Dispersive and Strichartz estimates for the wave equation inside cylindrical convex domains

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École Doctorale de Sciences Fondamentales et Appliquées

Unité de recherche: Laboratoire de Mathématiques J.A.Dieudonné

Thèse de doctorat

présentée en vue de l'obtention du

grade de docteur en **Mathématiques**

de

l'Université Côte D'Azur

par

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Estimations De Dispersion Et De Strichartz Dans Un Domaine Cylindrique Convexe
Dispersive and Strichartz Estimates for The Wave Equation Inside Cylindrical Convex Domains

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Soutenue le 29 juin 2017

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Résumé

Estimations de dispersion et de Strichartz dans un domaine cylindrique convexe: Dans ce travail, nous allons établir des estimations de dispersion et des applications aux inégalités de Strichartz pour les solutions de l'équation des ondes dans un domaine cylindrique convexe $\Omega \subset \mathbb{R}^3$ à bord C^∞, $\partial\Omega \neq \emptyset$. Les estimations de dispersion sont classiquement utilisées pour prouver les estimations de Strichartz. Dans un domaine Ω général, des estimations de Strichartz ont été démontrées par Blair, Smith, Sogge [6, 7]. Des estimations optimales ont été prouvées dans [29] lorsque Ω est strictement convexe. Le cas des domaines cylindriques que nous considérons ici généralise les resultats de [29] dans le cas où la courbure positive dépend de l'angle d'incidence et s'annule dans certaines directions.

Mots Clés: estimations de dispersion, estimations de Strichartz, L'équation des ondes, domaines cylindriques.

Abstract

Dispersive and Strichartz Estimates for The Wave Equation Inside Cylindrical Convex Domains

by **Len MEAS**

In this work, we establish local in time dispersive estimates and its application to Strichartz estimates for solutions of the model case Dirichlet wave equation inside cylindrical convex domains $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega \neq \emptyset$. Let us recall that dispersive estimates are key ingredients to prove Strichartz estimates. Strichartz estimates for waves inside an arbitrary domain Ω have been proved by Blair, Smith, Sogge [6, 7]. Optimal estimates in strictly convex domains have been obtained in [29]. Our case of cylindrical domains is an extension of the result of [29] in the case where the nonnegative curvature radius depends on the incident angle and vanishes in some directions.

Keywords: dispersive estimates, Strichartz estimates, wave equation, cylindrical convex domains.

Acknowledgements

First of all, I would like to express my sincerest gratitude to my thesis advisors Gilles LEBEAU and Oana IVANOVICI for proposing the subject of this thesis and for motivating, guiding and inspiring through this work.

The research was supported by the ERC project SCAPDE.

Special thanks go to the rest of my thesis committee: Nicolas Burq, Nikolay Tzvetkov, András Vasy, Maciej Zworski for generously offering their time, support, guidance and for their insightful comments throughout the preparation and review of the manuscript.

My sincere thank also goes to colleaques at Laboratoire J.A.Dieudonné, École doctorale de Sciences Fondamentales et Appliquées of the Université Nice Sophia Antipolis for their kind support and for all of opportunities they have extended to me during my research.

I am grateful to my family, to all my teachers who put their faith in me and motivated me to do better.

Contents

Chapter 1

Introduction

Dispersive phenomena, which informally refer to the spread out of the wave packet as the time goes by, often play a crucial role in the study of evolution partial differential equations. Mathematically, exhibiting dispersion often amounts to proving a decay estimate for L^∞ norm of the solution at time t in terms of some (negative) power of t and of L^1 norm of the data. The dispersive inequality provides two types of information. The first concerns the precise decay rate of L^∞ norm of solution as $t \to \infty$ while the second provides information about the regularity of L^∞ norm of solution for $t > 0$. In many cases, proving these estimates relies on the (possibly degenerate) stationary phase theorem and on explicit representation of the solution.

The dispersive estimates for the wave equation in $\mathbb{R}^d$ or on a smooth Riemannian manifolds without boundary are well known. In these cases, we can get the pointwise decay estimates for the kernel of parametrix, which may be constructed in a suitable way by Fourier integral operators whose phase function is nondegenerate. In domains with boundary, the difficulties arise from the behaviour of the wave flow near the points of the boundary. In the case of a concave boundary, dispersive estimates follow by using the Melrose and Taylor parametrix for Dirichlet wave equation and the approach by Smith, Sogge in [41].

Recently, in [29], Ivanovici, Lebeau, and Planchon have established the optimal local in time dispersive estimates with losses inside the strictly convex domain, and this is due to caustics generated in arbitrarily small time near the boundary. A main approach of the proof consists in a detailed description of wave front set of the solution near the boundary. The dispersion is optimal because of the presence of swallowtail type singularities in the wave front set of the solution.

The analysis of wave front set consists two main ingredients: location of singularities and direction they propagate, namely along bicharacteristics. It appears in problems of the propagation of singularities in the phase space. On manifold without boundary, this phase space is the contangent bundle. In the case with non-empty boundary, the main challenge arises from the behaviour of singularities near the boundary. In the interior of the domain, due to Hörmander rather general theorem, these singularities propagate along the bicharacteristic curves (optical rays). The simplest case is that the singularities striking the boundary transversely simply reflect according to the usual law of geometric optics ("angle of incident equals angle of reflection") for the reflection of bicharacteristics. Melrose and Sjöstrand introduced the notion of generalized bicharacteristic rays to proved the propagation of singularities near the boundary. The difficulties arise when dealing with the rays tangent to the boundary. They proved that, at these "diffractive points", the singularities may only propagate along certain generalized bicharacteristics. The theorem on propagation of singularities in strictly convex domains was proved by Eskin in [17] by the construction of the parametrix near tangential direction to the boundary and was proved independently by Andersson and Melrose in [1].

A simplest geometry of wave front set is a spherical wave front, it moves outward from the source point at a constant speed and the energy propagates equally in all directions. In the case of half space or concave boundary, the reflected waves do not generate caustics. Interior of a strictly convex domains, reflected waves generate infinitely many singularities such as cusps and swallowtails.

In domains whose boundaries have the order of tangency greater or equal 3, there are no known results concerning dispersion except the approach of doubling the metric across the boundary and considering a boundaryless manifold with a Lipschitz metric across the boundary. These arguments require to work on a very short time intervals in order to construct parametrix in the case of only one reflection. But this approach yields non sharp dispersive estimates since the metric is not smooth enough.

In this thesis, we will study a model case of cylindrical domains with a convex boundary with zero curvature along the axis of the cylinder. The main result in this thesis is that we have proved the optimal local in time dispersive estimates with losses. Our approach of construction the parametrix allows us to give a detailed description of the wave front set, which shows precisely that the caustics appear between the first and the second reflection of the wave on the boundary. The result for dispersion is optimal due to the presence of the swallowtail type singularities in the wave front set.

Let us recall that it is now well established that these dispersive estimates, combined with an abstract functional analysis argument—the TT^* argument— yield a number of inequalities involving space-time Lebesgue norms $L^q_t(L^r_x)$—the so-called Strichartz estimates. The Strichartz estimates have proven to be of great importance in the study of semilinear or quasilinear Schrödinger and wave equations, in particular mixed (in time and space) $L^q_t(L^r_x)$ estimates are often the key to proving well-posedness results.

In $\mathbb{R}^d$, the first global $L_t^q(L_x^r)$ estimates was proved by Strichartz for the wave equation [see [47]] first in the particular case $q = r$. The extension to the whole set of admissible indices was achieved by Ginibre and Velo in [19] for Schrödinger equations, where (q, r) are sharp admissible and $q > 2$; the estimates for the wave equations were obtained independently by Ginibre, Velo in [20] and Lindblad, Sogge in [35], following earlier works by Kapitanski [see [31]]. The endpoint case estimates for both equations was established later by Keel and Tao in [33]. The so-called Knapp wave provides counter examples away from the endpoint.

For general manifolds, phenomena such as the existence of trapped geodesics or finiteness of volume can preclude the development of global estimates, leading us to consider just local in time estimates. Only partial progress has been made in establishing these estimates on manifolds, domains or singular spaces such as cones. For the conic case, its singularity affects the flow of energy and complicates many of the known techniques for proving these inequalities.

In [5], Blair, Ford, and Marzuola proved the dispersive and scale invariant Strichartz estimates for the wave equation on the flat cones by using the explicit representation of the solution operator in regions related to flat wave propagation and diffraction by the cone point. They also proved the corresponding inequalities on wedge domains, polygons, and Euclidean surfaces with conic singularities.

In [52], Zhang proved the global-in-time Strichartz estimates for wave equations on the nontrapping asymptotically conic manifolds . These type of estimates was dealt with in [48] outside normally hyperbolic trapped on odd dimensional Euclidean space. In [9], Bouclet proved Strichartz estimates for the wave and Schrödinger on surface with cusps.

In the case of a compact manifold with boundary, the finite speed of propagation allows us to work in coordinate charts and to establish the local Strichartz estimates for the variable coefficients wave operators in $\mathbb{R}^d$. In this case, Kapitanski in [32] and Mockenhaupt, Seeger and Sogge in [39] established such inequalities for operators with smooth coefficients. Smith in [40] and Tataru in [50] have proven Strichartz estimates for operators with $C^{1,1}$ coefficients. Local and global in time Strichartz estimates for exterior in $\mathbb{R}^d$ to a compact set with smooth boundary under a nontrapping assumption were obtained by Smith, Sogge in [42] for the case of odd dimensions and Burq in [11], Metcalfe in [38] for the case of even dimensions.

Using the $L^r(\Omega)$ estimates for the spectral projectors obtained by Smith and Sogge in [43], Burq, Lebeau, Planchon in [13] established Strichartz estimates for bounded domains in $\mathbb{R}^3$ for a certain range of triples (q, r, γ). In [7], Blair, Smith, Sogge expanded the range of indices q and r obtained in [13] and generalized results to higher dimensions.

For manifold with smooth, strictly geodesically concave boundary, the Melrose and Taylor parametrix had been used by Smith and Sogge in [41] in order to obtain the non-endpoint Strichartz estimates for the wave equation with Dirichlet boundary condition.

Recently in [29], Ivanovici, Lebeau, and Planchon have deduced the usual Strichartz estimates from the optimal dispersive estimates inside strictly convex domains of dimensions $d \geq 2$ for a certain range of the wave admissibility.

1.1 The cylindrical model problem

Let $\Omega = \{x \geq 0, (y, z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega = \{x = 0\}$, and let P be the wave operator:

$$P = \partial_t^2 - (\partial_x^2 + (1 + x)\partial_y^2 + \partial_z^2).$$

We consider solutions of the linear Dirichlet-wave equation inside Ω

$$Pu = 0, \quad u_{|t=0} = \delta_a, \quad \partial_t u_{|t=0} = 0, \quad u_{|x=0} = 0, \tag{1.1.1}$$

with $u = u(t, x, y, z)$, and for $a > 0$, $\delta_a = \delta_{x=a,y=0,z=0}$. We use the notation $\tau = \frac{h}{i}\partial_t, \eta = \frac{h}{i}\partial_y, \xi = \frac{h}{i}\partial_x, \zeta = \frac{h}{i}\partial_z$ for the Fourier variables and $h \in (0, 1]$. The Riemannian manifold (Ω, Δ) with $\Delta = \partial_x^2 + (1 + x)\partial_y^2 + \partial_z^2$ can be locally seen as a cylindrical domain in $\mathbb{R}^3$ by taking cylindrical coordinates (r, θ, z), where $r = 1 - x/2, \theta = y$, and $z = z$. The problem is local near the boundary $\partial\Omega = \{x = 0\}$. Let $(a, 0, 0) \in \Omega, a > 0$. In local coordinates, a is the distance from the source point to the boundary. We assume a is small enough as we are interested only in highly reflected waves, which we do not observe if the waves do not have time to hit the boundary. This gives us interesting phenomena such as caustics near the boundary.

We remark that when there is no z variable (or when $y \in \mathbb{R}^n$ and ∂_y^2 is replaced by Δ_y), it is the Friedlander model. In this case, the optimal dispersive estimates were recently obtained by Ivanovici, Lebeau, and Planchon in [29].

Recall that at time $t > 0$, the waves propagating from the source of light highly concentrate around a sphere of radius t. For a variable coefficients metric, if two different light rays emanating from the source do not cross (that is, if t is smaller than the injectivity radius), one may then construct parametrices using oscillatory integrals where the phase encodes the geometry of wave front. In our scenario, the geometry of the wave front becomes singular in arbitrarily small times which depend on the frequency of the source and its distance to the boundary. In fact, a caustic appears between the first and the second reflection of the wave front. Let us give a brief overview of what caustics are [see [29] section 1.1]. Geometrically, caustics are defined as envelopes of light rays coming from the source of light. At the caustic point we expect the light to be singularly intense. Analytically, caustics can be characterized as points where usual bounds on oscillatory integrals are no longer valid. The classification of asymptotic behavior of the oscillatory integrals with caustics depends on the number and the order of their critical points that are real. Let us consider an oscillatory integral

$$u_h(z) = \frac{1}{(2\pi h)^{1/2}} \int e^{\frac{i}{h}\Phi(z,\zeta)} g(z,\zeta,h)d\zeta, \quad z \in \mathbb{R}^d, \quad \zeta \in \mathbb{R}, \quad h \in (0,1].$$

We assume that Φ is smooth and that g is compactly support in z and ζ. If $\partial_\zeta \Phi \neq 0$ in an open neighborhood of the support of g, the repeated integration by parts yields $|u_h(z)| = O(h^N)$ for any $N > 0$. If $\partial_\zeta \Phi = 0$ and $\partial_\zeta^2 \Phi \neq 0$ (nondegenerate critical points), then the stationary phase method yields $\|u_h\|_{L^\infty} = O(1)$. If there are degenerate critical points, we define them to be caustics, as $\|u_h\|_{L^\infty}$ is no longer uniform bounded. The order of a caustic κ is defined as the infimum of κ' such that $\|u_h\|_{L^\infty} = O(h^{-\kappa'})$. Let us give some useful examples of degenerate phase functions. The phase function of the form $\Phi_F(z,\zeta) = \frac{\zeta^3}{3} + z_1\zeta + z_2$ corresponds to a fold with order $\kappa = \frac{1}{6}$. A typical example is the Airy function. The next canonical form is given by the phase function of the form $\Phi_C(z,\zeta) = \frac{\zeta^4}{4} + z_1\frac{\zeta^2}{2} + z_2\zeta + z_3$, which corresponds to a cusp singularity with order $\kappa = \frac{1}{4}$. A swallowtail canonical form is given by the phase $\Phi_S(z,\zeta) = \frac{\zeta^5}{5} + z_1\frac{\zeta^3}{3} + z_2\frac{\zeta^2}{2} + z_3\zeta + z_4$ with order $\kappa = 3/10$.

The crucial result of this work is the extension of the result of [29] to the case of our model cylindrical convex domains which have the following property: the nonnegative curvature radius depends on the incident angle and vanishes in some directions.

The main goals of this work are:

- To construct a local parametrix and establish local in time dispersive estimates for solution u to (1.1.1).
- To prove the Strichartz estimates inside cylindrical domains for solution u to (1.1.1).

1.2 Some known results

The dispersive estimates for the wave equation in $\mathbb{R}^d$ follows from the representation of solution as a sum of Fourier integral operators [see [10, 20, 3]]. They read as follows:

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq Ch^{-d} \min\left\{1, \left(\frac{h}{|t|}\right)^{\frac{d-1}{2}}\right\}, \tag{1.2.1}$$

where $\Delta_{\mathbb{R}^d}$ is the Laplace operator in $\mathbb{R}^d$. Here and in the sequel, the function χ belongs to $C_0^\infty(]0,\infty[)$ and is equal to 1 on $[1,2]$ and $D_t = \frac{1}{i}\partial_t$.

Inside strictly convex domains Ω_D of dimensions $d \geq 2$, the optimal (local in time) dispersive estimates for the wave equations have been established by Ivanovici, Lebeau, and Planchon in [29]. More precisely, they have proved that

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_D}}\|_{L^1(\Omega_D)\to L^\infty(\Omega_D)} \leq Ch^{-d}\min\left\{1, \left(\frac{h}{|t|}\right)^{\frac{d-1}{2}-\frac{1}{4}}\right\}, \tag{1.2.2}$$

where Δ_D is the Laplace operator on Ω_D. Due to the caustics formation in arbitrarily small times, (1.2.2) induces a loss of 1/4 powers of $(h/|t|)$ factor compared to (1.2.1).

Let us also recall a few results about Strichartz estimates [see [29], section 1]: let (Ω, g) be a Riemannian manifold without boundary of dimensions $d \geq 2$. Local in time Strichartz estimates state that

$$\|u\|_{L^q((-T,T);L^r(\Omega))} \leq C_T\left(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)}\right), \tag{1.2.3}$$

where $\dot{H}^\beta$ denotes the homogeneous Sobolev space over Ω of order β and $2 \leq q, r \leq \infty$ satisfy

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta, \quad \frac{1}{q} \leq \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right).$$

Here $u = u(t,x)$ is a solution to the wave equation

$$(\partial_t^2 - \Delta_g)u = 0 \text{ in } (-T,T)\times\Omega, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x),$$

where Δ_g denotes the Laplace-Beltrami operator on (Ω, g). The estimates (1.2.3) hold on $\Omega = \mathbb{R}^d$ and $g_{ij} = \delta_{ij}$.

In [7], Blair, Smith, Sogge proved the Strichartz estimates for the wave equation on (compact or noncompact) Riemannian manifold with boundary. They proved that the Strichartz estimates (1.2.3) hold if Ω is a compact manifold with boundary and (q, r, β) is a triple satisfying

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta \text{ , for } \begin{cases} \frac{3}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}, \ d \leq 4, \\ \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, \ d \geq 4. \end{cases}$$

Recently in [29], Ivanovici, Lebeau, and Planchon have deduced a local in time Strichartz estimates (1.2.3) from the optimal dispersive estimates inside strictly convex domains of dimensions $d \geq 2$ for a triple (d, q, β) satisfying

$$\frac{1}{q} \leq \left(\frac{d-1}{2} - \frac{1}{4}\right)\left(\frac{1}{2} - \frac{1}{r}\right), \text{ and } \beta = d\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q}.$$

For $d \geq 3$ this improves the range of indices for which sharp Strichartz estimates do hold compared to the result by Blair, Smith, Sogge in [7]. However, the results in [7] apply to any domains or manifolds with boundary.

1.3 Main results

Our main results concerning the local in time dispersive estimates and Strichartz estimates inside the cylindrical convex domain Ω are stated below. Let $\mathcal{G}_a$ be the Green function for (1.1.1).

Theorem 1.3.1. *There exists C such that for every $h \in]0, 1]$, every $t \in [-1, 1]$ and every $a \in]0, 1]$ the following holds:*

$$\|\chi(hD_t)\mathcal{G}_a(t, x, y, z)\|_{L^\infty} \leq Ch^{-3} \min\left\{1, \left(\frac{h}{|t|}\right)^{3/4}\right\}. \tag{1.3.1}$$

As in [29], Theorem 1.3.1 states that a loss of 1/4 powers of $(h/|t|)$ appears compared to (1.2.1) . We will obtain in Theorems 1.4.1, 1.4.2, 1.4.3 better results, in particular near directions which are close to the axis of the cylinder.

As a consequence of Theorem 1.3.1, conservation of energy, interpolation and TT^* arguments, we obtain the following set of (local in time) Strichartz estimates.

Theorem 1.3.2. *Let (Ω, Δ) as before. Let u be a solution of the wave equation on Ω:*

$$\begin{aligned} &(\partial_t^2 - \Delta)u = 0 \quad \text{in } \Omega, \\ &u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \\ &u_{|x=0} = 0. \end{aligned}$$

Then for all T there exists C_T such that

$$\|u\|_{L^q((0,T);L^r(\Omega))} \leq C_T \left(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)}\right),$$

with

$$\frac{1}{q} \leq \frac{3}{4}\left(\frac{1}{2} - \frac{1}{r}\right), \text{ and the scaling } \beta = 3\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q}.$$

Theorem 1.3.2 improves the range of indices for which sharp Strichartz estimates do hold compared to [7]. Notice however that the results in [7] apply to arbitrary domains or manifolds with non-empty boundary. To prove the Strichartz estimates in Theorem 1.3.2, we first prove the frequency-localized Strichartz estimates by utilizing the frequency-localized dispersive estimates, interpolation and TT^* arguments. We then apply the Littlewood-Paley squarefunction estimates [see [4, 5, 30]] to get the Strichartz estimates [Theorem 1.3.2] in the context of cylindrical domains.

1.4 Green function and precise dispersive estimates

The proofs of frequency-localized dispersive estimates are based on the construction of parametrices for the fundamental solution of the wave equation (1.1.1) and (possibly degenerate) stationary phase method.

We begin with the construction of the local parametrix for (1.1.1) by utilizing the spectral analysis of $-\Delta$ with Dirichlet condition on the boundary to obtain first the Green function associated to (1.1.1). The Laplace operator we work with on the half space Ω is equal to

$$\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2,$$

with the Dirichlet condition on the boundary $\partial\Omega$. We notice that a useful feature of this particular Laplace operator is that the coefficients of the metric do not depend on the vairables y, z and therefore this allows us to take the Fourier transform in y and z. Now taking the Fourier transform in y, z-variables yields

$$-\Delta_{\eta,\zeta} = -\partial_x^2 + (1+x)\eta^2 + \zeta^2.$$

For $\eta \neq 0$, $-\Delta_{\eta,\zeta}$ is a self-adjoint, positive operator on $L^2(\mathbb{R}_+)$ with compact resolvent. Let $(e_k)_{k\geq 1}$ be an orthonormal basis in $L^2(\mathbb{R}_+)$ of Dirichlet eigenfunctions of $-\Delta_{\eta,\zeta}$ and let $(\lambda_k)_k$ be the associated eigenvalues. We get easily

$$e_k = e_k(x,\eta) = f_k \frac{|\eta|^{1/3}}{k^{1/6}} Ai(|\eta|^{2/3}x - \omega_k).$$

and

$$\lambda_k = \lambda_k(\eta,\zeta) = \eta^2 + \zeta^2 + \omega_k|\eta|^{4/3},$$

where $(-\omega_k)_k$ denote the zeros of Airy function in decreasing order and for all $k \geq 1$, f_k are constants so that $\|e_k(.,\eta)\|_{L^2(\mathbb{R}_+)} = 1$. Observe that $(f_k)_k$ is uniformly bounded in a fixed compact subset of $]0,\infty[$ since $\int_{-\omega_k}^{-2} Ai^2(\omega)d\omega \simeq \frac{1}{4\pi}\int_{-\omega_k}^{-2} |\omega|^{-1/2}(1+O(\omega^{-1}))d\omega \simeq |\omega|^{1/2}$ and $\omega_k \simeq \left(\frac{3}{2}\pi k\right)^{2/3}(1+O(k^{-1}))$.

For $a \in \Omega$, let $g_a(t,x,\eta,\zeta)$ be the solution of

$$(\partial_t^2 - (\partial_x^2 - (1+x)\eta^2 - \zeta^2))g_a = 0,$$
$$g_{a|x=0} = 0,\ g_{a|t=0} = \delta_{x=a},\ \partial_t g_{a|t=0} = 0.$$

We have

$$g_a(t,x,\eta,\zeta) = \sum_{k\geq 1} \cos(t\lambda_k^{1/2}) e_k(x,\eta) e_k(a,\eta). \tag{1.4.1}$$

Here $\delta_{x=a}$ denotes the Dirac distribution on $\mathbb{R}_+, a > 0$ and it reads as follows:

$$\delta_{x=a} = \sum_{k\geq 1} e_k(x,\eta) e_k(a,\eta).$$

Now taking the inverse Fourier transform, the Green function for (1.1.1) is given by

$$\begin{aligned}\mathcal{G}_a(t,x,y,z) &= \frac{1}{4\pi^2}\int e^{i(y\eta+z\zeta)} g_a(t,x,\eta,\zeta) d\eta d\zeta,\\ &= \frac{1}{4\pi^2 h^2}\sum_{k\geq 1}\int e^{i(y\eta+z\zeta)/h}\cos(t\lambda_k^{1/2}) e_k(x,\eta/h) e_k(a,\eta/h) d\eta d\zeta.\end{aligned} \tag{1.4.2}$$

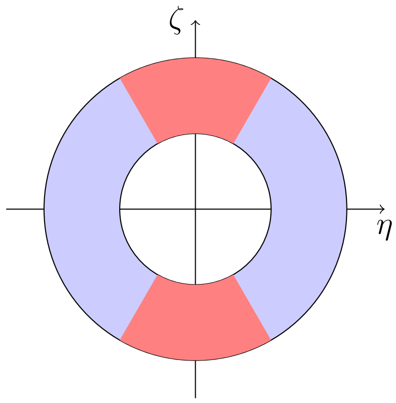
We thus get the following formula for $2\chi(hD_t)\mathcal{G}_a$

$$\begin{aligned}2\chi(hD_t)\mathcal{G}_a(t,x,y,z) = \frac{1}{4\pi^2 h^2}\sum_{k\geq 1}\int & e^{\frac{i}{h}(y\eta+z\zeta)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}} e_k(x,\eta/h) e_k(a,\eta/h)\\ & \chi((\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}) d\eta d\zeta.\end{aligned} \tag{1.4.3}$$

On the wave front set of the above expression, one has $\tau = (\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}$. In order to prove Theorem 1.3.1, we only need to work near tangential directions; therefore we will introduce an extra cutoff to insure $|\tau - (\eta^2+\zeta^2)^{1/2}|$ small, which is equivalent to $\omega_k h^{2/3}|\eta|^{4/3}$ small. Then, we are reduced to prove the dispersive estimate for $\mathcal{G}_{a,loc}$:

$$\begin{aligned}\mathcal{G}_{a,loc}(t,x,y,z) = \frac{1}{4\pi^2 h^2}\sum_{k\geq 1}\int & e^{\frac{i}{h}(y\eta+z\zeta)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}} e_k(x,\eta/h) e_k(a,\eta/h)\\ & \chi_0(\eta^2+\zeta^2)\chi_1(\omega_k h^{2/3}|\eta|^{4/3}) d\eta d\zeta,\end{aligned} \tag{1.4.4}$$

where the cut-off functions χ_0 and χ_1 are defined in section 2.1.



Phase space

To obtain the local in time dispersive estimates, we will cut the η integration in (1.4.4) in different pieces [as figure above]. More precisely, we write

$$\mathcal{G}_{a,loc} = \mathcal{G}_{a,c_0} + \sum_{\epsilon_0\sqrt{a}\le 2^m\sqrt{a}\le c_0} \mathcal{G}_{a,m} + \quad \mathcal{G}_{a,\epsilon_0}, \tag{1.4.5}$$

where $\mathcal{G}_{a,c_0}$ is associated with the integration for $|\eta| \ge c_0$, $\mathcal{G}_{a,m}$ is associated with the integration for $|\eta| \simeq 2^m\sqrt{a}$, and $\mathcal{G}_{a,\epsilon_0}$ is associated with the integration for $0 < |\eta| \le \epsilon_0\sqrt{a}$.

We will prove the following results. Let $\epsilon \in]0, 1/7[$.

Theorem 1.4.1. *There exists C such that for every $h \in]0, 1]$, every $t \in [h, 1]$ the following holds:*

$$\|\mathcal{G}_{a,c_0}(t, x, y, z)\|_{L^\infty(x\le a)} \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \gamma(t, h, a), \tag{1.4.6}$$

with

$$\gamma(t, h, a) = \begin{cases} \left(\frac{h}{t}\right)^{1/3} & \text{if } a \le h^{\frac{2}{3}(1-\epsilon)}, \\ \left(\frac{h}{t}\right)^{1/2} + a^{1/8}h^{1/4} & \text{if } a \ge h^{\frac{2}{3}-\epsilon'}, \epsilon' \in]0, \epsilon[. \end{cases}$$

Observe that in Theorem 1.4.1 we get the same estimate as in Ivanovici-Lebeau-Planchon [29].

Theorem 1.4.2. *There exists C such that for every $h \in]0, 1]$, every $t \in [h, 1]$ the following holds :*

$$\|\mathcal{G}_{a,m}(t, x, y, z)\|_{L^\infty(x\le a)} \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \gamma_m(t, h, a), \tag{1.4.7}$$

with

$$\gamma_m(t,h,a) = \begin{cases} \left(\frac{h}{t}\right)^{1/3} (2^m\sqrt{a})^{1/3} & \textit{if } a \leq \left(\frac{h}{2^m\sqrt{a}}\right)^{\frac{2}{3}(1-\epsilon)}, \\ \min\left\{\left(\frac{h}{t}\right)^{1/2}, 2^m\sqrt{a}|\log(2^m\sqrt{a})|\right\} + a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4} & \textit{if } a \geq \left(\frac{h}{2^m\sqrt{a}}\right)^{\frac{2}{3}-\epsilon'}, \\ \epsilon' \in]0,\epsilon[. \end{cases}$$

For $2^m\sqrt{a} \sim 1$, Theorem 1.4.2 yields the same result as in Theorem 1.4.1. We notice that the estimates get better when $|\eta|$ ($\sim 2^m\sqrt{a}$) decreases. This is compatible with the intuition: less curvature implies better dispersion.

Theorem 1.4.3. *There exists C such that for every $h \in]0,1]$, every $t \in [h,1]$, the following holds:*

$$\|\mathcal{G}_{a,\epsilon_0}(t,x,y,z)\|_{L^\infty(x\leq a)} \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \min\left\{(h/t)^{1/2}, \sqrt{a}|\log(a)|\right\}. \tag{1.4.8}$$

Let us verify that Theorem 1.3.1 is a consequence of Theorems 1.4.1, 1.4.2 and 1.4.3. We may assume $|t| \geq h$, since for $|t| \leq h$, the best bound for the dispersive estimate is equal to Ch^{-3} by Sobolev inequality. Then, by symmetry of the Green function, we may assume $t \in [h,1]$ and $x \leq a$. Then Theorem 1.3.1 is a consequence of $\sum_{m\leq M}(2^m\sqrt{a})^\nu \simeq (2^M\sqrt{a})^\nu$ for $\nu > 0$.

Chapter 2

Dispersive Estimates For The Model Problem

In this chapter, we prove Theorems 1.4.1, 1.4.2 and 1.4.3. This chapter is organized as follows:

In section 2.1, we prove Theorem 1.4.1. To do so, we use the representation of $\mathcal{G}_{a,c_0}$ as a sum over the eigenmodes which is used to prove the estimates for $a \leq h^{\frac{2}{3}(1-\epsilon)}$, $\epsilon \in]0, 1/7[$. Using the Airy-Poisson summation formula [see Lemma 2.1.4], $\mathcal{G}_{a,c_0}$ can be also represented as a sum over multiple reflections for $a \geq h^{\frac{2}{3}-\epsilon'}$, for $\epsilon' \in]0, \epsilon[$. These local parametrices can be written in terms of oscillatory integrals to which we can apply degenerate stationary phase results.

In section 2.2, we prove Theorem 1.4.2. To get the estimates for $\mathcal{G}_{a,m}$, we distinguish between two different cases. The first case is $a \leq \left(\frac{h}{2^m\sqrt{a}}\right)^{\frac{2}{3}(1-\epsilon)}$, $\epsilon \in]0, 1/7[$: here, we follow ideas in section 2.1 and construct a local parametrix as a sum over eigenmodes. The second case is $a \geq \left(\frac{h}{2^m\sqrt{a}}\right)^{\frac{2}{3}-\epsilon'}$, for $\epsilon' \in]0, \epsilon[$: there, the Airy-Poisson summation formula yields the representation of $\mathcal{G}_{a,m}$ as a sum over multiple reflections.

In section 2.3, we prove Theorem 1.4.3. Notice that as ϵ_0 is small, the estimates for $\mathcal{G}_{a,\epsilon_0}$ are in fact those in free case. To get that, we first compute the trajectories of the Hamiltonian flow for the operator P. At this frequency localization there is at most one reflection on the boundary of the cylinder. Moreover, we follow the techniques from section 2.1 and obtain an expression for $\mathcal{G}_{a,\epsilon_0}$ to which we apply the stationary phase method. It is particularly interesting that this localization gives us an oscillatory integral (the local parametrix) with nondegenerate phase function; this is due to the geometric study of the associated Lagrangian which rules out the cusps and swallowtails regimes for a given fixed time t, $|t| \leq 1$ if ϵ_0 is small.

In all these sections, we will assume that the integration with respect to η is restricted to $\eta > 0$, since the case $\eta < 0$ is exactly the same.

2.1 Dispersive Estimates for $|\eta| \geq c_0$.

In this section, we prove Theorem 1.4.1. The key ingredient is to construct local parametrices for the regimes $a \leq h^{\frac{2}{3}(1-\epsilon)}$ for $\epsilon \in]0, 1/7[$, and for $a \geq h^{\frac{2}{3}-\epsilon'}$, for $\epsilon' \in]0, \epsilon[$ respectively. These are oscillatory integrals to which we apply the (degenerate) stationary phase type arguments to get the desired estimates. The Airy-Poisson summation formula [see Lemma 2.1.4] gives us the parametrix as a sum over multiple reflections.

2.1.1 Dispersive Estimates for $0 < a \leq h^{\frac{2}{3}(1-\epsilon)}$, with $\epsilon \in]0, 1/7[$.

In this section, we prove local in time dispersive estimates for the function $\mathcal{G}_{a,c_0}$. In the regime $0 < a \leq h^{\frac{2}{3}(1-\epsilon)}$, with $\epsilon \in]0, 1/7[$, the parametrix reads as a sum over eigenmodes k. Taking into account the asymptotic behaviour of the Airy functions, we deal with different values of k as follows: for small values of k, we use Lemma 3.5[29] to get the estimates; for large values of k, we use the asymptotic expansion of the Airy functions. The last case, the parametrix is a sum of oscillatory integrals to which we apply Lemma 2.20[29]. Recall that the parametrix in this frequency localization and near tangential directions is equal to

$$\mathcal{G}_{a,c_0}(t, x, y, z) = \frac{1}{4\pi^2 h^2} \sum_{k \geq 1} \int e^{\frac{i}{h}(y\eta + z\zeta)} e^{i\frac{t}{h}(\eta^2 + \zeta^2 + \omega_k h^{2/3}|\eta|^{4/3})^{1/2}} e_k(x, \eta/h) e_k(a, \eta/h)$$
$$\times \chi_0(\zeta^2 + \eta^2)\psi_0(\eta)\chi_1(\omega_k h^{2/3}|\eta|^{4/3})(1 - \chi_1)(\varepsilon\omega_k) d\eta d\zeta. \qquad (2.1.1)$$

Here

- $\chi_0 \in C_0^\infty, 0 \leq \chi_0 \leq 1, \chi_0$ is supported in the neighborhood of 1.
- $\psi_0 \in C_0^\infty(c_0/2, \infty), 0 \leq \psi_0 \leq 1$, $\psi_0(\eta) = 1$ for $\eta \geq c_0$.
- $\chi_1 \in C_0^\infty, 0 \leq \chi_1 \leq 1, \chi_1$ is supported in $(-\infty, 2\varepsilon], \chi_1 = 1$ on $(-\infty, \varepsilon]$, for $\varepsilon > 0$ small. χ_1 is used to localize in tangential directions. Notice that on the support of χ_1, we have $\omega_k h^{2/3}|\eta|^{4/3} \leq 2\varepsilon$ and since $\omega_k \simeq k^{2/3}$; we obtain $k \leq \frac{\varepsilon}{h|\eta|^2}$. Thus since η is bounded from below, we may assume that $k \leq \varepsilon/h$. Moreover, we have $(1 - \chi_1)(\varepsilon\omega_k) = 1$ for every $k \geq 1$ since $\omega_1 \simeq 2.33$.

The main result of this section is the following proposition.

