Benjamini-Schramm convergence of locally symmetric spaces

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Résumé : Le sujet principal de ce mémoire est le comportement asymptotique de la géométrie et topologie des variétés localement symétriques $\Gamma\backslash X$ quand $\mathrm{Vol}(\Gamma\backslash X) \to \infty$. Notre premier résultat porte sur la convergence Benjamini-Schramm des 2 ou 3-variétés hyperboliques arithmétiques. Une suite d'espaces localement symétriques $(\Gamma_n\backslash X)$ converge Benjamini-Schramm vers l'espace symétrique X si pour chaque $R > 0$ on a $\lim_{n\to\infty} \frac{\mathrm{Vol}((\Gamma\backslash X)_{<R})}{\mathrm{Vol}(\Gamma\backslash X)} = 0$, où $(\Gamma\backslash X)_{<R}$ est la partie R-mince de l'espace $\Gamma\backslash X$. On montre qu'il existe une constante réelle $C = C_R$ satisfaisant la propriété suivante: pour chaque réseau arithmétique de congruence Γ de $\mathrm{PGL}(2,\mathbb{R})$ ou $\mathrm{PGL}(2,\mathbb{C})$ sans torsion on a

$$\mathrm{Vol}((\Gamma\backslash X)_{<R}) \leq C_R \mathrm{Vol}(\Gamma\backslash X)^{0.986}. \tag{0.1}$$

Il n'y a qu'un nombre fini de réseaux arithmétiques de covolume borné par une constante donc ce résultat implique la convergence Benjamini-Schramm pour des variétés arithmétiques de congruence. On donne aussi une version de (0.1) un peu plus faible qui reste vraie pour des réseaux arithmétiques qui ne sont pas de congruence. Les majorations de volume de la partie R-mince sont déduites d'une version forte de la propriété de la multiplicité limite satisfaite par les réseaux arithmétiques de $\mathrm{PGL}(2,\mathbb{R})$ et $\mathrm{PGL}(2,\mathbb{C})$. En utilisant nos résultats on confirme la conjecture de Gelander pour des 3-variétés arithmétiques hyperboliques: pour chaque telle variété M on construit un complexe simplicial $\mathcal{N}$ homotope à M dont le nombre des simplexes est $O(\mathrm{Vol}(M))$ et le degré des nœuds est uniformément borné par une constante absolue.

Dans la deuxième partie on s'intéresse aux espaces localement symétriques $\Gamma\backslash X$ où X est de rang supérieur ou égal à 2. Notre résultat principal affirme que la dimension du premier groupe d'homologie à coefficients dans $\mathbb{F}_2$ (corps avec 2 éléments) est sous-linéaire en le volume. Ce résultat est à comparer avec des travaux de Calegari et Emerton sur la cohomologie $\mathrm{mod}-p$ dans les tours $p-$adiques des 3-variétés et les résultats de Abert, Gelander et Nikolov sur le rang des sous-groupes d'un réseau de rang supérieur à angles droits. Le point fort de notre approche est qu'il n'y a pas besoin de travailler dans une seule classe de commensurabilité.

La troisième partie est indépendante des deux premières. Elle porte sur une extension du théorème de Kesten. Le théorème de Kesten affirme que si Γ est un groupe engendré par un ensemble fini symétrique S, N est un sous groupe normal de Γ alors N est moyennable si et seulement si les rayons spectraux du graphe de Cayley $\mathrm{Cay}(\Gamma, S)$ et du graphe de Scheier $\mathrm{Sch}(\Gamma/N, S)$ coïncident. En utilisant les techniques de Abert, Glasner et Virag on généralise le theorème de Kesten aux N-uniformément récurrents.

Le dernier chapitre est consacré aux bornes sur les valeurs des caractères irréductibles des groupes finis de type Lie et des groupes de la forme $\mathbb{G}(\mathbb{Z}_p)$ où $\mathbb{G}$ est un schéma en groupes lisse, réductif sur $\mathbb{Z}$. Soit G un groupe comme ci-dessus et soit χ le caractère d'une représentation lisse irréductible de G. On cherche des bornes de la forme suivante:

$$|\chi(\gamma)| \leq \chi(1)^{\delta} \text{ avec } \delta < 1,$$

où δ et la constante implicite ne dépendante que de γ. Les bornes de telle forme sont les ingrédients de la preuve de (0.1) dans le deuxième chapitre. On montre que dans un groupe réductif G défini sur un corps fini $\mathbb{F}_q$ pour chaque $\gamma \in G(\mathbb{F}_q)$ régulier semi-simple on a

$$|\chi(\gamma)| \leq |W|,$$

où W est le groupe de Weyl. L'idée principale est de trouver un foncteur de restriction de Deligne-Lusztig qui ne change pas la valeur des caractères sur γ.

Title : Benjamini-Schramm convergence of locally symmetric spaces

Keys words : Locally symmetric spaces, arithmetic groups, representation theory, random walks

Abstract : The main theme of this work is the study of geometry and topology of locally symmetric spaces $\Gamma\backslash X$ as $\mathrm{Vol}(\Gamma\backslash X) \to \infty$. Our first main result concerns the Benjamini-Schramm convergence for arithmetic hyperbolic 2 or 3-manifolds. A sequence of locally symmetric spaces $(\Gamma_n\backslash X)$ converges Benjamini-Schramm to X if and only if for every radius $R > 0$ we have $\lim_{n\to\infty} \frac{\mathrm{Vol}((\Gamma\backslash X)_{<R}}{\mathrm{Vol}(\Gamma\backslash X)} = 0$, where $(\Gamma\backslash X)_{<R}$ stands for the R-thin part of $\Gamma\backslash X$. We prove that there exists a positive constant $C = C_R$ with the following property: for every torsion free, uniform, congruence arithmetic lattice Γ in $\mathrm{PGL}(2,\mathbb{R})$ or $\mathrm{PGL}(2,\mathbb{C})$

$$\mathrm{Vol}((\Gamma\backslash X)_{<R}) \ll C\mathrm{Vol}(\Gamma\backslash X))^{0.986}. \tag{0.2}$$

There is only finitely many arithmetic lattices of covolume bounded by a constant so the result above implies the Benjamini-Schramm convergence for any sequence of congruence arithmetic hyperbolic 3-manifolds. We also prove a similar but slightly weaker inequality for non-congruence subgroups. Our results are deduced form a strong form of the limit multiplicity property that holds for arithmetic lattices in $\mathrm{PGL}(2,\mathbb{R})$ of $\mathrm{PGL}(2,\mathbb{C})$. As an application of our bounds we confirm Gelander's conjecture on the triangulations of arithmetic hyperbolic 3-manifolds: we show that every arithmetic hyperbolic 3-manifold M admits a triangulation with $O(\mathrm{Vol}(M))$ simplices and degrees of vertices bounded uniformly by an absolute constant.

Next, we move to the setting of higher rank locally symmetric spaces. Let $M_n = \Gamma_n\backslash X$ be a sequence of pairwise distinct locally symmetric spaces modeled after a higher rank symmetric space X. We show that the dimension of the first homology group with coefficients in $\mathbb{F}_2$ is sublinear in volume. This can be compared with the results of Calegari and Emerton on mod$-p$ homology growth in p-adic analytic towers of 3-manifolds as well as the results of Abert, Gelander and Nikolov on the rank gradient of right-angled lattices in higher rank Lie groups. The main strength of our theorem is that we do not need to assume that the manifolds in question are commensurable.

Our third result is independent of the first two. Kesten theorem asserts that if Γ is group generated by a finite symmetric set S and N is a normal subgroup of Γ then N is amenable if and only if the spectral radii of the Cayley graphs $\mathrm{Cay}(\Gamma, S)$ and the Schreier graph $\mathrm{Sch}(\Gamma/N, S)$ are equal. Building on the work of Abert, Glasner and Virag we extend Kesten's theorem to uniformly recurrent subgroups.

The last chapter is devoted to character bounds for finite simple Lie groups and groups of form $\mathbb{G}(\mathbb{Z}_p)$ where $\mathbb{G}$ is a reductive smooth group scheme over $\mathbb{Z}$. Let G be one of the above and let χ be the character of a smooth irreducible representation of G. We are interested in bounds of form

$$|\chi(\gamma)| \ll \chi(1)^{\delta} \text{ with } \delta < 1,$$

where δ and implicit constant depending only on γ. Such bounds proved very useful in proving the Limit multiplicity property for arithmetic lattices in the second chapter. We show that in a finite reductive group G for γ regular semi-simple one has

$$|\chi(\gamma)| \leq |W|.$$

The key idea is to find a Deligne-Lusztig restriction functor which does not change the character value on γ.

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Contents

CHAPTER 1

Introduction en français

1. Espaces localement symétriques

Dans la première partie de l'introduction, on rappelle la définition d'un espace localement symétrique et on décrit une construction à l'aide des groupes de Lie semi-simples. Après avoir parlé de réseaux arithmétiques, on présente la convergence Benjamini-Schramm et la propriété de la multiplicité limite. Ensuite, on présente nos résultats principaux des chapitres 2 et 3. Chaque chapitre a sa propre introduction où l'on donne l'idée des démonstrations et on fait une comparaison avec les résultats déjà connus. Le lecteur familier avec les espaces localement symétriques et les réseaux arithmétiques peut passer directement à la section 1.3.

1.1. Espaces symétriques. Une variété riemannienne (M, g) est un **espace symétrique** si, pour chaque point $x \in M$, il existe un automorphisme isométrique ι_x de M tel que $\iota_x(x) = x$ et $d\iota_x \colon T_xM \to T_xM$ est la multiplication par -1. Autrement dit, ι_x renverse les géodésiques qui passent par x. Un **espace localement symétrique** est défini par la même condition mais avec ι_x défini sur un voisinage ouvert de x. Un espace symétrique simplement connexe est appelé **irréductible** s'il n'est pas isométrique à un produit d'espaces localement symétriques. Un espace symétrique irréductible est dit **de type non-compact** s'il est de courbure sectionnelle négative ou nulle mais pas nulle. Finalement, en général un espace localement symétrique est dit de type non-compact s'il est un produit d'espaces irréductibles de type non-compact. Dans cette thèse, on ne s'intéresse qu'aux espaces de type non-compact.

Les espaces symétriques de type non compact peuvent être construits à l'aide des groupes de Lie semisimples. Soit G un groupe de Lie connexe semi-simple, soit $\mathfrak{g}$ son algèbre de Lie et $B(-,-)$ sa forme de Killing. Choisissons une involution de Cartan $\Theta : \mathfrak{g} \to \mathfrak{g}$, c'est une involution linéaire telle que la forme bilinéaire $(X, Y) \mapsto -B(X, \Theta Y)$ soit définie positive. On a une décomposition de $\mathfrak{g}$ en une somme orthogonale des espaces propres de Θ:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \text{ où } \Theta X = X \text{ pour } X \in \mathfrak{k} \text{ et } \Theta Y = -Y \text{ pour } Y \in \mathfrak{p}.$$

$\mathfrak{k}$ est une sous-algèbre de Lie, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ et $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$. Posons $K = \{g \in G | \mathrm{Ad}(g)\mathfrak{k} = \mathfrak{k}\}$. Alors, $\mathfrak{k}$ est l'algèbre de Lie de K et K est un sous-groupe maximal compact de G.[1]

Soit $X = G/K$. L'espace tangent T_KX s'identifie avec $\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{k}$. L'espace X est muni de l'unique métrique riemanienne g, G-invariante à gauche, telle que

$$g(X, Y) = B(X, Y) \text{ pour } X, Y \in \mathfrak{p}.$$

La variété riemannienne (X, g) est appelée *l'espace symétrique* de G, les involutions étant induites par des involutions de Cartan. Le groupe G agit sur X à gauche par isométries et cette action est propre i.e. les stabilisateur des points sont compacts.

Le tenseur de courbure de X au point K est donné par

$$R_K(X, Y)Z = [[X, Y], Z] \text{ pour } X, Y, Z \in \mathfrak{p}.$$

Notons que la courbure sectionnelle n'est jamais strictement positive: $g(R(X, Y)X, Y) = B([X, Y], [X, Y]) \leq 0$ car $[X, Y] \in \mathfrak{k}$. On termine cette section en disant pourquoi X

[1] K est le groupe des points réels d'un groupe algébrique linéaire sur $\mathbb{R}$ donc il n'a qu'un nombre fini de composantes connexes. La composante de l'identité est un groupe compacte car sa forme de Killing est définie négative.

est contractile. La décomposition d'Iwasawa $G = NAK$, où N est unipotent et A est isomorphe à $\mathbb{R}_+^r$, implique que X est homéomorphe à $NA \simeq \mathbb{R}^{\dim G - \dim K}$.

1.2. Réseaux arithmétiques. Étant donné un groupe de Lie connexe semi-simple G et son espace symétrique X, on sait produire des exemples d'espaces localement symétriques de type $\Gamma\backslash X$ où Γ est un sous-groupe discret de G sans torsion. Si on prend Γ avec torsion, on obtient un orbifold localement symétrique. Même si ce ne sont pas des vraies variétés riemanniennes, on va les appeler encore espaces localement symétriques. Dans cette thèse, on étudie les espaces localement symétriques de volume fini, ce qui revient à prendre des quotients $\Gamma\backslash X$ où Γ est un réseau de G.

Une façon de construire des réseaux dans un groupe de Lie semisimple G est d'utiliser l'arithmétique. Par exemple, si $G = \mathrm{SL}(n, \mathbb{R})$, le sous-groupe $\mathrm{SL}(n, \mathbb{Z})$ est un réseau. En général, le théorème de Borel–Harish-Chandra [**22**] affirme que pour chaque sous-groupe algébrique connexe semi-simple $\mathbb{G}$ de $\mathrm{GL}(n, \mathbb{Q})$ défini sur $\mathbb{Q}$ sans de cractère défini sur $\mathbb{Q}$ le sous-groupe $\mathbb{G}(\mathbb{Q}) \cap \mathrm{GL}(n, \mathbb{Z})$ est un réseau de $\mathbb{G}(\mathbb{R})$. Ces sont les exemples les plus simples de réseaux arithmétiques. Pour décrire une construction qui donne tous les réseaux arithmétiques on va utiliser les adèles. Ce point de vue est crucial dans le chapitre 2. Une référence classique sur les adèles et les groupes algèbriques est le livre de Weil [**104**]. Fixons un groupe de Lie connexe semi-simple G. Soit k un corps de nombres. Écrivons M pour l'ensemble des places de k, M_∞ pour les places archimédiennes et M_f pour les places finies. Soient $\mathcal{O}_k, \mathcal{O}_{k_\mathfrak{p}}$ les anneaux des entiers dans k et $k_\mathfrak{p}$ respectivement. L'anneau des adèles $\mathbb{A} = \mathbb{A}_k$ (on va supprimer l'indice k si le corps est fixé) est défini comme un produit restreint :

$$\mathbb{A} := \{(a_\nu)_\nu \in \prod_{\nu \in M_k} k_\nu \mid a_\mathfrak{p} \in \mathcal{O}_{k_\mathfrak{p}} \text{ pour presque tout } \mathfrak{p} \in M_f\}.$$

Il est muni de la topologie de produit restreint. Alors $\mathbb{A}$ est une algèbre sur k et son groupe additif est localement compact. Ce sera utile d'écrire $\mathbb{A}$ comme un produit $\mathbb{A}_\infty \times \mathbb{A}_f$ où $\mathbb{A}_\infty$ est le produit des complétions archimédiennes de k et $\mathbb{A}_f$ est le produit restreint de toutes les complétions p-adiques de k. Soit $\mathbb{G}$ un groupe algébrique linéaire connexe semi-simple défini sur k. On dit que $\mathbb{G}$ est **anisotrope** sur k s'il n'y a pas d'homomorphisme non-trivial $\mathbb{G}_m \to \mathbb{G}$ défini sur k. Si $\mathbb{G}$ n'est pas anisotrope, on dit qu'il est **isotrope** sur k. Le groupe $\mathbb{G}(\mathbb{A})$ avec la topologie induite de $\mathbb{A}$ est un groupe localement compact. Afin de voir $\mathbb{G}(\mathbb{A})$ comme un produit restreint fixons une représentation rationnelle $\rho\colon \mathbb{G} \to \mathrm{GL}(n, k)$. On a

$$\mathbb{G}(\mathbb{A}) := \{(g_\nu)_\nu \in \prod_{\nu \in M_k} \mathbb{G}(k_\nu) \mid \rho(g_\mathfrak{p}) \in \mathrm{GL}(n, \mathcal{O}_{k_\mathfrak{p}}) \text{ pour presque tout } \mathfrak{p} \in M_f\}.$$

Comme avec $\mathbb{A}$ on a $\mathbb{G}(\mathbb{A}) = \mathbb{G}(\mathbb{A}_\infty) \times \mathbb{G}(\mathbb{A}_f)$. Par le théorème de Mostow-Tamagawa, le groupe $\mathbb{G}(k)$ est un réseau de $\mathbb{G}(\mathbb{A})$, qui est cocompact si et seulement si $\mathbb{G}$ est anisotrope. Choisissons un sous-groupe ouvert compact U de $\mathbb{G}(\mathbb{A}_f)$. Alors, le groupe $\Gamma_U := \mathbb{G}(k) \cap \mathbb{G}(\mathbb{A}_\infty) \times U$ est un réseau dans $\mathbb{G}(\mathbb{A}_\infty) \times U$. La projection de Γ_U vers $\mathbb{G}(\mathbb{A}_\infty)$ reste un réseau. Si $\mathbb{G}(\mathbb{A}_\infty)$ admet une décomposition $G \times H$ avec H compact, l'image de Γ_U est un réseau dans G.

Définition 1.1. *Soit G un groupe de Lie semi-simple. Un réseau Γ dans G est arithmétique s'il existe un corps de nombres k, un groupe algébrique linéaire semi-simple $\mathbb{G}$ défini sur k, un sous-groupe ouvert compact U de $\mathbb{G}(\mathbb{A}_f)$ et un homomorphisme ρ de $\mathbb{G}(\mathbb{A}_\infty)$ vers G de noyau compact tel que $\rho(\Gamma_U)$ est commensurable avec Γ. Si, de plus, Γ contient $\rho(\Gamma_U)$, on dit que Γ est **de congruence**.*

Étant donné un réseau arithmétique de congruence Γ, on peut demander s'il existe $k, \mathbb{G}, U$ et ρ tels que $\Gamma = \rho(\Gamma_U)$. Les réseaux arithmétiques cocompacts maximaux dans $\mathrm{SL}(2, \mathbb{R})$ sont les normalisateurs des $\rho(\Gamma_U)$ pour un certain choix de $k, \mathbb{G}, U$ et ρ et ils sont strictement plus grands que $\rho(\Gamma_U)$ (voir [**21**] et [**94**, 11.4]). Par contre, la réponse à la dernière question est positive si G est un groupe de Lie semi-simple adjoint (voir Lemme 2.32 et Proposition 2.33 où on donne une démonstration pour $\mathrm{PGL}(2, \mathbb{R})$ et $\mathrm{PGL}(2, \mathbb{C})$).

Traitons un exemple de construction adélique d'un réseau arithmétique de $\mathrm{SL}(2,\mathbb{C})$. Soit $k=\mathbb{Q}(\sqrt{-5})$. Notons son unique place complexe ν. Son anneau des entiers est $\mathcal{O}_k=\mathbb{Z}[\sqrt{-5}]$. Il y a deux places finies de k ramifiées $(\sqrt{-5})$ et $(2,\sqrt{-5}+1)$, pour chaque nombre premier $p\equiv 11,13,17,19 \bmod 20$, il y a une place $\mathfrak{p}$ au dessus de p et pour chaque $p\equiv 1,3,7,9 \bmod 20$, il y a deux places $\mathfrak{p}_1,\mathfrak{p}_2$ au dessus de p. Notons par A l'algèbre des quaternions sur k définie par

$$A=k+\mathbf{i}k+\mathbf{j}k+\mathbf{ij}k$$

avec les relations $\mathbf{i}^2=\sqrt{-5}$, $\mathbf{j}^2=2$ et $\mathbf{ij}=-\mathbf{ji}$. La forme quadratique $\sqrt{-5}\,x^2+2y^2-z^2$ est anisotrope sur k donc A est un corps gauche [**94**, Thm 2.3.1]). La norme sur A est définie par $n(x+\mathbf{i}y+\mathbf{j}z+\mathbf{ij}t)=x^2-\sqrt{-5}y^2-2t^2+2\sqrt{-5}t^2$. Posons $\mathbb{G}=\{a\in A | n(a)=1\}$. Le groupe $\mathbb{G}$ est une k-forme de $\mathrm{SL}(2,k)$ c'est-à-dire $\mathbb{G}\simeq \mathrm{SL}(2)$ sur la clôture algébrique de k. On a

$$\mathbb{G}(\mathbb{A})=\mathbb{G}(k_\nu)\times\mathbb{G}(\mathbb{A}_f).$$

On se donne un sous-anneau $\mathcal{O}$ de A: $\mathcal{O}=\mathcal{O}_k+\mathbf{i}\mathcal{O}_k+\mathbf{j}\mathcal{O}_k+\mathbf{ij}\mathcal{O}_k$. C'est un **ordre** de A, c'est-à-dire un sous-$\mathcal{O}_k$-module de A de rang 4 qui est aussi un sous-anneau. On va s'en servir pour construire un sous-groupe ouvert U de $\mathbb{A}_f$. L'ordre $\mathcal{O}$ est un sous-anneau de la $\mathbb{A}_f$-algèbre $A\otimes_k\mathbb{A}_f$. Soit $\mathcal{O}_f$ la clôture de $\mathcal{O}$ dans $A\otimes_k\mathbb{A}_f$. C'est un sous-anneau compact ouvert. Posons $U=\mathcal{O}_f^1=\{s\in\mathcal{O}\,|n(a)=1\}$. La projection de $\Gamma_U=\mathbb{G}(k)\cap(\mathrm{SL}(2,\mathbb{C})\times U)$ est un réseau arithmétique de congruence dans $\mathrm{SL}(2,\mathbb{C})$. Explicitement $\rho(\Gamma_U)$ est:

$$\left\{\begin{pmatrix} x+\sqrt[4]{-5}y & z+\sqrt[4]{-5}t\\ 2(z-\sqrt[4]{-5}t) & x-\sqrt[4]{-5}y\end{pmatrix} \mid x,y,z,t\in\mathcal{O}_k \text{ et } x^2-\sqrt{-5}y^2-2z^2+2\sqrt{-5}t^2=1\right\}$$

et la projection ρ est définie par

$$\rho(1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\rho(\mathbf{i})=\begin{pmatrix}\sqrt[4]{-5}&0\\0&-\sqrt[4]{-5}\end{pmatrix} \text{ et } \rho(\mathbf{j})=\begin{pmatrix}0&1\\2&0\end{pmatrix}.$$

1.3. Propriété de la multiplicité limite. Soit G un groupe de Lie semi-simple et soit Γ un réseau de G. Le groupe G agit sur $L^2(\Gamma\backslash G)$ par des translations à droite:

$$(R_\Gamma g\Phi)(x)=\Phi(xg) \text{ for } \Phi\in L^2(\Gamma\backslash G), g\in G \text{ et } x\in\Gamma\backslash G.$$

Cette représentation est unitaire. Étant donnée une suite de réseaux $(\Gamma_n)_{n\in\mathbb{N}}$, on peut étudier les propriétés asymptotiques des représentations $L^2(\Gamma_n\backslash G)$ quand $n\to\infty$. Soit π une représentation irréductible unitaire de G, notons $m_\Gamma(\pi)$ sa multiplicité dans $L^2(\Gamma\backslash G)$ définie comme la dimension de $\mathrm{Hom}_G(\pi,L^2(\Gamma\backslash G)$. En 1978, DeGeorge et Wallach [**34**] ont démontré que sous certaines conditions raisonnables sur $(\Gamma_n)_{n\in\mathbb{N}}$, on a

$$\lim_{n\to\infty}\frac{m_{\Gamma_n}(\pi)}{\mathrm{Vol}(\Gamma\backslash G)}=d(\pi),$$

où $d(\pi)$ est le degré de π (qui est non-nul ssi π est une série discrète). La propriété de la multiplicité limite est encore plus forte que la convergence des multiplicités normalisées par le volume. Elle est définie en termes des distributions des sous-représentations irréductibles de $L^2(\Gamma\backslash G)$ dans l'espace dual unitaire[2] $\Pi(G)$ de G. Pour un réseau Γ de G, notons

$$\mu_\Gamma:=\frac{1}{\mathrm{Vol}(\Gamma\backslash G)}\sum_{\pi\in\Pi(G)}\delta_\pi.$$

On dit qu'une suite $(\Gamma_n)_{n\in\mathbb{N}}$ a la propriété de la multiplicité limite si et seulement si la condition suivante est satisfaite : pour chaque sous-ensemble mesurable U de $\Pi(G)$ tel que la mesure de Plancherel du bord $\overline{U}-U^o$ est 0, on a

$$\lim_{n\to\infty}\mu_{\Gamma_n}(U)=\mu^{Pl}(U),$$

[2] $\Pi(G)$ est l'ensemble des classes d'équivalence des représentations unitaires irréductibles de G, munie de la topologie de Fell [**39**].

où μ^{Pl} est la mesure de Plancherel (voir [**48**] pour une définition abstraite de μ^{Pl}). Sauvageot [**97**] a demontré qu'une suite $(\Gamma_n)_{n\in\mathrm{N}}$ a la propriété de la multiplicité limite si

$$\lim_{n\to\infty}\frac{\mathrm{tr}\, R_{\Gamma_n} f}{\mathrm{Vol}(\Gamma_n\backslash G)} = f(1),\ \text{pour toute } f\in C_c^\infty(G).$$

Notre premier résultat principal donne une borne quantitative:

Théorème 1.2 (Thm 2.8). *Soit $\mathbb{K}=\mathbb{R}$ où $\mathbb{C}$. Pour chaque $R>0$ la proposition suivante est vraie: Soit Γ un réseau arithmétique de congruence, cocompact sans torsion dans $\mathrm{PGL}(2,\mathbb{K})$. Pour chaque $f\in C(\mathrm{PGL}(2,\mathbb{K}))$ dont le support est contenu dans la boule de rayon R autour de 1, on a*

$$|\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{K}))f(1)-\mathrm{tr}R_\Gamma f|\ll_R \|f\|_\infty(\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{K}))^{0.986}. \tag{1.1}$$

Borel [**21**] a démontré que pour tout $V\in\mathbb{R}$, il n'y a qu'un nombre fini (à conjugaison prés) de réseaux arithmétiques Γ de $\mathrm{PGL}(2,\mathbb{K})$ avec $\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{K}))\le V$. Par conséquent, notre résultat montre que toute suite de réseaux cocompacts, arithmétiques de congruence sans torsion $(\Gamma_n)_{n\in\mathrm{N}}$ deux-à-deux non-conjugués a la propriété de la multiplicité limite. On montre aussi une version plus faible pour des réseaux qui ne sont pas de congruence. On en déduit le taux de croissance des nombres de Betti dans les 3-variétés hyperboliques arithmétiques de congruence:

Corollaire 1.3. *Soit $(\Gamma_n)_{n\in\mathrm{N}}$ une suite de réseaux de $\mathrm{PGL}(2,\mathbb{C})$ cocompacts, arithmétiques sans torsion deux-à-deux disjoints. Admettons que soit ils sont tous de congruence soit ils sont deux-à-deux non-commensurables. Alors pour $i=1,2$*

$$\lim_{n\to\infty}\frac{b_i(\Gamma_n\backslash\mathbb{H}^3)}{\mathrm{Vol}(\Gamma_n\backslash\mathbb{H}^3)} = b_i^{(2)}(\mathbb{H}^3)=0.$$

1.4. Convergence de Benjamini-Schramm. Soit $\mathcal{M}$ l'espace des espaces métriques localement compacts pointés muni de la topologie de Gromov-Hausdorff. Rappelons que dans cette topologie une suite d'espaces pointés (X_n,x_n) converge vers (Y,y) si et seulement si pour chaque $R>0$ et $\varepsilon>0$ il existe $N=N_{R,\varepsilon}$ satisfaisant la propriété suivante: pour $n\ge N$ il existe un espace métrique M et des plongements isométriques des boules $\pi_1: B_{X_n}(x_n,R)\to M$ et $\pi_2: B_Y(y,R)\to M$ tels que la distance de Hausdorff entre les images est inférieure à ε. De façon intuitive, (X_n,x_n) converge vers (Y,y) si pour chaque $R>0$ les R-boules autour de x_n ressemblent de plus en plus à la boule $B_Y(y,R)$. Munie de cette topologie, $\mathcal{M}$ est un espace séparé. Soit (X,μ) un espace métrique muni d'une mesure de probabilité borélienne μ. On associe à X une mesure de probabilité $\nu_X:=\int\delta_{(X,x)}d\mu(x)$ sur $\mathcal{M}$. La topologie de Benjamini-Schramm sur les espaces de probabilité métriques est induite par la topologie *-faible sur $\mathcal{PM}$, c'est-à-dire $(X_n,\nu_n)_{n\in\mathrm{N}}$ converge vers (Y,μ) si et seulement si ν_{X_n} converge *-faiblement vers ν_Y. Ce type de convergence a été introduit par Benjamini et Schramm dans [**15**] pour des graphes réguliers. Elle a été adapté aux espaces localement symétriques de volume fini par Abért, Bergeron, Biringer, Gelander, Nikolov, Raimbault et Samet dans [**2**] où ils ont étudié (parmi d'autres choses) la convergence de Benjamini-Schramm vers X. On peut démontrer qu'une suite d'espaces localement symétriques $(\Gamma_n\backslash X)_{n\in\mathrm{N}}$ converge vers X si et seulement si pour chaque $R>0$ on a

$$\lim_{n\to\infty}\frac{(\mathrm{Vol}((\Gamma_n\backslash X)_{<R})}{\mathrm{Vol}(\Gamma_n\backslash X)}=0,$$

où $(\Gamma_n\backslash X)_{<R}$ est la partie R-mince de $\Gamma_n\backslash X$. Autrement dit, pour n grand une R-boule autour d'un point typique de X_n est isométrique à la R-boule de X. Soit G un groupe de Lie semi-simple et soit X son espace symétrique. Le théorème [**3**, Thm 1.2] affirme que si G est de rang réel au moins 2 et a la propriété (T) alors chaque suite d'espaces localement symétriques $(\Gamma_n\backslash X)_{n\in\mathrm{N}}$ deux-à-deux non-isométriques converge vers X. Dans le contexte des réseaux arithmétiques, ils donnent une majoration du volume de la partie R-mince [**3**, Thm 1.12] qui reste valable pour les groupes de rang 1. Pour un réseau

arithmétique cocompact Γ de G, il existe des constantes $c, \mu > 0$ dépendantes de Γ telles que pour chaque sous-groupe de congruence $\Gamma' \subset \Gamma$ on a

$$\mathrm{Vol}((\Gamma'\backslash X)_{<R}) \leq e^{cR}\mathrm{Vol}(\Gamma'\backslash X)^{1-\mu} \text{ pour tout } R > 1.$$

On en déduit que les suites d'espaces $(\Gamma_n\backslash X)_{n\in\mathbb{N}}$ avec $\Gamma_n \subset \Gamma$ sous-groupes de congruence deux-à-deux non-conjugués convergent Benjamini-Schramm vers X.

La convergence de Benjamini-Schramm des espaces localement symétriques est reliée à la notion de **sous-groupe aléatoire invariant**. Soit Sub_G l'espace des sous-groupes fermés de G muni de la topologie de convergence de Hausdorff sur les sous-ensembles compacts de G. Alors Sub_G est un espace métrisable [**33**]. Le groupe G agit sur Sub_G par conjugaison. Un sous-groupe aléatoire invariant est une variable aléatoire H à valeurs dans Sub_G dont la distribution est invariante par G. Parfois les mesures de probabilité G-invariantes sur Sub_G sont aussi appelées les sous-groupes invariants aléatoires. Étant donné un réseau Γ de G, on lui associe une mesure

$$\nu_\Gamma := \frac{1}{\mathrm{Vol}(\Gamma\backslash G)}\int_{\Gamma\backslash G} \delta_{g^{-1}\Gamma g} dg.$$

Dans [**3**] les auteurs ont utilisé de façon cruciale le fait qu'une suite d'espaces localement symétriques $(\Gamma_n\backslash X)_{n\in\mathbb{N}}$ converge Benjamini-Schramm vers X si et seulement si la suite des sous-groupes invariants aléatoires ν_{Γ_n} tend *-faiblement vers $\delta_{\{1\}}$.

Notre résultat principal sur la convergence de Benjamini-Schramm porte sur les suites de réseaux arithmétiques de $\mathrm{PGL}(2,\mathbb{R})$ et $\mathrm{PGL}(2,\mathbb{C})$. L'énoncé est similaire à [**3**, Thm 1.12] mais on ne suppose pas que tous les réseaux sont commensurables.

Théorème 1.4 (Thm 2.9). *Soient $R > 0$ et $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Soit Γ un réseau arithmetique de congruence sans torsion de $\mathrm{PGL}(2,\mathbb{K})$. Alors*

$$\mathrm{Vol}((\Gamma\backslash X)_{<R}) \ll_R \mathrm{Vol}((\Gamma\backslash X))^{0.986}. \tag{1.2}$$

C'est une conséquence du théorème 1.2. On montre également une version pour des réseaux qui ne sont pas de congruence:

Théorème 1.5 (Thm 2.11). *Soient $R > 0$ et $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Soit Γ un réseau arithmétique de $\mathrm{PGL}(2,\mathbb{K})$. Alors pour chaque $f \in C(\mathrm{PGL}(2,\mathbb{K}))$ tel que $\mathrm{supp} f \subset B(1,R)$*

$$|\mathrm{Vol}(\Gamma\backslash X)f(1) - \mathrm{tr}R_\Gamma f| \ll_R \|f\|_\infty \Delta_k^{-0.0006}, \tag{1.3}$$

$$\frac{\mathrm{Vol}((\Gamma\backslash X)_{<R})}{\mathrm{Vol}((\Gamma\backslash X))} \ll_R \Delta_k^{-0.0006}, \tag{1.4}$$

où Δ_k est le discriminant du corps des traces k de Γ.

On applique théorème 1.5 à la démonstration de la conjecture de Gelander pour des 3-variétés arithmétiques hyperboliques [**53**, Conjecture 1.3].

Théorème 1.6 (Thm 2.16). *Il existe des constantes positives A, B telles que toute 3-variété arithmétique hyperbolique M est homotope à un complexe simplicial $\mathcal{N}$ ayant au plus $A\mathrm{Vol}(M)$ sommets dont les degrés sont uniformément bornés par B (dans le cas compact on peut prendre $B = 245$).*

1.5. Croissance de la dimension de l'homologie en rang supérieur. Dans le Chapitre 3, on entreprend une étude de la croissance de $\dim_{\mathbb{F}_2} H_1(\Gamma\backslash X, \mathbb{F}_2)$ où X est un espace symétrique de rang au moins 2, $\mathbb{F}_2$ et le corps à deux éléments et $\Gamma\backslash X$ est de volume fini. D'abord exhibons quelques résultats antécédents sur la croissance d'homologie et de la cohomologie des espaces localement symétriques. Pour l'instant, X est n'importe quel espace symétrique de type non-compact. Le théorème classique de Gromov [**56**] donne une constante $C = C(X)$ telle que

$$b_i(\Gamma\backslash X) := \dim_{\mathbb{Q}} H^i(\Gamma\backslash X, \mathbb{Q}) \leq C\mathrm{Vol}(\Gamma\backslash X),$$

pour chaque espace localement symétrique $\Gamma\backslash X$. Dans certains cas on sait dire plus. Soient G un groupe de Lie et X son espace symétrique. Une suite de réseaux $(\Gamma_n)_{m\in\mathbb{N}}$ de

G est dite **uniformément discrète** si la longueur des géodésiques fermées sur $(\Gamma_n\backslash X)$ est minorée uniformément en n. Dans [**3**], il est montré que si $(\Gamma_n)_{n\in\mathbb{N}}$ est une suite des réseaux uniformément discrète et si $(\Gamma_n\backslash X)$ converge Benjamini-Schramm vers X, alors

$$\lim_{n\to\infty}\frac{b_i(\Gamma_n\backslash X)}{\mathrm{Vol}(\Gamma_n\backslash X)}=b_i^{(2)}(X),$$

où $b_i^{(2)}(X)$ est le i-ème nombre de Betti L^2 (voir [**75**]). On sait qu'en rang supérieur les suites d'espaces localement symétriques deux-à-deux non-isométriques convergent vers X donc le résultat ci-dessus fournit un outil puissant pour étudier la croissance de la (co)homologie à coefficients rationnels. D'autre part, les techniques analytiques qui permettent de traiter les nombres de Betti rationnels ne donnent pas d'information sur les dimensions de la (co)homologie modulo p. Le théorème de Margulis sur les sous-groupes normaux ou bien la propriété (T) impliquent qu'en rang supérieur le groupe $H_1(\Gamma\backslash X,\mathbb{Z})$ est fini donc $b_1(\Gamma\backslash X)=0$ mais on ne sait même pas si la dimension $\dim_{\mathbb{F}_p}H_1(\Gamma\backslash X,\mathbb{F}_p)$ est sous-linéaire en $\mathrm{Vol}(\Gamma\backslash X)$. Gelander [**54**] démontre que pour chaque espace symétrique X (pas forcement de rang supérieur) il existe une constante $B=B(X)$ telle que

$$d(\Gamma)\leq B\mathrm{Vol}(\Gamma\backslash X),$$

où $d(\Gamma)$ est le cardinal minimal d'une partie génératrice de Γ. Par conséquent

$$\limsup_{\mathrm{Vol}(\Gamma\backslash X)\to\infty}\frac{\dim_{\mathbb{F}_p}H_1(\Gamma\backslash X,\mathbb{F}_p)}{\mathrm{Vol}(\Gamma\backslash X)}\leq B.$$

La croissance de la dimension de l'homologie est reliée à la notion du **gradient de rang** ("rank gradient" en anglais [**1**, **5**, **67**]) de Γ. Le gradient de rang d'un groupe Γ relatif à une suite de sous-groupes d'indice fini $(\Gamma_n)_{n\in\mathbb{N}}$ est donné par

$$\mathrm{RG}(\Gamma,(\Gamma_n)):=\lim_{n\to\infty}\frac{d(\Gamma_n)-1}{[\Gamma:\Gamma_n]},$$

si la limite existe. Dans [**1**] (voir aussi [**5**] pour le cas des réseaux de $\mathrm{SL}(2,\mathbb{C})$), il est montré que le gradient de rang des réseaux de rang supérieur qui sont "à angles droits" vaut 0. Un groupe est dit **à angles droits** s'il admet une partie génératrice $s_1,s_2,\ldots,s_d$ telle que $[s_i,s_{i+1}]=1$ pour $i=1,\ldots,d-1$. Il suit immédiatement que dans ce cas

$$\lim_{n\to\infty}\frac{\dim_{\mathbb{F}_p}H_1(\Gamma_n\backslash X,\mathbb{F}_p)}{\mathrm{Vol}(\Gamma_n\backslash X)}=0.$$

En fait, pour les réseaux de rang supérieur aux angles droits, on peut faire mieux [**1**]:

$$\lim_{n\to\infty}\frac{\log|H_1(\Gamma_n\backslash X)|}{\mathrm{Vol}(\Gamma_n\backslash X)}=0.$$

Rappelons que ces résultats ne sont valables que pour des suites de sous-groupes de Γ. On croit [**1**, Conjecture 3] que dans un groupe semi-simple G de rang supérieur

$$\lim_{\mathrm{Vol}(\Gamma\backslash X)\to\infty}\frac{d(\Gamma)-1}{\mathrm{Vol}(\Gamma\backslash X)}=0$$

uniformément en tous les réseaux de G.

Revenons à l'étude de la croissance de l'homologie modulo p. Dans le cas des groupes de rang 1, on dispose du résultat suivant de Calegari et Emerton [**30**]. Soit M une variété de dimension 3 et notons Γ son groupe fondamental. Soit $\phi\colon\Gamma\to\mathrm{GL}(n,\mathbb{Z}_p)$ et notons G la clôture de $\phi(\Gamma)$. Posons $G_k=G\cap(\ker\mathrm{GL}(n,\mathbb{Z}_p)\to\mathrm{GL}(n,\mathbb{Z}/p^k\mathbb{Z}))$ et $\Gamma_k=\phi^{-1}(G_n)$. Soit M_k le revêtement fini de M correspondant au sous groupe Γ_k de Γ. La suite des variétés $(M_k)_{k\in\mathbb{N}}$ obtenue comme ci-dessus est appelée une **tour p-adique** de 3-variétés. Soit d la dimension de G en tant que groupe $p-$adique analytique. Dans [**30**], les auteurs montrent que la dimension de $H_1(M_k,\mathbb{F}_p)$ satisfait une des conditions suivantes:

- $\dim_{\mathbb{F}_p}H_1(M_k,\mathbb{F}_p)=\lambda p^{dk}+O(p^{(d-1)k})$ (croissance linéaire).
- $\dim_{\mathbb{F}_p}H_1(M_k,\mathbb{F}_p)=\lambda p^{(d-1)k}+O(p^{(d-2)k})$.
- $d=2$ et $\dim_{\mathbb{F}_p}H_1(M_k,\mathbb{F}_p)=O(1)$.

- $d = 3$ et $\dim_{\mathbb{F}_p} H_1(M_k, \mathbb{F}_p) \leq 3$ pour k assez grand.

En particulier si M est hyperbolique la limite $\lim_{k\to\infty} \frac{\dim_{\mathbb{F}_p} H_1(M_k,\mathbb{F}_p)}{\mathrm{Vol}(M_k)}$ existe. Calegari et Emerton conjecturent que le premier cas n'est pas possible si $\bigcap_{n\in\mathbb{N}} \Gamma_n = \{1\}$. Si c'est vrai la dimension de l'homologie modulo p est sous-linéaire en le volume dans les tours p-adiques de 3-variétés qui convergent Benjamini-Schramm vers $\mathbb{H}^3$.

Notre résultat principal dans le chapitre 3 est le théorème suivant.

THÉORÈME 1.7. *Soit G un groupe semi-simple de rang supérieur et soit X l'espace symétrique associé. Alors*

$$\lim_{\mathrm{Vol}(\Gamma\backslash X)\to\infty} \frac{\dim_{\mathbb{F}_2} H_1(\Gamma\backslash X, \mathbb{F}_2)}{\mathrm{Vol}(\Gamma\backslash X)} = 0.$$

La caractéristique 2 du corps joue une rôle important dans la preuve.

2. Théorème de Kesten

Soit $G = (V, E)$ un graphe d-régulier avec ensemble des sommets V et ensemble des arêtes E. L'opérateur de Markov associé à la marche aléatoire (simple) sur G est donné par

$$M\Phi(x) = \frac{1}{d}\sum_{y\sim x} \Phi(y) \text{ pour } x \in V, \Phi \in L^2(V),$$

où la somme parcourt les sommets voisins de x. Le **rayon spectral** de G est défini comme la norme $\|M\| = \sup_{\|\Phi\|_2=1} \|M\Phi\|_2$. Le rayon spectral décrit la vitesse d'expansion dans G. Parmi les graphes infinis d-réguliers c'est l'arbre $\mathbb{T}_d$ qui possède le rayon spectral le plus petit: $\rho(\mathbb{T}_d) = \frac{2\sqrt{d-1}}{d}$. Un graphe d-régulier infini G est dit de **Ramanujan** si $\rho(G) = \rho(\mathbb{T}_d)$.

Soit Γ un groupe dénombrable engendré par une partie finie symétrique S. Le graphe de Cayley $\mathrm{Cay}(\Gamma, S)$ a Γ pour l'ensemble des sommets et $\{(g, gs)|g \in \Gamma, s \in S\}$ pour l'ensemble des arêtes. De la même manière, si X est un ensemble dénombrable muni d'une action de Γ à droite on lui associe son graphe de Schreier $\mathrm{Sch}(X, S)$ dont l'ensemble des sommets est X et l'ensemble des arêtes est $\{(x, xs) \mid x \in X, s \in S\}$.

Un théorème remarquable de Kesten **[65, 66]** dit qu'un sous groupe distingué N de Γ est moyennable si et seulement si $\rho(\mathrm{Cay}(\Gamma, S)) = \rho(\mathrm{Sch}(N\backslash\Gamma, S))$. Un sous-groupe H de Γ est dit *de Ramanujan* si $\rho(\mathrm{Cay}(\Gamma, S)) = \rho(\mathrm{Sch}(H\backslash\Gamma, S))$. Chaque sous-groupe moyennable est Ramanujan mais l'inverse n'est pas vrai. Dans **[6]** Abért, Glasner et Virag donnent une version probabiliste du théorème de Kesten. Ils ont démontré qu'un sous-groupe aléatoire invariant H est de Ramanujan presque surement si et seulement si H est moyennable presque surement.

Dans le chapitre 4 on étend le théorème de Kesten aux sous-groupes uniformément récurrents. Un sous-groupe H de Γ est **uniformément récurrent** (voir **[42, 43, 80]**) si la clôture de la G-orbite de H dans Sub_G est un système dynamique minimal. Ces groupes ont été récemment utilisés par Kennedy **[64]** pour caractériser les groupes C^*-simples.

THÉORÈME 1.8 (Thm 3.1). *Soit Γ un groupe engendré par une partie finie symétrique S. Un sous-groupe uniformement recurrent H de Γ est de Ramanujan si et seulement si il est moyennable.*

La démonstration repose sur l'argument de Abért, Glasner et Virag dans **[4, 6]**. En utilisant les ingrédients de la preuve, on redémontre un résultat récent de Lyons et Peres **[76]**. Soit $C(-, k)\colon V \to \{1, 0\}$ la fonction définie par $C(v, k) = 1$ si v fait partie d'un cycle non-contractile de longueur k et 0 sinon.

THÉORÈME 1.9 (Thm 4.1). *Soit G un graphe infini d-régulier de Ramanujan enraciné en x. Soit $k \geq 1$ et soit X_n la marche aléatoire simple sur G partant de x. On a*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\big[\, C(X_i, k)\,\big] = 0.$$

3. Bornes sur les caractères

Dans le cinquième chapitre, on étudie les bornes sur les caractères irréductibles des groupes finis de type de Lie et des groupes compacts de la forme $\mathbb{G}(\mathbb{Z}_p)$ où $\mathbb{G}$ est un schéma en groupes réductif lisse sur $\mathbb{Z}_p$. On s'intéresse aux majorations de la forme suivante:

$$|\chi(g)| \leq C\chi(1)^{1-\delta}, \text{ pour chaque caractère irréductible } \chi \tag{3.1}$$

où $C, \delta > 0$ ne dépendent que de g.

Expliquons la motivation provenant du chapitre 2. Soit Γ un sous-groupe discret. Une suite de sous-groupes d'indice fini $(\Gamma_n)_{n\in\mathbb{N}}$ de Γ est dite **une suite de Farber** si pour tout $g \in \Gamma \setminus Z(\Gamma)$

$$\lim_{n\to\infty} \frac{|\{h\Gamma_n \mid h^{-1}gh \in \Gamma_n\}|}{[\Gamma : \Gamma_n]} = 0. \tag{3.2}$$

On voit facilement que $(\Gamma_n)_{n\in\mathbb{N}}$ est une suite de Farber si et seulement si la suite des graphes de Schreier $\mathrm{Sch}(\Gamma_n\backslash\Gamma, S)$ converge Benjamini-Schramm vers $\mathrm{Cay}(\Gamma, S)$, pour une partie génératrice S de Γ. Si Γ est un réseau d'un groupe semi-simple G, la dernière condition est équivalente à la convergence Benjamaini-Schramm des orbifolds $\Gamma_n\backslash X$ vers X.

Soit ρ une représentation complexe de Γ de dimension finie. Notons χ_ρ son caractère. La condition de Farber (3.2) peut être exprimée en termes des caractères des représentations induites. Pour $g \in \Gamma \setminus Z(\Gamma)$, on a

$$\lim_{n\to\infty} \frac{|\chi_{\mathrm{Ind}_{\Gamma_n}^{\Gamma} 1}(g)|}{\chi_{\mathrm{Ind}_{\Gamma_n}^{\Gamma} 1}(1)} = 0.$$

Dans le chapitre 2, lemme 2.83, on montre que pour Γ à croissance des représentations polynomiale chaque borne de type

$$|\chi(g)| \leq C\chi(1)^{1-\delta} \text{ pour } \chi \text{ caratère irréductible de } \Gamma$$

où $\delta > 0$ implique

$$|\chi_{\mathrm{Ind}_{\Gamma_n}^{\Gamma} 1}(g)| \leq C'\chi_{\mathrm{Ind}_{\Gamma_n}^{\Gamma}}(1)^{1-\delta'}$$

pour un certain $\delta' > 0$. Les inégalités de cette forme sont parmi les ingrédients les plus importants de la preuve du théorème 1.2.

L'autre source de motivation pour étudier les bornes de type (3.1) provient des travaux de Liebeck, Shalev et al. **[74]** qui les ont appliquées à l'étude des marches aléatoires sur les groupes finis de type de Lie, au problème de la génération aléatoire et au problème de la génération par des mots. Les inégalités de Glück **[55]** permettent de trouver pour chaque groupe de Chevalley G des constantes $C, \delta > 0$ telles que pour $g \in G(\mathbb{F}_q)$ noncentral, on a:

$$|\chi(g)| \leq C\chi(1)^{1-\delta} \text{ pour chaque caractère irreductible } \chi.$$

Soulignons que cette borne est uniforme en q. La constante δ extraite des bornes de Glück est $\frac{1}{\dim G}$ ce qui est loin d'être optimal pour la plupart des éléments de G. Larsen a prouvé dans un article non-publié **[71]** que si G est un groupe réductif défini sur le corps fini $\mathbb{F}_q$ et $g \in G(\mathbb{F}_q)$ est régulier semi-simple alors $|\chi(g)| \leq |W|^2$ où W est le groupe de Weyl de G. Dans le chapitre 2, on adapte la méthode de Larsen au cas p-adique pour monter la majoration suivante.

Théorème 1.10 (Remark 2.62). *Soit G un groupe algébrique réductif défini sur un corps local non-archimèdien F. Soit K un sous-groupe compact ouvert de $G(F)$ et soit $\gamma \in K$ un élément régulier semi-simple. Alors pour chaque caractère irréductible χ de K, nous avons*

$$|\chi(\gamma)| \leq C,$$

où la constante C ne dépend que de γ et G (ni de K ni de χ).

Dans le chapitre 5, on démontre une légère amélioration de la borne de Larsen en utilisant les foncteurs de restriction de Deligne-Lusztig. On prouve que dans un groupe réductif G défini sur un corps fini, la valeur de chaque caractère irreductible sur un élément semi-simple régulier est bornée par $|W|$. Indépendamment, en poursuivant des idées similaires, Bezrukavnikov, Liebeck, Shalev et Tiep **[74]** ont amélioré les bornes de Glück. Notons

$$\alpha(L) := \max\left\{\frac{\dim u^L}{\dim u^G} \mid u \in L, u \neq 1 \text{ unipotent}\right\}.$$

THÉORÈME 1.11 (**[19]**). *Soit $\gamma \in G(\mathbb{F}_q)$ et soit L une composante de Levi rationnelle d'un sous-groupe rationnel parabolique de G tel que la composante connexe du centralisateur de γ est contenue dans L. Alors, pour chaque caractère irréductible χ de $G(\mathbb{F}_q)$, on a*

$$|\chi(\gamma)| \ll \chi(1)^{\alpha(L)},$$

où la constante implicite ne dépend que du rang de G.

CHAPTER 1

Introduction in English

1. Locally symmetric spaces

In this part of the introduction we recall the definition of a locally symmetric space and give a standard construction using semisimple Lie groups. After giving the construction of arithmetic lattices in semisimple Lie groups we briefly introduce the Benjamini-Schramm convergence and Limit Multiplicity property. Lastly we describe our results from Chapters 2 and 3. We keep this introduction quite short because every Chapter has its own introduction where we give more background for every result, outline the structure of the proof as well as make comparison with literature. Reader familiar with locally symmetric spaces and the adélic construction of arithmetic lattices may safely skip the first and the second part of this introduction.

1.1. Symmetric spaces. A Riemannian manifold (M, g) is a **symmetric space** if for every point $x \in M$ there exists an isometry ι_x of M fixing x such that $d\iota_x \colon T_xM \to T_xM$ is the multiplication by -1. In other words ι_x reverses the geodesics passing through x. A **locally symmetric space** is given by the same condition but ι_x needs to be defined only on an open neighborhood of x. A simply connected symmetric space is called **irreducible** if it is not a product symmetric spaces. An irreducible symmetric space is said to be of **non-compact type** if it has non-positive but non-zero sectional curvature and more generally a symmetric space of non-compact type is a product of irreducible symmetric spaces of non-compact type. Throughout this text we will only consider the symmetric spaces of non-compact type. Those spaces maybe realized as a quotient of a semisimple Lie group [**58**].

Let G be a connected real semisimple Lie group. Write $\mathfrak{g}$ for its Lie algebra and $B(-,-)\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ for the Killing form. Choose a Cartan involution $\Theta\colon \mathfrak{g} \to \mathfrak{g}$, recall that being Cartan means that the bilinear form $(X, Y) \mapsto -B(X, \Theta Y)$ is positive definite. We decompose $\mathfrak{g}$ into orthogonal sum of eigenspaces of Θ:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \text{ where } \Theta X = X, X \in \mathfrak{k} \text{ and } \Theta Y = -Y, Y \in \mathfrak{p}.$$

$\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Let $K = \{g \in G | \mathrm{Ad}(g)\mathfrak{k} = \mathfrak{k}\}$ be the stabilizer of $\mathfrak{k}$. Then $\mathfrak{k}$ is the lie algebra of K and K is a maximal compact subgroup of G[1].

Let $X = G/K$, the tangent space T_KX is canonically identified with $\mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$. X is endowed with the unique left G-invariant Riemannian metric g such that

$$g(X, Y) = B(X, Y) \text{ for } X, Y \in \mathfrak{p}.$$

The Riemannian manifold (X, g) is the symmetric space of G, the involutions inverting the geodesics are provided by Cartan involutions. The group G acts on X by isometries and this action is proper i.e. the stabilizers of points are compact.

The curvature tensor of X at the point K is given by

$$R_K(X, Y)Z = [[X, Y], Z] \text{ for } X, Y, Z \in \mathfrak{p}.$$

To see that the sectional curvature is non-positive we compute $g(R(X, Y)X, Y) = B([[X, Y], X], Y]) = B([X, Y], [X, Y])$, the last term is non-positive because $[X, Y] \in \mathfrak{k}$. The last property that

[1] It is the group of real points of an algebraic group so it has finitely many connected components [**24**]. Its connected component is compact since the Killing form restricted to $\mathfrak{k}$ is negative definite.

we are going to establish in this brief overview is contractibility of X. The Iwasawa decomposition $G = NAK$, where A is isomorphic to $\mathbb{R}^r_+$ and N is a connected nilpotent subgroup of G, yields a homeomorphism $\mathbb{R}^{\dim G - \dim K} \to G/K = X$.

1.2. Arithmetic lattices. Given a semisimple Lie group G and its symmetric space X we can produce examples of locally symmetric spaces of form $\Gamma\backslash X$ where Γ is a discrete torsion free subgroup of G. As the group is torsion free the resulting quotient space is a manifold, for a discrete group with torsion we get an orbifold. By a slight abuse of notation we will also call such spaces locally symmetric. In this thesis we are interested only in locally symmetric spaces of finite volume, which amounts to taking quotients $\Gamma\backslash X$ where Γ is a lattice.

One of the ways of constructing lattices in a semisimple Lie group G is via arithmetic. For example if $G = \mathrm{SL}(n, \mathbb{R})$ the group $\mathrm{SL}(n, \mathbb{Z})$ is a lattice. More generally whenever $\mathbb{G}$ is an algebraic subgroup of $\mathrm{GL}(n, \mathbb{Q})$ defined over $\mathbb{Q}$ without characters defined over $\mathbb{Q}$ then the group $\mathbb{G}(\mathbb{Q}) \cap \mathrm{GL}(n, \mathbb{Z})$ is a lattice in $\mathbb{G}(\mathbb{R})$ by virtue of Borel Harish-Chandra theorem [**22**]. Those are the simplest examples of arithmetic lattices. We are going to describe another construction using adèles. This point of view is crucial in the first chapter. For more in depth discussion on algebraic groups and adèles we refer to the classical book by Weil [**104**]. Let k be a number field. Write M for the set of all places of k, M_∞ for the archimedean places and M_f for the finite ones. We write $\mathcal{O}_k, \mathcal{O}_{k_\mathfrak{p}}$ for the rings of integers in $k, k_\mathfrak{p}$ respectively. The adèle ring of k denoted $\mathbb{A} = \mathbb{A}_k$ (we shall omit the index when we work with a fixed field) is given by the restricted product

$$\mathbb{A} := \{(a_\nu)_\nu \in \prod_{\nu \in M_k} k_\nu \mid a_\mathfrak{p} \in \mathcal{O}_{k_\mathfrak{p}} \text{ for almost all } \mathfrak{p} \in M_f\}.$$

It comes with a natural topology of the restricted product which makes $\mathbb{A}$ into locally compact algebra over k. It will be convenient to write $\mathbb{A} = \mathbb{A}_\infty \times \mathbb{A}_f$ where the $\mathbb{A}_\infty$ is the product over all (finitely many) archimedean places and $\mathbb{A}_f$ is the restricted product over infinite places. Let $\mathbb{G}$ be a simple linear algebraic group defined over k. We say that $\mathbb{G}$ is **anisotropic** over k if there are no non-trivial homomorphisms $\mathbb{G}_m \to \mathbb{G}$ defined over k and that $\mathbb{G}$ is **isotropic** otherwise. The group $\mathbb{G}(\mathbb{A})$ with the topology inherited from $\mathbb{A}$ is a locally compact group. In order to present it as restricted product let us fix a rational representation $\rho\colon \mathbb{G} \to \mathrm{GL}(n, k)$. Then

$$\mathbb{G}(\mathbb{A}) := \{(g_\nu)_\nu \in \prod_{\nu \in M_k} \mathbb{G}(k_\nu) \mid \rho(g_\mathfrak{p}) \in \mathrm{GL}(n, \mathcal{O}_{k_\mathfrak{p}}) \text{ for almost all } \mathfrak{p} \in M_f\}.$$

As with the ring of adèles we have $\mathbb{G}(\mathbb{A}) = \mathbb{G}(\mathbb{A}_\infty) \times \mathbb{G}(\mathbb{A}_f)$. By the Mostow-Tamagawa theorem [**83**] the group $\mathbb{G}(k)$ is a lattice of $\mathbb{G}(\mathbb{A})$, which is cocompact if and only if $\mathbb{G}$ is anisotropic. This is the starting point for the adélic construction of lattices in semisimple Lie groups. Let us choose an open compact subgroup of $\mathbb{G}(\mathbb{A}_f)$. Then the group $\Gamma_U := \mathbb{G}(k) \cap \mathbb{G}(\mathbb{A}_\infty) \times U$ is a lattice in $\mathbb{G}(\mathbb{A}_f) \times U$. Let us split the set archimedean places $M_\infty = \Sigma_1 \sqcup \Sigma_2$ where $\Sigma_1 = \{\nu \in M_\infty \mid \mathbb{G}(k_\nu) \text{ is not compact } \}$ and $\Sigma_2 = M_\infty \setminus \Sigma_1$. Then

$$\mathbb{G}(\mathbb{A}_\infty) = G_1 \times G_2 = \left(\prod_{\nu \in \Sigma_1} \mathbb{G}(k_\nu)\right) \times \left(\prod_{\nu \in \Sigma_2} \mathbb{G}(k_\nu)\right).$$

G_1 is a semisimple Lie group and the projection of Γ_U to G_1 is a lattice in G_1.

Definition 1.1. *Let G be a semisimple Lie group without compact factors. A lattice $\Gamma \subset G$ is an irreducible arithmetic lattice if there exists a number field k, simple algebraic group $\mathbb{G}$ defined over k, an open compact subgroup U of $\mathbb{G}(\mathbb{A}_f)$ and an isomorphism $\rho : G_1 \simeq G$ such that Γ is commensurable with $\rho(\Gamma_U)$. Moreover, if $\rho(\Gamma_U) \subset \Gamma$ then Γ is a congruence arithmetic lattice. Non irreducible arithmetic lattices are those which are commensurable with Cartesian products of irreducible arithmetic lattices.*

If Γ is a congruence arithmetic lattice one can ask if we can find $k, \mathbb{G}, U$ and ρ such that $\Gamma = \rho(\Gamma_U)$. In general this is not possible, for example the maximal cocompact arithmetic

lattices in $\mathrm{SL}(2,\mathbb{R})$ are normalizers of $\rho(\Gamma_U)$ for certain choices of $k, \mathbb{G}, U$ but they are strictly bigger than $\rho(\Gamma_U)$ (see [**21**] and [**94**, 11.4]). The answer is positive if we restrict to the adjoint semisimple Lie groups (see Lemma 2.32 and Proposition 2.33 where we prove this for maximal arithmetic lattices $\mathrm{PGL}(2,\mathbb{R}), \mathrm{PGL}(2,\mathbb{C})$).

We examine an example of the above construction in $\mathrm{SL}(2,\mathbb{C})$. Let $k = \mathbb{Q}(\sqrt{-5})$. This field has a single complex place ν. Its ring of integers is $\mathcal{O}_k = \mathbb{Z}[\sqrt{-5}]$. The finite places of k are parametrized as follows: unique ramified primes $(\sqrt{-5}), (2, 1+\sqrt{-5})$, single prime lying over every rational prime p with $p \equiv 11, 13, 17, 19 \bmod 20$, and two primes lying over every rational prime $p \equiv 1, 3, 7, 9 \bmod 20$. Consider the quaternion algebra A over k:

$$A = k + \mathbf{i}k + \mathbf{j}k + \mathbf{ij}k,$$

with relations $\mathbf{i}^2 = \sqrt{-5}, \mathbf{j}^2 = 2$ and $\mathbf{ij} = -\mathbf{ji}$. The quadratic form $-x^2 + \sqrt{-5}y^2 + 2z^2$ is anisotropic over k so the algebra A is a division ring ([**94**, Thm 2.3.1]). The norm on A is given by $n(x+\mathbf{i}y+\mathbf{j}z+\mathbf{ij}t) = x^2 - \sqrt{-5}y^2 - 2z^2 + 2\sqrt{-5}t^2$. We put $\mathbb{G} = \{a \in A \mid n(a) = 1\}$, the group $\mathbb{G}$ is a k-form of $\mathrm{SL}(2,k)$ i.e. $\mathbb{G} \simeq \mathrm{SL}(2)$ over the algebraic closure $\overline{k}$. We have

$$\mathbb{G}(\mathbb{A}) = \mathbb{G}(k_\nu) \times \mathbb{G}(\mathbb{A}_f) = \mathrm{SL}(2,\mathbb{C}) \times \mathbb{G}(\mathbb{A}_f).$$

Consider the subring of A given by $\mathcal{O} = \mathcal{O}_k + \mathbf{i}\mathcal{O}_k + \mathbf{j}\mathcal{O}_k + \mathbf{ij}\mathcal{O}_k$. It is an **order** in A i.e. an $\mathcal{O}_k$-module of rank 4 which is also a subring of A. We use it to construct an open compact subgroup U of $\mathbb{G}(\mathbb{A}_f)$. The algebra A and the order $\mathcal{O}$ are subrings of $A \otimes_k \mathbb{A}_f$. Let $\mathcal{O}_f$ be the closure of $\mathcal{O}$ in $A \otimes \mathbb{A}_f$, it is a compact open subring. We define U as the group of elements of $\mathcal{O}_f$ of norm 1. The projection of the group $\Gamma_U = \mathbb{G}(k) \cap (\mathrm{SL}(2,\mathbb{C}) \times U)$ is an arithmetic lattice in $\mathrm{SL}(2,\mathbb{C})$. Explicitly, $\rho(\Gamma_U)$ is the subgroup

$$\left\{ \begin{pmatrix} x + \sqrt[4]{-5}y & z + \sqrt[4]{-5}t \\ 2(z - \sqrt[4]{-5}t) & x - \sqrt[4]{-5}y \end{pmatrix} \mid x,y,z,t \in \mathcal{O}_k \text{ and } x^2 - \sqrt{-5}y^2 - 2z^2 + 2\sqrt{-5}t^2 = 1 \right\},$$

and the projection ρ is given by

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho(\mathbf{i}) = \begin{pmatrix} \sqrt[4]{-5} & 0 \\ 0 & -\sqrt[4]{-5} \end{pmatrix} \text{ and } \rho(\mathbf{j}) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

1.3. Limit Multiplicity property. Let G be a semisimple Lie group and let Γ be a lattice in G. The group G acts on $L^2(\Gamma\backslash G)$ by right translations:

$$(R_\Gamma g\Phi)(x) = \Phi(xg) \text{ for } \Phi \in L^2(\Gamma\backslash G), g \in G \text{ and } x \in \Gamma\backslash G.$$

This representation is unitary. Given a sequence of lattices (Γ_n) we can study the asymptotic properties of $L^2(\Gamma_n\backslash G)$ as $n \to \infty$. For example one can ask about the growth of the multiplicity $m_\Gamma(\pi) := \dim \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$ of a given irreducible unitary representation π in $L^2(\Gamma_n\backslash X)$. This question was considered by DeGeorge and Wallach in [**34**] where they proved that under certain reasonable conditions on (Γ_n) we have

$$\lim_{n\to\infty} \frac{m_{\Gamma_n}(\pi)}{\mathrm{Vol}(\Gamma\backslash G)} = d(\pi),$$

where $d(\pi)$ is defined as the degree of π if π is in discrete series and 0 otherwise. The limit multiplicity is stronger than the convergence of multiplicities, it deals with the distribution of irreducible subrepresentations of $L^2(\Gamma\backslash G)$ in the space $\Pi(G)$ of all irreducible unitary representations of G, equipped with the Fell topology (see [**39**]). For every lattice $\Gamma \subset G$ we construct a measure

$$\mu_\Gamma := \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \sum_{\pi \in \Pi(G)} \delta_\pi.$$

We say that a sequence (Γ_n) has the **limit multiplicity property** if the following condition is satisfied: for every measurable subset U of $\Pi(G)$ such that the Plancherel measure (see [**48**]) of the boundary $\overline{U} - U^o$ is 0 we have

$$\lim_{n\to\infty} \mu_{\Gamma_n}(U) = \mu^{Pl}(U),$$

where μ^{Pl} is the Plancherel measure. Sauvageot showed [**97**] that for cocompact lattices the Limit Multiplicity property holds if

$$\lim_{n\to\infty}\frac{\mathrm{tr}R_{\Gamma_n}f}{\mathrm{Vol}(\Gamma_n\backslash G)}=f(1),\ \text{for all } f\in C_c^\infty(G). \tag{1.1}$$

Our first main result gives the following quantitative bound:

THEOREM 1.2. *[Thm. 2.8] Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. There exists $a>0$ such that for any $R>0$ the following holds. Let Γ be a uniform torsion free, congruence arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$. For any $f\in C(\mathrm{PGL}(2,\mathbb{K}))$ with $\mathrm{supp}f\subset B(1,R)$*

$$|\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{K}))f(1)-\mathrm{tr}R_\Gamma f|\ll_R \|f\|_\infty(\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{K}))^{1-a}. \tag{1.2}$$

We can take $a\geq 0.014$.

