Systèmes quantiques multi-particulaires et localisation à basses énergies ou à faible interaction.

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Résumé

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Dans cette thèse, on étudie le phénomène de localisation d'Anderson des opérateurs de Schrödinger à N particules qui englobe aussi bien la localisation exponentielle des fonctions propres que la localisation dynamique.

Dans un premier temps, on considère le modèle d'Anderson discret multi-particulaire avec un potentiel aléatoire à valeurs indépendantes et identiquement distribuées i.i.d. dont la distribution commune est au moins log-Höldérienne et une interaction de courte portée. On établit pour ce modèle la localisation d'Anderson pour les énergies suffisamment proches du bas du spectre après avoir montré qu'il est non-aléatoire.

D'autre part, on montre que la localisation complète des systèmes mono-particulaires s'étend aux systèmes multi-particulaires ayant une interaction globale suffisamment faible et pour un désordre arbitraire. Pour ce résultat, l'existence d'une densité bornée de la distribution commune des variables aléatoires i.i.d. est nécessaire et on le montre pour des interactions ayant une forme très générale mais bornées.

Les résultats sont démontrés à l'aide de l'adaptation aux systèmes multi-particulaires de l'analyse multi-échelle à énergie variable qui permet de traiter des distributions singulières contrairement à la méthode des moments fractionnaires.

Les estimées de Wegner intervenant notamment dans l'analyse multi-échelle sont établies pour des cubes vérifiant une propriété de séparabilité en utilisant le Lemme de Stollmann. On démontre également l'estimée de Combes-Thomas qui joue un role important dans l'analyse des énergies extrêmes.

Mots-clefs

Opérateurs aléatoires, systèmes multi-particulaires, basses énergies, faibles interactions.

Multi-particle localization for disordered quantum systems at low energies or with weak interaction

Abstract

In this thesis, we study for the N-particles Schrodinger operators the Anderson localization phenomenon which consists of both exponential localization of eigenfunctions and dynamical localization.

We first consider the discrete multi-particle Anderson model with a short range interaction and a random potential whose values are independent and identically distributed i.i.d. with a log-Hölder continuous common probability distribution function. For such model, we show that the bottom of its spectrum is non-random and prove the Anderson localization for energies sufficiently close to the spectral edge.

On the other hand, we establish that the complete localization from single-particle systems extends to multi-particle systems with sufficiently weak interaction at arbitrary disorder and for absolute continuous probability distribution function of the i.i.d random variables.

The results are proved by an adaptation to multi-particle systems of the variable energy multi-scale analysis which allows singular distributions instead of the fractional moments method.

Wegner bounds, useful for the multi-scale analysis are proved for separable cubes using the Stollmann's Lemma. We also prove the Combes-Thomas estimate which plays an important role in the analysis of extreme energies.

Keywords

Anderson localization, random operators, multi-particle systems, low energies, weak interaction.

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Chapitre 1

Introduction et présentation des résultats

1.1 Introduction

En mécanique quantique l'évolution temporelle d'une particule représentée par un état ψ, est gouvernée par l'équation de Schrödinger

$$i\frac{\partial}{\partial t}\psi = H^{(1)}\psi, \tag{1.1}$$

où $H^{(1)}$ est le Hamiltonien définit par

$$H^{(1)} = -\frac{\hbar^2}{2m}\Delta + V,$$

agissant sur $\mathcal{H} = L^2(\mathbb{R}^d)$ ou $\ell^2(\mathbb{Z}^d)$ avec Δ le Laplacien modélisant l'énergie cinétique de la particule et V est un potentiel décrivant la structure physique du système et agissant comme opérateur de multiplication par la fonction $V(x)$. Les solutions de l'équation (1.1) sont de la forme

$$\begin{cases} \psi(t) = \mathrm{e}^{-itH^{(1)}}\psi_0 \\ \psi(0) = \psi_0. \end{cases}$$

Ainsi, pour décrire le comportement des états $\psi(t)$, il est utile d'analyser les propriétés spectrales de l'opérateur $H^{(1)}$. Dans une telle analyse la constante $\frac{\hbar^2}{2m}$ joue un role fictif et il est coutume de supposer $\frac{\hbar^2}{2m} = 1$.

Pour des systèmes parfaitement ordonnés à potentiels périodiques, i.e., $V(x - x_0) = V(x)$ pour tout $x \in \mathbb{Z}^d$ ou $\mathbb{R}^d$ et un certain $x_0 \in \mathbb{Z}^d$ fixé, il est établit [Eas73, RS78] que le spectre de l'opérateur de Schrödinger $H^{(1)}$ est absolument continue et constitué de bandes. Dans ce cas, la particule évolue librement dans tout l'espace.

Cependant, la présence de désordre induit des effets particuliers sur les particules quantiques. Pour les décrire, P.W. Anderson [And58], a introduit un modèle qui porte son nom et qui lui a valu le prix nobel de physique «le modèle d'Anderson». Dans le modèle d'Anderson, le désordre est pris en compte par le caractère aléatoire du potentiel externe V. Dans l'approximation monoparticulaire, il est décrit par l'opérateur de Schrödinger aléatoire

$$H^{(1)}(\omega) = -\Delta + V(x, \omega),$$

agissant sur $\Psi \cong \Psi(x) \in \ell^2(\mathbb{Z}^d)$ et où $\{V(x, \omega)\}$ constitue un champ aléatoire à valeurs indépendantes et identiquement distribuées relatif à un espace probabilisé $(\Omega, \mathfrak{B}, \mathbb{P})$.

Un phénomène important qui apparaît pour les opérateurs $H^{(1)}(\omega)$ est la *localisation d'Anderson* qui se traduit par la présence de spectre purement ponctuel avec décroissance exponentielle des fonctions propres à l'infini. Une autre propriété plus forte et très intéressante dans la description de l'évolution temporelle du système est la *localisation dynamique* qui exprime le fait que la particule reste concentrée au voisinage de l'origine uniformément en temps.

On rappelle (cf. [RS80]) que l'espace $\mathcal{H}$ se décompose en $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_c = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}$, où $\mathcal{H}_{pp}$, $\mathcal{H}_c$, $\mathcal{H}_{ac}$ et $\mathcal{H}_{sc}$ sont respectivement les sous-espaces correspondants à la composante purement ponctuelle, continue, absolument continue et singulièrement continue de la mesure spectrale associée à $H^{(1)}(\omega)$. On note les spectres des restrictions de $H^{(1)}(\omega)$ à ces différents sous-espaces par $\sigma_\bullet(H^{(1)}(\omega))$, avec $\bullet = pp, c, ac, sc$.

Un résultat basé sur les travaux de Ruelle [Rue69], Amrein, Georgescu [AG73] et Enss [Ens78] et connu comme Théorème de RAGE, donne une première idée de la dynamique associée aux opérateurs $H^{(1)}(\omega)$: si on note B_R la boule fermée de $\mathbb{Z}^d$ de centre 0 et de rayon R et $\Lambda \subset \mathbb{Z}^d$ un domaine borné, alors

(a) $\psi \in \mathcal{H}_{pp}$ implique

$$\lim_{R\to\infty} \sup_{t\geq 0} \left(\sum_{x \notin B_R} |\mathrm{e}^{-itH} \psi(x)|^2 \right) = 0,$$

(b) $\psi \in \mathcal{H}_c$ implique

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\sum_{x\in\Lambda} |\mathrm{e}^{-itH} \psi(x)|^2 \right) dt = 0.$$

Dans le cas (a), on dit que l'état ψ est localisé en opposition aux états diffus ou étendus correspondant au cas (b) où la particule se déplace dans tout l'espace. Les premiers résultats mathématiquement rigoureux sur la localisation d'Anderson ont été obtenus par L. Pastur, I. Golsheid and S. Molchanov [GM76, GMP77, Mol78] dans le cadre de la dimension 1 et pour l'opérateur de Sturm-Liouville. La localisation d'Anderson pour les opérateurs de type $H^{(1)}(\omega)$ a été initialement prouvée par Kunz et Souillard [KS80] en dimension 1 pour des distributions à densité bornée. Toujours en dimension 1 plusieurs autres résultats ont suivi incluant des potentiels singuliers de type Bernoulli [Pas80, PF92, Car82, Kot84, Kot86, KS87, CKM87, DSS02]. Les méthodes sont basées sur l'étude de l'exposant de Lyapounov et des fonctions de corrélations et ne s'étendent pas en dimension $d \geq 2$.

Pour $d = 2$ d'après des expériences physique, on s'attend encore à la localisation complète comme dans le cas $d = 1$ mais les bornes d'analyse multi-échelle pourraient être moins forte que pour le cas $d = 1$. Bref c'est un sujet encore en débats dans les mathématiques des systèmes désordonnés quantiques.

En dimension quelconque, le travail réalisé par Fröhlich et Spencer [FS83] dans le cas discret, basé sur l'analyse multi-échelle et établissant l'absence de diffusion dans le modèle d'Anderson à basses énergies ou à grand désordre fut une étape importante dans la localisation multi-dimensionnelle. Par la suite Martinelli et Scoppola [MS85] d'une part et Fröhlich et al. [FMSS85] d'autre part établissent respectivement l'absense de spectre absolument continu et la localisation d'Anderson à grand désordre ou à basses énergies. Plus tard, von Dreifus et Klein [vDK89] simplifient l'analyse multi-échelle et proposent par la même occasion une nouvelle preuve de la localisation d'Anderson. L'adaptation de l'analyse multi-échelle au cas continu a été réalisé dans [MH84, CH94, Klo95a, Klo95b, CHM96]. La localisation dynamique quant à elle a été initialement démontrée par Aizenmann et Molchanov [AM93, Aiz94] dans le cas discret via la méthode des moments fractionnaires et dans le cas continu dans [GB98, FK96, RJYS96]. La méthode des moments quand elle s'applique conduit naturellement à la localisation dynamique mais nécessite des distributions assez régulière (généralement absolument continue à densité bornée). Tandis que l'analyse multi-échelle permet de traiter un large spectre de potentiels aléatoires incluant des potentiels singuliers de type Bernoulli [Car82, CL90] en dimension 1. Cependant une analyse supplémentaire est requise afin d'en déduire la décroissance des fonctions propres et la localisation dynamique. La localisation dynamique par des méthodes reposant sur l'analyse multi-échelle a été au départ démontrée par Germinet et De Bièvre [GB98]. Plus tard, Damanik et Stollmann [DS01] montrent que les estimées d'analyse multi-échelle entraînent la localisation dynamique forte au sens de l'espérance. Une version plus fine (en norme Hilbert-Schmidt) de la localisation dynamique a également été prouvée par Germinet et Klein [GK01]. Dans cette thèse, on se servira de l'analyse multi-échelle pour établir la localisation d'Anderson et dynamique multi-particulaire.

L'analyse multi-échelle est un procédé inductif sur des échelles ou longueurs $\{L_k\}_{k\geq 0}$ avec comme relation de récurrence $L_{k+1} = \lfloor L_k^\alpha \rfloor + 1$, $1 < \alpha < 2$. On commence par restreindre l'opérateur $H^{(1)}(\omega)$ à des cubes $C^{(1)}_{L_k}(u)$ de côté L_k avec conditions aux bords nulles. L'analyse multi-échelle consiste alors à établir la décroissance (généralement exponentielle) à toutes les échelles, des éléments matriciels loin de la diagonale de l'opérateur restreint $H^{(1)}_{C^{(1)}_{L_k}(u)}(\omega)$ avec une probabilité convenable. L'étape inductive est basée sur l'identité géométrique de la résolvante qui permet d'établir une relation entre les éléments matriciels des résolvantes correspondant aux échelles k et $k+1$. En effet un résultat central pour l'analyse multi-échelle et qui utilise l'identité géométrique de la résolvante établit la décroissance des éléments matriciels dans un cube à partir de la décroissance dans des sous-cubes disjoints et de leur non-résonance. La résonance d'un cube $C^{(1)}_{L_k}(u)$ par rapport à une certaine énergie E caractérise le fait que cette énergie soit très proche du spectre de l'opérateur $H^{(1)}_{C^{(1)}_{L_k}(u)}(\omega)$. De ce fait, se servant de l'hypothèse inductive sur des échelles $\{L_k\}$, l'étape suivante consiste à estimer en probabilité les résonances. Cela se fait à l'aide des estimées de Wegner.

Le role des hypothèses telles *grand désordre* ou *voisinage des énergies extrêmes* est de pouvoir établir les estimées de l'échelle initiale. Dans le cas des basses énergies qui nous intéresse, cette estimée est basée d'une part sur l'estimée de Combes-Thomas qui fournit une décroissance des résolvantes locales pour des énergies en dehors du spectre et d'autre part sur l'étude des asymptotiques de Lifshitz de la densité d'états intégrée.

Le cas des potentiels aléatoires corrélés a été analysé par Dreifus et Klein [vDK91] qui ont démontré la localisation d'Anderson des champs Gaussiens à covariances décroissantes et des champs de Gibbs complètement analytiques. Krüger [Kru12], a quant à lui, établit la localisation pour des potentiels corrélés à corrélations exponentiellement décroissantes par une méthode reposant sur une forme d'analyse multi-échelle à estimée de Wegner inductive développée précédemment par Bourgain et al. [BGS02] et améliorée dans [Bou02, Bou07]. Plus récemment, Klopp [Klo12] a analysé les statistiques spectrales des valeurs propres de l'opérateur de Schrödinger à potentiels faiblement corrélés dans le régime localisé.

Rappelons le modèle d'Anderson dans l'approximation multi-particulaire avec un nombre de particule $N \geq 2$ fini. L'étude mathématique de tels opérateurs est relativement récente. La structure générale de l'Hamiltonien N-particulaire avec interaction est la suivante :

$$\mathbf{H}^{(N)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^{N} V(x_j, \omega) + \mathbf{U}, \tag{1.2}$$

il opère sur $\mathbf{\Psi} \cong \mathbf{\Psi}(x_1, \ldots, x_n) \in \ell^2((\mathbb{Z}^d)N)$, où $\mathbf{\Delta}$ est l'opérateur Laplacien sur $(\mathbb{Z}^d)^N \cong \mathbb{Z}^{Nd}$, $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ est un champ aléatoire relatif à un espace probabilisé $(\Omega, \mathbb{P})$ qui agit comme operateur de multiplication par la fonction $V(\mathbf{x}, \omega) = \sum_{i=1}^{N} V(x_i, \omega)$ et $\mathbf{U} : (\mathbb{Z}^d)^N \to \mathbb{R}$ est un potentiel d'interaction entre les particules quantiques et agit aussi comme opérateur de multiplication par la fonction $\mathbf{U}(\mathbf{x})$. Les premiers résultats sur la localisation d'Anderson des opérateurs $\mathbf{H}^{(N)}(\omega)$ ont été obtenus d'une part par Chulaevsky et Suhov [CS09a, CS09b] en étendant l'analyse multi-échelle aux systèmes multi-particulaires et d'autre part par Aizenmann et Warzel [AW09, AW10] qui ont réalisé cette extension pour la méthode des moments fractionnaires. Dans les deux cas, il a été supposé que le champ aléatoire V est à valeurs indépendantes et identiquement distribuées et que l'interaction $\mathbf{U}$ est de portée finie.

Dans [AW09, AW10], les auteurs établissent la localisation spectrale et dynamique sous le régime de grand désordre ou de faible interaction. Compte tenu de la spécifité de la méthode des moments, Aizenmann et Warzel ont supposé que la distribution commune des variables aléatoires admet une densité bornée vérifiant une condition technique supplémentaire. Tandis que Chulaevsky et Suhov [CS09b], ont obtenus la localisation spectrale multi-particulaire sous le régime de grand désordre et pour des distributions Höldériennes. Notre travail qui traite le cas des basses énergies complète ces précédents travaux.

Un problème important qui apparaît dans l'étude des systèmes multi-particulaires et notamment dans la localisation près du bas du spectre est celui de l'ergodicité car les résultats d'ergodicité de la théorie monoparticulaire ne s'appliquent pas ici. En effet, il n'y a plus ergodicité à partir des translations comme dans le système monoparticulaire. Telle est la raison pour laquelle l'ergodicité n'est pas utilisée dans notre stratégie générale concernant le fait que le bas du spectre est non-aléatoire. L'ergodicité permet d'après un résultat dû à Pastur, d'obtenir que presque sûrement, le spectre $\sigma(H^{(1)}(\omega))$ vérifie :

$$\sigma(H^{(1)}(\omega)) = \Sigma,$$

où Σ est un sous ensemble fermé de $\mathbb{R}$. De plus, il existe des sous-ensembles Σ_{pp},

Σ_{ac}, Σ_{sc} tels que avec $\mathbb{P}$-probabilité 1,

$$\sigma_{pp}(H^{(1)}(\omega)) = \Sigma_{pp} \quad \sigma_{ac}(H^{(1)}(\omega)) = \Sigma_{ac} \quad \sigma_{sc}(H^{(1)}(\omega)) = \Sigma_{sc}.$$

où $\sigma_{pp}(H^{(1)}(\omega))$, $\sigma_{ac}(H^{(1)}(\omega))$, $\sigma_{sc}(H^{(1)}(\omega))$ désignent respectivement le spectre purement ponctuel, absolument continu et singulièrement continu de $H^{(1)}(\omega)$. Comme conséquence, la borne inférieure du spectre $E_0 = \inf \sigma(H^{(1)}(\omega))$ est non-aléatoire.

Nous décrivons dans le paragraphe suivant les divers travaux de la thèse.

1.2 Présentation des travaux de la thèse

Dans cette étude on considère l'espace de probabilité $(\Omega, \mathfrak{B}, \mathbb{P})$ donné par

$$\Omega = \mathbb{R}^{\mathbb{Z}^d}, \quad \mathfrak{B} = B(\mathbb{R})^{\mathbb{Z}^d},$$

(où $B(\mathbb{R})$ est la tribu borélienne de $\mathbb{R}$) et un potentiel externe $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ défini par $V(x,\omega) = \omega_x$ pour $\omega = (\omega_i)_{i\in\mathbb{Z}^d}$. On s'intéresse alors aux propriétés spectrales des Hamiltoniens N-particulaire $\{\mathbf{H}^{(N)}(\omega), \omega \in \Omega\}$. Soit $N \geq 2$ un entier fini qui peut être arbitrairement grand. L'analyse multi-échelle nécessite de considérer aussi les Hamiltoniens $\mathbf{H}^{(n)}(\omega)$ correspondant aux sous-systèmes à n particules pour tout $n = 1, \ldots, N$. Ainsi, on introduit les notations suivantes pour tout $n = 1, \ldots, N$.

Soit $\nu \geq 1$, on considère la norme max $|\cdot|$ sur $\mathbb{Z}^\nu$ définie par

$$|x| = \max\{|x_1|, \ldots, |x_\nu|\}, \tag{1.3}$$

pour $x = \big(x_1, \ldots, x_\nu\big) \in \mathbb{Z}^\nu$. Occasionnellement (par exemple dans la définition du Laplacien), on utilisera la norme $|\cdot|_1$ définie par

$$|x|_1 = |x_1| + \cdots + |x_\nu|. \tag{1.4}$$

La configuration d'un système de n particules quantiques distinctes sur $\mathbb{Z}^d$ où $d \geq 1$ est un entier fixé, est représenté par un vecteur

$$\mathbf{x} = (x_1, \ldots, x_n) \in (\mathbb{Z}^d)^n \cong \mathbb{Z}^{nd},$$

où $x_j = (x_j^{(1)}, \ldots, x_j^{(d)}) \in \mathbb{Z}^d$, $j = 1, \ldots, n$ et $\mathbf{\Delta}$ est le Laplacien sur $\mathbb{Z}^{nd}$ défini par

$$(\mathbf{\Delta}\mathbf{\Psi})(\mathbf{x}) = \sum_{\substack{\mathbf{y}\in\mathbb{Z}^{nd}\\ |\mathbf{y}-\mathbf{x}|_1=1}} (\mathbf{\Psi}(\mathbf{y}) - \mathbf{\Psi}(\mathbf{x})) = \sum_{\substack{\mathbf{y}\in\mathbb{Z}^{nd}\\ |\mathbf{y}-\mathbf{x}|_1=1}} \mathbf{\Psi}(\mathbf{y}) - 2dn\mathbf{\Psi}(\mathbf{x}), \tag{1.5}$$

pour tout $\mathbf{\Psi} \in \ell^2(\mathbb{Z}^{nd})$, $\mathbf{x} \in \mathbb{Z}^{nd}$. Noter que $\mathbf{\Delta}$ est borné et $-\mathbf{\Delta}$ est positif. On considérera les opérateurs de Schrödinger aléatoire $\mathbf{H}^{(n)}(\omega)$ pour $n = 1, \ldots, N$ de la forme suivante déjà vue en (1.2)

$$\mathbf{H}^{(n)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^{n} V(x_j, \omega) + \mathbf{U}$$

agissant sur l'espace de Hilbert $\mathcal{H}^{(n)} = \ell^2(\mathbb{Z}^{dn})$. (1.3) et (1.4) sont compatibles avec l'identification $\mathbb{Z}^{dn} \cong (\mathbb{Z}^d)^n$, i.e.,

$$|\mathbf{x}| = \max_{1 \le j \le n} |x_j|,$$
$$|\mathbf{x}|_1 = |x_1|_1 + \cdots + |x_n|_1,$$

pour $\mathbf{x} = (x_1, \ldots, x_n) \in (\mathbb{Z}^d)^n$. Pour le régime faiblement interactif, on introduit un paramètre $h > 0$ qui permet de mesurer la force de l'interaction. Plus précisément on a l'Hamiltonien suivant :

$$\mathbf{H}_h^{(n)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^{n} V(x_j, \omega) + h\mathbf{U},$$

où la dépendance avec la force de l'interaction est visible selon que $h > 0$ est petit ou non.

