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# W-entropy formulas on super Ricci flows and matrix Dirichlet processes.

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**HAL Id: tel-01616853**

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# THÈSE

En vue de l'obtention du

## DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*  
Cotutelle internationale *Fudan University*

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Présentée et soutenue le *04/12/2015* par :

SONGZI LI

*W*-entropy formulas on super Ricci flows and matrix Dirichlet processes

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### École doctorale et spécialité :

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# Acknowledgements

Five years have passed since I became a PhD student in 2010. During these years, I have studied in three different places: Fudan University (FDU), Chinese Academy of Sciences (CAS), and Université Paul Sabatier (UPS), and there are a lot of people to whom I would like to express my gratitude.

Back in 2009, I was a 3rd-year undergraduate in Fudan University with a "not so good" academic performances. After reading the book "Chance and Chaos" written by David Ruelle, I began to have a quite "naive" ambition to study "how probability works in statistical mechanics". However, concerned with my poor academic performance, I did not know if it was possible for me to pursue further study. So I took the course on "Stochastic Analysis" given by Prof. X.-D. Li to see whether I should decide to study the theory of Probability. It turned out that I should, and Prof. X.-D. Li gave me a great encouragement for further study. After an entrance exam in August, 2009, he agreed to be my supervisor. During the last year as a undergraduate student, I began to take courses for PhD students, and did a memoir on "Optimal Transportation" under the supervision of Prof. X.-D. Li.

In 2010, Prof. J.-G. Ying was so kind to accept me as his PhD student in FDU, after Prof. X.-D. Li left for CAS. My first knowledge of Probability came from Prof. J.-G. Ying, by taking his two courses on Probability, for undergraduates and for graduates. In my opinion, Prof. Ying has a clear aesthetic of neat and beautiful mathematics. This also applies to his courses, which influenced me in understanding Probability since then and pushed me to learn more about it.

In 2011, I went to CAS to begin my research under the supervision of Prof. X.-D. Li after taking courses in FDU for one year. Prof. X.-D. Li had a great patience with me, teaching me all the elementary knowledge and skills to do mathematics and to be a qualified researcher. The most important influence of Prof. X.-D. Li to me is that he has a very broad view of mathematics and has always ambition to study problems in a new area even he was not familiar in it. He encouraged me to study Ricci flows, optimal transportation, nonlinear PDEs and even physics.

In 2013, I became a PhD student of Prof. D. Bakry in UPS after receiving a scholarship from China Scholarship Council, with the support from FDU, Prof. X.-D. Li and Prof. J.-G. Ying. There is no need to say how good Prof. D. Bakry is as a mathematician. As far as I can see, he is also a mathematician with a strong personality of doing mathematics: he is passionate and persistent in his own interest and preference in mathematics, optimistic about the results for most of the time and patient with all my stupid mistakes. His preference and persistence in mathematics has a great impact on my view in understanding mathematics and the way I do my own research work.

Except for the influence on mathematics, all of three supervisors helped me a lot to ensure me have a nice environment for study. In particular, Prof. X.-D. Li and Prof. J.-G. Ying recommended me to get the scholarship from China Scholarship Council, and Prof. D. Bakry gave me a lot of help from the beginning I arrived in Toulouse and made a great effort to help me deal with all the administrative issues in France. I would like to express my deepest gratitude

to all my supervisors!

I am also indebted to Prof. J.-X. Hong, Prof. Z.-M. Ma, Prof. J.-A. Yan, Prof. F.-Z. Gong, Prof. Z. Dong, Prof. F.-Y. Wang, Prof. K.-Y. Guo, Prof. L.M. Wu, Prof. J.-M. Bismut, Prof. D. Chafaï, Prof. A.-B. Cruzeiro, Prof. I. Gentil, Prof. K. Kuwae, Prof. M. Ledoux, Prof. T. Lyons, Prof. A. Thalmaier, and Dr. K. He, Dr. Y. Liu, Dr. D. J. Luo, Dr. B. Qian, Dr. B. Wu, for their supports and discussions. I would also like to thank Prof. B.-L. Chen, Prof. R. Speicher, Prof. S. Peszat, Prof. P. He, Dr. R. Wang and Dr. X. Chen for inviting me to visit their universities and for providing me very nice opportunities to present my research works.

I would also like to thank Prof. D. Chafaï, Prof. F.-Z. Gong, Prof. K.-T. Sturm, Prof. A. Thalmaier, Prof. F.-Y. Wang for taking the responsibility of being the referee of my thesis, and Prof. X. Bressaud, Prof. D. Chafaï, Prof. F.-Z. Gong, Prof. H.-Q. Li, Prof. A. Thalmaier and Prof. F.-Y. Wang for accepting to be the members of jury for the defense of my PhD thesis.

I would also like to thank my friends for their helps, including X.-C. Wang, L.-P. Zhang, F. Zhang, D. Zhang, G.-P. Liu, Y.-X. Xie, S.-Q. Jian, Y.-Z. Wang, Nil, Tatiana, Stephane and Claire and all the friends in FDU, CAS and UPS.

Finally I would like to thank my parents for their support and encouragement all these years long. Special thanks go to my fiancé, for all his love.

Songzi Li  
October 21, 2015

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**Part I**

**Introduction**



# Chapter 1

## Introduction

This thesis relies on four papers (accepted, submitted or preprint) and a study of matrix Dirichlet processes (in preparation). In this introduction, we briefly describe the various and relevant topics which are the content of this thesis. In the second Part, we present the basic materials including framework, examples and previous results in order to better describe our results. Then, in the third Part, we provide a more detailed description of the results that we have obtained in accepted and submitted papers, and we refer the readers for the details to the papers (which are at the end of this thesis). In the fourth Part, we develop the full extent of our work on matrix Dirichlet process and related work. In the fifth part, we give the proofs of the main results in our published paper and the proofs of our new results which have not appeared in any preprints or publications.

### 1.1 Preliminaries

Since Itô's fundamental contribution to stochastic analysis, the interaction between probability theory, PDEs, geometry, quantum field theory and statistical mechanics has been one of the main research areas in mathematics. In particular, diffusion processes, diffusion operators, and their Markov semigroups have been important tools to describe time evolution phenomena and stochastic dynamic systems. They have been the central subjects studied in stochastic analysis and probability theory, PDEs, geometric analysis, and more recently, in random matrices theory and orthogonal polynomials, etc.

There is a huge literature on symmetric diffusion operators, since half a century ago. The interplay between analytic, probabilistic and geometric aspects of diffusion operators has been extensively studied. The relevant topics range from heat kernel bounds, longtime behavior such as the convergence to equilibrium and its rate, to functional inequalities and differential geometry for operators on Riemannian manifolds, and more recently optimal transportation theory with its application to metric measure spaces. See [14, 114, 115] and reference therein.

This thesis aims to study the diffusion operator from both geometric and algebraic views, involving Harnack inequalities,  $W$ -entropy formulas, diffusions on matrices, non-associative algebra etc. In this introduction, we describe the background and motivation of our work.

In this thesis, we consider symmetric diffusion operators, which naturally appear as infinitesimal generators to stochastic differential equations. More precisely, consider a Markov diffusion process  $(X_t)_{t \geq 0}$  with continuous trajectories on an open set of  $\mathbb{R}^d$  or a Riemannian manifold. The Markov semigroup  $(P_t)_{t \geq 0}$  generated by the diffusion process  $(X_t)_{t \geq 0}$  is given by the following

probabilistic representation formula

$$P_t(f)(x) = \mathbb{E}(f(X_t)|X_0 = x),$$

where  $f$  is a bounded measurable function or more general measurable functions with suitable integrability or regularity condition. The corresponding infinitesimal generator  $L$  is given by

$$Lf = \lim_{t \rightarrow \infty} \frac{P_t f - f}{t},$$

for  $f$  such that the limit exists in suitable sense (for example, in  $L^2$  with respect to a reference measure on  $\mathbb{R}^n$  or a given Riemannian manifold). Moreover,  $P_t f$  is the unique solution to the following heat equation associated with the operator  $L$

$$\partial_t P_t f = L P_t f = P_t L f, \quad P_0 f = f.$$

A typical example is the heat semigroup generated by the Laplace operator on  $\mathbb{R}^n$

$$P_t f(x) = \int_{\mathbb{R}^n} f(y) p_t(x, y) dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(y) e^{-\frac{|x-y|^2}{4t}} dy, \quad t > 0, \quad x \in \mathbb{R}^n,$$

in which  $p_t(x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$  is the fundamental solution to the heat equation  $\partial_t u = \Delta u$  on  $\mathbb{R}^n$ , and  $P_t f$  is the unique solution to the heat equation  $\partial_t u = \Delta u$  with initial data  $f$ , where  $f$  is a bounded measurable function on  $\mathbb{R}^n$ .

The operator  $L$  is a second order differential operator with no zero component, semi-elliptic and of the form

$$L(f) = \sum g^{ij}(x) \partial_{ij}^2 f + \sum_i b^i(x) \partial_i f, \quad (1.1.1)$$

where the symmetric matrix  $(g^{ij}(x))$  is everywhere non negative. The fact that  $L$  is symmetric means that  $L$  is a self-adjoint operator with respect to some measure  $\mu$ . When  $\mu$  has a density  $\rho$  with suitable regularity, we may write

$$L(f) = \frac{1}{\rho} \sum_{ij} \partial_i (g^{ij} \rho \partial_j f). \quad (1.1.2)$$

Given  $E$  a smooth manifold,  $\mathcal{A}$  the space of smooth compactly supported functions on  $E$ , and the diffusion operator  $L$ , we define its carré du champ operator and the  $\Gamma_2$  operator as follows

$$\Gamma(f, g) = \frac{1}{2} (L(fg) - fL(g) - L(f)g), \quad (1.1.3)$$

and

$$\Gamma_2(f) = \frac{1}{2} (L\Gamma(f, f) - 2\Gamma(f, Lf)), \quad (1.1.4)$$

where  $f, g \in \mathcal{A}$ .

In this thesis, we will always identify the coordinate function with the process written in coordinates. For example, for a standard Brownian motion  $X_t = (X^1, \dots, X^d)$  on  $\mathbb{R}^d$ , we write its diffusion operators as

$$\Gamma(X^i, X^j) = \delta_{ij}, \quad L(X^i) = 0.$$

Following Bakry and Emery [12], we say that the curvature-dimension condition  $CD(\rho, m)$  with  $\rho \in \mathbb{R}$  and  $m \in [1, \infty]$  is satisfied, if for all the function  $f \in \mathcal{A}$ ,

$$\Gamma_2(f) \geq \rho \Gamma(f) + \frac{1}{m} (Lf)^2.$$

Parts of results in this thesis are proved for the weighted Laplacian. In fact, any elliptic diffusion operator which is symmetric with respect to an invariant measure can be uniquely transformed into a weighted Laplacian. By Itô's SDE theory, given a weighted Laplacian  $L = \Delta - \nabla\phi \cdot \nabla$  on a Riemannian manifold  $(M, g)$ , we can construct a diffusion process  $(X_t, t \in [0, T])$  on  $M$  by solving the following Itô's stochastic differential equation

$$dX_t = \sqrt{2}dW_t - \nabla\phi(X_t)dt,$$

such that  $L$  is the infinitesimal generator of  $(X_t, t \in [0, T])$ . Here  $W_t$  denotes the Brownian motion on  $(M, g)$ .

The invariant measure of the weighted Laplacian  $L = \Delta - \nabla\phi \cdot \nabla$  on  $(M, g)$  is given by  $d\mu(x) = e^{-\phi}dvol(x)$ , where  $dvol$  is the Riemannian volume element determined by the Riemannian metric  $g$ , i.e.,  $dvol(x) = \sqrt{\det g(x)}dx$ . By the Bochner Formula we have for a smooth function  $f$  on  $M$

$$\Gamma_2(f) = |\text{Hess}f|^2 + \text{Ric}(L), \quad (1.1.5)$$

where  $\text{Ric}(L) = \text{Ric} + \text{Hess}\phi$  is the so-called Bakry-Emery Ricci curvature associated with the weighted Laplacian  $L = \Delta - \nabla\phi \cdot \nabla$  on  $(M, g)$ . The curvature-dimension condition  $CD(\rho, m)$  is equivalent to

$$\text{Ric} + \text{Hess}\phi \geq \rho g + \frac{\nabla\phi \otimes \nabla\phi}{m-n},$$

where  $m \geq n$ ,  $n$  is the dimension of the manifold  $M$ , and  $m \in [n, \infty]$  is a constant. See [12]. Moreover, we denote the  $m$ -dimensional Bakry-Emery Ricci curvature by  $\text{Ric}_{m,n}(L) = \text{Ric} + \text{Hess}\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n}$ . Here we use the convention that  $m = n$  if and only if  $\phi$  is a constant, and when  $m = \infty$ ,  $\text{Ric}_{m,n}(L) = \text{Ric}(L)$ .

## 1.2 Harnack inequalities for $K$ -super Ricci flows and $(K, m)$ -super Ricci flows

The transition probability density function  $p_t(x, y)$  of a diffusion process  $X_t$  with infinitesimal generator  $L$  is the fundamental solution (i.e., the heat kernel) to the heat equation  $\partial_t u = Lu$ . Thus, it is a fundamental problem in stochastic analysis to study the heat kernel and its properties for a given diffusion process (or a given diffusion operator  $L$ ). One important and effective way to do so is to use the gradient estimate and the Harnack inequality for the positive solution of the heat equation  $\partial_t u = Lu$  and to use the geometric or analytic conditions on Riemannian manifolds to derive the heat kernel estimates. There are extensive references in the literature in the study of the Harnack inequality and the heat kernel, see e.g. [42, 14] and reference therein.

Let  $M$  be an  $n$  dimensional complete Riemannian manifold,  $u$  be a positive solution to the heat equation

$$\partial_t u = \Delta u.$$

In their famous paper [119], P. Li and S.T. Yau proved that, if  $\text{Ric} \geq -K$ ,  $K \geq 0$ , then the following Li-Yau differential Harnack inequality holds: for any  $\alpha > 1$ ,

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 K}{\sqrt{2}(\alpha-1)}. \quad (1.2.6)$$

For some improvements of the above inequality, see Davies [42] and Bakry-Qian [18]. If  $\text{Ric} \geq 0$ , taking  $\alpha \rightarrow 1$ , the following Li-Yau differential Harnack inequality holds for positive solutions

to the heat equation  $\partial_t u = \Delta u$  on complete Riemannian manifolds with non-negative Ricci curvature

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}. \quad (1.2.7)$$

On the other hand, R. Hamilton [59] proved a dimension free Harnack inequality for positive and bounded solutions to the heat equation  $\partial_t u = \Delta u$  on compact Riemannian manifolds, which is different from the Li-Yau differential Harnack inequality (1.2.6) under the same condition. More precisely, suppose that there exists a constant  $K \geq 0$  such that  $\text{Ric} \geq -K$ , then for any positive and bounded solution  $u$  to the heat equation  $\partial_t u = \Delta u$  the following Harnack inequality holds:

$$|\nabla \log u|^2 \leq \left( \frac{1}{t} + 2K \right) \log(A/u), \quad \forall x \in M, t > 0, \quad (1.2.8)$$

where  $A := \sup\{u(t, x) : x \in M, t \geq 0\}$ .

Under the same condition  $\text{Ric} \geq -K$ , Hamilton [59] also proved the following Li-Yau type Harnack inequality for any positive solution to the heat equation  $\partial_t u = \Delta u$

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{n}{2t} e^{4Kt}. \quad (1.2.9)$$

There is a huge literature on Li-Yau inequality and Li-Yau-Hamilton inequality, see [12, 18, 15, 52, 54, 75, 78, 80, 6, 7] and references therein. In [116], F.Y. Wang proved a dimension free Harnack inequality for the weighted Laplacian from a logarithmic Sobolev inequality on complete Riemannian manifolds satisfying the curvature-dimension condition  $CD(K, \infty)$ . In [15], Bakry and Ledoux derived the Li-Yau Harnack inequality (1.2.7) for positive solution of the heat equation  $\partial_t u = \Delta u$  on Riemannian manifolds with non-negative Ricci curvature from an improved logarithmic Sobolev inequality for the heat semigroup  $P_t = e^{t\Delta}$ . In [75, 79, 74], the Li-Yau Harnack inequality (1.2.7) has been extended to positive and bounded solutions to the heat equation associated to the weighted Laplacian  $L = \Delta - \nabla\phi \cdot \nabla$  on complete Riemannian manifolds with both  $CD(0, m)$ ,  $m > n$  and  $CD(-K, \infty)$ ,  $K > 0$  condition. In [74], an improved version of the Hamilton Harnack inequality (1.2.8) has been proved for positive solution of the heat equation  $\partial_t u = \Delta u$  on complete Riemannian manifolds with  $\text{Ric} \geq -K$  and for  $\partial_t u = Lu$  for more general weighted Laplacian  $L = \Delta - \nabla\phi \cdot \nabla$  on complete Riemannian manifolds with  $\text{Ric}(L) = \text{Ric} + \text{Hess}\phi \geq -K$ . In a more recent paper by Bakry, Gentil and Bolley [13], they extended the idea in Bakry and Ledoux [15] and proved a Li-Yau Harnack inequality for general Markov semigroup on complete Riemannian manifolds under  $CD(\rho, n)$  condition, where  $\rho \neq 0$  is a constant,  $n \geq 1$ . Their results is sharper than the well-known Li-Yau type Harnack inequalities.

In this thesis, we aim to extend the Li-Yau Harnack inequality (1.2.7) and the Li-Yau-Hamilton Harnack inequality (1.2.8) to the heat equation of the Laplacian or the weighted Laplacian on Riemannian manifolds with time-dependent metrics and potentials, in particular, to the  $K$ -super Perelman Ricci flow. The Ricci flow was introduced by R. Hamilton [58] in 1982, as an approach to prove the Poincaré conjecture and Thurston's geometrization conjecture. It allows to deform the Riemannian metrics on a given manifold along the flow of the Ricci curvature tensor

$$\partial_t g = -2\text{Ric}. \quad (1.2.10)$$

In [58], Hamilton proved that the Cauchy problem of the above equations admit a unique local solution defined on  $M \times [0, T]$  on any compact manifold, and on 3-dimensional compact manifold with initial Riemannian metric whose Ricci curvature is strictly positive, the Ricci flow equation has a unique global solution defined on  $M \times [0, \infty)$  with given initial condition. Deforming the metric along the Ricci flow, one can produce canonical geometric structures starting from rather

general initial metric and prove certain analytical, geometrical, and topological results about the canonical metric on Riemannian manifolds. The most famous success of the Ricci flow is the proof of the Poincaré conjecture and Thurston’s geometrization conjecture by G. Perelman [99] in 2002. See also [90, 63, 34, 36] and related references.

Thus it is very natural to develop geometric analysis and stochastic analysis on Riemannian manifolds with time dependent metrics. For example, to study the gradient estimates, Harnack inequalities and functional inequalities for diffusion processes on manifolds equipped with the Ricci flow or modified Ricci flows. Since Hamilton and Perelman’s seminal works, there have been extensively work on this topic, for example the classical results by Hamilton [59], Perelman [99], see also [40] and reference therein. In [5], Arnaudon, Coulibaly and Thalmaier constructed the Brownian motion on manifolds with time dependent metrics and give a probabilistic characterization of the Ricci flow using the damped parallel transport along the path of Brownian motions on these manifolds. In [55], Guo, Phillipowski and Thalmaier proved the Hamilton type Harnack inequality and the Boltzmann entropy dissipation formula for positive solution of the backward heat equation  $\partial_t u + \Delta_{g(t)} u = 0$  on compact manifolds equipped with the backward super Ricci flow  $\frac{\partial g}{\partial t} \leq 2Ric$  using a stochastic analysis approach. In her PhD thesis [37] and related work [39, 38], L.-J. Cheng extended the stochastic analysis approach and proved the HWI inequality and related inequalities on Riemannian manifolds equipped with the super Ricci flows. Let us mention the very recent work by Sturm [108, 109, 110] for the introduction of the notion of super Ricci flows on metric measure spaces and relevant study in geometric analysis on metric measure spaces.

In this thesis (see Chapter 5), we first prove that the logarithmic Sobolev inequalities for the heat equation associated with the weighted Laplacian on Riemannian manifolds is equivalent to the fact that the metric satisfies the  $K$ -super Perelman Ricci flow (Theorem 5.2.1). As a consequence, we are able to prove the Hamilton type Harnack inequality (Theorem 5.2.4) by extending the method used in [74]. Moreover, we prove the Li-Yau-Hamilton type Harnack inequality to the heat equation associated with the weighted Laplacian on compact Riemannian manifolds with the  $(K, m)$ -super Ricci flow (Theorem 5.2.6), and on complete Riemannian manifolds with fixed metric satisfying the  $CD(K, m)$  condition (Theorem 5.2.7). These results play an important role in the study of the  $W$ -entropy formulas for the heat equation of the weighted Laplacian on Riemannian manifolds equipped with the  $K$ -super Perelman Ricci flow or the  $(K, m)$ -super Ricci flow. See the next subsection.

### 1.3 $W$ -entropy formulas on $K$ -super Ricci flows and $(K, m)$ -super Ricci flows

The physical notion of entropy was first introduced by R. Clausius in 1865 in his study of the Carnot cycle in thermodynamics. In 1872, L. Boltzmann [25] introduced the evolution equation (now called the Boltzmann equation) for the probability distribution density of the ideal gas in the phase space. He also introduced the  $H$ -quantity (now called the Boltzmann entropy) and proved the  $H$ -theorem for the Boltzmann equation in kinetic theory of gas. In 1877, Boltzmann [26] gave the statistical interpretation of the  $H$ -quantity using the probability theory.

The  $H$ -entropy introduced by Boltzmann is an important tool in many mathematical fields, such as partial differential equation, probability, statistical mechanics, etc. It has been founded as an important tool in the study of information and communication theory by Shannon [101]. It plays also important role in J. Nash’s seminal work [91] on the regularity of the solution to the parabolic equation and elliptic equation of second order elliptic operators with measurable

coefficient related to the 19th Hilbert problem. The notion of the Kolmogorov-Sinai entropy [64, 104], which is a variant of the Boltzmann entropy, has been also an important tool in the study of the ergodicity theory for dynamical systems. Moreover, the notion of the entropy has been the source for P. Lax and other people to introduce the notion of the entropy solution for hyperbolic systems and the fluid equations which have singularity structure, such as the Euler equation and the Navier-Stokes equations. See Evans [50, 51] and references therein.

In his seminal paper [99] in 2002, Perelman introduced  $W$ -entropy for the Ricci flow and proved its mononicity along the conjugate heat equation, which enabled him to prove the no local collapsing theorem for the Ricci flow and "remove the major stumbling block in Hamilton's approach" to the Poincaré conjecture and Thurston's geometrization conjecture. More precisely, let  $M$  be an  $n$ -dimensional compact Riemannian manifold equipped with the Ricci flow

$$\partial_t g = -2\text{Ric}, \quad (1.3.11)$$

Perelman first introduced the so-called conjugate heat equation

$$\partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau},$$

related to the Ricci flow, and then introduced the  $W$ -entropy as follows

$$W(g, u, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n] u \, dvol, \quad (1.3.12)$$

where  $R$  denotes the scalar curvature of the metric  $g$ ,  $dvol$  is the volume element,  $\partial_t \tau = -1$ ,  $u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$ . Moreover, he proved the following  $W$ -entropy formula

$$\frac{d}{dt} W(g, f, \tau) = 2 \int_M \tau \left| \text{Ric} + \text{Hess} f - \frac{g}{2\tau} \right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}} dvol. \quad (1.3.13)$$

As a consequence, the  $W$ -entropy is monotone along the solutions to the Ricci flow and the conjugate heat equation.

The importance of the  $W$ -entropy lies in the fact that it reveals the evolution of the system and gives the description of the equilibrium state. More precisely, from the mononicity formula (1.3.13) we know that  $\frac{d}{dt} W(g, f, \tau) = 0$  at some time  $t = t_0$  if and only if  $(M, g(t_0), f(t_0))$  is a shrinking gradient Ricci soliton

$$\text{Ric} + \text{Hess} f = \frac{g}{2\tau}. \quad (1.3.14)$$

Moreover, the  $W$ -entropy is closely related to the logarithmic Sobolev inequality. Indeed, the mononicity of  $W$ -entropy implies that the optimal constant  $\mu(\tau)$  in the logarithmic Sobolev inequality, defined by

$$\begin{aligned} \mu(\tau) &= \inf \left\{ W(u, \tau), \int_M u \, dvol = 1 \right\} \\ &= \inf \left\{ \int_M [\tau(4|\nabla \omega|^2 + R\omega^2) - \omega^2 \log \omega^2] d\mu \right\} \end{aligned}$$

is decreasing in time along the conjugate heat equation. For details, see Perelman [99].

In [93, 92], Ni studied the  $W$ -entropy for the linear heat equation  $\partial_t u = \Delta u$  on complete Riemannian manifolds and proved that the  $W$ -entropy is decreasing on complete Riemannian manifolds with non-negative Ricci curvature.

In [76, 78], X.-D. Li introduced the  $W$ -entropy for the heat equation  $\partial_t u = Lu$  of the weighted Laplacian  $L = \Delta - \nabla\phi \cdot \nabla$  on complete Riemannian manifolds and proved the  $W$ -entropy formula. The monotonicity of the  $W$ -entropy is proved on complete Riemannian manifolds with non-negative  $m$ -dimensional Bakry-Emery Ricci curvature, which extends the above mentioned result due to Ni [93]. Moreover, a rigidity theorem is proved for the  $W$ -entropy in the corresponding context.

It is natural to ask the questions (raised by experts during the past years) what happens for the heat equation  $\partial_t u = Lu$  of the weighted Laplacian  $L = \Delta - \nabla\phi \cdot \nabla$  on complete Riemannian manifolds with  $Ric_{m,n}(L) \geq K$ , equivalently, the  $CD(K, m)$  condition holds, for  $K \in \mathbb{R}$  and  $m \in [n, \infty)$ , and what happens when  $K \in \mathbb{R}$  and  $m = \infty$ , equivalently, the  $CD(K, \infty)$  condition holds. Moreover, it is also interesting to know what happens for the heat equation  $\partial_t u = Lu$  on Riemannian manifolds with its metric satisfying the  $(K, m)$ -super Ricci flow or the  $K$ -super Ricci flow.

In this thesis, we give the complete answers to the above mentioned questions. The main tool in our work is the Bakry-Emery Ricci curvature. In fact, during the past decades many results have been proved in the study of geometric and stochastic analysis on Riemannian manifolds with weighted measure using the Bakry-Emery Ricci curvature. See e.g. Ané et al. [4], Bakry-Gentil-Ledoux [14], F.-Y. Wang [116, 117], X.-D. Li [75, 52, 54, 74] and references therein.

The Bakry-Emery Ricci curvature has been essentially used in Perelman's work [99]. More precisely, let  $\mathcal{M} = \{\text{Riemannian metric } g \text{ on } M\}$ . Perelman introduced the  $\mathcal{F}$ -entropy functional by

$$\mathcal{F}(g, f) = \int (R + |\nabla f|^2) e^{-f} dv$$

on  $\mathcal{M} \times \mathcal{C}^\infty(M)$ , where  $R$  is the scalar curvature and  $dv$  is the volume element. Under the restriction that  $d\mu = e^{-f} dv$  is a fixed measure on  $M$ , he proved that the gradient flow of the  $\mathcal{F}$ -entropy functional with respect to the standard  $L^2$ -metric on  $\mathcal{M} \times \mathcal{C}^\infty(M)$  is given by the following modified Ricci flow (called Perelman's Ricci flow throughout this thesis)

$$\partial_t g = -2(\text{Ric} + \text{Hess}f) \tag{1.3.15}$$

and the conjugated heat equation

$$\partial_t f = -\Delta f - R. \tag{1.3.16}$$

By [99], under a family of time-dependent diffeomorphisms on  $M$ , the modified Ricci flow is equivalent to Ricci flow (1.3.11). Moreover, it is proved by Perelman that

$$\frac{d}{dt} \mathcal{F}(g(t), f(t)) = 2 \int_M |\text{Ric} + \text{Hess}f|^2 e^{-f} d\text{vol}.$$

Thus, the  $\mathcal{F}$ -entropy functional is increasing along the Perelman Ricci flow and the conjugate heat equation, and  $\frac{d}{dt} \mathcal{F}(g(t), f(t)) = 0$  at some time  $t = t_0$  if and only if  $(M, g(t_0), f(t_0))$  is a steady gradient Ricci soliton

$$\text{Ric} + \text{Hess}f = 0. \tag{1.3.17}$$

We now introduce the definition of  $(K, m)$ -super Ricci flow: Let  $M$  be an  $n$ -dimensional manifold equipped with a family of time dependent metric  $g(t)$  and potentials  $\phi(t)$ , By definition,  $(M, g(t), \phi(t), t \in [0, T])$  is called a  $(K, m)$ -super Ricci flow if the the following inequality holds

$$\frac{1}{2} \partial_t g_t + \text{Ric}_{m,n}(L) \geq K, \quad t \in [0, T],$$

where  $K \in \mathbb{R}$  and  $m \in [n, \infty]$  are two constants. When  $K = 0$  and  $m = n$ , it is the super Ricci flow; when  $m = n$ ,  $K \in \mathbb{R}$ , it is called the  $K$ -super Ricci flow. When  $K \in \mathbb{R}$  and  $m = \infty$ , we call  $(M, g(t), \phi(t))$  a  $K$ -super Perelman Ricci flow, i.e.,

$$\frac{1}{2} \partial_t g_t + \text{Ric}(L) \geq K, \quad t \in [0, T].$$

In this thesis, based on the Harnack inequalities we proved in Chapter 5 (see also Section 1.2 above), we introduce the  $W$ -entropy functional and prove its monotonicity formula for the  $(K, m)$ -super Ricci flow and for the  $(K, \infty)$ -super Ricci flow. More precisely, in Section 6.3.1, we prove the  $W$ -entropy formulas on manifolds with time dependent metrics and potentials (Theorem 6.3.2), as a corollary, on compact manifolds equipped with the  $(0, m)$ -Perelman Ricci flow we prove the monotonicity theorem for the optimal logarithmic Sobolev constant (Theorem 6.3.3); in Section 6.3.2, for  $K \in \mathbb{R}$ , we prove the  $W$ -entropy formula on complete manifolds with the  $CD(K, m)$  condition (Theorem 6.3.4) and the  $W$ -entropy formula on compact manifolds with the  $(K, m)$ -super Ricci flow (Theorem 6.3.6); in Section 6.3.3, for  $K \in \mathbb{R}$ , we prove the  $W$ -entropy formula on complete manifolds with the  $CD(K, \infty)$  condition (Theorem 6.3.7) and the  $W$ -entropy formula on compact manifolds with the  $(K, \infty)$ -super Ricci flow (Theorem 6.3.9). Moreover, using the warped product interpretation of the  $m$ -dimensional Bakry-Emery Ricci curvature, we give a new proof of the  $W$ -entropy formula for the weighted Laplacian on manifolds with fixed metric, and give a natural geometric interpretation of all the quantities in the  $W$ -entropy formulas. For details, see Chapter 6.

## 1.4 $W$ -entropy formulas on Wasserstein space over Riemannian manifolds

During the past decades, inspired by the seminal works of G. Monge [89] and L.V. Kantorovich [61], the theory of the optimal transportation have been intensively developed in probability theory, PDEs, geometric analysis, and other related areas in mathematics. By the works of Brenier [29], McCann [85], Caffarelli [32, 31], and Caffarelli-Feldman-McCann [33], the existence, uniqueness and regularity of the Monge-Kantorovich problem with the quadratic cost function have been completely solved. Inspired by the work of V.I. Arnold [8] on the incompressible Euler equation, F. Otto [96] introduced an infinite dimensional Riemannian metric on the Wasserstein space of probability measures over Euclidean space, and proved that the Fokker-Planck equation and the porous media equation can be realized as the gradient flow of the Boltzmann entropy and the Renyi entropy on the Wasserstein space. In view of this, he proved the contraction property of the  $W_2$ -Wasserstein metric between the Fokker-Planck diffusions and the solutions to the porous media equation. In [97], Otto and Villani proved the HWI inequality for the heat equation of the weighted Laplacian on manifolds with the  $CD(K, \infty)$ -condition. Sturm [105], Sturm and von Renesse [111] extended Otto's results to the Wasserstein space over Riemannian manifolds, and proved the contraction property of the  $W_2$ -Wasserstein metric between the Fokker-Planck diffusions is indeed equivalent to the  $CD(0, \infty)$ -condition for the Bakry-Emery Ricci curvature. In [35], Carrillo, McCann and Villani proved the  $K$ -displacement convexity for the free energy functional associated to the nonlinear Fokker-Planck equation in a porous media with self-interaction between particles. More recently, Lott, Villani [82] and Sturm [106, 107] have independently developed a study of geometric analysis on the metric measure spaces using the theory of the optimal transportation. See Villani [114, 115], Ambrosio-Gigli-Savare [2] and Ambrosio-Gigli [1] and extensive reference therein.

To better describe our work, let us give more details about the above mentioned results. Let  $(M, g)$  be a complete Riemannian manifold equipped with a smooth measure  $d\mu = e^{-\phi} d\text{vol}$ , where  $\phi \in C^2(M)$ , and  $\mathcal{P}_2(M, \mu)$  be the Wasserstein space of all probability measures  $\rho(x)d\mu(x)$  with density function  $\rho$  on  $M$  such that  $\int_M d^2(o, x)\rho(x)d\mu(x) < \infty$ , where  $d(o, \cdot)$  denotes the distance function from a fixed point  $o \in M$ . For any two probability measures  $\mu_0, \mu_1 \in \mathcal{P}_2(M, \mu)$ , the Wasserstein distance  $W_2(\mu_0, \mu_1)$  is defined by

$$W_2^2(\mu_0, \mu_1) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} \int_M d^2(x, y)d\pi(x, y), \quad (1.4.18)$$

where  $\Pi(\mu_0, \mu_1)$  is a probability measure on  $M \times M$ , whose marginal measures are  $\mu_0$  and  $\mu_1$ . Let  $L = \Delta - \nabla\phi \cdot \nabla$ , and let  $u$  be a positive solution to the heat equation  $\partial_t u = Lu$ , then it is well known that the  $CD(0, \infty)$ -condition, i.e.,  $\text{Ric} + \nabla^2\phi \geq 0$ , implies that the relative Boltzmann entropy

$$\text{Ent}(u\mu|\mu) = \int_M u \log u d\mu$$

is convex in time  $t$  along the heat equation  $\partial_t u = Lu$ . See Bakry and Emery [12].

Let  $d\mu_0 = u_0 d\mu$  and  $d\mu_1 = u_1 d\mu$  be two probability measures on  $M$ , and let  $\{\mu_t, t \in [0, 1]\}$  be the Wasserstein geodesic between  $\mu_0$  and  $\mu_1$  in the Wasserstein space  $\mathcal{P}_2(M, \mu)$  equipped with Otto's infinite dimensional Riemannian metric, and define  $\text{Ent}(\mu_t|\mu)$  to be the relative Boltzmann entropy of the probability measure  $\mu_t$  with respect to  $\mu$ , i.e.,

$$\text{Ent}(\mu_t|\mu) = \int_M \frac{d\mu_t}{d\mu} \log \frac{d\mu_t}{d\mu} d\mu.$$

Under the  $CD(K, \infty)$  condition, i.e.,  $\text{Ric} + \nabla^2\phi \geq K$ , Sturm [105, 106] and Lott-Villani [82] independently proved that the relative Boltzmann entropy  $\text{Ent}(\mu_t|\mu)$  is  $K$ -convex along the Wasserstein geodesic on the Wasserstein space  $\mathcal{P}_2(M, \mu)$ . More precisely, if  $\text{Ric} + \nabla^2\phi \geq K$ , then

$$\text{Ent}(\mu_t|\mu) \leq (1-t)\text{Ent}(\mu_0|\mu) + t\text{Ent}(\mu_1|\mu) - \frac{K}{2}t(1-t)W_2^2(\mu_0, \mu_1), \quad t \in [0, 1].$$

In [82, 106, 107, 115], the  $K$ -convexity along the Wasserstein geodesic has been used as the definition property for the curvature-dimension  $CD(K, \infty)$  condition on metric measure spaces which usually have singularity and loss the smooth regularity for the development of a nice geometric analysis in a standard way.

The above results has further been extended by McCann-Topping [86] and Lott [81] to compact manifolds equipped with the super Ricci flow or the Ricci flow. In [86], McCann and Topping proved that the Brownian diffusions on compact Riemannian manifolds equipped with the super Ricci flow has the  $W_2$ -contraction property. On the other hand, Lott [81] proved two convexity results for the Boltzmann type entropy functionals along the  $\mathcal{L}$ -geodesics on the Wasserstein space over  $(M, g(t))$  equipped with the Ricci flow, which are closely related to Perelman's results on the monotonicity of the  $\mathcal{F}$  and  $\mathcal{W}$ -entropy for the Ricci flow. Moreover, using the Li-Yau-Hamilton differential Harnack inequality for Ricci flow due to Perelman [99], Lott proved that the monotonicity of Perelman's reduced volume is a consequence of the convexity property of a Boltzmann entropy type functional related with the Wasserstein space associated with Perelman's reduced distance. In [66], Kuwada and Phillipowski derived McCann-Topping's  $W_2$ -contraction property using the coupling of Brownian motions on manifolds equipped with the super Ricci flow.

In this thesis, inspired by the all the works mentioned above, and following Perelman [99] and [93, 76, 78, 72], we first prove an analogue of McCann-Topping's  $W_2$ -contraction for the

Fokker-Planck diffusions and Lott's convexity theorem for suitably defined Boltzmann type entropy functionals on compact manifolds equipped with the Perelman modified Ricci flow (1.3.15) and the conjugate heat equation (1.3.16), see Section 7.1 and Section 7.3. We then introduce the notion of the  $W$ -entropy for the geodesic flow on the Wasserstein space  $\mathcal{P}_2(M, \mu)$  and prove its monotonicity formula (Theorem 7.3.1). We find that there is an essential similarity between the  $W$ -entropy formula for the geodesic flow on the Wasserstein space and the  $W$ -entropy formula for the heat equation on the underlying manifolds. Inspired by J.-M. Bismut's works [22, 23, 24] on the deformation of the Witten Laplacian on the cotangent bundle over finite dimensional Riemannian manifold, and extending his idea to the infinite dimensional Wasserstein space over Riemannian manifolds, we introduce the Langevin deformation of geometric flows on the Wasserstein space, which interpolates the geodesic flow and gradient flow on the Wasserstein space, and prove the Boltzmann entropy dissipation formula along the Langevin deformation of flows (Theorem 7.5.2). Moreover, we prove the  $W$ -entropy inequality for the Langevin deformation flows on Wasserstein space over the manifolds with the entropic curvature-dimension  $CD_{\text{Ent}}(K, m)$  condition, which was introduced very recently by Erbar-Kawada-Sturm [46].

## 1.5 Spectrum processes on the octonion algebra

We now describe the motivation of our study on the matrix-valued diffusion processes, which can be viewed as diffusion processes taking values in Riemannian manifolds of matrices satisfying some algebraic or geometric constraint conditions. The study of the law of the spectrum has been one of the most important topics in random matrix theory. One may consider stochastic diffusion processes on specific matrices, for example symmetric or Hermitian matrices. Usually when one considers the empirical measure of the spectrum, it satisfies again a stochastic differential equation, hence defines a diffusion process, called the Dyson type diffusion process, see e.g. the seminal works of Wigner [118], Mehta [87], Dyson [45], more recently Anderson-Guionnet-Zeitouni [3], Erdős et al. [47, 49, 48], Forrester [53] and references therein. In [73], the Dyson type diffusion processes has been studied for real symmetric or Hermitian matrices with general external potential functions.

In the paper [19] by Bakry and Zani, the authors considered real symmetric matrices whose elements are independent Brownian motions depending on some associative algebra structure of the Clifford type. To study the law of the spectrum of the matrices, they consider the processes on the characteristic polynomials  $P(X)$ , leading to the spectrum of the matrices which reflects the structure of the algebra, known as Bott periodicity. Their results have shown that the related diffusion operator provides an efficient method to study the spectrum of random matrices, and even the structure of the algebra, since the Bott periodicity even appears as a consequence of the study of the spectrum of the matrices.

The previous study on Dyson Brownian motion, including the work of Bakry and Zani [19] on the Clifford algebras, mainly concentrated on the case where the underlying algebra is associative. It is therefore worth understanding how important this property is in the study of the related Dyson processes. The octonion algebra, which is nonassociative but only alternative, provides a good example for us to start with. In his book [53] (Section 1.3.5), Forrester mentioned that the distribution  $C_{\beta, n} \exp(-\frac{\beta}{2} \sum_{j=1}^n \lambda_j^2) \prod_{1 \leq j < k \leq n} |\lambda_k - \lambda_j|^\beta$  with  $\beta = 8$  can be realized by the law of the spectrum of the  $2 \times 2$  matrices on octonions, with Gaussian entries. It is therefore worth to look at the associated Dyson process, which could also provide the result in Forrester [53] through the study of its reversible measure.

There are only four normed division algebras:  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  and  $\mathbb{O}$ . We are familiar with  $\mathbb{R}$ ,  $\mathbb{C}$ , and while quaternions  $\mathbb{H}$  are noncommutative but associative, octonions are nonassociative but

only alternative. Even though their properties are not so nice, octonions have some important connections to different fields of mathematics, such as geometry, topology and algebra. One interesting example is its role in the classification of simple Lie algebras. There are 3 infinite families of simple Lie algebras, coming from the isometry groups of the projective spaces  $\mathbb{R}P^n$ ,  $\mathbb{C}P^n$  and  $\mathbb{H}P^n$ . The remaining 5 simple Lie algebras were later discovered to be in a connection with octonions: they come from the isometry groups of the projective planes over  $\mathbb{O}$ ,  $\mathbb{O} \otimes \mathbb{C}$ ,  $\mathbb{O} \otimes \mathbb{H}$ ,  $\mathbb{O} \otimes \mathbb{O}$  and the automorphism group of octonions. It is also worth to mention that, according to the independent work by Kervaire [62] and Bott-Milnor [27] in 1958, saying that there are only 4 parallelizable spheres:  $S^0$ ,  $S^1$ ,  $S^3$  and  $S^7$ , which correspond precisely to elements of unit norm in the normed division algebras of the real numbers, complex numbers, quaternions, and octonions. See more examples in the paper by J. Baez [9].

In our work we consider Brownian motions on symmetric matrices of octonions. Due to the fact that octonions are nonassociative, and in contrast with the Clifford case, the dimension of the matrices plays a specific role. To study the spectrum of the matrices, we consider the processes on the characteristic polynomials  $P(X)$ , as introduced in the paper [19]. Because of the specific structure of octonions, the traditional way to compute the law of the spectrum turns out to be quite hard, while computation of diffusion operators on the process of  $P(X)$  provides a simpler and more efficient method to see things clearly. We provide two specific models on octonions, one is in dimension 2 (Section 8.2.1) and the other one in any dimension (Section 8.2.2), and study the diffusion of their spectrum (Proposition 8.2.1, Proposition 8.2.2). We prove that in both cases the spectrum of the matrix is a process itself and we describe the multiplicity of the eigenvalues and the invariant measures. In particular, our model in dimension 2 implies the distribution of spectrum of  $2 \times 2$  matrices on octonions with Gaussian entries, as mentioned in Forrester [53].