Proposition 2.1.1. *Let $\epsilon \in]0, 1/7[$. There exists C such that for every $h \in]0, 1]$, every $0 < a \leq h^{\frac{2}{3}(1-\epsilon)}$, and every $t \in [h, 1], y \in \mathbb{R}, z \in \mathbb{R}$, the following holds:*

$$\|\mathcal{G}_{a,c_0}(t, x, y, z)\|_{L^\infty(x \leq a)} \leq Ch^{-3} \left(\frac{h}{t}\right)^{5/6}. \tag{2.1.2}$$

Proof. First, we study the integration in ζ. Let

$$J = \int e^{i\frac{t}{h}\phi_k} \chi_0(\zeta^2 + \eta^2) d\zeta.$$

Recall that $\chi_0 \in C_0^\infty$ is supported near 1. The phase function ϕ_k is given by

$$\phi_k(\zeta) = \frac{z}{t}\zeta + \left(\eta^2 + \zeta^2 + \gamma\eta^2\right)^{1/2},$$

with $\gamma = h^{2/3}\omega_k |\eta|^{-2/3} > 0$. We introduce a change of variables $\zeta = |\eta|\tilde{\zeta}, z = t\tilde{z}$. Then we obtain

$$\phi_k(\zeta) = |\eta| \left(\tilde{z}\tilde{\zeta} + (1 + \tilde{\zeta}^2 + \gamma)^{1/2}\right).$$

Differentiating with respect to $\tilde{\zeta}$, we get

$$\partial_{\tilde{\zeta}}\phi_k = |\eta| \left(\tilde{z} + \frac{\tilde{\zeta}}{(1 + \tilde{\zeta}^2 + \gamma)^{1/2}}\right).$$

Since η is bounded from below, $\tilde{\zeta} = \zeta/|\eta|$ is bounded, therefore we have $\left|\frac{\tilde{\zeta}}{(1+\tilde{\zeta}^2+\gamma)^{1/2}}\right| \leq 1 - 2\delta_1$, for some $\delta_1 > 0$ small. Then if $|\tilde{z}| \geq 1 - \delta_1$, the contribution of $\tilde{\zeta}$-integration is $O_{C^\infty}((h/t)^\infty)$ by integration by parts. Thus we may assume that $|\tilde{z}| \leq 1 - \delta_1$. In this case, the phase ϕ_k has a unique critical point on the support of χ_0. It is given by $\tilde{\zeta}_c = -\frac{\tilde{z}(1+\gamma)^{1/2}}{\sqrt{1-\tilde{z}^2}}$ and this critical point is nondegenerate since

$$\partial^2_{\tilde{\zeta}}\phi_k = |\eta| \left(\frac{1 + \gamma}{(1 + \tilde{\zeta}^2 + \gamma)^{3/2}}\right) > 0.$$

Then we obtain by the stationary phase method (as $|\tilde{z}| < 1 - \delta_1$)

$$J = \left(\frac{h}{t}\right)^{1/2} e^{i\frac{t}{h}|\eta|\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2}} \tilde{\chi}_0,$$

where $\tilde{\chi}_0$ is a classical symbol of order 0 with small parameter h/t. Hence we get

$$\mathcal{G}_{a,c_0}(t,x,y,z) = \frac{1}{4\pi^2 h^2}\left(\frac{h}{t}\right)^{1/2} \sum_{k\geq 1} \int e^{\frac{i}{h}\left(y\eta+|\eta|t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2}\right)} e_k(x,\eta/h)e_k(a,\eta/h) \\ \times \tilde{\chi}_0\psi_0(\eta)\chi_1(\gamma|\eta|^2)(1-\chi_1)(\varepsilon\gamma h^{-2/3}|\eta|^{2/3})d\eta. \tag{2.1.3}$$

Next, we observe that $\mathcal{G}_{a,c_0}$ contains Airy functions which behave differently depending on the various values of k. To deal with it, we split the sum over k into $\mathcal{G}_{a,c_0} = \mathcal{G}_{a,<L} + \mathcal{G}_{a,>L}$, where in $\mathcal{G}_{a,<L}$ only the sum over $1 \leq k \leq L$ is considered. To get the estimates for $\mathcal{G}_{a,<L}$, we need the next lemma, which follows from the bound $|Ai(s)| \leq C(1+|s|)^{-1/4}$.

Lemma 2.1.2. *(Lemma 3.5[29]) There exists C_0 such that for $L \geq 1$, the following holds:*

$$\sup_{\mathbf{b}\in\mathbb{R}} \left(\sum_{1\leq k\leq L} k^{-1/3} Ai^2(\mathbf{b}-\omega_k)\right) \leq C_0 L^{1/3}.$$

We use the Cauchy-Schwarz inequality for (2.1.3) and Lemma 2.1.2 to get

$$\begin{aligned}\|\mathcal{G}_{a,<L}\|_{L^\infty} &\lesssim h^{-2}\left(\frac{h}{t}\right)^{1/2} \sum_{1\leq k\leq L} h^{-2/3}k^{-1/3}Ai(|\eta/h|^{2/3}x-\omega_k)Ai(|\eta/h|^{2/3}a-\omega_k), \\ &\lesssim h^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}\left(\sum_{1\leq k\leq L} k^{-1/3}Ai^2(h^{-2/3}|\eta|^{2/3}x-\omega_k)\right)^{1/2} \\ &\qquad\qquad \times \left(\sum_{1\leq k\leq L} k^{-1/3}Ai^2(h^{-2/3}|\eta|^{2/3}a-\omega_k)\right)^{1/2}, \\ &\lesssim h^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}L^{1/3}.\end{aligned}$$

We only have to prove (2.1.2) for $t > h$. Let $\epsilon \in]0,1/3[$ and $L = h^{-\epsilon}$. If $t \leq h^\epsilon$, then $L \leq \frac{1}{t}$, hence

$$\|\mathcal{G}_{a,<L}(t,x,y,z)\|_{L^\infty} \leq Ch^{-3}\left(\frac{h}{t}\right)^{5/6}.$$

We are reduced to the case $t > h^\epsilon \geq h^{1/3}$. Then we apply the stationary phase for η-integration of the form

$$\int e^{\frac{i}{h}\Phi_k} Ai(h^{-2/3}|\eta|^{2/3}x-\omega_k)Ai(h^{-2/3}|\eta|^{2/3}a-\omega_k)d\eta,$$

with the phase function

$$\Phi_k(\eta) = \eta(y + t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2}).$$

To deal with this integral, we rewrite $\Phi_k = h\lambda\Psi_k$ where $\lambda = t\omega_k h^{-1/3}$ is a large parameter. We have $|\partial_\eta^2 \Psi_k| \geq c > 0$. To apply the stationary phase, we need to check that one has for some $\nu > 0$ one has

$$|\partial_\eta^j Ai(h^{-2/3}|\eta|^{2/3}x - \omega_k)| \leq C_j \lambda^{j(1/2-\nu)}.$$

Since one has $\sup_{b\geq 0} |b^l Ai^{(l)}(b-\omega_k) \leq C_l \omega_k^{3l/2}$, it is sufficient to check that there exists $\epsilon > 0$ such that for $t > h^\epsilon$ and $k \leq h^{-\epsilon}$,

$$\omega_k^{3/2} \leq (t\omega_k h^{-1/3})^{(1/2-\nu)}$$

This holds if $\epsilon < 1/7$. Therefore the estimate for $\epsilon < 1/7$ and $t > h^\epsilon$ is

$$\begin{aligned}
\|\mathbb{1}_{x\leq a}\mathcal{G}_{a,<L}(t,x,y,z)\|_{L^\infty} &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{1/3}\sum_{1\leq k\leq h^{-\epsilon}} k^{-1/3}\lambda^{-1/2}],\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{1/3}\sum_{1\leq k\leq h^{-\epsilon}} k^{-1/3}(t\omega_k h^{-1/3})^{-1/2}],\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left[\left(\frac{h}{t}\right)^{1/2} h^{-\epsilon/3}\right],\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\frac{h}{t}\right)^{1/3}.
\end{aligned}$$

We now deal with large values of $k, L \leq k \leq \varepsilon/h$ with $L \geq D\max\{h^{-\epsilon}, 1/t\}, D > 0$ large constant. We are left to prove (2.1.2) holds true for $\mathcal{G}_{a,>L}$, defined by the sum over $L \leq k \leq \frac{\varepsilon}{h}$. For $k > Dh^{-\epsilon}$ and $0 \leq x \leq a \leq h^{\frac{2}{3}(1-\epsilon)}$, we have

$$\omega_k - |\eta|^{2/3}h^{-2/3}x > \omega_k/2.$$

Therefore we can use the asymptotic expansion of the Airy function (see Appendix A)

$$Ai(\vartheta) = \sum_{\pm} \omega^{\pm} e^{\mp\frac{2}{3}i(-\vartheta)^{3/2}}(-\vartheta)^{-1/4}\Psi_{\pm}(-\vartheta) \text{ for } -\vartheta > 1, \text{where } \omega^{\pm} = e^{\pm i\pi/4}$$

and where $\Psi_\pm$ are given in the Appendix. By the definition of e_k, we have

$$\begin{aligned} e_k(x,\eta/h) &= f_k \frac{|\eta|^{1/3}h^{-1/3}}{k^{1/6}} Ai(h^{-2/3}|\eta|^{2/3}x-\omega_k), \\ &= f_k \frac{|\eta|^{1/3}h^{-1/3}}{k^{1/6}} \sum_{\pm} \omega^{\pm} e^{\mp\frac{2}{3}i(\omega_k-|\eta|^{2/3}h^{-2/3}x)^{3/2}} \frac{\Psi_{\pm}(\omega_k-|\eta|^{2/3}h^{-2/3}x)}{(\omega_k-|\eta|^{2/3}h^{-2/3}x)^{1/4}}. \end{aligned}$$

We can rewrite $\mathcal{G}_{a,>L}$ as follows:

$$\mathcal{G}_{a,>L}(t,x,y,z) = \sum_{L\le k\le\frac{\varepsilon}{h}} \frac{1}{4\pi^2h^2}\left(\frac{h}{t}\right)^{1/2} \sum_{\pm,\pm} \int e^{\frac{i}{h}\Phi_k^{\pm,\pm}} \sigma_k^{\pm,\pm} d\eta, \tag{2.1.4}$$

with the phase functions are defined by

$$\Phi_k^{\pm,\pm}(t,x,y,z,a;\eta) = y\eta + |\eta|t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2} \pm \frac{2}{3}|\eta|(\gamma-x)^{3/2} \pm \frac{2}{3}|\eta|(\gamma-a)^{3/2},$$

and the symbols are given by

$$\begin{aligned} \sigma_k^{\pm,\pm}(x,a,h;\eta) &= h^{-1/3}|\eta|^{1/3}\tilde{\chi}_0\chi_1(\gamma\eta^2)(1-\chi_1)(\varepsilon\gamma h^{-2/3}|\eta|^{2/3})\frac{f_k^2}{k^{1/3}}\omega^{\pm}\omega^{\pm} \\ &\quad \times (\gamma-x)^{-1/4}(\gamma-a)^{-1/4}\Psi_{\pm}(|\eta|^{2/3}h^{-2/3}(\gamma-x))\Psi_{\pm}(|\eta|^{2/3}h^{-2/3}(\gamma-a)). \end{aligned}$$

We have $3\eta\partial_\eta = -2\gamma\partial_\gamma$ and for $0\le x\le a\le 2\gamma$,

$$|(\gamma\partial_\gamma)^j((\gamma-x)^{-1/4})| \le C_j\gamma^{-1/4} \le C_j'(hk)^{-1/6};$$

moreover, $\Psi_{\pm}$ are classical symbols of order 0 at infinity which is true in this case since we have

$$|\eta^{2/3}h^{-2/3}(\gamma-x)| \ge \omega_k/2 \ge Ch^{-2\epsilon/3},$$

since $k\ge L\ge h^{-\epsilon}$. Hence we obtain that for all j, there exists C_j such that

$$|\partial_\eta^j \sigma_k^{\pm,\pm}(x,a,h;\eta)| \le C_j(hk)^{-2/3},$$

since in the symbols $\sigma_k^{\pm,\pm}$ there is a factor $(hk)^{-1/3}$ and we apply η derivatives to the product $(\gamma-x)^{-1/4}(\gamma-a)^{-1/4}$ to get another factor $(hk)^{-1/3}$.

Now we study the oscillatory integral of the form

$$\int e^{\frac{i}{h}\Phi_k^{\pm,\pm}}\sigma_k^{\pm,\pm}d\eta.$$

To get the estimates for this integral, we set $\Phi_k^{\pm,\pm} = h\lambda\psi_k^{\pm,\pm}$, where $\lambda = t\omega_k h^{-1/3}$. It defines a new large parameter since $\lambda\ge c>0$ as $\omega_k\simeq k^{2/3}, k\ge 1/t,$ and $t\ge h$. The following result gives an estimate of these oscillatory integrals.

Proposition 2.1.3. *Let $\epsilon \in]0,1/7[$. For small ε, there exists a constant C independent of $a \in (0, h^{\frac{2}{3}(1-\epsilon)}]$, $t \in [h,1], x \in [0,a], y \in \mathbb{R}, z \in \mathbb{R}$ and $k \in [L, \frac{\varepsilon}{h}]$ such that the following holds:*

$$\left| \int e^{i\lambda\psi_k^{\pm,\pm}} \sigma_k^{\pm,\pm} d\eta \right| \leq C(hk)^{-2/3}\lambda^{-1/3}.$$

Proof of Proposition 2.1.3. Since $(hk)^{2/3}\sigma_k^{\pm,\pm}$ are classical symbols of degree 0 compactly supported in η, we apply the stationary phase method to an integral of the form

$$J_1 = \int e^{i\lambda\psi_k^{\pm,\pm}} (hk)^{2/3}\sigma_k^{\pm,\pm} d\eta.$$

We have to prove that the following inequality holds uniformly with respect to the parameters:

$$|J_1| \leq C\lambda^{-1/3}.$$

Let us recall that

$$h\lambda\psi_k^{\pm,\pm}(t,x,y,z;\eta) = y\eta + |\eta| t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2} \pm \frac{2}{3}|\eta|(\gamma - x)^{3/2} \pm \frac{2}{3}|\eta|(\gamma-a)^{3/2}.$$

We compute

$$h\lambda\partial_\eta\psi_k^{\pm,\pm} = y + t\sqrt{1-\tilde{z}^2}\frac{1+\frac{2}{3}\gamma}{\sqrt{1+\gamma}} \pm \frac{2}{3}x(\gamma-x)^{1/2} \pm \frac{2}{3}a(\gamma-a)^{1/2},$$

and we need to consider four cases. Let $\delta = \frac{x}{a} \in [0,1], \alpha = \frac{a}{\omega_k h^{2/3}} \in [0, \alpha_0]$. Indeed, since $\omega_k \simeq k^{2/3}$, $k \geq Dh^{-\epsilon}$ and $a \leq h^{\frac{2}{3}(1-\epsilon)}$, we have $\alpha = ak^{-2/3}h^{-2/3} \leq D^{-2/3}ah^{-\frac{2}{3}(1-\epsilon)} \leq D^{-2/3} := \alpha_0$. Let $\rho = |\eta|^{-2/3}$, $V = \frac{y+t\sqrt{1-\tilde{z}^2}}{t\omega_k h^{2/3}}$ and define the function $F(\gamma)$ by

$$\frac{1+\frac{2}{3}\gamma}{\sqrt{1+\gamma}} = 1 + \gamma F(\gamma), \quad F(\gamma) = \frac{1}{6} + \frac{\gamma}{24} + O(\gamma^2).$$

With these notations we get:

$$\partial_\eta\psi_k^{\pm,\pm} = V + \sqrt{1-\tilde{z}^2}\rho F(h^{2/3}\omega_k\rho) + \frac{2}{3}\mu\left(\pm\delta(\rho-\delta\alpha)^{1/2} \pm (\rho-\alpha)^{1/2}\right),$$

where $\mu = \frac{ah^{-1/3}}{t\omega_k^{1/2}}$; it satisfies $0 \leq \mu \leq \frac{h^{\frac{1}{3}(1-\epsilon)}}{t}\min\{1, h^{-\epsilon/3}t^{1/3}\}$ and thus μ may be small or arbitrary large. In fact, if $t \geq h^\epsilon$, $\mu \leq h^{\frac{1}{3}(1-\epsilon)}t^{-1} \leq h^{1/3-4\epsilon/3}$, which is small if $\epsilon \leq 1/4$. If $t \leq h^\epsilon$, we have $\mu \leq h^{1/3-2\epsilon/3}t^{-2/3}$ which could be large when $t \leq h^{1/2-\epsilon}$.
First, we consider the case where μ is bounded. We now study the critical points. We

take $\rho = |\eta|^{-2/3}$ as variable, we get

$$\partial_\rho \partial_\eta \psi_k^{\pm,\pm} = \sqrt{1-\tilde{z}^2}(F(\gamma) + \gamma F'(\gamma)) + \frac{\mu}{3}\big(\pm \delta(\rho - \delta\alpha)^{-1/2} \pm (\rho-\alpha)^{-1/2}\big),$$
$$\partial_\rho^2 \partial_\eta \psi_k^{\pm,\pm} = \sqrt{1-\tilde{z}^2} h^{2/3}\omega_k(2F'(\gamma) + \gamma F''(\gamma)) - \frac{\mu}{6}\big(\pm \delta(\rho - \delta\alpha)^{-3/2} \pm (\rho-\alpha)^{-3/2}\big).$$

For ε small enough, there exists $c > 0$ independent of $k \le \frac{\varepsilon}{h}$ such that

$$|\partial_\rho \partial_\eta \psi_k^{\pm,\pm}| + |\partial_\rho^2 \partial_\eta \psi_k^{\pm,\pm}| \ge c. \tag{2.1.5}$$

Indeed, we observe that $(\rho-\alpha)^{-1/2} \ge \delta(\rho - \delta\alpha)^{-1/2}$ and $F(\gamma) + \gamma F'(\gamma) \simeq \frac{1}{6}$. Thus we get $|\partial_\rho \partial_\eta \psi_k^{\pm,+}| \ge c_1 > 0$. Other cases, $\partial_\rho \partial_\eta \psi_k^{\pm,-}$ could vanish and when this happens we have

$$|\partial_\rho \partial_\eta \psi_k^{\pm,-}| \le 1/100 \Longrightarrow \frac{\mu}{3}(\rho-\alpha)^{-1/2} \ge 0.05.$$

Then we have $|\partial_\rho^2 \partial_\eta \psi_k^{-,-}| \ge c_2 > 0$. Moreover, for any function f, we have

$$f(\rho-\alpha) - \delta f(\rho - \delta\alpha) = (1-\delta) f(\rho - \delta\alpha) - \int_0^{\alpha(1-\delta)} f'(\rho - \delta\alpha - t)dt. \tag{2.1.6}$$

Taking $f(t) = t^{-1/2}$, we get that

$$|\partial_\rho \partial_\eta \psi_k^{+,-}| \le 1/100 \Longrightarrow \mu(1-\delta) \ge c > 0.$$

Applying (2.1.6) with $f(t) = t^{-3/2}$, we obtain $|\partial_\rho^2 \partial_\eta \psi_k^{+,-}| \ge c/2 > 0$. As a consequence of (2.1.5) together with Lemma 2.20 in [29][see Appendix], we get that the proposition holds true for μ bounded.

It remains to study the case where μ is large. For $(+,+)$ or $(-,+)$ case, we study again the critical points and we take $\Lambda = \lambda\mu$ as a large parameter. Since $\delta(\rho - \delta\alpha)^{-1/2} + (\rho-\alpha)^{-1/2} \ge c > 0$, we have $|\partial_\rho \partial_\eta \psi_k^{\pm,+}| \ge c > 0$. Hence $|J_1| \le C(\lambda\mu)^{-1/2}$. For $(+,-)$ and $(-,-)$ cases, we can use (2.1.6). We distinguish between two cases: if $\mu(1-\delta)$ is bounded, the computation of the derivatives of the phase functions $\psi_k^{\pm,-}$ yields the inequality (2.1.5) and the conclusion follows the Lemma 2.20 [29]. If $\mu(1-\delta)$ is large, we take $\Lambda' = \lambda\mu(1-\delta)$ as a large parameter in J_1 . Since by (2.1.6), we have

$$|(\rho-\alpha)^{-1/2} - \delta(\rho - \delta\alpha)^{-1/2}| \ge c(1-\delta)$$

with $c > 0$. We get that $|\partial_\rho \partial_\eta \psi_k^{\pm,-}| \ge c > 0$ and hence $|J_1| \le C(\lambda\mu(1-\delta))^{-1/2}$. □

To summarize, the Proposition 2.1.3 yields the dispersive estimates for the large values of k, $L \le k \le \varepsilon/h$ as follows:

$$\begin{aligned}
\|\mathbb{1}_{x\le a}\mathcal{G}_{a,>L}(t,x,y,z)\|_{L^\infty} &\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2}\sum_{k\le\frac{\varepsilon}{h}}(hk)^{-2/3}\lambda^{-1/3},\\
&\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2}\sum_{k\le\frac{\varepsilon}{h}}(hk)^{-2/3}(t\omega_k h^{-1/3})^{-1/3},\\
&\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2}\sum_{k\le\frac{\varepsilon}{h}}(hk)^{-2/3}t^{-1/3}k^{-2/9}h^{1/9},\\
&\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\frac{h}{t}\right)^{1/3}h^{1/9}\left(\sum_{k\le\frac{\varepsilon}{h}}k^{-8/9}\right),\\
&\le Ch^{-3}\left(\frac{h}{t}\right)^{5/6},
\end{aligned}$$

where we used $\lambda = t\omega_k h^{-1/3}$ in the second line, and $\omega_k \simeq k^{2/3}$ in the third line.

This concludes the proof of Proposition 2.1.1. □

2.1.2 Airy-Poisson Summation Formula.

Let $A_\pm(z) = e^{\mp i\pi/3}Ai(e^{\mp i\pi/3}z)$, we have $Ai(-z) = A_+(z) + A_-(z)$. For $\omega \in \mathbb{R}$, set

$$L(\omega) = \pi + i\log\left(\frac{A_-(\omega)}{A_+(\omega)}\right).$$

As in Lemma 2.7 in [27], the function L is analytic, strictly increasing and satisfies

$$L(0) = \pi/3, \quad \lim_{\omega\to-\infty} L(\omega) = 0, \quad L(\omega) = \frac{4}{3}\omega^{3/2} - B(\omega^{3/2}),\ \text{for } \omega \ge 1,$$

with

$$B(\omega) \simeq \sum_{j\ge1} b_j\omega^{-j}, \quad b_j \in \mathbb{R}, \quad b_1 \ne 1,$$

and for all $k \ge 1$, the following holds

$$L(\omega_k) = 2\pi k \Leftrightarrow Ai(-\omega_k) = 0, \quad L'(\omega_k) = 2\pi\int_0^\infty Ai^2(x-\omega_k)dx.$$

Recall that f_k are constants such that $\|e_k(.,\eta)\|_{L^2(\mathbb{R}_+)} = 1$. This yields

$$\int_0^\infty Ai^2(x-\omega_k)dx = \frac{k^{1/3}}{f_k^2} = \frac{L'(\omega_k)}{2\pi}.$$

The next lemma, whose proof is in the Appendix, is the key tool to transform the sum over the eigenmodes k to the sum over N.

Lemma 2.1.4 (Airy-Poisson Summation Formula). *The following equality holds true in $\mathcal{D}'(\mathbb{R}_\omega)$,*

$$\sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)}\delta_{\omega=\omega_k}.$$

Now we rewrite (1.4.4) and we replace the factor $\frac{f_k^2}{k^{1/3}}$ by $\frac{2\pi}{L'(\omega_k)}$. We get

$$\begin{aligned}
\mathcal{G}_{a,c_0}(t,x,y,z) &= \frac{1}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} \sum_{k\geq 1} \frac{|f_k|^2}{k^{1/3}} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3}\chi_0(\eta^2+\zeta^2) \\
&\quad\times \psi_0(\eta)\chi_1(\omega_k h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega_k) Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega_k\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega_k\Big) d\eta d\zeta, \\
&= \frac{1}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} \sum_{k\geq 1} \frac{2\pi}{L'(\omega_k)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega_k h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3}\chi_0(\eta^2+\zeta^2) \\
&\quad\times \psi_0(\eta)\chi_1(\omega_k h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega_k) Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega_k\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega_k\Big) d\eta d\zeta, \\
&= \frac{1}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} 2\pi \sum_{k\geq 1} \frac{\delta_{\omega=\omega_k}}{L'(\omega_k)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3}\chi_0(\eta^2+\zeta^2) \\
&\quad\times \psi_0(\eta)\chi_1(\omega h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega) Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega\Big) d\omega d\eta d\zeta.
\end{aligned}$$

Using Lemma 2.1.4, $\mathcal{G}_{a,c_0}$ becomes

$$\begin{aligned}
\mathcal{G}_{a,c_0}(t,x,y,z) &= \frac{1}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} \sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3}\chi_0(\zeta^2+\eta^2) \\
&\quad\times \psi_0(\eta)\chi_1(\omega h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega) Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega\Big) d\omega d\eta d\zeta.
\end{aligned}$$

$$\begin{aligned}
&= \sum_{N\in\mathbb{Z}} \frac{(-1)^N}{(2\pi)^2 h^{8/3}} \int e^{\frac{i}{h}(y\eta+z\zeta)} e^{i\frac{t}{h}(\eta^2+\zeta^2+\omega h^{2/3}|\eta|^{4/3})^{1/2}} |\eta|^{2/3}\chi_0(\zeta^2+\eta^2)\psi_0(\eta)\chi_1(\omega h^{2/3}|\eta|^{4/3}) \\
&\qquad\times (1-\chi_1)(\varepsilon\omega) \left(\frac{A_-(\omega)}{A_+(\omega)}\right)^N Ai\Big(h^{-2/3}|\eta|^{2/3}x-\omega\Big) Ai\Big(h^{-2/3}|\eta|^{2/3}a-\omega\Big) d\omega d\eta d\zeta, \\
&= \sum_{N\in\mathbb{Z}} \frac{(-i)^N}{(2\pi)^4 h^{10/3}} \int e^{\frac{i}{h}\Big(y\eta+z\zeta+t(\eta^2+\zeta^2+\omega h^{2/3}|\eta|^{4/3})^{1/2}+\frac{s^3}{3}+s(|\eta|^{2/3}x-\omega h^{2/3})+\frac{\sigma^3}{3}+\sigma(|\eta|^{2/3}a-\omega h^{2/3})\Big)} \\
&\qquad\times |\eta|^{2/3}\chi_0(\zeta^2+\eta^2)\psi_0(\eta)\chi_1(\omega h^{2/3}|\eta|^{4/3})(1-\chi_1)(\varepsilon\omega) e^{-\frac{4}{3}iN\omega^{3/2}+iNB(\omega^{3/2})} ds d\sigma d\omega d\eta d\zeta,
\end{aligned} \tag{2.1.7}$$

where we used the definition of the Airy function [see Appendix A]

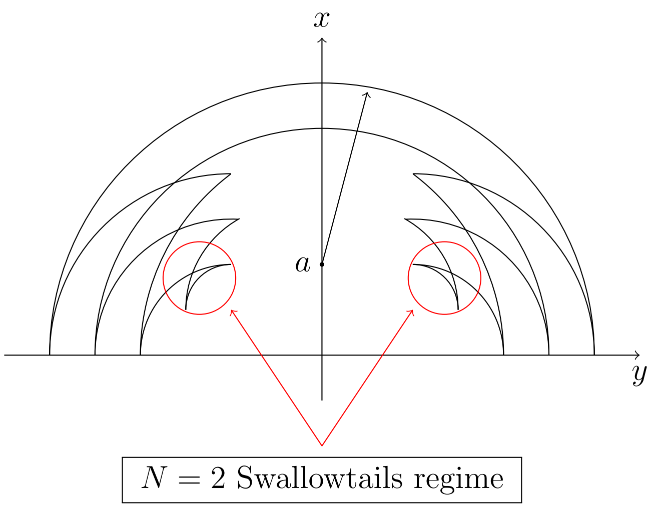
$$Ai(-\tilde{z}) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(s^3/3 - s\tilde{z})}ds, \quad \text{and} \quad \left(\frac{A_-(\omega)}{A_+(\omega)}\right)^N = i^N e^{-\frac{4}{3}iN\omega^{3/2}} e^{iNB(\omega^{3/2})},$$

where for $\tilde{z} \in \mathbb{R}_+$, we recall that $B(\tilde{z}) \in \mathbb{R}$ and $B(\tilde{z}) \sim \sum_{j\geq 1} b_j \tilde{z}^{-j}$ for $\tilde{z} \to +\infty$ and $b_1 \neq 0$. From the second to the third line, we made a change of variables $s = Sh^{-1/3}, \sigma = \Sigma h^{-1/3}$ in the Airy functions; but for simplicity we keep the notations s, σ.

Therefore, (2.1.7) is a local parametrix that reads as a sum over N. Notice that our parametrix coincides with the constructed sum over reflected waves in [29] since each term has essentially the same phase. In the sequel, we refer N as multiple reflections.

2.1.3 Dispersive Estimates for $a \geq h^{\frac{2}{3}-\epsilon'}, \epsilon' \in]0, \epsilon[$.

In this section, we establish the local in time dispersive estimates for the parametrix in the form (2.1.7) as a sum over multiple reflections on the boundary in the regime $a \geq h^{\frac{2}{3}-\epsilon'}$, for $\epsilon' \in]0, \epsilon[$. Recall that our local parametrix under the form (2.1.7) is constructed from (1.4.4) together with the Lemma 2.1.4. It is a sum of oscillatory integrals with phase functions containing an Airy type terms with degenerate critical points.



To deal with (2.1.7), we introduce a change of variables $a\tilde{\omega} = h^{2/3}\omega|\eta|^{-2/3}, x = aX$, $\zeta = |\eta|\tilde{\zeta}, s = a^{1/2}|\eta|^{1/3}\tilde{s}, \sigma = a^{1/2}|\eta|^{1/3}\tilde{\sigma}$. Then we can rewrite $\mathcal{G}_{a,c_0}$ as follows:

$$\mathcal{G}_{a,c_0}(t,x,y,z) = \sum_{N\in\mathbb{Z}} G_{a,N}, \tag{2.1.8}$$

with for each $N \in \mathbb{Z}$,

$$G_{a,N}(t,x,y,z) = \frac{(-i)^N a^2}{(2\pi)^4 h^4} \int e^{\frac{i}{h}\Phi_{N,a,h}} |\eta|^3 \chi_0(\eta^2(1+|\tilde{\zeta}|^2))\psi_0(\eta)\chi_1(a\tilde{\omega}\eta^2) \times (1-\chi_1)(\varepsilon a h^{-2/3}|\eta|^{2/3}\tilde{\omega}) d\tilde{s}d\tilde{\sigma}d\tilde{\omega}d\tilde{\zeta}d\eta, \quad (2.1.9)$$

with the phase function $\Phi_{N,a,h} = \Phi_{N,a,h}(t,x,y,z;\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\zeta},\eta)$,

$$\Phi_{N,a,h} = y\eta + |\eta|z\tilde{\zeta} + |\eta|t(1+\tilde{\zeta}^2 + a\tilde{\omega})^{1/2} + a^{3/2}|\eta|\Bigg(\frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{h}{a^{3/2}|\eta|}NB\big(\tilde{\omega}^{3/2}a^{3/2}|\eta|/h\big)\Bigg).$$

The main result of this section is Theorem 2.1.5. It gives the estimate of the sum over N of the oscillatory integrals of the form (2.1.9) by using the stationary phase type estimates with degenerate critical points.

Theorem 2.1.5. *Let $\alpha < 2/3$. There exists C such that for all $h \in]0,h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0,1]$, all $T \in]0, a^{-1/2}]$, all $Y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left|\sum_{0\le N\le C_0 a^{-1/2}} G_{a,N}(T,X,Y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\left(\frac{h}{t}\right)^{1/2} + a^{1/8}h^{1/4}\right). \quad (2.1.10)$$

Notice that the first part on the right hand side of (2.1.10) corresponds to the free space estimates in $\mathbb{R}^3$, while the contribution in the second part appears as a consequence of the presence of caustics (cusps and swallowtails type).

First of all, we observe that when $N = 0, G_{a,0}$ satisfies $PG_{a,0} = 0$ and the associated data at time $t = 0$ is a localized Dirac at $x = a, y = 0, z = 0$. Therefore, $G_{a,0}$ satisfies the classical dispersive estimate for the wave equation in three-dimensional space; that is,

$$|G_{a,0}(T,X,Y,z,h)| \le Ch^{-3}\left(\frac{h}{t}\right).$$

Thus it remains to prove the theorem for the sum over $1 \le N \le C_0 a^{-1/2}$.

Lemma 2.1.6. *One has*

$$J_{N,a,h} = \int e^{\frac{i}{h}|\eta|(z\tilde{\zeta}+t(1+\tilde{\zeta}^2+a\tilde{\omega})^{1/2})}\chi_0(\eta^2(1+|\tilde{\zeta}|^2))d\tilde{\zeta} = \left(\frac{h}{t}\right)^{1/2} e^{\frac{i}{h}|\eta|\sqrt{t^2-z^2}(1+a\tilde{\omega})^{1/2}}\tilde{\chi}_0,$$

where $\tilde{\chi}_0$ is a classical symbol of order 0 with small parameter h/t.