1.2.1 Hypothèses générales

(I1). Le potentiel d'interaction

$$\mathbf{U} \colon (\mathbb{Z}^d)^n \to \mathbb{R}$$

est borné et de la forme

$$\mathbf{U}(\mathbf{x}) = \sum_{1 \le i < j \le n} \Phi(|x_i - x_j|), \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^{nd}, \tag{1.6}$$

où $\Phi \colon \mathbb{N} \to \mathbb{R}_+$ est une fonction positive à support compact

$$\exists\, r_0 \in \mathbb{N} : \quad \operatorname{supp} \Phi \subset [0, r_0]. \tag{1.7}$$

L'étude du régime faiblement interactif ne nécessite pas l'interaction $\mathbf{U}$ d'être positive, pour cette raison on introduit l'hypothèse plus générale suivante.

(I2). L'interaction

$$\mathbf{U}\colon (\mathbb{Z}^d)^n \to \mathbb{R}$$

satisfait pour tout $n = 1, \dots, N$

$$\mathbf{U}(\mathbf{x}) = \sum_{1\le i<j\le n} \Phi(|x_i - x_j|), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^{nd},$$

où $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{Z}^d)^n$ et la fonction Φ n'est pas nécessairement positive. De plus,

$$\exists r_0 \in \mathbb{N}: \quad \operatorname{supp} \Phi \subset [0, r_0].$$

Pour décrire les hypothèses sur le potentiel aléatoire

$$V\colon \mathbb{Z}^d \times \Omega \to \mathbb{R},$$

on a besoin d'un certain nombre de notations. On désigne par F_V la distribution sur $\mathbb{R}$ des variables aléatoires i.i.d. $V(x,\omega)$. Soit $E \in \mathbb{R}$

$$F_V(E) := \mathbb{P}\left\{V(0,\omega) \le E\right\}$$

et soit μ la mesure associée sur $\mathbb{R}$, i.e,

$$\mu([a,b]) = F_V(b) - F_V(a).$$

On pose

$$s(\mu,\varepsilon) = \sup_{a\in\mathbb{R}} \mu([a, a+\varepsilon]) = \sup_{a\in\mathbb{R}} (F_V(a+\varepsilon) - F_V(a)).$$

On définit aussi

$$L^\infty((1+|x|)^{1+\kappa}) := \{f : \mathbb{R} \to \mathbb{R} | \sup_{x\in\mathbb{R}} |f(x)(1+|x|)^{1+\kappa}| < \infty\},$$

pour un certain $\kappa \in (0,1)$.

Le support d'une mesure μ est défini par

$$\operatorname{supp} \mu = \{x \in \mathbb{R} \,\Big|\, \mu\left((x-\varepsilon, x+\varepsilon)\right) > 0, \forall \varepsilon > 0\}. \tag{1.8}$$

(P1). La distribution commune μ des variables aléatoires i.i.d. $V(x,\omega)$ vérifie $0 \in \operatorname{supp} \mu \subset [0, +\infty)$. De plus, elle est log-Höldérienne, à savoir :

$$s(\mu,\varepsilon) := \sup_{a\in\mathbb{R}} \mathbb{P}\left\{V(0,\omega) \in [a, a+\varepsilon]\right\} \le \frac{C}{|\ln \epsilon|^{2A}} \tag{1.9}$$

où $C \in (0,\infty)$ et $A > 0$ est suffisamment grand.

(P2). Le potentiel aléatoire $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ est i.i.d. et borné, i.e., il existe $M \in (0,\infty)$ tel que $\operatorname{supp} \mu \subset [-M, M]$. De plus , la mesure de probabilité μ possède une densié bornée, i.e., $\mathbb{P}\left\{V(0,\omega) \in A\right\} = \mu(A) = \int_A \rho(\lambda) d\lambda$, et $\rho \in L^\infty((1+|x|)^{1+\kappa})$.

Les résultats suivants ont été l'objet de projets d'articles [Eka11, Eka12].

1.2.2 Détermination du bas du spectre

Comme dit plus haut, sans avoir recours à l'ergodicité, il n'est pas trivial d'établir que le bas du spectre $E_0^{(n)}$ des Hamiltoniens $\mathbf{H}^{(n)}(\omega)$ est presque sûrement constant. Or, un tel résultat est crucial pour la localisation d'Anderson. Sous des hypothèses adéquates mais très générales (**I1**) et (**P1**) sur l'interaction $\mathbf{U}$ et le potentiel aléatoire V, on montre que presque sûrement, toutes les quantités $E_0^{(n)}$ sont identiquement nulles.

Théorème 1.1. *Soit $n = 1, \dots, N$. Sous les hypothèses* (**I1**) *et* (**P1**), *avec probabilité* 1 *:*

$$[0, 4nd] \subset \sigma(\mathbf{H}^{(n)}(\omega)) \subset [0, +\infty).$$

Par conséquent,

$$E_0^{(n)} := \inf \sigma(\mathbf{H}^{(n)}(\omega)) = 0 \quad \text{p.s.}$$

Les idées de la preuve apparaissent déjà sous une forme différente dans le livre de P. Stollmann [Sto01] dans le cas des opérateurs de Schrödinger monoparticulaire dans $L^2(\mathbb{R}^d)$. Dans notre cas, la présence de l'interaction entraîne des changements majeures dans la preuve. L'argument repose sur le Lemme de Borel-Cantelli et sur le fait qu'en réalité l'intervalle $[0, 4nd]$ est le spectre du Laplacien dans $\ell^2(\mathbb{Z}^{nd})$. En effet, étant donné que l'interaction est de portée finie on peut trouver une suite de cubes dans $\mathbb{Z}^{nd}$ sur lesquels l'interaction est identiquement nulle et ayant des projections totales disjointes sur $\mathbb{Z}^d$. Les évènements relatifs à de telles projections sont indépendants. Ensuite, étant donné une suite de Weyl pour le Laplacien et à support dans un cube centré en l'origine, on la translate de façon à ce qu'elle soit à support contenu dans l'un des termes de la suite des cubes où l'interaction est nulle. La suite translatée reste une suite de Weyl pour le Laplacien et un argument probabiliste sur le potentiel aléatoire permet de conclure. Pour plus de détails sur ce résultat, on renvoie au Chapitre 3.

Dans le modèle d'Anderson, pour tout $E \in \mathbb{R}$, la limite

$$\lim_{L \to \infty} L^{-nd} \operatorname{card} \left\{ n : E_n(\mathbf{H}^{(n)}(\omega)|_{\mathbf{C}_L^{(n)}(\mathbf{0})}) \leq E \right\},$$

existe et est appelée *densité d'états intégrée* de $\mathbf{H}^{(n)}(\omega)$.

Sous des hypothèses adéquates, d'après un résultat dû à Klopp et Zenk [KZ09], la densité d'états intégrée de l'Hamiltonien n-particulaire $\mathbf{H}^{(n)}(\omega)$ existe et coincide avec celle de l'Hamiltonien monoparticulaire $H^{(1)}(\omega)$. De ce fait, si un résultat analoque est établi dans le cadre discret, d'après la théorie sur le modèle d'Anderson à potentiel aléatoire ergodique, nous savons que la denstité d'états intégrée

$\mathcal{N}$ admet un comportement de Lifshitz au voisinage du bas du spectre (voir par exemple le Théorème 6.1 dans [Kir08]) à savoir :

$$\lim_{E \searrow E_0^{(1)}} \frac{\ln |\ln \mathcal{N}(E)|}{\ln(E - E_0^{(1)})} = -\frac{d}{2}.$$

Etant donné que pour tous les Hamiltoniens $\mathbf{H}^{(n)}(\omega)$, $n = 1, \ldots, N$, $E_0^{(n)} = 0$, on peut en déduire que tous les Hamiltoniens $\mathbf{H}^{(n)}(\omega)$ admettent un comportement de Lifshitz au voisinage de 0.

Les résultats suivants concernant le régime des basses énergies sont démontrés au Chapitre 3.

1.2.3 Analyse multi-échelle multi-particulaire à basses énergies

Comme dans sa version monoparticulaire, l'analyse multi-échelle dans sa version multi-particulaire initialement développé par Chulaevsky et Suhov [CS09b] dans le cas de grand désordre reste une induction sur les échelles $\{L_k\}$. Mais à cela s'ajoute une induction sur le nombre de particules $n = 1, \ldots, N$. Cependant, les estimées de Wegner servant à contrôler de façon adaptée les probabilités des résonances sont établies à partir du Lemme de Stollmann [Sto00, Sto01] et de l'information sur les projections des cubes (n-particulaire) sur $\mathbb{Z}^d$. En effet, des évenèments relatifs à deux cubes disjoints dans $\mathbb{Z}^{nd}$ ne sont pas forcément indépendants comme dans le cas monoparticulaire. C'est pour cette raison qu'on introduit la notion de cubes *séparables* où la projection partielle (voir totale) de l'un des cubes est disjointe de la réunion des autres projections et de celles de l'autre cube. Plus précisément :

Définition 1.2. On dit qu'un cube $\mathbf{C}_L^{(n)}(\mathbf{x}) = C_L^{(1)}(x_1) \times \cdots \times C_L^{(1)}(x_n)$ est $\mathcal{J}$-séparable d'un autre $\mathbf{C}_L^{(n)}(\mathbf{y}) = C_L^{(1)}(y_1) \times \cdots \times C_L^{(1)}(y_n)$ s'il existe un sous ensemble non-vide $\mathcal{J} \subset \{1, \cdots, n\}$ tel que

$$\left(\bigcup_{j \in \mathcal{J}} C_L^{(1)}(x_j) \right) \cap \left(\bigcup_{j \notin \mathcal{J}} C_L^{(1)}(x_j) \cup \left(\bigcup_{\ell=1}^{n} C_L^{(1)}(y_\ell)\right) \right) = \emptyset,$$

La paire $(\mathbf{C}_L^{(n)}(\mathbf{x}), \mathbf{C}_L^{(n)}(\mathbf{y}))$ est dite séparable si $|\mathbf{x} - \mathbf{y}| > 7NL$ et si l'un des cubes est $\mathcal{J}$-séparable de l'autre.

Cette propriété permet par indépendance et conditionnement de fixer certaines variables aléatoires afin d'appliquer le Lemme de Stollmann. La séparabilité est une propriété vraiment essentielle, car on aurait pu penser à des cubes suffisamment distants, mais le fait d'avoir des cubes à distance suffisamment grande ne garantit pas leur séparabilité. Comme exemple, on se place en dimension $d = 1$ et $n = 3$ et on pose $\mathbf{x} = (a, a, b) \in \mathbb{Z}^3$ et $\mathbf{y} = (a, b, b) \in \mathbb{Z}^3$ alors pour tout $L \geq 0$, les cubes

$$\mathbf{C}_L^{(3)}(\mathbf{x}) = ([-L + a, a + L] \times [-L + a, a + L] \times [-L + b, b + L]) \cap \mathbb{Z}^3$$

et

$$\mathbf{C}_L^{(3)}(\mathbf{y}) = ([-L + a, a + L] \times [-L + b, b + L] \times [-L + b, b + L]) \cap \mathbb{Z}^3$$

bien qu' à distance $|a - b|$ qui peut être arbitrairement grande si $a \neq b$ ne sont pas séparables car l'une des projections d'un cube intersecte toujours la projection totale de l'autre cube.

On peut aussi se demander pourquoi une induction sur le nombre de particules et pas une preuve directe pour tout $n = 1, \ldots, N$ pour des cubes séparables ? L'induction sur les échelles de k à $k+1$ utilise le fait que les projections totales des cubes sont disjointes afin d'utiliser l'indépendance et évaluer la probabilité d'une intersection d'évènements relatifs à ces projections. Cependant, cette dernière propriété est plus restrictive que la séparabilité et restreindre l'analyse multi-échelle uniquement à de tels cubes n'est pas suffisant pour établir la localisation d'Anderson via le schéma que nous utilisons et qui remonte à von Dreifus et Klein [vDK89]. En effet, pour un certain cube dans $\mathbb{Z}^{nd}$, il existe une infinité de cubes à distance suffisamment grande et dont les projections totales intersectent celle du premier cube. Autrement dit, il n'est pas possible de confiner dans une région bornée tous les cubes dont les projections totales intersectent celle du premier. Or, tel est le cas concernant les cubes non-séparables. Cela nous conduit de ce fait à distinguer différentes sortes de cubes : ceux proches de la diagonale principale $\mathbb{D}$ de $\mathbb{Z}^{nd}$

$$\mathbb{D} = \{\mathbf{x} \in (\mathbb{Z}^d)^n : \mathbf{x} = (x, \ldots, x), x \in \mathbb{Z}^d\},$$

dont les centres ont un petit diamètre et qu'on appellera par la suite dans notre schéma cubes *totalement interactifs* et ceux suffisamment éloingés de la diagonale principale $\mathbb{D}$ dont les centres peuvent avoir des diamètres arbitrairement grand qui seront appélés cubes *partiellement interactifs*.

Les cubes totalement interactifs et à distance suffisante sont ceux dont les projections totales sont disjointes permettant ainsi de reproduire l'induction sur les échelles comme dans le cas monoparticulaire, c'est ce qui justifie la condition de distance $|\mathbf{x} - \mathbf{y}| > 7NL$ dans la définition 1.2. Par contre, pour l'analyse des cubes partiellement interactifs, il est nécessaire de définir judicieusement cette notion de façon à ce que l'interaction se décompose sur un tel cube éloigné de la diagonale. Comme conséquence, l'Hamiltonien se décompose lui aussi comme somme de deux Hamiltoniens ayant un nombre de particules strictement plus petit. L'idée est donc de déduire des informations sur l'Hamiltonien originel à partir des informations sur les sous-Hamiltoniens : d'où l'induction sur le nombre de particules.

Le schéma d'induction présenté par Chulaevsky et Suhov [CS09b] sous le régime de grand désordre ne peut pas être directement utilisé dans le cas des basses énergies et a dû être modifié. Par exemple ce schéma utilise le fait qu'un certain paramètre $p(N,g) \to +\infty$ quand le désordre $|g| \to +\infty$. Pour démarrer l'analyse multi-échelle à basses énergies, on se sert de l'estimée de Combes-Thomas et d'une estimée de grande déviation pour le processus $\{V(x,\omega)\}_{x\in\mathbb{Z}^d}$ à valeurs indépendantes et identiquement distribuées.

Soit $1 \leq n \leq N$. Etant donné $m > 0$ et une énergie $E \in \mathbb{R}$, le cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}) \subset \mathbb{Z}^{nd}$ est dit (E,m)-non-singulier ((E,m)-NS) si la résolvante $(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u})} - E)^{-1}$ vérifie

$$\max_{\substack{\mathbf{v}\in\mathbb{Z}^{nd}\\ |\mathbf{v}-\mathbf{u}|=L_k}} \left|(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u})} - E)^{-1}(\mathbf{u},\mathbf{v})\right| \leq \mathrm{e}^{-m(1+L_k^{-1/8})^{N-n+1}L_k},$$

dans le cas contraire, on dit que le cube est (E,m)-singulier ((E,m)-S). Nous avons le Théorème suivant qui établit la validité des estimées d'analyse multi-échelles pour toutes les échelles :

Théorème 1.3. *Soit $n = 1,\dots,N$. Sous les hypothèses* (**I1**) *et* (**P1**), *pour tout $p > 6Nd$, il existe $m > 0$ et $E^* > E_0^{(N)}$ tel que pour toute paire de cubes séparable $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ et $\mathbf{C}^{(n)}_{L_k}(\mathbf{v})$*

$$\mathbb{P}\left\{\exists E \in I : \mathbf{C}^{(n)}_{L_k}(\mathbf{u}) \text{ et } \mathbf{C}^{(n)}_{L_k}(\mathbf{v}) \text{ soient } (E,m)\text{-}S\right\} \leq L_k^{-2p\,4^{N-n}}, \quad \text{pour tout } k \geq 0,$$

où $I = (-\infty, E^*]$.

1.2.4 Localisation d'Anderson à basses énergies

Ce résultat utilise les estimées de l'analyse multi-échelle et le schéma que nous proposons qui donne une décroissance exponentielle des fonctions propres en terme de norme-max, étend celui devéloppé par von Dreifus et Klein [vDK89] aux systèmes multi-particulaires.

Théorème 1.4. *Sous les hypothèses* (**I1**) *et* (**P1**)*, il existe* $E^* > E_0^{(N)}$ *tel que avec* $\mathbb{P}$*-probabilité* 1*,*

(i) *le spectre de* $\mathbf{H}^{(N)}(\omega)$ *dans* $[E_0^{(N)}, E^*]$ *est non-vide et purement ponctuel ;*

(ii) *les fonctions propres* $\mathbf{\Psi}_i(\mathbf{x}, \omega)$ *correspondant aux valeurs propres* $E_i(\omega) \in [E_0^{(N)}, E^*]$ *décroissent exponentiellement vite à l'infini : il existe* $m > 0$ *et* $C_i(\omega) > 0$ *tels que*
$$|\mathbf{\Psi}_i(\mathbf{x}, \omega)| \leq C_i(\omega)\mathrm{e}^{-m|\mathbf{x}|}. \tag{1.10}$$

1.2.5 Localisation dynamique à basses énergies

On obtient plus précisément la localisation dynamique en terme de la norme Hilbert-Schmidt. Cette version est plus fine que les autres formes de localisation dynamique précédemment établies [GB98, DS01] et a été initialement dévéloppée par Germinet et Klein [GK01] dans le contexte des opérateurs différentiels dans $\mathbb{R}^d$.

On note $\mathcal{B}_1$ l'ensemble des fonctions mesurables bornées $f : \mathbb{R} \to \mathbb{R}$ telles que $\|f\|_\infty \leq 1$.

Théorème 1.5. *Sous les hypothèses* (**I1**) *et* (**P1**)*, il existe* $E^* > E_0^{(N)}$ *et* $s^* > 0$ *tel que pour toute région bornée* $\mathbf{K} \subset \mathbb{Z}^{Nd}$ *et tout* $0 < s < s^*$ *:*
$$\mathbb{E}\left[\sup_{f \in \mathcal{B}_1} \left\| |\mathbf{X}|^{\frac{s}{2}} f(\mathbf{H}^{(N)}(\omega)) \mathbf{P}_I(\mathbf{H}^{(N)}(\omega)) \mathbf{1}_{\mathbf{K}} \right\|_{HS}^2\right] < \infty, \tag{1.11}$$
où $(|\mathbf{X}|\mathbf{\Psi})(\mathbf{x}) := |\mathbf{x}|\mathbf{\Psi}(\mathbf{x})$*,* $\mathbf{P}_I(\mathbf{H}^{(N)}(\omega))$ *est le projecteur spectral de* $\mathbf{H}^{(N)}(\omega)$ *sur l' intervalle* $I := [E_0^{(N)}, E^*]$*.*

Il est possible d'étendre les résultats ci-dessus aux potentiels aléatoires corrélés vérifiant la condition d'indépendance à grande distance (IAD). Cela impose de changer de notion de séparabilité et supposer les projections à distance suffisante. Un autre point important est l'estimée de grande déviation pour le processus $\{V(x, \omega)\}_{x \in \mathbb{Z}^d}$ qui n'est pas automatique si les variables aléatoires $V(x, \omega)$ sont corrélées.