Borel [**21**] showed that for every $V>0$ there exists only finitely many arithmetic lattices $\Gamma\subset\mathrm{PGL}(2,\mathbb{K})$ with $\mathrm{Vol}(\Gamma\backslash G)\leq V$ so our result implies that any sequence (Γ_n) of pairwise disjoint, cocompact torsion free arithmetic congruence lattices one has the Limit Multiplicity Property. As a corollary we can control the growth of Betti numbers in sequences of arithmetic congruence hyperbolic 3-manifolds:

COROLLARY 1.3 (Corollary 2.18). *Let (Γ_n) be a sequence of pairwise distinct arithmetic, torsion free lattices in $\mathrm{PGL}(2,\mathbb{C})$. Assume that either the are congruence or they are pairwise non-commensurable. Then for $i=1,2$*

$$\lim_{n\to\infty}\frac{b_i(\Gamma_n\backslash\mathbb{H}^3)}{\mathrm{Vol}(\Gamma_n\backslash\mathbb{H}^3)}=b_i^{(2)}(\mathbb{H}^3)=0.$$

1.4. Benjamini-Schramm convergence. Let $\mathcal{M}$ be the space of rooted locally compact metric spaces equipped with the pointed Gromov-Hausdorff topology (see [**57**]). In that topology a sequence of rooted metric spaces (X_n,x_n) converges to (Y,y) if for every $R>0$ and $\varepsilon>0$ there exists $N=N_{R,\varepsilon}$ with the following property: for $n\geq N$ there exists a metric space M and isometric embeddings of R-balls $\pi_1\colon B_{X_n}(x_n,R)\to M,\pi_2\colon B_Y(y,R)\to M$ such that the Hausdorff distance between $\pi_1(B_{X_n}(x_n,R))$ and $\pi_2(B_Y(y,R))$ is at most ε. Intuitively (X_n,x_n) converges to (Y,y) if R-balls around x_n look like an R-ball around y for big enough n. With that topology, the space $\mathcal{M}$ is a Hausdorf topological space. For a metric space X with a probanility measure ν we send it to a probability measure $\int\delta_{(X,x)}d\mu(x)\in\mathcal{P}\mathcal{M}$ where $\mathcal{P}M$ is the space of all Borel probability measures on $\mathcal{M}$. The Benjamini-Schramm topology is the topology on metric probability spaces induced by the weak-* convergence in $\mathcal{P}M$. Originally this notion of convergence was considered by Benjamini and Schramm for regular graphs in [**15**]. It was adapted to the setting of locally symmetric spaces of finite volume by Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault and Samet in [**3**]. The question of particular interest is when a sequence of locally symmetric spaces $\Gamma_n\backslash X$ converges to X. The condition in that case is much simpler than the abstract definition of Benjamini-Schramm convergence: $(\Gamma_n\backslash X)$ converges to X if and only if for every $R>0$ we have

$$\lim_{n\to\infty}\frac{\mathrm{Vol}((\Gamma_n\backslash X)_{<R})}{\mathrm{Vol}(\Gamma_n\backslash X)}=0.$$

$(\Gamma_n\backslash X)_{<R}$ is the R-thin part of the orbifold $\Gamma_n\backslash X$. This means that in the sequences of orbifolds convergent Benjamini-Schramm to X the R-balls around typical points look like an R-ball in the universal cover X. It is known [**3**, Thm 1.5] that if G has property (T) and is of real rank at least 2 then every sequence of pairwise non-conjugate lattices (Γ_n) converges to X in the Benjamini-Schramm topology. For an arithmetic, cocompact lattice $\Gamma\subset G$ of arbitrary rank, there are constants $c,\mu>0$ such that for every congruence subgroup $\Gamma'\subset\Gamma$ and $R>1$ one has [**3**, Thm 1.12]:

$$\mathrm{Vol}((\Gamma'\backslash X)_{<R})\leq e^{cR}\mathrm{Vol}(\Gamma'\backslash G)^{1-\mu}.$$

It implies that for every sequence of pairwise distinct congruence subgroups (Γ_n) of Γ the orbifolds $(\Gamma_n\backslash X)$ converge to X in the Benjamini-Schramm topology.

Benjamini-Schramm convergence of locally symmetric spaces is related to **invariant random subgroups**. Let Sub_G be the space of closed subgroups of G equipped with the topology of Haussdorff convergence on compact sets. It makes Sub_G into a metrizable space [**33**]. The group G acts on Sub_G by conjugation. An invariant random subgroup H is a random variable taking values in Sub_G with Borel probability distribution invariant by conjugation by G. Sometimes we will work directly with G-invariant Borel probability measures on Sub_G and call them invariant random subgroups. Given a lattice in G we construct an invariant random subgroup

$$\nu_G := \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \delta_{g^{-1}\Gamma g} dg.$$

One of the key steps in [**3**] is the observation that $(\Gamma_n\backslash X)$ converges Benjamini-Schramm to X if and only if ν_{Γ_n} converges weakly-* to $\delta_{\{1\}}$.

Our main result on Benjamini-Schramm convergence deals with sequences of arithmetic latices in $\mathrm{PGL}(2,\mathbb{R})$ and $\mathrm{PGL}(2,\mathbb{C})$, it is similar to [**3**, Thm 1.12] but the big difference is that we do not need to assume that lattices are contained in a single arithmetic lattice.

THEOREM 1.4 (Thm 2.9). *There exists $a > 0$ such that for any $R > 0$ the following holds. Let Γ be a torsion free, congruence arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$. Then*

$$\mathrm{Vol}((\Gamma\backslash X)_{<R}) \ll_R \mathrm{Vol}((\Gamma\backslash X))^{1-a}. \tag{1.3}$$

We can take $a \geq 0.014$.

It is derived from Theorem 1.2. For non cocompact lattices it was shown by Raimbault in [**92**]. We also prove a version for non-congruence subgroups

THEOREM 1.5 (Thm 2.11). *There exists $c > 0$ such that for any $R > 0$ the following holds. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let Γ be a torsion free, arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$. Then for any $f \in C(\mathrm{PGL}(2,\mathbb{K}))$ with $\mathrm{supp} f \subset B(1,R)$*

$$|\mathrm{Vol}(\Gamma\backslash X) f(1) - \mathrm{tr} R_\Gamma f| \ll_R \|f\|_\infty \Delta_k^{-c}, \tag{1.4}$$

$$\frac{\mathrm{Vol}((\Gamma\backslash X)_{<R})}{\mathrm{Vol}((\Gamma\backslash X))} \ll_R \Delta_k^{-c}, \tag{1.5}$$

Where Δ_k is the discriminant of the trace field k of Γ and $c \geq 0.0006$.

We use Theorem 1.5 to prove Gelander's conjecture for arithmetic hyperbolic 3-manifolds [**53**, Conjecture 1.3]:

THEOREM 1.6. *[Theorem 2.16] There exist positive constants A, B such that every arithmetic, hyperbolic 3-manifold M is homotopically equivalent to a simplicial complex with at most $A\mathrm{Vol}(M)$ vertices and each vertex has degree bounded by B (if M is compact we can take $B = 245$).*

1.5. Growth of homology in higher rank. In Chapter 3 we investigate the growth of the dimension of $H_1(\Gamma\backslash X, \mathbb{F}_2)$ for finite volume locally symmetric spaces $\Gamma\backslash X$, where X is a higher rank symmetric space. Let us review what is known about the homology and cohomology growth in locally symmetric spaces. A classical theorem of Gromov [**56**] asserts that there exists a positive constant $C = C(X)$ such that

$$b_i(\Gamma\backslash X) := \dim_{\mathbb{Q}} H^i(\Gamma\backslash X, \mathbb{Q}) \leq C\mathrm{Vol}(\Gamma\backslash X)$$

for every finite volume locally symmetric space $\Gamma\backslash X$. In some cases Gromov's result may be improved to a precise asymptotic. A sequence of cocompact lattices (Γ_n) is uniformly discrete if the length of closed geodesics on $(\Gamma_n\backslash X)$ is bounded from below uniformly in n. In [**3**] it is proved that for a uniformly discrete sequence of cocompact lattices (Γ_n) such that $\Gamma_n\backslash X$ converges to X in Benjamini-Schramm topology we have

$$\lim_{n\to\infty} \frac{b_i(\Gamma_n\backslash X)}{\mathrm{Vol}(\Gamma_n\backslash X)} = b_i^{(2)}(X)$$

where $b_i^{(2)}(X)$ is the $i-$th L^2-Betti number of X. Combined with [**3**, Thm 1.2] it gives a powerful tool to study the growth of Betti numbers in higher rank symmetric spaces. The aforementioned results give satisfactory control over the growth of homology or cohomology with rational coefficients but give little information about the mod$-p$ homology or cohomology. For higher rank symmetric spaces X of dimension d the Margulis normal subgroup theorem or property (T) implies that $H_1(\Gamma\backslash X,\mathbb{Z})$ is finite so $b_1(\Gamma\backslash X)=b_{d-1}(\Gamma\backslash X)=0$ but we do not even know whether $\dim_{\mathbb{F}_p} H_1(\Gamma\backslash X,\mathbb{F}_p)$ is sublinear in volume. In [**54**] Gelander showed that for every symmetric space X (not necessarily higher rank) there exists a positive constant $B=B(X)$ such that

$$d(\Gamma)\leq B\mathrm{Vol}(\Gamma\backslash X),$$

where $d(\Gamma)$ is the minimal cardinality of a generating set. The result of Gelander implies that

$$\limsup_{\mathrm{Vol}(\Gamma\backslash X)\to\infty}\frac{\dim_{\mathbb{F}_p} H_1(\Gamma\backslash X,\mathbb{F}_p)}{\mathrm{Vol}(\Gamma\backslash X)}\leq B.$$

The question on the growth of dimensions of mod$-p$ homology groups is related to the rank gradient [**1**, **5**, **67**]. The rank gradient of a group Γ with respect to a sequence of subgroups (Γ_n) is defined as

$$\mathrm{RG}(\Gamma,(\Gamma_n)):=\lim_{n\to\infty}\frac{d(\Gamma_n)-1}{[\Gamma,\Gamma_n]},$$

provided that the limit exists. In [**1**] (see [**5**] for a similar problem for fundamental groups of hyperbolic 3-manifolds) it is shown that if Γ is a higher rank lattice which admits a set of generators[2] $s_1,s_2,\ldots,s_n$ such that $[s_i,s_{i+1}]=1$ then $RG(\Gamma,(\Gamma_n))=0$ for every sequence of pairwise distinct finite index subgroups (Γ_n). We can deduce immediately that

$$\lim_{n\to\infty}\frac{\dim_{\mathbb{F}_p} H_1(\Gamma_n\backslash X,\mathbb{F}_p)}{\mathrm{Vol}(\Gamma_n\backslash X)}=0.$$

In fact for right-angled groups an even stronger conclusion holds ([**1**]):

$$\lim_{\mathrm{Vol}(\Gamma\backslash X)\to\infty}\frac{\log|H_1(\Gamma\backslash X,\mathbb{Z})|}{\mathrm{Vol}(\Gamma\backslash X)}=0.$$

It is conjectured [**1**, Conjecture 3] that

$$\lim_{n\to\infty}\frac{d(\Gamma_n)-1}{\mathrm{Vol}(\Gamma_n\backslash X)}=0$$

for any sequence of pairwise distinct lattices in a higher rank semisimple Lie group.

Regarding the growth of mod$-p$ homology groups in rank one lattices we have the following result of Calegari and Emerton [**30**]. Let M be a 3-manifold with fundamental group Γ. Fix a map $\phi:\Gamma\to GL(n,\mathbb{Z}_p)$ and write G for the closure of $\phi(\Gamma)$. Put $G_k:=G\cap(\ker\left[\mathrm{GL}(n,\mathbb{Z}_p)\to\mathrm{GL}(n,\mathbb{Z}/p^k\mathbb{Z})\right])$ and $\Gamma_k:=\phi^{-1}(G_k)$. Associated to each Γ_k we have a finite connected cover M_k of M such that $\pi_1(M_k)=\Gamma_k$. The sequence of 3-manifolds obtained in this way is called a $p-$**adic analytic tower**. Let d be the dimension of G as a $p-$adic analytic group. In [**30**] it is proved that the dimensions of the homology groups $H_1(M_k,\mathbb{F}_p)$ may grow only in one of the following ways:

- $\dim_{\mathbb{F}_p} H_1(M_k,\mathbb{F}_p)=\lambda p^{dk}+O(p^{(d-1)k})$ (linear growth).
- $\dim_{\mathbb{F}_p} H_1(M_k,\mathbb{F}_p)=\lambda p^{(d-1)k}+O(p^{(d-2)k})$.
- $d=2$ and $\dim_{\mathbb{F}_p} H_1(M_k,\mathbb{F}_p)=O(1)$.
- $d=3$ and $\dim_{\mathbb{F}_p} H_1(M_k,\mathbb{F}_p)\leq 3$ for k large enough.

In particular when M_k are hyperbolic then the limit $\lim_{k\to\infty}\frac{\dim_{\mathbb{F}_p} H_1(M_k,\mathbb{F}_p)}{\mathrm{Vol}(M_k)}$ exists. Calegari and Emerton conjecture that the first possibility can not occur if $\bigcap\Gamma_k=\{1\}$. If that is true then the dimension of mod$-p$ homology groups would grow sublinearly in $p-$adic analytic towers of hyperbolic 3-manifolds which converge Benjamini-Schramm to $\mathbb{H}^3$.

[2] Such groups are called **right-angled**.

Our main result from Chapter 3 deals with mod-2 homology in higher rank spaces:

THEOREM 1.7. *Let X be the symmetric space of a higher rank group G and let (Γ_n) be a sequence of pairwise distinct torsion free lattices in G. Then*

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{F}_2} H_1(\Gamma_n\backslash X, \mathbb{F}_2)}{\mathrm{Vol}(\Gamma_n\backslash X)} = 0.$$

The argument uses non-trivially the characteristic 2 and any extension to odd characteristic would require substantial modifications.

2. Kesten theorems

For a d-regular graph $G = (V, E)$ the Markov averaging operator $M\colon L^2(V) \to L^2(V)$ associated to the random walk on G is given by

$$M\Phi(x) = \frac{1}{d}\sum_{y\sim x} \Phi(y) \text{ for } \Phi \in L^2(V), x \in V,$$

where the sum is taken over all the neighbors of x. The **spectral radius** of G is defined as the operator norm $\|M\| = \sup_{\|\phi\|_2=1} \|M\phi\|_2$. The spectral radius measures the rate at which random walk expands in the graphs. Among infinite d regular graphs the one with the lowest spectral radius is the d-regular tree $\mathbb{T}_d$ whose spectral radius is $\rho(\mathbb{T}_d) = \frac{2\sqrt{d-1}}{d}$. An infinite d-regular graph G is called **Ramanujan graph** if $\rho(G) = \rho(\mathbb{T}_d)$.

Let Γ be a countable group with a finite symmetric set of generators S. The Cayley graph $\mathrm{Cay}(\Gamma, S)$ is the graph with vertex set Γ and edges $\{(g, gs) \mid g \in \Gamma, s \in S\}$. Similarly for any countable set X with right action of Γ we define the Schreier graph $\mathrm{Sch}(X, S)$ as the graph with vertex set X and edges $\{(x, xs) \mid g \in G, s \in S\}$.

A well known theorem of Kesten **[65, 66]** asserts that a normal subgroup N of Γ is amenable if and only if $\rho(\mathrm{Cay}(\Gamma, S)) = \rho(\mathrm{Sch}(\mathrm{N}\backslash\Gamma, S))$. A subgroup H is called Ramanujan if $\rho(\mathrm{Sch}(H\backslash\Gamma, S)) = \rho(\mathrm{Cay}(\Gamma, S))$. Every amenable subgroup is Ramanujan but there are examples of non-amenable not normal subgroups (e.g. free group on 2 generators inside a free group on 5 generators). In **[6]** Abert, Glasner and Virag proved a probabilistic analogue of Kesten's theorem. They prove that if an invariant random subgroup H of Γ is Ramanujan almost surely then it is amenable almost surely. This allows to extend Kesten's theorem beyond normal subgroups.

Our main result from Chapter 4 is the extension of Kesten's theorem to uniformly recurrent subgroups. A subgroup H of Γ is called **uniformly recurrent** (see **[42, 43, 80]**) if the closure of the G-orbit of H in Sub_Γ is a minimal dynamical system. Such groups proved useful in the recent characterization of C^*-simplicity by Kennedy **[64]**. We show

THEOREM 1.8 (Thm 3.1). *Let Γ be a group generated by finite symmetric set S. A uniformly recurrent subgroup H of Γ is Ramanujan if and only if it is amenable.*

The proof is based on the argument of Abert, Glasner and Virag from **[4, 6]**. Using similar techniques we give a short proof of a recent result of Lyons and Peres **[76]**. Define the function $C(-, k)\colon G \to \{1, 0\}$ by $C(v, k) = 1$ if v is contained in a non-backtracking cycle of length k and 0 otherwise.

THEOREM 1.9 (Thm 4.1). *Let G be a d-regular rooted infinite Ramanujan graph. Let (X_i) be the standard random walk on G. Then for any $k \geq 1$*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\big[\, C(X_i, k)\,\big] = 0.$$

3. Character bounds

In Chapter 5 we study character bounds for the finite groups of Lie type as well as the compact p-adic reductive groups. We are interested in bounds of the form

$$|\chi(g)| \leq C\chi(1)^{1-\delta}, \text{ for every irreducible character } \chi \tag{3.1}$$

where $C, \delta > 0$ depend only on g.

Let us begin by explaining the motivation coming from Chapter 2. Let Γ be a discrete group. We say that a sequence of finite index subgroups (Γ_n) of Γ is a **Farber sequence** if for every $g \in \Gamma \setminus Z(\Gamma)$

$$\lim_{n\to\infty} \frac{|\{h\Gamma_n \mid h^{-1}gh \in \Gamma_n\}|}{[\Gamma : \Gamma_n]} = 0. \tag{3.2}$$

It is not hard to see that a sequence of subgroups (Γ_n) is a Farber sequence if and only if the sequence of Schreier graphs $\mathrm{Sch}(\Gamma_n\backslash\Gamma, S)$ converges to $\mathrm{Cay}(\Gamma, S)$ in Benjamini-Schramm topology, for any generating set S. In case if Γ is a lattice in a semisimple Lie group the last condition is equivalent to convergence of orbifolds $\Gamma_n\backslash X$ to X in Benjamini-Schramm topology.

For a finite dimensional representation ρ we write χ_ρ for its character. The Farber condition (3.2) can be reformulated in terms of characters of the induced representations. For every non-central $g \in \Gamma$ we have

$$\lim_{n\to\infty} \frac{|\chi_{\mathrm{Ind}_{\Gamma_n}^{\Gamma} 1}(g)|}{\chi_{\mathrm{Ind}_{\Gamma_n}^{\Gamma} 1}(1)} = 0.$$

In Chapter 2, Lemma 2.83 we show that for Γ with a polynomial representation growth any bound of the form

$$|\chi(g)| \leq C\chi(1)^{1-\delta} \text{ for } \chi \text{ irreducible}$$

with $\delta > 0$ implies

$$|\chi_{\mathrm{Ind}_{\Gamma_n}^{\Gamma} 1}(g)| \leq C'\chi_{\mathrm{Ind}_{\Gamma_n}^{\Gamma} 1}(1)^{1-\delta'}$$

for certain positive δ'. Inequalities of this type are one of the crucial components of the proof of Theorem 1.2.

Another motivation to study the bounds of the form (3.1) comes from work of Liebeck, Shalev et al. [**74**] who successfully applied them to study random walks on finite groups of Lie type, random generation problem and word generation problem. Inequalities obtained by Glück [**55**] imply that for every Chevalley group G there exists $C, \delta > 0$ such that for every $g \in G(\mathbb{F}_q)$ non-central we have

$$|\chi(g)| \leq C\chi(1)^{1-\delta} \text{ for every irreducible character } \chi.$$

Note that this bound is uniform in q. The constant $\delta 4 =$ resulting from Glück's estimates is $\frac{1}{\dim G}$ which is highly non-optimal for most elements of G. Larsen proved in an unpublished note [**71**] that if G is a reductive group defined over a finite field $\mathbb{F}_q$ and $g \in G(\mathbb{F}_q)$ is regular semsimple then $|\chi(g)| \leq |W|^2$ where W is the absolute Weyl group of G. In Chapter 2 we adapted Larsen's method to the p-adic setting and proved the following:

THEOREM 1.10. *Let G be a reductive group defined over a non-archimedean local field F. Let K be a compact open subgroup of $G(F)$, let $\gamma \in K$ be a regular semisimple element. Then for every irreducible character χ of K:*

$$|\chi(\gamma)| \leq C,$$

where constant C depends only on γ and G (not on K or χ).

In Chapter 5 we prove a slight improvement of Larsen's bound using Deligne-Lusztig restriction functors. We prove that in a reductive group G over a finite field the value of an irreducible character on any semisimple regular element is bounded by $|W|$. A similar approach using restriction functors was developed independently by Bezrukavnikov, Liebeck, Shalev and Tiep [**74**]. They use it to strengthen Glück's bound. Write

$$\alpha(L) := \max\left\{\frac{\dim u^L}{\dim u^G} \mid u \in L, u \neq 1 \text{ unipotent}\right\}.$$

THEOREM 1.11 (**[19]**). *Let $\gamma \in G(\mathbb{F}_q)$ and let L be a rational Levi subgroup of some rational parabolic subgroup of G such that $G_\gamma^0 \subset L$. Then, for every irreducible character χ of $G(\mathbb{F}_q)$ we have*

$$|\chi(\gamma)| \ll \chi(1)^{\alpha(L)},$$

where the implicit constant depends only on the rank of G.

CHAPTER 2

Strong limit multiplicity for arithmetic hyperbolic surfaces and 3-manifolds

1. Introduction

1.1. Limit multiplicity. Let G be a semisimple Lie group and let Γ be a cocompact lattice in G. We fix a Haar measure on G. The group G acts on $L^2(\Gamma\backslash G)$ by right translations which makes it a unitary representation of G. If Γ is cocompact, the space $L^2(\Gamma\backslash G)$ decomposes as a direct sum of its irreducible subrepresentations, possibly with multiplicities. A natural question one can pose is how the distribution of the irreducible components of $L^2(\Gamma\backslash G)$ changes as we vary the lattice Γ. First major results in this direction were obtained by DeGeorge and Wallach in [**34**]. They showed that if $(\Gamma_i)_{i\in\mathrm{N}}$ is a tower of cocompact lattices in G, i.e a sequence satisfying the following three conditions: $\Gamma_{i+1} \subset \Gamma_i$, $\Gamma_i \lhd \Gamma_1$, $\bigcap_{i=1}^{\infty} \Gamma_i = \{1\}$, then the asymptotic multiplicities of the discrete series representations are proportional to the volume $\mathrm{Vol}(\Gamma\backslash G)$. More precisely, for every irreducible unitary representation π of G they prove that

$$\lim_{i\to\infty} \frac{m_\Gamma(\pi)}{\mathrm{Vol}(\Gamma\backslash G)} = \begin{cases} d_\pi & \text{if } \pi \text{ is in the discrete series,} \\ 0 & \text{otherwise,} \end{cases}$$

where $m_\Gamma(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$ is the multiplicity of π in $L^2(\Gamma\backslash G)$. In the same paper DeGeorge and Wallach conjectured a stronger result called the limit multiplicity property (see [**37**]). It states that under the same conditions, the distribution of irreducible components of $L^2(\Gamma\backslash G)$ counted with multiplicity divided by the covolume of Γ tends to the Plancherel measure. Let us recall the definition of the Plancherel measure.

Let $\Pi(G)$ denote the set of the irreducible unitary representations up to equivalence. For $\pi \in \Pi$ we write H_π for the underlying Hilbert space. For a function $f \in L^1(G)$ we can consider its Fourier transform, given by

$$\hat{f}(\pi) := \pi(f) := \int_G f(g)\pi(g)dg.$$

It is a bounded linear operator on H_π [**48**, Chapters 7.4,7.5]. If $f \in C_c^\infty(G)$ the operator $\hat{f}(\pi)$ is a Hilbert-Schmidt operator [**87**, Proof of Theorem 8.2]. Thus we can treat $\hat{f}(\pi)$ as an element of $H_\pi \otimes H_\pi^*$. The Plancherel measure is the unique measure μ^{pl} on $\Pi(G)$ for which

$$\|f\|_{L^2}^2 = \int_{\Pi(G)} \|\hat{f}(\pi)\|_{H_\pi\otimes H_\pi^*}^2 d\mu^{pl} \text{ for all } f \in C_c^\infty(G).$$

Equivalently it is the unique measure for which

$$\langle f, g\rangle = \int_{\Pi(G)} \langle \hat{f}(\pi), \hat{g}(\pi)\rangle_{H_\pi\otimes H_\pi^*} d\mu^{pl} = \int_{\Pi(G)} \mathrm{tr}(\pi(f)\pi(g)^*) d\mu^{pl}.$$

By approximating the Dirac delta at the identity by functions from $C_c^\infty(G)$ we can deduce yet another identity

$$f(1) = \int_{\Pi(G)} \mathrm{tr}(\pi(f)) d\mu^{pl} \text{ for all } f \in C_c^\infty(G).$$

The last equality can be also used as the definition of the Plancherel measure. The tempered spectrum $\Pi^{\mathrm{temp}}(G)$ is defined as the support of the Plancherel measure.

Let us return to the limit multiplicity property. For any lattice Γ (not necessarily cocompact) we define the measure μ_Γ on $\Pi(G)$ by

$$\mu_\Gamma = \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \sum_{\pi\in\Pi(G)} m_\Gamma(\pi)\delta_\pi.$$

We say that a sequence of lattices $(\Gamma_i)_{i\in\mathbb{N}}$ has the *limit multiplicity property* if for every bounded function $\phi \in C(\Pi^{\mathrm{temp}}(G))$ [1]

$$\lim_{i\to\infty} \int_{\Pi(G)} \phi(\pi) d\mu_{\Gamma_i} = \int_{\Pi(G)} \phi(\pi) d\mu^{pl}$$

and for every bounded set B in $\Pi(G)\setminus\Pi^{\mathrm{temp}}(G)$ the measures $\mu_{\Gamma_i}(B)$ tend to 0. Sauvageot showed in **[97]** that it is enough to test the convergence for functions of the form $\pi \mapsto \mathrm{tr}(\pi(f))$ where f is a smooth, compactly supported function on G (see also **[101]**).

THEOREM 2.1 (Sauvageot Density Principle). *A sequence of lattices (Γ_i) has the limit multiplicity property if and only if, for every $f \in C_c^\infty(G)$ we have*

$$\lim_{i\to\infty} \int_{\Pi(G)} \mathrm{tr}\pi(f) d\mu_{\Gamma_i} = f(1).$$

We remark that this theorem holds for uniform as well as for non-uniform lattices. Write $R_\Gamma f$ for the operator

$$R_\Gamma f(\Phi)(x) := \int_G f(g)\Phi(xg)dg \text{ for } \Phi \in L^2(\Gamma\backslash G). \tag{1.1}$$

The Sauvageot density principle is especially convenient in the uniform case, because then $L^2(\Gamma\backslash G)$ decomposes discretely and we have

$$\int_{\Pi(G)} \mathrm{tr}\pi(f) d\mu_\Gamma = \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \sum_{\pi\in\Pi(G)} m_\Gamma(\pi)\mathrm{tr}\pi(f) = \frac{\mathrm{tr}R_\Gamma f}{\mathrm{Vol}(\Gamma\backslash G)}.$$

In 1979 DeGeorge and Wallach **[35]** proved the limit multiplicity property for towers of cocompact lattices in semisimple Lie groups of real rank 1 and in 1986 Delorme **[37]** settled the question for cocompact towers in arbitrary semsimple Lie groups. For sequences of non-uniform lattices Sarnak **[96]** showed that any sequence of principal congruence subgroups $\Gamma(N_i) \in \mathrm{SL}(2,\mathbb{Z})$ with $N_i \to \infty$ has the limit multiplicity property. Analogues in the non uniform case of the results of DeGeorge and Wallach about the multiplicites of irreducible unitary representations were obtained by DeGeorge **[36]**, Barbasch—Moscovici **[14]** for groups of real rank one, and by Clozel **[32]** for general groups. The strongest results for asymptotic multiplicities in towers of arithmetic lattices were obtained by Rohlfs—Spehr **[95]** and Savin **[98]**. These results were followed by works of Finis, Lapid and Mueller **[47]** who proved the limit multiplicity for principal congruence subgroups of $\mathrm{SL}(n,\mathcal{O}_k)$ where $\mathcal{O}_k$ is the ring of integers in a number field k. By the Borel-Harish-Chandra Theorem those are lattices in the semisimple Lie group $\mathrm{SL}(n, k\otimes_{\mathbb{Q}}\mathbb{R})$. This result was later extended in **[46]** to arbitrary sequences $(\Gamma_i)_{i\in\mathbb{N}}$ of congruence subgroups of $\mathrm{SL}(n,\mathcal{O}_k)$.

In the uniform case, a substantial breakthrough was obtained in **[2]**. Using invariant random subgroups and the notion of Benjamini-Schramm convergence it was shown that if G has real rank at least 2 and Kazhdan's property (T), then every sequence of cocompact lattices $(\Gamma_i)_{i\in\mathbb{N}}$ which are pairwise non conjugate and whose injectivity radius is uniformly bounded away from 0 has the limit multiplicity property. It was the first result dealing with sequences of not necessarily commensurable lattices. In 2013 Jean Raimbault **[92**, Corollary 1.3.5] obtained very general results on sequences of maximal lattices in $\mathrm{SL}(2,\mathbb{C})$ which are all defined over quadratic or cubic number fields. Recently Jasmin Matz **[81]** proved that the Limit Multiplicity hold for groups $G = \mathrm{SL}(2,\mathbb{R})^{r_1}\times\mathrm{SL}(2,\mathbb{C})^{r_2}$ and sequences of arithmetic lattices of form $\mathrm{SL}(2,\mathcal{O}_F)$ where F is a number field with r_1 real and r_2 complex places.

[1] We mean functions continuous with respect to the Fell topology on $\Pi(G)$ **[39**, 18.1]

1.2. Benjamini-Schramm convergence. Let G be a semisimple Lie group and $\Gamma \subset G$ a cocompact lattice. Let K be a maximal compact subgroup of G. Let $X = G/K$ be the symmetric space of G endowed with the G-invariant Riemannian metric defined by the Killing form of G (see [**58**]). Let d_G be associated left invariant Riemannian metric on G and write $B_G(\Sigma, R)$ for the open R-ball around the set $\Sigma \subset G$. We can define the R-thin part of the orbifold $\Gamma\backslash X$ as

$$(\Gamma\backslash X)_{<R} := \left\{\Gamma g K \mid B_G(K, R) \cap g^{-1}\Gamma g \neq \{1\}\right\} \tag{1.2}$$

Let $x \in \Gamma\backslash X$ and let $\tilde{x} \in X$ be a lift of x. Recall We that the **injectivity radius** $\operatorname{injrad} x$ is defined as the supremum of real numbers R such that the projection map $B(\tilde{x}, R) \to B(x, R)$ is injective. Given a sequence of lattices $(\Gamma_i)_{\in \mathbb{N}}$ we consider the sequence of locally symmetric spaces $(\Gamma_i\backslash X)_{i\in\mathbb{N}}$. We say that the sequence $(\Gamma_i\backslash X)_{i\in\mathbb{N}}$ converges Benjamini-Schramm (or *B-S converges*) to G/K if for every $R > 0$ we have

$$\lim_{i\to\infty} \frac{\operatorname{Vol}((\Gamma_i\backslash X)_{<R})}{\operatorname{Vol}(\Gamma_i\backslash X)} = 0 \tag{1.3}$$

For brevity we shall say that $(\Gamma_i)_{i\in\mathbb{N}}$ has property B-S if the sequence $(\Gamma_i\backslash X)_{i\in\mathbb{N}}$ converges Benjamini-Schramm to X. The notion of Benjamini-Schramm convergence originates from the paper [**15**] where they defined it for the sequences of graphs of bounded degree. For locally symmetric spaces it was defined and studied in [**2**]. It is a special case of Benjamini-Schramm convergence for metric spaces with probability measures (see [**2**, Chapter 3]).

It is well-known that the Limit Multiplicity Property implies the Benjamini-Schramm convergence for sequences of cocompact lattices (see Section 11).

One of the main results of [**2**] is the following:

THEOREM 2.2 ([**2**, Theorem 1.5]). *Let G be a real semisimple Lie group of real rank at least 2 and with Kazhdan's property (T). Then every sequence of pairwise non-conjugate lattices $(\Gamma_i)_{i\in\mathbb{N}}$ in G has property B-S.*

For a sequence of congruence subgroups of a fixed uniform arithmetic lattice they showed a stronger quantitative version of B-S convergence:

THEOREM 2.3 ([**2**, Theorem 1.12]). *Let Γ_0 a uniform arithmetic lattice in G. Then there exist positive constants c, μ depending only on Γ_0 such that for any congruence subgroup $\Gamma \subset \Gamma_0$ and any $R > 0$ we have*

$$\operatorname{Vol}((\Gamma\backslash X)_{<R}) \leq e^{cR}\operatorname{Vol}(\Gamma\backslash X)^{1-\mu}$$

The result concerning the limit multiplicity property that we referred to in the last section is a consequence of Theorem 2.2 and the following:

THEOREM 2.4 ([**2**, Theorem 1.2]). *Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a sequence of lattices with property B-S, such that the injectivity radius $\boldsymbol{inj\ rad}(\Gamma_i\backslash X)$ is uniformly bounded away from 0* [2]. *Then $(\Gamma_i)_{i\in\mathbb{N}}$ has the limit multiplicity property.*

The Margulis injectivity radius conjecture [**77**] predicts that for a fixed semisimple Lie group G the injectivity radius of $\Gamma\backslash G/K$ is bounded away from 0 uniformly for all arithmetic lattices. This would be implied by the Lehmer conjecture and would itself imply the Salem number conjecture. So far no decisive progress has been made towards the proof of the Margulis conjecture. Finally we should mention the results of Jean Raimbault [**92**] for arithmetic lattices in $\mathrm{SL}(2, \mathbb{C})$ and associated arithmetic 3-orbifolds, we shall describe them in greater detail in the next paragraph devoted to sequences of arithmetic lattices.

[2]The actual condition is: the lengths of closed geodesics are uniformly bounded from below and the order of torsion elements of Γ_i is uniformly bounded from above.

1.3. Sequences of arithmetic lattices. In this section we shall discuss what are the reasonable conditions that we should put on a sequence of arithmetic lattices $(\Gamma_i)_{i\in_N}$ to expect that it has the limit multiplicity property or the B-S property. The first obvious condition is

(1) $\lim_{i\to\infty} \mathrm{Vol}(\Gamma_i\backslash G) = \infty$,

By Theorem 2.2 this is sufficient for property B–S in higher rank groups with property (T). If G is a rank 1 Lie group, then this is not enough. For example if Γ is a torsion free, uniform arithmetic lattice in $\mathrm{SL}(2,\mathbb{R})$ and $M_0 := \Gamma\backslash\mathbb{H}$ is the corresponding compact hyperbolic surface, we can take $(M_i)_{i\in\mathbb{N}}$ a sequence of cyclic covers of M_0. Clearly $(M_i)_{i\in\mathbb{N}}$ does not converge Benjamini-Schramm to $\mathbb{H}$ because the radius of injectivity is uniformly bounded by $5\mathrm{diam}M_0$[3]. Hence the sequence of fundamental groups $\Gamma_i = \pi_1(M_i)$ does not have property B-S. There is a similar counter-example constructed by Sarnak and Phillips [**88**] for the limit multiplicity property. In both of these constructions the sequences contained lattices which were not congruence lattices. Hence, it seems reasonable to add the second condition

(2) Γ_i is a congruence lattice for all $i \in \mathbb{N}$.

With these conditions we expect that at least the weak version of the following conjecture holds:

CONJECTURE 2.5.
- *(Weak version) Let G be a semisimple Lie group with trivial center and $(\Gamma_i)_{i\in\mathbb{N}}$ a sequence of arithmetic lattices in G satisfying conditions (1) and (2). Then $(\Gamma_i)_{i\in\mathbb{N}}$ has the limit multiplicity property and the B-S property.*
- *(Strong version) Let $X = G/K$ be the Riemannian symmetric space of a center-free semisimple group G. There exists a $\delta > 0$ such that for any $R > 0$ and any congruence arithmetic lattice Γ in G and any $f \in C(G)$ with $\mathrm{supp}f \subset B(1,R)$ we have*

$$\mathrm{Vol}((\Gamma\backslash X)_{<R}) \ll_R \mathrm{Vol}(\Gamma\backslash X)^{1-\delta}, \tag{1.4}$$

$$\left| f(1) - \frac{\mathrm{tr}R_\Gamma f}{\mathrm{Vol}(\Gamma\backslash G)} \right| \ll_R \|f\|_\infty \mathrm{Vol}(\Gamma\backslash G)^{-\delta}. \tag{1.5}$$

Cases of this conjecture for maximal lattices and for lattices defined over fields of bounded degree were present in [**92**, Section 1.1.2] and that a similar statement holds was conjectured in [**2**, Conjecture 6.1]. For $G = \mathrm{SL}(2,\mathbb{C})$ and lattices defined over a cubic or quadratic field this was settled by Jean Raimbault [**92**, Theorem A]. He has shown, among other things, that there exists $\delta > 0$ with the following property. For any maximal lattice Γ in $\mathrm{SL}(2,\mathbb{C})$ defined over a cubic or quadratic number field put $M = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})/K$ then

$$\mathrm{Vol}((M)_{<R}) \ll_R \mathrm{Vol}(M)^{1-\delta}.$$

This implies that any sequence of maximal lattices $(\Gamma_i)_{i\in\mathbb{N}}$ defined over a cubic field has property B-S if $\mathrm{Vol}(\Gamma_i\backslash G)$ tends to infinity. In particular, he solved completely the case of sequences of non uniform lattices as all arithmetic non-uniform lattices in $\mathrm{SL}(2,\mathbb{C})$ are defined over a quadratic imaginary number field. For multiple factors of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ Jasmin Matz [**81**] proved a reasonable analogue of Strong Limit Multiplicity property for certain sequences of non-uniform arithmetic congruence lattices. Raimbault has also addressed Conjecture 2.5 for sequences of lattices defined over fields of bounded degree. He obtained the following

THEOREM 2.6. *[**92**, Theorem B] Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a sequence of lattices in $\mathrm{SL}(2,\mathbb{C})$ with fields of definition F_i such that:*
- *F_i is a quadratic extension of a totally real subfield B_i,*

[3]Let $a, b \in \pi_1(M_0)$ be such that $c = aba^{-1}b^{-1} \neq 0$ and let γ be a geodesic on M_0 corresponding to c. Since the covering $M_i \to M_0$ is abelian for all i the geodesic γ lifts to a closed geodesic on M_i. Every point x in M_i is at distance at most $\mathrm{diam}M_0$ to a lift of γ so $\mathrm{injrad}\,(x) \leq \mathrm{diam}M_0 + l(\gamma)$ where $l(\gamma)$ is the length of γ. We can pick a, b such that the loops representing them are of lengths smaller than $\mathrm{diam}M_0$. If follows that $l(\gamma) \leq 4\mathrm{diam}M_0$ and consequently $\mathrm{injrad}\,(x) \leq 5\mathrm{diam}M_0$.

- *the relative discriminants* Δ_{F_i/B_i} *go to infinity,*
- *the absolute degree* $[F_i:\mathbb{Q}]$ *is bounded.*

Then $(\Gamma_i)_{i\in\mathbb{N}}$ *has property B-S.*

The second condition is reminiscent of the assumptions of the Brauer Siegel theorem, but the proof in [**92**] does not use it. The argument follows the strategy of [**2**] via Invariant Random Subgroups (IRS) and the Borel density Theorem for IRS's [**2**, Theorem 2.6].

Let us explain how one could deduce the weak version of Conjecture 2.5 from the statement that holds only for maximal lattices. We shall use Theorems 2.3 and 2.4 ([**2**, Theorems 1.12,1.2]).

LEMMA 2.7. *Let* G *be a semisimple Lie group. Suppose that every sequence of maximal arithmetic lattices* $(\Gamma_i)_{i\in\mathbb{N}}$ *such that* $\mathrm{Vol}(\Gamma_i\backslash G)\to\infty$ *has the limit multiplicity property and property B-S. Then every sequence of congruence arithmetic lattices* $(\Lambda_i)_{i\in\mathbb{N}}$ *satisfying* $\lim_{i\to\infty}\mathrm{Vol}(\Lambda_i\backslash G)=\infty$ *has the limit multiplicity property and property B-S.*

PROOF. Let $(\Lambda_i)_{i\in\mathbb{N}}$ be as in the statement. It will be enough to show that we can always find a subsequence with the desired properties. For any i choose a maximal arithmetic lattice Γ_i containing Λ_i. We consider two cases, either $\mathrm{Vol}(\Gamma_i\backslash G)$ goes to ∞ or not. In the first case, the sequence (Γ_i) has the limit multiplicity property and the B-S property. For any $R>0$ and $f\in C_c(G)$ we have

$$\frac{\mathrm{Vol}((\Lambda_i\backslash G/K)_{<R})}{\mathrm{Vol}(\Lambda_i\backslash G/K)}\leq\frac{\mathrm{Vol}((\Gamma_i\backslash G/K)_{<R})}{\mathrm{Vol}(\Gamma_i\backslash G/K)} \tag{1.6}$$

$$\left|f(1)-\frac{\mathrm{tr}R_{\Lambda_i}f}{\mathrm{Vol}(\Lambda_i\backslash G)}\right|\leq\left|f(1)-\frac{\mathrm{tr}R_{\Gamma_i}f}{\mathrm{Vol}(\Gamma_i\backslash G)}\right|. \tag{1.7}$$

Inequality (1.6) follows from the fact that the R-thin part of $\Lambda_i\backslash G/K$ covers only the R-thin part of $\Gamma_i\backslash G/K$ so $\mathrm{Vol}((\Lambda_i\backslash G/K)_{<R})\leq[\Gamma_i:\Lambda_i]\mathrm{Vol}((\Gamma_i\backslash G/K)_{<R})$. To prove (1.7) note that for every Γ_i-conjugacy class $[\gamma]_{\Gamma_i}$ we have

$$\sum_{[\gamma']_{\Lambda_i}\subset[\gamma]_{\Gamma_i}}\mathrm{Vol}(\Lambda_{i,\gamma'}\backslash G_{\gamma'})\mathcal{O}_{\gamma'}(f)\leq[\Gamma_i:\Lambda_i]\mathrm{Vol}(\Gamma_{i,\gamma}\backslash G_\gamma)\mathcal{O}_\gamma(f).$$

Now (1.7) follows from the Selberg trace formula and the identity $\mathrm{Vol}(\Lambda_i\backslash G/K)=[\Gamma_i:\Lambda_i]\mathrm{Vol}(\Gamma_i\backslash G/K)$.

Hence $(\Lambda_i)_{i\in\mathbb{N}}$ also has the limit multiplicity property and the B-S property. In the second case we invoke the result of Borel and Prasad [**23**, Theorem A] on the finiteness of the number of arithmetic lattices of bounded volume. It follows that there exists an infinite subsequence $(n_i)_{i\in\mathbb{N}}$ such that $\Gamma_{n_i}=\Gamma$ for some fixed maximal Γ. The lattices in the sequence $(\Lambda_{n_i})_{i\in\mathbb{N}}$ are all contained in Γ so by Theorem 2.3 it has property B-S. Moreover the radius of injectivity of $\Lambda_{n_i}\backslash G/K$ is at least as big as injrad $(\Gamma_{n_i}\backslash G/K)$ so by Theorem 2.4 $(\Lambda_{n_i})_{i\in\mathbb{N}}$ has the limit multiplicity property. □

Note that even if the strong version of the Conjecture 2.5 holds for maximal lattices the argument above does not yield the strong version in the general case. The reason for that is that the implicit constant and the exponent in Theorem 2.3 depend of the lattice in a non-explicit way. [4]

1.4. Main results. Our main results deal with sequences of arbitrary, torsion free congruence arithmetic lattices in $PGL(2,\mathbb{R})$ and $\mathrm{PGL}(2,\mathbb{C})$. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and let K be a maximal compact subgroup of $\mathrm{PGL}(2,\mathbb{K})$. Write X for the symmetric space $\mathrm{PGL}(2,\mathbb{K})/K$ equipped with the Riemannian metric induced by the Killing form.

[4] It looks difficult to extract sufficient dependence of Γ from the proofs of [**2**, Thm 1.2] and the analogous result from [**46**]. One of the main innovation of the present paper is an alternative approach to [**2**, Thm 1.2] which gives very explicit bounds.

1.4.1. ***Strong Limit Multiplicity and Benjamini-Schramm convergence.***

THEOREM 2.8. *[Strong Limit Multiplicity] There exists $a > 0$ such that for any $R > 0$ the following holds. Let Γ be a uniform torsion free, congruence arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$. For any $f \in C(\mathrm{PGL}(2,\mathbb{K}))$ with $\mathrm{supp} f \subset B(1,R)$*

$$|\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))f(1) - \mathrm{tr} R_\Gamma f| \ll_R \|f\|_\infty (\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))^{1-a}. \tag{1.8}$$

We can take $a \geq 0.014$.

THEOREM 2.9. *[Strong Benjamini Schramm convergence] There exists $a > 0$ such that for any $R > 0$ the following holds. Let Γ be a torsion free, congruence arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$. Then*

$$\mathrm{Vol}((\Gamma\backslash X)_{<R}) \ll_R \mathrm{Vol}((\Gamma\backslash X))^{1-a}. \tag{1.9}$$

We can take $a \geq 0.014$.

REMARK 2.10. *This settles both versions of Conjecture 2.5 for lattices in $\mathrm{PGL}(2,\mathbb{R})$ and $\mathrm{PGL}(2,\mathbb{C})$ with an additional assumption that they are torsion free. The method also applies to congruence lattices without torsion of small order. Lattices with 2-torsion elements are the hardest case, where the bounds we can obtain are not sufficient to prove the Strong Limit Multiplicity.*

Without assuming the congruence condition we have

THEOREM 2.11. *There exists $c > 0$ such that for any $R > 0$ the following holds. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let Γ be a torsion free, arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$. Then for any $f \in C(\mathrm{PGL}(2,\mathbb{K}))$ with $\mathrm{supp} f \subset B(1,R)$*

$$|\mathrm{Vol}(\Gamma\backslash X)f(1) - \mathrm{tr} R_\Gamma f| \ll_R \|f\|_\infty \Delta_k^{-c}, \tag{1.10}$$

$$\frac{\mathrm{Vol}((\Gamma\backslash X)_{<R})}{\mathrm{Vol}(\Gamma\backslash X)} \ll_R \Delta_k^{-c}, \tag{1.11}$$

where Δ_k is the discriminant of the trace field k of Γ and $c \geq 0.0006$.

As a corollary of the proof we will get

COROLLARY 2.12. *Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a sequence of torsion free arithmetic lattices in $\mathrm{PGL}(2,\mathbb{K})$. Then either infinitely many Γ_i's are commensurable or the sequence $(\Gamma_i)_{i\in\mathbb{N}}$ has property B–S.*

By using a different argument Corollary 2.12 is vastly improved in **[49]** (see Corollary 2.15).

REMARK 2.13. *An element g is called $\mathbb{R}$-regular (**[16, 91]**) if $\mathrm{Ad}\, g$ has no eigenvalues on the unit circle. Define*

$$\mathrm{tr}^{rr} R_\Gamma f = \sum_{\substack{[\gamma]\in\Gamma \\ \gamma\, \mathbb{R}-regular}} \mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f).$$

In the proof Theorem 2.8 we actually show that for any congruence lattice Γ (possibly non-uniform or with torsion)

$$\left|\frac{\mathrm{tr}^{rr} R_\gamma f}{\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))}\right| \ll_R \|f\|_\infty \mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))^{(1-a)}. \tag{1.12}$$

Theorem 2.8 follows because in the torsion free lattices of $\mathrm{PGL}(2,\mathbb{K})$ every nontrivial element is $\mathbb{R}$-regular.

Several steps of the proofs of Theorems 2.8, 2.9 and 2.11 work also for other simple Lie groups. The difficulties arise when we want to estimate the adelic volumes of the centralizers (c.f. Proposition 2.93) and the values of irreducible congruence characters at semisimple, non-regular elements (c.f. Theorem 2.61). For general semisimple Lie group G the methods from the present paper should be mutatis mutandis enough to prove that the

contribution to the geometric side of the trace formula coming from $\mathbb{R}$-regular elements (see [**16**, **91**]) of a congruence lattice Γ is bounded by $\mathrm{Vol}(\Gamma\backslash G)^{1-\alpha}$ for some $\alpha > 0$. For torsion free uniform lattices in $\mathrm{PGL}(2,\mathbb{R}), \mathrm{PGL}(2,\mathbb{C})$ all nontrivial elements are $\mathbb{R}$-regular so $\mathrm{tr}R_\Gamma f - f(1)\mathrm{Vol}(\Gamma\backslash G) = \mathrm{tr}^{\mathrm{rr}} R_\Gamma f$ and we can prove the strong form of limit multiplicity. In the forthcoming work with Jean Raimbault [**49**] we prove:

LEMMA 2.14. *[49] Let G be simple Lie group and let $(\Gamma_i)_{i\in\mathbb{N}}$ be a sequence of lattices in G such that for every $f \in C_c(G)$ we have*

$$\lim_{i\to\infty} \frac{\mathrm{tr}^{rr} R_{\Gamma_i} f}{\mathrm{Vol}(\Gamma\backslash G)} = 0.$$

Then the sequence of locally symmetric spaces $\Gamma_i\backslash X$ tends to X in Benjamini-Schramm convergence.

As a corollary of (1.12) and the above lemma one gets

COROLLARY 2.15. *[49] Let be $(\Gamma_i)_{i\in\mathbb{N}}$ be any sequence of pairwise non-conjugate arithmetic congruence lattices in $\mathrm{PGL}(2,\mathbb{K})$. Then $\Gamma_i\backslash X$ converges B-S to X.*

1.4.2. ***Triangulations of arithmetic hyperbolic 3-manifolds.*** As an application of above results we prove Gelander conjecture [**53**, Conjecture 1.3] for arithmetic 3-manifolds:

THEOREM 2.16. *There exist absolute positive constants A, B such that every arithmetic, hyperbolic 3-manifold M is homotopically equivalent to a simplicial complex with at most $A\mathrm{Vol}(M)$ vertices and each vertex has degree bounded by B (if M is compact we can take $B = 245$).*

As a simple corollary we obtain:

COROLLARY 2.17. *There exists a constant $C > 0$ such that any arithmetic lattice Γ in $\mathrm{PGL}(2,\mathbb{C})$ admits a presentation*

$$\Gamma = \langle S \mid \Sigma \rangle$$

where the size of $|S|, |\Sigma|$ is bounded by $C\mathrm{Vol}(M)$ and all relations in Σ are of length at most 3.

1.4.3. ***Growth of Betti numbers.*** Mathsushima's formula [**18**, **79**] provides a link between the spectral decomposition of $L^2(\Gamma\backslash\mathrm{PGL}(2,\mathbb{K})$ and dimensions of cohomology groups $H^i(\Gamma\backslash X, \mathbb{C})$. We use standard notation $b_i(\Gamma\backslash X) := \dim_{\mathbb{C}} H^i(\Gamma\backslash X, \mathbb{C})$ and we write $b_i^{(2)}(X)$ for the L^2-Betti numbers of X. Using theorems 2.8 and 2.11 we deduce:

COROLLARY 2.18. *Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a sequence of pairwise distinct arithmetic, torsion free lattices in $\mathrm{PGL}(2,\mathbb{K})$. Assume that either the are congruence or they are pairwise non-commensurable. Then*

$$\lim_{i\to\infty} \frac{b_i(\Gamma\backslash X)}{\mathrm{Vol}(\Gamma\backslash X)} = b_i^{(2)}(X) = \begin{cases} \frac{1}{2\pi} & X = \mathbb{H}^2, i = 1 \\ 0 \text{ otherwise.} \end{cases}$$

1.5. Baby Case. The proof of Theorem 2.8 is quite long and does not split well into separate steps. Before giving the outline for the general case we will give a detailed sketch of the proof for a very particular type of arithmetic lattices. The baby case deals with the class of "nice" lattices in $\mathrm{PGL}(2,\mathbb{R})$ (we will define them shortly) which are very close to being maximal. This example already involves lattices with trace fields of unbounded degree so it does not follow from the results of Matz [**81**] nor from the work of Raimbault [**92**]. The only reason why we can not work with maximal lattices is that they contain torsion elements and some of our arguments break down for such. We recommend the reader to get acquainted with our notations (Section 2), preliminaries on the quaternion algebras (Section 3) and the construction of arithmetic lattices (Section 4.2, Definitions 2.35,2.48,2.34) before reading this sketch.

Let k be a totally real number field of odd degree. We assume that the ring of integers $\mathcal{O}_k$ has a prime ideal $\mathfrak{p}_0$ such that $\mathcal{O}_k/\mathfrak{p}_0 \simeq \mathbb{F}_2$. Fix a real place ν_0 of k. Let D be the

quaternion algebra over k with the ramification set $\mathrm{Ram}D = M_k^\infty \setminus \{\nu_0\}$. By Proposition 2.29 such D exists and is unique up to k-isomorphism. Write $\mathrm{PD}^\times$ for the projective multiplicative group of D. The group of adeles decomposes as the restricted product

$$\mathrm{PD}^\times(\mathbb{A}) \simeq \mathrm{PGL}(2,\mathbb{R}) \times \mathrm{PO}(3,\mathbb{R})^{[k:\mathbb{Q}]-1} \times \prod_{\mathfrak{p}\in M_k^f}{}^{*} \mathrm{PGL}(2,k_{\mathfrak{p}}). \tag{1.13}$$

We fix a maximal open compact subgroup U of $\mathrm{PGL}(2,\mathbb{A}_f)$:

$$U = \prod_{\mathfrak{p}\in M_k^f} \mathrm{PGL}(2,\mathcal{O}_{k_{\mathfrak{p}}}). \tag{1.14}$$

We put $K_0 = \{k_{\mathfrak{p}_0}^\times x \mid x \in \mathrm{GL}(2,\mathcal{O}_{k_{\mathfrak{p}_0}}), x \equiv \mathrm{Id} \mod \mathfrak{p}_0^2\}$ and $U_0 = K_0 \times \prod_{\substack{\mathfrak{p}\in M_k^f \\ \mathfrak{p}\neq\mathfrak{p}_0}} \mathrm{PGL}(2,\mathcal{O}_{k_{\mathfrak{p}}})$. Since $\mathcal{O}_k/\mathfrak{p}_0 \simeq \mathbb{F}_2$ we have $\mathrm{PGL}(2,\mathcal{O}_{k_{\mathfrak{p}_0}}) \simeq \mathrm{PGL}(2,\mathbb{Z}_2)$ and

$$K_0 \simeq \{\mathbb{Z}_2^\times x \mid x \in \mathrm{GL}(2,\mathbb{Z}_2), x \equiv \mathrm{Id} \mod 4\}$$

is a torsion-free subgroup of $\mathrm{PGL}(2,\mathcal{O}_{k_{\mathfrak{p}_0}})$ of index 48. We put $\Gamma := \mathrm{PD}^\times(k) \cap \big(\mathrm{PD}^\times(\mathbb{A}_\infty) \times U_0\big)$ and identity Γ with its projection to $\mathrm{PD}^\times(k_{\nu_0}) \simeq \mathrm{PGL}(2,\mathbb{R})$. It is a congruence arithmetic torsion-free lattice in $\mathrm{PGL}(2,\mathbb{R})$. We will call the lattices constructed in this way **nice**. In the baby case we will show that for any $f \in \mathcal{C}_c(\mathrm{PGL}(2,\mathbb{R}))$ and a nice lattice Γ we have

$$|\mathrm{tr}R_\Gamma f - \mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{R}))f(1)| \ll_f \mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{R}))^{1-a}, \tag{1.15}$$

for some positive constant a[5]. Since nice lattices are subgroups of maximal lattices of uniformly bounded index, the argument will be much simpler than in the general case. In particular, we will not need to use the representation zeta functions and the machinery developed in Section 8.

Step 1. We would like to bound the LHS of (1.15) from above using the adelic version of Selberg trace formula and give a lower bound on the volume $\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{R})$ using our variant of Borel volume formula (Corollary 2.92). Put $f_{\mathbb{A}} = f \otimes (\mathbf{1}_{\mathrm{PO}(3,\mathbb{R})})^{[k:\mathbb{Q}]-1} \otimes (48\cdot\mathbf{1}_{U_0})$ (see Section 4.5 for the explanation why we choose $f_{\mathbb{A}}$ in this way) and put the standard Haar measure (see Section 2.4) on $\mathrm{PD}^\times(\mathbb{A})$. We recall that the standard measure depends implicitly on the choice of the subgroup U. By Section 4.5, Corollary 2.54 we have the following estimate

$$|\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{R}))f(1) - \mathrm{tr}R_\Gamma f| \leq \frac{2}{|\mathrm{cl}\,(U_0)|} \sum_{\substack{[\gamma]_{PD^\times(k)} \\ \gamma\neq 1}} \mathrm{Vol}(PD_\gamma^\times(k)\backslash\mathrm{PD}_\gamma^\times(\mathbb{A}))\mathcal{O}_\gamma(|f|_{\mathbb{A}}), \tag{1.16}$$

and by Corollary 2.92

$$\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{R})) = \frac{[U:U_0]}{|\mathrm{cl}\,(U_0)|} \frac{|\Delta_k|^{3/2}\zeta_k(2)}{\pi(4\pi^2)^{[k:\mathbb{Q}]-1}} \geq \frac{48|\Delta_k|^{3/2}}{|\mathrm{cl}\,(U_0)|(4\pi^2)^{[k:\mathbb{Q}]}}. \tag{1.17}$$

Note that in the case of nice lattices the ramification set of the quaternion algebra consists only of archimedean places and the set S of places $\mathfrak{p}$ where $U \not\simeq \mathrm{PGL}(2,O_{k_{\mathfrak{p}}})$ is empty so the above formulas are simpler than in the general case. The proofs of Corollaries 2.54 and 2.92 are as hard for the baby case as they are in general.

Step 2. Note that $\mathcal{O}_\gamma(|f|_{\mathbb{A}}) \neq 0$ implies that the conjugacy class of γ in $\mathrm{PGL}(2,\mathbb{R})$ intersects the support of f. We endow $\mathrm{PGL}(2,\mathbb{R})$ with a left-invariant group metric $d(x,y) = \|1 - \mathrm{Ad}\,(y^{-1}x)\|$ where $\|A\| = \sqrt{\mathrm{tr}(A^TA)}$. Assume that $\mathrm{supp}f$ is contained in a ball $B(1,R)$. In Lemma 2.69 we show that if the conjugacy class of γ intersects $B(1,R)$ then the logarithmic Mahler measure $m(\gamma)$ of γ [6] is bounded by R. The Mahler

[5] Our proof in this special case yields $a \geq 1/2$ and this is not optimal. Note that the bound in general case is much weaker, with a≥ 0.014.

[6] For the definition of Mahler measure of an algebraic number see Section 5, $m(\gamma)$ is defined as $m(\lambda)$ where λ is a non-trivial eigenvalue of $\mathrm{Ad}\,\gamma$.

measure controls many arithmetic quantities related to γ including the orbital integrals and the adelic volumes appearing on the right hand side of 1.16. It is important to keep in mind that the non-vanishing of $\mathcal{O}_\gamma(|f|_{\mathbb{A}})$ gives a uniform bound on $m(\gamma)$.

The estimate of RHS of (1.16) can be split in three parts. First we need to estimate the number of the conjugacy classes that bring a non-zero contribution to the sum, secondly we need a uniform upper bound on $\mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A}))$ for those γ with $\mathcal{O}_\gamma(|f|_{\mathbb{A}}) \neq 0$ and finally a uniform upper bound on $\mathcal{O}_\gamma(|f|_{\mathbb{A}})$.

Step 3. We give an upper bound on the number $N = N(f, \Gamma)$ of rational conjugacy classes γ such that $\mathcal{O}_\gamma(|f|_{\mathbb{A}}) \neq 0$. In Section 7 we prove (Theorem 2.77) that $N = \exp(o([k : \mathbb{Q}]))$. The argument does not simplify for the baby case so for the details we refer to Section 7. The uniform bound on the logarithmic Mahler measure $m(\gamma)$ allows to use Bilu equidistribution theorem for the eigenvalues of γ. In the proof of Theorem 2.77 we apply bounds of Kabatianski-Levenstein on the number of almost orthogonal vectors in an Euclidean space to estimate the number of possible eigenvalues of $\mathrm{Ad}\,\gamma$ by $\exp(o([k : \mathbb{Q}]))$. In the projective group $\mathrm{PD}^\times(k)$ the eigenvalues of $\mathrm{Ad}\,\gamma$ practically determine the conjugacy class of γ (see Proposition 2.26) so we are done.

Step 4. Let γ be a regular non-torsion semisimple element of $\mathrm{PD}^\times(k)$ such that $\mathcal{O}_\gamma(|f|_{\mathbb{A}}) \neq 0$. To bound the volume $\mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A}))$ we express it using the completed Artin L-functions. Again, we will use the fact that $\mathcal{O}_\gamma(|f|_{\mathbb{A}}) \neq 0$ implies a bound on the logarithmic Mahler measure of γ. Let λ be one of the non-trivial eigenvalues of $\mathrm{Ad}\,\gamma$. Put $l = k(\lambda)$, it is a quadratic extension of k. Let $\xi_k(s), \xi_l(s)$ be the completed Dedekind zeta functions of k and l respectively. Put $\Lambda(s, \chi_{l/k}) = \xi_l(s)/\xi_k(s)$ [7]. By Proposition 2.93 we have

$$\mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A})) \ll \frac{\Lambda(1, \chi_{l/k})}{(2\pi)^{[k:\mathbb{Q}]}}. \tag{1.18}$$

With the help of Theorem 2.59 we can prove that $|\Lambda(s, \chi_{l/k})| \ll_s \exp(o(k : \mathbb{Q}))$. In this step we have to use the assumption that γ is non-torsion and that $m(\gamma) < R$. Next, we mimic the complex-analytic proof of the Brauer-Siegel theorem to get

$$\mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A})) \ll_{\varepsilon,R} \frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}}. \tag{1.19}$$

Inequality (1.19) is the contens of Proposition 2.95.