Proof. We apply the classical stationary phase method for $J_{N,a,h}$. First we make a change of variable $z = t\tilde{z}$. Let the phase function ϕ be

$$\phi(\tilde{\zeta};\tilde{z},\tilde{\omega},a) = \tilde{z}\tilde{\zeta} + (1+\tilde{\zeta}^2 + a\tilde{\omega})^{1/2}.$$

Differentiating with respect to $\tilde{\zeta}$, we get

$$\partial_{\tilde{\zeta}}\phi = \tilde{z} + \frac{\tilde{\zeta}}{(1+\tilde{\zeta}^2+a\tilde{\omega})^{1/2}}.$$

On the support of χ_0, we have $\left|\frac{\tilde{\zeta}}{(1+\tilde{\zeta}^2+a\tilde{\omega})^{1/2}}\right| \leq 1-2\delta_1$ for some $\delta_1 > 0$ small. If $|\tilde{z}| \geq 1-\delta_1$, then the contribution of $\tilde{\zeta}$-integration is $O_{C^\infty}((h/t)^\infty)$ by integration by parts. Thus we may assume that $|\tilde{z}| < 1-\delta_1$. In this case, the phase ϕ admits a unique critical point on the support of χ_0. It is given by $\tilde{\zeta}_c = -\frac{\tilde{z}(1+a\tilde{\omega})^{1/2}}{\sqrt{1-\tilde{z}^2}}$ and this critical point is nondegenerate since

$$\partial^2_{\tilde{\zeta}}\phi = \frac{1+a\tilde{\omega}}{(1+\tilde{\zeta}^2+a\tilde{\omega})^{3/2}} > 0.$$

Then by the stationary phase method (as $|\tilde{z}| < 1-\delta_1$),

$$J_{N,a,h} = \left(\frac{h}{t}\right)^{1/2} e^{i|\eta|\frac{t}{h}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{1/2}}\tilde{\chi}_0.$$

□

By Lemma 2.1.6,(2.1.8) becomes:

$$\mathcal{G}_{a,c_0}(t,x,y,z) = \sum_{N\in\mathbb{Z}} \frac{(-i)^N a^2}{(2\pi)^6 h^4}\left(\frac{h}{t}\right)^{1/2}\int e^{\frac{i}{h}\tilde{\Phi}_{N,a,h}}|\eta|^3\tilde{\chi}_0\psi_0\chi_1(1-\chi_1)d\tilde{s}d\tilde{\sigma}d\tilde{\omega}d\eta, \quad (2.1.11)$$

where $\tilde{\Phi}_{N,a,h} = \Phi_{N,a,h}(.,\tilde{\zeta}_c,.)$; that is,

$$\tilde{\Phi}_{N,a,h} = y\eta + |\eta|t\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{1/2} + a^{3/2}|\eta|\Big(\frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{h}{a^{3/2}|\eta|}NB\big(\tilde{\omega}^{3/2}a^{3/2}|\eta|/h\big)\Big). \quad (2.1.12)$$

First we study geometrically the set of critical points $\mathcal{C}_{N,a,h}$ of the associated Lagrangian manifold $\mathbf{\Lambda}_{N,a,h}$ for the phase function $\tilde{\Phi}_{N,a,h}$. The set of critical points is defined by

$$\mathcal{C}_{a,N,h} = \{(t,x,y,\tilde{s},\tilde{\sigma},\tilde{\omega},\eta)|\partial_{\tilde{s}}\tilde{\Phi}_{N,a,h} = \partial_{\tilde{\sigma}}\tilde{\Phi}_{N,a,h} = \partial_{\tilde{\omega}}\tilde{\Phi}_{N,a,h} = \partial_{\eta}\tilde{\Phi}_{N,a,h} = 0\}.$$

Hence $\mathcal{C}_{a,N,h}$ is defined by a system of equations

$$\begin{aligned}
&\frac{x}{a} = \tilde\omega - \tilde s^2,\\
&\tilde\omega = 1 + \tilde\sigma^2,\\
a^{-1/2}t &= \frac{2(1+a\tilde\omega)^{1/2}}{\sqrt{1-\tilde z^2}}\left(\tilde s + \tilde\sigma + 2N\tilde\omega^{1/2}\left(1 - \frac{3}{4}B'\Big(\tilde\omega^{3/2}\lambda\Big)\right)\right),\\
a^{-3/2}y &= -a^{-3/2}t\sqrt{1-\tilde z^2}(1+a\tilde\omega)^{1/2} - \frac{\tilde s^3}{3} - \tilde s(\frac{x}{a} - \tilde\omega) - \frac{\tilde\sigma^3}{3} - \tilde\sigma(1-\tilde\omega)\\
&\qquad + N\tilde\omega^{3/2}\left(\frac{4}{3} - B'\Big(\tilde\omega^{3/2}\lambda\Big)\right).
\end{aligned}$$

Let $\mathbf{\Lambda}_{a,N,h} \subset T^*\mathbb{R}^3$ be the image of $\mathcal{C}_{a,N,h}$ by the map

$$(t,x,y,\tilde s,\tilde\sigma,\tilde\omega,\eta) \longmapsto (x,t,y,\xi = \partial_x\tilde\Phi_{N,a,h}, \tau = \partial_t\tilde\Phi_{N,a,h}, \eta = \partial_y\tilde\Phi_{N,a,h}).$$

Then $\mathbf{\Lambda}_{a,N,h} \subset T^*\mathbb{R}^3$ is a Lagrangian submanifold parametrized by $(\tilde s,\tilde\sigma,\eta)$

$$\begin{aligned}
&\frac{x}{a} = \tilde\omega - \tilde s^2,\\
&\tilde\omega = 1 + \tilde\sigma^2,\\
a^{-1/2}t &= \frac{2(1+a\tilde\omega)^{1/2}}{\sqrt{1-\tilde z^2}}\left(\tilde s + \tilde\sigma + 2N\tilde\omega^{1/2}\left(1 - \frac{3}{4}B'\Big(\tilde\omega^{3/2}\lambda\Big)\right)\right),\\
a^{-3/2}y &= -a^{-3/2}t\sqrt{1-\tilde z^2}(1+a\tilde\omega)^{1/2} - \frac{\tilde s^3}{3} - \tilde s(\frac{x}{a} - \tilde\omega) - \frac{\tilde\sigma^3}{3} - \tilde\sigma(1-\tilde\omega)\\
&\qquad + N\tilde\omega^{3/2}\left(\frac{4}{3} - B'\Big(\tilde\omega^{3/2}\lambda\Big)\right),\\
&\xi = \eta\tilde s a^{1/2},\\
&\tau = \eta\sqrt{1-\tilde z^2}(1+a+a\tilde\sigma^2)^{1/2},\\
&\eta = \eta.
\end{aligned}$$

Now we introduce $t = a^{1/2}T, y + t\sqrt{1-\tilde z^2} = a^{3/2}Y, (1+a\tilde\omega)^{1/2} - 1 = a\gamma_a(\tilde\omega) = \frac{a\tilde\omega}{1+(1+a\tilde\omega)^{1/2}}$, and $\lambda = \frac{a^{3/2}}{h}|\eta|$. We get (2.1.12) as follows:

$$\begin{aligned}
\tilde\Phi_{N,a,h} = a^{3/2}|\eta|\Bigg\{ & Y + T\sqrt{1-\tilde z^2}\gamma_a(\tilde\omega) + \frac{\tilde s^3}{3} + \tilde s(X-\tilde\omega) + \frac{\tilde\sigma^3}{3} + \tilde\sigma(1-\tilde\omega)\\
&- \frac{4}{3}N\tilde\omega^{3/2} + \frac{h}{a^{3/2}|\eta|}NB\big(\tilde\omega^{3/2}a^{3/2}|\eta|/h\big)\Bigg\}. \qquad (2.1.13)
\end{aligned}$$

Then $\mathcal{C}_{a,N,h}$ is now defined by a system of equations

$$
\begin{aligned}
X &= \tilde{\omega} - \tilde{s}^2,\\
\tilde{\omega} &= 1 + \tilde{\sigma}^2,\\
T &= \frac{2(1+a\tilde{\omega})^{1/2}}{\sqrt{1-\tilde{z}^2}} \left(\tilde{s} + \tilde{\sigma} + 2N\tilde{\omega}^{1/2} \left(1 - \frac{3}{4} B'\Big(\tilde{\omega}^{3/2}\lambda\Big)\right)\right),\\
Y &= -T\sqrt{1-\tilde{z}^2}\gamma_a(\tilde{\omega}) - \frac{\tilde{s}^3}{3} - \tilde{s}(X - \tilde{\omega}) - \frac{\tilde{\sigma}^3}{3} - \tilde{\sigma}(1-\tilde{\omega}) + N\tilde{\omega}^{3/2}\left(\frac{4}{3} - B'\Big(\tilde{\omega}^{3/2}\lambda\Big)\right).
\end{aligned}
$$

We may parametrize $\mathcal{C}_{a,N,h}$ by $(\tilde{s}, \tilde{\sigma})$ near origin:

$$
\begin{aligned}
X &= 1 + \tilde{\sigma}^2 - \tilde{s}^2,\\
\tilde{\omega} &= 1 + \tilde{\sigma}^2,\\
T &= \frac{2}{\sqrt{1-\tilde{z}^2}}(1+a+a\tilde{\sigma}^2)^{1/2} \left(\tilde{s} + \tilde{\sigma} + 2N(1+\tilde{\sigma}^2)^{1/2} \left(1 - \frac{3}{4} B'\Big((1+\tilde{\sigma}^2)^{3/2}\lambda\Big)\right)\right),\\
Y &= H_1(a,\tilde{\sigma})(\tilde{s}+\tilde{\sigma}) + \frac{2}{3}(\tilde{s}^3 + \tilde{\sigma}^3) + \frac{4}{3} N H_2(a,\tilde{\sigma}) \left(1 - \frac{3}{4} B'\Big((1+\tilde{\sigma}^2)^{3/2}\lambda\Big)\right),
\end{aligned}
$$

with

$$
\begin{aligned}
H_1(a,\tilde{\sigma}) &= -(1+\tilde{\sigma}^2)\frac{(1+a+a\tilde{\sigma}^2)^{1/2}}{1+(1+a+a\tilde{\sigma}^2)^{1/2}},\\
H_2(a,\tilde{\sigma}) &= (1+\tilde{\sigma}^2)^{3/2}\frac{-3-4a-4a\tilde{\sigma}^2}{2+a+a\tilde{\sigma}^2+3(1+a+a\tilde{\sigma}^2)^{1/2}}.
\end{aligned}
$$

The Lagrangian submanifold $\mathbf{\Lambda}_{a,N,h} \subset T^*\mathbb{R}^3$ is parametrized by $(\tilde{s}, \tilde{\sigma}, \eta)$

$$
\begin{aligned}
X &= 1 + \tilde{\sigma}^2 - \tilde{s}^2,\\
T &= \frac{2}{\sqrt{1-\tilde{z}^2}}(1+a+a\tilde{\sigma}^2)^{1/2} \left(\tilde{s} + \tilde{\sigma} + 2N(1+\tilde{\sigma}^2)^{1/2} \left(1 - \frac{3}{4} B'\Big((1+\tilde{\sigma}^2)^{3/2}\lambda\Big)\right)\right),\\
Y &= H_1(a,\tilde{\sigma})(\tilde{s}+\tilde{\sigma}) + \frac{2}{3}(\tilde{s}^3 + \tilde{\sigma}^3) + \frac{4}{3} N H_2(a,\tilde{\sigma}) \left(1 - \frac{3}{4} B'\Big((1+\tilde{\sigma}^2)^{3/2}\lambda\Big)\right),\\
\xi &= \eta\tilde{s}a^{1/2},\\
\tau &= \eta\sqrt{1-\tilde{z}^2}(1+a+a\tilde{\sigma}^2)^{1/2},\\
\eta &= \eta.
\end{aligned}
$$

On $\mathcal{C}_{a,N,h}$, we have $\tilde{\omega} = 1 + \tilde{\sigma}^2$, thus the projection of $\mathbf{\Lambda}_{a,N,h}$ onto $\mathbb{R}^3$ is

$$\begin{aligned}
&X = 1 + \tilde{\sigma}^2 - \tilde{s}^2,\\
&T = \frac{2}{\sqrt{1-\tilde{z}^2}}(1 + a + a\tilde{\sigma}^2)^{1/2}\left(\tilde{s} + \tilde{\sigma} + 2N(1+\tilde{\sigma}^2)^{1/2}\left(1 - \frac{3}{4}B'\Big((1+\tilde{\sigma}^2)^{3/2}\lambda\Big)\right)\right),\\
&Y = H_1(a,\tilde{\sigma})(\tilde{s} + \tilde{\sigma}) + \frac{2}{3}(\tilde{s}^3 + \tilde{\sigma}^3) + \frac{4}{3}NH_2(a,\tilde{\sigma})\left(1 - \frac{3}{4}B'\Big((1+\tilde{\sigma}^2)^{3/2}\lambda\Big)\right).
\end{aligned}$$

As in [29], we rewrite the system (2.1.14) in the following form

$$\begin{aligned}
&X = 1 + \tilde{\sigma}^2 - \tilde{s}^2, \qquad\qquad (2.1.14)\\
&Y = H_1(a,\tilde{\sigma})(\tilde{s} + \tilde{\sigma}) + \frac{2}{3}(\tilde{s}^3 + \tilde{\sigma}^3) + \frac{2}{3}H_2(a,\tilde{\sigma})(1+\tilde{\sigma}^2)^{-1/2}\left(\frac{T\sqrt{1-\tilde{z}^2}}{2(1+a+a\tilde{\sigma}^2)^{1/2}} - \tilde{s} - \tilde{\sigma}\right),
\end{aligned}$$

and

$$2N\left(1 - \frac{3}{4}B'\Big(\tilde{\omega}^{3/2}\lambda\Big)\right) = (1+\tilde{\sigma}^2)^{-1/2}\left(\frac{T\sqrt{1-\tilde{z}^2}}{2(1+a+a\tilde{\sigma}^2)^{1/2}} - \tilde{s} - \tilde{\sigma}\right). \qquad (2.1.15)$$

Remark 2.1.7. *Notice that from (2.1.15) in the range of $T \in]0, a^{-1/2}]$, we can reduce the sum over $N \in \mathbb{Z}$ of $G_{a,N}$ in* (2.1.8) *to the sum over* $1 \leq N \leq C_0 a^{-1/2}$.

For a given a and $(X,Y,T) \in \mathbb{R}^3$, (2.1.14) is a system of two equations for unknown $(\tilde{s},\tilde{\sigma})$ and (2.1.15) gives an equation for N. We are looking for a solutions of (2.1.14) in the range

$$a \in [h^\alpha, a_0], \alpha < 2/3, a|\tilde{\sigma}|^2 \leq \epsilon_0, 0 < T \leq a^{-1/2}, X \in [0,1] \quad \text{with } a_0, \epsilon_0 \text{ small.}$$

Then for a given point $(X,Y,T) \in [-2,2] \times \mathbb{R} \times [0, a^{-1/2}]$, let us denote by $\mathcal{N}(X,Y,T)$ the set of integers $N \geq 1$ such that (2.1.14) admits at least one real solution $(\tilde{s},\tilde{\sigma},\lambda)$ with $a|\tilde{\sigma}|^2 \leq \epsilon_0$ and $\lambda \geq \lambda_0$.

We observe that (2.1.15) implies for $N_0 > 0$ independent of (X,Y,T) that $\mathcal{N}(X,Y,T) \subset [1, T/2 + N_0]$. For all $(X,Y,T) \in [0,1] \times \mathbb{R} \times [0,a^{-1/2}]$, there exists a constant C_0 such that $|\mathcal{N}(X,Y,T)| \leq C_0$. Set

$$\mathcal{N}_1(X,Y,T) = \bigcup_{|Y'-Y|+|T'-T| \leq 1, |X'-X| \leq 1} \mathcal{N}(X',Y',T').$$

In fact from (2.1.15) we have if $N, N' \in \mathcal{N}_1$,

$$2|N - N'| \leq C_0(1 + T\lambda^{-2}\tilde{\omega}^{-3}).$$

Hence we deduce a better estimate as follows [see Appendix E]:

$$|\mathcal{N}_1(X,Y,T)| \leq C_0(1+T\lambda^{-2}\tilde{\omega}^{-3}).$$

We notice that for $\tilde{\omega} \leq 3/4$, we get rapid decay in λ by integration by part in $\tilde{\sigma}$. In particular, we may replace $1-\chi_1$ by 1 in (2.1.11). Moreover, the swallowtails will appear when $\tilde{s} = \tilde{\sigma} = 0$ i.e for $\tilde{\omega} = 1$. For this reason, we introduce a cutoff function $\chi_2(\tilde{\omega}) \in C_0^\infty(]1/2, 3/2[), 0 \leq \chi_2 \leq 1, \chi_2 = 1$ on $]\frac{3}{4}, \frac{5}{4}[$ in the integral (2.1.11) and we denote by $G_{a,N,2}$ the corresponding integral. This $G_{a,N,2}$ corresponds to the regime of swallowtails. We write $G_{a,N} = G_{a,N,1} + G_{a,N,2}$. $G_{a,N,1}$ is defined by introducing χ_3 in (2.1.11). We will have $\tilde{\omega} \geq 5/4$ on the support of χ_3.
To summarize, we have $\mathcal{G}_{a,c_0}$ as follows:

$$\mathcal{G}_{a,c_0} = \sum_{1\leq N\leq C_0 a^{-1/2}} G_{a,N} = \sum_{1\leq N\leq C_0 a^{-1/2}} (G_{a,N,1} + G_{a,N,2}),$$

where

$$G_{a,N,1} = \frac{(-i)^N a^2}{(2\pi)^6 h^4}\left(\frac{h}{t}\right)^{1/2} \int e^{\frac{i}{h}\tilde{\Phi}_{N,a,h}} |\eta|^3 \tilde{\chi}_0 \psi_0 \chi_1 \chi_3(\tilde{\omega}) d\tilde{s} d\tilde{\sigma} d\tilde{\omega} d\eta,$$

$$G_{a,N,2} = \frac{(-i)^N a^2}{(2\pi)^6 h^4}\left(\frac{h}{t}\right)^{1/2} \int e^{\frac{i}{h}\tilde{\Phi}_{N,a,h}} |\eta|^3 \tilde{\chi}_0 \psi_0 \chi_1 \chi_2(\tilde{\omega}) d\tilde{s} d\tilde{\sigma} d\tilde{\omega} d\eta.$$

In what follows, we get the estimates for these oscillatory integrals based on the (degenerate) stationary phase type result which consists in the precise study of where the phase $\tilde{\Phi}_{N,a,h}$ may be stationary.

The Analysis of $G_{a,N,1}$

Let us recall that the $G_{a,N,1}$ is the oscillatory integral which corresponds to the regime where there are no swallowtails. Our main results of this subsection are Proposition 2.1.8 and Proposition 2.1.9.

Proposition 2.1.8. *There exists C such that for all $h \in]0, h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0,1]$, all $T \in]0, a^{-1/2}]$, all $Y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left| \sum_{2\leq N\leq C_0 a^{-1/2}} G_{a,N,1}(T,X,Y,z;h) \right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}.$$

Proof. First of all, we apply the stationary phase method to $(\tilde{s}, \tilde{\sigma})$-integrations since on the support of χ_3 we have $\tilde{\omega} > 1$. Let I be defined by

$$\begin{aligned}
I &= \int e^{i\lambda\left(\frac{\tilde{s}^3}{3} - \tilde{s}(\tilde{\omega}-X) + \frac{\tilde{\sigma}^3}{3} - \tilde{\sigma}(\tilde{\omega}-1)\right)} d\tilde{s}d\tilde{\sigma} \\
&= (\tilde{\omega}-X)^{1/2}(\tilde{\omega}-1)^{1/2} \int e^{i\lambda(\tilde{\omega}-X)^{3/2}\left(\frac{\tilde{s}^3}{3}-\tilde{s}\right)} e^{i\lambda(\tilde{\omega}-1)^{3/2}\left(\frac{\tilde{\sigma}^3}{3}-\tilde{\sigma}\right)} d\tilde{s}d\tilde{\sigma},
\end{aligned}$$

where in the second line we made a change of variables $\tilde{s} = (\tilde{\omega}-X)^{1/2}\bar{s}, \tilde{\sigma} = (\tilde{\omega}-1)^{1/2}\bar{\sigma}$ but for simplicity, we keep the notations $\tilde{s}, \tilde{\sigma}$. Thus by the stationary phase near the critical points $\tilde{s} = \pm 1, \tilde{\sigma} = \pm 1$ and integration by parts in $\tilde{s}, \tilde{\sigma}$ elsewhere we get

$$I = \lambda^{-1}(\tilde{\omega}-X)^{-1/4}(\tilde{\omega}-1)^{-1/4} e^{i\lambda\left(\pm\frac{2}{3}(\tilde{\omega}-X)^{3/2} \pm \frac{2}{3}(\tilde{\omega}-1)^{3/2}\right)} b_\pm c_\pm + O(\lambda^{-\infty}),$$

with $b_\pm, c_\pm$ are classical symbols of degree 0 in large parameter $\lambda(\tilde{\omega}-X)^{3/2}$ and $\lambda(\tilde{\omega}-1)^{3/2}$ respectively. Notice that I is a part of the $G_{a,N,1}$ corresponding to the integrations in $\tilde{s}, \tilde{\sigma}$. Therefore, we obtain

$$\begin{aligned}
G_{a,N,1}(T,X,Y,z;h) &= \frac{(-i)^N a^2 \lambda^{-1}}{(2\pi)^6 h^4} \left(\frac{h}{t}\right)^{1/2} \int e^{i\frac{a^{3/2}}{h}Y\eta} |\eta|^3 \tilde{G}_{a,N,1} d\eta, \\
\tilde{G}_{a,N,1}(T,X,Y,z;h) &= \sum_{\epsilon_1,\epsilon_2} \int e^{i\lambda\tilde{\Phi}_{N,\epsilon_1,\epsilon_2}} \Theta_{\epsilon_1,\epsilon_2} d\tilde{\omega} + O(\lambda^{-\infty}),
\end{aligned}$$

where $\epsilon_j = \pm$, $\Theta_{\epsilon_1,\epsilon_2}(\tilde{\omega}, a, \lambda) = \tilde{\chi}_0\psi_0\chi_1\chi_3(\tilde{\omega})(\tilde{\omega}-X)^{-1/4}(\tilde{\omega}-1)^{-1/4} b_{\epsilon_1}c_{\epsilon_2}$ which satisfy $\left|\tilde{\omega}^l \partial_{\tilde{\omega}}^l \Theta_{\epsilon_1,\epsilon_2}\right| \leq C_l \tilde{\omega}^{-1/2}$, and the phase functions are given by

$$\begin{aligned}
\tilde{\Phi}_{N,\epsilon_1,\epsilon_2}(T,X,z,\tilde{\omega};a,\lambda) = T\sqrt{1-\tilde{z}^2}\gamma_a(\tilde{\omega}) &+ \frac{2}{3}\epsilon_1(\tilde{\omega}-X)^{3/2} + \frac{2}{3}\epsilon_2(\tilde{\omega}-1)^{3/2} \\
&- \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{N}{\lambda}B(\tilde{\omega}^{3/2}\lambda). \quad (2.1.16)
\end{aligned}$$

Let us denote

$$\begin{aligned}
G_{a,N,1,\epsilon_1,\epsilon_2}(T,X,Y,z;h) &= \frac{(-i)^N a^2 \lambda^{-1}}{(2\pi)^6 h^4} \left(\frac{h}{t}\right)^{1/2} \int e^{i\frac{a^{3/2}}{h}Y\eta} |\eta|^3 \tilde{G}_{a,N,1,\epsilon_1,\epsilon_2} d\eta, \quad (2.1.17) \\
\tilde{G}_{a,N,1,\epsilon_1,\epsilon_2}(T,X,z;\lambda) &= \int e^{i\lambda\tilde{\Phi}_{N,\epsilon_1,\epsilon_2}} \Theta_{\epsilon_1,\epsilon_2}(\tilde{\omega},a,\lambda) d\tilde{\omega}.
\end{aligned}$$

We are reduced to proving the following inequality:

$$\left| \sum_{2 \leq N \leq C_0 a^{-1/2}} G_{a,N,1,\epsilon_1,\epsilon_2}(T,X,Y,z;h) \right| \leq C h^{-3} \left(\frac{h}{t}\right)^{1/2} h^{1/3}, \quad (2.1.18)$$

with a constant C independent of $h \in]0,h_0], a \in [h^{2/3}, a_0], X \in [0,1], T \in [0, a^{-1/2}]$. For convenience, let $\Omega = \tilde{\omega}^{3/2}$ be a new variable of integration and we get

$$\tilde{G}_{a,N,1,\epsilon_1,\epsilon_2}(T,X,z;\lambda) = \int e^{i\lambda\tilde{\Phi}_{N,\epsilon_1,\epsilon_2}}\tilde{\Theta}_{\epsilon_1,\epsilon_2}(\Omega,a,\lambda)d\Omega; \tag{2.1.19}$$

$\tilde{\Theta}_{\epsilon_1,\epsilon_2}(\Omega,a,\lambda)$ are smooth functions with compact support in Ω. Since $d\tilde{\omega} = \frac{2}{3}\Omega^{-1/3}d\Omega$, we get $\left|\Omega^l\partial_\Omega^l\tilde{\Theta}_{\epsilon_1,\epsilon_2}\right| \leq C_l\Omega^{-2/3}$ with C_l independent of a, λ and the phases (2.1.16) become

$$\begin{aligned}\tilde{\Phi}_{N,\epsilon_1,\epsilon_2}(T,X,z,\tilde{\omega};a,\lambda) = T\sqrt{1-\tilde{z}^2}\gamma_a(\Omega) + \frac{2}{3}\epsilon_1(\Omega^{2/3}-X)^{3/2} + \frac{2}{3}\epsilon_2(\Omega^{2/3}-1)^{3/2}\\ -\frac{4}{3}N\Omega + \frac{N}{\lambda}B(\Omega\lambda).\end{aligned}$$

We now study the critical points. We have

$$\partial_\Omega\tilde{\Phi}_{N,\epsilon_1,\epsilon_2} = \frac{2}{3}\left(H_{a,\epsilon_1,\epsilon_2}(T,X,z;\Omega) - 2N\Big(1-\frac{3}{4}B'(\Omega\lambda)\Big)\right), \tag{2.1.20}$$

$$H_{a,\epsilon_1,\epsilon_2} = \Omega^{-1/3}\Big(\frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a\Omega^{2/3})^{-1/2} + \epsilon_1(\Omega^{2/3}-X)^{1/2} + \epsilon_2(\Omega^{2/3}-1)^{1/2}\Big),$$

$$\begin{aligned}\partial_\Omega H_{a,\epsilon_1,\epsilon_2} = \frac{1}{3}\Omega^{-4/3}\Big(-\frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a\Omega^{2/3})^{-3/2}(1+2a\Omega^{2/3}) + \epsilon_1X(\Omega^{2/3}-X)^{-1/2}\\ +\epsilon_2(\Omega^{2/3}-1)^{-1/2}\Big).\end{aligned}$$

We will first prove that (2.1.18) holds true in the case $(\epsilon_1,\epsilon_2) = (+,+)$. We have that the equation $\partial_\Omega H_{a,+,+}(\Omega) = 0$ admits a unique solution $\Omega_q = \Omega_q^+(T,X,z,a) > 1$ such that

$$\lim_{T\to\infty}\Omega_q^+(T,X,z,a) = 1 \text{ uniformly in } X,z,a. \tag{2.1.21}$$

$$\begin{aligned}0 > \frac{9}{2}\Omega_q^{5/3}\partial_\Omega^2 H_{a,+,+}(\Omega_q) = -\frac{aT}{2}\sqrt{1-\tilde{z}^2}\big(1+a\Omega_q^{2/3}\big)^{-5/2}\big(\frac{1}{2}-a\Omega_q^{2/3}\big)\\ -\frac{1}{2}\big(\Omega_q^{2/3}-1\big)^{-3/2} - \frac{1}{2}X\big(\Omega_q^{2/3}-X\big)^{-3/2}.\end{aligned}$$

Therefore the function $H_{a,+,+}(\Omega)$ is strictly increasing on $[1,\Omega_q[$ and strictly decreasing on $]\Omega_q,\infty[$. Observe that

$$H_{a,+,+}(1) = \frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a)^{-1/2} + (1-X)^{1/2}, \quad \lim_{\Omega\to\infty}H_{a,+,+} = 2. \tag{2.1.22}$$

For all k, we have

$$\forall\Omega\geq 1, \quad |\partial_\Omega^k(NB'(\Omega\lambda))| \leq C_kN\lambda^{-2}\Omega^{-(k+2)}. \tag{2.1.23}$$

Let $T_0 \gg 1$. First suppose that $0 \leq T \leq T_0$. Since $H_{a,+,+}(\Omega) \leq C(1+T)$ and for $N \geq N(T_0) = C(1+T_0)$, we get $|\partial_\Omega\tilde{\Phi}_{N,+,+}| \geq c_0N$ with $c_0 > 0$. Then by integration by parts, we get $|\tilde{G}_{a,N,1,+,+}| \in O(N^{-\infty}\lambda^{-\infty})$ and this implies

$$\sup_{T\le T_0, X\in[0,1], Y\in\mathbb{R}, z\in\mathbb{R}} \left| \sum_{N(T_0)\le N\le C_0 a^{-1/2}} G_{a,N,1,+,+}(T,X,Y,z) \right| \in O(h^\infty).$$

Next for $0 \le T \le T_0$ and $2 \le N \le N(T_0)$, we may estimate the sum by the sup of each term. In this case, we see that $\tilde{\Phi}_{N,+,+}$ has at most a critical point of order 2 near $\Omega = \Omega_q$ and

$$|\partial_\Omega \tilde{\Phi}_{N,+,+}| + |\partial_\Omega^2 \tilde{\Phi}_{N,+,+}| + |\partial_\Omega^3 \tilde{\Phi}_{N,+,+}| \ge c > 0.$$

Moreover if $N \ge 2$, we have a positive lower bound for $|\partial_\Omega \tilde{\Phi}_{N,+,+}(\Omega)|$ for large values of Ω; thus the contribution of $\tilde{G}_{a,N,1,+,+}$ is $O_{C^\infty}(\lambda^{-\infty})$ for large values of Ω. Near the critical point of order 2 $\Omega = \Omega_q$, the estimate of $\tilde{G}_{a,N,1,+,+}$ is given by the Lemma 2.20 [29] which yields $|\tilde{G}_{a,N,1,+,+}(T,X,z;\lambda)| \le C\lambda^{-1/3}$ with C independent of $T \in [0,T_0], X \in [0,1]$. Hence from (2.1.17), we get

$$\begin{aligned}\sup_{X\in[0,1], Y\in\mathbb{R}, z\in\mathbb{R}} \left| \sum_{2\le N\le N(T_0)} G_{a,N,1,+,+}(T,X,Y,z,h) \right| &\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} [h^{-1}a^2\lambda^{-1}\lambda^{-1/3}], \\ &\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}.\end{aligned}$$

Then we prove that (2.1.18) holds true for $T_0 \le T \le a^{-1/2}$. As before, we may assume $N \le C_1 T$ with C_1 large, the contribution of the sum on N such that $C_1 T \le N \le C_0 a^{-1/2}$ being negligible. From (2.1.21), we may choose T_0 large enough so that $\Omega_q^+(T,X,z,a) < \Omega_0$ with $\Omega_0 > 1$ for $T \ge T_0$ and we may assume with a constant $c > 0$ that

$$|\partial_\Omega^2 \tilde{\Phi}_{N,+,+}(\Omega)| \ge cT\Omega^{-4/3}, \quad \forall\Omega \ge \Omega_0, \forall T \ge T_0, \forall N \le C_0 a^{-1/2}.$$

Therefore, on the support of $\tilde{\Theta}_{+,+}$, the phase $\tilde{\Phi}_{N,+,+}$ admits at most one critical point $\Omega_c = \Omega_c(T,X,z,N,\lambda,a)$ and this critical point is nondegenerate. Since $N \ge 2$, from the first item of (2.1.20) we get $\Omega_c^{1/3} \le T$ and this implies $\Omega_c^{1/3} \simeq T/N$. As a consequence, if T/N is bounded then Ω_c is bounded. By stationary phase method, we get

$$|\tilde{G}_{a,N,1,+,+}(T,X,z;\lambda)| \le C\lambda^{-1/2}T^{-1/2} \quad \text{with } C \text{ independent of } N.$$

If T/N is large, then we perform the change of variable $\Omega = \tilde{\Omega}(T/N)^3$ in (2.1.19); the unique critical point $\tilde{\Omega}_c$ remains in a fixed compact interval of $]0,\infty[$. We have

$$\partial_{\tilde{\Omega}}^k \tilde{\Theta}_{+,+}(\tilde{\Omega}(T/N)^3, a, \lambda) \le c_k (N/T)^2 \tilde{\Omega}^{-2/3-k}.$$

Thus by the stationary phase method, we get

$$\sup_{2\le N\le C_1 T, X\in[0,1], z\in\mathbb{R}} |\tilde{G}_{a,N,1,+,+}(T,X,z;\lambda)| \le C\lambda^{-1/2}T^{-1/2}.$$

It remains to estimate the sum

$$\left| \sum_{2\leq N\leq C_0 a^{-1/2}} G_{a,N,1,+,+}(T,X,Y,z;h) \right|.$$

Let $G_N(T,X,z,\lambda,a) = \tilde{\Phi}_{N,+,+}(T,X,z,\Omega_c(T,X,z,N,\lambda,a),\lambda,a)$. Then by the stationary phase method at the critical point $\Omega_c = \Omega_c(T,X,z,N,\lambda,a)$ in (2.1.19) we get

$$\tilde{G}_{a,N,1,+,+}(T,X,z,h) = \lambda^{-1/2}T^{-1/2}e^{i\lambda G_N(T,X,z,\lambda,a)}\psi_N(T,X,\lambda,a),$$

with $\psi_N(T,X,\lambda,a)$ is a classical symbol of order 0 in λ. Hence with $\tilde{\lambda} = a^{3/2}/h = \lambda/\eta$, we have

$$G_{a,N,1,+,+}(T,X,Y,z;h) = \frac{(-i)^N a^2\lambda^{-1}}{(2\pi)^6 h^4}\left(\frac{h}{t}\right)^{1/2}\lambda^{-1/2}T^{-1/2}\int e^{i\tilde{\lambda}|\eta|(Y+G_N(T,X,z,\tilde{\lambda}\eta,a))}\psi_N|\eta|^3 d\eta. \tag{2.1.24}$$

It is an oscillatory integral with large parameter $\tilde{\lambda}$ and phase

$$L_N(T,X,Y,z,\eta\tilde{\lambda}) = |\eta|\left(Y + G_N(T,X,z,\tilde{\lambda}\eta,a)\right).$$

By construction, the equation

$$\partial_\eta L_N = Y + G_N(T,X,z,\lambda,a) + \lambda\partial_\lambda G_N(T,X,z,\lambda,a) = 0$$

implies that (X,Y,T) belongs to the projection of $\mathbf{\Lambda}_{a,N,h}$ on $\mathbb{R}^3$. As in the proof of Proposition 2.14 [29], we see that the contribution of $G_{a,N,1,+,+}$ for the sum over N such that $N \notin \mathcal{N}_1(X,Y,T)$ is $O(\lambda^{-\infty})$. Thus it remains to estimate the sum

$$\left| \sum_{N\in\mathcal{N}_1(X,Y,T)} G_{a,N,1,+,+}(T,X,Y,z,h) \right|. \tag{2.1.25}$$

We apply the stationary phase method for η-integral with the phase function L_N. We have

$$\partial_\eta L_N = Y + G_N + \lambda\partial_\lambda G_N,$$

with

$$\lambda\partial_\lambda G_N = \lambda\partial_\lambda\tilde{\Phi}_{N,+,+}(T,X,\Omega_c,a,\lambda) = \frac{N}{\lambda}\big(- B(\lambda\Omega_c) + \lambda\Omega_c B'(\lambda\Omega_c)\big).$$

Then we obtain

$$\partial_\eta^2 L_N = \frac{N}{\eta}(\lambda\Omega_c)\partial_\lambda(\lambda\Omega_c)B''(\lambda\Omega_c).$$

On the other hand, $\partial_\lambda \Omega_c$ satisfies

$$\partial_\lambda \Omega_c \partial_\Omega^2 \tilde{\Phi}_{N,+,+}(\Omega_c) = -\partial_\lambda \partial_\Omega \tilde{\Phi}_{N,+,+}(\Omega_c) = -N\Omega_c B''(\lambda\Omega_c).$$

As we have $\partial_\Omega^2 \tilde{\Phi}_{N,+,+}(\Omega_c) \geq cT\Omega_c^{-4/3}$, $\Omega_c^{1/3} \simeq T/N$, and for ω large, we have $B''(\omega) \simeq c\omega^{-3}$. We get

$$|\partial_\lambda \Omega_c| \leq cT^{-1}\Omega_c^{4/3} N\Omega_c(\lambda^{-3}\Omega_c^{-3}) \leq c\lambda^{-3}\Omega_c^{-1}.$$

This yields

$$|\partial_\lambda(\lambda\Omega_c)| = |\lambda\partial_\lambda\Omega_c + \Omega_c| \geq c\Omega_c(1 - c\lambda^{-2}\Omega_c^{-2}) \geq c'\Omega_c.$$

Hence we deduce that

$$|\partial_\eta^2 L_N| \geq CN\lambda^{-2}\Omega_c^{-1}.$$

Therefore η-integration produces a factor $q^{-1/2}$ with $q = N\lambda^{-1}\Omega_c^{-1}$. Let us recall that

$$|\mathcal{N}_1(X,Y,T)| \leq C_0(1 + T\lambda^{-2}\Omega_c^{-2}).$$

We get the estimates of the sum in (2.1.25) by distinguishing between many cases which depend on whether there are contributions from η-integration and $|\mathcal{N}_1(X,Y,T)|$ as follows: First case, if $\Omega_c^{1/3} \simeq T/N$ is bounded, then $T \sim N$ and

- if $N \leq \lambda$, then there is no contribution from η-integration and we have $|\mathcal{N}_1| \leq C_0$. Hence the estimate is

$$\begin{aligned}\left|\sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+}\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}],\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} a^{-1/4}h^{1/2},\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3},\end{aligned}$$

  since $a^{-1/4}h^{1/2} \leq h^{1/3}$ when $a \geq h^{2/3}$.

- if $\lambda < N \leq \lambda^2$, then there is a contribution $q^{-1/2}$ factor from η-integration and we also have $|\mathcal{N}_1| \leq C_0$. We get

$$\begin{aligned}\left|\sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+}\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}N^{-1/2}\lambda^{1/2}],\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}a^2\lambda^{-2}],\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}.\end{aligned}$$

- if $N > \lambda^2$, then there are contributions from both $q^{-1/2}$ factor from η-integration and $|\mathcal{N}_1| \leq C_0 T\lambda^{-2}$. Thus the estimate is

$$\begin{aligned}
\left|\sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+}\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\sum_{N\in\mathcal{N}_1}[h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}N^{-1/2}\lambda^{1/2}],\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}\lambda^{-1}a^2T^{-1}|\mathcal{N}_1(X,Y,T)|],\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[a^{-5/2}h^2],\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}.
\end{aligned}$$

Second case, if T/N is large then Ω_c is large. We have

- if $N \leq \lambda\Omega_c$, then there is no contribution from η-integration. Moreover we have $|\mathcal{N}_1| \leq C_0$. To see this point, assume by contradiction $T \geq \lambda^2\Omega_c^2$; this implies $\Omega_c^{1/3} \simeq T/N \geq \lambda\Omega_c$ which is impossible since Ω_c is large. Thus the estimate is

$$\begin{aligned}
\left|\sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+}\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}],\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}a^{-1/4}h^{1/2},\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}.
\end{aligned}$$

- if $N > \lambda\Omega_c$ and $\lambda\Omega_c^{2/3} < T \leq \lambda^2\Omega_c^2$, then there is a contribution $q^{-1/2}$ factor from η-integration and we also have $|\mathcal{N}_1| \leq C_0$. We get

$$\begin{aligned}
\left|\sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+}\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}N^{-1/2}\lambda^{1/2}\Omega_c^{1/2}],\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}a^2\lambda^{-2}],\\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}.
\end{aligned}$$

- if $N > \lambda\Omega_c$ and $T > \lambda^2\Omega_c^2$, then there are contributions from both $q^{-1/2}$ factor from η-integration and $|\mathcal{N}_1| \leq C_0 T\lambda^{-2}\Omega_c^{-2}$. We get

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,N,1,+,+}\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\sum_{N\in\mathcal{N}_1}[h^{-1}\lambda^{-1}a^2\lambda^{-1/2}T^{-1/2}N^{-1/2}\lambda^{1/2}\Omega_c^{1/2}],$$
$$\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}\lambda^{-1}a^2T^{-1}\Omega_c^{2/3}|\mathcal{N}_1(X,Y,T)|],$$
$$\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}a^2\lambda^{-3}](T/N)^{-4},$$
$$\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}.$$

Next, we prove that (2.1.18) holds true in the case $(\epsilon_1,\epsilon_2)=(+,-)$. In this case, from the last item of (2.1.20), $X\in[0,1]$, and $B''(\lambda\Omega)=O(\lambda^{-3}\Omega^{-3})$ we get that for $T>0$, $\partial_\Omega H_{a,+,-}(\Omega)+\frac{3N}{2}\lambda B''(\lambda\Omega)<0$; that is, the function $H_{a,+,-}(\Omega)+\frac{3N}{2}B'(\lambda\Omega)$ decreases on $[1,\infty[$ from $H_{a,+,-}(1)+\frac{3N}{2}B'(\lambda)=\frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a)^{-1/2}+(1-X)^{1/2}+\frac{3N}{2}B'(\lambda)$ to $\left(H_{a,+,-}+\frac{3N}{2}B'(\lambda.)\right)(\infty)=0$. The equation $\partial_\Omega\Phi_{N,+,-}=0$ admits a unique solution Ω_c and it is nondegenerate; thus we can argue as $(+,+)$ case. Finally, the case $(\epsilon_1,\epsilon_2)=(-,+)$ is similar to $(+,+)$ case and $(\epsilon_1,\epsilon_2)=(-,-)$ is similar to $(+,-)$ case. The proof of proposition is complete. □

Now we prove the estimates for $N=1$.

