V}_1(X))$  and that of  $LR_1(X)$ , with inverse given by  $I \mapsto I + \bar{I}$ .*

*In particular,  $LR_1(X)$  a finite union of linear subspaces of  $F^1(X)$ .*

### 3.4 Case 3: $X$ being a punctured curve

Now we consider the simplest non-compact case, namely  $X = C \setminus \{p_0, \dots, p_n\}$  is an  $(n + 1)$ -punctured curve, where  $C$  is a complete smooth curve of genus  $g$ .

The Gysin model of  $X$  is given by

$$\begin{cases} A^0 = H^0(C) = \mathbb{C}, \\ A^1 = H^1(C) \oplus \bigoplus_{j=0}^n H^0(p_j) = \mathbb{C}^{2g} \oplus \mathbb{C}^{n+1}, \\ A^2 = H^2(C) = \mathbb{C}. \end{cases}$$

The differential  $d : A^0 \rightarrow A^1$  is zero and  $d : A^1 \rightarrow A^2$  is given by

$$d(\eta, (b_1, \dots, b_{n+1})) = \sum_{j=1}^{n+1} b_j$$

where  $\eta \in H^1(C)$ .

Let

$$W_2 = \ker(d : \bigoplus_j H^0(p_j) \rightarrow A^2),$$

then one has the following

$$\begin{cases} H^0(X) = H^0(C) = \mathbb{C}, \\ H^1(X) = H^1(C) \oplus W_2 = H^{0,1}(C) \oplus H^{1,0}(C) \oplus W_2, \\ H^2(X) = 0, \end{cases}$$

and

$$\begin{cases} F^0(X) = H^0(C), \\ F^1(X) = H^{1,0}(C) \oplus W_2, \\ F^2(X) = 0. \end{cases}$$

Notice that  $\dim F^1(X) = g + n$  and  $\dim A^1 = 2g + n + 1$ .

1. If  $g + n > 1$ ,  $\dim F^1(X) > \dim F^0(X) + \dim F^2(X)$  and  $\dim A^1 > \dim A^0 + \dim A^2$ , in this case

$$L\mathcal{R}_1(X) = F^1(X), \quad \text{and} \quad TC_1(\mathcal{V}_1(X)) = H^1(X),$$

we are done.

2. If  $g + n = 0$ , then  $g = 0$  and  $n = 0$ , then  $X$  is a once-punctured sphere, hence is simply connected, and thus  $H^1(X) = 0$ , there is nothing to prove, because in this case  $TC_1(\mathcal{V}_1(X)) = 0$  and  $L\mathcal{R}_1(X) = 0$ .
3. Let  $g + n = 1$ , then  $\dim F^1(X) = 1$  and  $L\mathcal{R}_1(X) = 0$ .

There are two possibilities for  $TC_1(\mathcal{V}_1(X))$ :

- (i) if  $g = 0$  and  $n = 1$ . Then

$$\begin{cases} A^0 = \mathbb{C}, \\ A^1 = \mathbb{C}^2, \\ A^2 = \mathbb{C}. \end{cases}$$

The differential  $d : A^1 \rightarrow A^2$  is given by  $d(b_1, b_2) = b_1 + b_2$ . An easy calculation gives  $\mathcal{R}_1(A) = 0$ .

- (ii) if  $g = 1$  and  $n = 0$ . Then

$$\dim A^1 = 2g + n + 1 = 3 > 2 = \dim A^0 + \dim A^2,$$

therefore  $\mathcal{R}_1(A) = H^1(X) \neq 0$ .

Therefore, we have the following

**Corollary 3.4.1.** *Let  $X$  be a punctured curve. Then unless  $X$  is an elliptic curve with one point deleted, the natural mapping  $E \mapsto F^1 E$  gives a one-to-one correspondence between the set irreducible components of the tangent cone  $TC_1(\mathcal{V}_1(X))$  and that of  $L\mathcal{R}_1(X)$ , with inverse given by  $I \mapsto I + \bar{I}$ .*

*In any case,  $L\mathcal{R}_1(X)$  is a union of linear subspaces of  $F^1(X)$ .*

**Remark 3.4.2.** When  $X$  is a once-punctured elliptic curve, the computations above also shows that

$$L\mathcal{R}_1(X) = 0, \quad \text{and} \quad TC_1(\mathcal{V}_1(X)) = H^1(X)$$

Moreover,  $TC_1(\mathcal{V}_1(X)) = H^1(X)$  is 2-dimensional and pure of weight 1.

Let  $X$  be a quasi-projective manifold of higher dimension. Then irreducible components of  $TC_1(\mathcal{V}_1(X))$  correspond to irrational pencils  $f : X \rightarrow S$  with  $S$  a smooth curve of negative Euler characteristic, see Theorem 2.2.6. The discussions about the first logarithmic resonance varieties for punctured curves above may convince one to pay attention to the irrational pencils onto a once-punctured curve or irreducible components of  $TC_1(\mathcal{V}_1(X))$  having dimension 2 and pure weight 1.

### 3.5 Case 4: general setting

Now suppose  $X$  is a smooth quasi-projective algebraic variety of dimension  $m > 1$ , and  $\bar{X}$  is a compactification of  $X$  with  $D = \bar{X} - X = \bigcup_{j=1}^n D_j$  a normal crossing divisor such that each  $D_j$  is a smooth hypersurface in  $\bar{X}$ .

We explore in a detailed fashion of the relations between  $L\mathcal{R}_1(X)$  and  $TC_1(\mathcal{V}_1(X))$  via the Gysin model of  $X$ . Denote  $\iota_j : D_j \rightarrow \bar{X}$  the natural inclusion and  $\omega$  a Kähler form of  $\bar{X}$ .

The Gysin model of  $X$  is given by

$$\begin{cases} A^0 = H^0(\bar{X}, \mathbb{C}) = \mathbb{C}, \\ A^1 = H^1(\bar{X}, \mathbb{C}) \oplus \bigoplus_j H^0(D_j) = (H^{0,1}(\bar{X}) + H^{1,0}(\bar{X})) \oplus \mathbb{C}^n, \\ A^2 = H^2(\bar{X}, \mathbb{C}) \oplus \bigoplus_j H^1(D_j) \oplus \bigoplus_{j < k} H^0(D_j \cap D_k). \end{cases}$$

An element  $\alpha \in A^1$  will be written as  $(\bar{\beta} + \gamma, (b_j))$  with  $\beta, \gamma \in H^{1,0}(\bar{X})$  holomorphic forms and  $(b_j) \in \mathbb{C}^n$  an  $n$ -tuple of complex numbers.

The differential  $d : A^0 \rightarrow A^1$  is the zero map while  $d : A^1 \rightarrow A^2$  is given by

$$d(\bar{\beta} + \gamma, (b_j)) = \left( \sum_j \iota_{j!}(b_j), 0, 0 \right).$$

So

$$\begin{cases} H^0(X) = H^0(\bar{X}), \\ H^1(X) = H^{0,1}(\bar{X}) \oplus H^{1,0}(\bar{X}) \oplus W_2, \end{cases}$$

where

$$W_2 = \{(b_1, \dots, b_n) \in \mathbb{C}^n \mid \sum_j \iota_{j!}(b_j) = 0\} \subseteq A^1,$$

and in addition,

$$\begin{cases} F^0(X) = H^0(\bar{X}), \\ F^1(X) = H^{1,0}(\bar{X}) \oplus W_2. \end{cases}$$

Given a nonzero irreducible component  $E$  of  $TC_1(\mathcal{V}_1(X))$ , we shall find conditions on the divisor  $D$  such that  $F^1 E \subseteq L\mathcal{R}_1(X)$ . To this end, let  $0 \neq (\alpha, (b_j)) \in F^1 E \subseteq F^1(X)$ , we first have by definition  $\sum \iota_{j!}(b_j) = 0$ , i.e.,  $(b_j) \in W_2$ .

Since  $E$  is a mixed Hodge substructure of  $H^1(X)$ ,  $(\bar{\alpha} + \alpha, (b_j)) \in E$  and thus by definition, there exists

$$(\bar{\beta} + \gamma, (c_j)) \in A^1 \setminus \mathbb{C}(\bar{\alpha} + \alpha, (b_j))$$

such that

$$(d + (\bar{\alpha} + \alpha, (b_j)) \wedge)(\bar{\beta} + \gamma, (c_j)) = 0,$$

that is,

$$\left( \sum \iota_{j!}(c_j) + \bar{\alpha} \wedge \gamma + \alpha \wedge \bar{\beta} + \bar{\alpha} \wedge \bar{\beta} + \alpha \wedge \gamma, (\iota_j^*(\bar{\alpha} + \alpha)c_j + b_j \iota_j^*(\bar{\beta} + \gamma)), (b_j c_k D_j \cdot D_k) \right) = 0,$$

where  $D_j \cdot D_k$  is the intersection product of  $D_j$  and  $D_k$  as homology classes and  $\iota_j^* : H^1(X) \rightarrow H^1(D_j)$  is the induced map on cohomology by the inclusion  $\iota_j : D_j \rightarrow \bar{X}$ .

Therefore, we obtain

$$(1) \quad \alpha \wedge \beta = 0, \quad \alpha \wedge \gamma = 0,$$

$$(2) \sum \iota_{j!}(c_j) + \bar{\alpha} \wedge \gamma + \alpha \wedge \bar{\beta} = 0,$$

$$(3') \iota_j^*(\bar{\alpha} + \alpha)c_j + b_j \iota_j^*(\bar{\beta} + \gamma) = 0,$$

$$(4) (b_j c_k D_j \cdot D_k) = 0.$$

If the map  $\iota_j^* : H^1(\bar{X}) \rightarrow H^1(D_j)$  is injective for each  $j$ , then by (3'), it follows that

$$(\bar{\alpha} + \alpha)c_j + b_j(\bar{\beta} + \gamma) = 0,$$

or

$$(3) \quad c_j \bar{\alpha} = b_j \bar{\beta}, \quad c_j \alpha = b_j \gamma.$$

Here we identified  $b_j, c_j \in H^0(D_j)$  with their corresponding images in  $H^0(X)$  via the isomorphism  $\iota_j^* : H^0(\bar{X}) \simeq H^0(D_j)$ , and also used the fact that the product in  $A^1$  is anti-commutative. We shall consider the following three cases.

(i) If  $(b_j) = 0$ , then  $\alpha \neq 0$  and

$$\begin{cases} (5) & \alpha \wedge \beta = 0, \quad \alpha \wedge \gamma = 0, \\ (6) & \sum \iota_{j!}(c_j) + \bar{\alpha} \wedge \gamma + \alpha \wedge \bar{\beta} = 0, \\ (7) & c_j \bar{\alpha} = 0, \quad c_j \alpha = 0 \end{cases}$$

From (7),  $c_j \alpha$  is an exact form. Since  $\alpha \neq 0$ , we have that  $c_j = 0$ , thus  $(c_j) = 0$ . Therefore, we have

$$\begin{cases} \alpha \wedge \beta = 0, \quad \alpha \wedge \gamma = 0, \\ \bar{\alpha} \wedge \gamma + \alpha \wedge \bar{\beta} = 0, \\ (\bar{\beta} + \gamma, 0) \notin \mathbb{C}(\bar{\alpha} + \alpha, 0). \end{cases}$$

A similar argument as in the compact case gives that  $\beta \notin \mathbb{C}\alpha$  or  $\gamma \notin \mathbb{C}\alpha$ , and note that  $(\beta, 0) \wedge (\alpha, 0) = 0$  and  $(\gamma, 0) \wedge (\alpha, 0) = 0$ , so  $(\alpha, 0) \in L\mathcal{R}_1(X)$ .

(ii) If  $b_j \neq 0$  for all  $j$ , then from (3), we obtain

$$c_j \bar{\alpha} = b_j \bar{\beta}, \quad c_j \alpha = b_j \gamma,$$

so

$$\gamma = A_j \alpha, \quad \bar{\beta} = A_j \bar{\alpha}$$

with  $A_j = c_j/b_j$ .

• If  $\alpha \neq 0$ , then

$$A_1 = \cdots = A_n = A$$

with

$$A = \int_{\bar{X}} \gamma \wedge \bar{\alpha} \wedge \omega^{m-1} / \int_{\bar{X}} \alpha \wedge \bar{\alpha} \wedge \omega^{m-1},$$

and so

$$(\bar{\beta} + \gamma, (c_j)) = A(\bar{\alpha} + \alpha, (b_j)) \in \mathbb{C}(\bar{\alpha} + \alpha, (b_j)),$$

contradicting the choice of  $(\bar{\beta} + \gamma, (c_j))$ .

- If  $\alpha = 0$ , then

$$\sum \iota_{j!}(c_j) = 0, \quad b_j \bar{\beta} = 0, \quad b_j \gamma = 0.$$

From the last two equations, we get  $\gamma = \beta = 0$  since  $b_j \neq 0$ .

By assumption,  $(c_j) \notin \mathbb{C}(b_j)$  and from  $\sum \iota_{j!}(c_j) = 0$ , one has  $(0, (c_j)) \in F^1(M)$ , therefore  $(\alpha, (b_j)) = L\mathcal{R}_1(X)$ .

- (iii) If  $b_1 = \dots = b_p \neq 0$  and  $b_{p+1} = \dots = b_n = 0$  for some  $1 < p < n$ .

If  $\alpha \neq 0$ , then from the equations

$$c_j \bar{\alpha} = 0, \quad c_j \alpha = 0, \quad j = p + 1, \dots, n$$

we get  $c_j = 0$  for  $j = p + 1, \dots, n$ , and moreover, from  $b_j \gamma = c_j \alpha, j = 1, \dots, p$ , one obtains

$$c_j/b_j = A = \int_{\bar{X}} \gamma \wedge \bar{\alpha} \wedge \omega^{m-1} / \int_{\bar{X}} \alpha \wedge \bar{\alpha} \wedge \omega^{m-1}$$

for  $j = 1, \dots, p$ . Hence,  $\bar{\beta} = A\bar{\alpha}$  and  $\gamma = A\alpha$ , so

$$(\bar{\beta} + \gamma, (c_j)) = A(\bar{\alpha} + \alpha, (b_j))$$

contradiction.

Therefore, in this case, we must have  $\alpha = 0$ , and then

$$\sum \iota_{j!}(c_j) = 0$$

and hence  $(0, (c_j)) \in F^1(X)$ . Furthermore,  $b_1 \bar{\beta} = 0$  and  $b_1 \gamma = 0$  imply that  $\beta = \gamma = 0$ , so  $(c_j) \notin \mathbb{C}(b_j)$  by assumption. Thus,  $(0, (b_j)) \in L\mathcal{R}_1(X)$ .

In conclusion, if  $(\alpha, (b_j)) \in F^1 E$  for an irreducible component  $E$  of  $TC_1(\mathcal{V}_1(X))$ , one has  $(\alpha, (b_j)) \in L\mathcal{R}_1(X)$ , under the condition that  $\dim X > 1$  and each  $\iota_j^* : H^1(\bar{X}) \rightarrow H^1(D_j)$  is injective. By Proposition 3.1.2, we have

**Proposition 3.5.1.** *Let  $X$  be a (connected) quasi-projective manifold such that  $X = \bar{X} \setminus D$  as above of dimension  $> 1$ . Suppose each  $\iota_j^* : H^1(\bar{X}) \rightarrow H^1(D_j)$  is injective, then the natural mapping  $E \mapsto F^1 E$  gives a one-to-one correspondence between the set of irreducible components of the tangent cone  $TC_1(\mathcal{V}_1(X))$  and that of  $L\mathcal{R}_1(X)$ , with inverse given by  $I \mapsto I + \bar{I}$ .*

**Corollary 3.5.2.** *If  $X = \bar{X} - \bigcup_j D_j$  is a quasi-projective manifold of dimension  $n > 1$  and  $H_{2n-1}(\bar{X} - D_j) = 0$  for each  $j$ , then the correspondence in Proposition 3.5.1 holds. In particular, this is the case if for each  $j$ ,  $X - D_j$  is affine, for instance,  $D_j$  is an hyperplane section of the projective manifold  $\bar{X}$ .*

*Proof.* The first assertion follows from the cohomology exact sequence associated to the pair  $(\bar{X}, D_j)$ :

$$\dots \rightarrow H^1(\bar{X}, D_j) \rightarrow H^1(\bar{X}) \rightarrow H^1(D_j) \rightarrow \dots$$

and the duality isomorphism  $H^1(\bar{X}, D_j) \simeq H_{2n-1}(\bar{X} - D_j)$ . If  $\bar{X} - D_j$  is affine, just use the fact that a smooth affine variety of complex dimension  $n$  has the homotopy type of a CW complex of real dimension  $\leq n$ , see [Voi2], Theorem 1.22.  $\square$

The interested reader may have found that in all cases we have discussed above,  $L\mathcal{R}_1(X)$  is always a finite union of linear subspaces of  $F^1(X)$ . We have also emphasized the speciality of the once-punctured elliptic curve in our discussions about punctured curves. These phenomena are not isolated.

## Isotropic Subspace Theorems and Inequalities of Hodge Numbers

In this chapter, we aim at giving some inequalities of Hodge numbers, as extensions of those in [BHPV]. In [BHPV], the authors give some inequalities concerning the Hodge numbers  $h^{1,1}(X)$ ,  $h^{1,0}(X)$ ,  $h^{2,0}(X)$  etc. for a compact complex surface  $X$  that does not admit any irrational pencil. As is known that irrational pencils have close relations with the characteristic variety  $\mathcal{V}_1(X)$ , it is natural that more general inequalities on Hodge numbers can be obtained by taking into account the dimension of  $\mathcal{V}_1(X)$  for a more general quasi-projective manifold  $X$ .

### 4.1 Castelnuovo-De Franchis theorems

The existence of an irrational pencil can be ensured by the celebrated Castelnuovo-De Franchis (CDF) theorem for a compact surface, a Kähler manifold or a quasi-projective manifold. Plainly speaking, such theorems say that existence of two linearly independent holomorphic 1-forms with wedge product zero implies existence of an irrational pencil.

In this section, we shall give detailed proofs of several versions of CDF theorems. None of the theorems and proofs is new and they are in fact given in many books and articles; however, it is still difficult to find a reference containing all of them, that is why we include the statements and proofs here.

**Theorem 4.1.1** (CDF Theorem for compact Kähler manifolds). *Let  $X$  be a compact Kähler manifold, and  $\omega_1, \omega_2 \in H^0(X, \Omega_X^1)$  be two linearly independent 1-forms such that  $\omega_1 \wedge \omega_2 = 0$ .*

*Then there exists an irrational pencil  $f : X \rightarrow C$  from  $X$  to a complete smooth curve of genus  $\geq 2$  such that  $\omega_1, \omega_2 \in f^*H^1(C, K_C)$ .*

*Proof.* We first give an outline of the proof containing several claims, then we continue to prove our claims.

**Step 1: Outline of proof** By assumption,  $\omega_1 \wedge \omega_2 = 0$ , we have

**Claim 4.1.2.** *There exists a nonconstant meromorphic function  $f_1$  on  $X$  such that  $\omega_2 = f_1\omega_1$ .*

So we have a rational morphism  $f_1 : X \dashrightarrow \mathbb{P}^1$ . By resolving the indeterminacies via blowups, we obtain a proper morphism  $\tau : \tilde{X} \rightarrow X$  and a holomorphic map  $f_2 : \tilde{X} \rightarrow \mathbb{P}^1$ . Then the Stein factorization gives  $\tilde{X} \xrightarrow{f_3} C \xrightarrow{h} \mathbb{P}^1$  such that each fiber of  $f_3$  is connected. The

above process can be summarized by the following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f_3} & C \\ \tau \downarrow & \searrow f_2 & \downarrow h \\ X & \xrightarrow{f_1} & \mathbb{P}^1 \end{array}$$

By Stein Factorization Theorem,  $C$  is normal (since  $\tilde{X}$  is smooth), and hence smooth since  $C$  is 1-dimensional.

Since on a Kähler manifold, holomorphic forms are closed,  $d\omega_2 = 0$ , so we have

$$0 = d\omega_2 = d(f_1\omega_1) = df_1 \wedge \omega_1.$$

Let  $\tilde{\omega}_i = \tau^*\omega_i, i = 1, 2$ , then  $df_2 \wedge \tilde{\omega}_1 = 0$ . Hence,  $\tilde{\omega}_1 = g_1 df_2$  for some meromorphic function  $g_1$  on  $\tilde{X}$ .

**Claim 4.1.3.**  $g_1 \in f_3^*(\mathcal{M}(C))$ , namely,  $g_1$  is constant along the general fiber of  $f_3$ . Here  $\mathcal{M}(C)$  denotes the field of meromorphic functions on  $C$ .

In fact, since  $\tilde{X}$  is Kähler, we have  $0 = d\tilde{\omega}_1 = dg_1 \wedge df_2$ . Let  $C' \subseteq C \times \mathbb{P}^1$  be the image of the map  $(f_3, g_1) : \tilde{X} \dashrightarrow C \times \mathbb{P}^1$ , then  $C'$  has dimension 1. Moreover, since  $f_3 = pr_1 \circ (f_3, g_1) : \tilde{X} \rightarrow C$  has connected fibers ( $pr_1$  is the projection of  $C \times \mathbb{P}^1$  onto the first factor),  $pr_1|_{C'} : C' \rightarrow C$  is of degree one. Since  $C$  is smooth,  $C'$  is isomorphic to  $C$  by Zariski Main Theorem. From the isomorphism  $pr_1 : C' \simeq C$ , we obtain that  $g_1 \in f_3^*(\mathcal{M}(C))$ .

Similarly, we also have  $\tilde{\omega}_2 = g_2 df_2$ , with  $g_2 \in f_3^*(\mathcal{M}(C))$ . Indeed,  $\tilde{\omega}_2 = f_2 \tilde{\omega}_1 = f_2 g_1 df_2 := g_2 df_2$  with  $g_2 = f_2 g_1$ .

More precisely, from the discussions above, one obtains that  $C'$  is the graph of a map  $g'_1 : C \rightarrow \mathbb{P}^1$ , so  $g_1 = g'_1 \circ f_3$ . Thus,

$$\tilde{\omega}_1 = g_1 df_2 = f_3^*(g'_1) f_3^* dh = f_3^*(g'_1 dh).$$

Let  $\eta_1 = g'_1 dh$ . Then  $\eta_1$  is a meromorphic form on  $C$ .

**Claim 4.1.4.**  $\eta_1 \in H^0(C, K_C)$ , namely,  $\eta_1$  is holomorphic.

So  $\tilde{\omega}_1 \in f_3^* H^0(C, K_C)$ . Similarly,  $\tilde{\omega}_2 \in f_3^* H^0(C, K_C)$ .

We are done once we show the following

**Claim 4.1.5.** The blowup  $\tau : \tilde{X} \rightarrow X$  is not needed, namely,  $f_3 \circ \tau^{-1} : X \dashrightarrow C$  is a morphism.

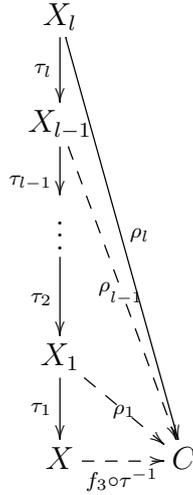
**Step 2: Proof of Claim 4.1.2** Indeed, given  $p$  such that  $\omega_1(p) \neq 0$ , we may choose local holomorphic coordinates  $(z_1, z_2, \dots)$  around  $p$  on  $X$  such that  $\omega_1 = dz_1$ , by the fact that  $d\omega_1 = 0$  and the holomorphic Poincaré Lemma. Then  $\omega_1 \wedge \omega_2 = 0$  gives  $\omega_2 = f_1 \omega_1$  for some locally defined holomorphic function  $f_1$ . So one obtains that  $f_1 = \omega_2/\omega_1$  is a well-defined meromorphic function on  $X$ . Clearly,  $f_1$  is not constant since  $\omega_1$  and  $\omega_2$  are linearly independent.

**Step 3: Proof of Claim 4.1.4** Choose a point  $p \in C$ , and local coordinate  $t$  around  $p$ . Further, choose a smooth point  $x \in f^{-1}(p)_{\text{red}}$ . Then there are local coordinates  $(z_1, \dots, z_n)$  around  $x$  such that  $t \circ f_3 = z_1^k$ . If  $\eta_1 = \phi_1(t) dt$ , then

$$\tilde{\omega}_1 = f_3^* \eta_1 = \phi(z_1^k) k z_1^{k-1} dz_1.$$

Since  $\tilde{\omega}_1$  is holomorphic,  $\phi_1$  and hence  $\eta_1$  is holomorphic.

**Step 4: Proof of Claim 4.1.5** Since  $\tau$  is a composition of a sequence of blowups, we have



with  $X_l = \tilde{X}$ ,  $\tau = \tau_1 \circ \tau_2 \circ \dots \circ \tau_l$ ,  $\rho_l = f_3$  and  $\tau_j : X_j \rightarrow X_{j-1}$  ( $X_0 = X$ ) a blowup along a smooth submanifold of  $X_{j-1}$ . In particular,  $X_j$  is smooth for all  $j$ .

Since  $\dim f^*H^0(C, K_C) \geq 2$ , we have that  $g(C) \geq 2$  and so the universal cover  $\tilde{C}$  is isomorphic to  $D = \{z \in \mathbb{C} : |z| < 1\}$  by the uniformization theorem. For  $m > 0$ , any morphism  $\mathbb{P}^m \rightarrow C$  factors through  $\mathbb{P}^m \rightarrow D$ , hence must be constant by the maximum principle.

Now consider the final blowup  $\tau_l : X_l \rightarrow X_{l-1}$ . Each fiber of  $\tau_l$  is either one point or some  $\mathbb{P}^m$  for some  $m > 0$ , hence each fiber of  $\tau_l$  is mapped to a point under  $\rho_l$ , thus  $\rho_{l-1}$  is a morphism and the blowup  $\tau_l$  is not needed. By a decreasing induction on  $l$ , we obtain that, none of the blowups  $\tau_l$  is needed and  $f_3 \circ \tau^{-1}$  is a morphism. We are done.  $\square$

**Theorem 4.1.6** (CDF Theorem for surfaces). *Let  $X$  be a compact smooth surface and  $\omega_1, \omega_2$  two linearly independent holomorphic 1-forms such that  $\omega_1 \wedge \omega_2 = 0$ .*

*Then there exists an irrational pencil  $f : X \rightarrow C$  with  $C$  a complete smooth curve of genus  $\geq 2$  such that  $\omega_1, \omega_2 \in f^*H^0(C, K_C)$ .*

*Proof.* By checking the proof of the previous theorem, it suffices to show that any holomorphic 1-form is closed.

Indeed, let  $\omega$  be a holomorphic 1-form, one has

$$0 = \int_X d(\omega \wedge d\bar{\omega}) = \int_X d\omega \wedge d\bar{\omega},$$

so  $d\omega = 0$ . (Let  $d\omega = fdz_1 \wedge dz_2$ , then

$$d\omega \wedge d\bar{\omega} = |f|^2 dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 = 4|f|^2 \text{dvol.}) \quad \square$$

**Remark 4.1.7.** A compact surface is not necessarily Kähler. In fact, a surface is Kähler if and only if the first Betti number  $b_1$  is even, see [BHPV].

**Theorem 4.1.8** (Log CDF Theorem, see [Bau]). *Let  $X$  be a quasi-projective manifold with a smooth compactification  $\bar{X} : D = \bar{X} - X$  being a divisor with normal crossings, and  $\omega_1, \omega_2 \in H^0(\bar{X}, \Omega_{\bar{X}}^1(\log D))$  be two linearly independent 1-forms with  $\omega_1 \wedge \omega_2 = 0$  in  $H^0(\bar{X}, \Omega_{\bar{X}}^2(\log D))$ .*

Then there exists an irrational pencil  $f : X \rightarrow C$  with  $C$  a quasi-projective curve such that  $\omega_1, \omega_2 \in f^*H^0(\overline{C}, \Omega_{\overline{C}}^1(\log(\overline{C} - C)))$ . Moreover, we have the following diagram

$$\begin{array}{ccc} X & \longrightarrow & \overline{X} \\ \downarrow f & & \downarrow \bar{f} \\ C & \longrightarrow & \overline{C} \end{array}$$

where  $\overline{C}$  is a smooth compactification of  $C$ .

**Remark 4.1.9.** Let  $C$  be a non-complete smooth curve and  $\overline{C}$  its smooth compactification. Then we call  $g(C) = g(\overline{C})$  the *genus* of  $C$  and  $g^*(C) = \dim H^0(\overline{C}, \Omega_{\overline{C}}^1(\log(\overline{C} - C)))$  the *logarithmic genus* of  $C$ . Clearly,  $b_1(C) = g(C) + g^*(C)$ .

The irrational pencil  $f : X \rightarrow C$  obtained above satisfies  $g^*(C) \geq 2$ , which is a stronger condition than  $\chi(C) < 0$ . Such irrational pencils are called *logarithmic irrational pencils*. If  $C$  is not complete, the pencil  $f$  is called *strictly logarithmic*.

*Proof.* As in the proof of Theorem 4.1.1, we have

- (i)  $\omega_2 = f_1\omega_1$  for some rational function  $f_1$  on  $\overline{X}$ .

$f_1$  defines a rational map  $\overline{X} \dashrightarrow \mathbb{P}^1$ , and hence by resolving the indeterminacies and applying Stein factorization we have the following commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f_3} & \overline{C} \\ \tau \downarrow & \searrow f_2 & \downarrow h \\ \overline{X} & \xrightarrow{f_1} & \mathbb{P}^1 \end{array}$$

as in the proof of Theorem 4.1.1.

- (ii) Since elements in  $H^0(\overline{X}, \Omega_{\overline{X}}^1(\log D))$  are closed by mixed Hodge theory, a similar argument as in the proof of Theorem 4.1.1 gives

$$\tau^*\omega_1 = \tilde{\omega}_1 = f_3^*\eta_1, \quad \tau^*\omega_2 = \tilde{\omega}_2 = f_3^*\eta_2,$$

with  $\eta_1, \eta_2$  rational 1-forms on  $\overline{C}$ .

**Claim 4.1.10.**  $\eta_1$  is a logarithmic form.

Indeed, as in the proof of Claim 4.1.4, we have here  $\omega_1 = \phi_1(z_1^k)kz_1^{k-1}dz_1$ . Since  $\omega_1$  has at most logarithmic poles by assumption, so does  $\eta_1$ .

Similarly, we have that  $\eta_2$  has at most logarithmic poles.

The smooth curve  $C = \overline{C} - B$  with  $B \subseteq \overline{C}$  a finite set such that  $\eta_1, \eta_2 \in H^0(\overline{C}, \Omega_{\overline{C}}^1(\log B))$ .

- (iii) The blowup  $\tau : \tilde{X} \rightarrow \overline{X}$  is not needed.

If  $g(\overline{C}) \geq 1$ , we are done by a similar argument as in the proof of Claim 4.1.5, since the universal cover of  $\overline{C}$  is either  $\mathbb{C}$  or  $D = \{z \in \mathbb{C} : |z| < 1\}$ . Observe also that each exceptional fiber of  $\tau$  is covered by various projective spaces  $\mathbb{P}^m$ 's ( $m > 0$ ) and is connected by Zariski Main Theorem.

Now we assume  $\overline{C} = \mathbb{P}^1$ . If there exists an  $x \in \overline{X}$  such that  $\omega_i$ ,  $i = 1$  or  $2$  is regular at  $x$  and a  $\mathbb{P}^m \subseteq \tau^{-1}(x)$  for some  $m > 0$ , then since  $\omega_i$  is regular at  $x$ ,  $\tilde{\omega}_i$  is regular along  $\mathbb{P}^m$ . If  $f_3(\mathbb{P}^m) = \mathbb{P}^1$ , then  $\eta_1$  is regular, we get a contradiction since  $H^0(\mathbb{P}^1, K_{\mathbb{P}^1}) = 0$ . Hence  $f_3(\mathbb{P}^m)$  is a point and the blowup is not necessary.

Therefore, the base locus of  $f_3 \circ \tau^{-1}$  is contained in  $Y \subseteq D$  where neither  $\omega_1$  nor  $\omega_2$  is regular. Note that the base locus of  $f_3 \circ \tau^{-1}$  is of codimension 2: in fact, since  $\mathbb{P}^1$  is projective, the base locus has codimension  $\geq 2$ ; since the map  $f_3 \circ \tau^{-1}$  can be written as  $(f_3^1, f_3^2) : X \dashrightarrow \mathbb{P}^1$ , the base locus is  $V(f_3^1, f_3^2)$ , namely the common zeros of  $f_3^1, f_3^2$ , hence it has codimension  $\leq 2$ . On the other hand, since  $C$  is equal to  $\mathbb{P}^1$  minus at least three points (because  $\dim H^0(C, \Omega_C^1(\log(\overline{C}-C))) \geq 2$ ), the base locus is contained in the intersection of at least three fibers of  $f_3$  and thus three distinct irreducible components of  $D$ , hence it has codimension  $\geq 3$  since  $D$  has normal crossings, contradiction.  $\square$

**Theorem 4.1.11** (CDF Theorem for several forms, see [Cat],[Bau]). *The following hold:*

1. *Let  $X$  be a compact Kähler manifold, or a compact surface, and  $\omega_1, \dots, \omega_r, r \geq 2$  linearly independent forms in  $H^0(X, \Omega_X^1)$  such that  $\omega_i \wedge \omega_j = 0$  in  $H^0(X, \Omega_X^2)$  for any pair  $(i, j)$ .*

*Then there exists an irrational pencil  $f : X \rightarrow C$  with  $C$  a complete smooth curve of genus  $\geq 2$  such that  $\omega_i \in f^*H^0(C, K_C)$ ,  $i = 1, \dots, r$ .*

2. *Let  $X$  be a quasi-projective manifold with a smooth compactification  $\overline{X} : D = \overline{X} - D$  is a divisor with normal crossings, and  $\omega_1, \dots, \omega_r \in H^0(\overline{X}, \Omega_{\overline{X}}^1(\log D))$  linearly independent forms with  $\omega_i \wedge \omega_j = 0$  in  $H^0(\overline{X}, \Omega_{\overline{X}}^2(\log D))$  for any pair  $(i, j)$ .*

*Then there exists a logarithmic irrational pencil  $f : X \rightarrow C$  with  $C$  a quasi-projective curve such that  $\omega_i \in f^*H^0(\overline{C}, \Omega_{\overline{C}}^1(\log(\overline{C}-C)))$ ,  $i = 1, \dots, r$ . Moreover, we have the following diagram*

$$\begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ \downarrow f & & \downarrow \bar{f} \\ C & \hookrightarrow & \overline{C} \end{array}$$

*where  $\overline{C}$  is a smooth compactification of  $C$ .*

The proof is completely the same as Theorem 4.1.1, Theorem 4.1.6 and Theorem 4.1.8.

## 4.2 Isotropic subspace theorems

In this section, we prove several isotropic subspace theorems, as natural extensions and applications of CDF theorems.

**Theorem 4.2.1** (See [Cat]). *Let  $X$  be a compact Kähler manifold, then there exists a irrational pencil  $f : X \rightarrow C$ , where  $C$  is a complete smooth curve of genus  $g \geq 2$ , if and only if there is a  $g$ -dimensional maximal isotropic subspace  $V$  of  $H^1(X)$ . Here saying  $V$  is isotropic means that its image is zero under the cup product map  $H^1(X) \wedge H^1(X) \rightarrow H^2(X)$ , i.e.  $\bigwedge^2 V = 0$ .*

*Moreover, any maximal isotropic subspace  $V$  as above occurs as a pull-back  $f^*V'$  of a maximal isotropic subspace  $V'$  of  $H^1(C)$  for some  $f : X \rightarrow C$  as above.*

*Proof.* Let  $V$  be a isotropic subspace of dimension  $g \geq 2$ , and  $\phi_1, \dots, \phi_g$  a basis of  $V$ . Since  $H^1(X, \mathbb{C}) = H^0(X, \Omega_X^1) \oplus \overline{H^0(X, \Omega_X^1)}$ , we can write  $\phi_1 = \omega_1 + \overline{\eta_1}, \dots, \phi_g = \omega_g + \overline{\eta_g}$  with  $\omega_i, \eta_j \in H^0(X, \Omega_X^1)$  for  $i, j = 1, \dots, g$ . Let

$$U = \mathbb{C}\langle \omega_1, \dots, \omega_g \rangle, \quad W = \mathbb{C}\langle \eta_1, \dots, \eta_g \rangle.$$

By assumption, one has, for any  $i, j$

$$(\omega_i + \overline{\eta_i}) \wedge (\omega_j + \overline{\eta_j}) = 0,$$

so

$$\omega_i \wedge \eta_i = 0, \quad \eta_i \wedge \eta_j = 0, \quad \omega_i \wedge \overline{\eta_j} + \overline{\eta_i} \wedge \omega_j = 0.$$

In particular,  $\bigwedge^2 U = 0$  and  $\bigwedge^2 W = 0$ . The proof is divided into three cases:

1. If  $\dim U \geq 2$  and  $\dim W \geq 2$ , then we can first apply CDF Theorem to  $U$  and  $W$  respectively to get two irrational pencils  $f : X \rightarrow C$  and  $f' : X \rightarrow C'$ . Consider now the product  $\phi : (f, f') : X \rightarrow C \times C'$ .

- (i) If the image of  $\phi$  is a curve  $C''$ , we are done by Stein factorization, since  $V$  pulls back from  $H^1(C'')$ .
- (ii) Otherwise,  $\phi$  is surjective. Then  $\phi^* : H^1(C \times C') \rightarrow H^1(X)$  is injective since  $X$  is Kähler (see [Voi1], Lemma 7.28) and from Künneth formula, one has  $H^\bullet(C \times C') = H^\bullet(C) \otimes H^\bullet(C')$ . In this case, the relations between  $\omega_i$ 's and  $\eta_j$ 's can not hold, and we are done.

Indeed, assume WLOG  $\omega_1 \neq 0$ , we have

$$\omega_1 \wedge \overline{\eta_j} + \omega_j \wedge \overline{\eta_1} = 0$$

in  $\phi^* H^\bullet(C \times C') = f^* H^\bullet(C) \otimes f'^* H^\bullet(C') \cong H^\bullet(C) \otimes H^\bullet(C')$ . If for some  $j \geq 2$ ,  $\omega_j, \omega_1$  are linearly independent, we obtain  $\omega_1 \wedge \overline{\eta_j} = 0$  and thus  $\eta_j = 0$ . Therefore, we can choose a basis of  $V$  among which an element is either completely contained in  $U$  or completely contained in  $\overline{W}$ . But for a nonzero element of  $U$  and nonzero element of  $\overline{W}$ , their wedge product cannot be zero by the Künneth formula for  $H^\bullet(C \times C')$ . This contradicts that fact that  $V$  is isotropic and that  $\dim U \geq 2$  and  $\dim W \geq 2$ .

2. If  $\dim U = 1$ , we may assume  $\omega_2 = \dots = \omega_g = 0$ , then  $\omega_1 \wedge \overline{\eta_j} = 0$  for  $j \geq 2$ , hence  $\omega_1 \wedge \eta_j = 0$ . In fact, let  $\omega$  be any Kähler form on  $X$ ,

$$0 = \int_X (\omega_1 \wedge \overline{\eta_j}) \wedge \overline{\omega_1 \wedge \overline{\eta_j}} \wedge \omega^{\dim X - 2} = - \int_X (\omega_1 \wedge \eta_j) \wedge \overline{\omega_1 \wedge \eta_j} \wedge \omega^{\dim X - 2},$$

so  $\omega_1 \wedge \eta_j = 0$ .

We also have  $\omega_1 \wedge \eta_1 = 0$ . Indeed, from  $\eta_1 \wedge \eta_j = 0$  for  $j \geq 2$ , we have that  $\eta_1$  is a meromorphic multiple of  $\eta_j$  if  $\eta_j \neq 0$ . Therefore, from  $\omega_1 \wedge \eta_j = 0$ , we get  $\omega_1 \wedge \eta_1 = 0$ .

Thus, the CDF Theorem can be applied to the subspace  $U + W$ , whose dimension is  $\geq 2$ , hence  $V$  pulls-back from a curve of genus  $\geq 2$ . To see why  $\dim(U + W) \geq 2$ , we assume the contrary, then all  $\eta_j$ 's are scalar multiples of  $\omega_1$ , and thus we have  $\omega_1 \wedge \overline{\omega_1} = 0$ , which implies that  $\omega_1 = 0$ , contradiction.

The genus of the curve must be equal to  $g$  by the maximality of  $V$ . In fact, let  $f : X \rightarrow C$  be the map with  $C$  of genus  $h$ . Then since  $X$  is Kähler,  $f^* : H^1(C) \rightarrow H^1(X)$  is injective and note that any isotropic subspace of  $f^*H^1(C) \subseteq H^1(X)$  has dimension  $h$ . By our settings,  $V \subseteq f^*H^1(C)$  is isotropic, hence it is contained in a maximal isotropic subspace  $V'$  of  $f^*H^1(C)$ . On the other hand,  $V'$  is also an isotropic subspace of  $H^1(X)$ , hence  $V' \subseteq V''$  for a maximal isotropic subspace  $V''$  of  $H^1(X)$ . Therefore we have  $V \subseteq V' \subseteq V''$ . The equalities hold because  $V$  is also maximal isotropic subspace for  $H^1(X)$ . Consequently,  $g = \dim V = \dim V' = h$ .