Step 5. Finally we need to bound the orbital integrals $\mathcal{O}_\gamma(|f|_{\mathbb{A}})$. This is the part where the baby case is much easier than the general case. Note that

$$|f|_{\mathbb{A}} = |f| \otimes (\mathbf{1}_{\mathrm{PO}(3,\mathbb{R})})^{\otimes[k:\mathbb{Q}]-1} \otimes 48\mathbf{1}_{K_0} \otimes \bigotimes_{\mathfrak{p}\neq\mathfrak{p}_0} \mathbf{1}_{\mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})},$$

so

$$\mathcal{O}_\gamma(|f|_{\mathbb{A}}) = 48\mathcal{O}_\gamma(|f|)\mathcal{O}_\gamma(\mathbf{1}_{K_0}) \prod_{\mathfrak{p}\neq\mathfrak{p}_0} \mathcal{O}_\gamma(\mathbf{1}_{\mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})}).$$

By inequality (6.16) from the proof of Proposition 2.65 we deduce that

$$|\mathcal{O}_\gamma(\mathbf{1}_{\mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})}) \leq 3^e |\Delta(\gamma)|_\mathfrak{p}^{-1/2}, \tag{1.20}$$

where $e = 0$ if $|\Delta(\gamma)|_\mathfrak{p} = 1$ and $e = 1$ otherwise. Recall that $\Delta(\gamma)$ is the Weyl discriminant of γ and that $|\cdot|_\mathfrak{p}$ stands for the multiplicative $\mathfrak{p}$-adic valuation, normalized so that $|\pi|_\mathfrak{p} = N(\mathfrak{p})^{-1}$ where π is the uniformiser of $\mathcal{O}_{k_\mathfrak{p}}$.

We obviously have $\mathcal{O}_\gamma(\mathbf{1}_{K_0}) \leq \mathcal{O}_\gamma(\mathbf{1}_{\mathrm{PGL}(2,\mathcal{O}_{k_{\mathfrak{p}_0}})})$ so

$$\mathcal{O}_\gamma(|f|_{\mathbb{A}}) \leq 48 \cdot 3^N \prod_\mathfrak{p} |\Delta(\gamma)|_\mathfrak{p}^{-1/2} = \mathcal{O}_\gamma(|f|_{\mathbb{A}}) \leq 48 \cdot 3^N |N_{k/\mathbb{Q}}(\Delta(\gamma))|^{1/2},$$

where N is the number of primes $\mathfrak{p}$ for which $|\Delta(\gamma)|_\mathfrak{p} \neq 1$. If $\mathcal{O}_\gamma(|f|) \neq 0$ then by Lemma 2.58 $|N_{k/\mathbb{Q}}(\Delta(\gamma))| = \exp(o([k : \mathbb{Q}]))$. This is one of the key ways we use Bilu

[7] This is completed Artin L-function attached to the unique non-trivial character $\chi_{l/k}$ of $\mathrm{Gal}(l/k)$.

equidistribution theorem. Note that

$$3^N < 2^{2N} \leq \prod_{\mathfrak{p}} |\Delta(\gamma)|^{-2} = |N_{k/\mathbb{Q}}(\Delta(\gamma)|^2 = \exp(o([k:\mathbb{Q}])).$$

It remains to deal with the archimedean orbital integral. By Corollary 2.70 $\mathcal{O}_\gamma(|f|_{\mathbb{A}}) \ll [k:\mathbb{Q}]^4 = \exp(o([k:\mathbb{Q}]))$. Putting those estimates together we get

$$\mathcal{O}_\gamma(|f|_{\mathbb{A}}) \ll \exp(o([k:\mathbb{Q}])).$$

Step 6. We apply the bounds from the last three steps to (1.16):

$$|\mathrm{tr}R_\Gamma f - \mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{R})| \ll \frac{2}{|\mathrm{cl}\,(U_0)|}\frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}}\exp(o([k:\mathbb{Q}])) \ll \frac{|\Delta_k|^{1/2+2\varepsilon}}{|\mathrm{cl}\,(U_0)|(2\pi)^{[k:\mathbb{Q}]}}.$$

In the last inequality we have used Minkowski's lower bound on Δ_k which tells us that Δ_k grows exponentially in the degree $[k:\mathbb{Q}]$. On the other hand the volume is bounded from below by

$$\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{R})) \gg \frac{|\Delta_k|^{3/2}}{|\mathrm{cl}\,(U_0)|(4\pi^2)^{[k:\mathbb{Q}]}}.$$

We look for $a > 0$ such that

$$\frac{|\Delta_k|^{1/2+2\varepsilon}}{|\mathrm{cl}\,(U_0)|(2\pi)^{[k:\mathbb{Q}]}} \ll \left(\frac{|\Delta_k|^{3/2}}{|\mathrm{cl}\,(U_0)|(4\pi^2)^{[k:\mathbb{Q}]}}\right)^{1-a}.$$

$$\frac{(4\pi^2)^{(1-a)[k:\mathbb{Q}]}}{|\mathrm{cl}\,(U_0)|^a(2\pi)^{[k:\mathbb{Q}]}} \ll |\Delta_k|^{(1-a)3/2-1/2}.$$

Minkowski lower bound on the discriminant[8] yields $|\Delta_k| \gg e^{2[k:\mathbb{Q}]-\log[k:\mathbb{Q}]} \gg e^{1.99[k:\mathbb{Q}]}$ so it would be enough to take $a > 0$ such that

$$\left(\frac{(4\pi^2)^{(1-a)}}{(2\pi)}\right)^{[k:\mathbb{Q}]} \ll e^{1.99[k:\mathbb{Q}][(1-a)3/2-1/2]}.$$

If we take $a = 1/2$ the left hand side is equal to 1 while the right hand side equals $e^{1.99[k:\mathbb{Q}]/4}$. The Strong Limit Multiplicity for the baby case follows with $a = 1/2$.

1.6. Outline of the proofs.

1.6.1. *From Limit Multiplicity to Benjamini-Schramm convergence.* The Strong Benjamini-Schramm convergence will be deduced from the Strong Limit Multiplicity property in the Section 11. The proof is just an application of Strong Limit Multiplicity to the characteristic function of the ball of radius R around the identity. Section 11 contains also the proofs of Theorem 2.11 and Corollary 2.12.

1.6.2. *Gelander Conjecture.* We construct appropriate simplical complex as nerve of a covering of M by balls. We use Dobrowolski lower bound on Mahler measure [**40**] to control the injectivity radius in terms of the degree of the trace field and Theorem 2.11 to estimate the volume of the thin part of the manifold. Quantitative control over both allows to deduce Theorem 2.16. Proof occupies Section 12.

1.6.3. *Strong Limit Multiplicity.* The proof of Theorem 2.8 is divided into several steps.

(1) **Adelic Trace Formula.** The first step is to express the trace of $R_\Gamma f$ using the adelic Arthur-Selberg trace formula. For brevity write $G = \mathrm{PGL}(2,\mathbb{K})$. The ordinary Selberg trace formula yields the equality

$$\mathrm{tr}R_\Gamma f = \sum_{[\gamma]_\Gamma \subset \Gamma} \mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)dg.$$

We shall write $\mathcal{O}_\gamma(f)$ for the orbital integral $\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)dg$. The problem with this trace formula is that we sum orbital integrals over conjugacy classes

[8] Minkowski gives $\sqrt{\Delta_k} \geq \frac{n^n}{n!}\left(\frac{\pi}{4}\right)^{r_2}$ where $n = [k:\mathbb{Q}]$ and r_2 is the number of complex places of k. As the field k is totally real the desired bound follows from Stirling's approximation.

in Γ. A priori it is not clear how to parametrize conjugacy classes in Γ. The adelic version of the trace formula allows to replace Γ by the k-points of a certain algebraic group $\mathrm{PD}^\times$ and to sum over rational conjugacy classes in $\mathrm{PD}^\times(k)$. The latter are easy to classify, we do this in Section 3.4.

In Section 4 we recall the construction of congruence arithmetic lattices and develop a suitable trace formula (Theorem 2.50) that allows to express $\mathrm{tr}R_\Gamma$ as a weighted sum of orbital integrals. For example, if D is a quaternion algebra defined over $\mathbb{Q}$ which splits over $\mathbb{R}$, $\mathcal{O}$ is a maximal order in D, then the group of units of norm 1 denoted by $\Gamma = \mathcal{O}^1$ is an arithmetic lattice in $\mathrm{SL}(2,\mathbb{R})$. We can choose an open, compact subgroup U of $D^1(\mathbb{A}_f)$ such that $\Gamma = D^1(\mathbb{Q}) \cap U$. Write R for the right regular action of $\mathbb{D}^1(\mathbb{A})$ on $L^2(D^1(k)\backslash D^1(\mathbb{A}))$. The trace formula reads

$$\mathrm{tr}R_\Gamma f = \mathrm{tr}Rf_{\mathbb{A}} = \sum_{[\gamma]\in D^1(\mathbb{Q})} \mathrm{Vol}(D^1_\gamma(\mathbb{Q})\backslash D^1_\gamma(\mathbb{A})) \int_{D^1_\gamma(\mathbb{A})\backslash D^1(\mathbb{A})} f_{\mathbb{A}}(x^{-1}\gamma x) d\mu_{\mathbb{A}}(x), \quad (1.21)$$

where $f_{\mathbb{A}} \in C_c^\infty(D^1(\mathbb{A}))$ is a tensor product of f and the characteristic function of U and $\mu_{\mathbb{A}}$ is an appropriate measure $D^1(\mathbb{A})$. To get the first equality one has to follow the argument from the proof of Lemma 2.42 plus the fact that D^1 satisfies the strong approximation property. A similar result holds for congruence subgroups which are derived from orders of quaternion algebras over number fields in the sense of [**94**].

Unfortunately not all congruence lattices can be constructed this way i.e. using simply connected algebraic groups (by the work of Borel [**21**] we know that each commensurability class of arithmetic lattices in $\mathrm{SL}(2,\mathbb{C})$ contains infinitely many maximal elements while the construction forom [**94**] provides only finitely many candidates for maximal lattices). Not all of them can be constructed as groups $\mathcal{O}^1$ for an order in a quaternion algebra. To construct and parametrize all congruence lattices we switch to the projective groups $\mathrm{PD}^\times$. For an admissible quaternion algebra D defined over a number field k the congruence lattices are obtained as intersections $F_V = \mathrm{PD}^\times(k) \cap V$ where V is an open compact subgroup of $\mathrm{PD}^\times(\mathbb{A}_f)$. The details of the construction are summarized in Sections 4.1, 4.2 and 4.3.

As we see in Lemma 2.42 the formula (1.21) does not hold for $\mathrm{PD}^\times$ and the adelic trace is a sum of traces $R_{\Gamma'_V} f$ where Γ'_V are lattices from the same packet (cf. Definition 2.34) as Γ_V. To isolate the trace of $R_{\Gamma_V} f$ we introduce twisted operators $R^\chi f_{\mathbb{A}}$ (cf. Definition 2.43). Using Fourier inversion on the class group of D (cf. Definitions 2.35 and 2.48) we express the trace of $R_{\Gamma_V} f$ as a linear combination of traces $\mathrm{tr}R^\chi f_{\mathbb{A}}$. This occupies Sections 4.4 and 4.5 and the final result is Theorem 2.50:

$$\mathrm{tr}R_{\Gamma_V} f = \sum_{[\gamma]_{\mathrm{PD}^\times(k)}} \mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash \mathrm{PD}^\times_\gamma(\mathbb{A})) \Xi^V_\gamma(f_{\mathbb{A}}), \quad (1.22)$$

where $\Xi^V_\gamma(f_{\mathbb{A}})$ is a combination of twisted orbital integrals defined in the statement of Theorem 2.50. Using basic Class Field Theory (Lemma 2.52) we get the following estimate (Corollary 2.54)

$$|\Xi^V_\gamma(h)| \leq \frac{2}{|\mathrm{cl}\,(V)|} |\mathcal{O}_\gamma(h)|, \quad (1.23)$$

for any smooth compactly supported function $h \in C_c^\infty(\mathrm{PD}^\times(\mathbb{A}))$ and regular semisimple $\gamma \in \mathrm{PD}^\times(k)$. The class group $\mathrm{cl}\,(V)$ is defined in Definition 2.48.

(2) **Normalization.** This step is crucial if we want to treat all congruence subgroups Γ_V not just the maximal ones. We fix a maximal compact subgroup U of $\mathrm{PD}^\times(\mathbb{A})$ containing V. To prove the Strong Limit Multiplicity we will have to bound the orbital integrals $\mathcal{O}_\gamma(f_{\mathbb{A}})$ appearing in the trace formula. The function $f_{\mathbb{A}}$ is given by a tensor product $f_{\mathbb{A}} = f_{\mathbb{A}_\infty} \otimes [U : V]\mathbb{1}_V$. We need to reprove the Limit

Multiplicity analogue of Theorem 2.3 for $\Gamma_0 = \Gamma_U$ with constants depending explicitly on D and U. The first step is to observe that the orbital integrals $\mathcal{O}_\gamma(f_{\mathbb{A}})$ are linear and invariant under conjugation by U. This allows us to replace $[U:V]\mathbb{1}_V$ by $\chi_{\mathrm{Ind}\,_V^U \mathbb{1}}$. Next, using Frobenius reciprocity, we get

$$\chi_{\mathrm{Ind}\,_V^U \mathbb{1}} = \sum_{\rho \in \mathrm{Irr} U} \langle \mathrm{Ind}\,_V^U \mathbb{1}, \rho \rangle \chi_\rho.$$

Recall that $\langle \mathrm{Ind}\,_V^U \mathbb{1}, \rho \rangle = \dim W_\rho^V$ where W_ρ is the space on which ρ acts and W_ρ^V is the subspace fixed by V. We have (cf. Lemma 2.55 and Corollary 2.54)

$$|\Xi_\gamma^V(f_{\mathbb{A}})| \leq \frac{2|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})|}{|\mathrm{cl}\,(V)|} \sum_{\rho \in \mathrm{Irr}(U)} \dim W_\rho^V |\mathcal{O}_\gamma(\chi_\rho)|$$

By the trace formula (Theorem 2.50) we get

$$|\mathrm{tr} R_{\Gamma_V} f - \mathrm{Vol}(\Gamma_V \backslash \mathrm{PGL}(2,\mathbb{K})) f(1)| \leq \sum_{1 \neq [\gamma] \in \mathrm{PD}^\times(k)} \mathrm{Vol}(\mathrm{PD}_\gamma^\times(k) \backslash \mathrm{PD}_\gamma^\times(\mathbb{A})) \frac{2|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})|}{|\mathrm{cl}\,(V)|} \sum_{\rho \in \mathrm{Irr}(U)} \dim W_\rho^V |\mathcal{O}_\gamma(\chi_\rho)|.$$

After reversing the order of summation the right hand side reads

$$\sum_{\rho \in \mathrm{Irr}(U)} \dim W_\rho^V \left[\sum_{1 \neq [\gamma] \in \mathrm{PD}^\times(k)} \mathrm{Vol}(\mathrm{PD}_\gamma^\times(k) \backslash \mathrm{PD}_\gamma^\times(\mathbb{A})) \frac{2|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})|}{|\mathrm{cl}\,(V)|} |\mathcal{O}_\gamma(\chi_\rho)| \right].$$

The passage from $[U:V]\mathbb{1}_V$ to a sum of irreducible characters will be referred to as *normalization*. Our next step is bounding the orbital integrals of irreducible characters appearing on the right side. To this end we first extend the character bounds of Larsen **[71]** (see Section 6.1) to the case of p-adic analytic groups and deduce that the orbital integral of χ is bounded by $\chi(1)^{1-\delta}$ for some absolute $\delta > 0$ (see Section 6.5). Given such bound we can use the special representation zeta function and Lemma 2.83 to bound the sum of orbital integrals over all irreducible characters. After that we will estimate the adelic volumes and the number of conjugacy classes with nontrivial contribution.

(3) **Orbital integrals.** Recall that U was a maximal open compact subgroup of $\mathrm{PD}^\times(\mathbb{A}_f)$. In Section 6 we give an upper bound on $|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})||\mathcal{O}_\gamma(\chi_\rho)|$ for an irreducible representation ρ of U. Since $U = \prod_{\mathfrak{p} \in M_k^f} U_\mathfrak{p}$ the character χ_ρ can be written as $\chi_\rho = \bigotimes_{\mathfrak{p} \in M_k^f} \chi_\rho^\mathfrak{p}$ where $\chi_\rho^\mathfrak{p}$ are irreducible characters of $U_\mathfrak{p}$. We have

$$|\mathcal{O}_\gamma(\chi_\rho)| = \prod_{\mathfrak{p} \in M_k^f} |\mathcal{O}_\gamma(\chi_\rho^\mathfrak{p})|.$$

The problem is now reduced to estimates on the local orbital integrals. This is one of the advantages of normalization since, in general, the characteristic function of a subgroup V of U does not admit a factorization over finite places. For a finite place $\mathfrak{p}$ we show in Proposition 2.65 that

$$|\mathcal{O}_\gamma(\chi_\rho^\mathfrak{p})| \ll |\Delta(\gamma)|_\mathfrak{p}^{-3/2}$$

where $\Delta(\gamma)$ is the Weyl discriminant of γ (see Notations 2.2) and the implicit constant is 1 for all but finitely many places. In the actual proof we have to control the precise value of the constant in terms of γ because we have to multiply this inequality over all finite places. After doing so we get that for δ sufficiently small we have (Proposition 2.74):

$$|\mathcal{O}_\gamma(\chi_\rho)| \ll 25^{[k:\mathbb{Q}]} 8^{\delta |\mathrm{Ram}^f(D)|} \chi_\rho(1)^{1-\delta},$$

if γ is torsion and (Proposition 2.75):

$$|\mathcal{O}_\gamma(\chi_\rho)| \ll \exp(o([k:\mathbb{Q}]))2^{\delta|\mathrm{Ram}^f D|}\chi_\rho(1)^{1-\delta}$$

if γ has infinite order. In the non torsion case we rely crucially on Bilu's Equidistribution theorem (see Theorem 2.56 and Section 5).

The archimedean orbital integrals are estimated in Section 6.3. Recall that we work with the assumption that f is supported on a ball of radius R. In this case we show that the integral is bounded by

$$|\mathcal{O}_\gamma(f)| \ll_R [k:\mathbb{Q}]^2\|f\|_\infty,$$

if γ is hyperbolic and

$$|\mathcal{O}_\gamma(f)| \ll_R [k:\mathbb{Q}]^4\|f\|_\infty,$$

if γ is elliptic. Their contribution is polynomial in the degree of k, which will turn out to be negligible. Combining these two we get that when γ is non-torsion then

$$|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})||\mathcal{O}_\gamma(\chi_\rho)| \ll_R \exp(o([k:\mathbb{Q}]))2^{\delta|\mathrm{Ram}^f D|}\chi_\rho(1)^{1-\delta}.$$

(4) **An estimate on the adelic volumes.** To control the size of $\mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A}))$ we show in Proposition 2.93 that

$$\mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A})) \leq \Lambda(1,\chi_{l/k})$$

where l is the quadratic extension of k generated by γ in D, $\chi_{l/k}$ is the unique nontrivial character of the Galois group $\mathrm{Gal}(l/k)$ and $\Lambda(s,\chi_{l/k})$ is the completed Artin L-function associated to $\chi_{l/k}$. This is probably well known to experts. We give a self contained proof using periods of Eisenstein series. Next, using Bilu's equidistribution theorem and the maximum principle, we show in Proposition 2.95 that when γ is not torsion and $\mathcal{O}_\gamma(f) \neq 0$ then for any $\varepsilon > 0$ we have

$$\mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A})) \ll_\varepsilon \exp(o([k:\mathbb{Q}]))\frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}}.$$

(5) **Number of conjugacy classes.** The estimates on orbital integrals and the adelic volume allow us to give a uniform bound on the contribution of a single conjugacy class. Now we need to bound the number of classes with non-trivial contribution. As we show in Section 3.4 the conjugacy classes are either 2-torsion or they are determined by their eigenvalues. By Lemma 2.69 from Section 6.3 the eigenvalues of non-torsion elements are Salem numbers. Then one can use Bilu equidistribution theorem and some geometric arguments to show that the number of classes with nontrivial contribution is of order $\exp(o([k:\mathbb{Q}]))$.

(6) **Conclusion.** Putting the three last steps together we get that

$$\sum_{\substack{[\gamma]\in\mathrm{PD}^\times(k)\\ \text{torsion free}}} \mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A}))2|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})||\mathcal{O}_\gamma(\chi_\rho)| \ll \tag{1.24}$$

$$\|f\|_\infty \exp(o([k:\mathbb{Q}]))\frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}}2^{\delta|\mathrm{Ram}^f D|}\chi_\rho(1)^{1-\delta}. \tag{1.25}$$

From which it follows that for a torsion free lattice Γ_V we have

$$|\mathrm{tr}R_{\Gamma_V}f - \mathrm{Vol}(\Gamma_V\backslash\mathrm{PGL}(2,\mathbb{K}))f(1)| \ll \tag{1.26}$$

$$\frac{\|f\|_\infty}{|\mathrm{cl}\,(V)|}\sum_{\rho\in\mathrm{Irr}U}\dim W_\rho^V\left[\exp(o([k:\mathbb{Q}]))\frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}}2^{\delta|\mathrm{Ram}^f D|}\chi_\rho(1)^{1-\delta}\right]. \tag{1.27}$$

Using a variant of Lemma 2.83 we deduce that there exist positive constants α and b (depending only on δ) such that

$$|\mathrm{tr}R_{\Gamma_V} f - \mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K}))f(1)| \ll \tag{1.28}$$

$$\|f\|_\infty \exp(o([k:\mathbb{Q}]))\frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}} 2^{\delta|\mathrm{Ram}^f D|}\zeta_U^*(b)^\alpha \frac{[U:V]^{1-\alpha}}{|\mathrm{cl}\,(V)|^{1-\alpha}}, \tag{1.29}$$

where ζ_U^* stands for the special representation zeta function of U. Note that in this bound, the only factor that depends on V is $\frac{[U:V]^{1-\alpha}}{|\mathrm{cl}\,(V)|^{1-\alpha}}$. By comparing the estimate with the Borel Volume formula for maximal lattices (see [**94**, Chapter 11] and Section 9), Corollary 2.51 and a lower bound on discriminants due to Odlyzko ([**85**]) we get the inequality

$$|\mathrm{tr}R_{\Gamma_V} f - \mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K}))f(1)| \ll \|f\|_\infty \mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K}))^{1-\alpha}.$$

This concludes the proof of Strong Limit Multiplicity for torsion free congruence lattices in $\mathrm{PGL}(2,\mathbb{K})$.

1.6.4. *Short geodesics and their consequences.* Let Γ be an arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$ and let X be the symmetric space of $\mathrm{PGL}(2,\mathbb{K})$. One of the key aspects of our proof is exploiting the properties of the trace field forced by the presence of short closed geodesics in $\Gamma\backslash X$. By short we mean shorter that some fixed positive constant R. Both the Strong Limit Multilicity property and Strong Benjamini-Schramm convergence hold trivially for compact quotients if there are no short closed geodesics so in our argument we may assume that there are short geodesics on $\Gamma\backslash X$. A primitive closed geodesic on $\Gamma\backslash X$ of length ℓ corresponds to a unique conjugacy class $[\gamma]_\Gamma$ such that $\mathrm{Ad}\,\gamma$ has an eingenvalue λ with $\log|\lambda| = \ell$. Lemma 2.69 tells us then that the logarithmic Mahler measure of λ is bounded by ℓ. Eigenvalue λ generates a quadratic extension over k so form the presence of short geodesics we can infer that k has quadratic extensions containing numbers of small logarithmic Mahler measure. Using the machinery developed in Section 5 we extract nontrivial information on the distribution of prime ideals of small norm in $\mathcal{O}_k$ (c.f. Theorem 2.59 and Corollary 2.60). This information is put to use in Section 10 where the volume formulas and our bounds are very sensitive to the presence of ideals of small norm in $\mathcal{O}_k$.

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2. Notation

2.1. Analysis. We shall use Vinogradov notation. If f, g are two functions dependent, among others on a variable X we write $f \ll_X g$ if there exists a constant C dependent on X such that $f \leq Cg$ similarly we write $f \gg_X g$ if the opposite inequality is true. We will write $f = o_X(g)$ if $\lim \frac{f}{g} = 0$ and the speed of convergence depends on X. Logarithms are always in base e. We write $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ for the fields of rational, real and complex numbers respectively. Throughout the text $\mathbb{K}$ will mean either $\mathbb{R}$ or $\mathbb{C}$. For any function f we write $\|f\|_\infty$ for the supremum norm and $\|f\|_{L^p}$ for the L^p norm whenever the latter can be defined. If z is a complex number we write $|z|$ for its modulus.

2.2. Groups. Let G be a group acting on a set X. For a subset $S \in G$ we write X^S for the set of points fixed by S, Gx for the orbit of G containing x and $\mathrm{Stab}_G x$ for the stabilizer of x. If H is a subgroup of G and $\gamma \in G$ we write $[\gamma]_H$ for the H-conjugacy class of γ. If $H = G$ we may omit the subscript and write $[\gamma]$. For two elements $x, y \in G$ we shall write $x \sim_H y$ if $y \in [x]_H$.

If G acts by automorphisms on the group X then we write $H^1(G, X)$ for the first cohomology set and if X is abelian we write $H^i(G, X)$ for the i-th cohomology group.

If G is a Lie group defined over a field F we shall write $\mathfrak{g}$ for its Lie algebra and $\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g})$ for the adjoint action. We write $X^*(G)$ for the group of characters of G i.e. homomorphisms to F^*. If G is semisimple and T is a maximal torus of G we write $\Phi(G,T)$ for the set of roots of G with respect to T. For a semisimple element $\gamma \in G$ we define the Weyl discriminant $\Delta(\gamma)$ of γ as 0 if γ is not regular semisimple and otherwise as

$$\Delta(\gamma) = \prod_{\lambda \in \Phi(G,T)} (1 - \lambda(\gamma)),$$

where T is the maximal torus containing γ.

2.3. Number Theory. Throughout the text, the letter k will usually mean a number field. For a given number field k we write

- $\mathcal{O}_k$ for the ring of integers, M_k for the set places of k, M_k^f for the set of finite places and M_k^∞ for the set of infinite (archimedean) places, for any finite set $S \in M_k$ containing M_k^∞ write $\mathcal{O}_{k,S}$ for the ring of S-integers;
- For an ideal $\mathfrak{a}$ in $\mathcal{O}_k$ we write $N(\mathfrak{a})$ for the norm of $\mathfrak{a}$. It is defined as the cardinality of $\mathcal{O}_k/\mathfrak{a}$;
- Usually we shall use letters ν, ω to denote infinite or general places and $\mathfrak{p}$ is reserved for finite places. We identify the set of finite places with prime ideals of $\mathcal{O}_k$ and write $q = N(\mathfrak{p})$ for the cardinality of the residue field $\mathcal{O}_k/\mathfrak{p}$;
- For every $\nu \in M_k$ we write k_ν for the completion of k with respect to ν. For $x \in k_\nu$ we write $|x|_\nu$ for the valuation of x. The $\mathfrak{p}$-adic valuation is normalized so that $|\pi|_\mathfrak{p} = q^{-1}$ for the uniformizer π;
- Let l/k be a finite extension. We write $N_{l/k}: l \to k$ for the norm and $\mathrm{tr}_{l/k}: l \to k$ for the trace of the extension l/k;
- Write Δ_k for the discriminant of k and $\Delta_{l/k}$ for the relative discriminant of extension l/k. We have $\Delta_l = \Delta_k^{[l:k]} N_{k/\mathbb{Q}}(\Delta_{l/k})$.
- For a non-archimedean local field F which is an extension of $\mathbb{Q}_p$ of degree d we write Δ_F for the ideal

  $$\langle \det((\mathrm{tr}_{F/\mathbb{Q}_p} x_i x_j)_{ij}) \mid x_1, \dots, x_d \in \mathcal{O}_F \rangle.$$

  We have

  $$|\Delta_k| = \prod_{\mathfrak{p} \in M_k^f} |\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^{-1}.$$

- We write $\mathbb{A}_k$ for the ring of adeles of k. For most time we work with a single number field k so we omit the subscript and write $\mathbb{A}$ instead of $\mathbb{A}_k$;
- If S, F are subsets of M_k such that $M_k^\infty \subset S$ we write $\mathbb{A}_F^S$ for the ring of S-integral F-adeles which is defined as

  $$\mathbb{A}_F^S = \prod_{\nu \in S \cap F}^{*} k_\nu \times \prod_{\nu \in F \setminus S} \mathcal{O}_{k_\nu}, \tag{2.1}$$

  where * means that almost all coordinates are integers. When using this convention we replace M_k^∞ with symbol ∞ and M_k^f with f. For example $\mathbb{A}_f$ is the ring of finite adeles, $\mathbb{A}_\infty$ is the product of archimedean completions of k and $\mathbb{A}_f^\infty$ is the ring of finite adeles integral on all coordinates;
- For a Galois extension l/k we write $\mathrm{Gal}(l/k)$ for the Galois group. Usually we will denote the nontrivial elements of the Galois group by σ and write x^σ for the result of acting by σ on $x \in l$;
- For a number field l we denote the Dedekind Zeta function of l by $\zeta_l(s)$ and completed Dedekind Zeta function by $\xi_l(s)$;
- For a character χ of $\mathrm{Gal}(\overline{k}/k)$ we write $L(s,\chi)$ for the associated Artin L-function, $\mathfrak{f}_\chi$ for the conductor of χ and $\Lambda(s,\chi)$ for the completed L-function. We will recall their definitions in Section 9 devoted to volumes of adelic quotients;

- Whenever D is a quaternion algebra over the field k we write $k_D^\times$ for $n(D^\times) \subset k^\times$ and $\mathbb{A}_D^\times$ for $n(D^\times(\mathbb{A})) \subset \mathbb{A}^\times$. Recall that n, tr stand for the reduced norm and the reduced trace respectively (see Section 3.2).
- Let D be a quaternion algebra defined over a non-archimedean local field $F/\mathbb{Q}_p$. Choose a maximal order $\mathcal{O}$ in D. We define $\Delta_{D/F}$ as the ideal $\langle \det((\mathrm{tr}(x_i x_j))_{ij}) \mid x_1, \ldots, x_4 \in \mathcal{O} \rangle$. As all maximal orders in D are conjugate [**94**, Theorems 6.4.1, 6.5.3] the definition does not depend on the choice of $\mathcal{O}$.

2.4. Volume conventions. Let X be a topological space with a measure μ and let Γ be a group acting properly discontinuously on X and preserving μ. We write $\mathrm{Vol}_\mu(\Gamma\backslash X)$ for the measure of a fundamental domain of Γ. When X is a Riemannian manifold we usually take μ to be the volume form on X in which case we omit the subscript μ and write $\mathrm{Vol}(\Gamma\backslash X)$ for the Riemannian volume of $\Gamma\backslash X$.

Let G be a reductive algebraic group defined over a local field F. We shall define a canonical Haar measure on G which will be called the standard measure of G. If F is archimedean then this measure will be defined uniquely while in the non-archimedean case it will depend on the choice of a maximal compact subgroup. Let us start with the archimedean semisimple case.

Definition 2.19. *Choose a maximal compact subgroup K of G. Let $X = G/K$ be the symmetric space equipped with the left G-invariant Riemannian metric associated to the Killing form of G (see [**59**]). Write dx for the volume form on X and dk for the normalized Haar measure on K. We define the standard Haar measure on G as the unique measure for which*

$$\int f d\mu(g) = \int_X \int_K f(xk) dk dx \text{ for any } f \in C_c(G) \tag{2.2}$$

As all maximal compact subgroups are conjugate this definition does not depend on the choice of K.

When G is reductive let $H = [G^o, G^o]$ be the maximal connected semisimple subgroup of G. The connected component $T_1 := (G/H)^o$ of $T_0 := G/H$ is isomorphic to $(\mathbb{R}_+^\times)^a \times (\mathbb{C}^\times)^b$. We endow T_1 with a Haar measure

$$dt = \prod_{i=1}^{a} \frac{dt_i}{t_i} \times \prod_{j=1}^{b} \frac{dx_i dy_i}{2\pi(x_i^2 + y_i^2)^{1/2}}$$

and extend it to T_0 by putting $dt/[T_0 : T_1]$ on each connected component. Now we are ready to define the standard measure on arbitrary reductive algebraic group over an archimedean field.

Definition 2.20. *The standard measure on G is defined as the unique measure dg for which*

$$\int f d\mu(g) = \int_{T_0} \int_H f(th) dh dt \text{ for any } f \in C_c(G), \tag{2.3}$$

where dh stands for the standard measure on H.

In the reductive non-archimedean case the definition is analogous but a bit simpler because maximal compact subgroups are open.

Definition 2.21. *Choose a maximal compact subgroup U of G. The quotient $X = G/U$ is a discrete space. Write dx for the counting measure on X and dk for the normalized Haar measure on U i.e. the one for which $\mathrm{Vol}(U) = 1$. We define the standard Haar measure on G as the unique measure for which*

$$\int f d\mu(g) = \int_X \int_K f(xk) dk dx \text{ for any } f \in C_c(G) \tag{2.4}$$

*The definition depends on the choice of maximal compact subgroup. There are finitely many conjugacy classes of maximal compact subgroups in G (cf. [**89**, Chapter 3.4]).*

The definition of the standard measure extends to the adele groups. Let k be a number field and let G be a reductive algebraic group defined over k. The group $G(\mathbb{A})$ is a locally compact group. Let us fix a maximal compact subgroup U of the group of finite adeles $G(\mathbb{A}_f)$. By maximality U decomposes as a product $U = \prod_{\mathfrak{p}\in M_k^f} U_\mathfrak{p}$ with $U_\mathfrak{p}$ maximal in $G(k_\mathfrak{p})$. Then the standard measure on $G(\mathbb{A})$ is defined as $\bigotimes_{\nu\in M_k} \mu_\nu$ where μ_ν is the standard Haar measure on $G(k_\nu)$.

2.5. Representation Theory. For any topological group G, write Irr$G$ for the set of equivalence classes of irreducible, continuous unitary representations of G. Whenever ρ is a unitary representation of G, we shall write W_ρ for the underlying Hilbert space. For a finite dimensional representation, we write χ_ρ for the character of ρ. For any closed subgroup H of G and representations ρ_1, ρ_2 of G, we write $\langle \rho_1, \rho_2\rangle = \dim_{\mathbb{C}} \mathrm{Hom}_H(\rho_1, \rho_2)$. If ρ_1, ρ_2 are finite dimensional then we also write $\langle \chi_{\rho_1}, \chi_{\rho_2}\rangle = \langle \rho_1, \rho_2\rangle$.

3. Preliminaries on quaternion algebras

3.1. Quaternion algebra. Throughout this section let F be an arbitrary field of characteristic different than 2.

Definition 2.22. *An associative unital algebra A over F is called a quaternion algebra if it is 4 dimensional and there exist $\mathbf{i}, \mathbf{j}, \mathbf{k} \in A$ such that:*

- $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ *is a basis of A over F;*
- $\mathbf{i^2}, \mathbf{j^2}, \mathbf{k^2} \in F^\times$;
- $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$.

If A satisfies the above conditions the values $\mathbf{i}^2 = a, \mathbf{j}^2 = b$ determine A up to F-isomorphism and we write $A = \left(\frac{a,b}{F}\right)$.

Let E be a an extension of F, we say that a quaternion algebra A splits over E if $A \otimes_F E \simeq M(2, E)$. If A splits over F we just say that it splits.

Proposition 2.23. *[**94**, Chapter 2] The following conditions are equivalent:*

1. *A quaternion algebra $\left(\frac{a,b}{F}\right)$ splits;*
2. *the quadratic form $x^2 - ay^2 - bz^2 + abt^2$ is isotropic;*
3. *$-ay^2 - bz^2 - abt^2$ is isotropic.*

In particular every quaternion algebra A over F splits over the algebraic closure $\overline{F}$.

3.2. Norm, trace and involution. Let $A = \left(\frac{a,b}{f}\right)$ be a quaternion algebra over F. Let $x = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4$. The **standard involution** on A is defined as

$$\overline{x} = x_1 - \mathbf{i}x_2 - \mathbf{j}x_3 - \mathbf{k}x_4.$$

The **trace** on A is defined as

$$\mathrm{tr}(x) = x + \overline{x} = 2x_1.$$

The **norm** on A is defined as

$$n(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) = x\overline{x} = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2.$$

For any element $x \in A$ we have $x^2 - \mathrm{tr}(x)x + n(x) = 0$. In particular if $F(x)$ is a subfield of A then $n(x) = N_{F(x)/F}(x)$ and $\mathrm{tr}(x) = \mathrm{Tr}_{F(x)/F}(x)$. An element $x \in A$ is invertible if and only if $n(x) \neq 0$. If that is the case, we have $x^{-1} = \frac{\overline{x}}{n(x)}$.

3.3. Quaternion algebras and algebraic groups. Given a quaternion algebra A we can associate to it three linear algebraic groups:

$$A^\times = \{x \in A | n(x) \neq 0\} \tag{3.1}$$

$$A^1 = \{x \in A | n(x) = 1\} \tag{3.2}$$

$$PA^\times = A^\times / Z(A^\times) = A^\times / \mathbb{G}_m \tag{3.3}$$

If A is split then they are isomorphic to $GL(2,F), SL(2,F), PGL(2,F)$ respectively. In particular the last two are simple algebraic groups of type A_1. Recall that we write G_γ for the centralizer of γ in G.

LEMMA 2.24. *Let γ be a non-central, semisimple element of $PA^\times(F)$ and let $\tilde{\gamma}$ be one of its preimages in $A^\times$ and let ω_1, ω_2 be the roots of the polynomial $X^2 - \mathrm{tr}(\tilde{\gamma})X + n(\tilde{\gamma})$. If $\omega_1, \omega_2 \notin F$ put $E = F(\omega_1)$. Then*

1. *$F[\gamma] \simeq F^2$ if $\omega_1, \omega_2 \in F$ and $F[\gamma] \simeq F[\omega]$ otherwise;*
2. *$PA^\times_\gamma \simeq \mathbb{G}_m$ if $\omega_1, \omega_2 \in F$ and $PA^\times_\gamma \simeq \mathrm{Res}^1_{E/F}\mathbb{G}_m$ otherwise;*
3. *$PA^\times_\gamma$ is a maximal torus of $PA^\times$. If λ_1, λ_2 are the roots of $PA^\times$ with respect to $PA^\times_\gamma$ then up to a change of enumeration $\lambda_1(\gamma) = \frac{\omega_1}{\omega_2}$ and $\lambda_2(\gamma) = \frac{\omega_2}{\omega_1}$;*
4. *The Weyl discriminant of γ is given by*

$$\Delta(\gamma) = (1 - \frac{\omega_1}{\omega_2})(1 - \frac{\omega_2}{\omega_1})$$

The same statement holds for $\gamma \in A^1(F)$.

PROOF. (1) follows from the Cayley-Hamilton theorem. For (2) observe that $F[\tilde{\gamma}]$ is the centralizer of $\tilde{\gamma}$ in A so on the level of F-points we have $PA^\times_\gamma \simeq F[\tilde{\gamma}]^\times / F^\times$. If $\omega_1, \omega_2 \in F$ then by (1) $PA^\times_\gamma \simeq \mathbb{G}_m \times \mathbb{G}_m / \Delta\mathbb{G}_m \simeq \mathbb{G}_m$, where Δ is the diagonal embedding. If $E = F[\tilde{\gamma}]$ is a quadratic extension of F then $PA^\times_\gamma \simeq \mathrm{Res}_{E/F}\mathbb{G}_m / \mathbb{G}_m$. By Hilbert's Theorem 90 the latter is isomorphic to $\mathrm{Res}^1_{E/F}\mathbb{G}_m$ via the map $x \mapsto x^{-1}x^\sigma$ where σ is the generator of $\mathrm{Gal}(E/F)$. For (3) recall that $\lambda_1(\gamma), \lambda_2(\gamma)$ are the nontrivial eigenvalues of $\mathrm{Ad}\,\gamma$. (4) follows directly from (3) and the definition of the Weyl discriminant (see Notations 2.2). □

3.4. Conjugacy classes.

THEOREM 2.25 (Skolem-Noether). *Let A be a central simple algebra over F and B a simple algebra over F. Any two nontrivial F-homomorphisms $\phi_1 \colon B \to A$ and $\phi_2 \colon B \to A$ are conjugate by an element of $A^\times$*

From this we can deduce:

PROPOSITION 2.26. *Let A be a non split quaternion algebra over a field F. Let $\gamma \in PA^\times \setminus \{1\}$ and let λ be one of the non-trivial eigenvalues of $\mathrm{Ad}\,\gamma$.*

1. *γ and γ^{-1} are conjugate in $PA^\times$;*
2. *γ is 2-torsion if and only if $\lambda = -1$. Otherwise $\lambda \notin F^\times$ and the values of λ, λ^{-1} determine the $PA^\times$-conjugacy class of γ;*
3. *Write $n \colon PA^\times \to F^\times/(F^\times)^2$ for the map induced by the norm. The map n induces an injective map from the set of 2-torsion conjugacy classes in $PA^\times$ to $n(A^\times)/(F^\times)^2$. Moreover for any lift $\tilde{\gamma}$ of γ we have $F[\tilde{\gamma}] \simeq F[\sqrt{-n(\gamma)}]$.*

PROOF. (1) Let $\tilde{\gamma}$ be a lift of γ to $A^\times$ and $E = F(\tilde{\gamma})$ be the subfield of A generated by $\tilde{\gamma}$. It is a quadratic extension of F. Let σ be the generator of $\mathrm{Gal}(E/F)$, it acts on E by an F-automorphism so by Skolem-Noether theorem there exists an $a \in A^\times$ such that $x^\sigma = axa^{-1}$ for all $x \in E$. In particular

$$\tilde{\gamma}a\tilde{\gamma}a^{-1} = \tilde{\gamma}\tilde{\gamma}^\sigma = n(\tilde{\gamma}) \in F^\times.$$

But this means that $a\gamma a^{-1} = \gamma^{-1}$.

(2) If $\lambda = -1$ then the eigenvalues of γ^2 are both equal to 1 so γ is 2-torsion. Conversely, if $\gamma^2 = 1$ then $\lambda = \lambda^{-1} = -1$. Assume that γ is not 2-torsion. We claim that $F(\lambda)$ is a quadratic extension of F. Choose an element $\tilde{\gamma} \in A^\times$ lifting γ. Write ω_1, ω_2 for the roots of the characteristic polynomial of $\tilde{\gamma}$. We have $\omega_1 = \lambda\omega_2$ and

$$(1+\lambda)\omega_2 = \omega_1 + \omega_2 = \mathrm{tr}\tilde{\gamma} \in F.$$

Since γ is not 2-torsion $\lambda+1 \neq 0$. If $\lambda \in F$ then $\omega_1, \omega_2 \in F^\times$. This cannot happen unless A is split. Thus, we have $\lambda \notin F$. We have $\lambda \in F(\omega_1)$ so by comparing degrees we get $F(\lambda) = F(\omega_1) \simeq F(\gamma) \subset A$.

Now we prove that if $\lambda \neq -1$ then it determines uniquely the conjugacy class of γ in $PA^\times$. Let $\gamma_1, \gamma_2 \in PA^\times$ be elements with same eigenvalues λ, λ^{-1}, different than -1. Let $\tilde{\gamma}_1, \tilde{\gamma}_1$ be their lifts to $A^\times$. We have shown that $F(\tilde{\gamma}_1) \simeq F(\tilde{\gamma}_2) \simeq F(\lambda)$. Thus by the Skolem-Noether theorem (Thm 2.25) we find $a \in A^\times$ such that $a\tilde{\gamma}_2 a^{-1} \in F(\tilde{\gamma}_1)$. We can replace γ_2 by $a\gamma_2 a^{-1}$. The equality of eigenvalues and a simple computation using characteristic polynomials implies that either $\gamma_1 = \gamma_2$ or $\gamma_1 = \gamma_2^{-1}$. By the point (1) of present proposition γ_1 and γ_2 are conjugate.

(3) Let γ be a 2-torsion element in $PA^\times$. We claim that $n(\gamma) \in F^\times/(F^\times)^2$ determines uniquely the $PA^\times$ conjugacy class of γ. Let $\tilde{\gamma}$ be a lift of γ to $A^\times$ and $E = F(\tilde{\gamma})$ be the subfield of A generated by $\tilde{\gamma}$. It is a quadratic extension of F. Let σ be the generator of $\mathrm{Gal}(E/F)$. We have $(\tilde{\gamma}^\sigma/\tilde{\gamma})^2 = 1$ so $\tilde{\gamma}^\sigma = -\tilde{\gamma}$, otherwise $\tilde{\gamma}$ would be in $F^\times$. Thus $n(\tilde{\gamma}) = -\tilde{\gamma}^2$ and consequently $E \simeq F(\sqrt{-n(\tilde{\gamma})})$. The isomorphism class of E depends only on the class of $n(\tilde{\gamma})$ modulo $(F^\times)^2$ so it is determined uniquely by the value of $n(\gamma)$. If $\gamma_1, \gamma_2 \in PA^\times$ are 2-torsion and $n(\gamma_1) = n(\gamma_2)$ then $F(\tilde{\gamma}_1) \simeq F(\tilde{\gamma}_2)$ so by Skolem-Noether $\tilde{\gamma}_2 \in F(\tilde{\gamma}_1)$ up to conjugation. But then $n(\gamma_1) = n(\gamma_2)$ implies that $\tilde{\gamma}_1^2 \equiv \tilde{\gamma}_1^2$ modulo $(F^\times)^2$ so $\gamma_1 = \gamma_2$.

□

3.5. Quaternion algebras over local fields.

PROPOSITION 2.27. *Let F be a local field. If F is non-archimedean or $\mathbb{R}$ there exist exactly two isomorphism classes of quaternion algebras over F. If F is non-archimedean there exists a unique unramified quadratic extension E/F.Let π be the uniformizer of F and σ the nontrivial Galois automorphism of E/F. The unique division quaternion algebra over F is given by*

$$\begin{pmatrix} x & y \\ \pi y^\sigma & x^\sigma \end{pmatrix} \text{ where } x, y \in E$$

If $F = \mathbb{R}$ then the unique division quaternion algebra is the Hamilton quaternion algebra

$$\mathbf{H} = \mathbb{R} + \mathbf{i}\mathbb{R} + \mathbf{j}\mathbb{R} + \mathbf{k}\mathbb{R} \text{ where } \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \text{ and } \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

LEMMA 2.28 (cf. [**94**, Exercise 7.5.2]). *If F is a non-archimedean local field of residue field of size q and D is a quaternion algebra over F then $|\Delta_{D/F}| = 1$ if D is split and q^{-2} if D is a division algebra.*

3.6. Quaternion algebras over number fields. Let k be a number field and let D be a quaternion algebra defined over k. By previous section for a place ν of k the algebra $D \otimes_k k_\nu$ is either isomorphic to $M(2,k)$ or to the unique, up to isomorphism, quaternion divison algebra over k_ν. In the second case we say that D is ramified in ν. Write $\mathrm{Ram}D$ for the set of places where D is ramified. We put $\mathrm{Ram}^f D := \mathrm{Ram}D \cap M_k^f$ and $\mathrm{Ram}^\infty D = \mathrm{Ram}D \cap M_k^\infty$.

PROPOSITION 2.29 ([**94**, Chapter 7]). *Let k be a number field and let D be a quaternion algebra defined over k. The set $\mathrm{Ram}D$ is finite of even cardinality and it determines the isomorphism class of D. Conversely, for every subset S of M_k of even cardinality there exists a unique up to isomorphism quaternion algebra D such that $\mathrm{Ram}D = S$.*

4. Trace formula for congruence lattices

Let $G = \mathrm{SL}(2,\mathbb{K})$ or $\mathrm{PGL}(2,\mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In this section we give a parametrization of maximal arithmetic lattices in G, and prove trace formulas for $\Gamma\backslash G$ for Γ an arithmetic congruence lattice. The first section follows the exposition from [**94**, Chapter 11.4]. In the next we switch to the adelic setting and give a construction of maximal lattices, originally due to Borel. In Sections 3, 4 and 5 we develop a trace formula valid for arbitrary congruence arithmetic lattices in $\mathrm{PGL}(2,\mathbb{K})$.

4.1. Commensurability classes. Let us begin with the parametrization of the commensurability classes of irreducible arithmetic lattices of G, those will be in one-to-one correspondence with certain quaternion algebras defined over number fields.

Let k be a number field with at most one complex place such that at least one archimedean completion of k is isomorphic to $\mathbb{K}$. We enumerate its infinite places $M_k^\infty = \{\nu_1,\dots,\nu_{r_1+r_2}\}$ in such a way that $k_{\nu_1} = \mathbb{K}$. Let D be a quaternion algebra over k (i.e. $D = \left(\frac{a,b}{k}\right)$ for some $a,b \in k^\times$) which splits only in one infinite place ν_1. Every algebra D satisfying above conditions will be called *admissible*. To shorten the formulas we will write $d = r_1 + r_2$. We introduce two isogeneous algebraic groups

$$D^1 = \{x \in D \mid n(x) = 1\} \text{ and } \mathrm{PD}^\times = D^\times / ZD^\times$$

where $ZD^\times$ stands for the center of $D^\times$. We have

$$D^1(k\otimes_{\mathbb{Q}}\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{K})\times \mathrm{SO}(3,\mathbb{R})^{d-1} = G\times \mathrm{SO}(3,\mathbb{R})^{d-1}$$
$$PD^\times(k\otimes_{\mathbb{Q}}\mathbb{R}) \simeq \mathrm{PGL}(2,\mathbb{K})\times \mathrm{PO}(3,\mathbb{R})^{d-1}$$

The classical procedure for constructing an arithmetic lattice in G is to take an order $\mathcal{O}$ in D and project $\mathcal{O}^1$ to G. The resulting subgroup, which we denote by $\Gamma_{\mathcal{O}^1}$ is an arithmetic lattice. Changing the order $\mathcal{O}$ gives a commensurable lattice so every admissible algebra D gives a well defined commensurability class, for which we will write $\mathcal{C}(D)$.

Definition 2.30. *A lattice Γ of G is an irreducible arithmetic lattice if it is, up to automorphisms of G, commensurable with $\Gamma_{\mathcal{O}^1}$ for certain admissible D.*

The assignment $D \mapsto \mathcal{C}(D)$ is a bijection between the set of admissible quaternion algebras and the commensurability classes of arithmetic lattices of G.

4.2. Maximal lattices. It has been observed by Borel, that even if we take $\mathcal{O}$ to be a maximal order in D the resulting lattice $\Gamma_{\mathcal{O}^1}$ is not maximal in the set-theoretic sense. Instead we have to look for maximal lattices among the normalizers of $\Gamma_{\mathcal{O}^1}$'s. Alternatively we may construct the maximal lattices using a projection from $\mathrm{PD}^\times$ rather that D^1. [9] Write PG for the group $\mathrm{PGL}(2,\mathbb{K})$ and π for the projection $\pi\colon G \to PG$. A lattice Γ in G is maximal if and only if $\pi^{-1}(\Gamma)$ is maximal in PG. Similarly Γ is arithmetic if and only if $\pi^{-1}(\Gamma)$ is. Thus, we may study the problem for PG. Instead of working with orders it will be more instructive to switch to the adelic setting. We have

$$\mathrm{PD}^\times(\mathbb{A}) \simeq PG \times PO(3,R)^{d-1} \times \mathrm{PD}^\times(\mathbb{A}_f)$$

The group $\mathrm{PD}^\times(k)$ is embedded diagonally in $\mathrm{PD}^\times(\mathbb{A})$ as a lattice. For any open compact subgroup U of $\mathrm{PD}^\times(\mathbb{A}_f)$ we put $\Gamma_U = PD^\times(k)\cap \mathrm{PD}^\times(\mathbb{A}_\infty)U$ and by abuse of notation we write the same for the projection of Γ_U to PG. Then Γ_U is an arithmetic lattice in PG and all arithmetic lattices are commensurable with ones constructed in this fashion.

Definition 2.31. *A lattice Γ of $\mathrm{PGL}(2,\mathbb{K})$ is called arithmetic if it is commensurable with Γ_U for some choice of U.*

[9] The reason why most sources start with a construction using D^1 is that this algebraic group is simply connected so it satisfies the Strong Approximation Property which makes it easier to work with.

Let Γ be a maximal arithmetic irreducible lattice in PG. We argue that Γ is of form Γ_U for some maximal open compact subgroup U of $\mathrm{PD}^\times(\mathbb{A}_f^\infty)$. Pick V open and compact, such that $\Gamma \cap \Gamma_V$ has finite index in Γ and Γ_V, we can do this by the arithmeticity of Γ. We claim that elements of Γ have rational coefficients.

LEMMA 2.32. *If Γ and Γ_V are commensurable, then Γ is contained in the projection of $\mathrm{PD}^\times(k)$ onto PG.*

PROOF. We identify PG with $\mathrm{PD}^\times(k_{\nu_1})$. The lattice Γ_V by construction lies in $\mathrm{PD}^\times(k)$. We want to show that the same holds for Γ. Let $\gamma \in \Gamma$ and let $\Lambda := \gamma^{-1}\Gamma_V\gamma \cap \Gamma_V$. The group Λ has a finite index in Γ so it is a lattice in PG. By Borel's density theorem Λ is a Zariski dense subset of $\mathrm{PD}^\times(k_{\nu_1})$ contained in $\mathrm{PD}^\times(k)$. The map $x \mapsto \gamma x \gamma^{-1}$ is an automorphism of $\mathrm{PD}^\times(k_{\nu_1})$ defined over k_{ν_1} which maps Λ into a subset of Γ_V. In particular it maps a Zariski dense subset of points defined over k to points defined over k, which by [**106**, Proposition 3.1.10] implies that $x \mapsto \gamma x \gamma^{-1}$ is defined over k. As $\mathrm{PD}^\times$ is an adjoint group we deduce that $\gamma \in \mathrm{PD}^\times(k)$. □

Once we know that Γ is contained in $\mathrm{PD}^\times(k)$ we can see it as a subset of $\mathrm{PD}^\times(\mathbb{A})$. Γ and Γ_V were commensurable so for every finite place $\mathfrak{p} \in M_k^f$ the $\mathfrak{p}$-adic closure of Γ denoted $\Gamma^\mathfrak{p}$ is an open compact subgroup of $DP^\times(k_\mathfrak{p})$. Hence for any $\mathfrak{p}$ we can find an open compact subset $U_\mathfrak{p}$ of $DP^\times(k_\mathfrak{p})$ such that $\Gamma \subset \mathrm{PD}^\times(k) \cap \mathrm{PD}^\times(\mathbb{A}_\infty) \prod_{\mathfrak{p} \in M_k^f} U_\mathfrak{p}$. By maximality of Γ the last inclusion has to be an equality. We obtain the following:

PROPOSITION 2.33. *Let Γ be a maximal arithmetic lattice in $\mathrm{SL}(2,\mathbb{K})$ lying in the commensurability class $\mathcal{C}(D)$. Then there exists a maximal open compact subgroup U of $\mathrm{PD}^\times(\mathbb{A}_f)$ such that $\Gamma = \pi^{-1}(\Gamma_U)$.*

4.3. Packets of maximal lattices. For this section we assume familiarity with the theory of Bruhat-Tits trees of $\mathrm{SL}(2,F)$ where F is a non-archimedean local field. For necessary background see [**94**, Chapter 5.2.1]. We shall write $X(SL(2,F))$ for the tree associated to $\mathrm{SL}(2,F)$. Recall that the adjoint group $\mathrm{PGL}(2,F)$ also acts on $X(SL(2,F))$. This action is transitive on the vertices and on the edges. In order to use Proposition 2.33 we have to describe the set of maximal open compact subgroups of $\mathrm{PD}^\times(\mathbb{A}_f)$. Such a group U can always be written as a product

$$U = \prod_{\mathfrak{p} \in M_k^f} U_\mathfrak{p},$$

where $U_\mathfrak{p}$ is a maximal compact subgroup of $\mathrm{PD}^\times(k_\mathfrak{p})$ and for almost all places $U_\mathfrak{p} = \mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})$. If D is ramified at $\mathfrak{p}$ then $\mathrm{PD}^\times(k_\mathfrak{p})$ is compact so $U_\mathfrak{p} = PD^\times(k_\mathfrak{p})$. If that is not the case then $\mathrm{PD}^\times(k_\mathfrak{p}) \simeq \mathrm{PGL}(2,k_\mathfrak{p})$ and by Tits fixed point theorem and maximality, $U_\mathfrak{p}$ is either a stabilizer of a vertex or of an edge. The set S of places where $U_\mathfrak{p}$ is a stabilizer of an edge determines the $\mathrm{PD}^\times(\mathbb{A})$ conjugacy class of U. Indeed, if $U_\mathfrak{p}$ is the stabiliser of a vertex or edge v then $g^{-1}U_\mathfrak{p}g$ is the stabilizer of vg. The action of $\mathrm{PD}^\times(k_\mathfrak{p})$ is transitive both on vertices and on edges of $X(\mathrm{SL}(2,k_\mathfrak{p}))$ so all stabilizers of vertices are conjugate and likewise all stabilizers of edges are conjugate.

DEFINITION 2.34. *Let S be a finite subset of $M_k^f \setminus \mathrm{Ram}^f(D)$. We write $\mathcal{C}(D,S)$ for the set of conjugacy classes of arithmetic lattices Γ_U with U maximal open compact subgroup of $\mathrm{PD}^\times(\mathbb{A}_f)$ given by*

$$U = \prod_{\mathfrak{p} \in M_k^f} U_\mathfrak{p}$$

where $U_\mathfrak{p}$ stabilizes an edge of $X(\mathrm{SL}(2,k_\mathfrak{p}))$ if and only if $\mathfrak{p} \in S$ and $U_\mathfrak{p} = \mathrm{PD}^\times(k_\mathfrak{p})$ for $\mathfrak{p} \in \mathrm{Ram}^f D$. The set $\mathcal{C}(D,S)$ will be called a packet of arithmetic lattices.

Even though all subgroups U satisfying the conditions imposed in the definition are conjugate in $\mathrm{PD}^\times(\mathbb{A})$ the resulting lattices Γ_U need not be conjugate in $\mathrm{PGL}(2,\mathbb{K})$. That

is why a packet may contain multiple conjugacy classes. We also point out that depending on the choice of S, a packet may consist of non-maximal lattices.

We shall parametrize the conjugacy classes inside $\mathcal{C}(D,S)$ using certain class group. The result is similar to the classification of maximal orders in a quaternion algebra from [**94**, Chapter 6.7]. Recall that subscript D means that we take only elements positive on all archimedean places of k where D is ramified. Recall that $k_D = n(D)$, $\mathbb{A}_D = n(D(\mathbb{A}))$ and $k^F = \prod_{\nu\in F} k_\nu$ for any finite subset F of M_k. We will write $\mathbb{A}_{\infty,D}$ for $n(D(\mathbb{A}_\infty))$.

Definition 2.35. *For any finite subset $S \subset M_k^f \setminus \mathrm{Ram}^f(D)$ we define the class group $\mathrm{cl}_S(D) = \mathrm{cl}_S$ (we omit D when it is clear from the context) by*

$$\mathrm{cl}_S := \mathrm{PD}^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})/\mathrm{PD}^\times(\mathbb{A}_\infty)U \simeq \mathbb{A}_D^\times/k_D^\times\mathbb{A}_{\infty,D}^\times(k^{S\cup\mathrm{Ram}^f(D)})^\times.$$

The second isomorphism is induced by the norm map. We explain where it comes from in the proof of Proposition 2.37.

Remark 2.36. *The class group cl_S is a quotient of the narrow class group $\mathrm{cl}^+(k)$ of exponent 2.*

Proposition 2.37. *There is a surjective map from the group cl_S to the set $\mathcal{C}(D,S)$. In particular the packets are finite.*

For the proof we will need:

Lemma 2.38. *Lattices Γ_U and Γ_V are conjugate in $\mathrm{PGL}(2,\mathbb{K})$ if and only if they are conjugate in $\mathrm{PD}^\times(k)$.*

Proof. Like in the proof of 2.32 any automorphism mapping Γ_U into Γ_V has to be defined over k. Thus, every g such that $g\Gamma_U g^{-1} = \Gamma_V$ lies in $\mathrm{PD}^\times(k)$. □

By abuse of notation we shall also denote by n the map $n\colon \mathrm{PD}^\times(\mathbb{A}) \to \mathbb{A}^\times/(\mathbb{A}^\times)^2$ induced by the norm map.

Lemma 2.39. *Let $U_\mathfrak{p}$ be a maximal open compact subgroup of $\mathrm{PD}^\times(k_\mathfrak{p})$. All matrices written below are to be understood as their image in the projective group.*

- *If D splits over $k_\mathfrak{p}$ and $U_\mathfrak{p}$ stabilizes a vertex of $X(\mathrm{SL}(2,k_\mathfrak{p}))$ then, up to conjugacy $U_\mathfrak{p} = \mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})$. We have $n(U_\mathfrak{p}{k_\mathfrak{p}}^\times) = \mathcal{O}_{k_\mathfrak{p}}^\times(k_\mathfrak{p}^\times)^2$.*
- *If D splits over $k_\mathfrak{p}$ and $U_\mathfrak{p}$ stabilizes an edge then, up to conjugacy*

$$U_\mathfrak{p} = \left\langle \begin{pmatrix} \mathcal{O}_{k_\mathfrak{p}}^\times & \mathcal{O}_{k_\mathfrak{p}} \\ \mathfrak{p} & \mathcal{O}_{k_\mathfrak{p}}^\times \end{pmatrix}, \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix} \right\rangle,$$

  where π is the uniformizer of $k_\mathfrak{p}$. We have $n(U_\mathfrak{p}{k_\mathfrak{p}}^\times) = k_\mathfrak{p}^\times$.
- *If $\mathfrak{p} \in \mathrm{Ram}^f(D)$ then $U_\mathfrak{p} = PD^\times(k_\mathfrak{p})$. Let E be the unique unramified quadratic extension of $k_\mathfrak{p}$. Then we can represent $U_\mathfrak{p}$ as*

$$U_\mathfrak{p} = \left\{ \begin{pmatrix} x & y \\ \pi y^\sigma & x \end{pmatrix} \mid x,y \in E \right\},$$

  where σ is the unique non-trivial element of $\mathrm{Gal}(E/k_\mathfrak{p})$. We have $n(U_\mathfrak{p}{k_\mathfrak{p}}^\times) = k_\mathfrak{p}^\times$.

The proof follows from the description of maximal compact subgroups as stabilizers of a point or an edge in the Bruhat-Tits tree. To see the last part note that $N_{E/k_\mathfrak{p}}$ is surjective ([**68**]).

Theorem 2.40 (Strong Approximation Property, [**89**, 7.1]). *The group $D^1(k)$ is dense in $D^1(\mathbb{A}_f)$.*

Proposition 2.41 ([**94**, Theorem 7.41]). *The image of the norm map $k_D^\times := n(D^\times(k))$ is equal to $\{x \in k^\times \mid (x)_\nu > 0 \text{ for all } \nu \in \mathrm{Ram}^\infty(D)\}$.*

Proof of Proposition 2.37. By Lemma 2.38 we can look at $\mathrm{PD}^\times(k)$ conjugacy classes of lattices in $\mathcal{C}(D,S)$. Fix a maximal open compact subgroup U of $\mathrm{PD}^\times(\mathbb{A}_f)$ stabilizing edges only for $\mathfrak{p}\in S$. We start with the parametrization of $\mathrm{PD}^\times(k)$-conjugacy classes of subgroups U' conjugate to U under $\mathrm{PD}^\times(\mathbb{A})$, for simplicity we refer to this set as $\mathcal{C}_1(D,S)$. By maximality, U is equal to its normalizer in $\mathrm{PD}^\times(\mathbb{A}_f)$. Hence

$$\mathcal{C}_1(D,S)\simeq \mathrm{PD}^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})/PD^\times(\mathbb{A}_\infty)U\simeq D^\times(k)\backslash D^\times(\mathbb{A})/D^\times(\mathbb{A}_\infty)\mathbb{A}_f^\times U. \tag{4.1}$$

Put $U^1=\{x\in(\mathbb{A}_f)^\times U\mid n(x)=1\}$. By Strong Approximation

$$D^1(k)\backslash D^1(\mathbb{A})/D^1(\mathbb{A}_\infty)U^1=1.$$

Hence, the map from (4.1) induced by the norm

$$n\colon D^\times(k)\backslash D^\times(\mathbb{A})/D^\times(\mathbb{A}_\infty)\mathbb{A}_f^\times U\to k_D^\times\backslash\mathbb{A}_D^\times/(\mathbb{A}_{\infty,D})^\times(\mathbb{A}^\times)^2n(\mathbb{A}_f^\times U)$$

is an isomorphism. By Lemma 2.39

$$n(\mathbb{A}_f^\times U)=\prod_{\mathfrak{p}\in M_k^f\backslash S}(k_\mathfrak{p}^\times)^2\mathcal{O}_{k_\mathfrak{p}}^\times\times\prod_{\mathfrak{p}\in S\cup\mathrm{Ram}^f(D)}k_\mathfrak{p}^\times.$$

Hence,

$$\mathcal{C}_1(D,S)\simeq k_D^\times\backslash\mathbb{A}^\times/(\mathbb{A}^\infty)^\times(\mathbb{A}^\times)^2n((\mathbb{A}_f)^\times U)\simeq\mathbb{A}_D^\times/k_D^\times\mathbb{A}_D^{S\cup\mathrm{Ram}^f(D)}(\mathbb{A}^\times)^2=\mathrm{cl}_S. \tag{4.2}$$

If two groups U,U' are conjugate under $\mathrm{PD}^\times(k)$ then the lattices Γ_U and $\Gamma_{U'}$ are also conjugate under $\mathrm{PD}^\times(k)$. This gives us a well defined surjective map from $\mathrm{cl}_S\simeq\mathcal{C}_1(D,S)\to\mathcal{C}(D,S)$. □

4.4. Trace formula for maximal lattices. Recall that the geometric side of the Selberg Trace formula yields that for any cocompact lattice $\Gamma\subset G$ and any function $f\in C_c(G)$ we have

$$\mathrm{tr}R_\Gamma f=\sum_{[\gamma]_\Gamma}\mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G/G_\gamma}f(x^{-1}\gamma x)dx,$$

where R_Γ is the right translation on $L^2(\Gamma\backslash G)$ In this section we develop an adelic version of the trace formula for maximal lattices which allows to express the geometric side as a sum over rational conjugacy classes in $\mathrm{PD}^\times(k)$. In the next section we generalize this procedure to arbitrary congruence lattices. **Throughout the following sections we assume that D is not split** so there will be no continuous spectrum.