Proposition 2.1.9. *There exists C such that for all $h\in]0,h_0]$, all $a\in[h^\alpha,a_0]$, all $X\in[0,1]$, all $T\in]0,a^{-1/2}]$, all $Y\in\mathbb{R}$, all $z\in\mathbb{R}$, the following holds:*

$$|G_{a,1,1}(T,X,Y,z;h)|\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\left(\frac{h}{t}\right)^{1/2}+h^{1/3}\right).$$

Proof. Let us recall that

$$G_{a,1,1}=\frac{(-i)a^2\lambda^{-1}}{(2\pi)^6h^4}\left(\frac{h}{t}\right)^{1/2}\int e^{i\frac{a^{3/2}}{h}Y\eta}|\eta|^3\tilde{G}_{a,1,1}d\eta,$$
$$\tilde{G}_{a,1,1}=\sum_{\epsilon_1,\epsilon_2}\int e^{i\lambda\tilde{\Phi}_{1,\epsilon_1,\epsilon_2}}\Theta_{\epsilon_1,\epsilon_2}d\tilde{\omega}+O_{C^\infty}(h^\infty).$$

The only difference with the case $N\ge 2$ is in the study of the phase $\tilde{\Phi}_{1,+,+}$ since in the case $N=1$ we may have a critical point $\tilde{\omega}_c$ large. Let

$$\tilde{G}_{a,1,1,+,+}=\int e^{i\lambda\tilde{\Phi}_{1,+,+}}\Theta_{+,+}(\tilde{\omega},a,\lambda)d\tilde{\omega}, \tag{2.1.26}$$

with the phase function

$$\tilde{\Phi}_{1,+,+}(T,X,z;\tilde{\omega}) = T\sqrt{1-\tilde{z}^2}\gamma_a(\tilde{\omega}) + \frac{2}{3}(\tilde{\omega}-X)^{3/2} + \frac{2}{3}(\tilde{\omega}-1)^{3/2} - \frac{4}{3}\tilde{\omega}^{3/2} + \frac{1}{\lambda}B(\lambda\tilde{\omega}^{3/2}),$$

and $\Theta_{+,+}(\tilde{\omega},a,\lambda)$ is a classical symbol of order $-1/2$ with respect to $\tilde{\omega}$. Let $\chi_3(\tilde{\omega}) \in C_0^\infty(]\tilde{\omega}_1,\infty[)$ with $\tilde{\omega}_1$ large and set

$$\tilde{J}_{1,+,+} = \int e^{i\lambda\tilde{\Phi}_{1,+,+}}\Theta_{+,+}(\tilde{\omega},a,\lambda)\chi_3(\tilde{\omega})d\tilde{\omega}. \tag{2.1.27}$$

To prove the proposition, we just have to verify $|\tilde{J}_{1,+,+}| \le C\lambda^{-1/2}T^{-1/2}$. We have

$$\begin{aligned}
\partial_{\tilde{\omega}}\tilde{\Phi}_{1,+,+} &= \frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{-1/2} - \frac{\tilde{\omega}^{-1/2}}{2}(1+X) + O(\tilde{\omega}^{-3/2}),\\
\partial^2_{\tilde{\omega}\tilde{\omega}}\tilde{\Phi}_{1,+,+} &= \frac{-Ta}{4}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{-3/2} + \frac{\tilde{\omega}^{-3/2}}{4}(1+X) + O(\tilde{\omega}^{-5/2}).
\end{aligned}$$

Therefore, to get a large critical point $\tilde{\omega}_c, T$ must be small. Then we have $\tilde{\omega}_c^{-1/2} \simeq T$ and thus $\partial^2_{\tilde{\omega}}\tilde{\Phi}_{1,+,+}(\tilde{\omega}_c) \simeq T^3$. We make a change of variable $\tilde{\omega} = T^{-2}\tilde{v}$ in (2.1.27). Since $\Theta_{+,+}(\tilde{\omega},a,\lambda)$ is a classical symbol in $\tilde{\omega}$ of order $-1/2$; thus $T^{-1}\tilde{v}^{1/2}\Theta_{+,+}(T^{-2}\tilde{v},a,\lambda)$ is a symbol of order 0 in $\tilde{v} \ge \tilde{v}_0 > 0$ uniformly in $T \in]0,T_0]$ and we also have $\partial^2_{\tilde{v}\tilde{v}}\tilde{\Phi}_{1,+,+} \simeq T^{-1}$. The stationary phase method yields $|\tilde{J}_{1,+,+}| \le C\lambda^{-1/2}T^{-1/2}$. □

The Analysis of $G_{a,N,2}$

Recall that the $G_{a,N,2}$ is a sum of oscillatory integrals which corresponds to the swallow-tails regime. Our result of this subsection is Proposition 2.1.10.

Proposition 2.1.10. *There exists C such that for all $h \in]0,h_0]$, all $a \in [h^\alpha, a_0]$, all $X \in [0,1]$, all $T \in]0,a^{-1/2}]$, all $Y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left|\sum_{1\le N\le C_0 a^{-1/2}} G_{a,N,2}(T,X,Y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} a^{1/8}h^{1/4}.$$

Proof. First, we rewrite $G_{a,N,2}$ in the form

$$\begin{aligned}
G_{a,N,2} &= \frac{(-i)^N a^2}{(2\pi)^6 h^4}\left(\frac{h}{t}\right)^{1/2}\int e^{i\frac{a^{3/2}}{h}Y\eta}|\eta|^3\tilde{G}_{a,N,2}d\eta, \qquad (2.1.28)\\
\tilde{G}_{a,N,2} &= \int e^{i\lambda\tilde{\phi}_{N,a,h}}\tilde{\chi}_0\psi_0\chi_1\chi_2(\tilde{\omega})d\tilde{s}d\tilde{\sigma}d\tilde{\omega},
\end{aligned}$$

with the phase

$$\tilde{\phi}_{N,a,h}(T,X,z;\tilde{s},\tilde{\sigma},\tilde{\omega}) = T\sqrt{1-\tilde{z}^2}\gamma_a(\tilde{\omega}) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{N}{\lambda}B(\tilde{\omega}^{3/2}\lambda).$$

Since $\tilde{\omega}$ is close to 1 on the support of χ_2, we may localize $\tilde{s},\tilde{\sigma}$ in a compact set. Let $K = \{\tilde{s},\tilde{\sigma} \in [-1,1], \tilde{\omega} = 1\}$ and K_1 be a suitable neighborhood of K depending on the support of χ_2. Introduce a cutoff function $\chi_4(\tilde{s},\tilde{\sigma},\tilde{\omega}) \in C_0^\infty$ equal to 1 near K_1. Then the contribution of $\tilde{G}_{a,N,2}$ outside K_1 is $O(\lambda^{-\infty})$ as a result of integration by parts. Therefore we obtain

$$\begin{aligned}\tilde{G}_{a,N,2}(T,X,z,h) &= \int e^{i\lambda\tilde{\phi}_{N,a,h}}\chi(\tilde{s},\tilde{\sigma},\tilde{\omega},a)d\tilde{s}d\tilde{\sigma}d\tilde{\omega} + O(\lambda^{-\infty}), \\ \chi(\tilde{s},\tilde{\sigma},\tilde{\omega},a,h) &= \tilde{\chi}_0\psi_0\chi_1\chi_2(\tilde{\omega})\chi_4(\tilde{s},\tilde{\sigma},\tilde{\omega}),\end{aligned} \tag{2.1.29}$$

with $O(\lambda^{-\infty})$ uniform in T,X,z,N,a and χ is a classical symbol of order 0 in h with support near K_1. We first perform the integration with respect to $\tilde{\omega}$. We have

$$\begin{aligned}\partial_{\tilde{\omega}}\tilde{\phi}_{N,a,h} &= \frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{-1/2} - \tilde{s} - \tilde{\sigma} - 2N\tilde{\omega}^{1/2}\Big(1 - \frac{3}{4}B'(\tilde{\omega}^{3/2}\lambda)\Big), \\ \partial^2_{\tilde{\omega}\tilde{\omega}}\tilde{\phi}_{N,a,h} &= -N\tilde{\omega}^{-1/2}(1 + O(\lambda^{-2}\tilde{\omega}^{-3})) + O(a^{1/2}).\end{aligned}$$

Since $\partial^2_{\tilde{\omega}\tilde{\omega}}\tilde{\phi}_{N,a,h} < 0$, then $\partial_{\tilde{\omega}}\tilde{\phi}_{N,a,h}$ decreases from $\partial_{\tilde{\omega}}\tilde{\phi}_{N,a,h}(1) > 0$ to $\partial_{\tilde{\omega}}\tilde{\phi}_{N,a,h}(\infty) < 0$. Therefore $\tilde{\phi}_{N,a,h}$ admits a unique nondegenerate critical point $\tilde{\omega}_c$ and we are interested in the values of parameters such that $\tilde{\omega}_c$ close to 1; then we must have $\tilde{T} = T/4N \in$ compact set of $\mathbb{R}_+$, say $[1/2, 3/2]$. In addition, from the equation $\partial_{\tilde{\omega}}\tilde{\phi}_{N,a,h} = 0$, we get

$$\frac{T}{2}\sqrt{1-\tilde{z}^2}(1+a\tilde{\omega})^{-1/2} = \tilde{s} + \tilde{\sigma} + 2N\tilde{\omega}^{1/2}\big(1 - \frac{3}{4}B'(\tilde{\omega}^{3/2}\lambda)\big). \tag{2.1.30}$$

Now we study the solution of (2.1.30) with $\lambda = \infty$; in this case, we have

$$\tilde{\omega}^{1/2}(1+a\tilde{\omega})^{1/2} = \tilde{T}\sqrt{1-\tilde{z}^2} - \frac{1}{2N}(\tilde{s}+\tilde{\sigma})(1+a\tilde{\omega})^{1/2}.$$

The solution of this equation is of the form $\tilde{\omega}_c = \sum f_k(a,\tilde{T},\tilde{s}/N,\tilde{\sigma}/N)$ where f_k are homogeneous function of degree k in $(\tilde{s}/N,\tilde{\sigma}/N)$. By comparing the terms with the same homogeneous degree in $(\tilde{s}/N,\tilde{\sigma}/N)$, we get

$$f_0(1+af_0) = \tilde{T}^2(1-\tilde{z}^2) \text{ which gives } F_0 =: f_0 = \frac{2\tilde{T}^2(1-\tilde{z}^2)}{1+\sqrt{1+4a\tilde{T}^2(1-\tilde{z}^2)}},$$

and

$$(1+2aF_0)f_1 = -\frac{\tilde{T}}{N}\sqrt{1-\tilde{z}^2}(\tilde{s}+\tilde{\sigma})(1+aF_0)^{1/2}.$$

We define

$$\begin{aligned} F_1 :&= f_1 = -\frac{E_0}{N}(\tilde{s}+\tilde{\sigma})(1+aF_0)^{1/2}, \\ E_0^{-1} :&= \sqrt{F_0}\sqrt{1+aF_0}\left(\frac{1}{F_0}+\frac{a}{1+aF_0}\right). \end{aligned}$$

Therefore $\tilde{\omega}_c = F_0 + F_1 + \mathcal{O}_2$ with the notation $\mathcal{O}_j$ means any function of the form $f = \sum_{k\geq j} f_k$. Then by the implicit function theorem, we get that the equation

$$\tilde{\omega}^{1/2}(1+a\tilde{\omega})^{1/2}\big(1-\frac{3}{4}B'(\tilde{\omega}^{3/2}\lambda)\big) = \tilde{T}\sqrt{1-\tilde{z}^2} - \frac{1}{2N}(\tilde{s}+\tilde{\sigma})(1+a\tilde{\omega})^{1/2}$$

has solution of the form $\tilde{\omega}_c = F_0 + F_1 + \mathcal{O}_2 + \frac{g_0}{\lambda^2}$ with g_0 is a function of degree 0 in λ. Substituting $\tilde{\omega}_c$ into $\tilde{\phi}_{N,a,h}$, we get a phase function denoted by $\tilde{\Psi}_{N,a,h} = \tilde{\phi}_{N,a,h}(.,\tilde{\omega}_c,.)$. It is given by

$$\begin{aligned} \tilde{\Psi}_{N,a,h} = T\sqrt{1-\tilde{z}^2}\gamma_a(F_0) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-F_0) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-F_0) + \frac{E_0}{N}(1+aF_0)^{1/2}(\tilde{s}+\tilde{\sigma})^2 \\ - \frac{1}{4N^2}(\tilde{s}+\tilde{\sigma})^3 + aN\mathcal{O}_3 + \frac{g_0}{\lambda^2} + N\left(-\frac{4}{3}F_0^{3/2} + \frac{g_1}{\lambda^2}\right). \end{aligned}$$

Hence by applying the stationary phase method for (2.1.29), we get

$$\tilde{G}_{a,N,2} = \frac{1}{\sqrt{\lambda N}}\int e^{i\lambda\tilde{\Psi}_{N,a,h}}\tilde{\chi}(\tilde{T},\tilde{s},\tilde{\sigma},1/N,a,h)d\tilde{s}d\tilde{\sigma} + O(\lambda^{-\infty}),$$

with $\tilde{\chi}$ is a classical symbol of order zero in h. Now with $\tilde{\lambda} = \lambda/\eta$, (2.1.28) becomes

$$G_{a,N,2} = \frac{(-i)^N a^2}{(2\pi)^6 h^4}\left(\frac{h}{t}\right)^{1/2}\frac{1}{\sqrt{\lambda N}}\int e^{i\tilde{\lambda}|\eta|(Y+\tilde{\Psi}_{N,a,h})}|\eta|^3\tilde{\chi}d\tilde{s}d\tilde{\sigma}d\eta + O(\lambda^{-\infty}).$$

We study the η-integration with the phase function $L_N = \eta(Y+\tilde{\Psi}_{N,a,h})$ and a large parameter $\tilde{\lambda}$. Follow the arguments in the proof of Proposition 2.1.8, we have

$$\partial_\eta L_N = Y + \tilde{\Psi}_{N,a,h} + \lambda\partial_\lambda\tilde{\Psi}_{N,a,h} = 0$$

implies that (X,Y,T) belongs to the projection of $\mathbf{\Lambda}_{N,a,h}$ on $\mathbb{R}^3$ and the sum for N such that $N \notin \mathcal{N}_1(X,Y,T)$ gives $O(\lambda^{-\infty})$ [see Lemma 2.24 [29]]. Hence it remains to estimate the sum

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,N,2}(T,X,Y,z;h)\right|.$$

We also have $|\partial_\eta^2 L_N| \geq CN\lambda^{-2}\tilde{\omega}_c^{-3/2}$. Hence by the stationary phase method, the η-integration gives a factor $q^{-1/2}$ with $q = N\lambda^{-1}$ since $\tilde{\omega}_c \approx 1$. It yields

$$G_{a,N,2} = \frac{(-i)^N a^2}{(2\pi)^6 h^4}\left(\frac{h}{t}\right)^{1/2}\frac{1}{\sqrt{\lambda N}}\lambda^{1/2}N^{-1/2}\int e^{i\tilde{\lambda}L_N(\eta_c)}|\eta|^3\tilde{\chi}_1 d\tilde{s}d\tilde{\sigma} + O(\lambda^{-\infty}). \quad (2.1.31)$$

We observe that the phase function $L_N(\eta_c)$ satisfies $\partial_{\tilde{s}}L_N(\eta_c) = \eta_c\partial_{\tilde{s}}\tilde{\Psi}_{N,a,h}$, $\partial_{\tilde{\sigma}}L_N(\eta_c) = \eta_c\partial_{\tilde{\sigma}}\tilde{\Psi}_{N,a,h}$. Moreover, when $\partial_{\tilde{s}}L_N(\eta_c) = \partial_{\tilde{s}}^2 L_N(\eta_c) = 0$; that is, when $\partial_{\tilde{s}}\tilde{\Psi}_{N,a,h} = \partial_{\tilde{s}}^2\tilde{\Psi}_{N,a,h} = 0$, we have $\partial_{\tilde{s}}^3 L_N(\eta_c) = \eta_c\partial_{\tilde{s}}^3\tilde{\Psi}_{N,a,h}$ and similar for $\tilde{\sigma}$. Thus the study the critical points of the phase $L_N(\eta_c)$ in $(\tilde{s},\tilde{\sigma})$-integrations is the same as ones with the phase $\tilde{\Psi}_{N,a,h}$. As in [29], to avoid multiplication of symbol by a classical symbol of order 0 in λ, we can replace $\tilde{\Psi}_{N,a,h}$ by $\tilde{\psi}_{a,N,h}$, where

$$\begin{aligned}\tilde{\psi}_{N,a,h}(T,X;\tilde{s},\tilde{\sigma}) = T\sqrt{1-\tilde{z}^2}\gamma_a(F_0) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-F_0) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-F_0)\\ + \frac{G_0}{N}(1+aF_0)^{1/2}(\tilde{s}+\tilde{\sigma})^2 - \frac{1}{4N^2}(\tilde{s}+\tilde{\sigma})^3 + aN\mathcal{O}_3.\end{aligned}$$

In what follows, we get the estimates of the oscillatory integral associated with the phase function $\tilde{\psi}_{a,N,h}$ for different values of N, namely for $N \geq \lambda^{1/3}$ and $N < \lambda^{1/3}$. Our results are Lemma 2.1.11 and Lemma 2.1.12.

Lemma 2.1.11. *There exists C such that for all $N \geq \lambda^{1/3}$,*

$$\frac{1}{\sqrt{N}}\left|\int e^{i\lambda\tilde{\psi}_{N,a,h}}\tilde{\chi}_1 d\tilde{s}d\tilde{\sigma}\right| \leq C\lambda^{-5/6}. \quad (2.1.32)$$

Here remark that C is any constant that is independent of $N \geq 1, X \in [0,1], T \in]0,a^{-1/2}], a \in [h^\alpha, a_0]$ and $\lambda \in [\lambda_0,\infty[$ with a_0 small and λ_0 large.

Proof. Adapting the arguments in the proof of Lemma 2.25 [29]. It is sufficient to prove that for all $N \geq \lambda^{1/3}$,

$$\left|\int e^{i\lambda\tilde{\psi}_{N,a,h}}\tilde{\chi}_1 d\tilde{s}d\tilde{\sigma}\right| \leq C\lambda^{-2/3}. \quad (2.1.33)$$

Set $X - F_0 = -A\lambda^{-2/3}, 1 - F_0 = -B\lambda^{-2/3}, \tilde{s} = \lambda^{-1/3}x', \tilde{\sigma} = \lambda^{-1/3}y'$. It remains to prove that

$$\left|\int e^{i\hat{\psi}_{N,a,h}}\tilde{\chi}_1(\lambda^{-1/3}x',\lambda^{-1/3}y',...)dx'dy'\right| \leq C, \quad (2.1.34)$$

with the phase function $\hat{\psi}_{N,a,h}$ given by

$$\hat{\psi}_{N,a,h} = T\lambda\sqrt{1-\tilde{z}^2}\gamma_a(F_0) - Ax' + \frac{x'^3}{3} - By' + \frac{y'^3}{3} + \frac{E_0\lambda^{1/3}}{N}(1+aF_0)^{1/2}(x'+y')^2 - \frac{1}{4N^2}(x'+y')^3 + aN\mathcal{O}_3.$$

Then (2.1.34) is an oscillatory integral over a domain of integration of size $\lambda^{1/3}$ with parameters $F_0, E_0, \lambda^{1/3}/N$ are bounded and we will prove that the constant C is uniform in $(A,B) = (r\cos\theta, r\sin\theta)$ with $r \le c_0\lambda^{2/3}$. We have

$$\partial_{x'}\hat{\psi}_{N,a,h} = -A + x'^2 + \frac{2E_0}{N}(1+aF_0)^{1/2}\lambda^{1/3}(x'+y') - \frac{3}{4N^2}(x'+y')^2 + aO\big((x',y')^2\big),$$
$$\partial_{y'}\hat{\psi}_{N,a,h} = -B + y'^2 + \frac{2E_0}{N}(1+aF_0)^{1/2}\lambda^{1/3}(x'+y') - \frac{3}{4N^2}(x'+y')^2 + aO\big((x',y')^2\big).$$

Moreover, the compactly support of $\tilde{\chi}_1$ in $(\tilde{s},\tilde{\sigma})$ yields

$$\sup_{(x',y')}\left|\partial^\alpha_{(x',y')}\tilde{\chi}_1(\lambda^{-1/3}x', \lambda^{-1/3}y', ...)\right| \le C_\alpha(1+|x'|+|y'|)^{-|\alpha|},$$

with C_α independent of T, a, N, λ. Therefore for $r \in [0, r_0], \forall r_0$ the oscillatory integral is bounded by integration by parts for large (x,y).
For $r \in [r_0, c_0\lambda^{2/3}]$, we rescale variables $(x',y') = r^{1/2}(x'',y'')$ and we set $\hat{\psi}_{N,a,h} = r^{3/2}\psi^*_{N,a,h}$ and $\chi'(x'',y'',...) = \tilde{\chi}_1(r^{1/2}\lambda^{-1/3}x'', r^{1/2}\lambda^{-1/3}y'', ...)$. Since $r^{1/2}\lambda^{-1/3}$ is bounded, we still have

$$\sup_{(x'',y'')}\left|\partial^\alpha_{(x'',y'')}\chi'\right| \le C_\alpha(1+|x''|+|y''|)^{-|\alpha|}.$$

It remains to prove

$$r\left|\int e^{ir^{3/2}\psi^*_{N,a,h}}\chi' dx''dy''\right| \le C. \tag{2.1.35}$$

Now we study the critical points of $\psi^*_{N,a,h}$. We have

$$\partial_{x''}\psi^*_{N,a,h} = -\cos\theta + x''^2 - \frac{3}{4N^2}(x''+y'')^2 + O(r^{-1/2}+a),$$
$$\partial_{y''}\psi^*_{N,a,h} = -\sin\theta + y''^2 - \frac{3}{4N^2}(x''+y'')^2 + O(r^{-1/2}+a).$$

For small a and large r_0, we may localize the integral to a compact set in (x'',y'') as a result of integration by parts for large (x'',y''). The Hessian of $\psi^*_{N,a,h}$,

$$\mathcal{H}_N(x'',y'') = 4x''y'' - \frac{3}{N^2}(x''+y'')^2 + O(r^{-1/2}+a).$$

Thus for $N \ge 2, a$ small and r_0 large, outside $(x'',y'') = (0,0)$, define a smooth curve

$\Gamma = \{(x'', y'')$ such that $\mathcal{H}_N(x'', y'') = 0\}$; that is, Γ is close to the union of two lines $c(x'' + y'') \pm (x'' - y'') = 0, c^2 = \frac{N^2-3}{N^2} \in [1/4, 1]$. Then we have 2 cases to consider

- The contribution of points (x'', y'') outside Γ to the integral is $O(r^{-3/2})$ by the usual stationary phase method and we get

$$r \left| \int e^{ir^{3/2}\psi^*_{N,a,h}} \chi' dx'' dy'' \right| \leq Cr^{-1/2}.$$

- The contribution of points (x'', y'') close to Γ is given by Lemma 2.21 [29]. For any values of θ, the hypothesis of part (a) Lemma 2.21 [29] holds true, then we get

$$r \left| \int e^{ir^{3/2}\psi^*_{N,a,h}} \chi' dx'' dy'' \right| \leq Cr(r^{3/2})^{-5/6} = Cr^{-1/4}.$$

Hence in any cases, (2.1.35) is satisfied. $\square$

To summarize, recall that $T \sim N$ and $|\mathcal{N}_1(X, Y, T)| \leq C_0(1 + T\lambda^{-2})$ in this case. We deduce the estimates for the sum of $G_{a,N,2}$ with Lemma 2.1.11 for $N \geq \lambda^{1/3}$ as follows:

- If $\lambda^{1/3} \leq N \leq \lambda$, there is no contribution from η- integration and we have $|\mathcal{N}_1| \leq C_0$. We obtain

$$\begin{aligned}
\left| \sum_{N \in \mathcal{N}_1} G_{a,N,2}(T, X, Y, z; h) \right| &\leq Ch^{-3} \left(\frac{h}{t}\right)^{1/2} [h^{-1}a^2\lambda^{-1/2}\lambda^{-5/6}], \\
&\leq Ch^{-3} \left(\frac{h}{t}\right)^{1/2} h^{1/3}.
\end{aligned}$$

- If $\lambda \leq N \leq \lambda^2$, then there is a $q^{-1/2}$ factor contribution from η-integration and we also have $|\mathcal{N}_1| \leq C_0$. We get

$$\begin{aligned}
\left| \sum_{N \in \mathcal{N}_1} G_{a,N,2}(T, X, Y, z; h) \right| &\leq Ch^{-3} \left(\frac{h}{t}\right)^{1/2} [h^{-1}a^2T^{-1/2}\lambda^{-5/6}], \\
&\leq Ch^{-3} \left(\frac{h}{t}\right)^{1/2} [h^{-1}a^2\lambda^{-1/2}\lambda^{-5/6}], \\
&\leq Ch^{-3} \left(\frac{h}{t}\right)^{1/2} h^{1/3}.
\end{aligned}$$

- If $N > \lambda^2$, then there are contributions from both $q^{-1/2}$ from η-integration and

$|\mathcal{N}_1| \leq C_0 T\lambda^{-2}$. We get

$$\begin{aligned}\left|\sum_{N\in\mathcal{N}_1} G_{a,N,2}(T,X,Y,z;h)\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\sum_{N\in\mathcal{N}_1}\left[h^{-1}a^2\frac{1}{N}\lambda^{-2/3}\right],\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}a^2\lambda^{-2/3}T^{-1}|\mathcal{N}_1(X,Y,T)|],\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[a^{-2}h^{5/3}],\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}.\end{aligned}$$

Lemma 2.1.12. *There exists C such that for all $N < \lambda^{1/3}$,*

$$\frac{1}{\sqrt{N}}\left|\int e^{i\lambda\tilde{\psi}_{N,a,h}}\tilde{\chi}_1 d\tilde{s}d\tilde{\sigma}\right| \leq CN^{-1/4}\lambda^{-3/4}. \tag{2.1.36}$$

Notice that Lemma 2.1.12 says that for N large it gives a better estimate and it is compatible with the estimate (2.1.32) for $N \simeq \lambda^{1/3}$.

Proof. Let $\frac{\lambda}{N^3} = \Lambda \geq 1$ and we take Λ as a new large parameter. To get the estimates of our oscillatory integral, we set $X - F_0 = -pN^{-2}, 1 - F_0 = -qN^{-2}, \tilde{s} = -\bar{x}/N, \tilde{\sigma} = -\bar{y}/N$. It yields $\tilde{\psi}_{N,a,h} = N^{-3}\bar{\psi}_{N,a,h}$. Then it remains to prove that

$$\left|\int e^{i\Lambda\bar{\psi}_{N,a,h}}\tilde{\chi}_1(\bar{x}/N,\bar{y}/N,...)d\bar{x}d\bar{y}\right| \leq C\Lambda^{-3/4}, \tag{2.1.37}$$

with the phase $\bar{\psi}_{N,a,h}$ takes the form

$$\begin{aligned}\bar{\psi}_{N,a,h} = p\bar{x} - \frac{\bar{x}^3}{3} + q\bar{y} - \frac{\bar{y}^3}{3} + E_0(1+aF_0)^{1/2}(\bar{x}+\bar{y})^2 + \frac{1}{4N^2}(\bar{x}+\bar{y})^3\\ + TN^3\sqrt{1-\tilde{z}^2}\gamma_a(F_0) + aN^{-2}O\big((\bar{x},\bar{y})^3\big).\end{aligned}$$

We have

$$\begin{aligned}\partial_{\bar{x}}\bar{\psi}_{N,a,h} &= p - \bar{x}^2 + 2E_0(1+aF_0)^{1/2}(\bar{x}+\bar{y}) + \frac{3}{4N^2}(\bar{x}+\bar{y})^2 + aO\big((\bar{x},\bar{y})^2\big), \qquad (2.1.38)\\ \partial_{\bar{y}}\bar{\psi}_{N,a,h} &= q - \bar{y}^2 + 2E_0(1+aF_0)^{1/2}(\bar{x}+\bar{y}) + \frac{3}{4N^2}(\bar{x}+\bar{y})^2 + aO\big((\bar{x},\bar{y})^2\big),\end{aligned}$$

and the Hessian of $\bar{\psi}_{N,a,h}$ is

$$\mathcal{H}_N(\bar{x},\bar{y},a) = 4\bar{x}\bar{y} - 4E_0(1+aF_0)^{1/2}(\bar{x}+\bar{y}) - \frac{3}{N^2}(\bar{x}+\bar{y})^2 + aO\big((\bar{x},\bar{y})\big).$$

Lemma 2.1.13. *There exist r_0 and C such that for all (p,q) with $|(p,q)| \geq r_0$,*

$$\left| \int e^{i\Lambda \bar{\psi}_{N,a,h}} \tilde{\chi}_1(\bar{x}/N, \bar{y}/N, ...) d\bar{x} d\bar{y} \right| \leq C\Lambda^{-5/6}. \tag{2.1.39}$$

Proof of Lemma 2.1.13. Apply the arguments in the proof of Lemma 2.26 [29]. Set $(p,q) = (r\cos\theta, r\sin\theta)$ with $r \geq r_0$. Let $\chi \in C_0^\infty(|(\bar{x},\bar{y})| < c)$ with small c and $\chi = 1$ near 0. Then from (2.1.38), we get by integration by parts in $(\bar{x},\bar{y})$, for all k,

$$\left| \int e^{i\Lambda \bar{\psi}_{N,a,h}} \chi(r^{-1/2}(\bar{x},\bar{y})) \tilde{\chi}_1(\bar{x}/N, \bar{y}/N, ...) d\bar{x} d\bar{y} \right| \leq C r^{-k} \Lambda^{-k}.$$

For $(\bar{x},\bar{y})$ large, we make a change of variable $(\bar{x},\bar{y}) = r^{1/2}(x',y')$ and set $\bar{\psi}'_{N,a,h} = r^{-3/2}\bar{\psi}_{N,a,h}$. Then it remains to prove

$$\left| r \int e^{ir^{3/2}\Lambda \bar{\psi}'_{N,a,h}} (1-\chi)(x',y') \tilde{\chi}_1(r^{1/2}x'/N, r^{1/2}y'/N, ...) dx'dy' \right| \leq C\Lambda^{-5/6}.$$

We observe that since $(1-\chi)(x',y') = 0$ near $0, (1-\chi)(x',y') = 1$ for $|(x',y')| \geq c$ and $\tilde{\chi}_1$ is compactly support, we still have

$$\sup_{(x',y')} \left| \partial^\alpha_{(x',y')} (1-\chi)(x',y') \tilde{\chi}_1(r^{1/2}x'/N, r^{1/2}y'/N, ...) \right| \leq C_\alpha (1 + |x'| + |y'|)^{-|\alpha|}.$$

The phase $\bar{\psi}'_{N,a,h}$ is of the form

$$\bar{\psi}'_{N,a,h} = \cos\theta x' - \frac{x'^3}{3} + \sin\theta y' - \frac{y'^3}{3} + \frac{1}{4N^2}(x'+y')^3 + \frac{TN^3}{r^{3/2}}\sqrt{1-\tilde{z}^2}\gamma_a(F_0) + O(r^{-1/2}+a).$$

We get that

$$\begin{aligned} \partial_{x'}\bar{\psi}'_{N,a,h} &= \cos\theta - x'^2 + \frac{3}{4N^2}(x'+y')^2 + O(r^{-1/2}+a), \\ \partial_{y'}\bar{\psi}'_{N,a,h} &= \sin\theta - y'^2 + \frac{3}{4N^2}(x'+y')^2 + O(r^{-1/2}+a). \end{aligned}$$

Thus for small a and large r_0, by integration by parts, we may localize the integral to a compact set in (x',y'). The Hessian of $\bar{\psi}'_{N,a,h}$ is

$$\mathcal{H}'_N(x',y',a) = 4x'y' - \frac{3}{N^2}(x'+y')^2 + O(r^{-1/2}+a).$$

The same argument as before, for $N \geq 2, a$ small and r_0 large, outside $(x',y') = (0,0)$, we set $\Gamma = \{(x',y') \text{ such that } \mathcal{H}'_N(x',y') = 0\}$ and there are 2 cases to consider:

- The contribution of points (x',y') outside Γ to the integral is $O(r^{-3/2}\Lambda^{-1})$ by the usual stationary phase method; that is,

$$\left| r \int e^{ir^{3/2}\Lambda\bar{\psi}'_{N,a,h}} (1-\chi)(x',y')\tilde{\chi}_1(r^{1/2}x'/N, r^{1/2}y'/N, ...)dx'dy' \right| \le Cr^{-1/2}\Lambda^{-1}.$$

- The contribution of points (x', y') close to Γ given by Lemma 2.21[29]. For any values of θ, the hypothesis of part (a) Lemma 2.21[29] holds true, then we get

$$\begin{aligned} \left| r \int e^{ir^{3/2}\Lambda\bar{\psi}'_{N,a,h}} (1-\chi)(x',y')\tilde{\chi}_1(r^{1/2}x'/N, r^{1/2}y'/N, ...)dx'dy' \right| &\le Cr(r^{3/2}\Lambda)^{-5/6}, \\ &\le Cr^{-1/4}\Lambda^{-5/6}. \end{aligned}$$

□

Lemma 2.1.14. *There exist r_0 and C such that for all (p,q) with $|(p,q)| \le r_0$,*

$$\left| \int e^{i\Lambda\bar{\psi}_{N,a,h}} \tilde{\chi}_1(\bar{x}/N, \bar{y}/N, ...)d\bar{x}d\bar{y} \right| \le C\Lambda^{-3/4}. \tag{2.1.40}$$

Proof of Lemma 2.1.14. Now we consider the case $|(p,q)| \le r_0$. There exists $c > 0$ independent of $N \ge 2$ such that

$$\forall (\bar{x},\bar{y}) \in \mathbb{R}^2, \quad \left| \bar{x}^2 - \frac{3}{4N^2}(\bar{x}+\bar{y})^2 \right| + \left| \bar{y}^2 - \frac{3}{4N^2}(\bar{x}+\bar{y})^2 \right| \ge c(\bar{x}^2+\bar{y}^2). \tag{2.1.41}$$

Then by integration by parts, (2.1.38) gives a contribution $O_{C^\infty}(\Lambda^{-\infty})$ to the integral (2.1.37) for large values $(\bar{x},\bar{y})$. Then we may assume that $(\bar{x},\bar{y})$ is in compact set. It remains to prove

$$\left| \int e^{i\Lambda\bar{\psi}_{N,a,h}} \tilde{\chi}_1 d\bar{x}d\bar{y} \right| \le C\Lambda^{-3/4},$$

with the phase

$$\bar{\psi}_{N,a,h} = p\bar{x} - \frac{\bar{x}^3}{3} + q\bar{y} - \frac{\bar{y}^3}{3} + \tilde{T}(\bar{x}+\bar{y})^2 + \frac{1}{4N^2}(\bar{x}+\bar{y})^3 + TN^3\sqrt{1-\tilde{z}^2}\gamma_a(F_0) + O(a).$$

We have

$$\begin{aligned} \partial_{\bar{x}}\bar{\psi}_{N,a,h} &= p - \bar{x}^2 + 2\tilde{T}(\bar{x}+\bar{y}) + \frac{3}{4N^2}(\bar{x}+\bar{y})^2 + O(a), \\ \partial_{\bar{y}}\bar{\psi}_{N,a,h} &= q - \bar{y}^2 + 2\tilde{T}(\bar{x}+\bar{y}) + \frac{3}{4N^2}(\bar{x}+\bar{y})^2 + O(a), \end{aligned}$$

and the Hessian of $\bar{\psi}_{N,a,h}$ is

$$\mathcal{H}_N(\bar{x},\bar{y},\tilde{T},a) = 4\bar{x}\bar{y} - 4\tilde{T}(\bar{x}+\bar{y}) - \frac{3}{N^2}(\bar{x}+\bar{y})^2 + O(a).$$

For a small, the set $\Gamma = \{(\bar{x},\bar{y})$ such that $\mathcal{H}_N(\bar{x},\bar{y}) = 0\}$ is a smooth curve that is close to the elliptic $4\bar{x}\bar{y} - 4\tilde{T}(\bar{x}+\bar{y}) - 3(\bar{x}+\bar{y})^2 = 0$ for $N = 1$ and close to hyperbola $4\bar{x}\bar{y} - 4\tilde{T}(\bar{x}+\bar{y}) - \frac{3}{N^2}(\bar{x}+\bar{y})^2 = 0$ for $N \geq 2$. It remains to use Lemma 2.21 [29] [see Appendix] for $(\bar{x},\bar{y})$ near (p,q) with $|(p,q)| \leq r_0$. Then there are 3 cases to consider:

- If (p,q) is outside Γ, then the contribution to the integral is $O(\Lambda^{-1})$ by usual stationary phase method.
- If $(0,0) \neq (p,q)$ is close to Γ, the contribution to the integral is given by Lemma 2.21[29]. Since the hypothesis of part (a) in Lemma 2.21[29] holds true, then near (p,q) the contribution to the integral is $O(\Lambda^{-5/6})$.
- If $(p,q) = (0,0)$, we have (x,y) near $(0,0)$ and hypothesis of part (b) in Lemma 2.21 [29] holds true. Then the contribution to the integral is $O(\Lambda^{-3/4})$.