The same argument applies if  $\dim W = 1$ .

3. If  $\dim U = 0$  or  $\dim W = 0$ , then we can directly apply the CDF Theorem and complete the proof.  $\square$

From the proof above, one can see that on a compact surface, we have the following.

**Theorem 4.2.2.** *Let  $X$  be a compact surface, then there exists an irrational pencil  $f : X \rightarrow C$ , where  $C$  is a complete smooth curve of genus  $g \geq 2$ , if and only if there is a  $g$ -dimensional maximal isotropic subspace  $V$  of  $H^1(X)$ .*

*Moreover, any maximal isotropic subspace  $V$  as above occurs as a pull-back  $f^*V'$  of a maximal subspace  $V'$  of  $H^1(C)$  for some  $f : X \rightarrow C$  as above.*

*Proof.* By following the proof of Theorem 4.2.1, it suffices check the following:

- (i) If  $\omega \wedge \eta = 0$  for  $\omega \in H^0(X, \Omega_X^1)$  and  $\eta \in H^1(X, \mathcal{O}_X)$ , then  $\omega \wedge \bar{\eta} = 0$ .

Indeed, we have

$$0 = \int_X \omega \wedge \eta \wedge \overline{\omega \wedge \eta} = \int_X \omega \wedge \bar{\eta} \wedge \overline{\omega \wedge \bar{\eta}},$$

so we are done.

- (ii) For any holomorphic map  $f : X \rightarrow C$  with a connected generic fiber,  $f^* : H^1(C) \rightarrow H^1(X)$  is injective.

In fact, we have that  $f_* : \pi_1(X) \rightarrow \pi_1(C)$  is surjective.  $\square$

**Remark 4.2.3.** (1) In the proof above, we have implicitly used the Hodge decomposition

$$H^1(X) = H^0(X, \Omega_X^1) \oplus H^1(X, \mathcal{O}_X)$$

for any compact smooth surface  $X$  and represented the cohomology in  $H^0(X, \Omega_X^1)$  or  $H^1(X, \mathcal{O}_X)$  by differential forms via Dolbeault theorem; see also [BHPV], Theorem 2.10.

- (2) The above isotropic subspace theorem for a compact smooth surface seems new; despite of the similarity of the isotropic subspaces theorems in [Cat], we do not need the Kähler condition on the manifold.

By applying the logarithmic CDF theorem, we can obtain similarly an isotropic subspace theorem for a quasi-projective manifold. Nevertheless, complexity arises in this case due to the non-compactness of the base manifold. Recall that in the proofs above, we used repeatedly integration of forms over the compact base manifold.

**Theorem 4.2.4** (See [Bau],[Dim08]). *Let  $X$  be a quasi-projective manifold,  $X = \bar{X} \setminus D$  with  $\bar{X}$  smooth projective and  $D$  a normal crossing divisor. Then the following hold:*

1. Every real maximal isotropic subspace  $V$  of  $H^1(X, \mathbb{R})$  either of dimension  $\geq 3$  or of dimension 2 but not coming from a non-isotropic subspace  $V'$  of  $H^1(\overline{X}, \mathbb{R})$  determines a unique logarithmic pencil  $f : X \rightarrow C$  onto a curve  $C$  with logarithmic genus  $g^* \geq 2$ .

The curve  $C$  is projective if and only if  $V \subseteq H^1(\overline{X}, \mathbb{R})$  and is isotropic there, otherwise,  $V = f^*H^1(C, \mathbb{R})$ , and one says that the pencil is strictly logarithmic.

2. There is a bijective correspondence equivalence between the maximal real isotropic subspaces  $V \subseteq H^1(X)$  of dimension  $\beta \geq 3$ , which are not contained in  $H^1(\overline{X})$ , and the set of strictly logarithmic irrational pencils  $f : X \rightarrow C$  with  $b_1(C) = \beta$ .

**Remark 4.2.5.** We will not give the proof of the above theorem, but mention here an application.

Let  $X$  be a quasi-projective manifold with a smooth compactification  $\overline{X}$  and let  $0 \neq E \subseteq TC_1(\mathcal{V}_1(X))$  be an irreducible component completely contained in  $Gr_2^W H^1(X) = H^{1,1}(X)$ , then it determines an irrational pencil  $f : X \rightarrow C$  with  $E = f^*H^1(C)$  by Theorem 2.2.6. Note that  $f^*W_1H^1(C) = W_1H^1(E) = 0$ , so  $C$  is obtained from  $\mathbb{P}^1$  deleting  $\geq 3$  points. Also,  $E = f^*H^1(C)$  is real isotropic for the wedge product on  $H^1(X)$ .

We claim that  $E$  is a maximal real isotropic subspace. Indeed, if not, there exists a maximal one  $E' \supsetneq E$ . Clearly,  $\dim E' \geq 2$  and  $E' \not\subseteq H^1(\overline{X})$ , so by the theorem above, there exists an irrational pencil  $f' : X \rightarrow C'$  such that  $E' = f'^*H^1(C')$ . Therefore, by Theorem 2.2.6 again, we have that  $E'$  is an irreducible component of  $TC_1(\mathcal{V}_1(X))$  strictly containing another irreducible component  $E$ , contradiction.

Also, from the proof we see that  $\dim E \geq 2$  for any irreducible component  $E$  of  $TC_1(\mathcal{V}_1(X))$ . See also the discussion following Lemma 5.4.1 in the next Chapter.

The isotropic subspace theorems are indeed very tasteful ingredients in the theory of irrational pencils and cohomology jump loci, for deeper issues concerning isotropic subspace theorem, see [Dim08].

## 4.3 Morphisms from products of projective spaces

In order to deduce some Hodge number inequalities via isotropic subspaces, we will make use of some properties of morphisms from products of projective spaces, which are also interesting for their own right.

**Lemma 4.3.1.** *Let*

$$f : \mathbb{P}^n \rightarrow \mathbb{P}^m$$

*be a morphism between projective spaces. Then the image of  $f$  has either dimension 0 or  $n$ .*

**Lemma 4.3.2.** *Let*

$$f : \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \rightarrow \mathbb{P}^m$$

*be a morphism, then the image of  $f$  has dimension 0,  $n_1, n_2$  or  $n_1 + n_2$ .*

*If  $\dim \text{Im} f = 0$ , then  $f$  is a constant map.*

*If  $\dim \text{Im} f < n_1 + n_2$ , then  $f$  factors through the first projection  $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \rightarrow \mathbb{P}^{n_1}$  or the second projection  $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \rightarrow \mathbb{P}^{n_2}$ .*

**Remark 4.3.3.** (1) Clearly, Lemma 4.3.2 can be extended for the source space being  $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \dots \times \mathbb{P}^{n_k}$ .

(2) Lemma 4.3.1 and its proof below are those of Proposition 6, §7, [Mum2]; Lemma 4.3.2 is used without proof in [BHPV] when a section named “some inequalities on Hodge numbers” is given, and there the authors give the reference [RV]. However, [RV] is written in German, which prevent us from understanding the proof. So we give a proof ourselves here.

**Proof of Lemma 4.3.1** Let  $Y = \text{Im}f$ , then  $Y$  is a closed subvariety of  $\mathbb{P}^m$ . Let  $r = \dim Y$  and we may assume  $0 < r < n$ .

Choose general  $r + 1$  linear functions  $L_1, \dots, L_{r+1}$  of  $\mathbb{P}^m$  such that

$$V(L_1, \dots, L_{r+1}) \cap Y = \emptyset.$$

Then  $V(f^*L_i) \neq \emptyset$ ,  $i = 1, \dots, r + 1$  since  $V(L_i) \cap Y \neq \emptyset, \forall i$  and

$$V(f^*L_1, \dots, f^*L_{r+1}) = \emptyset.$$

But  $r + 1 \leq n$  by assumption and  $V(f^*L_1, \dots, f^*L_{r+1})$  has codimension at most  $r + 1$ , so it cannot be empty. Contradiction.  $\square$

**Proof of Lemma 4.3.2** We first give an outline of the proof containing some claims, then we finish the proof by proving our claims.

*Step 1: Outline of proof* Let  $Y = \text{Im}f$  be the closed subvariety of  $\mathbb{P}^m$  and  $r = \dim Y$ . Assume WLOG that  $0 < r < n_1 + n_2$ .

Note that  $f : \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \rightarrow Y$  is a surjective morphism, so there exists a Zariski open set  $V$  of  $Y$  such that  $\text{res } f : f^{-1}(V) \rightarrow V$  is smooth, i.e., for any  $p \in f^{-1}(V)$ ,  $f(p)$  is a smooth point of  $V$  and the differential map  $f_{*,p} : T_p(f^{-1}(V)) \rightarrow T_{f(p)}V$  is surjective. In addition, there exist open sets in strong topology  $U_1 \subseteq \mathbb{P}^{n_1}$  and  $U_2 \subseteq \mathbb{P}^{n_2}$  such that

$$U_1 \times U_2 \subseteq f^{-1}(V).$$

In particular,  $\text{res } f : U_1 \times U_2 \rightarrow f(U_1 \times U_2) := U$  is a holomorphic submersion.

Now consider the tangent map

$$f_* : T(U_1 \times U_2) = TU_1 \times TU_2 \rightarrow TU.$$

Let

$$\ker f_* = \{\xi \in TU_1 \times TU_2 \quad : \quad f_*\xi = 0\}.$$

Since  $r = \dim Y < n_1 + n_2$ ,  $\ker f_* \neq 0$ .

Let  $pr_1 : T(U_1 \times U_2) = TU_1 \times TU_2 \rightarrow TU_1$  and  $pr_2 : TU_1 \times TU_2 \rightarrow TU_2$  be the natural projections, then we obtain two projections

$$\text{res } pr_1 : \ker f_* \rightarrow TU_1$$

and

$$\text{res } pr_2 : \ker f_* \rightarrow TU_2.$$

Moreover, at any point  $(x, y) \in U_1 \times U_2$ ,  $\text{res } pr_{1,(x,y)}$  and  $\text{res } pr_{2,(x,y)}$  cannot simultaneously be zero.

Without loss of generality, assume that there exists  $(x_0, y_0) \in U_1 \times U_2$  satisfying

$$\text{res } pr_{1,(x_0,y_0)} \neq 0.$$

Then by continuity, there exists an open set  $U'_1 \subseteq U_1$  such that for any  $x \in U'_1$  we have  $\text{res } pr_{1,(x,y_0)} \neq 0$ .

We shall prove the following.

**Claim 4.3.4.** *With the settings above, we have  $\dim f(\mathbb{P}^{n_1} \times \{y_0\}) < n_1$ .*

Hence by Lemma 4.3.1,  $f(\mathbb{P}^{n_1} \times \{y_0\})$  is one point. Then we are done by applying the following.

**Lemma 4.3.5.** *Let  $X, Y, Z$  be irreducible varieties with  $X$  proper, and  $f : X \times Y \rightarrow Z$  a morphism (seen as a family of morphisms from  $X$  to  $Z$  with parameter space  $Y$ ).*

*If there exists  $y_0 \in Y$  such that  $f(X \times \{y_0\}) \subseteq Z$  is one point, then for any  $y \in Y$ ,  $f(X \times \{y\}) \subseteq Z$  is one point, namely, the morphism  $f$  factors through the projection  $X \times Y \rightarrow Y$ .*

*Step 2: Proof of Claim 4.3.4* With our settings, we have that for any  $x \in U'_1$ ,

$$\text{res } pr_1 : \ker f_{*,(x,y_0)} \rightarrow T_x U'_1$$

is nonzero.

Given any  $u \in \text{Im}(\text{res } pr_1)$ , there exists  $v \in T_{y_0} U_2$  such that  $f_{*,(x,y_0)}(u, v) = 0$  by definition of  $\ker f_*$ . We claim that  $f_{*,(x,y_0)}(u, 0) = 0$ . Indeed, consider the following diagram

$$\begin{array}{ccc} \mathbb{C}^{n_1+1} \setminus \{0\} \times \mathbb{C}^{n_2+1} \setminus \{0\} & & \\ \downarrow \pi & \searrow F & \\ \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} & \xrightarrow{f} & Y \subseteq \mathbb{P}^m \end{array}$$

where  $\pi$  is the natural projection. We shall denote any pre-image of  $(x, y)$  under  $\pi$  by  $(X, Y)$ .

By assumption, given any  $(X, Y_0)$  such that  $\pi(X, Y_0) = (x, y_0)$  and  $(U, V) \in \mathbb{C}^{n_1+1} \times \mathbb{C}^{n_2+1}$  satisfying  $\pi_{*,(X,Y_0)}(U, V) = (u, v)$ , we have

$$F_{*,(X,Y_0)}(U, V) = 0.$$

Choose a smooth curve  $\gamma(t) = (\gamma_1(t), \gamma_2(t))$  defined on  $(-\epsilon, \epsilon)$ ,  $\epsilon > 0$  such that  $\gamma(0) = (X, Y_0)$  and  $\frac{d\gamma}{dt}(0) = (U, V)$ , then

$$\frac{dF(\gamma_1(t), \gamma_2(t))}{dt}(0) = 0.$$

Now for any  $\lambda \in \mathbb{C}^*$ , we clearly have  $F(\gamma_1(t), \gamma_2(t)) = F(\lambda \cdot \gamma_1(t), \gamma_2(t))$ . Therefore taking derivatives on both sides at  $t = 0$ , we obtain that

$$F_{*,(\lambda X, Y_0)}(\lambda U, V) = 0;$$

but  $F_{*,(X,Y_0)}$  is homogeneous of positive degree in  $X$ , so  $F_{*,(X,Y_0)}(\lambda^p u, v) = 0$  for some  $p > 0$ , thus applying  $\pi_*$ , we have

$$f_{*,(x,y_0)}(\lambda u, v) = 0, \quad \forall \lambda \in \mathbb{C}^*,$$

i.e.,  $(\lambda u, v) \in \ker f_{*,(x,y_0)}$  for all  $\lambda \in \mathbb{C}^*$ . Since  $\ker f_{*,(x,y_0)}$  is a linear subspace of  $T_x U'_1 \times T_{y_0} U_2$ , we deduce that  $(u, 0) \in \ker f_{*,(x,y_0)}$ .

In conclusion, for the map  $\text{res } f : \mathbb{P}^{n_1} \times \{y_0\} \rightarrow \mathbb{P}^m$ , we have

$$\ker f_{*,x} \neq 0, \quad \forall x \in U'_1.$$

Thus,  $f(\mathbb{P}^{n_1} \times \{y_0\})$  has dimension  $< n_1$ . For otherwise,  $\text{res } f : \mathbb{P}^{n_1} \times \{y_0\} \rightarrow f(\mathbb{P}^{n_1} \times \{y_0\})$  is generically finite to one and its differential would be generically injective, contradicting the existence of the nonempty  $U'_1$  above.

*Step 3: Proof of Lemma 4.3.5* Recall that we are given a map  $f : X \times Y \rightarrow Z$  with  $f(X \times \{y_0\})$  being one point. Consider the incidence variety

$$\mathcal{X} = \{(x, y, z) \in X \times Y \times Z \quad : \quad z = f(x, y)\}$$

with  $p_{12} : \mathcal{X} \rightarrow X \times Y$  and  $p_{23} : \mathcal{X} \rightarrow Y \times Z$  the natural projections. Then  $p_{12}$  is an isomorphism, and hence  $\mathcal{X}$  is irreducible.

Let  $\mathcal{Y} = p_{23}(\mathcal{X})$ , then  $\mathcal{Y}$  is closed, since  $X$  is proper. Moreover, it is also irreducible, as the image of the irreducible variety  $\mathcal{X}$ . Furthermore, let  $p_Y : \mathcal{Y} \rightarrow Y$  be the projection to  $Y$ , then  $p_Y$  is surjective.

Now by assumption  $p_Y^{-1}(y_0) = f^{-1}(X \times \{y_0\})$  is just one point, we see that  $\dim \mathcal{Y} = \dim Y$ . Choose any point  $x \in X$ , then the map  $s : Y \ni y \mapsto (y, f(x, y)) \in \mathcal{Y}$  gives a section of  $p_Y$ . Moreover, first the image  $s(Y)$  is isomorphic to  $Y$ , so it is irreducible; second, since the projection  $p_Y : \mathcal{Y} \rightarrow Y$  is separated,  $s$  is also a closed immersion. So by dimension consideration, we have  $s(Y) = \mathcal{Y}$ , i.e.,  $f(X \times \{y\}) = f(x, y)$  for any  $y \in Y$ . We are done.  $\square$

## 4.4 Inequalities of Hodge numbers

We now deduce some inequalities of Hodge numbers by considering the dimensions of isotropic subspaces.

### 4.4.1 Compact case

We first give the following definition.

**Definition 4.4.2.** Let  $X$  be a compact Kähler manifold or a compact surface. A subspace  $T$  of  $H^{1,0}(X) = H^0(X, \Omega_X^1)$  is called *totally non-isotropic* if for any linearly independent  $\omega_1, \omega_2 \in T$ ,  $\omega_1 \wedge \omega_2 \neq 0$  in  $H^{2,0}(X) = H^0(X, \Omega_X^2)$ .

We immediately have the following observation.

**Theorem 4.4.3.** *Let  $X$  be a compact Kähler manifold or a compact surface. If there exists a totally non-isotropic subspace  $T \subseteq H^{1,0}(X)$  of dimension  $t \geq 2$ , then*

$$(i) \quad h^{2,0}(X) \geq 2t - 3;$$

$$(ii) \quad h^{1,1}(X) \geq 2t - 1.$$

*Proof.* (i) Consider the morphism

$$\phi : T \times T \rightarrow H^{2,0}(X), \quad \phi(\omega, \eta) = \omega \wedge \eta.$$

We will mainly concern the dimension of the image. To this end, we compute the rank of the differential  $\phi_*$  at a general element  $(\omega, \eta)$ . We have

$$\phi_{*,(\omega,\eta)}(\omega', \eta') = \omega \wedge \eta' + \omega' \wedge \eta.$$

Choosing  $(\omega, \eta)$  generically, we see that  $\ker \phi_{*,(\omega,\eta)}$  is a linear combination of  $(0, \omega)$ ,  $(\eta, 0)$ ,  $(\omega, -\eta)$  with coefficients in  $H^0(X, \mathcal{O}_X) = \mathbb{C}$ . Indeed, choose a generic point  $p \in X$  and local holomorphic coordinates  $(z_1, z_2, \dots)$  around  $p$  such that  $\omega = dz_1, \eta = dz_2$ . One easily verifies that the result holds locally, hence globally, since the forms under consideration are holomorphic.

Therefore  $\phi_{*,(\omega,\eta)}$  has rank  $2t - 3$  and so  $\dim \text{Im } \phi = 2t - 3$  and  $h^{2,0} \geq \dim \text{Im } \phi = 2t - 3$ .

(ii) Consider the map

$$\psi : T \times T \rightarrow H^{1,1}(X), \quad \psi(\omega, \eta) = \omega \wedge \bar{\eta}.$$

If  $\omega, \eta \in T$  satisfy  $\psi(\omega, \eta) = 0$ , i.e.,  $\omega \wedge \bar{\eta} = 0$ ,  $\omega \wedge \eta = 0$ , see the proof of Theorem 4.2.1 and Theorem 4.2.2. Hence,  $\eta$  and  $\omega$  are linearly dependent since by assumption  $T$  is totally non-isotropic. If, in addition, neither of  $\omega$  nor  $\eta$  is zero, then  $\eta = A\omega$  for some  $A \in \mathbb{C}^*$  and moreover,  $\omega \wedge \bar{\eta} = 0$ . Therefore  $\omega \wedge \bar{\omega} = 0$ , implying that  $\omega = 0$ .

Consequently,  $\psi$  induces a map

$$\mathbb{P}(\psi) : \mathbb{P}(T) \times \mathbb{P}(T) \rightarrow \mathbb{P}(H^{1,1}(X)).$$

When restricted to each slice  $\mathbb{P}(T) \times \{\eta\}$ ,  $\{\omega\} \times \mathbb{P}(T)$ , the map  $\mathbb{P}(\psi)$  is injective. In fact, given  $\omega \neq 0$ , if  $\eta, \eta' \in T$  such that  $\omega \wedge \bar{\eta} = \omega \wedge \bar{\eta}'$ , i.e.,  $\omega \wedge \overline{\eta - \eta'} = 0$ . We have  $\omega \wedge \overline{\eta - \eta'} = 0$  and thus  $\eta - \eta' = B\omega$  for some  $B \in \mathbb{C}$  by the discussion above. From the equation  $\omega \wedge \overline{\eta - \eta'} = 0$  and  $\omega \neq 0$ , one sees that  $B = 0$ , i.e.,  $\eta = \eta'$ .

Therefore, the image of  $\mathbb{P}(\psi)$  has dimension  $2(t-1)$  by Lemma 4.3.2. Hence

$$h^{1,1}(X) - 1 = \dim \mathbb{P}(H^{1,1}(X)) \geq \dim \text{Im} \mathbb{P}(\psi) = 2(t-1),$$

namely,  $h^{1,1}(X) \geq 2t - 1$ .

Alternatively, we can show that the image of  $\psi$  has dimension  $2t - 1$  by computing the differential  $\psi_*$ . For generic  $\omega, \eta \in T$ , we have

$$\psi_{*,(\omega,\eta)}(\omega', \eta') = \omega' \wedge \bar{\eta} + \omega \wedge \bar{\eta}',$$

so the kernel of  $\psi_{*,(\omega,\eta)}$  is spanned by  $(\omega, -\eta)$  with coefficients in  $H^0(X, \mathcal{O}_X) = \mathbb{C}$ , and the image of  $\psi$  has dimension  $2t - 1$ . The details of the computation are similar as in the proof of (i), that is why we omit them.  $\square$

**Lemma 4.4.4.** *Let  $X$  be a compact Kähler manifold or a compact surface. Then  $X$  admits a totally non-isotropic subspace  $T$  of dimension  $h^{1,0}(X)$ , or equivalently,  $H^{1,0}(X)$  is totally non-isotropic if and only if  $X$  admits no irrational pencils.*

*Proof.* By the CDF theorem, we see that existence of a nontrivial isotropic subspace of  $H^{1,0}(X)$  is equivalent to the existence of an irrational pencil. Thus  $H^{1,0}(X)$  is itself totally non-isotropic is equivalent to the non-existence of irrational pencils.  $\square$

So we have reproved the inequalities of Hodge numbers in [BHPV], namely

**Corollary 4.4.5** (Prop. 5.2 & 5.3 in [BHPV]). *Let  $X$  be a compact Kähler manifold or a compact surface. If  $X$  admits no irrational pencils, then*

$$(i) \quad h^{2,0}(X) \geq 2h^{1,0}(X) - 3;$$

$$(ii) \quad h^{1,1}(X) \geq 2h^{1,0}(X) - 1.$$

**Remark 4.4.6.** The above inequalities are titled “some inequalities for Hodge numbers” in [BHPV], but we have already given different proofs.

Inspired by the intuition that isotropic subspaces have close relation with the first characteristic variety, we obtain the following.

**Lemma 4.4.7.** *Let  $X$  be a compact Kähler manifold, and  $T \subseteq H^{1,0}(X)$ . Then  $T$  is totally non-isotropic if and only if  $T \cap TC_1(\mathcal{V}_1(X))$  has dimension at most 1.*

*Proof.* Observe first that  $X$  is formal and moreover, each irreducible component of  $TC_1(\mathcal{V}_1(X))$  is a linear subspace of  $H^1(X)$ .

Suppose  $\dim T \cap TC_1(\mathcal{V}_1(X)) \leq 1$ . If  $T$  is not totally non-isotropic, then there exist linearly independent  $\omega_1, \omega_2 \in T$  such that  $\omega_1 \wedge \omega_2 = 0$ . Therefore  $\omega_1, \omega_2$  are contained in a maximal isotropic subspace of  $H^{1,0}(X)$ , which corresponds to an irrational pencil by CDF Theorem, and hence an irreducible component of  $TC_1(\mathcal{V}_1(X))$  by Theorem 2.2.6; this irreducible component is unique, since any two irreducible components of  $TC_1(\mathcal{V}_1(X))$  intersect trivially. Consequently,  $T \cap TC_1(\mathcal{V}_1(X))$  contains a linear subspace spanned by  $\omega_1, \omega_2$  whose dimension is at least 2, contradiction.

Conversely, if  $\dim T \cap TC_1(\mathcal{V}_1(X)) \geq 2$ , then  $\dim T \cap I \geq 2$  for an irreducible component  $I \subseteq TC_1(\mathcal{V}_1(X))$ , hence there exists linearly independent  $\omega_1, \omega_2 \in T \cap I$ . Since  $I$  is an isotropic subspace by the correspondences between isotropic subspaces, irrational pencils and irreducible components of  $TC_1(\mathcal{V}_1(X))$ , we have  $\omega_1 \wedge \omega_2 = 0$  since  $\omega_1, \omega_2 \in I$ , and thus  $T$  is not totally non-isotropic, since  $\omega_1, \omega_2 \in T$  are linearly independent.  $\square$

**Lemma 4.4.8.** *Let  $X$  be a compact Kähler manifold and  $s = \dim \mathcal{V}_1(X) > 0$ , i.e.,  $s$  is the largest dimension of the irreducible components of  $\mathcal{V}_1(X)$  passing through 1. Then  $X$  admits a totally non-isotropic subspace  $T$  of dimension  $h^{1,0}(X) - s/2 + 1$ .*

*Proof.* Each irreducible component of  $TC_1(\mathcal{V}_1(X))$  has the form  $U \oplus \bar{U}$ , for  $H \subseteq H^{1,0}(X)$ , by Theorem 2.2.6, so  $TC_1(\mathcal{V}_1(X)) \cap H^{1,0}(X)$  is a union of linear subspaces of dimension  $\leq s/2$ .

We can choose a linear subspace  $T \subseteq H^{1,0}(X)$  of dimension  $h^{1,0} - s/2 + 1$  such that  $\dim T \cap TC_1(\mathcal{V}_1(X)) \cap H^{1,0}(X) = \dim T \cap TC_1(\mathcal{V}_1(X)) \leq 1$ . By Lemma 4.4.7,  $T$  is totally non-isotropic. We are done.  $\square$

**Corollary 4.4.9.** *Let  $X$  be a compact Kähler manifold and  $s = \dim \mathcal{V}_1(X) > 0$ . Then*

$$(i) \quad h^{2,0}(X) \geq 2h^{1,0}(X) - s - 1;$$

$$(ii) \quad h^{1,1}(X) \geq 2h^{1,0}(X) - s + 1.$$

**Remark 4.4.10.** 1. When  $X$  is a compact Kähler manifold, either  $s = \dim \mathcal{V}_1(X) = 0$  or  $s \geq 4$ , since for any irrational pencil  $f : X \rightarrow C$ ,  $C$  is a complete smooth curve with genus  $g \geq 2$ .

Therefore, although when  $s = 2$ , the inequalities we get are formally the same as the case  $s = 0$ , such a coincidence can never occur.

2. We conjecture that the above inequalities still hold for a compact surface  $X$  with  $\dim \mathcal{V}_1(X) > 0$ . However, if  $X$  is not Kähler, no special properties of  $\mathcal{V}_1(X)$ , such as, its relations with irrational pencils and isotropic subspaces, are known: all these prevent us from giving extension for compact surfaces.

#### 4.4.11 Quasi-projective case

If  $X$  is a quasi-projective manifold, we may consider the first logarithmic resonance variety  $L\mathcal{R}_1(X)$  defined in [Dim10]. Note that the irreducible components of  $L\mathcal{R}_1(X)$  have close relations with isotropic subspaces in  $F^1(X)$  and irrational pencils. So we can get some inequalities of Hodge numbers by a similar method as in the compact case.

**Definition 4.4.12.** Let  $X$  be a quasi-projective manifold. A subspace  $T$  of  $F^1(X) = H^0(\overline{X}, \Omega_{\overline{X}}^1(\log D))$  is called *totally non-isotropic* if for any linearly independent  $\omega_1, \omega_2 \in T$ ,  $\omega_1 \wedge \omega_2 \neq 0$  in  $F^2(X) = H^0(\overline{X}, \Omega_{\overline{X}}^2(\log D))$ .

The definition above coincides with the compact case when  $X$  is itself compact, hence we used the same notation. A similar argument as in the previous section gives the following, with a little difference in the proof.

**Theorem 4.4.13.** *Let  $X$  be a quasi-projective manifold. If  $X$  admits a totally non-isotropic subspace  $T$  of dimension  $t$ , then*

$$\dim F^2(X) \geq 2t - 3.$$

*Proof.* The proof is different from the compact case due to the following observation: for general  $(\omega, \eta) \in T \times T$ , if  $(\omega', \eta')$  satisfies

$$\omega' \wedge \eta + \omega \wedge \eta' = 0,$$

then we obtain  $(\omega', \eta')$  is a linear combination of  $(0, \omega)$ ,  $(\eta, 0)$  and  $(\omega, -\eta)$  with coefficients in  $H^0(X, \mathcal{O}_X)$ . However, since  $X$  is not compact,  $H^0(X, \mathcal{O}_X) \neq \mathbb{C}$  in general. So we can only deduce that  $(\omega', \eta')$  is of the form  $(h_1\eta, 0)$ ,  $(0, h_2\omega)$  or  $(h_3\omega, -h_3\eta)$ , where  $h_1, h_2, h_3$  are holomorphic functions on  $X$ .

Therefore, it suffices to show that for a general element  $\xi \in T$ , if  $h\xi \in T$  for some holomorphic function  $h$  on  $X$ , then  $h$  is constant. Indeed,  $(h\xi) \wedge \xi = 0$ ; if  $h$  is not constant,  $h\xi$  and  $\xi$  are linearly independent, contradicting the totally non-isotropic assumption of  $T$ . We are done.  $\square$

Let  $s' = \dim L\mathcal{R}_1(X) = \max_I \dim I$  where  $I$  runs over all irreducible components of  $L\mathcal{R}_1$  and  $f^1 = \dim F^1(X)$ . Then clearly, there exists a linear subspace  $T$  such that  $T \cap L\mathcal{R}_1(X)$  is a union of linear subspaces of  $F^1(X)$  and has dimension at most 1.

Similar to Lemma 4.4.7, we have

**Lemma 4.4.14.** *Let  $X$  be a quasi-projective manifold. and  $T \subseteq F^1(X)$ . Then  $T$  is totally non-isotropic if and only if  $T \cap L\mathcal{R}_1(X)$  has dimension at most 1.*

**Corollary 4.4.15.** *Let  $X$  be a quasi-projective manifold. If  $s' = \dim L\mathcal{R}_1(X) > 0$ , then*

$$\dim F^2(X) \geq 2(f^1 - s' + 1) - 3 = 2(f^1 - s') - 1 = 2 \cdot \text{codim } L\mathcal{R}_1(X) - 1.$$

*Proof.* Clearly, there exists a linear subspace  $T$  of dimension  $t = f^1 - s' + 1$  such that  $T \cap L\mathcal{R}_1$  has dimension  $\leq 1$ , then  $T$  is totally non-isotropic. We are done by applying the results above.  $\square$

**Corollary 4.4.16.** *If  $X$  be a quasi-projective manifold and  $s' = \dim L\mathcal{R}_1(X)$ , then the following hold:*

1. If  $s' = 0$ , then  $\dim F^2(X) \geq 2 \cdot \dim F^1(X) - 3$ ;
2. If  $s' > 0$ , then  $\dim F^2(X) \geq 2 \cdot (\dim F^1(X) - s') - 1$ .

**Remark 4.4.17.**  $L\mathcal{R}_1(X)$  cannot be 1-dimensional. Indeed, any positive dimensional irreducible components of  $L\mathcal{R}_1(X)$  corresponds to an isotropic subspace of  $F^1(X)$  of positive dimension, whose dimension is at least 2.

## Geometry of Milnor Fibers of Line Arrangements

Given a nonzero homogeneous polynomial  $Q \in \mathbb{C}[x_0, \dots, x_n]$ ,  $n \geq 1$  of degree  $d \geq 2$ , by the well-known Milnor Fibration Theorem, the map

$$Q : \mathbb{C}^{n+1} \setminus \{Q = 0\} \rightarrow \mathbb{C}^*$$

is a smooth locally trivial fiber bundle. For a proof of Milnor Fibration Theorem, see [Mil], and for deeper properties of the Milnor fibration, see [Dim92]. The affine hypersurface

$$F = Q^{-1}(1)$$

in  $\mathbb{C}^{n+1}$  is called the (affine) Milnor fiber of  $Q$ . It is a smooth manifold of complex dimension  $n$ , and has the homotopy type of a finite CW complex of real dimension  $\leq n$ .

Associated to  $Q$ , one also obtains the projective hypersurface

$$V(Q) = \{Q = 0\} \subseteq \mathbb{P}^n$$

and moreover its complement

$$M = M(Q) = \mathbb{P}^n \setminus V(Q).$$

The natural projection  $\mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$  restricted to  $F$  gives a covering map

$$p : F \rightarrow M$$

with

$$\mu_d = \{z \in \mathbb{C} : z^d = 1\} = \mathbb{Z}/d\mathbb{Z}$$

as the group of Deck transformations. Let  $\xi = \exp(2\pi\sqrt{-1}/d)$ , then we have the *geometric monodromy*

$$h : F \rightarrow F$$

defined by  $h((x_0, \dots, x_n)) = (\xi x_0, \dots, \xi x_n)$ , which induces the *algebraic monodromy* in cohomology

$$h^* : H^i(F, \mathbb{Q}) \rightarrow H^i(F, \mathbb{Q}).$$

for all  $i$ . Moreover,  $H^i(F, \mathbb{Q})$  admits a natural MHS and  $h^*$  is in fact a MHS morphism because  $h$  is algebraic. In addition, denote

$$H^i(F)_a = \ker(a \cdot \text{Id} - h^* \quad : \quad H^i(F) \rightarrow H^i(F))$$

the eigenspace of  $h^*$  for  $a \in \mathbb{C}$  and  $i \geq 0$ .

Furthermore, when  $Q$  is square-free and  $V(Q)$  has only isolated singularities,  $H^{n-1}(F)$  has only two weights:  $n - 1$  and  $n$ , as is proved in [DP11]. In fact, in [DP11], the authors show that

$$W_{n-1}H^{n-1}(F) = H^{n-1}(F)_{\neq 1}$$

and

$$Gr_n^W H^{n-1}(F) = H^{n-1}(F)_1,$$

where

$$H^{n-1}(F)_{\neq 1} = \ker(h^{*d-1} + \dots + h^* + \text{Id}) = \bigoplus_{a \neq 1} H^{n-1}(F)_a.$$

When  $Q$  defines a hyperplane arrangement, its complement and Milnor fiber can have more interesting properties, see Preliminary of this thesis. In this chapter, we investigate the Milnor fiber from topological as well as geometric points of view. In particular, we discuss specifically irrational pencils and 1-formality, by considering the relations between the Milnor fiber and the complement.

## 5.1 Topological properties of Milnor fibers

Given a homogenous polynomial  $Q \in \mathbb{C}[x_0, \dots, x_n]$  with decomposition of its linear factors

$$Q = L_1^{k_1} L_2^{k_2} \dots L_l^{k_l}$$

then it is well known that the Milnor fiber  $F : Q = 1$  is connected if and only if  $\gcd(k_1, \dots, k_l) = 1$ . Since we do not find a good reference, we first give a proof of it for a more general setting.

**Lemma 5.1.1.** *Let  $Q \in \mathbb{C}[x_0, \dots, x_n]$  be a homogenous polynomial of degree  $d \geq 2$  and*

$$Q = Q_1^{k_1} Q_2^{k_2} \dots Q_l^{k_l}$$

*with  $Q_1, \dots, Q_l$  irreducible polynomials and  $k_i > 0$  for  $i = 1, \dots, l$ . Then the Milnor fiber  $F : Q = 1$  in  $\mathbb{C}^{n+1}$  has exactly  $\gcd(k_1 \deg Q_1, \dots, k_l \deg Q_l)$  connected components.*

*In particular, the following hold.*

(i)  *$F$  is connected if and only if  $\gcd(k_1 \deg Q_1, \dots, k_l \deg Q_l) = 1$ .*

(ii) *If  $F$  is completely reducible, namely,  $\deg Q_i = 1$  for  $i \geq 1$ ,  $F$  is connected if and only if  $\gcd(k_1, \dots, k_l) = 1$ .*

*Proof.* Let  $M = \mathbb{P}^n - V(Q)$  be the complement, then we have a covering  $p : F \rightarrow M$  with  $\mu_d$  as its group of Deck transformations. From the end of the homotopy exact sequence, we have

$$\pi_1(M) \rightarrow \mu_d \rightarrow \pi_0(F) \rightarrow 0.$$

The sequence ends with 0 because  $M$  is connected. Hence,  $\pi_0(F)$  is equal to the index in  $\mu_d$  of the image of the homomorphism  $\pi_1(M) \rightarrow \mu_d$ .

Note that the map  $\pi_1(M) \rightarrow \mu_d$  is exactly the topological monodromy associated to the covering  $p : F \rightarrow M$ . Also it factors through

$$H_1(M, \mathbb{Z}) = \pi_1(M) / [\pi_1(M), \pi_1(M)] \rightarrow \mu_d$$

where  $[\bullet, \bullet]$  denotes the commutator subgroup, since  $\mu_d$  is abelian.

For any  $i = 1, \dots, l$ , choose a small disk  $D_i$  in  $\mathbb{P}^n$  that intersects transversely with  $V(Q_i)$  but does not intersect other  $D_j$ 's. Let  $\gamma_i = \partial D_i$  with orientation naturally determined by those of  $V(Q_i)$  and  $D_i$  around the intersection point, then  $\gamma_i$ 's provide a basis for  $H_1(M, \mathbb{Z})$  and the map  $H_1(M, \mathbb{Z}) \rightarrow \mu_d$  is given by

$$\gamma_i \mapsto k_i \deg Q_i.$$

Hence  $\text{Im}(\pi_1(M) \rightarrow \mu_d) \cong \mu_{d/d'}$  where  $d' = \gcd(k_1 \deg Q_1, \dots, k_l \deg Q_l)$ , which has index  $d'$  in  $\mu_d$ . We are done.  $\square$

Recall that a set of generators of  $\pi_1(M)$  or  $H_1(M)$  can be provided by the  $\gamma_i$ 's, as in the proof above. This construction will be applied without mentioning again in the sequel.

**Corollary 5.1.2.** *Let  $\mathcal{A} : Q = 0$  be a hyperplane arrangement consisting of  $d$  hyperplanes, then the Milnor fiber  $F : Q = 1$  is a connected smooth manifold.*

*In particular, the topological monodromy map  $\pi_1(M) \rightarrow \mu_d$  is surjective, where  $M = M(\mathcal{A})$  is the complement.*

### 5.1.3 Morphisms with connected fibers

We can extend the above discussion about connectivity for more polynomials.

Given two homogenous polynomials  $Q_1, Q_2 \in \mathbb{C}[x_0, \dots, x_n]$  of degree  $d$ , we denote by

$$\mathcal{P}_{Q_1, Q_2} = \{\lambda Q_1 + \mu Q_2 \quad : \quad (\lambda, \mu) \in \mathbb{P}^1\}$$

the pencil determined by them. We shall call an element in  $\mathcal{P}_{Q_1, Q_2}$  a *fiber* of it.

**Lemma 5.1.4.** *Let  $n \geq 2$  and  $d \geq 3$ . Suppose  $Q_1, Q_2 \in \mathbb{C}[x_0, \dots, x_n]$  are two homogeneous polynomials of degree  $d$  such that  $Q_1$  is decomposed as  $Q_1 = q_1 q_2$  where  $q_i, i = 1, 2$  are irreducible, coprime and  $\deg q_1 \neq \deg q_2$ , and  $\gcd(Q_1, Q_2) = 1$ .*

*Then a generic fiber of the map*

$$\rho : M = \mathbb{P}^n \setminus V(Q_1 Q_2) \longrightarrow C, \quad \rho = (Q_1, Q_2)$$

*is connected, where  $C = \mathbb{P}^1 \setminus B$  with  $B = \{(1, 0), (0, 1)\}$ .*

*Proof.* Consider the Stein factorization: there exists a finite map  $p_0 : C_0 \rightarrow C$  and a morphism  $\rho_0 : M \rightarrow C_0$  such that a generic fiber of  $\rho_0$  is connected, and  $\rho = p_0 \circ \rho_0$ . Note that  $C_0$  is also a noncompact curve, so it has the form  $C_0 = \overline{C_0} \setminus B_0$  where  $\overline{C_0}$  is complete smooth curve and  $B_0 \subseteq \overline{C_0}$  is a finite set of points.

In the sequel, we first give an outline of the proof containing some claims, then we prove our claims.