Let S be a finite subset of $M_k^f\setminus\mathrm{Ram}^f(D)$. We fix U a maximal compact subgroup of $\mathrm{PD}^\times(\mathbb{A}_f)$ stabilizing an edge for $\mathfrak{p}\in S$. We fix the standard Haar measure (see Section 2.4) $\mu_\mathbb{A}$ on $\mathrm{PD}^\times(\mathbb{A})$ given by

$$\mu_\mathbb{A}=\mu_{\nu_1}\times\ldots\mu_{\nu_d}\times\mu_{\mathbb{A}_f} \tag{4.3}$$

Recall that the standard measure $\mu_{\mathbb{A}_f}$ depends on the choice of U. Where μ_{ν_1} is the volume on $\mathrm{PGL}(2,\mathbb{K})$, μ_{ν_i} is the normalized Haar measure on $\mathrm{PO}(3,\mathbb{R})$ for $d\geq i>1$ and $\mu_{\mathbb{A}_f}$ is a Haar measure on $\mathrm{PD}^\times(\mathbb{A}_f)$ normalized so that $\mu_{\mathbb{A}_f}(U)=1$. We write $R\colon\mathrm{PD}^\times(\mathbb{A})\to\mathcal{U}(L^2(PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})))$ for the right translation $(R(g)\phi)(x)=\phi(xg)$.

For any $f\in C_c(\mathrm{PGL}(2,\mathbb{K}))$ we define $f_\mathbb{A}\in C_c(PD^\times(\mathbb{A}))$ as $f_\mathbb{A}=f_{\mathbb{A}_\infty}\otimes f_{\mathbb{A}_f}$ where

$$f_{\mathbb{A}_\infty}=f\otimes\bigotimes_{i=2}^d\mathbb{1}_{PO(3,\mathbb{R})}$$
$$f_{\mathbb{A}_f}=\mathbb{1}_U$$

Using $f_\mathbb{A}$ we can relate the traces of $R_{\Gamma_U}f$ and $Rf_\mathbb{A}$. The following lemma illustrates this principle.

Lemma 2.42. *Let $1=c_1,c_2,\ldots,c_k$ be representatives of cl_S in $\mathrm{PD}^\times(\mathbb{A}_f)$ and put $U_i=c_iUc_i^{-1}$. With $f_\mathbb{A}$ and measure $\mu_\mathbb{A}$ defined as above, we have $\sum_{i=1}^k\mathrm{tr}R_{\Gamma_{U_i}}f=\mathrm{tr}Rf_\mathbb{A}$.*

PROOF. To shorten notation, write $\tilde{U}$ for $\prod_{i=2}^{d} \mathrm{PD}^\times(k_{\nu_i}) \times U$. Observe that the image of $Rf_{\mathbb{A}}$ lies in the $\tilde{U}$-fixed vectors of $L^2(\mathrm{PD}^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A}))$, so $\mathrm{tr}Rf_{\mathbb{A}} = \mathrm{tr}Rf_{\mathbb{A}}\mid_{L^2(PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A}))^{\tilde{U}}}$. We have a $\mathrm{PD}^\times(k_{\nu_1})$-equivariant isometry

$$L^2(PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A}))^{\tilde{U}} \simeq L^2(PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})/\tilde{U}).$$

By repeating the arguments from the proof of 2.37 we get a $\mathrm{PD}^\times(k_{\nu_1})$-equivariant bijection

$$\mathrm{PD}^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})/\tilde{U} \simeq \bigsqcup_{i=1}^{k} \Gamma_{U_i}\backslash\mathrm{PD}^\times(k_{\nu_1}),$$

so

$$L^2(PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})/\tilde{U}) \simeq \bigoplus_{i=1}^{k} L^2(\Gamma_{U_i}\backslash\mathrm{PD}^\times(k_{\nu_1}))$$

as unitary representations of $\mathrm{PGL}(2, k_{\nu_1})$. Thus $\mathrm{tr}Rf_{\mathbb{A}} = \sum_{i=1}^{k} \mathrm{tr}R_{\Gamma_{U_i}} f$. □

As a first step to isolate the trace $\mathrm{tr}R_{\Gamma_U} f$ we shall consider a family of modified operators on $L^2(PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A}))$.

DEFINITION 2.43. *Let $\chi : PD^\times(\mathbb{A}) \to \mathbb{C}^1$ be a smooth character vanishing on $\mathrm{PD}^\times(k)$. For any continuous, compactly supported $f \in C_c(PD^\times(\mathbb{A}))$ we define an operator $R^\chi f : L^2(PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})) \to L^2(PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A}))$ by*

$$(R^\chi f(\phi))(x) = (Rf(\phi))(x)\chi(x).$$

PROPOSITION 2.44.

$$\mathrm{tr}R^\chi f = \sum_{\substack{[\gamma]_{PD^\times(k)} \\ \chi|_{\mathrm{PD}^\times_\gamma(\mathbb{A})}=1}} \mathrm{Vol}(PD^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A})) \int_{PD^\times_\gamma(\mathbb{A})\backslash\mathrm{PD}^\times(\mathbb{A})} \chi(x) f(x^{-1}\gamma x)dx$$

PROOF. We have

$$\begin{aligned}(R^\chi f(\phi))(x) &= \chi(x) \int_{PD^\times(\mathbb{A})} f(y)\phi(xy)dy \\ &= \chi(x) \int_{PD^\times(\mathbb{A})} f(x^{-1}y)\phi(y)dy \\ &= \int_{PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})} \left[\sum_{\gamma\in\mathrm{PD}^\times(k)} \chi(x) f(x^{-1}\gamma y)\right] \phi(y)dy\end{aligned}$$

So the operator $(R^\chi f(\phi))$ is given by the kernel $K^\chi(x,y) = \sum_{\gamma\in\mathrm{PD}^\times(k)} \chi(x) f(x^{-1}\gamma y)$. The algebra D is not split so the quotient $\mathrm{PD}^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})$ is compact. For a compact quotient, the kernel is integrable on the diagonal and we have

$$\begin{aligned}\mathrm{tr}R^\chi f &= \int_{PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})} K^\chi(x,x)dx \\ &= \sum_{\gamma\in PD^\times(k)} \int_{PD^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})} \chi(x) f(x^{-1}\gamma x)dx \\ &= \sum_{[\gamma]_{PD^\times(k)}} \int_{PD^\times_\gamma(k)\backslash\mathrm{PD}^\times(\mathbb{A})} \chi(x) f(x^{-1}\gamma x)dx \\ &= \sum_{[\gamma]_{PD^\times(k)}} \int_{PD^\times_\gamma(\mathbb{A})\backslash\mathrm{PD}^\times(\mathbb{A})} \int_{PD^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A})} \chi(zx) f(x^{-1}\gamma x)dzdx \\ &= \sum_{\substack{[\gamma]_{PD^\times(k)} \\ \chi|_{\mathrm{PD}^\times_\gamma(\mathbb{A})}=1}} \mathrm{Vol}(PD^\times_\gamma(k)\backslash\mathrm{PD}^\times_\gamma(\mathbb{A})) \int_{PD^\times_\gamma(\mathbb{A})\backslash\mathrm{PD}^\times(\mathbb{A})} \chi(x) f(x^{-1}\gamma x)dx.\end{aligned}$$

□

LEMMA 2.45. *Let* $1 = c_1, c_2, \ldots, c_k$ *be representatives of* cl_S *chosen from* $\mathrm{PD}^\times(\mathbb{A}_f)$ *and put* $U_i = c_i U c_i^{-1}$. *Let* χ *be a smooth character of* $\mathrm{PD}^\times(\mathbb{A})$ *that factors through* cl_S *or equivalently, vanishes on* $\mathrm{PD}^\times(\mathbb{A}_\infty)U$ *and* $\mathrm{PD}^\times(k)$. *We have* $\mathrm{tr}R^\chi f_\mathbb{A} = \sum_{i=1}^k \chi(c_i)\mathrm{tr}R_{\Gamma_{U_i}} f$

PROOF. As in the proof of the Lemma 2.42 we will exploit the fact that the image $R^\chi f$ lies in the subspace of $\tilde{U}$-fixed vectors (for the definition of $\tilde{U}$ see the proof of Lemma 2.42), so by recalling the proof of Lemma 2.42 we get

$$PD^\times(\mathbb{A}) = \bigsqcup_{i=1}^k \mathrm{PD}^\times(k) c_i \tilde{U} \mathrm{PD}^\times(k_{\nu_1}). \tag{4.4}$$

Hence

$$L^2(PD^\times(k)\backslash \mathrm{PD}^\times(\mathbb{A}))^{\tilde{U}} \simeq \bigoplus_{i=1}^k L^2(\Gamma_{U_i}\backslash \mathrm{PD}^\times(k_{\nu_1}))$$

Using Equation (4.4) we get under that the last isomorphism the character χ is given by

$$\chi = \sum_{i=1}^k \chi(c_i) \mathbb{1}_{\Gamma_{U_i}\backslash \mathrm{PD}^\times(k_{\nu_1})}.$$

It follows that

$$\mathrm{tr}R^\chi f_\mathbb{A} = \sum_{i=1}^k \chi(c_i)\mathrm{tr}R_{\Gamma_{U_i}} f.$$

□

COROLLARY 2.46.

$$\mathrm{tr}R_{\Gamma_{U_i}} f = \frac{1}{|\mathrm{cl}_S|} \sum_{\chi \in \widehat{\mathrm{cl}_S}} \overline{\chi(c_i)} \mathrm{tr}R^\chi f_\mathbb{A}$$

PROOF. By Lemma 2.45, for any character χ of cl_S we have

$$\mathrm{tr}R^\chi f_\mathbb{A} = \sum_{i=1}^k \chi(c_i)\mathrm{tr}R_{\Gamma_{U_i}} f.$$

Thus

$$\sum_{\chi \in \widehat{\mathrm{cl}_S}} \overline{\chi(c_i)} \mathrm{tr}R^\chi f_\mathbb{A} = \sum_{j=1}^k \sum_{\chi \in \widehat{\mathrm{cl}_S}} \overline{\chi(c_i)}\chi(c_j)\mathrm{tr}R_{\Gamma_{U_i}} f = |\mathrm{cl}_S| \mathrm{tr}R_{\Gamma_{U_i}} f.$$

□

Combining Corollary 2.46 and Proposition 2.44 we get:

PROPOSITION 2.47. *With* $U, f, f_\mathbb{A}$ *as before, we have*

$$\mathrm{tr}R_{\Gamma_{U_i}} f = \sum_{[\gamma]_{PD^\times(k)}} \mathrm{Vol}(PD^\times_\gamma(k)\backslash \mathrm{PD}^\times_\gamma(\mathbb{A}))\Xi_\gamma(f_\mathbb{A}),$$

where, for $h \in C_c(PD^\times(\mathbb{A}))$

$$\Xi_\gamma(h) := \frac{1}{|\mathrm{cl}_S|} \sum_{\substack{\chi \in \widehat{\mathrm{cl}_S} \\ \chi|_{PD^\times_\gamma(\mathbb{A})} = 1}} \int_{PD^\times_\gamma(\mathbb{A})\backslash \mathrm{PD}^\times(\mathbb{A})} \chi(x) h(x^{-1}\gamma x) dx.$$

PROOF. By Corollary 2.46 and Proposition 2.44 we get

$$\begin{aligned}
\mathrm{tr}R_{\Gamma_{U_i}} f =& \frac{1}{|\mathrm{cl}\ S|} \sum_{\chi \in \widehat{\mathrm{cl}\ S}} \mathrm{tr}R^{\chi} f_{\mathbb{A}} && (4.5)\\
=& \sum_{[\gamma]_{PD^{\times}(k)}} \mathrm{Vol}(PD^{\times}_{\gamma}(k)\backslash \mathrm{PD}^{\times}_{\gamma}(\mathbb{A})) && (4.6)\\
& \frac{1}{|\mathrm{cl}\ S|} \sum_{\substack{\chi \in \widehat{\mathrm{cl}\ S} \\ \chi|_{\mathrm{PD}^{\times}_{\gamma}(\mathbb{A})}=1}} \int_{PD^{\times}_{\gamma}(\mathbb{A})\backslash \mathrm{PD}^{\times}(\mathbb{A})} \chi(x) f_{\mathbb{A}}(x^{-1}\gamma x)dx && (4.7)\\
=& \sum_{[\gamma]_{PD^{\times}(k)}} \mathrm{Vol}(PD^{\times}_{\gamma}(k)\backslash \mathrm{PD}^{\times}_{\gamma}(\mathbb{A}))\Xi_{\gamma}(f) && (4.8)
\end{aligned}$$

□

4.5. Trace formula for congruence lattices. The reasoning will be very close to one we conducted for maximal lattices. We decided to split this into two parts to avoid an overload of technicalities. We keep the notation from previous sections, in particular D is still an admissible quaternion algebra defined over a number field k. Any congruence lattice in the commensurability class $\mathcal{C}(D)$ defined by D is of the from $\Gamma = \Gamma_V$ for some open compact subgroup $V \subset \mathrm{PD}^{\times}(\mathbb{A}_f)$. From now on we fix such V and we fix a maximal open compact U containing V. We use U to define a measure $\mu_{\mathbb{A}}$ on $\mathrm{PD}^{\times}(\mathbb{A})$ the same way we did in Equation (4.3). Let $f \in C_c(\mathrm{PGL}(2,\mathbb{K}))$, we define

$$f^V_{\mathbb{A}} = f_{\mathbb{A}_\infty} \otimes f^V_{\mathbb{A}_f},$$

where

$$\begin{aligned}
f_{\mathbb{A}_\infty} =& f \otimes \mathbb{1}_{PD^{\times}(k_{\nu_2})} \otimes \ldots \otimes \mathbb{1}_{PD^{\times}(k_{\nu_d})} && (4.9)\\
f^V_{\mathbb{A}_f} =& [U : V]\mathbb{1}_V && (4.10)
\end{aligned}$$

DEFINITION 2.48. *We define the class group* $\mathrm{cl}\,(D,V) = \mathrm{cl}\,(V)$ *(we omit* D *when it is clear from the context) as the quotient* $\mathrm{PD}^{\times}(k)\backslash \mathrm{PD}^{\times}(\mathbb{A}_f)/V$.

With this notation, for U maximal we have $\mathrm{cl}\,(U) = \mathrm{cl}\ S$ where S is the set of places where $U_{\mathfrak{p}}$ fixes an edge of the Bruhat-Tits tree.

LEMMA 2.49. *The norm map induces an isomorphism*

$$\mathrm{cl}\,(V) \simeq \mathbb{A}^{\times}_D / k^{\times}_D n(\mathbb{A}^{\times} V) \simeq \mathbb{A}^{\times}_f / k^{\times}_D n(\mathbb{A}^{\times}_f V).$$

In particular $\mathrm{cl}\,(V)$ *is finite and has an abelian group structure.* [10]

PROOF. Let V^1 denote the preimage of V in $D^1(\mathbb{A}_f)$. By the Strong Approximation Theorem (Thm. 2.40) the product $D^1(k)V^1$ equals $D^1(\mathbb{A}_f)$. Thus the map induced by the norm

$$n\colon \mathrm{PD}^{\times}(k)\backslash \mathrm{PD}(\mathbb{A}_f)/V \to k^{\times}_D \backslash \mathbb{A}^{\times}_D / n(V\mathbb{A}^{\times}) \tag{4.11}$$

is an isomorphism. The second isomorphism is a consequence of the Weak Approximation Property for the multiplicative group $\mathbb{G}_m$ (see [**89**, Chapter 7]). □

The last Lemma allows us to identify the characters of $\mathbb{A}^{\times}_f / k^{\times}_D (\mathbb{A}^{\times}_f)^2 n(V)$ with the characters of $\mathrm{PD}^{\times}(\mathbb{A})$ vanishing on V and $\mathrm{PD}^{\times}(k)$. Repeating the exact same argument as in the proof of Proposition 2.44 we get

[10] Note, that while $n(V)$ alone is not well defined because it is a subset of $\mathrm{PD}^{\times}(\mathbb{A}_f)$ not $D^{\times}(\mathbb{A}_f)$ the product $(\mathbb{A}^{\times}_f)^2 n(V) = n(\mathbb{A}^{\times}_f V)$ is.

THEOREM 2.50. *Let D be an admissible quaternion division algebra over k and let V be an open compact subset of $\mathrm{PD}^\times(\mathbb{A}_f)$. For any continuous, compactly supported function f on $\mathrm{SL}(2,\mathbb{K})$ we have*

$$\mathrm{tr}R_{\Gamma_V} f = \sum_{[\gamma]_{\mathrm{PD}^\times(k)}} \mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash \mathrm{PD}^\times_\gamma(\mathbb{A}))\Xi^V_\gamma(f_{\mathbb{A}}), \tag{4.12}$$

Where, for $h \in C_c(\mathrm{PD}^\times(\mathbb{A}))$

$$\Xi^V_\gamma(h) := \frac{1}{|\mathrm{cl}\,(V)|} \sum_{\substack{\chi \in \widehat{\mathrm{cl}\,(V)} \\ \chi|_{\mathrm{PD}^\times_\gamma(\mathbb{A})}=1}} \int_{\mathrm{PD}^\times_\gamma(\mathbb{A})\backslash \mathrm{PD}^\times(\mathbb{A})} \chi(x)h(x^{-1}\gamma x)dx. \tag{4.13}$$

As a corollary we get the formula for the volume

COROLLARY 2.51.

$$\mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K})) = \frac{\mathrm{Vol}(\mathrm{PD}^\times(k)\backslash \mathrm{PD}^\times(\mathbb{A}))[U:V]}{|\mathrm{cl}\,(V)|} \tag{4.14}$$

PROOF. Take a sequence (f_i) of functions with support contained in smaller and smaller balls around 1 and with $f_i(1) = 1$. By the usual Selberg trace formula $\lim_{i\to\infty} \mathrm{tr}R_{\Gamma_V} f_i = \mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K}))$. On the other hand by Theorem 2.50 we have

$$\lim_{i\to\infty} \mathrm{tr}R_{\Gamma_V} f_i = \frac{\mathrm{Vol}(\mathrm{PD}^\times(k)\backslash \mathrm{PD}^\times(\mathbb{A}))[U:V]}{|\mathrm{cl}\,(V)|}. \tag{4.15}$$

□

LEMMA 2.52. *With D, V as before and $\gamma \in \mathrm{PD}^\times(k), \gamma \neq 1$ there are at most two characters $\chi \in \widehat{\mathrm{cl}\,(V)}$ which vanish on $\mathrm{PD}^\times_\gamma(\mathbb{A})$.*

PROOF. Let $\chi \in \widehat{\mathrm{cl}\,(V)}$ be a character vanishing on $\mathrm{PD}^\times_\gamma(\mathbb{A})$. By Lemma 2.49 we can write $\chi(g) = \theta(n(g))$ where θ is a character of $\mathbb{A}_D^\times$ vanishing on $k_D^\times$ and $n(\mathbb{A}^\times V)$. Let l be the quadratic extension of k generated by the eigenvalues of γ. The image of $D^\times_\gamma(\mathbb{A})$ via the norm map equals $N_{l/k}(\mathbb{A}_l^\times)$. Hence, θ factors through

$$\mathbb{A}_D^\times/k_D^\times n(\mathbb{A}^\times V)N_{l/k}(\mathbb{A}_l^\times).$$

By the weak approximation theorem for the multiplicative group we have $\mathbb{A}^\times = k^\times \mathbb{A}_D$ and hence $\mathbb{A}_D^\times/k_D^\times \simeq \mathbb{A}^\times/k^\times$. The last quotient can be rewritten as

$$\mathbb{A}^\times/k^\times n(\mathbb{A}^\times V)N_{l/k}(\mathbb{A}_l^\times).$$

We claim that this is a group of order at most two. To do this we use Class Field Theory. By the Reciprocity Law [**82**, Theorem 5.3] there exists a homomorphism (Artin map) $\phi_k : \mathbb{A}^\times \to \mathrm{Gal}(k^{ab}/k)$ satisfying the following properties

- $\phi(k^\times) = 1$
- for every abelian extension l/k map ϕ_k defines an isomorphism

$$\phi_{l/k}\colon \mathbb{A}^\times/k^\times N_{l/k}(\mathbb{A}_l^\times) \to \mathrm{Gal}(l/k).$$

In particular, if l/k is a quadratic extension then $|\mathbb{A}^\times/k^\times N_{l/k}(\mathbb{A}_l^\times)| = 2$. □

REMARK 2.53. *An alternative way to finish the proof is to use the Second Inequality of the Class Field Theory [**82**, Theorem 5.1.a] which states that $|\mathbb{A}^\times/k^\times N_{l/k}(\mathbb{A}_l)|$ divides $[l:k]$ for **any** finite Galois extension l/k. In particular, this part of the proof should generalize to the other semisimple Lie groups without greater difficulty.*

COROLLARY 2.54. *For any $h \in C_c(PD^\times(\mathbb{A}))$ we have*

$$|\Xi^V_\gamma(h)| \leq \frac{2}{|\mathrm{cl}\,(V)|}|\mathcal{O}_\gamma(|h|)|.$$

In particular

$$|\mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K}))f(1) - \mathrm{tr}R_{\Gamma_V} f| \leq \frac{2}{|\mathrm{cl}\,(V)|} \sum_{\substack{[\gamma]_{PD^\times(k)} \\ \gamma\neq 1}} \mathrm{Vol}(PD^\times_\gamma(k)\backslash \mathrm{PD}^\times_\gamma(\mathbb{A}))\mathcal{O}_\gamma(|f|_{\mathbb{A}}), \tag{4.16}$$

4.6. Normalization. The maximal arithmetic lattices of $\mathrm{SL}(2,\mathbb{K})$ or $\mathrm{PGL}(2,\mathbb{K})$ admit a fairly explicit description (cf. Proposition 2.33). Therefore proving a limit multiplicity property (or property B-S) for sequences of maximal lattices is a priori much easier than doing it in full generality. In this section we develop methods to prove the strong version of Conjecture 2.5 for arbitrary congruence lattices, not necessarily maximal. The idea is to replace the characteristic function $\mathbb{1}_V$ in the definition of $f_{\mathbb{A}_f}$ by a function invariant by conjugation by some maximal compact subgroup U containing V and then express it as a combination of finite dimensional characters of U.

Let us fix an admissible quaternion algebra D, an open compact subgroup V of $\mathrm{PD}^\times(\mathbb{A}_f^\infty)$ and a maximal open compact subgroup U containing V. Recall that we write $\mathrm{Irr}G$ for the set of irreducible complex representations of a group G, and whenever ρ is such a representation we write χ_ρ for its character and W_ρ for the vector space on which G acts. If G is a group acting on a space W we write W^G for the set of fixed points. For $f \in C_c(\mathrm{PGL}(2,\mathbb{K}))$ let $f_{\mathbb{A}} = f^V_{\mathbb{A}}, f_{\mathbb{A}_\infty}$ and $f^V_{\mathbb{A}_f}$ be defined by Equations (4.9).

Lemma 2.55. *Let* $\gamma \in \mathrm{PD}^\times(k)$

$$|\Xi^V_\gamma(f_{\mathbb{A}})| \leq \frac{2|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})|}{|\mathrm{cl}\,(V)|} \sum_{\rho\in \mathrm{Irr}(U)} \dim W^V_\rho |\mathcal{O}_\gamma(\chi_\rho)| \tag{4.17}$$

Proof. By corollary 2.54 we have

$$|\Xi^V_\gamma(f_{\mathbb{A}})| \leq \frac{2|\mathcal{O}_\gamma(f_{\mathbb{A}})|}{|\mathrm{cl}\,(V)|} \tag{4.18}$$

Let $\tilde{f}_{\mathbb{A}_f} := \int_U f_{\mathbb{A}_f}(u^{-1}xu)du$. Then

$$\begin{aligned}
\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})\mathcal{O}_\gamma(\tilde{f}_{\mathbb{A}_f}) =& \mathcal{O}_\gamma(f_{\mathbb{A}_\infty}) \int_U \int_{PD^\times_\gamma(\mathbb{A}_f)\backslash \mathrm{PD}^\times(\mathbb{A}_f)} f((ux)^{-1}\gamma xu)dudx && (4.19)\\
=& \mathcal{O}_\gamma(f_{\mathbb{A}_\infty})\mathcal{O}_\gamma(f_{\mathbb{A}_f}) && (4.20)\\
=& \mathcal{O}_\gamma(f_{\mathbb{A}}) && (4.21)
\end{aligned}$$

On the other hand

$$\tilde{f}_{\mathbb{A}_f} = \int_U [U:V]\mathbb{1}_{uVu^{-1}}du = \sum_{u\in U/V} \mathbb{1}_{uVu^{-1}} = \chi_{\mathrm{Ind}\,^U_V 1}. \tag{4.22}$$

By the Frobenius reciprocity $\langle \mathrm{Ind}\,^U_V 1, \rho\rangle_U = \dim W^V_\rho$ so

$$\chi_{\mathrm{Ind}\,^U_V 1} = \sum_{\rho\in \mathrm{Irr}U} \dim W^V_\rho \chi_\rho. \tag{4.23}$$

Hence

$$\begin{aligned}
|\Xi^V_\gamma(f_{\mathbb{A}})| \leq& \frac{2|\mathcal{O}_\gamma(f_{\mathbb{A}})|}{|\mathrm{cl}\,(V)|} && (4.24)\\
\leq& \frac{2|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})|}{|\mathrm{cl}\,(V)|} |\mathcal{O}_\gamma(f_{\mathbb{A}_f^U})| && (4.25)\\
\leq& \frac{2|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})|}{|\mathrm{cl}\,(V)|} \sum_{\chi\in \mathrm{Irr}(U)} \dim W^V_\rho |\mathcal{O}_\gamma(\chi_\rho)|. && (4.26)
\end{aligned}$$

□

Section 6 will be devoted to estimates of orbital integrals appearing on the right hand side of Lemma 2.55. We will aim at an estimate of the form $|\mathcal{O}_\gamma(\chi_\rho)| \leq C\chi_\rho^{1-\delta}$ where $\delta > 0$ and C is a constant depending on the lattice in an explicit way (see Propositions 2.74 and 2.75).

5. Short geodesics and equidistribution

5.1. Bilu equidistribution theorem. Recall that the **logarithmic Mahler measure** of an algebraic number α is given by $m(\alpha) = \sum_{\nu \in M_{\mathbb{Q}(\alpha)}} \log^+ |\alpha|_\nu$, where $\log^+ t = \max\{\log t, 0\}$ for $t > 0$. We shall use the following result due to Bilu **[20]**

THEOREM 2.56. *Let (α_n) be a sequence of algebraic integers such that $\frac{m(\alpha_n)}{[\mathbb{Q}(\alpha_n):\mathbb{Q}]}$ tends to 0 as n goes to infinity, where $m(\alpha_n)$ is the logarithmic Mahler measure. Define the sequence of probability measures*

$$\mu_n := \frac{1}{[\mathbb{Q}[\alpha_n] : \mathbb{Q}]} \sum_{\sigma \in \mathrm{Hom}(\mathbb{Q}(\alpha_n), \mathbb{C})} \delta_{(\alpha_n)_\sigma}.$$

Then for any bounded continuous function $f : \mathbb{C} \to \mathbb{C}$ we have

$$\lim_{n\to\infty} \int f d\mu_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

The following corollary is crucial to estimate the orbital integrals in the non torsion case (see Section 6).

COROLLARY 2.57. *Let (α_n) be a sequence of algebraic integers with bounded logarithmic Mahler measure and degree tending to infinity. Then*

$$|N_{\mathbb{Q}(\alpha_n)/\mathbb{Q}}(1 - \alpha_n)| \ll e^{o([\mathbb{Q}(\alpha_n):\mathbb{Q}])}.$$

PROOF. Let μ_n be probability measures defined as in Theorem 2.56. Then

$$\frac{1}{[\mathbb{Q}(\alpha_n) : \mathbb{Q}]} \log |N_{\mathbb{Q}(\alpha_n)/\mathbb{Q}}(1 - \alpha_n)| = \int \log |1 - x| d\mu_n$$

Pick $A > 0$ big. We have

$$\lim_{n\to\infty} \int \log |1 - x| d\mu_n \leq \lim_{n\to\infty} \int \max\{-A, \log |1 - x|\} d\mu_n$$

We assume that the logarithmic Mahler measure of α_n is uniformly bounded so the support of μ_n lies in a fixed compact set. Hence, the function on the right hand side is uniformly bounded on the support of all μ_n's. By Bilu's equidistribution theorem, the limit on the right hand side equals

$$\frac{1}{2\pi} \int_0^{2\pi} \max\{-A, \log |1 - e^{i\theta}|\} d\theta.$$

By Lebesgue dominated convergence we have

$$\lim_{A\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \max\{-A, \log |1 - e^{i\theta}|\} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}| d\theta.$$

Pick the branch of complex logarithm that is defined on $\mathbb{C} \setminus (-\infty, 0]$. Let C be the contour of the unit circle. For any $\tau > 0$ we have

$$\int_0^{2\pi} \log |1 + \tau - e^{i\theta}| d\theta = \mathrm{Im} \int_C \log(1 + \tau - z) \frac{dz}{z} = 0,$$

taking $\tau \to 0$ we get

$$\int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0.$$

It follows that

$$\lim_{n\to\infty} \frac{1}{[\mathbb{Q}(\alpha_n) : \mathbb{Q}]} \log |N_{\mathbb{Q}(\alpha_n)/\mathbb{Q}}(1 - \alpha_n)| = 0$$

which ends the proof. □

We shall show that

LEMMA 2.58. *Let Γ be an arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$ ($\mathbb{K}=\mathbb{R},\mathbb{C}$) with trace field k. Let $\gamma\in\Gamma$ non-torsion and $m(\gamma)\leq M$. Recall that we define $m(\gamma)$ as $m(\lambda_1)=m(\lambda_2)$ where λ_1,λ_2 are the non-trivial eigenvalues of $\mathrm{Ad}\,\gamma$. Then*

$$|N_{k/\mathbb{Q}}(\Delta(\gamma))|\ll e^{o_M([k:\mathbb{Q}])}.$$

$\Delta(\gamma)$ was defined in Section 2.2.

PROOF. Note that $\lambda_1\lambda_2=1$. Let $L=k(\lambda_i)$. Then

$$|N_{k/\mathbb{Q}}(\Delta(\gamma))|^{[L:k]}=\prod_{i=1}^{2}|N_{L/\mathbb{Q}}(1-\lambda_i)|=e^{o([L:\mathbb{Q}])}=e^{o([k:\mathbb{Q}])}.$$

The inequalities follow from Corollary 2.57 and the fact that $\mathbb{Q}(\lambda_i)=k(\lambda_i)$ by Lemma 2.69. □

THEOREM 2.59. *Let (α_i) be a sequence of algebraic integers such that $\frac{m(\alpha_i)}{[\mathbb{Q}(\alpha_i):\mathbb{Q}]}\to 0$ and $[\mathbb{Q}(\alpha_i):\mathbb{Q}]\to\infty$ as i tends to infinity.*

(1) *Let $k_i=\mathbb{Q}(\alpha_i)$ and π_{k_i} be the prime counting function for the field k_i i.e.*

$$\pi_{k_i}(x)=\#\{\mathfrak{p}\text{ prime ideal of }\mathcal{O}_{k_i}\mid N(\mathfrak{p})\leq x\}.$$

Then, for any $x>0$ we have

$$\lim_{i\to\infty}\frac{\pi_{k_i}(x)}{[k_i:\mathbb{Q}]}=0.$$

(2) *Let $N\geq 1$ be a natural number and for each $i\in\mathbb{N}$ let k_i be a subfield of $\mathbb{Q}(\alpha_i)$ with $[\mathbb{Q}(\alpha_i):k_i]\leq N$. Then, for any $x>0$ we have*

$$\lim_{i\to\infty}\frac{\pi_{k_i}(x)}{[k_i:\mathbb{Q}]}=0.$$

PROOF. We start with the proof of (1). Since for a fixed x we have only finitely many rational primes p less than x the theorem is equivalent to the following statement. For every rational prime p, let $n_{p,i}(x)$ be the number of prime ideals $\mathfrak{p}$ of $\mathcal{O}_{k_i}$ lying above p with $N(\mathfrak{p})\leq x$. Then

$$\lim_{i\to\infty}\frac{n_{p,i}(x)}{[k_i:\mathbb{Q}]}=0.$$

Let L be the largest natural number such that $p^L\leq x$. For every prime ideal $\mathfrak{p}$ of $\mathcal{O}_{k_i}$ with $\mathfrak{p}\mid p$ and $N(\mathfrak{p})\leq x$ we have $\alpha_i^{p^{L!}}-\alpha_i\in\mathfrak{p}$. Hence

$$\prod_{\substack{\mathfrak{p}\mid p\\ N(\mathfrak{p})\leq x}}\mathfrak{p}\mid\alpha_i^{p^{L!}}-\alpha_i, \tag{5.1}$$

so

$$p^{n_{p,i}(x)}\leq N(\alpha_i^{p^{L!}}-\alpha_i)=N(\alpha_i)N(\alpha_i^{p^{L!-1}}-1)\leq e^{o([k_i:\mathbb{Q}])}. \tag{5.2}$$

The last inequality is a consequence of Corollary 2.57.

To deduce (2) observe that $\pi_{k_i}(x)\leq\pi_{\mathbb{Q}(\alpha_i)}(x^{[\mathbb{Q}(\alpha_i):k_i]})\leq\pi_{\mathbb{Q}(\alpha_i)}(x^N)$ so

$$\lim_{i\to\infty}\frac{\pi_{k_i}(x)}{[k_i:\mathbb{Q}]}\leq N\lim_{i\to\infty}\frac{\pi_{\mathbb{Q}(\alpha_i)}(x^N)}{[\mathbb{Q}(\alpha_i):\mathbb{Q}]}=0. \tag{5.3}$$

□

As a simple corollary of Theorem 2.59 we get

COROLLARY 2.60. *Let N be a natural number, let α_i be a sequence of algebraic integers such that $\frac{m(\alpha_i)}{[\mathbb{Q}(\alpha_i):\mathbb{Q}]}\to 0$ as i tends to infinity. For every $i\in\mathbb{N}$ let k_i be a subfield of $\mathbb{Q}(\alpha_i)$ such that $[\mathbb{Q}(\alpha_i):k_i]\leq N$. Then for every $\sigma>1$ and $\mathrm{Re}(s)\geq\sigma$ we have $|\zeta_{k_i}(s)|=\exp(o_\sigma([k_i:\mathbb{Q}]))$.*

PROOF. Since $|\zeta_{k_i}(\mathrm{Re}(s))| \geq |\zeta_{k_i}(s)|$ we may assume that s is real. For $M > 0$ define

$$A_M(s) = \sum_{\substack{p^m-\text{prime power} \\ p^m \geq M}} \sum_{n=1}^{\infty} \frac{n^{-1}}{p^{mns}},$$

where the sum is taken over powers of rational primes. For $s \geq \sigma > 1$ we have

$$\log \zeta_{k_i}(s) = \sum_{\mathfrak{p} \in M_{k_i}^f} \sum_{n=1}^{\infty} \frac{n^{-1}}{N(\mathfrak{p})^{ns}} \leq \sum_{\substack{\mathfrak{p} \in M_{k_i}^f \\ N(\mathfrak{p}) \leq M}} \sum_{n=1}^{\infty} \frac{n^{-1}}{N(\mathfrak{p})^{ns}} + [k : \mathbb{Q}] A_M(s). \tag{5.4}$$

By Theorem 2.59

$$\lim_{i \to \infty} \frac{|\log \zeta_{k_i}(s)|}{[k_i : \mathbb{Q}]} \leq A_M(s).$$

But for $\mathrm{Re} s > 1$ $\lim_{M \to \infty} A_M(s) = 0$. The rate of convergence of $A_M(s)$ to 0 depends on σ, so we get that $\log \zeta_{k_i}(s) = o_\sigma([k : \mathbb{Q}])$. □

6. Orbital integrals

In this section we give upper bounds for the orbital integrals. We start with the local considerations. For non-archimedean places we consider the integrals of the form $\mathcal{O}_\gamma(\chi_\rho)$ where ρ is an irreducible representation of a maximal compact open subgroup U of $\mathrm{PD}^\times(\mathbb{A}_f)$ and D is an admissible division algebra. For archimedean places we estimate $\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})$ where $f_{\mathbb{A}_\infty}$ has uniformly bounded support.

6.1. Values of irreducible characters at regular elements. Let G be a simple group of type A_1 defined over $k_\mathfrak{p}$. Let U be a maximal compact subgroup of $G(k_\mathfrak{p})$. In this section we give an upper bound on the values of irreducible characters of U on the regular elements of U. The main result of this section is

THEOREM 2.61. *Let χ be a character of an irreducible representation of U. Then for all $\gamma \in U$ regular we have*

$$|\chi(\gamma)| \leq 8 |\Delta(\gamma)|_\mathfrak{p}^{-1}.$$

If γ is torsion of order $2, 3$ or 4 the constant 8 can be improved to $4, 8\sqrt{3}/3$ and $4\sqrt{2}$ respectively.

Following Serre[11] [**100**, Formule (21)] we shall use the Weyl integration formula. We choose a Haar measure dg on $G(k_\mathfrak{p})$ and for any torus T we choose compatible measures dx, dt on $G(k_\mathfrak{p})/T(k_\mathfrak{p}), T(k_\mathfrak{p})$ respectively. For a torus T defined over $k_\mathfrak{p}$ put $W(T) = N(T(k_\mathfrak{p}))/T(k_\mathfrak{p})$. For any continuous compactly supported function ϕ on $G(k_\mathfrak{p})$ we have

$$\int \phi(g) dg = \sum_{[T]_{G(k_\mathfrak{p})} \subset G} \frac{1}{|W(T)|} \int_{G(k_\mathfrak{p})/T(k_\mathfrak{p})} \int_{T(k_\mathfrak{p})} |\Delta(t)|_\mathfrak{p} \phi(xtx^{-1}) dt\, dx \tag{6.1}$$

Where the sum is taken over the set of $G(k_\mathfrak{p})$-conjugacy classes of maximal tori defined over $k_\mathfrak{p}$. We normalize dg so that the measure of U is 1 and dt so that the measure of the maximal compact subgroup of $T(k_\mathfrak{p})$ is 1. From it we can deduce a similar formula for U. A directed torus is a pair (T, c) consisting of a torus and a Weyl chamber $c \subset X_*(T) \otimes \mathbb{R}$. The group $G(k_\mathfrak{p})$ acts on the set of directed tori by conjugation and the stabilizer of (T, c) is precisely $T(k_\mathfrak{p})$. To distinguish directed tori from the usual ones we will usually denote

[11] The formula is stated there for central simple division algebras but the same statement holds for any reductive algebraic group.

them by $\overrightarrow{T}$. Note that the number of directed conjugacy classes $[\overrightarrow{T}]_{G(k_\mathfrak{p})}$ is a single class $[T]_{G(\mathfrak{p})}$ equals $\frac{W(G)}{W(T)}$. From (6.1) we deduce

$$\int_U \phi(g)dg = \frac{1}{|W_G|} \sum_{[\overrightarrow{T}]_{G(k_\mathfrak{p})} \subset G} \int_{G(k_\mathfrak{p})/T(k_\mathfrak{p})} \int_{T(k_\mathfrak{p})} |\Delta(t)|_\mathfrak{p} \phi(xtx^{-1}) dt\, dx \tag{6.2}$$

$$= \frac{1}{|W_G|} \sum_{[\overrightarrow{T}]_U \subset G} \int_{U/(T(k_\mathfrak{p})\cap U)} \int_{T(k_\mathfrak{p})\cap U} |\Delta(t)|_\mathfrak{p} \phi(utu^{-1}) dt\, du \tag{6.3}$$

To get the second equality we split the integral over a single $G(k_p)$-conjugacy class of directed tori into sum of integrals over U-conjugacy classes. For a class function on U this simplifies to

$$\int_U \phi(g)dg = \frac{1}{|W_G|} \sum_{[\overrightarrow{T}]_U \subset G} \frac{1}{\mu_T(U \cap T(k_\mathfrak{p}))} \int_{T(k_\mathfrak{p})\cap U} |\Delta(t)|_\mathfrak{p} \phi(t) dt \tag{6.4}$$

To simplify the notation, for any directed torus $\overrightarrow{T}$ we shall normalize the measure μ_T on $T(k_\mathfrak{p})$ so that $\mu_T(U \cap T) = 1$. Now we are ready to start the proof of Theorem 2.61. The proof is inspired by **[71]** where M. Larsen has proven an analogous result for finite groups of Lie type.

Proof. Let χ be a character of an irreducible representation of U. By the formula (6.4) we get

$$1 = \int_U |\chi(u)|^2 du = \frac{1}{|W_G|} \sum_{[\overrightarrow{T}]_U \in G} \int_{T(k_\mathfrak{p})\cap U} |\Delta(\gamma)|_\mathfrak{p} |\chi(t)|^2 dt \tag{6.5}$$

In particular

$$|W_G| \geq \int_{T(k_\mathfrak{p})\cap U} |\Delta(t)|_\mathfrak{p} |\chi(t)|^2 dt, \tag{6.6}$$

for any directed torus $\overrightarrow{T}$. Now take T such that $\gamma \in T(k_\mathfrak{p})$. We can do so because γ is regular. We shall approximate $|\Delta(t)|_\mathfrak{p}$ by an integral combination of characters of $T(k_\mathfrak{p}) \cap U$. Write $\lambda_1, \ldots, \lambda_r$ for the positive roots of T. For any $t \in T(k_\mathfrak{p})$ the Weyl discriminant is given by

$$\Delta(t) = \prod_{i=1}^{r} (1 - \lambda_i(t))(1 - \lambda_i(t)^{-1})$$

(cf. Notation). Note that the image of $U \cap T(k_\mathfrak{p})$ via any character of $T(k_\mathfrak{p})$ has to be compact so $\lambda_i(U \cap T(k_\mathfrak{p})) \subset O_{k_\mathfrak{p}}^\times$ for $i = 1, \ldots, r$. For any $i = 1, \ldots, r$ we pick a character $\theta_i \colon \mathcal{O}_{k_\mathfrak{p}}^\times \to \mathbb{C}^\times$ such that $\theta_i(\lambda_i(\gamma)) \neq 1$ but $\theta_i(\lambda_i(t)) = 1$ for any t satisfying

$$|1 - \lambda_i(t)|_\mathfrak{p} < |1 - \lambda_i(\gamma)|_\mathfrak{p}. \tag{6.7}$$

Define $\Theta : T(k_\mathfrak{p}) \cap U \to \mathbb{C}$ by

$$\Theta = \prod_{i=1}^{r} (1 - \theta_i \circ \lambda_i).$$

We can choose characters θ_i is such a way that $|\Theta(\gamma)| \geq 1$ and at least $2, \sqrt{3}, \sqrt{2}$ if γ is $2, 3, 4$ torsion respectively. Because of condition (6.7) $\Theta(t) = 0$ for all t with $|\Delta(t)|_\mathfrak{p} < |\Delta(\gamma)|_\mathfrak{p}$. It follows that $|\Theta(t)|^2/|\Delta(t)|_\mathfrak{p} \leq |\Theta(t)|^2/|\Delta(\gamma)|_\mathfrak{p}$. We combine this inequality with (6.6) to get

$$\int_{T(k_\mathfrak{p})\cap U} |\Theta(t)|^2 |\chi(t)|^2 dt \leq \sup_{t \in T(k_p)\cap U} |\Theta(t)|^2 |\Delta(\gamma)|_\mathfrak{p}^{-1} |W_G|. \tag{6.8}$$

The function $\Theta(t)\chi(t)$ is an integral combination of characters of $T(k_\mathfrak{p}) \cap U$. We can write

$$\Theta(t)\chi(t) = \sum_{\zeta \in \Pi(T(k_\mathfrak{p})\cap U)} c_\zeta \zeta(t) \text{ with } c_\zeta \in \mathbb{Z} \tag{6.9}$$

Function Θ can be trivially bounded from above by 2^r where r is the number of positive roots. The inequality (6.8) implies that $\sum_{\zeta\in\Pi(T(k_\mathfrak{p})\cap U)} c_\zeta^2 \leq 2^{2r}|\Delta(\gamma)|_\mathfrak{p}^{-1}|W_G|$ so in particular

$$|\chi(\gamma)| \leq |\Theta(\gamma)\chi(\gamma)| = \left|\sum_{\zeta\in\Pi(T(k_\mathfrak{p})\cap U)} c_\zeta\zeta(\gamma)\right| \leq \sum_{\zeta\in\Pi(T(k_\mathfrak{p})\cap U)} |c_\zeta| \leq 2^{2r}|\Delta(\gamma)|_\mathfrak{p}^{-1}|W_G|. \tag{6.10}$$

In case when G if of type A_1 we have $r=1$ and $|W_G|=2$ which ends the proof. □

REMARK 2.62. *Note that we use the assumption that G is of type A_1 only in the last line of the proof. In particular the bound (6.10) holds for every reductive group G over $k_\mathfrak{p}$, any open compact subgroup U of $G(k_\mathfrak{p})$. The regularity assumption on γ is essential, otherwise $\Delta(\gamma)=0$.*

6.2. Estimates on the non-archimedean local orbital integrals. The aim of this section is to estimate the local orbital integrals of the characteristic functions of maximal open compacts subsets of $\mathrm{PD}^\times(k_\mathfrak{p})$ for $\mathfrak{p}\in M_k^f$. Note that in the local case $\mathrm{PD}^\times(k_\mathfrak{p})$ is either anisotropic and compact or isomorphic to $\mathrm{PGL}(2,k_\mathfrak{p})$ and we know the explicit description of the maximal compact subgroups. In the anisotropic case

$$\mathcal{O}_\gamma(\mathbb{1}_{U_\mathfrak{p}}) = \int_{PD_\gamma^\times(k_\mathfrak{p})\backslash \mathrm{PD}^\times(k_\mathfrak{p})} 1dx = 1$$

so we are left to deal with the case $\mathrm{PD}^\times(k_\mathfrak{p}) \simeq \mathrm{PGL}(2,k_\mathfrak{p})$. We start by recalling the basic properties of the Bruhat-Tits tree (for necessary background see [**94**, Chapter 5.2.1]). We write $X(\mathrm{SL}(2,k_\mathfrak{p}))$ for the Bruhat-Tits tree of $\mathrm{SL}(2,k_\mathfrak{p})$. It is a regular $q+1$-valent tree. We endow it with the natural metric. The group $\mathrm{PGL}(2,k_\mathfrak{p})$ acts on $X(\mathrm{SL}(2,k_\mathfrak{p}))$ by graph automorphisms. This action is transitive on vertices and on edges. Every maximal subgroup of $\mathrm{PGL}(2,k_\mathfrak{p})$ is a stabilizer of a vertex or of an edge. By a geodesic on $X(\mathrm{SL}(2,k_\mathfrak{p}))$ we mean a path on $X(\mathrm{SL}(2,k_\mathfrak{p}))$ infinite in both directions. If T is a maximal split torus of $\mathrm{PGL}(2,k_\mathfrak{p})$ then $T(k_\mathfrak{p})$ stabilizes a unique geodesic on $X(\mathrm{SL}(2,k_\mathfrak{p}))$, we shall call it the apartment of T. The argument below follows the exposition from [**8**, Chapter 5] (see also [**70**]). We shall write ν for the normalized additive $\mathfrak{p}$-adic valuation on $k_\mathfrak{p}$ and ν_E for the unique extension ν to a finite field extension $E/k_\mathfrak{p}$.

LEMMA 2.63. *Let γ be a regular element of $\mathrm{PGL}(2,k_\mathfrak{p})$ and let T be the connected component of its centralizer. Let F_γ be the subset of $X(\mathrm{SL}(2,k_\mathfrak{p}))$ fixed by γ. If $\Delta(\gamma)\notin\mathcal{O}_{k_\mathfrak{p}}$ then $F_\gamma=\emptyset$.*

1. *If T is split over $k_\mathfrak{p}$ then F_γ is a strip of radius $\nu(\Delta(\gamma))/2$ around the apartment of T.*
2. *If T is split over an unramified quadratic extension $E/k_\mathfrak{p}$ then F_γ is a ball of radius $\nu(\Delta(\gamma))/2$ around a vertex of $X(\mathrm{SL}(2,k_\mathfrak{p}))$.*
3. *If T is split by a tamely ramified quadratic extension $E/k_\mathfrak{p}$ then F_γ is a ball of radius $\nu(\Delta(\gamma))/2$ around the midpoint of an edge of $X(\mathrm{SL}(2,k_\mathfrak{p}))$.*
4. *If T is split over a wildly ramified extension $E/k_\mathfrak{p}$. Let σ be the generator of the Galois group $\mathrm{Gal}(E/k_\mathfrak{p})$. Put $w=\min\{\nu_E(\frac{a^\sigma}{a}-1)\mid a\in E^\times\}$ then F_γ is contained in a ball of radius $\nu(\Delta(\gamma))/2-w/2$ around the midpoint of an edge in $X(\mathrm{SL}(2,k_\mathfrak{p}))$.*

PROOF. By abuse of notation let us replace γ with its lift to $\mathrm{GL}(2,k_\mathfrak{p})$ so that we can write it down as an explicit matrix. If $\Delta(\gamma)\notin\mathcal{O}_{k_\mathfrak{p}}$ then the group generated by γ is not compact so it does not stabilize any point of $X(\mathrm{SL}(2,k_\mathfrak{p}))$. From now on assume $\Delta(\gamma)\in\mathcal{O}_{k_\mathfrak{p}}$. We will write d for the distance on tree, A_T for the apartment of T and q for the cardinality of the residue field of $k_\mathfrak{p}$.

1. **Split case** Without loss of generality assume that T is the group of diagonal matrices and that the entries of γ are in $\mathcal{O}_{k_\mathfrak{p}}$. We can do it because $\Delta(\gamma)\in\mathcal{O}_{k_\mathfrak{p}}$. Fix a vertex v of $X(\mathrm{SL}(2,k_\mathfrak{p}))$ stabilized by $K=\mathrm{GL}(2,\mathcal{O}_{k_\mathfrak{p}})$ and assume that gv

is fixed by γ for certain $g \in \mathrm{GL}(n, k_\mathfrak{p})$. Using the Iwasawa decomposition we write g as the product $g = ank$ where $a \in T(k)$ n is upper triangular unipotent and $k \in \mathrm{GL}(2, k_\mathfrak{p})$. The vertex gv is stabilized by γ if only if nv is. This is equivalent to saying that $n\gamma n^{-1} \in \mathrm{GL}(2, \mathcal{O}_{k_\mathfrak{p}})$ or $\gamma^{-1}n\gamma n^{-1} \in \mathrm{GL}(2, \mathcal{O}_{k_\mathfrak{p}})$.

$$\text{if } \gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ then } \gamma^{-1}n\gamma n^{-1} = \begin{pmatrix} 1 & x(1-\frac{a}{b}) \\ 0 & 1 \end{pmatrix}. \tag{6.11}$$

It follows that $\gamma^{-1}n\gamma n^{-1} \in \mathrm{GL}(2, \mathcal{O}_{k_\mathfrak{p}})$ if and only if $\nu(x) \geq -\nu(1-\frac{a}{b}) = -\nu(\Delta(\gamma))/2$. The set of vertices of form anv with $a \in T(k_\mathfrak{p})$ and $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ subject to the condition $\nu(x) \geq -\nu(1-\frac{a}{b}) = -\nu(\Delta(\gamma))/2$ forms a strip of radius $\nu(\Delta(\gamma))/2$ around the apartment of T which ends the proof.

(2) **Unramified case** By [**26**, Théorème 5.1.25] there is a natural embedding $\iota\colon X(\mathrm{SL}(2, k_\mathfrak{p})) \to X(\mathrm{SL}(2, E))$ and an action of the Galois group on $X(\mathrm{SL}(2, E))$ such that

$$X(\mathrm{SL}(2, E))^{\mathrm{Gal}(E/k_\mathfrak{p})} = \iota(X(\mathrm{SL}(2, k_\mathfrak{p}))).$$

Moreover ι is an isometry. The set of points in $X(\mathrm{SL}(2, E))$ fixed under the action of γ is a strip of radius $\nu(\Delta(\gamma))/2$ around the apartment of T which we will call $\mathcal{A}_T$. To finish the proof of this case it will be sufficient to show that $\mathcal{A}_T$ intersects $\iota(X(\mathrm{SL}(2, k_\mathfrak{p}))$ in a single vertex v. Then, since $\iota(X(\mathrm{SL}(2, k_\mathfrak{p}))$ is a totally geodesic subspace, its intersection with this strip will be a ball around v of radius $\nu(\Delta(\gamma))/2$.

T is defined over $k_\mathfrak{p}$ so the apartment $\mathcal{A}_T$ is stable under $\mathrm{Gal}(E/k_\mathfrak{p})$. The torus T is not split over $k_\mathfrak{p}$ so the Galois group acts non-trivially on $\mathcal{A}_T$. Thus the only possible action is an orientation reversing isometry which has a unique fixed point x. Thus $\mathcal{A}_T^{\mathrm{Gal}}(E/k_\mathfrak{p}) = \mathcal{A}_T \cap \iota(X(\mathrm{SL}(2, k_\mathfrak{p}))$ consists of a single point which has to be a vertex, because of the geometry of a tree. This ends the proof in the unramified case.

(3) **Tamely ramified case** The argument is the same as in the unramified case except that $\iota\colon X(\mathrm{SL}(2, k_\mathfrak{p})) \to X(\mathrm{SL}(2, E))$ is no longer an isometry. The image $\iota(X(\mathrm{SL}(2, k_\mathfrak{p}))$ contains vertices which are not images of a vertex. More precisely, the images of midpoints of edges in $X(\mathrm{SL}(2, k_\mathfrak{p}))$ are vertices in $X(\mathrm{SL}(2, E))$ and those are the only new vertices in $\iota(X(\mathrm{SL}(2, k_\mathfrak{p}))$. In particular $d(\iota(x), \iota(y)) = 2d(x, y)$ for any two vertices x, y. By the work of Prasad [**90**] for a tamely ramified extension $E/k_\mathfrak{p}$ we have $\iota(X(\mathrm{SL}(2, k_\mathfrak{p}))) = X(\mathrm{SL}(2, E))^{\mathrm{Gal}(E/k_\mathfrak{p})}$ so the argument from the unramified case, tells us that $\mathcal{A}_T^{\mathrm{Gal}(E:k_\mathfrak{p})}$ consists of a single point which has to be a vertex v inside $\iota(X(\mathrm{SL}(2, k_\mathfrak{p}))$. We argue that it has to be an image of midpoint of an edge. Assume the contrary that $v = \iota(x)$. We know that x has $q+1$ adjacent edges. The extension is totally ramified so $\iota(x)$ also has $q+1$ adjacent edges. It follows that all edges adjacent to $\iota(x)$ are $\mathrm{Gal}(E/k_\mathfrak{p})$ stable which would mean that $\mathcal{A}_T$ contains an edge fixed by $\mathrm{Gal}(E/k_\mathfrak{p})$. This contradicts what we already know. It follows that F_γ is a ball of radius $\nu(\Delta(\gamma))/2$ around a midpoint of an edge.

(4) **Wildly ramified case** We argue as in the tamely ramified case. The only difference is that there exists $w > 0$ such that $X(\mathrm{SL}(2, E))^{\mathrm{Gal}(E/k_\mathfrak{p})}$ contains a tubular neighborhood of radius w around $\iota(X(\mathrm{SL}(2, k_\mathfrak{p}))$. As before $\mathcal{A}_T$ has a unique fixed point fixed by $\mathrm{Gal}(E/k_\mathfrak{p})$ and by uniqueness it has to coincide with the vertex of $\mathcal{A}_T$ closest to $\iota(X(\mathrm{SL}(2, k_\mathfrak{p}))$. We write v for the unique fixed point of A_T and v' for the vertex of $\iota(X(\mathrm{SL}(2, k_\mathfrak{p}))$. As in the tamely ramified case we can show that v' is a midpoint of an edge and that $d(v, v') \geq w$. It follows that F_γ is contained a ball of radius $[\nu(\Delta) - w]/2$ around a midpoint of an edge. It remains to compute w which we do in the Lemma 2.64.

□

LEMMA 2.64. *Let E be a wildly ramified quadratic extension of $k_\mathfrak{p}$ and $w = \min\{\nu_E(a - a^\sigma) \mid a \in \mathcal{O}_E\}$. Let ι be the natural injection $\iota\colon X(\mathrm{SL}(2,k_\mathfrak{p})) \to X(\mathrm{SL}(2,E))$. Then*

$$B(\iota(X(\mathrm{SL}(2,k_\mathfrak{p})),w) \subset X(\mathrm{SL}(2,E))^{\mathrm{Gal}(E/k_\mathfrak{p})}$$

PROOF. To prove the inclusion it is sufficient to show that for every apartment A in $X(\mathrm{SL}(2,k_\mathfrak{p}))$ the tubular neighborhood $B(\iota(A),w)$ is pointwise fixed by $\mathrm{Gal}(E/k_\mathfrak{p})$. Since all apartments are in a single $\mathrm{SL}(2,k_\mathfrak{p})$ orbit we may without loss of generality assume that A is the apartment associated to the diagonal torus. In this case we have a vertex $\mathfrak{o} \in A$ stabilized by $PGL(2,\mathcal{O}_{k_\mathfrak{p}})$. We abuse the notation and also write $\mathfrak{o}$ for the image of $\mathfrak{o}$ in $X(\mathrm{SL}(2,E))$ via ι, this one is stabilized by $PGL(2,\mathcal{O}_E)$. All vertices of $\iota(A)$ are of form $t\mathfrak{o}$ where $t = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $a \in E^\times$. Write π_E for the uniformiser of E and π_k for the uniformiser of $k_\mathfrak{p}$. For any real number R and vertex $v \in B(\iota(A),R)$ there exists $x \in E$ with $v_E(x) \geq -R$ and $n \in \mathbb{Z}$ such that

$$v = \begin{pmatrix} \pi_E^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mathfrak{o}$$

Write σ for the generator of $\mathrm{Gal}(E/k_\mathfrak{p})$. Vertex $\mathfrak{o}$ is $\mathrm{Gal}(E/k_\mathfrak{p})$ invariant so $v \in X(\mathrm{SL}(2,E))^{\mathrm{Gal}(E:k_\mathfrak{p})}$ if and only if

$$\begin{pmatrix} \pi_E^{-n} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\pi_E^\sigma)^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x^\sigma \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}(2,\mathcal{O}_E) \tag{6.12}$$

$$\begin{pmatrix} (\pi_E^{-1}\pi_E^\sigma)^n & (\pi_E^{-1}\pi_E^\sigma)^n x^\sigma - x \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}(2,\mathcal{O}_E). \tag{6.13}$$

As $(\pi_E^{-1}\pi_E^\sigma) \in \mathcal{O}_E$ this is equivalent to $(\pi_E^{-1}\pi_E^\sigma)^n x^\sigma - x \in \mathcal{O}_E$. We have to check that $\nu_E((\pi_E^{-1}\pi_E^\sigma)^n x^\sigma - x) \geq 0$.

$$\nu_E((\pi_E^{-1}\pi_E^\sigma)^n x^\sigma - x) = \nu_E\left(\frac{(\pi_E^n x)^\sigma}{\pi_E^n x} - 1\right) + \nu_E(x) \tag{6.14}$$

$$\geq w + \nu_E(\pi^n x). \tag{6.15}$$

Thus v is fixed by $\mathrm{Gal}(E/k_\mathfrak{p})$ as long as $\nu(x) \geq -w$. It follows that all vertices in the tubular neighborhood $B(\iota(A),w)$ are fixed. Since $X(\mathrm{SL}(2,k_\mathfrak{p}))$ is the sum of its apartments we get that $B(\iota(X(\mathrm{SL}(2,k_\mathfrak{p})),w)$ is fixed by the Galois group.

□

Using 2.61 and 2.63 we can now give an upper bound on the orbital integrals of type $\mathcal{O}_\gamma(\chi)$ where γ is a regular element and χ is the character of an irreducible representation of a maximal compact subgroup of $\mathrm{PD}^\times(k_\mathfrak{p})$.

PROPOSITION 2.65. *Let G be an adjoint group of type A_1 defined over $k_\mathfrak{p}$, let $\gamma \in G(k_\mathfrak{p})$ be a regular element and U be a maximal compact subgroup of $G(k_\mathfrak{p})$. Put $M = \min\{8|\Delta(\gamma)|_\mathfrak{p}^{-1}, \chi(1)\}$. Let q be the cardinality of the residue field of $k_\mathfrak{p}$. Then for any irreducible character χ of U we have*

$$|\mathcal{O}_\gamma(\chi)| \leq M\left(|\Delta(\gamma)|_\mathfrak{p}^{-1/2} + b\frac{|\Delta(\gamma)|_\mathfrak{p}^{-1/2}}{q-1}\right),$$

where $b = 0$ if the centralizer of γ is split or $|\Delta(\gamma)|_\mathfrak{p} = 1$ and $b = 2$ otherwise. If G is anisotropic then we have

$$|\mathcal{O}_\gamma(\chi)| \leq M.$$

Moreover if γ is $2,3,4$ torsion the constant 8 can be improved to $4, 8\sqrt{3}/3$ and $4\sqrt{2}$ respectively.

PROOF. We start with the case $G \simeq \mathrm{PGL}(2,k_\mathfrak{p})$. We shall prove that

$$|\mathcal{O}_\gamma(\mathbb{1}_U)| \leq |\Delta(\gamma)|_\mathfrak{p}^{-1/2} + b\frac{|\Delta(\gamma)|_\mathfrak{p}^{-1/2}}{q-1}. \tag{6.16}$$

When dealing with orbital integrals we fix the standard Haar measure μ_G on $G(k_\mathfrak{p})$ giving mass 1 to U and the standard Haar measure μ_{G_γ} on $G_\gamma(k_\mathfrak{p})$ giving mass 1 to the maximal compact subgroup of $G_\gamma(k_\mathfrak{p})$. The integration will be done against a measure on $G_\gamma(k_\mathfrak{p})\backslash G(k_\mathfrak{p})$ compatible with those two. We have

$$\int_{G_\gamma(k_\mathfrak{p})\backslash G(k_\mathfrak{p})} \mathbb{1}_U(x^{-1}\gamma x)dx = \sum_{g\in G_\gamma(k_\mathfrak{p})\backslash G(k_\mathfrak{p})/U} \frac{\mathbb{1}_U(g^{-1}\gamma g)}{\mu_{G_\gamma}(G_\gamma(k_\mathfrak{p})\cap gUg^{-1})} \tag{6.17}$$

We shall give an interpretation of the right hand side in terms of action of γ on the Bruhat-Tits tree $X(\mathrm{SL}(2,k_\mathfrak{p}))$. U is a maximal compact subgroup of $G(k_\mathfrak{p})$ so it stabilizes a vertex or an edge of $X(\mathrm{SL}(2,k_\mathfrak{p}))$. Note that $\mathbb{1}_U(g^{-1}\gamma g)=1$ if and only if $\gamma gU=gU$. If we let $\mathfrak{o}$ be the vertex (edge) stabilized by U then $\mathbb{1}_U(g^{-1}\gamma g)=1$ if and only if $\mathfrak{o}$ is stabilized by γ. Write T for the maximal compact subgroup of $G_\gamma(k_\mathfrak{p})$. We have chosen μ_{G_γ} so that $\mu_{G_\gamma}(T)=1$ so $\mu_{G_\gamma}(G_\gamma(k_\mathfrak{p})\cap gUg^{-1})^{-1}=[T:G_\gamma(k_\mathfrak{p})\cap gUg^{-1}]$. The group $G_\gamma(k_\mathfrak{p})\cap gUg^{-1}$ is the stabilizer of $g\mathfrak{o}$ under the action of T so we have

$$\mu_{G_\gamma}(G_\gamma(k_\mathfrak{p})\cap gUg^{-1}) = \#\{Tg\mathfrak{o}\}.$$

Thus the right hand side of 6.17 is the sum of sizes of orbits of T fixed by γ lying in pairwise disjoint $G_\gamma(k_\mathfrak{p})$ orbits. Let S be a T-invariant subset of vertices (edges) of $X(\mathrm{SL}(2,k_\mathfrak{p}))$ fixed by γ such that

- every $G_\gamma(k_\mathfrak{p})$ orbit fixed by γ has nonempty intersection with S;
- if $x,y\in S$ are in the same $G_\gamma(k_\mathfrak{p})$ orbit then they are in the same T-orbit.