## 1.6 Matrix Dirichlet process

Let  $a = (a_0, a_1, a_2, \dots, a_d) \in \mathbb{R}_+^{d+1}$ , where  $a_i \in \mathbb{R}_+$ ,  $i = 0, 1, \dots, d$  are all positive. The Dirichlet distribution  $D(a_0, a_1, a_2, \dots, a_d)$  on the simplex  $\Delta_d = \{(x_i)_{i=0}^d \in \mathbb{R}_+^{d+1} : \sum_{i=0}^d x_i = 1\}$  is given by

$$\frac{1}{B_a} x_1^{a_1-1} \dots x_d^{a_d-1} (1 - x_1 - \dots - x_d)^{a_0-1} \mathbf{1}_{\Delta_d} (1 - \sum_{i=1}^d x_i, x_1, \dots, x_d) dx_1 \dots dx_d$$

where  $B_a = \frac{\Gamma(a_0) \dots \Gamma(a_d)}{\Gamma(a_0 + \dots + a_d)}$ . It is the multivariate generalization of the beta distribution, and plays an important role in statistics, such as prior distributions in Bayesian statistics, machine learning, natural language processing. It also has an application in population biology, for example the Wright-Fisher model.

The matrix Dirichlet measure, as an analogy of Dirichlet distribution, was first introduced by Gupta and Richards [57], as a special case of matrix Liouville measure. It has been deeply studied, by Olkin and Rubin [95], Gupta, Letac [68, 67] and the reference therein. In this thesis we introduce the matrix Dirichlet process whose invariant measure is matrix Dirichlet measure. On one hand the matrix Dirichlet measure is important in statistics, on the other hand it provides a model of multiple random matrices, which is related with orthogonal polynomials, integration formulas, etc. The problems such as the Dyson process of matrix Dirichlet process are worth to study. Moreover, it also reflects the geometry of spaces of matrices, see [60].

Our interest of this topic not only lies in its importances in statistics and random matrices, but also in the fact that it provides a polynomial model of multiple matrices. A detailed description of the polynomial models will be given in Section 2.5, but let us mention that in a polynomial

model, the diffusion operator can be diagonalized by the orthogonal polynomials, leading to the algebraic description of the boundary; Such polynomial models are quite rare: up to affine transformation, there are 3 models in  $\mathbb{R}$  [16] and 11 models on compact domains in  $\mathbb{R}^2$  [17]. More recently, Bakry and Bressaud [11] provided new models in dimension 2 and dimension 3, by investigating the finite groups of  $O(3)$  and their invariant polynomials. The difference between the paper [17] and the paper [11] is that in the first one, the polynomials are ranked with respect to their natural degree, whereas in the second, this hypothesis is relaxed, allowing for more examples. In fact, our construction of the diffusion operators of matrix Dirichlet process relies on the boundary equation of polynomial models introduced in [17].

It is worth to mention that in the PhD thesis of Y. Doumerc [43], in which he studied the matrix Jacobi process, which can be seen as a one matrix case of matrix Dirichlet process. The matrix Jacobi process is given by a stochastic differential equation. By Itô's SDE theory, one can prove the existence and uniqueness of the matrix Jacobi process.

In our work, we first provide a new point of view for understanding the classical results on the simplex, which comes from the images of the diffusion on the sphere; then we give the description of all the polynomial models on the simplex with Dirichlet distribution (Theorem 11.0.4), by using the arguments of the boundary equations in [17]; then we introduce the matrix Dirichlet process (Theorem 12.1.1), whose invariant measure is the matrix Dirichlet measure. Moreover, we give two interpretations for the matrix Dirichlet process: one is from the unitary matrices in the polar decomposition of complex matrices (Section 12.2), and the other one comes from the projection on  $SU(d)$  (or  $SO(d)$ ) (Section 12.3), which can be considered as a higher dimension analogy of the construction of the matrix Dirichlet process on the simplex.

**Part II**

**Basics and Examples**



# Chapter 2

## Basics on diffusion operators

In this section we give an introduction to the symmetric diffusion operators and provide some classical examples related to this thesis. For more details, we refer to the book by D.Bakry, I.Gentil and M.Ledoux [14].

### 2.1 Diffusion operators, integration by parts formula

Let  $\Omega$  be a smooth manifold, endowed with a  $\sigma$ -finite measure  $\mu$ . Let  $\mathcal{A}$  be the space of smooth compactly supported functions on  $\Omega$ . For any linear operator  $L : \mathcal{A} \mapsto \mathcal{A}$ , we define its carré du champ operator as

$$\Gamma(f, g) = \frac{1}{2} \left( L(fg) - fL(g) - gL(f) \right).$$

**Definition 2.1.1.** A symmetric diffusion operator is a linear operator  $L : \mathcal{A} \oplus 1 \mapsto \mathcal{A}$ , such that

1. Given a system of coordinates  $x = (x_1, \dots, x_n)$ , a smooth function  $f : \mathbb{R}^n \mapsto \mathbb{R}$ ,

$$L(f) = \sum_{i,j} \Gamma(x_i, x_j) \partial_{ij}^2 f(x) + \sum_i L(x_i) \partial_i f(x).$$

2.  $\forall f, g \in \mathcal{A} \oplus 1, \int fL(g) d\mu = \int gL(f) d\mu,$

3.  $\forall f \in \mathcal{A}, \Gamma(f, f) \geq 0,$

A direct consequence of the above definition leads to the change of variable formulas. For some smooth function  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f = (f_1, \dots, f_n)$ ,  $f_i \in \mathcal{A}$  for each  $i$ , we have

$$L(\Phi(f)) = \sum_i \partial_i \Phi(f) L(f_i) + \sum_{ij} \partial_{ij}^2 \Phi(f) \Gamma(f_i, f_j).$$

Another observation from the definition is the integration by parts formula, for any  $f, g \in \mathcal{A}$  we have

$$\int_{\Omega} \Gamma(f, g) d\mu = - \int_{\Omega} fLg d\mu.$$

If  $\mu(dx) = \rho(x)dx$  where  $\rho$  is a smooth positive function on  $\Omega$ ,  $dx$  is Lebesgue measure, then we can write

$$L(f) = \frac{1}{\rho} \sum_{ij} \partial_i (\Gamma(x_i, x_j) \rho \partial_j f).$$

which gives for each  $i$ ,

$$L(x_i) = \sum_j \Gamma(x_i, x_j) \partial_j \log \rho + \partial_j \Gamma(x_i, x_j).$$

Given  $L$  is an elliptic diffusion operator, it provides an easy way to compute the density  $\rho$  of the associated invariant measure.

Suppose  $\Omega$  is relatively compact and admits an piecewise smooth boundary  $\partial\Omega$ . For an arbitrary point  $x_0 \in \partial\Omega$ , consider a smooth function  $f$  which is compactly supported in  $B(x_0, r)$ , and a 1-form  $\omega_f$  given by

$$\omega_f = \sum_j g^{1j} f \rho dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_n,$$

where  $\widehat{dx_j}$  denotes the omission of  $dx_j$  at the  $j$ -th place. Then

$$d\omega_f = \rho(\Gamma(f, x_1) + fL(x_1))dx,$$

implying that  $\int_{\Omega} d\omega_f = 0$ . By Stokes formula, we have

$$\int_{\partial\Omega} \omega_f = \int_{\partial\Omega} \sum_j g^{1j} f \rho n_j dx = 0,$$

where  $n_j$  is the normal vector on the boundary. Since  $f$  is an arbitrary smooth function, we have on  $\partial\Omega \cap B(x_0, r)$

$$\sum_j g^{1j} n_j = 0.$$

If there exists smooth functions  $\{a_1, \dots, a_k\}$  such that for some smooth function  $G^{ij}$ ,  $B^i$

$$L(a_i) = B^i(a_1, \dots, a_k), \quad \Gamma(a_i, a_j) = G^{ij}(a_1, \dots, a_k)$$

Then we get

$$L(f(a)) = \sum_{ij} G^{ij} \partial_{ij}^2 f + \sum_k B^k \partial_k f$$

and we say  $\{a_1, \dots, a_k\}$  form a closed system, which enables us to find diffusions, as we will show later.

The spectral decomposition of the diffusion operator  $L$  is an efficient tool to analysis the associated semigroup  $P_t$ , especially when the spectrum is discrete. When the measure  $\mu$  is carried by an open subset of  $\mathbb{R}^d$ , then we may even expect to find those eigenvectors to be polynomials. Then, one is able to find an orthonormal basis of  $\mathcal{L}^2(\mu)$  formed with orthogonal polynomials which are eigenvectors of the operator  $L$ . In fact, let us assume that  $\mathcal{L}^2(\mu)$  admits an orthonormal basis  $\{e_n, n \in \mathbb{N}\}$  on a domain  $\Omega$  in  $\mathbb{R}^d$ , and  $L$  can be diagonalized by the basis such that

$$Le_n = -\lambda_n e_n.$$

Then we have an explicit formula of  $P_t$

$$P_t f(x) = \int f(y) p_t(x, y) d\mu(y),$$

where

$$p_t(x, y) = \sum_n e^{-\lambda_n t} e_n(x) e_n(y).$$

Although it is still needed to decide whether the series converges, only the existence of eigenvectors would provide more information on the diffusion models.

## 2.2 SDE related to diffusion operators

Consider a stochastic differential equation on  $\mathbb{R}^n$ :

$$\begin{aligned} dX_t &= \sigma(X_t)dB_t + b(X_t)dt \\ X_0 &= x \end{aligned}$$

where  $(B_t)_{t \geq 0}$  is the standard Brownian motion on  $\mathbb{R}^n$ ,  $\sigma$  and  $b$  are functions on  $\mathbb{R}^n$ . Let  $X_t$  be a solution to this stochastic differential equation. Then by Itô's formula, for a smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have

$$f(X_t) - f(x) = \int_0^t \partial_i f(X_s) \sigma_j^i(X_s) dB_s^j + \int_0^t \left( \frac{1}{2} \sum_k \sigma_k^i \sigma_k^j \partial_{ij} f(X_s) + b^i(X_s) \partial_i f \right) ds$$

Denote

$$\mathcal{L}f = \frac{1}{2} \sum_{ijk} \sigma_k^i \sigma_k^j \partial_{ij} f(X_s) + \sum_i b^i \partial_i f$$

Then

$$f(X_t) - f(x) = \int_0^t \partial_i f(X_s) \sigma_j^i(X_s) dB_s^j + \int_0^t \mathcal{L}f(X_s) ds$$

Since  $\int_0^t \partial_i f(X_s) \sigma_j^i(X_s) dB_s^j$  is a martingale, we have

$$\mathbb{E}f(X_t) = f(x) + \int_0^t \mathbb{E}(\mathcal{L}f(X_s)) ds$$

The semigroup operator  $P_t$  is defined as  $P_t f(x) = \mathbb{E}(f(X_t) | X_0 = x)$ , the above formula gives rise to

$$\partial_t P_t f = \mathcal{L}P_t f, \quad P_0(f)(x) = f(x)$$

On the other hand, from the above discussion, it is easy to see that given the stochastic differential equation 2.2.1, we can decide that

$$\begin{aligned} \Gamma(x_i, x_j) &= \frac{1}{2} \sum_k \sigma_k^i \sigma_k^j \\ \mathbf{L}(x_i) &= b^i \end{aligned}$$

## 2.3 Curvature-dimension inequality

We start with the definition of  $\Gamma_2$ , cf. [12]:

**Definition 2.3.1.** *The operator  $\Gamma_2$  is a bilinear map  $\mathcal{A}_0 \times \mathcal{A}_0 \rightarrow \mathcal{A}_0$  defined as:*

$$\Gamma_2(f, g) = \frac{1}{2} (\mathbf{L}\Gamma(f, g) - \Gamma(f, \mathbf{L}g) - \Gamma(\mathbf{L}g, f))$$

for all  $(f, g) \in \mathcal{A}_0 \times \mathcal{A}_0$ .

Here we use the same notation as in the previous section,  $\mathcal{A}_0$  is the space of all the smooth functions with compact support on  $\mathbb{R}^d$ . For simplicity we write  $\Gamma_2(f, f) = \Gamma_2(f)$ .

The formula of change of variable  $\Gamma_2$  is given by

$$\Gamma_2(\psi(f)) = \psi'(f)^2 \Gamma_2(f) + \psi'(f) \psi''(f) \Gamma(f, \Gamma(f)) + \psi''(f)^2 \Gamma(f)^2$$

where  $f \in \mathcal{A}_0$  and  $\psi \in C^2(\mathcal{A}_0, \mathcal{A}_0)$ .

**Definition 2.3.2.** We say that the operator  $L$  satisfies the curvature-dimension inequality  $CD(\rho, n)$ , with  $\rho \in \mathbb{R}$  and  $n \in [1, \infty]$ , if for all the function  $f \in \mathcal{A}_0$ ,

$$\Gamma_2(f) \geq \rho\Gamma(f) + \frac{1}{n}(Lf)^2$$

A general elliptic differential operator  $L$  on manifold with dimension  $d$  can be uniquely decomposed as  $L = \Delta + \nabla \log V \cdot \nabla$ , where  $\Delta$  is the Laplacian associated to the Riemannian metric  $g$ ,  $V = \rho - \sqrt{\det(g_{ij})}$  where  $\rho$  is the density of the invariant measure of  $L$ .  $\Gamma_2(f)$  can be decomposed to

$$\Gamma_2(f) = |\nabla^2 f|^2 + (\text{Ric}_g - \nabla^2 \log V)(\nabla f, \nabla f)$$

Then the curvature-dimension inequality  $CD(\rho, n)$  holds if and only if

$$\text{Ric}_g - \nabla^2 \log V \geq \rho g + \frac{1}{n-d} \log V \otimes \log V.$$

When  $L = \Delta$ , it is quite easy to get  $\Gamma_2(f) \geq \frac{1}{d}(\Delta f)^2$ ; On the other hand, when an operator satisfies  $CD(0, d)$ , we must have  $\log V = 0$ , such that this operator is exactly  $\Delta$ .

## 2.4 Examples

In this section we provide some classical examples of diffusion operators. For more details, see [10].

### 2.4.1 Brownian motion on the sphere

To give the explicit form of the diffusion operators of Brownian motion on the sphere  $\{\mathbb{S}^{d-1} : \sum_i^d x_i^2 = 1\}$ , we consider the sphere as a Riemannian manifold, and the upper(lower) half sphere has  $\mathbb{B}^{d-1} = \{x = (x_1, \dots, x_{d-1}), \sum_i x_i^2 \leq 1\}$  as its local chart, each point  $x \in \mathbb{B}^{d-1}$  correspond to  $(x, \sqrt{1 - |x|^2}) \in \mathbb{S}^{d-1}$ . Then the image of Euclidean metric of  $\mathbb{R}^d$  restricted on  $\mathbb{S}^{d-1}$  is

$$g_{\mathbb{S}^{d-1}} = \sum_{ij}^{d-1} \left( \delta_{ij} + \frac{x_i x_j}{1 - |x|^2} \right) dx_i dx_j$$

whose inverse is  $\Gamma(x_i, x_j) = \delta_{ij} - x_i x_j$ . And we also have  $L(x_i) = -(d-1)x_i$ . Then the Laplacian on the sphere is given by

$$\Delta_{\mathbb{S}^{d-1}} f(x) = \sum_{i,j \leq d-1} (\delta_{ij} - x_i x_j) \partial_i \partial_j f - (d-1) \sum_{i \leq d-1} x_i \partial_i f.$$

Consider the sphere as an embedded submanifold in  $\mathbb{R}^d$ , define the Euler operator  $V = \sum x_i \partial_i$ , then

$$\Gamma_{\mathbb{S}^d}(f, f) = \Gamma_{\mathbb{R}^d} f - (Vf)^2, \quad \Delta_{\mathbb{S}^d}(f) = \Delta_{\mathbb{R}^d} f - V^2 f - (d-2)Vf$$

In fact, if the diffusion process starts from the sphere  $\{\mathbb{S}^{d-1} : \sum_i^d x_i^2 = 1\}$ , we have

$$\Gamma(|x|^2, |x|^2) = 0, \quad L(|x|^2) = 0$$

which implies that for any smooth function  $L(f(|x|^2)) = 0$ , indicating that the diffusion process always stays on the sphere.

Now consider the projection  $\pi : \mathbb{S}^{d-1} \rightarrow \mathbb{B}^p$ , for  $p \leq d-1$ , i.e.  $\pi(x_1, \dots, x_{d-1}) = (x_1, \dots, x_p)$ . In this case the diffusion operators do not change, moreover, we may consider

$$\Gamma(x_i, x_j) = \delta_{ij} - x_i x_j, \quad L(x_i) = -(d-1)x_i,$$

Then the invariant measure is  $C(1 - \|x\|^2)^{\frac{d-p-2}{2}}$ . When  $p = d-1$ , it is  $C(1 - \|x\|^2)^{-\frac{1}{2}}$ , where  $1 - \|x\|^2 = 0$  is indeed the boundary of  $\mathbb{B}^{d-1}$ .

## 2.4.2 Brownian motion on $SO(d)$ and $SU(d)$

Most results in this part are adapted from Zribi [120].

The Lie group we consider here are all groups of matrices. Now let  $G$  be the  $n$  dimension Lie group with its Lie algebra  $\mathcal{G}$ , then on  $\mathcal{G}$  we have a bilinear Lie bracket  $[\cdot, \cdot]$ , which is antisymmetric  $[X, Y] = -[Y, X]$  and satisfies Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

On Lie algebra  $\mathcal{G}$ , we have adjoint action:  $ad(X)(Y) = [X, Y]$ . Then we introduce the Killing form on  $\mathcal{G}$ : For  $X, Y \in \mathcal{G}$ ,

$$\langle X, Y \rangle = \text{trace}(ad(X)ad(Y)).$$

If the Killing form is non-degenerated, we call  $\mathcal{G}$  a semi-simple Lie algebra.

For semi-simple Lie algebra  $\mathcal{G}$ , we can define its Casimir operator, which commutes with all the elements in Lie algebra  $\mathcal{G}$  and gives the concept of the Laplacian on semi-simple Lie group. More precisely, let  $\{A_i\}_{i=1 \dots n}$  be a basis of  $\mathcal{G}$ . Then define  $g_{ij} = \langle A_i, A_j \rangle$ , with  $g^{ij} = (g_{ij})^{-1}$ . The Casimir operator with respect to a representation  $\rho$  of  $\mathcal{G}$  is defined as

$$\Omega_\rho = \sum_{ij} g^{ij} \rho(A_i) \rho(A_j).$$

For any  $A \in \mathcal{G}$ , we have a right-invariant vector field  $X_A$  on  $G$ , which is given by

$$X_A(f)(z) = \frac{\partial}{\partial t} f(ze^{tA}) \Big|_{t=0}$$

Since  $X_{[A, B]} = [X_A, X_B]$  for  $A, B \in \mathcal{G}$ , the operation  $X : A \rightarrow X_A$  is a representation of  $\mathcal{G}$ . Then the Casimir operator can be written as  $\sum_i g^{ij} X_{A_i} X_{A_j}$ .

In this section we consider Lie group  $SO(d)$  and  $SU(d)$ . For  $SO(d)$  with  $d \geq 3$  and  $SU(d)$  with  $d \geq 2$ , they are all compact, semi-simple Lie groups. So we are able to define their Casimir operator, which has Haar measure as its invariant measures.

Now we consider  $SU(d)$  with its Lie algebra  $su(d) = \{X \in GL(d), X + X^* = 0, \text{trace}(X) = 0\}$ . The Killing form on  $su(d)$  is

$$\langle X, Y \rangle = 2d \text{trace}(XY^*) = -2d \text{trace}(XY).$$

Denote  $E_{ij}$  the matrix satisfying  $E_{ij,kl} = \delta_{ik} \delta_{jl}$ , and we define the following matrices:

$$\begin{aligned} R_{ij} &= E_{ij} - E_{ji} \\ S_{ij} &= i(E_{ij} + E_{ji}) \\ D_{ij} &= i(E_{ii} - E_{jj}) \end{aligned}$$

First it is easy to notice that

$$su(d) = \text{span}\{R_{ij}, S_{ij}, D_{ij}, i < j\};$$

then with the Killing form  $\langle X, Y \rangle = -2d \text{trace}(XY)$ , we have

$$\begin{aligned} \langle R_{ij}, R_{kl} \rangle &= 4d\delta_{ik}\delta_{jl}, & \langle S_{ij}, S_{kl} \rangle &= 4d\delta_{ik}\delta_{jl} & \langle D_{ij}, D_{kl} \rangle &= 2d(\delta_{ik} - \delta_{jk} - \delta_{il} + \delta_{jl}) \\ \langle R_{ij}, S_{kl} \rangle &= 0, & \langle R_{ij}, D_{kl} \rangle &= 0, & \langle S_{ij}, D_{kl} \rangle &= 0 \end{aligned}$$

In this setting, only  $\{D_{ij}\}$  are not orthogonal. To simplify the computation, notice that  $D_{ij}$  can be realized by  $\sqrt{2d}(x_i - x_j)$  where  $\{x_i\}$  are the coordinates on Euclid space  $\mathbb{R}^d$ , since

$$\langle x_i - x_j, x_k - x_l \rangle = \delta_{ik} - \delta_{jk} - \delta_{il} + \delta_{jl}.$$

To meet the restriction that  $\sum x_i = 0$ , define  $y_i = \sqrt{2d}(x_i - \frac{1}{d}\sum x_i)$ , then we have  $\Gamma_{\mathbb{R}^d}(y_i, y_j) = \frac{1}{2d}(\delta_{ij} - \frac{1}{d})$ . Therefore, by definition the Casimir operator given by  $\text{span}\{D_{ij}, i < j\}$  is

$$\sum_{ij} \frac{1}{2d}(\delta_{ij} - \frac{1}{d})y_i y_j = \frac{1}{2d^2} \sum_{i < j} (x_i - x_j)^2 = \frac{1}{2d^2} \sum_{i < j} X_{D_{ij}}^2$$

Therefore the Casimir operator can be written as

$$L_{SU(d)} = \frac{1}{4d} \sum_{i < j} (X_{R_{ij}}^2 + X_{S_{ij}}^2 + \frac{2}{d} X_{D_{ij}}^2), \quad (2.4.1)$$

whose its carre du champ operator is given by

$$\Gamma_{SU(d)}(f, g) = \frac{1}{4d} \left( \sum_{i < j} X_{R_{ij}}(f) X_{R_{ij}}(g) + X_{S_{ij}}(f) X_{S_{ij}}(g) + \frac{2}{d} X_{D_{ij}}(f) X_{D_{ij}}(g) \right). \quad (2.4.2)$$

To have the explicit formula of  $\Gamma$  and  $L$ , the direct way is to compute all the operators. However, since they are bi-invariant, it suffices to compute them at identity. First we have

**Lemma 2.4.1.** *At  $z = \text{Id}$ ,*

$$\begin{aligned} X_{R_{ij}} &= \partial_{z_{ij}} - \partial_{z_{ji}} + \partial_{\bar{z}_{ij}} - \partial_{\bar{z}_{ji}}, \\ X_{S_{ij}} &= i(\partial_{z_{ij}} + \partial_{z_{ji}} - \partial_{\bar{z}_{ij}} - \partial_{\bar{z}_{ji}}), \\ X_{D_{ij}} &= i(\partial_{z_{ii}} - \partial_{z_{jj}} - \partial_{\bar{z}_{ii}} + \partial_{\bar{z}_{jj}}). \end{aligned}$$

*Proof.* — The results is obtained directly by the formulae

$$\begin{aligned} X_A f(\text{Id}) &= \frac{\partial}{\partial t} f(ze^{tA}) \Big|_{t=0, z=\text{Id}} \\ &= \sum_{ij} \frac{\partial}{\partial m_{ij}} f(m)(zA)_{ij} \Big|_{z=\text{Id}} = \sum_{ij} \frac{\partial}{\partial m_{ij}} f(m) A_{ij}, \end{aligned}$$

■

Then at  $z = \text{Id}$ ,

$$\Gamma_{SU(d)}(z_{ij}, z_{kl}) = -\frac{1}{2d}\delta_{il}\delta_{kj} + \frac{1}{2d^2}\delta_{ij}\delta_{kl}.$$

since we are on  $SU(d)$ ,

$$\Gamma((zz^*)_{ij}, z_{kl}) = 0,$$

which leads to at  $z = \text{Id}$

$$\Gamma(\bar{z}_{ij}, z_{kl}) = -\Gamma(z_{ji}, z_{kl}) = \frac{1}{2d}\delta_{jl}\delta_{ik} - \frac{1}{2d^2}\delta_{ij}\delta_{kl}.$$

**Remarks 2.4.2.** From the above lemma, we can deduce that at identity,  $\Gamma_{SU(d)}(z_{ij}, z_{kl})$  are of the form  $A\delta_{il}\delta_{kj} + B\delta_{ik}\delta_{jl} + C\delta_{ij}\delta_{kl}$ . Of course by direct computation we are able to determine the constants  $A, B, C$ , but in which ways these constants characterize the property of  $\Gamma_{SU(d)}$ ? First we know that  $\Gamma(z_{ij}, z_{kl})(z) = \Gamma(z_{ip}, z_{kq})(\text{Id})z_{pj}z_{ql}$ , which implies  $B = 0$ ; Then by the fact that  $\det(z) = 1$ , we have  $\Gamma(\det(z), z_{ij}) = 0$ , and this gives rise to  $A + Cd = 0$ , so

$$\Gamma_{SU(d)}(z_{ij}, z_{kl}) = A(\delta_{il}\delta_{kj} - \frac{1}{d}\delta_{ij}\delta_{kl}),$$

and  $A$  is decided by the normalizing constant in Killing form.

Now we compute  $L$ . Take  $L(z)(\text{Id})$  as a matrix, since it is bi-invariant, we have  $L(z)(\text{Id}) = z^*L(z)(\text{Id})z$  for any  $z \in SU(d)$ . Then there exists a constant  $D$  such that  $L(z_{ij}) = D\delta_{ij}$  and  $L(\bar{z}_{ij}) = \bar{D}\delta_{ij}$ . Due to the fact that  $L((zz^*)_{ij}) = 0$ , we have

$$0 = \sum_p L(z_{ip}\bar{z}_{jp}) = \sum_p 2\Gamma(z_{ip}, \bar{z}_{jp}) + z_{ip}L(\bar{z}_{jp}) + L(z_{ip})\bar{z}_{jp}.$$

Then consider the equation at  $z = \text{Id}$ , we obtain  $Re(D) = -\frac{d^2-1}{d^2}$ . Notice that  $\det(\bar{z}) = 1$ , then at  $z = \text{Id}$ , and

$$\begin{aligned} 0 &= L(\log \det(\bar{z})) = \sum -z_{il}z_{kj}\Gamma(\bar{z}_{ij}, \bar{z}_{kl}) + z_{ij}L(\bar{z}_{ij}) \\ &= \frac{d^2-1}{2d} + d\bar{D}. \end{aligned}$$

Therefore,  $D = \bar{D} = -\frac{d^2-1}{2d^2}$ , i.e.  $L(z_{ij}) = -\frac{d^2-1}{2d^2}z_{ij}$ .

The above discussion leads to the following:

**Proposition 2.4.3.** At  $z \in SU(d)$ , we have the following formulae for Casimir operator:

$$\begin{aligned} \Gamma_{SU(d)}(z_{ij}, z_{kl}) &= -\frac{1}{2d}z_{il}z_{kj} + \frac{1}{2d^2}z_{ij}z_{kl} \\ \Gamma_{SU(d)}(z_{ij}, \bar{z}_{kl}) &= \frac{1}{2d}\delta_{ik}\delta_{jl} - \frac{1}{2d^2}z_{ij}\bar{z}_{kl} \\ L_{SU(d)}(z_{ij}) &= -\frac{d^2-1}{2d^2}z_{ij}, \quad L(\bar{z}_{ij}) = -\frac{d^2-1}{2d^2}\bar{z}_{ij} \end{aligned}$$

Now we consider  $SO(d)$ . It has Lie algebra  $so(d) = \{X \in GL(d), X + X^t = 0, \text{trace}(X) = 0\}$ , and Killing form  $\langle X, Y \rangle = -(d-2)\text{trace}(XY)$ , for  $d > 2$ . It is simpler than the  $SU(d)$  case, since  $so(d) = \text{span}\{R_{ij}, i < j\}$ .

Then by the same computation, we know that the Casimir operator can be written as

$$L_{SO(d)} = \frac{1}{2(d-2)} \sum_{ij} X_{R_{ij}}^2, \quad (2.4.3)$$

and its carré du champ operator

$$\Gamma_{SO(d)}(f, g) = \frac{1}{2(d-2)} \sum_{i < j} X_{R_{ij}}(f) X_{R_{ij}}(g) \quad (2.4.4)$$

which gives rise to

$$\Gamma_{SO(d)}(m_{ij}, m_{kl}) = \frac{1}{2(d-2)} (-\delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl})$$

By the same method, we yield at identity  $L_{SO(d)}(m_{ij}) = -\frac{d-1}{d-2}\delta_{ij}$ .

**Proposition 2.4.4.** *At  $m \in SO(d)$ , we have the following formulae for Casimir operator:*

$$\begin{aligned} \Gamma_{SO(d)}(m_{ij}, m_{kl}) &= \frac{1}{2(d-2)} (\delta_{ik}\delta_{jl} - m_{il}m_{kj}) \\ L_{SO(d)}(m_{ij}) &= -\frac{d-1}{2(d-2)} m_{ij} \end{aligned}$$

**Remarks 2.4.5.** *We can write Brownian motion on  $SU(d)$  and  $SO(d)$  as a stochastic differential equation. On  $SU(d)$ , we have*

$$dz_t = z_t dW_t - \frac{d^2-1}{d^2} z_t dt$$

where  $W + \bar{W}^t = 0$ , and for  $i \geq j$ ,

$$W_{ij} = \begin{cases} \sqrt{\frac{2}{d}}(X_{ij} + iY_{ij}), & i \neq j; \\ i\sqrt{\frac{2}{d}}(C_i - \frac{1}{d}\sum_k C_k), & i = j. \end{cases}$$

$\{X_{ij}\}$ ,  $\{Y_{ij}\}$  and  $\{C_i\}$  are all independent standard Brownian motion,  $X + X^t = 0$  and  $Y = Y^t$ . On  $SO(d)$ , we have

$$dm_t = m_t dV_t - \frac{d-1}{d-2} m_t dt.$$

where  $V + V^t = 0$ , and for  $i \geq j$ ,

$$V_{ij} = \begin{cases} \sqrt{\frac{2}{d-2}} B_{ij}, & i \neq j; \\ 0, & i = j. \end{cases}$$

where  $\{B_{ij}\}$  are all independent standard Brownian motion, satisfying  $B + B^t = 0$ .

**Remarks 2.4.6.** The classical result [88] by J. Milnor states that the Ricci curvature on compact Lie group is  $\frac{1}{4}$ . In fact, this result can also be proved by diffusion operators. Given a compact Lie group  $G$  of dimension  $d$  and its Lie algebra  $\mathcal{G}$ , let  $\{X_i\}$  be the basis of  $\mathcal{G}$  with respect to Killing form. Then the Casimir operator can be written as

$$Lf = \sum_i X_i^2 f$$

and

$$\begin{aligned}\Gamma(f, f) &= \sum_i X_i(f)^2 \\ \Gamma_2(f, f) &= \sum_{ij} (X_i X_j f)^2\end{aligned}$$

The last line comes from the fact that for the Casimir operator  $L$  we have  $[L, X_i] = 0$ , for any  $i$ . Then

$$\Gamma_2(f) \geq \frac{1}{d}(Lf)^2$$

which indicates that the Casimir operator is a Laplacian. By Bochner formula, we know that  $\Gamma_2(f, f) = |\text{Hess}f|^2 + \text{Ric}(f)$ .

Due to the fact that Killing form is bi-invariant, we have  $\nabla_Z \langle \cdot, \cdot \rangle = 0$  for any vector field  $Z$ . Then for any  $X_i, X_j$ , by the fact that  $\text{ad}_X$  is skew-symmetric, we have

$$\begin{aligned}2\langle \nabla_{X_i} X_j, Z \rangle &= \nabla_{X_i} \langle X_j, Z \rangle + \nabla_{X_j} \langle X_i, Z \rangle - \langle X_j, \nabla_{X_i} Z \rangle - \langle X_i, \nabla_{X_j} Z \rangle + \langle [X_i, X_j], Z \rangle \\ &= -\langle X_j, [X_i, Z] \rangle + \langle X_j, \nabla_Z X_i \rangle - \langle X_i, [X_j, Z] \rangle + \langle X_i, \nabla_Z X_j \rangle + \langle [X_i, X_j], Z \rangle \\ &= -\langle X_j, [X_i, Z] \rangle - \langle X_i, [X_j, Z] \rangle + \langle [X_i, X_j], Z \rangle \\ &= \langle [X_i, X_j], Z \rangle,\end{aligned}$$

implying that

$$\nabla_{X_i} X_j = \frac{1}{2}[X_i, X_j]. \quad (2.4.5)$$

Notice that

$$\text{Hess}f(X_i, X_j) = X_i X_j f - \nabla_{X_i} X_j f$$

with formula 2.4.5,

$$\nabla_{X_i} X_j f = -\nabla_{X_j} X_i f = \frac{1}{2}[X_i, X_j]f$$

we can write  $\text{Hess}f(X_i, X_j) = \frac{1}{2}(X_i X_j f + X_j X_i f)$ . Therefore,

$$\begin{aligned}\text{Ric}(f, f) &= \Gamma_2(f) - |\text{Hess}f|^2 \\ &= \sum_{(i,j)} (X_i X_j f)^2 + (X_j X_i f)^2 - \frac{1}{2}(X_i X_j f + X_j X_i f)^2 \\ &= \frac{1}{2} \sum_{(i,j)} ([X_i, X_j]f)^2\end{aligned}$$

To prove Milnor's theorem, we only need to prove that

$$\frac{1}{2} \sum_{(i,j)} ([X_i, X_j]f)^2 = \frac{1}{4} \sum_i (X_i f)^2$$

For any  $(i, j, k)$ , define  $c_{ij}^k$  as the structure constant

$$[X_i, X_j] = c_{ij}^k X_k$$

then

$$\sum_{(i,j)} ([X_i, X_j]f)^2 = \frac{1}{2} \sum_{i,j} \left( \sum_k c_{ij}^k X_k f \right)^2 = \frac{1}{2} \sum_{ijkl} c_{ij}^k c_{ij}^l X_k f X_l f$$

By the fact that  $\{X_i\}$  are the basis of the Lie algebra with respect to Killing form, we have

$$\sum_{kl} c_{jk}^l c_{il}^k = -\delta_{ij}$$

Also

$$\langle \text{ad}_{X_i} X_j, X_k \rangle = -\langle X_j, \text{ad}_{X_i} X_k \rangle$$

then  $c_{ij}^k = -c_{ik}^j$ , which implies that

$$\sum_{kl} c_{kl}^j c_{kl}^i = \delta_{ij}$$

Hence

$$\sum_{(i,j)} ([X_i, X_j]f)^2 = \frac{1}{2} \sum_{ijkl} c_{ij}^k c_{ij}^l X_k f X_l f = \frac{1}{2} \sum_i (X_i f)^2$$

which leads to the conclusion that  $\text{Ric}(f) = \frac{1}{4}\Gamma(f)$ , for any  $f \in \mathcal{A}_0$ . This ends the proof.

**Remarks 2.4.7.** Since compact Lie groups admit a positive constant curvature, Theorem 5.2.5 and Theorem 6.3.4 apply here.

## 2.5 Polynomial models

Given a diffusion operator  $L$  there are very few examples where we can give the explicit formula of the semigroup operator  $P_t$  generated by  $L$ . Recall that if one knows that  $L$  has countable basis of eigenvectors, then  $p_t$  can be expressed in terms of them. We will focus on the situation that those eigenvectors are polynomials.

First, we need to answer the fundamental question whether there exists an orthogonal polynomial basis in  $\mathcal{L}^2(\mu)$  to diagonalize  $L$ , we recall the main results on boundary equation in the paper by Bakry, Zani and Orekov [17]. We start with the following definition:

**Definition 2.5.1.** *Let  $\Omega$  be an open set in  $\mathbb{R}^d$  with piecewise smooth boundary,  $\mu$  be a probability measure on it with a positive density  $\rho$  with respect to Lebesgue measure.  $L$  is a symmetric diffusion operator in  $\Omega$ . The triple  $(\Omega, \Gamma, \mu)$  is called a polynomial model if  $L$  has polynomial eigenvectors which forms a basis in  $\mathcal{L}^2(\mu)$ .*

If we focus on the cases where  $\Omega$  is a bounded domain, Bakry-Orekov-Zani [17] have the following theorem

**Theorem 2.5.2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$ ,  $\mu(x) = \rho(x)dx$  is a probability measure on  $\Omega$ . Suppose that  $(\Omega, \Gamma, \mu)$  is a polynomial model. Then,*

1.  $\partial\Omega$  is an algebraic surface, satisfying the equation  $\{P(x) = 0\}$ , where  $P$  is a polynomial;
2.  $\Gamma(x_i, x_j)$  is a polynomial with  $\deg(\Gamma(x_i, x_j)) \leq 2$ ;
3. Denote  $\Gamma = (\Gamma(x_i, x_j))$ , the boundary equation  $P$  divides  $\det(\Gamma)$ ;
4. If  $P = P_1 \cdots P_k$ , where  $P_i$  are distinct irreducible polynomials, then for any  $i = 1, \dots, d$ , any  $r = 1, \dots, k$ , there exists a polynomial  $L_i^r$  with  $\deg(L_i^r) \leq 1$  such that

$$\Gamma(x_i, \log(P_r)) = L_i^r(x) \tag{2.5.6}$$

5. Moreover, the invariant measure can be written as  $\mu(dx) = C_{a_1, \dots, a_k} P_1^{a_1} \cdots P_k^{a_k} dx$ , where  $a_1, \dots, a_k$  are all real numbers and  $C_{a_1, \dots, a_k}$  is the normalizing constant.

Conversely, if  $\Omega$  is bounded and has an algebraic surface  $\{\partial\Omega = P(x) = 0\}$ , where  $P(x)$  can be written into distinct irreducible polynomials  $P = P_1 \cdots P_k$ ; suppose that there exists an elliptic  $\Gamma = (\Gamma(x_i, x_j))$ ,  $\Gamma(x_i, x_j)$  is a polynomial with  $\deg(\Gamma(x_i, x_j)) \leq 2$ , such that the boundary equation holds: for any  $i = 1, \dots, d$ , any  $r = 1, \dots, k$ , there exists a polynomial  $L_i^r$  with  $\deg(L_i^r) \leq 1$

$$\Gamma(x_i, \log(P_r)) = L_i^r(x)$$

then given a probability measure  $\mu(dx) = C_{a_1, \dots, a_k} P_1^{a_1} \cdots P_k^{a_k} dx$  for any real numbers  $a_1, \dots, a_k$  and normalizing constant  $C_{a_1, \dots, a_k}$ ,  $(\Omega, \Gamma, \mu)$  is a polynomial model.

The existence of a symmetric diffusion operator in a domain  $\Omega \subset \mathbb{R}^d$  having polynomial eigenvectors is indeed a very strong constraint condition on the domain  $\Omega$  itself. In Bakry, Orekov and Zani [17], they study the polynomial models on  $\mathbb{R}^2$  and find that up to affine transformation, there are 11 models on compact domains in  $\mathbb{R}^2$ . For a compact domain with regular boundary, the analysis of  $L$  with polynomial eigenvectors leads to the algebraic description of the boundary.

### 2.5.1 Polynomial models in dimension 1

In the paper [16] by Bakry and Mazet, the authors give a complete description of such polynomial model on  $\mathbb{R}$ . More precisely, let  $\mu$  be a probability measure which is absolutely continuous with respect to Lebesgue measure on an open interval  $I$  of  $\mathbb{R}$ , and  $\mu(x) = \rho(x)dx$ . Let  $\{e_n, n \in \mathbb{N}\}$  be a family of orthogonal polynomials in  $\mathbb{R}$ . Assume there exists some elliptic diffusion operator  $L$  on  $I$  which has  $\mu$  as its invariant measure and moreover,

$$Le_n = -\lambda_n e_n.$$

Then up to affine transformation, there are only 3 cases:

1. The Ornstein-Uhlenbeck operator on  $I = \mathbb{R}$

$$H = \frac{d^2}{dx^2} - x \frac{d}{dx},$$

the associated measure is Gaussian measure  $\mu(dx) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$ .  $\{e_n\}$  are the Hermite polynomials and  $\lambda_n = n$ .

2. The Laguerre operator on  $I = \mathbb{R}^+$

$$L_a = x \frac{d^2}{dx^2} + (a - x) \frac{d}{dx}, \quad a > 0,$$

the associated measure is gamma distribution  $\mu_a(dx) = C_a x^{a-1} e^{-x} dx$ .  $\{e_n\}$  are the Laguerre polynomials and  $\lambda_n = n$ .

3. The Jacobi operator on  $I = (-1, 1)$

$$J_{a,b} = (1 - x^2) \frac{d^2}{dx^2} - (a - b + (a + b)x) \frac{d}{dx}, \quad a, b > 0$$

the associated measure is  $\mu_{a,b}(dx) = C_{a,b} (1 - x)^{a-1} (1 + x)^{b-1} dx$ .  $\{e_n\}$  are the Jacobi polynomials and  $\lambda_n = n(n + a + b - 1)$ .

In fact, the Ornstein-Uhlenbeck case and the Laguerre operator case can be realized as the limits of Jacobi case, by choosing the values of  $a$  and  $b$  and scale the space variable  $x$ .

It is also worth to mention that when  $d$  and  $p, q$  are integers, the Laguerre operator  $L_d$  can be seen as the Ornstein-Uhlenbeck operator in dimension  $d$ , and the Jacobi operator  $J_{\frac{p}{2}, \frac{q}{2}}$  can be considered as an image of the Laplace operator on the sphere. More precisely, consider Ornstein-Uhlenbeck operator in dimension  $d$   $H_d = \Delta_d - x \cdot \nabla$ , and its associated process  $X_t = (X_1, \dots, X_d)$ , and define  $R(X) = \frac{1}{2} \sum_i^d X_i^2$ . Then for any smooth function  $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ , we have

$$H_d(F(R)) = 2R \partial_R^2 F(R) + 2 \left( \frac{d}{2} - \frac{R}{2} \right) \partial_R F(R) = 2L_{\frac{d}{2}} F(R),$$

which indicates that if  $X_t$  is a  $d$ -dimensional Ornstein-Uhlenbeck process, then  $|X_t|^2$  is a Laguerre process with parameter  $d$ .

Similarly, consider the Laplace operator  $\Delta_{\mathbb{S}^{p+q-1}}$  on  $\mathbb{S}^{p+q-1} \in \mathbb{R}^{p+q}$ , for integers  $p, q$ . Let  $R_p = X_1^2 + \dots + X_p^2$ ,  $Y_p = 2R_p - 1$ , and let  $f$  be a smooth function on  $[-1, 1]$ . Then

$$\begin{aligned} \Delta_{\mathbb{S}^{p+q-1}} f(Y_p) &= 4(1 - Y_p^2) \partial_{Y_p}^2 f - (2q - 2p + 2(p + q)Y_p) \partial_{Y_p} f \\ &= 4J_{\frac{q}{2}, \frac{p}{2}} f(Y_p). \end{aligned}$$

This interpretation comes from Zernike and Brinkman [30] and Braaksma and Meulenbeld [28], see also Koornwinder [65]. This notion of image processes and generators will play a major role in our study of matrix Dirichlet processes, and this why we show here on a simple example how it works.



## Chapter 3

# Projections on sub matrices: real and complex case

Starting from  $SO(d)$  and  $SU(d)$ , we may extract the first  $p$  lines and the first  $q$  columns from the orthogonal or unitary matrix, which yields a sub-matrix that is not orthogonal or unitary any more. It is known that for a Haar distributed matrix on  $U(d)$ (resp.  $O(d)$ ) and fixed  $p = q$ , the sub-matrix converges in law, up to a renormalization constant, to a complex(resp. real) Gaussian matrix, as the dimension  $d$  tends to infinity. In his thesis [41], B. Collins considered the Haar measure on the unitary group and gave an explicit formula for the projected Haar measure on the sub-matrix. Later Y. Doumerc [43] recovered Collins' result by studying matrix Jacobi processes(which we will discuss in the next section) and its invariant measure.