**Step 1: Outline** First, we have

**Claim 5.1.5.**  $\overline{C_0} = \mathbb{P}^1$ .

So  $\rho_0 : M \rightarrow C_0$  is given by  $\rho_0 = (P_1, P_2)$  for two coprime homogeneous polynomials  $P_1, P_2$ . Moreover,  $p_0 : C_0 \subseteq \mathbb{P}^1 \rightarrow C \subseteq \mathbb{P}^1$  extends to a morphism  $p'_0 : \overline{C_0} = \mathbb{P}^1 \rightarrow \mathbb{P}^1$ , which is necessarily of the form

$$p'_0(u, v) = \left( \prod_{j=1}^m (a_{1,j}u + b_{1,j}v), \prod_{j=1}^m (a_{2,j}u + b_{2,j}v) \right)$$

where  $(a_{1,j}, b_{1,j}) \neq (a_{2,k}, b_{2,k}), j, k = 1, \dots, m$  in  $\mathbb{P}^1$ .

If  $m = 1$ , then  $p'_0$  is an isomorphism and we are done. So suppose  $m > 1$ . Then from  $\rho = p_0 \circ \rho_0$ , we have that

$$(Q_1, Q_2) = \left( \prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2), \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2) \right).$$

and furthermore, we deduce that

**Claim 5.1.6.**

$$\begin{cases} Q_1 = c \prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2) \\ Q_2 = c \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2) \end{cases}$$

for some  $c \in \mathbb{C}$ .

By assumption,  $m > 1$  and  $Q_1$  has exactly two irreducible factors  $q_1, q_2$ , so  $m = 2$  and there exist complex numbers  $c_{i,j}, i, j = 1, 2$  such that  $q_i = c_{i,1}P_1 + c_{i,2}P_2$  for  $i = 1, 2$ . But then  $\deg q_1 = \deg q_2$ , contradiction.

**Step 2: Proof of Claim 5.1.5** Since  $\rho_0 : M \rightarrow C_0$  has a connected generic fiber, the induced map between fundamental groups  $\rho_{0,*} : \pi_1(M) \rightarrow \pi_1(C_0)$  is surjective, hence  $\rho_0^* : H^1(C_0) \rightarrow H^1(M)$  is injective. Note that  $\rho_0^*$  is a morphism of mixed Hodge structures, so

$$\rho_0^* : W_1 H^1(C_0) \rightarrow W_1 H^1(M)$$

is also injective.

Now we have  $W_1 H^1(C_0) = \text{Im}(H^1(\overline{C_0}) \rightarrow H^1(C_0)) \cong H^1(\overline{C_0})$  and

$$W_1 H^1(M) = \text{Im}(H^1(\mathbb{P}^n) \rightarrow H^1(M)) \cong H^1(\mathbb{P}^n) = 0,$$

therefore  $H^1(\overline{C_0}) = 0$  and  $\overline{C_0} \cong \mathbb{P}^1$ .

**Step 3: Proof of Claim 5.1.6** The main point is to show

$$\prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2) \quad \text{and} \quad \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2)$$

are coprime. First, the polynomials  $a_{1,j}P_1 + b_{1,j}P_2, a_{2,k}P_1 + b_{2,k}P_2$  are mutually coprime for any  $j, k = 1, \dots, m$ . Indeed,  $\gcd(a_{1,j}P_1 + b_{1,j}P_2, a_{2,k}P_1 + b_{2,k}P_2) = \gcd(P_1, P_2) = 1$  since  $(a_{1,j}, b_{1,j}) \neq (a_{2,k}, b_{2,k})$  in  $\mathbb{P}^1$ . Hence

$$\gcd\left(\prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2), \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2)\right) = 1.$$

Now, from the equality

$$(Q_1, Q_2) = \left( \prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2), \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2) \right)$$

we get that  $Q_2 \cdot \prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2) - Q_1 \cdot \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2)$  vanishing identically on  $M = \mathbb{P}^n \setminus V(Q_1 Q_2)$ , thus for some  $s$  sufficiently large,

$$(Q_1 Q_2)^s \left( Q_2 \cdot \prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2) - Q_1 \cdot \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2) \right)$$

vanishes identically on  $\mathbb{P}^n$  (see [Ha], Lemma 5.14), hence it is zero as a polynomial, and

$$Q_2 \cdot \prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2) = Q_1 \cdot \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2)$$

as an equality of polynomials in  $\mathbb{C}[x_0, \dots, x_n]$ . Furthermore,

$$\gcd(Q_1, Q_2) = 1, \quad \gcd\left(\prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2), \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2)\right) = 1,$$

we obtain that

$$Q_1 \left| \prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2), \quad \text{and} \quad \prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2) \right| Q_1,$$

so  $Q_1$  is a multiple of  $\prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2)$ ; similar for  $Q_2$ .  $\square$

**Corollary 5.1.7.** *Under the assumption of Lemma 5.1.4, the pencil  $\mathcal{P}_{Q_1, Q_2}$  contains at most  $d^2 - 1$  reducible fibers.*

*Proof.* This follows from the main theorem in [Vis], and we just need to show that a fiber  $V(h)$ , for  $h \in \mathcal{P}_{Q_1, Q_2}$  generic, is irreducible.

Indeed, since  $\gcd(Q_1, Q_2) = 1$ , the base locus  $V(Q_1, Q_2)$  of the pencil  $\mathcal{P}_{Q_1, Q_2}$  has codimension 2, and  $V(h)$  is pure of codimension 1. So it suffices to prove  $V(h) \setminus V(Q_1, Q_2)$  is irreducible. It is connected by Lemma 5.1.4, and smooth by Bertini Theorem, so it is irreducible as desired.  $\square$

**Remark 5.1.8.** The proofs of Lemma 5.1.4 and Corollary 5.1.7 provide a prototype to show that a morphism admits a connected generic fiber. Here we summarize the key points in the proofs as follows:

- (i) showing that  $\overline{C}_0$  is  $\mathbb{P}^1$ .

In this step, no restrictions on  $Q_1, Q_2$  are needed.

- (ii) showing that  $Q_1$  and  $Q_2$  are of the form

$$\begin{cases} Q_1 = c \prod_{j=1}^m (a_{1,j}P_1 + b_{1,j}P_2) \\ Q_2 = c \prod_{j=1}^m (a_{2,j}P_1 + b_{2,j}P_2) \end{cases}$$

for some coprime  $P_1$  and  $P_2$ .

In this step, we only need that  $Q_1$  and  $Q_2$  are coprime, we do not even need that  $Q_1, Q_2$  are reduced.

- (iii) showing a generic fiber of the pencil  $\mathcal{P}_{P_1, P_2}$  is irreducible.

In this step, we only need that  $\gcd(P_1, P_2) = 1$  and that a generic fiber of the morphism  $\rho_0 = (P_1, P_2)$  is connected.

Given a homogenous polynomial  $Q$  with decomposition into its irreducibles

$$Q = Q_1^{k_1} Q_2^{k_2} \cdots Q_l^{k_l}.$$

We call  $Q$  *inhomogeneous for irreducibles* if  $\deg Q_i, i = 1, \dots, l$  are not all the same integer.

**Lemma 5.1.9.** *Let  $Q_1, Q_2 \in \mathbb{C}[x_0, \dots, x_n]$  be two homogenous polynomials of degree  $d \geq 2$  such that*

(i)  $\gcd(Q_1, Q_2) = 1$ ;

(ii) *the pencil  $\mathcal{P}_{Q_1, Q_2}$  contains  $k \geq d^2/4$  reducible fibers, i.e., there exist  $k$  distinct points  $(\alpha_i, \beta_i) \in \mathbb{P}^1$  such that*

$$Q_i = \alpha_i Q_1 + \beta_i Q_2, \quad i = 1, \dots, k$$

*are reducible, where  $(\alpha_1, \beta_1) = (1, 0)$  and  $(\alpha_2, \beta_2) = (0, 1)$ ;*

(iii) *for all  $i = 1, \dots, k$ ,  $Q_i$  is inhomogeneous for irreducibles.*

Let  $Q = Q_1 Q_2 \cdots Q_k$ , then the map

$$\rho : M = M(Q) \rightarrow C, \quad \rho = (Q_1, Q_2)$$

with  $C = \mathbb{P}^1 \setminus \{(-\beta_j, \alpha_j) : j = 1, \dots, k\}$ , has a connected generic fiber.

*In particular, the pencil  $\mathcal{P}_{Q_1, Q_2}$  contains at most  $d^2 - 1$  reducible fibers.*

*Proof.* Consider the Stein factorization: there exists a finite map  $p_0 : C_0 \rightarrow C$  and a morphism  $\rho_0 : M \rightarrow C_0$  such that a generic fiber of  $\rho_0$  is connected, and  $\rho = p_0 \circ \rho_0$ . Note that  $C_0$  is also a noncompact curve, so it has the form  $C_0 = \overline{C_0} \setminus B_0$  where  $\overline{C_0}$  is complete smooth curve and  $B_0 \subseteq \overline{C_0}$  is a finite set of points.

As in the proof of Lemma 5.1.4, we have  $\overline{C_0} = \mathbb{P}^1$  and a morphism  $p'_0 : \overline{C_0} \rightarrow \mathbb{P}^1$ . Let  $m = \deg \rho'_0$ , and without loss of generality, we may suppose  $m > 1$ . Then there exists  $2m$  points  $(a_{i,l}, b_{i,l})$ ,  $i = 1, 2, l = 1, \dots, m$  points in  $\mathbb{P}^1$  and two coprime polynomials  $P_1$  and  $P_2$  such that

$$\begin{cases} Q_1 = c \prod_{l=1}^m (a_{1,l} P_1 + b_{1,l} P_2) \\ Q_2 = c \prod_{l=1}^m (a_{2,l} P_1 + b_{2,l} P_2) \end{cases}$$

for some constant  $c$ .

For each  $i$ , composing  $f$  with the morphism  $(u, v) \mapsto (u, \alpha_i u + \beta_i v)$ , we obtain that other  $Q_i$ 's have similar decompositions as  $Q_1, Q_2$ . Therefore, there exists  $km$  points  $(a_{i,l}, b_{i,l})$ ,  $i = 1, \dots, k, l = 1, \dots, m$  and two coprime polynomials  $P_1$  and  $P_2$  such that

$$Q_i = c \prod_{l=1}^m (a_{i,l} P_1 + b_{i,l} P_2), \quad i = 1, \dots, k,$$

for some constant  $c$ .

Fix  $i$ , since  $Q_i$  is inhomogeneous for irreducibles, we have that  $\deg P_1 = \deg P_2 = d/m > 1$  and moreover, there exists an  $j_i \in \{1, \dots, l\}$  such that  $a_{i,j_i} P_1 + b_{i,j_i} P_2$  is reducible. Thus, in this way, we obtain  $k$  reducible fibers in the pencil  $\mathcal{P}_{P_1, P_2}$ . In addition, a generic fiber in the pencil  $\mathcal{P}_{P_1, P_2}$  is irreducible, by a similar argument as in the proof of Corollary 5.1.7. This is impossible by applying the main theorem in [Vis], since by assumption

$$(d/m)^2 - 1 \leq d^2/4 - 1 < k.$$

□

□

Recall that a homogenous polynomial  $Q$  is called *completely reducible* if all its irreducible factors are linear forms.

**Lemma 5.1.10.** *Let  $n \geq 2$  and  $d \geq 3$ . Let  $Q_1, Q_2 \in \mathbb{C}[x_0, \dots, x_n]$  be two completely reducible polynomials such that  $\gcd(Q_1, Q_2) = 1$ . Assume there exists another completely reducible polynomial  $Q_3 = \lambda_0 Q_1 + \mu_0 Q_2$  in  $\mathcal{P}_{Q_1, Q_2}$  and that  $V(Q_1)$  and  $V(Q_2)$  are not cones over the same linear subspace of codimension 2.*

*Let  $M = \mathbb{P}^n \setminus V(Q_1 Q_2 Q_3)$  be the complement of the hypersurface  $V(Q_1 Q_2 Q_3)$ . Then the map defined by  $Q_1, Q_2$*

$$\rho : M \rightarrow C, \quad \rho(x_0, \dots, x_n) = (Q_1(x_0, \dots, x_n), Q_2(x_0, \dots, x_n))$$

*has a connected generic fiber, where  $C = \mathbb{P}^1 - B$  with  $B = \{(1, 0), (0, 1), (\mu_0, -\lambda_0)\}$  a set of three points in  $\mathbb{P}^1$ .*

*In particular, there are at most 4 completely reducible fibers in  $\mathcal{P}_{Q_1, Q_2}$ .*

*Proof.* Consider the Stein factorization: there exists a finite map  $p_0 : C_0 \rightarrow C$  and a morphism  $\rho_0 : M \rightarrow C_0$  such that a generic fiber of  $\rho_0$  is connected, and  $\rho = p_0 \circ \rho_0$ . Note that  $C_0$  is also a noncompact curve, so it has the form  $C_0 = \overline{C_0} \setminus B_0$  where  $\overline{C_0}$  is complete smooth curve and  $B_0 \subseteq \overline{C_0}$  is a finite set of points.

We have  $\overline{C_0} = \mathbb{P}^1$  and a morphism  $p'_0 : \overline{C_0} \rightarrow \mathbb{P}^1$ . Let  $m = \deg p'_0$ , and without loss of generality, suppose  $m > 1$ . Proceeding as in the proof of Lemma 5.1.9, we obtain that

$$Q_i = c \prod_{l=1}^m (a_{i,l} P_1 + b_{i,l} P_2), \quad i = 1, \dots, 3,$$

for  $3m$  points  $(a_{i,l}, b_{i,l})$ ,  $i = 1, 2, 3, l = 1, \dots, m$  in  $\mathbb{P}^1$ , two coprime polynomials  $P_1$  and  $P_2$  and a constant  $c$ . In this way, we obtain  $3m \geq 6$  completely reducible fibers in the pencil  $\mathcal{P}_{P_1, P_2}$ .

Note that  $\deg P_1 = \deg P_2 > 1$ , for otherwise  $V(Q_1)$  and  $V(Q_2)$  are cones over  $V(P_1, P_2)$  which is a linear subspace of codimension 2, contradicting our assumption. Hence by [PY] or [Yuz], there are at most 4 completely reducible fibers in the pencil  $\mathcal{P}_{P_1, P_2}$ . Note that the irreducibility of a generic fiber in  $\mathcal{P}_{P_1, P_2}$  holds well.

Therefore, we have obtained the desired contradiction.  $\square$

**Remark 5.1.11.** The Lemmas 5.1.4 and 5.1.10 have a remarkable application in the study of tangential deformations of homogenous polynomials, see [Wang].

## 5.2 Morphisms associated to multinets

For the theory of multinets, we refer to [FY] and Preliminaries of this thesis. Let  $\mathcal{A} = \{L_1, \dots, L_d\}$  be a line arrangement in  $\mathbb{P}^2$  with defining polynomial  $Q$ , which we always assume is centerless, i.e.,  $\cap_j L_j = \emptyset$ . Note that we identify a line  $L_i$  with its defining equation, i.e., a linear form  $L_i = L_i(x, y, z)$ .

Given a  $(k, e)$ -multinet  $\mathcal{M} = (\mathcal{A}; m; B)$  on  $\mathcal{A}$ , we obtain two important invariants:

- (1) a multiplicity function  $m : \{1, \dots, d\} \rightarrow \mathbb{Z}_{>0}$ .

Let  $m_i = m(i)$ .

- (2) a partition

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots \cup \mathcal{A}_k$$

into  $k \geq 3$  sets such that

$$\sum_{L_j \in \mathcal{A}_i} m_j = e$$

independent of  $j$ .

(3)  $\gcd(m_i : i = 1, \dots, d) = 1$ . (as we always assume)

Let

$$Q_i = \prod_{L_j \in \mathcal{A}_i} L_j^{m_j}, \quad i = 1, \dots, k$$

and

$$Q' = Q_1 Q_2 \cdots Q_k.$$

Then the vector space  $\mathbb{C}\langle Q_1, \dots, Q_k \rangle$  has dimension 2, namely,  $Q_i \in \mathcal{P}_{Q_1, Q_2}$  for all  $i$ .

Assume  $Q_i = \alpha_i Q_1 + \beta_i Q_2$ ,  $i = 1, \dots, k$  where  $(\alpha_1, \beta_1) = (1, 0)$  and  $(\alpha_2, \beta_2) = (0, 1)$  and define

$$f : M = M(Q) \rightarrow C, \quad f = (Q_1, Q_2)$$

with  $C = \mathbb{P}^1 \setminus \{(-\beta_j, \alpha_j) : j = 1, \dots, k\}$ .

We call  $f$  the *morphism (defined up to isomorphism) associated to the given multinet* or more briefly *the associated morphism*. Then from Lemma 5.1.10, we obtain the following.

**Proposition 5.2.1.** *Let  $\mathcal{A}$  be a centerless line arrangement in  $\mathbb{P}^2$ .*

*Then any associated morphism  $f$  for a  $(k, e)$ -multinet of  $\mathcal{A}$  has a connected generic fiber, in other words, an irrational pencil of  $M$ .*

*In particular, a generic fiber of  $f$  is irreducible, and hence  $k \leq 4$  by [Yuz].*

Conversely, it turns out any irrational pencil is associated to a multinet, see [FY] for more details.

**Remark 5.2.2.** Contrary to the conventional notion where a map from a variety to  $\mathbb{P}^1$  is called a *rational pencil*, in this thesis, we call a morphism onto  $\mathbb{P}^1$  with  $\geq 3$  points deleted an *irrational pencil*. Recall also that by definition, an irrational pencil is a morphism onto a smooth curve with negative Euler characteristic (under suitable equivalence relations).

Now let  $F' : Q' = 1$  be the Milnor fiber of  $Q'$  in  $\mathbb{C}^3$ , and moreover

$$G(u, v) = \prod_{i=1}^k (\alpha_i u + \beta_i v) = uv \prod_{i=3}^k (\alpha_i u + \beta_i v)$$

and  $H : G = 1$  the Milnor fiber of  $G$  in  $\mathbb{C}^2$  and  $p' : H \rightarrow C$  the natural projection. Let  $g' : F' \rightarrow H$  be defined as  $g'(x, y, z) = (Q_1(x, y, z), Q_2(x, y, z))$ , then we have the following diagram

$$\begin{array}{ccc} F' & \xrightarrow{g'} & H \\ p \downarrow & & \downarrow p' \\ M & \xrightarrow{f} & C. \end{array}$$

Note that  $g'$  is well defined since  $Q' = \prod_{i=1}^k (\alpha_i Q_1 + \beta_i Q_2)$ .

The base locus of the pencil  $\mathcal{P}_{Q_1, Q_2}$  is  $V(Q_1, Q_2)$ , which is a finite set. For any  $i = 1, \dots, k$  and  $b \in V(Q_1, Q_2)$ , let  $n_{b,i}$  be the multiplicity at  $b$  of the (possibly nonreduced) scheme  $Q_i = 0$ , namely,  $n_{b,i} = \sum_{L_j \in \mathcal{A}_i} m_j$ , then by the definition of a multinet,  $n_{b,1} = n_{b,2} = \dots = n_{b,k}$ , which we shall simply denote by  $n_b$ . Then the multiplicity at  $b$  of  $Q' = 0$  is  $kn_b$ .

**Proposition 5.2.3.** *With the notations above, assume  $\gcd(n_b : b \in V(Q_1, Q_2)) = 1$ , then a generic fiber of  $g'$  is connected.*

*Furthermore,  $g'$  is an irrational pencil of  $F'$ .*

*Proof.* First observe that  $F'$  is connected since  $\gcd(m_1, \dots, m_d) = 1$ . So for the covering  $p : F' \rightarrow M$  of degree  $ke$ , the monodromy map is surjective, i.e.,

$$\pi_1(M) \rightarrow \mu_{ke} = \mathbb{Z}/ke\mathbb{Z}$$

is surjective. Recall also that  $\pi_1(M)$  has generators provided by  $\gamma_1, \dots, \gamma_d$ , where  $\gamma_i$  is the boundary of a small disk intersecting transversely to  $L_i$ , and the monodromy map above is given by  $\gamma_i \mapsto m_i \bmod ke$ .

Let  $K$  be a generic fiber of  $f$ , then by Proposition 5.2.1,  $K$  is connected. Now consider the covering map  $p$  restricted to  $p^{-1}(K)$ , then by the proof of Lemma 5.1.1, the number of connected components is equal to the index in  $\mu_{ke}$  of the group

$$\text{Im}(\pi_1(K) \rightarrow \mu_{ke}),$$

which is the same as the image of the composition

$$\pi_1(K) \rightarrow \pi_1(M) \rightarrow \mu_{ke}.$$

Note that the map  $\pi_1(K) \rightarrow \mu_{ke}$  factors through  $H_1(K, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z}) \rightarrow \mu_{ke}$ .

Now for any  $b \in V(Q_1, Q_2)$ , we can choose a small topological disk  $D$  in  $\mathbb{P}^2$  such that  $D \setminus \{b\}$  is contained in  $K$ . Indeed, the closure of  $K$  contains the base locus  $V(Q_1, Q_2)$  and  $K$  is locally contractible as an algebraic variety. Moreover, choose a tubular product  $N \cong (D \setminus \{b\}) \times [-\epsilon, \epsilon]$  for  $\epsilon > 0$  small enough such that  $N \cap K \cong (D \setminus \{b\}) \times \{0\}$  and the boundary  $\partial N$  is completely contained in  $M$ .

Then the boundary  $\partial D \subseteq K$  is homologous in  $M$  to  $\partial(D \setminus \{b\}) \times \{\epsilon\}$ , which is mapped to  $kn_b \bmod ke$  under the map  $\pi_1(M) \rightarrow \mu_{ke}$ . Hence the image of the map  $\pi_1(K) \rightarrow \mu_{ke}$  contains at least the subgroup

$$\langle kn_b \bmod ke : b \in V(Q_1, Q_2) \rangle \cong \mu_e,$$

since  $\gcd(n_b : b \in V(Q_1, Q_2)) = 1$  by assumption. Therefore,  $p^{-1}(K)$  has at most  $k$  connected components.

Let  $f(K) = x$  and  $p'^{-1}(x) = (x_1, \dots, x_k)$ . Then  $p^{-1}(K) = \bigsqcup_l g^{-1}(x_l)$ . Since  $p'^{-1}(f(K))$  consists of  $k$  distinct points,  $p^{-1}(K)$  has at least  $k$  connected components.

Consequently,  $p^{-1}(K)$  has exactly  $k$  connected components, one for each preimage under  $p'$  of  $f(K)$ . Since  $K$  is generically chosen, we have that a generic fiber of  $g$  is connected.

To see that  $g'$  is an irrational pencil, we only need to check  $\chi(H) < 0$ . Indeed,  $\chi(H) = k \cdot \chi(C) = k \cdot (2 - k) < 0$  since  $k \geq 3$ .  $\square$

When the multinet is a reduced, namely,  $m_i = 1$  for all  $i = 1, \dots, k$ , and then  $d = ke$ . Recall also that a reduced multinet  $\mathcal{M} = (\mathcal{A}; m; B)$  is a net if any point  $b \in B$  satisfies  $n_b = 1$ .

From the discussions above, we have the following.

**Proposition 5.2.4.** *Let  $\mathcal{A} = \{L_1, \dots, L_d\}$  be a centerless line arrangement such that*

(i) *there exists a partition*

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots \cup \mathcal{A}_k$$

*into  $k \geq 3$  sets and each  $\mathcal{A}_i$  contains  $e = d/k$  lines.*

(ii)

$$Q_i = \prod_{L_j \in \mathcal{A}_i} L_j, \quad i = 1, \dots, k$$

*such that the vector space  $\mathbb{C}\langle Q_1, \dots, Q_k \rangle$  is of dimension 2.*

Let moreover  $Q = \prod_{j=1}^d L_j$ ,  $F : Q = 1$  the Milnor fiber and  $M = \mathbb{P}^2 \setminus V(Q)$  the complement; let  $Q_i = \alpha_i Q_1 + \beta_i Q_2$  for  $i = 1, \dots, k$  with  $(\alpha_1, \beta_1) = (1, 0)$  and  $(\alpha_2, \beta_2) = (0, 1)$ ,  $G = \prod_{i=1}^k (\alpha_i u + \beta_i v)$ ,  $H : G = 1$  the Milnor fiber in  $\mathbb{C}^2$  and  $C = \mathbb{P}^1 \setminus V(G)$ .

Then for the diagram

$$\begin{array}{ccc} F & \xrightarrow{g} & H \\ p \downarrow & & \downarrow p' \\ M & \xrightarrow{f} & C \end{array}$$

where  $f = (Q_1, Q_2)$ ,  $g = (Q_1, Q_2)$  and  $p, p'$  are the natural projections, the following hold:

1.  $f$  is an irrational pencil of  $M$ .

2. For  $b \in V(Q_1, Q_2)$ , let  $n_b$  be the numbers of lines in  $\mathcal{A}_1$  passing through  $b$ . If  $\gcd(n_b : b \in V(Q_1, Q_2)) = 1$ , then  $g$  is an irrational pencil of  $F$ .

In particular, the conclusion holds if some  $b \in V(Q_1, Q_2)$  is a smooth point of  $V(Q_1)$ .

This is the case if the constructions above are from a  $(k, e)$ -net of  $\mathcal{A}$ .

### 5.3 1-formality of Milnor fiber

Given a centerless line arrangement  $\mathcal{A} = \{L_1, \dots, L_d\}$ . We suppose there exists a partition  $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots \cup \mathcal{A}_k$  into  $k \geq 3$  sets and each  $\mathcal{A}_i$  contains  $e = d/k$  lines. Moreover,  $Q_i = \prod_{L_j \in \mathcal{A}_i} L_j$ ,  $i = 1, \dots, k$  such that the vector space  $\mathbb{C}\langle Q_1, \dots, Q_k \rangle$  is of dimension 2. Then as above, we have the following diagram

$$\begin{array}{ccc} F & \xrightarrow{g} & H \\ p \downarrow & & \downarrow p' \\ M & \xrightarrow{f} & C. \end{array}$$

Now we also take the monodromy  $h^* : H^i(F) \rightarrow H^i(F)$  into account, induced by  $h : F \rightarrow F$ ,  $h((x, y, z)) = (\xi x, \xi y, \xi z)$  with  $\xi = \exp(2\pi\sqrt{-1}/d)$ . Denote

$$H^i(F)_a = \ker(a \cdot \text{Id} - h^* : H^i(F) \rightarrow H^i(F))$$

and

$$H^i(F)_{\neq 1} = \bigoplus_{a \neq 1} H^i(F)_a.$$

Then we have  $W_1 H^1(F) = H^1(F)_{\neq 1}$  while  $H^1(F)_1 = p^* H^1(M)$ .

Moreover, we have the following

**Proposition 5.3.1.** *With notations as above, we have  $W_1 H^1(F) \subseteq \mathcal{R}_1(F)$ .*

*Proof.* Without loss of generality, assume  $W_1 H^1(F) = H^1(F)_{\neq 1} \neq 0$ . It suffices to consider the following two cases:

- (i) For any  $a^2 \neq 1$ , if  $H^1(F)_a \neq 0$ , then also  $H^1(F)_{a^{-1}} \neq 0$ , because  $a^{-1}$  and  $a$  generate the same subgroup of  $\mu_d$ , see [CS95]. Moreover, for any nonzero elements  $\alpha_1 \in H^1(F)_a$  and  $\alpha_2 \in H^1(F)_{a^{-1}}$ , we claim that  $\alpha_1 \cup \alpha_2 = 0$  in  $H^2(F)$ , hence  $H^1(F)_a \subseteq \mathcal{R}_1(F)$ .

Indeed,  $\alpha_1 \cup \alpha_2 \in H^2(F)_1 \cap W_2 H^2(F) = p^* H^2(M) \cap W_2 H^2(F)$ . Since  $H^2(M)$  is pure of type  $(2, 2)$ , we have that  $W_2 H^2(F) \cap p^* H^2(M) = 0$  since  $p^*$  is a MHS morphism.

- (ii) If  $H^1(F)_{-1} \neq 0$ , then it has even dimension, since  $H^1(F)_{\neq 1}$  a MHS of pure weight 1. So for any  $\alpha_3 \in H^1(F)_{-1}$ , we can find  $\alpha_4 \in H^1(F)_{-1}$  which is linearly independent to  $\alpha_3$ . Just as in the first case, we have  $\alpha_3 \cup \alpha_4 = 0$ , and hence  $\alpha_3 \in \mathcal{R}_1(F)$ .  $\square$

The following proof of non-formality of the Milnor fiber can date back to [Zub].

**Theorem 5.3.2.** *With notations as above. If  $g$  admits a connected generic fiber and*

$$g^*W_1H^1(H) \subsetneq W_1H^1(F),$$

*then  $F$  is not 1-formal.*

*Proof.* Suppose on the contrary that  $F$  is 1-formal, then  $g^*H^1(H)$  is an irreducible component of  $\mathcal{R}_1(F)$ . Moreover,  $g^* : H^1(H) \rightarrow H^1(F)$  is injective and

$$\dim W_1H^1(H) = \dim H^1(\overline{H})$$

where  $\overline{H} : G(u, v) + t^k = 0$  in  $\mathbb{P}^2$ . Note that  $\overline{H}$  is a smooth plane curve of degree  $k$ . By the degree-genus formula,  $g(\overline{H}) = (k-1)(k-2)/2 > 0$  since  $k \geq 3$ , so we have that

$$\dim W_1H^1(H) = 2g(\overline{H}) > 0.$$

Furthermore, since  $W_1H^1(F) \subseteq \mathcal{R}_1(F)$  by Proposition 5.3.1, there exists an irreducible component, say  $E$ , of  $\mathcal{R}_1(F)$  such that  $E \supseteq W_1H^1(F)$ .

Since by assumption  $g^*W_1H^1(H) \subsetneq W_1H^1(F)$ , we obtain that  $E \neq g^*H^1(H)$  and also  $E \cap g^*H^1(H) \supseteq g^*W_1H^1(H)$  is nonzero. Therefore, we obtain two distinct irreducible components of  $\mathcal{R}_1(F)$ , namely,  $E$  and  $g^*H^1(H)$ , intersecting nontrivially. This contradicts Theorem 2.2.6 (iii).  $\square$

**Example 5.3.3.** Consider the arrangement  $\mathcal{A}(m, m, 3)$ :

$$Q = (x^m - y^m)(y^m - z^m)(z^m - x^m),$$

where  $m \equiv 0 \pmod{3}$  and  $Q_1 = x^m - y^m, Q_2 = y^m - z^m, Q_3 = z^m - x^m$ .

Proceeding as above, we have that the map  $g$  is an irrational pencil for the Milnor fiber  $F : Q = 1$ , since  $V(Q_1, Q_2)$  are all smooth points of  $V(Q_1)$ . Moreover,  $k = 3$ , so  $\dim W_1H^1(H) = 2$ ; on the other hand,  $\dim W_1H^1(F) = \dim H^1(F)_{\neq 1} = 4$  by the example given in Section 2.3.10.

So  $F$  is not 1-formal in this case. In particular, when  $m = 3$ , we obtain the example given in [Zub].

## 5.4 Cohomology jump loci of Milnor fiber

With the settings before Theorem 5.3.2, assume further  $g$  is an irrational pencil and  $g^*W_1H^1(H) = W_1H^1(F)$ . Then  $g^*H^1(H)$  is an irreducible component of  $TC_1(\mathcal{V}_1(F))$ . Suppose there exists another irreducible component  $0 \neq E \subseteq TC_1(\mathcal{V}_1(F))$ , then necessarily,  $E \subseteq Gr_2^W H^1(F) = H^1(F)_1 = p^*H^1(M)$  since  $0 = E \cap g^*H^1(H) \supseteq E \cap W_1H^1(F)$ . Hence,  $E$  is a MHS having pure type  $(1, 1)$ . All these give sufficient motivations to study the irreducible components of  $TC_1(\mathcal{V}_1(F))$  that are pure of type  $(1, 1)$ . We first propose an interesting lemma about morphisms to curves (compare [DPS09], Lemma 6.10).

**Lemma 5.4.1.** *Let  $\rho : Y \rightarrow C$  be a dominating morphism from a quasi-projective manifold  $Y$  to a smooth curve  $C$ . Then the induced homomorphism in cohomology  $\rho^*$  is injective on  $W_1H^1$ , namely*

$$\rho^* : W_1H^1(C) \rightarrow W_1H^1(Y)$$

*is injective.*

*Proof.* By Stein Factorization, it suffices to prove the case where  $Y$  is itself a curve. Indeed, let  $Y \xrightarrow{\rho'} C' \xrightarrow{p_0} C$  be the Stein factorization for  $\rho$  such that a generic fiber of  $\rho'$  is connected,  $p_0 : C' \rightarrow C$  is finite and  $\rho = p_0 \circ \rho'$ . Since  $\rho'$  has a connected generic fiber, the induced map  $\rho'_* : \pi_1(Y) \rightarrow \pi_1(C')$  is surjective, hence  $\rho'^*$  is injective on  $H^1$ , a fortiori, on  $W_1H^1$ . If  $p_0^*$  is injective on  $W_1H^1$ , hence so is  $\rho^*$  because  $\rho^* = \rho'^* \circ p_0^*$ .

Now assume  $Y$  is also a curve. We claim that we can assume one step further that  $Y$  and  $C$  are both complete smooth curves. Indeed, let  $i_1 : Y \hookrightarrow \bar{Y}$  and  $i_2 : C \hookrightarrow \bar{C}$  be two smooth compactifications, then we obtain a rational morphism

$$\bar{\rho} = i_2 \circ \rho : \bar{Y} \dashrightarrow \bar{C}$$

which must be a morphism by Zariski Main Theorem, because it is a morphism between two complete smooth curves. Hence we have the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & C \\ \downarrow i_1 & & \downarrow i_2 \\ \bar{Y} & \xrightarrow{\bar{\rho}} & \bar{C}, \end{array}$$

and the induced homomorphisms on  $W_1H^1$ :

$$\begin{array}{ccc} W_1H^1(Y) & \xleftarrow{\rho^*} & W_1H^1(C) \\ \uparrow i_1^* & & \uparrow i_2^* \\ W_1H^1(\bar{Y}) = H^1(\bar{Y}) & \xleftarrow{\bar{\rho}^*} & W_1H^1(\bar{C}) = H^1(\bar{C}). \end{array}$$

Clearly  $i_1^*, i_2^*$  above are isomorphisms. If  $\bar{\rho}^*$  is injective, so is  $\rho^*$ .

Therefore, we are reduced to consider a surjective morphism  $\rho : Y \rightarrow C$  between two complete smooth curves. Indeed,  $\rho$  is a finite map and  $\rho_*\rho^* = \deg \rho \cdot \text{Id}$  where  $\rho_*$  is the Gysin map, see [Voi1], Section 7.3.2. Alternatively, we can also apply [Voi1], Lemma 7.28, since in our situation,  $Y$  is a compact Kähler manifold. We are done.  $\square$

Now let  $\mathcal{A}$  be a centerless line arrangement,  $F$  its Milnor fiber and  $M$  its complement. Suppose  $E \subseteq TC_1(\mathcal{V}_1(F))$  is a positive dimensional irreducible component pure of type  $(1, 1)$ , then  $E$  is a maximal isotropic subspace of  $H^1(F)$  by Remark 4.2.5, a fortiori, of  $p^*H^1(M)$ . Hence  $E = p^*E'$  for some maximal isotropic subspace of  $H^1(M)$ ; by correspondence between maximal isotropic subspaces and irrational pencils, we have that  $E'$  is also an irreducible component of  $TC_1(\mathcal{V}_1(M)) = \mathcal{R}_1(M)$ . Therefore, there exists a  $(k_E, e_E)$ -multinet  $\mathcal{M}_E = (\mathcal{A}; m_E; B_E)$  on  $\mathcal{A}$  with associated morphism  $f = (Q_1, Q_2) : M \rightarrow C$  with  $f^*H^1(C) = E'$ . In particular, we have the following diagram

$$\begin{array}{ccc} F & & \\ p \downarrow & \searrow^{g=f \circ p} & \\ M & \xrightarrow{f} & C. \end{array}$$

**Lemma 5.4.2.** *With notations as above, a generic fiber of  $g$  is connected, i.e.,  $g$  is an irrational pencil of  $F$ .*

*Proof.* Consider the Stein Factorization

$$\begin{array}{ccc} F & \xrightarrow{g'_0} & C' \\ p \downarrow & \searrow^{g=f \circ p} & \downarrow p_0 \\ M & \xrightarrow{f} & C \end{array}$$

where a generic fiber of  $g'_0$  is connected and  $p_0$  is a finite map. Let  $\overline{C'}$  be a smooth compactification of  $C'$ .

First,  $\overline{C'} \cong \mathbb{P}^1$ . Indeed,  $g'^*H^1(C')$  is a connected component of  $TC_1(\mathcal{V}_1(X))$  containing both  $g^*H^1(C) = p^*f^*H^1(C) = E$  and  $g^*W_1H^1(C')$ . If  $g(\overline{C'}) > 0$ , then  $\dim g'^*W_1H^1(C') = 2g(\overline{C'}) > 0$  by Lemma 5.4.1. Hence  $g'^*H^1(C')$  strictly contains  $E$ , contradiction.

Second,  $p_0$  extends to  $p'_0 : \overline{C'} \cong \mathbb{P}^1 \rightarrow \mathbb{P}^1$ . Let  $\deg p'_0 = m$  and without loss of generality, suppose  $m > 1$ . Then

$$\chi(C') \leq m \cdot \chi(C) \leq 2 \cdot \chi(C) < \chi(C),$$

and thus  $\dim H^1(C') > \dim H^1(C)$ . Therefore,  $g'^*H^1(C')$  is an irreducible component of  $TC_1(\mathcal{V}_1(F))$  strictly containing  $g^*H^1(C) = E$ , contradiction.  $\square$

**Theorem 5.4.3** (Compare [PaS], Theorem 1.5). *The multinet  $\mathcal{M}_E$  is not reduced, namely, there exists a  $j$  satisfying  $m_E(j) > 1$ . In particular,  $k_E = 3$  and  $\dim E = 2$ .*

*Proof.* Otherwise, the map  $g$  will match a diagram as before Theorem 5.3.2:

$$\begin{array}{ccc} F & \xrightarrow{g} & H \\ p \downarrow & & \downarrow p' \\ M & \xrightarrow{f} & C, \end{array}$$

hence each fiber of  $g$  contains at least  $k_E \geq 3$  connected components, contradicting the above lemma. For other statements, just apply the main theorem in [Yuz].  $\square$

As an obvious corollary of the above theorem, we have the following.

**Corollary 5.4.4.** *Assume there exists a net structure  $\mathcal{M} = (\mathcal{A}; m; B)$  on  $\mathcal{A}$  with associated morphisms  $f : M \rightarrow C$  and  $g : F \rightarrow H$  such that  $g^*W_1(H) = W_1(F)$ . For any irreducible component  $E' \neq f^*H^1(C)$  of  $TC_1(\mathcal{V}_1(M))$  of positive dimension, we have  $E' = f_{E'}^*H^1(C_{E'})$  where  $f_{E'} : M \rightarrow C_{E'}$  is the associated morphism to a multinet  $\mathcal{M}_{E'} = (\mathcal{A}; m_{E'}; B_{E'})$ .*

*Then,  $\mathcal{M}_{E'}$  is not a net.*

*Therefore, there is only one net structure on  $\mathcal{A}$ .*

## Surfaces associated to line arrangements and Chern numbers

One of the main open questions in hyperplane arrangement theory is whether the topology of the Milnor fiber  $F$ , e.g., its Betti numbers or the characteristic polynomials of the monodromy action on each  $H^m(F)$ , are determined by the combinatorics. In this chapter, we consider the compactification  $\overline{F}$  of  $F$  in  $\mathbb{P}^3$  in the case of a line arrangement  $\mathcal{A}$  in  $\mathbb{P}^2$ . If  $\tilde{F}$  is the minimal desingularization of  $\overline{F}$ , then we compute the Chern numbers of  $\tilde{F}$  in terms of the combinatorics. The hope is that using the classification theory of smooth projective surfaces, we can better understand the topology of the Milnor fiber  $F$  in this case.

### 6.1 Basic topology

Let  $Z$  be a topological space. We can define its singular cohomology groups  $H^*(Z, \mathbb{Z})$  and homology groups  $H_*(Z, \mathbb{Z})$ . Moreover, we have the cohomology groups and homology groups with compact support, see [PeS], Appendix B.