We will refer to these conditions as to $(*)$. It is easy to see that the right hand side of 6.17 must be equal to $\#S$. It remains to construct such a set and compute its cardinality. In the case when G_γ is anisotropic the group $G_\gamma(k_\mathfrak{p})$ is compact so we have $T=G_\gamma(k_\mathfrak{p})$ and we can take S to be the set of vertices (edges) fixed by γ. let E be the quadratic extension of $k_\mathfrak{p}$ generated by the eigenvalues of γ and let q be the cardinality of the residue field of $k_\mathfrak{p}$. Lemma 2.63 implies that

- if $E:k_\mathfrak{p}$ is unramified then

$$\#S = |\Delta(\gamma)|_\mathfrak{p}^{-1/2} + 2\frac{|\Delta(\gamma)|_\mathfrak{p}^{-1/2}-1}{q-1} \qquad \text{(vertices)}, \tag{6.18}$$

$$\#S = |\Delta(\gamma)|_\mathfrak{p}^{-1/2} + 2\frac{|\Delta(\gamma)|_\mathfrak{p}^{-1/2}-1}{q-1} - 1 \qquad \text{(edges)}. \tag{6.19}$$

- if $E/k_\mathfrak{p}$ is ramified then

$$\#S \le 2\frac{|\Delta(\gamma)]|^{-1/2}-q^{-1/2}}{q^{3/2}-q^{1/2}} \qquad \text{(vertices)}, \tag{6.20}$$

$$\#S \le 2\frac{|\Delta(\gamma)]|^{-1/2}-q^{-1/2}}{q^{3/2}-q^{1/2}} - 1 \qquad \text{(edges)}. \tag{6.21}$$

$$\tag{6.22}$$

  with equalities if $E/k_\mathfrak{p}$ is tamely ramified.

In the case when G_γ is split the set of points fixed by γ is too big. In order to choose a subset satisfying $(*)$ we proceed as follows. Let A be the apartment of $X(\mathrm{SL}(2,k_\mathfrak{p}))$ associated to G_γ. Fix a vertex $\mathfrak{o}\in A$ and an edge $\mathfrak{e}\in A$ adjacent to $\mathfrak{o}$. Let S' be the subset of vertices (edges) v of $X(\mathrm{SL}(2,k_\mathfrak{p}))\setminus A$ fixed by γ such that the vertex of A closest to v is $\mathfrak{o}$. We let $S=\mathfrak{o}\cup S'$ if U fixes a vertex and $S=\{\mathfrak{e}\}\cup S'$ if U stabilizes an edge. We verify that S satisfies $(*)$. Note that T fixes A pointwise so by definition S is T invariant. Let u be any vertex (edge) stabilized by γ and let w be the vertex of A closest to u. For any $t\in G_\gamma(k_\mathfrak{p})$ the vertex tw is the point of A closest to tu. The group $G_\gamma(k_\mathfrak{p})$ acts transitively on the set of vertices of A so there exists a $t_0\in G_\gamma(k_\mathfrak{p})$ such that $t_0u\in S$. This proves that every orbit of $G_\gamma(k_\mathfrak{p})$ fixed by γ intersects S. It remains to check that if two elements x,y of S are in the same G_γ orbit then they lie in the same T orbit. If $x=ty$ for some

$t \in G_\gamma(k_\mathfrak{p})$ then $t\mathfrak{o} = \mathfrak{o}$, but the stabilizer of $\mathfrak{o}$ is T so the condition $(*)$ holds. By Lemma 2.63 we get

$$\#S = |\Delta(\gamma)|_\mathfrak{p}^{-1/2} \qquad \text{(vertices)}, \tag{6.23}$$

$$\#S = |\Delta(\gamma)|_\mathfrak{p}^{-1/2} \qquad \text{(edges)}. \tag{6.24}$$

Combining the inequalities above we get inequality (6.16). Now we can invoke Theorem 2.61 to give the estimate:

$$|\mathcal{O}_\gamma(\chi)| \leq |\mathcal{O}_\gamma(\mathbb{1}_{U_\mathfrak{p}})| \max_{\gamma' \in [\gamma]_{G(k_\mathfrak{p})}} |\chi(\gamma')| \leq \left(|\Delta(\gamma)|_\mathfrak{p}^{-1/2} + b \frac{|\Delta(\gamma)|_\mathfrak{p}^{-1/2} - 1}{q-1} \right) M_\gamma. \tag{6.25}$$

We recall that $M_\gamma = \min\{8|\Delta(\gamma)|_\mathfrak{p}^{-1}, \chi(1)\}$. If G is anisotropic over $k_\mathfrak{p}$ then $U = G(k_\mathfrak{p})$ and we trivially have $\mathcal{O}_\gamma(\mathbb{1}_U) = 1$ so by Theorem 2.61

$$|\mathcal{O}_\gamma(\chi)| \leq M_\gamma. \tag{6.26}$$

If γ is $2, 3, 4$ torsion the constant 8 can be improved to $4, 8\sqrt{3}/3$ and $4\sqrt{2}$ respectively. $\square$

6.3. Archimedean Orbital Integrals. Let $G = PGL(2, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, let γ be a regular semisimple element of G. For a compactly supported continuous function f of G we shall bound $|\mathcal{O}_\gamma(f)|$ in terms of $\|f\|_\infty, |\Delta(\gamma)|$ and the size of the support of f. To speak about size we have to choose a metric on G. We fix a norm on G given as follows. For $g \in G$ we choose a representative $A \in \mathrm{GL}(2, \mathbb{K})$

$$\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.27}$$

with $|\det A| = 1$. We define the Frobenius norm of g as $\|g\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 = \mathrm{tr} A\, {}^\dagger A$. Let K be the maximal compact subgroup of G fixed by the conjugate transpose. As ${}^\dagger k = k^{-1}$ for any $k \in K$ the norm $\|\cdot\|$ is bi-K invariant. We fix a metric $d\colon G \times G \to \mathbb{R}_+$ by setting

$$d(g, h) = \|g^{-1}h\|.$$

Proposition 2.66. *Fix $R > 0$. Let γ be a regular semisimple element of G such that G_γ splits over $\mathbb{K}$. For any continuous function $f \in C_c(G)$ with $\mathrm{supp} f \subset B(1, R)$ we have*

$$|\mathcal{O}_\gamma(f)| \ll_R |\Delta(\gamma)|^{-1/2} \|f\|_\infty. \tag{6.28}$$

Proof. Put $f^K(g) := \int_K f(k^{-1}gk)dk$, then $\mathcal{O}_\gamma(f) = \mathcal{O}_\gamma(f^K)$, $\|f^K\|_\infty \leq \|f\|_\infty$ and $\mathrm{supp} f^K \subset B(1, R)$ because the metric is bi-K invariant. Hence it is enough to show the theorem for functions which are constant on the K-conjugacy classes. From now on assume that f is constant on K conjugacy classes. Choose a measurable bounded function α on G such that $\int_{G_\gamma} \alpha(tg)dt = 1$ for all $g \in G$. Then

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx = \int_G \alpha(g) f(g^{-1}\gamma g)dg. \tag{6.29}$$

Since G_γ is split, it is conjugate to the subgroup of diagonal matrices A. Since our problem is conjugation invariant we may assume $G_\gamma = A$. We have Iwasawa decomposition $G = ANK$ where N is the group of unipotent upper triangular matrices. By [**69**, Theorem 2.5.1] for any integrable h we have

$$\int_G h(g)dg = \int_A \int_N \int_K h(ank)dadndk. \tag{6.30}$$

Hence

$$\mathcal{O}_\gamma(f) = \int_A \int_N \int_K \alpha(ank) f(k^{-1}n^{-1}a^{-1}\gamma ank) da dn dk \tag{6.31}$$

$$= \int_A \int_N \int_K \alpha(ank) f(n^{-1}\gamma n) da dn dk \tag{6.32}$$

$$= \int_N f(n^{-1}\gamma n) dn \tag{6.33}$$

$$= \int_N f(\gamma(\gamma^{-1}n^{-1}\gamma n)) dn. \tag{6.34}$$

The map $\phi\colon n \mapsto \gamma^{-1}n^{-1}\gamma n$ is a differentiable automorphism of N. If γ has a lift $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ then it is easy to check that the Jacobian of ϕ is constant and equal to $1 - \frac{b}{a}$. Thus

$$\mathcal{O}_\gamma(f) = \int_N f(\gamma(\gamma^{-1}n^{-1}\gamma n)) dn = \left|1 - \frac{b}{a}\right|^{-1} \int_N f(\gamma n) dn \ll_R \left|1 - \frac{b}{a}\right|^{-1} \|f\|_\infty. \tag{6.35}$$

The last inequality stems from the fact that the support of f is contained in a ball of radius R so the rightmost integral can be bounded by $\sqrt{2}R\|f\|_\infty$. Elements γ and γ^{-1} are conjugate under the Weyl group so $\mathcal{O}_\gamma(f) = \mathcal{O}_{\gamma^{-1}}(f)$. From this we deduce that

$$|\mathcal{O}_\gamma(f)| \ll_R \min\left\{\left|1 - \frac{b}{a}\right|^{-1}, \left|1 - \frac{a}{b}\right|^{-1}\right\} \|f\|_\infty \tag{6.36}$$

$$\leq \left|\left(1 - \frac{b}{a}\right)\left(1 - \frac{a}{b}\right)\right|^{-1/2} \|f\|_\infty \tag{6.37}$$

$$= |\Delta(\gamma)|^{-1/2} \|f\|_\infty. \tag{6.38}$$

□

We have an analogous inequality for elements with anisotropic centralizer. Note that this can only happen in $\mathrm{PGL}(2,\mathbb{R})$ since $\mathbb{C}$ is algebraically closed.

Proposition 2.67. *Let $G = \mathrm{PGL}(2,\mathbb{R})$ and let γ be a regular semisimple element with anisotropic centralizer. Then for any continuous function $f \in C_c(G)$ with $\mathrm{supp} f \subset B(1,R)$ we have*

$$|\mathcal{O}_\gamma(f)| \ll_R |\Delta(\gamma)|^{-1} \|f\|_\infty. \tag{6.39}$$

Proof. Note that in $\mathrm{PGL}(2,\mathbb{R})$ all semisimple elements with anisotropic centralizer are conjugate to an element of the group

$$K = \left\langle \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid \theta \in \mathbb{R} \right\rangle,$$

and their connected centralizer is conjugate to the group

$$K_0 = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}.$$

Without loss on generality we may assume this is the case for γ. As in the proof of Proposition 2.66 we can assume that f is constant on the K conjugacy classes. We have the Cartan decomposition

$$G = K_0 A^+ K \text{ where } A^+ = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \geq 1 \right\}.$$

Define the function $J\colon A^+ \to \mathbb{R}_+$ by

$$J\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \left(\frac{a + a^{-1}}{2}\right)^2.$$

Then, by **[63**, p.37] we have

$$\int_G h(g)dg = \int_{K_0}\int_{A^+}\int_K J(a)h(k_0ak)dk_0dadk.$$

Fix a function $\alpha\colon G \to \mathbb{R}$ such that $\int_{K_0}\alpha(tg)dt = 1$ for all $g \in G$. Then

$$\mathcal{O}_\gamma(f) = \int_{K_0\backslash G} f(x^{-1}\gamma x)dx = \int_G \alpha(g)f(g^{-1}\gamma g)dg \tag{6.40}$$

$$= \int_{K_0}\int_{A^+}\int_K J(a)f(k^{-1}a^{-1}k_0^{-1}\gamma k_0 ak)dk_0dadk \tag{6.41}$$

$$= \int_{K_0}\int_{A^+}\int_K J(a)f(a^{-1}\gamma a)dk_0dadk \tag{6.42}$$

$$= \int_1^\infty \left(\frac{t+t^{-1}}{2}\right)^2 f\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\right)\frac{dt}{t}. \tag{6.43}$$

Write $\gamma = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$, we have

$$\begin{pmatrix} t^{-1} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} \cos\theta & t^{-1}\sin\theta \\ -t\sin\theta & \cos\theta\end{pmatrix}. \tag{6.44}$$

The distance of the last matrix to 1 is given by $\left(2(\cos\theta-1)^2 + (t^2+t^{-2})\sin^2\theta\right)^{1/2} \geq t|\sin\theta|$. The support of f lies in the ball $B(1,R)$ so from equation 6.40 we get

$$|\mathcal{O}_\gamma(f)| \leq \|f\|_\infty \int_1^{\frac{R}{|\sin\theta|}} \left(\frac{t+t^{-1}}{2}\right)^2 \frac{dt}{t} \ll_R |\sin\theta|^2. \tag{6.45}$$

It remains to relate $|\sin\theta|$ and $|\Delta(\gamma)|$. In $\mathrm{PGL}(2,\mathbb{C})$ the element γ is conjugate to the matrix $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$. Therefore $\Delta(\gamma) = (1-e^{2i\theta})(1-e^{-2i\theta}) = (e^{-i\theta}-e^{i\theta})(e^{i\theta}-e^{-i\theta}) = -\sin^2\theta$. Combining this with inequality 6.45 we get

$$|\mathcal{O}_\gamma(f)| \ll_R \|f\|_\infty |\Delta(\gamma)|^{-1}. \tag{6.46}$$

□

Now we shall combine previous estimates with number theoretic input to obtain estimates on the archimedean orbital integrals for regular elements of an arithmetic lattice. The key ingredient is a theorem due to Dobrowolski **[40]**

THEOREM 2.68. *Let α be an algebraic number of degree d. Then*

$$m(\alpha) \gg \left(\frac{\log\log d}{\log d}\right)^3.$$

LEMMA 2.69. *Let γ be a semisimple regular element of an arithmetic lattice γ in $\mathrm{PGL}(2,\mathbb{K})$ and let λ be one of the eigenvalues of $\mathrm{Ad}\gamma$. Let k be the trace field of Γ i.e. the field generated by traces of $\mathrm{Ad}g$ for $g \in \Gamma$. Then λ is an algebraic integer and if $\gamma \in B(1,R)$ then $m(\lambda) \leq R$. Moreover if γ is not torsion , then there exists a unique place ω of $k(\lambda)$ such that $|\lambda|_\omega > 1$. In such case we have $|\lambda| = |\lambda|_\omega^{\pm 1}$ and $\mathbb{Q}(\lambda)$ is an extension of k of degree at most 2.*

Since $k(\lambda) \simeq k(\gamma) \subset D$ we will write $m(\gamma)$ for the Mahler measure of γ seen as algebraic number over k. Clearly $m(\gamma) = m(\lambda)$.

PROOF. Without loss of generality assume that $\gamma \in \Gamma$ and Γ is a maximal arithmetic lattice. The fact that γ is an algebraic number follows straight from the construction of maximal arithmetic lattices (c.f. Proposition 2.33). There exists an admissible quaternion algebra D and family of open compacts subsets $U_\mathfrak{p} \subset \mathrm{PD}^\times(k_\mathfrak{p})$ for $\mathfrak{p} \in M_k^f$ such that Γ is the projection of $\mathrm{PD}^\times(k) \cap \prod_\mathfrak{p} U_\mathfrak{p}$ to $\mathrm{PGL}(2,\mathbb{K})$. In particular, the eigenvalue λ lies in $\mathcal{O}_{k_\mathfrak{p}}$

because of compactness of $U_{\mathfrak{p}}$. It follows that λ is an algebraic integer. Recall that the logarithmic Mahler measure is given by

$$m(\lambda) = \sum_{\omega \in M_{\mathbb{Q}(\lambda)}} \log^+ |\lambda|_\omega.$$

Let ν_1 be the unique archimedean place where D splits. For any place ν of k except ν_1 the element γ is contained in a compact subgroup of $\mathrm{PD}^\times(k_\nu)$ so for any ω extending ν we have $|\lambda|_\omega = 1$. It follows that

$$m(\lambda) = \frac{[\mathbb{Q}(\lambda):\mathbb{Q}]}{[k(\lambda):\mathbb{Q}]} \sum_{\omega \in M_{k(\lambda)}} \log^+ |\lambda|_\omega = \frac{[\mathbb{Q}(\lambda):\mathbb{Q}]}{[k(\lambda):\mathbb{Q}]} \sum_{\substack{\omega|\nu_1 \\ \omega \in M_{k(\lambda)}}} \log^+ |\lambda|_\omega.$$

Note that for any $\omega|\nu_1$ we have $|\lambda| = |\lambda|_\omega^{\pm}$. Among all extensions w of ν_1 to $k(\lambda)$ there is at most one with $|\lambda|_\omega > 1$. If there is none, then by Kronecker lemma λ is a root of identity and γ is torsion, in which case $m(\lambda) = 0$.

We have $d(\gamma, 1) = \|1 - \gamma\| \leq R$. The matrix $1 - \gamma$ has eignvalues $1 - \lambda, 1 - \lambda^{-1}$ so the last inequality implies that

$$|1 - \lambda| \leq R \tag{6.47}$$

$$|\lambda| \leq R + 1 \leq e^R \tag{6.48}$$

$$\log^+ |\lambda| \leq R. \tag{6.49}$$

Hence $m(\lambda) \leq R$. In the case $|\lambda|_\omega \neq 1$ we can easily show that $k(\lambda)$ is at most a quadratic extension of $\mathbb{Q}(\lambda)$. □

Combining Lemma 2.69, Theorem 2.68 and Propositions 2.66, 2.67 we get

Corollary 2.70. *Fix $R > 0$. Let γ be a regular element of an arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$ defined over a number field k and let $f \in C_c(\mathrm{PGL}(2,\mathbb{K}))$ be a continuous function supported on a ball $B(1,R)$. Then*

$$|\mathcal{O}_\gamma(f)| \ll_R \left(\frac{\log[k:\mathbb{Q}]}{\log\log[k:\mathbb{Q}]}\right)^3 \|f\|_\infty \qquad \text{if } \gamma \text{ is loxodromic or hyperbolic,} \tag{6.50}$$

$$|\mathcal{O}_\gamma(f)| \ll_R [k:\mathbb{Q}]^2 \|f\|_\infty \qquad \text{if } \gamma \text{ is elliptic and } \mathbb{K} = \mathbb{C}, \tag{6.51}$$

$$|\mathcal{O}_\gamma(f)| \ll_R [k:\mathbb{Q}]^4 \|f\|_\infty \qquad \text{if } \gamma \text{ is elliptic and } \mathbb{K} = \mathbb{R}. \tag{6.52}$$

$$\tag{6.53}$$

Proof. Note that if γ is loxodromic, hyperbolic or $\mathbb{K} = \mathbb{C}$ then its centralizer splits. In that case $m(\lambda) = \log|\lambda|$ where λ is the unique eigenvalue of $\mathrm{Ad}\,\gamma$ of modulus greater than 1. For the first inequality observe that Lemma 2.69 and Theorem 2.68 give

$$|\Delta(\gamma)| = |1 - \lambda||1 - \lambda^{-1}| \geq m(\lambda)^2 \gg \left(\frac{\log[k:\mathbb{Q}]}{\log\log[k:\mathbb{Q}]}\right)^{-6}. \tag{6.54}$$

This combined with estimate from Proposition 2.66 gives (6.50). In the elliptic case the eigenvalues λ, λ^{-1} of γ are roots of unity. Let n be the order of γ. Since λ lies in a quadratic extension of k we have $[\mathbb{Q}(\lambda):\mathbb{Q}] = \phi(n) \leq 2[k:\mathbb{Q}]$ where ϕ is the Euler's totient function. Using a crude estimate $n \ll \phi(n)^2$ we get $n \ll [k:\mathbb{Q}]^2$. Thus $|1 - \lambda| \gg [k:\mathbb{Q}]^{-2}$. If the centralizer of γ is split (which happens if and only if $\mathbb{K} = \mathbb{C}$) we get, by Proposition 2.66

$$|\mathcal{O}_\gamma(f)| \ll_R [k:\mathbb{Q}]^2 \|f\|_\infty.$$

If $\mathbb{K} = \mathbb{R}$ the centralizer of γ is anisotropic so

$$|\mathcal{O}_\gamma(f)| \ll_R [k:\mathbb{Q}]^4 \|f\|_\infty,$$

by Proposition 2.67. □

6.4. Global orbital integrals - general case. In this section we combine the results from Sections 6.3 and 6.2 to get a bound on $|\mathcal{O}_\gamma(f_{\mathbb{A}_f})|$. We start the discussion by a general bound resulting from what we did in the previous sections. We leave the non-torsion for the next section. To deal with the non-archimedean part we shall use a classical result of Frobenius

THEOREM 2.71 (Frobenius). *Any irreducible representation of* $\mathrm{PGL}(2,\mathbb{F}_q)$ *of dimension bigger than* 1 *has dimension at least* $q-1$.

For the anisotropic case we will use a result of Carayol on the dimensions of irreducible representations of quaternion algebras over a local field.

THEOREM 2.72 ([**31**, Proposition 6.5]). *Let* A *be a division quaternion algebra defined over* $k_\mathfrak{p}$. *Let* π *be a smooth irreducible representation of* $A^\times$ *of minimal level* c *and let* $r = gcd(2,c)$. *Then if* $c \geq 2$ *the dimension of* π *is given by the formula*

$$\dim \pi = r\frac{q^2-1}{q^r-1}q^{\frac{1}{2}(c+r-4)}. \tag{6.55}$$

For the definition of minimal level we refer to [**31**]. If the level is 1 then the representation π is of dimension 1 because it factors through an abelian group. Any complex representation of $PA^\times$ can be seen as a representation of $A^\times$ with trivial central character. As a consequence[12] we get:

COROLLARY 2.73. *Let* $U_\mathfrak{p}$ *be a maximal compact subgroup of* $\mathrm{PD}^\times(k_\mathfrak{p})$. *If* $\mathfrak{p} \notin \mathrm{Ram}^f D$ *and* $U \simeq \mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})$ *then any non-trivial irreducible representation of* $U_\mathfrak{p}$ *of dimension* > 1 *has dimension at least* $N(\mathfrak{p})-1$ *and if* $U_\mathfrak{p} \not\simeq \mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})$ *then every irreducible representation of dimension* > 2 *has dimension at least* $N(\mathfrak{p})-1$. *If* $\mathfrak{p} \in \mathrm{Ram}^f D$ *then any non-trivial representation of dimension* > 2 *has dimension at least* $N(\mathfrak{p})+1$.

We briefly recall the notation. $D^\times$ is an admissible quaternion algebra defined over a field k, U is a maximal compact subgroup of $\mathrm{PD}^\times(\mathbb{A}_f)$ and χ is the character of an irreducible representation of U. We have a decomposition $U = \prod_{\mathfrak{p}\in M_k^f} U_\mathfrak{p}$ so the character χ can be written as $\chi = \bigotimes_{\mathfrak{p}\in M_k^f} \chi_\mathfrak{p}$. As a direct application of the Proposition 2.65 we get

PROPOSITION 2.74. (1)

$$|\mathcal{O}_\gamma(\mathbb{1}_U)| \leq |N_{k/\mathbb{Q}}(\Delta(\gamma))|^{1/2} \prod_{\substack{\mathfrak{p}|\Delta(\gamma)\\ k_\mathfrak{p}[\gamma]\, quadratic}} \left(\frac{N(\mathfrak{p})+1}{N(\mathfrak{p})-1}\right).$$

(2) *For the character* χ *of an irreducible representation of* U *we have*

$$|\mathcal{O}_\gamma(\chi)| \leq |N_{k/\mathbb{Q}}(\Delta(\gamma))|^{3/2} \prod_{\substack{\mathfrak{p}|\Delta(\gamma)\\ k_\mathfrak{p}[\gamma]\, quadratic}} \left(\frac{N(\mathfrak{p})+1}{N(\mathfrak{p})-1}\right) \prod_{\substack{p\in M_k^f\\ 8|\Delta(\gamma)|_\mathfrak{p}^{-1}<\chi_\mathfrak{p}(1)}} 8 \prod_{\substack{p\in M_k^f\\ 8|\Delta(\gamma)|_\mathfrak{p}^{-1}\geq\chi_\mathfrak{p}(1)}} \chi_\mathfrak{p}(1),$$

If γ *is* $2,3,4$ *torsion the constant* 8 *can be improved to* $4, 8\sqrt{3}/3$ *and* $4\sqrt{2}$ *respectively.*

PROOF. For any χ we have

$$|\mathcal{O}_\gamma(\chi)| = \prod_{\mathfrak{p}\in M_k^f} |\mathcal{O}_\gamma(\chi_\mathfrak{p})|$$

For the first inequality put $\chi = \mathbb{1}_U$. If $\mathfrak{p}$ divides $\Delta(\gamma)$ and the extension $k_\mathfrak{p}[\gamma]/k_\mathfrak{p}$ is quadratic then by Proposition 2.65 (more precisely (6.16)) we have

$$|\mathcal{O}_\gamma(\mathbb{1}_{U_\mathfrak{p}})| \leq |\Delta(\gamma)|_\mathfrak{p}^{-1/2}\frac{N(\mathfrak{p})+1}{N(\mathfrak{p})-1},$$

[12]For the rigorous proof of lower bounds on dimensions in the unramified case see Section 8.2.

otherwise $|\mathcal{O}_\gamma(\mathbb{1}_{U_\mathfrak{p}})| \leq |\Delta(\gamma)|_\mathfrak{p}^{-1/2}$. Observe that $|N_{k/\mathbb{Q}}(\Delta(\gamma))| = \prod_{\mathfrak{p}\in M_k^f} |\Delta(\gamma)|_\mathfrak{p}^{-1}$ so we get

$$|\mathcal{O}_\gamma(\mathbb{1}_U)| \leq |N_{k/\mathbb{Q}}(\Delta(\gamma))|^{1/2} \prod_{\substack{\mathfrak{p}|\Delta(\gamma)\\ k_\mathfrak{p}[\gamma] \text{ quadratic}}} \frac{N(\mathfrak{p})+1}{N(\mathfrak{p})-1}.$$

The proof of the second inequality is completely analogous. □

We need to find more practical bounds on the factors $|N_{k/\mathbb{Q}}(\Delta(\gamma))|$,

$$\prod_{\substack{\mathfrak{p}|\Delta(\gamma)\\ k_\mathfrak{p}[\gamma]\text{quadratic}}} \left(\frac{N(\mathfrak{p})+1}{N(\mathfrak{p})-1}\right) \text{ and } \prod_{\substack{p\in M_k^f\\ 8|\Delta(\gamma)|_\mathfrak{p}^{-1}<\chi_\mathfrak{p}(1)}} 8 \prod_{\substack{p\in M_k^f\\ 8|\Delta(\gamma)|_\mathfrak{p}^{-1}\geq\chi_\mathfrak{p}(1)}} \chi_\mathfrak{p}(1)$$

At this point it is clear that presence of many prime ideals of small norm makes the problem harder. In the next section we will show that if γ is non-torsion and the archimedean orbital integral $\mathcal{O}_\gamma(f_{\mathbb{A}f})$ does not vanish then all three factors are of order $\exp(o([k:\mathbb{Q}]))$. This is more that enough for our purposes.

6.5. Global orbital integrals - non torsion case. In the non torsion case we shall use the Bilu equidistribution theorem to control the norms of Weyl discriminants and the distribution of prime ideals of small norm. This yields a much better bound than in the general case. Throughout this section let R be a fixed real positive number. The main result is

PROPOSITION 2.75. *Let Γ be a uniform arithmetic lattice of $\mathrm{PGL}(2,\mathbb{K})$ defined using a quaternion algebra D over a number field k and a maximal open compact subgroup U of $\mathrm{PD}^\times(\mathbb{A}_f)$. Let S be the set of finite places $\mathfrak{p}$ such that $U_\mathfrak{p} \not\simeq \mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})$. Let $f \in C_c(\mathrm{PGL}(2,\mathbb{K})$ with $\mathrm{supp} f \in B(1,R)$ and $\gamma \in \Gamma\setminus\{1\}$. If $\mathcal{O}_\gamma(f) \neq 0$ then for any irreducible character χ of U and any $0<\delta<1/2$ we have*

$$|\mathcal{O}_\gamma(\chi)| \leq \chi(1)^{1-\delta} 2^{\delta|S|} \exp(o_R([k:\mathbb{Q}])). \tag{6.56}$$

PROOF. Using Proposition 2.74 we get

$$|\mathcal{O}_\gamma(\chi)| \leq |N_{k/\mathbb{Q}}(\Delta(\gamma))|^{3/2} \prod_{\substack{\mathfrak{p}|\Delta(\gamma)\\ k_\mathfrak{p}[\gamma]\text{quadratic}}} \left(\frac{N(\mathfrak{p})+1}{N(\mathfrak{p})-1}\right) \prod_{\substack{p\in M_k^f\\ 8|\Delta(\gamma)|_\mathfrak{p}^{-1}<\chi_\mathfrak{p}(1)}} 8 \prod_{\substack{p\in M_k^f\\ 8|\Delta(\gamma)|_\mathfrak{p}^{-1}\geq\chi_\mathfrak{p}(1)}} \chi_\mathfrak{p}(1).$$

We shall estimate the right hand side factor by factor. The fact that $\mathcal{O}_\gamma(f)\neq 0$ implies that $m(\gamma)\leq R$. By Lemma 2.58 $|N_{k_\mathbb{Q}}(\Delta(\gamma))| = \exp(o([k:\mathbb{Q}]))$. By Theorem 2.59 the number of prime ideals of bounded norm is sublinear in $[k:\mathbb{Q}]$ so

$$\prod_{\substack{\mathfrak{p}|\Delta(\gamma)\\ k_\mathfrak{p}[\gamma]\text{quadratic}}} \left(\frac{N(\mathfrak{p})+1}{N(\mathfrak{p})-1}\right) = \exp(o([k:\mathbb{Q}])).$$

Choose δ between 0 and 1. By Corollary 2.73 the primes $\mathfrak{p}$ with $\chi_\mathfrak{p}(1)>8$ and $8\geq\chi_\mathfrak{p}(1)^{1-\delta}$ satisfy $(N(\mathfrak{p})-1)^{1-\delta}\leq 8$. Again using Theorem 2.59 we get that the number of such primes

is sublinear in $[k:\mathbb{Q}]$. It follows that

$$|\mathcal{O}_\gamma(\chi)| \leq \exp(o([k:\mathbb{Q}])) \prod_{\substack{p\in M_k^f \\ 8<\chi_\mathfrak{p}(1)}} 8 \prod_{\substack{p\in M_k^f \\ 8\geq\chi_\mathfrak{p}(1)}} \chi_\mathfrak{p}(1) \tag{6.57}$$

$$\leq \exp(o([k:\mathbb{Q}]))\chi(1)^{1-\delta} \prod_{\substack{p\in M_k^f \\ 8<\chi_\mathfrak{p}(1)}} \frac{8}{\chi_\mathfrak{p}(1)^{1-\delta}} \prod_{\substack{p\in M_k^f \\ 8\geq\chi_\mathfrak{p}(1)}} \chi_\mathfrak{p}(1)^\delta \tag{6.58}$$

$$\leq \exp(o([k:\mathbb{Q}]))\chi(1)^{1-\delta} \prod_{\substack{p\in M_k^f \\ 8\geq\chi_\mathfrak{p}(1)}} \chi_\mathfrak{p}(1)^\delta. \tag{6.59}$$

By Corollary 2.73 ideals $\mathfrak{p}$ with $1<\chi_\mathfrak{p}(1)\leq 8$ and $\mathfrak{p}\notin S$ satisfy $N(\mathfrak{p})-1\leq 8$ (note that it is not true for primes in S as $\mathbb{U}_\mathfrak{p}$ has irreducible representations of dimension 2). By 2.59 we conclude that

$$|\mathcal{O}_\gamma(\chi)| \leq \exp(o([k:\mathbb{Q}]))\chi(1)^{1-\delta} \prod_{\substack{p\in S \\ 8\geq\chi_\mathfrak{p}(1)}} \chi_\mathfrak{p}(1)^\delta \tag{6.60}$$

$$\leq \exp(o([k:\mathbb{Q}]))\chi(1)^{1-\delta} 2^{\delta|S|}. \tag{6.61}$$

The factor 2 appears because by Theorem 2.72 for $\mathfrak{p}\in S$ the dimension of an irreducible representation of $\mathrm{PD}^\times(k_\mathfrak{p})$ is either smaller or equal to 2 or at is greater or equal to $N(\mathfrak{p})-1$. □

7. Number of conjugacy classes with nontrivial contribution

Throughout this section fix $f\in C_c(\mathrm{PGL}(2,\mathbb{K}))$ with $\mathrm{supp} f\subset B(1,R)$. In Section 6.3, Lemma 2.69 we showed that if λ is an eigenvalue of a semisimple, non-torsion conjugacy class $[\gamma]$ in an arithmetic lattice and $[\gamma]_{\mathrm{PGL}(2,\mathbb{K})}\cap B(1,R)\neq\emptyset$ then the logarithmic Mahler measure satisfies $m(\lambda)\leq R$. Moreover, if Γ is defined over k then there exists a unique place ω of $k(\lambda)$ such that $|\lambda|_\omega>1$. In this section we will estimate the number of possible values of λ and in this way give an upper bound on the number of torsion free, rational conjugacy classes $[\gamma]\in\mathrm{PD}^\times(k)$ with $\mathcal{O}_\gamma(f_\infty)\neq 0$. We will write $m(\gamma)$ for $m(\lambda)$, this does not depend on the choice of eigenvalue since they have the same minimal polynomial over $\mathbb{Q}$.

Remark 2.76. *The problem reduces to counting the possible eigenvalues. If λ is such an eigenvalue, it is an algebraic integer, its logarithmic Mahler measure is bounded by R and $\lambda+\lambda^{-1}\in\mathcal{O}_k$. A priori the number of all algebraic integers satisfying those conditions may be much smaller than the number of all algebraic integers α of degree at most $2[k:\mathbb{Q}]$ and with $m(\alpha)\leq R$. The size of the latter set was bounded by Dubickas and Konyagin in [41]:*

$$|\{\alpha\in\mathbb{Z}^{alg} | [\mathbb{Q}(\alpha):\mathbb{Q}]\leq d \text{ and } m(\alpha)\leq R\}| \ll e^{R([k:\mathbb{Q}]+o([k:\mathbb{Q}]))}. \tag{7.1}$$

This bound is exponential in the degree while we need a sub-exponential one.

The main result of this section is

Theorem 2.77. *Fix $R>0$. Let D be a $\mathbb{K}$-admissible quaternion algebra defined over a number field k. The number of conjugacy classes $[\gamma]\subset\mathrm{PD}^\times(k)$ with $m(\gamma)\leq R$ is of order $\exp(O_R(\log^2[k:\mathbb{Q}]))$.*

We always assume that D is a $\mathbb{K}$-admissible quaternion algebra. Before proving Theorem 2.77 let us prove some auxiliary results.

For $z\in\mathbb{C}, |z|\leq 1$ and $t\geq 0$ define the function

$$\log_t|1-z| = \frac{1}{2}\sum_{n=1}^{\infty}\frac{-e^{-nt}(z^n+\overline{z}^n)}{n}. \tag{7.2}$$

Note that for $t = 0$ the series converge absolutely only if $|z| < 1$. In that case we have $\log_0 |1 - z| = \log |1 - z|$.

LEMMA 2.78. (1) *for $t < 1$ we have $|\log_t 0| \leq -\log t + O(1)$.*
(2) *for $|z| = 1, z \neq 1$ we have $\log |1 - z| \leq \frac{t}{2} + \log_t |1 - z|$.*

PROOF. (1)

$$0 \geq \log_t 0 = \sum_{n=1}^{\infty} \frac{-e^{-nt}}{n} \tag{7.3}$$

$$= \log |1 - e^{-t}| \tag{7.4}$$

$$\geq \log t + O(1). \tag{7.5}$$

(2) Note that $|z| = 1$ implies $\overline{z}^n = z^{-n}$.

$$\log |1 - z| - \log_t |1 - z| = \int_0^t \frac{\partial}{\partial s} \log_s |1 - z| ds \tag{7.6}$$

$$= \frac{1}{2} \int_0^t \frac{\partial}{\partial s} \left(\sum_{n \neq 0} \frac{-e^{-|n|s} z^n}{|n|} \right) ds \tag{7.7}$$

$$= \frac{1}{2} \int_0^t \left(\sum_{n \neq 0} -e^{-|n|s} z^n \right) ds \tag{7.8}$$

$$= \frac{t}{2} - \frac{1}{2} \int_0^t \left(\sum_{n=-\infty}^{\infty} e^{-|n|s} z^n \right). \tag{7.9}$$

To finish the proof it is enough to show that the sum $\sum_{n=-\infty}^{\infty} e^{-|n|s} z^n$ is positive for all z on the unit circle and $s > 0$. We have

$$\sum_{n=-\infty}^{\infty} e^{-|n|s} z^n = \frac{1}{1 - e^{-s} z} + \frac{1}{1 - e^{-s} \overline{z}} - 1 \tag{7.10}$$

$$= \frac{1 - e^{-2s}}{|1 - z e^{-s}|^2} \tag{7.11}$$

$$> 0. \tag{7.12}$$

□

LEMMA 2.79. *Let $\alpha \in \overline{\mathbb{Q}}$ be an algebraic integer of degree N such that $m(\alpha) \leq R$ and there exists at most C embeddings $\rho \colon \mathbb{Q}(\alpha) \to \mathbb{C}$ such that $|\rho(\alpha)| \neq 1$. Then*

$$\left| \frac{\mathrm{tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}} \alpha}{N} \right| \ll_{R,C} \left(\frac{\log N}{N} \right)^{1/2}. \tag{7.13}$$

PROOF. The proof follows the methods of **[44]** and **[12]**. Let $\alpha_1, \ldots, \alpha_N$ be the roots of the minimal polynomial of α. We divide them in two parts $\{\alpha_1, \ldots, \alpha_N\} = H_1 \sqcup H_2$ where $H_1 = \{\alpha_i \mid |\alpha_i| = 1\}$ and $H_2 = \{\alpha_i \mid |\alpha_i| \neq 1\}$. The discriminant of the minimal polynomial of α is a non zero integer so

$$\sum_{1 \leq i \neq j \leq n} \log |\alpha_i - \alpha_j| \geq 0. \tag{7.14}$$

Hence

$$\sum_{\omega \neq \omega' \in H_1} \log |\omega - \omega'| \geq -2 \sum_{\substack{\omega \in H_1 \\ \omega' \in H_2}} \log |\omega - \omega'| - \sum_{\omega \neq \omega' \in H_2} \log |\omega - \omega'| \tag{7.15}$$

$$\geq -2 |H_1| |H_2| \log(e^R + 1) - \binom{|H_2|}{2} \log(2e^R) \tag{7.16}$$

$$\gg_{R,C} -N. \tag{7.17}$$

We have used the fact that $m(\alpha) \leq R$ to guarantee that $|\omega| \leq e^R$ for any $\omega \in H_2$. The left hand side of the inequality can be rewritten as $\sum_{\omega \neq \omega' \in H_1} \log|1-\omega\overline{\omega'}|$. Using the last inequality and Lemma 2.78 we get for $t \in]0,1[$

$$\sum_{\omega,\omega' \in H_1} \log_t|1-\omega\overline{\omega'}| \geq |H_1|\log t + O(|H_1|) - \frac{t}{2}|H_1|^2 - O_{R,C}(N) \tag{7.18}$$

$$\gg_{R,C} -N(\frac{tN}{2} - \log t + O(1)). \tag{7.19}$$

Note that in the above sum we allow $\omega = \omega'$ hence the term $|H_1|\log t$. For $t = 1/N$ we get

$$\sum_{\omega,\omega' \in H_1} \log_t|1-\omega\overline{\omega'}| \gg_{R,C} -N(\log N + O(1)). \tag{7.20}$$

We multiply both sides by -1 and use the Fourier expansion of $\log_t|1-z|$ to get

$$N(\log N + O(1)) \gg_{R,C} \sum_{n\neq 0} \frac{e^{-|n|/N}}{|n|} \sum_{\omega,\omega \in H^1} \omega^n \overline{\omega'}^n \tag{7.21}$$

$$= \sum_{n\neq 0} \frac{e^{-|n|/N}}{|n|} \left|\sum_{\omega \in H_1} \omega^n\right|^2. \tag{7.22}$$

We deduce

$$\left|\sum_{\omega \in H_1} \omega\right|^2 \ll_{R,C} e^{1/N} N(\log N + O(1)) \tag{7.23}$$

$$\left|\sum_{\omega \in H_1} \omega\right| \ll_{R,C} N^{1/2}(\log N)^{1/2}. \tag{7.24}$$

We turn back to the main inequality of the lemma:

$$\left|\frac{\mathrm{tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}\alpha}{N}\right| \leq \left|\frac{\sum_{\omega \in H_1} w}{N}\right| + \left|\frac{\sum_{\omega \in H_2} w}{N}\right| \tag{7.25}$$

$$\ll_{R,C} \left(\frac{\log N}{N}\right)^{1/2} + \frac{Ce^R}{N} \ll_{R,C} \left(\frac{\log N}{N}\right)^{1/2}. \tag{7.26}$$

□

We will need a simplified version of Kabatjianksi-Levenstein bound on the number of almost orthogonal vectors on the unit sphere.

Lemma 2.80. *[T.Tao **[102]**] Let $v_1, v_2, \ldots, v_m$ be unit vectors in an n-dimensional Euclidean space such that for every $i \neq j$ $|\langle v_i, v_j\rangle| \leq An^{-1/2}$ for some $\frac{1}{2} < A < \frac{1}{2}\sqrt{n}$. Then $m \leq \left(\frac{Cn}{A^2}\right)^{CA^2}$ for some absolute constant C.*

No we can proceed to the proof:

Proof of Theorem 2.77. By Lemma 2.26 if γ is not 2-torsion then any nonzero eigenvalue λ of γ determines the conjugacy class $[\gamma]$. The eigenvalue λ is an algebraic integer of logarithmic Mahler measure at most R. We will prove the theorem by showing that there are at most $\exp(O(\log^2[k:\mathbb{Q}]))$ possible choices of λ. Let us enumerate the archimedean places of k by $\nu_1, \nu_2, \ldots, \nu_{r_1+r_2}$ is such a way that $k_{\nu_1} = \mathbb{K}$ and $\{\nu_2, \ldots, \nu_{r_1+r_2}\} = \mathrm{Ram}^\infty D$. Note that in our setting the number of complex places r_2 is 1 if $\mathbb{K} = \mathbb{C}$ and 0 otherwise.

Define the set

$$S_R = \{\lambda \in \overline{\mathbb{Q}} | \lambda+\lambda^{-1} \in k, m(\lambda) \leq R, (\lambda+\lambda^{-1})_{\nu_1} \notin [-2,2] \text{ and } |\lambda+\lambda^{-1}|_{\nu_i} \leq 2 \text{ for } i = 2, \ldots, r_1+r_2\} \tag{7.27}$$

By Lemma 2.69 the eigenvalues of γ lie in S_R. We want to show that

$$|S_R| \leq \exp(O(\log^2[k:\mathbb{Q}])). \tag{7.28}$$

Consider the set

$$A_R = \{\lambda + \lambda^{-1} | \lambda \in S_R \text{ and } \mathrm{Re}(\lambda + \lambda^{-1})_{\nu_1} > 0\}. \tag{7.29}$$

Here x_{ν_1} denotes the image of x under any embedding $\rho\colon k \to \mathbb{C}$ s.t. $|x|_{\nu_1} = |\rho(x)|$. Such an embedding is defined up to complex conjugation so the real part is well defined. We have $|S_R| \leq 4|A_R|$. Let us fix a Dirichlet embedding $\iota : k \to \mathbb{C}^{r_2} \times \mathbb{R}^{r_1} := V$ given by $x \mapsto (x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_{r_1+r_2}})$. We introduce a scalar product on V given by

$$\langle x, y\rangle = \begin{cases} \frac{1}{[k:\mathbb{Q}]}\left(2x_1\overline{y_1} + \sum_{i=2}^{r_1+r_2} x_i y_i\right) & \text{if } \mathbb{K} = \mathbb{C} \\ \frac{1}{[k:\mathbb{Q}]}\sum_{i=1}^{r_1+r_2} x_i y_i & \text{if } \mathbb{K} = \mathbb{R} \end{cases} \tag{7.30}$$

By means of the Dirichlet embedding we think of k as a subset of V. We claim that for every pair $x, y \in A_R$ we have $|\langle x, y\rangle| \ll (\log[k:\mathbb{Q}]/[k:\mathbb{Q}])^{1/2}$ and $|\langle x, x\rangle| = 2 + O((\log[k:\mathbb{Q}]/[k:\mathbb{Q}])^{1/2})$. We postpone the proofs of these inequalities to Lemma 2.81. For every $x \in A_R$ put $\tilde{x} = x/\sqrt{\langle x, x\rangle}$. V is a Euclidean space of dimension $[k:\mathbb{Q}]$ and the set $\{\tilde{x} | x \in A_R\}$ consists of unit vectors. For any distinct $\tilde{x}, \tilde{y}$ we have $\langle \tilde{x}, \tilde{y}\rangle \leq C_1 \log^{1/2}[k:\mathbb{Q}]/[k:\mathbb{Q}]^{1/2}$. We apply Lemma 2.80 to get:

$$|A_R| \ll \left(\frac{C[k:\mathbb{Q}]}{C_1^2 \log[k:\mathbb{Q}]}\right)^{CC_1^2 \log[k:\mathbb{Q}]} \ll \exp(CC_1^2 \log^2[k:\mathbb{Q}]). \tag{7.31}$$

It follows that $|S_R| = \exp(O(\log^2[k:\mathbb{Q}]))$. Since the conjugacy class of γ is determined by its eigenvalues that gives the desired bound on the number of conjugacy classes. □

LEMMA 2.81. *Put $N = [k:\mathbb{Q}]$. With the notation from the proof of Theorem 2.77, for every $x \neq y \in A_R$ we have*

(1)
$$|\langle x, y\rangle| \ll \left(\frac{\log N}{N}\right)^{1/2}$$

(2)
$$\langle x, x\rangle = 2 + O\left(\left(\frac{\log N}{N}\right)^{1/2}\right)$$

PROOF. Let us treat the case $\mathbb{K} = \mathbb{R}$, the proofs for $\mathbb{K} = \mathbb{C}$ are nearly identical. Write $x = \lambda_1 + \lambda_1^{-1}$ and $y = \lambda_2 + \lambda_2^{-1}$ with $\lambda_1, \lambda_2 \in S_R$. The conditions imposed on A_R ensure that $\lambda_1 \neq \pm\lambda_2^{\pm 1}$.

Claim. $\mathbb{Q}(\lambda_1^{\pm 1}\lambda_2^{\pm 1}) = \mathbb{Q}(\lambda_1, \lambda_2)$. To prove this statement we are going to use some basic Galois theory. Assume $\mathbb{Q}(\lambda_1\lambda_2) \subsetneq \mathbb{Q}(\lambda_1, \lambda_2)$. Then, there exists an automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing $\mathbb{Q}(\lambda_1\lambda_2)$ but not $\mathbb{Q}(\lambda_1, \lambda_2)$. So $\sigma(\lambda_1\lambda_2) = \lambda_1\lambda_2$ but $\sigma(\lambda_1) \neq \lambda_1$ or $\sigma(\lambda_2) \neq \lambda_2$. Let us examine the possibilities. The numbers λ_1, λ_2 are Salem numbers (see Lemma 2.69) so $\sigma(\lambda_i)$ is either complex non-real of modulus 1 or $\sigma(\lambda_i) = \lambda_i^{\pm 1}$. Since $\sigma(\lambda_1)\sigma(\lambda_2) = \lambda_1\lambda_2$ is a real number the only possibility is that both $\sigma(\lambda_i)$ are real or that they are complex of modulus 1 and $\lambda_1\lambda_2 = 1$. Both scenarios lead quickly to the contradiction with the condition $\lambda_1 \neq \lambda_2^{\pm 1}$. The Claim follows.

In particular, the Claim implies that $\mathbb{Q}(\lambda_1^{\pm 1}\lambda_2^{\pm 1})$ contains k and $4 \geq [\mathbb{Q}(\lambda_1^{\pm 1}\lambda_2^{\pm 1}) : \mathbb{Q}) : k] \geq 2$. We have

$$\langle x, y\rangle = \frac{\mathrm{tr}_{k:\mathbb{Q}} xy}{[k:\mathbb{Q}]} = \sum_{\pm,\pm} \frac{[k(\lambda_1^{\pm 1}\lambda_2^{\pm 1}) : k]\mathrm{tr}_{\mathbb{Q}(\lambda_1^{\pm 1}\lambda_2^{\pm 1})/\mathbb{Q}}(\lambda_1^{\pm 1}\lambda_2^{\pm 1})}{[\mathbb{Q}(\lambda_1^{\pm 1}\lambda_2^{\pm 1}) : \mathbb{Q}]} \tag{7.32}$$

$$\leq 4 \sum_{\pm,\pm} \frac{\mathrm{tr}_{\mathbb{Q}(\lambda_1^{\pm 1}\lambda_2^{\pm 1})/\mathbb{Q}}\lambda_1^{\pm 1}\lambda_2^{\pm 1}}{[\mathbb{Q}(\lambda_1^{\pm 1}\lambda_2^{\pm 1}) : \mathbb{Q}]} \tag{7.33}$$

$$\ll_R \left(\frac{\log[k:\mathbb{Q}]}{[k:\mathbb{Q}]}\right)^{1/2}. \tag{7.34}$$

In the last passage we have used Lemma 2.79. Now we move to the proof of the second equality.

$$\langle x, x\rangle = \frac{\mathrm{tr}_{k/\mathbb{Q}} x^2}{[k:\mathbb{Q}]} = \frac{\mathrm{tr}_{k/\mathbb{Q}} 2 + \mathrm{tr}_{k/\mathbb{Q}}(\lambda_1^2 + \lambda_1^{-2})}{[k:\mathbb{Q}]} \tag{7.35}$$

$$= 2 + \sum_{\pm} \frac{[k(\lambda_1^{\pm 2}) : k]\mathrm{tr}_{\mathbb{Q}(\lambda_1^{\pm 2})/\mathbb{Q}} \lambda_1^{\pm 2}}{[\mathbb{Q}(\lambda_1^{\pm 2}) : \mathbb{Q}]} \tag{7.36}$$

$$= 2 + O_R\left(\left(\frac{\log[k:\mathbb{Q}]}{[k:\mathbb{Q}]}\right)^{1/2}\right). \tag{7.37}$$

□

8. Representation Zeta functions

8.1. Definitions and motivation. Let G be a topological group. The representation zeta function of G denoted ζ_G is defined as the formal Dirichlet series

$$\zeta_G(s) = \sum_{\rho \in \mathrm{Irr} G} (\dim \rho)^{-s} \tag{8.1}$$

The series are well defined only for rigid groups i.e. those which have finitely many irreducible representations of bounded dimension. For our purposes we will be interested in the representations of a maximal compact subgroup U of $\mathrm{PD}^\times(\mathbb{A}_f)$, where D is an admissible quaternion algebra. By maximality U decomposes as a product of local factors $U = \prod_{\mathfrak{p} \in M_k^f} U_{\mathfrak{p}}$ so formally we have

$$\zeta_U = \prod_{\mathfrak{p} \in M_k^f} \zeta_{U_{\mathfrak{p}}}.$$

Unfortunately for us, the group U is not rigid and the representation zeta function is not well defined. Indeed, the image of U via the norm map $n(U) \subset \mathbb{A}^\times/(\mathbb{A}^\times)^2$ is infinite so U has infinitely many one dimensional representations. For this reason we define the special zeta function:

Definition 2.82. *Let G be a topological group. We identify the group of one dimensional characters of G with $\mathrm{Irr} G^{ab}$. $\mathrm{Irr} G^{ab}$ acts on $\mathrm{Irr} G$ by tensor product. The special representation zeta function of G denoted ζ_G^* is defined as the formal Dirichlet series*

$$\zeta_G^*(s) = \sum_{\rho \in \mathrm{Irr} G/\mathrm{Irr} G^{ab}} (\dim \rho)^{-s}. \tag{8.2}$$

For U we have the product decomposition:

$$\zeta_U^* = \prod_{\mathfrak{p} \in M_k^f} \zeta_{U_{\mathfrak{p}}}^*.$$

We will show in Section 8.2 that this function is well defined and can be explicitly bounded. Our motivation to study the representation zeta functions is illustrated by the following abstract principle.

Lemma 2.83. *Let G be a group and let H be a subgroup of finite index. Let $\gamma \in G$ be an element such that there exists $1 \geq \delta > 0$ such that $|\chi(\gamma)| \leq (\dim \rho)^{1-\delta}$ for all irreducible characters of G.*

(1) *Let $b \in \mathbb{R}$ be such that $\zeta_G(b-1)$ is finite and put $a = \delta/(1+b)$. Then*

$$\chi_{\mathrm{Ind}_H^G \mathbb{1}}(\gamma) = \sum_{\rho \in \mathrm{Irr} G} \dim W_\rho^H \chi_\rho(\gamma) \leq [G:H]^{1-a} \zeta_G(b-1)^a. \tag{8.3}$$

(2) *Let $b \in \mathbb{R}$ be such that $\zeta_G^*(b-1)$ is finite and put $a = \delta/(1+b)$. Write A for the image of H in G^{ab}. Then*

$$\chi_{\mathrm{Ind}\,\frac{G}{H}\mathbb{1}}(\gamma) \leq [G:H]^{1-a}\zeta_G^*(b-1)^a[G^{ab}:A]^a. \tag{8.4}$$

PROOF. (1) By Hölder's inequality we have

$$\left|\sum_{\rho\in\mathrm{Irr}G} \dim W_\rho^H \chi_\rho(\gamma)\right| \leq \sum_{\rho\in\mathrm{Irr}G} \dim W_\rho^H (\dim\rho)^{1-\delta} \tag{8.5}$$

$$= \sum_{\rho\in\mathrm{Irr}G} (\dim W_\rho^H \dim\rho)^{1-a} \left(\frac{\dim W_\rho^H}{(\dim\rho)^b}\right)^a \tag{8.6}$$

$$\leq \left(\sum_{\rho\in\mathrm{Irr}G} \dim W_\rho^H \dim\rho\right)^{1-a} \left(\sum_{\rho\in\mathrm{Irr}G} \frac{\dim W_\rho^H}{(\dim\rho)^b}\right)^a \tag{8.7}$$

$$\leq [G:H]^{1-a}\zeta_G(b-1)^a \tag{8.8}$$

(2) Dividing by a finite index normal subgroup $N \subset H$ we may assume without loss on generality that G is finite. For an irreducible representation ρ let $S_\rho = \#\{\theta \in \mathrm{Irr}G^{ab} \mid \rho \simeq \rho\otimes\theta\}$[13]. We have

$$\chi_{\mathrm{Ind}\,\frac{G}{H}1}(\gamma) = \sum_{\rho\in IrrG} \langle 1,\rho\rangle_H \chi_\rho(\gamma) \tag{8.9}$$

$$= \sum_{\rho\in\mathrm{Irr}G/\mathrm{Irr}G^{ab}} \frac{1}{S_\rho} \sum_{\theta\in\mathrm{Irr}G^{ab}} \langle 1,\rho\otimes\theta\rangle_H \chi_\rho(\gamma)\theta(\gamma) \tag{8.10}$$

$$= \sum_{\rho\in\mathrm{Irr}G/\mathrm{Irr}G^{ab}} \frac{\chi_\rho(\gamma)}{S_\rho} \left(\sum_{\theta\in\mathrm{Irr}G^{ab}} \langle\theta,\rho\rangle_H \overline{\theta}(\gamma)\right) \tag{8.11}$$

If $\theta_1|_A = \theta_2|_A$ then[14] $\langle\theta_1,\rho\rangle_H = \langle\theta_2,\rho\rangle_H$. It follows that

$$\sum_{\theta\in\mathrm{Irr}G^{ab}} \langle\theta,\rho\rangle_H = [G^{ab}:A] \sum_{\theta\in\mathrm{Irr}A} \langle\theta,\rho\rangle_H \tag{8.12}$$

Put $C(\rho) := \sum_{\theta\in\mathrm{Irr}A}\langle\theta,\rho\rangle_H$. Then using inequality $|\chi_\rho(\gamma)| \leq (\dim\rho)^{1-\delta}$ and the Hölder inequality we get

$$\chi_{\mathrm{Ind}\,\frac{G}{H}1}(\gamma) \leq [G^{ab}:A] \sum_{\rho\in\mathrm{Irr}G/\mathrm{Irr}G^{ab}} \frac{|\chi_\rho(\gamma)|C(\rho)}{S_\rho} \tag{8.13}$$

$$\leq [G^{ab}:A] \sum_{\rho\in\mathrm{Irr}G/\mathrm{Irr}G^{ab}} \frac{(\dim\rho)^{1-\delta}C(\rho)}{S_\rho} \tag{8.14}$$

$$= [G^{ab}:A] \sum_{\rho\in\mathrm{Irr}G/\mathrm{Irr}G^{ab}} \left(\frac{(\dim\rho)C(\rho)}{S_\rho}\right)^{1-a} \left(\frac{C(\rho)}{S_\rho(\dim\rho)^b}\right)^a \tag{8.15}$$

$$\leq [G^{ab}:A] \left(\sum_{\rho\in\mathrm{Irr}G/\mathrm{Irr}G^{ab}} \frac{(\dim\rho)C(\rho)}{S_\rho}\right)^{1-a} \left(\sum_{\rho\in\mathrm{Irr}G/\mathrm{Irr}G^{ab}} \frac{C(\rho)}{S_\rho(\dim\rho)^b}\right)^a \tag{8.16}$$

By definition $C(\rho) \leq \dim\rho$ and $S_\rho \geq 1$ so we can bound the last expression by

$$\leq [G^{ab}:A]^a \left(\sum_{\rho\in\mathrm{Irr}G/\mathrm{Irr}G^{ab}} \frac{(\dim\rho)C(\rho)[G^{ab}:A]}{S_\rho}\right)^{1-a} \zeta_G^*(b-1)^a \tag{8.17}$$

[13] The only reason why we replace G with a finite quotient is that we need S_ρ to be finite.

[14] Recall that A is the image of H in G^{ab}.

By plugging $\gamma = 1$ in (8.9-8.12) we see that

$$\left(\sum_{\rho\in \mathrm{Irr}G/\mathrm{Irr}G^{ab}} \frac{(\dim\rho)C(\rho)[G^{ab}:A]}{S_\rho}\right) = \chi_{\mathrm{Ind}\,\frac{G}{H}1}(1) = [G:H] \tag{8.18}$$

which ends the proof.

□

We will not be using Lemma 2.83 directly but apply a similar reasoning to bound the right hand side in Lemma 2.55. The upper bounds on the values of irreducible characters will be replaced by upper bounds on orbital integrals from Propositions 2.74 and 2.75.

8.2. Special Zeta function of maximal compact subgroup. Throughout this section U is a maximal compact subgroup of $\mathrm{PD}^\times(\mathbb{A}_f)$. It decomposes as a product $U = \prod_{\mathfrak{p}\in M_k^f} U_\mathfrak{p}$ where $U_\mathfrak{p}$ is a maximal compact subgroup of $\mathrm{PD}^\times(k_\mathfrak{p})$. We have seen in Lemma 2.39 that the structure of $U_\mathfrak{p}$ is known explicitly. Representation zeta functions for compact p-adic groups like $U_\mathfrak{p}$ have been studied in [**9**, **10**, **61**, **72**] and many others but it seems that the exact type of bound that we need is not present in the literature. The formula for the representation zeta function of $\mathrm{SL}(2,\mathcal{O}_{k_\mathfrak{p}})$ is known explicitly by the work of Jaikin-Zapirain [**61**]. Let $q = N(\mathfrak{p})$ be odd, then

$$\begin{aligned}\zeta_{\mathrm{SL}(2,\mathcal{O}_{k_\mathfrak{p}})}(s) =& 1 + q^{-s} + \frac{q-3}{2}(q+1)^{-s} + 2\left(\frac{q+1}{2}\right)^{-s} + \frac{q-1}{2}(q-1)^{-s} + 2\left(\frac{q-1}{2}\right)^{-s} \\ &+ \frac{4q\left(\frac{q^2-1}{2}\right)^{-s} + \frac{q^2-1}{2}(q^2-q)^{-s} + \frac{(q-1)^2}{2}(q^2+q)^{-s}}{1-q^{-s+1}}.\end{aligned}$$

We will use the following lemma

Lemma 2.84. *Let G be a group and H a normal subgroup such that G/H is abelian. Then for every $s > 0$ where $\zeta_G^*(s)$ and $\zeta_H(s)$ are well defined we have $\zeta_G^*(s) \leq \zeta_H(s)$.*

Proof. For $\rho \in \mathrm{Irr}H$ let $\mathrm{Irr}(G|\rho)$ denote the set of irreducible representations of G whose restriction to H contains ρ. For any $\pi \in \mathrm{Irr}G$ write a_π for $\#\{\pi\otimes\theta | \theta \in \mathrm{Irr}G^{ab}\}$. If $\rho_1, \rho_2 \in \mathrm{Irr}H$ we write $\rho_1 \sim \rho_2$ if they are conjugate by G. We have

$$\zeta_G^*(s) = \sum_{\rho\in\mathrm{Irr}H/\sim}\left(\sum_{\pi\in\mathrm{Irr}(G|\rho)}\frac{(\dim\pi)^{-s}}{a_\pi}\right) \leq \sum_{\rho\in\mathrm{Irr}H/\sim}\left(\sum_{\pi\in\mathrm{Irr}(G|\rho)/\mathrm{Irr}(G/H)}(\dim\pi)^{-s}\right). \tag{8.19}$$

In the last inequality we use the fact that G/H is abelian and $\mathrm{Irr}G/H$ acts on $\mathrm{Irr}(G|\rho)$ by tensor product. Write $K = K_\rho$ for the stabilizer of ρ in G. By Clifford's theorem [**60**, 6.2, 6.11, 6.17, and 11.22] for any $\pi \in \mathrm{Irr}(G|\rho)$ there exists e such that $e\rho$ extends to an irreducible representation $\tilde\rho$ of K and $\pi = \mathrm{Ind}\,\frac{G}{K}\tilde\rho$. For now let us fix π. Direct computation of characters gives

$$\sum_{\theta\in\mathrm{Irr}K/H}\chi_{\theta\otimes\tilde\rho} = [K:H]\mathbb{1}_H\chi_{\tilde\rho} = e\chi_{\mathrm{Ind}\,\frac{K}{H}\rho} \tag{8.20}$$

so every irreducible constituent of $\mathrm{Ind}\,\frac{K}{H}\rho$ is of form $\theta\otimes\tilde\rho$ for some character θ of K/H. In particular by Frobenius reciprocity every $\pi' \in \mathrm{Irr}(G|\rho)$ is of form $\pi' = \mathrm{Ind}\,\frac{G}{K}\theta\otimes\tilde\rho$ for some $\theta \in \mathrm{Irr}K/H$. Now if $\theta_2 \in \mathrm{Irr}G/H$ then $\theta_2 \otimes \mathrm{Ind}\,\frac{G}{K}\tilde\rho = \mathrm{Ind}\,\frac{G}{K}(\theta_2|_K)\otimes\tilde\rho$. This together with the previous remark shows that the action of $\mathrm{Irr}G/H$ on $\mathrm{Irr}(G|\rho)$ is transitive. This means that the right side of (8.19) equals

$$\sum_{\rho\in\mathrm{Irr}H/\sim}([G:K_\rho]\dim\rho)^{-s} \leq \zeta_H(s) \tag{8.21}$$

□

A corollary of the proof is

COROLLARY 2.85. *Let G be a group and H a normal subgroup such that G/H is abelian. Let ρ be an irreducible representation of H. Write K_ρ for the stabilizer of ρ under conjugation. Then all representations π of G such that $\pi|H$ contains ρ are in a single* $\mathrm{Irr}G/H$ *orbit and* $\dim\pi \geq [G:K_\rho]\dim\rho$. *In particular*

$$\zeta_G^*(s) \leq \sum_{\rho\in \mathrm{Irr}H/\sim} ([G:K_\rho]\dim\rho)^{-s}.$$

Now we can proceed to the estimates on $\zeta_{U_\mathfrak{p}}^*(s)$. We divide the reasoning in three parts according to the isomorphism class of $U_\mathfrak{p}$. In general the argument will be as follows. First we find a normal subgroup H of $U_\mathfrak{p}$ with abelian quotient. For a representation ρ of H of level n we prove a lower bound on $[U_\mathfrak{p} : \mathrm{Stab}_{U_\mathfrak{p}}\rho]\dim\rho$ of order q^n. We use a crude bound on the number of irreducible representations of level n of order $(q-1)q^{3(n-1)}$. Using Corollary 2.85 we will get a satisfactory bound for $s \geq 7$ (see Corollary 2.86).