In this section, by using diffusion operators inherited from the Casimir operator on  $SO(d)$  and  $SU(d)$ , we study the invariant measure of diffusion process on sub-matrices, which are expected as the same as the result of Collins [41] and Doumerc [43]. The sub-matrix inherits the same  $\Gamma$  and  $L$  from  $SO(d)$ (or  $SU(d)$ ), but does not satisfy the orthogonal(unitary)restriction  $mm^t = \text{Id}(zz^* = \text{Id})$ , which is the reason for all the difference.

For simplicity, in the rest of this thesis, we will use the normalized diffusion operators for Brownian motion on  $SO(d)$  and  $SU(d)$ . More precisely, for  $m \in SO(d)$ ,

$$\begin{aligned}\Gamma_{SO(d)}(m_{ij}, m_{kl}) &= \delta_{ik}\delta_{jl} - m_{il}m_{kj}, \\ L_{SO(d)}(m_{ij}) &= -(d-1)m_{ij}.\end{aligned}$$

For  $z \in SU(d)$ ,

$$\begin{aligned}\Gamma_{SU(d)}(z_{ij}, z_{kl}) &= -dz_{il}z_{kj} + z_{ij}z_{kl}, \\ \Gamma_{SU(d)}(z_{ij}, \bar{z}_{kl}) &= d\delta_{ik}\delta_{jl} - z_{ij}\bar{z}_{kl}, \\ L_{SU(d)}(z_{ij}) &= -(d^2-1)z_{ij}, \quad L(\bar{z}_{ij}) = -(d^2-1)\bar{z}_{ij}.\end{aligned}$$

For extracted matrix  $v$  from  $SO(d)$ , the diffusion process lives on the domain  $\{vv^t \leq \text{Id}\}$ . If  $\rho_v$  is the density of its invariant measure, then

$$\Gamma(\log \rho_v, v_{ij}) = (p + q - d + 1)v_{ij},$$

and

$$\Gamma(\log \det(\text{Id} - vv^t), v_{ij}) = -2v_{ij}.$$

Therefore if it exists, the invariant measure should be of the form  $C \det(\text{Id} - vv^t)^{\frac{d-p-q-1}{2}}$ . Notice the measure only makes sense on  $\{vv^t \leq \text{Id}\}$  when  $p+q < d+1$ , that is  $p+q \leq d$ .

The  $SU(d)$  case. Similarly, denote the extracted matrix by  $w$  and the density of its invariant  $\rho_w$ , then

$$\Gamma(\log \rho_w, w_{ij}) = d(p+q-d)w_{ij},$$

and

$$\Gamma(\log \det(\text{Id} - ww^*), w_{ij}) = -dw_{ij}.$$

Then the invariant measure should be of the form  $C \det(\text{Id} - ww^*)^{d-p-q}$ . On the domain  $\{ww^t \leq \text{Id}\}$ , this density exists when  $p+q \leq d$ , which is the same condition as that in the  $SO(d)$  case.

**Remarks 3.0.1.** In Hua's book [60], he studied the harmonic analysis of the classical domain  $\{zz^* < I_m\}$ , where  $z$  is a  $m \times n$  complex matrix and gave the kernel function on this domain, which is  $C \det(\text{Id} - ww^*)^{-m-n}$ .

**Remarks 3.0.2.** Notice that in both  $SU(d)$  and  $SO(d)$  case, when  $p=1$  or  $q=1$ , the extracted line or column is just a process on the sphere. However, here the complex spherical process can not be viewed as a projection of  $2d$  real spherical process. In fact, consider the extracted column in  $SO(2d)$ , let  $v_i = m_{i1}$

$$\begin{aligned} \Gamma_{SO(2d)}(v_i, v_j) &= \frac{1}{2d-2}(\delta_{ij} - v_i v_j), \\ L_{SO(2d)}(v_i) &= -\frac{2d-1}{2d-2}v_i. \end{aligned}$$

It is exactly the same as diffusion operator of Brownian motion on the sphere up to a constant.

On the other hand, consider  $w_i = z_{i1}$ ,

$$\begin{aligned} \Gamma_{SU(d)}(w_i, w_j) &= \frac{1-d}{d^2}w_i w_j, \\ \Gamma_{SU(d)}(w_i, \bar{w}_j) &= \frac{1}{d}\delta_{ij} - \frac{1}{d^2}w_i \bar{w}_j. \end{aligned}$$

write  $w_i = x_i + iy_i$ , then

$$\begin{aligned} \Gamma(x_i, x_j) &= \frac{1}{2d}\delta_{ij} - \frac{1}{2d}x_i x_j - \frac{2-d}{2d^2}y_i y_j, \\ \Gamma(y_i, y_j) &= \frac{1}{2d}\delta_{ij} - \frac{1}{2d}y_i y_j - \frac{2-d}{2d^2}x_i x_j, \\ \Gamma(x_i, y_j) &= -\frac{1}{2d}x_i y_j + \frac{2-d}{2d^2}x_j y_i. \end{aligned}$$

It is quite obvious that the extracted column of  $SU(d)$  does not come from that of  $SO(2d)$ , nor from the projection of real sphere. In fact, it comes from a polynomial model on the sphere. Let  $g_0^{ij} = \delta_{ij} - x_i x_j$  be the projected Euclidean metric from  $\mathbb{R}^{2d}$  on the sphere  $\mathbb{S}^{2d-1}$ . For any smooth function  $f$  on  $\mathbb{S}^{2d-1}$ , define  $\Gamma$  on  $\mathbb{S}^{2d-1} = \{(x_1, \dots, x_d, y_1, \dots, y_d), \sum_i^d x_i^2 + y_i^2 = 1\}$  as

$$\begin{aligned} & \sum \Gamma(x_i, x_j) \partial_{x_i} \partial_{x_j} f + \Gamma(y_i, y_j) \partial_{y_i} \partial_{y_j} f + \Gamma(x_i, y_j) \partial_{x_i} \partial_{y_j} f \\ = & \frac{1}{2d} \sum_{ij} [g_0^{ij} \partial_i \partial_j f + \frac{2-d}{2d} ((x_i \partial_{y_i} f - x_j \partial_{y_j} f)^2 + (y_i \partial_{x_i} f - y_j \partial_{x_j} f)^2 - 2(x_i \partial_{y_i} f - y_j \partial_{x_j} f)^2)], \end{aligned}$$

since we have

$$\begin{aligned} \sum (x_i \partial_{y_i} - y_j \partial_{x_j})^2 f &= d \sum_i x_i^2 \partial_{y_i}^2 f + d \sum_i y_i^2 \partial_{x_i}^2 f - 2 \sum y_i x_j \partial_{x_i} \partial_{y_j} f, \\ \sum (x_i \partial_{y_i} - x_j \partial_{y_j})^2 f &= 2d \sum_i x_i^2 \partial_{y_i}^2 f - 2 \sum x_i x_j \partial_{y_i} \partial_{y_j} f, \\ \sum (y_i \partial_{x_i} - y_j \partial_{x_j})^2 f &= 2d \sum_i y_i^2 \partial_{x_i}^2 f - 2 \sum y_i y_j \partial_{x_i} \partial_{x_j} f. \end{aligned}$$

which indicates that the defined  $\Gamma$  is exactly the one on  $SU(d)$ .



## Chapter 4

# Matrix Jacobi process

Consider a Brownian motion on the sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ , then for integer  $p \leq n$ , one can prove that the coordinates  $\{x^i\}_{i=1,\dots,p}$  is a diffusion process and so is  $J = \sum_i^p \|x^i\|^2$ , which is Jacobi process. In his paper [43], Y. Doumerc discussed matrix Jacobi process as follows: consider Brownian motion on  $SO(d)$ , let  $X$  be the extracted sub-matrix of size  $p \times q$ , then  $J = XX^*$  is the matrix Jacobi process. He also discussed Hermitian Jacobi process which comes from unitary group  $SU(d)$  instead of  $SO(d)$ .

In this section, we try to deal with matrix Jacobi process using diffusion operators on both  $SO(d)$  and  $SU(d)$  cases.

### 4.1 The $SO(d)$ case

For  $m \in SO(d)$ , split  $d = p + q$  and for  $m < d$ , let  $m_1$  be the extracted  $m \times p$  matrix,  $m_2$  be the matrix of size  $m \times q$  which is in the same lines as  $m_1$ . Then we have

$$m_1 m_1^t + m_2 m_2^t = \text{Id}_{m \times m}$$

Now write  $M = m_1 m_1^t$ , then  $M$  is a symmetric matrix satisfying  $0 \leq M \leq \text{Id}$ . We can derive the metric on  $M$  from that on  $SO(d)$  as follows:

$$\begin{aligned}\Gamma(M_{ij}, M_{kl}) &= \delta_{ik} M_{jl} + \delta_{il} M_{jk} + \delta_{jk} M_{il} + \delta_{jl} M_{ik} - 2M_{ik} M_{jl} - 2M_{il} M_{jk}, \\ \text{L}(M_{ij}) &= -2dM_{ij} + 2p\delta_{ij}.\end{aligned}$$

Assume  $\rho_1$  to be the density of the invariant measure of  $M$ . Then it should satisfy

$$\Gamma(\log \rho_1, M_{ij}) = 2(2(m+1) - d)M_{ij} + 2(p - m - 1)\delta_{ij}.$$

Notice that

$$\begin{aligned}\Gamma(\log \det(M), M_{ij}) &= 4\delta_{ij} - 4M_{ij} \\ \Gamma(\log \det(\text{Id} - M), M_{ij}) &= -4M_{ij}\end{aligned}$$

Thus if the invariant measure exists, it should be of the following form

$$\rho_1 = C \det(M)^{\frac{p-m-1}{2}} \det(\text{Id} - M)^{\frac{d-p-m-1}{2}}$$

Since  $0 \leq M \leq \text{Id}$ , to make the invariant measure exist we should have  $m - 1 < p < d - m + 1$ , from which we deduce that  $m < \frac{d}{2} + 1$ .

**Remarks 4.1.1.** *In his thesis, Doumerc gives the stochastic differential equation for matrix Jacobi process. Consider  $X$  as the extracted matrix of dimension  $m \times p$  from  $SO(d)$ , then the matrix Jacobi process  $J = XX^t$  satisfying the following:*

$$dJ = \sqrt{J} dB \sqrt{\text{Id} - J} + \sqrt{\text{Id} - J} dB^t \sqrt{J} + (p\text{Id} - dJ)dt$$

where  $B$  is a Brownian motion on real matrix of dimension  $m \times m$ .

## 4.2 The $SU(d)$ case

Similar to the  $SO(d)$  case, let  $v$  be the extracted  $m \times p$  matrix from the unitary matrix  $z$  and  $V = vv^*$ , then  $V$  is a  $m \times m$  Hermitian matrix.

$$\begin{aligned} \Gamma(V_{ij}, V_{kl}) &= d\delta_{il}V_{kj} + d\delta_{kj}V_{il} - 2dV_{kj}V_{il}, \\ \text{L}(V_{ij}) &= -2d^2V_{ij} + 2dp\delta_{ij}. \end{aligned}$$

Also notice that

$$\Gamma(\log \rho, V_{ij}) = -2d^2V_{ij} + 2dp\delta_{ij} - 2dm(\delta_{ij} - 2V_{ij}) = 2d(p - m)\delta_{ij} + 2d(2m - d)V_{ij},$$

and

$$\begin{aligned} \Gamma(\log \det V, V_{ij}) &= 2d(\delta_{ij} - V_{ij}), \\ \Gamma(\log \det(\text{Id} - V), V_{ij}) &= -2dV_{ij}. \end{aligned}$$

So let  $\rho_2$  be the density of the invariant measure of  $V$ , if it exists, it should be of the form

$$C \det(V)^{p-m} \det(\text{Id} - V)^{d-p-m}.$$

Since  $0 \leq V \leq \text{Id}$ , the condition of existence is  $m - 1 < p < d - m + 1$ , which is again the same as that in the  $SO(d)$  case.

**Remarks 4.2.1.** *We now give a very brief discussion on the integral of the invariant measure of the matrix Jacobi process. For further information, we refer the readers to the book by Gupta and Nagar [56].*

Define  $\beta_d(a, b)$  as the following integral on symmetric matrices of dimension  $d \times d$

$$\beta_d(a, b) = \int_{0 < A < \text{Id}} \det(A)^{a - \frac{1}{2}(d+1)} \det(\text{Id} - V)^{b - \frac{1}{2}(d+1)} dA,$$

where  $a, b$  are two constants  $a > \frac{1}{2}(p - 1)$ ,  $b > \frac{1}{2}(p - 1)$ .  $\beta_d(a, b)$  is called multivariate beta function. In fact, it can be explicitly computed out by the multivariate gamma function, which is defined by

$$\Gamma_d(a) = \int_{A > 0} e^{-\text{trace } A} \det(A)^{a - \frac{1}{2}(d+1)} dA$$

and we have  $\Gamma_d(a) = \pi^{\frac{1}{4}d(d-1)} \prod_{i=1}^d \Gamma(a - \frac{1}{2}(i-1))$  (For detail of the proof, see the book by Gupta, Nagar [56]).

Just as in the classical case, we have

$$\beta_d(a, b) = \frac{\Gamma_d(a)\Gamma_d(b)}{\Gamma_d(a+b)}$$

In fact, let  $W = A + B$  and  $V = W^{-\frac{1}{2}}BW^{-\frac{1}{2}}$ , then making the transformation  $A \rightarrow W$ ,  $B \rightarrow V$  (where the Jacobian is  $\det(W)^{\frac{1}{2}(d+1)}$ ), we get

$$\begin{aligned} & \Gamma_d(a)\Gamma_d(b) \\ &= \int_{A>0} e^{-\text{trace } A} \det(A)^{a-\frac{1}{2}(p+1)} dA \int_{B>0} e^{-\text{trace } B} \det(B)^{b-\frac{1}{2}(p+1)} dB \\ &= \int_{A>0} \int_{B>0} e^{-\text{trace } W} \det(W)^{a+b-\frac{1}{2}(p+1)} \det(\text{Id} - V)^{a-\frac{1}{2}(p+1)} \det(V)^{b-\frac{1}{2}(p+1)} dW dV \\ &= \Gamma_d(a+b)\beta_d(a, b) \end{aligned}$$

Similarly, we may also define the above integrals on Hermitian matrices:

$$\begin{aligned} \tilde{\Gamma}_d(\alpha) &= \int_{A>0} e^{-\text{trace } A} \det(A)^{\alpha-d} dA \\ \tilde{\beta}_d(\alpha, \beta) &= \int_{0<A<\text{Id}} \det(A)^{\alpha-d} \det(\text{Id} - V)^{\beta-d} dA \end{aligned}$$

where  $\alpha > d-1$ ,  $\beta > d-1$ . Here we also have

$$\tilde{\beta}_d(\alpha, \beta) = \frac{\tilde{\Gamma}_d(\alpha)\tilde{\Gamma}_d(\beta)}{\tilde{\beta}_d(\alpha, \beta)}$$

and  $\tilde{\Gamma}_d(\alpha) = \pi^{\frac{1}{2}d(d-1)} \prod_{i=1}^d \Gamma(a - (i-1))$ . For a more detailed and general discussion on the complex case, see A.Mathai, S.Provost [84].

### 4.3 Spectrum of matrix Jacobi process

Now we give the description of the spectrum of matrix Jacobi process, again in both  $SU(d)$  and  $SO(d)$  cases. We use the same notations as those in the previous section.

**Proposition 4.3.1.** *Let  $v$  be the extracted sub-unitary matrix of dimension  $m \times p$  on  $SU(d)$ , and  $V = vv^*$ .  $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$  are the eigenvalues of  $V$ . Then  $\{\lambda_i\}_{i=1}^m$  is a diffusion process, satisfying*

$$\begin{aligned} \Gamma(\lambda_i, \lambda_j) &= 2d\lambda_i(1-\lambda_i)\delta_{ij}, \\ L(\lambda_i) &= -4d \sum_{j \neq i} \frac{\lambda_i^2 - \lambda_j}{\lambda_i - \lambda_j} - 2d(d-2m+2)\lambda_i - 2d(m-p-1), \end{aligned}$$

with its invariant measure

$$\rho = C \prod_{(k,j), j \neq k} |\lambda_k - \lambda_j|^2 \left( \prod_j \lambda_j \right)^{p-m} \left( \prod_j (1-\lambda_j) \right)^{d-p-m}.$$

*Proof.* — Let  $P(X) = \det(X\text{Id} - V)$  be the characteristic function of  $V$ . We have

$$\begin{aligned} & \Gamma(\log P(X), \log P(Y)) \\ &= 2d(\text{trace}((X\text{Id} - V)^{-1}V(Y\text{Id} - V)^{-1}) - \text{trace}((X\text{Id} - V)^{-1}V(Y\text{Id} - V)^{-1}V)) \\ &= 2d\left(\frac{X(1-X)P'(X)}{Y-X} \frac{P'(X)}{P(X)} - \frac{Y(1-Y)P'(Y)}{Y-X} \frac{P'(Y)}{P(Y)} - m\right). \end{aligned}$$

Therefore

$$\sum_{i,j=1}^m \frac{\Gamma(\lambda_i, \lambda_j)}{(X - \lambda_i)(Y - \lambda_j)} = 2d \sum_i^m \frac{-\lambda_i^2 + \lambda_i}{(X - \lambda_i)(Y - \lambda_i)}.$$

Hence

$$\Gamma(\lambda_i, \lambda_j) = 2d\lambda_i(1 - \lambda_i)\delta_{ij},$$

$$\begin{aligned} \mathbf{L}(\log P(X)) &= 2d[X(X-1)\frac{P'^2}{P^2} + (m-p + (d-2m)X)\frac{P'}{P} + m(m-d)] \\ &= 2d\left[\sum_i \frac{\lambda_i^2 - \lambda_i}{(X - \lambda_i)^2} + 2\sum_i \frac{\lambda_i^2 - \lambda_i}{X - \lambda_i} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} + (d-2m+2)\sum_i \frac{\lambda_i}{X - \lambda_i} \right. \\ &\quad \left. + \sum_i \frac{m-p-1}{X - \lambda_i}\right], \\ \mathbf{L}(P(X)) &= P(X)(\Gamma(\log P(X), \log P(X)) + \mathbf{L}(\log P(X))) \\ &= 2dX(X-1)P'' + 2d(m-p-1 + (d-2m+2)X)P' + 2dm(m-d-1)P. \end{aligned}$$

Therefore

$$\mathbf{L}(\lambda_i) = -4d \sum_{j \neq i} \frac{\lambda_i^2 - \lambda_i}{\lambda_i - \lambda_j} - 2d(d-2m+2)\lambda_i - 2d(m-p-1).$$

Let  $\rho$  be the density of the invariant measure. Then

$$\Gamma(\log \rho, \lambda_i) = 4d \sum_{j \neq i} \frac{\lambda_i - \lambda_i^2}{\lambda_i - \lambda_j} + 2d(2m-d)\lambda_i - 2d(m-p).$$

Notice that

$$\begin{aligned} \Gamma(\log(\prod_{(k,j), j \neq k} (\lambda_k - \lambda_j)^2), \lambda_i) &= \sum_{j \neq i} \frac{4d\lambda_i(1 - \lambda_i)}{\lambda_i - \lambda_j}, \\ \Gamma(\log(\prod_j \lambda_j), \lambda_i) &= 2d(1 - \lambda_i), \\ \Gamma(\log \prod_j (1 - \lambda_j), \lambda_i) &= -2d\lambda_i. \end{aligned}$$

Therefore the measure is of the form  $C \prod_{(k,j), j \neq k} |\lambda_k - \lambda_j|^2 (\prod_j \lambda_j)^{p-m} (\prod_j (1 - \lambda_j))^{d-p-m}$ . ■

**Proposition 4.3.2.** *Let  $m_1$  be the extracted sub-unitary matrix of dimension  $m \times p$  on  $SO(d)$ , and  $M = m_1 m_1^t$ .  $\{\eta_1, \eta_2, \dots, \eta_m\}$  are the eigenvalues of  $M$ . Then  $\{\eta_i\}_{i=1}^m$  is a diffusion process, satisfying*

$$\begin{aligned}\Gamma(\eta_i, \eta_j) &= 4\eta_i(1 - \eta_i)\delta_{ij}, \\ \mathsf{L}(\eta_i) &= -4 \sum_{j \neq i} \frac{\eta_i^2 - \eta_i}{\eta_i - \eta_j} - 2(d - 2m - 2)\eta_i - 2(m - p + 1),\end{aligned}$$

with its invariant measure

$$\rho = C \prod_{(k,j), j \neq k} |\eta_k - \eta_j| \left( \prod_j \eta_j \right)^{-\frac{m+3-p}{2}} \left( \prod_j (1 - \eta_j) \right)^{\frac{d+p-3(m+3)}{2}}.$$

*Proof.* — Let  $P(X) = \det(X\text{Id} - M)$  be the characteristic function of  $M$ . We have

$$\begin{aligned}\Gamma(\log P(X), \log P(Y)) &= 4 \left( \frac{X(1-X)}{Y-X} \frac{P'(X)}{P(X)} - \frac{Y(1-Y)}{Y-X} \frac{P'(Y)}{P(Y)} - m \right), \\ \mathsf{L}(\log P(X)) &= 2 \left[ X(1-X) \frac{P''}{P} + 2X(X-1) \frac{P'^2}{P^2} + (m-p+1 + (d-2m-2)X) \frac{P'}{P} \right. \\ &\quad \left. + m(m-d+1) \right].\end{aligned}$$

Hence

$$\mathsf{L}(P(X)) = 2(m-p-1 + (d-2m+2)X)P' + 2m(m-d-1)P + 2X(X-1)P'',$$

and

$$\begin{aligned}\Gamma(\eta_i, \eta_j) &= 4\eta_i(1 - \eta_i)\delta_{ij}, \\ \mathsf{L}(\eta_i) &= -4 \sum_{j \neq i} \frac{\eta_i^2 - \eta_i}{\eta_i - \eta_j} - 2(d - 2m - 2)\eta_i - 2(m - p + 1).\end{aligned}$$

Moreover, Let  $\rho$  be the density of the invariant measure. Then

$$\Gamma(\log \rho, \eta_i) = 4 \sum_{j \neq i} \frac{\eta_i - \eta_i^2}{\eta_i - \eta_j} + 2(2m - d + 6)\eta_i - 2(m - p + 3).$$

Notice that

$$\begin{aligned}\Gamma(\log \prod_{(k,j), j \neq k} |\eta_k - \eta_j|^2, \eta_i) &= \sum_{j \neq i} \frac{8\eta_i(1 - \eta_i)}{\eta_i - \eta_j}, \\ \Gamma(\log \left( \prod_j \eta_j \right), \eta_i) &= 4(1 - \eta_i), \\ \Gamma(\log \left( \prod_j (1 - \eta_j) \right), \eta_i) &= -4\eta_i.\end{aligned}$$

Therefore the measure is  $C \prod_{(k,j), j \neq k} |\eta_k - \eta_j| \left( \prod_j \eta_j \right)^{-\frac{m+3-p}{2}} \left( \prod_j (1 - \eta_j) \right)^{\frac{d+p-3(m+3)}{2}}$ . ■

**Remarks 4.3.3.** Notice that for the matrix Jacobi process  $M$  on  $SO(d)$ , let  $P(X) = \det(X\text{Id} - M)$ , we have

$$L(P(X)) = 2X(X-1)P''(X) + 2(m-p-1 + (d-2m+2)X)P'(X) + 2m(m-d-1)P(X),$$

while the matrix Jacobi process  $V$  on  $SU(d)$ ,  $Q(X) = \det(X\text{Id} - V)$ ,

$$L(Q(X)) = 2dX(X-1)Q''(X) + 2d(m-p-1 + (d-2m+2)X)Q'(X) + 2dm(m-d-1)Q(X).$$

In both cases, notice that  $L(P)$  and  $L(Q)$  are almost the same up to a constant. Recall that in order to let their invariant measures make sense, we should have

$$m-1 < p < d-m+1,$$

which implies  $m < \frac{d}{2} + 1$ . In fact the restriction  $m \leq p$  ensures that  $\det(V) \neq 0$ .

Take  $L(P(X))$  as an example. When  $d = p + m - 1$ , we have the following:

$$L(P(X)) = 2X(X-1)P'' + 2(p-m+1)(X-1)P' - 2mpP.$$

Therefore on  $\{P(1) = 0\}$ , we have  $L(P(1)) = 0$ , which means that when  $d = p + m - 1$ , starting from a point on  $\{P(1) = 0\}$  the diffusion process will stay on this submanifold.

If  $d = p + m - 1 - \alpha$ , where  $\alpha > 0$ , we have

$$\begin{aligned} L(P(X)) &= 2X(X-1)P'' + 2(m-p-1 + (p-m+1-\alpha)X)P' + 2m(-p+\alpha)P \\ L(P'(X)) &= 2X(X-1)P''' + 2(m-p-2 + (p-m+3-\alpha)X)P'' \\ &\quad + 2(m(-p+\alpha) + p-m+3-\alpha)P' \end{aligned}$$

which means that on  $\{P(1) = 0\}$ ,  $L(P(1)) = -2\alpha P'(1)$ , thus if the diffusion process is on  $\{P(1) = 0\}$ , then it is on  $\{P'(1) = 0\} \cap \{P(1) = 0\}$ ; then on this domain  $L(P'(1)) = 2(1-\alpha)P''(1)$ , so we must have  $P''(1) = 0$  to ensure that the diffusion process is on  $\{P'(1) = 0\} \cap \{P(1) = 0\}$ . In fact

$$L(P^i(1)) = 2(i-\alpha)P^{i+1}(1),$$

for  $i+1 \leq m$ . Then if  $0 < \alpha < m$  is an integer, we have  $L(P(1)) = \dots = L(P^\alpha(1)) = 0$ , and the diffusion process is on the domain  $\{P(1) = P'(1) = \dots = P^\alpha(1) = 0\}$ . Otherwise, continue this procedure we arrive at  $P^d(1) = 0$  and the diffusion process is on  $\{P(1) = P'(1) = \dots = P^{d-1}(1) = 0\}$ , which characterizes  $SO(d)$ .

## Part III

# Main results in Publications and Preprints



## Chapter 5

# Harnack inequalities on manifolds with time-independent metric

This chapter is adapted from the papers [72, 70]. We start with a brief introduction of the Ricci flow, and then present the main results in this paper. We refer the readers to [72] for details of the proof.

### 5.1 Ricci flow

The notion of Ricci flow was introduced by Hamilton [58] in 1982, as an approach to prove the Poincaré conjecture and the geometrization conjectures. Its aim is to produce canonical geometric structures by deforming the Riemannian metric and proving certain geometrical, analytical results about the solution to Ricci flow. The most famous success of Ricci flow is the proof of Poincaré's conjecture by Perelman [99] in 2002. There are a huge amount of research and literature on this topic, we refer the readers to Chow, Lu and Ni [40] for a comprehensive knowledge of this theory, see also [90, 34, 63]. Here we only state some basic results that will be used in this thesis.

Let  $M$  be a  $n$ -dimensional complete Riemannian manifold. The Ricci flow on  $M$  is the following evolution equation of Riemannian metric  $g(t)$  on  $M \times [0, T]$ :

$$\begin{cases} \partial_t g_{ij}(x, t) = -2R_{ij}(x, t), & x \in M, \quad t > 0; \\ g_{ij}(x, 0) = g_{ij}^0, & x \in M. \end{cases}$$

On compact manifold, we have the existence and uniqueness of the solution of the equation (5.1.1), proved by Hamilton [58].

**Theorem 5.1.1.** *Let  $(M, g^0)$  be a compact Riemannian manifold. Then there exists  $T > 0$  such that the initial value problem of (5.1.1) admits a unique smooth solution  $g(x, t)$  on  $M \times [0, T]$ .*

On complete manifold with bounded curvature, there are also existence and uniqueness results for the initial value problem of Ricci flow, which we will not mention here.

The gradient estimate of curvature is proved by W.X. Shi [102, 103]:

**Theorem 5.1.2.** *There exist positive constants  $C_m$ ,  $m = 1, 2, \dots$  only depending on the dimension  $n$ , such that if the solution to Ricci flow satisfies*

$$|R_{ijkl}| \leq K, \quad \forall x \in M, t \in (0, \frac{1}{K}],$$

Then we have

$$|\nabla^m R_{ijkl}(x, t)| \leq C_m \frac{K}{t^{\frac{m}{2}}}, \quad \forall x \in M, t \in (0, \frac{1}{K}],$$

Some formula of geometric quantities we will need later:

**Lemma 5.1.3.** *If  $g_t$  is the solution to Ricci flow (5.1.1), then*

$$\begin{aligned} \partial_t dvol_t &= -R_t dvol_t \\ \partial_t R_t &= \Delta R_t + 2|\text{Ric}|^2 \end{aligned}$$

where  $dvol_t$  is the volume element with respect to  $g_t$  on  $M$ ,  $R_t$  is the scalar curvature of  $g_t$ .

For the proof, see the book by Chow, Lu and Ni [40].

## 5.2 Main results

We now state the main results of this part. We will focus on the weighted Laplacian  $L = \Delta - \nabla\phi \cdot \nabla$  for a potential function  $\phi$  with suitable regularity, and consider the solution of the heat equation associated with  $L$ . If the manifold is equipped with time-dependent metric, then the weight Laplacian is also time dependent. Recall that the Bakry-Emery Ricci curvatures associated to  $L$  are defined as follows

$$\begin{aligned} \text{Ric}(L) &= \text{Ric} + \nabla^2\phi, \\ \text{Ric}_{m,n}(L) &= \text{Ric} + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n}. \end{aligned}$$

Inspired by Bakry-Ledoux [15], we first prove the following logarithmic Sobolev inequality and the reversal logarithmic Sobolev inequality.

**Theorem 5.2.1.** *Let  $M$  be a complete Riemannian manifold equipped with a family of time dependent metrics and  $C^2$ -potentials  $(g(t), \phi(t), t \in [0, T])$ . Let  $L = \Delta_{g(t)} - \nabla_{g(t)}\phi(t) \cdot \nabla_{g(t)}$  is the time dependent weighted Laplacian on  $(M, g(t), \phi(t))$ ,  $u(\cdot, t) = P_t f$  be a positive solution to the heat equation  $\partial_t u = Lu$  with the initial condition  $u(\cdot, 0) = f$ , where  $f$  is a measurable positive function on  $M$ . Then  $(g(t), \phi(t), t \in [0, T])$  satisfies a  $K$ -super Perelman Ricci flow equation*

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq -K, \quad (5.2.1)$$

where  $K \geq 0$  is a constant, if and only if for  $0 \leq s < t \leq T$ , the following logarithmic Sobolev inequality holds

$$P_{s,t}(f \log f) - P_{s,t} f \log P_{s,t} f \leq \frac{e^{2K(t-s)} - 1}{2K} P_{s,t} \left( \frac{|\nabla f|^2}{f} \right), \quad (5.2.2)$$

and the reversal logarithmic Sobolev inequality holds

$$\frac{|\nabla P_{s,t} f|^2}{P_{s,t} f} \leq \frac{2K}{1 - e^{-2K(t-s)}} (P_{s,t}(f \log f) - P_{s,t} f \log P_{s,t} f). \quad (5.2.3)$$

Indeed, we can further prove the Poincaré inequality, the reversal Poincaré inequality as well as Bakry-Ledoux's Gromov-Lévy isoperimetric inequality on the super Ricci flow (5.2.1). In [108, 109, 110], Sturm introduced the super Ricci flow on metric measure space, and proved the equivalence between the super Ricci flow and the Poincaré inequality.

**Remark 5.2.2.** In Bakry and Ledoux [15], the authors introduced a finite dimensional version of Logarithm-Sobolev inequality, which implies and improve Li-Yau inequality. The main results in [15] state that the time homogeneous diffusion operator  $L$  satisfies the curvature-dimension condition  $CD(0, n)$  if and only if for any  $f \in \mathcal{A}$  and  $t > 0$ , the following log-Sobolev type inequality holds true

$$P_t f L(\log P_t f) \geq P_t (f L(\log f)) \left(1 + \frac{2t}{n} L(\log P_t f)\right), \quad (5.2.4)$$

which implies the Li-Yau inequality (1.2.8) under the same condition, i.e.  $\text{Ric} \geq 0$ .

On the super Ricci flows, we prove the following Li-Yau Harnack inequality by using the maximum principle.

**Theorem 5.2.3.** Let  $M$  be a compact manifold with a family of metrics and potentials  $(g(t), \phi(t))$ ,  $t \in [0, T]$ . Let  $\partial_t g = 2h$ . Let  $\alpha > 1$ . Assume that there exist some constants  $A > 0$ ,  $B > 0$ ,  $C > 0$  and  $K \in \mathbb{R}^+$  such that

$$\frac{1}{2}(1 - \alpha)\partial_t g + \text{Ric}(L) - \frac{\nabla\phi \otimes \nabla\phi}{2(m-n)} \geq -K,$$

and  $|h| \leq A$ ,  $|2\text{div}h - \nabla \text{Tr}_g h + \nabla \partial_t \phi| \leq B$ ,  $|\nabla \phi| \leq C$ . Then for all  $t \in (0, T]$  we have

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{m\alpha^2}{2t} \left[ 1 + \sqrt{1 + \frac{4T^2}{m^2\alpha^2} \left( \frac{\alpha^2(2mK + D)^2}{4(\alpha - 1)^2} + m\alpha^2 A^2 + D \right)} \right].$$

where

$$D = \frac{m\alpha(2AC + B)}{2}.$$

However, even in the case  $K = 0$ , due to the fact that  $L_t$  is time dependent, we cannot use the similar arguments as in [15] to derive the Li-Yau Harnack inequality from the Bakry-Ledoux type logarithmic Sobolev inequality on  $(0, m)$ -super Ricci flow.

By using the method in [74], we prove the following Hamilton's Harnack inequality for the time dependent weighted Laplacian on manifolds with  $K$ -super Ricci flow.

**Theorem 5.2.4.** Let  $(M, g(t), \phi(t))$  and  $u$  be the same as in Theorem 5.2.1, we have for all  $x \in M$  and  $t > 0$ ,

$$\frac{|\nabla u|^2}{u^2} \leq \frac{2K}{1 - e^{-2Kt}} \log(A/u), \quad (5.2.5)$$

where

$$A := \sup\{u(t, x) : x \in M, t \geq 0\}.$$

In particular, the Hamilton differential Harnack inequality holds

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K\right) \log(A/u). \quad (5.2.6)$$

In the case  $K = 0$ , i.e.,  $(M, g(t), \phi(t), t \in [0, T])$  is a complete Riemannian manifold equipped with the super Perelman Ricci flow

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq 0,$$

we have

$$\frac{|\nabla u|^2}{u^2} \leq \frac{1}{t} \log \frac{A}{u}. \quad (5.2.7)$$

As the corollary of Theorem 5.2.4, we have the following Harnack inequality,

**Corollary 5.2.5.** *Under the same condition as in Theorem 5.2.4, for any  $\delta > 0$ , and for all  $x, y \in M$ ,  $0 < t < T$ , we have*

$$u(x, t) \leq u(y, t)^{\frac{1}{1+\delta}} A^{\frac{\delta}{1+\delta}} \exp \left\{ \frac{1+\delta^{-1}}{4(1+\delta)} \frac{2K}{1-e^{-2Kt}} d^2(x, y) \right\}. \quad (5.2.8)$$

Under some bounded condition, we extend the Li-Yau-Hamilton type Harnack inequality to the compact Riemannian manifold with  $(K, m)$ -super Ricci flow.

**Theorem 5.2.6.** *Let  $(M, g(t), \phi(t), t \in [0, T])$  be a compact Riemannian manifold with a  $K$ -super Perelman Ricci flow with respect to the  $m$ -dimensional Bakry-Emery Ricci curvature  $\text{Ric}_{m,n}(\mathbf{L})$ , i.e.,*

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(\mathbf{L}) \geq -K,$$

where  $m \geq n$  and  $K > 0$  are two constants. Let  $u$  be a positive solution of the heat equation  $\partial_t u = Lu$ . Assume that there exist constants  $A, B, C > 0$ , such that on  $[0, T] \times M$ , we have  $|\frac{1}{2}\partial_t g| \leq A$ ,  $|\text{div}\partial_t g - 2\langle \nabla \text{Tr}_g, \partial_t g \rangle + \nabla \partial_t \phi| \leq B$  and  $|\nabla \phi| \leq C$ . Then for all  $t \in (0, T]$ ,

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{(m+n)e^{4Kt}}{4t} \left[ 2 + T \sqrt{\frac{8}{m+n}} \sqrt{A^2 + 2AC^2 + \frac{B^2}{4}} + \max_{t \in [0, T]} \frac{(4A+1)\sqrt{8t}}{1-e^{-2Kt}} \right].$$

We also extends the Li-Yau-Hamilton type Harnack inequality (1.2.9) to positive solutions of the heat equation  $\partial_t u = Lu$  on complete Riemannian manifolds with fixed metrics and potentials satisfying the  $CD(-K, m)$  condition. Here we would like to mention that Bakry, Bolley and Gentil [13] obtained an improved version of the Li-Yau type Harnack inequality for the heat equation  $\partial_t u = Lu$  on complete Riemannian manifolds with fixed metrics and potentials satisfying the  $CD(-K, m)$  condition. Our work is independent of [13], and our method is different from [13] and can be extended to the case of time dependent metrics and potentials.

**Theorem 5.2.7.** *Let  $(M, g)$  be a complete Riemannian manifold with a  $C^2$ -potential  $\phi$ . Suppose that there exist some constants  $m \geq n$  and  $K \geq 0$  such that  $\text{Ric}_{m,n}(\mathbf{L}) \geq -K$ . Let  $u$  be a positive solution of the heat equation  $\partial_t u = Lu$ . Then the Li-Yau-Hamilton differential Harnack inequality holds*

$$\frac{\partial_t u}{u} - e^{-2Kt} \frac{|\nabla u|^2}{u^2} + e^{2Kt} \frac{m}{2t} \geq 0. \quad (5.2.9)$$

In particular, if  $K = 0$ , i.e.,  $\text{Ric}_{m,n}(\mathbf{L}) \geq 0$ , then the Li-Yau differential Harnack inequality holds

$$\frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} + \frac{m}{2t} \geq 0. \quad (5.2.10)$$

## Chapter 6

# $W$ -entropy formulas on super Ricci flows

This chapter is adapted from the papers [72, 70]. We start with a discussion of canonical ensemble in statistical mechanics, then give an introduction to the probabilistic interpretation of  $W$ -entropy. In the end, we present the main results in this paper without the details of proof.

### 6.1 Canonical ensemble

To better understand  $W$ -entropy, we give a very brief introduction of canonical ensemble in statistical mechanics. This section is based on X.-D.Li [77] and Evans [50].

To start with, we are given a triple  $(\Omega, \mathcal{F}, \pi)$ , where  $\Omega$  is a set consisting of a precise description of microstate of some physical system. A point  $\omega \in \Omega$  is a microstate.  $\mathcal{F}$  is a  $\sigma$ -algebra of subsets of  $\Omega$ .  $\pi$  is a nonnegative measure defined on  $\mathcal{F}$ . The family of all  $\pi$ -measurable functions  $\rho : \Omega \rightarrow [0, \infty]$ , such that  $\rho d\pi$  is a probability measure is called microstate measure. A  $\pi$ -measurable function  $H : \Omega \rightarrow \mathbb{R}^n$  is the Hamiltonian function.

For a given  $\rho d\pi$ , we have  $\mathbb{E}H = \int_{\Omega} H(\omega)\rho(\omega)d\pi(\omega)$ , which indicates the macroscopic state of the physical system. Then the fundamental question is that given a macroscopic description  $\mathbb{E}H = H_0$ , how to determine the microstate distribution  $\rho d\pi$ ?

Now we introduce entropy as a measurement of uncertainty in our framework, which is a function of microstate density  $\rho d\pi$ , defined by

$$S(\rho) = -k \int_{\Omega} \rho \log \rho d\pi,$$

where  $k$  is the Boltzmann constant.

As a consequence of entropy maximization principle, under the restriction that  $\mathbb{E}H = H_0$ , we choose  $\sigma d\pi$  which maximizes the entropy  $S$  as the microstate distribution of the equilibrium state. In fact, the maximizer distribution is given by

$$\sigma = \frac{e^{-\beta H}}{\int_{\Omega} e^{-\beta H} d\pi},$$

where  $\beta \in \mathbb{R}$  is a constant. This is the canonical ensemble,  $\sigma d\pi$  is the so-called Gibbs measure, and  $Z_{\beta} = \int_{\Omega} e^{-\beta H} d\pi$  is the partition function.

Given the canonical distribution, all the relevant quantities such as temperature, energy, entropy can be expressed with respect to partition function  $Z_\beta$ . In particular, the entropy functional is

$$S = k(-\beta \frac{\partial}{\partial \beta} (\log Z) + \log Z).$$

## 6.2 Probabilistic interpretation

In the paper [99], Perelman gave an explanation of his  $W$ -entropy in the view of statistical mechanics:  $W$ -entropy is indeed the corresponding entropy for certain partition function on some canonical ensemble. We use the same notation as in the Section 6.1: let  $M$  be an  $n$ -dimensional compact Riemannian manifold with Ricci flow 5.1.1,  $f$  be the solution to the conjugate heat equation  $\partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}$ , and  $u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$ . Now assume there exists a canonical ensemble with a density of states measure for which the partition function is given by

$$\log Z_\beta = \int_M (\frac{n}{2} - f) u dv$$

Then the Boltzmann entropy formula gives that the entropy  $S$  is

$$S = \log Z_\beta - \beta \frac{\partial}{\partial \beta} \log Z_\beta$$

and Perelman formally proved that  $S = -W$ .

A probabilistic interpretation of  $W$ -entropy is given in the paper [79]. In fact, we have

$$\int (\frac{n}{2} - f) u dv = \int u \log u dv + \frac{n}{2} (1 + \log(4\pi\tau)).$$

Notice that the entropy of Gaussian heat kernel  $d\gamma_t^n(x) = \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{\frac{n}{2}}}$  is

$$H(\gamma_t^n) = \frac{n}{2} (1 + \log(4\pi\tau)).$$

Now define

$$H(g, \phi, u, \tau) = - \int u \log u d\mu - \frac{n}{2} (1 + \log(4\pi\tau)).$$

which is the difference between the Boltzmann entropy of  $u$  and that of Gaussian heat kernel measure. By direct computation we have

$$W = \frac{\partial}{\partial \tau} (\tau H(g, \phi, u, \tau))$$

This gives a probabilistic interpretation of Perelman's  $W$ -entropy. Similar interpretation also works in both heat equation [93] and heat equation associated with weighted Laplacian on complete manifolds with fixed metric, see Section 7 in [79]. This is the way in which we formulate the  $W$ -entropy formula in various contexts throughout this thesis.

### 6.3 Main results

First we recall the  $W$ -entropy formula for the weighted Laplacian  $L = \Delta - \nabla\phi \cdot \nabla$  on the manifolds [76, 79]

**Theorem 6.3.1.** *Let  $M$  be a compact Riemannian manifold. Let  $u$  be a positive solution of the heat equation  $\partial_t u = Lu$ . Define the  $H_m$ -entropy and the  $W_m$ -entropy as follows*

$$\begin{aligned} H_m(u, t) &= \text{Ent}(u(t)) + \frac{m}{2}(1 + \log(4\pi t)), \\ W_m(u, t) &= \frac{d}{dt}(tH_m(u, t)). \end{aligned}$$

Then

$$\frac{d}{dt}H_m(u, t) = - \int_M \left( L \log u + \frac{m}{2t} \right) u d\mu, \quad (6.3.1)$$

and

$$\begin{aligned} \frac{d}{dt}W_m(u, t) &= -2t \int_M \left[ \left| \text{Hess} \log u + \frac{g}{2t} \right|^2 + \text{Ric}_{m,n}(L)(\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad - \frac{2t}{m-n} \int_M \left| \nabla \log u \cdot \nabla \phi - \frac{m-n}{t} \right|^2 u d\mu. \end{aligned}$$

In particular, if  $\text{Ric}_{m,n}(L) \geq 0$ , then  $W_m(u, t)$  is decreasing in time  $t$  along the heat equation  $\partial_t u = Lu$ .