Les résultats suivant qui concernent le régime des faibles interactions sont démontrés au Chapitre 4.

1.2.6 Analyse multi-échelle multi-particulaire à faible interaction

Dans l'étude du régime faiblement interactif, il est judicieux d'introduire un paramètre permettant de controler la force de l'interaction $\mathbf{U}$. Ainsi, on réecrit les Hamiltoniens n-particulaire $\mathbf{H}^{(n)}(\omega)$, $n = 1, \ldots, N$ de la façon suivante :

$$\mathbf{H}^{(n)}(\omega) = -\boldsymbol{\Delta} + \sum_{j=1}^{n} V(x_j, \omega) + h\mathbf{U}, \quad h \in \mathbb{R}.$$

L'approche est différente par rapport à celle proposée en Section 1.2.3. En effet c'est le démarrage de l'estimée de longueur initiale L_0 qui crée la différence. Dans le premier cas de figure l'on se sert de l'estimée de Combes-Thomas pour établir l'analyse multi-échelle, tandis que dans le régime faiblement interactif l'on se sert des résultats de localisation mono-particulaire en dimension 1. Le régime faiblement interactif requiert une légère modification la propriété de non-singularité. En effet ici on fait dépendre cette propriété par le paramètre h permettant de controler la force de l'interaction. Lorsque $h = 0$, le système multiparticulaire est supposé sans interaction. Le cube est donc dit (E, m, h)-non singulier par rapport à l'Hamiltonien $\mathbf{H}_h^{(n)}(\omega)$. La stratégie dans ce régime repose sur plusieurs étapes :

(1)

(1) A partir de l'estimée du système sans interaction on établit celle du système faiblement interactif par un argument perturbatif utilisant les identités de la résolvante.

(2) Enfin, on développe l'induction pour le système faiblement interactif à énergie fixe et on endéduit par la suite l'estimée à énergie variable.

On a que sous les hypothèses (**I2**) and (**P2**), avec $\mathbb{P}$-probabilité 1

$$\sigma(\mathbf{H}_h^{(N)}(\omega)) \subset [-N(4d + M) - |h|\|\mathbf{U}\|, N(4d + M) + |h|\|\mathbf{U}\|].$$

Dans le reste de la section on pose

$$I = [-1 - N(4d + M) - |h|\|\mathbf{U}\|, N(4d + M) + |h|\|\mathbf{U}\| + 1]$$

et on définit pour un certain $L > 0$ et des points $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$:

$$\mathrm{F}_{\mathbf{x},\mathbf{y}}(\mathrm{E}) = |(\mathbf{H}_{\mathbf{C}_{\mathrm{L}}^{(\mathrm{N})}} - \mathrm{E})^{-1}(\mathbf{x}, \mathbf{y})|, \quad \mathrm{F}_{\mathbf{x}}(\mathrm{E}) = \max_{\mathbf{y} \in \partial^- \mathbf{C}_{\mathrm{L}}^{(\mathrm{N})}(\mathbf{x})} \mathrm{F}_{\mathbf{x},\mathbf{y}}(\mathrm{E}).$$

le résultat est le suivant :

Théorème 1.6. *Soient $n = 1, \ldots, N$ et $\theta \in (0, 1/3)$. Sous les hypothèses* (**I2**) *et* (**P2**)*, il existe $h^* > 0$ tel que pour tout $|h| \leq h^*$ et pour toute paire de cubes séparables $\mathbf{C}_{L_k}^{(n)}(\mathbf{x})$, $\mathbf{C}_{L_k}^{(n)}(\mathbf{y})$ on a :*

$$\mathbb{P}\left\{\exists E \in I : \min\{\mathrm{F}_{\mathbf{x}}(\mathrm{E}),\ \mathrm{F}_{\mathbf{y}}(\mathrm{E})\} \geq 2\mathrm{L}_{\mathrm{k}}^{-\frac{\mathrm{p}(1+\theta)^{\mathrm{k}}}{5}}\right\} \leq C(|I|, N, d) L_k^{-\frac{p(+\theta)^k}{5} + (2N+1)d},$$

pour tout $k \geq 0$, où $I = [-1 - N(4d + M) - |h|\|\mathbf{U}\|, N(4d + M) + |h|\|\mathbf{U}\| + 1]$ et $p > 6Nd/(1 - 3\theta)$.

1.2.7 Localisation d'Anderson complète à faible interaction

Théorème 1.7. *Sous les hypothèses* (**I2**) *and* (**P2**), *il existe* $h^* > 0$ *tel que pour tout* $h \in (-h^*, h^*)$ *l' Hamiltonian* $\mathbf{H}_h^{(N)}$, *avec interaction d'amplitude* $|h|$, *admet la localisation d'Anderson complète,* i.e., *avec* $\mathbb{P}$*-probabilité* 1,

(i) *le spectre de* $\mathbf{H}^{(n)}$ *purement ponctuel,*

(ii) *les fonctions propres* $\mathbf{\Psi}_i(\mathbf{x}, \omega)$ *relatif aux valeurs propres* $E_i(\omega) \in I$ *décroissent rapidement à l'infini : pour chaque* $\mathbf{\Psi}_j$ *pour tout* $\mathbf{x} \in \mathbb{Z}^{Nd}$ *et des constantes* $a, c, C_i(\omega) > 0$
$$|\mathbf{\Psi}_i(\mathbf{x}, \omega)| \leq C_i(\omega) \mathrm{e}^{-a(\ln |\mathbf{x}|)^{1+c}}.$$

1.2.8 Localisation dynamique à faible interaction

On définit $\mathcal{B}_1$ l'ensemble des fonctions bornées mesurables $f : \mathbb{R} \to \mathbb{R}$ satisfaisant $\|f\|_\infty \leq 1$. On a

Théorème 1.8. *Sous les hypothèses* (**I2**) *et* (**P2**), *il existe* $h^* > 0$ *tel que pour tout* $h \in (-h^*, h^*)$ *et toute fonction borélienne* $f : \mathbb{R} \to \mathbb{R}$, *toute région bornée* $\mathbf{K} \subset \mathbb{Z}^{Nd}$*et tout* $s > 0$ *on a :*
$$\mathbb{E}\left[\sup_{f \in \mathcal{B}_1} \left\| |\mathbf{X}|^{\frac{s}{2}} f(\mathbf{H}^{(N)}(\omega)) \mathbf{P}_I(\mathbf{H}_h^{(N)}(\omega)) \mathbf{1}_{\mathbf{K}} \right\|_{HS}^2 \right] < \infty. \tag{1.12}$$

où $(|\mathbf{X}|\mathbf{\Psi})(\mathbf{x}) := |\mathbf{x}|\mathbf{\Psi}(\mathbf{x})$, $\mathbf{P}_I(\mathbf{H}^{(N)}(\omega))$ *est la projection spectrale de* $\mathbf{H}^{(N)}(\omega)$ *dans l'intervalle* I.

Chapter 2

Multi-scale analysis scheme

In this chapter, we present the main tools which are used in the multi-scale analysis in both the low energy and the weak interaction regimes. Only fairly general assumptions are needed: the interaction should be of short range and the distribution of the i.i.d. random potential at least log-Hölder continuous, see assumptions (**I1**) and(**P1**), or very singular see assumptions (**I2**) and (**P2**) from Section 1.2.1.

2.1 Basic notations and geometric resolvent equations

Given any finite subset $\mathbf{\Lambda} \subset \mathbb{Z}^{nd}$, we define its *internal boundary*

$$\partial^{-}\mathbf{\Lambda} = \left\{\mathbf{v} \in \mathbb{Z}^{nd} : \text{dist}\left(\mathbf{v}, \mathbb{Z}^{nd} \setminus \mathbf{\Lambda}\right) = 1\right\}, \tag{2.1}$$

its *external boundary*

$$\partial^{+}\mathbf{\Lambda} = \left\{\mathbf{v} \in \mathbb{Z}^{nd} \setminus \mathbf{\Lambda} : \text{dist}\left(\mathbf{v}, \mathbf{\Lambda}\right) = 1\right\}, \tag{2.2}$$

and its *boundary*

$$\partial\mathbf{\Lambda} = \left\{(\mathbf{v}, \mathbf{v}') \in \mathbb{Z}^{nd} \times \mathbb{Z}^{nd} \,\middle|\, |\mathbf{v} - \mathbf{v}'|_1 = 1 \text{ and either } \mathbf{v} \in \mathbf{\Lambda},\ \mathbf{v}' \notin \mathbf{\Lambda} \text{ or } \mathbf{v} \notin \mathbf{\Lambda},\ \mathbf{v}' \in \mathbf{\Lambda}\right\}. \tag{2.3}$$

The distance between two subsets $\mathbf{\Lambda}, \mathbf{\Lambda}' \subset \mathbb{Z}^{nd}$ is defined by

$$\text{dist}(\mathbf{\Lambda}, \mathbf{\Lambda}') = \min_{x \in \mathbf{\Lambda}, x' \in \mathbf{\Lambda}'} |x - x'|.$$

We will mostly work with the restriction $\mathbf{\Delta}_{\mathbf{\Lambda}}$ of the Laplacian $\mathbf{\Delta}$ to a subset $\mathbf{\Lambda} \subset \mathbb{Z}^{nd}$ with *simple boundary conditions*, i.e., its matrix elements $\mathbf{\Delta}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}) = \mathbf{\Delta}(\mathbf{x}, \mathbf{y})$ whenever $\mathbf{x}, \mathbf{y} \in \mathbf{\Lambda}$ and $\mathbf{\Delta}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}) = 0$ otherwise. For the initial step of the multiscale analysis at low energy, it will be convenient to consider first the

Neumann restriction $\mathbf{\Delta}_{\mathbf{\Lambda}}^{\mathrm{N}}$ whose quadratic form is given by

$$\left\langle \varphi, (-\mathbf{\Delta})_{\mathbf{\Lambda}}^{\mathrm{N}} \phi \right\rangle = \frac{1}{2} \sum_{\substack{\mathbf{x},\mathbf{y} \in \mathbf{\Lambda} \\ |\mathbf{x}-\mathbf{y}|_1 = 1}} \overline{(\varphi(\mathbf{x}) - \varphi(\mathbf{y}))} (\phi(\mathbf{x}) - \phi(\mathbf{y})) \tag{2.4}$$
$$\forall \varphi, \phi \in \ell^2(\mathbf{\Lambda}),$$

and then use the inequality $\mathbf{\Delta}_{\mathbf{\Lambda}}^{\mathrm{N}} \leq \mathbf{\Delta}_{\mathbf{\Lambda}}$. For more information on boundary conditions in the lattice case we refer to the book by Kirsch [Kir08]. We denote the spectrum of $\mathbf{H}_{\mathbf{\Lambda}}^{(n)}$ by $\sigma(\mathbf{H}_{\mathbf{\Lambda}}^{(n)})$ and its resolvent by

$$\mathbf{G}_{\mathbf{\Lambda}}(E) := \left(\mathbf{H}_{\mathbf{\Lambda}}^{(n)} - E \right)^{-1}, \quad E \in \mathbb{R} \setminus \sigma\left(\mathbf{H}_{\mathbf{\Lambda}} \right). \tag{2.5}$$

Its matrix elements $\mathbf{G}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}; E)$ are usually called the *Green functions* of the operator $\mathbf{H}_{\mathbf{\Lambda}}^{(n)}$. Let $\mathbf{\Lambda} \subset \mathbb{Z}^{nd}$. Define the boundary operator $\mathbf{\Gamma}_{\mathbf{\Lambda}}$ by

$$\mathbf{\Gamma}_{\mathbf{\Lambda}}(\mathbf{x}, \mathbf{y}) = \begin{cases} -1 & \text{if } (\mathbf{x}, \mathbf{y}) \in \partial \mathbf{\Lambda} \\ 0 & \text{otherwise.} \end{cases}$$

The Hamiltonian $\mathbf{H}^{(n)}$ can be written as follows:

$$\mathbf{H}^{(n)} = \mathbf{H}_{\mathbf{\Lambda}}^{(n)} \oplus \mathbf{H}_{\mathbf{\Lambda}^{\mathrm{c}}}^{(n)} + \mathbf{\Gamma}_{\mathbf{\Lambda}}, \tag{2.6}$$

where we identified $\ell^2(\mathbb{Z}^{nd})$ with $\ell^2(\mathbf{\Lambda}) \oplus \ell^2(\mathbf{\Lambda}^{\mathrm{c}})$. The matrix elements of $\mathbf{H}_{\mathbf{\Lambda}}^{(n)} \oplus \mathbf{H}_{\mathbf{\Lambda}^{\mathrm{c}}}^{(n)}$ are given by:

$$(\mathbf{H}_{\mathbf{\Lambda}}^{(n)} \oplus \mathbf{H}_{\mathbf{\Lambda}^{\mathrm{c}}}^{(n)})(\mathbf{x}, \mathbf{y}) = \begin{cases} \mathbf{H}_{\mathbf{\Lambda}}^{(n)}(\mathbf{x}, \mathbf{y}) & \text{if } \mathbf{x}, \mathbf{y} \in \mathbf{\Lambda}, \\ \mathbf{H}_{\mathbf{\Lambda}^{\mathrm{c}}}^{(n)}(\mathbf{x}, \mathbf{y}) & \text{if } \mathbf{x}, \mathbf{y} \in \mathbf{\Lambda}^{\mathrm{c}}, \\ 0 & \text{otherwise.} \end{cases}$$

Consider two subsets $\mathbf{\Lambda}_1 \subset \mathbf{\Lambda}_2 \subset \mathbb{Z}^{nd}$ and define the relative boundary

$$\partial_{\mathbf{\Lambda}_2} \mathbf{\Lambda}_1 = \left\{ (\mathbf{x}, \mathbf{y}) \,\middle|\, |\mathbf{x} - \mathbf{y}|_1 = 1 \text{ and } \mathbf{x} \in \mathbf{\Lambda}_1, \mathbf{y} \in \mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1 \text{ or } \mathbf{x} \in \mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1, \mathbf{y} \in \mathbf{\Lambda}_1 \right\}.$$

We then have the following important relation

$$\mathbf{H}_{\mathbf{\Lambda}_2}^{(n)} = \mathbf{H}_{\mathbf{\Lambda}_1}^{(n)} \oplus \mathbf{H}_{\mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1}^{(n)} + \mathbf{\Gamma}_{\mathbf{\Lambda}_1}^{\mathbf{\Lambda}_2}, \tag{2.7}$$

with

$$\mathbf{\Gamma}_{\mathbf{\Lambda}_1}^{\mathbf{\Lambda}_2}(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y} \text{ and } \mathbf{x} \in \mathbf{\Lambda}_1, \\ 0 & \text{if } \mathbf{x} = \mathbf{y} \text{ and } \mathbf{x} \in \mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1, \\ -1 & \text{if } (\mathbf{x}, \mathbf{y}) \in \partial_{\mathbf{\Lambda}_2} \mathbf{\Lambda}_1, \\ 0 & \text{otherwise.} \end{cases}$$

We summarize in Theorem 2.1 the geometric resolvent equations for Green functions and eigenfunctions which play an important role in the inductive step of the multiscale analysis and the decay of eigenfunctions respectively.

Theorem 2.1. (A) If $\mathbf{\Lambda}_1 \subset \mathbf{\Lambda}_2$ and $\mathbf{x} \in \mathbf{\Lambda}_1$, $\mathbf{y} \in \mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1$ and if $E \notin (\sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_1}) \cup \sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2}))$, then

$$\mathbf{G}^{(n)}_{\mathbf{\Lambda}_2}(\mathbf{x},\mathbf{y};E) = \sum_{\substack{(\mathbf{v},\mathbf{v}')\in\partial\mathbf{\Lambda}_1\\ \mathbf{v}\in\mathbf{\Lambda}_1,\mathbf{v}'\in\mathbf{\Lambda}_2}} \mathbf{G}^{(n)}_{\mathbf{\Lambda}_1}(\mathbf{x},\mathbf{v};E)\mathbf{G}^{(n)}_{\mathbf{\Lambda}_2}(\mathbf{v}',\mathbf{y};E). \tag{2.8}$$

(B) Let $\mathbf{\Lambda} \subset \mathbb{Z}^{nd}$ and $E \notin \sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}})$. If $\mathbf{\Psi}$ is a solution of the equation $\mathbf{H}^{(n)}\mathbf{\Psi} = E\mathbf{\Psi}$, then for any $\mathbf{x} \in \mathbf{\Lambda}$

$$\mathbf{\Psi}(\mathbf{x}) = \sum_{\substack{(\mathbf{y},\mathbf{z})\in\partial\mathbf{\Lambda}\\ \mathbf{y}\in\partial^-\mathbf{\Lambda},\mathbf{z}\in\partial^+\mathbf{\Lambda}}} \mathbf{G}^{(n)}_{\mathbf{\Lambda}}(\mathbf{x},\mathbf{y};E)\mathbf{\Psi}(\mathbf{z}). \tag{2.9}$$

Proof. (A) Let $E \notin (\sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_1}) \cup \sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2}) \cup \sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2\setminus\mathbf{\Lambda}_1}))$. Assertion (2) of Proposition A.1 combined with (2.7) imply by a calculation

$$(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2} - E)^{-1} = (\mathbf{H}^{(n)}_{\mathbf{\Lambda}_1} \oplus \mathbf{H}^{(n)}_{\mathbf{\Lambda}_2\setminus\mathbf{\Lambda}_1} - E)^{-1} - (\mathbf{H}^{(n)}_{\mathbf{\Lambda}_1} \oplus \mathbf{H}^{(n)}_{\mathbf{\Lambda}_2\setminus\mathbf{\Lambda}_1} - E)^{-1}\mathbf{\Gamma}^{\mathbf{\Lambda}_2}_{\mathbf{\Lambda}_1}(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2} - E)^{-1}. \tag{2.10}$$

one sees that $(\mathbf{H}^{(n)}_{\Lambda_1} \oplus \mathbf{H}^{(n)}_{\mathbf{\Lambda}_2\setminus\mathbf{\Lambda}_1} - E)^{-1} = (\mathbf{H}^{(n)}_{\mathbf{\Lambda}_1} - E)^{-1} \oplus (\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2\setminus\mathbf{\Lambda}_1} - E)^{-1}$. Hence if $\mathbf{x} \in \mathbf{\Lambda}_1$, $\mathbf{y} \in \mathbf{\Lambda}_2 \setminus \mathbf{\Lambda}_1$, we have $(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_1} \oplus \mathbf{H}^{(n)}_{\mathbf{\Lambda}_2\setminus\mathbf{\Lambda}_1} - E)^{-1}(\mathbf{x},\mathbf{y}) = 0$. Thus it follows from (2.10) that

$$\begin{aligned}
&(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2} - E)^{-1}(\mathbf{x},\mathbf{y})\\
&\quad = -\sum_{\mathbf{v},\mathbf{v}'\in\mathbf{\Lambda}_2} (\mathbf{H}^{(n)}_{\mathbf{\Lambda}_1} \oplus \mathbf{H}^{(n)}_{\mathbf{\Lambda}_2} - E)^{-1}(\mathbf{x},\mathbf{v})\mathbf{\Gamma}^{\mathbf{\Lambda}_2}_{\mathbf{\Lambda}_1}(\mathbf{v},\mathbf{v}')(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2} - E)^{-1}(\mathbf{v}',\mathbf{y})\\
&\quad = \sum_{\substack{(\mathbf{v},\mathbf{v}')\in\partial\mathbf{\Lambda}_1\\ \mathbf{v}\in\mathbf{\Lambda}_1,\mathbf{v}'\in\mathbf{\Lambda}_2}} \mathbf{G}^{(n)}_{\mathbf{\Lambda}_1}(\mathbf{x},\mathbf{v};E)\mathbf{G}^{(n)}_{\mathbf{\Lambda}_2}(\mathbf{v}',\mathbf{y};E).
\end{aligned}$$