(1) $U_\mathfrak{p} \simeq \mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})$. Put $H = \mathrm{PSL}(2,\mathcal{O}_{k_\mathfrak{p}})$. The quotient G/H is abelian and we even have $H = G^{ab}$ unless $q = 2,3$. Write K_n for the kernel of the reduction map $\mathrm{PSL}(2,\mathcal{O}_{k_\mathfrak{p}}) \to \mathrm{PSL}(2,\mathcal{O}_{k_\mathfrak{p}}/\mathfrak{p}^n)$. We call K_n the n-th principal congruence subgroup. Recall that an irreducible representation ρ of H is of level n if $K_n \subset \ker\rho$ and $K_{n-1} \not\subset \ker\rho$. We will write $\mathrm{Stab}\rho$ for the stabilizer of ρ under the action of $U_\mathfrak{p}$ by conjugation.

Step 1. All open normal subgroups of $\mathrm{PSL}(2,\mathcal{O}_\mathfrak{p})$ contained in K_1 are of form K_n. Indeed let $N \subset K_1$ be an open normal subgroup of $\mathrm{PSL}(2,\mathcal{O}_\mathfrak{p})$. Let n be a minimal natural number such that $N \subset K_n$. Put $N' = K_{n+1}N$. By definition it is a normal subgroup contained in K_n. The quotient K_n/K_{n+1} is naturally identified with $\mathfrak{sl}(2,\mathbb{F}_q)$ and the action of $\mathrm{PSL}(2,\mathcal{O}_{k_\mathfrak{p}})$ factors through the adjoint action of $\mathrm{PSL}(2,\mathbb{F}_q)$. The quotient N'/K_{n+1} is a non zero subspace of $\mathfrak{sl}(2,\mathbb{F}_q)$ invariant by $\mathrm{PSL}(2,\mathbb{F}_q)$. But $\mathfrak{sl}(2,\mathbb{F}_q)$ is an irreducible $\mathrm{PSL}(2,\mathbb{F}_q)$ module so $N'/K_{n+1} = K_n/K_{n+1}$. Now the fact that $[K_1,K_l] = K_{l+1}$ and a simple inductive argument show that $N = K_n$.

Step 2. Let ρ be an irreducible representation of H of level n. Then $[U_\mathfrak{p} : \mathrm{Stab}\rho]\dim\rho \geq (q-1)q^{n-1}$. The proof is similar to the original argument of Frobenius. For any natural a let $N_a := \begin{pmatrix} 1 & \mathcal{O}_{k_\mathfrak{p}}\mathfrak{p}^a \\ 0 & 1 \end{pmatrix}$. By Step 1 the normal closure of N_a in $\mathrm{PSL}(2,\mathcal{O}_{k_\mathfrak{p}})$ equals K_a. It follows that the restriction $\rho|N_{n-1}$ is nontrivial. In particular there exists an irreducible character θ of $\mathcal{O}_{k_\mathfrak{p}}$ contained in $\rho|N_0$ nontrivial on N_{n-1}. The subgroup $B := \begin{pmatrix} \mathcal{O}_{k_\mathfrak{p}}^\times & 0 \\ 0 & \mathcal{O}_{k_\mathfrak{p}}^\times \end{pmatrix}$ of $U_\mathfrak{p}$ normalizes N_0 and $[G:\mathrm{Stab}\rho]\dim\rho \geq |B\theta| = |\mathcal{O}_{k_\mathfrak{p}}^\times\theta|$. Since θ is of level n we have $|\mathcal{O}_{k_\mathfrak{p}}^\times\theta| = (q-1)q^{n-1}$.

Step 3. Here we bound the number of $U_\mathfrak{p}$ orbits of irreducible representations of H of level n. It is easy to see that the number of $U_\mathfrak{p}$-orbits of irreducible representations of H/K_n equals the number of $U_\mathfrak{p}$ conjugacy classes in $H/K_n = \mathrm{PSL}(2,\mathcal{O}_{k_\mathfrak{p}}/\mathfrak{p}^n)$. We have

$$\begin{aligned}
&\#\{U_\mathfrak{p} \text{ conjugacy classes in } \mathrm{PSL}(2,\mathcal{O}_{k_\mathfrak{p}}/\mathfrak{p}^n)\} && (8.22)\\
\leq&\#\{U_\mathfrak{p} \text{ conjugacy classes in } \mathrm{PSL}(2,\mathbb{F}_q)\}[K_1:K_n] && (8.23)\\
\leq&\frac{q+3}{2}q^{3n-3} && \text{for } q \text{ odd}, \quad (8.24)\\
\leq&(q-1)q^{3n-3} && \text{for } q \text{ even}. \quad (8.25)\\
& && (8.26)
\end{aligned}$$

Step 4. If $q \geq 3$ then $\frac{q-1}{(q-1)^5} \leq \frac{1}{q^2}$ and $\frac{q+3}{2(q-1)^5} \leq \frac{1}{q^2}$ so steps 2, 3 and Corollary 2.85 combined give

$$\zeta^{\times}_{U_{\mathfrak{p}}}(5) \leq 1 + \sum_{n=1}^{\infty} \frac{q^{3(n-1)}}{q^2 q^{5(n-1)}} = 1 + \sum_{n=1}^{\infty} \frac{1}{q^{2n}} = (1-q^{-2})^{-1} \tag{8.27}$$

If $q = 2$ the groups H and $U_{\mathfrak{p}}$ are equal. We have $H/K_1 = \mathrm{PSL}(2, \mathbb{F}_2) = S_3$ so the contribution of characters of level 1 to $\zeta^*_{U_{\mathfrak{p}}}(s)$ equals $1+2^{-s}$. Using steps 2,3 we can estimate the contribution of higher level representations by $\sum_{n=2}^{\infty} \frac{(q-1)q^{3n-3}}{(q-1)^s q^{s(n-1)}} = \sum_{n=1}^{\infty} 2^{-s-3n}$. Hence we have $\zeta^*_{U_{\mathfrak{p}}}(6) \leq (1-q^{-2})^{-1}$

(2)

$$U_{\mathfrak{p}} \simeq \left\langle \begin{pmatrix} \mathcal{O}^{\times}_{k_{\mathfrak{p}}} & \mathcal{O}_{k_{\mathfrak{p}}} \\ \mathfrak{p} & \mathcal{O}^{\times}_{k_{\mathfrak{p}}} \end{pmatrix}, \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix} \right\rangle,$$

In this case we will not be using Corollary 2.85. If A is a subgroup of $U_{\mathfrak{p}}$ we will write A^1 for the intersection $A \cap \mathrm{PSL}(2, \mathcal{O}_{k_{\mathfrak{p}}})$. Put

$$U' = \begin{pmatrix} \mathcal{O}^{\times}_{k_{\mathfrak{p}}} & \mathcal{O}_{k_{\mathfrak{p}}} \\ \mathfrak{p} & \mathcal{O}^{\times}_{k_{\mathfrak{p}}} \end{pmatrix} \tag{8.28}$$

and

$$L_n := \begin{pmatrix} 1+\mathfrak{p}^{n+1} & \mathfrak{p}^n \\ \mathfrak{p}^{n+1} & 1+\mathfrak{p}^{n+1} \end{pmatrix}^1. \tag{8.29}$$

Those are normal subgroups of $U_{\mathfrak{p}}$ which will play a role analogous to the principal congruence subgroups K_n from the previous case. We say that a representation ρ of H is of level n if $L_n \in \ker \rho$ and $L_{n-1} \notin \ker \rho$. Note that this definition of level differs from the one used in the previous case.

Step 1. Let $N_a = \begin{pmatrix} 1 & \mathcal{O}_{k_{\mathfrak{p}}}\mathfrak{p}^a \\ 0 & 1 \end{pmatrix}$. Let M_a be the normal closure of N_a in $U_{\mathfrak{p}}$. Then $M_a = L_a$. To see this, first note that

$$\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \pi x & 1 \end{pmatrix}. \tag{8.30}$$

Hence M_a contains the group

$$\left\langle \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^{a+1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathfrak{p}^{a+1} \\ 0 & 1 \end{pmatrix} \right\rangle. \tag{8.31}$$

We identify the quotient K_{a+1}/K_{a+2} with $\mathfrak{sl}(2, \mathbb{F}_q)$. Put $M'_a = M_a \cap K_{a+1}$. The image M'_a/K_{a+2} in K_{a+1}/K_{a+2} is an U'-invariant subspace of $\mathfrak{sl}(2, \mathbb{F}_q)$ containing the vectors $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. A simple calculation shows that the only U' invariant subspace containing them is $\mathfrak{sl}(2, \mathbb{F}_q)$. It follows that $M'_a K_{a+2} = K_{a+1}$. M'_a is normal in U' and $K_1 \subset U'$. We have $[K_1, K_n] = K_{n+1}$ for any n so a simple inductive argument shows that $M'_a = K_{a+1}$. Finally we conclude that $M_a = N_a K_{a+1} = L_a$.

Step 2. Any irreducible representation of $U_{\mathfrak{p}}$ of level n has dimension at least $(q-1)q^{n-1}$. The proof is identical to the proof of Step 2 from the previous case. Note that we adapted the definition of level so that the argument still works.

Step 3. The group $U_{\mathfrak{p}}/L_0$ is isomorphic to the dihedral group D_{q-1} so it has $\frac{q+5}{2}$ conjugacy classes if q odd and $\frac{q+2}{2}$ if q is even (c.f. [**99**, 5.3]). Hence we can bound the number of conjugacy classes in $U_{\mathfrak{p}}/L_n$ by

$$\frac{q+5}{2}[L_0 : L_n] = \frac{q+5}{2} q^{3n} \qquad \text{for } q \text{ odd} \tag{8.32}$$

$$\frac{q+2}{2}[L_0 : L_n] = \frac{q+2}{2} q^{3n} \qquad \text{for } q \text{ even.} \tag{8.33}$$

Step 4. We start with the case $q \geq 3$. We have $U_{\mathfrak{p}}/L_0 \simeq D_{q-1}$. The dihedral group D_{q-1} has 4 representations of dimension 1 and $\frac{q-3}{2}$ representations of dimension 2 if q is odd and 2 representations of dimension 1 and $\frac{q-2}{2}$ representations of dimension 2 if q is even (c.f. [**99**, 5.3]). The contribution of representations of level 0 to $\zeta^*_{U_{\mathfrak{p}}}(s)$ is at most $1+\frac{q-3}{2}2^{-s}$ for q odd and at most $1+\frac{q-2}{2}$ for q even. Combining this with steps 2,3 we get that for q odd

$$\zeta^*_{U_{\mathfrak{p}}}(s) \leq 1+\frac{q-3}{2}2^{-s}+\sum_{n=1}^{\infty}\frac{(q+3)q^{3n}}{2(q-1)^s q^{s(n-1)}}. \tag{8.34}$$

If follows that $\zeta^*_{U_{\mathfrak{p}}}(7) \leq (1+q^{-2})^{-1}+\frac{q-3}{2}2^{-7} \leq (1-q^{-2})^{-1}(1+q)$. For q even we get

$$\zeta^*_{U_{\mathfrak{p}}}(s) \leq 1+\frac{q-2}{2}2^{-s}+\sum_{n=1}^{\infty}\frac{(q+2)q^{3n}}{2(q-1)^s q^{s(n-1)}}. \tag{8.35}$$

If $q \geq 4$ then as before $\zeta^*_{U_{\mathfrak{p}}}(7) \leq (1+q^{-2})^{-1}+\frac{q-2}{2}2^{-7} \leq (1+q^{-2})^{-1}(1+q)$. For $q=2$ there are additional complications because our method from the second step gives a trivial bound on the representations of level 1. In this case we have to compute the contribution of the representations of levels 0 and 1 by hand. The group $U_{\mathfrak{p}}/L_0$ is isomorphic to $D_1 \simeq \mathbb{Z}/2\mathbb{Z}$ with generator $\begin{pmatrix}0&1\\2&0\end{pmatrix}$ and $L_0/L_1 \simeq (\mathbb{Z}/2\mathbb{Z})^2$ with generators $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\2&1\end{pmatrix}$. Group $U_{\mathfrak{p}}/L_0$ acts on $L_0/L_1 \simeq (\mathbb{Z}/2\mathbb{Z})^2$ be swapping coordinates so $U_{\mathfrak{p}}/L_0 \simeq (\mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})^2$. We conclude that $U_{\mathfrak{p}}/L_1$ has 4 representations of dimension 1 and a single representation of dimension 2. Together with Steps 2 and 3 this yields the estimate

$$\zeta^*_{U_{\mathfrak{p}}}(s) \leq 1+2^{-s}+\sum_{n=2}^{\infty}\frac{2^{3n+2}}{2^{1+(n-1)s}}. \tag{8.36}$$

We get $\zeta^*_{U_{\mathfrak{p}}}(s) \leq 1+2^{-7}+1+2^{-4}+2^{-8}+\ldots \leq (1-q^{-2})^{-1}(1+q)$.

(3) D is ramified in $\mathfrak{p}$ and $U = PD^{\times}(k_{\mathfrak{p}})$. Write M_i for the subgroup $\{x \in D^{\times}(k_{\mathfrak{p}}) | n(x-1) \in \mathfrak{p}^i\}$ and $M_i^1 = M_i \cap D^1(k_{\mathfrak{p}})$. We have a surjective map $D(k_{\mathfrak{p}})^{\times} \to \mathrm{PD}^{\times}(k_{\mathfrak{p}})$ with abelian kernel so $\zeta^*_{\mathrm{PD}^{\times}(k_{\mathfrak{p}})}(s) \leq \zeta_{D^{\times}(k_{\mathfrak{p}})^*}(s)$ for every s where they are well defined. We will use Carayol's formula for the dimensions of irreducible representations in term of levels ([**31**], see theorem 2.72 for the statement). For every class c in $\mathrm{Irr}D^{\times}(k_{\mathfrak{p}})/\mathrm{Irr}D^{\times}(k_{\mathfrak{p}})^{ab}$ we choose a representative ρ_c of minimal level. Recall that in a finite group G the number of classes in $\mathrm{Irr}G/\mathrm{Irr}G^{ab}$ coincides with the number of conjugacy classes in $[G,G]$. It follows that the number of classes $c \in \mathrm{Irr}D^{\times}(k_{\mathfrak{p}})/\mathrm{Irr}D^{\times}(k_{\mathfrak{p}})^{ab}$ such that ρ_c is of level n is at most the number of conjugacy classes in $D^1(k_{\mathfrak{p}})/M^1_{n-1}$. As in Theorem 2.72 we put $r=\gcd(n,2)$, then $D^1(k_{\mathfrak{p}})/M^1_n = (q+1)q^{\frac{1}{2}(3n+r-4)}$. Using the dimension formula we get

$$\zeta^*_{D^{\times}(k_{\mathfrak{p}})} \leq 1+2^{-s}(q+1)+\sum_{n\geq 3}\frac{(q+1)q^{\frac{1}{2}(3n+r-4)}}{\left(r(q^2-1)(q^r-1)^{-1}q^{\frac{1}{2}(n+r-4)}\right)^s} \tag{8.37}$$

For $s \geq 7$ the left hand side is bounded by $\frac{q+1}{2^7}+(1+q^{-2})^{-1} \leq (1+q^{-2})^{-1}(1+q)$.

We wrap up the estimates on special representation zeta functions for maximal compact subgroups. Rather than aiming at an optimal estimate we will try to give simpler version with application to Strong Limit Multiplicity problem in mind.

COROLLARY 2.86. *Let D be an admissible quaternion algebra defined over a number field k. Let $U = \prod_{\mathfrak{p}\in M_k^f} U_{\mathfrak{p}}$ be a maximal compact subgroup of $\mathrm{PD}^{\times}(\mathbb{A}_f)$ and let S be the*

set of places where $U_{\mathfrak{p}} \not\simeq \mathrm{PGL}(2, \mathcal{O}_{k_{\mathfrak{p}}})$. *Then for* $s \geq 7$ *we have*

$$\zeta_U^*(s) \leq \zeta_k(2) \prod_{\mathfrak{p} \in S} (N(\mathfrak{p}) + 1) \tag{8.38}$$

8.3. Abelianization of maximal compact subgroup. Let U be a maximal compact subgroup in $\mathrm{PD}^\times(\mathbb{A}_f)$ and V an open subgroup. To prove the Strong Limit Multiplicity for a lattice $\Gamma_V = \mathrm{PD}^\times(k) \cap V$ we will use a variation of the second part of the Lemma 2.83. With this application in mind we need to control the index of the image of V in the abelianization of U. Write $\phi \colon U \to U^{ab}$, we will prove:

Proposition 2.87.

$$[U^{ab} : \phi(V)] \leq 2^{3[k:\mathbb{Q}]+|S|} |\mathrm{cl}\,(k)| \frac{|\mathrm{cl}\,(V)|}{|\mathrm{cl}\,(U)|}, \tag{8.39}$$

where $\mathrm{cl}\,(V), \mathrm{cl}\,(U)$ *are the class groups defined in 2.48 and* $\mathrm{cl}\,(k)$ *is the usual class group of* k.

Proof. We will use a simple fact that whenever $H \subset G$ are groups such that G/H is finite and ι is a homomorphism of G with finite kernel, then $[G : H] \leq |\ker \iota| [\iota G : \iota H]$.

Step 1. The norm induces a map $n : U^{ab} \to n(U)$. Write U^1 for the kernel of $n : U \to n(U)$. We have $U^1 = \prod_{\mathfrak{p} \in M_k^f} U_{\mathfrak{p}}^1$. If $\mathfrak{p} \notin \mathrm{Ram}^f D$ and $U_{\mathfrak{p}} \simeq \mathrm{PGL}(2, \mathcal{O}_{k_{\mathfrak{p}}})$ then $[U_{\mathfrak{p}}, U_{\mathfrak{p}}] = U_{\mathfrak{p}}^1$ unless[15] $N(\mathfrak{p}) = 2, 3$. If $N(\mathfrak{p}) = 2, 3$ then $[U_{\mathfrak{p}}, U_{\mathfrak{p}}]$ is a subgroup of $U_{\mathfrak{p}}^1$ of index 2. If $\mathfrak{p} \notin \mathrm{Ram}^f D$ and $U_{\mathfrak{p}} \not\simeq \mathrm{PGL}(2, \mathcal{O}_{k_{\mathfrak{p}}})$ then $U_{\mathfrak{p}}$ is of the second type in 2.39. For such groups we have $[U_{\mathfrak{p}}, U_{\mathfrak{p}}] = U_{\mathfrak{p}}^1$.

Finally if $\mathfrak{p} \in \mathrm{Ram}^f D$ then $[U_{\mathfrak{p}}, U_{\mathfrak{p}}] = U_{\mathfrak{p}}^1$. It follows that $|\ker \left[U^{ab} \to n(U)\right]| \leq 2^{n_2+n_3}$ where n_i is the number of prime ideals in k of norm i. There are at most $[k : \mathbb{Q}]$ prime ideals above every rational prime so we deduce $[U^{ab} : \phi(V)] \leq 2^{2[k:\mathbb{Q}]} [n(U) : n(V)]$.

Step 2. Our task is reduced to finding an upper bound on $[n(U) : n(V)]$. Let S be the set of finite places $\mathfrak{p}$ where $U_{\mathfrak{p}} \not\simeq \mathrm{PGL}(2, \mathcal{O}_{k_{\mathfrak{p}}})$. At this point it is good to recall that $U \subset \mathrm{PD}^\times$ so $n(U) \subset (\mathbb{A}_f^S)^\times / ((\mathbb{A}_f^\times)^S)^2$. Let us consider the quotient map $(\mathbb{A}_f^S)^\times / ((\mathbb{A}_f^S)^\times))^2 \to A_f^\times / k^\times \times ((\mathbb{A}_f)^\times)^2$ and write W_U, W_V respectively for the image of $n(U), n(V)$. We have an exact sequence

$$1 \longrightarrow (\mathbb{A}_f^S)^\times / \mathcal{O}_{k,S}^\times \longrightarrow A_f^\times / k^\times \longrightarrow A_f^\times / k^\times (\mathbb{A}_f^S)^\times \longrightarrow 1 \tag{8.40}$$

Let us call the last group by $\mathrm{cl}^S(k)$, it is a quotient of the class group of k. We tensor the exact sequence by $\mathbb{Z}/2\mathbb{Z}$ to get

$$[\mathrm{cl}^S(k)]_2 \longrightarrow (\mathbb{A}_f^S)^\times / \mathcal{O}_{k_D}^\times ((\mathbb{A}_f^S)^\times)^2 \longrightarrow A_f^\times / k_D^\times (\mathbb{A}_f^\times)^2 \longrightarrow \mathrm{cl}^S(k)/2\mathrm{cl}^S(k) \longrightarrow 1. \tag{8.41}$$

Then the kernel of the map $n(U) \to (\mathbb{A}_f^S)^\times / \mathcal{O}_k^\times ((\mathbb{A}_f^S)^\times)^2$ is a subgroup of $\mathcal{O}_{k,S}^\times / (\mathcal{O}_{k,S}^\times)^2$. By Dirichlet's Unit Theorem the latter is cardinality at most $2^{r_1+r_2+|S|}$ so the kernel of the map $n(U) \to W_U$ is of cardinality at most $2^{r_1+r_2+|S|} |[\mathrm{cl}^S(k)]_2|$. We deduce that $[n(U) : n(V)] \leq 2^{[k:\mathbb{Q}]+|S|} |\mathrm{cl}\,(k)| [W_U : W_V]$.

[15] The groups $\mathrm{PSL}(2, \mathbb{F}_q)$ are simple if $q \neq 2, 3$.

Step 3. Consider the commutative diagram

$$\begin{array}{ccccccccc} & & 1 & & 1 & & \ker[\mathrm{cl}\,(V)\to \mathrm{cl}\,(U)] & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & W_V & \longrightarrow & \mathbb{A}_f^\times/k_D^\times(\mathbb{A}_f^\times)^2 & \longrightarrow & \mathrm{cl}\,(V) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & W_U & \longrightarrow & \mathbb{A}_f^\times/k_D^\times(\mathbb{A}_f^\times)^2 & \longrightarrow & \mathrm{cl}\,(U) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & W_U/W_V & & 1 & & 1 & & \end{array} \tag{8.42}$$

The rows and columns are exact so by snake lemma $W_U/W_V \simeq \ker[\mathrm{cl}\,(V)\to \mathrm{cl}\,(U)]$. It follows that

$$[U^{ab}:\phi(V)] \leq 2^{3[k:\mathbb{Q}]+|S|}|\mathrm{cl}\,(k)|\frac{|\mathrm{cl}\,(V)|}{|\mathrm{cl}\,(U)|}. \tag{8.43}$$

□

In the Section 9 we will show that for fixed $f\in C_c(\mathrm{PGL}(2,\mathbb{K}))$ if a congruence lattice Γ_V contains non torsion elements with nonzero contribution to $\mathrm{tr}R_{\Gamma_V}f$ then $|\mathrm{cl}\,(k)| \ll_\varepsilon |\Delta_k|^{1/2+\varepsilon}$.

9. Adelic Volumes

9.1. Volumes of projective division algebras. In this section we compute the volume of $\mathrm{PD}^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})$ with respect to the standard measure (cf. Section 2.4). The standard measure depends on the choice of maximal compact subgroup U of $\mathrm{PD}^\times(\mathbb{A})$. The reasoning follows [**94**, Chapters 7,10,11]) but we work with $\mathrm{PD}^\times$ instead of D^1. Let us start by recalling the definition of a Tamagawa measure on an algebraic group.

Let G be an algebraic group defined over a number field k. Fix a left invariant gauge form ω on G defined over k. For any place ν in M_k the form ω induces a left invariant Haar measure on $G(k_\nu)$ denoted $|\omega|_\nu$. For the construction see [**105**, Section 2.2]. For any $a\in k^\times$ we have $|a\omega|_\nu = |a|_\nu|\omega|_\nu$. A sequence of positive real numbers $(\lambda_\nu)_{\nu\in M_k}$ is called a set of convergence factors if the product $\bigotimes_{\nu\in M_k}(\lambda_\nu^{-1}|\omega|_\nu)$ is a left invariant Haar measure on $G(\mathbb{A})$. If that is the case the measure $|\Delta_k|^{-\dim G/2}\bigotimes_{\nu\in M_k}(\lambda_\nu^{-1}|\omega|_\nu)$ is called the Tamagawa measure for G derived from the convergence factors (λ_ν) and we denote it by $(\omega,(\lambda_\nu)_{\nu\in M_k})$. Tamagawa measures do not depend on the choice of ω. If G is a semisimple algebraic group then $(1)_{\nu\in M_k}$ is a set of convergence factors and the Tamagawa measure derived from $(1)_{\nu\in M_k}$ is called the canonical Tamagawa measure on G. We shall write μ_{Tam} for the Tamagawa measure on a semisimple group G. We have the following result on the Tamagawa measure of $G(k)\backslash G(\mathbb{A})$ when G is the projective group of a division algebra:

Theorem 2.88. *[Weil, [**105**, Theorem 3.2.1]] Let A be a central division algebra of dimension n^2 over a number field k. The canonical Tamagawa measure of the quotient $PA^\times(k)\backslash PA^\times(\mathbb{A})$ equals n.*

Computing Tamagawa measures straight from definition is not very convenient. We shall use an explicit description of Tamagawa measures on $D^\times(\mathbb{A})$ and $\mathbb{A}^\times$ as products of local measures defined without reference to any gauge form. The constructed local measures coincide with those from [**94**, Chapter 7.5]. For this, we fix a maximal order $\mathcal{O}$ in D and put $\mathcal{O}_\mathfrak{p} = \mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k_\mathfrak{p}}$ for each $\mathfrak{p}\in M_k^f$ and proceed as in [**94**, Chapter 7]. Fix a set of convergence factors $(\lambda_\nu)_\nu$ with $\lambda_\nu = 1$ if $\nu\in M_k^\infty$ and $\lambda_\mathfrak{p} = (1-N(\mathfrak{p})^{-1})^{-1}$ for $\mathfrak{p}\in M_k^f$. First let us describe additive measures. The Tamagawa measures on $\mathbb{A}$ and $D(\mathbb{A})$ are given by $\mu^{\mathrm{Tam}}_{D(\mathbb{A})} = \prod_{\nu\in M_k}\mu^{\mathrm{Tam}}_{D(k_\nu)}$ and $\mu^{\mathrm{Tam}}_{\mathbb{A}} = \prod_{\nu\in M_k}\mu^{\mathrm{Tam}}_{k_\nu}$. At a non-archimedean place the local measures $\mu^{\mathrm{Tam}}_{k_\mathfrak{p}}, \mu^{\mathrm{Tam}}_{D(k_\mathfrak{p})}$ are defined as the unique Haar measures giving

masses $|\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^{1/2}, |\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^2 |\Delta_{D/k_\mathfrak{p}}|_\mathfrak{p}^{1/2}$ respectively to $\mathcal{O}_{k_\mathfrak{p}}$ and $\mathcal{O}_\mathfrak{p}$. At an archimedean place ν, $d\mu_{k_\nu}^{\mathrm{Tam}}$ is defined as Lebesgue measure if $k_\nu \simeq \mathbb{R}$ and if $k_\nu \simeq \mathbb{C}$ and $z = x + iy$ then $d\mu_{k_\nu}^{\mathrm{Tam}}(z) = 2dxdy$. For the quaternion algebra we put $d\mu_{D(k_\nu)}^{\mathrm{Tam}}(x) =$

- $4dx_1dx_2dx_3dx_4$ if $D(k_\nu) \simeq \mathbf{H}$ and $x = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4$;
- $dx_1dx_2dx_3dx_4$ if $D(k_\nu) \simeq M(2,\mathbb{R})$ and $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$;
- $16dx_1x_2\dots dx_8$ if $D(k_\nu) \simeq M(2,\mathbb{C})$ and $x = \begin{pmatrix} x_1+ix_2 & x_3+ix_4 \\ x_5+ix_6 & x_7+ix_8 \end{pmatrix}$.

The local multiplicative Tamagawa measures on $D^\times(k_\nu)$ and $k_\nu^\times$ derived from the set of convergence factors $(\lambda_\nu)_\nu$ are given by $d\mu_{k_\nu^\times}^{\mathrm{Tam}}(a) = |a|_\nu^{-1} d\mu_{k_\nu}^{\mathrm{Tam}}$ and $d\mu_{D^\times(k_\nu)}^{\mathrm{Tam}}(x) = |n(x)|_\nu^{-2} d\mu_{D(k_\nu)}^{\mathrm{Tam}}(x)$ respectively. We have to pay particular attention to the case $k_\nu \simeq \mathbb{C}$ where the valuation $|x|_\nu$ is the **square** of complex modulus. The measures $\mu_{D^\times(\mathbb{A})}^{\mathrm{Tam}} = \prod_{\nu\in M_k} \mu_{D^\times(k_\nu)}^{\mathrm{Tam}}$ and $\mu_{\mathbb{A}^\times}^{\mathrm{Tam}} = \prod_{\nu\in M_k} \mu_{k_\nu^\times}^{\mathrm{Tam}}$ are the Tamagawa measures derived from the set of convergence factors $(\lambda_\nu)_\nu$. Let us recall the definitions of matching gauge form and matching Haar measures. Let

$$1 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 1,$$

be an exact sequence of algebraic or topological locally compacts groups. Let $\omega_A, \omega_B, \omega_C$ be invariant gauge forms on A, B, C. We say that $\omega_A, \omega_B, \omega_C$ *match together algebraically* if $\omega_B = \iota_*(\omega_A) \wedge \pi^*(\omega_C)$. Let da, db, dc be Haar measures on A, B, C respectively. We say that dA, dB, dC *match together topologically* if, for every integrable f

$$\int_B f(b)db = \int_C \left(\int_A f(ac)da \right) dc.$$

To handle the canonical Tamagawa measure on $\mathrm{PD}^\times(\mathbb{A})$ we use

Proposition 2.89. *[Weil, [**105**, Theorem 2.4.3]] Let G be a connected algebraic group defined over k and let N be a normal closed subgroup. Put $H = G/N$. Let dg, dn, dh be left invariant gauge forms on G, H, N respectively, defined over k and matching together algebraically (cf. [**105**, p 24]). Let $(a_\nu)_{M_k}, (b_\nu)_{M_k}, (c_\nu)_{M_k}$ be respective sets of factors such that $a_\nu = b_\nu c_\nu$. Then:*

1. *If two of three sets $(a_\nu)_{M_k}, (b_\nu)_{M_k}, (c_\nu)_{M_k}$ are sets of convergence factors, so is the third one.*
2. *If (1) holds then the measures $(dx, (a_\nu)_{M_k}), (dn, (b_\nu)_{M_k}), (dh, (c_\nu)_{M_k})$ match together topologically.*

For $R = \mathbb{A}$ or $R = k_\nu, \nu \in M_k$ consider the exact sequence

$$1 \longrightarrow R^\times \longrightarrow D^\times(R) \longrightarrow \mathrm{PD}^\times(R) \longrightarrow 1$$

Define $\mu_{\mathrm{PD}^\times(k_\nu)}^{\mathrm{Tam}}$ as the unique measure compatible with $\mu_{k_\nu^\times}^{\mathrm{Tam}}$ on $k_\nu^\times$ and $\mu_{D^\times(k_\nu)}^{\mathrm{Tam}}$ on $D^\times(k_\nu)$. Then the product measure $\prod_{\nu\in M_k} \mu_{\mathrm{PD}^\times(k_\nu)}^{\mathrm{Tam}}$ is compatible with with $\mu_{\mathbb{A}^\times}^{\mathrm{Tam}}$ and $\mu_{D^\times(\mathbb{A})}^{\mathrm{Tam}}$. By Proposition 2.89

$$\prod_{\nu\in M_k} \mu_{\mathrm{PD}^\times(k_\nu)}^{\mathrm{Tam}} = \mu_{\mathrm{PD}^\times(\mathbb{A})}^{\mathrm{Tam}}$$

where $\mu_{\mathrm{PD}^\times(\mathbb{A})}^{\mathrm{Tam}}$ is the canonical Tamagawa measure. Be definition (see Section 2.4) the standard measure admits a similar decomposition

$$\mu_{PD^\times(\mathbb{A})}^{\mathrm{st}} = \prod_{\nu\in M_k} \mu_{\mathrm{PD}^\times(k_\nu)}^{\mathrm{st}}.$$

It depends implicitly on the choice of maximal compact subgroup U of $\mathrm{PD}^\times(\mathbb{A}_f)$.

To compute the standard volume of $\mathrm{PD}^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})$ we need to find the ratio

$$\frac{\mu^{\mathrm{Tam}}_{\mathrm{PD}^\times(\mathbb{A})}}{\mu^{\mathrm{st}}_{\mathrm{PD}^\times(\mathbb{A})}} = \prod_{\nu\in M_k}\frac{\mu^{\mathrm{Tam}}_{\mathrm{PD}^\times(k_\nu)}}{\mu^{\mathrm{st}}_{\mathrm{PD}^\times(k_\nu)}}.$$

The remaining part of this section is devoted to computations of local ratios.

Case $D(k_\nu)\simeq M(2,\mathbb{R})$. Put $f(A)=e^{-\pi\mathrm{tr}AA^t}|\det(A)|^2$. Then

$$\int_{GL(2,\mathbb{R})} f(g)d\mu^{\mathrm{Tam}}_{\mathrm{GL}(2,\mathbb{R})}(g) = \int_{\mathbb{R}^4} e^{-\pi(x_1^2+x_2^2+x_3^2+x_4^2)}dx_1dx_2dx_3dx_4 \tag{9.1}$$

$$= \left(\int_{\mathbb{R}} e^{-\pi x^2}dx\right)^4 = 1 \tag{9.2}$$

$$\tag{9.3}$$

By compatibility of $\mu^{\mathrm{Tam}}_{R^\times},\mu^{\mathrm{Tam}}_{\mathrm{GL}(2,\mathbb{R})}$ and $\mu^{\mathrm{Tam}}_{\mathrm{PD}^\times(\mathbb{R})}$ the same integral may be rewritten as

$$\int_{GL(2,\mathbb{R})} f(g)d\mu^{\mathrm{Tam}}_{\mathrm{GL}(2,\mathbb{R})}(g) = \int_{\mathrm{PGL}(2,\mathbb{R})}\left(\int_{\mathbb{R}^\times} f(gt)\frac{dt}{|t|}\right)d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{R})}(g\mathbb{R}^\times) \tag{9.4}$$

$$= \int_{\mathrm{PGL}(2,\mathbb{R})}\left(|\det g|^2\int_{\mathbb{R}^\times} e^{-\pi t^2(\mathrm{tr}gg^t)}|t|^3dt\right)d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{R})} \tag{9.5}$$

$$= \frac{1}{\pi^2}\int_{\mathrm{PGL}(2,\mathbb{R})}\frac{|\det g|^2}{(\mathrm{tr}gg^t)^2}d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{R})}(g\mathbb{R}^\times) \tag{9.6}$$

Hence $\int_{\mathrm{PGL}(2,\mathbb{R})}\frac{|\det g|^2}{(\mathrm{tr}gg^t)^2}d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{R})}(g\mathbb{R}^\times)=\pi^2$. We integrate the same function against the standard measure. Choose the maximal compact subgroup $K=\mathrm{PO}(2,\mathbb{R})$ and write A for the subgroup of positive diagonal matrices and N for the group upper triangular unipotent matrices. We have the Iwasawa decomposition $\mathrm{PGL}(2,\mathbb{R})=NAK$, which induces a diffeomorphism $NA\simeq\mathbb{H}^2$ given explicitly by

$$NA\ni\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y&0\\0&1\end{pmatrix}\mapsto x+iy\in\mathbb{H}^2.$$

The function $\varphi(g)=\frac{|\det g|^2}{(\mathrm{tr}gg^t)^2}$ is right K-invariant so

$$\int_{\mathrm{PGL}(2,\mathbb{R})}\varphi(g)d\mu^{\mathrm{st}}_{\mathrm{PGL}(2,\mathbb{R})}(g) = \int_{\mathbb{H}^2}\varphi(g)dgK \tag{9.7}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}_+}\varphi\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}\right)\frac{dydx}{y^2} \tag{9.8}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}_+}\frac{1}{(1+y^2+x^2)^2}dydx \tag{9.9}$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^\infty\frac{r}{(1+r^2)^2}drd\theta \tag{9.10}$$

$$= \frac{\pi}{2} \tag{9.11}$$

It follows that $\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{R})}/\mu^{\mathrm{st}}_{\mathrm{PGL}(2,\mathbb{R})}=2\pi$

Remark 2.90. *Using the same method one can show that*

$$\mu^{\mathrm{Tam}}_{\mathrm{PGL}(n,\mathbb{R})}/\mu^{\mathrm{st}}_{\mathrm{PGL}(n,\mathbb{R})} = \frac{\pi^{\frac{n^2+n}{4}}}{2^{n-1}\Gamma(1/2)\Gamma(2/2)\dots\Gamma(n/2)}.$$

Case $D(k_\nu)\simeq M(2,\mathbb{C})$. Put $f(A)=e^{-\pi\mathrm{tr}AA^*}|\det g|^4$ where A^* is the Hermitian transpose of A. Note that for $x\in k_\nu$, $|x|_\nu=|x|^2$ i.e. the valuation ν is not the complex

modulus but its square. We have

$$\int f(g)d\mu^{\mathrm{Tam}}_{GL(2,\mathbb{C})} = 16\int_{\mathbb{R}^8} e^{-\pi(x_1^2+\ldots+x_8^2)}dx_1\ldots dx_8 \tag{9.12}$$

$$= 16\left(\int_{\mathbb{R}} e^{-\pi x^2}dx\right)^8 = 16 \tag{9.13}$$

By compatibility we can rewrite this integral as

$$\int_{GL(2,\mathbb{C})} f(g)d\mu^{\mathrm{Tam}}_{\mathrm{GL}(2,\mathbb{C})}(g) = \int_{\mathrm{PGL}(2,\mathbb{C})}\left(\int_{\mathbb{C}^\times} f(gz)d\mu^{\mathrm{Tam}}_{\mathbb{C}^\times}(z)\right)d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{C})}(g) \tag{9.14}$$

$$= \int_{\mathrm{PGL}(2,\mathbb{C})}\left(\int_{\mathbb{C}^\times} f(g(x+iy))\frac{2dxdy}{x^2+y^2}\right)d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{C})}(g) \tag{9.15}$$

$$= \int_{\mathrm{PGL}(2,\mathbb{C})}\left(\int_{\mathbb{R}^+} r\int_0^{2\pi} f(gre^{i\theta})\frac{2d\theta dr}{r^2}\right)d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{C})}(g) \tag{9.16}$$

$$= \int_{\mathrm{PGL}(2,\mathbb{C})} 4\pi|\det g|^4\left(\int_{\mathbb{R}^+} e^{-\pi r^2\mathrm{tr}gg^*}r^7dr\right)d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{C})}(g) \tag{9.17}$$

$$= 4\pi\int_{\mathrm{PGL}(2,\mathbb{C})}\frac{|\det g|^4\Gamma(4)}{2\pi^4(\mathrm{tr}gg^*)^4}d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{C})}(g) \tag{9.18}$$

$$= \frac{12}{\pi^3}\int_{\mathrm{PGL}(2,\mathbb{C})}\frac{|\det g|^4}{(\mathrm{tr}gg^*)^4}d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{C})}(g) \tag{9.19}$$

Hence $\int_{\mathrm{PGL}(2,\mathbb{C})}\frac{|\det g|^4}{(\mathrm{tr}gg^*)^4}d\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{C})}(g) = \frac{4\pi^3}{3}$. We compute the same integral against $\mu^{\mathrm{st}}_{\mathrm{PGL}(2,\mathbb{C})}$. Let $\phi(g) = \frac{|\det g|^4}{(\mathrm{tr}gg^*)^4}$. Let $K = \mathrm{PU}(2)$, A be the group of positive diagonal matrices and let N the group of upper triangular unipotents. We use the upper halfspace model to represent $\mathbb{H}^3$. Using the Iwasawa decomposition we identify NA with $\mathbb{H}^3$ via the map

$$\begin{pmatrix}1 & x_1+ix_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} \mapsto (x_1,x_2,y)\in\mathbb{H}^3.$$

The function ϕ is right K-invariant so

$$\int_{\mathrm{PGL}(2,\mathbb{R})}\phi(g)d\mu^{\mathrm{st}}_{\mathrm{PGL}(2,\mathbb{C})}(g) = \int_{\mathbb{H}^3}\phi(g)dgK \tag{9.20}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\phi\left(\begin{pmatrix}y & x_1+ix_2\\ 0 & 1\end{pmatrix}\right)\frac{dydx_1dx_2}{y^3} \tag{9.21}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\frac{y}{(1+y^2+x_1^2+x_2^2)^4}dydx_1dx_2 \tag{9.22}$$

$$= \frac{1}{6}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(1+x_1^2+x_2^2)^2}dx_1dx_2 \tag{9.23}$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\infty}\frac{r}{(1+r^2)^2}drd\theta \tag{9.24}$$

$$= \frac{\pi}{6} \tag{9.25}$$

Hence $\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,\mathbb{C})}/\mu^{\mathrm{st}}_{\mathrm{PGL}(2,\mathbb{C})} = \frac{4\pi^3}{3}\frac{6}{\pi} = 8\pi^2$

Case $D(k_\nu)\simeq\mathbb{H} = \mathbb{R}+\mathbf{i}\mathbb{R}+\mathbf{j}\mathbb{R} = \mathbf{k}\mathbb{R}$. Put $f(x) = e^{-\pi n(x)}n(x)^2$. Then

$$\int_{H^\times} f(g)d\mu^{\mathrm{Tam}}_{H^\times}(g) = \int_{\mathbb{R}^4} e^{-\pi(x_1^2+x_2^2+x_3^2+x_4^2)}4dx_1dx_2dx_3dx_4 \tag{9.26}$$

$$= 4\left(\int_{\mathbb{R}} e^{-\pi x^2}dx\right)^4 = 4 \tag{9.27}$$

On the other hand:

$$\int_{H^\times} f(g)d\mu_{H^\times}^{\mathrm{Tam}}(g) = \int_{\mathrm{PH}^\times}\left(\int_{\mathbb{R}^\times} f(gt)\frac{dt}{|t|}\right)d\mu_{\mathrm{PH}^\times}^{\mathrm{Tam}}(g) \tag{9.28}$$

$$= \int_{\mathrm{PH}^\times}\left(n(g)^2\int_{\mathbb{R}^\times} e^{-\pi n(g)t^2}|t|^3dt\right)d\mu_{\mathrm{PH}^\times}^{\mathrm{Tam}}(g) \tag{9.29}$$

$$= \int_{\mathrm{PH}^\times}\frac{\Gamma(2)}{\pi^2}d\mu_{\mathrm{PH}^\times}^{\mathrm{Tam}}(g) \tag{9.30}$$

$$= \frac{1}{\pi^2}\mathrm{Vol}_{\mathrm{Tam}}(\mathrm{PH}^\times), \tag{9.31}$$

so $\mathrm{Vol}_{\mathrm{Tam}}(\mathrm{PH}^\times) = 4\pi^2$. Since the standard volume of a compact group is 1 we get $\mu_{\mathrm{PH}^\times}^{\mathrm{Tam}}/\mu_{\mathrm{PH}^\times}^{\mathrm{st}} = 4\pi^2$.

We proceed to finite places where there are only 2 cases:

Case $\mathfrak{p} \in M_k^f$ and $D(k_\mathfrak{p}) \simeq M(2, k_\mathfrak{p})$. Let $\mathcal{O}_\mathfrak{p}$ be the same maximal order of $D(k_\mathfrak{p})$ as the one used to define $\mu_{D(k_\mathfrak{p})}^{\mathrm{Tam}}$. Up to conjugation we may assume $\mathcal{O}_\mathfrak{p} = M(2, \mathcal{O}_{k_\mathfrak{p}})$. Define $f\colon \mathrm{GL}(2, k_\mathfrak{p}) \to \mathbb{R}$ by

$$f(g) = \begin{cases} |n(g)|^2 & \text{if } g \in \mathcal{O}_\mathfrak{p} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_{\mathrm{GL}(2,k_\mathfrak{p})} f(g)d\mu_{\mathrm{GL}(2,k_\mathfrak{p})}^{\mathrm{Tam}}(g) = \frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\int_{M(2,k_\nu)} \mathbf{1}_{\mathcal{O}_\mathfrak{p}}d\mu_{M(2,k_\mathfrak{p})}^{\mathrm{Tam}} \tag{9.32}$$

$$= \frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}|\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^2 \tag{9.33}$$

By compatibility of Tamagawa measures we have

$$\int_{\mathrm{GL}(2,k_\mathfrak{p})} f(g)d\mu_{\mathrm{GL}(2,k_\mathfrak{p})}^{\mathrm{Tam}}(g) = \int_{\mathrm{PGL}(2,k_\mathfrak{p})}\left(\int_{k_\mathfrak{p}^\times} f(gt)d\mu_{k_\mathfrak{p}^\times}^{\mathrm{Tam}}(t)\right)d\mu_{\mathrm{PGL}(2,k_\mathfrak{p})}^{\mathrm{Tam}}(g) \tag{9.34}$$

$$= \int_{\mathrm{PGL}(2,k_\mathfrak{p})}\frac{|\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^{1/2}|n(g)|_\mathfrak{p}^2 N(\mathfrak{p})}{N(\mathfrak{p})-1}\left(\int_{k_\mathfrak{p}}\mathbf{1}_{\mathcal{O}_\mathfrak{p}}(gt)|t|_\mathfrak{p}^3dt\right)d\mu_{\mathrm{PGL}(2,k_\mathfrak{p})}^{\mathrm{Tam}}(g). \tag{9.35}$$

Here dt stands for Haar measure on $k_\mathfrak{p}$ giving mass 1 to $\mathcal{O}_{k_\mathfrak{p}}$. Let us write $\|g\|_\mathfrak{p}$ for the maximal valuation of coefficients of g. That is $\|g\|_\mathfrak{p} := \min_{g\in t\mathcal{O}_\mathfrak{p}}|t|_\mathfrak{p}$ or $\|g\|_\mathfrak{p} = N(\mathfrak{p})^{-k}$ where $k = \min\{i \mid g \in \pi^i\mathcal{O}_\mathfrak{p}\}$. We have

$$\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\int_{k_\mathfrak{p}}\mathbf{1}_{\mathcal{O}_\mathfrak{p}}(gt)|t|_\mathfrak{p}^3dt = \frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\sum_{n\in\mathbb{Z}}\mathbf{1}_{\mathcal{O}_\mathfrak{p}}(g\pi^n)|\pi|_\mathfrak{p}^{3n}\mathrm{Vol}(\pi^n\mathcal{O}_{k_\mathfrak{p}}^\times) \tag{9.36}$$

$$= \sum_{n\in\mathbb{Z}}\mathbf{1}_{\mathcal{O}_\mathfrak{p}}(g\pi^n)N(\mathfrak{p})^{-4n} \tag{9.37}$$

$$= \sum_{n=-k}^{\infty}N(\mathfrak{p})^{-4n} = \frac{N(\mathfrak{p})^k}{1-N(\mathfrak{p})^{-4}} = \frac{\|g\|_\mathfrak{p}^4}{1-N(\mathfrak{p})^{-4}}. \tag{9.38}$$

We deduce that

$$\int_{\mathrm{PGL}(2,k_\mathfrak{p})}\frac{|\det g|^2}{\|g\|_\mathfrak{p}^4}d\mu_{\mathrm{PGL}(2,k_\mathfrak{p})}^{\mathrm{Tam}} = |\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^{3/2}(1-N(\mathfrak{p})^{-4})(1-N(\mathfrak{p})^{-1})^{-1}. \tag{9.39}$$

Let us compute the same integral against the standard measure. We will assume some familiarity with Bruhat-Tits trees. The standard measure depends on the choice of a maximal compact subgroup so let us start with $U_\mathfrak{p} = \mathrm{PGL}(2, \mathcal{O}_{k_\mathfrak{p}})$. Function $g \mapsto \frac{|\det g|_\mathfrak{p}^2}{\|g\|_\mathfrak{p}^4}$

is bi $U_{\mathfrak{p}}$-invariant so

$$\int_{\mathrm{PGL}(2,k_{\mathfrak{p}})} \frac{|\det g|^2}{\|g\|_{\mathfrak{p}}^4} d\mu^{\mathrm{st}}_{\mathrm{PGL}(2,k_{\mathfrak{p}})} = \sum_{g\in \mathrm{PGL}(2,k_{\mathfrak{p}})/U_{\mathfrak{p}}} \frac{|\det g|_{\mathfrak{p}}^2}{\|g\|_{\mathfrak{p}}^4} \tag{9.40}$$

The cosets $\mathrm{PGL}(2,k_{\mathfrak{p}})/U_{\mathfrak{p}}$ may be identified with vertices of the Bruhat-Tits tree $X(\mathrm{SL}(2,K_{\mathfrak{p}})$. Let v_0 be the vertex stabilized by $U_{\mathfrak{p}}$ and put $\phi(gv_0) = \frac{|\det g|_{\mathfrak{p}}^2}{\|g\|_{\mathfrak{p}}^4}$. Because of bi-$U_{\mathfrak{p}}$ invariance the value of $\phi(v)$ depends only on the $U_{\mathfrak{p}}$ orbit of v, but by Cartan decomposition the latter depends only on the distance to v_0. More precisely the set of vertices v with $d(v,v_0)=n$ is a single $U_{\mathfrak{p}}$ orbit of size $N(\mathfrak{p})^{n-1}(N(\mathfrak{p})+1)$ or 1 if $n=0$, with a representative $\begin{pmatrix}\pi^n & 0\\ 0 & 1\end{pmatrix} v_0$. Consequently

$$\begin{aligned}
\sum_{g\in \mathrm{PGL}(2,k_{\mathfrak{p}})/U_{\mathfrak{p}}} \frac{|\det g|_{\mathfrak{p}}^2}{\|g\|_{\mathfrak{p}}^4} =& 1 + (N(p)+1)\sum_{n=1}^{\infty} N(\mathfrak{p})^{n-1} N(\mathfrak{p})^{-2n} && (9.41)\\
=& 1 + (N(\mathfrak{p})+1)\sum_{n=0}^{\infty} N(\mathfrak{p})^{-n-2} && (9.42)\\
=& (1+N(\mathfrak{p})^{-2})(1-N(\mathfrak{p})^{-1})^{-1} && (9.43)
\end{aligned}$$

Hence

$$\frac{\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,k_{\mathfrak{p}})}}{\mu^{\mathrm{st}}_{\mathrm{PGL}(2,k_{\mathfrak{p}})}} = |\Delta_{k_{\mathfrak{p}}}|_{\mathfrak{p}}^{3/2}(1-N(\mathfrak{p})^{-2}).$$

If $U_{\mathfrak{p}} \not\simeq \mathrm{PGL}(2,\mathcal{O}_{k_{\mathfrak{p}}})$ then $[\mathrm{PGL}(2,\mathcal{O}_{k_{\mathfrak{p}}}) : U_{\mathfrak{p}}] = \frac{N(\mathfrak{p})+1}{2}$ so in this case

$$\frac{\mu^{\mathrm{Tam}}_{\mathrm{PGL}(2,k_{\mathfrak{p}})}}{\mu^{\mathrm{st}}_{\mathrm{PGL}(2,k_{\mathfrak{p}})}} = \frac{2|\Delta_{k_{\mathfrak{p}}}|_{\mathfrak{p}}^{3/2}}{N(\mathfrak{p})+1}(1-N(\mathfrak{p})^{-2}).$$

Case $\mathfrak{p} \in \mathrm{Ram}^f D$. Recall that in this case $\mathcal{O}_{\mathfrak{p}} = \{x \in D(k_\nu) \mid |n(x)|_{\mathfrak{p}} \le 1\}$. We define $f : D(k_{\mathfrak{p}}) \to \mathbb{R}$ by the same formula as in the split case. By Lemma 2.28 we have

$$\int_{D^\times(k_{\mathfrak{p}})} f(g) d\mu^{\mathrm{Tam}}_{D^\times(k_{\mathfrak{p}})}(g) = |\Delta_{k_{\mathfrak{p}}}|_{\mathfrak{p}}^2 |\Delta_{D/k_{\mathfrak{p}}}|_{\mathfrak{p}}^{1/2} \frac{N(\mathfrak{p})}{N(\mathfrak{p})-1} = \frac{|\Delta_{k_{\mathfrak{p}}}|_{\mathfrak{p}}^2}{N(\mathfrak{p})-1}. \tag{9.44}$$

By compatibility of Tamagawa measures we have

$$\begin{aligned}
\int_{D^\times(k_{\mathfrak{p}})} f(g) d\mu^{\mathrm{Tam}}_{D^\times(k_{\mathfrak{p}})} =& \int_{\mathrm{PD}^\times(k_{\mathfrak{p}})} \left(\int_{k_{\mathfrak{p}}^\times} f(gt) d\mu^{\mathrm{Tam}}_{k_{\mathfrak{p}}^\times}(t) \right) d\mu^{\mathrm{Tam}}_{\mathrm{PD}^\times(k_{\mathfrak{p}})}(g) && (9.45)\\
=& \int_{\mathrm{PD}^\times(k_{\mathfrak{p}})} \left(\int_{k_{\mathfrak{p}}^\times} f(t) d\mu^{\mathrm{Tam}}_{k_{\mathfrak{p}}^\times}(t) \right) d\mu^{\mathrm{Tam}}_{\mathrm{PD}^\times(k_{\mathfrak{p}})} && (9.46)\\
& && (9.47)
\end{aligned}$$

Let $\mathrm{PD}^\times(k_\mathfrak{p}) = A \cup B$ where $A = \{xk_\mathfrak{p}^\times \mid |n(x)|_\mathfrak{p} \in N(\mathfrak{p})^{2\mathbb{Z}}\}$ and $B = \{xk_\mathfrak{p}^\times \mid |n(x)|_\mathfrak{p} \in N(\mathfrak{p})^{2\mathbb{Z}+1}\}$. Then $\mathrm{Vol}_{\mathrm{Tam}}(A) = \mathrm{Vol}_{\mathrm{Tam}}(B) = \frac{1}{2}\mathrm{Vol}_{\mathrm{Tam}}(\mathrm{PD}^\times(k_\mathfrak{p}))$. We have

$$\int_A \left(\int_{k_\mathfrak{p}^\times} f(t) d\mu^{\mathrm{Tam}}_{k_\mathfrak{p}^\times}(t) \right) d\mu^{\mathrm{Tam}}_{\mathrm{PD}^\times(k_\mathfrak{p})} = \mathrm{Vol}_{\mathrm{Tam}}(A) \left(\frac{|\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^{1/2} N(\mathfrak{p})}{N(\mathfrak{p})-1} \int_{\mathcal{O}_{k_\mathfrak{p}}} |t|_\mathfrak{p}^3 dt \right), \tag{9.48}$$

$$= \frac{\mathrm{Vol}(\mathrm{PD}^\times(k_\mathfrak{p}))}{2} \frac{|\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^{1/2}}{1 - N(\mathfrak{p})^{-4}} \tag{9.49}$$

$$\int_B \left(\int_{k_\mathfrak{p}^\times} f(t) d\mu^{\mathrm{Tam}}_{k_\mathfrak{p}^\times}(t) \right) d\mu^{\mathrm{Tam}}_{\mathrm{PD}^\times(k_\mathfrak{p})} = \frac{\mathrm{Vol}(\mathrm{PD}^\times(k_\mathfrak{p}))}{2} \frac{|\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^{1/2} N(\mathfrak{p})^{-2}}{1 - N(\mathfrak{p})^{-4}} \tag{9.50}$$

$$\int_{D^\times(k_\mathfrak{p})} f(g) d\mu^{\mathrm{Tam}}_{D^\times(k_\mathfrak{p})} = \mathrm{Vol}(\mathrm{PD}^\times(k_\mathfrak{p})) \frac{|\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^{1/2}}{2(1 - N(\mathfrak{p})^{-2})} \tag{9.51}$$

$$\tag{9.52}$$

It follows that $\mathrm{Vol}_{\mathrm{Tam}}(\mathrm{PD}^\times(k_\mathfrak{p})) = \frac{2|\Delta_{k_\mathfrak{p}}|_\mathfrak{p}^{3/2}}{N(\mathfrak{p})-1}(1 - N(\mathfrak{p})^{-2})$.

Proposition 2.91. *Let D be an admissible quaternion algebra defined over k i.e. one that can be used to construct arithmetic lattices in $\mathrm{PGL}(2,\mathbb{K}), \mathbb{K} = \mathbb{R}, \mathbb{C}$. Let U be a maximal compact subgroup of $\mathrm{PD}^\times(\mathbb{A}_f)$ and let $\mu^{\mathrm{st}}_{\mathrm{PD}^\times(\mathbb{A})}$ be the standard measure relative to U. Denote by S the set finite places $\mathfrak{p}$ such that $U_\mathfrak{p} \not\simeq \mathrm{PGL}(2, \mathcal{O}_{k_\mathfrak{p}})$. The standard volume of $\mathrm{PD}^\times(k)\backslash\mathrm{PD}^\times(\mathbb{A})$ equals*

$$\frac{2|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathfrak{p}\in \mathrm{Ram}^f D}(N(\mathfrak{p})-1)\prod_{\mathfrak{p}\in S\setminus \mathrm{Ram}^f D}(N(\mathfrak{p})+1)}{2\pi(4\pi^2)^{[k:\mathbb{Q}]-1}2^{|S|}}, \tag{9.53}$$

if $\mathbb{K} = \mathbb{R}$ and

$$\frac{2|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathfrak{p}\in \mathrm{Ram}^f D}(N(\mathfrak{p})-1)\prod_{\mathfrak{p}\in S\setminus \mathrm{Ram}^f D}(N(\mathfrak{p})+1)}{8\pi^2(4\pi^2)^{[k:\mathbb{Q}]-2}2^{|S|}}, \tag{9.54}$$

if $\mathbb{K} = \mathbb{C}$.

Proof. We just need to put together the local ratio computations with Theorem 2.88. □

Using Corollary 2.51 we get:

Corollary 2.92. *With U as before let V be an open subgroup of U. Then*

$$\mathrm{Vol}(\Gamma_V\backslash\mathbb{H}^2) = \frac{[U:V]}{|\mathrm{cl}\,(V)|}\frac{|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathfrak{p}\in \mathrm{Ram}^f D}(N(\mathfrak{p})-1)\prod_{\mathfrak{p}\in S\setminus \mathrm{Ram}^f D}(N(\mathfrak{p})+1)}{\pi(4\pi^2)^{[k:\mathbb{Q}]-1}2^{|S|}}, \tag{9.55}$$

if $\mathbb{K} = \mathbb{R}$ and

$$\mathrm{Vol}(\Gamma_V\backslash\mathbb{H}^3) = \frac{[U:V]}{|\mathrm{cl}\,(V)|}\frac{|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathfrak{p}\in \mathrm{Ram}^f D}(N(\mathfrak{p})-1)\prod_{\mathfrak{p}\in S\setminus \mathrm{Ram}^f D}(N(\mathfrak{p})+1)}{4\pi^2(4\pi^2)^{[k:\mathbb{Q}]-2}2^{|S|}}, \tag{9.56}$$

if $\mathbb{K} = \mathbb{C}$.

9.2. Volumes of algebraic tori. Let l be a quadratic extension of a number field k and let $T = \mathrm{Res}^1_{l/k}\mathbb{G}_m$ be the norm torus. In this section we prove an upper bound on the volume of $T(k)\backslash T(\mathbb{A})$ with respect to the standard measure (see Section 2.4). We will start with an exact formula for the volume which is probably well known to experts. For a representation ρ of the Galois group $\mathrm{Gal}(\overline{k}/k$ the function $\Lambda(s,\rho)$ denotes the completed Artin L-function

Proposition 2.93. *Let $\chi_{l/k}$ be the unique nontrivial character of the Galois group l/k. We have*

$$\mathrm{Vol}_{st}(T(k)\backslash T(\mathbb{A})) = \frac{2\Lambda(1,\chi_{l/k})}{2^a e(l/k)},$$

where $a = r_{1,l} - r_{1,k} + r_{2,l} - r_{2,k}$ *and* $e(l/k) = \prod_{\mathfrak{p}\in M_k^f} e(l_\mathfrak{p}/k_\mathfrak{p})$ *is the global ramification index.*

The idea of the proof is to see T as a subgroup of $\mathrm{PGL}(2,k)$ and consider the period integral of an Eisenstein series along $T(k)\backslash T(\mathbb{A})$. The residue of Eisenstein series at 1 is constant so it is enough to compute the residue of the integral to determine the volume of $T(k)\backslash T(\mathbb{A})$. Before moving to the proof we will show a simple lemma on compatibility of standard measures on certain algebraic tori:

LEMMA 2.94. *Let* k *be a number field and* l/k *a finite extension. Put* $T = \mathrm{Res}_{l/k}\mathbb{G}_m/\mathbb{G}_m$ *so that* $T(k) \simeq l^\times/k^\times$ *canonically. Then* $T(\mathbb{A}) \simeq \mathbb{A}_l^\times/\mathbb{A}^\times$ *and for every function* $f \in C_c(\mathbb{A}_l^\times)$ *we have*

$$\int_{\mathbb{A}_l^\times} f(g)d\mu^{\mathrm{st}}_{\mathbb{A}_l^\times} = e(l/k)\int_{T(\mathbb{A})}\left(\int_{\mathbb{A}^\times} f(gt)d\mu^{\mathrm{st}}_{\mathbb{A}^\times}(t)\right)d\mu^{\mathrm{st}}_{T(\mathbb{A})}(g),$$

where $e(l/k) = \prod_{\mathfrak{p}\in M_k^f} e(l_\mathfrak{p}/k_\mathfrak{p})$ *is the product of local ramification indices.*

PROOF. It is clear that the lemma will follow from the corresponding local statement: For any $\nu \in M_k$ and some $f \in C_c(l_\nu^\times)$ we have

$$\int_{l_\nu^\times} f(g)d\mu^{\mathrm{st}}_{l_\nu^\times} = e(l_\nu/k_\nu)\int_{T(k_\nu)}\left(\int_{k_\nu^\times} f(gt)d\mu^{\mathrm{st}}_{k_\nu^\times}(t)\right)d\mu^{\mathrm{st}}_{T(k_\nu)}(g),$$

with $e(l_\nu/k_\nu)$ defined to be 1 for all archimedean places.

If the extension l_ν/k_ν is split i.e. $l_\nu \simeq k_\nu^2$ then the assertion is clear. If $l_\nu \simeq \mathbb{C}$ and $k_\nu \simeq \mathbb{R}$ then the desired inequality may be shown by integrating the characteristic function of an annulus. It remains to treat the case $\mathfrak{p} \in M_k^f$ and $l_\mathfrak{p}$ quadratic extension of $k_\mathfrak{p}$. Put $f = \mathbf{1}_{\mathcal{O}_{l_\mathfrak{p}}^\times}$. The group $\mathcal{O}_{l_\mathfrak{p}}^\times$ is the maximal compact subgroup of $l_\mathfrak{p}^\times$ so

$$\int_{l_\mathfrak{p}^\times} f(g)d\mu^{\mathrm{st}}_{l_\mathfrak{p}^\times}(g) = 1.$$

On the other hand

$$\int_{l_\mathfrak{p}^\times/k_\mathfrak{p}^\times}\left(\int_{k_\mathfrak{p}^\times} f(tg)d\mu^{\mathrm{st}}_{k_\mathfrak{p}^\times}\right)d\mu^{\mathrm{st}}_{l_\mathfrak{p}^\times/k_\mathfrak{p}^\times}(g) = \int_{l_\mathfrak{p}^\times/k_\mathfrak{p}^\times}\begin{cases}1 & \text{if } |g|_\mathfrak{p} \in |k_\mathfrak{p}^\times|_\mathfrak{p}\\ 0 & \text{otherwise}\end{cases} d\mu^{\mathrm{st}}_{l_\mathfrak{p}^\times/k_\mathfrak{p}^\times}(g) \tag{9.57}$$

$$= e(l_\mathfrak{p}/k_\mathfrak{p})^{-1} \tag{9.58}$$

□

PROOF OF PROPOSITION 2.93. Note that $T \simeq \mathrm{Res}_{l/k}\mathbb{G}_m/\mathbb{G}_m$ so $T(k) \simeq l^\times/k^\times$. Fix an isomorphism $l \simeq k^2$. Multiplication by an element of $l^\times$ gives rise to an embedding $l^\times \hookrightarrow \mathrm{GL}(2,k)$ which induces an embedding $T(k) \hookrightarrow \mathrm{PGL}(2,k)$. From now on think of T as of subgroup of $\mathrm{PGL}(2,k)$. To compute the standard volume $\mathrm{Vol}_{\mathrm{st}}(T(k)\backslash T(\mathbb{A}))$ we consider a period integral

$$P(s) := \int_{T(k)\backslash T(\mathbb{A})} E(s,g)d\mu^{\mathrm{st}}_{T(\mathbb{A})}(g),$$

where $E(s,g)$ is an explicit Eisenstein series that we are going to define shortly. The residue of $E(s,g)$ in $s=1$ is a constant function on $\mathrm{PGL}(2,k)\backslash\mathrm{PGL}(2,\mathbb{A})$ so

$$\mathrm{Vol}_{\mathrm{st}}(T(k)\backslash T(\mathbb{A})) = \frac{\mathrm{res}_{s=1}P(s)}{\mathrm{res}_{s=1}E(s,g)}.$$

We will choose $E(s,g)$ in such a way that $P(s) = C\xi_l(s)$ i.e. a constant times the completed zeta function of l. Let P be the parabolic subgroup consisting of upper triangular matrices in $\mathrm{PGL}(2,k)$. Put

$$E(s,g) = \sum_{\gamma\in P(k)\backslash\mathrm{PGL}(2,k)} \tau(\gamma g, s),$$

where

$$\tau(g,s) = \|detg\|_{\mathbb{A}}^s \int_{\mathbb{A}^\times} g_0((0,1)tg)\|t\|_{\mathbb{A}}^{2s} d\mu_{\mathbb{A}^\times}^{\mathrm{Tam}}(t),$$

where $g_0 : \mathbb{A}^2 \simeq \mathbb{A}_l \to \mathbb{C}$ is defined as in [**68**, p. 298] for the number field l. With this choice of τ the Eisenstein series $E(s,g)$ have simple poles on 0 and 1 with residues respectively $-\frac{1}{2}\mathrm{Vol}_{\mathrm{Tam}}(\mathbb{A}^1/k^\times)g_0(0)$ and $\frac{1}{2}\mathrm{Vol}_{\mathrm{Tam}}(\mathbb{A}^1/k^\times)\hat{g}_0(0)$.