Notice that the probabilistic interpretation also applies here: the definition of  $H_m$  is indeed the difference between the entropy of  $\mu$  and that of the Gaussian measure  $\gamma^m$ . Another observation is that in the formula (6.3.1), the integrand is exactly the form of Li-Yau inequality for the heat equation associated with weighted Laplacian under  $CD(0, m)$  condition [75],

$$L \log u + \frac{m}{2t} \geq 0.$$

This is also the case in the  $W$ -entropy for linear heat equation on manifolds with non-negative Ricci curvature.

It is also worth to mention that the  $m$ -dimensional Bakry-Emery Ricci curvature

$$\text{Ric}_{m,n}(L) = \text{Ric} + \nabla^2 \phi - \frac{\nabla \phi \otimes \nabla \phi}{m-n},$$

where  $m \geq n$ ,  $m = n$  if and only if  $\phi$  is a constant, has a natural geometric interpretation by using warped product metric, as mentioned in [14, 80]. More precisely, let  $q = m - n$ , consider the warped product metric on  $\widetilde{M} = S^q \times M$ ,

$$g_{\widetilde{M}} = g_M + e^{-\frac{2\phi}{q}} g_{S^q}$$

Denote  $\text{Ric}_{\widetilde{M}}$  the Ricci curvature on  $\widetilde{M}$ . Let  $\bar{X}$  be the horizontal lift of a vector field  $X$  on  $M$  to  $\widetilde{M}$ . The  $m$ -dimensional Bakry-Emery Ricci curvature  $\text{Ric}_{m,n}^M(L)$  is exactly  $\text{Ric}_{\widetilde{M}}$  restricted on  $M$ , i.e

$$\text{Ric}_{m,n}^M(L)(X, X) = \text{Ric}_{\widetilde{M}}(\bar{X}, \bar{X}).$$

Applying Ni's  $W$ -entropy for the heat equation on  $\widetilde{M} = S^q \times M$ , we are able to give a new proof of Theorem 6.3.1 and find that

$$|\nabla^2 \log u + \frac{g}{2t}|^2 + \frac{2}{q} (\nabla \phi \cdot \nabla \log u - \frac{m-n}{2t})^2 = |\widetilde{\nabla}^2 \log \widetilde{u} + \frac{\widetilde{g}}{2t}|^2.$$

This gives a natural geometric interpretation for the quantities appearing in the monotonicity formula (6.3.2).

Now we introduce our main results in this part: the  $W$ -entropy in various contexts and their monotonicity formulas. The results in this section are included in [72, 70].

### 6.3.1 Case I: $(0, m)$ -super Ricci flow

**Theorem 6.3.2.** *Let  $(M, g(t), t \in [0, T])$  be a family of compact Riemannian manifolds with potential functions  $\phi(t) \in C^\infty(M)$ ,  $t \in [0, T]$ . Suppose that  $g(t)$  and  $\phi(t)$  satisfy the conjugate equation*

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right).$$

Let  $L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}$  be the time dependent weighted Laplacian on  $(M, g(t), \phi(t))$ . Let  $u$  be a positive solution of the heat equation  $\partial_t u = Lu$  with initial data  $u(0)$  satisfying  $\int_M u(0) d\mu(0) = 1$ . Define

$$H_m(u, t) = - \int_M u \log u d\mu - \frac{m}{2} (1 + \log(4\pi t)).$$

and

$$W_m(u, t) = \frac{d}{dt} (tH_m(u)).$$

Then

$$\begin{aligned} \frac{d}{dt} W_m(u, t) &= -2t \int_M \left| \nabla^2 \log u + \frac{g}{2t} \right|^2 u d\mu - \frac{2t}{m-n} \int_M \left( \nabla \phi \cdot \nabla \log u - \frac{m-n}{2t} \right)^2 u d\mu \\ &\quad - 2 \int_M t \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) \right) (\nabla \log u, \nabla \log u) u d\mu. \end{aligned}$$

In particular, if  $\{g(t), \phi(t), t \in (0, T]\}$  satisfies the  $m$ -dimensional Perelman super Ricci flow and the conjugate equation

$$\begin{aligned} \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) &\geq 0, \\ \frac{\partial \phi}{\partial t} &= \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right), \end{aligned}$$

then  $W_m(u, t)$  is decreasing in  $t \in (0, T]$ , i.e.,

$$\frac{dW_m(u, t)}{dt} \leq 0, \quad \forall t \in (0, T].$$

As an application of Theorem 6.3.2, we prove that the optimal logarithmic Sobolev constant associated with the weighted Laplacian on compact manifolds equipped with the  $m$ -dimensional Perelman's super Ricci flow is decreasing in time.

**Theorem 6.3.3.** *Let  $(M, g(t), \phi(t), t \in [0, T])$  be as in Theorem 6.3.2. Then, for any fixed  $t \in [0, T]$ , there exists a positive and smooth function  $u = e^{-\frac{v}{2}}$  such that  $v$  achieves the optimal logarithmic Sobolev constant  $\mu(t)$  defined by*

$$\mu(t) = \inf \left\{ W_m(u, t) : \int_M \frac{e^{-v}}{(4\pi t)^{m/2}} d\mu = 1 \right\}.$$

Indeed,  $u = e^{-\frac{v}{2}}$  is a solution to the nonlinear PDE

$$-4t\text{Lu} - 2u \log u - mu = \mu(t)u.$$

Moreover, if  $\{g(t), \phi(t), t \in [0, T]\}$  satisfies the  $m$ -dimensional super Ricci flow (6.3.2) and the conjugate equation (6.3.2), then  $\mu(t)$  is decreasing in  $t$  on  $[0, T]$ .

### 6.3.2 Case II: $CD(K, m)$ and $(K, m)$ -super Ricci flow

Let  $m \geq n$  and  $K$  be two constants, define

$$H_{m,K}(u, t) = - \int_M u \log u d\mu - \frac{m}{2}(1 + \log(4\pi t)) - \frac{m}{2}Kt \left(1 + \frac{1}{6}Kt\right), \quad (6.3.2)$$

and

$$W_{m,K}(u, t) = \frac{d}{dt}(tH_{m,K}(u)). \quad (6.3.3)$$

We now prove the monotonicity of  $W_{m,K}$  in two different contexts. First, on complete manifolds with fixed metric, we have the following

**Theorem 6.3.4.** *Let  $m \geq n$  and  $K$  be two constants. Let  $M$  be a complete Riemannian manifold with bounded geometry condition, and  $u$  be a positive solution of the heat equation  $\partial_t u = \text{Lu}$ . Denote  $u = \frac{e^{-f}}{(4\pi t)^{m/2}}$ . Then*

$$W_{m,K}(u, t) = \int_M \left( t|\nabla f|^2 + f - m \left(1 + \frac{1}{2}Kt\right)^2 \right) u d\mu, \quad (6.3.4)$$

and

$$\begin{aligned} \frac{d}{dt}W_{m,K}(u, t) &= -2t \int_M \left( \left| \nabla^2 f - \left( \frac{1}{2t} + \frac{K}{2} \right) g \right|^2 + (\text{Ric}_{m,n}(\text{L}) + Kg)(\nabla f, \nabla f) \right) u d\mu \\ &\quad - \frac{2t}{m-n} \int_M \left( \nabla \phi \cdot \nabla f + (m-n) \left( \frac{1}{2t} + \frac{K}{2} \right) \right)^2 u d\mu. \end{aligned}$$

**Remark 6.3.5.** *In fact, we can derive a rigidity theorem: if  $\text{Ric}_{m,n}(\text{L}) \geq -K$ , then by Theorem 6.3.4*

$$\frac{d}{dt}W_{m,K}(u, t) = 0$$

if and only if

$$\text{Ric}_{m,n}(\text{L}) = -Kg, \quad \nabla^2 f = \left( \frac{1}{2t} + \frac{K}{2} \right) g, \quad \nabla \phi \cdot \nabla f = -(m-n) \left( \frac{1}{2t} + \frac{K}{2} \right).$$

In particular, if  $m = n$  and  $\phi = C$  is a constant, then  $(M, g)$  is an Einstein manifold with  $\text{Ric} = -K$ , and the potential  $f$  satisfies the shrinking gradient Ricci soliton equation

$$\frac{1}{2}\text{Ric} + \nabla^2 f = \frac{g}{2t}.$$

Then on compact manifolds with time dependent metrics and potentials, we have

**Theorem 6.3.6.** *Let  $m \geq n$  and  $K$  be two constants. Let  $M$  be a compact Riemannian manifold. Let  $u$  be a positive solution of the heat equation  $\partial_t u = Lu$ , set  $u = \frac{e^{-f}}{(4\pi t)^{m/2}}$ . Then*

$$W_{m,K}(u, t) = \int_M \left[ t|\nabla f|^2 + f - m\left(1 + \frac{1}{2}Kt\right)^2 \right] u d\mu, \quad (6.3.5)$$

and

$$\begin{aligned} \frac{d}{dt} W_{m,K}(u, t) &= -2t \int_M \left| \nabla^2 f - \frac{1}{2t}g - \frac{K}{2}g \right|^2 \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu \\ &\quad - 2t \int_M \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) + Kg \right) (\nabla f, \nabla f) \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu \\ &\quad - \frac{2t}{m-n} \int_M \left( \nabla \phi \cdot \nabla f + (m-n) \left( \frac{1}{2t} + \frac{K}{2} \right) \right)^2 \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu. \end{aligned}$$

In particular, if  $\{g(t), \phi(t), t \in (0, T]\}$  is the  $K$ -super  $m$ -dimensional Bakry-Emery Ricci flow and satisfies the conjugate equation

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) \geq -Kg, \quad (6.3.6)$$

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right), \quad (6.3.7)$$

then  $W_{m,K}(u, t)$  is decreasing in  $t \in (0, T]$ , i.e.,

$$\frac{d}{dt} W_{m,K}(u, t) \leq 0, \quad \forall t \in (0, T].$$

We can also prove a rigidity theorem similar to Remark 6.3.5 in this case, see [72].

### 6.3.3 Case III: $CD(K, \infty)$ and $K$ -super Ricci flow

Based on the reversal logarithmic Sobolev inequality on complete Riemannian manifolds with both fixed metric [15] and time-dependent metric (Theorem 5.2.1), we introduce the revised Boltzmann entropy,

$$H_K(f, t) = D_K(t) \int_M (P_t(f \log f) - P_t f \log P_t f) d\mu, \quad (6.3.8)$$

where  $D_0(t) = \frac{1}{t}$  and  $D_K(t) = \frac{1}{|1 - e^{-2Kt}|}$  for  $K \neq 0$ . Define the  $W$ -entropy by the revised Boltzmann entropy formula

$$W_K(f, t) = H_K(f, t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt} H_K(f, t). \quad (6.3.9)$$

In this section, we prove the  $W$ -entropy formula for weighted Laplacian  $L = \Delta - \nabla \phi \cdot \nabla$  on complete manifolds with  $CD(K, \infty)$  condition, and also for weighted Laplacian on compact Riemannian manifolds with  $K$ -super Ricci flow.

We now state our first result.

**Theorem 6.3.7.** *Let  $M$  be a complete Riemannian manifold with bounded geometry condition,  $\phi \in C^4(M)$  with  $\nabla\phi \in C_b^3(M)$ . Suppose that  $\text{Ric} + \nabla^2\phi \geq K$ , where  $K \in \mathbb{R}$  is a constant. Let  $u(\cdot, t) = P_t f$  be a positive solution to the heat equation  $\partial_t u = Lu$  with  $u(\cdot, 0) = f$ ,  $f$  is a positive and measurable function on  $M$ . Then, for all  $K \in \mathbb{R}$  and  $t > 0$ ,*

$$\frac{d}{dt} H_K(f, t) \leq 0, \quad (6.3.10)$$

and

$$\frac{d^2}{dt^2} H_K(t) + 2K \coth(2Kt) \frac{d}{dt} H_K(t) \leq -2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu, \quad (6.3.11)$$

implying that

$$\begin{aligned} \frac{d}{dt} W_K(f, t) &= -\frac{\sinh(2Kt)}{K} D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\ &\quad - \frac{\sinh(2Kt)}{K} D_K(t) \int_M (\text{Ric}(L) - Kg) \langle \nabla \log P_t f, \nabla \log P_t f \rangle P_t f d\mu, \end{aligned}$$

In particular, for all  $K \in \mathbb{R}$ , we have

$$\frac{d}{dt} W_K(f, t) \leq 0, \quad \forall t > 0.$$

**Remark 6.3.8.** *In particular, under the assumptions of Theorem 6.3.7, we see that  $\frac{d}{dt} W_K(f, t) + \frac{\sinh(2Kt)}{K} D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu = 0$  if and only if  $(M, g, f)$  is a gradient shrinking Ricci soliton, i.e.,*

$$\text{Ric}(L) = Kg.$$

The following theorem extends the  $W$ -entropy formula to the time dependent weighted Laplacian on compact Riemannian manifolds with  $K$ -super Perelman Ricci flow.

**Theorem 6.3.9.** *Let  $(M, g(t), \phi(t), t \in [0, T])$  be a compact Riemannian manifold with a family of time dependent metrics  $g(t)$  and potentials  $\phi(t)$ , and  $\phi \in C^{2,1}(M \times [0, T])$ . Let  $L = \Delta_{g(t)} - \nabla_{g(t)}\phi(t) \cdot \nabla_{g(t)}$  be the time dependent Witten Laplacian on  $(M, g(t), \phi(t))$ . Suppose that*

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq Kg, \quad \frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \frac{\partial g}{\partial t}$$

where  $K \in \mathbb{R}$  is a constant. Let  $u(\cdot, t) = P_t f$  be a positive solution to the heat equation  $\partial_t u = Lu$  with  $u(\cdot, 0) = f$ ,  $f$  is a positive and measurable function on  $M$ . Then, for all  $K \in \mathbb{R}$  and  $t \in (0, T]$ , we have

$$\frac{d}{dt} H_K(f, t) \leq 0, \quad \forall t \in (0, T], \quad (6.3.12)$$

and

$$\frac{d^2}{dt^2} H_K(t) + 2K \coth(2Kt) \frac{d}{dt} H_K(t) \leq -2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu, \quad (6.3.13)$$

implying that

$$\begin{aligned} \frac{d}{dt} W_K(f, t) &= -\frac{\sinh(2Kt)}{K} D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\ &\quad - \frac{\sinh(2Kt)}{K} D_K(t) \int_M \left( \frac{1}{2} \partial_t g + \text{Ric}(L) - Kg \right) \langle \nabla \log P_t f, \nabla \log P_t f \rangle P_t f d\mu, \end{aligned}$$

In particular, for all  $K \in \mathbb{R}$ , we have

$$\frac{d}{dt}W_K(f, t) \leq 0, \quad \forall t \in (0, T].$$

**Remark 6.3.10.** In particular, under the assumptions of Theorem 6.3.9, we see that  $\frac{d}{dt}W_K(f, t) + \frac{\sinh(2Kt)}{K}D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu = 0$  if and only if  $(M, g(t), \phi(t))$  satisfies Perelman's  $K$ -Ricci flow equation and the conjugate heat equation, i.e.,

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) = Kg, \quad \frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \frac{\partial g}{\partial t}.$$

## Chapter 7

# $W$ -entropy formulas on Wasserstein space over Riemannian manifolds

This chapter is adapted from the paper [71] and some new results that have not been published. We will present the main results without the details of proof, and the paper and the proof will be given in Chapter 13.

### 7.1 Wasserstein distance between Fokker-Planck diffusions: time-dependent case

Under the convexity condition of the potential  $V$ , Otto [96] proved that the Wasserstein distance between two family of probabilities measures  $d\mu_0(t) = u_0(t)dx$  and  $d\mu_1(t) = u_1(t)dx$ , where  $u_i(t)$  is the positive solution to the Fokker-Planck equation  $\partial_t u_i = \Delta u_i - \nabla \cdot (u_i \nabla f)$  with initial date  $u_i(0)$ ,  $i = 0, 1$ , is non-increasing in time  $t$ . See also Otto and Westdickenberg [98].

Otto's result has been further extended by Sturm and Renesse [111] to complete Riemannian manifolds with smooth weighted measures. Let  $(M, g)$  be a complete Riemannian manifold with a weighted volume measure  $d\mu(x) = e^{-f} dvol$ , where  $f \in C^2(M)$ , let  $u_i(t)$  be the positive solution the heat equation  $\partial_t u_i = \Delta u_i - \nabla \cdot (u_i \nabla f)$  with initial date  $u_i(0)$ ,  $i = 0, 1$ . Then they proved that the Bakry-Emery Ricci curvature associated with  $(M, g, e^{-f} dvol)$  is bounded below by a constant  $K \in \mathbb{R}$ , i.e., the  $CD(K, \infty)$  condition holds if and only if the Wasserstein distance between the measures  $\mu_0(t) = u_0(t, x)d\mu$  and  $\mu_1(t) = u_1(t)d\mu$  satisfies the differential inequality

$$\frac{d}{dt} W_2(\mu_0(t), \mu_1(t)) \leq e^{-Kt} W_2(\mu_0(t), \mu_1(t)), \quad \forall t > 0.$$

In particular,  $Ric + \nabla^2 f \geq 0$  if and only if  $W_2(\mu_0(t), \mu_1(t))$  is non-increasing in  $t$ .

The above results has been further extended to Riemannian manifolds equipped with time-dependent metric. Let  $M$  be a compact Riemannian manifold on which  $\{g(t), t \geq 0\}$  satisfies Ricci flow (5.1.1). Let  $u_i(t)$ ,  $t \in [0, T]$ ,  $i = 0, 1$ , be the positive solution of the conjugate (backward) heat equation

$$\partial_t u_i = -\Delta u_i + R u_i$$

with initial date  $u_i(0)$ . McCann and Topping[86], Topping [113], Lott[81] proved that the Wasserstein distance  $W_2(\mu_0(t), \mu_1(t))$  is non-decreasing in  $t$ .

One of our main results is to extend the above results to the heat equation associated with weighted Laplacian  $L = \Delta_{g_t} - \nabla_{g_t} f(t) \cdot \nabla_{g_t}$  on closed Riemannian manifolds  $(M, g(t), f(t))$  equipped with Perelman's modified Ricci flow (1.3.15) with potential  $f(t)$  which satisfies (1.3.16). Let  $u(t)$  be a positive solution to the backward Fokker-Planck equation on  $(M, g(t))$

$$\partial_t u = -Lu. \quad (7.1.1)$$

Since  $\partial_t d\mu = 0$ , by the integration by parts formula, we have  $\frac{\partial}{\partial t} \int_M u(t) d\mu(t) = - \int_M Lu d\mu = 0$ . Hence  $\{u(t) d\mu, t \geq 0\}$  is a family of probability measures on  $M$  provide that  $\int_M u(0) d\mu = 1$ . Then we have the following theorem:

**Theorem 7.1.1.** *Let  $(M, g(t), f(t))$  be a compact manifold equipped with modified Ricci flow (1.3.15) and the conjugate heat equation (1.3.16). Let  $d\mu_i(t) = u_i(t) d\mu$ ,  $i = 0, 1$ , and  $t \in [0, T]$ , where  $u_i(t)$  is the positive solution of the backward heat equation on  $(M, g(t))$ :*

$$\partial_t u_i(t) = -Lu_i(t)$$

*with initial date  $u_i(0)$  satisfying  $\int_M u_i(0) d\mu(0) = 1$ ,  $i = 0, 1$ . Then the Wasserstein distance  $W_2(\mu_0(t), \mu_1(t))$  is non-decreasing in  $t$ .*

## 7.2 Geodesics on Wasserstein space

We now introduce our main results by starting with the geodesic of Wasserstein space.

Let  $\mathcal{P}_2^\infty(M, \nu)$  be the Wasserstein space of all probability measures  $\rho(x) dvol(x)$  on  $M$  such that  $\int_M d^2(o, x) \rho(x) dvol(x) < \infty$ , where  $d(o, \cdot)$  denotes the distance function from a fixed point  $o \in M$ . By Otto [96], the tangent space  $T_{\rho d\mu} \mathcal{P}_2^\infty(M, \mu)$  is identified as follows

$$T_{\rho d\mu} \mathcal{P}_2^\infty(M, \mu) = \{s = -\nabla_\mu^*(\rho \nabla \phi) : \phi \in C^\infty(M)\},$$

where  $\nabla_\mu^*$  denotes the  $L^2$ -adjoint of the Riemannian gradient  $\nabla$  with respect to the weighted volume measure  $d\mu$  on  $(M, g)$ . For  $s_i = -\nabla_\mu^*(\rho \nabla \phi_i) \in T_{\rho d\mu} \mathcal{P}_2^\infty(M, \mu)$ , Otto [96] introduced the following infinite dimensional Riemannian metric on  $\mathcal{P}_2^\infty(M, \mu)$

$$\langle s_1, s_2 \rangle = \int_M \nabla \phi_1 \cdot \nabla \phi_2 \rho d\mu.$$

provided that  $\|s_i\|^2 = \int_M |\nabla \phi_i|^2 \rho d\mu < \infty$ ,  $i = 1, 2$ . Then the Wasserstein space  $\mathcal{P}_2^\infty(M, \mu)$  can be considered as an infinite dimensional Riemannian manifold with Otto's metric.

By Benamou and Brenier [21], for any given  $\mu_i = \rho_i d\mu \in \mathcal{P}_2^\infty(M, \mu)$ ,  $i = 0, 1$ , then we have

$$W_2^2(\mu_0, \mu_1) = \inf_{\substack{\rho_t \in \mathcal{P}_2^\infty(M, \mu), \\ t \in [0, 1]}} \left\{ \frac{1}{2} \int_0^1 \int_M |\nabla \phi(x, t)|^2 \rho(x, t) d\mu, \partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0 \right\}, \quad (7.2.2)$$

where  $\rho(0) = \rho_0$ ,  $\rho(1) = \rho_1$ . In particular, if the geodesic connecting  $\mu_0$  and  $\mu_1$  lies entirely in  $\mathcal{P}_2^\infty(M, \mu)$ , (7.2.2) implies that the Wasserstein distance can be achieved by the geodesic curve  $(\rho, \phi) \in T^* \mathcal{P}_2^\infty(M, \mu)$ , which satisfies the transport equation and the Hamilton-Jacobi equation

$$\partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0, \quad (7.2.3)$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0, \quad (7.2.4)$$

with the boundary condition  $\rho(0) = \rho_0$  and  $\rho(1) = \rho_1$ . In view of this, the transport equation (7.2.3) and the Hamilton-Jacobi equation (7.2.4) describe the geodesic flow on the cotangent bundle  $T^*\mathcal{P}_2^\infty(M, \mu)$  over the Wasserstein space  $\mathcal{P}_2^\infty(M, \mu)$ .

### 7.3 Main results I

Our first main result in this part is introducing the  $W$ -entropy for the geodesic flows in  $\mathcal{P}_2(M, \mu)$  and proving its monotonicity formula under  $CD(0, m)$  condition.

**Theorem 7.3.1.** *Let  $(M, g)$  be a compact Riemannian manifold,  $f \in C^2(M)$ ,  $d\mu = e^{-f} \text{dvol}$ . Let  $\rho : M \times [0, T] \rightarrow \mathbb{R}^+$  and  $\phi : M \times [0, T] \rightarrow \mathbb{R}$  be smooth solutions to the geodesic equation (7.2.3) (7.2.4). For any  $m \geq n$ , define the  $H_m$ -entropy and  $W_m$ -entropy for the geodesic flow  $(\rho, \phi)$  on  $T^*\mathcal{P}_2^\infty(M, \mu)$  as follows*

$$H_m(\rho, t) = \text{Ent}(\rho(t)) + \frac{m}{2} (1 + \log(4\pi t^2)), \quad (7.3.5)$$

and

$$W_m(\rho, t) = \frac{d}{dt}(tH_m(\rho, t)). \quad (7.3.6)$$

Then for all  $t > 0$ , we have

$$\begin{aligned} \frac{d}{dt}W_m(\rho, t) &= t \int_M \left[ \left| \text{Hess}\phi - \frac{g}{t} \right|^2 + \text{Ric}_{m,n}(\text{L})(\nabla\phi, \nabla\phi) \right] \rho d\mu \\ &\quad + \frac{t}{m-n} \int_M \left| \nabla\phi \cdot \nabla f - \frac{m-n}{t} \right|^2 \rho d\mu. \end{aligned}$$

In particular, if  $\text{Ric}_{m,n}(\text{L}) \geq 0$ , then  $W_m(\rho, t)$  is increasing in time  $t$  along the geodesic flow on  $T^*\mathcal{P}_2^\infty(M, \mu)$ .

As a corollary of Theorem 7.3.1, we recapture the following result due to Lott-Villani [81, 83].

**Corollary 7.3.2.** *Let  $M$  be a compact Riemannian manifold. Suppose that  $\text{Ric}_{m,n}(L) \geq 0$ . Then  $t\text{Ent}(\rho(t)) + m\log t$  is convex in time  $t$  along the geodesic on  $\mathcal{P}_2^\infty(M, \mu)$ .*

Moreover, when  $m \in \mathbb{N}$ , we can easily check that the following  $(\rho_m, \phi_m)$

$$\begin{aligned} \phi_m(x, t) &= \frac{\|x\|^2}{2t}, \\ \rho_m(x, t) &= \frac{1}{(4\pi t^2)^{m/2}} e^{-\frac{\|x\|^2}{4t^2}}, \end{aligned}$$

where  $t > 0, x \in \mathbb{R}^m$ , is a solution to the geodesic equation (7.2.4) on  $\mathbb{R}^m$  equipped with the standard Lebesgue measure. Moreover, the Boltzmann entropy of  $\rho_m$  is given by

$$\text{Ent}(\rho_m(t)) = -\frac{m}{2}(1 + \log(4\pi t^2)).$$

Then the  $H_m$ -entropy and  $W_m$ -entropy for the geodesic flow on Wasserstein space  $\mathcal{P}_2(M, \mu)$  can be reformulated as

$$\begin{aligned} H_m(\rho(t)) &= \text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t)) \\ W_m(\rho, t) &= \frac{d}{dt}(t(\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t)))) \end{aligned}$$

which exactly follows the probabilistic interpretation of  $W$ -entropy of heat equation on manifolds, as stated in [79] and Section 6.2.

Moreover, we prove an analogue of Lott's convexity theorem of the Boltzmann type functional for compact manifolds equipped with the Perelman's modified Ricci flow and the conjugate heat equation. The following results provide the reasonable forms of the Boltzmann entropy type functionals which are convex along the geometric interpolation equations on the Wasserstein space over such manifolds.

**Theorem 7.3.3.** *Let  $M$  be a compact manifold with a family of Riemannian metrics and potential functions  $(g(t), f(t), t \in [0, T])$  which satisfy Perelman's Ricci flow equation (1.3.15) and the conjugate heat equation (1.3.16). Let  $\rho$  and  $\phi$  be solutions of the following equations*

$$\partial_t \rho = -\nabla_\mu^*(\rho \nabla \phi). \quad (7.3.7)$$

$$\partial_t \phi = -\frac{1}{2}|\nabla \phi|^2 - \frac{1}{2}R_f \quad (7.3.8)$$

where

$$R_f = R + 2\Delta f - |\nabla f|^2.$$

Then

$$\frac{d^2}{dt^2} \int_M (\rho \log \rho - \phi \rho) d\mu = \int_M |\text{Ric} + \text{Hess} f - \text{Hess} \phi|^2 \rho d\mu.$$

In particular, the free energy functional  $\int_M \rho \log \rho d\mu - \int_M \phi \rho d\mu$  is convex in  $t$  on  $[0, T]$ , and the convexity is strict unless that  $g$  is a steady Ricci soliton

$$\text{Ric} = \text{Hess}(\phi - f).$$

From the kinetic point of view, the equation (7.3.8) is the Hamilton-Jacobi equation for the velocity  $\phi$  of the particle moving in an external field with a potential  $R_f$ , and the equation (7.3.7) is the transport equation for the density  $\rho$  of particles with respect to the weighted measure  $\mu$ . According to Perelman [99], the quantity  $R_f = R - 2\Delta f + |\nabla f|^2$  is the scalar curvature term appeared in the modified Bochner-Licherowicz formula for the weighted Dirac operator with respect to the weighted volume measure  $\mu = e^{-f} d\text{vol}$  on  $M$ . In view of this, the equations in Theorem 7.3.3 are naturally related to the Perelman's modified Ricci flow (1.3.15) and the conjugate heat equation (1.3.16), and the convexity of the functional  $\int_M \rho \log \rho d\mu - \int_M \phi \rho d\mu$  is a corresponding result of the Perelman's monotonicity of the  $\mathcal{F}$ -functional along the Ricci flow and the conjugate heat equation.

Moreover, we also have the following convexity result, which is the natural corresponding of a result proved by Lott [81] for the Ricci flow.

**Theorem 7.3.4.** *Let  $M$  be a compact manifold with a family of Riemannian metrics and potential functions  $(g(t), f(t), t \in [0, T])$  as in Theorem 7.3.3, and  $\rho$  and  $\phi$  be positive solutions to*

$$\begin{aligned} \partial_t \rho &= -\nabla_\mu^*(\rho \nabla \phi), \\ \partial_t \phi &= -\frac{1}{2}|\nabla \phi|^2 - \frac{1}{2}R_f - \frac{1}{2\tau}\phi - \frac{1}{2\tau}f. \end{aligned}$$

Let  $\tau = T - t$ , then

$$\left(\tau^{\frac{3}{2}} \frac{d}{d\tau}\right)^2 \left(\int_M (\rho \log \rho + \phi) \rho d\mu + \frac{n}{2} \log(4\tau)\right) = \tau^3 \int_M \left|\text{Ric} + \text{Hess} \phi + \text{Hess} f - \frac{g}{2\tau}\right|^2 \rho d\mu.$$

In particular,  $\int_M \rho \log \rho d\mu + \int_M \phi \rho d\mu + \frac{n}{2} \log(4\tau)$  is convex in  $\tau^{-\frac{1}{2}}$ .

## 7.4 Langevin flows on Wasserstein space

Compare the  $W$ -entropy formula for the weighted Laplacian (6.3.2) and  $W$ -entropy for the geodesics in the Wasserstein space (7.3.7), we find that they have a great similarity. As we have explained in the previous section, the  $W$ -entropy for the geodesic in the Wasserstein space  $\mathcal{P}_2^\infty(M, \mu)$  has the same probabilistic interpretation. Then a natural question arises: how to understand the similarity between the  $W$ -entropy formulas in Theorem 7.3.1 and Theorem 6.3.1? One of the possible approaches is to use the vanishing viscosity limit method from the heat equation to the Hamilton-Jacobi equation. However, it seems that this approach does not work.

Inspired by J.-M. Bismut's work (see [22, 23]) on the deformation of hypoelliptic Laplacians on the cotangent bundle over Riemannian manifolds, we introduce a deformation of geometric flows on the cotangent bundle of the Wasserstein space over compact Riemannian manifolds to link the geodesic flows and the gradient flows.

We now describe how to introduce the deformation of geometric flows on  $T^*\mathcal{P}_2^\infty(M, \mu)$ . For any  $c \geq 0$ , define the geometric flow  $(\rho, \phi) : [0, T] \rightarrow T^*\mathcal{P}_2^\infty(M, \mu)$  by solving the following equations on  $T^*\mathcal{P}_2^\infty(M, \mu)$

$$\partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0, \quad (7.4.9)$$

$$c^2 \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) = -\phi + \log \rho + 1. \quad (7.4.10)$$

Notice that in the particular case  $M = \mathbb{R}^m$  and  $\mu = dx$ , the above system is the compressible Euler equation with damping in the isentropic fluid

$$\partial_t u + u \cdot \nabla u = -\frac{1}{c^2} u + \frac{1}{c^2} \frac{\nabla \rho}{\rho}. \quad (7.4.11)$$

Indeed,  $\rho$  is the density of the fluid,  $u = \nabla \phi$  is the velocity, and  $\gamma = \frac{1}{c^2}$  is the friction constant,  $p(\rho) = \log \rho + 1$  is the pressure. In general, we consider (7.4.9), (7.4.10) as the potential flow of the compressible Euler equation with damping in the isentropic fluid on the compact Riemannian manifold  $(M, g)$  equipped with the reference measure  $d\mu$ . On Euclidean space or compact Riemannian manifolds, with a suitable condition of its initial value, we can prove that there exists  $T > 0$  such that the compressible Euler equation with damping (7.4.9), (7.4.10) admits a unique smooth solution on  $[0, T] \times M$ . Moreover, on Euclidean space or compact Riemannian manifolds with the first de Rham cohomology group  $H_1(M) = \text{Ker } d / \text{Im } d = 0$ , we can prove that the Cauchy problem for (7.4.9), (7.4.10) admits a unique smooth solution on  $[0, T] \times M$ . We call the geometric flow  $(\rho, \phi)$  the Langevin flow.

The limiting cases  $c \rightarrow 0$  and  $c \rightarrow \infty$  can be specific as follows. When  $c = 0$ , from (7.4.10) we have  $\phi = \log \rho + 1 = \frac{\delta \text{Ent}(\rho)}{\delta \rho}$  and  $\partial_t \rho = -L\rho$ , showing that  $(\rho, \phi)$  can be regarded as the backward gradient flow of the Boltzmann entropy on  $\mathcal{P}_2^\infty(M, \mu)$  equipped with Otto's infinite dimensional Riemannian metric. When  $c \rightarrow \infty$ , to make the sense of the equation (7.4.10),  $\rho$  and  $\phi$  must satisfy the transport equation (7.2.3) and the Hamilton-Jacobi equation (7.2.4), i.e.,  $(\rho, \phi)$  is the geodesic flow on the cotangent bundle over the Wasserstein space  $\mathcal{P}_2^\infty(M, \mu)$ . Thus, the family of flows  $\{(\rho, \phi) : c \in [0, \infty]\}$  is a deformation of geometric flows on  $\mathcal{P}_2^\infty(M, \mu)$  which interpolate the backward gradient flow of the Boltzmann entropy and the geodesic flow.

Moreover, on  $(\mathbb{R}^m, dx)$ ,  $m \in \mathbb{N}$  there is a special solution to the deformation flow (7.4.9), (7.4.10). More precisely, let  $T > 0$ , let  $u : (0, T) \rightarrow (0, \infty)$  be a smooth solution to the ODE

$$c^2 u'' + u' = -\frac{1}{2u},$$

with a given date  $u(T) > 0$ . Let  $\alpha(t) = \frac{u'(t)}{u(t)}$ , and let  $\beta(t)$  be a smooth function such that

$$c^2 \dot{\beta}(t) = -\beta(t) - m \log u(t) - \frac{m}{2} \log(4\pi) + 1.$$

For  $x \in \mathbb{R}^m$  and  $t > 0$ , let us introduce

$$\begin{aligned} \phi_m(x, t) &= \frac{\alpha(t)}{2} \|x\|^2 + \beta(t), \\ \rho_m(x, t) &= \frac{1}{(4\pi u^2(t))^{m/2}} e^{-\frac{\|x\|^2}{4u^2(t)}}. \end{aligned}$$

Then  $(\rho_m, \phi_m)$  is a smooth solution to (7.4.9), (7.4.10) on  $\mathbb{R}^m$ .

## 7.5 Main results II

Our main results of this part is  $W$ -entropy formulas for the deformation flows. To follow the probabilistic interpretation of  $W$ -entropy, we first introduce a Hamiltonian energy functional

$$H(\rho, \phi) = \frac{c^2}{2} \int_M |\nabla \phi|^2 \rho d\mu + \int_M \rho \log \rho d\mu, \quad (7.5.12)$$

and prove its convexity along the deformation flows under  $CD(0, \infty)$  condition as follows.

**Theorem 7.5.1.** *Let  $(M, g)$  be Euclidean space or a compact Riemannian manifold,  $f \in C^2(M)$ ,  $d\mu = e^{-f} dv$ . For any  $c \geq 0$ , let  $(\phi, \rho)$  be a smooth solution to the deformation flows (7.4.9), (7.4.10). Then we have*

$$\frac{d^2}{dt^2} H(\rho, \phi) = 2 \int_M [c^{-2} |\nabla \phi - \nabla \log \rho|^2 + |\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu. \quad (7.5.13)$$

*In particular, if the  $CD(0, \infty)$ -condition holds, i.e.,  $\text{Ric}(L) = \text{Ric} + \text{Hess} f \geq 0$ , then  $H(\rho, \phi)$  is convex along the deformed flows (7.4.9), (7.4.10).*

The following result can be viewed as a variant of the  $W$ -entropy formula for the deformed flow on  $T^* \mathcal{P}_2^\infty(M, \mu)$ , and interpolate the  $W$ -entropy formula for the geodesic flow on  $T^* \mathcal{P}_2^\infty(M, \mu)$  and the backward gradient flow of the Boltzmann entropy on  $\mathcal{P}_2^\infty(M, \mu)$ .

**Theorem 7.5.2.** *Let  $(M, g)$  be Euclidean space or a compact Riemannian manifold,  $f \in C^2(M)$ ,  $d\mu = e^{-f} dv$ . For any  $c \geq 0$ , let  $(\phi, \rho)$  be a smooth solution to the deformation flows (7.4.9), (7.4.10). Then*

$$\begin{aligned} & \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left( 2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + m\alpha^2(t) \\ &= \int_M \left[ |\text{Hess} \phi - \alpha(t)g|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu \\ & \quad + (m-n) \int_M \left| \alpha(t) + \frac{\nabla \phi \cdot \nabla f}{m-n} \right|^2 \rho d\mu + \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu. \end{aligned}$$

*In particular, if the  $CD(0, m)$ -condition holds, i.e.,  $\text{Ric}_{m,n}(L) \geq 0$ , we have*

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left( 2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + m\alpha^2(t) \geq \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu.$$

*Moreover, under the  $CD(0, m)$ -condition, the equality holds if and only if  $M = \mathbb{R}^m$ ,  $\rho = \rho_m$  and  $\phi = \phi_m$ .*

## 7.6 $W$ -entropy inequalities under Erbar-Kawada-Sturm's entropic curvature-dimension condition

In [46], Erbar-Kuwada-Sturm introduced a new definition of the curvature-dimension condition on metric-measure spaces, called the entropic curvature-dimension condition. By [46], the entropic curvature-dimension condition, denoted by  $CD_{\text{Ent}}(K, m)$ , holds if the Boltzmann entropy Ent satisfies

$$\text{HessEnt} - \frac{1}{N} \nabla \text{Ent}^{\otimes 2} \geq K,$$

where  $K \in \mathbb{R}$ ,  $N \geq n$  are two constants.

The purpose of this section is to prove the  $W$ -entropy inequalities for the geodesic flow, the gradient flows as well as the Langevin deformation of flows on the Wasserstein space over complete Riemannian manifolds with Erbar-Kawada-Sturm's  $CD_{\text{Ent}}(K, N)$  condition. This might bring some new insights to the study of geometric analysis on metric measure spaces.

**Theorem 7.6.1.** *Let  $M$  be a complete Riemannian manifold of dimension  $n$ . Suppose that Erbar-Kawada-Sturm's  $CD_{\text{Ent}}(K, N)$  condition holds, i.e.,*

$$\text{HessEnt} - \frac{1}{N} \nabla \text{Ent}^{\otimes 2} \geq K,$$

where  $K \in \mathbb{R}$ ,  $N \geq n$  are two constants. Then

(i) for geodesic flow  $(\rho(t), \phi(t))$  on  $T^*P_2^\infty(M, \mu)$ , we have

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{N}{t^2} \geq \frac{1}{N} \left| \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{t} \right|^2 + K |\dot{\rho}(t)|^2.$$

(ii) for the backward gradient flow  $\dot{\rho}(t) = \nabla \text{Ent}(\rho(t))$  on  $P_2^\infty(M, \mu)$ , we have<sup>1</sup>

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{N}{2} \left( K + \frac{1}{t} \right)^2 \geq \frac{2}{N} \left| \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{2} \left( K + \frac{1}{t} \right) g \right|^2.$$

**Theorem 7.6.2.** *Let  $c \in [0, \infty)$ , and let  $(\rho(t), \phi(t))$  be the Langevin deformation of flows on  $T^*P_2^\infty(M, \mu)$ . Suppose that Erbar-Kawada-Sturm's  $CD_{\text{Ent}}(K, N)$ -condition holds for some constants  $K \in \mathbb{R}$  and  $N \in \mathbb{N}$  with  $N \geq n$ , i.e.,*

$$\text{HessEnt} - \frac{1}{N} \nabla \text{Ent}^{\otimes 2} \geq K.$$

Let  $\alpha(t) = (\log u)'$  be as in Section 7.4 with  $m = N$ . Then

$$\begin{aligned} \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left( 2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + N\alpha^2(t) + \frac{1}{c^2} |\nabla \text{Ent}(\rho(t))|^2 \\ \geq \frac{1}{N} |\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha(t)|^2 + K |\dot{\rho}(t)|^2. \end{aligned}$$

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<sup>1</sup>Note that  $\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle = \int_M |\nabla \log \rho|^2 \rho d\mu$ .



## Chapter 8

# Spectrum processes on the octonion algebra

This chapter is adapted from the paper [69]. We will present the main results in this paper without the details of proof.

### 8.1 The octonion algebra

In this section, we recall some facts about the octonion algebra, and we refer to [9] for more details. We start with a few definitions.

**Definition 8.1.1.** *An algebra  $A$  is a division algebra if for any  $a, b \in A$ , with  $ab = 0$ , then either  $a = 0$  or  $b = 0$ . A normed division algebra is a division algebra that is also a normed vector space with  $\|ab\| = \|a\|\|b\|$ .*

**Definition 8.1.2.** *An algebra  $A$  is alternative if the subalgebra generated by any two elements is associative. By a theorem of Artin [100], this is equivalent to the fact that for any  $a, b \in A$ ,  $(aa)b = a(ab)$ ,  $(ba)a = b(aa)$ .*

As mentioned earlier, there are only four normed division algebras,  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$ ,  $\mathbb{O}$ . There is a nice way called "Cayley-Dickson construction" to produce this sequence of algebras: the complex number  $a + ib$  can be seen as a pair of real numbers  $(a, b)$ ; the quaternions can be defined as a pair of complex number; and similarly the octonions is a pair of quaternions. As the construction proceeds, the property of the algebra becomes worse and worse: the quaternions are noncommutative but associate while the octonions are only alternative but not associative.

Since octonions and Clifford algebra are both the algebra with the dimension  $2^n$  (in this case  $n = 3$ ), which share some special property, we can use the presentation provided in Bakry and Zani [19] to describe the algebra structure on a basis of octonions, in order to simplify the computations. This presentation is not classical, and we shall therefore use the table below.

Define  $E = \{1, 2, 3\}$ , and let  $\mathcal{P}(E)$  denote the set of the subsets of  $E$ . For every set  $A \in \mathcal{P}(E)$ , we associate a basis element  $\omega_A$  in the octonion algebra, with  $\omega_\emptyset = \text{Id}$ , the identity element. Then an element  $x \in \mathbb{O}$  can be written in the form

$$x = x_A \omega_A,$$

and the product of two elements  $x$  and  $y$  is given by

$$xy = \sum_{A,B} x_A y_B \omega_A \omega_B.$$

and it remains to define  $\omega_A \omega_B$  for given  $A, B \in \mathcal{P}(E)$ , which is given through the following rule: denote by  $A.B$  the symmetric difference  $A \cup B \setminus (A \cap B)$ , then  $\omega_A \omega_B = (A|B)\omega_{A.B}$ , where  $(A|B)$  takes value in  $\{-1, 1\}$ . Then, the multiplication rule in the octonion algebra is defined by a sign table, which is as follows :

	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\emptyset$	1	1	1	1	1	1	1	1
$\{1\}$	1	-1	1	1	-1	-1	1	-1
$\{2\}$	1	-1	-1	1	1	-1	-1	1
$\{3\}$	1	-1	-1	-1	1	1	1	-1
$\{1,2\}$	1	1	-1	-1	-1	-1	1	1
$\{1,3\}$	1	1	1	-1	1	-1	-1	-1
$\{2,3\}$	1	-1	1	-1	-1	1	-1	1
$\{1,2,3\}$	1	1	-1	1	-1	1	-1	-1

In this table, the element  $(i, j)$  is the sign  $(A_i|A_j)$ , where  $A_i$  is the  $i$ th element in the first column,  $A_j$  is the  $j$ th element in the first row.