This proves (2.8) for $E \notin \sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2\setminus\mathbf{\Lambda}_1})$. Now, observe that the latter equation does not depend on the domain $\Lambda_2 \setminus \Lambda_1$ and both sides of (2.8) exist and are analytic outside of $\sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_1}) \cup \sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2})$, so the formula holds true for all $E \notin \sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_1}) \cup \sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}_2})$.
(B) Let $\mathbf{\Psi}$ be a solution of the equation $\mathbf{H}^{(n)}\mathbf{\Psi} = E\mathbf{\Psi}$, then by (2.6):

$$0 = (\mathbf{H}^{(n)} - E)\mathbf{\Psi} = (\mathbf{H}^{(n)}_{\mathbf{\Lambda}} \oplus \mathbf{H}^{(n)}_{\mathbf{\Lambda}^c} + \mathbf{\Gamma}_{\mathbf{\Lambda}} - E)\mathbf{\Psi},$$

so that $(\mathbf{H}^{(n)}_{\mathbf{\Lambda}} \oplus \mathbf{H}^{(n)}_{\mathbf{\Lambda}^c} - E)\mathbf{\Psi} = -\mathbf{\Gamma}_{\mathbf{\Lambda}}\mathbf{\Psi}$ for $E \notin \sigma(\mathbf{H}^{(n)}_{\mathbf{\Lambda}})$. So for any $\mathbf{x} \in \mathbf{\Lambda}$,

$$\mathbf{\Psi}(\mathbf{x}) = -[(\mathbf{H}^{(n)}_{\mathbf{\Lambda}} - E)^{-1}\mathbf{\Gamma}_{\mathbf{\Lambda}}\mathbf{\Psi}](\mathbf{x}) = \sum_{(\mathbf{y},\mathbf{z})\in\partial\mathbf{\Lambda}} \mathbf{G}^{(n)}_{\mathbf{\Lambda}}(\mathbf{x},\mathbf{y};E)\mathbf{\Psi}(\mathbf{z}). \qquad \square$$

According to the general structure of the MSA, we will consider lattice *rectangles*, for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}^{nd}$ and given a sequence of lengths $\{L_i : i = 1, \dots, n\}$,

$$\mathbf{C}^{(n)}(\mathbf{u}) = \prod_{i=1}^{n} C^{(1)}_{L_i}(u_i), \tag{2.11}$$

where

$$C^{(1)}_{L_i}(u_i) = \left\{x \in \mathbb{Z}^d : |x - u_i| \le L_i\right\}.$$

By $\mathbf{C}^{(n)}_L(\mathbf{u})$ we denote the n-particle cube, i.e.,

$$\mathbf{C}^{(n)}_L(\mathbf{u}) = \left\{\mathbf{x} \in \mathbb{Z}^{nd} : |\mathbf{x} - \mathbf{u}| \le L\right\}.$$

Clearly, $|\mathbf{C}^{(n)}_L(\mathbf{u})| := \operatorname{card} \mathbf{C}^{(n)}_L(\mathbf{u}) = (2L+1)^{nd}$. We will often use the simpler estimate $|\mathbf{C}^{(n)}_L(\mathbf{u})| \le (3L)^{nd}$. For any $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}^{nd}$, we define its *projection* $\varPi\mathbf{u}$ by

$$\varPi\mathbf{u} := \{u_1, \dots, u_n\} \subset \mathbb{Z}^d, \quad \text{for } \mathbf{u} = (u_1, \dots, u_n).$$

More generally, for any nonempty subset $\mathcal{J} \subset \{1, \dots, n\}$ we define the $\mathcal{J}$-projection by

$$\varPi_{\mathcal{J}}\mathbf{u} := \{u_j, j \in \mathcal{J}\} \subset \mathbb{Z}^d.$$

(Note that for $\mathcal{J} = \{1, \dots, n\}$ we have $\varPi_{\mathcal{J}} = \varPi$.) Respectively, for an n-particle rectangle

$$\mathbf{C}^{(n)}(\mathbf{u}) = C_{L_1}(u_1) \times \cdots \times C_{L_n}(u_n) \subset \mathbb{Z}^{nd}$$

we can consider its $\mathcal{J}$-projection, for any $\mathcal{J} \subseteq \{1, \dots, n\}$:

$$\varPi_{\mathcal{J}}\mathbf{C}^{(n)}(\mathbf{u}) = \bigcup_{j\in\mathcal{J}} C_{L_j}(u_j) \subset \mathbb{Z}^d. \tag{2.12}$$

Again, for $\mathcal{J} = \{1, \dots, n\}$ we will write $\varPi\mathbf{C}^{(n)}(\mathbf{u})$ instead of $\varPi_{\mathcal{J}}\mathbf{C}^{(n)}(\mathbf{u})$.

2.2 Separability

The notion of separability is crucial for the proof of the Wegner estimates which are important to bound the probability of resonances in a good way. Separability was initially introduced in the framework of the multi-particle multi-scale analysis in [CS09b] in the discrete case and in [BCSS10] in the continuum.

Definition 2.2. Let $\mathbf{C}^{(n)}(\mathbf{x}) = \prod_{i=1,\dots,n} C^{(1)}_{L_i}(x_i)$ and $\mathbf{C}^{(n)}(\mathbf{y}) = \prod_{i=1,\dots,n} C^{(1)}_{L'_i}(y_i)$ be two rectangles.

(i) $\mathbf{C}^{(n)}(\mathbf{x})$ is $\mathcal{J}$-pre-separable from $\mathbf{C}^{(n)}(\mathbf{y})$ if there exists a nonempty subset $\mathcal{J} \subset \{1,\cdots,n\}$ such that

$$\left(\bigcup_{j\in\mathcal{J}} \Pi_j \mathbf{C}^{(n)}(\mathbf{x})\right) \cap \left(\bigcup_{j\notin\mathcal{J}} \Pi_j \mathbf{C}^{(n)}(\mathbf{x}) \cup \Pi \mathbf{C}^{(n)}(\mathbf{y})\right) = \emptyset.$$

(ii) A pair $(\mathbf{C}^{(n)}(\mathbf{x}), \mathbf{C}^{(n)}(\mathbf{y}))$ is pre-separable if one of the rectangle is $\mathcal{J}$-pre-separable from the other.

(iii) Set $L = \max\{L_i, L'_i : i = 1,\dots,n\}$. A pair $(\mathbf{C}^{(n)}(\mathbf{x}), \mathbf{C}^{(n)}(\mathbf{y}))$ is separable if and only if it is pre-separable and $|\mathbf{x}-\mathbf{y}| > 7NL$.

Now, we state some geometric facts for pairs of separable cubes.

Lemma 2.3. *Let $L > 1$ and $\mathbf{x} \in \mathbb{Z}^{nd}$.*

(A) *There exists a collection of n-particle cubes $\mathbf{C}^{(n)}_{2nL}(\mathbf{x}^{(\ell)})$ with $x^{(\ell)}_k \in \{x_1,\dots,x_n\}$ for all $k = 1,\cdots,n$, $\ell = 1,\dots,\kappa(n)$, $\kappa(n) \le n^n$ such that if a configuration $\mathbf{y}$ satisfies $|\mathbf{y}-\mathbf{x}| > 7NL$ and*

$$\mathbf{y} \notin \bigcup_{\ell=1}^{\kappa(n)} \mathbf{C}^{(n)}_{2nL}(\mathbf{x}^{(\ell)}),$$

then the cubes $\mathbf{C}^{(n)}_L(\mathbf{x})$ and $\mathbf{C}^{(n)}_L(\mathbf{y})$ are separable.

(B) *Let $\mathbf{C}^{(n)}_L(\mathbf{y}) \subset \mathbb{Z}^{nd}$ be an n-particle cube. Any cube $\mathbf{C}^{(n)}_L(\mathbf{x})$ with $|\mathbf{y}-\mathbf{x}| > \max_{1\le i,j\le n}|y_i - y_j| + 5NL$ is $\mathcal{J}$-separable from $\mathbf{C}^{(n)}_L(\mathbf{y})$ for some $\mathcal{J} \subset \{1,\dots,n\}$.*

Proof. We proceed as in [BCSS10]. (A) Let $L > 0$, $\emptyset \ne \mathcal{J} \subset \{1,\dots,n\}$ and $\mathbf{y} \in \mathbb{Z}^{nd}$. $\{y_j\}_{j\in\mathcal{J}}$ is called an L-cluster if the union

$$\bigcup_{j\in\mathcal{J}} C^{(1)}_L(y_j)$$

cannot be decomposed into two non-empty disjoint subsets. Next, given two configurations $\mathbf{x},\mathbf{y} \in \mathbb{Z}^{nd}$, we proceed as follows:

1. We decompose the vector $\mathbf{y}$ into maximal L-clusters $\Gamma_1,\dots,\Gamma_M$ (each of diameter $\le 2nL$) with $M \le n$.

2. Each position y_i corresponds to exactly one cluster Γ_j, $j = j(i) \in \{1, \dots, M\}$.
3. If there exists $j \in \{1, \dots, M\}$ such that $\Gamma_j \cap \Pi\mathbf{C}_L^{(n)}(\mathbf{x}) = \emptyset$, then cubes $\mathbf{C}_L^{(n)}(\mathbf{y})$ and $\mathbf{C}_L^{(n)}(\mathbf{x})$ are separable.
4. If (3) is wrong, then for all $k = 1, \dots, M$, $\Gamma_k \cap \Pi\mathbf{C}_L^{(n)}(\mathbf{x}) \neq \emptyset$. Thus for all $k = 1, \dots, M$, $\exists i = 1, \dots, n$ such that $\Gamma_k \cap \mathbf{C}_L^{(1)}(x_i) \neq \emptyset$. Now for any $j = 1, \dots, n$ there exists $k = 1, \dots, M$ such that $y_j \in \Gamma_k$. Therefore for such k, by hypothesis there exists $i = 1, \dots, n$ such that $\Gamma_k \cap C_L^{(1)}(x_i) \neq \emptyset$. Next let $z \in \Gamma_k \cap C_L^{(1)}(x_i)$ so that $|z - x_i| \leq L$. We have that

$$\begin{aligned}|y_j - x_i| &\leq |y_j - z| + |z - x_i| \\ &\leq 2nL - L + L = 2nL\end{aligned}$$

since $y_j, z \in \Gamma_k$. Notice that above we have the bound $|y_j - z| \leq 2nL - L$ instead of $2nL$ because y_j is a center of the L-cluster Γ_k. Hence for all $j = 1, \dots, n$ y_j must belong to one of the cubes $C_{2nL}^{(1)}(x_i)$ for the n positions $(y_1, \dots, y_n)$. Set $\kappa(n) = n^n$. For any choice of at most $\kappa(n)$ possibilities, $\mathbf{y} = (y_1, \dots, y_n)$ must belong to the Cartesian product of n cubes of size $2nL$ i.e. to an nd-dimensional cube of size $2nL$, the assertion then follows.

(B) Set $R(\mathbf{y}) = \max_{1 \leq i,j \leq n} |y_i - y_j| + 5NL$ and consider a cube $\mathbf{C}_L^{(n)}(\mathbf{x})$ with $|\mathbf{y} - \mathbf{x}| > R(\mathbf{y})$. Then there exists $i_0 \in \{1, \dots, n\}$ such that $|y_{i_0} - x_{i_0}| > R(\mathbf{y})$. Consider the maximal connected component $\Lambda_{\mathbf{x}} := \bigcup_{i \in \mathcal{J}} C_L^{(1)}(x_i)$ of the union $\bigcup_i C_L^{(1)}(x_i)$ containing x_{i_0}. Its diameter is bounded by $2nL$. We have

$$\operatorname{dist}(\Lambda_{\mathbf{x}}, \Pi\mathbf{C}_L^{(n)}(\mathbf{y})) = \min_{u,v} |u - v|,$$

now since

$$|x_{i_0} - y_{i_0}| \leq |x_{i_0} - u| + |u - v| + |v - y_{i_0}|,$$

then

$$\begin{aligned}\operatorname{dist}(\Lambda_{\mathbf{x}}, \Pi\mathbf{C}_L^{(n)}(\mathbf{y})) &= \min_{u,v} |u - v| \\ &\geq |x_{i_0} - y_{i_0}| - \operatorname{diam}(\Lambda_{\mathbf{x}}) - \max_{v, y_{i_0}} |v - y_{i_0}|.\end{aligned}$$

Recall that $\operatorname{diam}(\boldsymbol{\Lambda}_{\mathbf{x}}) \leq 2nL$ and

$$\max_{v, y_{i_0}} |v - y_{i_0}| \leq \max_{v} |v - y_j| + \max_{y_{i_0}} |y_j - y_{i_0}|,$$

for some $j = 1, \cdots n$ such that $v \in C_L^{(1)}(y_j)$. Finally we get

$$\mathrm{dist}(\mathbf{\Lambda_x}, \mathit{\Pi}\mathbf{C}_L^{(n)}(\mathbf{y})) > R(\mathbf{y}) - \mathrm{diam}(\Lambda_{\mathbf{x}}) - (2L + \mathrm{diam}(\mathit{\Pi}\mathbf{y})) > 0,$$

this implies that $\mathbf{C}_L^{(n)}(\mathbf{x})$ is $\mathcal{J}$-separable from $\mathbf{C}_L^{(n)}(\mathbf{y})$ with $\mathcal{J}$ the index subset appearing in the definition of $\Lambda_{\mathbf{x}}$. □

2.3 Wegner estimates

The Wegner estimates initially introduced by Wegner [Weg81] play a crucial role in the inductive step of the multiscale analysis. The proof which uses Stollmann's Lemma was originally developped by Stollmann for single-particle Anderson models [Sto01] and later adapted to the two-particle systems by Chulaevsky and Suhov [CS08]. Notice that Stollmann's Lemma was extended to correlated random potentials by Chulaevsky [Chu08]. Kirsch [Kir07], obtained optimal multiparticle Wegner estimates for absolutely continuous distributions with a bounded density.

Let J be a non-empty set of finite cardinality $|J| = p$. J is identified with $\{1, \ldots, p\}$. Let $\mathbb{R}^J \cong \mathbb{R}^p$, the Euclidean space with canonical basis $(e_1, \ldots, e_p)$. Consider

$$\mathbb{R}_+^J = \{q = (q_1, \ldots, q_p) \in \mathbb{R}^J : q_j \geq 0, j = 1, \ldots, p\}.$$

Definition 2.4. $\Phi : \mathbb{R} \to \mathbb{R}$ is called *diagonally monotone* if

(i) for all $r \in \mathbb{R}_+^J$ and $v \in \mathbb{R}^J$,

$$\Phi(v + r) \geq \Phi(v).$$

(ii) Moreover for $e = e_1 + \cdots + e_p \in \mathbb{R}^J$, for all $v \in \mathbb{R}^J$ and $t > 0$,

$$\Phi(v + te) - \Phi(v) \geq t.$$

Stollmann's Lemma was initially formulated for general product of measures [Sto00, Sto01]. Here since we are concerned with i.i.d. random potentials with a common probability distribution μ, we formulate it directly as in [CS08, Chu08] for product measures of the form $\mu^J = \mu \times \cdots \times \mu$ on $\mathbb{R}^J$.

Lemma 2.5 (Stollmann's lemma). *Let J be an index set of finite cardinality $|J| = p$, μ a probability measure on $\mathbb{R}$, $\mu^J = \mu \times \cdots \times \mu$ a product measure on $\mathbb{R}^J$. If the function $\Phi : \mathbb{R}^J \to \mathbb{R}$ is diagonally monotone, then for any open interval $I \subset \mathbb{R}$ we have:*

$$\mu^J \{q : \Phi(q) \in I\} \leq p \cdot s(F_V, |I|).$$

Proof. Let $I = (a, b)$, $b - a = \varepsilon > 0$. Set

$$A = \{q : \Phi(q) \le a\},$$

$$A_0^\varepsilon = A,\ A_j^\varepsilon = A_{j-1}^\varepsilon + [0, \varepsilon]e_j := \big\{q + te_j : q \in A_{j-1}^\varepsilon, t \in [0, \varepsilon]\big\}.$$

Clearly, A_j^ε is an increasing sequence in j. The diagonal monotonicity hypothesis of Φ implies

$$\{q : \Phi(q) < b\} \subset A_p^\varepsilon.$$

Indeed, if $\Phi(q) < b$ then for $r := q - \varepsilon \cdot e$,

$$\Phi(r) \le \Phi(r + \varepsilon e) - \varepsilon = \Phi(q) - \varepsilon \le b - \varepsilon \le a.$$

Thus $r \in \{\Phi \le a\} = A$. Hence,

$$q = r + \varepsilon \cdot e \in A_p^\varepsilon.$$

Now we have that

$$\begin{aligned}\{q : \Phi(q) \in I\} &= \{q : \Phi(q) \in (a, b)\}\\ &= \{q : \Phi(q) < b\} \setminus \{q : \Phi(q) \le a\}\\ &\subset A_p^\varepsilon \setminus A.\end{aligned}$$

Moreover

$$\begin{aligned}\mu^p\{q : \Phi(q) \in I\} &\le \mu^p\left(A_p^\varepsilon \setminus A\right)\\ &= \mu^p\left(\bigcup_{j=1}^p \left(A_j^\varepsilon \setminus A_{j-1}^\varepsilon\right)\right)\\ &\le \sum_{j=1}^p \mu^p\left(A_j^\varepsilon \setminus A_{j-1}^\varepsilon\right)\end{aligned}$$

For $q'_{\ne 1} = (q_2, \cdots, q_p) \in \mathbb{R}^{p-1}$, set

$$I_1(q'_{\ne 1}) = \{q_1 \in \mathbb{R} : (q_1, q'_{\ne 1}) \in A_1^\varepsilon \setminus A\}.$$

By definition, $(q_1, q'_{\ne 1}) \in A_1^\varepsilon \setminus A$ if there exist $(a_1, \ldots, a_p) \in A_1^\varepsilon \setminus A$ such that $q'_{\ne 1} = (a_2, \ldots, a_2)$ and $q_1 = a_1 + t$ for some $t \in]0, \varepsilon]$. Therefore $I_1(q'_{\ne 1})$ is an interval of length $\le \varepsilon$. Since μ^p is a product measure

$$\mu^p\left(A_1^\varepsilon \setminus A\right) = \int d\mu^{p-1}(q'_{\ne 1}) \int_{I_1} d\mu(q_1) \le s(F_V, \varepsilon). \tag{2.13}$$

Similarly, we obtain for $j = 2, \cdots, p$,

$$\mu^p(A_j^\varepsilon \setminus A_{j-1}^\varepsilon) \le s(F_V, \varepsilon).$$

Therefore

$$\mu^p \{q : \Phi(q) \in I\} \le \sum_{j=1}^{p} \mu^p(A_j^\varepsilon \setminus A_{j-1}^\varepsilon) \le p \cdot s(F_V, \varepsilon). \qquad \square$$

Definition 2.6. Let $n \ge 1$, $\beta = 1/2$, and $E \in \mathbb{R}$ be given. Consider a rectangle $\mathbf{C}^{(n)}(\mathbf{u}) = \prod_{i=1}^n C_{L_i}^{(1)}(u_i)$ and set $L = \min_{i=1,\ldots,n}\{L_i\}$. $\mathbf{C}^{(n)}(\mathbf{u})$ is called E-resonant (E-R) if

$$\operatorname{dist}\Big[E, \sigma\big(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}\big)\Big] < \mathrm{e}^{-L^\beta}. \tag{2.14}$$

Otherwise it is called E-non-resonant (E-NR).

Definition 2.7. Let $E \in \mathbb{R}$ be given. A cube $\mathbf{C}_L^{(n)}(\mathbf{v}) \subset \mathbb{Z}^{nd}$ of size $L \ge 2$ is called E-completely non-resonant (E-CNR) if it does not contain any E-R cube of size $\ge L^{2/3}$. In particular, $\mathbf{C}_L(\mathbf{v})$ is itself E-NR.