It remains to compute the period integral. We follow the notes by Garrett [**51**] but keep track of all the constants.

$$P(s) = \int_{T(k)\backslash T(\mathbb{A})} E(s,g) d\mu_{T(\mathbb{A})}^{\mathrm{st}}(g) \tag{9.59}$$

$$= \int_{T(k)\backslash T(\mathbb{A})} \sum_{\gamma \in P(k)\backslash \mathrm{PGL}(2,k)} \tau(\gamma g, s) d\mu_{T(\mathbb{A})}^{\mathrm{st}}(g) \tag{9.60}$$

$$= \int_{T(\mathbb{A})} \tau(\gamma g, s) d\mu_{T(\mathbb{A})}^{\mathrm{st}}(g) \tag{9.61}$$

The last transition holds because $T(k)$ acts freely transitively on $l^\times/k^\times \simeq \mathbf{P}^1(k) = P(k)\backslash \mathrm{PGL}(2,k)$. We unwind the expression for τ and use compatibility of Tamagawa measures to get:

$$\int_{T(\mathbb{A})} \|detg\|_{\mathbb{A}}^s \int_{\mathbb{A}^\times} g_0((0,1)tg)\|t\|_{\mathbb{A}}^{2s} d\mu_{\mathbb{A}^\times}^{\mathrm{Tam}}(t) d\mu_{T(\mathbb{A})}^{\mathrm{st}}(g) \tag{9.62}$$

$$= \frac{\mu_{T(\mathbb{A})}^{\mathrm{st}}}{\mu_{T(\mathbb{A})}^{\mathrm{Tam}}} \int_{T(\mathbb{A})} \int_{\mathbb{A}^\times} g_0((0,1)tg)\|tg\|_{\mathbb{A}_l^\times}^s d\mu_{\mathbb{A}^\times}^{\mathrm{Tam}}(t) d\mu_{\mathbb{T}(\mathbb{A})}^{\mathrm{Tam}}(g) \tag{9.63}$$

$$= \frac{\mu_{T(\mathbb{A})}^{\mathrm{st}}}{\mu_{T(\mathbb{A})}^{\mathrm{Tam}}} \int_{\mathbb{A}_l^\times} g_0((0,1)a)\|a\|_{\mathbb{A}_l}^\times d\mu_{\mathbb{A}_l^\times}^{\mathrm{Tam}}(a). \tag{9.64}$$

By [**68**, Corollary 3, p.300] the integral is equal to $\xi_l(s)$ – the completed zeta function of l. Let $r_{1,k}, r_{1,l}$ denote the number of real places of k, l respectively and $r_{2,k}, r_{2,l}$ the number of complex places of k, l respectively. In explicit terms

$$\xi_l(s) = (2^{-2r_{2,l}}\pi^{-[l:\mathbb{Q}]}|\Delta_l|)^{s/2}\Gamma^{r_{1,l}}(s/2)\Gamma^{r_{2,l}}(s)\zeta_l(s).$$

On the other hand by [**68**, Proposition 11, p. 298] $g_0(0) = \hat{g}_0(0) = |\Delta_l|^{1/2}(2\pi)^{-r_{2,l}}$ and $\mathrm{Vol}_{\mathrm{Tam}}(\mathbb{A}^1/k^\times) = \mathrm{res}_{s=1}\zeta_k(s) = \rho_k$ so

$$\mathrm{res}_{s=1}E(s,g) = \rho_k|\Delta_l|^{1/2}(2\pi)^{-r_{2,l}}2^{-1}. \tag{9.65}$$

We turn to the residue of the period integral. By Lemma 2.94 we have

$$\frac{\mu_{T(\mathbb{A})}^{\mathrm{st}}}{\mu_{T(\mathbb{A})}^{\mathrm{Tam}}} = \frac{\mu_{\mathbb{A}_l^\times}^{\mathrm{st}}}{\mu_{\mathbb{A}_l^\times}^{\mathrm{Tam}}} \frac{\mu_{\mathbb{A}^\times}^{\mathrm{Tam}}}{\mu_{\mathbb{A}^\times}^{\mathrm{st}}} e(l/k)^{-1} \tag{9.66}$$

$$= \frac{|\Delta_l|^{1/2}}{2^{r_{1,l}}(4\pi)^{r_{2,l}}} \frac{2^{r_{1,k}}(4\pi)^{r_{2,k}}}{|\Delta_k|^{1/2}} e(l/k)^{-1} \tag{9.67}$$

$$= \frac{|\Delta_k|^{1/2}|N_{k/\mathbb{Q}}(\Delta_{l/k})|^{1/2}}{2^{r_{1,l}-r_{1,k}}(4\pi)^{r_{2,l}-r_{2,k}}e(l/k)}. \tag{9.68}$$

We put the formulas together to get

$$\mathrm{Vol}_{\mathrm{st}}(T(k)\backslash T(\mathbb{A})) = 2\frac{|\Delta_k|^{1/2}|N_{k/\mathbb{Q}}(\Delta_{l/k})|^{1/2}\rho_l\rho_k^{-1}}{2^{r_{1,l}-r_{1,k}+r_{2,l}-r_{2,k}}(2\pi)^{r_{2,l}-r_{2,k}}e(l/k)} \tag{9.69}$$

$$= 2\frac{\Lambda(1,\chi_{l/k})}{2^a e(l/k)}, \tag{9.70}$$

where $a = r_{1,l} - r_{1,k} + r_{2,l} - r_{2,k}$. □

In the following proposition we assume that the lattice is of the form Γ_V and $f_{\mathbb{A}}$ is as in Section 4.5 with $\mathrm{supp} f \in B(1,R)$.

PROPOSITION 2.95. *Let γ be a non-torsion element in an arithmetic lattice Γ in $\mathrm{PGL}(2,\mathbb{K})$, such that $\mathcal{O}_\gamma(f_{\mathbb{A}}) \neq 0$. Let D be the quaternion algebra used to define Γ and let k be its field of definition. Let T be the centralizer of γ in $\mathrm{PD}^\times(k)$. Then*

$$\mathrm{Vol}_{\mathrm{st}}(T(k)\backslash T(\mathbb{A})) \ll_{\varepsilon,R} \frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}}.$$

PROOF. By virtue of Proposition 2.93 and the fact that for admissible quaternion algebras $a = 1$ it is enough to show that

$$|\Lambda(1,\chi_{l/k})| \ll_\varepsilon \frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}} \tag{9.71}$$

The function $\Lambda(s,\chi_{l/k})$ is symmetric with respect to transformation $s \mapsto 1-s$ and bounded in vertical strips ([**27**, Theorem 3.1.2]). Hence, by the maximum principle, we have

$$|\Lambda(1,\chi_{l/k})| \leq \sup_{t\in\mathbb{R}} |\Lambda(1+\delta+it,\chi_{l/k})|, \tag{9.72}$$

for any $\delta > 0$. Choose $\delta > 0$ such that $2^\delta \Gamma(1+\delta/2) \leq 60^{\varepsilon/3}$. For $\mathbb{K} = \mathbb{R}$ we have $r_{1,k} = [k:\mathbb{Q}], r_{2,k} = 0$ and $r_{1,l} = 2, r_{2,l} = [k:\mathbb{Q}] - 1$. For simplicity we put $d = [k:\mathbb{Q}]$. We have

$$|\Lambda(1+\delta+it,\chi_{l/k})| = \left(2^{-2(d-1)}\pi^{-d}\Delta_k N_{k/\mathbb{Q}}(\Delta_{l/k})\right)^{\frac{1+\delta}{2}} \frac{|\Gamma(1+\delta+it)|^{d-1}}{|\Gamma(\frac{1+\delta+it}{2})|^{d-2}} |L(1+\delta+it,\chi_{l/k})|. \tag{9.73}$$

Using Legendre's duplication formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ we get

$$\left|\frac{\Gamma(1+\delta+it)}{\Gamma(\frac{1+\delta+it}{2})}\right| = \left|2^\delta \Gamma(1+\frac{\delta}{2}+\frac{it}{2})\right| \pi^{-1/2} \tag{9.74}$$

$$\leq 2^\delta \Gamma(1+\frac{\delta}{2})\pi^{-1/2} \tag{9.75}$$

$$\leq 60^{\varepsilon/3}\pi^{-1/2} \tag{9.76}$$

Odlyzko's lower bound [**85**] on the discriminant Δ_k yields [16]

$$\frac{|\Gamma(1+\delta+it)|^{d-1}}{|\Gamma(\frac{1+\delta+it}{2})|^{d-2}} \ll |\Delta_k|^{\varepsilon/3}\pi^{-d/2}. \tag{9.77}$$

Now we estimate the absolute value of $N_{k/\mathbb{Q}}(\Delta_{l/k})$. The relative discriminant $\Delta_{l/k}$ is defined as the ideal of $\mathcal{O}_k$ generated by the set

$$\left\{ \det \begin{pmatrix} a & b \\ a^\sigma & b^\sigma \end{pmatrix}^2 \mid a,b \in \mathcal{O}_l \right\} \tag{9.78}$$

Let λ be one of non zero eigenvalues of $\mathrm{Ad}\gamma$. By choosing $a = 1, b = \lambda$ we get

$$N_{k/\mathbb{Q}}(\Delta_{l/k}) \leq |N_{k/\mathbb{Q}}((\lambda-\lambda^\sigma)^2)| = N_{l/\mathbb{Q}}(\lambda-\lambda^{-1}) \tag{9.79}$$

$$= N_{l/\mathbb{Q}}(1-\lambda^2). \tag{9.80}$$

We have used the fact that $\lambda^\sigma = \lambda^{-1}$ and that λ is a unit in $\mathcal{O}_l$ (see the proof of Proposition 2.26). By Lemma 2.69 non vanishing of $\mathcal{O}_\gamma(f)$ implies that $m(\lambda) \leq R$ so $m(\lambda^2) \leq 2R$. By Corollary 2.57 we have $N_{l/k}(1-\lambda^2) \ll_R \exp(o([k:\mathbb{Q}]))$. Consequently

$$|N_{k/\mathbb{Q}}(\Delta_{l/k})| \ll \exp(o([k:\mathbb{Q}])) \ll |\Delta_k|^{\varepsilon/3}. \tag{9.81}$$

[16] Here we could use as well Minkowski's weaker bound.

By Corollary 2.60 we also have

$$|L(1+\delta+it,\chi_{l/k})| = \prod_{\mathfrak{p}\in M_k^f}\left|1-\frac{\chi_{l/k}(\mathfrak{p})}{N(\mathfrak{p})^{1+\delta+it}}\right|^{-1} \tag{9.82}$$

$$\leq |\zeta_k(1+\delta)| = \exp(o([k:\mathbb{Q}])) \ll |\Delta_k|^{\varepsilon/3}. \tag{9.83}$$

Note that at this point we crucially use the fact that γ is not torsion as otherwise we can not say that the Weil height goes to 0 as the degree $[k:\mathbb{Q}]$ grows. Putting everything together we get

$$|\Lambda(1,\chi_{l/k})| \ll_\varepsilon \frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}}. \tag{9.84}$$

□

10. Proof of Strong Limit Multiplicity

In this section we prove Theorem 2.8 stating that for a uniform torsion free arithmetic congruence lattice Γ in $\mathrm{PGL}(2,\mathbb{K})$, $K=\mathbb{R},\mathbb{C}$ and any $f\in C_c(\mathrm{PGL}(2,\mathbb{K}))$ supported in the ball $B(1,R)$ we have

$$|\mathrm{tr}R_\Gamma f - f(1)\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{K}))| \ll_R \|f\|_\infty \mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{K}))^{1-a}, \tag{10.1}$$

for some absolute constant $a>0$. We start with a lower bound on the covolume of a maximal arithmetic lattice.

Proposition 2.96. *Let $R>0$. Let $\Gamma=\Gamma_U$ be a maximal arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$ with the trace field k and associated quaternion algebra D defined over k. Write S for the set of finite places $\mathfrak{p}$ of k where $U_\mathfrak{p}\not\simeq\mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})$. Then either Γ does not contain any non-torsion elements γ with $[\gamma]_{\mathrm{PGL}(2,\mathbb{K})}\cap B(1,R)\neq\emptyset$ or*

$$\mathrm{Vol}(\Gamma\backslash\mathrm{PGL}(2,\mathbb{K})) \gg |\Delta_k|^{0.044}\prod_{\mathfrak{p}\in\mathrm{Ram}^f D}\frac{N(\mathfrak{p})-1}{2}\prod_{\mathfrak{p}\in S\backslash\mathrm{Ram}^f D}\frac{N(\mathfrak{p})+1}{2}. \tag{10.2}$$

Proof. Suppose that there exists $\gamma\in\Gamma$ of infinite order, such that $[\gamma]_{\mathrm{PGL}(2,\mathbb{K})}\cap B(1,R)\neq\emptyset$. By Corollary 2.92, (10.2) reduces to the estimate

$$|\Delta|_k^{0.044} \ll \frac{|\Delta_k|^{3/2}\zeta_k(2)}{|\mathrm{cl}\,(U)|(2\pi)^{2[k:\mathbb{Q}]}}.$$

Let us start by bounding $|\mathrm{cl}\,(U)|$. By Lemma 2.49 we have $|\mathrm{cl}\,(U)| = |\mathbb{A}_f^\times/(\mathbb{A}_f^\times)^2 k_D^\times n(U)$. From Equation 4.2 we deduce that $n(U)$ contains the group $\mathbb{A}_f^{\infty\times}/(\mathbb{A}_f^{\infty\times})^2$. In particular $\mathrm{cl}\,(U)$ is a quotient of the narrow class group $\mathrm{cl}^+(k)$. It is well known that $|\mathrm{cl}^+(k)|\leq 2^{[k:\mathbb{Q}]}|\mathrm{cl}\,(k)|$ so we can deduce that $|\mathrm{cl}\,(U)|\leq 2^{[k:\mathbb{Q}]}|\mathrm{cl}\,(k)|$. It remains to find a good upper bound on the class number $|\mathrm{cl}\,(k)|$. Recall that we write R_k for the regulator of k. The analytic proof of the Brauer-Siegel theorem (see **[68]** or Section 9.2) yields the estimate

$$|\mathrm{cl}\,(k)|R_k \ll_\varepsilon |\Delta_k|^{1/2+\varepsilon/2}|\zeta_k(1+\varepsilon/2)|. \tag{10.3}$$

Let λ,λ^{-1} be the non-trivial eigenvalues of γ. By Lemma 2.69 we know that $k\subset\mathbb{Q}(\lambda)$, $[\mathcal{O}(\lambda):k]\leq 2$ and the logarithmic Mahler measure of λ satisfies $m(\lambda)\leq R$. We apply Corollary 2.60 to get $|\zeta_k(1+\varepsilon/2)| = \exp(o_\varepsilon([k:\mathbb{Q}]))$. We combine it with the inequality 10.3 to get

$$|\mathrm{cl}\,(U)| \leq |\mathrm{cl}\,(k)|2^{[k:\mathbb{Q}]} \ll_\varepsilon 2^{[k:\mathbb{Q}]}|\Delta_k|^{1/2+\varepsilon}R_k^{-1}. \tag{10.4}$$

We invoke a lower bound on regulator due to Zimmert **[107]**. It states that $R_k\gg\exp(0.46r_{1,k}+0.1r_{2,k})$ where $r_{1,k}$ and $r_{2,k}$ are the numbers of respectively real and complex

places of k. In our case we get $R_k \gg \exp(0.46[k:\mathbb{Q}]) > 1.58^{[k:\mathbb{Q}]}$ because all places of k except possibly 1 are real. We get

$$\frac{|\Delta_k|^{3/2}\zeta_k(2)}{|\mathrm{cl}\,(U)|(2\pi)^{2[k:\mathbb{Q}]}} \gg \frac{|\Delta_k|^{1-\varepsilon}R_k}{2^{[k:\mathbb{Q}]}(2\pi)^{2[k:\mathbb{Q}]}} \tag{10.5}$$

$$\gg \frac{|\Delta_k|^{1-\varepsilon}}{50^{[k:\mathbb{Q}]}}. \tag{10.6}$$

Using Odlyzko's lower bound [17] $|\Delta_k| \gg 60^{[k:\mathbb{Q}]}$ (see **[85]**) we can estimate the last expression by $|\Delta_k|^{1-\log 50/\log 60-\varepsilon} \geq |\Delta_k|^{0.0445-\varepsilon}$. To end the proof we note that we can take $\varepsilon \leq 0.0005$. □

Proof of Theorem 2.8. Without loss of generality we may assume that $\|f\|_\infty \leq 1$. Write $\Gamma = \Gamma_V$ for an open compact subgroup V of $\mathrm{PD}^\times(\mathbb{A}_f)$ and let Γ_U, $V \subset U$ be a maximal lattice containing Γ. By Theorem 2.50 and Corollary 2.51 we have

$$\mathrm{tr}R_\Gamma f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K})) = \sum_{1\neq[\gamma]\in \mathrm{PD}^\times(k)} \mathrm{Vol}(PD^\times_\gamma(k)\backslash PD^\times_\gamma(\mathbb{A}))\Xi^V_\gamma(f_\mathbb{A}). \tag{10.7}$$

Since Γ is torsion free we can sum only over the non-torsion conjugacy classes. Using Lemma 2.55 we get a bound

$$|\mathrm{tr}R_\Gamma f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \leq \tag{10.8}$$

$$\sum_{\substack{[\gamma]\in \mathrm{PD}^\times(k)\\ \gamma \text{ torsion free}}} \mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash \mathrm{PD}^\times_\gamma(\mathbb{A}))\frac{2|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})|}{|\mathrm{cl}\,(V)|}\sum_{\rho\in \mathrm{Irr}(U)} \dim W^V_\rho |\mathcal{O}_\gamma(\chi_\rho)|. \tag{10.9}$$

By virtue of Proposition 2.95 the adelic volume $\mathrm{Vol}(\mathrm{PD}^\times_\gamma(k)\backslash \mathrm{PD}^\times_\gamma(\mathbb{A}))$ is uniformly bounded[18] for conjugacy classes $[\gamma]$ for which the orbital integral does not vanish. We get

$$|\mathrm{tr}R_\Gamma f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \ll_{R,\varepsilon} \sum_{\substack{[\gamma]\in \mathrm{PD}^\times(k)\\ \gamma \text{ torsion free}}} \frac{|\Delta_k|^{1/2+\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}}\frac{|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})|}{|\mathrm{cl}\,(V)|}\sum_{\rho\in \mathrm{Irr}(U)} \dim W^V_\rho |\mathcal{O}_\gamma(\chi_\rho)|. \tag{10.10}$$

If the orbital integral $\mathcal{O}_\gamma(f_\infty)$ does not vanish then the conjugacy class of γ intersects the ball $B(1,R)$. Hence $m(\gamma) \leq R$. By Theorem 2.77 the number of such classes is of order $\exp(O(\log^2[k:\mathbb{Q}])) = \exp(o([k:\mathbb{Q}]) \ll |\Delta_k|^\varepsilon$. We get

$$|\mathrm{tr}R_\Gamma f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \ll_{R,\varepsilon} \frac{|\Delta_k|^{1/2+2\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}}\frac{|\mathcal{O}_\gamma(f_{\mathbb{A}_\infty})|}{|\mathrm{cl}\,(V)|}\sum_{\rho\in \mathrm{Irr}(U)} \dim W^V_\rho |\mathcal{O}_\gamma(\chi_\rho)|. \tag{10.11}$$

From Corollary 2.70 we deduce the bound $|\mathcal{O}_\gamma(f_\infty)| = \exp(o_f([k:\mathbb{Q}]))$ and from Proposition 2.75 for any $\delta > 0$ and Γ torsion free we have

$$|\mathcal{O}_\gamma(\chi)| \leq \chi(1)^{1-\delta}\exp(o_{f,\delta}([k:\mathbb{Q}]))2^{\delta|S|}, \tag{10.12}$$

where S is the set of finite places where $U_\mathfrak{p} \not\simeq \mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})$. Note that by Minkowski's bound $\exp(o_{R,\delta}([k:\mathbb{Q}])) \ll_{R,\delta,\varepsilon} |\Delta_k|^\varepsilon$. We apply those inequalities to get

$$|\mathrm{tr}R_\Gamma f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \ll_{R,\delta,\varepsilon} 2^{\delta|S|}\frac{|\Delta_k|^{1/2+3\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}|\mathrm{cl}\,(V)|}\sum_{\rho\in \mathrm{Irr}(U)} \dim W^V_\rho \chi(1)^{1-\delta} \tag{10.13}$$

[17] Note that at this point we use the full strength of Odlyzko's bound, Minkowski's basic bound is not sufficient.

[18] independently of Γ and γ.

Pick $b \in \mathbb{R}$ such that $\zeta_U^*(b-1)$ converges and let $a = \delta/(b+1)$. Write $\rho\colon U \to U^{ab}$ for the abelianization map. Using the exact same argument as one in the proof of Lemma 2.83 (2) we get

$$|\mathrm{tr} R_\Gamma f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \ll_{R,\delta,\varepsilon} 2^{|S|} \frac{|\Delta_k|^{1/2+3\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}|\mathrm{cl}\,(V)|}[U:V]^{1-a}\zeta_U^*(b-1)^a[U^{ab}:\rho(V)]^a. \tag{10.14}$$

Lemma 2.97. *Assume the lattice Γ_U contains a non-torsion element γ such that $\mathcal{O}_\gamma(f_\mathbb{A}) \neq 0$. Then for $c = 46 > 2/0.044$ we have $\zeta_U^*(7)[U^{ab}:\rho(V)] \ll \mathrm{Vol}(\Gamma_U\backslash \mathrm{PGL}(2,\mathbb{K}))^c \frac{|\mathrm{cl}\,(V)|}{|\mathrm{cl}\,(U)|}$.*

Proof. Let us write S for the set of finite places $\mathfrak{p}$ where $U_\mathfrak{p} \not\simeq \mathrm{PGL}(2,\mathcal{O}_{k_\mathfrak{p}})$. By Corollary 2.86 and Proposition 2.87 we have

$$\zeta_U^*(7)[U^{ab}:\rho(V)]\frac{|\mathrm{cl}\,(U)|}{|\mathrm{cl}\,(V)|} \ll \zeta_k(2)\prod_{\mathfrak{p}\in S}(N(\mathfrak{p})+1)2^{3[k:\mathbb{Q}]+|S|}|\mathrm{cl}\,(k)|. \tag{10.15}$$

We have shown in the proof of Proposition 2.96 that $|\mathrm{cl}\,(k)| \ll [\Delta_k|^{1/2+\varepsilon}$. By Proposition 2.96 we get

$$\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K})) \gg |\Delta_k|^{0.044}\prod_{\mathfrak{p}\in \mathrm{Ram}^f D}\frac{N(\mathfrak{p})-1}{2}\prod_{\mathfrak{p}\in S\backslash \mathrm{Ram}^f D}\frac{N(\mathfrak{p})+1}{2}. \tag{10.16}$$

$c > 2/0.044$ so $0.044c \geq max\{\frac{1}{2}+2\varepsilon+3\frac{\log 2}{\log 60}, 2\}$ for ε small enough. Then Odlyzko's lower bound yields[19]

$$\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))^c \gg |\Delta_k|^{\frac{1}{2}+2\varepsilon+3\frac{\log 2}{\log 60}}\prod_{\mathfrak{p}\in \mathrm{Ram}^f D}\left(\frac{N(\mathfrak{p})-1}{2}\right)^2\prod_{\mathfrak{p}\in S\backslash \mathrm{Ram}^f D}\left(\frac{N(\mathfrak{p})+1}{2}\right)^2 \tag{10.17}$$

$$\geq |\Delta_k|^{\frac{1}{2}+\varepsilon}2^{3[k:\mathbb{Q}]}|\Delta_k|^{\varepsilon}\prod_{\mathfrak{p}\in S}\left(\frac{N(\mathfrak{p})-1}{2}\right)^2. \tag{10.18}$$

By Theorem 2.59 we get that $\prod_{\mathfrak{p}\in S}(N(\mathfrak{p})+1) \geq \exp(o_R([k:\mathbb{Q}]))32^{|S|}$ and we always have $\prod_{\mathfrak{p}\in S}(N(\mathfrak{p})-1) \geq \prod_{\mathfrak{p}\in S}\frac{N(\mathfrak{p})+1}{2}$. It follows that

$$|\Delta_k|^{\varepsilon}\prod_{\mathfrak{p}\in S}\left(\frac{N(\mathfrak{p})-1}{2}\right)^2 \geq |\Delta_k|^{\varepsilon}\prod_{\mathfrak{p}\in S}(N(\mathfrak{p})+1)\prod_{\mathfrak{p}\in S}\frac{N(\mathfrak{p})+1}{16} \tag{10.19}$$

$$\geq |\Delta_k|^{\varepsilon}\exp(o_R([k:\mathbb{Q}]))\prod_{\mathfrak{p}\in S}(N(\mathfrak{p})+1)\,2^{|S|} \tag{10.20}$$

$$\gg_R \zeta_k(2)2^{|S|}\prod_{\mathfrak{p}\in S}(N(\mathfrak{p})+1). \tag{10.21}$$

In the last passage we have used the Corollary 2.60 together with Minkowski's lower bound on the discriminant. Finally we conclude that

$$\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))^c \gg_R \zeta_k(2)|\Delta_k|^{\frac{1}{2}+\varepsilon}2^{3[k:\mathbb{Q}]+|S|}\prod_{\mathfrak{p}\in S}(N(\mathfrak{p})+1) \gg_R \zeta_U^*(7)[U^{ab}:\rho(V)]\frac{|\mathrm{cl}\,(U)|}{|\mathrm{cl}\,(V)|}. \tag{10.22}$$

The lemma follows. □

Lemma 2.98. *Suppose that Γ_U contains a non-torsion semisimple element γ such that $\mathcal{O}_\gamma(f_\mathbb{A}) \neq 0$. Then for any $\varepsilon > 0$*

$$2^{|S|} \ll_R \mathrm{Vol}(\Gamma_U\backslash \mathrm{PGL}(2,\mathbb{K}))^\varepsilon$$

[19] With a bigger c this part of the argument would work with Minkowski's bound .

Proof. By Proposition 2.96

$$\mathrm{Vol}(\Gamma_U\backslash \mathrm{PGL}(2,\mathbb{K})) \gg \prod_{\mathfrak{p}\in S}\frac{N(\mathfrak{p})-1}{2} \tag{10.23}$$

From Theorem 2.59 we infer that

$$2^{|S|} = \left(\prod_{\mathfrak{p}\in S}\frac{N(\mathfrak{p})-1}{2}\right)^{o(1)}, \tag{10.24}$$

which proves the Lemma. □

We return to the proof of Strong Limit Multiplicity. We use Lemma 2.97 and Lemma 2.98:

$$|\mathrm{tr}R_{\Gamma_V}f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \ll_{R,\delta,\varepsilon} \frac{|\Delta_k|^{1/2+3\varepsilon}[U:V]^{1-a}}{(2\pi)^{[k:\mathbb{Q}]}|\mathrm{cl}\,(V)|}\mathrm{Vol}(\Gamma_U\backslash \mathrm{PGL}(2,\mathbb{K}))^{ac+\varepsilon}\left(\frac{|\mathrm{cl}\,(V)|}{|\mathrm{cl}\,(U)|}\right)^a \tag{10.25}$$

$$\ll_{R,\delta,\varepsilon} \frac{|\Delta|^{1/2+3\varepsilon}[U:V]^{1-a}|\mathrm{cl}\,(U)|^{1-a}}{(2\pi)^{[k:\mathbb{Q}]}|\mathrm{cl}\,(U)||\mathrm{cl}\,(V)|^{1-a}}\mathrm{Vol}(\Gamma_U\backslash \mathrm{PGL}(2,\mathbb{K}))^{ca+\varepsilon} \tag{10.26}$$

By Corollary 2.51 we have

$$\frac{[U:V]|\mathrm{cl}\,(U)|}{|\mathrm{cl}\,(V)|} = \frac{\mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K}))}{\mathrm{Vol}(\Gamma_U\backslash \mathrm{PGL}(2,\mathbb{K}))}. \tag{10.27}$$

Hence

$$|\mathrm{tr}R_{\Gamma_V}f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \ll_{R,\delta,\varepsilon} \frac{|\Delta_k|^{1/2+3\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}|\mathrm{cl}\,(U)|}\frac{\mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K}))^{1-a}}{\mathrm{Vol}(\Gamma_U\backslash \mathrm{PGL}(2,\mathbb{K}))^{1-a-ca-\varepsilon}} \tag{10.28}$$

By Corollary 2.92 we have $\frac{|\Delta_k|^{1/2+3\varepsilon}}{(2\pi)^{[k:\mathbb{Q}]}|\mathrm{cl}\,(U)|} \ll_R \mathrm{Vol}(\Gamma_U\backslash \mathrm{PGL}(2,\mathbb{K}))^{1/3+2\varepsilon}$. Therefore

$$|\mathrm{tr}R_{\Gamma_V}f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \ll_{R,\delta,\varepsilon} \frac{\mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K}))^{1-a}}{\mathrm{Vol}(\Gamma_U\backslash \mathrm{PGL}(2,\mathbb{K}))^{2/3-3\varepsilon-a-ca}} \tag{10.29}$$

For δ and ε small we will have $2/3-3\varepsilon-a-ca \geq 0$ so finally

$$|\mathrm{tr}R_{\Gamma_V}f - f(1)\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \ll_{R,\delta,\varepsilon} \mathrm{Vol}(\Gamma_V\backslash \mathrm{PGL}(2,\mathbb{K}))^{1-a}, \tag{10.30}$$

this proves the Strong Limit Multiplicity property for torsion free, cocompact arithmetic congruence lattices. Recall that in Lemma 2.97 for ε small enough we can take $c = 46$ so the inequality works with $a = 0.014 < \frac{2}{3\cdot 47} - \frac{2\varepsilon}{47}$. □

11. Proof of Strong Benjamini-Schramm Convergence

The aim of this section is to prove Theorem 2.9. Let $\Gamma = \Gamma_V$ be a congruence arithmetic lattice in $\mathrm{PGL}(2,\mathbb{K})$ as defined in Section 4.5. The case of non-uniform arithmetic lattices[20] was treated in [**92**, Theorem A] so we may assume that Γ is a uniform lattice. Throughout this section the Haar measure on $\mathrm{PGL}(2,\mathbb{K})$ is the standard measure (see Section 2.4). Fix an identification $X \simeq \mathrm{PGL}(2,\mathbb{K})/K$ where K is a maximal compact subgroup of $\mathrm{PGL}(2,\mathbb{K})$. Choose a bi K-invariant metric on $\mathrm{PGL}(2,\mathbb{K})$ such that the quotient metric on X coincides with the Riemannian metric.

[20] If $\mathbb{K} = \mathbb{R}$ they are defiend over $\mathbb{Q}$ and if $\mathbb{K} = \mathbb{C}$ the are defined over a quadratic imaginary number field.

Proof of Theorem 2.9. To prove the Strong Benjamini-Schramm convergence we need to show that for every $R > 0$ we have

$$\mathrm{Vol}((\Gamma\backslash X)_{<R}) \ll_R \mathrm{Vol}(\Gamma\backslash X)^{1-a} \tag{11.1}$$

for some absolute positive constant $a \geq 0.014$. To this end we apply Theorem 2.8 to the lattice Γ and the bi-K-invariant function $f = \mathbb{1}_{B(K,R)}$ - the characteristic function of the set of points at length at most R from K. This function is not continuous but it can be approximated from above by continuous compactly supported functions so the estimate from Theorem 2.8 is still valid. We have

$$|\mathrm{tr}R_\Gamma f - \mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))| \ll_R \mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))^{1-a}. \tag{11.2}$$

Since we are working with standard measure we have $\mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K})) = \mathrm{Vol}(\Gamma\backslash X))$. On the other hand unfolding the proof of Selberg Trace formula for compact quotients (see e.g. [**8**, p. 9, second equality]) gives

$$\mathrm{tr}R_\Gamma f = \int_{\Gamma\backslash \mathrm{PGL}(2,\mathbb{K})} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)dx \tag{11.3}$$

$$= \int_{\Gamma\backslash X} |\{B(K,R) \cap x^{-1}\Gamma x\}|dx. \tag{11.4}$$

$$= \mathrm{Vol}(\Gamma\backslash X) + \int_{\Gamma\backslash X} \left[|\{B(K,R) \cap x^{-1}\Gamma x\}| - 1\right] dx \tag{11.5}$$

The last two integrals are well defined because of the bi-K invariance of the metric on $\mathrm{PGL}(2,\mathbb{K})$ and the last one is non-negative. The set of points $x \in X$ whose injectivity radius is smaller than R can be described as $\{xK \in X \mid |B(K,R) \cap x^{-1}\Gamma x| \geq 2\}$. Hence

$$\mathrm{tr}R_\Gamma f - \mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K})) \geq \mathrm{Vol}((\Gamma\backslash X)_{<R}). \tag{11.6}$$

By Strong Limit Multiplicity $\mathrm{Vol}((\Gamma\backslash X)_{<R}) \ll_R \mathrm{Vol}(\Gamma\backslash X)^{1-a}$ for some absolute positive constant $a \geq 0.014$. This ends the proof of Strong Benjamini-Schramm convergence for cocompact, torsion free, congruence arithmetic lattices. □

Remark 2.99. *This argument is very general and can be used to show that the Limit Multiplicity property implies property B–S for arbitrary sequences of cocompact lattices in any semisimple Lie group G.*

Proof of Theorem 2.11. Let Γ be a torsion free arithmetic lattice with the trace field k. Like in the in the argument above it is enough to show that for some positive constant c and for $f = \mathbb{1}_{B(1,R)}$ we have

$$\mathrm{tr}R_\Gamma f - \mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K})) \ll_R \mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))|\Delta_k|^{-c} \tag{11.7}$$

We cannot apply Theorem 2.8 directly because we do not assume that Γ is a congruence lattice. Let Γ' be a maximal lattice containing Γ. Maximal lattices are all congruence lattices of form $\Gamma' = \Gamma_U$ so by the proof of Theorem 2.8 we have

$$\sum_{\substack{[\gamma]_{\Gamma'} \\ \text{not torsion}}} \mathrm{Vol}(\Gamma'_\gamma\backslash \mathrm{PGL}(2,\mathbb{K})_\gamma)\mathcal{O}_\gamma(f) \ll_R \mathrm{Vol}(\Gamma'\backslash \mathrm{PGL}(2,\mathbb{K}))^{1-a}. \tag{11.8}$$

The function f is non-negative and every conjugacy class in Γ' splits into at most $[\Gamma' : \Gamma]/[\Gamma'_\gamma : \Gamma_\gamma]$ conjugacy classes in Γ. It follows that

$$\mathrm{tr}R_\Gamma f - \mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K})) = \sum_{[\gamma]_\Gamma} \mathrm{Vol}(\Gamma_\gamma\backslash \mathrm{PGL}(2,\mathbb{K})_\gamma)\mathcal{O}_\gamma(f) \tag{11.9}$$

$$\leq [\Gamma' : \Gamma] \sum_{\substack{[\gamma]_{\Gamma'} \\ \text{not torsion}}} \mathrm{Vol}(\Gamma'_\gamma\backslash \mathrm{PGL}(2,\mathbb{K})_\gamma)\mathcal{O}_\gamma(f) \tag{11.10}$$

$$\ll_R [\Gamma' : \Gamma]\mathrm{Vol}(\Gamma'\backslash \mathrm{PGL}(2,\mathbb{K}))^{1-a} \tag{11.11}$$

$$= \mathrm{Vol}(\Gamma\backslash \mathrm{PGL}(2,\mathbb{K}))\mathrm{Vol}(\Gamma'\backslash \mathrm{PGL}(2,\mathbb{K}))^{-a}. \tag{11.12}$$

Now we consider two cases. Either Γ does not contain any non-torsion conjugacy classes $[\gamma]$ such that $[\gamma]_{\mathrm{PGL}(2,\mathbb{K})}$ intersects a ball of radius R or by Proposition 2.96 we have

$$\mathrm{Vol}(\Gamma'\backslash \mathrm{PGL}(2,\mathbb{K})) \gg_R |\Delta_k|^{0.044} \prod_{\mathfrak{p}\in \mathrm{Ram}^f D} \frac{N(\mathfrak{p})-1}{2}. \tag{11.13}$$

By Theorem 2.59 the product $\prod_{\mathfrak{p}\in \mathrm{Ram}^f D} \frac{N(\mathfrak{p})-1}{2}$ is bounded from below by $\exp(-o([k:\mathbb{Q}]))$ so by Odlyzko's bound[21] we get

$$\mathrm{Vol}(\Gamma'\backslash \mathrm{PGL}(2,\mathbb{K})) \gg_R |\Delta_k|^{0.043}. \tag{11.14}$$

This proves the theorem with $c = 0.043a \geq 0.0006$. □

Now we can prove Corollary 2.12.

Proof. Put $f = \mathbb{1}_{B(1,R)}$. Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a sequence of cocompact arithmetic (not necessarily congruence) lattices in $\mathrm{PGL}(2,\mathbb{K})$. By Theorem 2.11 either $(\Gamma_i)_{i\in\mathbb{N}}$ has property B–S or infinitely many lattices share the same trace field, say k. For the sake of the proof we may assume that the trace field of all Γ_i's is equal to k and the lattices are pairwise non commensurable. Let D_i be the quaternion algebra defined over the field k determining the commensurability class of Γ_i (see Section 4.1). Let Γ_i' be a maximal lattice containing Γ_i. From the inequalities (11.9)–(11.13) we get

$$\begin{aligned}
\mathrm{Vol}((\Gamma_i\backslash \mathrm{PGL}(2,\mathbb{K})_{<R}) &\leq \mathrm{tr}R_{\Gamma_i} f - \mathrm{Vol}(\Gamma_i\backslash \mathrm{PGL}(2,\mathbb{K})) && (11.15)\\
&\ll_{k,R} \mathrm{Vol}(\Gamma_i\backslash \mathrm{PGL}(2,\mathbb{K}))\mathrm{Vol}(\Gamma_i'\backslash \mathrm{PGL}(2,\mathbb{K}))^{-a} && (11.16)\\
&\ll_{k,R} \mathrm{Vol}(\Gamma_i\backslash \mathrm{PGL}(2,\mathbb{K})) \left(\prod_{\mathfrak{p}\in \mathrm{Ram}^f D_i} \frac{2}{N(\mathfrak{p})-1}\right)^a. && (11.17)
\end{aligned}$$

We know that the isomorphism class of D_i is uniquely determined by the set $\mathrm{Ram}D_i$. Hence $\lim_{i\to\infty} |\mathrm{Ram}D_i| = \infty$ and

$$\lim_{i\to\infty} \frac{\mathrm{Vol}((\Gamma_i\backslash \mathrm{PGL}(2,\mathbb{K})_{<R})}{\mathrm{Vol}(\Gamma_i\backslash \mathrm{PGL}(2,\mathbb{K}))} = 0. \tag{11.18}$$

□

12. Applications

12.1. Gelander conjecture. The entirety of this section is devoted to Theorem 2.16.

Proof of Theorem 2.16. Let $\Gamma \subset \mathrm{PGL}(2,\mathbb{C})$ be a torsion free arithmetic lattice with the trace field k. Put $M = \Gamma\backslash\mathbb{H}^3$. Gelander already proved the conjecture for non-uniform arithmetic lattices so we shall assume that Γ is uniform[22] . Following the method from **[53]** we will construct a simplicial complex $\mathcal{N}$ homotopic to M as a nerve of a covering of M by certain balls. We are able to bound the number of simplices in $\mathcal{N}$ because both the size of the thin part of the manifold and its injectivity radius can be controlled by the degree $[k:\mathbb{Q}]$. Let ε be the Margulis constant for $\mathbb{H}^3$.

For $x \in \Gamma\backslash\mathbb{H}^3$ or $x \in \mathbb{H}^3$ we will write $B(x,R)$ for the ball of radius R centered in x. Define $i(x) = \min\{\mathrm{injrad}\, x, 1\}$ for $x \in M$. Let $\mathcal{B}$ be a maximal with respect to inclusion set of points in M satisfying the following conditions: For any distinct $x, y \in \mathcal{B}$ we have $B(x,i(x)/16) \cap B(y,i(y)/16) = \emptyset$ ($B(x,i(x)/16)$ is to be replaced by B^2_x if $x \in \mathcal{C}$) .

Claim 1.

$$\bigcup_{c\in\mathcal{C}} B^1_c \cup \bigcup_{x\in\mathcal{B}} B(x,i(x)/5) = M.$$

[21] Alternatively we could use Minkowski's lower bound.

[22] The key feature he used is that non-uniform lattices are all defined over a qudratic imaginary field. This implies a uniform lower bound on the lengths of closed geodesics on such manifolds.

PROOF. Let $y \in M$. The proof for non-compact case is completely analogous. By maximality of $\mathcal{B}$ there exists $x \in \mathcal{B}$ such that $B(x, i(x)/16) \cap B(y, i(y)/16) \neq \emptyset$. Hence $d(x,y) < \frac{i(x)+i(y)}{16}$. If $i(x) \geq i(y)$ then $d(x,y) < i(x)/8$ so $y \in B(x, i(x)/5)$. We shall use crucially the fact that $i(x)$ is a 1-Lipschitz function, this is easy to see using the definition of injectivity radius as the maximal radius of a ball around a lift of x which maps injectively to $\Gamma\backslash\mathbb{H}^3$. If $i(x) < i(y)$ then

$$i(y) - i(x) \leq d(x,y) < \frac{i(x)+i(y)}{16},$$

so $i(y) < \frac{17}{15} i(x)$. Then

$$d(x,y) < \frac{i(x)+i(y)}{16} < \frac{2}{15} i(x)$$

and $y \in B(x, i(x)/5)$. □

Claim 2. For every $y \in \mathcal{B}$ the number of $x \in \mathcal{B}$ such that $B(x, i(x)/5) \cap B(y, i(y)/5) \neq \emptyset$ is at most 245.

PROOF. If $x \in \mathcal{B}\backslash\mathcal{C}$ and $B(x, i(x)/5) \cap B(y, i(y)/5) \neq \emptyset$ then $B(x, i(x)/5) \subset B(y, \frac{i(y)+2i(x)}{5})$. Note that

$$i(x) \leq i(y) + d(x,y) < i(y) + \frac{i(x)+i(y)}{5},$$

so $i(x) < \frac{3}{2} i(y)$. Hence $B(x, i(x)/5) \subset B(y, \frac{i(x)+2i(y)}{5}) \subset B(y, 4i(y)/5)$. On the other hand $i(x) \geq i(y) - d(x,y) > i(y) - \frac{i(x)+i(y)}{5}$ so $\frac{i(x)}{5} > \frac{2i(y)}{15}$. By comparing the volumes of $B(x, i(x)/5)$ and $B(y, 4i(y)/5)$ we get

$$|\{x \in \mathcal{B} \mid B(x, i(x)/5) \cap B(y, i(y)/5) \neq \emptyset\}| \leq \frac{\mathrm{Vol}(B(y, 4i(y)/5))}{\mathrm{Vol}(B(y, 2i(y)/15))} \leq \frac{\mathrm{Vol}(B(y, 4/5))}{\mathrm{Vol}(B(y, 2/15))} \approx 244.52 < 245$$

The last inequality is a consequence of the formula for the volume of a ball in hyperbolic 3-space $\mathrm{Vol}(B(x,R)) = \pi(\sinh 2R - 2R)$ [**93**, p.83 Ex 3.4.5] and an elementary calculation. In the non compact case we may start with $y \in \mathcal{C}$. □

Let $\mathcal{U}$ be the open cover $M = \bigcup_{x \in \mathcal{B}} B(x, i(x)/5)$, by the first claim it is indeed a cover of M. Any nonempty intersection of sets in $\mathcal{U}$ is a convex set so it is contractible. It follows that the cover $\mathcal{U}$ is "good" in the terminology of [**25**]. By [**25**, Theorem 13.4] the nerve $\mathcal{N}$ of $\mathcal{U}$ is homotopy equivalent to M. By definition the vertices in $\mathcal{N}$ correspond to the open sets in $\mathcal{U}$ and k-simplices correspond to unordered k-tuples in $\mathcal{U}$ with nonempty intersection. Using the second claim we deduce that the degree of vertices in $\mathcal{N}$ is bounded by 245.

It remains to bound the number of vertices in $\mathcal{N}$ which is equal to $|\mathcal{B}|$. We will bound the size separately for $\mathcal{B}_1 := \mathcal{B} \cap M_{\geq 1}$ and $\mathcal{B}_2 := \mathcal{B} \cap M_{<1}$. The union $\bigsqcup_{x \in B_1} B(x, 1/16)$ is disjoint so

$$|\mathcal{B}_1| \leq \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(B(x, 1/16))} \ll \mathrm{Vol}(M).$$

For any semisimple $\gamma \in \Gamma$ the minimal displacement[23] of γ is given by $m(\gamma)$. In this case $m(\gamma)$ is the half of the logarithmic Mahler measure of the characteristic polynomial of γ. For a short argument see [**53**, p. 39]. By Lemma 2.69 and Dobrowolski Theorem [**40**] we get that for $\gamma \neq 1$

$$m(\gamma) \gg (\log[k : \mathbb{Q}])^{-3}.$$

We can deduce that the injectivity radius of M is bounded below by $C(\log[k : \mathbb{Q}])^{-3} < 1$ for some absolute positive constant C. The disjoint union $\bigsqcup_{x \in B_2} B(x, C \log[k : \mathbb{Q}])^{-3}/16)$ lies in $M_{<17/16}$ so

$$|\mathcal{B}_2| \leq \frac{\mathrm{Vol}(M_{<5/4})}{\mathrm{Vol}(B(x, C \log[k : \mathbb{Q}])^{-3}/16))} \ll \mathrm{Vol}(M_{<5/4})(\log[k : \mathbb{Q}])^9).$$

[23] Recall that the minimal displacement of $g \in G$ is defined as the infimum of $d(x, gx)$ over all $x \in X$, where X is the symmetric space of G.

By Theorem 2.11 and Odlyzko's lower bound [**85**] (or Minkowski's weaker bound) we get $|\mathcal{B}_2| \ll \mathrm{Vol}(M)60^{-0.0006[k:\mathbb{Q}]}\log[k:\mathbb{Q}])^3) = o(\mathrm{Vol}(M))$. Hence $|\mathcal{B}| \ll \mathrm{Vol}(M) + o(\mathrm{Vol}(M)) \ll \mathrm{Vol}(M)$. This proves that the number of vertices in $\mathcal{N}$ is at most linear in the volume of M. □

To prove Corollary 2.17 one just has to repeat the steps of the proof of [**53**, Theorem 11.2]. Another consequence of Theorem 2.16 (actually Corollary 2.17) is the following bound on the size of the torsion part of $H_1(\Gamma\backslash\mathbb{H}^3)$:

COROLLARY 2.100. *Let Γ be a torsion free, arithmetic lattice in* $\mathrm{PGL}(2,\mathbb{C})$. *Then*

$$\log|H_1(\Gamma\backslash\mathbb{H}^3,\mathbb{Z})_{\mathrm{tors}}| \ll \mathrm{Vol}(\Gamma\backslash\mathbb{H}^3).$$

In a forthcoming paper [**11**] Bader, Bergeron, Gelander and Sauer prove the analogous bound for the torsion of homology groups of higher rank symmetric spaces. Theorem 2.16 will be used as an ingredient in their proof.

12.2. Growth of Betti numbers.

PROOF OF COROLLARY 2.18. In this section we follow closely the exposition from [**17**, p.11-17]. Before the proof of Corollary 2.18 let us set up some notations. Write $G = \mathrm{PGL}(2,\mathbb{K})$, $\mathfrak{g} = \mathrm{Lie}(G)$. Let K be a maximal compact subgroup of G and let $X = G/K$ be the associated symmetric space. Lie algebra $\mathfrak{g}$ can be written as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{p}$ is the space orthogonal to $\mathfrak{k}$ via the Killing form [24]. The restriction of the Killing form to $\mathfrak{p}$ is a positive definite and is preserved by the action of K so we will think of $\mathfrak{p}$ as a unitary representation of K. Let $\mathcal{H}^k(\Gamma\backslash X)$ denote the space of harmonic differential k-forms on the locally symmetric space $\Gamma\backslash X$. By Hodge theory we know that $H^k(\Gamma\backslash X,\mathbb{C}) \simeq \mathcal{H}^k(\Gamma\backslash X)$. The dimension of the space of harmonic k-forms can be read from the decomposition of $L^2(\Gamma\backslash G)$ into irreducible unitary representations. The Casimir operator $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$ is given by

$$\Omega = \sum_{i=1}^{\dim\mathfrak{g}} e_i \otimes e_i^*,$$

where (e_i) is a basis of $\mathfrak{g}$ and (e_i^*) is the basis of $\mathfrak{g}$ dual to (e_i) via the Killing form.

Let $\pi \in \Pi(G)$ be an irreducible unitary representation of G acting on a Hilbert space H_π. The number $\Omega(\pi)$ is the unique real number such that for every smooth vector $v \in H_\pi$ we have

$$\Omega v := \sum_{i=1}^{\dim\mathfrak{g}} D_{e_i} D_{e_i^*} v = \Omega(\pi) v.$$

The main tool to prove Corollary 2.18 is a special case of the Matsushima's formula [**18**].

THEOREM 2.101.

$$\dim\mathcal{H}^k(\Gamma\backslash X) = \sum_{\substack{\pi\in\Pi(G)\\ \Omega(\pi)=0}} m_\Gamma(\pi)\dim\mathrm{Hom}_K\left(\bigwedge{}^k\mathfrak{p},\pi\right). \tag{12.1}$$

Note that $\dim\mathfrak{p} = \dim X$ so the terms in the above sum vanish for $k > \dim X$. Irreducible representations π which have non-trivial contribution in the above sum for some $k \in \{1,\dots,\dim X\}$ are called the *cohomological representations*. The set of equivalence classes of cohomological representations of G is finite [**103**]. Once we admit this fact, the proof of Corollary 2.18 is a simple consequence of the Limit Multiplicity property. Indeed, let $\Sigma = \{\pi_1,\dots,\pi_n\}$ be the set of equivalence classes of the cohomological representations of G. By Theorem 2.8 or Theorem 2.11 we have

$$\lim_{i\to\infty}\frac{m_{\Gamma_i}(\pi_l)}{\mathrm{Vol}(\Gamma_i\backslash G)} = \begin{cases} d(\pi_l) & \text{if } \pi_l \text{ is discrete series}\\ 0 & \text{otherwise.}\end{cases}$$

[24] Note that Killing form is negative definite on $\mathfrak{k}$ so $\mathfrak{p}\cap\mathfrak{k} = 0$.

To shorten the formulas we shall extend the formal degree to all irreducible representations by putting $d(\pi) = 0$ whenever π is not discrete series. Using Theorem 2.101 and the fact that with our choice of measures we have $\mathrm{Vol}(\Gamma_i \backslash X) = \mathrm{Vol}(\Gamma_i \backslash G)$ we deduce that

$$\lim_{i \to \infty} \frac{b_k(\Gamma_i \backslash X)}{\mathrm{Vol}(\Gamma_i \backslash X)} = \sum_{l=1}^{n} d(\pi_l) \dim \mathrm{Hom}_K \left(\bigwedge^k \mathfrak{p}, \pi_l \right).$$

The last expression is known to be the $k-$th L^2-Betti number of X [**86**]. □

CHAPTER 3

Growth of mod-2 homology in higher rank locally symmetric spaces

1. Introduction

Throughout this paper G will stand for a semisimple Lie group, X for the symmetric space G/K where K is a maximal compact subgroup and Γ (possibly with index) for a torsion free lattice in G. In recent years the growth of homology groups of locally symmetric spaces, i.e. orbifolds of form $\Gamma\backslash X$ has been an area of active research. Let us mention two examples. In **[2]** Abert, Bergeron, Biringer, Gelander, Nikolov, Samet and Raimbault show that for any uniformly discrete[1] sequence of pairwise non-conjugate lattices (Γ_i) in a higher rank Lie group G the limit

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{C}} H_k(\Gamma_i\backslash X, \mathbb{C})}{\mathrm{Vol}(\Gamma_i\backslash X)}$$

exists and is equal to the L^2-Betti number $b_k^{(2)}(X)$. It is proved that such sequences have the limit multiplicity property (see **[45]**) and the convergence of normalized Betti numbers is then deduced from Matsushima's formula **[78]**. The second example is due to Calegari and Emerton **[30]**. Let Γ be a lattice in $\mathrm{SL}(2,\mathbb{C})$ and let p be a rational prime. Calegari and Emerton define **p-adic analytic towers** of covers of $\Gamma\backslash\mathbb{H}^3$ and study the growth of the first mod$-p$ homology group in such towers. Their results imply that in a p-adic analytic tower $(\Gamma_k\backslash\mathbb{H}^3)$ the limit

$$\lim_{k\to\infty} \frac{\dim_{\mathbb{F}_p} H^k(\Gamma_k\backslash X, \mathbb{F}_p)}{\mathrm{Vol}(\Gamma_k\backslash X)}$$

always exists. DeRham complex of differential forms on $\Gamma_k\backslash X$ doesn't afford the mod-p cohomology classes so the analytic methods are not accessible. Calegari and Emerton solve this problem by using the completed homology and cohomology groups (see **[29]**). In this paper we develop a new geometric approach to study the homology growth in non-commensurable case. We show:

THEOREM 3.1. *Let G be a real semisimple Lie group of real rank at least 2 and let X be the associated symmetric space. Then for any sequence of pairwise non-conjugate lattices (Γ_n) we have*

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{F}_2} H_1(\Gamma_n\backslash X, \mathbb{F}_2)}{\mathrm{Vol}(\Gamma_n\backslash X)} = 0.$$

Let us briefly review what is known on the growth of the mod$-p$ homology groups in higher rank groups. Margulis normal subgroup theorem implies that $H_1(\Gamma_n\backslash X, \mathbb{Z})$ is finite. Our result controls the size of the 2$-$torsion part of $H_1(\Gamma_n\backslash X, \mathbb{Z})$ in terms of $\mathrm{Vol}(\Gamma\backslash X)$. Conjecturally **[1]** a stronger statement should hold, it is expected that

$$\lim_{n\to\infty} \frac{d(\Gamma_n)}{\mathrm{Vol}(\Gamma_n\backslash X)} = \lim_{n\to\infty} \frac{|H_1(\Gamma_n\backslash X, \mathbb{Z})|}{\mathrm{Vol}(\Gamma_n\backslash X)} = 0 \tag{1.1}$$

where $d(\Gamma_n)$ is the rank of Γ_n i.e. the minimal number of generators of Γ_n. If the limit on the left-hand side exists it is called the **rank gradient** of (Γ_n). In **[1]** it was shown that (1.1) holds for all sequences of pairwise different subgroups $(\Gamma_n) \subset \Gamma$ where Γ is a

[1]The uniform discreteness assumption will be lifted in an upcoming work **[7]** by a subset of authors of **[2]**.

right-angled lattice. A group H is called **right angled** if it admits a finite sequence of generators (possibly with repetitions) $s_1, s_2, \ldots, s_d$ such that $[s_i, s_{i+}] = 1$. If the group Γ is a lattice in a higher rank group so every non-stationary sequence of subgroups is a Farber sequence, this is why we do not need to put additional conditions on the sequence of subgroups. The other instance where it is known to be true is for sequences of congruence subgroups Γ_1 of a fixed arithmetic lattice Γ_0. In that case we control the rank by logarithm of index:

$$d(\Gamma_1) \ll_\Gamma \log[\Gamma : \Gamma_0].$$

The implicit constant depends on Γ so even in the case of congruence lattices it wasn't previously known that the dimension of the first mod$-p$ homology group grows sublinear in volume if we consider sequences non-commensurable lattices. The argument in [**1**] is based on the relation between the rank gradient and the combinatorial cost (see [**50**, **73**] and [**5**]). Combinatorial cost is a powerful tool when one wants to study the rank gradient or similar limits associated to a sequence of subgroups of a given group. To the author's knowledge this method has not been yet adapted to handle non-commensurable sequences of lattices.

1.1. Idea of the proof. To simplify the argument let us assume in this sketch that the **fundamental rank** $\delta(G)$ of G is at least 2. The fundamental rank is the difference between the absolute rank of G and the maximal rank of a compact torus in G e.g. $\delta(\mathrm{SL}(2,\mathbb{C})) = 1, \delta(\mathrm{SL}(2n,\mathbb{R})) = n-1$ and $\delta(\mathrm{SL}(2n+1,\mathbb{R})) = n$. This assumption will ensure that the closed geodesics will lie in closed flats of dimension at least 2. The advantage of working with $\mathbb{F}_2$ is that every homology class is represented by a sum of unoriented cycles. We will show that elements of $H_1(\Gamma_i\backslash X, \mathbb{F}_2)$ can be reprenented by combinations of closed geodesics whose total length is $o(\mathrm{Vol}(\Gamma_i\backslash X)$ (see Proposition 3.5). This, together with few facts on triangulations of $\Gamma_i\backslash X$ constructed by Gelander [**53**] will yield sufficiently good bounds on $|H_1(\Gamma_i\backslash X, \mathbb{F}_2)|$ (see Proposition 3.2) to deduce Theorem 3.1. We define the "reduced representatives" of a homology class α in $H_1(\Gamma_i\backslash X, \mathbb{F}_2)$ as the representatives $c \in Z_1(\Gamma_i\backslash X, \mathbb{F}_2)$ of minimal total length. Because we do not need to care about the orientation it is easy to see that c is always a sum of uniformly separated closed geodesic (see Lemma 3.8). If the fundamental rank is at least 2 then every closed geodesic on $\Gamma\backslash X$ is contained in a closed totally geodesic flat subspace of dimension ≥ 2. We can move the geodesic components of c in their respective maximal flats without changing the homotopy class of c. Together with uniform separation of geodesics this yields the uniform separation of flats supporting c. In general case this argument is replaced by Lemma 3.9. Once we know that flats supporting c are uniformly speparated, it is enough to use the Benjamini-Schramm convergence of $(\Gamma_i\backslash X)$ to X established in [**2**] to deduce that c cannot "fill" $\Gamma_i\backslash X$ with positive density so $\ell(c) = o(\mathrm{Vol}(\Gamma_i\backslash X))$. In the actual proof we use R-lengths (see Equation 2.1) instead of ordinary lengths in order to avoid technical difficulties. Nevertheless we hope this sketch gives the right idea.

1.2. Outline. In Section 2 we establish the connection between the lengths of representatives of homology classes and the dimension of the first homology group. The main tools are the simplicial complexes constructed by Gelander in [**53**] and the Benjamini-Schramm convergence of higher rank locally symmetric spaces established in [**2**]. In Section 3 we prove that in a higher rank locally symmetric space M all the homology classes in $H_1(M, \mathbb{F}_2)$ are represented by a cycle whose length inside the thick part is sublinear in the volume of M. We also give the proof of the main theorem (Theorem 3.1).

2. Lengths of homology classes and the dimension

Throughout this section we write M for a Riemannian manifold of the form $M = \Gamma\backslash X$ where X is a higher rank symmetric space. The reasoning is carried out for any prime p, special properties of $p = 2$ play an important part only in the next section. We write

$Z_1(M,\mathbb{Z})$ for the module of 1-cycles on M. Any cycle $c \in Z_1(M,\mathbb{Z})$ can be represented as

$$c = \sum_{i\in I} a_i\gamma_i,$$

where I is finite set of indices, $a_i \in \mathbb{Z}$ and γ_i are oriented smooth differentiable curves $\gamma_i\colon \mathbb{S}^1 \to M$. Fix $R > 0$, the R-length of a cycle c is defined as

$$\ell^R(c) := \sum_{i\in I} |a_i|\ell^R(\gamma), \tag{2.1}$$

where $\ell^R(\gamma_i)$ stands for the length of $\gamma_i \cap M_{>R}$. We put $\ell(c) := \ell^0(c)$. The universal coefficients theorem says that any class in $H_1(M,\mathbb{F}_p)$ has a representative in $Z_1(M,\mathbb{Z})$, we define the **total length** (resp. total R-length) of $\alpha \in H_1(M,\mathbb{F}_p)$ by

$$\ell(\alpha) = \inf_{\substack{c\in Z_1(M,\mathbb{Z})\\ [c]=\alpha}} \ell(c) \tag{2.2}$$

$$\ell^R(\alpha) = \inf_{\substack{c\in Z_1(M,\mathbb{Z})\\ [c]=\alpha}} \ell^R(c) \tag{2.3}$$

Finally, the normalized R-**length** of M is defined as

$$\ell^R(M) = \frac{\sup_{\alpha\in H_1(M,\mathbb{F}_p)} \ell^R(\alpha)}{\mathrm{Vol}(M)}.$$

The following proposition is the main goal of this section:

Proposition 3.2. *For every $\delta > 0$ there exist $\delta' > 0$ such that for every manifold M with $\ell^R(M) \le \delta'$ and $\mathrm{Vol}(M)$ big enough we have*

$$\dim_{\mathbb{F}_2} H_1(M,\mathbb{F}_p) \le \delta\mathrm{Vol}(M).$$

We shall prove it after introducing some tools. In **[53]** Gelander constructed for every manifold $X = \Gamma\backslash X$ a simplicial complex $\mathcal{N}$, with $\pi_1(\mathcal{N}) \simeq \pi_1(M)$, with the number of vertices bounded by $A\mathrm{Vol}(M)$ and degrees bounded uniformly by B for certain constants $A = A(X), B = B(X)$ dependent only on X. We shall extract from his construction the following lemma:

Lemma 3.3. *Let M, X be as before and let $\mathcal{N}$ be a simplicial complex constructed in **[53]** and let $R > 0$ be bigger than the Margulis constant of X. There exists a constant $C_1 = C_1(X)$ such that any homology class $\alpha \in H_1(M,\mathbb{F}_p)$ is represented by an integral combination $\sum_{i\in I} a_ie_i$, where e_1 are edges of $\mathcal{N}$ and $\sum_{i\in I} |a_i| \le C_1\ell^R(\alpha) + O(\mathrm{Vol}(M_{<R}))$.*

Proof. Let us recall few details of the Gelander's construction. Write $d : X \times X \to X$ for the Riemannian metric on X and let $\varepsilon' = \varepsilon'_X$ be the Margulis constant of X [**13**, Thm 9.5]. Let $\varepsilon = \varepsilon'/10$, we decompose the manifold M into the thick part $M_{\ge\varepsilon}$ and the thin part $M_{<\varepsilon}$. In **[53]**, Gelander shows that there exists a closed submanifold N of codimension ≥ 3 of $M_{<\varepsilon}$ such that $M \setminus N$ retracts onto $M_{\ge\varepsilon}$. The inclusion induces a surjective map $i^*\colon \pi_1(M_{\ge\varepsilon}) \to \pi_1(M)$. There is a simplicial complex $\mathcal{N}$ homotopy equivalent to $M \setminus N$. Inside it, there is a subcomplex $\mathcal{N}_{\mathrm{thin}}$ which is homotopy equivalent to $M_{<\varepsilon'} \setminus N$. Moreover we can choose $\mathcal{N}_{\mathrm{thin}}$ which has only $O(\mathrm{Vol}(M_{<\varepsilon}))$ simplices **[52]**. The complex $\mathcal{N}$ is constructed as the nerve of a "good cover" $\mathcal{U}$ of $M_{\ge\varepsilon}$. By general theory **[25**, 13.4], $\mathcal{N}$ is homotopy equivalent to $M_{\ge\varepsilon}$. To construct $\mathcal{U}$ we choose a radius r take maximal $r/2$-separated family Σ of points in $M_{\ge\varepsilon}$ and put $\mathcal{U}_0 = \{B_{M_{\ge\varepsilon}}(x,r) \mid x \in \Sigma\}$. Next, in order to get a good cover we have to modify the covering close to the boundary of $M_{\ge\varepsilon}$. This delicate procedure is the bulk of **[53]**. Subcomplex $\mathcal{N}_{\mathrm{thin}}$ is the nerve of the covering $\mathcal{U}_{\mathrm{thin}} = \{U \in \mathcal{U} \mid U \cap \partial M_{\ge\varepsilon} \ne \emptyset\}$. We are going to use the fact that away from $\mathcal{N}_{\mathrm{thin}}$ the complex $\mathcal{N}$ looks like the nerve of $\mathcal{U}_0$. For $R > \varepsilon$ let us write $\mathcal{N}_{<R}$ for the part of the nerve coming from $M_{\ge\varepsilon} \cap M_{<R}$. The complex $\mathcal{N}_{<R}$ has at most $O(\mathrm{Vol}(M_{<R})$ simplices. Now we are ready to prove Lemma 3.3:

Step 1. Let γ be a simple closed geodesic on M. Write $\gamma = \gamma_1 \sqcup \gamma_2 \sqcup \gamma_3$ where $\gamma_1 = \gamma \cap M_{<\varepsilon}, \gamma_2 = \gamma \cap M_{\ge\varepsilon} \cap M_{<R}$ and $\gamma_3 = \gamma \cap M_{\ge R}$. By perturbing γ by an arbitrarily

small amount we can assume it is disjoint from N. Write $r : M \setminus N \to M_{\geq\varepsilon}$ for the retract defined by Gelander. Then $i^*(r(\gamma))$ represents the same homotopy class as γ. We have $r(\gamma) = r(\gamma_1) \cup \gamma_2 \cup \gamma_3$. Now, as γ_3 passes through the interior of $M_{\geq R}$ and $r(\gamma_1) \cup \gamma_2 \subset \mathrm{M}_{\geq\varepsilon} \cap M_{<R}$ we can find a finite families of balls $\mathcal{F}_i, i = 2, 3$ from the good cover $\mathcal{U}$ such that $\gamma_i \subset \bigcup_{U\in\mathcal{F}_i} U, |\mathcal{F}_2| = O(\mathrm{Vol}(M_{<R}))$ and $|\mathcal{F}_3| \leq C_0\ell(\gamma_3)$[2]. Thus, the homology class of γ can be represented by a sum of certain number of edges in $\mathcal{N}_{<R}$, and at most $Z_1\ell(\gamma_3) \leq Z_1\ell^R(\gamma)$ edges from $\mathcal{N}_{\geq\varepsilon}$.