□

Lemma 2.1.13 and Lemma 2.1.14 yield the proof of Lemma 2.1.12. □

Notice that when $N < \lambda^{1/3}$, there is no contribution from η-integration and we have $|\mathcal{N}_1| \leq C_0$. As a consequence, we obtain the estimates for the sum of $G_{a,N,2}$ for $N < \lambda^{1/3}$ as follows:

$$\begin{aligned}\left|\sum_{N\in\mathcal{N}_1} G_{a,N,2}(T,X,Y,z;h)\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}a^2\lambda^{-1/2}N^{-1/4}\lambda^{-3/4}],\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[a^{1/8}h^{1/4}N^{-1/4}].\end{aligned}$$

We notice that we get the same estimates for $N = 1$,

$$\begin{aligned}|G_{a,1,2}(T,X,Y,z;h)| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}a^2\lambda^{-1/2}\lambda^{-3/4}],\\ &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[a^{1/8}h^{1/4}].\end{aligned}$$

To summarize, putting these estimates together we proved that

$$\left|\sum_{1\leq N\leq C_0a^{-1/2}} G_{a,N,2}(T,X,Y,z;h)\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{1/3} + a^{1/8}h^{1/4}].$$

Notice that $h^{1/3} \leq a^{1/8}h^{1/4}$ when $a \geq h^{2/3}$; hence the proof of the Proposition 2.1.10 is complete. $\square$

Proof of Theorem 2.1.5. Putting the estimates in Proposition 2.1.8, 2.1.9 and 2.1.10 together yields the desired result. $\square$

2.2 Dispersive Estimates for $\epsilon_0\sqrt{a} \leq \eta \leq c_0$.

In this section, we prove Theorem 1.4.2. Recall that we have

$$\mathcal{G}_a(t,x,y,z) = \frac{1}{4\pi^2 h^2} \sum_{k\geq 1} \int e^{\frac{i}{h}\Phi_k} \sigma_k d\eta d\zeta, \tag{2.2.1}$$

where the phase Φ_k and the function σ_k are defined by

$$\Phi_k = y\eta + z\zeta + t(\eta^2 + \zeta^2 + \omega_k h^{2/3}\eta^{4/3})^{1/2},$$

$$\sigma_k = e_k(x,\eta/h)e_k(a,\eta/h)\chi_0(\zeta^2+\eta^2)\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k).$$

We have to get L^∞ estimates for $\mathcal{G}_a$ in the range $t \in [h,1]$, when the integral in (2.2.1) is restricted to values of $\eta \in [\epsilon_0\sqrt{a}, c_0]$ with c_0 small. Let μ^2 be defined by

$$\mu^2 = \eta^2 + \omega_k h^{2/3}\eta^{4/3}.$$

Observe that μ^2 is small since $\omega_k h^{2/3}\eta^{4/3}$ is small by the truncation χ_1 and η is small. Let $\chi_4 \in C_0^\infty]-1,1[$ with $\chi_4 = 1$ on $[-1/2,1/2]$ and $D \geq 1$. Let $\mathcal{N}_a(t,x,y,z)$ be defined by

$$\mathcal{N}_a(t,x,y,z) = \frac{1}{4\pi^2 h^2} \sum_{k\geq 1} \int e^{\frac{i}{h}\Phi_k} \chi_4\left(\frac{t\mu^2}{Dh}\right) \sigma_k d\eta d\zeta.$$

The following lemma tells us that $\mathcal{N}_a$ satisfies the free dispersive estimate.

Lemma 2.2.1. *There exists C independent of D such that*

$$|\mathcal{N}_a(t,x,y,z)| \leq Ch^{-3}\left(\frac{h}{t}\right)D.$$

Proof. On the support of χ_4, one has $\eta^2 \leq Dh/t$ and $h\omega_k^{3/2}\eta^2 \leq (Dh/t - \eta^2)^{3/2}$. This implies that the sum over k is restricted to $k \leq c_0\frac{(Dh/t-\eta^2)^{3/2}}{h\eta^2}$. Since $e_k(x,\eta/h) =$

$f_k k^{-1/6}(\eta/h)^{1/3} Ai((\eta/h)^{2/3}x - \omega_k)$, Lemma 2.1.2 gives

$$\begin{aligned} |\mathcal{N}_a(t,x,y,z)| &\le Ch^{-2} \int_{\eta^2 \le Dh/t} (\eta/h)^{2/3} \frac{(Dh/t - \eta^2)^{1/2}}{h^{1/3}\eta^{2/3}} d\eta \\ &= Ch^{-3} \int_{\eta^2 \le Dh/t} (Dh/t - \eta^2)^{1/2} d\eta \end{aligned} \tag{2.2.2}$$

and the result follows from $\int_{\eta^2 \le Dh/t} (Dh/t - \eta^2)^{1/2} d\eta = (Dh/t) \int_{x^2 \le 1} (1 - x^2)^{1/2} dx$. □

Observe that in the range $\eta \ge c_0$, one has $\mu^2 \ge c_0^2$, so the condition $t\mu^2/h \le D$ is equivalent to $t \le Ch$ and the above lemma is irrelevant. But in the range $\eta \in [\epsilon_0\sqrt{a}, c_0]$, the above lemma becomes useful since it tells us that we may now assume that $\lambda = t\mu^2/h$ is a large parameter. Since we allow some loss in the dispersive estimate with respect to the free case, we may even assume that we have $\lambda = t\mu^2/h \ge (\frac{h}{t})^{-\epsilon}$ for some $\epsilon > 0$ (take $D = (\frac{h}{t})^{-\epsilon}$), and therefore in the sequel a term like $O(\lambda^{-\infty})$ will be neglectible. We are now in position to eliminate the ζ integration in (2.2.1). This is the purpose of the following lemma. Recall that the truncation $\chi_0(\zeta^2 + \eta^2)$ localizes $\zeta^2 + \eta^2$ near 1. Therefore, for η small, ζ will be close to 1 or -1. In the sequel, we assume ζ near 1.

Lemma 2.2.2. *Let $\lambda = t\mu^2/h \ge 1$, $\tilde{z} = z/t$ and $\phi(\tilde{z}, \mu^2, \zeta) = \frac{1}{\mu^2}(\tilde{z}\zeta + (\zeta^2 + \mu^2)^{1/2})$. Let*

$$I(\tilde{z}, \mu^2, \eta; \lambda) = \int_{\zeta \simeq 1} e^{i\lambda\phi(\tilde{z},\mu^2,\zeta)} \chi_0(\zeta^2 + \eta^2) d\zeta.$$

There exists $0 < c_1 < C_1$ such that the following holds true.

$$\text{For } \tilde{z} \notin [-1 + c_1\mu^2, -1 + C_1\mu^2] \text{ one has } \sup_{\tilde{z},\mu^2,\eta} |I(\tilde{z}, \mu^2, \eta; \lambda)| \in O(\lambda^{-\infty}). \tag{2.2.3}$$

For $\tilde{z} \in [-1 + c_1\mu^2, -1 + C_1\mu^2]$, set $\tilde{z} = -1 + z^*\mu^2$. There exists a classical symbol of degree 0 in λ, $\sigma_0(z^*, \eta, \mu^2; \lambda)$, such that one has

$$I(\tilde{z}, \mu^2, \eta; \lambda) = \left(\frac{h}{t\mu^2}\right)^{1/2} e^{i\frac{t\mu}{h}(1-\tilde{z}^2)^{1/2}} \sigma_0(z^*, \eta, \mu^2; \lambda). \tag{2.2.4}$$

Proof. One has $\partial_\zeta \phi = \frac{1}{\mu^2}(\tilde{z} + \zeta(\zeta^2 + \mu^2)^{-1/2})$, $\partial_\zeta^2 \phi = (\zeta^2 + \mu^2)^{-3/2} \ge c > 0$ and $\partial_\zeta^j \phi$ is bounded for all $j \ge 2$. Since $\zeta(\zeta^2 + \mu^2)^{-1/2} = 1 - \frac{\mu^2}{2\zeta^2} + O(\mu^4)$, (2.2.3) follows by integration by parts. For $\tilde{z} \in [-1 + c_1\mu^2, -1 + C_1\mu^2]$, and with $\tilde{z} = -1 + z^*\mu^2$, one has $\phi = z^*\zeta + (\zeta + (\zeta^2 + \mu^2)^{1/2})^{-1}$ and a unique critical point $\zeta_c = -\mu\tilde{z}(1 - \tilde{z}^2)^{1/2}$ with critical value $\phi(\zeta_c) = \frac{\zeta_c}{\mu^2}(\tilde{z} - 1/\tilde{z}) = (1 - \tilde{z}^2)^{1/2}/\mu \in O(1)$. Therefore, by stationary phase we get that (2.2.4) holds true. □

Using Lemmas 2.2.1 and 2.2.2, we are now reduced to the study of

$$\frac{1}{4\pi^2 h^2}(h/t)^{1/2}\sum_{k\geq 1}\int e^{\frac{i}{h}(y\eta+t\mu(1-\tilde{z}^2)^{1/2})}\frac{\tilde{\sigma}_k}{\mu}d\eta \tag{2.2.5}$$

where $\tilde{\sigma}_k$ is defined by

$$\tilde{\sigma}_k=\sigma_0(z^*,\eta,\mu^2;\lambda)(1-\chi_4(\frac{t\mu^2}{Dh}))e_k(x,\eta/h)e_k(a,\eta/h)\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k).$$

To get L^∞ estimate for the parametrix in the range $\eta\in[\epsilon_0\sqrt{a},c_0]$, we will use a Litttlewood-Paley decomposition in η. We choose $\psi_1\in C_0^\infty(]0.5,2.5[),0\leq\psi_1\leq 1$ such that $\sum_{m\in\mathbb{Z}}\psi_1(2^m x)=1$ for all $x>0$, and we introduce the cut-off function $\psi_1(\frac{\eta}{2^m\sqrt{a}})$ in (2.2.5). In the sequel, we will therefore have

$$\epsilon_0\leq 2^m\leq c_0/\sqrt{a}.$$

We will use the notations

$$\eta=2^m\sqrt{a}\;\tilde{\eta},\quad h=2^m\sqrt{a}\;\tilde{h}$$

$$\mu^2=\eta^2+\omega_k h^{2/3}\eta^{4/3}=(2^m\sqrt{a})^2(\tilde{\eta}^2+\omega_k\tilde{h}^{2/3}\tilde{\eta}^{4/3})=(2^m\sqrt{a})^2\tilde{\mu}^2$$

$$\gamma=\omega_k h^{2/3}\eta^{-2/3}=\omega_k\tilde{h}^{2/3}\tilde{\eta}^{-2/3}$$

We define $\mathcal{G}_{a,m}$ by the formula

$$\mathcal{G}_{a,m}(t,x,y,z)=\frac{1}{4\pi^2h^2}(h/t)^{1/2}\sum_{k\geq 1}\int e^{\frac{i}{h}(y\eta+t\mu(1-\tilde{z}^2)^{1/2})}\psi_1\left(\frac{\eta}{2^m\sqrt{a}}\right)\frac{\tilde{\sigma}_k}{\mu}d\eta. \tag{2.2.6}$$

Observe that due to the truncation χ_1, we have $k\leq\frac{\varepsilon}{h\eta^2}$ in the above sum. Using the change of variable $\eta=2^m\sqrt{a}\tilde{\eta}$, we get with $\tilde{y}=y/t$, since $d\eta/\mu=d\tilde{\eta}/\tilde{\mu}$

$$\mathcal{G}_{a,m}(t,x,y,z)=\frac{1}{4\pi^2h^2}(h/t)^{1/2}\sum_{1\leq k\leq\frac{\varepsilon}{(2^m\sqrt{a})^3\tilde{h}}}\int e^{\frac{it}{\tilde{h}}(\tilde{y}\tilde{\eta}+\tilde{\mu}(1-\tilde{z}^2)^{1/2})}g_k\psi_1(\tilde{\eta})d\tilde{\eta}, \tag{2.2.7}$$

where g_k is defined by

$$g_k=\frac{1}{\tilde{\mu}}\sigma_0(z^*,\eta,\mu^2;\lambda)(1-\chi_4(\frac{t\mu^2}{Dh}))e_k(x,\tilde{\eta}/\tilde{h})e_k(a,\tilde{\eta}/\tilde{h})\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k).$$

Lemma 2.2.3. *Let $M\geq 1$ be given. There exists C_M such that for all m,a,h such that $2^m\sqrt{a}\leq hM$, the following holds true:*

$$|\mathcal{G}_{a,m}|\leq C_M h^{-3}(h/t)^{1/2}2^m\sqrt{a}|\log(2^m\sqrt{a})|. \tag{2.2.8}$$

Proof. One has $\tilde{h}\geq 1/M$ and therefore $|e_k(x,\tilde{\eta}/\tilde{h})|\leq Ck^{-1/6}(\frac{\tilde{\eta}}{\tilde{h}})^{1/3}\omega_k^{-1/4}$. Moreover, we have $\tilde{\mu} \geq \omega_k^{1/2} \tilde{h}^{1/3} \tilde{\eta}^{2/3}$. Therefore we get

$$\begin{aligned}|\mathcal{G}_{a,m}| &\leq C'' h^{-2}(h/t)^{1/2} \sum_{1 \leq k \leq \frac{\varepsilon}{(2^m\sqrt{a})^3 \tilde{h}}} \omega_k^{-1/2} \tilde{h}^{-1/3} k^{-1/3} (\frac{1}{\tilde{h}})^{2/3} \omega_k^{-1/2} \\ &\leq C' h^{-3}(h/t)^{1/2} 2^m \sqrt{a} |\log(2^{2m} ah)| \leq C h^{-3}(h/t)^{1/2} 2^m \sqrt{a} |\log(2^m \sqrt{a})|. \end{aligned} \quad (2.2.9)$$

□

From the above lemma, we get in the range $\tilde{h} \geq 1/M$ the estimate

$$|\mathcal{G}_{a,m}| \leq C_M h^{-3}(h/t)^{1/2}(2^m\sqrt{a})^{1/3}(hM)^{2/3}|\log(hM)|. \quad (2.2.10)$$

This estimate is even better than the free estimate $Ch^{-3}(h/t)$. Therefore, in the sequel we will assume $\tilde{h} \leq \tilde{h}_0$ with $\tilde{h}_0$ small. To establish the local in time estimates for the $\mathcal{G}_{a,m}$, we follow the strategy of section 2.1 . We distinguish between two different cases. First case, if $a \leq \tilde{h}^{\frac{2}{3}(1-\epsilon)}$, for a given $\epsilon \in]0, 1/7[$, we use the sum over eigenmodes. Second case, if $a \geq \tilde{h}^{\frac{2}{3}(1-\epsilon')}$, with $\epsilon' \in]0, \epsilon[$, we use the Airy-Poisson summation formula [see Lemma 2.1.4] and we rewrite $\mathcal{G}_{a,m}$ as a sum over multiple reflections.

2.2.1 Dispersive Estimates for $0 < a \leq \tilde{h}^{\frac{2}{3}(1-\epsilon)}$, with $\epsilon \in]0, 1/7[$.

The following Proposition 2.2.4 gives a local in time dispersive estimates for $\mathcal{G}_{a,m}$ and is the main result of this subsection.

Proposition 2.2.4. *Let $\epsilon \in]0, 1/7[$. There exists C such that for all $h \in]0, 1]$, all $0 < a \leq \tilde{h}^{\frac{2}{3}(1-\epsilon)}$, and all $t \in [h, 1]$, the following holds true:*

$$\|\mathbb{1}_{x \leq a} \mathcal{G}_{a,m}(t, x, y, z)\|_{L^\infty} \leq C h^{-3} (2^m\sqrt{a})^{1/3} \left(\frac{h}{t}\right)^{5/6}. \quad (2.2.11)$$

Proof. Recall that $\mathcal{G}_{a,m}$ is defined by

$$\mathcal{G}_{a,m}(t, x, y, z) = \frac{1}{4\pi^2 h^2}(h/t)^{1/2} \sum_{1 \leq k \leq \frac{\varepsilon}{(2^m\sqrt{a})^3 \tilde{h}}} \int e^{\frac{it}{\tilde{h}}(\tilde{y}\tilde{\eta} + \tilde{\mu}(1-\tilde{z}^2)^{1/2})} g_k \psi_1(\tilde{\eta}) d\tilde{\eta}, \quad (2.2.12)$$

with g_k equal to

$$g_k = \frac{1}{\tilde{\mu}} \sigma_0(z^*, \eta, \mu^2; \lambda)(1 - \chi_4(\frac{t\mu^2}{Dh})) e_k(x, \tilde{\eta}/\tilde{h}) e_k(a, \tilde{\eta}/\tilde{h}) \chi_1(\omega_k h^{2/3} \eta^{4/3})(1 - \chi_1)(\varepsilon \omega_k).$$

Recall from (2.2.10) that we may assume $\tilde{h} \leq \tilde{h}_0$ with $\tilde{h}_0$ small. Since $\mathcal{G}_{a,m}$ contains Airy functions which behave differently depending on the various values of k, we split the sum over k in (2.2.12) in two pieces. We fix a large constant D and we write $\mathcal{G}_{a,m} = \mathcal{G}_{a,m,<} + \mathcal{G}_{a,m,>}$, where in $\mathcal{G}_{a,m,<}$ only the sum over $1 \le k \le D\tilde{h}^{-\epsilon}$ is considered.

Proof of (2.2.11) for $\mathcal{G}_{a,m,<}$.

Recall the definition of $\mathcal{G}_{a,m,<}$:

$$\mathcal{G}_{a,m,<}(t,x,y,z) = \frac{1}{4\pi^2 h^2}(h/t)^{1/2} \sum_{1\le k\le D\tilde{h}^{-\epsilon}} \int e^{\frac{it}{\tilde{h}}(\tilde{y}\tilde{\eta}+\tilde{\mu}(1-\tilde{z}^2)^{1/2})} g_k \psi_1(\tilde{\eta}) d\tilde{\eta}. \tag{2.2.13}$$

$$g_k = f_k^2 k^{-1/3} (\frac{\tilde{\eta}^{2/3}}{\tilde{\mu}\tilde{h}^{2/3}}) \sigma_0(z^*,\eta,\mu^2;\lambda)(1-\chi_4(\frac{t\mu^2}{Dh}))\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k) n_k.$$

$$n_k = Ai((\tilde{\eta}/\tilde{h})^{2/3}x - \omega_k) Ai((\tilde{\eta}/\tilde{h})^{2/3}a - \omega_k).$$

Let us first assume $t2^m\sqrt{a} \le \tilde{h}^\epsilon$. Since we have $\tilde{\mu} = (\tilde{\eta}^2 + \omega_k \tilde{h}^{2/3}\tilde{\eta}^{4/3})^{1/2} \ge \tilde{\eta}$, we get the estimate

$$|g_k| \le C\tilde{h}^{-2/3}k^{-1/3} \left| Ai((\tilde{\eta}/\tilde{h})^{2/3}x - \omega_k) Ai((\tilde{\eta}/\tilde{h})^{2/3}a - \omega_k)\right|.$$

By Lemma 2.1.2, this implies

$$\sum_{1\le k\le D\tilde{h}^{-\epsilon}} |g_k| \le C\tilde{h}^{-2/3}(\tilde{h}^{-\epsilon})^{1/3} \le C(\tilde{h})^{-2/3}(t2^m\sqrt{a})^{-1/3} = Ch^{-1}(2^m\sqrt{a})^{1/3}\left(\frac{h}{t}\right)^{1/3}$$

and (2.2.11) follows from (2.2.13).

Let us now assume $t2^m\sqrt{a} \ge \tilde{h}^\epsilon$. Observe that in the range $k \le \tilde{D}h^{-\epsilon}$, we have $\omega_k\tilde{h}^{2/3} \le C\tilde{h}^{2/3(1-\epsilon)} \le C\tilde{h}_0^{2/3(1-\epsilon)}$ small. Hence $\gamma = \omega_k\tilde{h}^{2/3}\tilde{\eta}^{-2/3}$ is small and $\tilde{\mu} = \tilde{\eta}(1+\gamma)^{1/2} = \tilde{\eta} + \tilde{\eta}^{1/3}\omega_k\tilde{h}^{2/3}/2 + O((\omega_k\tilde{h}^{2/3})^2)$. Therefore we get $|\frac{\partial^2\tilde{\mu}}{\partial\tilde{\eta}^2}| \ge c\omega_k\tilde{h}^{2/3}$ with $c>0$, and for all $j \ge 2$, $|\frac{\partial^j\tilde{\mu}}{\partial\tilde{\eta}^j}| \le C_j\omega_k\tilde{h}^{2/3}$. We will apply the stationary phase in $\tilde{\eta}$ in each term of the sum in (2.2.13) with the phase function $\Phi_k(\tilde{\eta}) = \frac{t}{\tilde{h}}(\tilde{y}\tilde{\eta} + \tilde{\mu}(1-\tilde{z}^2)^{1/2})$. Let $\Lambda_k = t\tilde{h}^{-1/3}\omega_k 2^m\sqrt{a}$, and let $\Psi_k(\tilde{\eta})$ the phase function defined by

$$\frac{t\Phi_k}{\tilde{h}} = \Lambda_k\Psi_k.$$

Lemma 2.2.5. *Let $\tilde{g}_k = k^{1/3}\tilde{h}^{2/3}g_k$. There exists C such that for all $1 \le k \le D\tilde{h}^{-\epsilon}$, the following holds true:*

$$\left| \int e^{i\Lambda_k\Psi_k}\tilde{g}_k\psi_1(\tilde{\eta})d\tilde{\eta}\right| \le C\min\left\{1, \Lambda_k^{-1/2}\right\}. \tag{2.2.14}$$

Proof. We may assume $\Lambda_k \ge 1$ since we have $|\tilde{g}_k| \le C$. Recall from Lemma 2.2.2 that we may assume $\sqrt{1-\tilde{z}^2} \simeq \mu = 2^m\sqrt{a}\tilde{\mu} \simeq 2^m\sqrt{a}$. Therefore, there exists $c>0$ such that for all $1 \le k \le D\tilde{h}^{-\epsilon}$ one has

$$\left|\frac{\partial^2 \Psi_k}{\partial \tilde{\eta}^2}\right| = \left|\frac{t}{\tilde{h}\Lambda_k}\frac{\partial^2 \tilde{\mu}}{\partial \tilde{\eta}^2}\sqrt{1-\tilde{z}^2}\right| \ge c > 0$$

and for all $j \ge 2$, $\left|\frac{\partial^j \Psi_k}{\partial \tilde{\eta}^j}\right| \le C_j$. Thus, to apply the stationary phase, we just need to check that there exist $\nu > 0$ and for all j, a constant C_j such that

$$\left|\frac{\partial^j \tilde{g}_k}{\partial \tilde{\eta}^j}\right| \le C_j \Lambda_k^{j(1/2-\nu)}, \quad \forall k \le D\tilde{h}^{-\epsilon} \ . \tag{2.2.15}$$

In Lemma 2.2.2, z^* is defined by $\tilde{z} = -1 + z^*\mu^2$, but since we have here $\mu \simeq 2^m\sqrt{a}$ we may as well define z^* by $\tilde{z} = -1 + z^* 2^{2m} a$. Then z^* becomes independent of $\tilde{\eta}$. Recall $\lambda = t\mu^2/h$. Since $\eta = 2^m\sqrt{a}\tilde{\eta}$ and all the derivatives of γ and $\tilde{\mu}$ with respect to $\tilde{\eta}$ are bounded, we get $|\frac{\partial^j \lambda}{\partial \tilde{\eta}^j}| \le C_j\lambda$ for all j. Since λ is bounded on the support of derivatives of χ_4, the term

$$f_k^2 \tilde{\eta}^{2/3}\sigma_0(z^*, \eta, \mu^2; \lambda)(1-\chi_4(\frac{t\mu^2}{Dh}))\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k)$$

satisfies the estimate (2.2.15), and it remains to show that the function $Ai((\frac{\tilde{\eta}}{\tilde{h}})^{2/3}x - \omega_k)$ satisfies the estimate (2.2.15) uniformly in $x \in [0, a]$. Let $\theta = x\tilde{h}^{-2/3} \ge 0$ and $r = \tilde{\eta}^{2/3}$ which belongs to a compact subset of $]0, \infty[$. One has $\partial_r^l (Ai(r\theta - \omega_k)) \simeq (r\theta)^l Ai^{(l)}(r\theta - \omega_k)$. Since for all l one has

$$\sup_{b \ge 0} |b^l Ai^{(l)}(b - \omega_k)| \le C_l \omega_k^{3l/2}$$

we get that (2.2.15) holds true if

$$\exists \beta > 3, c > 0, \quad c\omega_k^\beta \le \Lambda_k = t\tilde{h}^{-1/3}\omega_k 2^m\sqrt{a}$$

We have $t2^m\sqrt{a} \ge \tilde{h}^\epsilon$, and $c\omega_k^2 \le \tilde{h}^{-4\epsilon/3}$, thus this holds for $\epsilon < 1/7$. $\square$

Therefore we get the following estimate for $\mathcal{G}_{a,m,<}$ and $t2^m\sqrt{a} \ge \tilde{h}^\epsilon$

$$\begin{aligned}
\|\mathbb{1}_{x\le a}\mathcal{G}_{a,m,<}(t,x,y,z)\|_{L^\infty} &\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2}\left[\sum_{1\le k\le D\tilde{h}^{-\epsilon}} k^{-1/3}\tilde{h}^{-2/3}(t\tilde{h}^{-1/3}\omega_k 2^m\sqrt{a})^{-1/2}\right] \\
&\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2}(t2^m\sqrt{a})^{-1/2}\tilde{h}^{-(1/2+\epsilon/3)} \\
&\le Ch^{-3}(2^m\sqrt{a})^{1/3}\left(\frac{h}{t}\right)^{5/6} \\
&\quad \times \left[h\left(\frac{t}{h}\right)^{1/3}(2^m\sqrt{a})^{-1/3}(t2^m\sqrt{a})^{-1/2}\tilde{h}^{-(1/2+\epsilon/3)}\right] .
\end{aligned}$$

This concludes the proof of Proposition 2.2.4 for $\mathcal{G}_{a,m,<}$ since $t2^m\sqrt{a} \geq \tilde{h}^{\epsilon}$ implies

$$h^{2/3}t^{-1/6}(2^m\sqrt{a})^{-5/6}\tilde{h}^{-(1/2+\epsilon/3)} \leq \tilde{h}^{1/6-\epsilon/2}.$$

Proof of (2.2.11) for $\mathcal{G}_{a,m,>}$.

For $k \geq D\tilde{h}^{-\epsilon}$ with D large and $a \leq \tilde{h}^{2/3(1-\epsilon)}$ one has

$$\omega_k - \tilde{h}^{-2/3}\tilde{\eta}^{2/3}a \geq \omega_k/2.$$

Since $\gamma = \omega_k\tilde{h}^{2/3}\tilde{\eta}^{-2/3}$, we get $\gamma - a \geq a$ and $\gamma - a \geq \gamma/2$. Then by the definition of e_k and asymptotic of the Airy functions, we obtain

$$\mathcal{G}_{a,m,>}(t,x,y,z) = \sum_{\tilde{h}^{-\epsilon} \leq k \leq \frac{\varepsilon}{(2^m\sqrt{a})^3\tilde{h}}} \frac{1}{4\pi^2h^2}\left(\frac{h}{t}\right)^{1/2}\sum_{\pm,\pm}\int e^{\frac{i}{\tilde{h}}\Phi_k^{\pm,\pm}}\sigma_k^{\pm,\pm}\psi_1(\tilde{\eta})d\tilde{\eta}, \qquad (2.2.16)$$

with phase functions defined by

$$\Phi_k^{\pm,\pm}(\tilde{\eta}) = \tilde{\eta}\Big[y + t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2} \pm \frac{2}{3}(\gamma - x)^{3/2} \pm \frac{2}{3}(\gamma - a)^{3/2}\Big], \qquad (2.2.17)$$

and the symbols are given by

$$\begin{aligned}\sigma_k^{\pm,\pm}(\tilde{\eta}) &= f_k^2k^{-1/3}\tilde{h}^{-1/3}\tilde{\eta}^{-2/3}\sigma_0(z^*,\eta,\mu^2,\lambda)(1-\chi_4(\frac{t\mu^2}{Dh}))\chi_1(\omega_kh^{2/3}\eta^{4/3})(1-\chi_1(\epsilon\omega_k))\\ &\times(\gamma-x)^{-1/4}(\gamma-a)^{-1/4}(1+\gamma)^{-1/2}\omega^{\pm}\omega^{\pm}\Psi_{\pm}\left(\tilde{\eta}^{2/3}\tilde{h}^{-2/3}(\gamma-x)\right)\Psi_{\pm}\left(\tilde{\eta}^{2/3}\tilde{h}^{-2/3}(\gamma-a)\right),\end{aligned}$$

where $\Psi_{\pm}$ are classical symbols of order 0 at infinity. In Lemma 2.2.2, z^* is defined by $\tilde{z} = -1 + z^*\mu^2$, but since we have here $\mu \simeq 2^m\sqrt{a}(1+\omega_k\tilde{h}^{2/3})^{1/2}$ we may as well define z^* by $\tilde{z} = -1 + z^*2^{2m}a(1+\omega_k\tilde{h}^{2/3})$. Then z^* becomes independent of $\tilde{\eta}$. Observe that for all j, there exists C_j, C_j' such that for all k one has

$$|\partial_{\tilde{\eta}}^j\gamma| \simeq C_j\gamma, \quad |\partial_{\tilde{\eta}}^j\tilde{\mu}| \leq C_j'\tilde{\mu}, \quad |\partial_{\tilde{\eta}}^j\mu^2| \leq C_j'\mu^2 \leq C_j'.$$

Since $\lambda = t\mu^2/h = \frac{t2^m\sqrt{a}}{\tilde{h}}(1+\gamma)$, we get $|\frac{\partial^j\lambda}{\partial\tilde{\eta}^j}| \leq C_j\lambda$ for all j. Finally, λ is bounded on the support of derivatives of χ_4 and there exists $c_1 > 0$ such that $\tilde{\eta}^{2/3}\tilde{h}^{-2/3}(\gamma - a) \geq c_1$. Since $\gamma \simeq (k\tilde{h})^{2/3}$, we get that for all j, there exists C_j such that for all k one has

$$|\partial_{\tilde{\eta}}^j\sigma_k^{\pm,\pm}(\tilde{\eta})| \leq C_j(k\tilde{h})^{-2/3}(1+\gamma)^{-1/2}. \qquad (2.2.18)$$

We notice that for the values of $k, D\tilde{h}^{-\epsilon} \leq k \leq \frac{1}{h\eta^2}$, we get $\gamma \in [2a, \frac{1}{2^{2m}a}]$. In what follows, we distinguish between the two cases: $\gamma \in [2a, 1]$ and $\gamma \in [1, \frac{1}{2^{2m}a}]$.

- The first case $\gamma \in [2a, 1]$ corresponds to $\tilde{h}^{-\epsilon} \le k \le \tilde{h}^{-1}$. Let $\Lambda_k = t2^m\sqrt{a}\omega_k\tilde{h}^{-1/3}$ and $\Phi_k^{\pm,\pm} = \tilde{h}\Lambda_k\Psi_k^{\pm,\pm}$.

  Proposition 2.2.6. *There exists a constant C independent of $a \in]0, \tilde{h}^{2/3(1-\epsilon)}]$, $t \in [h, 1]$, $x \in [0, a], y \in \mathbb{R}, z \in \mathbb{R}$, and $k \in [\tilde{h}^{-\epsilon}, \tilde{h}^{-1}]$ such that the following holds:*

$$\left| \int e^{i\Lambda_k \Psi_k^{\pm,\pm}} \sigma_k^{\pm,\pm} \psi_1(\tilde{\eta}) d\tilde{\eta} \right| \le C(\tilde{h}k)^{-2/3}\Lambda_k^{-1/3}.$$

  Proof of Proposition 2.2.6. By (2.2.18) , Proposition 2.2.6 is obvious for $\Lambda_k \le 1$. In the case $\Lambda_k \ge 1$, we use $\tilde{\mu} \simeq 2^m\sqrt{a}$ which implies $t\sqrt{1-\tilde{z}^2} \simeq t2^m\sqrt{a}$. Then the proof is the same as the proof of Proposition 2.1.3, if one replaces (h, t) in Proposition 2.1.3 by $(\tilde{h}, t2^m\sqrt{a})$. □

  Hence the corresponding estimate of $\mathcal{G}_{a,m,>}$ for $\tilde{h}^{-\epsilon} \le k \le \tilde{h}^{-1}$ is given by

$$\begin{aligned} \|\mathbb{1}_{x\le a}\mathcal{G}_{a,m,>}(t,x,y,z)\|_{L^\infty} &\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2} \sum_{\tilde{h}^{-\epsilon}\le k\le \tilde{h}^{-1}} (\tilde{h}k)^{-2/3}(t2^m\sqrt{a}\omega_k\tilde{h}^{-1/3})^{-1/3} \\ &\le Ch^{-2}\left(\frac{h}{t}\right)^{1/2} \tilde{h}^{-2/3}(t2^m\sqrt{a})^{-1/3}\tilde{h}^{1/9}\sum_{k\le 1/\tilde{h}} k^{-8/9} \\ &\le Ch^{-3}\left(\frac{h}{t}\right)^{5/6}(2^m\sqrt{a})^{1/3}. \end{aligned}$$

- The second case $\gamma \in [1, \frac{1}{2^{2m}a}]$ corresponds to $\tilde{h}^{-1} \le k \le \frac{1}{2^{2m}ah}$. We still define Λ_k and $\Psi_k^{\pm,\pm}$ by $\Lambda_k = t2^m\sqrt{a}\omega_k\tilde{h}^{-1/3}$ and $\Phi_k^{\pm,\pm} = \tilde{h}\Lambda_k\Psi_k^{\pm,\pm}$.