From the facts that for  $A, B \neq \emptyset$ ,  $\omega_A^2 = -1$  and  $\omega_A \omega_B = -\omega_B \omega_A$ , it is easy to get the following rules:

$$(A|A) = \begin{cases} -1, & A \neq \emptyset; \\ 1, & A = \emptyset. \end{cases}$$

$$(A|B) = -(B|A), \quad \text{for } B \neq A, A, B \neq \emptyset;$$

We can check from the table that the octonion is a division algebra, moreover non-associative but alternative.

Moreover,  $\mathbb{O}$  can be equipped with the Euclidean structure obtained by identifying  $\mathbb{O}$  as a 8 dimensional (real) vector space via

$$x = \sum_A x_A \omega_A \mapsto (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{3\}}, x_{\{1,2\}}, x_{\{1,3\}}, x_{\{2,3\}}, x_{\{1,2,3\}}),$$

so that the inner product and the norm are respectively :

$$\langle x, y \rangle = \sum_A x_A y_A, \quad \|x\| = \left( \sum_A x_A^2 \right)^{1/2}.$$

Let us recall that to prove  $\mathbb{O}$  is a division algebra, it is usual to introduce the conjugate

$$x = \sum_A x_A \omega_A \implies x^* = \sum_A x_A \omega_A (A|A)$$

and observe that  $(xy)^* = y^* x^*$ ,  $xx^* = x^* x$  and  $\|x\|^2 = xx^*$ , so that  $\|xy\|^2 = (xy)(xy)^* = (xy)(y^* x^*) = x(yy^*)x^* = \|x\|^2 \|y\|^2$ .

Although the previous table does not provide an associative algebra, the octonion algebra satisfies however some useful identities. In what follows, we shall make a strong use of Moufang identities, which are stated as follows : for elements  $x, y, z$  belongs to  $\mathbb{O}$ , we have

$$\begin{aligned} z(x(zy)) &= (zxx)y, \\ ((xz)y)z &= x(zyz), \\ (zx)(yz) &= (z(xy))z, \\ (zx)(yz) &= z((xy)z). \end{aligned}$$

For a  $n \times n$  matrix on octonions, write it as  $\mathcal{M} = \sum_A M^A \omega_A$ , where  $\{M^A\}$  are real  $n \times n$  matrices. For an  $n$  dimensional vector  $\sum_B X^B \omega_B$ ,

$$\left(\sum_A M^A \omega_A\right) \left(\sum_B X^B \omega_B\right) = \sum_{A,B} M^A X^B (A|B) \omega_{A.B} = \sum_{A,B} (A.B|B) M^{A.B} X^B \omega_A.$$

Therefore,  $\mathcal{M}$  can be expressed by the real  $8n \times 8n$  block matrix  $\{M_{ij}^{A,B}\}$ , where  $M_{ij}^{A,B} = (A.B|B) M_{ij}^{A.B}$ .

This leads to the following definition:

**Definition 8.1.3.** A  $(2^3 \times n) \times (2^3 \times n)$  block matrix  $M^{A,B}$  (where  $A, B \subset \{1, 2, 3\}$ ) is a real octonionic if  $M^{A,B} = (A.B|B) M^{A.B}$ , where  $M^A = M^{A,\emptyset}$  is a family of  $8n \times n$  square matrices. It is the real form of a matrix with octonionic entries. We shall denote it as  $\mathcal{M} = \sum_A M^A \omega_A$ .

Then, we shall say that an octonionic matrix is symmetric if its real form is symmetric. This corresponds to the fact that, for any  $A \in \mathcal{P}(E)$ ,  $(M^A)^t = (A|A) M^A$ .

That is to say,  $(M^{A,B})^t = (A.B|B) (M^{A.B})^t = M^{B,A} = (B.A|A) M^{A.B}$ . Due to property 2 of Lemma 2.3, this leads to that for any  $A \in \mathcal{P}(E)$ ,  $(M^A)^t = (A|A) M^A$ , i.e.  $M^\emptyset$  is symmetric while  $M^A$  is antisymmetric for any  $A \neq \emptyset$ .

It is worth to point out that since the octonion algebra is not associative, there is no matrix presentation of the algebra structure for the octonions, and therefore the matrix multiplication of the real octonionic matrices does not corresponds to the octonionic multiplication of the associated matrices with octonion entries. Even the product of octonionic matrices is not octonionic in general.

The inverse of an octonionic matrix is in general not octonionic, and its exact structure is not easy to decipher; the octonionic property may not be preserved. The following lemma gives a condition for this last property to hold, and will play an important role in our results.

**Lemma 8.1.4.** Let  $M = \sum M^A \omega_A$  be an octonionic matrix such that  $M^\emptyset$  is invertible. Assume moreover that, for any  $A, B \in \mathcal{P}(E)$

$$M^A (M^\emptyset)^{-1} M^B = M^B (M^\emptyset)^{-1} M^A. \quad (8.1.1)$$

and that  $\sum_C M^C (M^\emptyset)^{-1} M^C$  is invertible. Then,  $M$  is invertible and its inverse  $N$  is octonionic, satisfying  $N = \sum_A \omega_A N^A$ , with

$$N^A = -N^\emptyset M^A (M^\emptyset)^{-1}, \quad \text{for } A \neq \emptyset \quad (8.1.2)$$

$$N^\emptyset = \left(\sum_C M^C (M^\emptyset)^{-1} M^C\right)^{-1}. \quad (8.1.3)$$

In this paper we perform computations on the characteristic polynomial  $P(X) = \det(M - XId)$  of a matrix  $M$ . Assume that we have some diffusion operator acting on the entries of a matrix  $M$ , described by the values of  $L(M_{ij})$  and  $\Gamma(M_{ij}, M_{kl})$  for any  $(i, j, k, l)$ . Then, we have,

$$\begin{aligned}\Gamma(\log P(X), \log P(Y)) &= \sum \partial_{M_{ij}} \log(P(X)) \partial_{M_{kl}} \log(P(Y)) \Gamma(M_{ij}, M_{kl}), \\ L(\log P(X)) &= \sum \partial_{M_{ij}} \log(P(X)) L(M_{ij}) + \sum \partial_{M_{ij}} \partial_{M_{kl}} \log(P(X)) \Gamma(M_{ij}, M_{kl}).\end{aligned}$$

To compute  $\partial_{M_{ij}} \log(P(X))$  and  $\partial_{M_{ij}} \partial_{M_{kl}} \log(P(X))$  in the above formulae, we use the Lemma 6.1 in Bakry and Zani [19], which we quote here without proof.

**Lemma 8.1.5.** *Let  $M = (M_{ij})$  be a matrix and  $M^{-1}$  be its inverse, on the set  $\{\det M \neq 0\}$  we have*

$$\begin{aligned}\partial_{M_{ij}} \log \det M &= M_{ji}^{-1}, \\ \partial_{M_{ij}} \partial_{M_{kl}} \log \det M &= -M_{jk}^{-1} M_{li}^{-1}.\end{aligned}$$

Hence with  $M^{-1}(X) = (M - XId)^{-1}$ ,

$$\Gamma(\log P(X), \log P(Y)) = \sum M^{-1}(X)_{ji} M^{-1}(Y)_{lk} \Gamma(M_{ij}, M_{kl}), \quad (8.1.4)$$

$$L(\log P(X)) = \sum M_{ji}^{-1}(X) L(M_{ij}) - \sum M_{jk}^{-1}(X) M_{li}^{-1}(X) \Gamma(M_{ij}, M_{kl}). \quad (8.1.5)$$

According to Bakry and Zani [19], one can get the information from  $\Gamma(P(X), P(Y))$  and  $L(P(X))$  about the multiplicity of the eigenvalues and the invariant measure of the operator  $L(P(X))$ :

If for some constants  $\alpha_1, \alpha_2, \alpha_3$ ,

$$L(P) = \alpha_1 P'' + \alpha_2 \frac{P'^2}{P}, \quad \Gamma(\log P(X), \log P(Y)) = \frac{\alpha_3}{Y - X} \left( \frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)} \right). \quad (8.1.6)$$

and if there exists for some  $a \in \mathbb{R}$ ,  $a \neq 0$  which satisfies

$$a^2(\alpha_1 + \alpha_2) - a(\alpha_1 + \alpha_3) + \alpha_3 = 0, \quad (8.1.7)$$

Then

1. If  $a$  is a positive integer, it is the multiplicity of the eigenvalues of  $M$ ;
2. Write  $P(X) = \prod_{i=1}^n (X - x_i)^a$ , the invariant measure for the operator  $L$  in the Weyl chamber  $\{x_1 < \dots < x_n\}$  is

$$d\mu = \left( \prod_{i < j} (x_i - x_j)^2 \right)^{-\frac{a^2(\alpha_1 + \alpha_2)}{\alpha_3}} d\mu_0,$$

where  $d\mu_0$  is the Lebesgue measure.

For the eigenvalue problem of matrices on octonions, Y.G.Tian proved in his paper [112] that  $2 \times 2$  Hermitian matrix on octonions has 2 eigenvalues, each of them has multiplicity 8. For  $3 \times 3$  Hermitian octonionic matrix, Dray-Manogue [44] and Okubo [94] showed that it has 6 eigenvalues with multiplicity 4. For  $4 \times 4$  and  $5 \times 5$  Hermitian octonionic matrices, there are only numerical results, indicating that the eigenvalues have multiplicity 2 [112]. It is still unknown for matrices in higher dimension. Following the analysis of Bakry and Zani [19], one may expect that the study of probabilistic models on matrices of octonions could give new insights in these directions.

## 8.2 Main results: symmetric matrices on octonions

Our aim is to describe the law of the spectrum of the real symmetric matrix on octonions. The block matrix is  $\mathcal{M} = (M^{A,B}) = ((A.B|B)M^{A,B})$ , satisfying  $(M^A)^t = (A|A)M^A$  due to symmetry.

Now set  $P(X) = \det(\mathcal{M} - X\text{Id})$  and  $U(X) = (\mathcal{M} - X\text{Id})^{-1}$ . Then by the formulae 8.1.4, 8.1.5, we have

$$\Gamma(\log P(X), \log P(Y)) = \sum U_{ji}^{A,B}(X)U_{lk}^{C,D}(Y)\Gamma(M_{ij}^{A,B}, M_{kl}^{C,D}), \quad (8.2.8)$$

$$L(\log P(X)) = \sum U_{ji}^{A,B}(X)L(M_{ij}^{A,B}) - \sum U_{jk}^{B,C}(X)U_{li}^{D,A}(X)\Gamma(M_{ij}^{A,B}, M_{kl}^{C,D}). \quad (8.2.9)$$

We will focus on the two cases where the inverse matrix  $U(X)$  is octonionic, i.e. the symmetry condition 8.1.1 of matrix  $\mathcal{M} - X\text{Id}$  is satisfied, almost surely for the stochastic process under consideration.

### 8.2.1 The dimension 2 case

Consider  $M = \sum M^A \omega_A$ , where  $\{M^A\}$  are matrices whose elements are independent Brownian motions. For  $A \neq \emptyset$ , due to symmetry of  $\mathcal{M}$ ,  $(M^A)^t = (A|A)M^A = -M^A$ . Such matrices naturally satisfy the symmetry restriction (8.1.1) in dimension 2, since the  $2 \times 2$  antisymmetric matrices are all of the form  $\begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix}$ , and they are therefore all proportional to each other. However this is not true in higher dimensions. Set

$$\Gamma(M_{ij}^A, M_{kl}^B) = \frac{1}{2}\delta_{A,B}(\delta_{ik}\delta_{jl} + (A|A)\delta_{il}\delta_{jk}), \quad L(M_{ij}^A) = 0,$$

which reflects the symmetry of the matrices. Notice that the inverse matrix  $U(X)$  is also symmetric with  $(U^A)^t = (A|A)U^A$ . We have the following result

**Proposition 8.2.1.** *For the  $2 \times 2$  symmetric matrix  $M = \sum M^A \omega_A$ ,*

$$\begin{aligned} \Gamma(\log P(X), \log P(Y)) &= \frac{8}{Y - X} \left( \frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)} \right), \\ L(\log P) &= 3 \left( \frac{P'(X)^2}{P(X)^2} - \frac{P''(X)}{P(X)} \right) - \frac{1}{2} \frac{P'(X)^2}{P(X)^2}, \end{aligned}$$

such that

$$L(P) = \left(11 - \frac{1}{2}\right) \frac{P'(X)^2}{P(X)} - 11P''(X).$$

Then the multiplicity of the eigenvalues is 8. Assume  $\rho$  is the density of the invariant measure of  $L$  of the coordinates  $\{x_i\}$  in Weyl chamber, we have  $\rho = C \prod_{i < j} (x_i - x_j)^8$ .

### 8.2.2 Another model in any dimension

We now provide another set of random octonionic matrices for which the symmetry condition (8.1.1) is automatically satisfied.

Let  $M^\emptyset$  be a symmetric matrix with independent Brownian motions as its entries. For all  $A, B \neq \emptyset$ , let  $M^A = M^B = \mathcal{A}$  be a random antisymmetric matrix with independent Brownian motion off diagonal entries. Then consider  $M = M^\emptyset \omega_\emptyset + \mathcal{A} \sum_{C \neq \emptyset} \omega_C$ . This model is similar to the Hermitian case considered in Bakry and Zani [19] (see Remark 8.2.3). Similarly to the Hermitian case, we set

$$\begin{aligned}\Gamma(M_{ij}^\emptyset, M_{kl}^\emptyset) &= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \\ \Gamma(\mathcal{A}_{ij}, \mathcal{A}_{kl}) &= \frac{1}{14}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \\ \Gamma(M_{ij}^\emptyset, \mathcal{A}_{kl}) &= 0.\end{aligned}$$

and  $L(M_{ij}^A) = 0$  for any  $A$ . Also due to Lemma 8.1.4, for the inverse matrix  $U(X) = (M - XI)^{-1}$ , we have for every  $C \neq \emptyset$

$$U^C = -U^\emptyset M^C (M^\emptyset - XI)^{-1} = -U^\emptyset \mathcal{A} (M^\emptyset - XI)^{-1},$$

which means for all  $C \neq \emptyset$ ,  $U^C$  is the same, and we denote it by  $U_a$ .

**Proposition 8.2.2.** *For the matrix  $M = M^\emptyset \omega_\emptyset + \sum_{C \neq \emptyset} \mathcal{A} \omega_C$  on the octonions,*

$$\begin{aligned}\Gamma(\log P(X), \log P(Y)) &= \frac{8}{Y - X} \left( \frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)} \right), \\ L(\log P) &= -\frac{1}{8} \frac{P'(X)^2}{P(X)^2},\end{aligned}$$

such that

$$L(P) = \left(8 - \frac{1}{8}\right) \frac{P'(X)^2}{P} - 8P''(X).$$

The multiplicity is  $a = 8$ , while the density of the invariant measure of  $L$  is  $C \prod_{i < j} |x_i - x_j|^2$ .

**Remark 8.2.3.** *Recall that in Bakry and Zani [19], section 7.1, for a Hermitian matrix  $H = M + iA$  with independent Brownian motions as its entries (where  $M$  is symmetric,  $A$  is anti-symmetric), we have*

$$\begin{aligned}\Gamma(M_{ij}, M_{kl}) &= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ \Gamma(A_{ij}, A_{kl}) &= \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\end{aligned}$$

In our model  $M = M^\emptyset \omega_\emptyset + \mathcal{A} \sum_{C \neq \emptyset} \omega_C$ , denote  $e$  the specific element in the octonion algebra  $e = \sum_{C \neq \emptyset} \omega_C$ . Notice that

$$e^2 = -7.$$

which indicates that  $e$  works like  $i$  in the Hermitian matrices, just with a different variance. Therefore, this example is indeed similar to the case of Hermitian matrices.

Our two models provide examples where the multiplicity of eigenvalues and the exponent  $\beta$  in the law are not related, which is in accordance with the conclusion in Bakry and Zani [19], that the exponent reflects the structure of the algebra while the multiplicity of the eigenvalues is decided by the dimension of the eigenspaces.

As we have seen, the octonionic structure of the matrix plays an important role. For higher dimension, the problem may be studied by our method if we know the structure of the inverse matrix, which is not necessarily octonionic. The main obstacle is still the non-associativity, which prevents any matrix presentation for octonionic multiplication. Let us recall that the  $3 \times 3$  matrices on octonions have been studied by Dray and Manogue [44] and Okubo [94] using algebraic method, showing that there are 6 eigenvalues with multiplicity 4. It is still an open problem to provide a probabilistic model in this case which would lead to this conclusion.



## Part IV

# Matrix Dirichlet Process



In this part we present our work on matrix Dirichlet process. We will start with the spectrum process of Brownian motion on  $SO(d)$  and  $SU(d)$ , and study the polar decomposition of Brownian motion on complex(real) matrices. Then we recall the classical results of Dirichlet measure on the simplex and give our new interpretation. In the end, we study matrix Dirichlet process, and present two inteprations.



## Chapter 9

# Spectrum of Brownian motion on $SO(d)$ and $SU(d)$

In this section we study the spectrum process of Brownian motion on  $SU(d)$  and  $SO(d)$ .

### 9.1 $SU(d)$ case

We start with the  $SU(d)$  case. Recall in Section [?], for  $z \in SU(d)$ , the diffusion operators on  $SU(d)$  are

$$\begin{aligned}\Gamma_{SU(d)}(z_{ij}, z_{kl}) &= -dz_{il}z_{kj} + z_{ij}z_{kl}, \\ \Gamma_{SU(d)}(z_{ij}, \bar{z}_{kl}) &= d\delta_{ik}\delta_{jl} - z_{ij}\bar{z}_{kl}, \\ L_{SU(d)}(z_{ij}) &= -(d^2 - 1)z_{ij}, \quad L(\bar{z}_{ij}) = -(d^2 - 1)\bar{z}_{ij}.\end{aligned}$$

It is known that any  $U \in SU(d)$  can be diagonalized to the matrix of the form

$$\{e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_{d-1}}, e^{-i\sum_{j=1}^{d-1}\theta_j}\},$$

where  $\{0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_{d-1} \leq 2\pi\}$ . Then we arrive at the following proposition:

**Proposition 9.1.1.** *For two integers  $1 \leq p, q \leq d-1$ , let  $\theta_d = -\sum_{k=1}^{d-1}\theta_k$ ,*

$$\begin{aligned}\Gamma(\theta_p, \theta_q) &= d\delta_{pq} - 1, \\ L(\theta_p) &= -2d \sum_{q \neq p}^d \operatorname{Im}\left(\frac{1}{1 - e^{i(\theta_q - \theta_p)}}\right).\end{aligned}$$

*And the density of its invariant measure is*

$$\prod_{(p,q), 1 \leq p, q \leq d, p \neq q} (e^{i\theta_p} - e^{i\theta_q})^2.$$

*Proof.* — Write  $P(X) = \det(X\text{Id} - U)$ , we have

$$\Gamma(\log P(X), \log P(Y)) = XY \left( \frac{P'}{P}(X) \frac{P'}{P}(Y) + \frac{n}{X-Y} \left( \frac{P'}{P}(X) - \frac{P'}{P}(Y) \right) \right).$$

Now define  $\theta_d = -\sum_i^{d-1} \theta_i$ . Since  $P(X) = \prod_{p=1}^d (X - e^{i\theta_p})$ , we have  $\frac{P'}{P}(X) = \sum_{p=1}^n \frac{1}{X - e^{i\theta_p}}$  such that

$$\begin{aligned} \Gamma(\log P(X), \log P(Y)) &= XY \left( \sum_{pq} \frac{1}{(X - e^{i\theta_p})(Y - e^{i\theta_q})} - n \sum_k \frac{1}{(X - e^{i\theta_k})(Y - e^{i\theta_k})} \right) \\ &= \sum_{pq} \frac{e^{i(\theta_p + \theta_q)}}{(X - e^{i\theta_p})(Y - e^{i\theta_q})} - n \sum_k \frac{e^{i2\theta_k}}{(X - e^{i\theta_k})(Y - e^{i\theta_k})}. \end{aligned}$$

On the other hand, by direct computation

$$\begin{aligned} \Gamma(\log P(X), \log P(Y)) &= \sum_{p,q \leq n-1} \frac{-e^{i(\theta_p + \theta_q)} \Gamma(\theta_p, \theta_q)}{(X - e^{i\theta_p})(Y - e^{i\theta_q})} - \frac{e^{i2\theta_n} \Gamma(\theta_n, \theta_n)}{(X - e^{i\theta_n})(Y - e^{i\theta_n})} \\ &\quad + \sum_{p,q \leq n-1} \left( \frac{e^{i(\theta_p + \theta_n)} \Gamma(\theta_p, \theta_q)}{(X - e^{i\theta_p})(Y - e^{i\theta_n})} + \frac{e^{i(\theta_p + \theta_n)} \Gamma(\theta_p, \theta_q)}{(X - e^{i\theta_n})(Y - e^{i\theta_p})} \right). \end{aligned}$$

Comparing the two formulas we yield the result:

$$\begin{aligned} \Gamma(\theta_p, \theta_q) &= -1, \quad \text{for } p \neq q, \quad \Gamma(\theta_p, \theta_p) = d - 1, \\ \Gamma(\theta_d, \theta_d) &= d - 1, \quad \Gamma(\theta_d, \theta_p) = -1. \end{aligned}$$

which leads to the conclusion in the proposition.

For L,

$$\begin{aligned} \text{L}(\log P(X)) &= X^2 \frac{P''}{P} + (d-1)X^2 \frac{P'^2}{P^2} + (1-d^2)X \frac{P'}{P} \\ &= (d-1) \sum_p \frac{e^{i2\theta_p}}{(X - e^{i\theta_p})^2} + 2d \sum_p \frac{e^{i2\theta_p}}{(X - e^{i\theta_p})} \sum_{q \neq p} \frac{1}{e^{i\theta_p} - e^{i\theta_q}} + (d-1) \sum \frac{2e^{i\theta_p}}{X - e^{i\theta_p}} \\ &\quad + (1-d^2) \sum_p \frac{e^{i\theta_p}}{(X - e^{i\theta_p})}, \end{aligned}$$

and

$$\begin{aligned} \text{L}(\log P(X)) &= \sum_{1 \leq p \leq n} \frac{e^{2i\theta_p} \Gamma(\theta_p, \theta_p)}{(X - e^{i\theta_p})^2} - \sum_{1 \leq p \leq n} \frac{\text{L}(e^{i\theta_p})}{X - e^{i\theta_p}}, \\ \text{L}(e^{i\theta_p}) &= -2de^{i2\theta_p} \sum_{q \neq p} \frac{1}{e^{i\theta_p} - e^{i\theta_q}} + (d-1)^2 e^{i\theta_p}. \end{aligned}$$

Thus

$$\text{L}(\theta_p) = i(2d \sum_{q \neq p}^d \frac{1}{1 - e^{i(\theta_q - \theta_p)}} - (d-1)d).$$

Notice that for each  $q \neq p$ ,

$$\frac{1}{1 - e^{i(\theta_q - \theta_p)}} + \frac{1}{1 - e^{-i(\theta_q - \theta_p)}} = 1,$$

which means  $Re(\frac{1}{1 - e^{i(\theta_q - \theta_p)}}) = \frac{1}{2}$ . Therefore,  $L(\theta_p)$  is real and

$$\begin{aligned} L(\theta_p) &= -2d \sum_{q \neq p}^d Im\left(\frac{1}{1 - e^{i(\theta_q - \theta_p)}}\right) \\ &= i(2d \sum_{q \neq p}^d \frac{e^{i\theta_p}}{e^{i\theta_p} - e^{i\theta_q}} - (d-1)d), \end{aligned}$$

$$\begin{aligned} \Gamma(\log \rho, \theta_p) &= L(\theta_p) - \sum_q^{d-1} \partial_{\theta_q}(\Gamma(\theta_q, \theta_p)) \\ &= i(2d \sum_{q \neq p}^d \frac{e^{i\theta_p}}{e^{i\theta_p} - e^{i\theta_q}} - (d-1)d). \end{aligned}$$

Notice that

$$\begin{aligned} \sum_{q \neq p}^d \Gamma(\log(e^{i\theta_p} - e^{i\theta_q}), \theta_p) &= \sum_{q \neq p}^d i \frac{(d-1)e^{i\theta_p} + e^{i\theta_q}}{e^{i\theta_p} - e^{i\theta_q}} \\ &= i(d \sum_{q \neq p}^d \frac{e^{i\theta_p}}{e^{i\theta_p} - e^{i\theta_q}} - (d-1)), \\ \sum_{q \neq r, q, r \neq p} \Gamma(\log(e^{i\theta_q} - e^{i\theta_r}), \theta_p) &= -i(d-1)(d-2). \end{aligned}$$

Hence the invariant measure should be

$$\rho = \prod_{(p,q), 1 \leq p, q \leq d, p \neq q} (e^{i\theta_p} - e^{i\theta_q})^2.$$

■

**Remark 9.1.2.** This metric is indeed flat metric restricted on  $\sum^d x_i = 0$  up to the dimension constant  $d$ . In fact, for each  $x_i$  in  $\mathbb{R}^d$ , define  $y_i = x_i - \frac{1}{d} \sum_p x_p$ . Then it is easy to check that  $\Gamma(y_i, y_j) = \delta_{ij} - \frac{1}{d}$ .

## 9.2 $SO(d)$ case

Recall the diffusion operators of Brownian motion on  $m \in SO(d)$  in Section 2.4.2,

$$\begin{aligned} \Gamma_{SO(d)}(m_{ij}, m_{kl}) &= \delta_{ik} \delta_{jl} - m_{il} m_{kj}, \\ L_{SO(d)}(m_{ij}) &= -(d-1)m_{ij}. \end{aligned}$$

For  $O \in SO(d)$ , if  $d = 2n$ , its eigenvalues can be written in the form as  $\{e^{i\theta_1}, e^{-i\theta_1}, \dots, e^{i\theta_n}, e^{-i\theta_n}\}$ ; otherwise  $d = 2n + 1$ , its eigenvalues are  $\{1, e^{i\theta_1}, e^{-i\theta_1}, \dots, e^{i\theta_n}, e^{-i\theta_n}\}$ . We have the following result:

**Proposition 9.2.1.**

$$\begin{aligned}\Gamma(\theta_i, \theta_j) &= \delta_{ij}, \\ \mathbb{L}(\theta_i) &= 2\text{Im}\left(\frac{1}{e^{2i\theta_i} - 1}\right) - 2 \sum_{j \neq i} \text{Im}\left(\frac{e^{i\theta_j}}{e^{i\theta_i} - e^{i\theta_j}}\right).\end{aligned}$$

*Proof.* — First let  $\{\lambda_i\}_{i=1, \dots, n}$  be the eigenvalues of  $O$ . Notice that  $P(X) = \det(X\text{Id} - O) = \prod_i (X - \lambda_i) = \prod_i (X - \lambda_i^{-1})$ , by direct computation,

$$\begin{aligned}&\Gamma(\log P(X), \log P(Y)) \\ &= \frac{X}{XY - 1} \frac{P'(X)}{P(X)} + \frac{Y}{XY - 1} \frac{P'(Y)}{P(Y)} + n \frac{XY}{1 - XY} + \frac{X^2}{X - Y} \frac{P'}{P}(X) - \frac{Y^2}{X - Y} \frac{P'}{P}(Y) \\ &= \sum_i \frac{1}{(X - \lambda_i)(Y - \lambda_i^{-1})} - \sum_i \frac{\lambda_i^2}{(X - \lambda_i)(Y - \lambda_i)}.\end{aligned}$$

When  $d = 2n$ , the eigenvalues are  $\{e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_2}, e^{-i\theta_2}, \dots, e^{i\theta_n}, e^{-i\theta_n}\}$ ; When  $n = 2d + 1$ , eigenvalues are  $\{1, e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_2}, e^{-i\theta_2}, \dots, e^{i\theta_d}, e^{-i\theta_d}\}$ ; however in both cases we always have

$$\Gamma(\log P(X), \log P(Y)) = \sum_i \frac{1}{(X - e^{i\theta_i})(Y - e^{-i\theta_i})} - \frac{e^{i2\theta_i}}{(X - e^{i\theta_i})(Y - e^{i\theta_i})}.$$

Therefore,

$$\Gamma(\theta_i, \theta_j) = \delta_{ij}.$$

Moreover,

$$\mathbb{L}(\log P(X)) = X^2 \frac{P'^2}{P^2} - (d + 1 + \frac{2}{X^2 - 1})X \frac{P'}{P} + d \frac{X^2}{X^2 - 1},$$

and

$$\begin{aligned}&\mathbb{L}(\log P(X)) \\ &= \sum_i \frac{\lambda_i^2}{(X - \lambda_i)^2} + 2 \sum_i \frac{1}{X - \lambda_i} \sum_{j \neq i} \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} + (d - 1) \sum_i \frac{\lambda_i}{X - \lambda_i} - 2 \sum_i \frac{1}{\lambda_i - \lambda_i^{-1}} \frac{1}{X - \lambda_i}.\end{aligned}$$

Thus

$$\mathbb{L}(\lambda_i) = -2 \sum_{j \neq i} \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} - (d - 1)\lambda_i + 2 \frac{1}{\lambda_i - \lambda_i^{-1}},$$

and

$$i\mathbb{L}(\theta_i) = -2 \sum_{j \neq i} \frac{e^{i\theta_j}}{e^{i\theta_i} - e^{i\theta_j}} + 2 \frac{1}{e^{2i\theta_i} - 1} - (d - 2).$$

Similarly,

$$\operatorname{Re}\left(-2 \sum_{j \neq i} \frac{e^{i\theta_j}}{e^{i\theta_i} - e^{i\theta_j}} + 2 \frac{1}{e^{2i\theta_i} - 1}\right) = d - 2,$$

such that

$$\mathsf{L}(\theta_i) = 2\operatorname{Im}\left(\frac{1}{e^{2i\theta_i} - 1}\right) - 2 \sum_{j \neq i} \operatorname{Im}\left(\frac{e^{i\theta_j}}{e^{i\theta_i} - e^{i\theta_j}}\right),$$

$$\Gamma(\log \rho, \theta_i) = \mathsf{L}(\theta_i) = i\left(2 \sum_{j \neq i} \frac{e^{i\theta_j}}{e^{i\theta_i} - e^{i\theta_j}} - 2 \frac{1}{e^{2i\theta_i} - 1} + (d - 2)\right).$$

Notice that

$$\begin{aligned} \sum_{j \neq i} \Gamma(\log(e^{i\theta_i} - e^{i\theta_j}), \theta_i) &= \sum_{j \neq i} i \frac{e^{i\theta_i}}{e^{i\theta_i} - e^{i\theta_j}} \\ &= \sum_{j \neq i} i \left(1 + \frac{e^{i\theta_j}}{e^{i\theta_i} - e^{i\theta_j}}\right), \\ \Gamma(\log(e^{-i\theta_i} - e^{i\theta_i}), \theta_i) &= i \frac{-e^{-i\theta_i} - e^{i\theta_i}}{e^{-i\theta_i} - e^{i\theta_i}} \\ &= i \left(1 + \frac{2}{e^{2i\theta_i} - 1}\right), \end{aligned}$$

Therefore,

$$\rho = \prod_{(i,j), i \neq j} (e^{i\theta_i} - e^{i\theta_j})^2 \prod_i |e^{-i\theta_i} - e^{i\theta_i}|.$$

■

**Remark 9.2.2.** Notice that the Brownian motion (2.4.4) on  $SO(d)$  up to a constant depending on dimension constant has nothing to do with dimension  $d$ ; while the Brownian motion (2.4.3) on  $SU(d)$  depends on the dimension. The results in this section gives an explanation: the spectrum of Brownian motion on  $SO(d)$  is indeed decided by some independent Brownian motion on  $\mathbb{R}^d$ , while the spectrum of Brownian motion on  $SU(d)$  is indeed Brownian motion on  $\mathbb{R}^d$  restricted to the surface  $\sum_i^d x_i = 0$ , which depends on the dimension.



# Chapter 10

## Polar decomposition

### 10.1 Polar decomposition of complex matrices

It is well known that for every complex matrix  $m$ , there is a polar decomposition:  $m = VN$ , where  $V$  is a unitary matrix and  $N$  is a Hermitian matrix. In this section we aim at describing the polar decomposition of a Brownian motion on the complex matrix.

We start from a Brownian motion on complex matrix  $m \in \mathcal{M}^d(\mathbb{C})$ , which satisfies

$$\Gamma(m_{ij}, m_{kl}) = 0, \quad \Gamma(m_{ij}, \bar{m}_{kl}) = 2\delta_{ik}\delta_{jl}, \quad \mathbb{L}(m_{ij}) = 0.$$

Now define  $M = m^*m$ , then it is Hermitian satisfying  $M = N^2$ , and it has a decomposition  $M = UD^2U^*$ , where  $U$  is a unitary matrix. Therefore, we have  $N = UDU^*$ .

In our case here, for any complex matrix  $m$ , its law is invariant under both left and right unitary multiplications. Since we have  $M = m^*m = UD^2U^*$  and  $m = VN$ , the law of  $(V, N)$  remains same under the transformation  $l_{(V_0, U_0)} : (V, U) \rightarrow (V_0U_0VU_0^*, U_0U)$  for two fixed unitary matrix  $V_0, U_0$ . This property ensures that if we know the law at  $V = U = \text{Id}$ , then we can know it at any point. Specifically, for two functions  $f, g$  on  $(V, U)$  we have at the fixed point  $(V_0, U_0)$

$$\begin{aligned} \Gamma(f(V, U), g(V, U))(V_0, U_0) &= \Gamma(f(l_{(V_0, U_0)}(V, U)), g(l_{(V_0, U_0)}(V, U)))(\text{Id}), \\ \mathbb{L}(f(V, U))(V_0, U_0) &= \mathbb{L}(f(l_{(V_0, U_0)}(V, U)))(\text{Id}). \end{aligned}$$

**Proposition 10.1.1.** *Let  $m$  be the Brownian motion on a  $d \times d$  complex matrix. Let  $m = VN$  be its polar decomposition, where  $V$  is a unitary matrix and  $N$  is a Hermitian matrix. Moreover, define  $M = m^*m$  and  $N$  has a spectral decomposition  $N = UDU^*$ , where  $D = \text{diag}x_1, \dots, x_d$  is*

a diagonal matrix and  $U$  is a unitary matrix. Then, at arbitrary point  $V, U$  we have

$$\begin{aligned}
\Gamma(m_{ij}, m_{kl}) &= 0, & \Gamma(m_{ij}, \bar{m}_{kl}) &= 2\delta_{ik}\delta_{jl}, & \mathbf{L}(m_{ij}) &= 0, \\
\Gamma(M_{ij}, M_{kl}) &= 2(\delta_{jk}M_{il} + \delta_{il}M_{jk}), & \mathbf{L}(M_{ij}) &= 4d\delta_{ij}, \\
\Gamma(x_i, x_j) &= \delta_{ij}, & \mathbf{L}(x_i) &= \frac{1}{x_i} + 4x_i \sum_{j \neq i} \frac{1}{x_i^2 - x_j^2}, \\
\rho_{ij} &= 2\frac{x_i^2 + x_j^2}{(x_i + x_j)^2}, & \Gamma(N_{ij}, N_{kl}) &= \sum \rho_{pq} U_{ip} \bar{U}_{jq} U_{kq} \bar{U}_{lp}, & \mathbf{L}(N_{ij}) &= 4 \sum_{kr} \frac{x_r}{(x_k + x_r)^2} U_{ik} \bar{U}_{jk}, \\
r_{ij} &= -2\frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2}, & \Gamma(U_{ij}, U_{kl}) &= r_{lj} U_{il} U_{kj}, & \Gamma(U_{ij}, \bar{U}_{kl}) &= -\delta_{lj} \sum_p r_{pj} U_{ip} \bar{U}_{kp}, \\
\mathbf{L}(U_{ij}) &= U_{ij} \sum_{k \neq j} r_{jk}, & \mathbf{L}(\bar{U}_{ij}) &= \bar{U}_{ij} \sum_{k \neq j} r_{jk}, \\
C_{ij} &= -\frac{4}{(x_i + x_j)^2}, & \Gamma(V_{ij}, V_{kl}) &= \sum_{rs} C_{rs} (VU)_{ir} (VU)_{ks} \bar{U}_{js} \bar{U}_{lr}, \\
\Gamma(V_{ij}, \bar{V}_{kl}) &= -\sum_{rs} C_{rs} (VU)_{ir} (\bar{VU})_{kr} \bar{U}_{js} U_{ls}, \\
\mathbf{L}(V_{ij}) &= \sum_{rs} C_{rs} (VU)_{ir} \bar{U}_{jr}, & \mathbf{L}(\bar{V}_{ij}) &= \sum_{rs} C_{rs} (\bar{VU})_{ir} U_{jr} \\
a_{ij} &= \frac{2}{(x_i + x_j)^2}, & \Gamma(V_{ij}, U_{kl}) &= \sum_{l \neq q} a_{lq} (VU)_{il} \bar{U}_{jq} U_{kq}, \\
\Gamma(V_{ij}, x_k) &= 0, & \Gamma(U_{ij}, x_k) &= 0,
\end{aligned}$$

*Proof.* — We divide the proof into several steps:

1. To compute the diagonal part  $D = \{x_1, x_2, \dots, x_d\}$ . Since  $M = m^* m$ , the eigenvalues of  $M$  are  $\{X_i\}_{i=1}^d$  and  $X_i = x_i^2$ . We have

$$\Gamma(M_{ij}, M_{kl}) = 2(\delta_{il}M_{kj} + \delta_{jk}M_{il}), \quad \mathbf{L}(M_{ij}) = 4d\delta_{ij}.$$

Let  $P(X) = \det(X\text{Id} - M)$  be the characteristic polynomial. Then

$$\begin{aligned}
\Gamma(\log P(X), \log P(Y)) &= \frac{4}{Y - X} \left( \frac{X P'(X)}{P(X)} - \frac{Y P'(Y)}{P(Y)} \right), \\
\mathbf{L}(\log P) &= -4X \frac{P'^2}{P^2}.
\end{aligned}$$

Comparing it with the formulas in terms of the eigenvalues leads to

$$\Gamma(X_i, X_j) = 4X_i \delta_{ij}, \quad \mathbf{L}(X_i) = 4 \left( 1 + 2X_i \sum_{i \neq j} \frac{1}{X_i - X_j} \right),$$

that is

$$\Gamma(x_i, x_j) = \delta_{ij}, \quad \mathbf{L}(x_i) = \frac{1}{x_i} + 4x_i \sum_{j \neq i} \frac{1}{x_i^2 - x_j^2}.$$

2. To compute the diffusion on  $U$ . In fact, since the law of  $U$  is left-invariant, we only need to compute  $\Gamma$  and  $L$  of  $U$  at identity. First we compute  $\Gamma(U_{ij}, X_k)$ . Notice that

$$\Gamma(M_{ij}, \log P(X)) = \sum_k \frac{1}{X_k - X} \Gamma(M_{ij}, X_k) = \sum_{pq} (M - X)_{qp}^{-1} \Gamma(M_{ij}, M_{pq}),$$

Setting  $V_{ijp,k} = \Gamma(U_{ip} \bar{U}_{jp}, X_k)$ , we get

$$\sum_k \frac{1}{X_k - X} \Gamma(M_{ij}, X_k) = 4 \left( \frac{M}{M - X} \right)_{ij}.$$

that is

$$4U \frac{D}{D - X} U^* + \sum_{kp} \frac{X_p}{X_k - X} V_{ijp,k} = 4 \frac{M}{M - X}.$$

from which

$$\sum_{kp} \frac{X_p}{X_k - X} V_{ijp,k} = 0.$$

Therefore,

$$\begin{aligned} \Gamma(M_{ij}, X_k) &= 4U_{ik} X_k \bar{U}_{jk}, \\ \sum_p X_p V_{ijp,k} &= 0. \end{aligned}$$

Since  $U$  is unitary, we have  $\sum_p \Gamma(U_{ip} \bar{U}_{jp}, X_k) = 0$ . Valuing this formula at identity leads to  $\Gamma(U_{ij}, X_k) = -\Gamma(\bar{U}_{ji}, X)$ . Also taking  $\sum_p X_p V_{ijp,k} = 0$  at  $U = \text{Id}$ , we have

$$(X_i - X_j) \Gamma(U_{ij}, X_k) = 0,$$

which indicates that  $\Gamma(U_{ij}, X_k) = 0$ , for any  $i \neq j$  and  $k$ .

Now we are in the position to compute  $\Gamma(U_{ij}, U_{kl})$ . Once again, all the computation only needs to be done at  $U = \text{Id}$ , where we have  $\Gamma(M_{ij}, M_{kl}) = 2\delta_{il}\delta_{kj}(X_i + X_j)$ ; On the other hand,

$$\begin{aligned} 2\delta_{il}\delta_{kj}(X_i + X_j) &= 4\delta_{i=j=k=l}X_i + X_i X_k \Gamma(\bar{U}_{ji}, \bar{U}_{lk}) \\ &\quad + X_i X_l \Gamma(\bar{U}_{ji}, U_{kl}) + X_j X_l \Gamma(U_{ij}, U_{kl}) + X_j X_k \Gamma(U_{ij}, \bar{U}_{lk}). \end{aligned}$$

At  $U = \text{Id}$ ,  $\Gamma(\cdot, \bar{U}_{ij}) = -\Gamma(\cdot, U_{ji})$ , therefore

$$2\delta_{il}\delta_{kj}(X_i + X_j) = 4\delta_{i=j=k=l}X_i + (X_i - X_j)(X_k - X_l)\Gamma(U_{ij}, U_{kl}).$$

From which we deduce for  $i \neq j$

$$\Gamma(U_{ij}, U_{kl})(\text{Id}) = -2 \frac{(X_i + X_j)}{(X_i - X_j)^2} \delta_{il}\delta_{kj},$$

so we have

$$\Gamma(U_{ij}, \bar{U}_{kl})(\text{Id}) = -\Gamma(U_{ij}, U_{lk}) = 2 \frac{X_i + X_j}{(X_i - X_j)^2} \delta_{ik}\delta_{jl}.$$

Observe that  $\Gamma(U_{ii}, U_{ii}) = r_{ii}$  does not play any role in the computations. This comes from the fact that  $(U, D) \mapsto UDU^*$  is not a local homeomorphism, the choice of  $U$  is not unique. In fact for a Hermitian matrix  $H$  with distinct eigenvectors,  $H \rightarrow (D, [U])$  is one-to-one where  $[U] = \{VW, \forall V \in U(d)\}$ ,  $VW = \{VW_0, W_0 \in W\}$  and  $W = \text{diag}(e^{i\phi_1}, \dots, e^{i\phi_d})$ . We may choose the phase of  $U$  such that  $\det(U) = 1$ . Therefore, at  $U = \text{Id}$  for any  $i, j$ ,

$$0 = \Gamma(\log \det(U), U_{ij}) = \sum_k \Gamma(U_{kk}, U_{ij}) = r_{ii}\delta_{ij},$$

such that for any  $i$ ,  $r_{ii} = 0$ .