Theorem 2.8. *Given two pre-separable rectangles* $\mathbf{C}^{(n)}(\mathbf{u}) = \prod_{i=1}^n C_{L_i}^{(1)}(u_i)$, $\mathbf{C}^{(n)}(\mathbf{u}') = \prod_{i=1}^n C_{L_i'}^{(1)}(u_i')$, *we have:*

(A) *for any* $\varepsilon > 0$ *and* $E \in \mathbb{R}$

$$\mathbb{P}\left\{\operatorname{dist}\left[\sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}), E\right] < \varepsilon\right\} \le |\mathbf{C}^{(n)}(\mathbf{u})| \min_{i=1,\ldots,n} |\Pi_i \mathbf{C}^{(n)}(\mathbf{u})| s(F_V, 2\varepsilon),$$

(B) *for any* $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}&\left\{\operatorname{dist}\left[\sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}), \sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u}')})\right] < \varepsilon\right\}\\ &\le |\mathbf{C}^{(n)}(\mathbf{u}')| \cdot |\mathbf{C}^{(n)}(\mathbf{u})| \cdot \max_{i=1,\ldots,n} \max_{\mathbf{u},\mathbf{u}'}\{|\Pi_i \mathbf{C}^{(n)}(\mathbf{u})|, |\Pi_i \mathbf{C}^{(n)}(\mathbf{u}')|\} \cdot s(F_V, 2\varepsilon).\end{aligned}$$

Proof. (A) Without loss of generality, assume that $L_1 = \min_{i=1,\ldots,n}\{L_i\}$ and set $J = \Pi_1 \mathbf{C}^{(n)}(\mathbf{u})$, $p = |J|$. Assume that $J = \{y_1, \ldots, y_p\}$.

Let $\{\lambda^{(k)}_{\mathbf{C}^{(n)}(\mathbf{u})} : k = 1, \ldots, |\mathbf{C}^{(n)}(\mathbf{u})|\}$ be the eigenvalues of $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega)$. For any subset $\Lambda \subset \mathbb{Z}^d$, denote by $\mathbb{P}_\Lambda$ and $\mathbb{E}_\Lambda$, the probability and the expectation with respect to random variables $\{\omega_x : x \in \Lambda\}$. The random Hamiltonian $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}$

write as

$$\begin{aligned}\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega) &= -\mathbf{\Delta} + \mathbf{U}(\mathbf{x}) + \sum_{i=1}^{n} \omega_{x_i}\\ &= -\mathbf{\Delta} + \mathbf{U}(\mathbf{x}) + \sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{x_i,y_j}\omega_{y_j}\\ &+ \sum_{i=1}^{n} \sum_{y\in \Pi\mathbf{C}^{(n)}(\mathbf{u})\setminus J} \delta_{x_i,y}\omega_y.\end{aligned}$$

Now, we have

$$\begin{aligned}&\mathbb{P}\left\{\omega : \operatorname{dist}(E, \sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega))) \le \varepsilon\right\}\\ &\le \sum_{k=1}^{|\mathbf{C}^{(n)}(\mathbf{u})|} \mathbb{P}\left\{|E - \lambda^{(k)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega)| \le \varepsilon\right\}\\ &= \sum_{k=1}^{|\mathbf{C}^{(n)}(\mathbf{u})|} \cdot\mathbb{P}_{\Pi\mathbf{C}^{(n)}(\mathbf{u})\setminus)J} \otimes \mathbb{P}_J\{|E - \lambda^{(k)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega)| \le \varepsilon\}\\ &= \sum_{k=1}^{|\mathbf{C}^{(n)}(\mathbf{u})|} \mathbb{E}_{\Pi\mathbf{C}^{(n)}(\mathbf{u})\setminus J}[\mu^J((\omega_x)_{x\in J} : |E - \lambda^{(k)}(\omega)| \le \varepsilon)].\end{aligned}$$

Consider the function $\Phi(\omega) := \lambda^{(k)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega)$. We show that Φ is diagonally monotone:

$$\begin{aligned}\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega + t\cdot(1,\dots,1)) &= -\mathbf{\Delta} + \mathbf{U}(\mathbf{x}) + \sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{x_i,y_j}\omega_{y_j}\\ &+ t\sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{x_i,y_j}\\ &\ge \mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega) + t\cdot Id,\end{aligned}$$

since $x_1 \in J$ so that $\sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{x_i,y_j} \ge 1$. Therefore, by the min-max principle (cf. Theorem A.5) we have:

$$\lambda^{(k)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega + t\cdot(1,\dots,1) \ge \lambda^{(k)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega) + t.$$

We then apply the Stollmann's Lemma, i.e., Lemma 2.5 and obtain,

$$\mu^J(\omega = (\omega_x)_{x\in J} : |\lambda^{(k)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega) - E| \le \varepsilon) \le |J|\cdot s(\mu,\varepsilon).$$

Finally,

$$\mathbb{P}\left\{\operatorname{dist}(E, \sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega))) \leq \varepsilon\right\} \leq |\mathbf{C}^{(n)}(\mathbf{u})| \cdot |\Pi_1 \mathbf{C}^{(n)}(\mathbf{u})| \cdot s(F_V, 2\varepsilon),$$

which proves (A).

(B) Without loss of generality, we can assume that $\mathbf{C}^{(n)}(\mathbf{u})$ is pre-separable from $\mathbf{C}^{(n)}(\mathbf{u}')$:

$$\exists\, \mathcal{J} \subset \{1, \ldots, n\} : \quad \Pi_{\mathcal{J}} \mathbf{C}^{(n)}(\mathbf{u}) \cap \left(\Pi_{\mathcal{J}^c} \mathbf{C}^{(n)}(\mathbf{u}) \cup \Pi \mathbf{C}^{(n)}(\mathbf{u}')\right) = \emptyset.$$

(Otherwise, we exchange the roles of $\mathbf{C}^{(n)}(\mathbf{u})$ and $\mathbf{C}^{(n)}(\mathbf{u}')$.) Let

$$\{\lambda^k : k = 1, \ldots, |\mathbf{C}^{(n)}(\mathbf{u})|\}, \quad \{\lambda^{k'} : k' = 1, \cdots, |\mathbf{C}^{(n)}(\mathbf{u}')|\},$$

be the eigenvalues of $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}$ and $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u}')}$ respectively. For any $i \in \mathcal{J}$, we proceed as in (A) and set $J = \Pi_i \mathbf{C}^{(n)}(\mathbf{u})$. We also assume as in (A) that $J = \{y_1, \ldots, y_p\}$. Therefore, we have:

$$\begin{aligned}
&\mathbb{P}\left\{\operatorname{dist}(\sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}), \sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u}')})) \leq \varepsilon\right\}\\
&= \mathbb{P}_{\Pi \mathbf{C}^{(n)}(\mathbf{u}) \cup \Pi \mathbf{C}^{(n)}(\mathbf{u}') \setminus J} \otimes \mathbb{P}_J \{\operatorname{dist}(\sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}), \sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u}')})) \leq \varepsilon\}\\
&\leq \sum_{i=1}^{|\mathbf{C}^{(n)}(\mathbf{u})|} \cdot \sum_{j=1}^{|\mathbf{C}^{(n)}(\mathbf{u}')|} \mathbb{E}_{\Pi \mathbf{C}^{(n)}(\mathbf{u}) \cup \varpi \mathbf{C}^{(n)}(\mathbf{u}') \setminus J} [\mu^J ((\omega_x)_{x \in J} : |\lambda^{(i)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega) - \lambda^{(j)}_{\mathbf{C}^{(n)}(\mathbf{u}')}(\omega)| \leq \varepsilon)].
\end{aligned} \tag{2.15}$$

As in case (A) we have that the function $\Phi(\omega) := \lambda^{(i)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega)$ is diagonally monotone. Therefore the Stollmann's Lemma implies that

$$\mu^J\{\omega = (\omega_x)_{x \in J} : |\lambda^{(i)}_{\mathbf{C}^{(n)}(\mathbf{u})}(\omega) - \lambda^{(j)}_{\mathbf{C}^{(n)}(\mathbf{u}')}(\omega)| \leq \varepsilon\} \leq |J| \cdot s(F_V, 2\varepsilon),$$

Finally,

$$\begin{aligned}
&\mathbb{P}\left\{\operatorname{dist}(\sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u})}), \sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}(\mathbf{u}')})) \leq \varepsilon\right\}\\
&\quad \leq |\mathbf{C}^{(n)}(\mathbf{u}')| \cdot |\mathbf{C}^{(n)}(\mathbf{u})| \cdot \max_{i=1,\ldots,n} \max_{\mathbf{u},\mathbf{u}'} \{|\Pi_i \mathbf{C}^{(n)}(\mathbf{u})|, |\Pi_i \mathbf{C}^{(n)}(\mathbf{u}')|\} \cdot s(F_V, 2\varepsilon). \square
\end{aligned}$$

Corollary 2.9. *Let $\alpha = 3/2$ and $p > 6Nd$. Assume that the random potential satisfies assumption* (**P1**) *with $A > 4^N \alpha p + 6\alpha N d$, then*

(A) *for any $E \in \mathbb{R}$*

$$\mathbb{P}\left\{\mathbf{C}^{(n)}_L(\mathbf{x}) \textit{ is not } E\textit{-CNR}\right\} \leq L^{-4^N p}, \tag{2.16}$$

(B)

$$\mathbb{P}\left\{\exists E\in\mathbb{R} : \textit{neither } \mathbf{C}^{(n)}_L(\mathbf{x}) \textit{ nor } \mathbf{C}^{(n)}_L(\mathbf{y}) \textit{ is } E\textit{-CNR}\right\} \le L^{-4^N p}. \quad (2.17)$$

Proof. (A), we have

$$\begin{aligned}
&\mathbb{P}\left\{\mathbf{C}^{(n)}_L(\mathbf{x}) \text{ is not } E\text{-CNR}\right\}\\
&\quad= \mathbb{P}\left\{\exists \mathbf{y}\in\mathbf{C}^{(n)}_L(\mathbf{x}),\ \exists\ell : L^{1/\alpha}\le\ell\le L, \mathbf{C}^{(n)}_\ell(\mathbf{y}) \text{ is } E\text{-R}\right\}\\
&\quad\le |\mathbf{C}^{(n)}_L(\mathbf{x})|(L+1)\times\mathbb{P}\left\{\mathbf{C}^{(n)}_\ell(\mathbf{y}) \text{ is } E\text{-R}\right\}\\
&\quad\le (2L+1)^{nd}(L+1)\times\mathbb{P}\left\{\operatorname{dist}[E,\sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_\ell(\mathbf{y})})] < \mathrm{e}^{-\ell^\beta}\right\}\\
&\quad\le (3L)^{nd+1}\times|\mathbf{C}^{(n)}_\ell(\mathbf{y})|\times|C^{(1)}_\ell(y_1)|\times s(\mu, 2\mathrm{e}^{-\ell^\beta}) && (2.18)\\
&\quad\le Const\cdot L^{3nd+1}L^{-A/\alpha} < L^{-4^N p}, && (2.19)
\end{aligned}$$

where we used Theorem 2.8 (A) in equation (2.18) and assumption (**P1**) on log-Hölder continuity of the marginal distribution μ in equation (2.19), provided L is large enough.

(B) First, note that for any given pair of cubes $\mathbf{C}^{(n)}_{\ell_1}(\mathbf{u}^1)$ and $\mathbf{C}^{(n)}_{\ell_2}(\mathbf{u}^2)$ with $L^{1/\alpha}\le \ell_1,\ell_2\le L$, one has

$$\begin{aligned}
&\mathbb{P}\left\{\exists E\in\mathbb{R}\text{: } \mathbf{C}^{(n)}_{\ell_1}(\mathbf{u}^1) \text{ and } \mathbf{C}^{(n)}_{\ell_2}(\mathbf{u}^2) \text{ are } E\text{-R}\right\}\\
&\quad= \mathbb{P}\left\{\exists E\in\mathbb{R} : \operatorname{dist}[E,\sigma(\mathbf{H}^{(n)}_{\mathbf{C}_{\ell_j}(\mathbf{u}^j)})] < e^{-\ell_j^\beta}, j=1,2\right\}\\
&\quad\le \mathbb{P}\left\{\operatorname{dist}[\sigma(\mathbf{H}^{(n)}_{\mathbf{C}_{\ell_1}(\mathbf{u}^1)}),\sigma(\mathbf{H}^{(n)}_{\mathbf{C}_{\ell_2}(\mathbf{u}^2)})] < 2\mathrm{e}^{-L^{\beta/\alpha}}\right\}\\
&\quad\le (3L)^{(2n+1)d}\cdot Const\, L^{-A/\alpha},
\end{aligned}$$

by Theorem 2.8 and assumption (**P1**). Counting the number of possible centers $\mathbf{u}^1,\mathbf{u}^2$ in the respective cubes of radius L and the number of possible values ℓ_1,ℓ_2, we conclude that

$$\begin{aligned}
&\mathbb{P}\left\{\exists E\in\mathbb{R} : \text{neither } \mathbf{C}^{(n)}_L(\mathbf{x}) \text{ nor } \mathbf{C}^{(n)}_L(\mathbf{y}) \text{ is } E\text{-CNR}\right\}\\
&\quad= \mathbb{P}\{\ \exists E\in\mathbb{R}, \exists\mathbf{u}^1\in\mathbf{C}^{(n)}_L(\mathbf{x}), \exists\mathbf{u}^2\in\mathbf{C}^{(n)}_L(\mathbf{y}), \exists\ell_1,\ell_2 \text{ with } L^{1/\alpha}\le\ell_1,\ell_2\le L :\\
&\qquad\qquad \mathbf{C}^{(n)}_{\ell_1}(\mathbf{u}^1) \text{ and } \mathbf{C}^{(n)}_{\ell_2}(\mathbf{u}^2) \text{ are } E\text{-R}\}\\
&\quad\le (3L)^{2nd}\cdot L^2\cdot(3L)^{(2n+1)d}\cdot Const\, L^{-A/\alpha}\le L^{-4^N p},
\end{aligned}$$

since $A > 4^N p\alpha + 9Nd$. □

Corollary 2.10. *Assume that the random potential satisfies assumption* (**P2**)*, then*

(A) *for any* $E \in \mathbb{R}$

$$\mathbb{P}\left\{\mathbf{C}_L^{(n)}(\mathbf{x}) \text{ is not } E\text{-CNR }\right\} \leq \mathrm{e}^{-\tau_1 L^{1/2}}, \tag{2.20}$$

for some $\tau_1 \in (0,1)$.

(B)

$$\mathbb{P}\left\{\exists E \in \mathbb{R} : \text{neither } \mathbf{C}_L^{(n)}(\mathbf{x}) \text{ nor } \mathbf{C}_L^{(n)}(\mathbf{y}) \text{ is } E\text{-CNR}\right\} \leq \mathrm{e}^{-\tau_2 L^{1/2}}, \tag{2.21}$$

for some $\tau_2 \in (0,1)$.

Proof. The proof follows the same lines as that of Corollary 2.9 but the RHS bounds are sub-exponential decaying since used the Hölder continuity of the probability distribution instead of the log-Hölder continuity as in Corollary 2.9. □

2.4 Combes–Thomas estimate

The Combes–Thomas estimate proposed here is borrowed from [Kir08] and for the sake of completeness, we also give the proof. We begin with a general result on operator theory which is used in the proof (cf. [Wei80]).

Lemma 2.11. *Let* $\Lambda \subset \mathbb{Z}^d$ *and* A *be an operator on* $\ell^2(\Lambda)$ *with matrix elements* $A(u,v)$. *Then* A *is bounded if*

$$a_1 = \sup_{u\in\Lambda} \sum_{v\in\Lambda} |A(u,v)| < \infty$$
$$a_2 = \sup_{v\in\Lambda} \sum_{u\in\Lambda} |A(u,v)| < \infty.$$

Moreover we have

$$\|A\| \leq a_1^{1/2} a_2^{1/2}.$$

Proof. Let us set

$$B(u,v) = |A(u,v)|^{1/2}, \quad C(u,v) = |A(u,v)|^{1/2}\mathrm{sgn}A(u,v)$$

with sgn $A(u,v) = 0$ if $A(u,v) = 0$ and sgn $A(u,v) = \frac{A(u,v)}{|A(u,v)|}$ if $A(u,v) \neq 0$. Then we have

$$A(u,v) = B(u,v)C(u,v).$$

and

$$\sum_{v\in\Lambda} |B(u,v)|^2 \le a_1 < \infty$$
$$\sum_{u\in\Lambda} |C(u,v)|^2 \le a_2 < \infty.$$

Now

$$\begin{aligned}
\|Af\|^2 = \sum_u \left| \sum_v A(u,v) \langle \delta_v, f\rangle \right|^2 &= \sum_u \left| \sum_v B(u,v)C(u,v) \langle \delta_v, f\rangle \right|^2 \\
&\le \sum_u \left[\sum_v |B(u,v)|^2 \sum_v |C(u,v) \langle \delta_v, f\rangle |^2 \right] \\
&\le a_1 \sum_{u,v} |C(u,v)|^2 | \langle \delta_v, f\rangle |^2 \\
&\le a_1 a_2 \sum_v | \langle \delta_v, f\rangle |^2 \\
&= a_1 a_2 \|f\|^2
\end{aligned}$$

so $\|Af\| \le a_1^{1/2} a_2^{1/2} \|f\|$ for all $f \in \ell^2(\Lambda)$.

□

Theorem 2.12. *Consider a lattice Schrödinger operator*

$$H_\Lambda = -\Delta_\Lambda + W(x)$$

acting in $\ell^2(\Lambda)$, $\Lambda \subset \mathbb{Z}^\nu$ is a finite subset, $\nu \ge 1$, with an arbitrary[1] *potential $W\colon \Lambda \to \mathbb{R}$. Suppose that $E \in \mathbb{R}$ is such that* $\operatorname{dist}(E, \sigma(H)) \ge \eta$ *with $\eta \in (0,1]$. Then*

$$\forall\, x, y \in \Lambda \qquad \left| (H-E)^{-1}(x,y) \right| \le 2\eta^{-1} \mathrm{e}^{-\frac{\eta}{12\nu}|x-y|}. \tag{2.22}$$

Proof. Let $\mu > 0$ to be specified later. Fix $x_0 \in \Lambda$, define the operator of multiplication $F = F_{x_0}$ on $\ell^2(\Lambda)$ by

$$Fu(x) = F_{x_0}u(x) = e^{\mu|x_0-x|_1} u(x).$$

Then for any operator A:

$$\left(F_{x_0}^{-1} A F_{x_0}\right)(x,y) = e^{-\mu|x_0-x|_1} A(x,y) e^{\mu|x_0-y|_1}. \tag{2.23}$$

1. This includes the cases of single- and multi-particle operators, that differ only by their potentials.

Setting $F = F_x$

$$|(H-E)^{-1}(x,y)| = \mathrm{e}^{-\mu|x-y|_1}|F^{-1}(H-E)^{-1}F(x,y)|$$
$$= \mathrm{e}^{-\mu|x-y|_1}|(F^{-1}HF-E)^{-1}(x,y)|$$
$$\leq \mathrm{e}^{-\mu|x-y|_1}\|(F^{-1}HF-E)^{-1}\|. \tag{2.24}$$

Therefore using the resolvent equation (Proposition A.1), we have

$$(F^{-1}HF-E)^{-1} = (H-E)^{-1} - (F^{-1}HF-E)^{-1}(F^{-1}HF-H)(H-E)^{-1}$$

yielding

$$(F^{-1}HF-E)^{-1}(I+(F^{-1}HF-H)(H-E)^{-1}) = (H-E)^{-1}.$$

If $\|(F^{-1}HF-H)(H-E)^{-1}\| < 1$, then the operator $(I+(F^{-1}HF-H)(H-E)^{-1})$ is invertible and obtain

$$(F^{-1}HF-E)^{-1} = (H-E)^{-1}(I+(F^{-1}HF-H)(H-E)^{-1})^{-1}. \tag{2.25}$$

We now focus on the operator $F^{-1}HF-H$. Equation (2.23) implies

$$\sum_{v\in\Lambda}|(F_x^{-1}HF_x-H)(u,v)| \leq \sum_{v:|u-v|_1=1}|\mathrm{e}^{-\mu|x-u|_1}\mathrm{e}^{\mu|x-v|_1}-1|$$
$$\leq 2\nu\mu\mathrm{e}^{\mu}. \tag{2.26}$$

For the last inequality (2.26) we used that for $|u-v|_1 \leq 1$:

$$||x-u|_1 - |x-v|_1| \leq |u-v|_1 \leq 1.$$

Further, for $|a| \leq 1$ and $\mu > 0$,

$$|\mathrm{e}^{\mu a}-1| \leq |\mathrm{e}^{\mu}-1| = \mathrm{e}^{\mu}-1$$
$$\leq \int_0^1 \mu\mathrm{e}^{\mu t}dt \leq \mu\mathrm{e}^{\mu}$$

An analogous estimate to (2.26) for u is obtained by exchanging the roles of u and v. Hence Lemma 2.11 implies

$$\|F^{-1}HF-H\| \leq 2\nu\mu e^{\mu}.$$

Set $\mu = \frac{\eta}{12\nu}$. Since $\eta \leq 1$:

$$\|(F^{-1}HF-H)(H-E)^{-1}\| \leq \|(F^{-1}HF-H)\|\|(H-E)^{-1}\|$$
$$\leq 2\nu\mu\mathrm{e}^{\mu}\frac{1}{\eta}$$
$$= 2\nu\frac{\eta}{12\nu}\mathrm{e}^{\frac{\eta}{12\nu}}\frac{1}{\eta} \tag{2.27}$$
$$\leq \frac{1}{2}.$$

Above we used that $e^{\frac{\eta}{12\nu}} \le e \le 3$ since $\eta \le 1$. Therefore the operator $I + (F^{-1}HF - H)(H-E)^{-1}$ is invertible and using the Neumann series we conclude that

$$\|(I + (F^{-1}HF - H)(H-E)^{-1})^{-1}\| \le 2. \tag{2.28}$$

Equation (2.25) leads to

$$\begin{aligned}\|(F^{-1}HF - E)^{-1}\| &= \|(H-E)^{-1}(I + (F^{-1}HF - H)(H-E)^{-1})^{-1}\| \\ &\le \frac{2}{\eta}\end{aligned}$$

and by (2.24)

$$\begin{aligned}|(H-E)^{-1}(x,y)| &\le e^{-\mu|x-y|_1}\|(F^{-1}HF - E)^{-1}\| \\ &\le \frac{2}{\eta} e^{-\frac{\eta}{12\nu}|x-y|_1}.\end{aligned}$$

□

2.5 Fully and partially interactive cubes

Definition 2.13 (fully/partially interactive). An n-particle cube $\mathbf{C}_L^{(n)}(\mathbf{u}) \subset \mathbb{Z}^{nd}$ is called fully interactive (FI) if

$$\operatorname{diam} \Pi\mathbf{u} \equiv \max_{i \ne j} |u_i - u_j| \le n(2L + r_0), \tag{2.29}$$

and partially interactive (PI) otherwise.