By definition, the cohomology group of  $Z$  of degree  $k$  with compact support, denoted by  $H_c^k(Z, \mathbb{Z})$ , is

$$H_c^k(Z, \mathbb{Z}) = \varinjlim_{K \subseteq Z} H^k(Z, Z - K, \mathbb{Z}),$$

where  $K$  ranges over all *compact subsets* of  $Z$ , and the direct limit (or inductive limit) is taken with respect to the natural morphism  $H^k(Z, Z - K_1, \mathbb{Z}) \rightarrow H^k(Z, Z - K_2, \mathbb{Z})$  induced by the inclusion  $(Z, Z - K_2) \hookrightarrow (Z, Z - K_1)$  for any two compact subsets  $K_1 \subseteq K_2$  of  $Z$ . The corresponding compactly supported version for homology of dimension  $k$ , denoted by  $H_k^{BM}(Z, \mathbb{Z})$ , is usually called Borel-Moore homology, and is given by

$$H_k^{BM}(Z, \mathbb{Z}) = \varprojlim_{K \subseteq Z} H_k(Z, Z - K, \mathbb{Z}),$$

where  $K$  varies among all the compact subsets of  $Z$  and the projective limit (or inverse limit) is taken with respect to the morphism  $H_k(Z, Z - K_2, \mathbb{Z}) \rightarrow H_k(Z, Z - K_1, \mathbb{Z})$  induced by the inclusion  $(Z, Z - K_2) \hookrightarrow (Z, Z - K_1)$  for any two compact subsets  $K_1 \subseteq K_2$  of  $Z$ .

#### 6.1.1 Differential topology

When  $Z$  is a smooth manifold, the cohomology groups and homology groups admit quite interesting properties. Now assume  $Z$  is a complex manifold of complex dimension  $n$ , which

is sufficient for our purpose of applications. Then  $Z$  is an oriented differential manifold with a canonical orientation induced by the complex structure. The orientation class gives canonically a generator  $[Z] \in H_{2n}^{\text{BM}}(Z, \mathbb{Z}) \cong \mathbb{Z}$ , corresponding to  $1 \in \mathbb{Z}$ ; the class  $[Z]$  is also called the *fundamental class* for  $Z$ .

Taking cap-product with  $[Z]$  gives the Poincaré duality

$$[Z] \cap : H_c^k(Z, \mathbb{Z}) \simeq H_{2n-k}(Z, \mathbb{Z}), \quad k \leq 2n,$$

and also

$$[Z] \cap : H^k(Z, \mathbb{Z}) \simeq H_{2n-k}^{\text{BM}}(Z, \mathbb{Z}), \quad k \leq 2n;$$

in addition, we have a pairing

$$H_c^n(Z, \mathbb{Z}) \times H^n(Z, \mathbb{Z}) \rightarrow \mathbb{Z}, \quad (\xi, \eta) \mapsto [Z] \cap (\xi \cup \eta).$$

See [PeS], Appendix B. For simplicity, write  $\xi \cdot \eta = [Z] \cap (\xi \cup \eta)$ .

Since  $Z$  is smooth, by de Rham Theorem, the cohomology in  $H^k(Z) = H^k(Z, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$  can be represented by closed forms; correspondingly, the cohomology with compact support in  $H_c^k(Z) = H_c^k(Z, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$  can be represented by closed forms with compact support. Moreover, the Poincaré duality with complex coefficients can be represented as

$$H_c^k(Z) \times H^{2n-k}(Z) \rightarrow \mathbb{C}, \quad ([\omega_1], [\omega_2]) \mapsto \int_Z \omega_1 \wedge \omega_2, \quad (6.1.1)$$

where  $\omega_1, \omega_2$  are closed forms and  $\omega_1$  has compact support. The integration makes sense since  $\omega_1 \wedge \omega_2$  has compact support. The pairing in (6.1.1) is perfect by Poincaré duality, see also [BT], §5.

Clearly, when  $Z$  is itself compact, the homology or cohomology with compact support coincides with the usual ones.

## 6.2 Intersection theory on surfaces

### 6.2.1 Intersection theory for smooth surfaces

Let  $X$  be a connected smooth quasi-projective surface. Assume  $C, D$  are two reduced irreducible curves on  $X$  such that  $C$  is compact. Then for  $C$ , we have  $H_2(C, \mathbb{Z}) \simeq \mathbb{Z}$  with a generator  $[C]$  corresponding to  $1 \in \mathbb{Z}$ ; this is a classical result since topologically  $C$  is a finite CW complex of real dimension 2. In fact, let  $\text{Sing}(C)$  be the singular locus of  $C$ , then  $\text{Sing}(C)$  is a set consisting of finite points and the long exact sequence for homology groups associated to the pair  $(C, \text{Sing}(C))$  gives

$$H_2(\text{Sing}(C), \mathbb{Z}) = 0 \rightarrow H_2(C, \mathbb{Z}) \rightarrow H_2(C, \text{Sing}(C), \mathbb{Z}) \rightarrow H_1(\text{Sing}(C), \mathbb{Z}) = 0,$$

so

$$H_2(C, \mathbb{Z}) \simeq H_2(C, \text{Sing}(C), \mathbb{Z});$$

meanwhile, let  $\nu : \tilde{C} \rightarrow C$  be a normalization for  $C$ , then since complex analytic spaces are locally contractible, and  $\nu : \tilde{C} \setminus \nu^{-1}(\text{Sing}(C)) \rightarrow C \setminus \text{Sing}(C)$  is an isomorphism, by the Strong Excision Theorem (see [PeS], Theorem B. 3), we have

$$H_2(C, \text{Sing}(C), \mathbb{Z}) \simeq H_2(\tilde{C}, \nu^{-1}(\text{Sing}(C)), \mathbb{Z});$$

note that  $\nu^{-1}(\text{Sing}(C))$  is also a set of finite points, by considering the exact sequence of homology groups associated to the pair  $(\tilde{C}, \nu^{-1}(\text{Sing}(C)))$ , we get

$$H_2(\tilde{C}, \nu^{-1}(\text{Sing}(C)), \mathbb{Z}) \simeq H_2(\tilde{C}, \mathbb{Z}) \simeq \mathbb{Z},$$

where the last isomorphism follows from the fact that  $\tilde{C}$  is a smooth complete curve. Since all the isomorphisms above are canonical, we have in fact obtained an isomorphism

$$\nu_* : H_2(\tilde{C}, \mathbb{Z}) \simeq H_2(C, \mathbb{Z})$$

induced by the normalization  $\nu : \tilde{C} \rightarrow C$ , so  $[C] = \nu_*([\tilde{C}])$ , where  $[\tilde{C}] \in H_2(\tilde{C}, \mathbb{Z})$  is the fundamental class for  $\tilde{C}$ .

Similarly, for  $D$ , we have  $H_2^{BM}(D, \mathbb{Z}) \simeq \mathbb{Z}$ . Indeed, let  $D \subseteq D^*$  be a compactification of  $D$  such that  $D^* \setminus D$  is a set of finite points; let  $D^{**}$  be the space obtained from  $D^*$  by contracting  $D^* \setminus D$  to a point. Let  $\infty \in D^{**}$  be the image  $D^* \setminus D$  under the contraction. Then  $D^{**}$  is a one point compactification of  $D$  and

$$H_2^{BM}(D, \mathbb{Z}) \stackrel{1}{\cong} H_2(D^{**}, \infty, \mathbb{Z}) \stackrel{2}{\cong} H_2(D^*, D^* \setminus D, \mathbb{Z}) \stackrel{3}{\cong} H_2(D^*, \mathbb{Z}) \stackrel{4}{\cong} \mathbb{Z},$$

where  $\stackrel{1}{\cong}$  follows from Formula  $(B-2)$  in [PeS],  $\stackrel{2}{\cong}$  follows from the Strong Excision Theorem (see [PeS], Theorem B. 3), and  $\stackrel{3}{\cong}$  follows by considering the exact sequence for homology groups associated to the pair  $(D^*, D^* \setminus D)$  and the fact that  $D^* \setminus D$  is a set of finitely many points, and  $\stackrel{4}{\cong}$  follows from our above discussions about  $H_2(C, \mathbb{Z})$  and the fact that  $D^*$  is a compact curve. Denote by  $[D]$  the generator for  $H_2^{BM}(D, \mathbb{Z})$  corresponding to  $1 \in \mathbb{Z} \simeq H_2^{BM}(D, \mathbb{Z})$ .

Moreover, we have a natural map  $H_2(C, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z})$  induced by the inclusion  $C \hookrightarrow X$ . Denote, by abuse of notation, by  $[C] \in H_c^2(X, \mathbb{Z})$  the image of  $[C] \in H_2(C, \mathbb{Z})$  under the composition  $H_2(C, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z}) \cong H_c^2(X, \mathbb{Z})$ . Similarly, denote by  $[D] \in H^2(X, \mathbb{Z})$  the image of  $[D] \in H_2(D, \mathbb{Z})$  under the composition  $H_2^{BM}(D, \mathbb{Z}) \rightarrow H_2^{BM}(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ .

Finally, with the help of the pairing  $H_c^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$ , we define the *intersection number* of  $C$  and  $D$  by

$$C \cdot D = [C] \cdot [D].$$

We shall also write  $C^2 = C \cdot C$ .

Now a few points are worth to emphasize:

- (i) Since  $X$  is smooth,  $C, D$  can be seen as Cartier divisors and hence they correspond to two line bundles, i.e.,  $\mathcal{O}_X(C)$  and  $\mathcal{O}_X(D)$ , then the class  $[C]$  (resp.  $[D]$ ) is exactly the first Chern class  $c_1(\mathcal{O}_X(C))$  (resp.  $c_1(\mathcal{O}_X(D))$ ). Moreover,

$$C \cdot D = \deg(\nu^* \mathcal{O}_X(D)|_C),$$

where  $\nu : \tilde{C} \rightarrow C$  is the normalization of  $C$ .

Therefore, if  $C \cap D = \emptyset$ , we have  $C \cdot D = 0$ , since  $\mathcal{O}_X(D)|_C$  is a trivial bundle on  $C$ .

- (ii) If  $\dim C \cap D = 0$ , namely,  $C$  and  $D$  intersect at finitely many points. Let  $x \in C \cap D$  be any intersection point and  $\mathcal{O}_{X,x}$  be the local ring at  $x$  of  $X$ . Assume  $f, g \in \mathcal{O}_{X,x}$  are local equations for the germs at  $x$  of  $C, D$  respectively. Then we have the *intersection multiplicity* of  $C, D$  at  $x$

$$(C \cdot D)_x = \dim_{\mathbb{C}} \mathcal{O}_{X,x}/(f, g);$$

note that the right hand side makes sense since  $\mathcal{O}_{X,x}/(f, g)$  is an Artinian ring, hence a finite-dimensional vector space. Moreover,

$$C \cdot D = \sum_{x \in C \cap D} (C \cdot D)_x,$$

(see [BHPV], Chapter II) from which it obviously follows that  $C \cdot D = 0$  when  $C \cap D = \emptyset$ .

(iii) Now integration over the smooth locus of  $C$  defines a linear functional

$$H^2(X) \rightarrow \mathbb{C}, \quad [\omega] \mapsto \int_{C \setminus \text{Sing}(C)} \omega.$$

This is well defined, see [Voi1], Theorem 11.21, and hence by Poincaré duality gives an element in  $H_c^2(X)$ . By de Rham theorem, this element can be represented by a closed form  $\omega_C$  on  $X$  with compact support.

Similarly, integration over the smooth locus of  $D$  gives a linear functional

$$H_c^2(X) \rightarrow \mathbb{C}, \quad [\omega] \mapsto \int_{D \setminus \text{Sing}(D)} \omega$$

and hence, by Poincaré duality, it gives an element in  $H^2(X)$  and can be represented by a closed form  $\omega_D$  on  $X$ .

Then we have

$$C \cdot D = \int_X \omega_C \wedge \omega_D.$$

The integration above makes sense since  $\omega_C \wedge \omega_D$  is a 4-form with compact support.

(iv) When  $X$  is compact, then by the celebrated Hirzebruch-Atiyah-Singer Riemann-Roch Theorem (see [BHPV], Chapter I), we have

$$\chi(\mathcal{O}_X(m_1C + m_2D)) = \sum_{p=0}^2 (-1)^p \dim_{\mathbb{C}} H^p(X, \mathcal{O}_X(m_1C + m_2D))$$

is a polynomial of degree  $\leq 2$  of  $m_1, m_2$ , see also [Deb], and moreover, it admits the following asymptotic expression:

$$\chi(\mathcal{O}_X(m_1C + m_2D)) = \frac{1}{2}(m_1^2 C^2 + 2m_1 m_2 C \cdot D + m_2^2 D^2) + \dots;$$

hence  $C \cdot D$  is the coefficient of  $m_1 m_2$  in  $\chi(\mathcal{O}_X(m_1C + m_2D))$ .

Extending by linearity, we can define the intersection number of any two divisors, namely, let  $C, D$  are two divisors on  $X$  such that  $C = \sum n_i C_i$  and  $D = \sum m_j D_j$  where  $C_i, D_j$  are reduced irreducible curves and all  $C_i$ 's are compact, then define

$$C \cdot D = \sum n_i m_j C_i \cdot D_j.$$

### 6.2.2 Intersection theory for normal surfaces

Given a quasi-projective surface  $X$ . When  $X$  is not smooth, we assume that  $X$  is normal, namely, the local ring  $\mathcal{O}_{X,x}$  at any point  $x \in X$  is an integrally closed domain. In particular,  $X$  is regular in codimension 1, and thus we can consider the Weil divisors and Cartier divisors on  $X$ .

Let  $C$  be a reduced irreducible curve on  $X$  and  $D$  a Cartier divisor on  $X$  such that  $C$ , as a Weil divisor, has compact support, i.e.,  $C$  is compact. We define the intersection number of  $C$  and  $D$  by

$$C \cdot D = \deg(\nu^* \mathcal{O}_X(D)|_C), \quad (6.2.1)$$

where  $\nu : \tilde{C} \rightarrow C$  is the normalization. Extend this definition by linearity for a general Weil divisor with compact support and a general Cartier divisor, that is, let  $C = \sum n_i C_i$  where  $C_i$  are reduced irreducible curves and all  $C_i$ 's are compact and  $D$  is a Cartier divisor, then

$$C \cdot D = \sum n_i C_i \cdot D.$$

Recall that two Weil divisors  $D, D'$  on  $X$  are called *rationally equivalent*, denoted by  $D \sim D'$  if the difference  $D - D'$  is a principal divisor, i.e.,  $D - D' = (f)$  for some rational function  $f$  on  $X$ , where  $(f) = (f)_0 - (f)_\infty$  is the zeros of  $f$  minus the poles of  $f$ . If  $D, D'$  are two Cartier divisors and  $D \sim D'$ , we have, by Formula (6.2.1)

$$C \cdot D = C \cdot D',$$

for any Weil divisor  $C$  on  $X$  with compact support.

Moreover, if  $X$  is compact and  $C, D$  are both Cartier divisors, then  $C \cdot D = D \cdot C$ ; in addition, even if the compact normal surface  $X$  is not smooth, the number

$$\chi(\mathcal{O}_X(m_1 C + m_2 D)) = \sum_{p=0}^2 (-1)^p \dim H^p(X, \mathcal{O}_X(m_1 C + m_2 D))$$

still makes sense and is in fact a polynomial in  $m_1, m_2$  of degree  $\leq 2$ ; furthermore,  $C \cdot D$  is the coefficient of  $m_1 m_2$  in it, see [Deb].

Finally, we emphasize once more that if  $C, D$  has disjoint support, then  $C \cdot D = 0$ , since  $\mathcal{O}_X(D)|_C$  is a trivial line bundle.

### 6.2.3 Projection formula

Let  $\tilde{X}, X$  be two normal quasi-projective surfaces and  $\pi : \tilde{X} \rightarrow X$  be a proper surjective morphism. Two functors on divisors  $\pi^*$  and  $\pi_*$  are defined as follows:

- (i) For any reduced irreducible curve  $C$  on  $X$ , seen as a Weil divisor, we define  $\pi^* C = \pi^{-1}(C)$  as the scheme-theoretic inverse image of  $C$  on  $\tilde{X}$ . Note that  $\pi^* C$  can be regarded as a Weil divisor on  $\tilde{X}$ . Extending by linearity, we define  $\pi^* C$  for any Weil divisor  $C$  on  $X$ , a fortiori, for any Cartier divisor on  $X$ . Note that if  $C$  is a Cartier divisor on  $X$ , then  $\pi^* C$  is also a Cartier divisor on  $\tilde{X}$ .
- (ii) Given any reduced irreducible curve  $E$  on  $\tilde{X}$ , seen as a Weil divisor. If  $E$  is contracted by  $\pi$  to a point, we define  $\pi_* E = 0$ , as a Weil divisor. If  $\pi(E)$  is a curve, then define

$$\pi_* E = \deg(\pi|_E) \pi(E).$$

Extending by linearity, we can define  $\pi_* E$  for any Weil divisor  $E$  on  $\tilde{X}$ , with  $\pi_* E$  regarded as a Weil divisor on  $X$ .

The following property of intersection theory is of great importance in the sequel, see also [Deb].

**Theorem 6.2.4** (Projection formula). *Let  $\tilde{X}, X$  be normal quasi-projective surfaces and  $\pi : \tilde{X} \rightarrow X$  a proper surjective morphism between them. Let  $C$  be an reduced irreducible compact curve on  $X$  and  $D$  is a Cartier divisor on  $X$ . Then*

$$\pi^*C \cdot \pi^*D = (\deg \pi)C \cdot D.$$

*If moreover,  $E$  is a reduced irreducible compact curve on  $\tilde{X}$  such that either  $E$  is contracted by  $\pi$  or the restriction  $\pi|_E : E \rightarrow \pi(E)$  has degree  $\deg(\pi)$ , then*

$$E \cdot \pi^*D = \pi_*E \cdot D.$$

Thus, by linearity, we have the following corollary.

**Corollary 6.2.5.** *Let  $\tilde{X}, X$  be normal quasi-projective surfaces and  $\pi : \tilde{X} \rightarrow X$  a proper surjective morphism between them. Let  $C$  be a Weil divisor on  $X$  with compact support and  $D$  a Cartier divisor on  $X$ . Then*

$$\pi^*C \cdot \pi^*D = (\deg \pi)C \cdot D.$$

*If moreover,  $E$  is a Weil divisor with compact support on  $\tilde{X}$  such that for each irreducible component  $E_0$  of  $E$  with  $\dim \pi(E_0) = 1$ , the restriction  $\pi|_{E_0} : E_0 \rightarrow \pi(E_0)$  has degree  $\deg(\pi)$ , then*

$$E \cdot \pi^*D = \pi_*E \cdot D;$$

*in particular, this is the case if  $\pi$  is a birational morphism, namely  $\deg \pi = 1$ ; and moreover, if  $\dim \pi(E) = 0$ , then  $E \cdot \pi^*D = 0$ .*

The proof of Theorem 6.2.4 will be divided into the following several steps.

**Step 1:**  $C$  and  $C' = \pi^{-1}(C)$  are both smooth complete curves and the restriction  $\pi|_{C'} : C' \rightarrow C$  has degree  $\deg(\pi : \tilde{X} \rightarrow X)$

In this case,  $\pi^*C = C'$  and thus

$$\pi^*C \cdot \pi^*D = \deg(\pi|_{C'}^* \mathcal{O}_X(D)|_{C'}).$$

We are done by recalling the following proposition.

**Proposition 6.2.6.** *Let  $S, T$  be smooth irreducible complete curves and  $\rho : S \rightarrow T$  a morphism. Let  $\mathcal{L}$  be a line bundle over  $T$ , then*

$$\deg(\rho^*\mathcal{L}) = \deg \rho \deg(\mathcal{L}).$$

*Proof.* If  $\rho$  is not surjective, namely,  $\rho(S)$  is one point, then  $\rho^*\mathcal{L}$  is trivial and  $\deg \rho = 0$ , the equality obviously holds. So WLOG assume  $\rho$  is surjective.

Let  $s$  be any meromorphic section of  $\mathcal{L}$ , then by definition,  $\deg(\mathcal{L}) = \deg(s)$  and  $\deg(\rho^*\mathcal{L}) = \deg(\rho^*s)$ . Here  $(s)$  denotes the Cartier divisor associated to  $s$ .

Let  $p \in T$  be such that  $s(p) = 0$ , and let  $\rho^{-1}(p) = \{q_1, \dots, q_r\}$ . Fix  $i$ , choose local coordinate  $z$  centered at  $q_i$  and local coordinate  $w$  centered at  $p$  such that  $\rho$  can be represented as  $w = z^{k_i}$  and moreover, in the coordinate patch under consideration,  $s$  can be represented

as  $s = w^{j_p}$ . Then  $j_p = \text{ord}_p(s)$  and  $\sum_{i=1}^r k_i = \text{deg } \rho$ . In addition, around  $q_i$ ,  $\rho^*s = (z^{k_i})^{j_p}$ , hence  $\text{ord}_{q_i}(\rho^*s) = k_i \cdot j_p$ , hence

$$\sum_{i=1}^r \text{ord}_{q_i}(\rho^*s) = \text{deg } \rho \cdot \text{ord}_p(s).$$

Similarly, when  $p$  is a pole for  $s$ , we have the same equation, namely,

$$\sum_{q \in \rho^{-1}(p)} \text{ord}_q(\rho^*s) = \text{deg } \rho \cdot \text{ord}_p(s).$$

Therefore, we have

$$\begin{aligned} \text{deg}(\rho^*\mathcal{L}) &= \sum_{p \in \text{Supp}((s))} \sum_{q \in \rho^{-1}(p)} \text{ord}_q(\rho^*s) \\ &= \text{deg}(\rho) \sum_{p \in \text{Supp}((s))} \text{ord}_p(s) \\ &= \text{deg}(\rho) \text{deg}(\mathcal{L}). \quad \square \end{aligned}$$

### Step 2: Proof of the first statement

Let  $\pi^*C = \sum_i m_i C'_i$ , where  $C'_i$  are irreducible reduced compact curves in  $\tilde{X}$ . Let also  $\nu_i : \tilde{C}'_i \rightarrow C'_i$  be the normalization of  $C'_i$ , and  $\pi'_i : C'_i \rightarrow C$  be the restriction of  $\pi$  to  $C'_i$ . Then by definition (6.2.1),

$$\begin{aligned} \pi^*C \cdot \pi^*D &= \sum_i m_i \text{deg}(\nu_i^* \mathcal{O}_{\tilde{X}}(\pi^*D)|_{C'_i}) \\ &= \sum_i m_i \text{deg}(\nu_i^* \pi_i'^* \mathcal{O}_X(D)|_C) \\ &= \sum_i m_i \text{deg}((\pi'_i \circ \nu_i)^* \mathcal{O}_X(D)|_C). \end{aligned}$$

Now  $\nu_i \circ \pi_i' : \tilde{C}'_i \rightarrow C$  factors through the normalization  $\nu : \tilde{C} \rightarrow C$  since  $\tilde{C}'_i$  is normal, namely we have the following diagram

$$\begin{array}{ccc} \tilde{C}'_i & \xrightarrow{\pi'_i} & \tilde{C} \\ \downarrow \nu_i & & \downarrow \nu \\ C'_i & \xrightarrow{\pi'_i} & C; \end{array}$$

hence, by definition (6.2.1) and Proposition 6.2.6, we have

$$\begin{aligned} \text{deg}((\pi_i \circ \nu_i)^* \mathcal{O}_X(D)|_C) &= \text{deg}(\tilde{\pi}_i'^* \nu^* \mathcal{O}_X(D)|_C) \\ &= \text{deg}(\tilde{\pi}_i') \text{deg}(\nu^* \mathcal{O}_X(D)|_C) \\ &= \text{deg}(\tilde{\pi}_i')(C \cdot D) \\ &= \text{deg}(\pi'_i)(C \cdot D), \end{aligned}$$

therefore, we obtain that

$$\begin{aligned} \pi^*C \cdot \pi^*D &= \sum_i m_i \text{deg}((\pi'_i \circ \nu_i)^* \mathcal{O}_X(D)|_C) \\ &= \sum_i m_i \text{deg}(\pi'_i)(C \cdot D) \\ &= \left( \sum_i m_i \text{deg}(\pi'_i) \right) C \cdot D. \end{aligned}$$

Moreover, we also have  $\sum m_i \deg(\pi'_i) = \deg \pi$ , so,

$$\pi^* C \cdot \pi^* D = (\deg \pi) C \cdot D.$$

### Step 3: Conclusion

Now we prove the equality  $E \cdot \pi^* D = \pi_* E \cdot D$ .

If  $\pi(E)$  is one point, say  $\pi(E) = \{p\}$ , then choose an open neighborhood  $U$  of  $p$  such that  $\mathcal{O}_X(D)|_U$  is trivial, then

$$\mathcal{O}_{\tilde{X}}(\pi^* D)|_{\pi^{-1}(U)} = \pi^* \mathcal{O}_D(D)|_U$$

is a trivial bundle over  $\pi^{-1}(U)$ . Note that  $E \subseteq \pi^{-1}(U)$ , so  $\pi^* \mathcal{O}_X(D)|_E$  is trivial, and hence

$$E \cdot \pi^* D = \deg(\nu^*(\pi^* \mathcal{O}_X(D)|_E)) = 0,$$

where  $\nu: \tilde{E} \rightarrow E$  is the normalization.

By linearity,  $E \cdot \pi^* D = 0$  if  $E$  is a combination of curves that are contracted by  $\pi$ .

If  $E$  is not contracted by  $\pi$ , then, by assumption,

$$\pi^* \pi_* E = \deg(\pi) E + E',$$

where  $E'$  is a linear combination of curves contracted by  $\pi$ . By the above discussion,  $E' \cdot \pi^* D = 0$ , and hence

$$(\deg \pi) E \cdot \pi^* D = \pi^* \pi_* E \cdot \pi^* D = (\deg \pi) \pi_* E \cdot D,$$

hence, by Step 2,

$$E \cdot \pi^* D = \pi_* E \cdot D.$$

For more comprehensive descriptions on intersection theory, we refer to [Fu] and [EH].

## 6.3 Projective surfaces and the Miyaoka–Yau inequality

### 6.3.1 Chern classes and Chern numbers

Let  $X$  be a smooth projective surface. Then as a complex manifold,  $X$  admits a naturally defined holomorphic bundle rank 2, namely, the holomorphic tangent bundle of  $X$ , denoted by  $TX$ .

For any complex vector bundle  $\mathcal{V}$  over  $X$ , we have the associated Chern classes, denoted by  $c_i(\mathcal{V}) \in H^{2i}(X, \mathbb{Z})$ , of the bundle  $\mathcal{V}$ ; for details, see [BT]. Now considering the underlying complex bundle of the holomorphic tangent bundle  $TX$ , we thus have the *first Chern class*  $c_1(TX) \in H^2(X, \mathbb{Z})$  of  $X$  and the *second Chern class*  $c_2(TX) \in H^4(X, \mathbb{Z})$  of  $X$ . Furthermore, we call the number

$$c_1^2(X) = [X] \cap (c_1(TX) \cup c_1(TX)) \in \mathbb{Z}$$

the *first Chern number* of  $X$ , and

$$c_2(X) = [X] \cap c_2(TX) \in \mathbb{Z}$$

the *second Chern number* of  $X$ .

The line bundle  $K_X = \bigwedge^4 T^* X$  which is the dual of the line bundle  $\bigwedge^4 TX$ , is called the *canonical bundle* of  $X$ ; in addition,  $K_X$ , identified as an invertible sheaf on  $X$ , corresponds to

a Cartier divisor on  $X$ , still denoted by  $K_X$  by abuse of notation. We call  $K_X$  the *canonical divisor* of  $X$ .

We have

$$c_1^2(X) = K_X^2, \quad c_2(X) = \chi(X),$$

where  $\chi(X)$  denotes the (topological) Euler number of  $X$ ; for a proof of the latter equality, see [BT].

### 6.3.2 Classification of smooth projective surfaces

It is possible to classify smooth projective surfaces up to birational isomorphisms, see [BHPV]. Such a classification depends heavily on the numerical properties of the canonical divisor.

Let  $X$  be a smooth projective surface. The canonical divisor  $K_X$  is called *big* if

$$\dim H^0(X, \mathcal{O}_X(mK_X)) \geq cm^2$$

when  $m$  is sufficiently large for some constant  $c > 0$  (independent of  $m$ ).  $K_X$  is called *nef* if  $K_X \cdot C \geq 0$  for any reduced irreducible curve  $C$  on  $X$ . Moreover, a curve  $E$  on  $X$  is called a  $(-1)$ -curve if  $E$  is a smooth rational curve and  $E^2 = -1$ .  $X$  is called *minimal* if it does not contain any  $(-1)$ -curve.

Every smooth projective surface  $X$  has a birational minimal model  $X'$ ; by definition, this means that  $X'$  is a smooth projective surface which is birationally isomorphic to  $X$  and  $X'$  does not contain any  $(-1)$ -curve. In fact,  $X'$  can be obtained from  $X$  by successively blowing down  $(-1)$ -curves, which is the inverse process of blowing up.

The smooth projective surface  $X$  is said to be of *general type* if  $K_X$  is big. We have that  $X$  is of general type if and only if its minimal model  $X'$  is; in addition,  $X'$  is unique and  $K_{X'}$  is big and nef.

Furthermore, for a surface to be of general type, there are many topological restrictions, one of which is the celebrated *Miyaoka–Yau inequality*, namely, if  $X$  is a smooth projective surface of general type, we have

$$c_1^2(X) \leq 3c_2(X)$$

where equality holds if and only if  $X$  is a ball quotient. By definition, this means that  $X$  is biholomorphic to  $B/\Gamma$ , where  $B = \{(z^1, z^2) \in \mathbb{C}^2 : 1 - |z^1|^2 - |z^2|^2 > 0\}$  equipped with the Kähler metric whose Kähler form is given by  $\omega_B = -\sqrt{-1}\partial\bar{\partial}\log(1 - |z^1|^2 - |z^2|^2)$ , and  $\Gamma$  is a discrete subgroup of isometries of  $B$ .

For more properties of projective surfaces, the classification of surfaces and history and proof of the Miyaoka–Yau inequality, see the excellent book [BHPV].

### 6.3.3 Miyaoka–Yau number for a compact normal surface

Let  $X$  be a compact surface on a smooth projective threefold  $Y$ . When  $X$  is smooth, then we have, by the *adjunction formula* for the *canonical divisor* of  $X$ ,

$$K_X = (K_Y + X)|_X.$$

Indeed, the line bundle  $\mathcal{O}_Y(X)|_X \cong N_{X/Y}$ , the normal bundle of  $X$  in  $Y$ , hence

$$K_X \cong K_Y|_X \otimes N_{X/Y} = K_Y \otimes \mathcal{O}_Y(X)|_X = (K_Y + X)|_X.$$

When  $X$  is only normal, or equivalently, in our case  $X$  has only isolated singularities, there is also a well-defined canonical class  $K_X$  for  $X$ . In fact, on the smooth locus  $X \setminus \text{Sing}(X)$ , we can define the canonical bundle, and hence its associated Cartier divisor; by taking closure in  $X$  of this divisor, we obtain  $K_X$ . Usually, for a general normal surface  $X$ ,  $K_X$ , although can be always defined, is not a Cartier divisor but only a Weil divisor. But here  $X$  is a surface lying on the ambient threefold  $Y$ , the adjunction formula still holds

$$K_X = (K_Y + X)|_X.$$

This can be seen as follows: the equality obviously holds when restricted to  $X \setminus \text{Sing}(X)$ , which is a smooth submanifold of  $Y$ ; moreover,  $\text{Sing}(X)$  has codimension 2 in  $X$  because  $X$  is normal, hence any divisor on  $X$  is uniquely determined by its restriction to  $X \setminus \text{Sing}(X)$ , we are done.

Since  $Y$  is smooth, both  $K_Y$  and  $X$  are Cartier divisors on  $Y$ , hence  $K_X$  is a Cartier divisor on  $X$ . Thus we have the intersection number  $K_X^2 = K_X \cdot K_X$ . By Proposition 3.5 in [Deb], we have

$$K_X^2 = (K_Y + X) \cdot (K_Y + X) \cdot X, \quad (6.3.1)$$

where the RHS is the intersection number of three Cartier divisors  $K_Y + X, K_Y + X, X$  on  $Y$ .

Moreover, when  $X$  is smooth, its second Chern number is the topological Euler number, i.e.,  $c_2(X) = \chi(X)$ ; when  $X$  is only normal, we can still have  $c_2(X)$  as a homology class of  $X$ , however, the topic about Chern classes on singular varieties is quite complicated and thus instead we shall use  $\chi(X)$  as a substitute of  $c_2(X)$ .

Finally, in view of the Miyaoka-Yau inequality, we give the following definition.

**Definition 6.3.4.** Let  $X \subseteq Y$  be a normal surface on a smooth projective threefold  $Y$ , the *Miyaoka-Yau number* of  $X$  is defined by

$$MY(X) = 3\chi(X) - K_X^2.$$

Note that if  $X$  is smooth and of general type,  $MY(X) \geq 0$ ; and if moreover  $\pi : \tilde{X} \rightarrow X$  be a blowup at one point of  $X$ , then  $MY(\tilde{X}) = MY(X) + 4$ .

**Example 6.3.5.** Let  $\mathcal{A} = \{L_1, \dots, L_d\}$  with  $L_k : \ell_k = 0, k = 1, \dots, d$ , be a line arrangement in  $\mathbb{P}^2$  with defining polynomial  $Q = \ell_1 \ell_2 \dots \ell_d$ , then the surface

$$\overline{F} : Q(x, y, z) + t^d = 0$$

is a normal surface in  $\mathbb{P}^3$ , and is the most natural compactification for the Milnor fiber  $F : Q = 1$  of  $\mathcal{A}$ .

Then by definition,

$$K_{\overline{F}} = (K_{\mathbb{P}^3} + \overline{F})|_{\overline{F}} \sim (d-4)H|_{\overline{F}},$$

where  $H$  is a hyperplane section of  $\mathbb{P}^3$ . Indeed, we have  $\overline{F} \sim dH$  and  $K_{\mathbb{P}^3} \sim -4H$ . Therefore, using Formula (6.3.1), we have

$$K_{\overline{F}}^2 = d(d-4)^2. \quad (6.3.2)$$

Moreover, there is a natural projection

$$p : \overline{F} \rightarrow \mathbb{P}^2, \quad (x, y, z, t) \mapsto (x, y, z),$$

which is a branched covering of degree  $d$  with ramification locus  $V(Q) \subseteq \mathbb{P}^2$ , hence

$$\begin{aligned}\chi(\bar{F}) &= d\chi(\mathbb{P}^2 - V(Q)) + \chi(V(Q)) \\ &= d\chi(\mathbb{P}^2) - (d-1)\chi(V(Q)) \\ &= 3d - (d-1)\chi(V(Q)).\end{aligned}$$

Let  $C_d$  be a smooth curve of degree  $d$  in  $\mathbb{P}^2$ , then

$$\chi(V(Q)) = \chi(C_d) + \sum_{p \in V(Q)} \mu_p(V(Q)) = d(3-d) + \sum_r t_r \mu_r(V(Q)),$$

where  $t_r$  be the number of singular points of multiplicity  $r$  in  $V(Q)$  and  $\mu_p(V(Q))$  (resp.  $\mu_r(V(Q))$ ) is the Milnor number for  $V(Q)$  at the point  $p$  (resp. a point of multiplicity  $r$ ), see [Wal]. Note that if  $p$  is a singularity of multiplicity  $r$ , then locally around  $p$ , we have

$$(V(Q), p) \quad : \quad G_r(x, y) = 0,$$

where  $G_r(x, y)$  is a product of  $r$  distinct linear forms. Therefore, by definition,

$$\mu_p(V(Q)) = \mu_r(V(Q)) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(\partial G_r / \partial x, \partial G_r / \partial y)} = (r-1)^2.$$

So we have

$$\chi(V(Q)) = d(3-d) + \sum_r t_r (r-1)^2,$$

implying that

$$\begin{aligned}\chi(\bar{F}) &= 3d - (d-1) \left( d(3-d) + \sum_r t_r (r-1)^2 \right) \\ &= d(d^2 - 4d + 6) - (d-1) \sum_r t_r (r-1)^2\end{aligned}\tag{6.3.3}$$

Consequently,

$$\begin{aligned}MY(\bar{F}) &= 3\chi(\bar{F}) - K_{\bar{F}}^2 \\ &= 3d(d^2 - 4d + 6) - 3(d-1) \sum_r t_r (r-1)^2 - d(d-4)^2 \\ &= 2d(d-1)^2 - 3(d-1) \sum_r t_r (r-1)^2 \\ &= 2(d-1) \sum_r t_r r(r-1) - 3(d-1) \sum_r t_r (r-1)^2 \\ &= (d-1) \sum_r t_r (r-1)(3-r). \quad \square\end{aligned}$$

**Remark 6.3.6.** In the above calculations, we have used the equality  $\frac{d(d-1)}{2} = \sum_r t_r \frac{r(r-1)}{2}$ . This is well-known and can be shown as follows. We count the number of intersection points of the  $d$  lines in  $\mathcal{A}$  under the rule that any two lines intersect at one point. Then in total, we have  $\frac{d(d-1)}{2}$  intersection points. Meanwhile, we can count the number of intersections points in another way: a point of multiplicity  $r$  on  $V(Q)$  is counted  $\frac{r(r-1)}{2}$  times, hence we eventually have  $\sum_r t_r \frac{r(r-1)}{2}$  intersection points. Therefore, the desired equality holds.

**Example 6.3.7.** 1. Let  $\mathcal{A} : (x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$ , then  $MY(\overline{F}) = 0$ , since  $t_r \neq 0$  only if  $r = 3$ .

2. Denote the Milnor number of the line arrangement  $\mathcal{A}$  by

$$\mu(\mathcal{A}) = \sum_{p \in V(Q)} \mu_p(V(Q)) = \sum_r t_r (r-1)^2,$$

then

$$MY(\overline{F}) = 2d(d-1)^2 - 3(d-1)\mu(\mathcal{A}) = 3(d-1) \left[ \frac{2d(d-1)}{3} - \mu(\mathcal{A}) \right].$$

As is shown in [DIM], Example 4.6,  $\mu(\mathcal{A}) \geq d(d-1)/2$ . Indeed,

$$\mu(\mathcal{A}) = \sum_r t_r (r-1)^2 \geq \sum_r t_r \binom{r}{2} = \frac{d(d-1)}{2}$$

where the last equality follows from Remark 6.3.6. On the other hand, as

$$b_0(V(Q)) - b_1(V(Q)) + b_2(V(Q)) = \chi(V(Q)) = d(3-d) + \mu(\mathcal{A})$$

and we obviously have  $b_0(V(Q)) = 1$  and  $b_2(V(Q)) = d$ , it follows that

$$b_1(V(Q)) = (d-1)^2 - \mu(\mathcal{A}) \geq 0$$

implying that  $\mu(\mathcal{A}) \leq (d-1)^2$ .

Therefore,  $MY(\overline{F})$  can be positive for some line arrangement  $\mathcal{A}$ , for instance,  $\mathcal{A}$  is a generic line arrangement where  $\mu(\mathcal{A}) = d(d-1)/2$ ; meanwhile,  $MY(\overline{F})$  can also be negative, e.g.,  $\mathcal{A}$  is a pencil of  $d$  lines where  $\mu(\mathcal{A}) = (d-1)^2$ .

Let  $X$  have an isolated singularity  $0 \in X$ , and  $\pi : \tilde{X} \rightarrow X$  be a minimal resolution (see below) of the singularity  $0$ , given by successive embedded blowups. Let  $\pi' : \tilde{Y} \rightarrow Y$  be the effect of the successive blowups on  $Y$ . Then  $\tilde{Y}$  is a smooth projective threefold, on which  $\tilde{X}$  is a normal surface, hence we have the canonical divisor  $K_{\tilde{X}} = (K_{\tilde{Y}} + \tilde{X})|_{\tilde{X}}$  and the Miyaoka-Yau number of  $\tilde{X}$ :  $MY(\tilde{X}) = 3\chi(\tilde{X}) - K_{\tilde{X}}^2$ .

**Definition 6.3.8.** Three numerical invariant differences for the minimal resolution  $\pi : \tilde{X} \rightarrow X$  are defined as follows:

(i) The difference for the first Chern number is

$$DCI = K_{\tilde{X}}^2 - K_X^2;$$

(ii) The difference for the second Chern number is

$$DCII = \chi(\tilde{X}) - \chi(X);$$

(iii) The difference for the Miyaoka-Yau number is

$$DMY = MY(\tilde{X}) - MY(X) = 3DCII - DCI.$$

If  $(X, 0) : G_r(x, y) + t^d = 0$  around the local coordinates  $(x, y, t)$  centered at 0 on  $Y$  where  $G_r$  is a product of  $r$  distinct linear forms, then it turns out that the differences defined above are determined by  $r$  and  $d$ , and will be denoted by  $DCI_{r,d}$ ,  $DCII_{r,d}$  and  $DMY_{r,d} = 3DCII_{r,d} - DCI_{r,d}$ .

**Remark 6.3.9.** For the resolution  $\pi : \tilde{X} \rightarrow X$ , we have

$$K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i, \quad (6.3.4)$$

where  $E_i$  are the exceptional curves (i.e. curves that are contracted by  $\pi$ ). So by the projection formula,

$$\begin{aligned} DCI &= K_{\tilde{X}}^2 - K_X^2 \\ &= \left( \pi^* K_X + \sum a_i E_i \right)^2 - K_X^2 \\ &= \pi^* K_X^2 + 2 \sum a_i E_i \cdot \pi^* K_X + \left( \sum a_i E_i \right)^2 - K_X^2 \\ &= K_X^2 + 0 + \left( \sum a_i E_i \right)^2 - K_X^2 \\ &= \left( \sum a_i E_i \right)^2. \end{aligned}$$

Therefore, to compute  $DCI$ , we still need the intersection matrix  $(E_i \cdot E_j)$  and the coefficients  $a_i$ 's in Formula (6.3.4).