  Proposition 2.2.7. *There exists a constant C independent of $a \in]0, \tilde{h}^{2/3(1-\epsilon)}]$, $t \in [h, 1]$, $x \in [0, a], y \in \mathbb{R}, z \in \mathbb{R}$, and $k \in [\tilde{h}^{-1}, \frac{1}{2^{2m}ah}]$ such that the following holds:*

$$\left| \int e^{i\Lambda_k \Psi_k^{\pm,\pm}} \sigma_k^{\pm,\pm} \psi_1(\tilde{\eta}) d\tilde{\eta} \right| \le C(\tilde{h}k)^{-1}\Lambda_k^{-1/3}.$$

  Proof of Proposition 2.2.7 . One has $\gamma \simeq (k\tilde{h})^{2/3}$. Thus $\gamma \ge 1$ and (2.2.18) imply $|\partial_{\tilde{\eta}}^j \sigma_k^{\pm,\pm}(\tilde{\eta})| \le C_j(k\tilde{h})^{-1}$. Hence Proposition 2.2.7 is obvious for $\Lambda_k \le 1$. In the case $\Lambda_k \ge 1$ we proceed as in the Proposition 2.1.3 . Recall that $\tilde{z}$ is close to -1 and $\tilde{z} = -1 + z^*2^{2m}a(1 + \omega_k\tilde{h}^{2/3})$ with z^* in a compact set of $]0, \infty[$. We write

$$\frac{t\sqrt{1-\tilde{z}^2}}{\tilde{h}\Lambda_k}\frac{1+2\gamma/3}{(1+\gamma)^{1/2}} = \sqrt{z^*}(1-\tilde{z})^{1/2}\tilde{\eta}^{-2/3}\tilde{F}(\gamma), \quad \tilde{F}(\gamma) = \frac{2(1+\omega_k\tilde{h}^{2/3})^{1/2}}{3(1+\gamma)^{1/2}}(1+1/\gamma).$$

For γ large one has $\tilde{F}(\gamma) \simeq 1, \tilde{F}(\gamma) + \gamma \tilde{F}'(\gamma) \simeq 1$. Moreover, one has

$$\omega_k \tilde{h}^{2/3}(2\tilde{F}'(\gamma) + \gamma \tilde{F}''(\gamma)) \simeq \omega_k \tilde{h}^{2/3} \gamma^{-1} \simeq 1.$$

Hence the proof is the same as the proof of Proposition 2.1.3, if one replaces (h, F) in Proposition 2.1.3 by $(\tilde{h}, \tilde{F})$. □

Using Proposition 2.2.7, we get the estimate of $\mathcal{G}_{a,m,>}$ for $\tilde{h}^{-1} \leq k \leq \frac{1}{2^{2m} a h}$:

$$\begin{aligned}
\|\mathbb{1}_{x \leq a} \mathcal{G}_{a,m,>}(t,x,y,z)\|_{L^\infty} &\leq C h^{-2} \left(\frac{h}{t}\right)^{1/2} \sum_{\tilde{h}^{-1} \leq k} (\tilde{h} k)^{-1} (t 2^m \sqrt{a} \omega_k \tilde{h}^{-1/3})^{-1/3} \\
&\leq C h^{-2} \left(\frac{h}{t}\right)^{1/2} t^{-1/3} (2^m \sqrt{a})^{-1/3} \tilde{h}^{-8/9} \sum_{\tilde{h}^{-1} \leq k} k^{-11/9} \\
&\leq C h^{-3} \left(\frac{h}{t}\right)^{5/6} (2^m \sqrt{a})^{1/3}.
\end{aligned}$$

This concludes the proof of Proposition 2.2.4. □

2.2.2 Dispersive Estimates for $a \geq \tilde{h}^{\frac{2}{3}(1-\epsilon')}$, for $\epsilon' \in]0, \epsilon[$.

In this subsection, we assume $a \geq \tilde{h}^{\frac{2}{3}(1-\epsilon')}$, for some $\epsilon' \in]0, \epsilon[$ and we establish a local in time dispersive estimates for $\mathcal{G}_{a,m}$. Observe that $\Lambda = a^{3/2}/\tilde{h} \geq \tilde{h}^{-\epsilon'}$ is a large parameter. Recall from (2.2.12) that $\mathcal{G}_{a,m}$ is defined by

$$\mathcal{G}_{a,m}(t,x,y,z) = \frac{1}{4\pi^2 h^2}(h/t)^{1/2} \sum_{1 \leq k \leq \frac{\varepsilon}{(2^m \sqrt{a})^3 \tilde{h}}} \int e^{\frac{i}{\tilde{h}}(y\tilde{\eta} + t\tilde{\mu}(1-\tilde{z}^2)^{1/2})} g(\omega_k, \tilde{\eta}, \tilde{h}) \psi_1(\tilde{\eta}) d\tilde{\eta}, \tag{2.2.19}$$

with $g(\omega_k, \tilde{\eta}, \tilde{h})$ equal to

$$g = \frac{1}{\tilde{\mu}} \sigma_0(z^*, \eta, \mu^2; \lambda)(1 - \chi_4(\frac{t\mu^2}{Dh})) e_k(x, \tilde{\eta}/\tilde{h}) e_k(a, \tilde{\eta}/\tilde{h}) \chi_1(\omega_k h^{2/3} \eta^{4/3})(1 - \chi_1)(\varepsilon \omega_k),$$

and we recall $h = 2^m \sqrt{a} \tilde{h}, \eta = 2^m \sqrt{a} \tilde{\eta}, \mu = 2^m \sqrt{a} \tilde{\mu}$, and

$$\gamma = \omega \tilde{h}^{2/3} \tilde{\eta}^{-2/3}, \quad \tilde{\mu} = \tilde{\eta}(1+\gamma)^{1/2}.$$

We will use the same notations as in section 2.1, $t = a^{1/2} T, x = aX, y + t\sqrt{1-\tilde{z}^2} = a^{3/2} Y$. Let $\omega = \tilde{\eta}^{2/3} \tilde{h}^{-2/3} a \tilde{\omega}$. We get $\gamma = a\tilde{\omega}$ and $(1 + a\tilde{\omega})^{1/2} - 1 = a\gamma_a(\tilde{\omega}) = \frac{a\tilde{\omega}}{1+(1+a\tilde{\omega})^{1/2}}$. Then we use the Airy Poisson summation formula, and we get

$$\mathcal{G}_{a,m} = \sum_N G_{a,m,N}$$

with

$$G_{a,m,N}(t,x,y,z) = \frac{(-1)^N}{(2\pi)^4 h^4}(h/t)^{1/2} a^2 (2^m\sqrt{a})^2 \int e^{i\Lambda\Phi_N} f_m \tilde{\eta}^2 \psi_1(\tilde{\eta}) d\tilde{s} d\tilde{\sigma} d\tilde{\omega} d\tilde{\eta} \quad (2.2.20)$$

with the phase function

$$\begin{aligned}\Phi_N(\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\eta}) = \tilde{\eta}\Big[& Y + T(1-\tilde{z}^2)^{1/2}\gamma_a(\tilde{\omega}) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) \\ & - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{N}{\Lambda\tilde{\eta}} B\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right)\Big],\end{aligned}$$

and symbol $f_m(a,t,z;\tilde{\eta},\tilde{\omega},\tilde{h})$ equal to, with $\lambda = t2^m\sqrt{a}\tilde{\mu}^2/\tilde{h}$,

$$f_m = \frac{1}{\tilde{\mu}}\sigma_0(z^*,\eta,\mu^2;\lambda)(1-\chi_4(\lambda/D))\chi_1((2^m\sqrt{a})^2\tilde{\eta}^2 a\tilde{\omega})(1-\chi_1)(\varepsilon\tilde{\eta}^{2/3}\tilde{h}^{-2/3}a\tilde{\omega}). \quad (2.2.21)$$

Observe that we get the same phase function Φ_N as in section 2.1, but we have to take care of the fact that now $(1-\tilde{z}^2)^{1/2}$ may be small. Therefore, in order to use the results of section 2.1, we introduce the notation $\tilde{T} = T(1-\tilde{z}^2)^{1/2}$. Set

$$\mathcal{C}_{a,m,N,h} = \{(t,x,y,\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\eta}) \text{ such that } \partial_{\tilde{s}}\Phi_N = \partial_{\tilde{\sigma}}\Phi_N = \partial_{\tilde{\omega}}\Phi_N = \partial_{\tilde{\eta}}\Phi_N = 0\}.$$

Hence $\mathcal{C}_{a,m,N,h}$ is defined by the system of equations

$$\begin{aligned}& X = \tilde{\omega} - \tilde{s}^2, \\ & \tilde{\omega} = 1 + \tilde{\sigma}^2, \\ & \tilde{T} = 2(1+a\tilde{\omega})^{1/2}\left(\tilde{s} + \tilde{\sigma} + 2N\tilde{\omega}^{1/2}\left(1 - \frac{3}{4}B'\Big(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\Big)\right)\right), \\ & Y = -\tilde{T}\gamma_a(\tilde{\omega}) - \frac{\tilde{s}^3}{3} - \tilde{s}(X-\tilde{\omega}) - \frac{\tilde{\sigma}^3}{3} - \tilde{\sigma}(1-\tilde{\omega}) + N\tilde{\omega}^{3/2}\left(\frac{4}{3} - B'\Big(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\Big)\right).\end{aligned}$$

We define the Lagrangian submanifold $\mathbf{\Lambda}_{a,m,N,h} \subset T^*\mathbb{R}^3$ as the image of $\mathcal{C}_{a,m,N,h}$ by the map

$$(t,x,y,\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\eta}) \longmapsto (x,t,y,\xi = \partial_x\Phi_N, \tau = \partial_t\Phi_N, \eta = \partial_y\Phi_N).$$

Then the projection of $\mathbf{\Lambda}_{a,m,N,h}$ onto $\mathbb{R}^3$ is defined by the system of equations

$$X = 1 + \tilde{\sigma}^2 - \tilde{s}^2, \tag{2.2.22}$$
$$Y = H_1(a,\tilde{\sigma})(\tilde{s}+\tilde{\sigma}) + \frac{2}{3}(\tilde{s}^3+\tilde{\sigma}^3) + \frac{2}{3}H_2(a,\tilde{\sigma})(1+\tilde{\sigma}^2)^{-1/2}\left(\frac{\tilde{T}}{2(1+a+a\tilde{\sigma}^2)^{1/2}} - \tilde{s} - \tilde{\sigma}\right),$$

where H_1, H_2 are defined in Section 2.1 and

$$2N\left(1-\frac{3}{4}B'\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right)\right) = (1+\tilde{\sigma}^2)^{-1/2}\left(\frac{\tilde{T}}{2(1+a+a\tilde{\sigma}^2)^{1/2}} - \tilde{s} - \tilde{\sigma}\right). \tag{2.2.23}$$

Remark 2.2.8. *We notice from* (2.2.23) *in the range of* $T \in]0, a^{-1/2}]$*, we can still reduce the sum over* $N \in \mathbb{Z}$ *to the sum over* $1 \le N \le C_0 a^{-1/2}$ *since* $\tilde{T} \le T$*.*

This system yields $\mathcal{N}(X,Y,T) \le C_0$ and $\mathcal{N}_1(X,Y,T) \le C_0\left(1+\tilde{T}\Lambda^{-2}\tilde{\omega}^{-3}\right)$. Recall that here the notations $\mathcal{N}, \mathcal{N}_1$ are those defined in Section 2.1.

Our main result of this subsection is Theorem 2.2.9, which gives dispersive estimates for the sum over N of $G_{a,m,N}$.

Theorem 2.2.9. *Let* $\alpha < 2/3$*. There exists* C *such that for all* $h \in]0, h_0]$*, all* $a \in \left[\tilde{h}^\alpha, a_0\right]$*, all* $x \in [0,a]$*, all* $t \in]h, 1]$*, all* $y \in \mathbb{R}$*, all* $z \in \mathbb{R}$*, the following holds:*

$$\left|\sum_{1\le N\le C_0 a^{-1/2}} G_{a,m,N}(t,x,y,z)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\min\left\{\left(\frac{h}{t}\right)^{1/2}, 2^m\sqrt{a}\right\} + a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}\right).$$

We notice as in section 2.1, that for $\tilde{\omega} \le 3/4$, we get rapid decay in Λ by integration by parts in $\tilde{\sigma}$. In particular, we may replace $1-\chi_1$ by 1 in (2.2.21). As in section 2.1, we introduce a cutoff function $\chi_2(\tilde{\omega}) \in C_0^\infty(]1/2, 3/2[), 0 \le \chi_2 \le 1, \chi_2 = 1$ on $]\frac{3}{4}, \frac{5}{4}[$ and we denote by $G_{a,m,N,2}$ the corresponding integral. We get $G_{a,m,N} = G_{a,m,N,1} + G_{a,m,N,2} + O(\Lambda^{-\infty})$ where $G_{a,m,N,1}$ is defined by a cutoff χ_3 with $\tilde{\omega} \ge 5/4$ on the support of χ_3.

The Analysis of $G_{a,m,N,1}$

The main results in this subsection are Proposition 2.2.10 and Proposition 2.2.11.

Proposition 2.2.10. *Let* $\alpha < 2/3$*. There exists* C *such that for all* $h \in]0, h_0]$*, all* $a \in \left[\tilde{h}^\alpha, a_0\right]$*, all* $x \in [0,a]$*, all* $t \in]h, 1]$*, all* $y \in \mathbb{R}$*, all* $z \in \mathbb{R}$*, the following holds:*

$$\left|\sum_{2\le N\le C_0 a^{-1/2}} G_{a,m,N,1}(t,x,y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

Proof. On the support of χ_3, we can apply the stationary phase method for $(\tilde{s},\tilde{\sigma})$-integrations with large parameter $\Lambda\tilde{\eta}$; hence we get

$$G_{a,m,N,1} = \frac{(-1)^N a^2 \Lambda^{-1}}{(2\pi)^4 h^4}(2^m\sqrt{a})^2\left(\frac{h}{t}\right)^{1/2}\int e^{i\Lambda Y\tilde{\eta}}\tilde{\eta}\psi_1(\tilde{\eta})\tilde{G}_{a,m,N,1}d\tilde{\eta},$$
$$\tilde{G}_{a,m,N,1} = \sum_{\epsilon_1,\epsilon_2}\int e^{i\Lambda\tilde{\eta}\Phi_{N,m,\epsilon_1,\epsilon_2}}\Theta_{\epsilon_1,\epsilon_2}(1+a\tilde{\omega})^{-1/2}d\tilde{\omega},$$

with symbols $\Theta_{\epsilon_1,\epsilon_2}$ with support in $\tilde{\omega} \leq (2^m\sqrt{a})^{-2}/a$, and such that $|\tilde{\omega}^l\partial^l_{\tilde{\omega}}\Theta_{\epsilon_1,\epsilon_2}| \leq C_l\tilde{\omega}^{-1/2}$ with C_l independent of a, m, and where $\epsilon_j = \pm$. The phase functions are

$$\Phi_{N,m,\epsilon_1,\epsilon_2}(\tilde{\omega}) = \tilde{T}\gamma_a(\tilde{\omega})+\frac{2}{3}\epsilon_1(\tilde{\omega}-X)^{3/2}+\frac{2}{3}\epsilon_2(\tilde{\omega}-1)^{3/2}-\frac{4}{3}N\tilde{\omega}^{3/2}+\frac{N}{\Lambda\tilde{\eta}}B\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right).$$

Let us define

$$G_{a,m,N,1,\epsilon_1,\epsilon_2} = \frac{(-1)^N a^2 \Lambda^{-1}}{(2\pi)^4 h^4}(2^m\sqrt{a})^2\left(\frac{h}{t}\right)^{1/2}\int e^{i\Lambda Y\tilde{\eta}}\tilde{\eta}\psi_1(\tilde{\eta})\tilde{G}_{a,m,N,1,\epsilon_1,\epsilon_2}d\tilde{\eta},$$
$$\tilde{G}_{a,m,N,1,\epsilon_1,\epsilon_2} = \sum_{\epsilon_1,\epsilon_2}\int e^{i\Lambda\tilde{\eta}\Phi_{N,m,\epsilon_1,\epsilon_2}}\Theta_{\epsilon_1,\epsilon_2}(1+a\tilde{\omega})^{-1/2}d\tilde{\omega}.$$

We are reduce to prove the following inequality

$$\left|\sum_{2\leq N\leq C_0 a^{-1/2}} G_{a,m,N,1,\epsilon_1,\epsilon_2}(t,x,y,z,h)\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}(2^m\sqrt{a})^{2/3}, \qquad (2.2.24)$$

with a constant C independent of m, $h \in]0,h_0]$, $a \in \left[\tilde{h}^{2/3}, a_0\right]$, $x \in [0,a]$, $t \in [h,1]$. We proceed as in the proof of Proposition 2.1.8. Let us recall that on the support of χ_1 we have $a\tilde{\omega} \leq \varepsilon/2^{2m}a$; hence $a\tilde{\omega}$ could be small or large. We distinguish between two cases:

The first case is $a\tilde{\omega} \leq 1$. Let $\tilde{T}_0 \gg 1$. We get the following results:

- For $0 \leq \tilde{T} \leq \tilde{T}_0, N \geq N(\tilde{T}_0)$, then we apply the integration by parts to get $|\tilde{G}_{a,m,N,1,+,+}| \in O(N^{-\infty}\Lambda^{-\infty})$ and

$$\sup_{\tilde{T}\leq\tilde{T}_0, X\in[0,1],(y,z)\in\mathbb{R}^2}\left|\sum_{N(\tilde{T}_0)\leq N\leq Ca^{-1/2}} G_{a,m,N,1,+,+}\right| \in O(h^\infty).$$

- For $0 \leq \tilde{T} \leq \tilde{T}_0, 2 \leq N \leq N(\tilde{T}_0)$, Lemma 2.20[29] yields the following estimate

$|\tilde{G}_{a,m,N,1,+,+}| \leq C\Lambda^{-1/3}$ and

$$\sup_{\tilde{T}\leq\tilde{T}_0, X\in[0,1],(y,z)\in\mathbb{R}^2} \left|\sum_{2\leq N\leq N(\tilde{T}_0)} G_{a,m,N,1,+,+}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(h^{-1}a^2(2^m\sqrt{a})^2\Lambda^{-4/3}\right),$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

- For $\tilde{T}_0 \leq \tilde{T} \leq a^{-1/2}(1-\tilde{z}^2)^{1/2}$, we use the same notation as before $\Omega = \tilde{\omega}^{3/2}$; we have $|\partial_\Omega^2 \Phi_{N,m,+,+}| \geq c\tilde{T}\Omega^{-4/3}$ and a nondegenerate critical point Ω_c which satisfies for $N \geq 2$, $\Omega_c^{1/3} \simeq \frac{\tilde{T}}{N}$. We have also either $\tilde{T}/N$ bounded or large, the stationary phase yields $|\tilde{G}_{a,m,N,1,+,+}| \leq C\Lambda^{-1/2}\tilde{T}^{-1/2}$. Moreover, the $\tilde{\eta}$ -integration produces a $q^{-1/2}$ factor contribution with $q = N\Lambda^{-1}\Omega_c^{-1}$ when $q \geq 1$. Thus, we get the estimates as follows:

If $\tilde{T}/N$ is bounded, Ω_c stays in a compact subset of $[1,\infty[$, and we get $\tilde{T} \simeq N$.

- If $N \leq \Lambda^2$, we have $|\mathcal{N}_1| \leq C_0$. Hence the estimate is

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\Lambda^{-1/2}\tilde{T}^{-1/2}]$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} a^{-1/4}h^{1/2}(2^m\sqrt{a})^{1/2}\tilde{T}^{-1/2}$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3},$$

since $\tilde{T} \geq \tilde{T}_0$ and $a^{-1/4}h^{1/2} \leq h^{1/3}(2^m\sqrt{a})^{1/6}$ when $a \geq \tilde{h}^{2/3}$.

- If $N > \Lambda^2$, then there is the contribution $q^{-1/2}$ from $\tilde{\eta}$-integration and $|\mathcal{N}_1| \leq C_0\tilde{T}\Lambda^{-2}$. Thus the estimate is

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\sum_{N\in\mathcal{N}_1}[h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\Lambda^{-1/2}\tilde{T}^{-1/2}N^{-1/2}\Lambda^{1/2}]$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\tilde{T}^{-1}|\mathcal{N}_1(X,Y,T)|]$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[a^{-5/2}\tilde{h}^2 2^m\sqrt{a}]$$
$$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\tilde{h}^{1/3}2^m\sqrt{a} = Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

Next, if $\tilde{T}/N$ is large then Ω_c is large.

- If $N \leq \Lambda\Omega_c$, then there is no contribution from $\tilde{\eta}$-integration. Moreover we have $|\mathcal{N}_1| \leq C_0$ since $\tilde{T} \geq \Lambda^2\Omega_c^2$ implies $\Omega_c^{1/3} \simeq \tilde{T}/N \geq \lambda\Omega_c$ which is impossible since Ω_c is large. Thus the estimate is

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

- If $N > \Lambda\Omega_c$ and $\tilde{T} \leq \Lambda^2\Omega_c^2$, we also have $|\mathcal{N}_1| \leq C_0$. Thus we get the estimate

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

- If $N > \lambda\Omega_c$ and $\tilde{T} > \Lambda^2\Omega_c^2$, then there is the contribution $q^{-1/2}$ from $\tilde{\eta}$-integration and $|\mathcal{N}_1| \leq C_0\tilde{T}\Lambda^{-2}\Omega_c^{-2}$. We get

$$\begin{aligned}
\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,1,+,+}\right| &\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} \sum_{N\in\mathcal{N}_1}[h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\tilde{T}^{-1/2}N^{-1/2}\Omega_c^{1/2}] \\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} [h^{-1}\Lambda^{-1}a^2(2^m\sqrt{a})^2\tilde{T}^{-1}\Omega_c^{2/3}|\mathcal{N}_1(X,Y,T)|] \\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} [h^{-1}a^2(2^m\sqrt{a})^2\Lambda^{-3}] \\
&\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.
\end{aligned}$$

The result of the other cases of (ϵ_1, ϵ_2) can be achieved by proceeding along the same lines as in the proof for $G_{a,N,1}$ in section 2.1.

If $a\tilde{\omega} \geq 1$, then a critical point Ω_c will satisfy $\Omega_c^{1/3}(1 + a\Omega_c^{2/3})^{1/2} \simeq \frac{\tilde{T}}{N}$ for $N \geq 2$. This yields, since $T \geq C\tilde{T}$ with C large,

$$T \geq C\tilde{T} \geq CN\Omega_c^{1/3} = CN\omega_c^{1/2} \geq CNa^{-1/2}$$

which contradicts $t \leq 1$.

$\square$

Now we prove the following estimate for $N = 1$.

Proposition 2.2.11. *Let $\alpha < 2/3$. There exists C such that for all $h \in]0, h_0]$, all $a \in [\tilde{h}^\alpha, a_0]$, all $x \in [0, a]$, all $t \in [h, 1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$|G_{a,m,1,1}(t,x,y,z;h)| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\min\left\{\left(\frac{h}{t}\right)^{1/2}, 2^m\sqrt{a}|\log(2^m\sqrt{a})|\right\} + h^{1/3}(2^m\sqrt{a})^{2/3}\right).$$

Proof. Let us recall

$$G_{a,m,1,1} = \frac{(-1)a^2\Lambda^{-1}}{(2\pi)^4h^4}(2^m\sqrt{a})^2\left(\frac{h}{t}\right)^{1/2}\int e^{i\Lambda Y\tilde{\eta}}\tilde{\eta}\psi_1(\tilde{\eta})\tilde{G}_{a,m,1,1}d\tilde{\eta},$$
$$\tilde{G}_{a,m,1,1} = \sum_{\epsilon_1,\epsilon_2}\int e^{i\Lambda\tilde{\eta}\Phi_{1,m,\epsilon_1,\epsilon_2}}\Theta_{\epsilon_1,\epsilon_2}(1+a\tilde{\omega})^{-1/2}d\tilde{\omega}.$$

The only difference with the case $N \geq 2$ is in the study of the phase $\Phi_{1,m,+,+}$ since in the case $N = 1$ we may have a critical point $\tilde{\omega}_c$ large. Let

$$\tilde{G}_{a,m,1,1,+,+} = \int e^{i\Lambda\Phi_{1,m,+,+}}\Theta_{+,+}(1+a\tilde{\omega})^{-1/2}d\tilde{\omega}, \tag{2.2.25}$$

with the phase function

$$\Phi_{1,m,+,+} = \tilde{T}\gamma_a(\tilde{\omega}) + \frac{2}{3}(\tilde{\omega}-X)^{3/2} + \frac{2}{3}(\tilde{\omega}-1)^{3/2} - \frac{4}{3}\tilde{\omega}^{3/2} + \frac{1}{\Lambda\tilde{\eta}}B(\Lambda\tilde{\omega}^{3/2}\tilde{\eta}),$$

and $\Theta_{+,+}$ is a classical symbol of order $-1/2$ with respect to $\tilde{\omega}$ which satisfy $|\tilde{\omega}^l\partial_{\tilde{\omega}}^l\Theta_{+,+}| \leq C_l\tilde{\omega}^{-1/2}$. Let $\chi_3(\tilde{\omega}) \in C_0^\infty(]\tilde{\omega}_1, \infty[)$ with $\tilde{\omega}_1$ large and set

$$J = \int e^{i\Lambda\Phi_{1,m,+,+}}\Theta_{+,+}\chi_3(\tilde{\omega})(1+a\tilde{\omega})^{-1/2}d\tilde{\omega}. \tag{2.2.26}$$

To prove the proposition, we just have to verify

$$a^{1/2}2^m\sqrt{a}|J| \leq C\min\left\{\left(\frac{h}{t}\right)^{1/2}, 2^m\sqrt{a}|\log(2^m\sqrt{a})|\right\}. \tag{2.2.27}$$

We first observe that on the support of the integral in (2.2.26), one has $a\tilde{\omega} \leq (2^m\sqrt{a})^{-2} = L$. Hence we get

$$|J| \leq C\left(1 + \int_1^{L/a}\frac{1}{\sqrt{x(1+ax)}}dx\right) = C\left(1 + a^{-1/2}\int_a^L\frac{1}{\sqrt{y(1+y)}}dy\right) \leq Ca^{-1/2}\log L.$$

This implies

$$a^{1/2}2^m\sqrt{a}|J| \leq C2^m\sqrt{a}|\log(2^m\sqrt{a})|.$$

We have

$$\partial_{\tilde\omega}\Phi_{1,m,+,+} = \frac{\tilde T}{2}(1+a\tilde\omega)^{-1/2} - \frac{\tilde\omega^{-1/2}}{2}(1+X) + O(\tilde\omega^{-3/2}),$$
$$\partial^2_{\tilde\omega\tilde\omega}\Phi_{1,m,+,+} = \frac{-\tilde T a}{4}(1+a\tilde\omega)^{-3/2} + \frac{\tilde\omega^{-3/2}}{4}(1+X) + O(\tilde\omega^{-5/2}).$$

At a large critical point we have $\tilde T^2 \simeq (a+\tilde\omega_c^{-1})(1+X)^2$. Hence $\tilde T$ is small and $\partial^2_{\tilde\omega}\Phi_{1,+,+}(\tilde\omega_c) \simeq \tilde T^3(1+a\tilde\omega_c)^{-5/2}$. Let $S = (\tilde T/(1+X))^2 - a$. Then we have $S \simeq \tilde\omega_c^{-1}$, and by stationary phase we will get

$$|J| \le C(1+a\tilde\omega_c)^{3/4}\Lambda^{-1/2}\tilde T^{-3/2}S^{1/2}.$$

We have to take care in this section that $a\tilde\omega_c$ may be large. In the case $a\tilde\omega_c \le 1$, we have $S \simeq \tilde T^2$, and we get as before $|J| \le C\Lambda^{-1/2}\tilde T^{-1/2}$, which gives

$$a^{1/2}2^m\sqrt{a}|J| \le C\left(\frac{h}{t}\right)^{1/2}.$$

In the case $a\tilde\omega_c \ge 1$, we must have $\tilde T \simeq \sqrt{a}$, and $S = a\rho$ with $\rho > 0$ small. We get $|J| \le C\rho^{-1/4}a^{-1/4}\Lambda^{-1/2}$. This gives

$$a^{1/2}2^m\sqrt{a}|J| \le Ch^{1/2}\left((2^m\sqrt{a})^{1/2}a^{-1/2}\rho^{-1/4}\right)$$

Finally, we observe that we have

$$\sqrt{a} \simeq \tilde T \simeq ta^{-1/2}2^m\sqrt{a}(1+a\tilde\omega_c)^{1/2} \Rightarrow t \simeq a(2^m\sqrt{a})^{-1}\rho^{1/2},$$

which gives $a^{1/2}2^m\sqrt{a}|J| \le C(h/t)^{1/2}$. The proof of Proposition 2.2.11 is complete. □

The Analysis of $G_{a,m,N,2}$

The main result in this subsection is Proposition 2.2.12.

Proposition 2.2.12. *Let $\alpha < 2/3$. There exists C such that for all $h \in]0,h_0]$, all $a \in [\tilde h^\alpha, a_0]$, all $x \in [0,a]$, all $t \in]h,1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left|\sum_{1\le N\le C_0a^{-1/2}} G_{a,m,N,2}(t,x,y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}.$$

Proof. Recall

$$G_{a,m,N,2}(t,x,y,z) = \frac{(-1)^N}{(2\pi)^4h^4}(h/t)^{1/2}a^2(2^m\sqrt{a})^2\int e^{i\Lambda\Phi_N}f_m\tilde\eta^2\psi_1(\tilde\eta)\chi_2(\tilde\omega)d\tilde s d\tilde\sigma d\tilde\omega d\tilde\eta \tag{2.2.28}$$

with the phase function

$$\Phi_N(\tilde{s},\tilde{\sigma},\tilde{\omega},\tilde{\eta}) = \tilde{\eta}\Big[Y + \tilde{T}\gamma_a(\tilde{\omega}) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{N}{\Lambda\tilde{\eta}}B\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right)\Big].$$

To start with, we rewrite $G_{a,m,N,2}$ in the following form

$$G_{a,m,N,2} = \frac{(-1)^N}{(2\pi)^4 h^4}(h/t)^{1/2}a^2(2^m\sqrt{a})^2\int e^{i\Lambda Y\tilde{\eta}}\tilde{\eta}^2\psi_1(\tilde{\eta})\tilde{G}_{a,m,N,2}d\tilde{\eta},$$
$$\tilde{G}_{a,m,N,2} = \int e^{i\Lambda\tilde{\eta}\tilde{\phi}_{N,m}}f_m\chi_2(\tilde{\omega})d\tilde{s}d\tilde{\sigma}d\tilde{\omega},$$

with the phase function

$$\tilde{\phi}_{N,m}(\tilde{s},\tilde{\sigma},\tilde{\omega}) = \tilde{T}\gamma_a(\tilde{\omega}) + \frac{\tilde{s}^3}{3} + \tilde{s}(X-\tilde{\omega}) + \frac{\tilde{\sigma}^3}{3} + \tilde{\sigma}(1-\tilde{\omega}) - \frac{4}{3}N\tilde{\omega}^{3/2} + \frac{N}{\Lambda\tilde{\eta}}B\left(\tilde{\omega}^{3/2}\Lambda\tilde{\eta}\right).$$

Now we can proceed as in the analysis of $G_{a,N,2}$ in section 2.1. More precisely, we apply the stationary phase method for $\tilde{\omega},\tilde{\eta}$-integrations. It yields $\Lambda^{-1/2}$ and $q^{-1/2}$ with $q = N\Lambda^{-1}$ respectively. We have the following facts [see Section 2.1]:

- Lemma 2.1.11: For $N \geq \Lambda^{1/3}$, there exists C such that

  $$\left|\int e^{i\Lambda\tilde{\psi}_{N,m}}\tilde{\chi}d\tilde{s}d\tilde{\sigma}\right| \leq C\Lambda^{-2/3}, \text{and } \frac{1}{\sqrt{N}}\left|\int e^{i\Lambda\tilde{\psi}_{N,m}}\tilde{\chi}d\tilde{s}d\tilde{\sigma}\right| \leq C\Lambda^{-5/6},$$

  with $\tilde{\psi}_{N,m}$ is a perturbation of the phase function obtained from $\tilde{\phi}_{N,m}$ at the critical point $\tilde{\omega}_c$. Hence we obtain the following estimates:

  - When $|\mathcal{N}_1(X,Y,T)| \leq C_0$, we get

    $$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,2}\right| \leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left[(2^m\sqrt{a})^2h^{-1}a^2\Lambda^{-1/2}\Lambda^{-5/6}\right],$$
    $$\leq Ch^{-3}\left(\frac{h}{t}\right)^{1/2}h^{1/3}(2^m\sqrt{a})^{2/3}.$$

  - When $|\mathcal{N}_1(X,Y,T)| \simeq C_0\tilde{T}\Lambda^{-2}$, the $q^{-1/2}$ factor contributes to the $\tilde{\eta}$-integration, and we get

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,2}\right| \le \sum_{N\in\mathcal{N}_1} Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left[(2^m\sqrt{a})^2 h^{-1}a^2 N^{-1}\Lambda^{-2/3}\right],$$
$$\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left[(2^m\sqrt{a})^2 h^{-1}a^2\Lambda^{-8/3}\right],$$
$$\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}(2^m\sqrt{a})^{2/3}.$$

Recall that we used $N \simeq \tilde{T}$, $|\mathcal{N}_1| \le C_0(1+\tilde{T}\Lambda^{-2})$ and $a \ge \tilde{h}^{2/3}$.

- Lemma 2.1.12: For $N \le \Lambda^{1/3}$, we have

$$\frac{1}{\sqrt{N}}\left|\int e^{i\Lambda\tilde{\psi}_{N,m}}\tilde{\chi}\,d\tilde{s}\,d\tilde{\sigma}\right| \le CN^{-1/4}\Lambda^{-3/4}.$$

Therefore, the estimate in this case is given by

$$\left|\sum_{N\in\mathcal{N}_1} G_{a,m,N,2}\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left[(2^m\sqrt{a})^2 h^{-1}a^2\Lambda^{-1/2}\Lambda^{-3/4}\right],$$
$$\le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}.$$

Hence putting these estimates together, we get

$$\left|\sum_{1\le N\le C_0 a^{-1/2}} G_{a,m,N,2}\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}[h^{1/3}(2^m\sqrt{a})^{2/3} + a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}].$$

We notice that $h^{1/3}(2^m\sqrt{a})^{2/3} \le a^{1/8}h^{1/4}(2^m\sqrt{a})^{3/4}$ when $a \ge \left(\frac{h}{2^m\sqrt{a}}\right)^{2/3}$. The proof of the Proposition 2.2.12 is complete. □

Proof of Theorem 2.2.9. The estimate follows from Propositions 2.2.10, 2.2.11, 2.2.12. □

2.3 Dispersive Estimates for $|\eta| \le \epsilon_0\sqrt{a}$.

In this section, we prove Theorem 1.4.3. We first compute the trajectories of the Hamiltonian flow for the operator P. At this frequency localization there is at most one reflection on the boundary. Moreover, we follow the techniques from section 2.1 and

2.2. It is particularly interesting that at this localization, $\mathcal{G}_{a,\epsilon_0}$ is an oscillatory integral with nondegenerate phase function; this is due to the geometric study of the associated Lagrangian which rules out the swallowtails regime for $|t| \leq 1$ if ϵ_0 is small enough.