Partially interactive and fully interactive cubes are used in the framework of the inductive step of the multiscale analysis which will be done in the nexts two Chapters. We clarify the notion of PI cubes in the following simple statement.

Lemma 2.14. *If a cube $\mathbf{C}_L^{(n)}(\mathbf{u})$ is PI, then there exists a subset $\mathcal{J} \subset \{1, \dots, n\}$ with $1 \le \operatorname{card} \mathcal{J} \le n-1$ such that*

$$\operatorname{dist}\left(\Pi_{\mathcal{J}}\mathbf{C}_L^{(n)}(\mathbf{u}), \Pi_{\mathcal{J}^c}\mathbf{C}_L^{(n)}(\mathbf{u})\right) > r_0.$$

Proof. It is convenient to use the canonical injection $\mathbb{Z}^d \hookrightarrow \mathbb{R}^d$; then the notion of connectedness in $\mathbb{R}^d$ induces its analog for lattice cubes. Set $R := 2L + r_0$ and assume that $\operatorname{diam} \Pi\mathbf{u} = \max_{i,j} |u_i - u_j| > nR$. If the union of cubes $\mathbf{C}_{R/2}^{(n)}(u_i)$, $1 \le i \le n$, were not decomposable into two (or more) disjoint groups, then it would be connected, hence its diameter would be bounded by $n(2(R/2)) = nR$, hence $\operatorname{diam} \Pi\mathbf{u} \le nR$ which contradicts the hypothesis. Therefore, there exists an index subset $\mathcal{J} \subset \{1, \dots, n\}$ such that $|u_{j_1} - u_{j_2}| > 2(R/2)$ for all $j_1 \in \mathcal{J}$, $j_2 \in \mathcal{J}^c$, this implies that

$$\begin{aligned}
\operatorname{dist}\left(\Pi_{\mathcal{J}}\mathbf{C}_L^{(n)}(\mathbf{u}), \Pi_{\mathcal{J}^c}\mathbf{C}_L^{(n)}(\mathbf{u})\right) &= \min_{j_1\in\mathcal{J}, j_2\in\mathcal{J}^c} \operatorname{dist}\left(C_L^{(1)}(u_{j_1}), C_L^{(1)}(u_{j_2})\right)\\
&\geq \min_{j_1\in\mathcal{J}, j_2\in\mathcal{J}^c} |u_{j_1} - u_{j_2}| - 2L > r_0.
\end{aligned}$$

□

If $\mathbf{C}_L^{(n)}(\mathbf{u})$ is a PI cube by the above Lemma, we can write it as

$$\mathbf{C}_L^{(n)}(\mathbf{u}) = \mathbf{C}_L^{(n')}(\mathbf{u}') \times \mathbf{C}_L^{(n'')}(\mathbf{u}''), \tag{2.30}$$

with

$$\operatorname{dist}\left(\Pi\mathbf{C}_L^{(n')}(\mathbf{u}'), \Pi\mathbf{C}_L^{(n'')}(\mathbf{u}'')\right) > r_0, \tag{2.31}$$

where $\mathbf{u}' = \mathbf{u}_{\mathcal{J}} = (u_j : j \in \mathcal{J})$, $\mathbf{u}'' = \mathbf{u}_{\mathcal{J}^c} = (u_j : j \in \mathcal{J}^c)$, $n' = \operatorname{card}\mathcal{J}$ and $n'' = \operatorname{card}\mathcal{J}^c$. Throughout, when we decompose the cube $\mathbf{C}_L^{(n)}(\mathbf{u})$ as (2.30), we will implicitly assume that the decomposition (2.31) holds.

The main outcome of the previous Lemma is that on a PI cube $\mathbf{C}_L^{(n)}(\mathbf{u})$ the interparticle interaction can be decompose as follows, $\mathbf{U}(\mathbf{u}) = \mathbf{U}(\mathbf{u}') + \mathbf{U}(\mathbf{u}'')$. This allows the induction on the number particles using information on n'-particle subsystems for any $n' < n$. Now we turn to geometrical properties of FI cubes.

Lemma 2.15. *Let $n \geq 2$, $L > 2r_0$ and consider two n-particle fully interactive cubes $\mathbf{C}_L^{(n)}(\mathbf{u})$ and $\mathbf{C}_L^{(n)}(\mathbf{v})$ with $|\mathbf{x} - \mathbf{y}| > 7\,nL$. Then*

$$\Pi\mathbf{C}_L^{(n)}(\mathbf{u}) \cap \Pi\mathbf{C}_L^{(n)}(\mathbf{v})) = \varnothing.$$

Proof. If for some $R > 0$,

$$R < |\mathbf{x} - \mathbf{y}| = \max_{1\leq j\leq n} |x_j - y_j|,$$

then there exists $1 \leq j_0 \leq n$ such that $|x_{j_0} - y_{j_0}| > R$. Since both cubes are fully interactive, we can use (2.29) for the centers $\mathbf{x}$, $\mathbf{y}$ and write:

$$\begin{aligned}
|x_{j_0} - x_i| &\leq \operatorname{diam}\Pi\mathbf{x} \leq n(2L + r_0),\\
|y_{j_0} - y_j| &\leq \operatorname{diam}\Pi\mathbf{y} \leq n(2L + r_0).
\end{aligned}$$

By triangle inequality, for any $1 \leq i, j \leq n$ and $R > 7\,nL > 6\,nL + 2\,nr_0$, we have

$$\begin{aligned}
|x_i - y_j| &\geq |x_{j_0} - y_{j_0}| - |x_{j_0} - x_i| - |y_{j_0} - y_j|\\
&> 6nL + 2nr_0 - 2n(2L + r_0) = 2nL.
\end{aligned}$$

Therefore, for any $1 \le i, j \le n$,

$$\min_{i,j} \operatorname{dist}(C_L^{(1)}(x_i), C_L^{(1)}(y_j)) \ge \min_{i,j} |x_i - y_j| - 2L > 2(n-1)L \ge 2L.$$

This means that

$$\operatorname{dist}(\mathit{\Pi}\mathbf{C}_L^{(n)}(\mathbf{x}), \mathit{\Pi}\mathbf{C}_L^{(n)}(\mathbf{y})) = \min_{i,j} \operatorname{dist}\left(C_L^{(1)}(x_i), C_L^{(1)}(y_j)\right) > 0. \qquad \square$$

Definition 2.16. Let $\alpha = 3/2$. The multiscale analysis is based on a length scale $\{L_k\}_{k\ge 0}$ which is chosen as follows. The length-scale $\{L_k\}_{k\ge 0}$ is a sequence of integers defined by the initial length-scale $L_0 > L_0^*(r_0, N, d)$ for some large enough $L_0(N, d, r_0) > 0$, and by the recurrence relation

$$L_{k+1} = \lfloor L_k^\alpha \rfloor + 1.$$

This length scale $\{L_k\}_{k\ge 0}$ is assumed to be chosen at the beginning of the multiscale analysis, except that in the course of the analysis it is often required that L_0 be large enough.

Definition 2.17. Let $m > 0$ and $E \in \mathbb{R}$ be given. A cube $\mathbf{C}_L^{(n)}(\mathbf{u}) \subset \mathbb{Z}^{nd}$, $1 \le n \le N$ is called (E, m)-nonsingular ((E, m)-NS) if $E \notin \sigma(\mathbf{H}^{(n)}_{\mathbf{C}_L^{(n)}(\mathbf{u})})$ and

$$\max_{\mathbf{v}\in\partial^-\mathbf{C}_L^{(n)}(\mathbf{u})} \left|\mathbf{G}_{\mathbf{C}_L^{(n)}(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E)\right| \le \mathrm{e}^{-\gamma(m,L,n)L}, \tag{2.32}$$

where $\gamma(m, L, n) = \gamma(m, L, n, N)$ is defined by

$$\gamma(m, L, n) = m(1 + L^{-1/8})^{N-n+1} > m. \tag{2.33}$$

Otherwise it is called (E, m)-singular ((E, m)-S).

Given an n-particle cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ and $E \in \mathbb{R}$, we denote

— by $M^{\mathrm{sep}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E)$ the maximal number of pairwise separable (E, m)-singular cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$

— by $M^{\mathrm{sep}}_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E)$ the maximal number of pairwise separable, (E, m)-singular PI cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$;

— by $M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E)$ the maximal number of (not necessarily separable) (E, m)-singular PI cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ with $|\mathbf{u}^{(j)} - \mathbf{u}^{(j')}| > 7NL_k$ for all $j \ne j'$;

— by $M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)$ the maximal number of (E,m)-singular cubes FI $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ with $|\mathbf{u}^{(j)}-\mathbf{u}^{(j')}|>7NL_k$ for all $j\neq j'$ [2].

— by $M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)$ the maximal number of (E,m)-singular cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ with $\operatorname{dist}(\mathbf{u}^{(j)},\partial^-\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}))>2L_k$ and $|\mathbf{u}^{(j)}-\mathbf{u}^{(j')}|>7NL_k$ for all $j\neq j'$.

Further, set

$$M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),I):=\sup_{E\in I} M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E). \tag{2.34}$$

$$M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),I):=\sup_{E\in I} M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E). \tag{2.35}$$

Clearly

$$M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)+M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)\geq M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E).$$

Lemma 2.18. (i) *If $M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)\geq \kappa(n)+2$ with $\kappa(n)=n^n$, then $M^{\mathrm{sep}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E$ 2.*

(ii) *If $M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)\geq \kappa(n)+2$ then $M^{\mathrm{sep}}_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)\geq 2$.*

Proof. Assume that $M^{\mathrm{sep}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)<2$, (i.e., there is no pair of separable cubes of radius L_k in $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$), but $M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)\geq\kappa(n)+2$. Then $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ must contain at least $\kappa(n)+2$ cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{v}_i)$, $0\leq i\leq\kappa(n)+1$ which are non separable but satisfy $|\mathbf{v}_i-\mathbf{v}_{i'}|>7NL_k$, for all $i\neq i'$. On the other hand, by Lemma 2.3 there are at most $\kappa(n)$ cubes $\mathbf{C}^{(n)}_{2nL_k}(\mathbf{y}_i)$, such that any cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{x})$ with $\mathbf{x}\notin\bigcup_{j=1}^{\kappa(n)}\mathbf{C}^{(n)}_{2nL_k}(\mathbf{y}_j)$ is separable from $\mathbf{C}^{(n)}_{L_k}(\mathbf{v}_0)$. Since for all $i\neq i'$, $|\mathbf{v}_i-\mathbf{v}_{i'}|>7NL_k$, there must be at most one center $\mathbf{v}_i$ per cube $\mathbf{C}^{(n)}_{2nL_k}(\mathbf{y}_j)$, $1\leq j\leq\kappa(n)$, hence we come to a contradiction:

$$\kappa(n)+1\leq\kappa(n)$$

indeed $\mathbf{v}_i\in\bigcup_j\mathbf{C}^{(n)}_{2nL_k}(\mathbf{y}_j)$, $i=1,\cdots,\kappa(n)+1$. The same analysis holds true if we consider only PI cubes. □

Definition 2.19. Let $1\leq n\leq N$. We say that two subsets $\mathbf{A},\mathbf{B}\subset\mathbb{Z}^{nd}$ *touch* if $\mathbf{A}\cap\mathbf{B}\neq\emptyset$ or if there exist $\mathbf{x}\in\mathbf{A}$ and $\mathbf{y}\in\mathbf{B}$ such that $|\mathbf{x}-\mathbf{y}|=1$.

2. Note that by lemma 2.15, two FI cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)})$ and $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j')})$ with $|\mathbf{u}^{(j)}-\mathbf{u}^{(j')}|>7NL_k$ are automatically separable

In the following, we adapt Lemma 4.2 from [vDK89] to our multi-particle case.

In the rest of the text and more precisely in the following Lemma parameter $J := n^n + 5$

Lemma 2.20. *Let* $m = (12Nd)L_0^{-1/2}$, $E \in \mathbb{R}$. *Suppose that*

(i) $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ *is* E*-CNR,*

(ii) $M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E) + M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E) \le J$.

Then there exists $\tilde{L}_2^*(J, N, d) > 0$ *such that if* $L_0 \ge \tilde{L}_2^*(J, N, d)$ *we have that* $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ *is* (E, m)*-NS.*

Proof. By (ii) there are at most J cubes of size L_k contained in $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ and with centers at distance $> 7NL_k$ that are (E, m)-S. Therefore we can find $\mathbf{x}_i \in \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ with
$\mathrm{dist}(\mathbf{x}_i, \partial^- \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})) \ge 2L_k$, $i = 1, \cdots, r \le J$ such that if

$$\mathbf{x} \in \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}) \setminus \bigcup_{i=1}^{r} \mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i),$$

then $\mathbf{C}^{(n)}_{L_k}(\mathbf{x})$ is (E, m)-NS. Observe that the cubes $\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i)$ and $\mathbf{C}^{(n)}_{2L_k}(x_{i'})$ for $i \ne i'$ do not touch, indeed

$$\begin{aligned}\mathrm{dist}(\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i), \mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_{i'})) &\ge |\mathbf{x}_i - \mathbf{x}_{i'}| - 4L_k \\ &> 7NL_k - 4L_k > 1.\end{aligned}$$

Note that, if $\mathbf{x} \in \partial^+ \mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_j)$ for some $j = 1, \cdots, r$ then $\mathbf{x} \notin \bigcup_{i=1}^{r} \mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i)$.

Next, let $\Lambda \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ and $\mathbf{x} \in \Lambda$, $\mathbf{y} \in \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}) \setminus \Lambda$, it follows from the resolvent identity that

$$\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E) = \sum_{(\mathbf{z}, \mathbf{z}') \in \partial\Lambda} \mathbf{G}^{(n)}_{\Lambda}(\mathbf{x}, \mathbf{z}; E)\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}', \mathbf{y}; E).$$

Thus

$$|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E)| \le \left[\sum_{\mathbf{z} \in \partial^- \Lambda} |\mathbf{G}^{(n)}_{\Lambda}(\mathbf{x}, \mathbf{z}; E)|\right] |\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}}(\mathbf{z}_1, \mathbf{y}; E)| \qquad (2.36)$$

for some $z_1 \in \partial^+ \Lambda$.

Fix $\mathbf{v} \in \partial^- \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ and let $\mathbf{x} \in \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$, with $\mathrm{dist}(\mathbf{x}, \partial^- \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})) \ge L_k$. We have two cases:

(a) $\mathbf{C}^{(n)}_{L_k}(\mathbf{x})$ is (E,m)-NS. In this case

$$\sum_{\mathbf{z}\in\partial^-\mathbf{C}^{(n)}_{L_k}(\mathbf{x})} |\mathbf{G}^{(n)}(\mathbf{x},\mathbf{z};E)| \le 2^{nd}\cdot nd\cdot L_k^{nd-1}\mathrm{e}^{-\gamma(m,L_k,n)L_k}, \tag{2.37}$$

and by (2.36)

$$|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{x},\mathbf{v};E)| \le 2^{nd}\cdot nd\cdot L_k^{nd-1}\mathrm{e}^{-\gamma(m,L_k,n)L_k}|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}}(\mathbf{z}_1,\mathbf{y};E)|, \tag{2.38}$$

for some $z_1\in\partial^+\mathbf{C}^{(n)}_{L_k}(\mathbf{x})$.

(b) $\mathbf{C}^{(n)}_{L_k}(\mathbf{x})$ is (E,m)-S. In this case we have that $\mathbf{x}\in\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i)$ for some $i=1,\cdots,r$. If $\operatorname{dist}(\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i),\partial^-\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}))\ge L_k+1$, then equation (2.36) gives

$$|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{x},\mathbf{v};E)| \le \left[\sum_{\mathbf{z}\in\partial^-\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i)} |\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i)}(\mathbf{x},\mathbf{z};E)|\right]|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}_1,\mathbf{v};E)|$$

for some $\mathbf{z}_1\in\partial^+\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i)$. Observe that $\operatorname{dist}(\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i),\partial^-\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}))\ge L_k+1$, implies

$$\operatorname{dist}(\mathbf{z}_1,\partial^-\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}))\ge L_k.$$

Since the cubes $\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i)$ are E-NR we have:

$$\begin{aligned}|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{x},\mathbf{v};E)| &\le 2^{Nd}\cdot Nd(2L_k)^{nd-1}\\ &\qquad\times \mathrm{e}^{(2L_k)^{1/2}}|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}_1,\mathbf{v};E)|\end{aligned}$$

and since the cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{z}_1)$ is (E,m)-NS, we apply (2.36) and (2.37) getting

$$\begin{aligned}|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{x},\mathbf{v};E)| &\le (2^{Nd}\cdot Nd)^2(2L_k)^{nd-1}L_k^{nd-1}\\ &\quad\times \mathrm{e}^{(2L_k)^{1/2}-\gamma(m,L_k,n)L_k}|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}_2,\mathbf{v};E)|\\ &\le (2^{Nd}\cdot Nd)^2(2)^{nd-1}(L_k+1)^{2(nd-1)}\\ &\quad\times \mathrm{e}^{(2L_k)^{1/2}-\gamma(m,L_k,n)L_k}|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{Lk+1}(\mathbf{u})}(\mathbf{z}_2,\mathbf{v};E)|\end{aligned}$$

for some $\mathbf{z}_2\in\partial^+\mathbf{C}^{(n)}_{L_k}(\mathbf{z}_1)$. Thus

$$|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{x},\mathbf{v};E)| \le \mathrm{e}^{-\gamma'(m,L_k,n)L_k}|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}_2,\mathbf{v};E)| \tag{2.39}$$

where

$$\begin{aligned}\gamma'(m, L_k, n) &= \gamma(m, L_k, n) - \frac{1}{L_k}[(2L_k)^{1/2} + 2(nd-1)\ln(L_k+1)\\ &\quad + \ln((2^{Nd}\cdot Nd)^2)(2)^{nd-1})]\\ &\geq \gamma(m, L_k, n) - \left[6Nd + 2\ln(2^{Nd}Nd)\right] L_k^{-1/2}\\ &> m - \left[6Nd + 2\ln(2^{Nd}Nd)\right] L_0^{-1/2} > 0,\end{aligned}$$

since $m = (12Nd)L_0^{-1/2}$.