## 6.4 Surfaces associated to line arrangements

Let  $\mathcal{A} = \{L_1, \dots, L_d\}$  with  $L_i : \ell_i = 0, i = 1, \dots, d$  be a line arrangement in  $\mathbb{P}^2$  with defining polynomial  $Q = Q(\mathcal{A}) = \ell_1 \ell_2 \cdots \ell_d$ .

**Definition 6.4.1.** Given  $r \geq 2$ . If a point  $x \in \mathbb{P}^2$  lies on exactly  $r$  lines in  $\mathcal{A}$ , or equivalently,  $x$  is a singular point of multiplicity  $r$  of the curve  $V(Q) \subseteq \mathbb{P}^2$ , we say  $x$  is of multiplicity  $r$ .

The number of points of multiplicity  $r$  will be denoted by  $t_r$ . Moreover, denote

$$f_0 = \sum t_r, \quad f_1 = \sum r t_r.$$

Now we begin to consider surfaces that can be constructed from the given line arrangement  $\mathcal{A}$ . The basic philosophy is that we first construct a possibly singular surface as a branched cover of  $\mathbb{P}^2$ , then we resolve the singularities.

### 6.4.2 Hirzebruch's construction

In the celebrated paper [Hi], F. Hirzebruch considers the Kummer cover of  $\mathbb{P}^2$  branched over the given line arrangement, and after suitable blowups, he obtains some smooth surfaces of general type and computes the Chern numbers  $c_1^2$  and  $c_2$ . With the help of Miyaoka–Yau inequality, he gives some numerical constraints on  $t_i$ 's for a given arrangement. In the sequel of this subsection, we make a review of his results about the constructions of surfaces.

Let  $\tau : \widehat{\mathbb{P}^2} \rightarrow \mathbb{P}^2$  be the blowup of  $\mathbb{P}^2$  along the intersection points of  $\mathcal{A}$ , and  $X$  the algebraic surface with function field

$$\mathbb{C}(z_1/z_0, z_2/z_0)((\ell_2/\ell_1)^{1/n}, \dots, (\ell_d/\ell_1)^{1/n})$$

which is an abelian extension of the function field  $\mathbb{C}(z_1/z_0, z_2/z_0)$  of  $\mathbb{P}^2$ . Then  $X$  is normal and we obtain a ramified cover

$$\pi : X \rightarrow \mathbb{P}^2$$

with the lines in  $\mathcal{A}$  as the ramification locus. Let  $Y$  be the fibred product

$$Y = X \times_{\mathbb{P}^2} \widehat{\mathbb{P}^2},$$

then as is shown in [Hi],  $Y$  is a smooth projective surface. Moreover, the following holds.

**Proposition 6.4.3** (See [Hi]). *Let  $Y$  be the algebraic surface associated to the line arrangement  $\mathcal{A} = \{L_1, \dots, L_d\}$  constructed as above, and  $t_r$  the number of points lying on exactly  $r$  lines in  $\mathcal{A}$  for  $r = 2, \dots, d$ , and  $f_0 = \sum t_r$ ,  $f_1 = \sum rt_r$ .*

*Then we have the following:*

(i) *The second Chern number or Euler number of  $Y$  satisfies*

$$\begin{aligned} c_2(Y)/n^{d-3} &= \chi(Y)/n^{d-3} \\ &= n^2(3 - 2d + f_1 - f_0) + 2n(d - f_1 + f_0) + f_1 - t_2. \end{aligned}$$

(ii) *The first Chern number  $c_1^2(Y)$  satisfies*

$$\begin{aligned} c_1^2(Y)/n^{d-3} &= n^2(-5d + 9 + 3f_1 - 4f_0) + 4n(d - f_1 + f_0) \\ &\quad + f_1 - f_0 + d + t_2. \end{aligned}$$

*Moreover, if in addition,  $d \geq 6, n \geq 2, t_d = t_{d-1} = t_{d-2} = 0$ , the following hold:*

(iii)  *$Y$  is of general type.*

(iv) *if  $\mathcal{A}$  cannot be divided into two pencils of lines,  $Y$  is minimal, i.e., it does not contain any  $(-1)$ -curves.*

In particular, we have

$$\begin{aligned} (3c_2(Y) - c_1^2(Y))/n^{d-3} &= n^2(f_0 - d) + n(-2f_1 + 2f_0 + 2d) \\ &\quad + 2f_1 + f_0 - d - 4t_2. \end{aligned} \tag{6.4.1}$$

By applying Proposition 6.4.3, when the above number is zero, we get several surfaces that are ball quotients, as are given in [Hi].

**Example 6.4.4.** We give the line arrangements provided in [Hi], so that the associated surfaces  $Y$  are ball quotients.

1. The (unique) arrangement  $A_1(6)$  with  $d = 6$ , and  $t_2 = 3, t_3 = 4$  and  $t_r = 0$  otherwise.  
In fact,  $A_1(6)$  is the arrangement  $\mathcal{A}(2, 2, 3)$  appearing in Section 2.3.10.
2. The Hesse arrangement,  $d = 12$  and  $t_2 = 12, t_4 = 9$  and  $t_r = 0$  otherwise.

3. The arrangement  $A_3^0(3)$  coming, by dualizing, from the nine inflection points on a cubic. We have  $d = 6$ ,  $t_3 = 12$  and  $t_r = 0$  otherwise.

Indeed,  $A_3^0(3)$  is exactly the Ceva arrangement or the arrangement  $\mathcal{A}(3, 3, 3)$  in Section 2.3.10.

More importantly, when the surface  $Y$  is of general type (which is indeed the case when the line arrangement is quite general by Proposition 6.4.3), from  $3c_2(Y) - c_1^2(Y) \geq 0$ , we can get remarkable inequalities about the numbers  $t_r$ 's. Here we do not give the details, however we will apply one inequality that was given by Hirzebruch in [Hi] in an essential way.

For interesting generalizations of Hirzebruch's method and more examples, see [Mat] and the references of it.

### 6.4.5 Compactifications of Milnor fibers of line arrangements

Let  $\mathcal{A} = \{L_1, \dots, L_d\}$ ,  $L_i : \ell_i = 0$  with  $d \geq 4$  be a line arrangement in  $\mathbb{P}^2$  with defining polynomial  $Q(x, y, z) = \ell_1 \ell_2 \cdots \ell_d$ .

Consider the Milnor fiber  $F : Q = 1$  in  $\mathbb{C}^3$ , for which we have a natural compactification

$$\overline{F} : Q(x, y, z) + t^d = 0$$

in  $\mathbb{P}^3$ .  $\overline{F}$  is a singular normal surface in  $\mathbb{P}^3$ ; a singular point of multiplicity  $r$  of  $V(Q)$  gives a singular point of multiplicity  $r$  of  $\overline{F}$ , and vice versa. Moreover, since  $Q$  is a product of linear forms, around a singular point of  $\overline{F}$  of multiplicity  $r$ , we have  $\overline{F} : G_r(x, y) + t^d = 0$  with  $G_r(x, y)$  a product of  $r$  distinct linear forms, whose resolution will be detailed investigated in next section.

Let  $\pi : \tilde{F} \rightarrow \overline{F}$  be a minimal resolution of  $\overline{F}$ , namely, the following three conditions hold:

- (i)  $\tilde{F}$  is a smooth surface and  $\pi$  is proper birational morphism;
- (ii)  $\pi : \tilde{F} \setminus \pi^{-1}(\text{Sing}(\overline{F})) \rightarrow \overline{F} \setminus \text{Sing}(\overline{F})$  is an isomorphism;
- (iii) there is no exceptional  $(-1)$ -curves on  $\tilde{F}$ , i.e., a rational curve  $E$  on  $\tilde{F}$  such that  $E^2 = -1$  and  $E$  is contracted to a point by  $\pi$ .

Such a resolution  $\pi$  can be obtained by successive embedded blowups, namely by blowing up along submanifolds of  $\mathbb{P}^3$  as well as the resulting manifolds in each step. Note that  $K_{\tilde{F}}$  is a Cartier divisor since  $\tilde{F}$  is smooth.

#### Determination of the canonical divisor

Let  $p_1, \dots, p_s$  be all the singular points of  $\overline{F}$  and  $r_i$  be the multiplicity of  $p_i$ . Let  $E_{i,1}, \dots, E_{i,v_i}$  be the irreducible components of  $\pi^{-1}(p_i)$  and  $M_{i,j,k} = E_{i,j} \cdot E_{i,k}$  be the intersection product of  $E_{i,j}$  and  $E_{i,k}$ . Let moreover,

$$\mathbf{M}_i = (M_{i,j,k})$$

be the intersection matrix of  $E_{i,j}$ 's for any fixed  $i$ . It is a  $v_i \times v_i$  matrix. In addition, let

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_1 & & & \\ & \mathbf{M}_2 & & \\ & & \ddots & \\ & & & \mathbf{M}_s \end{pmatrix}.$$

Set

$$\mathbf{E}_i = (E_{i,1}, E_{i,2}, \dots, E_{i,v_i})$$

as a  $1 \times v_i$  matrix and

$$\mathbf{E} = (E_1, \dots, E_s) = (E_{1,1}, \dots, E_{1,v_1}, E_{2,1}, \dots, E_{s,v_s})$$

as a  $1 \times (v_1 + \dots + v_s)$  matrix.

Then the canonical divisor  $K_{\tilde{F}}$  is of the following form

$$K_{\tilde{F}} = \pi^* K_{\overline{F}} + \sum_{i=1}^s \sum_{j=1}^{v_i} a_{i,j} E_{i,j}.$$

Let

$$\mathbf{A}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,v_i})^T$$

be a  $v_i \times 1$  matrix, where  $(\ )^T$  denotes the transpose of a matrix, and

$$\mathbf{A} = (a_{1,1}, \dots, a_{1,v_1}, a_{2,1}, \dots, a_{s,v_s})^T,$$

be a  $(v_1 + \dots + v_s) \times 1$  matrix, then  $K_{\tilde{F}}$  can be written as

$$\begin{aligned} K_{\tilde{F}} &= \pi^* K_{\overline{F}} + \mathbf{E} \mathbf{A} \\ &= \pi^* K_{\overline{F}} + \sum_{i=1}^s \mathbf{E}_i \mathbf{A}_i. \end{aligned} \quad (6.4.2)$$

Taking intersection product of  $K_{\tilde{F}}$  with the exceptional divisors  $E_{i,j}$ 's, we get, for all  $i = 1, \dots, s$  and  $k = 1, \dots, v_i$ ,

$$E_{i,k} \cdot K_{\tilde{F}} = E_{i,k} \cdot \pi^* K_{\overline{F}} + \sum_{j=1}^{v_i} a_{i,j} M_{i,k,j} + \sum_{i' \neq i} \sum_{j=1}^{v_{i'}} a_{i',j} E_{i,k} \cdot E_{i',j}.$$

Note also that all the surfaces under consideration are compact and all the divisors are Cartier. By the projection formula,  $E_{i,k} \cdot \pi^* K_{\overline{F}} = 0$  for  $k = 1, \dots, v_i$  and  $i = 1, \dots, s$ . Also,  $E_{i,k} \cdot E_{i',j} = 0$  when  $i \neq i'$  since  $E_{i,k} \cap E_{i',j} = \emptyset$  (they are contracted to different points by  $\pi$ ). Therefore, we have

$$E_{i,k} \cdot K_{\tilde{F}} = \sum_{j=1}^{v_i} M_{i,k,j} a_{i,j}.$$

Let

$$\mathbf{E}_i \cdot K_{\tilde{F}} = (E_{i,1} \cdot K_{\tilde{F}}, \dots, E_{i,v_i} \cdot K_{\tilde{F}}),$$

be a  $1 \times v_i$  matrix, then the above equality can be written as

$$\mathbf{M}_i \mathbf{A}_i = (\mathbf{E}_i \cdot K_{\tilde{F}})^T. \quad (6.4.3)$$

Now we list some basic results, in the form we shall use, concerning the resolution of singularities of a normal surface.

- i) For each  $i$ , the fiber  $\pi^{-1}(p_i)$  is connected. This follows from Zariski Main Theorem.
- ii) Each  $E_{i,j}$  is a smooth complete curve, see Theorem 6.5.2 below.

iii) For each  $i$ ,  $\mathbf{M}_i$  is a symmetric, negative definite  $v_i \times v_i$  matrix, see [Mum1]. So  $M_i$  is a fortiori invertible and we may define

$$\mathbf{N}_i = -\mathbf{M}_i^{-1}.$$

Then each  $\mathbf{N}_i$  is symmetric and positive definite.

Hence, from the equality (6.4.3), we have

$$\mathbf{A}_i = -\mathbf{N}_i(\mathbf{E}_i \cdot K_{\tilde{F}})^T,$$

and hence by equality (6.4.2), we have

$$K_{\tilde{F}} = \pi^* K_{\overline{F}} - \sum_{i=1}^s \mathbf{E}_i \mathbf{N}_i (\mathbf{E}_i \cdot K_{\tilde{F}})^T. \quad (6.4.4)$$

Furthermore, by the adjunction formula,

$$E_{i,j} \cdot K_{\tilde{F}} = 2g(E_{i,j}) - 2 - E_{i,j}^2. \quad (6.4.5)$$

We claim that, for all  $i, j$ , this number is always nonnegative. Indeed, if

$$2g(E_{i,j}) - 2 - E_{i,j}^2 < 0,$$

Since  $E_{i,j}^2$  is a diagonal entry for  $\mathbf{M}_i$ , we have  $E_{i,j}^2 \leq -1$ , so

$$2g(E_{i,j}) < 2 + E_{i,j}^2 \leq 1;$$

we deduce that  $g(E_{i,j}) = 0$ , i.e.,  $E_{i,j}$  is a rational curve. Moreover, from  $-2 - E_{i,j}^2 < 0$ ,  $E_{i,j}^2 \leq -1$ , we get  $E_{i,j}^2 = -1$ , thus  $E_{i,j}$  is a  $(-1)$ -curve, contradicting our assumption that  $\tilde{F}$  is a minimal resolution.

The adjunction formula (6.4.5) above together with (6.4.4) also shows that  $K_{\tilde{F}}$  is uniquely determined once we know the genera  $g(E_{i,j})$ 's and the intersection matrices  $\mathbf{M}_i$ 's.

Finally, by the projection formula, we have, by Formula (6.4.4),

$$\begin{aligned} K_{\tilde{F}}^2 - K_{\overline{F}}^2 &= \left( \sum_{i=1}^s \mathbf{E}_i \mathbf{N}_i (\mathbf{E}_i \cdot K_{\tilde{F}})^T \right)^2 \\ &= \sum_{i=1}^s \left( \mathbf{E}_i \mathbf{N}_i (\mathbf{E}_i \cdot K_{\tilde{F}})^T \right)^2. \end{aligned}$$

Note that the term  $\mathbf{E}_i \mathbf{N}_i (\mathbf{E}_i \cdot K_{\tilde{F}})^T$  involves only the resolution of the point  $p_i$ , hence the above formula motivates us to study carefully the resolution of only one singularity of a normal surface or more specifically, resolution of a normal surface germ.

### Miyaoka–Yau number

Recall that  $p_i$  is a singular point of  $\overline{F}$  of multiplicity  $r_i$ . By definition, we have

$$DCI_{r_i,d} = \left( \mathbf{E}_i \mathbf{N}_i (\mathbf{E}_i \cdot K_{\tilde{F}})^T \right)^2,$$

and hence

$$K_{\bar{F}}^2 - K_F^2 = \sum_{i=1}^s DCI_{r_i,d} = \sum_r t_r DCI_{r,d}.$$

Similarly, for the Euler number, we have

$$\chi(\tilde{F}) - \chi(\bar{F}) = \sum_{i=1}^s DCII_{r_i,d} = \sum_r t_r DCII_{r,d},$$

therefore, by Example 6.3.5,

$$\begin{aligned} MY(\tilde{F}) &= MY(\bar{F}) + \sum_r t_r (3DCII_{r,d} - DCI_{r,d}) \\ &= MY(\bar{F}) + \sum_r t_r DMY_{r,d} \\ &= \sum_r t_r (d-1)(r-1)(3-r) + \sum_r t_r DMY_{r,d} \\ &= \sum_r t_r ((d-1)(r-1)(3-r) + DMY_{r,d}). \end{aligned} \tag{6.4.6}$$

From this, we can see that the number  $(d-1)(r-1)(3-r) + DMY_{r,d}$  is of great importance in our computations of the Miyaoka-Yau numbers; for later convenience, we set

$$\mathcal{E}_{r,d} = (d-1)(r-1)(3-r) + DMY_{r,d}.$$

We use the notation  $\mathcal{E}_{r,d}$  since we will estimate it carefully in the sequel and of course this number depends only on  $r$  and  $d$ .

## 6.5 Resolution of singularities

Given a singular surface  $X$ , we can apply Hironaka's desingularization process to get a smooth surface  $\tilde{X}$  and a proper birational morphism  $\pi : \tilde{X} \rightarrow X$  which is an isomorphism away from the singular locus of  $X$ . Here we assume that  $X$  is normal. The normality of  $X$  implies that  $X$  has only isolated singularities. Typical examples of normal surfaces are hypersurfaces with isolated singularities in  $\mathbb{C}^3$  or  $\mathbb{P}^3$ .

Let  $(X, 0) \subseteq (\mathbb{C}^n, 0)$  be a germ of a singular surface with isolated singularity 0 and  $\pi : \tilde{X} \rightarrow X$  be a resolution of the singularity. This resolution  $\pi$  is called *very good* if

- (i)  $\tilde{X}$  is a smooth complex surface with boundary  $\partial\tilde{X} = \partial X$ ;
- (ii)  $\pi$  is a proper analytic morphism;
- (iii) the exceptional divisor  $E := \pi^{-1}(0) = \cup_{j=1}^s E_j$  has only normal crossings;
- (iv) moreover, each  $E_j$  is a smooth curve for  $j = 1, \dots, s$  and  $E_j \cdot E_k \leq 1$  for all  $j \neq k$ .

Recall also that a resolution  $\pi : \tilde{X} \rightarrow X$  is called *minimal* if the conditions above (i), (ii), (iv) holds and there is no exceptional  $(-1)$ -curve  $C$  on  $\tilde{X}$ , i.e.,  $C \subseteq \pi^{-1}(0)$ ,  $C \cong \mathbb{P}^1$  and  $C^2 = -1$ . Since  $X$  is normal,  $E = \pi^{-1}(0)$  is *connected* by Zariski Main Theorem; in particular, this is the case for a hypersurface with an isolated singularity in  $\mathbb{C}^3$ . Note that a minimal resolution is not necessarily very good, which is different from the definition of minimal resolution in [Dim92], p.50.

To a very good resolution  $\pi : \tilde{X} \rightarrow X$ , we can associate the *dual graph*: its vertices correspond to the exceptional curves  $E_j, j = 1, \dots, s$  and two vertices  $E_j$  and  $E_k$  are joined by an edge if and only if  $E_j \cdot E_k \neq 0$ . To include more information, one usually considers the *weighted dual graph*, where the vertex corresponding to  $E_j$  is weighted by  $-E_j^2$ .

Now let  $X$  be a hypersurface in a 3-dimensional projective manifold  $Y$ . Locally,  $X$  has the form  $X : f(u, v, t) = 0$  where  $(u, v, t)$  are the local coordinates of  $Y$  around the point under consideration. Assume from now on  $X$  has only isolated singularities and  $0 \in X$  is a singular point of  $X$ , and the above local coordinates  $(x, y, t)$  are centered at 0.

We shall consider singularities of the type  $(X, 0) : f(u, v, t) = 0$  with  $f(u, v, t) = G_r(u, v) + t^d$ , where  $G_r(u, v)$  is a product of  $r$  distinct linear forms in  $u, v$ . For instance, when we consider the compactification of the Milnor fiber of a line arrangement, each singular point is of this type. Such a type of singularity in fact belongs to a special class of singularities, namely weighted homogeneous singularities, whose resolutions are explicitly known.

### 6.5.1 Weighted homogenous singularities

Consider the  $\mathbb{C}^*$  action on  $\mathbb{C}^3$  given by

$$a \cdot (z_1, z_2, z_3) = (a^{w_1} z_1, a^{w_2} z_2, a^{w_3} z_3),$$

where the weights  $w_i = \text{weight}(z_i)$  are strictly positive integers satisfying

$$\gcd(w_1, w_2, w_3) = 1.$$

An isolated surface singularity  $(X', 0) : f'(z_1, z_2, z_3) = 0$  is called *weighted homogeneous* of degree  $N$  for the weights  $w_i$  if

$$a \cdot f'(z_1, z_2, z_3) = f'(a^{w_1} z_1, a^{w_2} z_2, a^{w_3} z_3) = a^N f'(z_1, z_2, z_3), \quad \forall a \in \mathbb{C}^*.$$

In our situation,  $(X, 0) : f(u, v, t) = G_r(u, v) + t^d = 0$  is weighted homogeneous with weights

$$\begin{cases} w_1 &= \text{weight}(u) = d / \gcd(r, d), \\ w_2 &= \text{weight}(v) = d / \gcd(r, d), \\ w_3 &= \text{weight}(t) = r / \gcd(r, d). \end{cases}$$

and with degree  $N = dr / \gcd(r, d) = \text{lcm}(r, d)$ . For the following theorem, see [Dim92], Section 4.10.

**Theorem 6.5.2.** *Let  $(X, 0) : f(u, v, t) = G_r(u, v) + t^d = 0, r \leq d$  be an isolated weighted homogeneous singularity of degree  $N = rd / \gcd(r, d)$ , where  $G_r(u, v)$  is a product of  $r$  distinct linear forms in  $u, v$ , for the weights*

$$\begin{cases} w_1 &= \text{weight}(u) = d / \gcd(r, d), \\ w_2 &= \text{weight}(v) = d / \gcd(r, d), \\ w_3 &= \text{weight}(t) = r / \gcd(r, d). \end{cases}$$

*Then there is a resolution  $\pi : \tilde{X} \rightarrow X$  such that:*

- (i) *there is a  $\mathbb{C}^*$  action on  $\tilde{X}$  under which the morphism  $\pi$  is equivariant.*

(ii) the exceptional divisor  $\pi^{-1}(0)$  has exactly one component, denoted by  $E_0$ , which is fixed pointwise by the  $\mathbb{C}^*$  action on  $\tilde{X}$ .

(iii)  $\pi^{-1}(0)$  has the following form

$$\pi^{-1}(0) = E_0 \cup E_1 \cup \dots \cup E_q,$$

where for each  $k = 1, \dots, q$ ,

$$E_k = E_k^1 \cup \dots \cup E_k^r$$

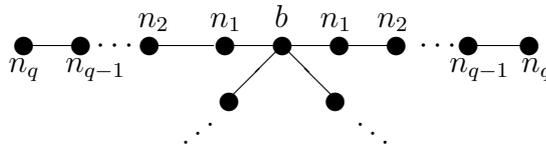
is a disjoint union of  $r$  curves, corresponding to vertices of distance  $k$  from the center in the dual graph below.

(iv) For each  $k = 1, \dots, q$  and  $j = 1, \dots, r$ , the curve  $E_k^j$  is a smooth rational irreducible curve and has self-intersection  $(E_k^j)^2 = -n_k \leq -2$  (independent of  $j$ ).

(v)  $E_0$  is isomorphic to  $C = (X - \{0\})/\mathbb{C}^*$ , a smooth curve of genus

$$\begin{aligned} g(C) &= \frac{1}{2} \left[ \frac{N^2}{w_1 w_2 w_3} - \sum_{i < j} \frac{N \gcd(w_i, w_j)}{w_i w_j} + \sum_i \frac{\gcd(N, w_i)}{w_i} - 1 \right] \\ &= \frac{1}{2} (r - 2)(\gcd(r, d) - 1). \end{aligned} \tag{6.5.1}$$

(vi) The components  $E_0, E_k^j$ 's meet transversally according to the following star-shaped graph



where the central vertex corresponds to  $E_0$  and there are exactly  $r$  arms, which have the same length  $q$  and the same weight sequences  $n_1, \dots, n_q$ .

(vii) Moreover, the above dual graph satisfies the following: if we index the arms  $1, 2, \dots, r$  from leftmost to right by the anticlockwise order and go along the arm indexed by  $j$  from the end closest to  $E_0$  to the one farthest to  $E_0$ , we get, in order, the vertices corresponding to the curves  $E_1^j, E_2^j, \dots, E_q^j$ .

(viii) Let  $\alpha = w_1 = d/\gcd(r, d)$  and  $b' = w_3 = r/\gcd(r, d)$ . When  $\alpha = 1$ , then there are in fact no arms, i.e.,  $q = 0$  and in this case, let  $\beta = 0$ . When  $\alpha > 1$ , choose  $0 < \beta < \alpha$  such that  $\beta b' \equiv -1 \pmod{\alpha}$ . Then the weights of the vertices of the dual graph are determined as follows:

- The weight of the central vertex is

$$b = \frac{N}{w_1 w_2 w_3} + r\beta/\alpha = \frac{\gcd(r, d)(1 + b'\beta)}{\alpha}.$$

- The weight sequence  $(n_1, \dots, n_q)$  along each arm is given by the following continued fraction decomposition

$$\frac{\alpha}{\beta} = n_1 - \frac{1}{n_2 - \frac{1}{\dots - \frac{1}{n_q}}}.$$

Let  $(X, 0) : f(u, v, t) = G_r(u, v) + t^d = 0$  be a surface germ in  $(\mathbb{C}^3, 0)$ , where  $G_r$  is a product of  $r$  distinct linear binary forms. Let  $\pi : \tilde{X} \rightarrow X$  be the resolution given in Theorem 6.5.2. With the notations in the theorem, we shall write the divisors on  $\tilde{X}$ ,

$$E_k = E_k^1 + E_k^2 \cdots + E_k^r, \quad k = 1, \dots, q.$$

Clearly, each  $E_k$  is a Cartier divisor with compact support on  $\tilde{X}$ , and

$$E_k^2 = -rn_k, \quad k = 1, \dots, q,$$

and

$$E_k \cdot E_{k'} = \begin{cases} r, & \text{if } k' = k \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also,, we can see that

$$\chi(\tilde{X}) - \chi(X) = -1 + \chi(E_0) + rq.$$

Indeed, essentially  $\tilde{X}$  is obtained from  $X$  by replacing 0 by  $(1 + rq)$  curves intersecting according to the dual graph;  $E_0$  contributes to  $\chi(E_0)$  for  $\chi(\tilde{X})$ ; each arm in the dual graph gives rise to a disjoint union of  $q$  copies of  $\mathbb{P}^1 \setminus \{\text{one point}\} \cong \mathbb{C}$ , and hence contributes  $q$  for  $\chi(\tilde{X})$ .

Moreover, note that  $K_{\tilde{X}}$  has the following form

$$K_{\tilde{X}} = \pi^* K_X + a_0 E_0 + \sum_{k,j} a_k^j E_k^j.$$

As before, set

$$\mathbf{E}^j = (E_1^j, \dots, E_q^j),$$

and

$$\mathbf{a}^j = (a_1^j, \dots, a_q^j),$$

then

$$K_{\tilde{X}} = \pi^* K_X + a_0 E_0 + \sum_j \mathbf{E}^j (\mathbf{a}^j)^T.$$

By considering the adjunction formula, we have

$$E_0 \cdot K_{\tilde{X}} = g(E_0) - 2 - E_0^2 = 2g(E_0) - 2 + b$$

and for all  $k, j$ ,

$$E_k^j \cdot K_{\tilde{X}} = g(E_k^j) - 2 - (E_k^j)^2 = -2 + n_k,$$

hence, by projection formula and Theorem 6.5.2, we get a systems of equations

$$\begin{cases} -ba_0 + (a_1^1 + \cdots + a_1^r) & = (r-2)(\gcd(r, d) - 1) - 2 + b \\ -n_k a_k^j + (a_{k-1}^j + a_{k+1}^j) & = -2 + n_k, \quad \forall k, j \end{cases} \quad (6.5.2)$$

where we have denoted  $a_0^j = a_0$  and  $a_{q+1}^j = 0$  for all  $j$ .

Recall that the intersection matrix of  $E_0, E_k^l$ 's is negative definite, so from (6.5.2) we can uniquely solve  $a_0, \mathbf{a}^j$ 's. Moreover, we can see that if  $(a_0, \mathbf{a}^1, \dots, \mathbf{a}^r)$  is a solution of the system (6.5.2),  $(a_0, \mathbf{a}^j, \mathbf{a}^2, \dots, \mathbf{a}^{j-1}, \mathbf{a}^j, \mathbf{a}^{j+1}, \dots, \mathbf{a}^r)$  is also a solution for any  $j > 1$ , hence by uniqueness of the solution, we have

$$\mathbf{a}^1 = \mathbf{a}^2 = \cdots = \mathbf{a}^r,$$

namely,

$$a_k^1 = a_k^2 = \cdots = a_k^r$$

for all  $k$ , hence  $K_{\tilde{X}}$  has the following form

$$K_{\tilde{X}} = \pi^*K_X + a_0E_0 + \sum_{k=1}^q a_kE_k = \pi^*K_X + \sum_{k=0}^q a_kE_k,$$

satisfying (following from (6.5.2))

$$\begin{cases} -ba_0 + ra_1 & = (r-2)(\gcd(r,d) - 1) - 2 + b \\ -n_k a_k + (a_{k-1} + a_{k+1}) & = -2 + n_k, \quad k = 1, \dots, q, \end{cases} \quad (6.5.3)$$

where  $a_{q+1} = 0$ .

## 6.6 Examples of resolutions of surfaces

Now we continue applying the notations in the above section, and considering the resolution given in Theorem 6.5.2 of the surface germ  $(X, 0) : G_r(x, y) + t^d = 0$ .

**Example 6.6.1.** When  $r = 2$ , then  $(X, 0)$  is a singularity of type  $A_{d-1}$ , and its minimal resolution  $\pi : \tilde{X} \rightarrow X$  is well-known: the dual graph is



where there are  $(d-1)$  vertices and each vertex has weight 2. Moreover,  $K_{\tilde{X}} = \pi^*K_X$  (see [Re]), so we have

$$DCI_{r,d} = 0, \quad DCII_{r,d} = d-1,$$

hence

$$DMY_{r,d} = 3(d-1),$$

and

$$\mathcal{E}_{r,d} = (d-1)(r-1)(3-r) + DMY_{r,d} = 4(d-1). \quad \square$$

Note that when  $r = 2$  and  $d = rp + 1$  for  $p \geq 1$ , the resolution given in Theorem 6.5.2 is not minimal. Indeed, the central curve  $E_0$  is a  $(-1)$ -curve, as can be seen as follows: we have

$$\gcd(r, d) = 1, \quad \alpha = d, \quad b' = 2,$$

and hence, by Formula (6.5.1)

$$g(E_0) = \frac{1}{2}(r-2)(\gcd(r,d) - 1) = 0,$$

namely,  $E_0$  is a rational curve; moreover, since  $0 < \beta < \alpha$  is chosen so that  $b'\beta \equiv -1 \pmod{\alpha}$  and by assumption  $\alpha = b'p + 1$ , we have

$$\beta = p.$$

Thus,

$$b = \frac{\gcd(r,d)(b'\beta + 1)}{\alpha} = 1,$$

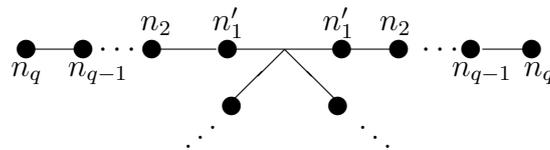
i.e.,  $E_0^2 = -1$ . That is,  $E_0$  is a  $(-1)$ -curve.

On the other hand, in this case,

$$\alpha/\beta = d/p = (2p + 1)/p,$$

considering the continued fraction decomposition in Theorem 6.5.2, we have  $n_1 = 3$ ; hence by blowing down the  $(-1)$ -curve  $E_0$ , we still have a smooth surface by Castelnuovo contraction theorem, and the resulting surface still gives a minimal resolution: in fact, the dual graph is obtained from that given in Theorem 6.5.2 by deleting the central vertex and replacing  $n_1$  by  $n_1 - 1$ .

In general, if  $r \geq 3$  and  $E_0$  is a  $(-1)$ -curve and  $n_1 \geq 3$ , after blowing down  $E_0$ , we still get a resolution surface of  $X$ . But now the dual graph is



where  $n'_1 = n_1 - 1$  and there is no central vertex, meaning that for the  $r$  exceptional curves  $E_1^1, \dots, E_1^r$  corresponding to the vertices of weight  $n'_1$ , we have  $E_1^j \cdot E_1^{j'} = 1$  for  $j \neq j'$ . In particular, the new exceptional divisor does not have normal crossings.

Now assume  $r \geq 3$ . We illustrate when the resolution in Theorem 6.5.2 is not minimal. This is the case only if  $E_0$  is a  $(-1)$ -curve, since other exceptional irreducible curves all have self-intersection  $\leq -2$ . Therefore, we have  $g(E_0) = 0$  and  $b = 1$ , namely,

$$\begin{cases} 0 &= g(E_0) = \frac{1}{2}(r - 2)(\gcd(r, d) - 1) \\ 1 &= b = \gcd(r, d)(b'\beta + 1)/\alpha \end{cases}$$

hence we have  $\gcd(r, d) = 1$  and  $b'\beta + 1 = \alpha$ . Now from  $\gcd(r, d) = 1$ , we have by definition  $\alpha = d/\gcd(r, d) = d$  and  $b' = r/\gcd(r, d) = r$ , so  $d = r\beta + 1$  with  $0 < \beta < d$ .

Consequently, if  $d$  cannot be written as  $d = rp + 1$  for some  $p \geq 1$ , the resolution given in Theorem 6.5.2 is already minimal. If, on the other hand,  $d = rp + 1$  for some  $p \geq 1$ , the resolution given in Theorem 6.5.2 is not minimal and  $E_0$  is a  $(-1)$ -curve. By blowing down  $E_0$ , we get another resolution  $\tilde{X}'$  of  $X$ , and moreover, since in this case  $\alpha = b'\beta + 1 = r\beta + 1$ , by performing the continued fraction decomposition of  $\alpha/\beta = (r\beta + 1)/\beta$ , we have  $n_1 = r + 1 \geq 4$ , hence  $\tilde{X}'$  is a minimal resolution of  $X$ . In addition, note that the way to obtain the dual graph for the resolution  $\tilde{X}' \rightarrow X$  has been discussed above.

In the sequel, by abuse of notation, we will not distinguish  $\tilde{X}$  and  $\tilde{X}'$  and always denote  $\tilde{X}$  the minimal resolution of  $X$  obtained, by blowing down the central curve  $E_0$  if necessary, from the resolution given in Theorem 6.5.2.

**Example 6.6.2.** Let  $r \geq 3$  and  $d = rp, p \geq 1$ . Then the resolution  $\pi : \tilde{X} \rightarrow X$  in Theorem 6.5.2 is minimal. So we immediately have

$$DCII_{r,d} = -1 + \chi(E_0) + rq, \tag{6.6.1}$$

and if

$$K_{\tilde{X}} = \pi^* K_X + \sum_{k=0}^q a_k E_k,$$

we have

$$DCI_{r,d} = \left( \sum_{k=0}^q a_k E_k \right)^2. \quad (6.6.2)$$

According to Theorem 6.5.2, we have

$$g(E_0) = 1/2(r-2)(\gcd(r,d)-1) = 1/2(r-2)(r-1);$$

moreover,

$$\alpha = d/\gcd(r,d) = p, \quad b' = r/\gcd(r,d) = 1;$$

since  $\beta$  is chosen so that  $0 < \beta < \alpha$  satisfying  $b'\beta \equiv -1 \pmod{\alpha}$ , we have  $\beta = \alpha - 1$ , therefore,

$$b = \gcd(r,d)(b'\beta + 1)/\alpha = r;$$

meanwhile, if  $\alpha > 1$ ,

$$\alpha/\beta = p/(p-1) = 2 - (p-2)/(p-1)$$

performing the continued fraction decomposition as in Theorem 6.5.2, we have

$$q = p - 1$$

and

$$n_1 = n_2 = \cdots = n_q = 2.$$

Hence, by Formula (6.6.1),

$$DCII_{r,d} = rq - 1 + \chi(E_0) = r(p-1) - 1 + r(3-r) = -1 + r(p+2-r).$$

By applying the equation (6.5.3), we obtain

$$\begin{cases} -ra_0 + ra_1 & = (r-2)(r-1) - 2 + r \\ -2a_1 + (a_0 + a_2) & = 0 \\ -2a_2 + (a_1 + a_3) & = 0 \\ \vdots & \\ -2a_q + a_{q-1} & = 0 \end{cases}$$

therefore, by considering from bottom equation to the second top one, we have

$$a_{q-1} = 2a_q, \quad a_{q-2} = 3a_q, \cdots, a_2 = (q-1)a_q, \quad a_1 = a_q, \quad a_0 = (q+1)a_q,$$

and hence from the first equation, we get  $a_q = -(r-2)$ . So if  $\alpha > 1$ , i.e.,  $q+1 = p > 1$ , we have

$$K_{\tilde{X}} = \pi^* K_X - (r-2)(E_{p-1} + 2E_{p-2} + \cdots + (p-1)E_1 + pE_0).$$

Similarly, if  $p = 1$ , we get  $q = 0$ , and

$$K_{\tilde{X}} = \pi^* K_X - (r-2)E_0.$$

Thus, whether  $p > 1$  or  $p = 1$ , we always have

$$K_{\tilde{X}} = \pi^* K_X - (r-2)(E_{p-1} + 2E_{p-2} + \cdots + (p-1)E_1 + pE_0).$$

Recall that  $E_0^2 = -b = -r$  and  $E_k^2 = -rn_k = -2r$ ; in addition,  $E_k \cdot E_{k'} = r$  for  $k' = k \pm 1$ ,  $=0$  otherwise. Therefore, by Formula 6.6.2,

$$\begin{aligned}
DCI_{r,d} &= (r-2)^2 \left( E_{p-1} + 2E_{p-2} + \cdots + (p-1)E_1 + pE_0 \right)^2 \\
&= (r-2)^2 \left( \sum_{i=1}^p i^2 E_{p-i}^2 + \sum_{i=1}^{p-1} i(i+1) E_{p-i} \cdot E_{p-i-1} \right) \\
&= (r-2)^2 \left( -2r \sum_{i=1}^{p-1} i^2 - rp^2 + 2r \sum_{i=1}^{p-1} i(i+1) \right) \\
&= (r-2)^2 (-rp) \\
&= -d(r-2)^2
\end{aligned}$$

So

$$\begin{aligned}
DMY_{r,d} &= 3DCII_{r,d} - DCI_{r,d} \\
&= 3(-1 + r(p+2-r)) + d(r-2)^2 \\
&= -3 + 3r(p+2-r) + d(r-2)^2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathcal{E}_{r,d} &= DMY_{r,d} + (d-1)(r-1)(3-r) \\
&= -3 + 3r(p+2-r) + d(r-2)^2 + (d-1)(r-1)(3-r) \\
&= -3 - 3r(r-2) + 3d + d(r-2)^2 - (d-1) \left( (r-2)^2 - 1 \right) \\
&= -3 - 3r(r-2) + 3d + (d-1) + (r-2)^2 \\
&= 4(d-1) + (r-2)(r-2-3r) \\
&= 4(d-1) - 2(r-2)(r+1) \\
&= 4(d-1) - 2(r^2 - r - 2) \\
&= 4d - 2r(r-1). \quad \square
\end{aligned}$$

**Example 6.6.3.** Let  $r \geq 3$  and  $d = rp + 1, p \geq 1$ . Then according to Theorem 6.5.2, we have

i)  $\gcd(r, d) = 1$ , and hence by Formula (6.5.1),

$$g(E_0) = 1/2(r-2)(\gcd(r, d) - 1) = 0,$$

so  $E_0$  is a rational curve.

ii)  $\alpha = d/\gcd(r, d) = d$  and  $b' = r/\gcd(r, d) = r$ , so

$$\alpha = b'p + 1;$$

since  $0 < \beta < \alpha$  is chosen so that  $b'\beta \equiv -1 \pmod{\alpha}$ , we have  $\beta = p$ , and thus

$$b = \gcd(r, d)(b'\beta + 1)/\alpha = 1,$$

therefore,  $E_0$  is a  $(-1)$ -curve. Blow it down, we get the minimal resolution  $\pi : \tilde{X} \rightarrow X$ .