2.3.1 Free Space Trajectories.

Recall that the operator P is given by $P(t,x,y,z,\partial_t,\partial_x,\partial_y,\partial_z) = \partial_t^2 - (\partial_x^2 + (1+x)\partial_y^2 + \partial_z^2)$. Now we compute the trajectories in the free space for the associated symbol

$$p = \xi^2 + \zeta^2 + (1+x)\eta^2 - \tau^2.$$

To do so, we start at t_0, x_0, y_0, z_0 with ξ_0 close to 0, $\eta_0 = \theta\zeta_0, |\theta| \leq \epsilon_0\sqrt{a}, \zeta_0 \sim 1$, $\tau_0 = 1, \xi_0^2 + (1+x_0)\eta_0^2 + \zeta_0^2 = 1$. The Hamilton Jacobi equation is

$$\begin{aligned} &\dot{x} = 2\xi; \quad \dot{y} = 2\eta(1+x); \quad \dot{z} = 2\zeta; \quad \dot{t} = -2\tau \\ &\dot{\xi} = -\eta^2; \quad \dot{\eta} = 0; \quad \dot{\zeta} = 0; \quad \dot{\tau} = 0. \end{aligned}$$

This yields

$$\begin{aligned} \tau(s) &= \tau_0; \\ \eta(s) &= \eta_0; \\ \zeta(s) &= \zeta_0; \\ \xi(s) &= \xi_0 - \eta_0^2 s; \\ t(s) &= t_0 - 2\tau_0 s; \\ z(s) &= z_0 + 2\zeta_0 s; \\ x(s) &= x_0 + 2\xi_0 s - \eta_0^2 s^2; \\ y(s) &= y_0 + 2\eta_0\big((1+x_0)s + \xi_0 s^2 - \frac{1}{3}\eta_0^2 s^3\big). \end{aligned}$$

In our case, we start at $t_0 = 0, x_0 = a, y_0 = z_0 = 0$; the system becomes

$$\begin{aligned} &\tau(s) = \tau_0; \quad \eta(s) = \eta_0; \quad \zeta(s) = \zeta_0; \quad \xi(s) = \xi_0 - \eta_0^2 s; \\ &t(s) = -2\tau_0 s;\ z(s) = 2\zeta_0 s;\ x(s) = a + 2\xi_0 s - \eta_0^2 s^2;\ y(s) = 2\eta_0\big((1+a)s + \xi_0 s^2 - \frac{1}{3}\eta_0^2 s^3\big). \end{aligned} \tag{2.3.1}$$

The Lagrangian $\Lambda \subset T^*(\mathbb{R}^4_{t,x,y,z})$: we have $\Lambda \subset \{p = 0\}$ is parametrized by the system (2.3.1) with parameters $(s, \xi_0, \eta_0, \zeta_0)$ together with $(\xi_0^2 + (1+a)\eta_0^2 + \zeta_0^2)^{1/2} = \tau_0$; s is homogeneous of degree -1. Since $t(s) = -2\tau_0 s \Rightarrow s = -\frac{t}{2\tau_0}$, we replace it in the system (2.3.1). Then (2.3.1) becomes an homogeneous system parametrizing the Lagrangian Λ

as follows:

$$
\begin{aligned}
x(t) &= a - \frac{\xi_0}{\tau_0}t - \frac{\eta_0^2}{4\tau_0^2}t^2, \\
y(t) &= \frac{\eta_0}{\tau_0}\Big(- (1+a)t + \frac{\xi_0}{2\tau_0}t^2 + \frac{\eta_0^2}{12\tau_0^2}t^3\Big), \\
z(t) &= -\frac{\zeta_0}{\tau_0}t, \\
\xi(t) &= \xi_0 + \frac{\eta_0^2}{2\tau_0}t, \\
\tau(t) &= \tau_0 = 1.
\end{aligned}
$$

The trajectories hit the boundary when $x(t) = 0$; that is ,

$$\frac{\eta_0^2}{4}t^2 + t\xi_0 - a = 0.$$

This yields the time t_* when $x(t_*) = 0$:

$$t_*\xi_0 = a - \frac{\zeta_0^2\theta^2}{4}t_*^2 \sim a.$$

We want to prove that at this frequency localization, the trajectories hit the boundary only once for a given fixed time $0 < t \leq 1$. To do this, suppose that the trajectory hit the boundary at $(x = 0, y_*, z_*, \xi_*, \eta_*, \zeta_0)$, which is given by the system (2.3.1). More precisely, $\xi_* = -(\xi_0 + \frac{\eta_0^2}{2}t_*)$ and we get

$$
\begin{aligned}
\xi(s) &= \xi_* - \eta_0^2 s, \\
x(s) &= 2\xi_* s - \eta_0^2 s^2, \\
t(s) &= t_* - 2s.
\end{aligned}
$$

Now we assume that the trajectory, issuing from the point $(x = 0, y_*, z_*, \xi_*, \eta_*, \zeta_0)$, hits the boundary; that is, $x(t) = 0$, then $t\eta_0^2 = 2\xi_*$. This yields

$$
\begin{aligned}
t\theta^2\zeta_0^2 &= -2\left(\xi_0 + \frac{\theta^2\zeta_0^2}{2}t_*\right) = -2\left(\xi_0 + \frac{\theta^2\zeta_0^2}{2}\Big(a - \frac{\theta^2\zeta_0^2}{4}t_*^2\Big)/\xi_0\right), \\
|t\theta^2\zeta_0^2| &\geq 4\sqrt{\frac{a\theta^2}{2}} \Rightarrow |t| \geq \frac{4\sqrt{a/2}}{\zeta_0^2|\theta|} \geq \frac{1}{\epsilon_0} \gg 1.
\end{aligned}
$$

Therefore, we can only see at most one reflection on the boundary of the cylinder for $0 < t \leq 1$ at this frequency location.

2.3.2 Dispersive Estimates for $|\eta| \leq \epsilon_0\sqrt{a}$.

In this part, we prove dispersive estimates for $\mathcal{G}_{a,\epsilon_0}$. The main result is Theorem 2.3.1.

Theorem 2.3.1 (Theorem 1.4.3)**.** *There exists C such that for every $h \in]0,1]$, every $t \in [h,1]$, the following holds:*

$$\|\mathcal{G}_{a,\epsilon_0}(t,x,y,z)\|_{L^\infty(x\leq a)} \leq Ch^{-3}(h/t)^{1/2}\min\left\{(h/t)^{1/2}, \sqrt{a}|\log(a)|\right\}. \tag{2.3.2}$$

We start as in section 2.2. Recall that we have

$$\mathcal{G}_{a,\epsilon_0}(t,x,y,z) = \frac{1}{4\pi^2h^2}\sum_{k\geq 1}\int e^{\frac{i}{h}\Phi_k}\sigma_k d\eta d\zeta, \tag{2.3.3}$$

where the phase Φ_k and the function σ_k are defined by

$$\Phi_k = y\eta + z\zeta + t(\eta^2+\zeta^2+\omega_k h^{2/3}\eta^{4/3})^{1/2},$$

$$\sigma_k = \psi_2(\eta/\sqrt{a})e_k(x,\eta/h)e_k(a,\eta/h)\chi_0(\zeta^2+\eta^2)\chi_1(\omega_k h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega_k),$$

with $\psi_2 \in C_0^\infty(]-2\epsilon_0, 2\epsilon_0[)$ equal to 1 on $[-\epsilon_0,\epsilon_0]$. We still use the notation $\mu^2 = \eta^2+\omega_k h^{2/3}\eta^{4/3}$. Let $\chi_4 \in C_0^\infty]-1,1[$ with $\chi_4 = 1$ on $[-1/2,1/2]$. The following lemma (for $|\eta| \leq \epsilon_0\sqrt{a}$) is a refinement of Lemma 2.2.1.

Lemma 2.3.2. *Let*

$$\mathcal{N} = \frac{1}{4\pi^2h^2}\sum_{k\geq 1}\int e^{\frac{i}{h}\Phi_k}\chi_4\left(\frac{t\mu^2}{h}\right)\sigma_k d\eta d\zeta.$$

There exists C such that

$$|\mathcal{N}| \leq Ch^{-3}(h/t)^{1/2}\min\left\{(h/t)^{1/2}, \sqrt{a}\right\}.$$

Proof. As in Lemma 2.2.1, and taking in account the cutoff $\psi_2(\eta/\sqrt{a})$, we get

$$|\mathcal{N}| \leq Ch^{-3}(h/t)\int_{-1}^{1}(1-x^2)^{1/2}\psi_2(x\ \sqrt{h/(ta)})dx$$

and the result follows from

$$(h/t)^{1/2}\int_{-1}^{1}(1-x^2)^{1/2}\psi_2(x\ \sqrt{h/(ta)})dx \leq \min\left\{(h/t)^{1/2}, \sqrt{a}\right\}\ .$$

$\square$

By the proof of Lemma 2.2.3, in the case $\sqrt{a} \leq Mh$, we get the estimate

$$|\mathcal{G}_{a,\epsilon_0}| \leq C_M h^{-3}(h/t)^{1/2}\sqrt{a}|\log(\sqrt{a})|,$$

hence we may assume in what follows that $\tilde{h} = h/\sqrt{a}$ is a small parameter.

Using Lemmas 2.3.2 and 2.2.2, we are now reduced to the study of

$$J_{a,\epsilon_0} = \frac{1}{4\pi^2 h^2}(h/t)^{1/2} \sum_{k\geq 1} \int e^{\frac{i}{h}(y\eta + t\mu(1-\tilde{z}^2)^{1/2})} \tilde{\sigma}(\omega_k)(\eta/h)^{2/3} \frac{2\pi}{L'(\omega_k)} \frac{\psi_2(\eta/\sqrt{a})}{\mu} d\eta, \quad (2.3.4)$$

with $\tilde{\sigma}(\omega)$ defined by

$$\tilde{\sigma} = \sigma_0(z^*, \eta, \mu^2; \lambda)(1-\chi_4(\lambda))Ai((\eta/h)^{2/3}x - \omega)Ai((\eta/h)^{2/3}a - \omega)\chi_1(\omega h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega),$$

where $\lambda = t\mu^2/h$. By Airy Poisson summation formula, we have $J_{a,\epsilon_0} = \sum_{N\in\mathbb{Z}} J_N$ with

$$J_N = \frac{1}{4\pi^2 h^2}(h/t)^{1/2} \int e^{\frac{i}{h}(y\eta + t\mu(1-\tilde{z}^2)^{1/2})} \tilde{\sigma}(\omega)(\eta/h)^{2/3} \frac{\psi_2(\eta/\sqrt{a})}{\mu} e^{-iNL(\omega)} d\omega d\eta. \quad (2.3.5)$$

By the preceding paragraph, we know that it is sufficient to prove an estimate on $J_{-1} + J_0 + J_1$. We will focus on J_1, since J_{-1} is similar and J_0 is simpler since it is the free wave. One has J_1 equal to:

$$J_1 = \frac{(h/t)^{1/2}}{(2\pi)^4 h^2}\, h^{-4/3} \int e^{\frac{i}{h}\phi_1} |\eta|^{2/3} f(\omega, \eta, \mu^2, \lambda, h) \frac{\psi_2(\eta/\sqrt{a})}{\mu} ds d\sigma d\eta d\omega, \quad (2.3.6)$$

with the phase function

$$\phi_1 = y\eta + t\mu(1-\tilde{z}^2)^{1/2} + \frac{s^3}{3} + s(|\eta|^{2/3}x - \omega h^{2/3}) + \frac{\sigma^3}{3} + \sigma(|\eta|^{2/3}a - \omega h^{2/3}) - hL(\omega),$$

and symbol

$$f(\omega, \eta, \mu^2, \lambda, h) = \sigma_0(z^*, \eta, \mu^2; \lambda)(1-\chi_4(\lambda))\chi_1(\omega h^{2/3}\eta^{4/3})(1-\chi_1)(\varepsilon\omega).$$

Recall that

$$L(\omega) = \frac{4}{3}\omega^{3/2} - B(\omega^{3/2}), \text{ for } \omega \geq 1,$$

with

$$B(\omega) \simeq \sum_{j\geq 1} b_j \omega^{-j}, \quad b_j \in \mathbb{R}, \quad b_1 \neq 1.$$

Lemma 2.3.3. *Let L as in section 2.1.2,*

$$L(\omega) = \pi + i \log\left(\frac{A_-(\omega)}{A_+(\omega)}\right).$$

Then for all $\omega \geq 0$, we have

$$L'(\omega) \geq 2\omega^{1/2}.$$

This lemma, whose proof is in the Appendix, is useful in the geometric study of the canonical set and the Lagrangian submanifold associated to the phase function of J_1.

Proof of theorem 2.3.1. To study J_1 in (2.3.6), we restrict the integral to $\eta > 0$ and we first make the change of variables $\omega = h^{-2/3}\eta^{2/3}\omega^*, s = \eta^{1/3}s^*, \sigma = \eta^{1/3}\sigma^*$, and we obtain, since $\mu = \eta(1+\omega^*)^{1/2}$

$$J_1 = \frac{(h/t)^{1/2}}{(2\pi h)^4}\int e^{\frac{i\eta}{h}(y+\tilde{\phi}_1)}\ \eta\ (1+\omega^*)^{-1/2}\ f\ \psi_2(\eta/\sqrt{a})ds^*d\sigma^*d\omega^*d\eta, \tag{2.3.7}$$

with the phase function $\tilde{\phi}_1$ equal to

$$\tilde{\phi}_1 = t(1-\tilde{z}^2)^{1/2}(1+\omega^*)^{1/2} + {s^*}^3/3 + s^*(x-\omega^*) + {\sigma^*}^3/3 + \sigma^*(a-\omega^*) - \frac{h}{\eta}L(\eta^{2/3}h^{-2/3}\omega^*). \tag{2.3.8}$$

We have

$$\begin{aligned}
\partial_{s^*}\tilde{\phi}_1 &= {s^*}^2 + x - \omega^*,\\
\partial_{\sigma^*}\tilde{\phi}_1 &= {\sigma^*}^2 + a - \omega^*,\\
\partial_{\omega^*}\tilde{\phi}_1 &= \frac{t(1-\tilde{z}^2)^{1/2}(1+\omega^*)^{-1/2}}{2} - (s^*+\sigma^*) - \frac{h^{1/3}}{\eta^{1/3}}L'(\eta^{2/3}h^{-2/3}\omega^*).
\end{aligned}$$

Therefore, at a stationary point in s^*, σ^*, ω^* of $\tilde{\phi}_1$, we must have, using Lemma 2.3.3, $|s^*| \le \sqrt{\omega^*}$ and $|\sigma^*| \le \sqrt{(\omega^* - a)}$:

$$t(1-\tilde{z}^2)^{1/2}(1+\omega^*)^{-1/2} \ge 2\big(\sqrt{\omega^*} - \sqrt{(\omega^*-a)}\big).$$

Since $(1-\tilde{z}^2) \simeq \mu = \eta(1+\omega^*)^{1/2}$, $t \le 1$ and $\eta \le \epsilon_0\sqrt{a}$ we get

$$\epsilon_0\sqrt{a} \ge \epsilon_0 t\sqrt{a} \ge 2\big(\sqrt{\omega^*} - \sqrt{(\omega^*-a)}\big),$$

and therefore, we may assume $\omega^* > Ma$ with M large if ϵ_0 is small. This proves that the swallowtail in the first reflection appears after a time $t > 1$. Hence we are reduced to study what happen before the first occurence of a swallowtail. This case corresponds to a regime where there are no swallowtails and no cusps. We are reduce to estimate the oscillatory integral J:

$$J = \frac{(h/t)^{1/2}}{(2\pi h)^4}\int e^{\frac{i\eta}{h}(y+\tilde{\phi}_1)}\ \eta\ (1+\omega^*)^{-1/2}\ f\ \psi_2(\eta/\sqrt{a})\kappa(\omega^*/(Ma))ds^*d\sigma^*d\omega^*d\eta, \tag{2.3.9}$$

where $\kappa \in C^\infty(]1/2,\infty[), 0 \le \kappa \le 1$, and κ equal to 1 on $[1,\infty[$. We then re-perform the $ds^*d\sigma^*$ integration using the definition of the Airy function, and we make the change of variables $\eta = \sqrt{a}\tilde{\eta}$ and $\omega^* = a\tilde{\omega}$. As in Proposition 2.2.11, we get with $\Lambda = a^{3/2}/\tilde{h} = a^2/h$

$$
\begin{aligned}
J &= \frac{a}{(2\pi)^4 h^3}\left(\frac{h}{t}\right)^{1/2}\int e^{i\Lambda Y\tilde{\eta}}\ \tilde{J}\ \psi_2(\tilde{\eta})d\tilde{\eta},\\
\tilde{J} &= \sum_{\pm,\pm}\int e^{i\Lambda\tilde{\eta}\Phi_{\pm,\pm}}\ \Theta_{\pm,\pm}\ \kappa(\tilde{\omega}/M)\ (1+a\tilde{\omega})^{-1/2}d\tilde{\omega},\\
\Phi_{\pm,\pm} &= \tilde{T}\gamma_a(\tilde{\omega}) \pm \frac{2}{3}(\tilde{\omega}-X)^{3/2} \pm \frac{2}{3}(\tilde{\omega}-1)^{3/2} - \frac{4}{3}\tilde{\omega}^{3/2} + \frac{1}{\Lambda\tilde{\eta}}B(\Lambda\tilde{\omega}^{3/2}\tilde{\eta}),\\
\Theta_{\pm,\pm} &= (\tilde{\omega}-1)^{-1/4}(\tilde{\omega}-X)^{-1/4}\Psi_\pm(\Lambda^{2/3}\tilde{\eta}^{2/3}(\tilde{\omega}-1))\Psi_\pm(\Lambda^{2/3}\tilde{\eta}^{2/3}(\tilde{\omega}-X))\ f\ ,
\end{aligned}
$$

where $\Psi_\pm(\tilde{z}) \in C^\infty([0,\infty[)$ are classical symbols of degree 0 in $\tilde{z}\to\infty$. Therefore it remains to prove

$$
\left|\int e^{i\Lambda Y\tilde{\eta}}\ a\tilde{J}\ \psi_2(\tilde{\eta})d\tilde{\eta}\right| \leq C\min\left\{(h/t)^{1/2}, \sqrt{a}|\log(a)|\right\}. \tag{2.3.10}
$$

Since on the support of f one has $\tilde{\omega} \leq \frac{1}{a^2\tilde{\eta}^2}$, we get

$$
|\tilde{J}| \leq C\int_1^{\frac{1}{a^2\tilde{\eta}^2}} \tilde{\omega}^{-1/2}(1+a\tilde{\omega})^{-1/2}d\tilde{\omega} \leq Ca^{-1/2}|\log(a\tilde{\eta}^2)|, \tag{2.3.11}
$$

and this implies

$$
\left|\int e^{i\Lambda Y\tilde{\eta}}\ a\tilde{J}\ \psi_2(\tilde{\eta})d\tilde{\eta}\right| \leq C\sqrt{a}|\log(a)|\ .
$$

Next, we have

$$
\partial_{\hat{\omega}}\tilde{\Phi}_{\pm,\pm} = \frac{\tilde{T}}{2}(1+a\hat{\omega})^{-1/2} \pm (\tilde{\omega}-X)^{1/2} \pm (\tilde{\omega}-1)^{1/2} - 2\tilde{\omega}^{1/2} + O(\tilde{\omega}^{-1/2}),
$$

and $(1+a\tilde{\omega})^{-1/2}\tilde{T} \simeq T\eta \leq \epsilon_0 t \leq 1$. Hence the phases $\Phi_{-,\pm}, \Phi_{+,-}$ have no critical points $\tilde{\omega} \geq M/2$ large, and this implies in particular for their contribution J^* to J the estimate $|J^*| \leq C(\Lambda\tilde{\eta})^{-1/2} = Ch^{1/2}\tilde{\eta}^{-1/2}a^{-1}$, which implies

$$
\left|\int e^{i\Lambda Y\tilde{\eta}}\ a\tilde{J}^*\ \psi_2(\tilde{\eta})d\tilde{\eta}\right| \leq Ch^{1/2}\int \tilde{\eta}^{-1/2}\psi_2(\tilde{\eta})d\tilde{\eta} \leq C(h/t)^{1/2}.
$$

For the contribution to J of the phase $\Phi_{+,+}$, we use the same proof as the proof of Proposition 2.2.11. We thus get the estimate

$$
a|\tilde{J}| \leq C(h/t)^{1/2}.
$$

This concludes the proof of Theorem 1.4.3 $\square$

Chapter 3

Strichartz Estimates For The Model Problem

In this chapter, we present details for the derivation of Strichartz estimates for the solutions u of the wave equation (1.1.1) in cylindrical convex domains (Ω, Δ). The key ingredients to prove Strichartz estimates are dispersive estimates, energy estimates, interpoloation and TT^* arguments. The main result in this section is Theorem 1.3.2.

We first prove the frequency-localized Strichartz estimates [Theorem 3.1.1] from the frequency-localized dispersive estimates. We then deduce the Theorem 1.3.2 by the Littlewood-Paley squarefunction estimates.

Let us recall some notations: For any $I \subset \mathbb{R}, \Omega \subset \mathbb{R}^d$, we define the mixed space-time norms

$$\|u\|_{L^q(I;L^r(\Omega))} := \left(\int_I \|u(t,.)\|^q_{L^r(\Omega)} dt\right)^{1/q}, \quad \text{if } 1 \leq q < \infty,$$
$$\|u\|_{L^\infty(I;L^r(\Omega))} := \operatorname*{ess\,sup}_{t \in I} \|u(t,.)\|_{L^r(\Omega)}.$$

3.1 Frequency-Localized Strichartz Estimates

In this section, we prove Theorem 3.1.2. The classical strategy is as follows: we begin by interpolating between the energy estimates and dispersive estimates. This yields a new estimate, which we further manipulate via a classical L^p inequality to establish (3.1.10). This last step imposes conditions on space-time exponent pair (q, r); these are precisely the wave admissibility criteria. The classical inequalities used are the Young, Hölder, and Hardy-Littlewood-Sobolev inequalities. Let us recall the Littlewood-Paley decomposition and some links with Sobolev spaces. For more details see the book of [[3], chapter 2].

Let $\chi \in C_0^\infty(\mathbb{R}^*)$ and equal to 1 on $[1/2, 2]$ such that

$$\sum_{j\in\mathbb{Z}} \chi(2^{-j}\lambda) = 1, \quad \lambda > 0.$$

We define the associated Littlewood-Paley frequency cutoffs $\chi(2^{-j}\sqrt{-\Delta})$ using spectral theorem for Δ and we have

$$\sum_{j\in\mathbb{Z}} \chi(2^{-j}\sqrt{-\Delta}) = Id : L^2(\Omega) \longrightarrow L^2(\Omega).$$

This decomposition takes a single function and writes it as a superposition of a countably infinite family of functions χ each one having a frequency of magnitude $\sim 2^j$, for $j \geq 1$. We have that a norm of the homogeneous Sobolev of $\dot{H}^\beta$ is defined as follows: for all $\beta \geq 0$,

$$\|u\|_{\dot{H}^\beta} := \left(\sum_{j\in\mathbb{Z}} 2^{2j\beta} \|\chi(2^{-j}D_t)u\|_{L^2}^2 \right)^{1/2}.$$

With this decomposition, the result about the Littlewood-Paley squarefunction estimate [see [4, 5, 30]] reads as follows: for $f \in L^r(\Omega), \forall r \in [2, \infty[$,

$$\|f\|_{L^r(\Omega)} \leq C_r \left\| \left(\sum_{j\in\mathbb{Z}} |\chi(2^{-j}\sqrt{-\Delta})f|^2 \right)^{1/2} \right\|_{L^r(\Omega)}. \tag{3.1.1}$$

The proof follows from the Stein classical argument involving Rademacher functions and an appropriate Mikhlin-Hörmander multiplier theorem.

We define the frequency localization v_j of u by $v_j = \chi(2^{-j}\sqrt{-\Delta})u$. Hence $u = \sum_{j\geq 0} v_j$. Let $h = 2^{-j}$. We deduce from Theorem 1.3.1 the frequency-localized dispersive estimates for the solution $v_j = \chi(hD_t)u$ of the (frequency-localized) wave equation

$$(\partial_t^2 - \Delta)v_j = 0 \text{ in } \Omega, \quad v_{j|t=0} = \chi(hD_t)u_0, \quad \partial_t v_{j|t=0} = \chi(hD_t)u_1, \quad v_{j|\partial\Omega} = 0 \tag{3.1.2}$$

read as follows:

$$\left\| \dot{\mathcal{U}}(t)\chi(hD_t)u_0 \right\|_{L^\infty} \lesssim h^{-3} \min\left\{ 1, \left(\frac{h}{t}\right)^{\frac{3}{4}} \right\} \|\chi(hD_t)u_0\|_{L^1}, \tag{3.1.3}$$

$$\|\mathcal{U}(t)\chi(hD_t)u_1\|_{L^\infty} \lesssim h^{-2} \min\left\{ 1, \left(\frac{h}{t}\right)^{\frac{3}{4}} \right\} \|\chi(hD_t)u_1\|_{L^1}, \tag{3.1.4}$$

where we use the notations

$$\mathcal{U}(t) := \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \qquad \text{and} \qquad \dot{\mathcal{U}}(t) := \cos\left(t\sqrt{-\Delta}\right).$$

These estimates yiled the following Strichartz estimates.

Theorem 3.1.1 (Frequency-localized Strichartz estimates)**.** *Let* (Ω, Δ) *as before. Let* v_j *be a solution of the (frequency-localized) wave equation* (3.1.2)*. Then for all* T *there exists* C_T *such that*

$$h^{\beta}\left\|\dot{\mathcal{U}}(t)\chi(hD_t)u_0\right\|_{L_t^q(L_x^r)} \lesssim \|\chi(hD_t)u_0\|_{L^2}, \tag{3.1.5}$$

$$h^{\beta-1}\left\|\mathcal{U}(t)\chi(hD_t)u_1\right\|_{L_t^q(L_x^r)} \lesssim \|\chi(hD_t)u_1\|_{L^2}, \tag{3.1.6}$$

with $q \in]2,\infty], r \in [2,\infty], \frac{1}{q} \leq \alpha_3\left(\frac{1}{2}-\frac{1}{r}\right), \alpha_3 = \frac{3}{4}$*, and the scaling* $\beta = 3\left(\frac{1}{2}-\frac{1}{r}\right) - \frac{1}{q}$*.*

Remark that if $\frac{1}{q} = \alpha_3\left(\frac{1}{2}-\frac{1}{r}\right)$ then $\beta = (3-\alpha_3)\left(\frac{1}{2}-\frac{1}{r}\right)$. Let us recall also the following facts:

- The Riesz-Thorin Interpolation Theorem [see [22]. Thm.7.1.12] states that if T is a linear map from $L^{p_1} \cap L^{p_2}$ to $L^{q_1} \cap L^{q_2}$ such that

$$\|Tf\|_{L^{q_j}} \leq M_j \|f\|_{L^{p_j}}, \ j = 1, 2$$

and if $\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, $\frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$, for some $\theta \in (0,1)$, then

$$\|Tf\|_{L^{q_\theta}} \leq M_1^{\theta} M_2^{1-\theta} \|f\|_{L^{p_\theta}}, \ f \in L^{p_1} \cap L^{p_2}. \tag{3.1.7}$$

- TT^* argument: let $\mathcal{H}$ be Hilbert space, $\mathcal{B}$ and its dual $\mathcal{B}^*$ be Banach spaces. Let $T : \mathcal{H} \to \mathcal{B}$ be linear operator and $T^* : \mathcal{B}^* \to \mathcal{H}$ its adjoint. The followings are equivalent:

  1. The operator T is continuous from $\mathcal{H}$ to $\mathcal{B}$, $\quad \|Tf\|_{\mathcal{B}} \leq C\|f\|_{\mathcal{H}}$,
  2. The operator T^* is continuous from $\mathcal{B}^*$ to $\mathcal{H}$, $\quad \|T^*g\|_{\mathcal{H}} \leq C\|g\|_{\mathcal{B}^*}$,
  3. The operator T^*T is continuous from $\mathcal{B}^*$ to $\mathcal{B}$, $\quad \|T^*Tg\|_{\mathcal{B}} \leq C^2\|g\|_{\mathcal{B}^*}$.

  In particular, let $(U(t))_{t\in\mathbb{R}}$ be a bounded family of continuous operators on $L^2(\mathbb{R}^d)$. Let T be the solution operator

$$T : u_0 \longmapsto [t \mapsto U(t)u_0],$$

then

$$T^* : \phi \longmapsto \int U^*(t)\phi(t)dt.$$

Moreover, TT^* coincides with the operator

$$\phi \longmapsto \left[t \mapsto \int_{\mathbb{R}} U(t)U^*(t')\phi(t')dt'\right].$$

We have

$$\begin{aligned}\|U(t)u_0\|_{L_t^q(L_x^r)} &= \sup_{\phi\in\mathcal{B}_{q,r}} \left|\int U(t)u_0(x)\phi(t,x)dtdx\right| \\ &= \sup_{\phi\in\mathcal{B}_{q,r}} \left|\int (U(t)u_0(x)|\phi(t))_{L^2}dt\right|\end{aligned}$$

where $\mathcal{B}_{q,r} := \{\phi \in \mathcal{D}(\mathbb{R}^{1+d};\mathbb{C})/\|\phi\|_{L_t^{q'}(L_x^{r'})} \leq 1\}$.
By the definition of the adjoint operator, we have

$$\|U(t)u_0\|_{L_t^q(L_x^r)} = \sup_{\phi\in\mathcal{B}_{q,r}} \left|\Big(u_0\Big|\int U^*(t)\phi(t)dt\Big)_{L^2}\right|.$$

Using the Cauchy-Schwarz inequality, we deduce that

$$\|U(t)u_0\|_{L_t^q(L_x^r)} \leq \|u_0\|_{L^2} \sup_{\phi\in\mathcal{B}_{q,r}} \left\|\int U^*(t)\phi(t)dt\right\|_{L^2}.$$

Moreover, we have

$$\begin{aligned}\left\|\int U^*(t)\phi(t)dt\right\|_{L^2}^2 &= \int (U^*(t')\phi(t')|U^*(t)\phi(t))_{L^2}dt'dt \\ &= \int (U(t)U^*(t')\phi(t')|\phi(t))_{L^2}dt'dt \\ &= \int \langle U(t)U^*(t')\phi(t'),\bar{\phi}(t)\rangle dt'dt \\ &\leq \left\|\int U(t)U^*(t')\phi(t')dt'\right\|_{L_t^qL_x^r} \|\bar{\phi}\|_{L_t^{q'}L_x^{r'}},\end{aligned}$$

by the Hölder inequality in t and x.

Proof of Theorem 3.1.1. We prove only (3.1.5) since (3.1.6) follows analogously. We have the frequency-localized dispersive estimates in Ω in (3.1.3) for $|t| \geq h$ as follows:

$$\|\dot{\mathcal{U}}(t)\chi(hD_t)u_0\|_{L^\infty} \lesssim h^{-3}\Big(\frac{h}{t}\Big)^{\alpha_3}\|\chi(hD_t)u_0\|_{L^1}, \tag{3.1.8}$$

and the energy estimates

$$\|\dot{\mathcal{U}}(t)\chi(hD_t)u_0\|_{L^2} \lesssim \|\chi(hD_t)u_0\|_{L^2} \tag{3.1.9}$$

We apply the Riesz-Thorin Interpolation Theorem (3.1.7) to the operator $\dot{\mathcal{U}}(t)\chi(hD_t)$ for fixed time $t \in \mathbb{R}$. Interpolating between (3.1.8) and (3.1.9) with $\theta = 1 - \frac{2}{r}$ yields

$$\|\dot{\mathcal{U}}(t)\chi(hD_t)u_0\|_{L^r} \leq Ch^{(-3+\alpha_3)(1-\frac{2}{r})}t^{-\alpha_3(1-\frac{2}{r})}\|\chi(hD_t)u_0\|_{L^{r'}}, \tag{3.1.10}$$

for all $2 \leq r \leq \infty$, where r' denote the exponent conjugate to r; that is, $\frac{1}{r} + \frac{1}{r'} = 1$.

Let T be the operator solution defined by

$$T : \phi_0 \in L^2 \longmapsto T\phi_0 = \dot{\mathcal{U}}(t)\chi(hD_t)\phi_0 \in L_t^q L_x^r,$$

the its adjoint is given by

$$T^* : \psi \in L_t^{q'} L_x^{r'} \longmapsto T^*\psi = \int \dot{\mathcal{U}}(t)\chi^*(hD_t)\psi(t)dt \in L^2.$$

Moreover, we have

$$T^*T : \psi \in L_t^{q'} L_x^{r'} \longmapsto T^*T\psi = \int \dot{\mathcal{U}}(t-s)\chi^*(hD_t)\chi(hD_t)\psi(s)ds \in L_t^q L_x^r.$$

By TT^* argument, it is sufficient to prove

$$\|T^*T\psi\|_{L_t^q L_x^r} \lesssim h^{-2\beta}\|\psi\|_{L_t^{q'} L_x^{r'}}.$$

We have

$$\begin{aligned}
\|T^*T\psi\|_{L_t^q L_x^r} &= \left\| \int \dot{\mathcal{U}}(t-s)\chi^*(hD_t)\chi(hD_t)\psi(s)ds \right\|_{L_t^q L_x^r}, \\
&\lesssim h^{-2(3-\alpha_3)(\frac{1}{2}-\frac{1}{r})} \left\| \int |t-s|^{-2\alpha_3(\frac{1}{2}-\frac{1}{r})}\|\psi\|_{L_x^{r'}} ds \right\|_{L_t^q}.
\end{aligned} \tag{3.1.11}$$

When $\frac{1}{q} < \alpha_3\left(\frac{1}{2} - \frac{1}{r}\right)$, we use Young's inequality, which states that for $1 \leq p, q \leq \infty$,

$$\|K * u\|_{L^q} \leq \|K\|_{L^{\tilde{r}}}\|u\|_{L^p} \tag{3.1.12}$$

for $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$. We apply (3.1.12) with $\tilde{r} = q/2, p = q'$ and $\frac{1}{q} + \frac{1}{q'} = 1$ to get an estimate

$$\left\| \int_h^\infty |t-s|^{-2\alpha_3(\frac{1}{2}-\frac{1}{r})} \|\psi\|_{L_x^{r'}} ds \right\|_{L_t^q} \leq \|\psi\|_{L_t^{q'}L_x^{r'}} \left\| t^{-2\alpha_3(\frac{1}{2}-\frac{1}{r})} \right\|_{L^{q/2}_{|t|\geq h}},$$
$$\leq h^{-2\alpha_3(\frac{1}{2}-\frac{1}{r})+\frac{2}{q}} \|\psi\|_{L_t^{q'}L_x^{r'}},$$

since $\frac{1}{q} < \alpha_3\left(\frac{1}{2}-\frac{1}{r}\right)$, we get that

$$\|t^{-2\alpha_3(\frac{1}{2}-\frac{2}{r})}\|_{L^{q/2}_{|t|\geq h}} = \left(\int_h^\infty t^{-2\alpha_3(\frac{1}{2}-\frac{2}{r})q/2} dt\right)^{2/q} \simeq h^{-2\alpha_3(\frac{1}{2}-\frac{1}{r})+\frac{2}{q}}.$$

Then (3.1.11) becomes

$$\|T^*T\psi\|_{L_t^qL_x^r} \lesssim h^{-2(3-\alpha_3)(\frac{1}{2}-\frac{1}{r})} \left\| \int |t-s|^{-2\alpha_3(\frac{1}{2}-\frac{1}{r})} \|\psi\|_{L_x^{r'}} ds \right\|_{L_t^q},$$
$$\lesssim h^{-2[3(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}]} \|\psi\|_{L_t^{q'}L_x^{r'}},$$
$$\lesssim h^{-2\beta} \|\psi\|_{L_t^{q'}L_x^{r'}}.$$

When $\frac{1}{q} = \alpha_3\left(\frac{1}{2}-\frac{1}{r}\right)$, we instead use Hardy-Littlewood-Sobolev inequality [see[22]. Thm. 4.5.3] which says that for $K(t) = |t|^{-1/\gamma}$ and $1 < \gamma < \infty$ that

$$\|K * u\|_{L^{\tilde{r}}(\mathbb{R})} \lesssim \|u\|_{L^{p'}(\mathbb{R})}, \tag{3.1.13}$$

for $\frac{1}{\gamma} = \frac{1}{p} + \frac{1}{\tilde{r}}$. Here, we apply (3.1.13) with $\tilde{r} = q, p = q$ and $\frac{1}{\gamma} = \frac{2}{q} = 2\alpha_3\left(\frac{1}{2}-\frac{1}{r}\right)$ to get that for $q > 2, t^{-2/q} * : L^{q'} \to L^q$ is bounded. Hence, we get from (3.1.11) that

$$\|T^*T\psi\|_{L_t^qL_x^r} \lesssim h^{-2(3-\alpha_3)(\frac{1}{2}-\frac{1}{r})} \|\psi\|_{L_t^{q'}L_x^{r'}},$$
$$\lesssim h^{-2\beta} \|\psi\|_{L_t^{q'}L_x^{r'}}.$$

□

3.2 Homogeneous Strichartz Estimates

Let us restate Theorem 1.3.2 as:

Theorem 3.2.1 (Theorem 1.3.2). *Let (Ω, Δ) as before. Let u be a solution of the following wave equation on Ω:*

$$(\partial_t^2 - \Delta)u = 0 \quad \text{in } \Omega, \quad u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \quad u_{|x=0} = 0. \tag{3.2.1}$$

Then for all T there exists C_T such that

$$\|u\|_{L^q((0,T);L^r(\Omega))} \leq C_T \left(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)} \right),$$

with

$$\frac{1}{q} \leq \frac{3}{4}\left(\frac{1}{2} - \frac{1}{r}\right), \ \textit{and } \beta = 3\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q}.$$

Proof. Using the square function estimates (3.1.1), we get that

$$\|u\|_{L^q_t L^r_x} \lesssim \left(\sum_j \|v_j\|^2_{L^q_t L^r_x} \right)^{1/2}. \tag{3.2.2}$$

Indeed, we have

$$\begin{aligned} \|u\|_{L^r(\Omega)} &\lesssim \left\| \left(\sum_{j\geq 0} |v_j|^2 \right)^{1/2} \right\|_{L^r(\Omega)} = \left\| \sum_{j\geq 0} |v_j|^2 \right\|^{1/2}_{L^{r/2}(\Omega)} \\ &\lesssim \left\{ \sum_{j\geq 0} \|v_j^2\|_{L^{r/2}(\Omega)} \right\}^{1/2} = \left\{ \sum_{j\geq 0} \|v_j\|^2_{L^r(\Omega)} \right\}^{1/2}, \end{aligned}$$

Hence, we get

$$\begin{aligned} \|u\|_{L^q_t L^r_x} &\lesssim \left\| \left\{ \sum_{j\geq 0} \|v_j\|^2_{L^r_x} \right\}^{1/2} \right\|_{L^q_t} = \left\{ \left\| \sum_{j\geq 0} \|v_j\|^2_{L^r_x} \right\|_{L^{q/2}_t} \right\}^{1/2}, \\ &\lesssim \left\{ \sum_{j\geq 0} \left\| \|v_j\|^2_{L^r_x} \right\|_{L^{q/2}_t} \right\}^{1/2} = \left\{ \sum_{j\geq 0} \|v_j\|^2_{L^q_t L^r_x} \right\}^{1/2}. \end{aligned}$$

The solution u to the wave equation (3.2.1) with localized initial in frequency $1/h = 2^j$ is given by

$$u = \sum_j v_j, \quad \text{where} \quad v_j = \dot{\mathcal{U}}(t)\chi(2^{-j}D_t)u_0 + \mathcal{U}(t)\chi(2^{-j}D_t)u_1.$$

Therefore, we get that

$$\begin{aligned}
\|u\|_{L^q_t L^r_x} &\lesssim \left(\sum_j \|\dot{\mathcal{U}}(t)\chi(2^{-j}D_t)u_0\|^2_{L^q_t L^r_x} + \|\mathcal{U}(t)\chi(2^{-j}D_t)u_1\|^2_{L^q_t L^r_x} \right)^{1/2}, \\
&\lesssim \left(\sum_j 2^{2j\beta}\|\chi(2^{-j}D_t)u_0\|^2_{L^2} + 2^{2j(\beta-1)}\|\chi(2^{-j}D_t)u_1\|^2_{L^2} \right)^{1/2}, \\
&\lesssim \left(\sum_j 2^{2j\beta}\|\chi(2^{-j}D_t)u_0\|^2_{L^2} \right)^{1/2} + \left(\sum_j 2^{2j(\beta-1)}\|\chi(2^{-j}D_t)u_1\|^2_{L^2} \right)^{1/2}, \\
&\lesssim \|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)}.
\end{aligned}$$

where we used Minkowski inequality in the third line. □

Appendix

A. Airy function

Let $z > 0$. The Airy function Ai is defined as follows:

$$Ai(-z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(s^3/3 - sz)} ds.$$

It satisfies the Airy equation

$$Ai''(z) - zAi(z) = 0, \text{denoted by (A)}$$

Let $\omega = e^{2i\pi/3}$. Obviously, $z \mapsto Ai(\omega z)$ is a solution to (A). Any two of these three solutions $Ai(z), Ai(\omega z), Ai(\omega^2 z)$ yield a basis of solutions to (A) and the linear relation between them is $\sum_{j\in\{0,1,2\}} \omega^j Ai(\omega^j z) = 0$. Then $Ai(z) = -\omega Ai(\omega z) - \bar{\omega} Ai(\bar{\omega} z)$, which we rewrite as follows:

$$Ai(-z) = e^{-i\pi/3} Ai(e^{-i\pi/3} z) + e^{i\pi/3} Ai(e^{i\pi/3} z) = A_+(z) + A_-(z),$$

where we set $A_\pm(z) = e^{\mp i\pi/3} Ai(e^{\mp i\pi/3} z)$. Notice that $A_-(z) = \overline{A_+(\bar{z})}$. We also have the following asymptotic expansions

$$A_-(z) = \frac{1}{2\sqrt{\pi} z^{1/4}} e^{i\pi/4} e^{-\frac{2}{3} i z^{3/2}} \exp \Upsilon(z^{3/2}) = \frac{1}{z^{1/4}} e^{i\pi/4} e^{-\frac{2}{3} i z^{3/2}} \Psi_-(z),$$

with $\exp \Upsilon(z^{3/2}) \sim (1 + \sum_{l\geq 1} c_l z^{-3l/2}) \sim 2\sqrt{\pi}\Psi_-(z)$ as $z \to +\infty$ and the corresponding expansion for A_+, where we define $\Psi_+(z) = \bar{\Psi}_-(\bar{z})$. Moreover, we have

$$\frac{A_-(z)}{A_+(z)} = ie^{-\frac{4}{3} i z^{3/2}} e^{iB(z^{3/2})}, \quad \text{with } iB = \Upsilon - \bar{\Upsilon}.$$

Notice that for $z \in \mathbb{R}_+, B(z) \in \mathbb{R}$ and $B(z) \sim \sum_{j\geq 1} b_j z^{-j}$ for $z \to +\infty$ and $b_1 \neq 0$.