If $\mathbf{x} \in \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$, $\operatorname{dist}\left(\mathbf{x}, \partial^- \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})\right) \geq L_k$, let

$$W(\mathbf{x}) = \begin{cases} 2^{nd}\cdot nd \cdot L_k^{nd-1}\cdot \mathrm{e}^{-\gamma(m,L_k,n)L_k} \text{ if } \mathbf{x} \text{ is as case (a)}\\ \\ \mathrm{e}^{-\gamma'(m,L_k,n)L_k} \text{ if } \mathbf{x} \text{ is as case (b).}\end{cases}$$

Then (2.38) for case (a) and (2.39) for case (b) say that

$$|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{x}, \mathbf{v}; E)| \leq W(\mathbf{x})|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}, \mathbf{v}; E)|,$$

for some $\mathbf{z} \in \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$.

To estimate $|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E)|$ for $\mathbf{v} \in \partial^- \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$, we start from $\mathbf{u}$ and find $\mathbf{z}_1, \mathbf{z}_2, \ldots$ by applying the above procedure repeatedly when possible, getting after r steps,

$$\begin{aligned}|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E)| &\leq W(\mathbf{u})|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}_1, \mathbf{v}; E)| \leq W(\mathbf{u})W(\mathbf{z}_1)|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}_2, \mathbf{v}; E)|\\ &\leq \cdots \leq W(\mathbf{u})W(\mathbf{z}_1)\cdots W(\mathbf{z}_{r-1})|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}_r, \mathbf{v}; E)|.\end{aligned}$$

For this to be possible, $\mathbf{z}_1, \ldots \mathbf{z}_{r-1}$ must satisfy the conditions of either (a) or (b). Observe that in the first step, i.e, for $\mathbf{x} = \mathbf{u}$, either case (a) or (b) is done and this means that the set of $\mathbf{z}_1, \ldots, \mathbf{z}_{r-1}$ is non-empty. Indeed, if $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is (E, m)-NS then we are in case (a). Otherwise $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is (E, m)-S. So $\mathbf{u} \in \mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i)$ for some i. Next, denote by $\operatorname{diam}(\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i))$ the diameter of $\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i))$. Then

$$\begin{aligned}\operatorname{dist}(\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i), \partial^- \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})) &\geq L_{k+1} - \operatorname{diam}(\mathbf{C}^{(n)}_{2L_k}(\mathbf{x}_i))\\ &\geq L_{k+1} - 4L_k > L_k + 1,\end{aligned}$$

if $L_0 > C(N,J)$ for some constant $C(N,J) > 0$. Thus the conditions of case (b) are satisfied.

Let r_1 and r_2 be the number of times we had the bounds (2.38) and (2.39), respectively. We have

$$|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{u},\mathbf{v};E)| \leq 2^{Nd}\cdot Nd\cdot L_k^{nd-1}(\mathrm{e}^{-\gamma(m,L_k,n)L_k})^{r_1}(\mathrm{e}^{-\gamma'(m,L_k,n)L_k})^{r_2} \times |\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{z}_{r_1+r_2+1},\mathbf{v};E)|.$$

Notice that we have a lower bound for r_1 corresponding to the case where all the bad cubes $\mathbf{C}^{(n)}_{\ell_i}$ are met, namely:

$$r_1 \geq \frac{L_{k+1}-4JL_k}{L_k} \geq L_{k+1}L_k^{-1} - 4J.$$

By the assumption (i) of the Lemma, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ is E-NR, so $\|\mathbf{G}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(E)\| < \mathrm{e}^{L_{k+1}^{1/2}}$. Since $\gamma'(m,L_k,n) > 0$, $\mathrm{e}^{-r_2\gamma'(m,L_k,n)L_k} \leq 1$. Therefore

$$\begin{aligned}|\mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{u},\mathbf{v};E)| &\leq \left(2^{Nd}\cdot Nd\cdot L_k^{nd-1}\mathrm{e}^{-\gamma(m,L_k,n)L_k}\right)^{r_1}\mathrm{e}^{L_{k+1}^{1/2}}\\ &\leq \mathrm{e}^{-m'L_{k+1}}\end{aligned}$$

where

$$m' = \frac{1}{L_{k+1}}\left(r_1\gamma(m,L_k,n)L_k - r_1\ln(2^{Nd}Nd)L_k^{nd-1}\right) - \frac{1}{L_{k+1}^{1/2}}.$$

Since $L_{k+1}L_k^{-1} - 4J \leq r_1 \leq L_{k+1}L_k^{-1}$ we obtain

$$\begin{aligned}m' &\geq \gamma(m,L_k,n) - \gamma(m,L_k,n)\frac{4JL_k}{L_{k+1}}\\ &\quad - \frac{1}{L_{k+1}}\frac{L_{k+1}}{L_k}\ln(2^{Nd}Nd)L_k^{nd-1}) - \frac{1}{L_{k+1}^{1/2}}\\ &\geq \gamma(m,L_k,n) - \gamma(m,L_k,n)\,4JL_k^{-1/2}\\ &\qquad - L_k^{-1}(\ln(2^{Nd}Nd)) + (nd-1)\ln(L_k)) - L_k^{-3/4}\\ &\geq \gamma(m,L_k,n)[1-(4J+\ln(2^{Nd}Nd)+Nd)L_k^{-1/2}]\end{aligned}$$

if $L_0 \geq L_2^*(J,N,d)$ for some $L_2^*(J,N,d) > 0$ large enough. Since $\gamma(m,L_k,n) = m(1+L_k^{-1/8})^{N-n+1}$,

$$\frac{\gamma(m,L_k,n)}{\gamma(m,L_{k+1},n)} = \left(\frac{1+L_k^{-1/8}}{1+L_k^{-3/16}}\right)^{N-n+1} \geq \frac{1+L_k^{-1/8}}{1+L_k^{-3/16}}$$

Therefore we can compute

$$\begin{aligned}\frac{\gamma(m,L_k,n)}{\gamma(m,L_{k+1},n)}&(1-(4J+\ln(2^{Nd}Nd)+Nd)L_k^{-1/2})\\&\geq\frac{1+L_k^{-1/8}}{1+L_k^{-3/16}}(1-(4J+\ln(2^{Nd}Nd)+Nd)L_k^{-1/2})>1,\end{aligned}$$

provided $L_0 \geq \tilde{L}_2^*(J,N,d)$ for some large enough $\tilde{L}_2^*(J,N,d) > 0$. Indeed, denote by $C(J,N,d)$ the constant $4J + \ln(2^{Nd}2Nd) + Nd$. Observe that if $L_0 \geq \tilde{L}_2^*(J,N,d)$ for some large enough $\tilde{L}_2^*(J,N,d) > 0$ then

$$1 > C(J,N,d)L_k^{-3/8} + C(J,N,d)L_k^{-5/8} + L_k^{-3/16},$$

so that

$$L_k^{-1/8} > C(J,N,d)L_k^{-1/2} + C(J,N,d)L_k^{-5/8} + L_k^{-3/16},$$

yielding

$$1 - C(J,N,d)L_k^{-1/2} + L_k^{-1/8} - C(J,N,d)L_k^{-5/8} > 1 + L_k^{-3/16},$$

thus

$$(1 + L_k^{-1/8})(1 - C(J,N,d)L_k^{-1/2}) > 1 + L_k^{-3/16},$$

finally, we have the desired bound:

$$\frac{1+L_k^{-1/8}}{1+L_k^{-3/16}}(1 - C(J,N,d)L_k^{-1/2}) > 1.$$

Therefore, we obtain that $m' > \gamma(m,l_{k+1},n)$ and $|\mathbf{G}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{u},\mathbf{v};E)| \leq \mathrm{e}^{-\gamma(m,L_{k+1},n)L_{k+1}}$. This completes the proof of Lemma 2.20. $\square$

Chapter 3

Multi-particle eigenfunctions decay and localization at low energies

In this chapter, we prove localization for multi-particle systems with a number of particles bounded by some integer $N < \infty$. The value of $N \geq 2$ may be arbitrary, but once it is chosen, it is fixed for the rest of the scaling analysis, and several important parameters used in our scheme depend upon N.

3.1 Assumptions and the main results

3.1.1 Assumptions

(I1). The potential of inter-particle interaction

$$\mathbf{U}\colon (\mathbb{Z}^d)^n \to \mathbb{R}$$

is bounded and of the form

$$\mathbf{U}(\mathbf{x}) = \sum_{1\leq i<j\leq n} \Phi(|x_i - x_j|), \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^{nd}, \tag{3.1}$$

where $\Phi\colon \mathbb{N} \to \mathbb{R}_+$ is a compactly supported non-negative function:

$$\exists\, r_0 \in \mathbb{N} : \quad \operatorname{supp} \Phi \subset [0, r_0]. \tag{3.2}$$

We will call r_0 the "range" of the interaction $\mathbf{U}$.

(P1). It is assumed that $0 \in \operatorname{supp} \mu \subset [0, +\infty)$. Further, the probability distribution function F_V is log-Hölder continuous. More precisely,

$$s(F_V, \varepsilon) := \sup_{a\in\mathbb{R}} (F_V(a+\varepsilon) - F_V(a)) \leq \frac{C}{|\ln \epsilon|^{2A}} \tag{3.3}$$

$$\text{for some } C \in (0, \infty) \text{ and } A > \frac{3}{2} \times 4^N p + 9Nd.$$

The last condition depends on the parameter p which will be introduced in property $(\mathbf{DS}.k, n, N)$ below.

3.1.2 The results

For any $n = 1, \dots, N$, we denote by $\sigma(\mathbf{H}^{(n)}(\omega))$ the spectrum of $\mathbf{H}^{(n)}(\omega))$ and $E_0^{(n)}$ the infimum of $\sigma(\mathbf{H}^{(n)}(\omega))$.

Theorem 3.1. *Let $1 \le n \le N$. Under assumptions* (**I1**) *and* (**P1**)*, we have with probability one:*

$$[0, 4nd] \subset \sigma(\mathbf{H}^{(n)}(\omega)) \subset [0, +\infty).$$

Consequently,

$$E_0^{(n)} := \inf \sigma(\mathbf{H}^{(n)}(\omega)) = 0 \quad \text{a.s.}$$

Theorem 3.2. *Under the assumptions* (**I1**) *and* (**P1**)*, there exists $E^* > E_0^{(N)}$ such that with $\mathbb{P}$-probability one,*

(i) *the spectrum of $\mathbf{H}^{(N)}(\omega)$ in $[E_0^{(N)}, E^*]$ is nonempty and pure point;*

(ii) *any eigenfunction $\mathbf{\Psi}_i(\mathbf{x}, \omega)$ with eigenvalue $E_i(\omega) \in [E_0^{(N)}, E^*]$ is exponentially decaying at infinity: there exist a non-random constant $m_+ > 0$ and a random constant $C_i(\omega) > 0$ such that*

$$|\mathbf{\Psi}_i(\mathbf{x}, \omega)| \le C_i(\omega) \mathrm{e}^{-m_+|\mathbf{x}|}. \tag{3.4}$$

Denote by $\mathcal{B}_1$ the set of bounded Borel functions $f : \mathbb{R} \to \mathbb{R}$ such that $\|f\|_\infty \le 1$.

Theorem 3.3. *Under the assumptions* (**I1**) *and* (**P1**)*, for any $s^* > 0$ there exists $E^* > E_0^{(N)}$ such that for any bounded set $\mathbf{K} \subset \mathbb{Z}^{Nd}$ and any $s \in (0, s^*)$, we have*

$$\mathbb{E}\left[\sup_{f \in \mathcal{B}_1} \left\| |\mathbf{X}|^{\frac{s}{2}} f(\mathbf{H}^{(N)}(\omega)) \mathbf{P}_I(\mathbf{H}^{(N)}(\omega)) \mathbf{1}_{\mathbf{K}} \right\|_{HS}^2 \right] < \infty, \tag{3.5}$$

where $(|\mathbf{X}|\mathbf{\Psi})(\mathbf{x}) := |\mathbf{x}|\mathbf{\Psi}(\mathbf{x})$, $\mathbf{P}_I(\mathbf{H}^{(N)}(\omega))$ is the spectral projection of $\mathbf{H}^{(N)}(\omega)$ onto the interval $I := [E_0^{(N)}, E^*]$.

We will carry out a finite induction on the number of particles n varying from 1 to N.

The inductive step from $n-1$ to n requires information on n'-particle systems for all $n' = 1, \dots, n-1$. This can be briefly explained by the fact that the induction step involves the analysis of the decompositions of an n-particle system into two subsystems with n' and $n'' = n - n'$ particles, for any $n' = 1, \dots, n-1$. So it is assumed that all necessary bounds are available for n'-particle systems with any $n' = 1, \dots, n-1$.

Since our main method is (a multi-particle adaptation of) the multiscale analysis, the induction step from $n-1$ to n involves also a scale induction, i.e., induction over a sequence of scales L_k, $k \geq 0$. The starting point of the scale induction requires key bounds of the multiscale analysis to be established at the initial scale L_0.

Throughout this Chapter, we assume that the initial scale length of the multiscale analysis satifies the bound.

$$L_0 \geq L_0^*(N, d, r_0).$$

for some large enough $L_0^*(N, d, r_0) > 0$. Given $n = 1, \dots, N$, the following property for some $m > 0$ and $E^* > 0$ will play an important role in our main strategy.

$(\mathbf{DS}.k, n, N)$. For any pair of *separable* cubes $\mathbf{C}_{L_k}^{(n)}(\mathbf{u})$ and $\mathbf{C}_{L_k}^{(n)}(\mathbf{v})$

$$\mathbb{P}\left\{\exists\, E \in I : \mathbf{C}_{L_k}^{(n)}(\mathbf{u}),\ \mathbf{C}_{L_k}^{(n)}(\mathbf{v}) \text{ are } (E, m)\text{-S}\right\} \leq L_k^{-2p\,4^{N-n}}, \tag{3.6}$$

where $m > 0$, $E^* > 0$, $p > 6Nd$, and $I = (-\infty, E^*]$ are fixed.

3.2 Initial multi-particle multi-scale analysis bounds

3.2.1 Initial scale bounds

For any $n = 1, \dots, N$, we first prove the following result which relies on the Combes-Thomas's Lemma and a large deviations estimate from the study of the Lifshitz asymptotics in the single-particle model.

Lemma 3.4. *Let $\mathbf{H}^{(n)}(\omega) = -\mathbf{\Delta} + V(x_1, \omega) + \cdots + V(x_n, \omega) + \mathbf{U}(\mathbf{x})$ be an n-particle random Schrödinger operator in $\ell^2(\mathbb{Z}^{nd})$ where $\mathbf{U}$ and V satisfy* $(\mathbf{I1})$ *and* $(\mathbf{P1})$ *respectively. Then, for any $C > 0$ there exist arbitrarily large $L_0(C) > 0$ and $C_1, c > 0$ such that for any cube $\mathbf{C}_{L_0}^{(n)}(\mathbf{u})$ the lowest eigenvalue $E_0^{(n)}(\omega)$ of $\mathbf{H}_{\mathbf{C}_{L_0}^{(n)}(\mathbf{u})}^{(n)}(\omega)$ satisfies*

$$\mathbb{P}\left\{E_0^{(n)}(\omega) \leq 2CL_0^{-1/2}\right\} \leq C_1 L_0^d \mathrm{e}^{-cL_0^{1/4}}. \tag{3.7}$$

Proof. Since the interaction potential $\mathbf{U}$ is nonnegative, it follows from the min-max principle that the lowest eigenvalue $E_0^{(n)}(\omega)$ of $\mathbf{H}_{\mathbf{C}_{L_0}^{(n)}}^{(n)}(\omega)$ is bounded from below by the lowest eigenvalue $\widetilde{E}_0^{(n)}(\omega)$ of the operator without interaction,

$$\widetilde{\mathbf{H}}_{\mathbf{C}_{L_0}^{(n)}(\mathbf{u})}^{(n)}(\omega) = -\mathbf{\Delta} + V(x_1, \omega) + \cdots + V(x_n, \omega).$$

Denote $\mathcal{H}_i = \ell^2(C^{(1)}_{L_0}(u_i))$. The above operator can be rewritten as follows:

$$\widetilde{\mathbf{H}}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega) = \sum_{j=1}^{n} \underbrace{\mathbf{1}_{\mathcal{H}_1} \otimes \cdots \otimes \mathbf{1}_{\mathcal{H}_{j-1}}}_{j-1 \text{ times}} \otimes H^{(1)}_j \otimes \underbrace{\mathbf{1}_{\mathcal{H}_{j+1}} \otimes \cdots \otimes \mathbf{1}_{\mathcal{H}_n}}_{n-j \text{ times}} \tag{3.8}$$

where $H^{(1)}_j(\omega) = -\Delta^{(1)}_{C^{(1)}_{L_0}(u_j)} + V(x_j,\omega)$, $j = 1,\ldots,n$, is acting in $\mathcal{H}_j$. Hence the lowest eigenvalue $\widetilde{E}^{(n)}_0(\omega)$ of $\widetilde{\mathbf{H}}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega)$ has the following form:

$$\widetilde{E}^{(n)}_0(\omega) = \sum_{j=1}^{n} E^{(1)}_{0,j}(\omega),$$

where $E^{(1)}_{0,j}$ is the lowest eigenvalue of $H^{(1)}_j$. All eigenvalues $E^{(1)}_{0,j}$ are nonnegative due to the nonnegativity of the operators $H^{(1)}_j$ with nonnegative external potential, so that for any $s \geq 0$,

$$\mathbb{P}\left\{\widetilde{E}^{(n)}_0(\omega) \leq s\right\} \leq \mathbb{P}\left\{E^{(1)}_{0,1}(\omega) \leq s\right\}.$$

Therefore, Lemma 3.4 follows from Theorem A.6 applied to the single-particle Schrödinger operator $H^{(1)}_1$. $\square$

Remark 3.5. It is to be emphasized that for the initial scale bound, the amplitude of the interaction potential is irrelevant, provided that it is nonnegative. In other words, a nonnegative interaction enhances the "Lifshitz tails" phenomenon.

Theorem 3.6 (initial scale estimate)**.** *Under assumptions* (**I1**) *and* (**P1**)*, for any $p > 0$, there exists arbitrarily large $L_0(N,d,p)$ such that if $m := 12NdL_0^{-1/2}$ and $E^* := (12Nd)(2^{N+1}m)$, then* (**DS**.$0,n,N$) *is valid for any $n = 1,\ldots,N$.*

Proof. Set $C = (12Nd)^2 \cdot 2^{N+1}$ and let $\mathbf{C}^{(n)}_{L_0}(\mathbf{x})$ be a cube in $\mathbb{Z}^{nd}$. Consider $\omega \in \Omega$ such that the first eigenvalue $E^{(n)}_0(\omega)$ of $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\omega)}$ satisfies $E^{(n)}_0(\omega) > 2CL_0^{-1/2}$. Then for all $E \leq CL_0^{-1/2} = E^*$ we have

$$\operatorname{dist}(E, \sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega))) = E^{(n)}_0(\omega) - E > CL_0^{-1/2} =: \eta > 0.$$

Therefore, for any $\mathbf{v} \in \partial^-\mathbf{C}^{(n)}_{L_0}(\mathbf{u})$ we obtain by the Combes-Thomas estimate (Theorem 2.12)

$$|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(E,\mathbf{u},\mathbf{v})| \leq 2C^{-1}L_0^{1/2}\, \mathrm{e}^{-\frac{CL_0^{-1/2}}{12Nd}L_0}.$$

Now observe that $\frac{CL_0^{-1/2}}{12Nd} = 2^{N+1}m$. Thus

$$\begin{aligned}|\mathbf{G}_{\mathbf{C}_{L_0}^{(n)}(\mathbf{u})}(E,\mathbf{u},\mathbf{v})| &\leq 2C^{-1}L_0^{1/2}\,\mathrm{e}^{-2^{N+1}mL_0}\\ &\leq 2C^{-1}L_0^{1/2}\mathrm{e}^{-2\gamma(m,L_0,n)L_0}\\ &\leq \mathrm{e}^{-\gamma(m,L_0,n)L_0},\end{aligned}$$

for L_0 large enough, since

$$\gamma(m,L,n) = m(1+L^{-1/8})^{N-n+1} < m \times 2^N.$$

This implies that $\mathbf{C}_{L_0}^{(n)}(\mathbf{u})$ is (E,m)-NS. Thus

$$\omega \in \{\omega \in \Omega \,\big|\, \mathbf{C}_L^{(n)}(\mathbf{u}) \text{ is } (E,m)\text{-NS } \forall E \leq E^*\}.$$

Therefore

$$\mathbb{P}\left\{\exists E \leq E^*,\, \mathbf{C}_{L_0}^{(n)}(\mathbf{u}) \text{ is } (E,m)\text{-S}\right\} \leq \mathbb{P}\left\{E_0^{(n)}(\omega) \leq 2CL_0^{-1/2}\right\},$$

and by Lemma 3.4,

$$\mathbb{P}\{E_0^{(n)}(\omega) \leq 2CL_0^{-1/2}\} \leq C_1L_0^d\mathrm{e}^{-cL_0^{1/4}}.$$

Finally, the quantity $C_1L_0^d\mathrm{e}^{-cL_0^d/4}$ for L_0 sufficiently large is less than $L_0^{-2p\,4^{N-n}}$. This proves the claim since the probability for two cubes to be singular at the same energy is bounded by the probability of either one of them to be singular. □

In the rest of the Chapter, the parameters m and E^* are as in Theorem 3.6 and we denote by I the interval $(-\infty, E^*]$.