The canonical divisor  $K_{\tilde{X}}$  has the following form

$$K_{\tilde{X}} = \pi^*K_X + \sum_{k=1}^q a_k E_k,$$

where, different from Example 6.6.2,  $E_1 = E_1^1 + \cdots + E_1^r$  such that  $(E_1^l)^2 = -n'_1 = -(n_1 - 1)$  and  $E_1^l \cdot E_1^{l'} = 1$  for  $l < l'$ .

iii) We have

$$\alpha/\beta = (rp + 1)/p,$$

so considering the continued fraction decomposition, we have

$$q = p,$$

and

$$n_1 = r + 1, \quad n_2 = n_3 = \cdots = n_q = 2.$$

Now taking the intersection product of  $K_{\tilde{X}}$  with  $E_k^l$ 's and applying the adjunction formula and projection formula, we have

$$\begin{cases} -n'_1 a_1 + a_2 + (r-1)a_1 & = -2 + n'_1 \\ -n_2 a_2 + (a_1 + a_3) & = -2 + n_2 \\ -n_3 a_2 + (a_2 + a_4) & = -2 + n_3 \\ \vdots & \\ -n_{q-1} a_{q-1} + (a_{q-2} + a_q) & = -2 + n_{q-1} \\ -n_q a_q + a_{q-1} & = -2 + n_q \end{cases}$$

that is,

$$\begin{cases} -r a_1 + a_2 + (r-1)a_1 & = -2 + r \\ -2a_2 + (a_1 + a_3) & = 0 \\ -2a_2 + (a_2 + a_4) & = 0 \\ \vdots & \\ -2a_{q-1} + (a_{q-2} + a_q) & = 0 \\ -2a_q + a_{q-1} & = 0. \end{cases}$$

Just as in Example 6.6.2, we have

$$a_q = -(r-2); \quad a_k = (q+1-k)a_q, \quad k = 1, \dots, q-1.$$

Therefore, we have

$$K_{\tilde{X}} = \pi^*K_X - (r-2)(E_p + 2E_{p-1} + \cdots + pE_1),$$

and thus,

$$DCI_{r,d} = (r-2)^2(E_p + 2E_{p-1} + \cdots + pE_1)^2.$$

Note that  $E_k^2 = -rn_k = -2r$  for  $k > 1$  and

$$\begin{aligned} E_1^2 &= (E_1^1 + \cdots + E_1^r)^2 \\ &= \sum_{l=1}^r (E_1^l)^2 + 2 \sum_{l < l'} E_1^l \cdot E_1^{l'} \\ &= r(-r) + 2 \cdot r(r-1)/2 \\ &= -r; \end{aligned}$$

moreover,  $E_k \cdot E_{k'} = r$  for  $k' = k \pm 1$ ,  $=0$  otherwise, hence a similar computation as in Example 6.6.2 gives

$$\begin{aligned} DCI_{r,d} &= (r-2)^2(E_p + 2E_{p-1} + \cdots + pE_1)^2 \\ &= -(r-2)^2rp \\ &= -(d-1)(r-2)^2. \end{aligned}$$

Clearly, we have

$$DCII_{r,d} = rq = rp = d-1,$$

and thus

$$\begin{aligned} DMY_{r,d} &= 3DCII_{r,d} - DCI_{r,d} \\ &= 3(d-1) + (d-1)(r-2)^2. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathcal{E}_{r,d} &= DMY_{r,d} + (d-1)(r-1)(3-r) \\ &= 3(d-1) + (d-1)(r-2)^2 + (d-1)(r-1)(3-r) \\ &= 3(d-1) + (d-1) \\ &= 4(d-1). \quad \square \end{aligned}$$

## 6.7 Numerical invariants for minimal resolutions: general case

Now we consider the general case for Theorem 6.5.2. Although our method applies for more general situations, we assume  $r \geq 3$  and  $d \not\equiv 0, 1 \pmod{r}$ , since otherwise we are done by Example 6.6.1, Example 6.6.2 and Example 6.6.3. In particular, the resolution  $\pi : \tilde{X} \rightarrow X$  given in Theorem 6.5.2 is a minimal resolution.

### 6.7.1 Continued fraction decomposition

In order to apply Theorem 6.5.2, we first deal with the continued fraction decomposition

$$\frac{\alpha}{\beta} = n_1 - \frac{1}{n_2 - \frac{1}{\dots - \frac{1}{n_q}}}.$$

Recall that  $\beta$  is chosen such that  $b'\beta \equiv -1 \pmod{\alpha}$ , hence  $\gcd(\alpha, \beta) = 1$ . Let

$$\alpha_0, \alpha_1, \dots, \alpha_{q-1}, \alpha_q = 1, \alpha_{q+1} = 0$$

be a sequence of natural numbers such that  $\gcd(\alpha_i, \alpha_{i+1}) = 1$  for  $i = 0, 1, \dots, q$  and

$$\frac{\alpha_i}{\alpha_{i+1}} = n_{i+1} - \frac{1}{n_{i+2} - \frac{1}{\dots - \frac{1}{n_q}}}, \quad i = 0, 1, \dots, q-1. \quad (6.7.1)$$

Clearly, the numbers  $\alpha_i$ 's are uniquely determined by the continued fraction decomposition above, and  $\alpha_i > 0$  for  $i < q+1$ .

Moreover, we have by definition (6.7.1)

$$\frac{\alpha_{i-1}}{\alpha_i} = n_i - \frac{1}{\alpha_i/\alpha_{i+1}} = \frac{n_i\alpha_i - \alpha_{i+1}}{\alpha_i},$$

hence

$$\alpha_{i-1} = n_i\alpha_i - \alpha_{i+1},$$

or in another more convenient formulation

$$\begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} = \begin{pmatrix} n_i & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_i \\ \alpha_{i+1} \end{pmatrix}. \quad (6.7.2)$$

Set for  $i = 1, \dots, q$ ,

$$\mathbf{G}_i = \begin{pmatrix} n_i & -1 \\ 1 & 0 \end{pmatrix} \quad (6.7.3)$$

be a  $2 \times 2$  matrix. Then the relation (6.7.2) can be formulated as

$$\begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} = \mathbf{G}_i \begin{pmatrix} \alpha_i \\ \alpha_{i+1} \end{pmatrix}.$$

Thus, we have

$$\begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} = \mathbf{G}_i \mathbf{G}_{i+1} \cdots \mathbf{G}_q \begin{pmatrix} \alpha_q \\ \alpha_{q+1} \end{pmatrix} = \mathbf{G}_i \mathbf{G}_{i+1} \cdots \mathbf{G}_q \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6.7.4)$$

for all  $i \geq 1$ .

Note also that by definition (6.7.1) and our conventions,  $\alpha_0 = \alpha$  and  $\alpha_1 = \beta$ .

Let

$$\mathbf{G} = \mathbf{G}_1 \mathbf{G}_2 \cdots \mathbf{G}_q,$$

then by (6.7.4), we have

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \mathbf{G} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So  $\mathbf{G}$  is of the form

$$\mathbf{G} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

for some integers  $\gamma, \delta$ . In fact, we have the following more precise result.

**Proposition 6.7.2.** *With the notations as above and in Theorem 6.5.2, we have*

$$\mathbf{G} = \begin{pmatrix} \alpha & b' - \alpha \\ \beta & \frac{1+b'\beta}{\alpha} - \beta \end{pmatrix},$$

namely,  $\gamma = b' - \alpha$  and  $\delta = -\beta + (1 + b'\beta)/\alpha$ .

*Proof.* First, we show

**Claim 6.7.3.**  $-\alpha < \gamma \leq 0$  and  $-\beta < \delta \leq 0$ .

Assuming the claim, note that by definition,

$$\det \mathbf{G} = \alpha\delta - \beta\gamma = 1,$$

hence  $\beta\gamma \equiv -1 \pmod{\alpha}$ . Recall also the  $b'\beta \equiv -1 \pmod{\alpha}$ , so we have  $\gamma = b' - \alpha$  since  $\gamma, b' - \alpha \in (-\alpha, 0]$  and the equation  $\beta x \equiv -1 \pmod{\alpha}$  admits a unique solution satisfying  $x \in (-\alpha, 0]$ . In addition,

$$\delta = \frac{1 + \beta\gamma}{\alpha} = \frac{1 + \beta(b' - \alpha)}{\alpha} = \frac{1 + b'\beta}{\alpha} - \beta.$$

*Proof of Claim 6.7.3:* For  $i \geq 1$ , let

$$\begin{pmatrix} \xi_i & \gamma_i \\ \eta_i & \delta_i \end{pmatrix} = \mathbf{G}_1 \mathbf{G}_2 \cdots \mathbf{G}_i,$$

then  $\xi_i, \eta_i, \gamma_i, \delta_i$  are all integers. It suffices to show the following:

- (i)  $\xi_i, \eta_i > 0$  for all  $i$ .
- (ii)  $\gamma_i \in (-\xi_i, 0]$  and  $\delta_i \in (-\eta_i, 0]$  for all  $i$ .

We prove this by induction on  $i$ . When  $i = 1$ , then we have

$$\begin{pmatrix} \xi_1 & \gamma_1 \\ \eta_1 & \delta_1 \end{pmatrix} = \mathbf{G}_1 = \begin{pmatrix} n_1 & -1 \\ 1 & 0 \end{pmatrix},$$

and the conclusion obviously holds. Now assuming the validity of the result for  $i$ , we have

$$\begin{pmatrix} \xi_{i+1} & \gamma_{i+1} \\ \eta_{i+1} & \delta_{i+1} \end{pmatrix} = \begin{pmatrix} \xi_i & \gamma_i \\ \eta_i & \delta_i \end{pmatrix} \mathbf{G}_{i+1} = \begin{pmatrix} \xi_i & \gamma_i \\ \eta_i & \delta_i \end{pmatrix} \begin{pmatrix} n_{i+1} & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore,

- i)  $\xi_{i+1} = n_{i+1}\xi_i + \gamma_i > 2\xi_i - \xi_i > 0$  since  $n_{i+1} \geq 2$  and by inductive hypothesis,  $\xi_i > 0$  and  $\gamma_i \in (-\xi_i, 0]$ . Similarly,  $\eta_{i+1} = n_{i+1}\eta_i + \gamma_i > 0$  by the inductive hypothesis  $\eta_i > 0$  and  $\gamma_i \in (-\eta_i, 0]$ .
- ii)  $\gamma_{i+1} = -\xi_i < 0$  since  $\xi_i > 0$ ; in addition,

$$\gamma_{i+1} + \xi_{i+1} = -\xi_i + (n_{i+1}\xi_i + \gamma_i) > (n_{i+1} - 2)\xi_i \geq 0,$$

since  $n_{i+1} \geq 2$  and  $\gamma_i > -\xi_i$  by the inductive hypothesis. Similarly,  $\delta_{i+1} = -\eta_i < 0$  and

$$\delta_{i+1} + \eta_{i+1} = -\eta_i + (n_{i+1}\eta_i + \delta_i) > (n_{i+1} - 2)\eta_i \geq 0.$$

We are done. □

### 6.7.4 Formulae about canonical divisor

As before, we assume

$$K_{\tilde{X}} = \pi^* K_X + \sum_{i=0}^q a_i E_i.$$

Therefore,

$$\begin{aligned} DCI_{r,d} &= \left( \sum_{i=0}^q a_i E_i \right)^2 \\ &= \sum_{i=0}^q a_i^2 E_i^2 + 2 \sum_{i=0}^{q-1} a_i a_{i+1} E_i \cdot E_{i+1}. \end{aligned}$$

Recall that  $E_0^2 = -b$  and  $E_i^2 = -rn_i$  for  $i > 0$ . In addition,  $E_i \cdot E_{i'} = r$  for  $i' = i \pm 1$ ,  $=0$  otherwise. Hence, we have

$$\begin{aligned} DCI_{r,d} &= -ba_0^2 - r \sum_{i=1}^q a_i^2 n_i + 2r \sum_{i=0}^{q-1} a_i a_{i+1} \\ &= -ba_0^2 - r \sum_{i=1}^q n_i a_i^2 + 2r \sum_{i=1}^q a_i a_{i-1} \\ &= -ba_0^2 + r \sum_{i=1}^q a_i (2a_{i-1} - n_i a_i). \end{aligned}$$

By (6.5.3), we have  $-n_i a_i + a_{i-1} + a_{i+1} = -2 + n_i$ , so

$$\begin{aligned} DCI_{r,d} &= -ba_0^2 + r \sum_{i=1}^q a_i (a_{i-1} - a_{i+1} + n_i - 2) \\ &= -ba_0^2 + r \sum_{i=1}^q (n_i - 2) a_i + r \left( \sum_{i=1}^q a_i a_{i-1} - \sum_{i=1}^q a_i a_{i+1} \right) \\ &= -ba_0^2 + r \left( \sum_{i=1}^q n_i a_i - 2 \sum_{i=1}^q a_i \right) + r \left( \sum_{i=0}^{q-1} a_i a_{i+1} - \sum_{i=1}^q a_i a_{i+1} \right) \\ &= -ba_0^2 + r \left( \sum_{i=1}^q n_i a_i - 2 \sum_{i=1}^q a_i \right) + r a_0 a_1 \\ &= a_0 (-ba_0 + r a_1) + r \left( \sum_{i=1}^q n_i a_i - 2 \sum_{i=1}^q a_i \right) \end{aligned} \tag{6.7.5}$$

By (6.5.3), we have  $-ba_0 + ra_1 = (r-2)(\gcd(r, d) - 1) - 2 + b$  and  $-n_i a_i + a_{i-1} + a_{i+1} = -2 + n_i$ , so

$$\begin{aligned}
\sum_{i=1}^q n_i a_i - 2 \sum_{i=1}^q a_i &= \sum_{i=1}^q (a_{i-1} + a_{i+1} - (n_i - 2)) - 2 \sum_{i=1}^q a_i \\
&= -\sum_{i=1}^q (n_i - 2) + \sum_{i=1}^q (a_{i-1} - a_i) - \sum_{i=1}^q (a_i - a_{i+1}) \\
&= -\sum_{i=1}^q (n_i - 2) + \sum_{i=0}^{q-1} (a_i - a_{i+1}) - \sum_{i=1}^q (a_i - a_{i+1}) \\
&= -\sum_{i=1}^q (n_i - 2) + (a_0 - a_1 - a_q);
\end{aligned}$$

consequently, by (6.7.5), we get

$$\begin{aligned}
DCI_{r,d} &= a_0((r-2)(\gcd(r, d) - 1) - 2 + b) - r \sum_{i=1}^q (n_i - 2) + r(a_0 - a_1 - a_q) \\
&= a_0 \left( \gcd(r, d)(r-2) + b - r \right) - r \sum_{i=1}^q (n_i - 2) \\
&\quad + r(a_0 - a_1 - a_q). \tag{6.7.6}
\end{aligned}$$

Now we are to compute  $a_0$ ,  $a_1$  and  $a_q$ . By equation (6.5.3), we have

$$\begin{cases}
-ba_0 + ra_1 &= (r-2)(\gcd(r, d) - 1) - 2 + b \\
-n_1 a_1 + (a_0 + a_2) &= -2 + n_1 \\
-n_2 a_2 + (a_1 + a_3) &= -2 + n_2 \\
\vdots & \\
-n_{q-1} a_{q-1} + (a_{q-2} + a_q) &= -2 + n_{q-1} \\
-n_q a_q + a_{q-1} &= -2 + n_q.
\end{cases}$$

Let  $a_i^* = a_i + 1$  for  $i = 0, 1, \dots, q+1$ . Recall also that  $a_{q+1} = 0$ . Then the above equations can be reformulated into a more convenient form:

$$\begin{cases}
-ba_0^* + ra_1^* &= \gcd(r, d)(r-2) \\
-n_1 a_1^* + (a_0^* + a_2^*) &= 0 \\
-n_2 a_2^* + (a_1^* + a_3^*) &= 0 \\
\vdots & \\
-n_{q-1} a_{q-1}^* + (a_{q-2}^* + a_q^*) &= 0 \\
-n_q a_q^* + (a_{q-1}^* + a_{q+1}^*) &= 0.
\end{cases}$$

With the help of the matrices  $\mathbf{G}_i$  defined in (6.7.3), we have

$$\begin{pmatrix} a_{i-1}^* \\ a_i^* \end{pmatrix} = \mathbf{G}_i \begin{pmatrix} a_i^* \\ a_{i+1}^* \end{pmatrix},$$

hence

$$\begin{pmatrix} a_0^* \\ a_1^* \end{pmatrix} = \mathbf{G}_1 \cdots \mathbf{G}_q \begin{pmatrix} a_q^* \\ a_{q+1}^* \end{pmatrix} = \mathbf{G} \begin{pmatrix} a_q^* \\ 1 \end{pmatrix}.$$

By Proposition 6.7.2, we thus have

$$\begin{cases} a_0^* &= \alpha a_q^* + (b' - \alpha) = \alpha a_q + b' \\ a_1^* &= \beta a_q^* + \left(\frac{1+b'\beta}{\alpha} - \beta\right) = \beta a_q + (1 + b'\beta)/\alpha, \end{cases} \quad (6.7.7)$$

so

$$\begin{aligned} -ba_0^* + ra_1^* &= -b\alpha a_q - bb' + r\beta a_q + r(1 + b'\beta)/\alpha \\ &= (r\beta - b\alpha)a_q + \left(-bb' + r(1 + b'\beta)/\alpha\right). \end{aligned}$$

Recall that  $b = \gcd(r, d)(1 + b'\beta)/\alpha$  and  $b' = r/\gcd(r, d)$ , we have

$$r\beta - b\alpha = r\beta - \gcd(r, d)(1 + b'\beta) = -\gcd(r, d),$$

and

$$-bb' + \frac{r(1 + b'\beta)}{\alpha} = -\frac{\gcd(r, d)(1 + b'\beta)}{\alpha} \cdot \frac{r}{\gcd(r, d)} + \frac{r(1 + b'\beta)}{\alpha} = 0;$$

therefore,  $-ba_0^* + ra_1^* = -\gcd(r, d)a_q$ .

From the equation  $-ba_0^* + ra_1^* = \gcd(r, d)(r - 2)$ , we thus get

$$a_q = -(r - 2).$$

Recall that  $a_i^* = a_i + 1$ , by (6.7.7), we obtain

$$\begin{cases} a_0 &= (2 - r)\alpha + b' - 1, \\ a_1 &= (2 - r)\beta + \frac{1+b'\beta}{\alpha} - 1. \end{cases}$$

As a conclusion, by (6.7.6), we get

$$\begin{aligned} DCI_{r,d} &= ((2 - r)\alpha + b' - 1)(\gcd(r, d)(r - 2) + b - r) - r \sum_{i=1}^q (n_i - 2) \\ &\quad + r \left( (2 - r)(\alpha - \beta) + b' - \frac{1 + b'\beta}{\alpha} \right) + r(r - 2) \\ &= -(r - 2)^2 \alpha \gcd(r, d) - r \sum_{i=1}^q (n_i - 2) \\ &\quad + \left( (b' - 1)(b - r) + r \left( b' - \frac{1 + b'\beta}{\alpha} \right) \right) \\ &\quad + (r - 2) \left( -\alpha(b - r) + \gcd(r, d)(b' - 1) - r(\alpha - \beta) + r \right). \end{aligned}$$

Recall that  $\alpha = d/\gcd(r, d)$ , so  $\alpha \gcd(r, d) = d$ . Recall also that  $b = \gcd(r, d)(1 + b'\beta)/\alpha$  and  $b' = r/\gcd(r, d)$ , so

$$\begin{aligned} (b' - 1)(b - r) &= \frac{r - \gcd(r, d)}{\gcd(r, d)} \cdot \left( \frac{\gcd(r, d)(1 + b'\beta)}{\alpha} - \gcd(r, d)b' \right) \\ &= (r - \gcd(r, d)) \left( \frac{1 + b'\beta}{\alpha} - b' \right), \end{aligned}$$

and thus,

$$(b' - 1)(b - r) + r \left( b' - \frac{1 + b'\beta}{\alpha} \right) = \gcd(r, d) \left( b' - \frac{1 + b'\beta}{\alpha} \right) = r - b.$$

Moreover, we have

$$\begin{aligned} & -\alpha(b - r) + \gcd(r, d)(b' - 1) - r(\alpha - \beta) + r \\ = & -\alpha \left( \frac{\gcd(r, d)(1 + b'\beta)}{\alpha} - r \right) + (r - \gcd(r, d)) - r(\alpha - \beta) + r \\ = & 2r - 2\gcd(r, d) + (r\beta - \gcd(r, d)b'\beta) \\ = & 2r - 2\gcd(r, d). \end{aligned}$$

Therefore, putting all these together, we have

$$DCI_{r,d} = -d(r - 2)^2 - r \sum_{i=1}^q (n_i - 2) + 2(r - 2)(r - \gcd(r, d)) + (r - b). \quad (6.7.8)$$

It is clear, in the above formula, that the first term  $-d(r - 2)^2$  is obviously negative, and also  $-r \sum_{i=1}^q (n_i - 2) \leq 0$  since  $n_i \geq 2$ . We can see that the remaining two terms are positive, and in fact we need to estimate them carefully later. Indeed,  $2(r - 2)(r - \gcd(r, d)) \geq 0$  since  $r \geq 3$  and  $r \geq \gcd(r, d)$  always; also

$$r - b = \gcd(r, d) \left( b' - \frac{1 + b'\beta}{\alpha} \right) = \frac{\gcd(r, d)}{\alpha} (b'(\alpha - \beta) - 1)$$

is also nonnegative since  $\alpha - \beta > 0$  and  $b' \geq 1$ .

### 6.7.5 Estimations of the Miyaoka-Yau numbers

We continue to consider the Miyaoka-Yau number of the minimal resolution  $\pi : \tilde{F} \rightarrow \bar{F}$ . As for Formula (6.7.8), we also need the formula for  $DCII_{r,d}$ ; but this is easy. In fact, we have

$$DCII_{r,d} = -1 + \chi(E_0) + rq = 1 + rq - (r - 2)(\gcd(r, d) - 1). \quad (6.7.9)$$

Hence, by definition,

$$\begin{aligned} DMY_{r,d} &= 3DCII_{r,d} - DCI_{r,d} \\ &= 3(1 + rq) - 3(r - 2)(\gcd(r, d) - 1) + d(r - 2)^2 + r \sum_{i=1}^q (n_i - 2) \\ &\quad - 2(r - 2)(r - \gcd(r, d)) - (r - b) \\ &= \left( 3(1 + rq) + r \sum_{i=1}^q (n_i - 2) \right) + d(r - 2)^2 \\ &\quad - \left( (r - 2)(\gcd(r, d) + 2r - 3) + (r - b) \right), \\ &= \left( 3(1 + rq) + r \sum_{i=1}^q (n_i - 2) \right) + (d - 1)(r - 2)^2 \\ &\quad - \left( (r - 2)(\gcd(r, d) + r - 1) + (r - b) \right), \end{aligned}$$

hence,

$$\begin{aligned}
\mathcal{E}_{r,d} &= DM Y_{r,d} + (d-1)(r-1)(3-r) \\
&= DM Y_{r,d} - (d-1)\left((r-2)^2 - 1\right) \\
&= \left(3(1+rq) + r \sum_{i=1}^q (n_i - 2)\right) + (d-1) \\
&\quad - \left((r-2)(\gcd(r,d) + r-1) + (r-b)\right) \\
&= \left(r \sum_{i=1}^q (n_i + 1)\right) + (d+2) \\
&\quad - \left((r-2)(\gcd(r,d) + r-1) + (r-b)\right). \tag{6.7.10}
\end{aligned}$$

Now we begin to estimate  $\mathcal{E}_{r,d}$ . First, we have

$$\begin{aligned}
&(r-2)(\gcd(r,d) + r-1) + (r-b) \\
&\leq (r-2)(2r-1) + r \\
&= 2r(r-2) + 2 \\
&= 2(r-1)^2,
\end{aligned}$$

so the following hold:

$$\begin{aligned}
\mathcal{E}_{r,d} &\geq \left(r \sum_{i=1}^q (n_i + 1)\right) + (d+2) - 2(r-1)^2 \\
&\geq -2(r-1)^2 \\
&> -2r(r-1). \tag{6.7.11}
\end{aligned}$$

**Remark 6.7.6.** 1. The above estimate is also true when  $d \equiv 1 \pmod r$  by Example 6.6.3 and when  $r|d$  by Example 6.6.2.

2. The above estimate is far from optimal, as one can easily see. However, it turns out this is enough for our purpose.

As an application of the above calculations, we give a new example of computing  $\mathcal{E}_{r,d}$  by directly using Formula (6.7.10).

**Example 6.7.7.** Let  $r \geq 3$  and  $d = r(p-1) + (r-1) = rp - 1$  for  $p \geq 2$ . Then in view of Formula (6.7.10), we have

(i)  $\gcd(r,d) = 1$ , so  $\alpha = d/\gcd(r,d) = d$  and  $b' = r/\gcd(r,d) = r$ . Since  $\alpha = b'p - 1$  and by assumption  $\beta$  is chosen so that  $0 \leq \beta < \alpha$  satisfying  $b'\beta \equiv -1 \pmod \alpha$ , we have  $\beta = \alpha - p = p(r-1) - 1$ .

(ii) We get

$$b = \frac{\gcd(r,d)(1+b'\beta)}{\alpha} = r-1.$$

(iii) We have

$$\frac{\alpha}{\beta} = \frac{d}{r(p-1)-1} = \frac{rp-1}{r(p-1)-1},$$

doing the continued fraction decomposition, we see that

$$q = p + r - 3,$$

and

$$n_1 = \cdots = n_{p-2} = 2, \quad n_{p-1} = 3, \quad n_p = n_{p+1} = \cdots = n_{p+r-3} = 2.$$

Therefore,

$$r \sum_{i=1}^q (n_i + 1) = r(3q + 1) = 3r(p + r) - 8r = 3(d + 1) + 3r^2 - 8r.$$

(iv) Eventually, by (6.7.8), we have

$$\begin{aligned} DCI_{r,d} &= -d(r-2)^2 - r + 2(r-2)(r-1) + (r - (r-1)) \\ &= -d(r-2)^2 + (2r-5)(r-1); \end{aligned}$$

by (6.7.9), we have

$$\begin{aligned} DCII_{r,d} &= 1 + rq - (r-2)(\gcd(r, d) - 1) \\ &= 1 + r(p + r - 3) \\ &= d + (r-1)(r-2); \end{aligned}$$

by (6.7.10), we obtain

$$\begin{aligned} \mathcal{E}_{r,d} &= \left( 3(d+1) + 3r^2 - 8r \right) + (d+2) - \left( r(r-2) + 1 \right) \\ &= 4(d+1) + 2r(r-3). \end{aligned}$$

Consequently, when  $r = 3$ , we have the following:

- (1) when  $3|d$ , we have  $\mathcal{E}_{3,d} = 4d - 12$  by Example 6.6.2;
- (2) when  $d \equiv 1 \pmod{3}$ , we have  $\mathcal{E}_{3,d} = 4(d-1)$  by Example 6.6.3;
- (3) when  $d \equiv 2 \pmod{3}$ , we have  $\mathcal{E}_{3,d} = 4(d+1)$  by the results above.

In particular, when  $d \geq 4$ , it is always true that  $\mathcal{E}_{3,d} \geq 4(d-3)$ .

## 6.8 The minimal resolution is not a ball quotient

Let  $\pi : \tilde{F} \rightarrow \bar{F}$  be the minimal resolution obtained in previous sections. Then we have the following.

**Theorem 6.8.1.** *Assume  $d = |\mathcal{A}| \geq 2$  and that for  $d = 3$ ,  $\mathcal{A}$  is not a pencil. Then  $MY(\tilde{F}) \neq 0$ .*

*In particular,  $\tilde{F}$  is not a ball quotient.*

*Proof.* The proof will be divided into three cases with respect to the values of  $t_d$  and  $t_{d-1}$ .

- i) When the lines in  $\mathcal{A}$  form a pencil, namely,  $t_d = 1$ , we have,  $d \neq 3$  by the assumption of Theorem 6.8.1; moreover, by Example 6.6.1, Example 6.6.2 and Formula (6.4.6),

$$\begin{aligned} MY(\tilde{F}) &= \mathcal{E}_{d,d} \\ &= 4d - 2d(d-1) \\ &= 2d(3-d) \\ &\neq 0. \end{aligned}$$

- ii) If  $t_d = 0$  while  $t_{d-1} \neq 0$ , then we have  $t_{d-1} = 1$  and  $t_2 = d-1$  (if  $d = 3$ ,  $t_2 = d = 3$ ). Moreover, by Example 6.6.1 and Example 6.6.3, in view of (6.4.6), we have

$$\begin{aligned} MY(\tilde{F}) &= \sum_r t_r \mathcal{E}_{r,d} \\ &= t_2 \mathcal{E}_{2,d} + t_{d-1} \mathcal{E}_{r,d} \\ &= (d-1) \left( 4(d-1) \right) + 4(d-1) \\ &= 4d(d-1) > 0. \end{aligned}$$

- iii) Now we consider the case  $t_d = 0, t_{d-1} = 0$ . Then by the estimation (6.7.11), we have

$$\begin{aligned} MY(\tilde{F}) &= \sum_r t_r \mathcal{E}_{r,d} \\ &= t_2 \mathcal{E}_{2,d} + t_3 \mathcal{E}_{3,d} + \sum_{r \geq 4} t_r \mathcal{E}_{r,d} \\ &\geq t_2 \mathcal{E}_{2,d} + t_3 \mathcal{E}_{3,d} - 2 \sum_{r \geq 4} t_r r(r-1) \\ &= \left( t_2 (\mathcal{E}_{2,d} + 4) + t_3 (\mathcal{E}_{3,d} + 12) \right) - 2 \sum_r t_r r(r-1). \end{aligned}$$

From Remark 6.3.6, we have  $\sum_r t_r r(r-1) = d(d-1)$ ; moreover, from Example 6.6.1 and the end of Example 6.7.7, we deduce that

$$\begin{aligned} MY(\tilde{F}) &\geq 4d(t_2 + t_3) - 2d(d-1) \\ &= 2d \left( 2(t_2 + t_3) - (d-1) \right). \end{aligned} \tag{6.8.1}$$

Now we use the celebrated inequality in the second remark added in proof of [Hi], which states that

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{r \geq 5} (r-4)t_r,$$

see also [Sa] or Appendix A of [Tr]. In particular,  $t_2 + t_3 \geq d$ . It follows immediately, by (6.8.1), that

$$MY(\tilde{F}) > 0.$$

The proof now is complete. □ □

**Remark 6.8.2.** When  $d = |\mathcal{A}| = 3$  and  $\mathcal{A}$  is a pencil, i.e.,  $t_3 = 1$ , then  $MY(\tilde{F}) = 0$ . Moreover, using Formula (6.7.8), a similar computation as in Example 6.6.2 gives  $DCI_{3,3} = -3$ . Hence, by Example 6.3.5, we have

$$c_1^2(\tilde{F}) = K_{\tilde{F}}^2 = K_F^2 + DCI_{3,3} = 3 \times (3 - 4)^2 - 3 = 0.$$

Moreover,  $c_2(\tilde{F}) = 0$  since  $MY(\tilde{F}) = 3c_2(\tilde{F}) - c_1^2(\tilde{F}) = 0$ .

Since  $c_2 > 0$  for a smooth projective surface of general type (see [BHPV], Chapter VII), it follows that  $\tilde{F}$  is not of general type. In particular,  $\tilde{F}$  is not a ball quotient.

**Remark 6.8.3.** Note that a ball quotient cannot admit any rational curves. In fact, for any smooth projective curve surface  $X$ , any given morphism  $f : \mathbb{P}^1 \rightarrow X$  lifts to a morphism  $\tilde{f} : \mathbb{P}^1 \rightarrow \tilde{X}$  since  $\mathbb{P}^1$  is simply connected, where  $\tilde{X}$  is the universal cover of  $X$ . If  $X$  is a ball quotient or equivalently  $\tilde{X}$  is biholomorphic to a ball, then  $\tilde{f}$  and hence  $f$  must be constant.

When the line arrangement  $\mathcal{A}$  is not a pencil, from Theorem 6.5.2,  $\tilde{F}$  clearly contains rational curves in the exceptional divisors; it follows immediately that  $\tilde{F}$  is not a ball quotient. However, we have showed above that  $MY(\tilde{F}) > 0$  when  $\mathcal{A}$  is not a pencil; this is much stronger than the non-ball-quotient property.

## Surfaces of general type associated to line arrangements

Let  $\mathcal{A}$  be a line arrangement in  $\mathbb{P}^2$ . It remains an interesting question whether the associated  $\tilde{F}$  is of general type. In fact, as we have shown in the proof of Theorem 6.8.1,  $MY(\tilde{F}) > 0$  when  $\mathcal{A}$  is not a pencil, hence this equation is quite natural in view of the Miyaoka-Yau inequality.

### 7.1 General type criteria

Recall that the smooth projective surface  $X$  is said to be of *general type* if  $K_X$  is big. Moreover,  $X$  is of general type if and only if its minimal model  $X'$  is; in addition, if  $X$  is of general type, then  $X'$  is unique and  $K_{X'}$  is big and nef.

From the Enriques-Kodaira classification of compact smooth surfaces (see [BHPV], Chapter VI), we see that the condition  $K_X^2 > 0$  is a very restrictive condition for a minimal surface  $X$ . Indeed, if  $X$  is a smooth minimal projective surface and  $K_X^2 > 0$ , then either  $X$  is of general type or  $X$  is birationally isomorphic to  $\mathbb{P}^2$  with the additional property that  $K_X^2 = 9, c_2(X) = 3$  or  $K_X^2 = 8, c_2(X) = 4$ .

In fact, we have the following criteria for a smooth projective surface to be of general type.

**Proposition 7.1.1.** *Let  $X$  be a smooth projective surface. If  $c_1^2(X) > 9$ , then  $X$  is of general type.*

*Proof.* Let  $X'$  be a minimal model of  $X$ . Then  $X'$  is obtained by successively blowing down  $(-1)$ -curves. Note that once we blow down a  $(-1)$ -curve,  $c_1^2$  increases by 1, so  $c_1^2(X) \geq c_1^2(X') > 9$ , hence by the Enriques-Kodaira classification of surfaces (see [BHPV], Chapter VI),  $X'$  is of general type, and thus so is  $X$ .  $\square$

A similar proof gives the following more useful criterion.

**Proposition 7.1.2.** *Let  $X$  be a smooth projective surface. If  $c_1^2(X) \geq k > 0$  and  $3c_2(X) - c_1^2(X) > 4(9 - k)$ , then  $X$  is of general type.*

*Proof.* We may assume  $k \leq 9$ , otherwise just apply Proposition 7.1.1.

Let  $X'$  be a minimal model of  $X$ . Then  $X'$  is obtained by successively blowing down  $(-1)$ -curves. Note also that once we blow down a  $(-1)$ -curve,  $c_1^2$  increases by 1 and  $c_2(X)$  decreases by 1, so  $3c_2 - c_1^2$  decreases by 4. In addition, we have  $c_1^2(X') \geq k > 0$ .

By the Enriques-Kodaira classification of surfaces (see [BHPV], Chapter VI), if  $X'$  is not of general type, then  $c_1^2(X')^2 = 9, c_2(X') = 3$  or  $c_1^2(X') = 8, c_2(X') = 4$ . So we consider the following cases:

- (i) if  $c_1^2(X')^2 = 9, c_2(X') = 3$ , then by considering the change of  $c_1^2$ , we need  $9 - c_1^2(X) \leq 9 - k$  blow-downs to obtain  $X'$  from  $X$ , hence

$$\begin{aligned} 3c_2(X') - c_1^2(X') &= (3c_2(X) - c_1^2(X)) - 4(9 - c_1^2(X)) \\ &> 4(9 - k) - 4(9 - k) \\ &= 0, \end{aligned}$$

while we known explicitly that  $c_1^2(X')^2 = 9, c_2(X') = 3$ . So we get a contraction.

- (ii) if  $c_1^2(X') = 8, c_2(X') = 4$ , then proceeding as in the first case, we get

$$\begin{aligned} 4 &= 3c_2(X') - c_1^2(X') \\ &= (3c_2(X) - c_1^2(X)) - 4(8 - c_1^2(X)) \\ &> 4(9 - k) - 4(8 - k) \\ &= 4, \end{aligned}$$

contradiction. □

For a generic line arrangement,  $V(Q)$  has only nodes as singularities, and by Example 6.6.1,  $DCI_{2,d} = 0$ , and hence, by Example 6.3.5

$$K_{\tilde{F}}^2 = K_{\overline{F}}^2 = d(d-4)^2 > 9$$

for  $d \geq 6$ , hence  $\tilde{F}$  is of general type by Proposition 7.1.1. On the other hand, if the line arrangement  $\mathcal{A}$  is a pencil, i.e.,  $t_d = 1$ , then by Example 6.6.2, we have

$$MY(\tilde{F}) = 2d(3-d) < 0$$

for  $d \geq 4$ , thus  $\tilde{F}$  is not of general type in view of the Miyaoka-Yau inequality.

Moreover, inspired by [Hi], it is natural to conjecture that  $\tilde{F}$  is of general type if  $\mathcal{A}$  is not too singular, i.e.,  $t_r = 0$  for  $r$  large compared with  $d$ .

Now we give an example before we proceed to investigate in detail. We will continue to apply, without mentioning again, the notations in the previous chapter.

**Example 7.1.3.** Let  $\mathcal{A}_1 : (x-y)(x-2y)(x-3y)(x-4y) = 0$ . Then  $\mathcal{A}_1$  is a pencil in  $\mathbb{P}^2$  consisting of 4 lines. Now add two more generic lines to  $\mathcal{A}_1$ , we get a line arrangement  $\mathcal{A}$  containing 6 lines with  $t_2 = 9, t_4 = 1$  and  $t_r = 0$  for other  $r$ 's. Let  $F = F(\mathcal{A})$ ,  $\overline{F}$  be the compactification of  $F$  and  $\pi : \tilde{F} \rightarrow \overline{F}$  be the minimal resolution given by Theorem 6.5.2.

**Step 1: Resolutions of singular points with  $r = 4, d = 6$**

Let  $p$  be the only singular point of  $\overline{F}$  of multiplicity 4. Then according to Theorem 6.5.2, we have

- (i)  $r = 4, d = 6$ , hence  $\gcd(r, d) = 2$  and

$$g(E_0) = 1/2(r-2)(\gcd(r, d) - 1) = 1.$$

- (ii)  $\alpha = d/\gcd(r, d) = 3, b' = r/\gcd(r, d) = 2$ , hence  $\beta = 1$  and thus,

$$b = \gcd(r, d)(b'\beta + 1)/\alpha = 2.$$

(iii)  $\alpha/\beta = 3$ , hence  $q = 1$  and  $n_1 = 3$ .

By (6.7.8), we have

$$\begin{aligned} DCI_{4,6} &= -d(r-2)^2 - r \sum_{i=1}^q (n_i - 2) + 2(r-2)(r - \gcd(r, d)) + (r-b) \\ &= -6 \times 4 - 4 \times 1 + 2 \times 2 \times (4-2) + (4-2) \\ &= -18; \end{aligned}$$

and by (6.7.9)

$$\begin{aligned} DCII_{4,6} &= 1 + rq - (r-2)(\gcd(r, d) - 1) \\ &= 1 + 4 \times 1 - 2 \times (2-1) \\ &= 3; \end{aligned}$$

and by (6.7.10),

$$\begin{aligned} \mathcal{E}_{4,6} &= r \sum_{i=1}^q (n_i + 1) + (d+2) - \left( (r-2)(\gcd(r, d) + r - 1) + (r-b) \right) \\ &= 4 \times 4 + 8 - (2 \times 5 + 2) \\ &= 12. \end{aligned}$$

**Step 2: Miyaoka–Yau number of  $\tilde{F}$**

By Example 6.6.1, we have  $\mathcal{E}_{2,6} = 4(d-1) = 20$ . Therefore,

$$MY(\tilde{F}) = t_2 \mathcal{E}_{2,6} + t_4 \mathcal{E}_{4,6} = 9 \times 20 + 1 \times 12 = 192.$$

Hence, as we have shown in Theorem 6.8.1,  $\tilde{F}$  is not a ball quotient.

**Step 3: Is  $\tilde{F}$  of general type?**

Note that possibly  $\tilde{F}$  is not a minimal surface, namely, it may contain a curve of self-intersection  $-1$ . Recall that even if  $\tilde{F}$  is a minimal resolution, hence no exceptional rational curve has self-intersection  $-1$ , it can still happen that there exists a rational  $(-1)$ -curve  $C$  on  $\tilde{F}$  such that  $C$  is not exceptional, i.e.,  $C$  is not contracted by  $\pi$ .

By Example 6.6.1,  $DCI_{2,6} = 0$ , hence

$$K_{\tilde{F}}^2 = K_{\bar{F}}^2 + t_4 \cdot DCI_{4,6} = 6 \times (6-4)^2 + (-18) = 6 > 0;$$

and we also have  $MY(\tilde{F}) = 192 > 4(9 - K_{\tilde{F}}^2)$ , so  $\tilde{F}$  is of general type by Proposition 7.1.2.  $\square$

Note that for a smooth projective surface  $X$ , we have

$$(3c_2(X) - c_1(X)) - 4(9 - c_1^2(X)) = 3(c_1^2(X) + c_2(X) - 12).$$

hence by Proposition 7.1.2,  $X$  is of general type if  $c_1^2(X) > 0$  and  $c_1^2(X) + c_2(X) > 12$ .