Step 2. Let $c = \sum_i a_i\gamma_i$ be a representative of α such that $\ell(c) \leq 2\ell(\alpha)$. By the first step we can represent c as $c = c_1 + c_2 + c_3$ where $c_1 + c_2 = \sum_{e\in\mathcal{N}_{<R}} a_e e$ and $c_3 = \sum_{e\in\mathcal{N}_{\geq R}} b_e e$ with $\sum_{e\in\mathcal{N}_{\geq R}} |b_e| \leq C_1\ell^R(\alpha)$. Since we are interested in homology mod$-p$ we can assume that in our representation all coefficients have absolute values less than p. Hence, $\sum_{e\in\mathcal{N}_{<\varepsilon}} |a_e| = O(\mathrm{Vol}(M_{<R}))$. We put the inequalities together to get the lemma. □

Lemma 3.4. *For every $0 < \delta < \frac{1}{2}$ and n big enough we have*

$$\sum_{i=1}^{[\delta n]} \binom{n}{i} (p-1)^i \ll p^{\delta(2-\log\delta)n}$$

Proof. By Stirling approximation

$$\sum_{i=1}^{[\delta n]} \binom{n}{i} (p-1)^i \leq (p-1)^{\delta n} \delta n \binom{n}{[\delta n]} \tag{2.4}$$

$$\ll (p-1)^{\delta n} \delta n \frac{n^{\delta n} e^{\delta n}}{(\delta n)^{\delta n}} \ll (p - \frac{1}{2})^{\delta n} \left(\frac{e^\delta}{\delta^\delta}\right)^n \tag{2.5}$$

$$= p^{\delta((\log p)^{-1} - \log\delta + 1)n} \leq p^{\delta(2-\log\delta)n}. \tag{2.6}$$

□

Proof of Proposition 3.2. Let $\mathcal{N}$ be the simplicial complex constructed by Gelander, with the property that $\pi_1(\mathcal{N}) \simeq \pi_1(M)$. Recall that there are constants A, B dependent only on the symmetric space X such that $\mathcal{N}$ has at most $A\mathrm{Vol}(M)$ vertices, all with degrees bounded by B. Let C_1 be as in Lemma 3.3, let $\delta > \frac{1}{2}$ and let $\delta' > 0$ be such that if we put $\delta'' = 2C_1\delta'/AB$ then $\delta''(2 - \log\delta'') \leq \delta/2ABC_1$. Assume that $\ell^R(M) \leq \delta'$. By Lemma 3.3 every class in $H_1(M, \mathbb{F}_p)$ can be represented as a sum at most $C_1\delta'\mathrm{Vol}(M) + O(\mathrm{Vol}(M_{<R}))$ 1-cells in $\mathcal{N}$. By [**2**, Theorem 1.5] we have $\mathrm{Vol}(M_{<R}) = o(\mathrm{Vol}(M))$. Hence, for big enough $\mathrm{Vol}(M)$ every class in $H_1(M, \mathbb{F}_p)$ is represented by a sum of at most $2C_1\delta'\mathrm{Vol}(M)$ 1-cells in $\mathcal{N}$. Total number of 1-cells in $\mathcal{N}$ is bounded by $AB\mathrm{Vol}(M)$. By applying lemma Lemma 3.4 with $n = AB\mathrm{Vol}(M)$ and δ'' we deduce that the number of such representatives is bounded by $p^{\delta\mathrm{Vol}(M)}$. We infer that for $\mathrm{Vol}(M)$ big enough we have $\dim_{\mathbb{F}_p} H_1(M, \mathbb{F}_p) \leq \delta\mathrm{Vol}(M)$. □

3. Reduced representatives

In the sequel M is a locally symmetric space of higher rank and $p = 2$. The aim of this section is to show that the assumption of Proposition 3.2 are automatically satisfied once $\mathrm{Vol}(M)$ and R are big enough:

Proposition 3.5.

$$\ell^R(M) \leq C_2 R^{-1/2}.$$

We postpone the proof until the end of this section. Once we have Proposition 3.5 our main result is a simple consequence:

[2]Set $\mathcal{F}_3$ can be taken as the set of all balls in $\mathcal{U}$ intersecting γ_3. By Gelander's contruction their cenetrs are uniformly spearated, hence the inequality $|\mathcal{F}_3| \leq C_0\ell(\gamma_3)$.

Proof of Theorem 3.1. Let $\delta > 0$. By Proposition 3.2 there exists δ' such that $\dim_{\mathbb{F}_2} H_1(M, \mathbb{F}_2) \leq \delta \mathrm{Vol}(M)$ for $\mathrm{Vol}(M)$ big enough and M such that $\ell^R(M) \leq \delta' \mathrm{Vol}(M)$. Pick $R \geq (\delta')^{-2} C_3^2$. By Proposition 3.5 we have $\ell^R(M) \leq \delta'$ so

$$\limsup_{\mathrm{Vol}(M)\to\infty} \frac{\dim_{\mathbb{F}_2} H_1(M, \mathbb{F}_2)}{\mathrm{Vol}(M)} \leq \delta.$$

To get the Theorem we let δ go to 0. □

Fix $R > 1$. Recall that a reduced representative of a homology class $\alpha \in H_1(M, \mathbb{F}_p)$ is a cycle $c \in Z_1(M, \mathbb{Z})$ such that $\ell(c) = \ell(\alpha)$. Standard compactness argument yields

Lemma 3.6. *Every class $\alpha \in H_1(M, \mathbb{F}_p)$ has a reduced representative. It is an integral combination of closed geodesics with coefficients bounded in absolute value by $\frac{p}{2}$. In general it is not unique.*

Remark 3.7. *We could define the R-reduced representatives in the same way but the would not have such a simple description. As it will turn out, any reduced representative c satisfies $\ell^R(c) = o(Vol(M))$ which is already enough to show that $\ell^R(M)$ tends to 0 as $\mathrm{Vol}(M) \to \infty$.*

From now on it will be important that we work with $p = 2$. Being a reduced representative of a mod-2 homology class forces strong geometric constraints on c. The following Lemma guarantees that whenever a cycle c has two geodesic components that are not κ_1-separated in the thick part $M_{>R}$ then there is a mod-2 homologous cycle c' with $\ell(c') \leq \ell(c) - \kappa_2$ for some positive constant κ_2. We will write $[a, b]$ for the shortest geodesic connecting points a and b.

Lemma 3.8. *There exist $\kappa_1, \kappa_2 > 0$ with following property. Then for any two closed, non-contractible curves γ_1, γ_2 on M such that $d_{M_{\geq R}}(\gamma_1, \gamma_2) \leq \kappa_1$ there exists a cycle $c \in Z_1(M, \mathbb{F}_2)$ such that $\ell(c) \leq \ell(\gamma_2) + \ell(\gamma_2) - \kappa_2$ and $[c] = [\gamma_1 + \gamma_2]$ in $H_1(M, \mathbb{F}_2)$.*

Proof. Let x_1, x_2 be points on $\gamma_1 \cap M_{\geq R}, \gamma_2 \cap M_{\geq R}$ respectively, such that $d(\gamma_1, \gamma_2) = d(x_1, x_2)$. Let y be the midpoint of the shortest geodesic connecting x_1, x_2. Fix some radius $R' < 1$ and consider the intersection of γ_1, γ_2 with $B_M(y, R')$. Note that $R' < R$ so the ball is isometric to an R'-ball in X. Since neither of γ_1, γ_2 is contractible, they have non-empty intersection with $S_M(y, R')$. Let p_i, q_i be intersection points of γ_i with the sphere $S_M(y, R')$ such that x_i lies on the segment of γ_i bounded by p_i and q_i, for $i = 1, 2$.

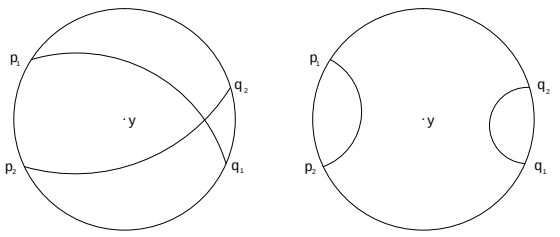

Figure 1. Before (left) and after (right) performing a surgery on close curves.

We replace segments $\gamma_1 \cap B(y, R')$ and $\gamma_2 \cap B(y, R')$ by geodesics $[p_1, p_2], [q_1, q_2]$ or $[p_1, q_2], [p_2, q_1]$ as in Figure 1. We always choose the pair with minimal total length. For R' small enough (how small depending only on X) the metric inside $B(y, R')$ is close to the flat euclidean metric so for κ_1 close to 0 it is evident that one of those operations will

reduce the total length by at least κ_2 for some positive constant κ_2. Note that this surgery does not change the mod-2 homology class. □

The second lemma says that in higher rank we have a lot of freedom to perturb closed geodesics into curves with a minimal increase in length. From this point onward we assume for technical reasons that $R > 2(1+\kappa_1+\kappa_2)$, this is not a problem since later we are going to let R tend to infinity anyway.

LEMMA 3.9. *Let γ be a closed geodesic on M. Let κ_1, κ_2 be as in Lemma 3.8. Put*

$$N(\gamma) = \{x \in M_{\geq R} \mid \exists \gamma' \text{ curve homotopic to } \gamma \text{ such that } d(x,\gamma') < \kappa_1/2 \text{ and } \ell(\gamma') < \ell(\gamma)+\kappa_2/2\}.$$

Then $\mathrm{Vol}(N(\gamma)) \geq C_3 \ell^R(\gamma) R^{\frac{1}{2}}$ *for some absolute positive constant* C_3.

PROOF. Write $\iota\colon X \to M$ for the covering map and $B(\Sigma, \varepsilon)$ for the open ε-neigborhood of a set Σ. Let $x_1, \ldots, x_N$ be a maximal R-separated subset of $\gamma \cap M_{>R}$. Clearly we have $N \geq \ell^R(\gamma)/2R$. Choose a lift $\tilde{\gamma}$ of γ to X and let $\tilde{x}_i$ be lifts of x_i's lying on $\tilde{\gamma}$. There exits a maximal flat F of X containing $\tilde{\gamma}$ [3]Since X is a higher rank symmetric space F is isometric to $\mathbb{R}^d$ with $d = \mathrm{rank}X \geq 2$ [**84**, p.152]. For every $i = 1, \ldots, N$ let $F_i = F \cap B(x_i, R - \kappa_1/2)$ and $G_i = B(F_i, \kappa_1/2)$. Note that covering map $\iota\colon X \to M$ is injective on G_i for every i and the images in M are pairwise disjoint. Let x_i', x_i'' be the pair of points on $\tilde{\gamma}$ at distance $R - \kappa_1/2 - \kappa_2$ from $\tilde{x}_i$. Consider the open ellipsoid E_i in F_i defined as $E_i = \{y \in F_i \mid d(y, x_i') + d(y, x_i'') < 2R + \kappa_2/2\}$ (see Figure 2).

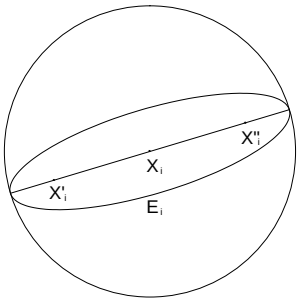

FIGURE 2. Image of $E_i \subset F_i$ under the covering map.

Note that for any point $y \in \iota(E_i)$ there exists a closed curve γ' homotopic to γ such that γ' passes through y and $\ell(\gamma') < \ell(\gamma) + \kappa_2/2$. Put $H_i = B(E_i, \kappa_1/2)$. We have $H_i \subset G_i$ so the images of H_i are pairwise disjoint. Formula for the volume of an ellipsoid yields $\mathrm{Vol}(H_i) \geq C_4 R^{\frac{d+1}{2}} \geq C_4 R^{\frac{3}{2}}$ for some positive constant C_4 depending only on X and κ_1. Hence

$$\mathrm{Vol}(\bigsqcup_{i=1}^{N} H_i) \geq C_3 \ell^R(\gamma) R^{\frac{1}{2}}.$$

By construction $\iota(H_i) \subset N(\gamma)$ for every $= 1, \ldots N$ which ends the proof. □

As a corollary of Lemmas 3.8 and 3.9 we get

[3]Flat F does not necessarily descend to a closed flat on M.

COROLLARY 3.10. *Let $\alpha \in H_1(M, \mathbb{F}_2)$ and let c be a reduced representative of α. Write $c = \sum_{i \in I} \gamma_i$ for some set of closed geodesics indexed by I. Then the sets $N(\gamma_i), i \in I$ defined as in Lemma 3.9 are pairwise disjoint.*

PROOF. We argue by contradiction. Let γ_1, γ_2 be two geodesic components of c such that $N(\gamma_1) \cap N(\gamma_2) \neq \emptyset$. Let $y \in N(\gamma_1) \cap N(\gamma_2)$. By definition $y \in M_{>R}$ and there exist closed curves γ_1', γ_2' homotopic to γ_1, γ_2 respectively such that $\ell(\gamma_i') < \ell(\gamma_i) + \kappa_2/2$ and $d(\gamma_i', y) < \kappa_1/2$ for $i = 1, 2$. Let c' be the cycle obtained from c by replacing γ_i by γ_i' for $i = 1, 2$. It represents the same homology class. We have $\ell(c') < \ell(c) + \kappa_2$. Curves γ_1', γ_2' satisfy $d_{M_{>R}}(\gamma_1', \gamma_2') < \kappa_1$ so we may perform the surgery from Lemma 3.8 to construct homologous cycle c'' with $\ell(c'') \leq \ell(c') - \kappa_2 < \ell(c)$. The last inequality contradicts the assumption that c was a reduced representative. □

PROOF OF PROPOSITION 3.5. Let $\alpha \in H_1(M, \mathbb{F}_2)$ and let c be a reduced representative of α. Write Write $c = \sum_{i \in I} \gamma_i$ for some set of closed geodesics indexed by I. By Corollary 3.10 the sets $N(\gamma_i)$ are pairwise disjoint and by Lemma 3.9 we have $\mathrm{Vol}(N(\gamma_i)) \geq C_3 \ell^R(\gamma_i) R^{\frac{1}{2}}$ so

$$\ell^R(c) = \sum_{i \in I} \ell^R(\gamma_i) \leq \frac{\mathrm{Vol}(M_{>R})}{C_3 R^{\frac{1}{2}}}.$$

We deduce that $\ell^R(M) \leq C_3^{-1} R^{-\frac{1}{2}}$.

□

CHAPTER 4

Kesten's theorem for uniformly recurrent subgroups

1. Introduction

1.1. Kesten theorems. The spectral radius of an infinite d-regular, countable, undirected graph G is defined as the ℓ^2-norm of the Markov averaging operator $M : \ell^2(G) \to \ell^2(G)$. Fix a vertex o of G. If we write $A_G(n)$ for the set of walks starting at o and returning to o after time n the spectral radius can be computed as the limit

$$\rho(G) = \lim_{n\to\infty} \left(\frac{|A_G(2n)|}{d^{2n}} \right)^{1/2n}.$$

If G is an infinite countable group generated by a symmetric set S and H is a sugbroup of G we write $\mathrm{Cay}(G, S)$ for its Cayley graph and $\mathrm{Sch}(H\backslash G, S)$ for the Schreier graph whose vertices are left H cosets. For a subgroup H of G write $\rho(G) = \rho(\mathrm{Cay}(G, S))$ and $\rho(H\backslash G) = \rho(H\backslash G, S)$ for the spectral radii of the random walk on the Cayley graph and the Schreier graph respectively. The subject of this paper has its origins in the criterion for amenability given by Kesten [**65**, **66**]

THEOREM 1.1 (Kesten). *Let G be a group generated by a finite symmetric set S. Then G is amenable if and only if $\rho(G) = 1$.*

The following result, also due to Kesten extends Theorem 1.1 and characterizes the amenability of a normal subgroup in terms of the spectral radius $\rho(H\backslash G)$.

THEOREM 1.2 (Kesten). *Let G be a group generated by a finite symmetric set S and let H be a normal subgroup of G. Then H is amenable if and only if $\rho(G) = \rho(H\backslash G)$.*

If H is amenable then $\rho(G) = \rho(H\backslash G)$ unconditionally but the converse does not hold in general. We shall say that a subgroup H is Ramanujan (with respect to S) if $\rho(G) = \rho(H\backslash G)$. In [**4**] Abert, Glasner and Virag proved a probabilistic version of Kesten's theorem:

THEOREM 1.3 (Aber,Glasner,Virag). *Let G be a group generated by a finite symmetric set S and let H be an invariant random subgroup of G. Then H is amenable almost surely if and only if H is Ramanujan almost surely.*

In other words an IRS is Ramanujan if and only if it is amenable. We refer to the article [**4**] for the definition and properties of IRS'ses. In the present paper we develop a quantitative version of Kesten's theorem that works for any subgroup H of G. We prove an inequality (Theorem 2.1) relying $\rho(G, S)$ and $\rho(H\backslash G, S)$ in terms of certain averages of the logarithms of spectral radii $\rho(H^g, H^g \cap S)$ for $g \in G$:

$$\log \rho(H\backslash G, S) - \log \rho(G, S) \geq \frac{\limsup_{n\to\infty} \int |H^g \cap S|(-\log \rho(H^g, S \cap H^g))d\mu_{2n}(g)}{|S|^2 \rho(G, S)^2}, \tag{1.1}$$

where μ_{2n} is averaging measure over traces of recurrent random walks of length $2n$. Section 2 is devoted to the proof of (1.1). We follow closely the argument from [**4**]. In fact our main contribution is just an application of the inequality between the arithmetic and the geometric means in the right place, similarly to how it was used in [**6**]. In Sections 3,4 we derive two consequences of the inequality 1.1. First one extends Kesten theorem to the uniformly recurrent subgroups and the second is a relatively short proof that [**76**, Theorem 1.2] holds on average.

1.2. Uniformly recurrent subgroups. Uniforlmy recurrent subgroup is the topological dynamical analogue of an ergodic IRS. Write Sub_G for the set of subgroups of a group G endowed with the Chabauty topology [**33**]. A closed subset X of Sub_G invariant under conjugation is called a **uniformly recurrent subgroup** (URS for short) if it is minimal as a dynamical G-system. The notion of URS was defined for the first time by Glasner and Weiss in [**43**] and was further studied in recent papers [**42, 80**]. It was used by Kennedy [**64**] in a new characterization of C^*-simplicity for countable groups.

THEOREM 1.4 (Kennedy). *A countable group G is C^*-simple if and only if it has no amenable URS.*

We prove that the natural extension of Kesten's theorem holds for URS'ses.

THEOREM 1.5. *[Theorem 3.1] Let G a countable group generated by a finite set S, let X be an URS of G and let $H \in X$. Then H is amenable (i.e. consists of amenable subgroups) if and only if $\rho(G) = \rho(H\backslash G)$.*

It was already shown in [**43**] that an URS is amenable if and only if it contains an amenable subgroup, similarly an URS is Ramanujan if and only if it contains a Ramanujan subgroup.

1.3. Cycle density in Ramanujan graphs. Let (G, x) be a d-regular Ramanujan graph and let $k \geq 1$ be a fixed integer. For any $n \geq 0$ write q_n for the probability that a random walk starting at x lies at time n on a cycle of length at most k. In [**76**] Lyons and Peres proved that $\lim_{n\to\infty} q_n = 0$. Their result was motivated by [**6**, Problem 11]. Using inequality 1.1 we show (Theorem 4.1) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} q_j = 0.$$

In other words the random walks on a Ramanujan graph do not spend much time in the short cycles. This gives a relatively simple proof that the conclusion of [**76**, Theorem 1.2] holds in average.

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2. Inequality on spectral radii

Let G be a group generated by a finite symmetric set S. If H is a subgroup of G write $\mathrm{Sch}(H\backslash G, S)$ for the Schreier graph encoding the action of generators from S on the coset space $H\backslash G$. Write $A(n, S), A_H(S, n)$ for the sets of walks on $\mathrm{Cay}(G, S)$ respectively $\mathrm{Sch}(H\backslash G, S)$ that return to the identity after n steps. We identify the walks of length n with elements of S^n. We will consider the right action of G on $\mathrm{Sch}(H\backslash G, S)$. For a walk $w = (a_1, \ldots, a_n)$ we write $w(i) = a_1 a_2 \ldots a_i$ for the position after i steps. Write $\rho(H\backslash G, S)$ for the spectral radius of the graph $\mathrm{Sch}(H\backslash G, S)$. If H is a subgroup of G and $F \subset H$ we adopt the convention that $\rho(H, F) = \rho(\langle H \cap F\rangle, F)$ in the case when F does not generate H. We will use abbreviation $H^g = g^{-1}Hg$.

Define the measure

$$\mu_n = \frac{1}{|A(n, S)|} \sum_{w\in A(n,S)} \left(\frac{1}{n} \sum_{i=1}^{n} \delta_{w(i)} \right).$$

Intuitively $\mu_n(g)$ tells us how large proportion of time is spent in g by a random recurrent walk of length n. We have

THEOREM 2.1. *Let* $I(H,S) = \limsup_{n\to\infty} \int |H^g \cap S|(-\log \rho(H^g, S\cap H^g))d\mu_{2n}(g)$. *Then*

$$\log \rho(H\backslash G, S) - \log \rho(G,S) \geq \frac{I(H,S)}{|S|^2 \rho(G,S)^2}. \tag{2.1}$$

PROOF. We follow closely the strategy from **[4]**. For a walk $w \in S^n$ we will write $T(w) = \{t \in \{1,\dots,n\} | Hw(t-1) = Hw(t)\}$. It is the set of times where a walks doesn't change the H-coset. For each walk we define its class $C(w)$ as

$$C(w) = \{w' \in S^n | T(w') = T(w) \text{ and } w'(t-1)^{-1}w'(t) = w(t-1)^{-1}w(t) \text{ for } t \notin T(w)\}.$$

Two walks are in the same class if they have the same trajectories on $H\backslash G$ and whenever they change H-coset they move by the same element of S. For every walk $w \in A(n,S)$ have $C(w) \in A_H(n,S)$ so

$$|A_H(n,S)| \geq \sum_{w\in A(n,S)} \frac{|C(w)|}{|C(w)\cap A(n,S)|}.$$

By lemmas 8 and 9 from **[4]** we have

$$\frac{|C(w)|}{|C(w)\cap A(n,S)|} \geq \prod_{t\in T(w)} \rho(H^{w(t)}, S\cap H^{w(t)})^{-1}.$$

Using the inequality between arithmetic and geometric means we get

$$\frac{|A_H(n,S)|}{|A(n,S)|} \geq \frac{1}{|A(n,S)|} \sum_{w\in A(n,S)} \prod_{t\in T(w)} \rho(H^{w(t)}, S\cap H^{w(t)})^{-1} \tag{2.2}$$

$$\geq \left(\prod_{w\in A(n,S)} \prod_{t\in T(w)} \rho(H^{w(t)}, S\cap H^{w(t)})^{-1} \right)^{\frac{1}{|A(n,S)|}}. \tag{2.3}$$

Take logarithms of both sides

$$\log|A_H(n,S)| - \log|A(n,S)| \geq \frac{1}{|A(n,S)|} \sum_{w\in A(n,S)} \sum_{t\in T(w)} -\log \rho(H^{w(t)}, H^{w(t)}\cap S) \tag{2.4}$$

$$= \frac{1}{|A(n,S)|} \sum_{t=1}^{n} \sum_{w\in A(n,S)} -\mathbf{1}_{T(w)}(t) \log \rho(H^{w(t)}, H^{w(t)}\cap S). \tag{2.5}$$

$$\tag{2.6}$$

We can estimate the rightmost sum by counting for each $t \in \{2,\dots,n\}$ only the walks of form $w = (s_1,\dots,s_{t-2},h,h^{-1},s_{t+1},\dots,s_n)$ with $h \in H^{w(t-2)}\cap S$ and $(s_1,\dots,s_{t-2},s_{t+1},\dots,s_n) \in A(n-2,S)$. Thus, for $t \in \{2,\dots,n\}$ we have

$$-\sum_{w\in A(n,S)} \mathbf{1}_{T(w)}(t) \log \rho(H^{w(t)}, H^{w(t)}\cap S) \geq -\sum_{w\in A(n-2,S)} |H^{w(t-2)}\cap S| \log \rho(H^{w(t-2)}, H^{w(t-2)}\cap S) \tag{2.7}$$

We plug it into our previous estimate to get

$$\log|A_H(n,S)| - \log|A(n,S)| \geq \frac{-1}{A(n,S)} \sum_{w\in A(n-2,S)} \sum_{t=1}^{n-2} |H^{w(t)}\cap S| \log \rho(H^{w(t)}, H^{w(t)}\cap S) \tag{2.8}$$

$$= \frac{-(n-2)|A(n-2,S)|}{|A(n,S)|} \int |H^{w(t)}\cap S| \log \rho(H^{w(t)}, H^{w(t)}\cap S) d\mu_{n-2}(g) \tag{2.9}$$

$$\tag{2.10}$$

We divide both sides by n to get

$$\frac{\log|A_H(n,S)|}{n}-\frac{\log|A(n,S)|}{n}\geq\frac{-(n-2)|A(n-2,S)|}{n|A(n,S)|}\int|H^{w(t)}\cap S|\log\rho(H^{w(t)},H^{w(t)}\cap S)d\mu_{n-2}(g) \tag{2.11}$$

Replace n by $2n$ and take limes superior of both sides as $n\to\infty$

$$\log\rho(H\backslash G,S)-\log\rho(G,S)\leq\limsup_{n\to\infty}\frac{-|A(2n-2,S)|}{|A(2n,S)|}\int|H^{w(t)}\cap S|\log\rho(H^{w(t)},H^{w(t)}\cap S)d\mu_{2n-2}(g) \tag{2.12}$$

Let $P:l^2(G)\to l^2(G)$ be the transition operator of the random walk on $\mathrm{Cay}(G,S)$. Then

$$|A(2n,S)|=|S|^{2n}\langle P^{2n}\mathbf{1}_e,\mathbf{1}_e\rangle=|S|^{2n}\|P^n\mathbf{1}_e\|_2^2\leq|S|^{2n}\|P\|^2\|P^{n-2}\mathbf{1}_e\|_2^2=|S|^2\rho(G,S)^2|A(2n-2,S)|$$

Hence

$$\log\rho(H\backslash G,S)-\log\rho(G,S)\geq\frac{\limsup_{n\to\infty}\int|H^{w(t)}\cap S|(-\log\rho(H^{w(t)},H^{w(t)}\cap S))d\mu_{2n}(g)}{|S|^2\rho(G,S)^2} \tag{2.13}$$

□

3. Application to uniformly recurrent subgroups

THEOREM 3.1. *Let G be a countable group generated by a finite symmetric set S and let $X\subset\mathrm{Sub}_G$ be a uniformly recurrent subgroup. The following conditions are equivalent*

1. *X contains an amenable subgroup;*
2. *X is amenable;*
3. *$\rho(H\backslash G,S)=\rho(G,S)$ for all H in X;*
4. *there exists $H\in X$ such that $\rho(H\backslash G,S)=\rho(G,S)$.*

For the proof define the probability measures ν_n as

$$\nu_n=\frac{1}{n}\sum_{k=n+1}^{2n}\mu_{2k}.$$

LEMMA 3.2. *For any $s\in S$ and any subset A of G we have*

$$\nu_n(As)\geq\frac{n\nu(A)}{(n+1)|S^2|\rho(G,S)^2}-\frac{1}{n}.$$

PROOF. First, let us show that for each $s\in S$ we have $\mu_{2k}(As)\geq\frac{k}{(k+1)|S^2|\rho(G,S)^2}\mu_{2(k+1)}(A)$. We have

$$\mu_{2(k+1)}(As)=\frac{1}{(2k+2)|A(2k+2,S)|}\sum_{t=1}^{2k+2}\sum_{w\in A(2k+2,S)}\delta_{w(t)}(A).$$

We estimate the leftmost sum from below by counting only walks of form $(a_1,\dots,a_{t-1},s,s^{-1},a_{t+2},\dots,a_{2k+2})$ with $(a_1,\dots,a_{t-1},a_{t+2},\dots,a_{2k+2})\in A(2k,S)$. For $2\leq t\leq 2k+1$

$$\sum_{w\in A(2k+2,S)}\delta_{w(t)}(As)\geq\sum_{w\in A(2k,S)}\delta_{w(t-1)}(A).$$

Hence

$$\mu_{2k+2}(As)\geq\frac{1}{(2k+2)|A(2k+2,S)|}\sum_{w\in A(2k,S)}\sum_{t=2}^{2k+1}\sum_{w\in A(2k,S)}\delta_{w(t-1)}(A)=\frac{(2k)|A(2k,S)|}{(2k+2)|A(2k+2,S)|}\mu_{2k}(A).$$

As

$$\frac{|A(2k,S)|}{|A(2k+2,S)|}\geq\frac{1}{|S|^2\rho(G,S)^2}$$

we get

$$\mu_{2k}(As)\geq\frac{k\mu_{2(k+1)}(A)}{(k+1)|S^2|\rho(G,S)^2}.$$

it follows that

$$\nu_n(As) + \frac{\mu_{2n+2}(As)}{n} \geq \frac{n\nu_n(A)}{(n+1)|S^2|\rho(G,S)^2}$$

$$\nu_n(As) \geq \frac{n\nu_n(A)}{(n+1)|S^2|\rho(G,S)^2} - \frac{1}{n}.$$

□

PROOF OF THEOREM 3.1. Equivalences between (1) ↔ (2) and (3) ↔ (4) follow form [**43**, Proposition 2.2] and the fact that the property of being amenable or Ramanujan is admissible in the sense of [**43**, Definition 2.1].

Implication (2) ⟹ (3) is an application of Kesten's theorem and (3) ⟹ (4) is obvious. The only nontrivial implication is (4) ⟹ (1). Let $H_0 \in X$ be a subgroup such that $\rho(H\backslash G, S) = \rho(G,S)$. By Theorem 2.1 we have $\limsup_{n\to\infty} \int |H_0^g \cap S|(-\log \rho(H_0^g, H_0^g \cap S) d\mu_{2n}(g) = 0$. Then we also have

$$\lim_{n\to\infty} \int |H_0^g \cap S|(-\log \rho(H_0^g, H_0^g \cap S) d\nu_n(g) = 0.$$

Let δ_{H_0} be the dirac mass in H_0 and let ω be a weak-* limit of measures $\delta_{H_0} * \nu_n$ as $n \to \infty$. Then

$$\int |H \cap S|(-\log \rho(H, H \cap S)) d\omega(H) = 0$$

so the set of H such that $\rho(H, H \cap S) > 0$ has measure 0. By Kesten's criterion this is precisely the set of H for which $\langle H \cap S \rangle$ is non-amenable. As S is finite the latter set is open . From Lemma 3.2 we deduce that ω is quasi-invariant (i.e. $\omega(E) = 0$ if and only if $\omega(gE) = 0$ for all $g \in G$). X is a minimal dynamical G-system so the support of ω has to be the whole X. In particular the only open set of measure 0 is the empty set. If follows that $\langle H \cap S \rangle$ is amenable for all $H \in X$. By taking $S' = (\{1\} \cup \{S\})^m$ and letting m go to infinity we show in this way that every $H \in X$ is amenable. □

COROLLARY 3.3. *Let G be countable C^*-simple group with a finite symmetric generating set S. If H is a Ramanujan subgroup of G (i.e. $\rho(G) = \rho(H\backslash G)$) then there exists a sequence $(g_i)_{i\in\mathbb{N}}$ such that H^{g_i} converges to the identity subgroup in the Chabauty topology.*

PROOF. The closure $X = \overline{\{H^g \mid g \in G\}}$ consists of Ramanujan subgroups. By the Zorn lemma there exists a minimal G-invariant closed subset $Y \subset X$. By Theorem 3.1 and Kennedy's criterion (Theorem 1.4) $Y = \{1\}$ which proves the assertion. □

4. Cycle density along random walks

Let G be a d-regular graph. For any vertex x and $k \geq 1$ let $C_G(x,k) = 1$ if there exists a non-backtracking cycle of length k starting at x and $C_G(x,k) = 0$ otherwise. Similarly let $D_G(x,k) = 1$ if there exist at least two independent[1] non-backtracking cycles starting at x and $C_G(x,k) = 0$ otherwise. In this section we prove:

THEOREM 4.1. *Let G be d-regular rooted Ramanujan graph. Let (X_i) be the standard random walk on G. Then for any $k \geq 1$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}\big[\, C_G(X_i, k) \,\big] = 0.$$

Write $\mathbb{T}_d$ for the d-regular rooted tree. If $G = (V,E)$ we shall write G^k for the graph (V, E^k) where the (multi)set of edges is given by

$$E^k = \{(e_0, e_k) | (e_0, e_1, \ldots, e_k) \text{ is a non-backtracking walk in } G\}.$$

G^k is always a $d(d-1)^{k-1}$-regular graph and $C_G(x,k) = C_{G^k}(x,1)$. We have

LEMMA 4.2. *G is a Ramanujan graph if and only G^k is.*

[1]Here independent means: non equal and one is not the inverse of another.

PROOF. Since $\mathbb{T}_d^k = T_{d(d-1)^{k-1}}$ it enough to observe that $\rho(G^k)$ is a strictly decreasing function of $\rho(G)$. □

We will use the notion of a **stationary random graph**. We think of the d-regular random graphs as the Borel probability measures on the space of isomorphism classes of rooted d-regular graphs. For more comprehensive introduction to random graphs we refer to **[6]**. A random, rooted d-regular graph $(\tilde{G}, \tilde{x})$ is called stationary if its probability distribution is invariant under replacing the root $\tilde{x}$ by a random neighbor. A good example of stationary random graphs are the graphs of form $\mathrm{Sch}(H\backslash G, S)$ where H is a random subgroup of a group G satisfying $\mathbf{E}\big[\, f(H)\,\big] = \frac{1}{|S|}\sum_{s\in S}\mathbf{E}\big[\, f(H^s)\,\big]$ for every continuous function f on Sub_G. Such random subgroups groups are called **stationary random subgroups**.

PROOF OF THEOREM 4.1. First note that if we can prove this theorem for some $k > 1$ then it is automatically true for 1 because $C(x,1) \le C(x,k)$. We will use that to assume that $k \ge 2$ in the third step. Since our methods wouldn't yield any quantitative estimates we will use an argument by contradiction to shorten the proof. The idea will be to take a graph for which the conclusion does not hold and construct a random limit object with contradictory properties.

Step 1. We replace G by a stationary random graph. Let

$$0 < \alpha = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\big[\, C_G(X_i, k)\,\big].$$

There exists an increasing sequence $(n_i)_{i\in\mathrm{N}}$ such that $\alpha = \lim_{i\to\infty}\frac{1}{n_i}\sum_{j=1}^{n_i}\mathbf{E}\big[\, C_G(X_j,k)\,\big]$. For each $i \ge 0$ (G, X_i) is a random graph isomorphic to G whose root is given by the position of the random walk at time i. Let $(\tilde{G}, \tilde{x})$ be any weak limit of the sequence of random graphs

$$\frac{1}{n_i}\sum_{j=1}^{n_i}(G, X_j).$$

Then $(\tilde{G}, \tilde{x})$ is a stationary random graph and $\mathbf{E}\big[\, C_{\tilde{G}}(\tilde{x}, k)\,\big] = \alpha > 0$. Moreover, since G was Ramanujan $\tilde{G}$ is Ramanujan almost surely.

Step 2. We show that there exists $a > 0$ such that $\mathbf{E}\big[\, D_{\tilde{G}^a}(\tilde{x}, k)\,\big] > 0$. Since $D_{\tilde{G}^a}(\tilde{x}, k) = D_{\tilde{G}^{ka}}(\tilde{x}, 1)$ we may assume in this step that $k = 1$. First, we claim that for A big enough the ball $B_{\tilde{G}}(\tilde{x}, A)$ contains at least two vertices y_1, y_2 such that $C_{\tilde{G}}(y_1, 1) = C_{\tilde{G}}(y_2, 1) = 1$. Let $(\tilde{X}_i)$ denote the random walk on $\tilde{G}$. By stationarity we have $(\tilde{G}, \tilde{x}) = (\tilde{G}, \tilde{X}_i)$ for all $i \in \mathrm{N}$. Since $\tilde{G}$ is Ramanujan almost surely, the random walk "flattens" the probability distribution uniformly fast. Hence, there exists an $A > 0$ such that: for almost all $(\tilde{G}, \tilde{x})$, for every vertex $v \in \tilde{G}$ we have $\mathbf{P}\big[\, \tilde{X}_A = v\,\big] < \frac{\alpha}{3}$. Then, the equality $\mathbf{E}\big[\, C_{\tilde{G}}(\tilde{X}_A, k)\,\big] = \mathbf{E}\big[\, C_{\tilde{G}}(\tilde{x}, k)\,\big] = \alpha$ implies that with positive probability $(\tilde{G}, \tilde{x})$ is such that there are at least 2 possible values for $\tilde{X}_A$ where $C(\tilde{X}_A, 1) = 1$. That proves the claim. If the ball $B_{\tilde{G}}(\tilde{x}, A)$ contains two distinct vertices with loops attached (loop = 1-cycle) then we can construct two independent non-backtracking cycles of length $2A + 1$ starting at $\tilde{x}$. Thus $\mathbf{E}\big[\, D_{\tilde{G}^{2A+1}}(\tilde{x}, 1)\,\big] > 0$.

Step 3. Let $k \ge 2$. Put $2d' = d(d-1)^{a-1}$. We are going to replace the random rooted graph $(\tilde{G}^a, \tilde{x})$ by a Schreier graph $\mathrm{Sch}(H\backslash F_{d'}, S)$ where $F_{d'}$ is the free group on d' generators, S the standard symmetric free generating set and H a random subgroup of $F_{d'}$. By Fatou lemma the graph $(\tilde{G}^a, \tilde{x})$ satisfies with positive probability the following property:

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} D_{\tilde{G}^a}(\tilde{X}_i, k) > 0. \tag{4.1}$$

In particular there exist (deterministic) d'-regular Ramanujan graph (G_1, x_1) with this property. Note that the degree of G_1 is even so by **[62]** it is isomorphic to a Schreier

graph. Hence, there exists $H_0 \subset F_{d'}$ such that $(G_1, x_1) \simeq \mathrm{Sch}(H_0 \backslash F_{d'}, S)$. We construct a stationary random subgroup H as a weak-* limit of $\frac{1}{n}\sum_{i=1}^{n} \frac{1}{|S|^i} \sum_{s \in S^i} H^s$ along a subsequence for which the limit superior (4.1) converges to a positive number. Then we have $\mathbf{E}\big[\, D_{\mathrm{Sch}(H\backslash F_{d'},S)}(H,k)\,\big] > 0$.

Step 4. We reinterpret the condition $\mathbf{E}\big[\, D_{\mathrm{Sch}(H\backslash F_{d'},S)}(H,k)\,\big] > 0$ in terms of the expected spectral radius. Let H_1 be any deterministic subgroup of $F_{d'}$. Any two independent non-backtracking k-cycles c_1, c_2 in $\mathrm{Sch}(F_{d'}/H_1, S)$ starting at H_1 give rise to two elements $a, b \in S^k \cap H$ generating a free subgroup. Hence there is $\beta = \beta(k, d') < 1$ such that $D_{\mathrm{Sch}(F_{d'}/H_1,S)}(H_1, k) = 1$ implies $\rho(H_1, H_1 \cap S^k) \leq \beta$. We deduce that $\mathbf{E}\big[\, -\log \rho(H_1, H_1 \cap S^k)\,\big] > 0$.

Step 5. We use Theorem 2.1 to get a contradiction. The graph $\mathrm{Sch}(F_{d'}/H, S)$ is Ramanujan almost surely so by Theorem 2.1

$$\lim_{n\to\infty} -\mathbf{E}\Big[\int_{F_{d'}} |H^g \cap S^k| \log \rho(H^g, H^g \cap S^k\rangle) d\mu_{2n}(g)\Big] = 0. \tag{4.2}$$

The density function of μ_{2n} for the free group and the standard symmetric generating is a spherical function on $F_{d'}$ (its value depends only on the distance from the root). Hence, we can use the property that H is stationary to get

$$-\mathbf{E}\Big[\int_{F_{d'}} |H^g \cap S^k| \log \rho(H^g, H^g \cap S^k\rangle) d\mu_{2n}(g)\Big] = -\mathbf{E}\big[\, |H \cap S^k| (-\log \rho(H_1, H_1 \cap S^k))\,\big],$$

which together with (4.2) contradicts the conclusion of the fourth step. □

Remark 4.3. *Steps 2-5 can be used to prove [**6**, Theorem 4, Theorem 5]. Indeed, any sequence of finite Ramanujan graph gives rise to an unimodular random graph which is Ramanujan almost surely. Unimodularity implies that such a graph is stationary. Steps 2-5 show that any Ramanujan stationary random graph has to be the regular tree.*

CHAPTER 5

Character bounds in finite groups of Lie type

1. Introduction

1.1. Notation. Let $\mathbf{G}$ be a reductive group defined over a finite field $\mathbb{F}_q$. We will write F for the Frobenius automorphism and $\mathbf{G}^F$ for the group of $\mathbb{F}_q$ points. We will write Ind_H^G and Res_H^G for the usual induction, restriction functors. When L is a rational Levi subgroup of G we write $R_L^G, {}^*R_L^G$ for the Deligne-Lusztig induction, restriction functors.

1.2. Motivation. Let χ be an irreducible character of $\mathbf{G}^F$ and let γ in $\mathbf{G}^F$ be a non-central element. We look for bounds of type $|\chi(\gamma)| \leq C_\gamma \chi(1)^{1-\delta}$ with $\delta > 0$ dependent only on the (dimension of) the centralizer $\mathbf{G}_\gamma$. At this point it is a bit hard to tell how δ should depend on $\mathbf{G}_\gamma$. When γ is regular i.e. $\dim \mathbf{G}_\gamma = \operatorname{rank}\mathbf{G}$ where we can take $\delta = 1$ so we have the best possible exponent. In general the lower the dimension of the centralizer the bigger δ we can take . The motivation for studying character bounds is twofold: First by the work of Shalev, Liebeck (see **[74]** for a survey article) and others we know that such bounds can be used to study questions on random walks on finite groups of Lie type as well as problems concerning the diameter and word generation. Secondly a simple argument using Frobenius reciprocity and Holder inequality show a bound on fixed point ratios $|(G/H)^\gamma| \ll |G/H|^{1-\delta'}$ where δ' depends only on δ and the type of $\mathbf{G}$. These estimates on fixed point ratios for the finite groups of Lie type can often be obtained by other means **[28]** and are relatively well understood. We hope that by giving a proof via character estimates we will be able to approach the problem of bounding the fixed point ratios on compact groups of type $\mathbb{S}L(n, \mathbb{Z}_p)$.

1.3. Known results. In an unpublished note **[71]** Larsen gives an elegant proof of the following estimate:

$$|\chi(\gamma)| \leq |W|^2$$

for any regular semisimple element $\gamma \in \mathbf{G}^F$ and irreducible character χ. The main property used in the proof is that a centralized of such γ is abelian and that the typical element of $\mathbf{G}_\gamma^F$ has the same centralizer as γ. His method would probably also yield an estimate by a constant for non-semisimple regular elements. The functorial approach we are going to use to bound the values at regular elements improves Larsens' results by giving the best possible bounds.

We have seen that for the regular elements the problem is more or less settled. On the other end of the spectrum are the elements with big centralizers. Here the Larsens' method cannot be used - the main problem being the fact that the centralizers are no longer abelian. For the the unipotent, non-regular elements (which can be seen as the most pathological examples) Glück **[55]** has shown that $|\chi(\gamma)| \ll |\chi(1)|/q^{1/2}$. While this implies the bound of desired type we see no way in which his proof would generalize to the compact p-adic context. Finally we note that in an upcoming work **[19]**, announced in **[74]** R. Bezrukavnikov, M. W. Liebeck, A. Shalev, and P. H. Tiep proved independently a very general bound for non-regular elements. Let L be a Levi subgroup of G. We write

$$\alpha(L) := \max\left\{ \frac{\dim u^L}{\dim u^G} \mid u \in L, u \neq 1 \text{ unipotent} \right\}.$$

THEOREM 5.1 (**[19]**). *Let $\gamma \in G(\mathbb{F}_q)$ and let L be rational Levi subgroup of some rational parabolic subgroup of G such that $G_\gamma^0 \subset L$. Then, for every irreducible character*

χ of $G(\mathbb{F}_q)$ we have

$$|\chi(\gamma)| \ll \chi(1)^{\alpha(L)},$$

where the implicit constant depends only on the rank of G.

The proof also uses the invariance of character values under certain parabolic restriction.

1.4. Our results. In Lemma 5.2 we establish a link between orbital integrals in p-adic groups and the "naîve" parabolic restriction functor on compact p-adic groups. In the context of finite groups we Lie type we have the following result: Let $\gamma \in \mathbf{G}^F$ and let $\gamma = su$ be the Jordan decomposition of γ. In Proposition 5.3 we show that for any rational Levi subgroup $\mathbf{L}$ containing the connected component of the centralizer $\mathbf{G}_s^0$ we have

$$\chi(\gamma) = {}^*R_L^G\chi(\gamma),$$

for every character χ. As an application we deduce a strengthening of Larsen's bound from [**71**]: for any regular semisimple element $\gamma \in G^F$ and any irreducible character χ we have

$$|\chi(\gamma)| \leq |W|.$$

1.5. Baby cases for finite groups. Here we show how one can get the optimal bound for regular the semisimple elements which have quasi-spit centralizer. A maximal rational torus $\mathbf{T}$ of $\mathbf{G}$ is called quasi-split if it is contained in a rational Borel subgroup. We will show that for such γ we have

$$|\chi(\gamma)| \leq |W|,$$

where W is the absolute Weyl group of G.

Proof. Let $\mathbf{T}$ be the centralizer of γ and let $\mathbf{B} = \mathbf{TU}$ be a Levi decomposition of a Borel subgroup B containing T. By the formula for parabolic restriction [**38**, p. 49]

$${}^*R_T^G\chi(\gamma) = \frac{1}{|\mathbf{U}^F|}\sum_{u\in\mathbf{U}^F}\chi(ul).$$

Since γ is regular the map $\mathbf{U}^F \to \mathbf{U}^F$ given by $u \mapsto ulu^{-1}l^{-1}$ is bijective. In particular for any $u \in \mathbf{U}^F$ there exists $v \in \mathbf{U}^F$ such that $u = vlv^{-1}l^{-1}$ but then $ul = vlv^{-1}$. We deduce that ${}^*R_T^G\chi(\gamma) = \chi(\gamma)$. Finding a suitable restriction functor which does not change the value of characters on γ will be the key ingredient in all our estimates. Now it remains to bound the dimension of ${}^*R_T^G\chi$.

$$\dim{}^*R_T^G\chi = \sum_{\theta\in\widehat{\mathbf{T}^F}}\langle\chi, R_T^G\theta\rangle_G.$$

The Mackey formula for parabolic induction [**38**, Theorem 5.1] implies that any irreducible character appears in $\sum_{\theta\in\widehat{\mathbf{T}^F}} R_T^G\theta$ with multiplicity at most $|W|$. It follows that $\dim{}^*R_T^G\chi \leq |W|$ and consequently $|\chi(\gamma)| = |{}^*R_T^G\chi(\gamma)| \leq |W|$. □

In the general semisimple case we shall replace parabolic restriction by the Deligne-Lusztig restriction and the Mackey formula has to be replaced by the results on geometric disjointness of induced characters. Also for non-regular elements instead of restricting to the centralizer we will have to restrict to a rational Levi subgroup containing $\mathbf{G}_\gamma$.

Now let us treat the unipotent "baby case". Let $\gamma \in \mathbf{G}^\gamma$ be a regular unipotent element. The proof is entirely different from the semisimple case, thought we also use a suitable functor. This time it will be the Alvin-Curtis duality functor. We will show that if γ is a regular unipotent element of $\mathbf{G}^F$, the center of $\mathbf{G}$ is connected and the characteristic is good (this excludes a finite (small) number of exceptions) then

$$|\chi(\gamma)| \leq 1.$$

Of course this is optimal in all applicable cases.

PROOF. If the center is connected and the characteristic is good there is only one conjugacy class of regular unipotent elements. Write $\psi = |\mathbf{G}_\gamma^F|\mathbf{1}_{[\gamma]}$. Then $\chi(\gamma) = \langle \chi, \psi \rangle_G$. The duality functor D_G (see [**38**, Chapter 8]) is a an isometry on the space of class function on $\mathbf{G}^F$, so $\chi(\gamma) = \langle D_G\chi, D_G\psi \rangle_G$. The (virtual) character $D_G\chi$ is an irreducible character up to sign. On the other hand the fuction $D_G\psi$ is, by [**38**, Corollary 14.37] the character of the Gelfand-Graev represenatation of $\mathbf{G}^F$, denoted Γ. By [**38**, Theorem 14.30] Gelfend-Graev representations are multiplicity free, hence $|\chi(\gamma)| = |\langle D_G\chi, \Gamma \rangle_G| \leq 1$. □

1.6. Bounds for compact p-adic groups. By a compact p-adic group we will mean a compact subgroup K of linear algebraic group $G(F)$ defined over a local non-archimedean field F. For example $\mathrm{SL}(n, \mathbb{Z}_p) \subset \mathrm{SL}(n, \mathbb{Q}_p)$ or $\mathrm{SL}(n, \mathbb{F}_q[[t]]) \subset \mathrm{SL}(n, \mathbb{F}_q((t))$. A modification of Larsen's method [**71**] yields near optimal bounds for values of irreducible characters at regular elements in compact p-adic groups (Chapter 2 see Theorem 2.61, Remark 2.62). The bounds on values of irreducible characters on non-central elements are useful in the context of the Limit Multiplicity problem where we need to estimate the orbital integrals of form

$$\int_{G_\gamma(F)\backslash G(F)} \chi(x^{-1}\gamma x)dx$$

For $G = \mathrm{PGL}(2, F)$ we used in Chapter 2 an argument involving the Bruhat-Tits buildings and character bounds to show that

$$\left| \int_{G_\gamma(F)\backslash G(F)} \chi(x^{-1}\gamma x)dx \right| \leq C|\Delta(\gamma)|_F^{-3/2},$$

where C is a constant dependent only on the type of G. We end this paragraph by showing the simplest instance of the functorial approach to estimating the orbital integrals. It is a p-adic analogue of the split regular case for finite groups. Let $G = \mathrm{SL}(2, \mathbb{Q}_p)$, $K = \mathrm{SL}(2, \mathbb{Z}_p)$, let T be the subgroup of diagonal matrices in G and let P, U be the groups of upper triangular resp. unipotent upper triangular matrices. Define the "naive" parabolic restriction ${}^*R_{T\cap K}^K$ as the composition of ordinary restriction $\mathrm{Res}\,_{K\cap P}^K$ with taking $U \cap K$-coinvariants. The action of parabolic restriction on the characters of K is given by

$${}^*R_{T\cap K}^K\chi(t) = \int_{U\cap K} \chi(tu)du.$$

The measure du on $U \cap K$ is the normalized Haar measure.

LEMMA 5.2. *Let χ be a character of K then for every regular element $\gamma \in T$ we have*

$$\int_{T(F)\backslash G(F)} \chi(x^{-1}\gamma x)dx = |\Delta(\gamma)|_F^{-1/2} {}^*R_{T\cap K}^K\chi(\gamma).$$

The Haar measures on $G(F), T(F)$ are chosen so that K respectively $K \cap T(F)$ have measure 1.

PROOF. Use the Iwasawa decomposition to get

$$\int_{T(F)\backslash G(F)} \chi(x^{-1}\gamma x)dx = \int_K \int U(F) f(k^{-1}u^{-1}\gamma uk)dudx \tag{1.1}$$

$$= \int_{U(F)} f(u^{-1}\gamma u) = \int_{U(F)} f(\gamma[\gamma^{-1}, u])du. \tag{1.2}$$

Since γ is regular, the map $u \mapsto [\gamma^{-1}, u]$ is a diffeomorphism with constant Jacobian equal to $|\Delta(\gamma)|_F^{1/2}$. Hence

$$\int_{T(F)\backslash G(F)} \chi(x^{-1}\gamma x)dx = |\Delta(\gamma)|^{-1/2} \int_{U(F)\cap K} f(\gamma u)du = |\Delta(\gamma)|_F^{-1/2} {}^*R_{T\cap K}^K\chi(\gamma).$$

□

We cannot repeat the ending of the proof in the finite case because this time the parabolic restriction of an irreducible character can have *a priori* unbounded number of irreducible constituents. To get the desired bound we need to study the properties of the functor ${}^*R^K_{T\cap K}$. For $G=\mathrm{PGL}(2,F)$ many properties of ${}^*R^K_{T\cap K}$ can be extracted from the well developed theory of smooth representations of $\mathrm{PGL}(2,F)$.

2. A property of Deligne-Lusztig restriction

In this section we prove proposition that will be the key component in the estimates for semisimple elements. Let l be a fixed prime number not dividing q.

PROPOSITION 5.3. *Let $\gamma\in\mathbf{G}^F$ and let $\gamma=su$ be the Jordan decomposition of γ. Then, for any rational Levi subgroup $\mathbf{L}$ containing the connected component of the centralizer $\mathbf{G}_s^0$ we have*

$$\chi(\gamma)={}^*R^G_L\chi(\gamma),$$

for every character χ.

Before the proof let us recall few facts on the Deligne-Lusztig induction. All module we consider will be over the field $\overline{\mathbb{Q}_l}$. We start with the notion of a generalized induction functor. Both DL-induction and restriction are the examples of such functor. Let G_0,H_0 be groups and let M be a G_0-module-H_0. We define the *generalized induction functor* $R^{G_0}_{H_0}$ associated to M by

$$R^{G_0}_{H_0}\colon \mathrm{Mod}_{H_0}\ni N\mapsto N\otimes_{\overline{\mathbb{Q}_l}[H_0]}M\in\mathrm{Mod}_{G_0}.$$

LEMMA 5.4. *Let H_0 be a subgroup of G_0 and let $\gamma\in H_0$. Let M be a H_0-module-G_0 and write $R^{H_0}_{G_0}$ for the associated generalized induction functor. Then the functor $R^{H_0}_{G_0}$ has the property that $R^{H_0}_{G_0}\chi(\gamma)=\chi(\gamma)$ for every character γ if and only if*

$$\mathrm{tr}((\gamma,g)\mid M)=\begin{cases}|(G_0)_\gamma| & \text{if } \gamma \text{ and } g^{-1} \text{ are conjugate in } G_0\\ 0 & \text{otherwise.}\end{cases}\tag{2.1}$$

PROOF. By [**38**, Proposition 4.5]

$$R^{H_0}_{G_0}\chi(\gamma)=|G_0|^{-1}\sum_{g\in G_0}\mathrm{tr}((\gamma,g^{-1})\mid M)\chi(g).$$

From that it is clear that if Equation 2.1 holds if and only if the functor $R^{H_0}_{G_0}$ does not change the value of characters on γ. □

If N is a G_0-module-H_0 we write $N^\wedge$ for the same $\mathbb{Z}$-module with left H_0-action and right G_0-action of $(h,g)\in H_0\times G_0$ given by $(h,g)\times m\mapsto g^{-1}mh^{-1}$. If $R^{G_0}_{H_0}$ is the generalized induction functor associated to a N then the adjoint functor ${}^*R^{G_0}_{H_0}$ is the generalized induction functor associated to the H_0-module-G_0 $N^\wedge$.

We will recall briefly the construction of Deligne-Lusztig induction functor. Let G be a reductive group defined over $\mathbb{F}_q$ with the Frobenius automorphism F. Write $\mathcal{L}:G\to G$ for the Lang-Steinberg map given by $x\mapsto x^{-1F}x$ (we denote the action by Frobenius endomorphism on the left to avoid confusion with taking the F-fixed points). Let P be a parabolic subgroup of G with Levi decomposition $P=LU$, where L is a Levi component and U is the unipotent radical. Assume that L is rational i.e. ${}^FL=L$. The preimage $\mathcal{L}^{-1}(U)$ is an affine subvariety of G. We have an $G^F\times L^F$ action on $\mathcal{L}^{-1}(U)$ given by $x\mapsto gxl$. The virtual module $H^*_c(\mathcal{L}^{-1}(U))$ is given by

$$H^*_c(\mathcal{L}^{-1}(U))=H^*_c(\mathcal{L}^{-1}(U),\overline{Q}_l)=\sum_{i=0}^{\infty}(-1)^iH^i_c(\mathcal{L}^{-1}(U),\overline{\mathbb{Q}_l}).$$

The left G^F-action and right L^F-action makes $H^*_c(\mathcal{L}^{-1}(U))$ a G^F-module-L^F. Deligne-Lusztig induction functor denoted by R^G_L is the generalized induction functor associated to $H^*_c(\mathcal{L}^{-1}(U))$. The restriction functor ${}^*R^G_L$ is the induction functor associated to

$H_c^*(\mathcal{L}^{-1}(U))^\wedge = H_c^*(\mathcal{L}^{-1}(U)^\wedge)$ where $L^F \times G^F$ acts on the variety $\mathcal{L}^{-1}(U)^\wedge$ by $(l,g)\times x \mapsto g^{-1}xl^{-1}$.

Proof of Proposition 5.3. Since all irreducible representations of G^F in characteristic 0 are defined over $\overline{\mathbb{Q}}$ we may prove it for $\overline{\mathbb{Q}}_l$-representations and the statement for complex ones will follow. Let $\gamma \in G^F$ with Jordan decomposition $\gamma = su$. Let L be a rational Levi subgroup containing G_s^0. By Lemma 5.4 we have to verify that

$$\mathrm{tr}((\gamma,g) \mid H_c^*(\mathcal{L}^{-1}(U)^\wedge)) = \begin{cases} |(G_0)_\gamma| & \text{if } \gamma \text{ and } g^{-1} \text{ are conjugate in } G_0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Let $g = rv$ be the Jordan decomposition of g. To shorten notation we will write $\mathcal{L}^{-1}(U)^\wedge = X$. By [**38**, Proposition 10.14] we have

$$\mathrm{tr}((\gamma,g) \mid H_c^*(X)) = \mathrm{tr}((u,v) \mid H_c^*(X^{(s,r)}).$$

Note that if $x \in X^{(s,r)}$ then $r^{-1}xs^{-1} = x$ so $xsx^{-1} = r^{-1}$. Since $r \in G^F$ we deduce that $\mathcal{L}(x) = x^{-1F}x \in G_s$. Since $\mathcal{L}(x)$ is unipotent it is contained in the connected component $G_s^0 \subset L$. But $L \cap U = 0$ so we must have $\mathcal{L}(x) = 0$ and consequently $x \in G^F$. This proves that if $X^{(s,r)}$ is non-empty if and only is s, r^{-1} are conjugate in G^F and in that case there exists $x \in G^F$ such that $X^{(s,r)} = xG_s^F$. The variety $X^{(s,r)} = xG_s^F$ is zero-dimensional so by [**38**, Proposition 10.8] $H_c^*(X^{(s,r)})$ is the permutation module $\overline{\mathbb{Q}}_l[xG_s^F]$. We have

$$\mathrm{tr}((u,v) \mid \overline{\mathbb{Q}}_l[xG_s^F]) = |\{y \in xG_s^F \mid u = y^{-1}v^{-1}y\}|.$$

The latter is equal to $|(G_s^F)_u|$ if u and $x^{-1}v^{-1}x$ are conjugate in G_s^F and 0 otherwise. It's easy to see that γ and g^{-1} are conjugate in G^F if and only if there exists $x \in G^F$ such that $xsx^{-1} = r^{-1}$ and u is conjugate to $x^{-1}v^{-1}x$ in G_s^F. As $G_\gamma^F = (G_s^F)_u$ we deduce (2.2). □

For the next part we will need the notion of geometric conjugacy. Let G be as before and let T, T' be two maximal rational tori and θ, θ' irreducible characters of T, T' respectively. We say that pairs (T,θ) and (T',θ') are geometrically conjugate if there exists $n \geq 0$, $g \in G^{f^n}$ such that $T' = {}^gT$ and

$$\theta(\mathrm{N}_{F^n/F}(t)) = \theta'(\mathrm{N}_{F^n/F}({}^gt)) \text{ for all } t \in T.$$

$N_{F^n/F}$ stands for the norm map defined as $x \mapsto x^F x^{F^2} x \ldots^{F^{n-1}} x$. For future use we prove a quick lemma:

Lemma 5.5. *For any algebraic torus T defied over $\mathbb{F}_q$ and any $n \geq 0$ the norm map is surjective.*

Proof. Let $x \in T$. By Lang-Steinberg theorem there exists $y \in T$ such that $x = y^{-1F^n}y$. Then $N_{F^n/F}(y^{-1F}y) = y^{-1F}y({}^Fy)^{-1F^2}y \ldots ({}^{F^{n-1}}y)^{-1F^n}y = y^{-1F^n}y = x$. □

As a quick application of Proposition 5.3 we get the following bound

Corollary 5.6. *Let G be a reductive group defined over $\mathbb{F}_q$ with Frobenius morphism F. Then for any regular semisimple element $\gamma \in G^F$ and any irreducible character χ we have*

$$|\chi(\gamma)| \leq |W|.$$

Proof. The proof will be similar to the "baby-case" of regular semisimple elements with split centralizer. Let T be the connected component of the centralizer of γ and let W be the Weyl group of T. As a first step let us prove that ${}^*R_T^G\chi(1) \leq |W|$ for any irreducible character χ of G^F. Let θ, θ' be irreducible characters of T^F such that θ is a constituent of ${}^*R_T^G\chi$. By [**38**, Proposition 10.15] we have

$$\langle R_T^G\theta, R_T^G\theta\rangle = |\{w \in W^F \mid {}^w\theta = \theta\}|.$$

Therefore the maximal multiplicity of θ in ${}^*R_T^G\chi$ is $|\mathrm{Stab}_{W^F}\theta|^{1/2}$. On the other hand, by [**38**, Proposition 13.3] the representations $R_T^G\theta$ and $R_T^G\theta'$ have no common irreducible

constituents unless (T,θ) and (T,θ') are geometrically conjugate. In that is the case then $\theta' = \theta_w$ where

$$\theta_w(\mathrm{N}_{F^n/F}({}^w t)) = \theta(\mathrm{N}_{F^n/F}(t)) = \text{ for all } t \in T.$$

This defines an irreducible representation of T^F because the norm map is surjective (Lemma 5.5). Note that if $w' \in \mathrm{Stab}_{W^F}\theta$ then $\theta_{w'w} = \theta_w$ so

$$\{\theta' \in \mathrm{Irr} T^F \mid \langle \theta, \chi \rangle \neq 0\langle\} \subset \{\theta_w \mid w \in \mathrm{Stab}_{W^F}\theta \backslash W\}.$$

We deduce that

$${}^*R_T^G\chi(1) \leq |\mathrm{Stab}_{W^F}\theta|^{1/2}||\mathrm{Stab}_{W^F}\theta\backslash W| \leq |W|.$$

We finish the proof using Proposition 5.3:

$$|\chi(\gamma)| = |{}^*R_T^G\chi(\gamma)| \leq {}^*R_T^G\chi(1) \leq |W|.$$

□

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