As for  $L(U_{ij})$ , notice that  $\sum_r U_{ir}\bar{U}_{jr} = \delta_{ij}$ , so at  $U = \text{Id}$ ,

$$L(\bar{U}_{ji}) + L(U_{ij}) + 2 \sum_r \Gamma(U_{ir}, \bar{U}_{jr}) = 0.$$

Since  $M = UD^2U^*$ ,  $M_{ij} = \sum_r U_{ir}X_r\bar{U}_{jr}$ , we have

$$4d\delta_{ij} = L(M_{ij}) = X_jL(U_{ij}) + X_iL(\bar{U}_{ji}) + L(X_i)\delta_{ij} + 2 \sum_r r_{ir}X_r\delta_{ij},$$

$$(X_j - X_i)L(U_{ij}) = (4d - 2 \sum_r r_{ir}X_r + 2 \sum_r X_i r_{ir} - L(X_i))\delta_{ij}.$$

Notice that  $4d - 2 \sum_r r_{ir}X_r + 2 \sum_r X_i r_{ir} - L(X_i) = 0$ , such that

$$(X_j - X_i)L(U_{ij}) = 0.$$

Therefore, we must have

$$L(U_{ij}) = L(\bar{U}_{ji}) = - \sum_{r \neq i} r_{ir}\delta_{ij} = -2 \sum_{r \neq i} \frac{x_i^2 + x_r^2}{(x_i^2 - x_r^2)^2} \delta_{ij}.$$

3. We now compute the diffusion on  $N$  and  $V$ . Since  $M = UD^2U^*$  and  $N = UDU^*$ , where  $D^2 = \text{diag}\{X_1, \dots, X_d\} = \{x_1^2, \dots, x_d^2\}$ , it is not difficult to compute that

$$\Gamma(N_{ij}, N_{kl}) = \delta_{ijkl} + \mathbb{1}_{i \neq j} 2\delta_{il}\delta_{jk} \frac{x_i^2 + x_j^2}{(x_i + x_j)^2} = 2\delta_{il}\delta_{kj} \frac{x_i^2 + x_j^2}{(x_i + x_j)^2}.$$

Let  $\widetilde{M} = mm^*$ , then

$$\begin{aligned} \Gamma(\widetilde{M}_{ij}, \widetilde{M}_{kl}) &= 2\delta_{il}M_{kj} + 2\delta_{jk}M_{il}, \\ L(\widetilde{M}_{ij}) &= 4d\delta_{ij}, \end{aligned}$$

which is exactly the same as  $M$ . Since  $\widetilde{M} = WD^2W^*$ , follow the procedure above, we can compute  $\Gamma$  and  $L$  of  $W$ , which is the same as  $U$ . However, we should pay attention here the spectral decomposition of  $M$  only gives  $\Gamma(U_{ij}, U_{kl})$  when  $i \neq j, k \neq l$ . We have at  $W = \text{Id}$ , when  $i \neq j, k \neq l$

$$\Gamma(W_{ij}, W_{kl}) = -2 \frac{(x_i^2 + x_j^2)}{(x_i^2 - x_j^2)^2} \delta_{il}\delta_{kj}.$$

and we also have  $\Gamma(W_{ij}, x_k) = 0$ .

Since  $m_{ij} = \sum_p W_{ip} x_p \bar{U}_{jp}$  and  $\Gamma(m_{ij}, m_{kl}) = 0$ ,  $\Gamma(m_{ij}, \bar{m}_{kl}) = 2\delta_{ik}\delta_{jl}$ , we get

$$\begin{aligned} 0 &= \Gamma(W_{ij}, W_{kl})x_j x_l + \Gamma(x_i, x_k)\delta_{ij}\delta_{kl} + \Gamma(U_{ij}, U_{kl})x_i x_k \\ &\quad - \Gamma(W_{ij}, U_{kl})x_j x_k - \Gamma(W_{kl}, U_{ij})x_i x_l, \\ 2\delta_{ik}\delta_{jl} &= -\Gamma(W_{ij}, W_{lk})x_j x_l + \Gamma(x_i, x_k)\delta_{ij}\delta_{kl} - \Gamma(U_{ij}, U_{lk})x_i x_k \\ &\quad + \Gamma(W_{ij}, U_{lk})x_j x_k + \Gamma(W_{lk}, U_{ij})x_i x_l, \end{aligned}$$

which leads to

$$\Gamma(U_{ij}, W_{kl}) = -\frac{4x_i x_j}{(x_i^2 - x_j^2)^2} \delta_{il} \delta_{kj}.$$

And when  $i = j$  or  $k = l$ ,

$$\Gamma(W_{ij}, W_{kl}) = -\frac{1}{x_i^2} \delta_{ijkl}.$$

4. Since  $W = VU$ , we can compute  $\Gamma$  and  $L$  of  $V$  from  $W$  and  $U$ . At  $V = U = \text{Id}$ ,

$$\begin{aligned} \Gamma(W_{ij}, W_{kl}) &= \Gamma(V_{ij}, V_{kl}) + \Gamma(U_{ij}, U_{kl}) + \Gamma(V_{ij}, U_{kl}) + \Gamma(V_{kl}, U_{ij}), \\ \Gamma(U_{ij}, W_{kl}) &= \Gamma(U_{ij}, V_{kl}) + \Gamma(U_{ij}, U_{kl}), \end{aligned}$$

which leads to

$$\begin{aligned} \Gamma(U_{ij}, V_{kl}) &= \frac{2}{(x_i + x_j)^2} \delta_{il} \delta_{kj}, \\ \Gamma(V_{ij}, V_{kl}) &= -\frac{4}{(x_i + x_j)^2} \delta_{il} \delta_{kj}. \end{aligned}$$

Also since  $N = UDU^*$ ,

$$\Gamma(V_{ij}, N_{kl}) = 2\frac{(x_i - x_j)}{(x_i + x_j)^2} \delta_{il} \delta_{jk} := S_{ij} \delta_{il} \delta_{jk}.$$

Since  $V$  is unitary, we have  $\Gamma(\bar{V}_{ij}, \cdot) = -\Gamma(V_{ji}, \cdot)$  at  $V = \text{Id}$ , such that

$$\Gamma(\bar{V}_{ij}, N_{kl}) = 2\frac{(x_i - x_j)}{(x_i + x_j)^2} \delta_{ik} \delta_{jl}.$$

Now compute  $L(N_{ij})$ ,  $N_{ij} = \sum_r U_{ir} x_r \bar{U}_{jr}$  at  $U = \text{Id}$ , for  $i \neq j$

$$\begin{aligned} L(N_{ij}) &= (L(x_i) + 2 \sum_{r \neq i} r_{ir} x_r - 2 \sum_{r \neq i} x_i r_{ir}) \delta_{ij} \\ &= \left( \frac{1}{x_i} + 4 \sum_{r \neq i} \frac{x_r}{(x_i + x_r)^2} \right) \delta_{ij} = 4 \sum_r \frac{x_r}{(x_i + x_r)^2} \delta_{ij}. \end{aligned}$$

Now for the fact that  $L(m_{ij}) = L(\bar{m}_{kl}) = 0$ , we can compute  $L(V_{ij})$  and  $L(\bar{V}_{ij})$ . At  $V = U = \text{Id}$ ,

$$\begin{aligned} L(m_{ij}) &= \sum_r L(V_{ir}N_{rj}) = \sum_r V_{ir}L(N_{rj}) + L(V_{ir})N_{rj} + 2\Gamma(V_{ir}, N_{rj}) \\ &= 4 \sum_r \frac{x_r}{(x_i + x_r)^2} \delta_{ij} + L(V_{ij})x_j + 2 \sum_r S_{ir} \delta_{ij}, \end{aligned}$$

so that

$$L(V_{ij}) = -4 \sum_r \frac{1}{(x_i + x_r)^2} \delta_{ij},$$

and

$$L(\bar{V}_{ij}) = L(V_{ij}) = -4 \sum_r \frac{1}{(x_i + x_r)^2} \delta_{ij}.$$

Now we have  $\Gamma$  and  $L$  for all the elements in the polar decomposition of the complex matrix  $m$  at  $V = U = \text{Id}$ ,

$$\begin{aligned} \Gamma(U_{ij}, U_{kl}) &= -2 \frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2} \delta_{il} \delta_{jk}, \quad \Gamma(U_{ij}, \bar{U}_{kl}) = 2 \frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2} \delta_{ik} \delta_{jl}, \\ L(U_{ij}) = L(\bar{U}_{ij}) &= -2 \sum_{k \neq i} \frac{x_i^2 + x_k^2}{(x_i^2 - x_k^2)^2} \delta_{ij}, \\ \Gamma(x_i, x_j) = \delta_{ij}, \quad L(x_i) &= \frac{1}{x_i} + 4x_i \sum_{j \neq i} \frac{1}{x_i^2 - x_j^2}, \\ \Gamma(N_{ij}, N_{kl}) &= 2 \frac{x_i^2 + x_j^2}{(x_i + x_j)^2} \delta_{il} \delta_{jk}, \quad L(N_{ij}) = 4 \sum_r \frac{x_r}{(x_i + x_r)^2} \delta_{ij}, \\ \Gamma(V_{ij}, V_{kl}) &= -\frac{4}{(x_i + x_j)^2} \delta_{il} \delta_{jk}, \quad \Gamma(V_{ij}, \bar{V}_{kl}) = \frac{4}{(x_i + x_j)^2} \delta_{ik} \delta_{jl}, \\ L(V_{ij}) = L(\bar{V}_{ij}) &= -4 \sum_r \frac{1}{(x_i + x_r)^2} \delta_{ij}, \\ \Gamma(V_{ij}, x_k) = 0, \quad \Gamma(U_{ij}, x_k) &= 0, \quad \Gamma(V_{ij}, U_{kl}) = \frac{2}{(x_i + x_j)^2} \delta_{il} \delta_{jk}. \end{aligned}$$

By the property of invariance under the transformation  $(V, N) \rightarrow (V_0 U_0 V U_0^*, (U_0 U) D (U_0 U)^*)$ , we have at arbitrary point  $V, U$

$$\begin{aligned} \Gamma(U_{ij}, U_{kl})(U) &= \sum U_{ip} U_{kq} \Gamma(U_{pj}, U_{ql})(\text{Id}), \\ \Gamma(V_{ij}, V_{kl})(V) &= \sum (VU)_{ip} (VU)_{kq} \bar{U}_{jr} \bar{U}_{ls} \Gamma(V_{pr}, V_{qs})(\text{Id}), \\ L(U_{ij}) &= \sum U_{ip} L(U_{pj}), \quad L(\bar{U}_{ij}) = \bar{U}_{ip} L(\bar{U}_{pj}), \end{aligned}$$

other terms such as  $\Gamma(V_{ij}, U_{kl})$ ,  $L(V_{ij})$  follow the same procedure. In the end, we get the conclusion in the proposition. ■

## 10.2 Polar decomposition of real matrices

Consider polar decomposition for Brownian motion on real matrix  $m$ . First we have

$$\Gamma(m_{ij}, m_{kl}) = \delta_{ik}\delta_{jl}, \quad \mathbf{L}(m_{ij}) = 0.$$

$M = m^t m$  is a symmetric matrix, assume it can be diagonalized as  $M = P^t D^2 P$ , where  $D = \text{diag}\{\lambda_1, \dots, \lambda_d\}$ ,  $P$  is an orthogonal matrix. Then  $m$  can be decomposed as  $m = VN = QDP$ , where  $N = M^{\frac{1}{2}}$ ,  $V$  is also an orthogonal matrix and  $Q = VP^t$ .

**Proposition 10.2.1.** *Let  $m$  be the Brownian motion on a  $d \times d$  real matrix. Let  $m = VN$  be its polar decomposition, where  $V$  is an orthogonal and  $N$  is a symmetric matrix. Moreover, define  $M = m^t m$  and  $N$  has a spectral decomposition  $N = PDP^*$ , where  $D = \text{diag}x_1, \dots, x_d$  is a diagonal matrix and  $P$  is an orthogonal matrix. At arbitrary  $V, P$ , we have*

$$\begin{aligned} \Gamma(M_{ij}, M_{kl}) &= \delta_{ik}M_{jl} + \delta_{kj}M_{il} + \delta_{il}M_{jk} + \delta_{jl}M_{ik}, \\ \mathbf{L}(M_{ij}) &= 2d\delta_{ij}, \\ \Gamma(\lambda_i, \lambda_j) &= 4\lambda_i\delta_{ij}, \quad \mathbf{L}(\lambda_i) = \sum_{j \neq i} \frac{4\lambda_j}{\lambda_i - \lambda_j} + 4d - 2, \\ \Gamma(x_i, x_j) &= \delta_{ij}, \quad \mathbf{L}(x_i) = \sum_{j \neq i} \frac{2x_i}{x_i^2 - x_j^2}, \\ \Gamma(P_{ij}, x_k) &= 0, \quad \Gamma(V_{ij}, x_k) = 0. \end{aligned}$$

Let  $a_{ij} = \frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2}$ ,  $c_{ij} = \frac{1}{(x_i + x_j)^2}$ ,

$$\begin{aligned} \Gamma(P_{ij}, P_{kl}) &= \delta_{ik} \sum_{r \neq i} a_{ir} P_{rj} P_{rl} - a_{ik} P_{kj} P_{il}, \quad \mathbf{L}(P_{ij}) = -P_{ij} \sum_{q \neq i} a_{iq}, \\ \Gamma(V_{ij}, V_{kl}) &= \sum_{pq} 2c_{pq} (VP^t)_{ip} (VP^t)_{kp} P_{qj} P_{ql} - \sum_{pq} 2c_{pq} (VP^t)_{ip} (VP^t)_{kq} P_{qj} P_{pl}, \\ \mathbf{L}(V_{ij}) &= - \sum_{p \neq q} 2c_{pq} (VP^t)_{ip} P_{pj}, \\ \Gamma(P_{ij}, V_{kl}) &= \sum_p c_{ip} P_{pj} P_{pl} (VP^t)_{ki} - \sum_p c_{ip} P_{pj} (VP^t)_{kp} P_{il}, \\ \Gamma(N_{ij}, N_{kl}) &= \sum_{p \neq q} a_{pq} (P_{pi} P_{qj} P_{pk} P_{ql} + P_{pi} P_{qj} P_{qk} P_{pl}), \\ \mathbf{L}(N_{ij}) &= 2 \sum_{p \neq q} \frac{x_p}{(x_q + x_p)^2} P_{pi} P_{pj}, \\ \Gamma(N_{ij}, V_{kl}) &= \sum_{p \neq q} a_{pq} (P_{pi} P_{qj} (VP^t)_{kp} P_{ql} - P_{pi} P_{qj} (VP^t)_{kq} P_{pl}). \end{aligned}$$

*Proof.* — First, to compute the spectrum of  $M$ .

$$\begin{aligned} \Gamma(M_{ij}, M_{kl}) &= \delta_{ik}M_{jl} + \delta_{kj}M_{il} + \delta_{il}M_{jk} + \delta_{jl}M_{ik}, \\ \mathbf{L}(M_{ij}) &= 2d\delta_{ij}. \end{aligned}$$

Let  $P(X) = \det(X\text{Id} - M)$ ,

$$\begin{aligned}\Gamma(\log P(X), \log P(Y)) &= 4\text{trace}(M^{-1}(X)MM^{-1}(Y)) = \frac{4}{Y-X}\left(X\frac{P'(X)}{P(X)} - Y\frac{P'(Y)}{P(Y)}\right), \\ \text{L}(\log P(X)) &= -2\text{trace}(M^{-1}(X)MM^{-1}(X)) - 2\text{trace}(M^{-1}(X))\text{trace}(M^{-1}(X)M) \\ &\quad - 2d\text{trace}(M^{-1}(X)) \\ &= 2X\frac{P''}{P}(X) - 4X\frac{P'(X)^2}{P(X)^2} + 2\frac{P'(X)}{P(X)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Gamma(\log P(X), \log P(Y)) &= \frac{\Gamma(\lambda_i, \lambda_j)}{(X - \lambda_i)(Y - \lambda_j)}, \\ \text{L}(\log P(X)) &= \sum_i -\frac{\Gamma(\lambda_i, \lambda_i)}{(X - \lambda_i)^2} - \frac{\text{L}(\lambda_i)}{(X - \lambda_i)}.\end{aligned}$$

Compare the two formulas, we get

$$\begin{aligned}\Gamma(\lambda_i, \lambda_j) &= 4\lambda_i\delta_{ij}, \\ \text{L}(\lambda_i) &= \sum_{j \neq i} \frac{4\lambda_j}{\lambda_i - \lambda_j} + 4d - 2.\end{aligned}$$

Since  $\lambda_i = x_i^2$  for any  $i$ ,

$$\begin{aligned}\Gamma(x_i, x_j) &= \delta_{ij}, \\ \text{L}(x_i) &= \sum_{j \neq i} \frac{2x_i}{x_i^2 - x_j^2}.\end{aligned}$$

Now we compute the diffusion on  $P$ . For the same reason as in the complex matrix case, we only need to compute it at  $P = \text{Id}$ .

$$\begin{aligned}\Gamma(M_{ij}, \log \det(X\text{Id} - M)) &= M^{-1}(X)_{lk}\Gamma(M_{ij}, M_{kl}) \\ &= 4(MM^{-1}(X))_{ij}.\end{aligned}$$

Moreover,

$$\Gamma(M_{ij}, \log \det(X\text{Id} - M)) = \sum_k 4\frac{\lambda_k}{X - \lambda_k} P_{ki}P_{kj} + \frac{\lambda_l}{X - \lambda_k} \Gamma(P_{li}P_{lj}, \lambda_k),$$

which ends in

$$\sum_{k,l} \frac{\lambda_l}{X - \lambda_k} \Gamma(P_{li}P_{lj}, \lambda_k) = 0,$$

such that

$$\sum_l \lambda_l \Gamma(P_{li}P_{lj}, \lambda_k) = 0.$$

Valuing the formula at  $P = \text{Id}$ , we have  $\Gamma(P_{ij}, \lambda_k) = 0$ .

Also notice that at  $P = \text{Id}$ ,

$$\begin{aligned}\Gamma(M_{ij}, M_{kl}) &= (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})(\lambda_i + \lambda_j), \\ \Gamma(P_{pi}P_{pj}, \cdot) &= \Gamma(P_{ji}, \cdot) + \Gamma(P_{ij}, \cdot) = 0,\end{aligned}$$

such that we have

$$\begin{aligned}\Gamma(M_{ij}, M_{kl}) &= \sum_{pq} \delta_{ij}\delta_{kl}\Gamma(\lambda_i, \lambda_k) + \lambda_p\lambda_q\Gamma(P_{pi}P_{pj}, P_{qk}P_{ql}) \\ &= 4\lambda_i\delta_{ijkl} + (\lambda_i - \lambda_j)(\lambda_k - \lambda_l)\Gamma(P_{ij}, P_{kl}).\end{aligned}$$

Therefore,

$$\begin{aligned}\Gamma(P_{ij}, P_{kl}) &= (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj})\frac{\lambda_i + \lambda_j}{(\lambda_i - \lambda_j)^2} \\ &= (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj})\frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2}.\end{aligned}$$

As for  $L(P_{ij})$ , at  $P = \text{Id}$

$$\begin{aligned}2d\delta_{ij} &= L(M_{ij}) = \sum_p L(P_{pi}\lambda_p P_{pj}) \\ &= \sum_p 2\lambda_p\Gamma(P_{pi}, P_{pj}) + P_{pi}P_{pj}L(\lambda_p) + P_{pi}\lambda_p L(P_{pj}) + L(P_{pi})\lambda_p P_{pj}.\end{aligned}$$

Also we have

$$0 = \sum_p L(P_{ip}P_{jp}) = 2\sum_p \Gamma(P_{ip}, P_{jp}) + L(P_{ij}) + L(P_{ji}).$$

Then

$$\begin{aligned}&(\lambda_i - \lambda_j)L(P_{ij}) \\ &= -2\delta_{ij}\sum_p \frac{\lambda_p(\lambda_i + \lambda_p)}{(\lambda_i - \lambda_p)^2} + 2\delta_{ij}\sum_p \frac{\lambda_i(\lambda_i + \lambda_p)}{(\lambda_i - \lambda_p)^2} - \delta_{ij}\left(\sum_{p \neq i} \frac{4\lambda_p}{\lambda_i - \lambda_p} + 4d - 2\right) + 2d\delta_{ij} = 0,\end{aligned}$$

which indicates that for any  $i, j$ , at  $P = \text{Id}$ , there exists a constant  $A_i$  such that we always have  $L(P_{ij}) = A_i\delta_{ij}$ . Therefore,

$$L(P_{ij}) = -\delta_{ij}\sum_p \frac{\lambda_i + \lambda_p}{(\lambda_i - \lambda_p)^2}.$$

Now we compute  $Q$ . Let  $\widetilde{M} = mm^t = QD^2Q^t$ , then

$$\Gamma(\widetilde{M}_{ij}, \widetilde{M}_{kl}) = \widetilde{M}_{jl}\delta_{ik} + \widetilde{M}_{jk}\delta_{il} + \widetilde{M}_{il}\delta_{jk} + \widetilde{M}_{ik}\delta_{jl},$$

which is all the same as  $\Gamma(M_{ij}, M_{kl})$ , implying  $Q_{ij}$  has the same  $\Gamma$  as  $P_{ij}$ :

$$\Gamma(Q_{ij}, Q_{kl}) = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj})\frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2}.$$

Moreover, at  $P = Q = \text{Id}$ ,

$$\begin{aligned}\delta_{ik}\delta_{lj} &= \Gamma(m_{ij}, m_{kl}) = \sum_{p,q} \Gamma(Q_{ip}x_pP_{pj}, Q_{kq}x_qP_{ql}) \\ &= \delta_{ijkl} + x_i x_k \Gamma(P_{ij}, P_{kl}) + x_j x_k \Gamma(Q_{ij}, P_{kl}) + x_i x_l \Gamma(P_{ij}, Q_{kl}) + x_j x_l \Gamma(Q_{ij}, Q_{kl}).\end{aligned}$$

Thus,

$$\begin{aligned}&x_j x_k \Gamma(Q_{ij}, P_{kl}) + x_i x_l \Gamma(P_{ij}, Q_{kl}) \\ &= -\frac{4x_i^2 x_j^2}{(x_i^2 - x_j^2)^2} \delta_{ik} \delta_{jl} + \frac{2x_i x_j (x_i^2 + x_j^2)}{(x_i^2 - x_j^2)^2} \delta_{il} \delta_{kj}.\end{aligned}$$

Exchange  $i$  and  $j$  and apply the fact that  $\Gamma(Q_{ij}, \cdot) = -\Gamma(Q_{ji}, \cdot)$  and  $\Gamma(P_{ij}, \cdot) = -\Gamma(P_{ji}, \cdot)$ ,

$$\begin{aligned}&-x_i x_k \Gamma(Q_{ij}, P_{kl}) - x_j x_l \Gamma(P_{ij}, Q_{kl}) \\ &= -\frac{4x_i^2 x_j^2}{(x_i^2 - x_j^2)^2} \delta_{jk} \delta_{il} + \frac{2x_i x_j (x_i^2 + x_j^2)}{(x_i^2 - x_j^2)^2} \delta_{jl} \delta_{ik}.\end{aligned}$$

Hence

$$\Gamma(P_{ij}, Q_{kl}) = -\frac{2x_i x_j}{(x_i^2 - x_j^2)^2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}).$$

And for  $L(Q_{ij})$ ,

$$\begin{aligned}0 &= L(m_{ij}) = \sum_p L(Q_{ip}x_pP_{pj}) \\ &= \sum_p 2x_p \Gamma(Q_{ip}, P_{pj}) + L(Q_{ip})x_pP_{pj} + Q_{ip}L(x_p)P_{pj} + Q_{ip}x_pL(P_{pj}).\end{aligned}$$

At  $P = Q = \text{Id}$ , this reduces to

$$0 = 2 \sum_p x_p \Gamma(Q_{ip}, P_{pj}) + L(Q_{ij})x_j + \delta_{ij}L(x_i) + x_iL(P_{ij}).$$

Therefore,

$$L(Q_{ij}) = -\delta_{ij} \sum \frac{x_i^2 + x_p^2}{(x_i^2 - x_p^2)^2}.$$

Now write  $Q = VP^t$ , where  $V$  is also an orthogonal matrix. At  $P = Q = \text{Id}$ ,

$$\begin{aligned}\Gamma(Q_{ij}, Q_{kl}) &= \Gamma(V_{ij}, V_{kl}) + \Gamma(V_{ij}, P_{lk}) + \Gamma(P_{ji}, V_{kl}) + \Gamma(P_{ji}, P_{lk}), \\ \Gamma(P_{ij}, Q_{kl}) &= \Gamma(P_{ij}, V_{kl}) + \Gamma(P_{ij}, P_{lk}).\end{aligned}$$

Therefore,

$$\begin{aligned}\Gamma(P_{ij}, V_{kl}) &= \frac{1}{(x_i + x_j)^2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}), \\ \Gamma(V_{ij}, V_{kl}) &= \frac{2}{(x_i + x_j)^2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}).\end{aligned}$$

As for L,

$$\begin{aligned} L(Q_{ij}) &= 2\Gamma(V_{ip}, P_{jp}) + L(V_{ip})P_{jp} + L(P_{jp})V_{ip} \\ &= \sum_{p \neq i} \frac{2}{(x_i + x_p)^2} \delta_{ij} + L(V_{ij}) + L(P_{ji}). \end{aligned}$$

Therefore,

$$L(V_{ij}) = - \sum_{p \neq i} \frac{2}{(x_i + x_p)^2} \delta_{ij}.$$

As for  $N = M^{\frac{1}{2}} = P^t D P$ , we have

$$\begin{aligned} \Gamma(N_{ij}, N_{kl}) &= \frac{x_i^2 + x_j^2}{(x_i + x_j)^2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj}), \\ L(N_{ij}) &= 2\delta_{ij} \sum_{p \neq i} \frac{x_p}{(x_i + x_p)^2}, \end{aligned}$$

and

$$\Gamma(N_{ij}, V_{kl}) = \frac{x_i - x_j}{(x_i + x_j)^2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}).$$

To sum up, we have

$$\begin{aligned} \Gamma(M_{ij}, M_{kl}) &= \delta_{ik} M_{jl} + \delta_{kj} M_{il} + \delta_{il} M_{jk} + \delta_{jl} M_{ik}, \\ L(M_{ij}) &= 2d\delta_{ij}, \\ \Gamma(\lambda_i, \lambda_j) &= 4\lambda_i \delta_{ij}, \quad L(\lambda_i) = \sum_{j \neq i} \frac{4\lambda_j}{\lambda_i - \lambda_j} + 4d - 2, \\ \Gamma(x_i, x_j) &= \delta_{ij}, \quad L(x_i) = \sum_{j \neq i} \frac{2x_i}{x_i^2 - x_j^2}. \end{aligned}$$

And at  $P = Q = \text{Id}$ ,

$$\begin{aligned} \Gamma(P_{ij}, P_{kl}) &= \Gamma(Q_{ij}, Q_{kl}) = (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}) \frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2}, \\ \Gamma(P_{ij}, Q_{kl}) &= - \frac{2x_i x_j}{(x_i^2 - x_j^2)^2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}), \\ L(P_{ij}) &= -\delta_{ij} \sum_{p \neq i} \frac{x_i^2 + x_p^2}{(x_i^2 - x_p^2)^2}, \\ L(Q_{ij}) &= -\delta_{ij} \sum_{p \neq i} \frac{x_i^2 + x_p^2}{(x_i^2 - x_p^2)^2}. \end{aligned}$$

$$\begin{aligned} \Gamma(P_{ij}, V_{kl}) &= \frac{1}{(x_i + x_j)^2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}), \\ \Gamma(V_{ij}, V_{kl}) &= \frac{2}{(x_i + x_j)^2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}). \end{aligned}$$

As for  $L$ ,

$$\begin{aligned} L(Q_{ij}) &= 2\Gamma(V_{ip}, P_{jp}) + L(V_{ip})P_{jp} + L(P_{jp})V_{ip} \\ &= \sum_{p \neq i} \frac{2}{(x_i + x_p)^2} \delta_{ij} + L(V_{ij}) + L(P_{ji}). \end{aligned}$$

Therefore,

$$L(V_{ij}) = - \sum_{p \neq i} \frac{2}{(x_i + x_p)^2} \delta_{ij}.$$

Since  $P$  is invariant by right multiplication,  $V$  is invariant under  $V \rightarrow V_0 P_0^t V P_0$ , then we get the conclusion in the proposition.  $\blacksquare$

### 10.3 Ricci curvature on the manifold of $m^*m$

In this section, we compute the curvature on the space of Hermitian matrices

$$\mathcal{H} = \{M = m^*m, m \text{ is a complex matrix with Euclidean metric}\}.$$

We use the same notations as those in the previous section. Recall that

$$\Gamma(M_{ij}, M_{kl}) = 2(\delta_{jk}M_{il} + \delta_{il}M_{kj}), \quad L(M_{ij}) = 4d\delta_{ij}.$$

We now compute the corresponding Riemannian metric  $G$  on  $\mathcal{H}$ . Notice that the metric on  $\mathcal{H}$  is invariant under unitary transformation, i.e. for any  $M \in \mathcal{H}$  with spectral decomposition  $M = UD^2U^*$ , where  $U$  is a unitary matrix, we have

$$\Gamma(M_{ij}, M_{kl})(U) = \sum_{pqrs} U_{ip} \bar{U}_{jq} U_{kr} \bar{U}_{ls} \Gamma(M_{pq}, M_{rs})(U = \text{Id}).$$

Then it suffices to compute the Riemannian metric at  $U = \text{Id}$ , where

$$\Gamma(M_{ij}, M_{kl})(U = \text{Id}) = 2(X_i + X_j) \delta_{il} \delta_{jk}.$$

Therefore,

$$G(M_{ij}, M_{kl})(U = \text{Id}) = \frac{1}{2(X_i + X_j)} \delta_{il} \delta_{jk},$$

such that at any point  $U$  we have

$$G(M_{ij}, M_{kl}) = \sum_{pqrs} \frac{1}{2(X_p + X_q)} \bar{U}_{ip} U_{jq} \bar{U}_{kr} U_{ls}.$$

We are now in the position to compute the Ricci curvature. The method to compute Ricci curvature follows from that for any  $f$  who is a smooth function  $M$ , we have

$$\Gamma_2(f) = \|\text{Hess}f\|^2 + \text{Ric}(\nabla f, \nabla f).$$

Therefore,

$$\text{Ric}(M_{ij}, M_{kl}) = \Gamma_2(M_{ij}, M_{kl}) - \langle \text{Hess}M_{ij}, \text{Hess}M_{kl} \rangle.$$

**Proposition 10.3.1.**

$$\begin{aligned}\Gamma_2(M_{ij}, M_{kl}) &= 8d\delta_{il}\delta_{jk}, \\ Ric_{ij,kl} &= 8\sum_a \left(1 - \frac{X_a^2}{(X_a + X_i)(X_a + X_j)}\right)\delta_{il}\delta_{jk} - 8\frac{X_i X_k}{(X_i + X_k)^2}\delta_{ij}\delta_{kl},\end{aligned}$$

and the Ricci curvature is positive.

*Proof.* — First we compute  $\Gamma_2(M_{ij}, M_{kl})$ . Recall that

$$\Gamma(M_{ij}, M_{kl}) = 2(\delta_{jk}M_{il} + \delta_{il}M_{kj}), \quad L(M_{ij}) = 4d\delta_{ij}.$$

Then by definition,

$$\Gamma_2(M_{ij}, M_{kl}) = 8d\delta_{il}\delta_{jk}.$$

Then by the formula that (see the book by Bakry, Gentil, Ledoux [14])

$$\text{Hess}f(h, g) = \frac{1}{2}(\Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(h, g))),$$

we are able to compute that

$$\begin{aligned}\text{Hess}M_{ij}(M_{pq}, M_{rs}) &= \frac{1}{2}(\Gamma(M_{pq}, \Gamma(M_{ij}, M_{rs})) + \Gamma(M_{rs}, \Gamma(M_{ij}, M_{pq})) - \Gamma(M_{ij}, \Gamma(M_{pq}, M_{rs}))) \\ &= 4(\delta_{is}\delta_{pj}M_{rq} + \delta_{rj}\delta_{iq}M_{ps}).\end{aligned}$$

Notice that  $\Gamma_2(M_{ij}, M_{kl})$  is invariant under the unitary transformation of  $M$ , while  $\text{Hess}M_{ij}$  is not in the same case. However,  $\langle \text{Hess}M_{ij}, \text{Hess}M_{kl} \rangle$  is unitary-invariant:

$$\begin{aligned}&\langle \text{Hess}M_{ij}, \text{Hess}M_{kl} \rangle \\ &= \sum G(M_{pq}, M_{rs})G(M_{ab}, M_{cd})\text{Hess}M_{ij}(M_{pq}, M_{ab})\text{Hess}M_{kl}(M_{rs}, M_{cd}) \\ &= \sum 16[G(M_{jq}, M_{rk})G(M_{ai}, M_{ld})M_{aq}M_{rd} + G(M_{pi}, M_{rk})G(M_{jb}, M_{ld})M_{pb}M_{rd} \\ &\quad + G(M_{jq}, M_{ls})G(M_{ai}, M_{ck})M_{aq}M_{cs} + G(M_{pi}, M_{ls})G(M_{jb}, M_{ck})M_{pb}M_{cs}].\end{aligned}$$

In fact,

$$\begin{aligned}\sum G(M_{jq}, M_{rk})G(M_{ai}, M_{ld})M_{aq}M_{rd} &= \sum_{uvh} \frac{X_v^2}{4(X_u + X_v)(X_v + X_h)} U_{ih}\bar{U}_{ju}U_{ku}\bar{U}_{lh} \\ \sum G(M_{pi}, M_{rk})G(M_{jb}, M_{ld})M_{pb}M_{rd} &= \sum_{uv} \frac{X_u X_v}{4(X_u + X_v)^2} U_{iv}\bar{U}_{jv}U_{ku}\bar{U}_{lu} \\ \sum G(M_{jq}, M_{ls})G(M_{ai}, M_{ck})M_{aq}M_{cs} &= \sum_{uv} \frac{X_u X_v}{4(X_u + X_v)^2} U_{iu}\bar{U}_{ju}U_{kv}\bar{U}_{lv} \\ \sum G(M_{pi}, M_{ls})G(M_{jb}, M_{ck})M_{pb}M_{cs} &= \sum_{uvh} \frac{X_u^2}{4(X_u + X_v)(X_h + X_u)} U_{iv}\bar{U}_{jh}U_{kh}\bar{U}_{lv}.\end{aligned}$$

Hence

$$\begin{aligned} & \langle \text{Hess}M_{ij}, \text{Hess}M_{kl} \rangle \\ &= 8 \sum_{uv} \left[ \sum_h \frac{X_h^2}{(X_h + X_v)(X_h + X_u)} U_{iu} \bar{U}_{jv} U_{kv} \bar{U}_{lu} + \frac{X_u X_v}{(X_u + X_v)^2} U_{iu} \bar{U}_{ju} U_{kv} \bar{U}_{lv} \right], \end{aligned}$$

which indicates that  $\langle \text{Hess}M_{ij}, \text{Hess}M_{kl} \rangle$  is invariant under unitary transformation.

In the end, we have

$$\text{Ric}_{ij,kl}(U) = 8d\delta_{il}\delta_{jk} - 8 \sum_{uv} \left[ \sum_h \frac{X_h^2}{(X_h + X_v)(X_h + X_u)} U_{iu} \bar{U}_{jv} U_{kv} \bar{U}_{lu} + \frac{X_u X_v}{(X_u + X_v)^2} U_{iu} \bar{U}_{ju} U_{kv} \bar{U}_{lv} \right].$$

At  $U = \text{Id}$ ,

$$\text{Ric}_{ij,kl}(U = \text{Id}) = 8 \sum_h \left( 1 - \frac{X_h^2}{(X_h + X_i)(X_h + X_j)} \right) \delta_{il}\delta_{jk} - 8 \frac{X_i X_k}{(X_i + X_k)^2} \delta_{ij}\delta_{kl}.$$

It is not quite obvious that the Ricci curvature is non-negative:

$$\begin{aligned} \sum_{ij,kl} \lambda_{ij} \bar{\lambda}_{kl} \text{Ric}_{ij,kl} &= 8 \sum_h \left( 1 - \frac{X_h^2}{(X_h + X_i)(X_h + X_j)} \right) \lambda_{ij} \bar{\lambda}_{ij} - 8 \sum_{ik} \frac{X_i X_k}{(X_i + X_k)^2} \lambda_{ii} \bar{\lambda}_{kk} \\ &= 8 \sum_{ij} \sum_h \frac{X_h(X_i + X_j) + X_i X_j}{(X_h + X_i)(X_h + X_j)} |\lambda_{ij}|^2 - 8 \sum_{ik} \frac{X_i X_k}{(X_i + X_k)^2} \lambda_{ii} \bar{\lambda}_{kk}. \end{aligned}$$

Notice that

$$\begin{aligned} -8 \sum_{ik} \frac{X_i X_k}{(X_i + X_k)^2} \lambda_{ii} \bar{\lambda}_{kk} &\geq -8 \sum_{ik} \frac{X_i X_k}{(X_i + X_k)^2} \frac{1}{2} (|\lambda_{ii}|^2 + |\lambda_{kk}|^2) \\ &\geq -8 \sum_{ik} \frac{X_i X_k}{(X_i + X_k)^2} |\lambda_{ii}|^2, \end{aligned}$$

such that

$$\begin{aligned} \sum_{ij,kl} \lambda_{ij} \bar{\lambda}_{kl} \text{Ric}_{ij,kl} &\geq 8 \sum_{ij} \sum_h \frac{X_h(X_i + X_j) + X_i X_j}{(X_h + X_i)(X_h + X_j)} |\lambda_{ij}|^2 - 8 \sum_{ik} \frac{X_i X_k}{(X_i + X_k)^2} |\lambda_{ii}|^2 \\ &\geq 8 \sum_{i \neq j} \sum_h \frac{X_h(X_i + X_j) + X_i X_j}{(X_h + X_i)(X_h + X_j)} |\lambda_{ij}|^2 > 0. \end{aligned}$$

Therefore, we prove that the Ricci curvature is positive.  $\blacksquare$

**Remark 10.3.2.** In fact, on the space of complex matrices of dimension  $d$ , we know that  $\text{Ric} = 0$ , therefore for any smooth function  $f$  on  $m$ ,

$$\Gamma_2 f(m) \geq 0.$$

Restricting  $f$  on  $M = m^*m$  such that there exist a smooth function  $\tilde{f}(M) = f(m^*m)$ , we have

$$\Gamma_2 \tilde{f}(M) = \Gamma_2 f(m^*m) \geq 0,$$

indicating that the Ricci curvature of the space of  $M$  is non-negative. The above formula gives its explicit formula.

# Chapter 11

## Dirichlet measure on the simplex

**Definition 11.0.1.** Let  $a = (a_0, a_1, a_2, \dots, a_d) \in \mathbb{R}^{d+1}$  and  $a_i, i = 0, 1, \dots, d$  are all positive. The Dirichlet distribution  $D(a_0, a_1, a_2, \dots, a_d)$  on the simplex

$$\Delta_d = \{x_i \geq 0, \sum_{i=0}^d x_i = 1, i = 0, 1, \dots, d\}$$

is given by

$$\frac{1}{B_a} x_1^{a_1-1} \dots x_d^{a_d-1} (1 - x_1 - \dots - x_d)^{a_0-1} \mathbf{1}_{\Delta_d}(x_1, \dots, x_d) dx_1 \dots dx_d,$$

where  $B_a = \frac{\Gamma(a_0) \dots \Gamma(a_d)}{\Gamma(a_0 + \dots + a_d)}$ .

The Dirichlet measure can be considered as a generalization of beta distribution,

$$\beta(a_0, a_1) = \frac{1}{B(a_0, a_1)} x^{a_0-1} (1-x)^{a_1-1} \mathbf{1}_{(0,1)}(x) dx,$$

which is indeed  $D(a_0, a_1)$ .

We quote the following proposition from [67], which gives a construction of Dirichlet random variable through gamma distributions given by

$$\gamma_{\alpha, \beta}(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} \mathbf{1}_{(0, \infty)}(x),$$

where  $\alpha, \beta > 0$ ,  $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ . It also gives a method to generate independent random variables.

**Proposition 11.0.2.** Consider independent random variables  $X_0, X_1, \dots, X_d$  such that each  $X_k$  has gamma distribution  $\gamma_{\alpha_k, \beta}$ . Define  $S = \sum_{k=0}^d X_k$ . Then  $S$  is independent of  $\frac{1}{S}(X_1, \dots, X_d)$ , and  $\frac{1}{S}(X_0, X_1, \dots, X_d)$  has the density  $D(\alpha_0, \alpha_1, \dots, \alpha_d)$ .

A simple application of this proposition gives a realization of Dirichlet random variables. Consider  $x = (x_1, x_2, \dots, x_d)$  to be standard Gaussian variable in  $\mathbb{R}^d$ , then  $y = (y_1, y_2, \dots, y_d) = \frac{x}{\|x\|}$  is on the unit sphere in  $\mathbb{R}^d$ . Each  $x_i^2$  is distributed as  $\gamma_{\frac{1}{2}, 2}$ , so the above proposition leads to the conclusion that  $(y_1^2, y_2^2, \dots, y_d^2)$  is distributed as  $D(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ . Moreover, if we define

$$X_1 = y_1^2 + \dots + y_{p_1}^2, \quad X_2 = y_{p_1+1}^2 + \dots + y_{p_1+p_2}^2, \quad \dots X_n = y_{\sum_{j=1}^{n-1} p_j+1}^2 + \dots + y_d^2,$$

where  $\sum_j^n p_j = d$ . Then  $\{X_1, \dots, X_n\}$  is distributed as  $D(\frac{p_1}{2}, \frac{p_2}{2}, \dots, \frac{p_n}{2})$ .

Our main concern in this section is the diffusion process with Dirichlet measure as its invariant measure. Let us start with a generalization of the above example. Consider Brownian motion on  $\mathbb{S}^{d-1}$  embedded in  $\mathbb{R}^d$ . Recall Section 2.4.1 that the Euclidean metric restricted on the sphere  $\mathbb{S}^{d-1}$  is

$$\Gamma_{\mathbb{S}^{d-1}}(x_i, x_j) = \delta_{ij} - x_i x_j, \quad L_{\mathbb{S}^{d-1}}(x_i) = -(d-1)x_i.$$

For each  $i = 1, \dots, d$ , define  $X_i = x_i^2$ . Then we have a simplex  $\{0 \leq X_i \leq 1, \sum_i X_i = 1\}$ , where

$$\Gamma(X_i, X_j) = 4X_i(\delta_{ij} - X_j), \quad L(X_i) = 2 - 2dX_i.$$

The invariant measure of  $\{X_i, i = 1, \dots, d\}$  is

$$\prod_{i=1}^{d-1} X_i^{-\frac{1}{2}} (1 - \sum_i X_i)^{-\frac{1}{2}},$$

which is the Dirichlet measure  $D(\frac{1}{2}, \dots, \frac{1}{2})$ .