B. Phase Integrals

Lemma (Lemma 2.20[29]). *Let $K \subset \mathbb{R}$, and let $a(\xi, \lambda)$ be a classical symbol of degree* 0 *in $\lambda \geq 1$ with $a(\xi, \lambda) = 0$ for $\xi \notin K$. Let $k \geq 2, c_0 > 0$ and $\Phi(\xi)$ a phase function such that*

$$\sum_{2 \leq j \leq k} |\Phi^{(j)}(\xi)| \geq c_0, \quad \xi \in K.$$

Then there exists C such that

$$\left| \int e^{i\lambda\Phi(\xi)} a(\xi, \lambda) d\xi \right| \leq C\lambda^{-1/k}, \quad \forall \lambda \geq 1.$$

Moreover, the constant C depends only on c_0 and on an upper bound of a finite number of derivatives of order ≥ 2 of Φ, a in a neighbourhood of K.

Let $H(\xi)$ be a smooth function defined in a neighbourhood of $(0, 0)$ in $\mathbb{R}^2$ such that $H(0) = 0$ and $\nabla H(0) = 0$. We assume that the Hessian H'' satisfies $\operatorname{rank}(H''(0)) = 1$ and $\nabla \det(H'')(0) \neq 0$. Then the equation $\det(H'')(\xi) = 0$ defines a smooth curve $\mathcal{C}$ near 0 in $\mathbb{R}^2$ with $0 \in \mathcal{C}$. Let $s \to \xi(s)$ be a smooth parametrization of $\mathcal{C}$, with $\xi(0) = 0$, and define the curve $X(s)$ in $\mathbb{R}^2$ by

$$X(s) = H'(\xi(s)).$$

Lemma (Lemma 2.21[29]). *Let $K = \{\xi \in \mathbb{R}^2, |\xi| \leq r\}$, and let $a(\xi, \lambda)$ be a classical symbol of degree* 0 *in $\lambda \geq 1$ with $a(\xi, \lambda) = 0$ for $\xi \notin K$. For $x \in \mathbb{R}^2$ close to* 0 *set*

$$I(x, \lambda) = \int e^{i\lambda(x.\xi - H(\xi))} a(\xi, \lambda) d\xi.$$

Then for $r > 0$ small enough, the followings hold true:

(a) If $X'(0) \neq 0$, there exists C such that for all x close to 0,

$$|I(x, \lambda)| \leq C\lambda^{-5/6}.$$

(b) If $X'(0) = 0$ and $X''(0) \neq 0$, there exists C such that for all x close to 0,

$$|I(x, \lambda)| \leq C\lambda^{-3/4}.$$

Moreover, if a is elliptic at $\xi = 0$, there exists C' such that

$$|I(0, \lambda)| \geq C'\lambda^{-3/4}.$$

Lemma. *Let $0 < \gamma < 1$ and let*

$$J(\lambda) = \int_0^\infty e^{i\lambda\xi^\gamma} f(\xi) d\xi,$$

where $f \in C^\infty(\mathbb{R})$, *supp*$f \subset [1,\infty[$, $|\xi^j \partial_\xi^j f(\xi)| \leq \xi^m$. *Then one has*

$$J(\lambda) \in O(\lambda^{-\infty}).$$

Proof. The result follows the integration by parts. We have

$$J(\lambda) = \int_0^\infty e^{i\lambda\xi^\gamma} L^n(f) d\xi,$$

with

$$L(f) = \frac{i}{\gamma\lambda}\frac{d}{dx}\left[\xi^{1-\gamma} f(\xi)\right],$$

which is a symbol of degree $m-\gamma$. Then we get

$$|J(\lambda)| \leq \frac{1}{\lambda^n}\left|\int_0^\infty e^{i\lambda\xi^\gamma} g_n(\xi) d\xi\right| \leq \frac{1}{\lambda^n}\int_0^\infty (1+|\xi|)^{m-n\gamma} d\xi,$$

with g_n is a symbol of degree $m-n\gamma$. Choose n large enough, then the result follows. □

C. Proof of Airy-Poisson Formula

Let us recall the Lemma 2.1.4:

Lemma. *The following equality holds true in* $\mathcal{D}'(\mathbb{R}_\omega)$,

$$\sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)} \delta_{\omega=\omega_k}.$$

Proof. Let $\phi \in C_0^\infty(]0,\infty[) \subset C_0^\infty(\mathbb{R})$ and $z = L(\omega) \Leftrightarrow \phi(z) = \omega$. The Poisson summation formula read as follows:

$$2\pi \sum_{k\in\mathbb{Z}} \phi(2\pi k) = \sum_{N\in\mathbb{Z}} \int e^{-iNz}\phi(z) dz.$$

$$2\pi \sum_{k\in\mathbb{Z}} \phi(2\pi k) = \sum_{N\in\mathbb{Z}} \int e^{-iNL(\omega)}\phi[L(\omega)]L'(\omega) d\omega.$$

Let $F(\omega) = Ai(-\omega)$. With $A_+(\omega) = \rho(\omega)e^{i\theta(\omega)}$, we get $F(\omega) = 2\rho(\omega)\cos(\theta(\omega))$. Therefore, the equation $F(\omega) = 0$ is equivalent to $\theta(\omega) = \pi/2 + l\pi, l \in \mathbb{Z}$, which is equivalent to $L(\omega) = 2\pi(1+l)$. Since L is a diffeomorphism from $\mathbb{R}$ onto $]0,\infty[$, one has for all integer $k \geq 1$, $Ai(-\omega_k) = 0$ iff $L(\omega_k) = 2\pi k$. Let $H(\omega) = \phi[L(\omega)]L'(\omega) \in C_0^\infty(\mathbb{R})$. Then we get

$$2\pi \sum_{k\in\mathbb{Z}} \frac{H(\omega_k)}{L'(\omega_k)} = \sum_{N\in\mathbb{Z}} \int e^{-iNL(\omega)} H(\omega) d\omega.$$

This is equivalent to

$$\sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)}\delta_{\omega=\omega_k} \text{ in } \mathcal{D}'(\mathbb{R}_\omega).$$

Finally, using the Airy equation $F''(y) + yF(y) = 0$ and integration by part, we get

$$\int_\infty^\omega F^2(y)dy = \omega F^2(\omega) - \int_{-\infty}^\omega 2yFF'\ dy = \omega F^2(\omega) + \int_{-\infty}^\omega 2F''F'\ dy = \omega F^2(\omega) + F'^2(\omega).$$

Since $F'(\omega_k) = 2\rho'(\omega_k)\cos(\theta(\omega_k)) + 2\rho(\omega_k)\theta'(\omega_k)\sin(\theta(\omega_k))$, we get

$$\int_0^\infty Ai^2(x-\omega_k)\,dx = F'^2(\omega_k) = 4\rho^2(\omega_k)\theta'^2(\omega_k) = \rho^2(\omega_k)L'^2(\omega_k) = c_0 L'(\omega_k)\ ,$$

From $2\pi Ai(0) = 3^{-1/6}\Gamma(1/3), 2\pi Ai'(0) = -3^{1/6}\Gamma(2/3)$ and the Euler reflection formula for the Γ function, $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$, we get $2\pi c_0 = 1$, thus

$$\int_0^\infty Ai^2(x-\omega_k)\,dx = \frac{L'(\omega_k)}{2\pi}.$$

□

D. Proof of Lemma 2.3.3

Proof. To prove the inequality in Lemma 2.3.3, let us recall that

$$\begin{aligned}
\delta_{x=a} &= \sum_{k\geq 1} e_k(x,\eta)e_k(a,\eta) \\
&= \sum_{k\geq 1} \frac{1}{L'(\omega_k)}|\eta|^{2/3}Ai(|\eta|^{2/3}x-\omega_k)Ai(|\eta|^{2/3}a-\omega_k), \\
&= \int \sum_{k\geq 1} \frac{\delta_{\omega=\omega_k}}{L'(\omega_k)}|\eta|^{2/3}Ai(|\eta|^{2/3}x-\omega)Ai(|\eta|^{2/3}a-\omega)d\omega, \\
&= \frac{1}{4\pi^2}\int \sum_{N\in\mathbb{Z}} e^{-iNL(\omega)}|\eta|^{2/3}e^{i\left(\frac{s^3}{3}+s(|\eta|^{2/3}x-\omega)+\frac{\sigma^3}{3}+\sigma(|\eta|^{2/3}a-\omega)\right)}d\omega.
\end{aligned}$$

Now we rewrite

$$\delta_{x=a} = \sum_{N\in\mathbb{Z}} T_N,$$

with

$$T_N = \frac{1}{4\pi^2}\int |\eta|^{2/3}e^{i\left(\frac{s^3}{3}+s(|\eta|^{2/3}x-\omega)+\frac{\sigma^3}{3}+\sigma(|\eta|^{2/3}a-\omega)-NL(\omega)\right)}d\omega.$$

Since we also have $\delta_{x=a} = T_0$, then the wave front set satisfies

$$\mathrm{WF}\big(\sum_{N\neq 0} T_N\big) \subset \{x \leq 0\}.$$

Moreover, $\mathrm{WF}(T_N)$ are pairwise disjoint, and we have

$$\forall N, \mathrm{WF}(T_N) \subset \{x \leq 0\}.$$

The Lagrangian submanifold associated to T_N is defined by a system of equations:

$$\begin{aligned}
&s^2 + |\eta|^{2/3}x - \omega = 0,\\
&\sigma^2 + |\eta|^{2/3}a - \omega = 0,\\
&s + \sigma = -NL'(\omega)
\end{aligned}$$

We have

$$\begin{aligned}
|\eta|^{2/3}x = \omega - s^2 &= \sigma^2 + |\eta|^{2/3}a - \left(\sigma + NL'(\sigma^2 + |\eta|^{2/3}a)\right)^2,\\
&= |\eta|^{2/3}a - 2\sigma NL'(|\eta|^{2/3}a + \sigma^2) - N^2[L'(|\eta|^{2/3}a + \sigma^2)]^2.
\end{aligned}$$

Since $x \leq 0$ on WT, then we get $\forall a > 0, \forall \sigma \in \mathbb{R}, \forall N \neq 0$,

$$\begin{aligned}
|\eta|^{2/3}a - 2\sigma NL'(|\eta|^{2/3}a + \sigma^2) - N^2[L'(|\eta|^{2/3}a + \sigma^2)]^2 \leq 0,\\
|\eta|^{2/3}a \leq \inf_{\sigma\in\mathbb{R}} \left\{ |N|L'(|\eta|^{2/3}a + \sigma^2)\left(|N|L'(|\eta|^{2/3}a + \sigma^2) + 2\sigma\frac{N}{|N|}\right)\right\}.
\end{aligned}$$

Let $\tilde{a} = |\eta|^{2/3}a$. It is equivalent to have $\forall \tilde{a} \geq 0, \forall r \geq 0, \forall N \geq 1$,

$$\tilde{a} \leq \inf_{r\geq 0} \left\{NL'(\tilde{a} + r^2)\left(NL'(\tilde{a} + r^2) - 2r\right)\right\}.$$

This reduces to $\forall \tilde{a} \geq 0, \forall r \geq 0$, since $N \geq 1$

$$\tilde{a} \leq \inf_{r\geq 0} \left\{L'(\tilde{a} + r^2)\left(L'(\tilde{a} + r^2) - 2r\right)\right\}.$$

For $\tilde{a} = 0$, since L is strictly increasing, it yields

$$L'(r^2) \geq 2r, \quad \forall r \geq 0.$$

Now let $\hat{a} = \tilde{a} + r^2$, then we have $\forall 0 \leq \tilde{a} \leq \hat{a}$,

$$\tilde{a} + 2L'(\hat{a})\sqrt{\hat{a} - \tilde{a}} \leq L'(\hat{a})^2.$$

Since $\sup_{0\leq\tilde{a}\leq\hat{a}} \tilde{a} + 2L'(\hat{a})\sqrt{\hat{a} - \tilde{a}} = 2L'(\hat{a})\sqrt{\hat{a}}$; we get the desired inequality.
The inequality is strict for large values of ω; that is, we have $L'(\omega) > 2\omega^{1/2}$. From the

asymptotic expansion of the Airy functions, we have

$$A_-(\omega) = \frac{1}{2\sqrt{\pi}\omega^{1/4}} e^{i\pi/4} e^{-\frac{2}{3}i\omega^{3/2}}(1 + \frac{\mathbf{b}}{\omega^{3/2}} + \cdots); \quad A_+(\omega) = \overline{A_-(\omega)}.$$

Then we get

$$\begin{aligned} L(\omega) &= \frac{\pi}{2} + \frac{4}{3}\omega^{3/2} - 2\frac{Im(\mathbf{b})}{\omega^{3/2}} + \cdots, \\ L'(\omega) &= 2\omega^{1/2} + 3Im(\mathbf{b})\omega^{-5/2} + \cdots \end{aligned}$$

To prove that $L'(\omega) > 2\omega^{1/2}$ for ω large, it is sufficient to prove that $Im(\mathbf{b}) > 0$ as follows:

$$\begin{aligned} A_-(\omega) = e^{i\pi/3} Ai(e^{i\pi/3}\omega) &= \frac{e^{i\pi/3}}{2\pi} \int_{\mathbb{R}} e^{i(\frac{s^3}{3} + s e^{i\pi/3}\omega)} ds \\ &= \frac{e^{i\pi/3}}{2\pi} \int_{\mathbb{R}} \omega^{1/2} e^{i\omega^{3/2}(\frac{X^3}{3} + X e^{i\pi/3})} dX, \quad (s = \omega^{1/2} W) \end{aligned}$$

Let $W = ie^{i\pi/6} + t$, then $\frac{W^3}{3} + We^{i\pi/3} = \frac{t^3}{3} + ie^{i\pi/6}t^2 - \frac{2}{3}$ and

$$\begin{aligned} A_-(\omega) &= \frac{e^{i\pi/3}}{2\pi}\omega^{1/2} e^{-\frac{2}{3}\omega^{3/2}} \int_{\mathbb{R}} e^{i\omega^{3/2}(\frac{t^3}{3} + ie^{i\pi/6}t^2)} dt, \\ &= \frac{e^{i\pi/4}}{2\pi}\omega^{1/2} e^{-\frac{2}{3}\omega^{3/2}} \int_{\mathbb{R}} e^{-\omega^{3/2}(t'^2 - ie^{-i\pi/4}\frac{t'^3}{3})} dt', \quad (t = e^{-i\pi/12}t'), \\ &= \frac{e^{i\pi/4}}{2\sqrt{\pi}\omega^{1/4}} e^{-\frac{2}{3}\omega^{3/2}} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(\tilde{t}^2 - ie^{-i\pi/4}\frac{\tilde{t}^3}{3\omega^{3/4}})} d\tilde{t}, \quad (t' = \omega^{-3/4}\tilde{t}), \\ &= \frac{e^{i\pi/4}}{2\sqrt{\pi}\omega^{1/4}} e^{-\frac{2}{3}\omega^{3/2}} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\tilde{t}^2}(1 + ie^{-i\pi/4}\frac{\tilde{t}^3}{3\omega^{3/4}} + \frac{i}{18}\frac{\tilde{t}^6}{\omega^{3/2}} + \cdots), \\ &= \frac{e^{i\pi/4}}{2\sqrt{\pi}\omega^{1/4}} e^{-\frac{2}{3}\omega^{3/2}} (1 + \frac{i}{18\omega^{3/2}} \int \tilde{t}^6 e^{-\tilde{t}^2} d\tilde{t} + \cdots). \end{aligned}$$

Hence, we get that

$$Im(\mathbf{b}) = \frac{1}{18}\int \tilde{t}^6 e^{-\tilde{t}^2} d\tilde{t} > 0.$$

□

E. Geometric Estimates

Let $f(a, a\tilde{\sigma}^2)$ be various analytic functions defined for a and $a\tilde{\sigma}^2$ small, with $f(a, b) \in \mathbb{R}$ for $(a, b) \in \mathbb{R}^2$. Recall that the projection of $\mathbf{\Lambda}_{a,N,h}$ onto $\mathbb{R}^3$ is given by

$$
\begin{aligned}
&X = 1 + \tilde{\sigma}^2 - \tilde{s}^2, && (3.2.3)\\
&\tilde{T} = 2(1 + a + a\tilde{\sigma}^2)^{1/2}\left(\tilde{s} + \tilde{\sigma} + 2N(1+\tilde{\sigma}^2)^{1/2}\left(1 - \frac{3}{4}B'\Big((1+\tilde{\sigma}^2)^{3/2}\lambda\Big)\right)\right),\\
&Y = H_1(a,\tilde{\sigma})(\tilde{s}+\tilde{\sigma}) + \frac{2}{3}(\tilde{s}^3 + \tilde{\sigma}^3) + \frac{4}{3}NH_2(a,\tilde{\sigma})\left(1 - \frac{3}{4}B'\Big((1+\tilde{\sigma}^2)^{3/2}\lambda\Big)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
H_1(a,\tilde{\sigma}) &= -(1+\tilde{\sigma}^2)\frac{(1+a+a\tilde{\sigma}^2)^{1/2}}{1+(1+a+a\tilde{\sigma}^2)^{1/2}},\\
H_2(a,\tilde{\sigma}) &= (1+\tilde{\sigma}^2)^{3/2}\frac{-3-4a-4a\tilde{\sigma}^2}{2+a+a\tilde{\sigma}^2+3(1+a+a\tilde{\sigma}^2)^{1/2}}.
\end{aligned}
$$

We can rewrite H_1, H_2 as follows:

$$
\begin{aligned}
H_1 &= f_0(a, a\tilde{\sigma}^2) + \tilde{\sigma}^2 f_1(a, a\tilde{\sigma}^2), f_0(0,0) = f_1(0,0) = -1/2,\\
(1+\tilde{\sigma}^2)^{-3/2}H_2 &= f_2(a, a\tilde{\sigma}^2), f_2(0,0) = -3/5.
\end{aligned}
$$

As in [29], we rewrite the system in the following form

$$
\begin{aligned}
&X = 1 + \tilde{\sigma}^2 - \tilde{s}^2, && (3.2.4)\\
&Y = H_1(a,\tilde{\sigma})(\tilde{s}+\tilde{\sigma}) + \frac{2}{3}(\tilde{s}^3+\tilde{\sigma}^3) + \frac{2}{3}H_2(a,\tilde{\sigma})(1+\tilde{\sigma}^2)^{-1/2}\left(\frac{\tilde{T}}{2(1+a+a\tilde{\sigma}^2)^{1/2}} - \tilde{s} - \tilde{\sigma}\right),
\end{aligned}
$$

and with $\tilde{\omega} = 1 + \tilde{\sigma}^2$,

$$
2N\left(1 - \frac{3}{4}B'\Big(\tilde{\omega}^{3/2}\lambda\Big)\right) = (1+\tilde{\sigma}^2)^{-1/2}\left(\frac{\tilde{T}}{2(1+a+a\tilde{\sigma}^2)^{1/2}} - \tilde{s} - \tilde{\sigma}\right). \tag{3.2.5}
$$

For a given a and $(X, Y, \tilde{T}) \in \mathbb{R}^3$, (3.2.4) is a system of two equations for unknown $(\tilde{s}, \tilde{\sigma})$ and (3.2.5) gives an equation for N. We are looking for a solutions of (3.2.4) in the range

$$
a \in [h^\alpha, a_0], \alpha < 2/3, a|\tilde{\sigma}|^2 \leq \epsilon_0, 0 < T \leq a^{-1/2}, X \in [0,1] \quad \text{with } a_0, \epsilon_0 \text{ small.}
$$

Let us recall Lemma 2.18 [29]. Let $R = 2(1 - 3Y/\tilde{T}), \tilde{T} \geq \tilde{T}_0 > 0, X \in [-2,2], Y \in \mathbb{R}$. There exists $\tilde{\sigma}_j(X, Y, \tilde{T}, a) \in \mathbb{C}, j = 1,2,3,4$ such that

$$
\{\tilde{\sigma} \in \mathbb{C}, a|\tilde{\sigma}|^2 \leq \epsilon_0, \exists \tilde{s} \in \mathbb{C}, (\tilde{s}, \tilde{\sigma}) \text{ is a solution of (3.2.4)}\} \subset \{\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\sigma}_4\}.
$$

Moreover, there exists a function $f_*(a, a\tilde{\sigma}^2)$ with $f_*(0,0) = 1$ and constants $C_0, C_1, C_2 >$

$0, R_0, M_0 > 0$ such that the following holds:

(a) If $|R| \geq R_0$, two of the $\tilde{\sigma}_j f_*(a, a\tilde{\sigma}_j^2)$ are in the complex disk $D(\sqrt{R}, A)$, the two others are in the complex disk $D(-\sqrt{R}, A)$ with $A = C_0 \left(\frac{1}{\tilde{T}} + \frac{a(1+|R|)}{\sqrt{|R|}} \right)$. Moreover, one has $\sqrt{|R|} \geq 2A$.

(b) If $|R| \leq R_0$ and $|R|\tilde{T} \geq M_0$, two of the $\tilde{\sigma}_j f_*(a, a\tilde{\sigma}_j^2)$ are in the complex disk $D(\sqrt{R}, A)$, the two others are in the complex disk $D(-\sqrt{R}, A)$ with $A = \frac{C_1}{\tilde{T}\sqrt{|R|}}$. Moreover, one has $\sqrt{|R|} \geq 2A$.

(c) If $|R| \leq R_0$ and $|R|\tilde{T} \leq M_0$, one has $|\tilde{\sigma}_j| \leq C_2 \tilde{T}^{-1/2}$ for all j.

Then for a given point $(X, Y, \tilde{T}) \in [-2, 2] \times \mathbb{R} \times [0, a^{-1/2}]$, let us denote by $\mathcal{N}(X, Y, \tilde{T})$ the set of integers $N \geq 1$ such that (3.2.4) has at least one real solution $(\tilde{s}, \tilde{\sigma}, \lambda)$ with $a|\tilde{\sigma}|^2 \leq \epsilon_0$ and $\lambda \geq \lambda_0$. We denote by $\mathcal{N}^{\mathbb{C}}(X, Y, \tilde{T})$ the set of complex N such that (3.2.4) has at least one complex solution $(\tilde{s}, \tilde{\sigma}, \lambda)$ with $a|\tilde{\sigma}|^2 \leq \epsilon_0$ and $\lambda \geq \lambda_0$. Let us now rewrite the equation (3.2.5) as

$$2N = \tilde{T}\Phi_a(\tilde{\sigma}) + O(\tilde{T}\lambda^{-2}\tilde{\omega}^{-3}) + O(1), \quad \Phi_a(\tilde{\sigma}) = \frac{1}{2\langle\tilde{\sigma}\rangle(1 + a + a\tilde{\sigma}^2)^{1/2}}, \tag{3.2.6}$$

with $\langle\tilde{\sigma}\rangle = (1+\tilde{\sigma}^2)^{1/2}$ and $\Phi_a(\tilde{\sigma})$ is bounded on the set $U = \{\tilde{\sigma} \in \mathbb{C}, |\tilde{\sigma}| \leq 1/2 \text{ or } |\mathrm{Im}(\tilde{\sigma})| \leq |\mathrm{Re}(\tilde{\sigma})|/\sqrt{3}\}$ and

$$|\Phi_a(\tilde{\sigma}) - \Phi_a(\tilde{\sigma}')| \leq \frac{C|\tilde{\sigma}^2 - \tilde{\sigma}'^2|}{\sup(\langle|\tilde{\sigma}|\rangle, \langle|\tilde{\sigma}'|\rangle)} \left(a + \frac{1}{\langle|\tilde{\sigma}|\rangle\langle|\tilde{\sigma}'|\rangle} \right), \forall \tilde{\sigma}, \tilde{\sigma}' \in U. \tag{3.2.7}$$

We observe that (3.2.6) implies for $N_0 > 0$ independent of $(X, Y, \tilde{T})$ that

$$\mathcal{N}(X, Y, \tilde{T}) \subset [1, \tilde{T}/2 + N_0]. \tag{3.2.8}$$

Lemma. *There exists a constant C_0 such that the followings hold:*

(a) For all $(X, Y, \tilde{T}) \in [0, 1] \times \mathbb{R} \times [0, a^{-1/2}]$, one has $|\mathcal{N}(X, Y, \tilde{T})| \leq C_0$, and $\mathcal{N}^{\mathbb{C}}(X, Y, \tilde{T})$ is a subset of the union of four disks of radius C_0.

(b) For all $(X, Y, \tilde{T}) \in [0, 1] \times \mathbb{R} \times [0, a^{-1/2}]$, the subset of $\mathbb{N}$,

$$\mathcal{N}_1(X, Y, \tilde{T}) = \bigcup_{|Y'-Y|+|\tilde{T}'-\tilde{T}|\leq 1, |X'-X|\leq 1} \mathcal{N}(X', Y', \tilde{T}').$$

satisfies

$$|\mathcal{N}_1(X, Y, \tilde{T})| \leq C_0(1 + \tilde{T}\lambda^{-2}\tilde{\omega}^{-3}).$$

Proof. Part (a) is a consequence of (3.2.4). Indeed, for a given $(X, Y, \tilde{T} \geq \tilde{T}_0)$, there are at most four possible values of $\tilde{\sigma}$ by Lemma 2.18 [29]. For $\tilde{T} \leq \tilde{T}_0$, we use (3.2.8). To prove part (b), we may assume that $\tilde{T} \geq \tilde{T}_1$, with $\tilde{T}_1$ large. Recall that $R = 2(1 - 3Y/\tilde{T})$. Let $(X', Y', \tilde{T}')$ be such that $|Y' - Y| + |\tilde{T}' - \tilde{T}| \leq 1, X' \in [0, 1]$. Set $R' = 2(1 - 3Y'/\tilde{T}')$. We have $|R - R'| \leq C(1 + |R|)/\tilde{T}$.

- If $|R| \geq 2R_0$, with R_0 as in Lemma 2.18 [29]. Since $\tilde{T}$ is large, we have $|R'| \geq R_0$ and $|R| \simeq |R'|$. Let $N' \in \mathcal{N}(X', Y', \tilde{T}')$ and $\tilde{\sigma}'$ such that (3.2.4) holds. By Lemma 2.18 [29] (a), we may assume $\tilde{\sigma}'^* \in D(\sqrt{R'}, A')$. Take $\tilde{\sigma}^* \in D(\sqrt{R}, A)$ associated to $(X, Y, \tilde{T})$. Since $\tilde{\sigma}'$ is real, we have $R' \geq R_0$, thus $R \geq 2R_0$, and $\tilde{\sigma} \in U$. Let $N \in \mathcal{N}^{\mathbb{C}}(X, Y, \tilde{T})$ associated to $\tilde{\sigma}$. Since $a^{1/2} \leq 1/\tilde{T}$, we get

  $$|\tilde{\sigma} - \tilde{\sigma}'| \leq C|\tilde{\sigma}^* - \tilde{\sigma}'^*| \leq C\left(A + A' + |\sqrt{R} - \sqrt{R'}|\right) \leq \frac{C(1 + |R|)}{\tilde{T}\sqrt{|R|}}.$$

  Since $|\tilde{\sigma}| + |\tilde{\sigma}'| \leq C\sqrt{|R|}$, we get

  $$|\tilde{\sigma}^2 - \tilde{\sigma}'^2| \leq \frac{C(1 + |R|)}{\tilde{T}}.$$

  By (3.2.7) and (3.2.6), we have since $a|R| \simeq a|\tilde{\sigma}'|^2 \leq \epsilon_0$,

  $$\begin{aligned} 2|N - N'| &\leq |\tilde{T} - \tilde{T}'|\Phi_a(\tilde{\sigma}') + \tilde{T}|\Phi_a(\tilde{\sigma}') - \Phi_a(\tilde{\sigma})| + O((|\tilde{T}| + |\tilde{T}'|)\lambda^{-2}\tilde{\omega}^{-3}) + O(1), \\ &\leq C(a + 1/|R|)\frac{(1 + |R|)}{\sqrt{|R|}} + O(\tilde{T}\lambda^{-2}\tilde{\omega}^{-3}) + O(1) \in O(\tilde{T}\lambda^{-2}\tilde{\omega}^{-3}) + O(1). \end{aligned}$$

- If $|R| \leq 2R_0$ and $\tilde{T}|R| \geq M_0 + 8$. Since $|R\tilde{T} - R'\tilde{T}'| \leq 8$, we get $|R'|\tilde{T}' \geq M_0$. Thus we may apply Lemma 2.18 [29] (b). Let $N' \in \mathcal{N}(X', Y', \tilde{T}')$ and $\tilde{\sigma}' \in \mathbb{R}$ such that (3.2.4) holds. Since $\tilde{\sigma}'$ is real, we have $R' > 0$. Thus $R'\tilde{T}' > M_0$ and this implies $R > 0$ (take M_0 large). Moreover, we have $|R - R'| \leq C(1 + |R|)/\tilde{T}, |R'| \leq 3R_0$, and $|R| \simeq |R'|$. Now, we get $|\tilde{\sigma} - \tilde{\sigma}'| \leq \frac{C(1+R_0)}{\tilde{T}\sqrt{R}}$ and since $|\tilde{\sigma}| + |\tilde{\sigma}'| \leq C\sqrt{|R|}$, we get $|\tilde{\sigma}^2 - \tilde{\sigma}'^2| \leq \frac{C(1+R_0)}{\tilde{T}}$. Therefore, we get

  $$2|N - N'| \leq C\tilde{T}(a + O(1))(1 + R_0)/\tilde{T} + O(\tilde{T}\lambda^{-2}\tilde{\omega}^{-3}) + O(1) \in O(\tilde{T}\lambda^{-2}\tilde{\omega}^{-3}) + O(1).$$

- If $|R| \leq 2R_0$ and $\tilde{T}|R| \leq M_0 + 8$. We have $\tilde{T}'|R'| \leq M_0 + 16$. Thus by Lemma 2.18 [29](c), we have $|\tilde{\sigma}_j'| \leq C\tilde{T}^{1/2}, |\tilde{\sigma}_j| \leq C\tilde{T}^{-1/2}$. We get $|\tilde{\sigma}^2 - \tilde{\sigma}'^2| \leq C/\tilde{T}$ and thus

  $$2|N - N'| \in O(\tilde{T}\lambda^{-2}\tilde{\omega}^{-3}) + O(1).$$

□

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Abstract

Dispersive and Strichartz Estimates for The Wave Equation Inside Cylindrical Convex Domains

by **Len MEAS**

In this work, we establish local in time dispersive estimates and its application to Strichartz estimates for solutions of the model case Dirichlet wave equation inside cylindrical convex domains $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega \neq \emptyset$. Let us recall that dispersive estimates are key ingredients to prove Strichartz estimates. Strichartz estimates for waves inside an arbitrary domain Ω have been proved by Blair, Smith, Sogge [6, 7]. Optimal estimates in strictly convex domains have been obtained in [29]. Our case of cylindrical domains is an extension of the result of [29] in the case where the nonnegative curvature radius depends on the incident angle and vanishes in some directions.

Keywords: dispersive estimates, Strichartz estimates, wave equation, cylindrical convex domains.

Résumé: Estimations de dispersion et de Strichartz dans un domaine cylindrique convexe. Dans ce travail, nous allons établir des estimations de dispersion et des applications aux inégalités de Strichartz pour les solutions de l'équation des ondes dans un domaine cylindrique convexe $\Omega \subset \mathbb{R}^3$ à bord C^∞, $\partial\Omega \neq \emptyset$. Les estimations de dispersion sont classiquement utilisées pour prouver les estimations de Strichartz. Dans un domaine Ω général, des estimations de Strichartz ont été démontrées par Blair, Smith, Sogge [6, 7]. Des estimations optimales ont été prouvées dans [29] lorsque Ω est strictement convexe. Le cas des domaines cylindriques que nous considérons ici généralise les resultats de [29] dans le cas où la courbure positive dépend de l'angle d'incidence et s'annule dans certaines directions.

Mots Clés: estimations de dispersion, estimations de Strichartz, l'équation des ondes, domaines cylindriques.

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