**Remark 7.1.4.** By Noether's formula (see [BHPV]), when  $X$  is a smooth projective surface, we have

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1^2(X) + c_2(X)),$$

so  $c_1^2(X) + c_2(X) > 12$  means exactly that  $\chi(X, \mathcal{O}_X) > 1$ .

## 7.2 Surfaces associated to line arrangements with only nodes and triple points

In the sequel, we consider surfaces associated to line arrangements such that  $t_r = 0$  whenever  $r \geq 4$ .

For  $r = 2$ , by Example 6.6.1, we have

$$DCI_{2,d} = 0.$$

When  $r = 3$ , we have

- (i) If  $3|d$ ,  $DCI_{3,d} = -d$ ,  $DCII_{3,d} = d - 4$  by Example 6.6.2;
- (ii) If  $d \equiv 1 \pmod{3}$ ,  $DCI_{3,d} = -(d - 1)$ ,  $DCII_{3,d} = d - 1$  by Example 6.6.3;
- (iii) If  $d \equiv 2 \pmod{3}$ , we have  $DCI_{3,d} = -(d - 2)$ ,  $DCII_{r,d} = d + 2$  by Example 6.7.7

### 7.2.1 Case $3|d$

When  $d = 3p$  with  $p \geq 2$ , we have as above that  $DCI_{3,d} = -d$ . Hence the first Chern number of  $\tilde{F}$  is

$$\begin{aligned} c_1^2(\tilde{F}) &= K_{\tilde{F}}^2 + \sum_r t_r DCI_{r,d} \\ &= d(d - 4)^2 - t_3 d \\ &= d((d - 4)^2 - t_3). \end{aligned}$$

By Remark 6.3.6, we have  $2t_2 + 6t_3 = d(d - 1)$ , hence  $t_3 \leq d(d - 1)/6$  and when  $d = 3p \geq 9$

$$(d - 4)^2 - t_3 \geq \frac{1}{6}(5d^2 - 47d + 96) \geq \frac{1}{6}(5 \cdot 9^2 - 47 \cdot 9 + 96) = 13.$$

When  $d = 3p = 6$ , where exists a line arrangement containing 6 lines such that  $t_3 \geq 4$ , so a direct computation gives  $c_1^2(\tilde{F}) \leq 0$ . Therefore, to apply Proposition 7.1.1 or Proposition 7.1.2, we assume  $d = 3p \geq 9$ .

Then as is shown above  $(d - 4)^2 - t_3 \geq 3$ , thus

$$c_1^2(\tilde{F}) = d((d - 4)^2 - t_3) \geq 9 \cdot 13 > 9.$$

Therefore  $\tilde{F}$  is of general type by Proposition 7.1.1.

In addition, by (6.3.3),

$$c_2(\tilde{F}) = \chi(\tilde{F}) + \sum_r t_r DCII_{r,d} = d(d^2 - 4d + 6) - 3t_3 d.$$

Hence

$$\frac{c_1^2(\tilde{F})}{c_2(\tilde{F})} = \frac{(d - 4)^2 - t_3}{d^2 - 4d + 6 - 3t_3} = \frac{1}{3} \left( 1 + \frac{2(d - 3)(d - 7)}{d^2 - 4d + 6 - 3t_3} \right).$$

**Example 7.2.2.** As in [Art], there are two line arrangements  $\mathcal{A}_1$  and  $\mathcal{A}_2$  in  $\mathbb{P}^2$  such that

$$b_1(F(\mathcal{A}_1)) = 10, \quad b_1(F(\mathcal{A}_2)) = 8$$

while  $\mathcal{A}_1, \mathcal{A}_2$  have the same number of multiple points, namely, for each of them,

$$d = 9, \quad t_2 = 9, \quad t_3 = 9, \quad t_r = 0, \quad r \geq 4.$$

Let  $\tilde{F}(\mathcal{A}_i)$  be the surfaces associated to  $\mathcal{A}_i$  for  $i = 1, 2$ . Then they are of general type by the discussion above.

Note also that  $\tilde{F}(\mathcal{A}_i)$  can be seen as a smooth compactification of  $F(\mathcal{A}_i)$ , hence by Theorem 2.1.9

$$\dim W_1 H^1(F(\mathcal{A}_i)) \simeq \dim H^1(\tilde{F}(\mathcal{A}_i)) = b_1(\tilde{F}(\mathcal{A}_i)), \quad i = 1, 2.$$

See also the derivations of Formula (7.3.5) in the next Chapter. Moreover, note that for each  $i$ ,  $H^1(F(\mathcal{A}_i))$  has two weights 1, 2 and  $Gr_2^W H^1(F(\mathcal{A}_i)) \simeq H^1(M(\mathcal{A}_i))$ , hence

$$\dim Gr_2^W H^1(F(\mathcal{A}_i)) = \dim H^1(M(\mathcal{A}_i)) = |\mathcal{A}_i| - 1 = 8.$$

Consequently,  $b_1(\tilde{F}(\mathcal{A}_1)) = 2$  while  $b_1(F(\mathcal{A}_2)) = 0$ . In particular, they have very different topology.

Note that the Chern numbers of  $\tilde{F}(\mathcal{A}_i)$ ,  $i = 1, 2$  are uniquely determined by the number of multiple points of  $\mathcal{A}_i$ , hence they are the same. Therefore, this example gives explicit smooth projective surfaces with the same Chern numbers but different first Betti numbers. Since the first Betti number is a birational invariant, we conclude that  $\tilde{F}(\mathcal{A}_i)$ ,  $i = 1, 2$  are not birationally equivalent.

### 7.2.3 Case $d \equiv 1 \pmod{3}$

When  $d = 3p + 1$  with  $p \geq 1$ , we have that  $DCI_{3,d} = -(d - 1)$ . Hence the first Chern number of  $\tilde{F}$  is

$$\begin{aligned} c_1^2(\tilde{F}) &= K_{\tilde{F}}^2 + \sum_r t_r DCI_{r,d} \\ &= d(d - 4)^2 - t_3(d - 1). \end{aligned}$$

Since  $2t_2 + 6t_3 = d(d - 1)$ , we have  $t_3 \leq d(d - 1)/6$  so, when  $p \geq 2$  or equivalently  $d \geq 7$

$$\begin{aligned} c_1^2(\tilde{F}) &\geq d(d - 4)^2 - \frac{1}{6}d(d - 1)^2 \\ &= \frac{1}{6}d(5d^2 - 46d + 95) \\ &\geq \frac{1}{6} \cdot 7 \cdot (5 \cdot 7^2 - 46 \cdot 7 + 95) \\ &= 21 > 9, \end{aligned}$$

hence,  $\tilde{F}$  is of general type by Proposition 7.1.1.

In addition,

$$c_2(\tilde{F}) = \chi(\tilde{F}) + \sum_r t_r DCII_{r,d} = d(d^2 - 4d + 6) - 3(d - 1)t_3.$$

Hence,

$$\frac{c_1^2(\tilde{F})}{c_2(\tilde{F})} = \frac{d(d - 4)^2 - (d - 1)t_3}{d(d^2 - 4d + 6) - 3(d - 1)t_3} = \frac{1}{3} \left( 1 + \frac{2d(d - 3)(d - 7)}{d(d^2 - 4d + 6) - 3(d - 1)t_3} \right).$$

### 7.2.4 Case $d \equiv 2 \pmod{3}$

When  $d = 3p + 2$  with  $p \geq 1$ , we have that  $DCI_{3,d} = -(d-2)$ . So the first Chern number of  $\tilde{F}$  is

$$\begin{aligned} c_1^2(\tilde{F}) &= K_{\tilde{F}}^2 + \sum_r t_r DCI_{r,d} \\ &= d(d-4)^2 - t_3(d-2). \end{aligned}$$

Since  $2t_2 + 6t_3 = d(d-1)$ , we have  $t_3 \leq d(d-1)/6$  so, when  $p \geq 2$  or equivalently  $d \geq 8$

$$\begin{aligned} c_1^2(\tilde{F}) &\geq d(d-4)^2 - \frac{1}{6}d(d-1)(d-2) \\ &= \frac{1}{6}d(5d^2 - 45d + 94) \\ &\geq \frac{1}{6} \cdot 8 \cdot (5 \cdot 8^2 - 45 \cdot 8 + 94) \\ &= 72 > 9, \end{aligned}$$

hence,  $\tilde{F}$  is of general type by Proposition 7.1.1. In addition,

$$c_2(\tilde{F}) = \chi(\tilde{F}) + \sum_r t_r DCII_{r,d} = d(d^2 - 4d + 6) - 3(d-2)t_3.$$

hence,

$$\frac{c_1^2(\tilde{F})}{c_2(\tilde{F})} = \frac{d(d-4)^2 - (d-2)t_3}{d(d^2 - 4d + 6) - 3(d-2)t_3} = \frac{1}{3} \left( 1 + \frac{2d(d-3)(d-7)}{d(d^2 - 4d + 6) - 3(d-2)t_3} \right).$$

We have the following.

**Theorem 7.2.5.** *Let  $\mathcal{A}$  be a line arrangement of  $\mathbb{P}^2$  containing  $d \geq 7$  lines such that  $t_r = 0$  for  $r \geq 4$ . Then the associated surface  $\tilde{F}$  is of general type.*

*All the Hodge numbers of  $\tilde{F}$  are determined by the combinatorics of  $\mathcal{A}$ . In addition,  $b_1(\tilde{F}) \leq 4$ .*

*Proof.* The first statement follows from the discussions above.

The last claim follows the fact that  $b_1(\tilde{F}) = 2\beta_3(\mathcal{A})$ , see Section 7.3 below, where  $\beta_3(\mathcal{A})$  is the Papadima-Suciu invariant introduced in [PaS], where the authors show that  $0 \leq \beta_3(\mathcal{A}) \leq 2$  for any line arrangement satisfying  $t_r = 0$  for  $r \geq 4$ .  $\square$

**Remark 7.2.6.** 1. The Theorem above cannot be extended to the case  $d = 6$ .

2. In any case,  $\frac{c_1^2(\tilde{F})}{c_2(\tilde{F})}$  is an increasing function in  $t_3$  with fixed  $d \geq 7$ . As  $t_3 \leq d(d-1)/6$ , it follows that

$$1 \leq \liminf_{d \rightarrow \infty} \frac{c_1^2(\tilde{F})}{c_2(\tilde{F})} \leq \limsup_{d \rightarrow \infty} \frac{c_1^2(\tilde{F})}{c_2(\tilde{F})} \leq \frac{5}{3}.$$

### 7.3 Hodge numbers and Chern numbers

One of motivation of our work is to understand whether the Hodge numbers of  $\tilde{F}$  are combinatorially determined, one of the main open question in the theory of line arrangements, see [PaS]. As is explained below, we have the following formulae

$$\begin{cases} h^{0,0}(X) &= h^{2,2}(X) = 1, \\ h^{0,1}(X) &= h^{1,0}(X) = h^{1,2}(X) = h^{2,1}(X) = q(X), \\ h^{0,2}(X) &= h^{2,0}(X) = \frac{1}{12}(c_1^2(X) + c_2(X)) - (1 - q(X)), \\ h^{1,1}(X) &= -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q(X). \end{cases}$$

where  $h^{p,q}(X)$ 's denote the Hodge numbers for a given smooth projective surface  $X$ ,  $c_1^2(X)$ ,  $c_2(X)$  denote the Chern numbers and  $q(X)$  the irregularity. For the associated surface  $\tilde{F}$ , the Chern numbers  $c_1^2(\tilde{F})$ ,  $c_2(\tilde{F})$  are determined by the combinatorics of  $\mathcal{A}$ . On the other hand, it follows from [DP11] that  $2q(\tilde{F}) = \dim H^1(\tilde{F})_{\neq 1}$ , which is known for many line arrangements, see [CS95]. In fact, in [PaS], a combinatorial formula for  $q$  is given when  $\mathcal{A}$  has only double or triple points; more examples are given in [Su1] where  $q(\tilde{F})$  is computed. [Su14] is a good and recent survey on the monodromy computations and in a recent preprint [DStenMF], an effective algorithm to compute  $q(\tilde{F})$  is provided.

To illustrate more numerical properties of the surface  $\tilde{F}$ , we give some examples in which we compute all the Hodge numbers of  $\tilde{F}$ . We also compute the Chern ratio. This numerical invariant is of special interest for algebraic surfaces, see for instance [MP],[MPP],[Naie],[Rou].

In [Hi], some line arrangements are given so that they give ball quotients through the construction via Kummer covers; such arrangements include the Hesse arrangement and the arrangement  $\mathcal{A}(2, 2, 3) : (x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0$ . By Theorem 6.8.1, these arrangements do not give ball quotient through our approach. Instead, we compute the Hodge numbers of the associated surfaces.

#### 7.3.1 Relations between Hodge numbers and Chern numbers

In this section, we shall fix a smooth projective surface  $X$ . Denote  $q(X) = h^{0,1}(X)$  its irregularity and  $p(X) = h^{0,2}(X)$  its geometric genus. Denote also  $b_i(X)$ ,  $i = 1, 2, 3, 4$  the Betti numbers of  $X$  and  $c_1^2(X)$ ,  $c_2(X)$  the Chern numbers, as well as  $h^{p,q}(X)$  the Hodge numbers.

Then by from Noether's formula (see [BHPV]), we first have

$$1 - q(X) + p(X) = \frac{1}{12}(c_1^2(X) + c_2(X)); \quad (7.3.1)$$

second, from the formula for Euler characteristic, we have

$$2 - 2b_1(X) + b_2(X) = c_2(X). \quad (7.3.2)$$

Moreover, from Hodge decomposition and Serre duality, we have

$$\begin{cases} b_1(X) &= 2q(X), \\ b_2(X) &= h^{0,2}(X) + h^{2,0}(X) + h^{1,1}(X) \\ h^{p,q}(X) &= h^{q,p}(X) = h^{2-p,2-q}(X), p, q = 0, 1, 2. \end{cases} \quad (7.3.3)$$

We may see the equalities (7.3.1), (7.3.2), (7.3.3) as equations for the Hodge numbers  $h^{p,q}(X)$ 's, with knowns  $c_1(X), c_2(X), q(X)$ , and we have the following solution:

$$\begin{cases} h^{0,0}(X) &= h^{2,2}(X) = 1, \\ h^{0,1}(X) &= h^{1,0}(X) = h^{1,2}(X) = h^{2,1}(X) = q(X), \\ h^{0,2}(X) &= h^{2,0}(X) = \frac{1}{12}(c_1^2(X) + c_2(X)) - (1 - q(X)), \\ h^{1,1}(X) &= -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q(X). \end{cases} \quad (7.3.4)$$

### 7.3.2 First Betti number for surfaces associated to line arrangements

As we have illustrated in Example 7.2.2, the first Betti number  $b_1(\tilde{F})$  can be computed from that of  $F$ . More precisely,  $\tilde{F}$  can clearly be seen as a smooth compactification of  $F$ , and let  $j : F \hookrightarrow \tilde{F}$  be the identified inclusion, then from Theorem 2.1.9, we have

$$W_1 H^1(F) = j^* H^1(\tilde{F}).$$

But the induced morphism between the fundamental groups  $j_* : \pi_1(F) \rightarrow \pi_1(\tilde{F})$  is surjective because  $F$  is identified via  $j$  with a Zariski open subset of  $\tilde{F}$ , so  $j^* : H^1(\tilde{F}) \rightarrow H^1(F)$  is injective. Therefore, we have

$$W_1 H^1(F) \simeq H^1(\tilde{F}). \quad (7.3.5)$$

Let  $M = M(\mathcal{A})$  be the complement of the arrangement  $\mathcal{A}$  and  $h^* : H^1(F) \rightarrow H^1(F)$  be the monodromy morphism. Then as is shown in [DP11], we have

$$W_1 H^1(F) = H^1(F)_{\neq 1}, \quad Gr_2^W H^1(F) = H^1(F)_1 = \ker(h^* - \text{Id}) \quad (7.3.6)$$

where  $H^1(F)_{\neq 1} = \bigoplus_{a \neq 1} \ker(h^* - a\text{Id})$ .

So from (7.3.5) and (7.3.6), we get that

$$b_1(\tilde{F}) = \dim H^1(F)_{\neq 1}.$$

For interesting line arrangements, the latter dimension is known explicitly, see Section 2.3.10 in the end of Chapter 1. Thus for these line arrangements,  $q(\tilde{F}) = \frac{1}{2}b_1(\tilde{F})$  can be obtained. Note also that we can also compute the Chern numbers  $c_1^2(\tilde{F}), c_2(\tilde{F})$ , hence all the Hodge numbers  $h^{p,q}(\tilde{F})$  can be calculated by Formula (7.3.4).

### 7.3.3 Computing Hodge numbers via Chern numbers and monodromy

In this section, we let  $\mathcal{A}$  denote a line arrangement in  $\mathbb{P}^2$  and  $\tilde{F}$  be the associated surface. We first give the formulae for the Chern numbers of  $\tilde{F}$ : by Example 6.3.5, we have

$$K_{\tilde{F}}^2 = d(d - 4)^2$$

and

$$\chi(\tilde{F}) = d(d^2 - 4d + 6) - (d - 1) \sum_r t_r (r - 1)^2.$$

In the examples below,  $t_r \neq 0$  only if  $r|d$ . By example 6.6.2, we have

$$DCI_{r,d} = -d(r - 2)^2, \quad DCII_{r,d} = d - (r - 1)^2.$$

The irregularity  $q$  is closely related to the monodromy of  $h^* : H^1(F) \rightarrow H^1(F)$ , which are given in Section 2.3.10.

**Example 7.3.4.** (Hesse arrangement) The Hesse arrangement is defined by

$$Q = xyz((x^3 + y^3 + z^3)^3 - 27x^3y^3z^3)$$

with  $d = 12$  and  $\xi = \exp(2\pi\sqrt{-1}/12)$ .  $V(Q)$  has 12 double points and 9 points of multiplicity 4. Moreover, we have  $\dim H^1(F)_a = 2$  for  $a = \xi^3, \xi^6, \xi^9$ , so  $q(\tilde{F}) = 3$ .

For the Chern numbers, we first have

$$K_{\tilde{F}}^2 = d(d-4)^2 = 12 \cdot 8^2 = 768.$$

Since  $DCI_{2,12} = 0$  and  $DCI_{4,12} = -48$ , we have

$$\begin{aligned} c_1^2(\tilde{F}) &= K_{\tilde{F}}^2 + \sum_r t_r DCI_{r,d} \\ &= 768 - 9 \cdot 48 \\ &= 336. \end{aligned}$$

So by Proposition 7.1.1,  $\tilde{F}$  is of general type. Moreover,

$$\begin{aligned} \chi(\tilde{F}) &= d(d^2 - 4d + 6) - (d-1) \sum_r t_r (r-1)^2 \\ &= 12 \cdot (144 - 48 + 6) - 11 \cdot (12 \cdot 1 + 9 \cdot 9) \\ &= 201. \end{aligned}$$

Since  $DCII_{2,12} = 11$  and  $DCII_{4,12} = 3$ , so

$$\begin{aligned} c_2(\tilde{F}) &= \chi(\tilde{F}) + \sum_r t_r DCII_{r,d} \\ &= 201 + (12 \times 11 + 9 \times 3) \\ &= 360. \end{aligned}$$

Finally, by Formula (7.3.4), we obtain

$$\begin{cases} h^{0,0}(\tilde{F}) = h^{2,2}(\tilde{F}) = 1, \\ h^{0,1}(\tilde{F}) = h^{1,0}(\tilde{F}) = h^{1,2}(\tilde{F}) = h^{2,1}(\tilde{F}) = q(\tilde{F}) = 3, \\ h^{0,2}(\tilde{F}) = h^{2,0}(\tilde{F}) = \frac{1}{12}(c_1^2(\tilde{F}) + c_2(\tilde{F})) - (1 - q(\tilde{F})) = 60, \\ h^{1,1}(\tilde{F}) = -\frac{1}{6}c_1^2(\tilde{F}) + \frac{5}{6}c_2(\tilde{F}) + 2q(\tilde{F}) = 250. \end{cases}$$

**Example 7.3.5.** Consider the arrangement  $\mathcal{A}(m, m, 3)$  defined by

$$Q = (x^m - y^m)(y^m - z^m)(z^m - x^m) = 0.$$

Then if  $m = 3$ , we have  $t_3 = 12$  and for  $m \neq 3$ , we have  $t_3 = m^2, t_m = 3$ . In addition, set  $\theta = \exp(2\pi\sqrt{-1}/3)$ , then we have two cases:

- (i) If  $m \equiv 0 \pmod{3}$ , then  $\dim H^1(F)_\theta = \dim H^1(F)_{\theta^2} = 2$ , so  $q(\tilde{F}) = 2$ ;
- (ii) If  $m \not\equiv 0 \pmod{3}$ , then  $\dim H^1(F)_\theta = \dim H^1(F)_{\theta^2} = 1$ , so  $q(\tilde{F}) = 1$ .

Moreover, by Example 6.6.2, the following hold:

$$DCI_{3,d} = -d = -3m, \quad DCII_{3,d} = d - 4 = 3m - 4$$

and

$$DCI_{m,d} = -d(m-2)^2 = -3m(m-2)^2, \quad DCII_{m,d} = d - (m-1)^2 = 3m - (m-1)^2.$$

Therefore,

$$\begin{aligned} c_1^2(\tilde{F}) &= K_{\tilde{F}}^2 + \sum_r t_r DCI_{r,d} \\ &= d(d-4)^2 - m^2 \cdot (3m) - 3 \cdot (3m(m-2)^2) \\ &= 3m(3m-4)^2 - 3m^3 - 9m(m-2)^2 \\ &= 3m(m-2)(5m-2). \end{aligned} \tag{7.3.7}$$

and

$$\begin{aligned} c_2(\tilde{F}) &= \chi(\tilde{F}) + \sum_r t_r DCII_{r,d} \\ &= d \cdot (d^2 - 4d + 6) - (d-1)(m^2 \cdot 2^2 + 3 \cdot (m-1)^2) \\ &\quad + (m^2 \cdot (3m-4) + 3 \cdot (3m - (m-1)^2)) \\ &= 3m(9m^2 - 12m + 6) - (3m-1)(4m^2 + 3(m-1)^2) \\ &\quad + (m^2(3m-4) + 9m - 3(m-1)^2) \\ &= 9m(m^2 - 2m + 2). \end{aligned} \tag{7.3.8}$$

(1) First consider the case where  $m = 2$ . Then  $q(\tilde{F}) = 1$  and we have from (7.3.7)

$$c_1^2(\tilde{F}) = 0$$

and from (7.3.8),

$$c_2(\tilde{F}) = 36.$$

Therefore, by Formula (7.3.4), we have

$$\begin{cases} h^{0,0}(\tilde{F}) = h^{2,2}(\tilde{F}) = 1, \\ h^{0,1}(\tilde{F}) = h^{1,0}(\tilde{F}) = h^{1,2}(\tilde{F}) = h^{2,1}(\tilde{F}) = q(\tilde{F}) = 1, \\ h^{0,2}(\tilde{F}) = h^{2,0}(\tilde{F}) = \frac{1}{12}(c_1^2(\tilde{F}) + c_2(\tilde{F})) - (1 - q(\tilde{F})) = 3, \\ h^{1,1}(\tilde{F}) = -\frac{1}{6}c_1^2(\tilde{F}) + \frac{5}{6}c_2(\tilde{F}) + 2q(\tilde{F}) = 32. \end{cases}$$

Note that in this case, if  $\tilde{F}$  is a minimal surface, then by the classification of surfaces (see [BHPV], Chapter VI),  $\tilde{F}$  has Kodaira dimension 1 and is a minimal properly elliptic surface, since we have  $c_1^2(\tilde{F}) = 0$  and  $c_2^2(\tilde{F}) = 36 > 24$ . Otherwise, let  $\tilde{F}'$  be a minimal model of  $\tilde{F}$ , then

$$c_1^2(\tilde{F}') > c_1^2(\tilde{F}) = 0,$$

and

$$c_1^2(\tilde{F}') + c_2(\tilde{F}') = c_1^2(\tilde{F}) + c_2(\tilde{F}) = 36 > 12,$$

it follows that  $\tilde{F}'$  and hence  $\tilde{F}$  is of general type by Proposition 7.1.2.

(2) Second, consider the case where  $m = 3$ , then  $q(\tilde{F}) = 2$  and we have from (7.3.7) that

$$c_1^2(\tilde{F}) = 117$$

implying that  $\tilde{F}$  is of general type by Proposition 7.1.1 and from (7.3.8),

$$c_2(\tilde{F}) = 135.$$

Therefore, by Formula (7.3.4), we have

$$\begin{cases} h^{0,0}(\tilde{F}) &= h^{2,2}(\tilde{F}) = 1, \\ h^{0,1}(\tilde{F}) &= h^{1,0}(\tilde{F}) = h^{1,2}(\tilde{F}) = h^{2,1}(\tilde{F}) = q(\tilde{F}) = 1, \\ h^{0,2}(\tilde{F}) &= h^{2,0}(\tilde{F}) = \frac{1}{12}(c_1^2(\tilde{F}) + c_2(\tilde{F})) - (1 - q(\tilde{F})) = 22, \\ h^{1,1}(\tilde{F}) &= -\frac{1}{6}c_1^2(\tilde{F}) + \frac{5}{6}c_2(\tilde{F}) + 2q(\tilde{F}) = 97. \end{cases}$$

(3) Consider the case where  $m > 3$  and  $m \not\equiv 0 \pmod{3}$ . Then  $q(\tilde{F}) = 1$ . From (7.3.7), we get  $c_1^2(\tilde{F}) = 3m(m-2)(5m-2)$ . Note that in our situation  $m \geq 4$ , hence  $c_1^2(\tilde{F}) \geq 3 \cdot 4 \cdot 2 \cdot 18 = 432 > 9$ , hence  $\tilde{F}$  is of general type by Proposition 7.1.1. In addition, from (7.3.8) we have  $c_2(\tilde{F}) = 9m(m^2 - 2m + 2)$ . Therefore, by Formula (7.3.4), we have

$$\begin{cases} h^{0,0}(\tilde{F}) &= h^{2,2}(\tilde{F}) = 1, \\ h^{0,1}(\tilde{F}) &= h^{1,0}(\tilde{F}) = h^{1,2}(\tilde{F}) = h^{2,1}(\tilde{F}) = q(\tilde{F}) = 1, \\ h^{0,2}(\tilde{F}) &= h^{2,0}(\tilde{F}) = \frac{1}{12}(c_1^2(\tilde{F}) + c_2(\tilde{F})) - (1 - q(\tilde{F})) \\ &= \frac{1}{2}m(m-1)(4m-5) \\ h^{1,1}(\tilde{F}) &= -\frac{1}{6}c_1^2(\tilde{F}) + \frac{5}{6}c_2(\tilde{F}) + 2q(\tilde{F}) \\ &= 5m^3 - 9m^2 + 13m + 2 \end{cases}$$

(4) Finally, we consider the case where  $m > 3$  and  $m \equiv 0 \pmod{3}$ . Then  $q(\tilde{F}) = 2$  and from (7.3.7), we have

$$c_1^2(\tilde{F}) = 3m(m-2)(5m-2) \geq 3 \cdot 6 \cdot (6-2) \cdot (5 \cdot 6 - 2) > 9,$$

hence  $\tilde{F}$  is of general type by Proposition 7.1.1. Moreover, from (7.3.8), we have  $c_2(\tilde{F}) = 9m(m^2 - 2m + 2)$ , hence by Formula (7.3.4), we have

$$\begin{cases} h^{0,0}(\tilde{F}) &= h^{2,2}(\tilde{F}) = 1, \\ h^{0,1}(\tilde{F}) &= h^{1,0}(\tilde{F}) = h^{1,2}(\tilde{F}) = h^{2,1}(\tilde{F}) = q(\tilde{F}) = 1, \\ h^{0,2}(\tilde{F}) &= h^{2,0}(\tilde{F}) = \frac{1}{12}(c_1^2(\tilde{F}) + c_2(\tilde{F})) - (1 - q(\tilde{F})) \\ &= \frac{1}{2}m(m-1)(4m-5) + 1 \\ h^{1,1}(\tilde{F}) &= -\frac{1}{6}c_1^2(\tilde{F}) + \frac{5}{6}c_2(\tilde{F}) + 2q(\tilde{F}) \\ &= 5m^3 - 9m^2 + 13m + 4 \end{cases}$$

Conclusion, for  $m \geq 3$ , the surface  $\tilde{F}$  is of general type. Furthermore, as  $m \rightarrow \infty$ , one can show that  $h^{0,2}, h^{1,1} \rightarrow \infty$  while other Hodge numbers remains 1 or 2. In addition, the Chern ration

$$\frac{c_1^2(\tilde{F})}{c_2(\tilde{F})} = \frac{3m(m-2)(5m-2)}{9m(m^2-2m+2)} \rightarrow \frac{5}{3} \quad \text{as } m \rightarrow \infty.$$

Note that we can also consider the signature  $\tau(\tilde{F})$  of  $\tilde{F}$  which depends only on the Chern numbers (see [BHPV], p.140)

$$\tau(\tilde{F}) = \frac{1}{3}(c_1^2(\tilde{F}) - 2c_2(\tilde{F})).$$

By (7.3.7) and (7.3.8), we have

$$\tau(\tilde{F}) = -m^3 - 8m,$$

hence  $\tau(\tilde{F})$  is always negative.

**Example 7.3.6.** Now we consider line arrangements which arise from restriction of higher dimensional hyperplane arrangements. The braid arrangement in  $\mathbb{P}^n$  is given by

$$\mathfrak{B}_n : \prod_{0 \leq i < j \leq n} (x_i - x_j) = 0,$$

consisting of  $\binom{n+1}{2}$  hyperplanes. Let  $E \subseteq \mathbb{P}^n$  be a *generic* projective plane and let  $\mathcal{A}_n = \mathfrak{B}_n|_E$  the restriction of  $\mathfrak{B}_n$  to  $E$ . Then  $\mathcal{A}_n$  is a line arrangements in the projective plane with only nodes and triple points such that

$$d = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

and

$$t_3 = \binom{n+1}{3}.$$

Indeed, any triple points of  $\mathcal{A}_n$  corresponds to the intersection of exactly three hyperplanes in  $\mathfrak{B}_n$ , which is thus of the form  $\{x_{i_1} = x_{i_2} = x_{i_3}\}$  for some  $i_1 < i_2 < i_3$ . Hence

$$t_3 = \#\{(i_1, i_2, i_3) : i_1, i_2, i_3 \in [0, n], i_1 < i_2 < i_3\} = \binom{n+1}{3}.$$

From Remark 6.3.6, we have

$$2t_2 + 6t_3 = d(d-1) = \binom{n+1}{2} \left( \binom{n+1}{2} - 1 \right),$$

hence

$$t_2 = \frac{d(d-1)}{2} - 3t_3 = \frac{n^2(n^2-1)}{4}.$$

Note that if  $n \equiv 1 \pmod{3}$ , then  $d \equiv 1 \pmod{3}$ , otherwise  $3|d$ , so we consider the following two cases:

- (i) If  $n \not\equiv 1 \pmod{3}$ , we have  $3|d$ . Moreover, if  $n = 2, 3$ , then  $q(\tilde{F}) = 1$ , otherwise  $q(\tilde{F}) = 0$  by [MP]. In addition,

$$DCI_{2,d} = 0, \quad DCII_{2,d} = d - 1$$

and

$$DCII_{3,d} = -d, \quad DCIII_{3,d} = d - 4.$$

Hence,

$$c_1^2(\tilde{F}) = K_F^2 + \sum_r t_r DCI_{r,d} = \frac{1}{24}n(n+1)(n-2)(n-3)(3n^2 + 19n + 32),$$

so if  $n \geq 4$ ,  $c_1^2(\tilde{F}) > 9$  and thus  $\tilde{F}$  is of general type by Proposition 7.1.1. Moreover,

$$c_2(\tilde{F}) = \chi(\bar{F}) + \sum_r t_r DCII_{r,d} = \frac{1}{8}n(n+1)(n-2)(n^3 + 2n^2 - 3n - 12).$$

(ii) If  $n \equiv 1 \pmod{3}$ , then  $d \equiv 1 \pmod{3}$  and  $q(\tilde{F}) = 0$  by [MP]. In addition,

$$DCI_{2,d} = 0, \quad DCII_{2,d} = d - 1$$

and by Example 6.6.3, we have

$$DCII_{3,d} = -(d - 1), \quad DCII_{3,d} = d - 1.$$

Thus,

$$\begin{aligned} c_1^2(\tilde{F}) &= K_F^2 + \sum_r t_r DCI_{r,d} \\ &= \frac{1}{24}n(n+1) \left( 3n^2(n^2 - 15) + 2n(2n^2 - 21) + 188 \right) \end{aligned}$$

so if  $n \geq 4$ ,  $c_1^2(\tilde{F}) > 9$  and thus  $\tilde{F}$  is of general type by Proposition 7.1.1. Moreover,

$$c_2(\tilde{F}) = \chi(\bar{F}) + \sum_r t_r DCII_{r,d} = \frac{1}{8}n(n+1)(n^4 - 7n^2 - 2n + 20).$$

The concrete formulae for the Hodge numbers by applying (7.3.4) are left to the reader. The monodromy is described for instance in [MP]. For the Chern ratio, we have

$$\lim_{n \rightarrow \infty} \frac{c_1^2(\tilde{F})}{c_2(\tilde{F})} = 1.$$

## 7.4 Extensions of irrational pencils of the Milnor fiber

As we have mentioned above,  $\tilde{F}$  can be seen as a compactification of the Milnor fiber  $F$ . In Chapter 5, we have extensively investigated the topological and geometric properties of  $F$ , especially the cohomology jump loci as well as the irrational pencils. Recall also that some of the irrational pencils of  $F$  appearing are of the following form (see Proposition 5.2.4):

$g : F \rightarrow H$ , with  $H \subseteq \mathbb{C}^2$  is defined by  $H : G(u, v) = \prod_{i=1}^k (\alpha_i u + \beta_i v) = 1$ , where  $k \geq 3$  and  $(\alpha_i, \beta_i), i = 1, \dots, k$  are  $k$  distinct points in  $\mathbb{P}^1$ .

Let  $\bar{H} : G(u, v) + t^k = 0$  in  $\mathbb{P}^2$  be the smooth compactification of  $H$ , we ask whether  $g$  can be extended to a morphism  $\bar{g} : \tilde{F} \rightarrow \bar{H}$ .

Clearly,  $\bar{H}$  is a smooth complete curve of genus  $g(\bar{H}) = \frac{1}{2}(k-1)(k-2) \geq 1$ , so the universal cover of  $\bar{H}$  is either  $\mathbb{C}$  or the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Moreover, the morphism  $g$  can be seen as a rational map  $\bar{g} : \tilde{F} \dashrightarrow \bar{H}$ , for which the indeterminacy locus has codimension

at least 2, since  $\overline{H}$  is complete. Thus,  $\overline{g} : \widetilde{F} \dashrightarrow \overline{H}$  is defined except at finitely many points. Resolving the indeterminacies via successive blowups, we get a morphism  $\overline{g}^* : \widetilde{F}^* \rightarrow \overline{H}$ , where  $\widetilde{F}^*$  is obtained by successive blowups of  $\widetilde{F}$ . Just as the discussions in the proof of Theorem 4.1.1, we can show that the blowups are not needed, and hence  $\overline{g} : \widetilde{F} \rightarrow \overline{H}$  is already a morphism. Indeed, we also apply Hartog's theorem to prove this result: locally  $\overline{g}$  is a holomorphic map  $\overline{g} : B \setminus \{0\} \rightarrow \overline{H}$ , where  $B$  is a small ball in  $\mathbb{C}^2$ , which lifts to a holomorphic map  $\overline{g}' : B \setminus \{0\} \rightarrow \mathbb{D}$  since  $B \setminus \{0\}$  is simply connected. Hence by Hartog's theorem,  $\overline{g}'$  and thus  $\overline{g}$  extends to all of  $B$ .

Note also that a general fiber of  $\overline{g} : \widetilde{F} \rightarrow \overline{H}$  is connected since this is so for  $g : F \rightarrow H$ . So  $\overline{g}$  is an irrational pencil of  $\widetilde{F}$  if and only if  $g(\overline{H}) > 1$ , namely,  $k > 3$ . Recall also that always,  $k = 3, 4$ , it follows that  $\overline{g} : \widetilde{F} \rightarrow \overline{H}$  is an irrational pencil if and only if  $k = 4$ .

The only known case with  $k = 4$  is the Hesse arrangement given in Section 2.3.10. It is an open question whether this is indeed the only line arrangement giving a pencil with  $k = 4$  totally reducible fibers.

## 7.5 Curve arrangements

Some discussions about line arrangements in this Chapter as well as the previous one can be extended to curve arrangements in  $\mathbb{P}^2$ , namely, a finite number of projective plane curves. Algebraic surfaces associated to curves arrangements have also been extensively studied, see for instance [Rou] and [Ur]. We make such an exploration by giving some examples concerning the Chern numbers.

To avoid any confusion, we first fix notations. We denote a curve arrangement by  $\mathcal{C}$ :

$$\mathcal{C} = \{C_1, \dots, C_s\},$$

where  $C_i : F_i(x, y, z) = 0$  is a projective plane curve. Denote

$$Q = F_1 F_2 \cdots F_s$$

the defining equation for  $\mathcal{C}$  and  $V(Q) \subseteq \mathbb{P}^2$  is the union of curves in  $\mathcal{C}$ . Moreover, the Milnor fiber of  $\mathcal{C}$  is

$$\mathcal{F} : Q = 1$$

in  $\mathbb{C}^3$ . A natural compactification  $\overline{\mathcal{F}}$  of  $\mathcal{F}$  in  $\mathbb{P}^3$  is given by

$$\overline{\mathcal{F}} : Q(x, y, z) + t^{\deg Q} = 0,$$

Moreover, we can also consider other compactifications of  $\mathcal{F}$  in some weighted projective space  $\mathbb{P}(\underline{w})$  where  $\underline{w} = (w_0, w_1, w_2, w_3)$  is a sequence of weights. In order to get some ball quotients, making compactifications in weighted projective spaces is essential, since we need to avoid the appearance of rational curves, see Theorem 6.5.2, Remark 6.8.3 and Remark 7.5.3 below. For definitions and some basic properties of weighted projective varieties, we refer to [Dim92], Appendix B or [Dol] for a relatively complete treatment. In addition, weighted projective spaces are simplicial complete toric varieties, thus they share interesting properties as toric varieties, for instance, any Weil divisor is  $\mathbb{Q}$ -Cartier, see [CLS].

Denote  $\overline{\mathcal{F}}$  be a compactification of  $\mathcal{F}$  in some weighted projective space  $\mathbb{P}(\underline{w})$  such that  $\overline{\mathcal{F}}$  has only isolated singularities. As in the case of line arrangements, we also need to resolve the singularities of  $\overline{\mathcal{F}}$ , for instance using Theorem 6.5.2, which we have used many times in Chapter 6 and will continue to use below.

Let  $\pi : \tilde{\mathcal{F}} \rightarrow \bar{\mathcal{F}}$  be a minimal resolution of  $\bar{\mathcal{F}}$ . Then  $\tilde{\mathcal{F}}$  is a smooth projective surface. We will do the same things for  $\tilde{\mathcal{F}}$  as for the surfaces associated to line arrangements, e.g., computing the Chern numbers and Hodge numbers. To achieve this goal, some singularity theory is indispensable.

Consider a plane curve germ with isolated singularity

$$(C, 0) : g(u, v) = g_r(u, v) + g_{r+1}(u, v) + \cdots + g_l(u, v)$$

where  $(u, v)$  are local coordinates in  $\mathbb{C}^2$  around 0 and  $g_i$  is a homogeneous polynomial of degree  $i$  for  $i = r, \dots, l$ . Moreover,  $r \geq 1$  and  $g_r \neq 0$ . We say that 0 is an ordinary  $r$ -multiple point if  $C$  has exactly  $r$  branches at 0 (necessarily smooth) with distinct tangent lines. By Hensel's Lemma, see [Dim87], p.172, this is equivalent to  $g_r$  being a product of distinct  $r$  linear forms. Typical examples are curves of the form  $V(Q) \subseteq \mathbb{P}^2$ , where  $Q$  is the defining equation of a line arrangement.

We say that  $(C, 0)$  is a strict ordinary  $r$ -multiple point if there is a local change of local coordinates, say  $(u, v) \mapsto (u', v')$ , at the origin of  $\mathbb{C}^2$  such that the equation of  $(C, 0)$  becomes  $g'(u', v') = 0$  for some homogeneous polynomial  $g'$ . It is clear that the results obtained in Chapter 6 about the computations of Chern numbers extend from line arrangements to curves in  $\mathbb{P}^2$  having only strict ordinary  $r$ -multiple points for any  $r$ . In fact, a strict ordinary  $r$ -multiple point is an ordinary  $r$ -multiple point which is in the same time weighted homogeneous. Note also that one can resolve any strict ordinary  $r$ -multiple point by using Theorem 6.5.2.