Moreover, let  $Y = X_1 + X_2$  then

$$\begin{aligned} \Gamma(Y, Y) &= 4(Y - Y^2), & \Gamma(Y, X_j) &= -4YX_j, \\ L(Y) &= 4 - 2dY, \end{aligned}$$

where  $j \neq 1, 2$ .  $\{Y, X_3, X_4, \dots, X_{d-1}\}$  has their invariant measure, whose density is

$$\prod_{i=3}^{d-1} X_i^{-\frac{1}{2}} (1 - Y - \sum_{i=3}^{d-1} X_i)^{-\frac{1}{2}}.$$

It is in fact the Dirichlet measure  $D(1, \frac{1}{2}, \dots, \frac{1}{2})$ . Such amalgamating process can be generalized to the following proposition:

**Proposition 11.0.3.** *Let  $p_1, p_2, \dots, p_n$  be a partition of  $\{1, 2, \dots, d\}$  such that  $\sum_k p_k = d$ , and denote  $q_k = \sum_{i=1}^k p_k$  for  $k = 1, \dots, n$ . Let  $X_1 = x_1^2 + x_2^2 + \dots + x_{q_1}^2$ ,  $X_2 = x_{q_1+1}^2 + \dots + x_{q_2}^2$ , ...,  $X_n = x_{q_{n-1}+1}^2 + \dots + x_{q_n}^2$ . Notice that  $\{X_i \geq 0, \sum_{i=1}^n X_i = 1\}$ , and*

$$\Gamma(X_i, X_j) = 4X_i(\delta_{ij} - X_j), \quad L(X_i) = 2p_i - 2dX_i,$$

which give the density of their invariant measure as  $D(\frac{p_1}{2}, \frac{p_2}{2}, \dots, \frac{p_n}{2})$ ,

$$\prod_{k=1}^{n-1} X_k^{\frac{p_k}{2}-1} (1 - \sum_{k=1}^{n-1} X_k)^{\frac{p_n}{2}-1}.$$

A more general model comes from the Laguerre operator on  $\mathbb{R}^+$  [16], which is given by

$$L_a = x \frac{d^2}{dx^2} + (a-x) \frac{d}{dx}, \quad (11.0.1)$$

for  $a > 0$ . Its invariant measure is  $\mu_a(dx) = C_a x^{a-1} e^{-x} dx$ , and has Laguerre polynomials as its eigenvectors.

Now consider  $X = (X_1, \dots, X_d)$  to be independent process on  $\mathbb{R}^+$ , and each  $X_i$  is associated with a Laguerre operator  $L_{a_i}$ , for some positive constant  $a_i$ . Then for  $1 \leq i, j \leq d$ , we have

$$\Gamma(X_i, X_j) = X_i \delta_{ij}, \quad L(X_i) = a_i - X_i.$$

Define  $S = \sum_i X_i$  and  $Y_i = \frac{X_i}{S}$ ,  $\{Y_1, \dots, Y_d\}$  constitutes a simplex  $\{0 \leq Y_i \leq 1, \sum_{i=1}^d Y_i = 1\}$ . Direct computation yields that for any  $i$ ,

$$\Gamma(S, Y_i) = 0,$$

which implies that  $S$  is independent of  $(Y_1, \dots, Y_d)$ . Then restricting on the submanifold  $\{S = 1\}$ , we have

$$\Gamma(Y_i, Y_j) = Y_i \delta_{ij} - Y_i Y_j, \quad L(Y_i) = a_i - \sum_k a_k Y_i,$$

with the invariant measure

$$C \prod_{i=1}^{d-1} Y_i^{a_i-1} (1 - \sum_{i=1}^{d-1} Y_i)^{a_d-1},$$

which is indeed the Dirichlet measure  $D(a_1, \dots, a_d)$  on the simplex.

Moreover, since the invariant measure of the Laguerre operator 11.0.1

$$\mu_a(dx) = C_a x^{a-1} e^{-x} dx$$

is indeed  $\gamma_{a,1}$ , the above model in fact gives a dynamical proof of Prop 11.0.2. In fact, let  $x = (x_1, \dots, x_d)$  be the initial value of  $X = (X_1, \dots, X_d)$ , which are independent and distributed as  $\gamma_{a_1,1}, \dots, \gamma_{a_d,1}$ . By the definition of invariant measure, for any bounded function  $f$  on  $\mathbb{R}^d$ ,

$$\int f(x) d\mu_{a_1}(x_1) \dots d\mu_{a_d}(x_d) = \int \mathbb{E}(f(X_t))(x) d\mu_{a_1}(x_1) \dots d\mu_{a_d}(x_d).$$

Similarly let  $y = (y_1, \dots, y_{d-1})$  be the initial value of  $Y_t = (Y_1, \dots, Y_{d-1})$ , distributed as the Dirichlet distribution  $D(a_1, \dots, a_d)$ , then

$$\int f(y, 1) dD(a_1, \dots, a_d)(y) = \int \mathbb{E}(f(Y_1, \dots, Y_{d-1}, 1))(y, 1) dD(a_1, \dots, a_d)(y).$$

Define  $\Phi : (X_1, \dots, X_d) \rightarrow (Y_1, \dots, Y_{d-1}, S)$ . (Since  $Y_1 + \dots + Y_d = 1$ , we use  $(Y_1, \dots, Y_{d-1}, S)$  as coordinates to ensure that  $\Phi$  is non-degenerated.) According to the above discussion and the fact that  $Y_t$  is independent of  $S$  such that

$$\mathbb{E}(f \circ \Phi(X_t))(x) = \mathbb{E}(f(Y_t, S))(y) = \mathbb{E}(\mathbb{E}(f(Y_t, 1) | S = 1))(y) = \mathbb{E}(f(Y_t, 1))(y),$$

we know that

$$\int \mathbb{E}(f \circ \Phi(X_t))(x) d\mu_{a_1}(x_1) \dots d\mu_{a_d}(x_d) = \int \mathbb{E}(f(Y_t, 1))(y) dD(a_1, \dots, a_d)(y).$$

Therefore,

$$\begin{aligned} \int f \circ \Phi(x) d\mu_{a_1}(x_1) \dots d\mu_{a_d}(x_d) &= \int \mathbb{E}(f \circ \Phi(X_t))(x) d\mu_{a_1}(x_1) \dots d\mu_{a_d}(x_d) \\ &= \int \mathbb{E}(f(Y_t, 1))(y) dD(a_1, \dots, a_d)(y) \\ &= \int f(y, 1) dD(a_1, \dots, a_d)(y), \end{aligned}$$

which proves Proposition 11.0.2.

In fact, we are able to describe all the polynomial models on the simplex with Dirichlet measure by applying the results of Boundary equation 2.5.2.

**Theorem 11.0.4.** *For every the polynomial model in the simplex  $\{V_i \geq 0, i = 1, \dots, n-1, 1 - \sum_{j=1}^{n-1} V_j \geq 0\}$  with Dirichlet distribution as its invariant measure, i.e.*

$$\rho = \prod_i V_i^{a_i} (1 - \sum_{k=1}^{n-1} V_k)^{a_n},$$

where  $\{a_i, i = 1, \dots, n\}$  are positive constants. Then their diffusion operator is given by

$$\Gamma(V_i, V_j) = A_{ij} V_i V_j - \delta_{ij} \sum_{k=1}^{n-1} A_{ik} V_k V_i - \delta_{ij} A_{in} V_n V_i, \quad (11.0.2)$$

where  $V_n = 1 - \sum_{j=1}^{n-1} V_j$ ,  $A_{ij} = A_{ji}$  for all  $1 \leq i, j \leq n$  are negative constants.

*Proof.* — According to Theorem 2.5.2, to be a polynomial model  $\Gamma(V_i, V_j)$  must be a polynomial no more than degree 2, and for  $1 \leq i, j \leq n-1$  satisfies

$$\begin{aligned} \Gamma(V_i, \log V_j^{a_j}) &= L^{ij}, \\ \Gamma(V_i, \log V_n^{a_n}) &= L^{in}, \end{aligned}$$

where  $\{L^{ij}, 1 \leq j \leq n-1\}$ ,  $L^{in}$  are polynomials of no more that degree 1.

From the first equality, we get

$$a_j \Gamma(V_i, V_j) = V_j L^{ij},$$

which indicates that there exist constants  $A_{ij}$ ,  $A_{in}$  and  $B_{ik}$  such that

$$\Gamma(V_i, V_j) = A_{ij} V_j V_i + \delta_{ij} \sum_{k \neq i}^{n-1} B_{ik} V_k V_i + \delta_{ij} A_{in} V_i,$$

where  $A_{ij} = A_{ji}$ ,  $A_{ij} = 0$  when  $i = j$ .

From the second equality,

$$a_n \sum_j^{n-1} \Gamma(V_i, V_j) = -V_n L^{in},$$

that is

$$a_n \left( \sum_{j \neq i}^{n-1} (A_{ij} + B_{ij}) V_j V_i + a_n A_{in} V_i \right) = -V_n L^{in},$$

which implies that  $A_{ij} + B_{ij} = -A_{in}$ . Then we have

$$\Gamma(V_i, V_j) = A_{ij} V_j V_i - \delta_{ij} \sum_{k \neq i}^{n-1} A_{ik} V_k V_i + \delta_{ij} A_{in} V_n V_i.$$

The ellipticity of  $\Gamma$  is a consequence of Theorem 12.1.1. ■

Moreover, by direct computation we get

$$L(V_i) = \sum_j (a_j + 1) A_{ij} V_i - \left( \sum_{k=1}^{n-1} A_{ik} V_k + C_i V_n \right) (a_i + 1) + a_n C_i V_i.$$

# Chapter 12

## Matrix Dirichlet process

### 12.1 General matrix Dirichlet Process

As we have explained before, the polynomial models are quite rare. However, when we study the polar decomposition of complex matrix, a matrix generalization of simplex appears:

$$\{\forall k, Z^k = (Z^k)^*, 0 \leq Z^k, \sum_{k=1}^n Z^k = \text{Id}\} \quad (12.1.1)$$

It indicates that there may exist a polynomial model of multiple matrices which has matrix simplex as its boundary, as in the classical case of Dirichlet measure. Similar to the Dirichlet measure on the simplex, we generalize the Dirichlet measure into the following form:

$$\rho = C \prod_i^{n-1} \det(Z^i)^{a_i} \det(\text{Id} - \sum_{k=1}^{n-1} Z^k)^{a_n}, \quad (12.1.2)$$

where  $\{a_i\}_{i=1}^n$  are all positive constants.

Now we consider  $\Gamma$  in the following form:

$$\Gamma(Z_{ij}^p, Z_{kl}^q) = A_{pq}(Z_{il}^q Z_{kj}^p + Z_{il}^p Z_{kj}^q) - \delta_{pq} \left[ \sum_s B_{sp}(Z_{il}^s Z_{kj}^p + Z_{kj}^s Z_{il}^p) + A_{np}(Z_{il}^n Z_{kj}^p + Z_{kj}^n Z_{il}^p) \right], \quad (12.1.3)$$

for  $1 \leq p, q \leq n-1$ ,  $A_{pq} = A_{qp}$ ,  $B_{pq} = B_{qp}$  are all constants. In fact, such form of  $\Gamma$  appears naturally when we study the polar decomposition of Brownian motion on complex matrices (see Chapter 10).

To find the corresponding diffusion operators on the matrix simplex, we have the following result:

**Theorem 12.1.1.** *For integers  $p, q$  satisfying  $1 \leq p, q \leq n-1$ , and  $A_{pq} = A_{qp}$  all negative constants, the diffusion operator given by*

$$\Gamma(Z_{ij}^p, Z_{kl}^q) = A_{pq}(Z_{il}^q Z_{kj}^p + Z_{il}^p Z_{kj}^q) - \delta_{pq} \left[ \sum_s A_{sp}(Z_{il}^s Z_{kj}^p + Z_{kj}^s Z_{il}^p) + A_{np}(Z_{il}^n Z_{kj}^p + Z_{kj}^n Z_{il}^p) \right], \quad (12.1.4)$$

together with the Dirichlet distribution (12.1.2), constitutes a polynomial model on the matrix simplex (12.1.1).

*Proof.* — The proof is separated into two steps, first we prove that to have a polynomial model, in formula (12.1.3) we must have  $A_{pq} = B_{pq}$ ; then we prove that  $\Gamma$  is elliptic if and only if for all  $1 \leq p, q \leq n$ ,  $A_{pq} < 0$ .

Although  $\Gamma$  is expressed in terms of  $\{Z^i, i = 1, \dots, n\}$ , we always use  $\{Z^i, i = 1, \dots, n-1\}$  as coordinates, since  $Z^n = \text{Id} - \sum_k^{n-1} Z^k$ .

Recall that to have a polynomial model, according to Theorem 2.5.2, we require  $\Gamma(Z_{ij}^p, \log \rho)$  to be a polynomial at most 1 degree, for each  $i, j, p$ . Notice that for  $1 \leq q \leq n-1$ ,

$$\begin{aligned}\Gamma(\log \det(Z^q), Z_{ij}^p) &= 2A_{qp}Z_{ij}^p - \delta_{pq} \sum_s 2B_{sp}Z_{ij}^s - \delta_{pq} 2A_{np}\delta_{ij}, \\ \Gamma(\log \det(Z^n), Z_{ij}^p) &= -(Z^n)_{lk}^{-1} \left[ \sum_q^{n-1} (A_{pq} - B_{pq})(Z_{il}^q Z_{kj}^p + Z_{il}^p Z_{kj}^q) \right] + 2A_{np}Z_{ij}^p.\end{aligned}$$

Therefore, to make  $\Gamma(\log \det(Z^n), Z_{ij}^p)$  a polynomial we should have  $A_{pq} = B_{pq}$ , which leads to formula (12.1.4).

Now we prove that on the matrix simplex  $\{0 \leq Z^k, \sum_{k=1}^n Z^k = \text{Id}\}$ ,  $\Gamma$  in the formula (12.1.4) is elliptic if and only if for  $p, q = 1, \dots, n$ ,  $A_{pq} < 0$ .

For the fixed  $(p, q)$ , consider  $\Gamma(Z_{ij}^p, Z_{kl}^q)$  as a  $d^2 \times d^2$  matrix with index  $(ij, kl)$ . Then  $(\Gamma(Z_{ij}^p, Z_{kl}^q))$  is a  $(n-1)d^2 \times (n-1)d^2$  block matrix with index  $(p, q)$ . Notice that

$$(\Gamma_{(p,q)}) = \sum_{(p,q), p \neq q} A_{pq} M^{pq} + CD,$$

Where  $M^{pq}$  is a  $(n-1)d^2 \times (n-1)d^2$  matrix with

$$\begin{aligned}(M^{(p,q)})_{pq,(ij,kl)} &= Z_{il}^q Z_{kj}^p + Z_{il}^p Z_{kj}^q, \\ (M^{(p,q)})_{qp,(ij,kl)} &= Z_{il}^q Z_{kj}^p + Z_{il}^p Z_{kj}^q, \\ (M^{(p,q)})_{pp,(ij,kl)} &= -(Z_{il}^q Z_{kj}^p + Z_{il}^p Z_{kj}^q), \\ (M^{(p,q)})_{qq,(ij,kl)} &= -(Z_{il}^q Z_{kj}^p + Z_{il}^p Z_{kj}^q),\end{aligned}$$

and other entries 0;  $D$  is a diagonal block matrix with  $D_{(p,p),(ij,kl)} = -(Z_{il}^n Z_{kj}^p + Z_{il}^p Z_{kj}^n)$  and other entries 0; and  $C$  is a diagonal matrix of size satisfying  $C_{(p,p)} = A_{pn} \text{Id}_{d^2 \times d^2}$ .

Notice that each  $p \neq q$ ,  $M^{(p,q)}$  is equivalent to  $\widetilde{M}$  by unitary transformation, which is defined as

$$(\widetilde{M}^{(p,q)})_{pp,(ij,kl)} = -(Z_{il}^q Z_{kj}^p + Z_{il}^p Z_{kj}^q),$$

with other entries 0. In fact we have for any non-zero matrix  $\Lambda = \{\lambda_{(ij)}\}$ ,

$$\begin{aligned}\sum_{ij,kl} \lambda_{ij} \bar{\lambda}_{kl} (\widetilde{M}_{pp,(ij,kl)}^{(p,q)}) &= \sum_{ij,kl} -\lambda_{ij} \bar{\lambda}_{kl} (Z_{ik}^q Z_{lj}^p + Z_{ik}^p Z_{lj}^q) \\ &= -\text{trace}(Z^q \bar{\Lambda} Z^p \Lambda^t) - \text{trace}(Z^p \bar{\Lambda} Z^q \Lambda^t).\end{aligned}$$

Since  $Z^p$  is Hermitian and positive for any  $p$ , we have

$$\text{trace}(Z^q \bar{\Lambda} Z^p \Lambda^t) > 0, \quad \text{trace}(Z^p \bar{\Lambda} Z^q \Lambda^t) > 0,$$

therefore  $M^{pq}$  are all negative-definite matrices. Similarly, we have  $D > 0$ . We may first fix  $p$ , and let other matrix  $Z^r = 0$ ,  $r \neq p$ . Then the fact that both  $Z^p$  are Hermitian makes sure that  $D < 0$ , to make  $\Gamma$  elliptic, we must have  $C_p < 0$ ; Then for fixed  $p, q$ , let  $Z^p$  and  $Z^q$  be two non-zero Hermitian matrices and  $Z^p + Z^q = \text{Id}$ , which means  $Z^n = 0$ ;  $Z^r = 0$ , for  $r \neq p, q$ . Then  $M^{pq} < 0$ , we must have  $A_{pq} < 0$  to make  $\Gamma > 0$ .

Therefore,  $\Gamma$  is positive-definite if and only if for any  $1 \leq p, q \leq n-1$ ,  $A_{pq} < 0$ ,  $A_{pn} < 0$ . ■

We also note that given  $\Gamma$ , together with matrix Dirichlet measure, we have

$$L(Z_{ij}^p) = \sum_q^n 2(a_q + d)A_{pq}Z_{ij}^p - \sum_q^n 2(a_p + d)A_{pq}Z_{ij}^q.$$

We may write stochastic differential equation for matrix Dirichlet process in complex matrix simplex. Let  $\{B^{pq}, 1 \leq p, q \leq n-1\}$  be standard Brownian motions on complex matrices of dimension  $d \times d$ , satisfying  $B^{pq} = -(B^{qp})^*$ ; and for different pair  $(p, q)$  and  $(r, s)$ ,  $B^{pq}$  and  $B^{rs}$  are independent. Define

$$W^{pq} = \sqrt{\frac{-A_{pq}}{2}} B^{pq},$$

since  $A_{pq} < 0$ . For fixed  $(p, q)$ , we have

$$\Gamma(W_{ij}^{pq}, W_{kl}^{pq}) = 0, \quad \Gamma(W_{ij}^{pq}, \bar{W}_{kl}^{pq}) = -A_{pq}\delta_{ik}\delta_{jl}.$$

Then  $\{Z^p, 1 \leq p \leq n-1\}$  satisfies the following stochastic differential equation:

$$\begin{aligned} dZ^p &= \sum_{k \neq p}^n (Z^k)^{\frac{1}{2}} dW^{kp} (Z^p)^{\frac{1}{2}} + (Z^p)^{\frac{1}{2}} d(W^{kp})^* (Z^k)^{\frac{1}{2}} \\ &\quad + \left( \sum_q^n 2(a_q + d)A_{pq}Z^p - \sum_q^n 2(a_p + d)A_{pq}Z^q \right) dt. \end{aligned}$$

It can be verified that for  $p \neq q < n$ ,

$$\begin{aligned} \Gamma(Z_{ij}^p, Z_{kl}^q) &= A_{pq}(Z_{il}^q Z_{kj}^p + Z_{kj}^q Z_{il}^p), \\ \Gamma(Z_{ij}^p, Z_{kl}^n) &= A_{pn}(Z_{il}^n Z_{kj}^p + Z_{kj}^n Z_{il}^p), \\ \Gamma(Z_{ij}^p, Z_{kl}^p) &= -\sum_{s \neq p}^n A_{sp}(Z_{il}^s Z_{kj}^p + Z_{kj}^s Z_{il}^p), \\ L(Z_{ij}^p) &= \sum_q^n 2(a_q + d)A_{pq}Z_{ij}^p - \sum_q^n 2(a_p + d)A_{pq}Z_{ij}^q. \end{aligned}$$

**Remarks 12.1.2.** According to the paper by Bakry and Zribi [20], for a polynomial model, let  $\rho$  be the density of its invariant measure, there exists a constant  $\mu$  such that for  $h = \frac{1}{\rho}$

$$L(h) = \mu h.$$

In our case, since  $\{a_i\}$  are all positive and the measure vanishes at the boundary, their theorem indicates that  $\mu$  is positive. Let  $Y_t = h(Z_t)$ , where  $Z_t$  is the matrix Dirichlet process, then by Itô's formula,  $e^{-\mu t} Y_t$  is a local martingale, so we have

$$\mathbb{E}(e^{-\mu t} Y_t) = Y_0.$$

Assume the starting point to be  $Y_0 = \frac{1}{\rho(Z_0)} = a$ . Now consider the hitting time  $T_\varepsilon$  of  $\rho(Z_{T_\varepsilon}) = \varepsilon$ , we have

$$\mathbb{E}(e^{-\mu T_\varepsilon}) = \varepsilon a.$$

When  $\varepsilon \rightarrow 0$ , we see that  $T_\varepsilon$  converges increasingly to infinity, which means that it will never hit the boundary.

Notice that for the general matrix Dirichlet process, the amalgamation property does not hold any more:  $\{Y = Z^1 + Z^2, Z^3, \dots, Z^{n-1}\}$  is not a process itself, for the reason that  $\Gamma(Y, Z_{ij}^p)$  does not satisfy a closed formula. However, we will see later that when if  $A_{pq}$  is the same constant for any  $p, q$ , the amalgamation property still holds.

## 12.2 General matrix Dirichlet process from polar decomposition

Let  $Z_{ij,k} = U_{ik}\bar{U}_{jk}$ , such that for fixed  $k$ ,  $(Z_{ij,k})$  is a Hermitian matrix:  $\bar{Z}_{ij,k} = \bar{U}_{ik}U_{jk} = Z_{ji,k}$ , and  $\{X_i\}$  be the eigenvalues of  $M$ . Also we have

$$\begin{aligned} \sum_k^d Z_{ij,k} &= \delta_{ij}, \\ \text{trace}(Z_{ij,k}) &= 1. \quad \text{for } \forall k \end{aligned}$$

Then

$$\begin{aligned} \Gamma(Z_{ij,p}, Z_{kl,q}) &= r_{qp}(Z_{il,q}Z_{kj,p} + Z_{il,p}Z_{kj,q}) - \delta_{pq} \sum_s r_{sp}(Z_{il,s}Z_{kj,p} + Z_{kj,s}Z_{il,p}), \\ L(Z_{ij,p}) &= 2 \sum_{p \neq q} r_{pq}(Z_{ij,p} - Z_{ij,q}). \end{aligned}$$

Now consider  $(Z_{ij,k})$  as a process. However, notice that for a fixed  $k$ , the matrix  $(Z_{ij})$  is of rank one, we first discuss the 1 dimension case. Let  $1 \leq i \leq d$  be fixed, then we have  $\{0 \leq Z_{ii,p}, \sum_{p=1}^d Z_{ii,p} = 1\}$ .

$$\begin{aligned} \Gamma(Z_{ii,p}, Z_{ii,q}) &= 2r_{qp}Z_{ii,p}Z_{ii,q} - \delta_{pq} \sum_s 2r_{sp}Z_{ii,s}Z_{ii,p}, \\ L(Z_{ii,p}) &= 2 \sum_{p \neq q} r_{pq}(Z_{ii,p} - Z_{ii,q}). \end{aligned}$$

Take  $Z_{ii}^1, Z_{ii}^2, \dots, Z_{ii}^{d-1}$  as coordinates. The metric turns to be

$$\begin{aligned} \Gamma(Z_{ii}^p, Z_{ii}^q) &= 2r_{qp}Z_{ii}^pZ_{ii}^q + \delta_{pq} \sum_s^{d-1} 2(r_{dp} - r_{sp})Z_{ii}^sZ_{ii}^p - \delta_{pq} 2r_{dp}Z_{ii}^p, \\ L(Z_{ii}^p) &= 2 \sum_{p \neq q} r_{pq}Z_{ii}^p + 2 \sum_{p \neq q} (r_{pd} - r_{pq})Z_{ii}^q - 2r_{dp}. \end{aligned}$$

This process is indeed a one-dimension Dirichlet process with the Lebesgue measure  $dZ_{ii}^1 \dots dZ_{ii}^{d-1}$  as its invariant measure. It inspires us to link the process of  $(Z_{ij,k})$  with Dirichlet process. However, since the matrices  $(Z_{ij,k})$  are too degenerated, we need to get rid of the restrictions on the matrices. This is the reason that we introduce the matrix simplex and the general  $\Gamma$  described in the previous chapter:

$$\begin{aligned} \Gamma(Z_{ij}^p, Z_{kl}^q) &= A_{pq}(Z_{il}^q Z_{kj}^p + Z_{il}^p Z_{kj}^q) \\ &\quad - \delta_{pq} \left( \sum (A_{sp} - C_p)(Z_{il}^s Z_{kj}^p + Z_{il}^p Z_{kj}^s) + C_p(\delta_{il} Z_{kj}^p + \delta_{kj} Z_{il}^p) \right). \end{aligned}$$

### 12.3 A construction from $SU(d)$ and $SO(d)$

We now give a natural construction of matrix Dirichlet process, which is a special case of the general matrix Dirichlet process. Following the splitting procedure in matrix Jacobi case, instead of taking one sub-matrix, we extract several matrices: for example starting from a Brownian motion on  $SU(d)$ , take the first  $m$  lines, and split  $d$  into  $n$  parts:  $d = a_1 + a_2 + \dots + a_n$ , then we get  $n$  matrices  $\{v^i\}$ , respectively of size  $m \times a_1, m \times a_2, \dots, m \times a_n$ . Moreover, let  $V_i = v_i v_i^*$ , such that each  $V_i$  is a Hermitian matrix of dimension  $m \times m$ , satisfying  $0 \leq V_i \leq \text{Id}$  for each  $i$  and  $\sum_i V_i = \text{Id}$ . Write  $V_n = \text{Id} - \sum_{i=1}^{n-1} V_i$ . By direct computation, for  $1 \leq p, q \leq n-1$ ,

$$\begin{aligned} \Gamma(V_{ij}^p, V_{kl}^q) &= d\delta_{pq}(\delta_{il} V_{kj}^p + \delta_{kj} V_{il}^p) - d(V_{kj}^p V_{il}^q + V_{il}^p V_{kj}^q), \\ \text{L}(V_{ij}^p) &= -2d^2 V_{ij}^p + 2da_p \delta_{ij}. \end{aligned}$$

Notice that this is a general matrix Dirichlet process when  $A_{pq} = -d$ . Now we study the invariant measure of this matrix process  $\{V_i\}_{i=1}^{n-1}$ . Suppose the invariant measure exists and let  $\rho$  be its density. Then

$$\begin{aligned} \Gamma(\log \rho, V_{ij}^p) &= 2d(a_p - m)\delta_{ij} + 2d(mn - d)V_{ij}^p, \\ \Gamma(\log \det V^q, V_{ij}^p) &= 2d[\delta_{pq}(\delta_{ij} - V_{ij}^p) - \delta_{p \neq q} V_{ij}^p], \\ \Gamma(\log \det(\text{Id} - \sum_k^{n-1} V^k), V_{ij}^p) &= -2dV_{ij}^p. \end{aligned}$$

Therefore, the invariant measure, if it exists, should be

$$C \prod_{p=1}^{n-1} \det(V^p)^{a_p - m} \det(\text{Id} - \sum_k^{n-1} V^k)^{d - \sum_{k=1}^{n-1} a_k - m}. \quad (12.3.5)$$

Notice that this invariant measure is a Dirichlet measure, which means the extracted matrices process is a matrix Dirichlet process.

**Proposition 12.3.1.** *Let  $p, q$  be two integers  $1 \leq p, q \leq n-1$ , and define  $Y = V_p + V_q$ . Then we get a new process  $\{V_i, Y, 1 \leq i \leq n-1, i \neq p, q\}$ , which is still a matrix Dirichlet process, with its invariant measure*

$$C \prod_{k=1, k \neq p, q}^{n-1} \det(V^k)^{a_k - m} \det(Y)^{a_p + a_q - m} \det(\text{Id} - \sum_{k \neq p, q}^{n-1} V^k - Y)^{d - \sum_{k=1}^{n-1} a_k - m}.$$

*Proof.* — The  $\Gamma$  and  $L$  are all the same for  $V_r$ ,  $r \neq p, q$ ,

$$\begin{aligned}\Gamma(V_{ij}^r, V_{kl}^s) &= d\delta_{rs}(\delta_{il}V_{kj}^r + \delta_{kj}V_{il}^r) - d(V_{kj}^r V_{il}^s + V_{il}^r V_{kj}^s), \\ L(V_{ij}^r) &= -2d^2V_{ij}^r + 2da_r\delta_{ij}.\end{aligned}$$

Except for  $Y$  we have

$$\begin{aligned}\Gamma(Y_{ij}, Y_{kl}) &= d(\delta_{il}Y_{kj} + \delta_{kj}Y_{il} - 2Y_{il}Y_{kj}), \\ \Gamma(Y_{ij}, V_{kl}^r) &= -dY_{kj}V_{il}^r - dY_{il}V_{kj}^r, \\ L(Y_{ij}) &= -2d^2Y_{ij} + 2d(a_p + a_q)\delta_{ij}.\end{aligned}$$

If the invariant measure exists, let  $\rho_a$  be its density, then

$$\begin{aligned}\Gamma(V_{ij}^r, \log \rho_a) &= 2d(a_r - m)\delta_{ij} + 2d(m(n-1) - d)V_{ij}^r, \\ \Gamma(Y_{ij}, \log \rho_a) &= 2d(a_p + a_q - m)\delta_{ij} + 2d(m(n-1) - d)V_{ij}^r.\end{aligned}$$

Notice that

$$\begin{aligned}\Gamma(V_{ij}^r, \log \det(V^s)) &= 2d(\delta_{rs}\delta_{ij} - V_{ij}^r), \\ \Gamma(V_{ij}^r, \log \det(Y)) &= -2dV_{ij}^r, \\ \Gamma(Y_{ij}, \log \det(V^r)) &= -2dY_{ij}, \\ \Gamma(Y_{ij}, \log \det(Y)) &= 2d(\delta_{ij} - Y_{ij}), \\ \Gamma(V_{ij}^r, \log \det(I - \sum_{k \neq p, q} V_k - Y)) &= -2dV_{ij}^r, \\ \Gamma(Y_{ij}, \log \det(I - \sum_{k \neq p, q} V_k - Y)) &= -2dY_{ij},\end{aligned}$$

which leads to the conclusion that

$$\rho_a = C \prod_{k=1, k \neq p, q}^{n-1} \det(V^k)^{a_k - m} \det(Y)^{a_p + a_q - m} \det(\text{Id} - \sum_{k \neq p, q}^{n-1} V^k - Y)^{d - \sum_{k=1}^{n-1} a_k - m}.$$

■

Equivalently, the matrix Dirichlet process can be also constructed on  $SO(d)$ . We use the same notation of partition in the previous section: for a matrix  $u$  on  $SO(d)$ , take the first  $m$  lines, and split  $d$  into  $n$  parts:  $d = a_1 + a_2 + \dots + a_n$ , then we get  $n$  matrices  $\{u^i\}$ , respectively of size  $m \times a_1, m \times a_2, \dots, m \times a_n$ . Let  $U_i = u_i u_i^t$ , such that each  $U_i$  is a symmetric matrix of dimension  $m \times m$ , satisfying  $0 \leq V_i \leq \text{Id}$  for each  $i$  and  $\sum_i^n U_i = \text{Id}$ . By direct computation, for  $1 \leq p, q \leq n-1$ ,

$$\begin{aligned}\Gamma(U_{ij}^p, U_{kl}^q) &= \delta_{pq}(\delta_{ik}U_{jl}^p + \delta_{il}U_{jk}^p + \delta_{jk}U_{il}^p + \delta_{jl}U_{ik}^p) - U_{kj}^p U_{il}^q - U_{jl}^p U_{ik}^q - U_{ik}^p U_{jl}^q - U_{il}^p U_{jk}^q, \\ L(U_{ij}^p) &= -2dU_{ij}^p + 2a_p\delta_{ij}.\end{aligned}$$

The density of their invariant measure is

$$\prod_{i=1}^{n-1} \det(U^i)^{\frac{a_i-m-1}{2}} \det(\text{Id} - \sum_i^{n-1} U^i)^{\frac{d-\sum_{k=1}^{n-1} a_k-m-1}{2}}.$$

The amalgamating property follows:

**Proposition 12.3.2.** *Let  $p, q$  be two integers  $1 \leq p, q \leq n-1$ , and define  $Y = U_p + U_q$ . Then we get a new process  $\{V_i, Y, 1 \leq i \leq n-1, i \neq p, q\}$ , which is still a matrix Dirichlet process, with its invariant measure*

$$C \prod_{k=1, k \neq p, q}^{n-1} \det(V^k)^{\frac{a_k-m-1}{2}} \det(Y)^{\frac{a_p+a_q-m-1}{2}} \det(\text{Id} - \sum_{k \neq p, q}^{n-1} V^k - Y)^{\frac{d-\sum_k^{n-1} a_k-m-1}{2}}.$$

**Remarks 12.3.3.** *The above result show that for any function  $f$  which is invariant under the amalgamation property, i.e. for any integers  $1 \leq p < q \leq n-1$ ,*

$$f(V^1, \dots, V^{n-1}) = f(V^1, \dots, V^{p-1}, V^p + V^q, V^{p+1}, \dots, V^{q-1}, V^{q+1}, \dots, V^{n-1}),$$

we have

$$\begin{aligned} & \int_{V^i \geq 0, \sum_p^{n-1} V^p \leq \text{Id}} f(V^1, \dots, V^{n-1}) \prod_k^{n-1} \det(V^k)^{a_k-m} \det(I - \sum_p^{n-1} V^p)^{a_n-m} dV^1 dV^2 \dots dV^{n-1} \\ &= C_1 \int f(Y, V^3, \dots, V^{n-1}) (\det(Y))^{a_1+a_2-m} \prod_{k=3, \dots, n-1} \det(V^k)^{a_k-m} \\ & \quad \times \det(I - Y - \sum_{k=3}^{n-1} V^k)^{a_n-m} dY dV^3 \dots dV^{n-1} \\ &= \dots \\ &= C_{n-1} \int_{0 \leq Y \leq \text{Id}} f(Y) (\det(Y))^{\sum_i^{n-1} a_i-m} \det(I - Y)^{a_n-m} dY, \end{aligned}$$

where  $a_n = d - \sum_i^{n-1} a_i$ ,  $\{C_1, C_2, \dots, C_{n-1}\}$  are all constants that can not be determined right now.

On the other hand, as discussed in remark 4.2.1, we have

$$\int_{0 \leq A \leq \text{Id}} \det(A)^{a_1-m} \det(\text{Id} - A)^{a_2-m} dA = \tilde{\beta}(a_1, a_2),$$

from which we can deduce the constants and prove the integral formulas. In fact,

$$\begin{aligned} & \int_{V^i \geq 0, \sum_p^{n-1} V^p \leq \text{Id}} f(V^1, \dots, V^{n-1}) \prod_k^{n-1} \det(V^k)^{a_k-m} \det(I - \sum_p^{n-1} V^p)^{a_n-m} dV^1 dV^2 \dots dV^{n-1} \\ &= \int \left( \int_{0 \leq V^1 \leq Y} \det(V^1)^{a_1-m} \det(Y - V^1)^{a_2-m} dV^1 \right) \\ & \quad \times f(Y, V^3, \dots, V^{n-1}) \prod_{k=3, \dots, n-1} \det(V^k)^{a_k-m} \det(I - Y - \sum_{k=3}^{n-1} V^k)^{a_n-m} dY dV^3 \dots dV^{n-1}. \end{aligned}$$

Notice that

$$\begin{aligned}
& \int_{0 \leq V^1 \leq Y} \det(V^1)^{a_1-m} (\det(Y - V^1))^{a_2-m} dV^1 \\
&= \det(Y)^{a_1+a_2-m} \int_{0 \leq Y^{-\frac{1}{2}} V^1 Y^{-\frac{1}{2}} \leq \text{Id}} \det(Y^{-\frac{1}{2}} V^1 Y^{-\frac{1}{2}})^{a_1-m} (\det(\text{Id} - Y^{-\frac{1}{2}} V^1 Y^{-\frac{1}{2}}))^{a_2-m} d(Y^{-\frac{1}{2}} V^1 Y^{-\frac{1}{2}}) \\
&= \det(Y)^{a_1+a_2-m} \tilde{\beta}(a_1, a_2),
\end{aligned}$$

such that

$$\begin{aligned}
& \int_{V^i \geq 0, \sum_p^{n-1} V^p \leq \text{Id}} f(V^1, \dots, V^{n-1}) \left( \prod_k^{n-1} \det(V^k)^{a_k-m} \right) \det(I - \sum_p^{n-1} V^p)^{a_n-m} dV^1 dV^2 \dots dV^{n-1} \\
&= \tilde{\beta}_m(a_1, a_2) \int f(Y, V^3, \dots, V^{n-1}) \det(Y)^{a_1+a_2-m} \\
&\quad \times \left( \prod_{k=3, \dots, n-1} \det(V^k)^{a_k-m} \right) \det(I - Y - \sum_{k=3}^{n-1} V^k)^{a_n-m} dY dV^3 \dots dV^{n-1} \\
&= \tilde{\beta}_m(a_1 + a_2, a_3) \tilde{\beta}_m(a_1, a_2) \int f(Y, V^4, \dots, V^{n-1}) \det(Y)^{a_1+a_2+a_3-m} \\
&\quad \times \left( \prod_{k=4, \dots, n-1} \det(V^k)^{a_k-m} \right) \det(\text{Id} - Y - \sum_{k=3}^{n-1} V^k)^{a_n-m} dY dV^4 \dots dV^{n-1} \\
&= \dots \\
&= \tilde{\beta}_m \left( \sum_i^{n-2} a_i, a_{n-1} \right) \dots \tilde{\beta}_m(a_1 + a_2, a_3) \tilde{\beta}_m(a_1, a_2) \int f(Y) \det(Y)^{\sum_{i=1}^{n-1} a_i - m} \det(\text{Id} - Y)^{a_n-m} dY.
\end{aligned}$$

In fact, the multivariate Dirichlet function is defined by (in complex case)

$$\tilde{\beta}_m(a_1, \dots, a_{n-1}; a_n) = \int_{V^i \geq 0, \sum_p^{n-1} V^p \leq \text{Id}} \prod_k^{n-1} \det(V^k)^{a_k-m} \det(I - \sum_p^{n-1} V^p)^{a_n-m} dV^1 dV^2 \dots dV^{n-1},$$

and the above discussion shows that

$$\tilde{\beta}_m(a_1, \dots, a_{n-1}; a_n) = \frac{\prod_i^n \tilde{\Gamma}_m(a_i)}{\tilde{\Gamma}_m(\sum_i a_i)}.$$

**Part V**

**Proofs of Theorems**



# Chapter 13

## Proofs

In this chapter, we give the proofs of Theorem 5.2.1, Theorem 6.3.7 and Theorem 6.3.9 which are included in the paper [70], and also the proofs of Theorem 6.3.2, Theorem 6.3.3, Theorem 6.3.4 and Theorem 6.3.6 in the published paper [72], Theorem 7.6.1 and Theorem 7.6.2 in [71], together with the proofs of Theorem 7.1.1, Theorem 7.3.3 and Theorem 7.3.4, which are not included in our preprints or publications.

### 13.1 Proofs of Theorem 5.2.1, Theorem 6.3.7 and Theorem 6.3.9

We first prove Theorem 5.2.1.

*Proof.* — Let  $P_{s,t}$  be the heat semigroup of the time dependent weighted Laplacian on  $L^2(M, \mu)$ , i.e.,  $u(t, \cdot) = P_{s,t}f(\cdot)$  is the unique solution of the heat equation  $\partial_t u = Lu$  in  $[s, T]$  with  $u(s, \cdot) = f$ . Similarly to Bakry and Ledoux [15] (where  $g$  and  $\phi$  are time independent), we introduce

$$h_s(t) = e^{2Kt} P_{s+T-t, T} \left( \frac{|\nabla P_{s, s+T-t} f|^2}{P_{s, s+T-t} f} \right), \quad t \in [s, T].$$

Note that, at time  $T - t + s$ , the generalized Bochner formula implies

$$(\partial_t + L) \frac{|\nabla u|^2}{u} = \frac{2}{u} |\nabla^2 u - u^{-1} \nabla u \otimes \nabla u|^2 + 2u^{-1} \left( \frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla u, \nabla u).$$

Hence

$$\begin{aligned} h'_s(t) &= 2Kh_s(t) + e^{2Kt} P_{s+T-t, T} \left[ \left( \frac{\partial}{\partial t} + L \right) \left( \frac{|\nabla P_{s, s+T-t} f|^2}{P_{s, s+T-t} f} \right) \right] \\ &= 2Kh_s(t) + e^{2Kt} P_{s+T-t, T} \left[ \frac{2}{u} |\nabla^2 u - u^{-1} \nabla u \otimes \nabla u|^2 + 2u^{-1} \left( \frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla u, \nabla u) \right] \\ &\geq 2e^{2Kt} P_{s+T-t, T} \left[ u^{-1} \left( \frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) + K \right) (\nabla u, \nabla u) \right]. \end{aligned}$$

If  $(g(t), \phi(t))$  is a  $(K, \infty)$ -super Ricci flow, i.e., if  $\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) + K \geq 0$ , then

$$h'_s(t) \geq 0.$$

Thus,  $t \rightarrow h(t)$  is increasing on  $[s, T]$ . This yields, for all  $t \in (s, T)$ ,

$$e^{2Ks} \frac{|\nabla P_{s,T} f|^2}{P_{s,T} f} \leq e^{2Kt} P_{s+T-t,T} \left( \frac{|\nabla P_{s,s+T-t} f|^2}{P_{s,s+T-t} f} \right) \leq e^{2KT} P_{s,T} \left( \frac{|\nabla f|^2}{f} \right).$$

Notice that

$$\begin{aligned} \frac{d}{dt} P_{s+T-t,T} (P_{s,s+T-t} f \log P_{s,s+T-t} f) &= P_{s+T-t,T} ((L_{s+T-t} + \partial_t)(P_{s,s+T-t} f \log P_{s,s+T-t} f)) \\ &= P_{s+T-t,T} \left( \frac{|\nabla P_{s,s+T-t} f|^2}{P_{s,s+T-t} f} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} P_{s,T}(f \log f) - P_{s,T} f \log P_{s,T} f &= \int_s^T \frac{d}{dt} P_{s+T-t,T} (P_{s,s+T-t} f \log P_{s,s+T-t} f) dt \\ &= \int_s^T P_{s+T-t,T} \left( \frac{|\nabla P_{s,s+T-t} f|^2}{P_{s,s+T-t} f} \right) dt \\ &\leq \frac{1}{2K} (e^{2K(T-s)} - 1) P_{s,T} \left( \frac{|\nabla f|^2}{f} \right). \end{aligned}$$

Thus the logarithmic Sobolev inequality holds on complete Riemannian manifolds equipped with a  $K$ -super Perelman Ricci flow

$$P_{s,T}(f \log f) - P_{s,T} f \log P_{s,T} f \leq \frac{e^{2K(T-s)} - 1}{2K} P_{s,T} \left( \frac{|\nabla f|^2}{f} \right). \quad (13.1.1)$$

Similarly to the above proof of the logarithmic Sobolev inequality (13.1.1), we have

$$\begin{aligned} P_{s,T}(f \log f) - P_{s,T} f \log P_{s,T} f &= \int_s^T \frac{d}{dt} P_{s+T-t,T} (P_{s,s+T-t} f \log P_{s,s+T-t} f) dt \\ &= \int_s^T P_{s+T-t,T} \left( \frac{|\nabla P_{s,s+T-t} f|^2}{P_{s,s+T-t} f} \right) dt \\ &\geq \int_s^T e^{2K(s-t)} \frac{|\nabla P_{s,T} f|^2}{P_{s,T} f} dt \\ &= \frac{1 - e^{2K(s-T)}}{2K} \frac{|\nabla P_{s,T} f|^2}{P_{s,T} f}. \end{aligned}$$

Thus, for all  $T > 0$ ,  $f \in C_b(M)$  with  $f > 0$ , the reversal logarithmic Sobolev inequality holds

$$\frac{|\nabla P_{s,T} f|^2}{P_{s,T} f} \leq \frac{2K}{1 - e^{2K(s-T)}} (P_{s,T}(f \log f) - P_{s,T} f \log P_{s,T} f). \quad (13.1.2)$$

It is then easy to prove that both (13.1.1) and (13.1.2) holds for  $P_{s,t}$  for all  $0 \leq s < t \leq T$ .