Using [Dim87], Proposition 7.38, p.116-117, it follows that for  $r = 2, 3, 4$ , any ordinary  $r$ -multiple point is in fact a strict ordinary  $r$ -multiple point. For  $r = 5$ , this is no longer true, for instance  $g(u, v) = u^5 + v^5 + u^3v^3$  is an ordinary 5-multiple point, but not a strict ordinary 5-multiple point.

Nevertheless, if  $\mathcal{P}_{P,Q} = \{\lambda P + \mu Q : (\lambda, \mu) \in \mathbb{P}^1\}$  is a pencil of degree  $e$  plane curves, such that the base locus  $V(P, Q) \subseteq \mathbb{P}^2$  consists of exactly  $e^2$  points, namely,  $V(P)$  and  $V(Q)$  intersect transversely only at smooth points, then at any intersection point  $p \in V(P, Q)$ , a union of  $r$  curves  $\lambda_i P + \mu_i Q$  in  $\mathcal{P}_{P,Q}$  will produce a strict ordinary  $r$ -multiple point for any  $r > 0$ . Indeed, at such a point, the functions germs corresponding  $P$  and  $Q$  can be taken as the good coordinates  $u', v'$  in the above definition. The polynomial  $g'(u', v')$  is just a product of linear forms  $\lambda_i u' + \mu_i v'$ .

As in Chapter 6, when considering Chern numbers, we denote  $DCI_{r,d}, DCII_{r,d}$  for the differences of Chern numbers after resolving a strict ordinary  $r$ -multiple point.

**Example 7.5.1.** Consider the curve arrangements consisting of all of singular curves in Section 2.3.10 (9). Namely,

$$\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5\}$$

where

$$\begin{cases} C_1 : & Q_1 = 0 \\ C_2 : & Q_2 = 0 \\ C_3 : & Q_1 - Q_2 = 0 \\ C_4 : & Q_1 - t_1 Q_2 = 0 \\ C_5 : & Q_1 - t_2 Q_2 = 0 \end{cases}$$

and  $Q_1 = 3xyz + y^3 + z^3, Q_2 = 3xyz + x^3 + z^3$  and  $t_1, t_2$  are the two roots of  $t^2 - 3t + 1 = 0$ . Let

$$Q = Q_1 Q_2 (Q_1 - Q_2) (Q_1 - t_1 Q_2) (Q_1 - t_2 Q_2)$$

be the defining equation of  $\mathcal{C}$  and  $\mathcal{F} : Q = 1$  be the Milnor fiber in  $\mathbb{C}^3$ .

Let

$$\overline{\mathcal{F}} : Q(x, y, z) + t^{15} = 0$$

be the compactification of  $\overline{\mathcal{F}}$  in  $\mathbb{P}^3$ .

Note that  $C_1, C_2$  are two nodal cubics intersecting transversely in 9 points,  $C_3$  is a union of three concurrent lines and each  $C_4, C_5$  is a union of three lines with 3 nodes. Thus  $V(Q)$  is a union of 2 nodal cubics and 9 lines. In addition, the line arrangement defined by  $(Q_1 - Q_2)(Q_1 - t_1 Q_2)(Q_1 - t_2 Q_2) = 0$  has 10 triple points and 6 nodes. See also [Bai16].

Let  $t_r$  be the number of singular points of multiplicity  $r$ . Then

$$t_2 = 8, \quad t_3 = 1, \quad t_5 = 9$$

and  $t_r = 0$  for other  $r$ 's. Moreover, all  $r$ -multiple points are strict ordinary  $r$ -multiple points as discussed above. Therefore, using Theorem 6.5.2, we get a minimal resolution

$$\pi : \tilde{\mathcal{F}} \rightarrow \overline{\mathcal{F}}.$$

To be consistent with the discussions in Chapter 6, set  $d = 15$ .

We divide our discussion into several steps:

1. Compute Chern numbers of  $\overline{\mathcal{F}}$ . By (6.3.2) and (6.3.3) we have

$$K_{\overline{\mathcal{F}}}^2 = d(d-4)^2 = 1815,$$

and

$$\chi(\overline{\mathcal{F}}) = d(d^2 - 4d + 6) - (d-1) \sum_r t_r (r-1)^2 = 381.$$

2. By Example 6.6.1, we have

$$DCI_{2,15} = 0, \quad DCII_{2,15} = d - 1 = 14.$$

By Example 6.6.2, we have

$$DCI_{3,15} = -15(3-2)^2 = -15, \quad DCII_{3,15} = -1 + 3(5+2-3) = 11$$

and

$$DCI_{5,15} = -15(5-2)^2 = -135, \quad DCII_{5,15} = -1 + 5(3+2-5) = -1.$$

Thus,

$$\begin{aligned} c_1^2(\tilde{\mathcal{F}}) &= K_{\overline{\mathcal{F}}}^2 + t_2 \cdot DCI_{2,15} + t_3 \cdot DCI_{3,15} + t_5 \cdot DCI_{5,15} \\ &= 1815 - 15 - 9 \cdot 135 \\ &= 585 \end{aligned}$$

and

$$\begin{aligned} c_2(\tilde{\mathcal{F}}) &= \chi(\overline{\mathcal{F}}) + t_2 \cdot DCII_{2,15} + t_3 \cdot DCII_{3,15} + t_5 \cdot DCII_{5,15} \\ &= 381 + 8 \cdot 14 + 11 + 9 \cdot (-1) \\ &= 495. \end{aligned}$$

We see that  $\tilde{\mathcal{F}}$  is of general type by proposition 7.1.1.

3. By the monodromy computed in [Bai16], we have

$$q(\tilde{\mathcal{F}}) = 6.$$

Thus, applying Formula (7.3.4), we obtain

$$\begin{cases} h^{0,0}(\tilde{\mathcal{F}}) &= h^{2,2}(\tilde{\mathcal{F}}) = 1, \\ h^{0,1}(\tilde{\mathcal{F}}) &= h^{1,0}(\tilde{\mathcal{F}}) = h^{1,2}(\tilde{\mathcal{F}}) = h^{2,1}(\tilde{\mathcal{F}}) = q(\tilde{\mathcal{F}}) = 6, \\ h^{0,2}(\tilde{\mathcal{F}}) &= h^{2,0}(\tilde{\mathcal{F}}) = \frac{1}{12}(c_1^2(\tilde{\mathcal{F}}) + c_2(\tilde{\mathcal{F}})) - (1 - q(\tilde{\mathcal{F}})) \\ &= 95, \\ h^{1,1}(\tilde{\mathcal{F}}) &= -\frac{1}{6}c_1^2(\tilde{\mathcal{F}}) + \frac{5}{6}c_2(\tilde{\mathcal{F}}) + 2q(\tilde{\mathcal{F}}) \\ &= 327. \end{cases}$$

This example is of some interest since  $b_1(\tilde{\mathcal{F}}) = 12$ , which is larger than the first Betti numbers of known surfaces associated to line arrangements. See also Section 2.3.10 for the monodromy of line arrangements.

In a similar fashion, one can consider the curve arrangement in Section 2.3.10 (10).

By Theorem 6.8.1, we cannot get any ball quotient from line arrangements; a deeper reason for such a failure is discussed in Remark 6.8.3. Our hope is that by considering curve arrangements and avoiding the appearance of rational curves during the desingularization process, some ball quotients can be obtained.

**Example 7.5.2.** Let  $\mathcal{C} : Q = 0$  be a curve arrangement in  $\mathbb{P}^2$ , with

$$Q = (x^3 + y^3 + z^3)(x^3 + y^3 + z^3 + xyz)(x^3 + y^3 + z^3 - xyz).$$

Then  $\mathcal{C}$  consists of 3 curves in the pencil

$$\mathcal{P} = \{\lambda(x^3 + y^3 + z^3) + \mu(x^3 + y^3 + z^3 + xyz) : (\lambda, \mu) \in \mathbb{P}^1\}.$$

The Milnor fiber is given by

$$\mathcal{F} : Q = 1.$$

Let

$$\overline{\mathcal{F}} : Q(x, y, z) + w^3 = 0$$

be a compactification of  $\mathcal{F}$  in the weighted projective space  $\mathbb{P} = \mathbb{P}(1, 1, 1, 3)$ , namely,  $\deg x = \deg y = \deg z = 1$  and  $\deg w = 3$ . Let

$$\pi : \tilde{\mathcal{F}} \rightarrow \overline{\mathcal{F}}.$$

be the minimal resolution.

Note that  $\mathbb{P}$  has only one singular point  $(0, 0, 0, 1)$  which does not belong to  $\overline{\mathcal{F}}$ , hence  $\overline{\mathcal{F}}$  has 9 singular points all of which are strict ordinary triple points, namely,  $t_3 = 9$  and  $t_r = 0$  for other  $r$ 's.

By applying Theorem 6.5.2, one sees easily that the exceptional fiber of a singular point is an elliptic curve (corresponding to  $r = 3, d = 3$ ) with self-intersection  $-3$ , see also Example 6.6.2. Recall also that

$$DCI_{3,3} = -3, \quad DCII_{3,3} = -1$$

To compute the Chern numbers of  $\tilde{\mathcal{F}}$ , we still need to compute  $K_{\overline{\mathcal{F}}}^2$  and  $\chi(\overline{\mathcal{F}})$ .

First, we need to show that  $K_{\overline{\mathcal{F}}}^2$  makes sense. Indeed,  $\mathbb{P}$  is a simplicial toric variety, hence any Weil divisor is  $\mathbb{Q}$ -Cartier, namely, some positive multiple of it is Cartier; moreover,  $\overline{\mathcal{F}}$  and  $\mathbb{P}$  are both normal varieties and the adjunction formula holds, i.e.,

$$K_{\overline{\mathcal{F}}} = (K_{\mathbb{P}} + \overline{\mathcal{F}})|_{\overline{\mathcal{F}}}.$$

because this equality holds on the smooth locus of  $\overline{\mathcal{F}}$ . Since  $K_{\mathbb{P}}$  and  $\overline{\mathcal{F}}$  are both  $\mathbb{Q}$ -Cartier divisors on  $\mathbb{P}$ ,  $K_{\overline{\mathcal{F}}}$  is  $\mathbb{Q}$ -Cartier on  $\overline{\mathcal{F}}$ , thus  $K_{\overline{\mathcal{F}}}^2$  makes sense in the following way: if  $A > 0$  is a positive integer such that  $AK_{\overline{\mathcal{F}}}$  is a Cartier divisor on  $\overline{\mathcal{F}}$ , then

$$K_{\overline{\mathcal{F}}}^2 = \frac{1}{A^2}(AK_{\overline{\mathcal{F}}})^2.$$

Now we continue to compute the Chern numbers. Let  $p : \mathbb{P}^3 \rightarrow \mathbb{P} = \mathbb{P}(1, 1, 1, 3)$  be the natural ramified covering given by

$$(x, y, z, t) \mapsto (x, y, z, w = t^3).$$

Then  $S = p^{-1}(\overline{\mathcal{F}})$  is given by

$$S : Q(x, y, z) + t^9 = 0$$

in  $\mathbb{P}^3$ . Moreover,  $p$  has degree 3 branched over  $B = \{w = 0\} \cap \overline{\mathcal{F}}$ . Let

$$R = p^{-1}(B) = \{t = 0\} \cap S.$$

Then we have

$$K_S = p^*K_{\overline{\mathcal{F}}} + 2R.$$

Note that  $p^*K_{\overline{\mathcal{F}}}$  is a  $\mathbb{Q}$ -Cartier divisor on  $S$  since  $K_{\overline{\mathcal{F}}}$  is a  $\mathbb{Q}$ -Cartier divisor on  $\overline{\mathcal{F}}$ ; in addition,  $R$  is a Cartier divisor on  $S$  since  $R = H|_S$  where  $H = \{t = 0\} \subseteq \mathbb{P}^3$  is a hyperplane which is a Cartier divisor on  $\mathbb{P}^3$ . Since  $K_S \sim (9 - 4)H|_S = 5H|_S$  and  $R \sim H|_S$ , it follows that

$$p^*K_{\overline{\mathcal{F}}} = K_S - 2R \sim 3H|_S.$$

hence by the projection formula

$$3K_{\overline{\mathcal{F}}}^2 = (\deg p)K_S^2 = p^*K_{\overline{\mathcal{F}}}^2 = 9H \cdot H \cdot S = 81,$$

i.e.,

$$K_{\overline{\mathcal{F}}}^2 = 27.$$

Recall that  $p$  is branched over  $B$  and  $B$  can be identified with the plane curve  $V(Q)$  in  $\mathbb{P}^2$ . Thus,

$$\chi(S) = 3\chi(\overline{\mathcal{F}}) - 2\chi(V(Q)),$$

so

$$\chi(\overline{\mathcal{F}}) = \frac{1}{3}(\chi(S) + 2\chi(V(Q))).$$

By the computations in Example 6.3.5, we obtain

$$\chi(S) = 9(9^2 - 4 \cdot 9 + 6) - (9 - 1) \cdot 9 \cdot (3 - 1)^2 = 171,$$

and

$$\chi(V(Q)) = 9(3 - 9) + 9 \cdot (3 - 1)^2 = -18,$$

therefore,

$$\chi(\overline{\mathcal{F}}) = \frac{1}{3}(171 - 2 \cdot 18) = 45.$$

It follows that

$$\begin{aligned} c_1^2(\tilde{\mathcal{F}}) &= K_{\tilde{\mathcal{F}}}^2 + t_3 \cdot DCI_{3,3} \\ &= 27 + 9 \cdot (-3) \\ &= 0. \end{aligned}$$

Moreover,

$$\begin{aligned} c_2(\tilde{\mathcal{F}}) &= \chi(\overline{\mathcal{F}}) + t_3 \cdot DCII_{3,3} \\ &= 45 + 9 \cdot (-1) \\ &= 36. \end{aligned}$$

Therefore,  $\tilde{\mathcal{F}}$  is not a ball quotient. In fact, we cannot even determine whether  $\tilde{\mathcal{F}}$  is of general type.

**Remark 7.5.3.** Note that in the above example, all the exceptional curves in the resolution  $\pi : \tilde{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  are elliptic curves. It turns out a ball quotient cannot contain any elliptic curves, hence we are doomed to fail to obtain a ball quotient from the beginning. Indeed, any morphism from an elliptic curve to a ball quotient lifts to a morphism from  $\mathbb{C}$  to a ball, hence constant. Compare Remark 6.8.3.

However, the above example illustrates a process to compute the Chern numbers for a hypersurface in a weighted projective space, which can be applied to other situations where the exceptional curves are neither rational curves nor elliptic curves. Note also in order to avoid rational curves or elliptic curves being exceptional curves, it is necessary, in the example below, to compactify the Milnor fiber in a weighted projective space, rather than the usual projective space  $\mathbb{P}^3$ .

**Example 7.5.4.** Let  $P_1, P_2 \in \mathbb{C}[x, y, z]$  be two homogeneous polynomials of degree  $e \geq 4$  such that  $V(P_1), V(P_2)$  are smooth curves in  $\mathbb{P}^2$ , intersecting transversely at  $e^2$  points. In particular, a generic fiber in the pencil  $\mathcal{P}_{P_1, P_2} = \{\lambda P_1 + \mu P_2 : (\lambda, \mu) \in \mathbb{P}^1\}$  is smooth. Choose  $(e - 2)$  generic fibers  $P_3, \dots, P_e$  in  $\mathcal{P}_{P_1, P_2}$  and let

$$Q = P_1 P_2 \cdots P_e.$$

Then  $Q = 0$  gives an arrangement of  $e$  curves in the pencil  $\mathcal{P}_{P_1, P_2}$ . The Milnor fiber is given by

$$\mathcal{F} : Q = 1.$$

Let

$$\overline{\mathcal{F}} : Q(x, y, z) + w^e = 0$$

be a compactification of  $\mathcal{F}$  in the weighted projective space  $\mathbb{P}(1, 1, 1, e)$ , namely,  $\deg x = \deg y = \deg z = 1$  and  $\deg w = e$ . Let

$$\pi : \tilde{\mathcal{F}} \rightarrow \overline{\mathcal{F}}.$$

be the minimal resolution given by Theorem 6.5.2. In this case, the exceptional fiber over a singular point of  $\overline{\mathcal{F}}$  is a smooth curve of genus  $1/2(e - 1)(e - 2) > 1$ , and therefore, rational curves and elliptic curves are avoided.

Just as in Example 7.5.2, let  $p : \mathbb{P}^3 \rightarrow \mathbb{P}(1, 1, 1, e)$  be the natural ramified covering given by

$$(x, y, z, t) \mapsto (x, y, z, w = t^e).$$

Then  $S = p^{-1}(\overline{\mathcal{F}})$  is given by

$$S : Q(x, y, z) + t^{e^2} = 0$$

in  $\mathbb{P}^3$ . Moreover,  $p$  has degree  $e$  branched over  $B = \{w = 0\} \cap \overline{\mathcal{F}}$ . Let

$$R = p^{-1}(B) = \{t = 0\} \cap S.$$

Then we have

$$K_S = p^*K_{\overline{\mathcal{F}}} + (e - 1)R.$$

Therefore, by the projection formula,

$$\begin{aligned} eK_{\overline{\mathcal{F}}}^2 &= (\deg p)K_{\overline{\mathcal{F}}}^2 \\ &= p^*K_{\overline{\mathcal{F}}}^2 \\ &= (K_S - (e - 1)R)^2. \end{aligned}$$

Note that  $K_S \sim (e^2 - 4)H|_S$  and  $R \sim H|_S$  where  $H$  is a hyperplane section of  $\mathbb{P}^3$ . Hence,

$$\begin{aligned} eK_{\overline{\mathcal{F}}}^2 &= (K_S - (e - 1)R)^2 \\ &= (e^2 - e - 3)^2 H|_S^2 \\ &= (e^2 - e - 3)^2 H \cdot H \cdot S \\ &= e^2(e^2 - e - 3)^2. \end{aligned}$$

It follows that

$$K_{\overline{\mathcal{F}}}^2 = e(e^2 - e - 3)^2.$$

Note that  $V(Q)$  has  $e^2$  singular points of multiplicity  $e$ , namely,  $t_e = e^2$  and  $t_r = 0$  for other  $r$ 's. Therefore, for the Euler characteristic, similar to the computations in Example 7.5.2, we have

$$\chi(S) = e^2(e^4 - 4e^2 + 6) - (e^2 - 1) \cdot e^2 \cdot (e - 1)^2 = e^2(2e^3 - 4e^2 - 2e + 7).$$

In addition,

$$\chi(V(Q)) = e^2(3 - e^2) + e^2(e - 1)^2 = 2e^2(2 - e).$$

It follows from  $\chi(S) = e\chi(\overline{\mathcal{F}}) - (e - 1)\chi(V(Q))$  that

$$\begin{aligned} \chi(\overline{\mathcal{F}}) &= \frac{1}{e} \left( \chi(S) + (e - 1)\chi(V(Q)) \right) \\ &= e(2e^3 - 6e^2 + 4e + 3). \end{aligned}$$

By Example 6.6.2, we have

$$DCI_{e,e} = -e(e - 2)^2, \quad DCII_{e,e} = -1 + e(3 - e).$$

It follows that

$$\begin{aligned} c_1^2(\tilde{\mathcal{F}}) &= K_{\overline{\mathcal{F}}}^2 + t_3 \cdot DCI_{3,3} \\ &= e(e^2 - e - 3)^2 - e^3(e - 2)^2 \\ &= e(2e^3 - 9e^2 + 6e + 9) \\ &= e(e - 3)(2e^2 - 3e - 3). \end{aligned}$$

Hence, when  $e \geq 4$ ,  $c_1^2(\tilde{\mathcal{F}}) \geq 68 > 9$ , hence  $\tilde{\mathcal{F}}$  is of general type by Proposition 7.1.1. Moreover,

$$\begin{aligned} c_2(\tilde{\mathcal{F}}) &= \chi(\overline{\mathcal{F}}) + t_3 \cdot DCII_{3,3} \\ &= e(2e^3 - 6e^2 + 4e + 3) + e^2(-1 + e(3 - e)) \\ &= e(e^3 - 3e^2 + 3e + 3). \end{aligned}$$

Therefore, the Miyaoka-Yau number is

$$\begin{aligned} MY(\tilde{\mathcal{F}}) &= 3c_2(\tilde{\mathcal{F}}) - c_1^2(\tilde{\mathcal{F}}) \\ &= e(e^3 + 3e) > 0. \end{aligned}$$

It follows from Miyaoka-Yau inequality that  $\tilde{\mathcal{F}}$  can never be a ball quotient!

**Remark 7.5.5.** Note that Example 7.5.4 can also be extended in the following way: let  $\mathcal{P}_{P_1, P_2}$  be a pencil consisting of degree  $e$  projective plane curves such that  $V(P_1), V(P_2)$  intersect transversely at  $e^2$  smooth points of each. Then we choose  $k$  general fibers  $P_1, P_2, \dots, P_k$  in  $\mathcal{P}_{P_1, P_2}$  and give a similar construction for  $Q = P_1 P_2 \cdots P_k$  in which the compactification is given by  $Q(x, y, z) + w^k = 0$  in  $\mathbb{P}(1, 1, 1, e)$ . Here  $k$  can be different from  $e$  and in addition,  $V(P_1), V(P_2)$  can be singular.

Recall also that a net structure on a line arrangement  $\mathcal{A}$  in some sense make  $\mathcal{A}$  split into product of  $k \geq 3$  fibers in a pencil  $\mathcal{P}_{Q_1, Q_2}$ ; see Chapter 5 of this thesis. Thus our construction mentioned above can also be seen as a generalization of net structures on line arrangements.

However, since we try to get ball quotients here and to this end, rational curves and elliptic curves should be avoided during the desingularization process. Therefore, some restrictive conditions must be put on  $P_1$  and  $P_2$  if they are singular. To get rid of such complicity, we just consider the case  $V(P_1), V(P_2)$  are smooth curves in the example below.

**Example 7.5.6.** Let  $P_1, P_2 \in \mathbb{C}[x, y, z]$  be two homogeneous polynomials of degree  $e \geq 2$  such that  $V(P_1), V(P_2)$  are smooth curves in  $\mathbb{P}^2$ , intersecting transversely at  $e^2$  points. In particular, a generic fiber in the pencil  $\mathcal{P}_{P_1, P_2} = \{\lambda P_1 + \mu P_2 : (\lambda, \mu) \in \mathbb{P}^1\}$  is smooth. Choose  $k - 2 \geq 2$  generic fibers  $P_3, \dots, P_k$  in  $\mathcal{P}_{P_1, P_2}$  and let

$$Q = P_1 P_2 \cdots P_k.$$

Then  $Q = 0$  gives an arrangement of  $e$  curves in the pencil  $\mathcal{P}_{P_1, P_2}$ . The Milnor fiber is given by

$$\mathcal{F} : Q = 1.$$

Let

$$\overline{\mathcal{F}} : Q(x, y, z) + w^k = 0$$

be a compactification of  $\mathcal{F}$  in the weighted projective space  $\mathbb{P}(1, 1, 1, e)$ , namely,  $\deg x = \deg y = \deg z = 1$  and  $\deg w = e$ . Let

$$\pi : \tilde{\mathcal{F}} \rightarrow \overline{\mathcal{F}}.$$

be the minimal resolution given by Theorem 6.5.2. In this case, the exceptional fiber over a singular point of  $\overline{\mathcal{F}}$  is a smooth curve of genus  $1/2(k-1)(k-2) > 1$ , and therefore, appearance rational curves and elliptic curves will not occur.

The computations below are analogues as those in Example 7.5.4, what is why we will omit some details below. Let  $p : \mathbb{P}^3 \rightarrow \mathbb{P}(1, 1, 1, e)$  be the natural ramified covering given by

$$(x, y, z, t) \mapsto (x, y, z, w = t^e).$$

Then  $S = p^{-1}(\overline{\mathcal{F}})$  is given by

$$S : Q(x, y, z) + t^{ke} = 0$$

in  $\mathbb{P}^3$ . Moreover,  $p$  has degree  $e$  branched over  $B = \{w = 0\} \cap \overline{\mathcal{F}}$ . Let

$$R = p^{-1}(B) = \{t = 0\} \cap S.$$

Then we have

$$K_S = p^*K_{\overline{\mathcal{F}}} + (e - 1)R.$$

Therefore, by the projection formula,

$$eK_{\overline{\mathcal{F}}}^2 = p^*K_{\overline{\mathcal{F}}}^2 = (K_S - (e - 1)R)^2.$$

Note that  $K_S \sim (ke - 4)H|_S$  and  $R \sim H|_S$  where  $H$  is a hyperplane section of  $\mathbb{P}^3$ . Hence,

$$eK_{\overline{\mathcal{F}}}^2 = (K_S - (e - 1)R)^2 = ke(ke - e - 3)^2.$$

It follows that

$$K_{\overline{\mathcal{F}}}^2 = k(ke - e - 3)^2.$$

Note that  $V(Q)$  has  $e^2$  singular points of multiplicity  $k$ , namely,  $t_k = e^2$  and  $t_r = 0$  for other  $r$ 's. Therefore, for the Euler characteristic, similar to the computations in Example 7.5.2, we have

$$\begin{aligned} \chi(S) &= ke(k^2e^2 - 4ke + 6) - (ke - 1) \cdot e^2 \cdot (k - 1)^2 \\ &= e(2k^2e^2 - ke^2 - 3k^2e - 2ke + 6k + e). \end{aligned}$$

In addition,

$$\chi(V(Q)) = ke(3 - ke) + e^2(k - 1)^2 = 2e^2(2 - e) = e(-2ke + 3k + e).$$

It follows from  $\chi(S) = e\chi(\overline{\mathcal{F}}) - (e - 1)\chi(V(Q))$  that

$$\begin{aligned} \chi(\overline{\mathcal{F}}) &= \frac{1}{e} \left( \chi(S) + (e - 1)\chi(V(Q)) \right) \\ &= 2k^2e^2 - 3ke^2 - 3k^2e + 3ke + e^2 + 3k. \end{aligned}$$

By Example 6.6.2, we have

$$DCI_{k,k} = -k(k - 2)^2, \quad DCII_{k,k} = -1 + k(3 - k).$$

It follows that

$$\begin{aligned} c_1^2(\tilde{\mathcal{F}}) &= K_{\overline{\mathcal{F}}}^2 + t_3 \cdot DCI_{3,3} \\ &= k(ke - e - 3)^2 - e^2k(k - 2)^2 \\ &= k(e - 3)(2ke - 3e - 3). \end{aligned}$$

Hence, when  $e \geq 4$  and  $k \geq 4$ ,  $c_1^2(\tilde{\mathcal{F}}) \geq 68 > 9$ , hence  $\tilde{\mathcal{F}}$  is of general type by Proposition 7.1.1. Moreover,

$$\begin{aligned} c_2(\tilde{\mathcal{F}}) &= \chi(\overline{\mathcal{F}}) + t_3 \cdot DCII_{3,3} \\ &= 2k^2e^2 - 3ke^2 - 3k^2e + 3ke + e^2 + 3k + e^2(-1 + k(3 - k)) \\ &= k^2e^2 - 3k^2e + 3ke + 3k. \end{aligned}$$

Therefore, the Miyaoka-Yau number is

$$\begin{aligned} MY(\tilde{\mathcal{F}}) &= 3c_2(\tilde{\mathcal{F}}) - c_1^2(\tilde{\mathcal{F}}) \\ &= ke(ke - 3k + 3e + 3) \\ &= ke \left( (e - 3)k + 3e + 3 \right) > 0 \end{aligned}$$

for  $k \geq 4$ . It follows from Miyaoka-Yau inequality that  $\tilde{\mathcal{F}}$  can never be a ball quotient!

## Bibliography

- [Ara] D. Arapura, *Geometry of cohomology support loci for local systems I*, J. Algebraic Geom. **6**(1997), 563–597. 1.3.2, 1.3.3, 2.2.2, 2.2.5, 3.1
- [Art] E. Artal-Bartolo, *Combinatorics and topology of line arrangements in the complex projective plane*, Proc. Amer. Math. Soc. **121**(2) (1994), 385–390. 7.2.2
- [Bai14] P. Bailet, *On the Monodromy of Milnor Fibers of Hyperplane Arrangements*, Canad. Math. Bull. **57** (2014), 697–707. 4, 7, 8
- [Bai16] P. Bailet, *On two pencils of cubic curves*, preprint. i, 9, 10, 7.5.1, 3
- [BHPV] W. Barth, K. Hulek, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Second enlarged edition, Springer 2004. 1.3.3, 4, 4.1.7, 1, 2, 4.4.1, 4.4.5, 4.4.6, ii, iv, 6.3.2, 6.8.2, 7.1, 7.1, 7.1, 7.1.4, 7.3.1, 1, 7.3.5
- [Bau] I. Bauer, *Irrational pencils on non-compact algebraic manifolds*, Inter. J. Math. **8**(1997), 441–450. 1.3.2, 2.2.5, 3.1, 4.1.8, 4.1.11, 4.2.4
- [BDS] N. Budur, A. Dimca, M. Saito, *First Milnor cohomology of hyperplane arrangements*, in: “Topology of Algebraic Varieties and Singularities”, Contemporary Mathematics **538**(2011), 279–292. 2
- [BS] N. Budur, M. Saito, *Jumping coefficients and spectrum of a hyperplane arrangement*, Math. Ann. **347** (2010), 545–579. 1.3.6
- [BT] R. Bott, L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics **82**, Springer-Verlag, 1982. 6.1.1, 6.3.1
- [Bri] E. Brieskorn, *Sur les groupes de tresses*, d’après V. I. Arnol’d, Sem. Bourbaki 24<sup>e</sup> année (1971/72), **401**, Lecture Notes in Math., **317**, Springer. 2.2.4, 4
- [BW] N. Budur, B. Wang, *Cohomology jump loci of differential graded Lie algebras*, Compos. Math. **151** (2015), no. 8, 1499–1528. 3.1
- [BWens] N. Budur, B. Wang, *Cohomology jump loci of quasi-projective varieties*, Ann. Sci. École Norm. Sup. (4) **48** (2015), no. 1, 227–236. 2.2.5

- [Cartan] E. Cartan, *Sur les nombres de Betti des espaces de groupes close*, C.R. Acad. Sci. Paris **187** (1928), 196—197. 2.2.4
- [Cat] F. Catanese, *Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations*, Invent. Math. **104**(1991), no. 2, 263–289(with an appendix by A. Beauville). 2.2.5, 4.1.11, 4.2.1, 2
- [CS95] D. Cohen, A. Suciu, *On Milnor fibrations of arrangements*, J. London Math. Soc. **51** (1) (1995), 105—119. 2.3.2, 1, 3, 6, 7, i, 7.3
- [CS99] D. Cohen, A. Suciu, *Characteristic varieties of arrangements*, Math. Proc. Cambridge Phil. Soc. **127** (1999), no. 1, 33–53. 3.1
- [CLS] D. Cox, J. Little, H. Schenck, *Toric Varieties*, Grad. Stud. Math., vol. **124**, Amer. Math. Soc., Providence, 2011. 7.5
- [Deb] O. Debarre, *Introduction to Mori theory*, notes in <http://www.math.ens.fr/debarre/M2.pdf>. 2010. iv, 6.2.2, 6.2.3, 6.3.3
- [Del71] P. Deligne, *Théorie de Hodge II*, Publ. Math. I.H.E.S **40**, 1971, 5–58. 1.1
- [Del74] P. Deligne, *Théorie de Hodge III*, Publ. Math. I.H.E.S **44**, 1974, 5–77. 1.1
- [DGMS] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274. 2.2.4
- [Dim87] A. Dimca, *Topics on Real and Complex Singularities*, Vieweg Advanced Lectures in Math., Vieweg 1987. 7.5
- [Dim92] A. Dimca, *Singularities and Topology of Hypersurfaces*, Springer-Verlag, 1992. 1.1, 2.1, 2.1.9, 2.3.2, 2.3.6, 5, 6.5, 6.5.1, 7.5
- [Dim08] A. Dimca, *On the isotropic subspace theorem*, Bull. Math. Soc. Sci. Math. Roumanie tome **51(99)** no. 4, 2008, 307—324 2.2.5, 2.2.5, 4.2.4, 4.2
- [Dim10] A. Dimca, *Characteristic varieties and logarithmic differential 1-forms*, Compositio Math. **146** (2010), 129–144. 1.3.2, 3.1, 3.1.1, 4.4.11
- [Dim10P] A. Dimca, *The Poincaré-Deligne polynomial of Milnor fibers of triple point line arrangements is combinatorially determined*, Canad. Math. Bull. **59** (2016), 279–286. 1.3.6
- [Dim] A. Dimca, *On the Milnor monodromy of the irreducible complex reflection arrangements*, arXiv:1606.04048. 4, 5
- [DIM] A. Dimca, D. Ibadula, D. Măcinic, *Numerical invariants and moduli spaces for line arrangements*, arXiv:1609.06551v2. 2
- [DP11] A. Dimca, S. Papadima, *Finite Galois covers, cohomology jump loci, formality properties, and multinets*, Annali Scuola Norm. Sup. Pisa **10** (2011), 253—268. 1.3.4, 2.3.4, 5, 7.3, 7.3.2
- [DP14] A. Dimca, S. Papadima, *Non-abelian cohomology jump loci from an analytic viewpoint*, Communications in Contemporary Mathematics, vol. **16**, no. 4 (2014) 1350025 (47 pages). 1.3.2, 2.2.1, 2.2.3, 2.2.5, 2.2.9, 2.2.5, 3.1, 3.1

- [DPS09] A. Dimca, S. Papadima, A. Suciu, *Topology and geometry of cohomology jump loci*, Duke Math. J. **148**(3), 405–457(2009). 1.3.2, 2.2.5, 3.1, 5.4
- [DPS15] A. Dimca, S. Papadima, A. Suciu, *Algebraic models, Alexander-type invariants, and Green-Lazarsfeld sets*, Bull. Math. Soc. Sci. Math. Roumanie, tome **58** (106) no. 3, 2015, 257–269. 3.1
- [DStenMF] A. Dimca, G. Sticlaru, *Computing the monodromy and pole order filtration on the Milnor fiber cohomology of plane curves*, arXiv:1609.06818. 7.3
- [Dol] I. Dolgachev, *Weighted projective varieties*, in: Group actions and vector fields, Lecture Notes in Math. **956**, 1982, pp. 34–71. 7.5
- [EH] D. Eisenbud, J. Harris, 3264 and all that: Intersection Theory in Algebraic Geometry, upcoming book, online version. 6.2.3
- [FY] M. Falk, S. Yuzvinsky, *Multinets, resonance varieties, and pencils of plane curves*, Compositio Math. **143** (2007), no. 4, 1069–1088. 2.3.7, 2.3.6, 5.2, 5.2
- [Fu] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **2**, Springer, Berlin, 1984. 6.2.3
- [FM] W. Fulton, R. MacPherson, *A compactification of configuration spaces*, Annals of Math. **139** (1994), 183–225. 2.2.8
- [Ha] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer-Verlag, New York (1977). 5.1.3
- [He] K. Hess, *Rational homotopy theory: a brief introduction*, arXiv:math/0604626v2. 2.2.4
- [Hi] F. Hirzebruch, *Arrangements of lines and algebraic surfaces*, Progress in Mathematics **36**, Birkhäuser Boston, 1983, 113–140. 1.1, 6.4.2, 6.4.3, 6.4.2, 6.4.4, 6.4.2, iii, 7.1, 7.3
- [LibF] A. Libgober, *First order deformations for rank one local systems with a non-vanishing cohomology*, Topology Appl. **118** (2002), no. 1-2, 159–168. 3.1
- [LibE] A. Libgober, *Eigenvalues for the monodromy of the Milnor fibers of arrangements*, in: Trends in Singularities, in: Trends Math., Birkhäuser, Basel, 2002, 141–150. 2.3.6
- [MP] A. Măcinic, S. Papadima, *On the monodromy action on Milnor fibers of graphic arrangements*, Topology and its Applications **156** (2009), 761–774. 7.3, i, ii, 7.3.6
- [MPP] A. Măcinic, S. Papadima, C. R. Popescu, *Modular equalities for complex reflexion arrangements*, arXiv:1406.7137. 7.3
- [MPPS] A. Măcinic, S. Papadima, R. Popescu, A. Suciu, *Flat connections and resonance varieties: from rank one to higher ranks*, arXiv:1312.1439v3. 2.2.1, 2.2.10
- [Mil] J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals Math. Studies **61**, Princeton, 1968. 5
- [Man] R. Mandelbaum, *Four-dimensional topology: an introduction*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), no. 1, 1–159. 1.1, 2

- [Mat] D. Matei, *On a class of algebraic surfaces constructed from arrangements of lines*, Revue Roumaine de Mathématiques Pures et Appliquées, **60** (2015), no. 3, 355–373. 6.4.2
- [Mor] J. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. **48** (1978), 137–204. 2.2.4, 2.2.8, 3.1, 3.2
- [Mum1] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math., No. 9 (1961), 5–22. iii
- [Mum2] D. Mumford, *The red book of varieties and schemes*, Lecture Notes in Math. **1358**, Springer, Berlin, 1988. 2
- [Naie] D. Naie, *The irregularity of cyclic multiple planes after Zariski*, Enseign. Math. (2) **53** (2007), 265–305. 7.3
- [OT] P. Orlik, H. Terao, *Arrangements of Hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **300**, Berlin: Springer-Verlag, 1992. 1.1, 1.2, 2.3, 2.3.5
- [PaS] S. Papadima, A. Suciu, *The Milnor fibration of a hyperplane arrangement: from modular resonance to algebraic monodromy*, arXiv:1401.0868. 5.4.3, 7.2.4, 7.3
- [PY] J. V. Pereira and S. Yuzvinsky, *Completely reducible hypersurfaces in a pencil*, Advances in Math. **219** (2008), 672–688. 5.1.3
- [PeS] C. Peters and J. Steenbrink, *Mixed Hodge Structures*, Springer, Berlin, 2008. 1.1, 2.1, 6.1, 6.1.1, 6.2.1
- [Re] M. Reid, *The Du Val singularities  $A_n, D_n, E_6, E_7, E_8$* , online notes. 6.6.1
- [RV] R. Remmert, A. Van de Ven, *Zur Funktionentheorie homogener komplexer Mannigfaltigkeiten*, Topology **2** (1963), 137–157. 2
- [Rou] X. Roulleau, *Divisor arrangements and algebraic surfaces*, Commentarii Mathematici Universitatis Sancti Pauli **57**, No. 1, 2008. 7.3, 7.5
- [Sa] F. Sakai, *Semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps*, Math. Ann. **254**, p. 89–120, 1980. iii
- [Ser] J. -P. Serre, *Sur la topologie des variétés algébriques en caractéristique  $p$* , Symposium Internacional de Topologia Algebraica, Universidad Nacional Autónoma de México, 1958, 24–53. 2
- [Su1] A. Suciu, *Fundamental groups of line arrangements: Enumerative aspects*, Contemporary Math., Amer. Math. Soc., **276** (2001), 43–79. 7.3
- [Su14] A. Suciu, *Hyperplane arrangements and Milnor fibrations*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), 417–481. 7.3
- [Sul] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47**(1977), no. 1, 269–331. 2.2.3

- [Tr] P. Tretkoff, *Complex Ball Quotients and Line Arrangements in the Projective Plane (MN-51)*, With an appendix by Hans-Christoph Im Hof, Princeton University Press, 2016. iii
- [Ur] G. Urzúa, *Arrangements of Curves and Algebraic Surfaces*, J. Algebraic Geom. **19** (2010), 335-365. 7.5
- [Vis] A. Vistoli, *The number of reducible hypersurfaces in a pencil*, Inventiones mathematicae **112.2** (1993), 247–262. 5.1.3, 5.1.3
- [Voi1] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I*, Studies in Advanced Mathematics **77**, Cambridge University Press 2003. 1.1, 1.2, 2.1.2, 2.1.3, 2.1.5, 2.1.6, 2.1.8, 1ii, 5.4, iii
- [Voi2] C. Voisin, *Hodge Theory and Complex Algebraic Geometry II*, Studies in Advanced Mathematics **77**, Cambridge University Press 2003. 3.5
- [Wal] C. T. C. Wall, *Singular points of plane curves*, London Mathematical Society Student Texts, **63**, Cambridge University Press, Cambridge, 2004. 6.3.5
- [Wang] Zhenjian Wang, *On tangential deformation of homogeneous polynomials*, accepted by Mathematische Nachrichten. 1.1, 5.1.11
- [Yuz] S. Yuzvinsky, *A new bound on the number of special fibers in a pencil of curves*, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1641—1648. 2.3.8, 5.1.3, 5.2.1, 5.4
- [Zub] H. Zuber, *Non-formality of Milnor fibres of line arrangements*, Bull. London Math. Soc. **42** (5): 905-911, 2010. 1.3.4, 5.3, 5.3.3