On the other hand, if for any  $0 \leq s < t \leq T$ , the log-Sobolev inequality for  $P_{s,t}$  holds, then applying (13.1.1) to  $1 + \varepsilon f$  and letting  $\varepsilon \rightarrow 0$ , we can obtain the Poincaré inequality

$$P_{s,t} f^2 - (P_{s,t} f)^2 \leq \frac{1}{K} (e^{2K(t-s)} - 1) P_{s,t} (|\nabla f|^2).$$

Notice that when  $s = t$ , we have

$$\begin{aligned} P_{s,t}f^2 - (P_{s,t}f)^2 - \frac{1}{K}(e^{2K(t-s)} - 1)P_{s,t}(|\nabla f|^2) &= 0, \\ \partial_s(P_{s,t}f^2 - (P_{s,t}f)^2 - \frac{1}{K}(e^{2K(t-s)} - 1)P_{s,t}(|\nabla f|^2)) &= 0, \end{aligned}$$

and at  $s = t$

$$\begin{aligned} \partial_s^2(P_{s,t}f^2 - (P_{s,t}f)^2 - \frac{1}{K}(e^{2K(t-s)} - 1)P_{s,t}(|\nabla f|^2)) \\ = -4[|Hessf|^2 + (2\partial_s g + Ric(L) + K)(\nabla f, \nabla f)]. \end{aligned}$$

Taking  $f$  to be normal coordinate functions, we get at any time  $t \in [0, T]$ ,

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) + K \geq 0.$$

This completes the proof of Theorem 5.2.1. ■

Based on the reversal logarithmic Sobolev inequality on complete Riemannian manifolds with fixed metrics and potentials, which is due to Bakry and Ledoux [15], we introduce the  $W$ -entropy and prove the  $W$ -entropy formula for the weighted Laplacian on complete Riemannian manifolds with fixed metric and potential satisfying the  $CD(K, \infty)$  condition (Theorem 6.3.7). We now give its proof.

*Proof.* — Let  $C_0(t) = \frac{1}{t}$ , and for  $K \neq 0$ ,  $C_K(t) = \frac{2K}{e^{2Kt} - 1}$ . Let  $D_0(t) = \frac{1}{t}$ ,  $D_K(t) = \frac{1}{|1 - e^{-2Kt}|}$ . Then  $D'_K(t) = -C_K(t)D_K(t)$  for all  $K \in \mathbb{R}$  and  $t > 0$ . We first introduce the revised Boltzmann entropy

$$H_K(f, t) = D_K(t) \int_M (P_t(f \log f) - P_t f \log P_t f) d\mu,$$

where  $f$  is a positive and measurable function on  $M$ . By direct calculation, we can prove

$$\begin{aligned} \frac{d}{dt} H_K(f, t) &= C_K(t)D_K(t) \int_M (P_t f \log P_t f - P_t(f \log f)) d\mu + D_K(t) \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu \\ &= D_K(t) \int_M \left[ \frac{|\nabla P_t f|^2}{P_t f} + C_K(t)(P_t f \log P_t f - P_t(f \log f)) \right] d\mu. \end{aligned} \quad (13.1.3)$$

Under the condition  $Ric(L) \geq K$ , by the reversal logarithmic Sobolev inequality due to Bakry and Ledoux [15], for all  $t > 0$ , we have

$$\frac{|\nabla P_t f|^2}{P_t f} \leq C_K(t)(P_t(f \log f) - P_t f \log P_t f). \quad (13.1.4)$$

Hence, for all  $K \in \mathbb{R}$ , we have

$$\frac{d}{dt} H_K(f, t) \leq 0, \quad \forall t > 0.$$

Taking the time derivative on the both sides of (13.1.3), we have

$$\begin{aligned} \frac{d^2}{dt^2} H_K(f, t) &= -C_K(t)D_K(t) \left[ \int_M \frac{|\nabla P_t f|^2}{P_t f} + C_K(t)P_t f \log P_t f - K(t)P_t(f \log f) \right] d\mu \\ &\quad + D_K(t) \left[ \frac{d}{dt} \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu - C_K(t) \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu \right] \\ &\quad + D_K(t) \frac{d}{dt} C_K(t) \int_M (P_t f \log P_t f - P_t(f \log f)) d\mu. \end{aligned}$$

By the fact that

$$\frac{d}{dt} \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu = -2 \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu - 2 \int_M Ric(L)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu,$$

we have

$$\begin{aligned} \frac{d^2}{dt^2} H_K(f, t) &= -C_K(t) \frac{d}{dt} H_K(t) - 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\ &\quad - D_K(t) \int_M (2Ric(L) + C_K(t))(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu \\ &\quad + D_K(t) \frac{d}{dt} C_K(t) \int_M (P_t f \log P_t f - P_t(f \log f)) d\mu. \end{aligned} \quad (13.1.5)$$

Note that, for all  $K \in \mathbb{R}$ , under the condition  $Ric(L) \geq K$ , we have

$$2Ric(L) + C_K(t) \geq 2K + \frac{2K}{e^{2Kt} - 1} = \frac{2K}{1 - e^{-2Kt}},$$

and

$$\frac{d}{dt} C_K(t) = \frac{d}{dt} \frac{2K}{e^{2Kt} - 1} = -\frac{2K}{1 - e^{-2Kt}} C_K(t).$$

Substituting these into (13.1.5), a simple calculation yields, for all  $K \in \mathbb{R}$ , and for all  $t > 0$ ,

$$\frac{d^2}{dt^2} H_K(f, t) \leq -2K \coth(2Kt) \frac{d}{dt} H_K(t) - 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu.$$

Indeed, from (13.1.5), we can prove

$$\begin{aligned} \frac{d^2}{dt^2} H_K(f, t) &= -2K \coth(2Kt) \frac{d}{dt} H_K(t) - 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\ &\quad - 2D_K(t) \int_M (Ric(L) - K)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu. \end{aligned} \quad (13.1.6)$$

Define the  $W$ -entropy by the revised Boltzmann entropy formula

$$W_K(f, t) := \frac{1}{\dot{\alpha}_K(t)} \frac{d}{dt} (\alpha_K(t) H_K(f, t)) = H_K + \frac{\alpha_K}{\dot{\alpha}_K} \dot{H}_K.$$

Then

$$\begin{aligned} \frac{d}{dt} W_K(f, t) &= \dot{H}_K + \left( \frac{\alpha_K}{\dot{\alpha}_K} \right)' \dot{H}_K + \frac{\alpha_K}{\dot{\alpha}_K} \ddot{H}_K \\ &= \left[ 2 - \frac{\alpha_K \ddot{\alpha}_K}{\dot{\alpha}_K^2} \right] \dot{H}_K + \frac{\alpha_K}{\dot{\alpha}_K} \ddot{H}_K \\ &= \frac{\alpha_K}{\dot{\alpha}_K} \left[ \ddot{H}_K + \frac{\dot{\alpha}_K}{\alpha_K} \left( 2 - \frac{\alpha_K \ddot{\alpha}_K}{\dot{\alpha}_K^2} \right) \dot{H}_K \right]. \end{aligned}$$

Choosing  $\alpha_K(t) = K \tanh(Kt)$ , we have

$$\frac{\dot{\alpha}_K}{\alpha_K} \left( 2 - \frac{\alpha_K \ddot{\alpha}_K}{\dot{\alpha}_K^2} \right) = 2K \coth(2Kt).$$

Therefore

$$W_K(f, t) = H_K(f, t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt} H_K(f, t),$$

and

$$\begin{aligned} \frac{d}{dt} W_K(f, t) &= -\frac{\sinh(2Kt)}{K} D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\ &\quad - \frac{\sinh(2Kt)}{K} D_K(t) \int_M (\text{Ric}(L) - Kg) \langle \nabla \log P_t f, \nabla \log P_t f \rangle P_t f d\mu. \end{aligned}$$

In particular, for all  $K \in \mathbb{R}$ , if  $\text{Ric}(L) \geq K$ , we have

$$\frac{d}{dt} W_K(f, t) \leq 0, \quad \forall t > 0.$$

In the case  $K = 0$ , (13.1.6) becomes

$$\frac{d^2}{dt^2} H_0(f, t) = -2 \frac{d}{dt} H_0(t) - \frac{2}{t} \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu - \frac{2}{t} \int_M \text{Ric}(L) \langle \nabla \log P_t f, \nabla \log P_t f \rangle P_t f d\mu.$$

Taking  $\alpha_0(t) = t$ , and defining the  $W$ -entropy by the standard Boltzmann entropy formula

$$W_0(f, t) = \frac{d}{dt} (t H_0(f, t)),$$

we have

$$W_0(f, t) = \frac{d}{dt} \int_M (P_t(f \log f) - P_t f \log P_t f) d\mu = \frac{d}{dt} \text{Ent}(P_t f) = \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu,$$

where  $\text{Ent}(f) = -\int_M f \log f d\mu$ , and we have

$$\frac{d}{dt} W_0(f, t) = -2 \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu - 2 \int_M \text{Ric}(L) \langle \nabla \log P_t f, \nabla \log P_t f \rangle P_t f d\mu.$$

This finishes the proof of Theorem 6.3.7. ■

Using our log-Sobolev inequality for super Ricci flow (Theorem 5.2.1), we can prove Theorem 6.3.9 by the similar method as used in the proof in Theorem 6.3.7.

## 13.2 Proofs of Theorem 6.3.2 and Theorem 6.3.3

First we prove the following dissipation formula for the Boltzmann-Shannon entropy associated with the weighted Laplacian on manifolds with time dependent metrics and potentials.

**Theorem 13.2.1.** *Let  $u$  be a positive solution to the heat equation  $\partial_t u = Lu$ . Let*

$$H(u, t) = - \int_M u \log u d\mu$$

*be the Boltzmann-Shannon entropy associated with the Witten Laplacian  $L$ . Then*

$$\frac{\partial^2}{\partial t^2} H(u, t) = -2 \int_M \left[ |\nabla^2 \log u|^2 + \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \right) \langle \nabla \log u, \nabla \log u \rangle \right] u d\mu.$$

*Proof.* — By direct calculation, we have

$$\frac{\partial}{\partial t}H(u, t) = - \int_M \partial_t u (\log u + 1) d\mu = - \int_M Lu (\log u + 1) d\mu.$$

Integrating by parts yields

$$\frac{\partial}{\partial t}H(u, t) = \int_M |\nabla \log u|_{g(t)}^2 u d\mu,$$

which further implies that, as  $\partial_t(d\mu) = 0$ , we have

$$\begin{aligned} \frac{\partial^2}{\partial t^2}H(u, t) &= \int_M \frac{\partial}{\partial t} (|\nabla \log u|_{g(t)}^2 u) d\mu \\ &= \int_M \left[ \frac{\partial}{\partial t} g^{ij} \nabla_i \log u \nabla_j \log u \right] u d\mu + \int_M \frac{\partial}{\partial t} \left[ \frac{|\nabla u|^2}{u} \right]_{g(t) \text{ fixed}} d\mu \\ &= \int_M \left[ -\frac{\partial}{\partial t} g_{ij} \nabla_i \log u \nabla_j \log u \right] u d\mu + \int_M \frac{\partial}{\partial t} \left[ \frac{|\nabla u|^2}{u} \right]_{g(t) \text{ fixed}} d\mu \\ &= \int_M \left( -\frac{\partial g}{\partial t} (\nabla \log u, \nabla u) + \frac{\partial}{\partial t} \left[ \frac{|\nabla u|^2}{u} \right]_{g(t) \text{ fixed}} \right) d\mu, \end{aligned} \quad (13.2.7)$$

where  $[\cdot]_{g(t) \text{ fixed}}$  means that the quantity  $|\nabla u|^2$  in  $[\cdot]$  is defined under a fixed metric  $g(t)$ . and we have used the facts  $|\nabla \log u|^2 = g^{ij} \nabla_i \log u \nabla_j \log u$  as well as  $\partial_t g^{ij} = -\partial_t g_{ij}$ .

By the well known entropy dissipation formula on manifolds with fixed metric, we have

$$\int_M \frac{\partial}{\partial t} \left[ \frac{|\nabla u|^2}{u} \right]_{g(t) \text{ fixed}} d\mu = -2 \int_M [|\nabla^2 \log u|^2 + Ric(L)(\nabla \log u, \nabla \log u)] u d\mu.$$

Combining (13.2.7) and (13.2.8), we finish the proof of Theorem 13.2.1. ■

According to [79], we introduce

$$W(u, t) = \frac{d}{dt}(tH(u, t)).$$

Direct computations yield

$$W(u, t) = \int_M [t|\nabla \log u|^2 - \log u] u d\mu,$$

and

$$\begin{aligned} \frac{d}{dt}W(u, t) &= -2 \int_M t \left[ |\nabla^2 \log u|^2 + \left( \frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad + 2 \int_M |\nabla \log u|^2 u d\mu. \end{aligned} \quad (13.2.8)$$

Let

$$H_m(u, t) = - \int_M u \log u d\mu - \frac{m}{2}(1 + \log(4\pi t)).$$

Similar to the case in [79], we define  $W_m(u, t)$  as follows

$$W_m(u, t) = \frac{d}{dt}(tH_m(u)). \quad (13.2.9)$$

It can be easily checked that  $W_m(u, t)$  is the same as the one in Theorem 6.3.2, i.e.,

$$W_m(u, t) = \int_M [t|\nabla \log u|^2 - \log u] u d\mu - \frac{m}{2}(2 + \log(4\pi t)).$$

We now prove Theorem 6.3.2.

*Proof.* — By Theorem 13.2.1, we have

$$\begin{aligned} \frac{dW_m(u)}{dt} &= -2 \int_M t \left[ |\nabla^2 \log u|^2 + \left( \frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad + 2 \int_M |\nabla \log u|^2 u d\mu - \frac{m}{2t}. \end{aligned} \quad (13.2.10)$$

Notice that

$$2t|\nabla^2 \log u|^2 + \frac{m}{2t} = 2t \left| \nabla^2 \log u + \frac{g}{2t} \right|^2 + \frac{m-n}{2t} - 2\Delta \log u.$$

Hence

$$\begin{aligned} \frac{d}{dt} W_m(u, t) &= -\frac{m-n}{2t} - 2t \int_M \left| \nabla^2 \log u + \frac{g}{2t} \right|^2 u d\mu + 2 \int_M |\nabla \log u|^2 u d\mu + 2 \int_M \Delta \log u u d\mu \\ &\quad - 2 \int_M t \left( \frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla \log u, \nabla \log u) u d\mu. \end{aligned}$$

Integrating by part yields

$$\begin{aligned} \int_M \Delta \log u u d\mu &= \int_M (L \log u + \nabla \phi \cdot \nabla \log u) u d\mu \\ &= - \int_M |\nabla \log u|^2 u d\mu + \int_M \nabla \phi \cdot \nabla \log u u d\mu, \end{aligned}$$

whence

$$\begin{aligned} \frac{d}{dt} W_m(u, t) &= -\frac{m-n}{2t} - 2t \int_M \left| \nabla^2 \log u + \frac{g}{2t} \right|^2 u d\mu + 2 \int_M \nabla \phi \cdot \nabla \log u u d\mu \\ &\quad - 2 \int_M t \left( \frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla \log u, \nabla \log u) u d\mu. \end{aligned}$$

Note that

$$\begin{aligned} &\frac{m-n}{2t} + 2t Ric(L)(\nabla \log u, \nabla \log u) - 2\nabla \phi \cdot \nabla \log u \\ &= 2t Ric_{m,n}(L)(\nabla \log u, \nabla \log u) + \frac{2t}{m-n} \left( \nabla \phi \cdot \nabla \log u - \frac{m-n}{2t} \right)^2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt} W_m(u, t) &= -2t \int_M \left| \nabla^2 \log u + \frac{g}{2t} \right|^2 u d\mu - \frac{2t}{m-n} \int_M \left( \nabla \phi \cdot \nabla \log u - \frac{m-n}{2t} \right)^2 u d\mu \\ &\quad - 2 \int_M t \left( \frac{1}{2} \frac{\partial g}{\partial t} + Ric_{m,n}(L) \right) (\nabla \log u, \nabla \log u) u d\mu. \end{aligned}$$

This proves the  $W$ -entropy formula in Theorem 6.3.2. The monotonicity result follows. The proof of Theorem 6.3.2 is completed.  $\blacksquare$

We now prove Theorem 6.3.3.

*Proof.* — By definition, we have

$$\mu(t) = \inf_u \left\{ \int_M [4t|\nabla u|^2 - u^2 \log u^2 - mu^2] (4\pi t)^{-\frac{m}{2}} d\mu \right\}, \quad (13.2.11)$$

where inf is taken among all the  $u$  such that

$$\int_M (4\pi t)^{-\frac{m}{2}} u^2 d\mu = 1.$$

Indeed,  $\mu(t)$  is the optimal constant in the following logarithmic Sobolev inequality: for all  $u$  satisfying the above condition,

$$\int_M u^2 \log u^2 (4\pi t)^{-\frac{m}{2}} d\mu \leq \mu(t) + m + 4 \int_M t|\nabla u|^2 (4\pi t)^{-\frac{m}{2}} d\mu.$$

By a similar argument as used in Perelman [99] and [79], we can prove that the minimization problem (13.2.11) has a non-negative minimizer  $u \in H^1(M, \mu)$ , which satisfies the Euler-Lagrange equation

$$-4tLu - 2u \log u - mu = \mu(t)u.$$

By the regularity theory of elliptic PDEs, we have  $u \in C^{1,\alpha}(M)$ . By an argument due to Rothaus (see [99, 79]), we can further prove that  $u$  is strictly positive and smooth. Hence  $v = -2 \log u$  is also smooth. Moreover, as a consequence of Theorem 6.3.2, we can derive that  $\mu(t)$  is a decreasing function in  $t \in [0, T]$ . The proof of Theorem 6.3.3 is completed.  $\blacksquare$

### 13.3 Proofs of Theorem 6.3.4 and Theorem 6.3.6

First we prove Theorem 6.3.4.

*Proof.* — By direct calculation and Theorem 6.3.2, we have

$$\begin{aligned} \frac{d}{dt} W_{m,K}(u, t) &= \frac{d}{dt} W(u, t) - \frac{m}{2t} - mK \left(1 + \frac{K}{2}t\right) \\ &= -2 \int_M t [|\nabla^2 \log u|^2 + Ric(L)(\nabla \log u, \nabla \log u)] u d\mu \\ &\quad + 2 \int_M |\nabla \log u|^2 u d\mu - \frac{m}{2t} - mK \left(1 + \frac{K}{2}t\right). \end{aligned}$$

Notice that

$$\begin{aligned} &2t|\nabla^2 \log u|^2 + \frac{m}{2t} + mK \left(1 + \frac{K}{2}t\right) \\ &= 2t \left| \nabla^2 \log u + \frac{1}{2t}g + \frac{K}{2}g \right|^2 - 2(1 + Kt)\Delta \log u + (m - n) \left( \frac{1}{2t} + K \left(1 + \frac{K}{2}t\right) \right). \end{aligned}$$

Integrating by part yields

$$\begin{aligned}
& \frac{d}{dt} W_{m,K}(u, t) \\
&= -2 \int_M t \left[ \left| \nabla^2 \log u + \frac{1}{2t} g + \frac{K}{2} g \right|^2 + (\text{Ric}(L) + Kg) (\nabla \log u, \nabla \log u) \right] u d\mu \\
&\quad + 2(1 + Kt) \int_M \nabla \log u \cdot \nabla \phi u d\mu + (m - n) \left( \frac{1}{2t} + K \left( 1 + \frac{K}{2} t \right) \right) \\
&= -2 \int_M t \left[ \left| \nabla^2 \log u + \frac{1}{2t} g + \frac{K}{2} g \right|^2 + (\text{Ric}_{m,n}(L) + Kg) (\nabla \log u, \nabla \log u) \right] u d\mu \\
&\quad - \frac{2t}{m - n} \int_M \left( \nabla \log u \cdot \nabla \phi - (m - n) \left( \frac{1}{2t} + \frac{K}{2} \right) \right)^2 u d\mu.
\end{aligned}$$

In particular, if  $\text{Ric}_{m,n}(L) \geq -Kg$ ,  $W_{m,K}(u, t)$  is monotone decreasing.  $\blacksquare$

Replacing  $\text{Ric}(L)$  by  $\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L)$  in the Formula (13.2.8), we can prove Theorem 6.3.6 by the similar method as used in the proof in Theorem 6.3.4.

## 13.4 Proof of Theorem 7.1.1

To prove Theorem 7.1.1, we first need the energy variational formula for the Fokker-Planck equation on Riemannian manifolds equipped with Perelman's Ricci flow and conjugate heat equation. The following result can be viewed as a generalization of a result due to Otto[96], McCann-Topping[86], Topping[113] and Lott[81]. The proof follows Lott [81].

**Theorem 13.4.1.** *Let  $(M, g(t), f(t))$  be a closed manifold equipped with Perelman's Ricci flow (1.3.15) and conjugate heat equation (1.3.16). Let  $\rho : [0, 1] \times [0, T] \rightarrow P^\infty(M)$  be positive solution to the transport equation*

$$\partial_s \rho = -\nabla \cdot (\rho \nabla \phi) + \langle \nabla f, \nabla \phi \rangle \rho, \quad (13.4.12)$$

where for any fixed  $t \in [0, T]$ ,  $\phi(\cdot, t) : [0, 1] \rightarrow C^\infty(M)$  can be viewed as the velocity vector of the smooth curve  $s \rightarrow \rho(s, t)$  in  $T_{\rho(\cdot, t)d\mu(t)} P^\infty(M)$ . Let

$$\begin{aligned}
Ent(\rho(s)) &= \int_M \rho \log \rho e^{-f} d\nu := \int_M \rho \log \rho d\mu, \\
E(c(t)) &= \frac{1}{2} \int_0^T \int_M |\nabla \phi|^2 \rho e^{-f} d\nu ds.
\end{aligned}$$

Then

$$\frac{d^2}{ds^2} Ent(\rho(s)) = \int_M \text{L}\rho (\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) d\mu + \int_M (|\text{Hess}\phi|^2 + \text{Ric}_f(\nabla \phi, \nabla \phi)) \rho d\mu,$$

and

$$\begin{aligned}
\frac{d}{dt} E(c(t)) &= \int_M \phi \left( \frac{\partial \rho}{\partial t} + \text{L}\rho \right) e^{-f} d\nu \Big|_0^T + \int_M |\text{Hess}\phi|^2 \rho e^{-f} d\nu \\
&\quad - \int_0^T \int_M (\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) (\partial_t \rho + \text{L}\rho) e^{-f} d\nu ds.
\end{aligned}$$

*Proof.* — By elementary calculation and Perelman's Ricci flow equation, we have

$$\frac{d}{dt}E(t) = \int_0^T \int_M [Ric_f(\nabla\phi, \nabla\phi)\rho + \langle \nabla\partial_t\phi, \nabla\phi \rangle\rho + \frac{1}{2}|\nabla\phi|^2\partial_t\rho]e^{-f} dv ds.$$

For a fixed function  $h \in C^\infty(M)$ , it holds

$$\begin{aligned} \int_M h \frac{\partial\rho}{\partial s} e^{-f} dv &= \int_M h(-\nabla \cdot (\rho\nabla\phi) + \langle \nabla f, \nabla\phi \rangle\rho) e^{-f} dv \\ &= \int_M \langle \nabla h, \nabla\phi \rangle \rho e^{-f} dv. \end{aligned}$$

Therefore,

$$\int_M h \frac{\partial^2\rho}{\partial s\partial t} e^{-f} dv = \int_M [2Ric_f(\nabla h, \nabla\phi)\rho + \langle \nabla h, \nabla\partial_t\phi \rangle\rho + \langle \nabla h, \nabla\phi \rangle\partial_t\rho]e^{-f} dv.$$

In particular, taking  $h = \phi$ , we obtain

$$\int_M \phi \frac{\partial^2\rho}{\partial s\partial t} e^{-f} dv = \int_M [2Ric_f(\nabla\phi, \nabla\phi)\rho + \langle \nabla\phi, \nabla\partial_t\phi \rangle\rho + |\nabla\phi|^2\partial_t\rho]e^{-f} dv.$$

Thus

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_0^T \int_M [\phi \frac{\partial^2\rho}{\partial s\partial t} - Ric_f(\nabla\phi, \nabla\phi)\rho - \frac{1}{2}|\nabla\phi|^2\partial_t\rho]e^{-f} dv ds \\ &= \int_M \phi \frac{\partial\rho}{\partial t} e^{-f} dv \Big|_0^T - \int_0^T \int_M (\partial_s\phi + \frac{1}{2}|\nabla\phi|^2)\partial_t\rho e^{-f} dv ds - \int_0^T \int_M Ric_f(\nabla\phi, \nabla\phi)\rho e^{-f} dv ds. \end{aligned}$$

Direct computation yields

$$\begin{aligned} \frac{d}{ds}Ent(\rho(s, t)) &= \int_M \langle \nabla\rho, \nabla\phi \rangle d\mu \\ \frac{d^2}{ds^2}Ent(\rho(s, t)) &= - \int_M L\rho(\partial_s\phi + \frac{1}{2}|\nabla\phi|^2)d\mu + \int_M \rho L(\frac{1}{2}|\nabla\phi|^2)d\mu - \int_M \rho \langle \nabla\phi, \nabla L\phi \rangle d\mu \\ &= - \int_M L\rho(\partial_s\phi + \frac{1}{2}|\nabla\phi|^2)d\mu + \int_M (|Hess\phi|^2 + Ric_f(\nabla\phi, \nabla\phi))\rho d\mu. \end{aligned}$$

Integrating  $\frac{d^2}{ds^2}Ent(\rho(s, t))$  from  $s = 0$  to  $s = T$ , we obtain

$$\int_M \langle \nabla\rho, \nabla\phi \rangle d\mu \Big|_0^T = - \int_0^T \int_M L\rho(\partial_s\phi + \frac{1}{2}|\nabla\phi|^2)d\mu ds + \int_0^T \int_M (|Hess\phi|^2 + Ric_f(\nabla\phi, \nabla\phi))\rho d\mu ds.$$

Combining the last equation with the previous one for  $\frac{dE(t)}{dt}$ , we have

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_M \phi \frac{\partial\rho}{\partial t} e^{-f} dv \Big|_0^T - \int_0^T \int_M \left( \partial_s\phi + \frac{1}{2}|\nabla\phi|^2 \right) \partial_t\rho e^{-f} dv ds - \int_0^T \int_M Ric_f(\nabla\phi, \nabla\phi)\rho e^{-f} dv ds \\ &= \int_M \phi \left( \frac{\partial\rho}{\partial t} + L\rho \right) e^{-f} dv \Big|_0^T + \int_M |Hess\phi|^2 \rho e^{-f} dv \\ &\quad - \int_0^T \int_M \left( \partial_s\phi + \frac{1}{2}|\nabla\phi|^2 \right) (\partial_t\rho + L\rho) e^{-f} dv ds. \end{aligned}$$

The proof of Theorem 13.4.1 is completed. ■

Now we are in a position to give the proof of Theorem 7.1.1.

*Proof.* — Fix  $t_0 \in (0, T]$ . According to the description of geodesic (7.2.2) in Wasserstein space by Brenier and Benamou [21], given  $\epsilon > 0$ , we can choose a smooth curve  $c : [0, 1] \rightarrow P^\infty(M)$  so that  $c(0) = c_0(t_0)$ ,  $c(1) = c_1(t_0)$  and

$$E(c) \leq \frac{1}{2}W_2(c_0(t_0), c_1(t_0))^2 + \epsilon.$$

For  $t \in [0, t_0]$ , define  $c_t : [0, 1] \rightarrow P^\infty(M)$  such that  $c_{t_0}(s) = c(s)$  and  $\partial_t c_t = -Lc_t$ . Writing  $c_t = \rho(t)e^{-f}d\nu$ , we have

$$\partial_t \rho(t) = -L\rho(t).$$

By Theorem 13.4.1,  $E(c_t)$  is nondecreasing in  $t$ . Hence

$$\frac{1}{2}W_2(c_0(t), c_1(t))^2 \leq E(c_t) \leq E(c_{t_0}) \leq \frac{1}{2}W_2(c_0(t_0), c_1(t_0))^2 + \epsilon.$$

Since  $\epsilon$  is arbitrary, Theorem 7.1.1 follows. ■

## 13.5 Proofs of Theorem 7.3.3, Theorem 7.3.4

There is no doubt that Theorem 7.3.3 and Theorem 7.3.4 can be proved by direct, and careful computation. We now provide a proof making use of the diffeomorphism that Perelman [99] introduced to show the equivalence of modified Ricci flow (1.3.15) and (5.1.1), which is exactly how we find the proper form of the entropy.

Before giving the proof of Theorem 7.3.3, we would like to point out that the Hamilton-Jacobi equation (7.3.8) is the geodesic equation for the  $W_2$ -distance with respect to the cost function  $L$  defined as follows: for  $\gamma : [0, T] \rightarrow M$

$$L(\gamma) = \int_0^T (|\dot{\gamma}|_{g(t)}^2 - R(\gamma(t), t) - 2\Delta_{g(t)}f + |\nabla f|_{g(t)}^2)dt.$$

i.e., (7.3.8) is the Euler-Lagrange equation of the Lagrangian

$$E(\rho) = \frac{1}{2} \int_0^T \int_M (|\nabla \phi|^2 - R - 2\Delta f + |\nabla f|^2)\rho d\mu dt.$$

We now give the proof of Theorem 7.3.3. *Proof.* — Following Perelman [99], we define a one-parameter family of diffeomorphisms  $\psi(t) : M \rightarrow M$  by

$$\begin{aligned} \frac{d}{dt}\psi(t) &= \nabla_{g(t)}f(t), \\ \psi(0) &= id_M. \end{aligned}$$

Then  $\bar{g}(t) := \psi(t)^*g(t)$ ,  $\bar{f}(t) := f \circ \psi(t)$  satisfy:

$$\begin{aligned} \frac{\partial \bar{g}_{ij}}{\partial t} &= -2\bar{R}_{ij}, \\ \frac{\partial \bar{f}}{\partial t} &= -\bar{\Delta}f + |\bar{\nabla}f|^2 - \bar{R}. \end{aligned}$$

Let  $u$  be the solution to the equation  $\partial_t u = -\Delta u - \nabla u \cdot \nabla f + Ru$ , then  $\bar{u}(t) := u(t) \circ \psi(t)$  satisfies

$$\partial_t \bar{u}(t) = -\bar{\Delta} \bar{u} + \bar{R} \bar{u}.$$

Now, let  $\bar{\rho} = \rho \circ \psi$ , and  $\bar{\phi} = \phi \circ \psi$ . Direct computation yields

$$\begin{aligned} \partial_t \bar{\rho} &= -\bar{\nabla} \cdot (\bar{\rho} \bar{\nabla} \bar{\phi}) + \bar{R} \bar{\rho}, \\ \partial_t \bar{\phi} &= -\frac{1}{2} |\bar{\nabla} \bar{\phi}|^2 + \frac{1}{2} \bar{R}. \end{aligned}$$

By Lott [81], we know that

$$\frac{d^2}{dt^2} \int_M (\bar{\rho} \log \bar{\rho} - \bar{\phi} \bar{\rho}) d\bar{vol} = \int_M |\bar{Ric} - \bar{Hess} \bar{\phi}|^2 \bar{\rho} d\bar{vol}.$$

Let  $\tilde{\rho} = \rho e^f$  and  $\tilde{\phi} = \phi + f$ . By direct calculation, we can prove that

$$\begin{aligned} \partial_t \tilde{\rho} &= -\nabla_\mu^* (\tilde{\rho} \nabla \tilde{\phi}). \\ \partial_t \tilde{\phi} &= -\frac{1}{2} |\nabla \tilde{\phi}|^2 - \frac{1}{2} R - \Delta f + \frac{1}{2} |\nabla f|^2. \end{aligned}$$

On the other hand, as  $\bar{\rho} = \rho \circ \psi$  and  $\bar{\phi} = \phi \circ \psi$ , using the invariance of the volume measure under diffeomorphism  $\psi$ , and making change of variable, we have

$$\int_M (\bar{\rho} \log \bar{\rho} - \bar{\phi} \bar{\rho}) d\bar{vol}_g = \int_M (\rho \log \rho - \phi \rho) d\text{vol}_g$$

Moreover, it is obvious to see that

$$\int_M (\rho \log \rho - \phi \rho) d\text{vol}_g = \int_M (\tilde{\rho} \log \tilde{\rho} - \tilde{\phi} \tilde{\rho}) d\mu.$$

Therefore

$$\begin{aligned} \frac{d^2}{dt^2} \int_M (\tilde{\rho} \log \tilde{\rho} - \tilde{\phi} \tilde{\rho}) d\mu &= \frac{d^2}{dt^2} \int_M (\rho \log \rho - \phi \rho) d\text{vol}_g \\ &= \int_M |\text{Ric}_f - \text{Hess} \tilde{\phi}|^2 \tilde{\rho} d\mu. \end{aligned}$$

This finishes the proof of Theorem 7.3.3. ■

Let  $\tau = T - t$  and set  $R_f = R(\gamma(\tau), \tau) - 2\Delta_{g(\tau)} f + |\nabla f|_{g(\tau)}^2 - \frac{f}{\tau}$ . Similarly, introducing the  $W_2$ -distance with respect to the cost function  $L$  defined as follows: for  $\gamma : [0, T] \rightarrow M$

$$L(\gamma) = \int_0^T \sqrt{\tau} (|\dot{\gamma}|_{g(\tau)}^2 - R(\gamma(\tau), \tau) - 2\Delta_{g(\tau)} f + |\nabla f|_{g(\tau)}^2 - \frac{f}{\tau}) d\tau,$$

we can prove that (7.3.9) is the Euler-Lagrange equation of the Lagrangian

$$E(\rho) = \frac{1}{2} \int_0^T \int_M \sqrt{\tau} (|\nabla \phi|^2 - R - 2\Delta f + |\nabla f|^2 - \frac{f}{\tau}) \rho d\mu d\tau.$$

*Proof.* — The proof follows that of Theorem 7.3.3. Indeed, define  $\Phi$  to be the diffeomorphism generated by  $-\nabla_{g(\tau)}f(\tau)$ , i.e.,

$$\begin{aligned}\frac{d}{d\tau}\Phi &= -\nabla_{g(\tau)}f(\tau, \Phi), \\ \Phi_0 &= \text{Id},\end{aligned}$$

and let  $\bar{g}(\tau) := \Phi(\tau)^*g(\tau)$ ,  $\bar{f}(\tau) := f \circ \Phi(\tau)$ . Then

$$\begin{aligned}\partial_\tau \bar{g} &= 2\bar{Ric}, \\ \partial_\tau \bar{f} &= \bar{\Delta}f - |\bar{\nabla}f|^2 + \bar{R}.\end{aligned}$$

and for  $\bar{\rho} = \rho \circ \psi$ ,  $\bar{\phi} = \phi \circ \psi$  we have

$$\begin{aligned}\partial_\tau \bar{\rho} &= -\bar{\nabla} \cdot (\bar{\rho} \bar{\nabla} \bar{\phi}) - \bar{R}\bar{\rho}, \\ \partial_\tau \bar{\phi} &= -\frac{1}{2}|\bar{\nabla} \bar{\phi}|^2 + \frac{1}{2}\bar{R} - \frac{1}{2\tau}\bar{\phi}.\end{aligned}$$

Also by Lott [81] we have

$$\begin{aligned}&\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2 \left(\int_M (\bar{\rho} \log \bar{\rho} + \bar{\phi} \bar{\rho}) d\bar{vol} + \frac{n}{2} \log(4\tau)\right) \\ &= \tau^3 \int_M \left| \bar{Ric} + \bar{\text{Hess}} \bar{\phi} - \frac{\bar{g}}{2\tau} \right|^2 \bar{\rho} d\bar{vol}.\end{aligned}$$

Hence,  $\int_M (\bar{\rho} \log \bar{\rho} + \bar{\phi} \bar{\rho}) d\bar{vol} + \frac{n}{2} \log(4\tau)$  is convex in  $\tau^{-\frac{1}{2}}$ . Let  $\tilde{\rho} = \rho e^f$  and  $\tilde{\phi} = \phi - f$ . Then

$$\begin{aligned}\partial_t \tilde{\rho} &= -\nabla_\mu^* (\tilde{\rho} \nabla \tilde{\phi}). \\ \partial_t \tilde{\phi} &= -\frac{1}{2}|\nabla \tilde{\phi}|^2 - \frac{1}{2}R - \Delta f + \frac{1}{2}|\nabla f|^2 - \frac{1}{2\tau}\tilde{\phi} - \frac{1}{2\tau}f.\end{aligned}$$

On the other hand, as  $\bar{\rho} = \rho \circ \psi$  and  $\bar{\phi} = \phi \circ \psi$ , using the invariance of the volume measure under diffeomorphism  $\psi$ , and making change of variable, we have

$$\int_M (\bar{\rho} \log \bar{\rho} + \bar{\phi} \bar{\rho}) d\bar{vol}_{\bar{g}} = \int_M (\rho \log \rho + \phi \rho) d\text{vol}_g$$

Moreover, it is obvious to see that

$$\int_M (\rho \log \rho + \phi \rho) d\text{vol}_g = \int_M (\tilde{\rho} \log \tilde{\rho} + \tilde{\phi} \tilde{\rho}) d\mu.$$

Therefore

$$\begin{aligned}&\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2 \left(\int_M (\tilde{\rho} \log \tilde{\rho} + \tilde{\phi} \tilde{\rho}) d\mu + \frac{n}{2} \log(4\tau)\right) \\ &= \tau^3 \int_M \left| Ric + \text{Hess} \tilde{\phi} + \text{Hess} f - \frac{g}{2\tau} \right|^2 \tilde{\rho} d\mu.\end{aligned}$$

This finishes the proof. ■

## 13.6 Proof of Theorem 7.6.1 and Theorem 7.6.2

*Proof of Theorem 7.6.1.* For geodesic flow  $(\rho(t), \phi(t))$  on  $T^*P_2^\infty(M, \mu)$ , we have

$$\begin{aligned}\frac{d}{dt}\text{Ent}(\rho(t)) &= \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle, \\ \frac{d^2}{dt^2}\text{Ent}(\rho(t)) &= \text{HessEnt}((\dot{\rho}(t), \dot{\rho}(t))).\end{aligned}$$

Thus

$$\begin{aligned}& \frac{d^2}{dt^2}\text{Ent}(\rho(t)) + \frac{2}{t}\frac{d}{dt}\text{Ent}(\rho(t)) + \frac{N}{t^2} \\ &= \text{HessEnt}(\rho(t)) + \frac{2}{t}\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{t^2} \\ &\geq \frac{1}{N}\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + K|\dot{\rho}(t)|^2 + \frac{2}{t}\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{t^2} \\ &= \frac{1}{N}\left|\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{t}\right|^2 + K|\dot{\rho}(t)|^2.\end{aligned}$$

On the other hand, for the backward gradient flow  $\dot{\rho}(t) = \nabla \text{Ent}(\rho(t))$  on  $P_2(M, \mu)$ , we have

$$\begin{aligned}\frac{d}{dt}\text{Ent}(\rho(t)) &= \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle \\ &= |\nabla \text{Ent}(\rho(t))|^2 = \int_M \frac{|\nabla \rho|^2}{\rho} d\mu, \\ |\dot{\rho}(t)|^2 &= |\nabla \text{Ent}(\rho(t))|^2 = \int_M \frac{|\nabla \rho|^2}{\rho} d\mu, \\ \frac{d^2}{dt^2}\text{Ent}(\rho(t)) &= 2\text{HessEnt}((\dot{\rho}(t), \dot{\rho}(t))).\end{aligned}$$

Thus

$$\begin{aligned}& \frac{d^2}{dt^2}\text{Ent}(\rho(t)) + \frac{2}{t}\frac{d}{dt}\text{Ent}(\rho(t)) + \frac{N}{2}\left(K + \frac{1}{t}\right)^2 \\ &= 2\text{HessEnt}(\rho(t)) + \frac{2}{t}\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{2}\left(K + \frac{1}{t}\right)^2 \\ &\geq \frac{2}{N}\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + 2K|\dot{\rho}(t)|^2 + \frac{2}{t}\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{2}\left(K + \frac{1}{t}\right)^2 \\ &= \frac{2}{N}\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + 2\left(K + \frac{1}{t}\right)\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{2}\left(K + \frac{1}{t}\right)^2 \\ &= \frac{2}{N}\left|\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{2}\left(K + \frac{1}{t}\right)g\right|^2\end{aligned}$$

This finishes the proof of Theorem 7.6.1.

*Proof of Theorem 7.6.2.* For the Langevin deformation of flows on  $T^*P_2^\infty(M, \mu)$ , we have

$$\begin{aligned}\frac{d}{dt}\text{Ent}(\rho(t)) &= \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle, \\ \frac{d^2}{dt^2}\text{Ent}(\rho(t)) &= \text{HessEnt}(\rho(t))(\dot{\rho}(t), \dot{\rho}(t)) + \langle \nabla \text{Ent}(\rho), \ddot{\rho} \rangle \\ &= \text{HessEnt}(\rho(t))(\dot{\rho}(t), \dot{\rho}(t)) - \frac{1}{c^2} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) + \nabla V(\rho) \rangle.\end{aligned}$$

Under Emar-Kawada-Sturm's  $CD_{\text{Ent}}(K, N)$ -condition, we have

$$\begin{aligned}& \frac{d^2}{dt^2}\text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt}\text{Ent}(\rho(t)) + N\alpha^2(t) \\ &= \text{HessEnt}(\rho(t))(\dot{\rho}(t), \dot{\rho}(t)) - \frac{1}{c^2} \langle \nabla \text{Ent}, \dot{\rho}(t) + \nabla V(\rho) \rangle + \left(2\alpha(t) + \frac{1}{c^2}\right) \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle \\ &\quad + N\alpha^2(t) \\ &\geq \frac{1}{N} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + K|\dot{\rho}(t)|^2 - \frac{1}{c^2} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) + \nabla V(\rho) \rangle \\ &\quad + \left(2\alpha(t) + \frac{1}{c^2}\right) \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha^2(t).\end{aligned}$$

Now  $V(\rho) = \text{Ent}(\rho) = \int_M \rho \log \rho d\mu$ . We have

$$\langle \nabla \text{Ent}(\rho(t)), \nabla V(\rho(t)) \rangle = \langle \nabla \text{Ent}(\rho(t)), \nabla \text{Ent}(\rho(t)) \rangle = |\nabla \text{Ent}(\rho(t))|^2 = \int_M \frac{|\nabla \rho|^2}{\rho} d\mu.$$

Hence

$$\begin{aligned}& \frac{d^2}{dt^2}\text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt}\text{Ent}(\rho(t)) + N\alpha^2(t) \\ &\geq \frac{1}{N} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + K|\dot{\rho}(t)|^2 - \frac{1}{c^2} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle - \frac{1}{c^2} |\nabla \text{Ent}(\rho(t))|^2 \\ &\quad + \left(2\alpha(t) + \frac{1}{c^2}\right) \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha^2(t) \\ &= \frac{1}{N} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + K|\dot{\rho}(t)|^2 + 2\alpha(t) \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha^2(t) - \frac{1}{c^2} |\nabla \text{Ent}(\rho(t))|^2 \\ &= \frac{1}{N} |\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha(t)|^2 - \frac{1}{c^2} |\nabla \text{Ent}(\rho(t))|^2 + K|\dot{\rho}(t)|^2.\end{aligned}$$

This finishes the proof of Theorem 7.6.2.



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