3.2.2 Single-particle multi-scale induction

We have just established the single particle initial scale bound property $(\mathbf{DS}.0,1,N)$. It remains to perform the inductive step $k \rightsquigarrow k+1$ for the single particle random Hamiltonian $H^{(1)}(\omega)$. Namely, we will prove

Theorem 3.7. *Under assumptions* $(\mathbf{I1})$ *and* $(\mathbf{P1})$*, there exists* $\widetilde{L}^*(d) > 0$ *such that if* $L_0 \geq \widetilde{L}^*$ *and if for* $k \geq 0$, $(\mathbf{DS}.k,1,N)$ *holds true, then* $(\mathbf{DS}.k+1,1,N)$ *holds true.*

Before we turn to the proof, we need an auxiliary result. Recall that $M(C_{L_{k+1}}^{(1)}(u);E)$ is the total number of (E,m)-S cubes $C_{L_k}^{(1)}(v^j) \subset C_{L_{k+1}}^{(1)}(u)$ with $|v^j - v^{j'}| > 7NL_k$ for all $j \neq j'$.

Lemma 3.8. *Under assumptions* (**I1**) *and* (**P1**), *let $k \geq 0$ and assume that property* $(\mathbf{DS}.k, 1, N)$ *holds true for all pairs of separable single particle cubes. Then for any $\ell \geq 1$*

$$\mathbb{P}\left\{M(C^{(1)}_{L_{k+1}}(u); E) \geq 2\,\ell\right\} \leq C(\ell, d) L_k^{2\ell\, d\alpha} L_k^{-2\ell\, p\, 4^{N-1}}. \tag{3.9}$$

Proof. If there exist 2ℓ pairwise separable single particle cubes $C^{(1)}_{L_k}(u^{(j)}) \subset C^{(1)}_{L_{k+1}}(u)$, $1 \leq j \leq 2\ell$, then those cubes are disjoints. So for any pair $C^{(1)}_{L_k}(u^{(2i-1)})$, $C^{(1)}_{L_k}(u^{(2i)})$ the Hamiltonians $H^{(1)}_{C^{(1)}_{L_k}(u^{(2i-1)})}$ and $H^{(1)}_{C^{(1)}_{L_k}(u^{(2i)})}$ are independent and so are their spectra and Green functions. For $i = 1, \cdots, \ell$, set

$$\mathrm{A}_i = \{\exists E \in I : C^{(1)}_{L_k}(u^{(2i-1)}) \text{ and } C^{(1)}_{L_k}(u^{(2i)}) \text{ are } (E, m)\text{-S}\}.$$

Using assumption $(\mathbf{DS}.k, 1, N)$ for $i = 1, \cdots, \ell$, we have

$$\mathbb{P}\{\mathrm{A}_i\} \leq L_k^{-2p\, 4^{N-1}},$$

and by independence of events $\mathrm{A}_1, \cdots, \mathrm{A}_\ell$,

$$\mathbb{P}\left\{\bigcap_{i=1}^{\ell} \mathrm{A}_i\right\} \leq \left(L_k^{-2p\, 4^{N-1}}\right)^{\ell}.$$

To finish, note that the number of different families of 2ℓ cubes $C^{(1)}_{L_k}(u^{(j)})$ is bounded by

$$\frac{1}{(2\ell)!} |C^{(1)}_{L_{k+1}}(u)|^{2\ell} \leq C(\ell, d) L_k^{2\ell\, d\alpha}.$$ $\square$

Proof of Theorem 3.7. Let $C^{(1)}_{L_{k+1}}(x)$, $C^{(1)}_{L_{k+1}}(y)$ be a pair of separable cubes. Set

$$\begin{aligned}
\mathrm{B}_{k+1} &= \{\exists E \in I : C^{(1)}_{L_{k+1}}(x) \text{ and } C^{(1)}_{L_{k+1}}(y) \text{ are } (E, m)\text{-S}\} \\
\Sigma &= \{\exists E \in I : \text{neither } C^{(1)}_{L_{k+1}}(x) \text{ nor } C^{(1)}_{L_{k+1}}(y) \text{ is } E\text{-CNR}\} \\
M_x &= \{\exists E \in I : M(C^{(1)}_{L_{k+1}}(x); E) \geq 6\} \\
M_y &= \{\exists E \in I : M(C^{(1)}_{L_{k+1}}(y); E) \geq 6\}
\end{aligned}$$

since $n^n + 5 = 6$ for $n = 1$. Let $\omega \in \mathrm{B}_{k+1} \setminus \Sigma \cup M_x$, then $\forall E \in I$ $C^{(1)}_{L_{k+1}}(x)$ or $C^{(1)}_{L_{k+1}}(y)$ is E-CNR and $M(C^{(1)}_{L_{k+1}}(x); E) \leq 3$. The cube $C^{(1)}_{L_{k+1}}(x)$ cannot be E-CNR: indeed by Lemma 2.20 it would be (E, m)-S. So the cube $C^{(1)}_{L_{k+1}}(y)$ is E-CNR and (E, m)-S. This implies again by Lemma 2.20 that $M(C^{(1)}_{L_{k+1}}(y); E) \geq 6$.

Thus $\omega \in M_y$ and $\mathrm{B}_{k+1} \subset \Sigma \cup M_x \cup M_y$. Using corollary 2.9 and Lemma 3.8, we finally get

$$\begin{aligned}
\mathbb{P}\left\{\mathrm{B}_{k+1}\right\} &\leq \mathbb{P}\left\{\Sigma\right\} + \mathbb{P}\left\{M_x\right\} + \mathbb{P}\left\{M_y\right\} \\
&\leq L_{k+1}^{-4^N p} + C(d) L_{k+1}^{-\frac{6}{\alpha}p\,4^{N-1}+6d} && (\text{for } \ell = 2) \\
&\leq \frac{1}{2} L_{k+1}^{-2p\,4^{N-1}} + \frac{1}{2} L_{k+1}^{-2p\,4^{N-1}} = L_{k+1}^{-2p\,4^{N-1}}.
\end{aligned}$$

Since $p > 4\alpha N d$ by assumption and provided $L_0 \geq \widetilde{L}^*$ for some large enough $\widetilde{L}^* > 0$. $\square$

3.3 Multi-particle multi-scale induction

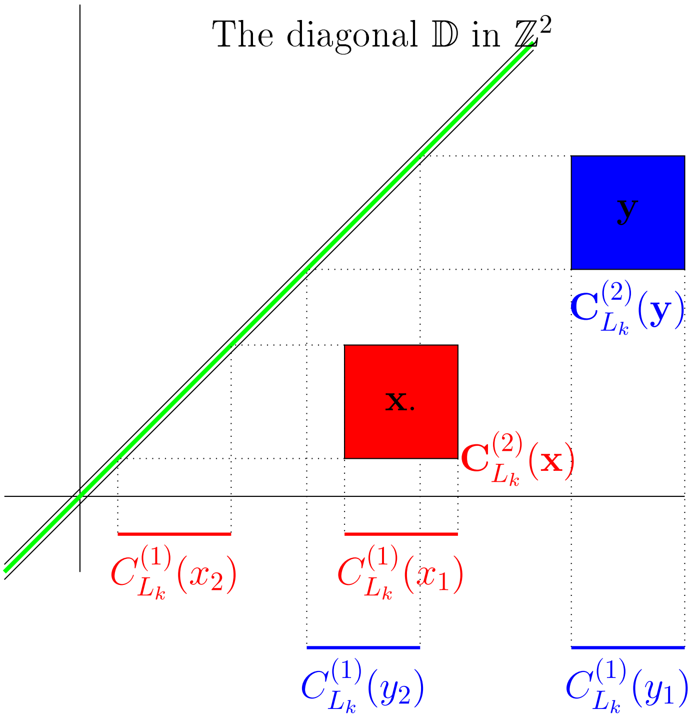
Given $n = 2, \ldots, N$, assuming $(\mathbf{DS}.k-1, n', N)$ for all $n' = 1, \ldots, n-1$ and $(\mathbf{DS}.k, n', N)$ for any $n' = 1, \ldots, n-1$, we now prove $(\mathbf{DS}.k+1, n, N)$, separately for the following three types of pairs of cubes:

(I) $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ are both PI-cubes.

(II) $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ are both FI-cubes.

(III) One of the cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ or $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is PI, while the other is FI.

3.3.1 Pairs of PI cubes



Pairs of PI cubes: for $d=1$, $n=2$, $\mathbf{x}=(x_1,x_2)$ and $\mathbf{y}=(y_1,y_2)$

We consider here a pair of separable partially interactive cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ and prove that $(\mathbf{DS}.k+1,n,N)$ holds for such a pair.

Let $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})=\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{u}')\times\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{u}'')$ be a PI-cube. We also write $\mathbf{x}=(\mathbf{x}',\mathbf{x}'')$ for any point $\mathbf{x}\in\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$, in the same way as $(\mathbf{u}',\mathbf{u}'')$. The corresponding Hamiltonian $\mathbf{H}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}$ has the form:

$$\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}\mathbf{\Psi}(\mathbf{x})=(-\mathbf{\Delta}\mathbf{\Psi})(\mathbf{x})+\left[\mathbf{U}(\mathbf{x}')+\mathbf{V}(\mathbf{x}',\omega)+\mathbf{U}(\mathbf{x}'')+\mathbf{V}(\mathbf{x}'',\omega)\right]\mathbf{\Psi}(\mathbf{x}) \tag{3.10}$$

or, in compact form

$$\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}=\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{u}')}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{u}'')}.$$

The representation (3.10) follows from Lemma 2.14.

We denote by $\mathbf{G}^{(n')}(\mathbf{u}',\mathbf{v}';E)$ and $\mathbf{G}^{(n'')}(\mathbf{u}'',\mathbf{v}'';E)$ the Green functions relative to $\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{u}')}$ and $\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{u}'')}$, respectively. Let

$$\{(\lambda_i,\varphi_i)\}_{i=1,\dots,|\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{u}')|}\quad\text{and}\quad\{(\mu_j,\phi_j)\}_{j=1,\dots,|\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{u}'')|}$$

be the eigenvalues and corresponding eigenfunctions of $\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{u}')}$ and $\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{u}'')}$. Then we can choose eigenvectors $\mathbf{\Psi}_{ij}$ of $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}$ as tensor products

$$\mathbf{\Psi}_{ij}(\mathbf{x}) = \varphi_i(\mathbf{x}') \otimes \phi_j(\mathbf{x}''), \tag{3.11}$$

while corresponding eigenvalues are given by the sums

$$E_{ij} = \lambda_i + \mu_j. \tag{3.12}$$

The eigenvectors of finite volume Hamiltonians appearing in subsequent arguments and calculations will be assumed normalized. Introduce the following

Definition 3.9 ([KN13])**.** Let $1 \le n \le N$ and $E \in \mathbb{R}$. Consider a PI cube $\mathbf{C}^{(n)}_L(\mathbf{u}) = \mathbf{C}^{(n')}_L(\mathbf{u}') \times \mathbf{C}^{(n'')}_L(\mathbf{u}'')$. Then $\mathbf{C}^{(n)}_L(\mathbf{u})$ is called E-highly non resonant (E-HNR) if

(i) for all $\mu_j \in \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_L(\mathbf{u}'')})$, the cube $\mathbf{C}^{(n')}_L(\mathbf{u}')$ is $(E-\mu_j)$-CNR and

(ii) for all $\lambda_i \in \sigma(\mathbf{H}^{(n')}_{\mathbf{H}^{(n')}_L(\mathbf{u}')})$ the cube $\mathbf{C}^{(n'')}_L(\mathbf{u}'')$ is $(E-\lambda_i)$-CNR.

Definition 3.10 ((E,m)-tunnelling)**.** Let $1 \le n \le N$, $E \in \mathbb{R}$ and $m > 0$. Consider a PI cube $\mathbf{C}^{(n)}_L(\mathbf{u}) = \mathbf{C}^{(n')}_L(\mathbf{u}') \times \mathbf{C}^{(n'')}_L(\mathbf{u}'')$.

Then $\mathbf{C}^{(n)}_L(\mathbf{u})$ is called

(i) (E,m) left-tunnelling ((E,m)-LT) if $\exists \mu_j \in \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_L(\mathbf{u}'')})$ such that $\mathbf{C}^{(n')}_L(\mathbf{u}')$ contains two separable $(E-\mu_j,m)$-S cubes $\mathbf{C}^{(n')}_l(\mathbf{v}_1)$ and $\mathbf{C}^{(n')}_l(\mathbf{v}_2)$ with $L = \lfloor l^\alpha \rfloor + 1$. Otherwise, it is called (E,m) non-left-tunnelling ((E,m)-NLT).

(ii) (E,m) right-tunnelling ((E,m)-RT) if $\exists \lambda_i \in \sigma(\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_L(\mathbf{u}')})$ such that $\mathbf{C}^{(n'')}_L(\mathbf{u}'')$ contains two separable $(E-\lambda_i,m)$-S cubes $\mathbf{C}^{(n'')}_l(\mathbf{v}_1)$ and $\mathbf{C}^{(n'')}_l(\mathbf{v}_2)$ with $L = \lfloor l^\alpha \rfloor + 1$. Otherwise, it is called (E,m) non-right-tunnelling ((E,m)-NRT).

(iii) (E,m)-tunnelling ((E,m)-T) if either it is (E,m)-LT or (E,m)-RT. Otherwise it is called (E,m)-non-tunnelling ((E,m)-NT).

We reformulate and prove Lemma 3.18 from [KN13] in our context.

Lemma 3.11. *Let $E \in \mathbb{R}$. If a PI cube $\mathbf{C}^{(n)}_L(\mathbf{u}) = \mathbf{C}^{(n')}_L(\mathbf{u}') \times \mathbf{C}^{(n'')}_L(\mathbf{u}'')$ is not E-HNR then*

(i) *either there exist* $L^{1/\alpha} \le \ell \le L$, $\mathbf{x} \in \mathbf{C}_L^{(n')}(\mathbf{u}')$ *such that the* n*-particle rectangle* $\mathbf{C}^{(n)} = \mathbf{C}_\ell^{(n')}(\mathbf{x}) \times \mathbf{C}_L^{(n'')}(\mathbf{u}'') \subset \mathbf{C}_L^{(n)}(\mathbf{u})$ *is* E*-R.*

(ii) *or there exist* $L^{1/\alpha} \le \ell \le L$, $\mathbf{x} \in \mathbf{C}_L^{(n'')}(\mathbf{u}'')$ *such that the* n*-particle rectangle* $\mathbf{C}^{(n)} = \mathbf{C}_L^{(n')}(\mathbf{u}') \times \mathbf{C}_\ell^{(n'')}(\mathbf{x}) \subset \mathbf{C}_L^{(n)}(\mathbf{u})$ *is* E*-R.*

Proof. By Definition 3.9, if $\mathbf{C}_L^{(n)}(\mathbf{u})$ is not E-HNR then either (a) there exists $\mu_j \in \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}_L^{(n'')}(\mathbf{u}'')})$ such that $\mathbf{C}_L^{(n')}(\mathbf{u}')$ is not $E-\mu_j$-CNR or (b) there exists $\lambda_i \in \sigma(\mathbf{H}^{(n')}_{\mathbf{C}_L^{(n')}(\mathbf{u}')})$ such that $\mathbf{C}_L^{(n'')}(\mathbf{u}'')$ is not $E-\lambda_i$-CNR. Let us first focus on case (a). Since $\mathbf{C}_L^{(n')}(\mathbf{u}')$ is not $E-\mu_j$-CNR there exists $L^{1/\alpha} \le \ell \le L$, $\mathbf{x} \in \mathbf{C}_L^{(n')}(\mathbf{u}')$ such that $\mathbf{C}_\ell^{(n')}(\mathbf{x}) \subset \mathbf{C}_L^{(n')}(\mathbf{u}')$ and $\mathbf{C}_\ell^{(n')}(\mathbf{x})$ is $E-\mu_j$-R. So $\operatorname{dist}(E-\mu_j, \sigma(\mathbf{H}^{(n')}_{\mathbf{C}_\ell^{(n')}(\mathbf{x})})) < \mathrm{e}^{-\ell^\beta}$. Therefore there exists $\eta \in \sigma(\mathbf{H}^{(n')}_{\mathbf{C}_\ell^{(n')}(\mathbf{x})})$ such that $|E-\mu_j-\eta| < \mathrm{e}^{-\ell^\beta}$. Now consider $\mathbf{C}^{(n)} = \mathbf{C}_\ell^{(n')}(\mathbf{x}) \times \mathbf{C}_L^{(n'')}(\mathbf{u}'')$, since the cube $\mathbf{C}_L^{(n)}(\mathbf{u})$ is PI we have $\sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}}) = \sigma(\mathbf{H}^{(n')}_{\mathbf{C}_\ell^{(n')}(\mathbf{x})}) + \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}(n'')_L(\mathbf{u}'')})$, hence

$$\operatorname{dist}(E, \sigma(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}})) \le |E-\mu_j-\eta| < \mathrm{e}^{-\ell^\beta}.$$

Thus $\mathbf{C}^{(n)}$ is E-R. The same arguments show that case (ii) arises when (b) occurs. $\square$

Lemma 3.12. *Let* $\alpha = 3/2$. *For any* $p > 6Nd$ *assume that the random potential satisfies* (**P1**) *with* $A > 4^N \alpha\, p + 6\alpha N d$. *Let* $\mathbf{C}_L^{(n)}(\mathbf{u}) = \mathbf{C}_L^{(n')}(\mathbf{u}') \times \mathbf{C}_L^{(n'')}(\mathbf{u}'')$ *and* $\mathbf{C}_L^{(n)}(\mathbf{v}) = \mathbf{C}_L^{(n')}(\mathbf{v}') \times \mathbf{C}_L^{(n'')}(\mathbf{v}'')$ *be two cubes:*

(A) *Suppose that* $\mathbf{C}_L^{(n)}(\mathbf{u})$ *and* $\mathbf{C}_L^{(n)}(\mathbf{v})$ *are PI cubes then*

$$\mathbb{P}\left\{\exists E \in \mathbb{R},\ \textit{neither } \mathbf{C}_L^{(n)}(\mathbf{u}) \textit{ nor } \mathbf{C}_L^{(n)}(\mathbf{v}) \textit{ is } E\textit{-HNR}\right\} \le L^{-4^N p}.$$

(B) *Suppose that* $\mathbf{C}_L^{(n)}(\mathbf{u})$ *is a PI cube while* $\mathbf{C}_L^{(n)}(\mathbf{v})$ *is a FI cube. Then*

$$\mathbb{P}\left\{\exists E \in \mathbb{R},\ \mathbf{C}_L^{(n)}(\mathbf{u}) \textit{ is not } E\textit{-HNR and } \mathbf{C}_L^{(n)}(\mathbf{v}) \textit{ is not } E\textit{-CNR}\right\} \le L^{-4^N p}.$$

Proof. (A) We have using Lemma 3.11

$$\begin{aligned}
&\mathbb{P}\{\exists E\in\mathbb{R},\text{ neither } \mathbf{C}^{(n)}_L(\mathbf{u}) \text{ nor } \mathbf{C}^{(n)}_L(\mathbf{v}) \text{ is } E\text{-HNR}\}\\
&\quad\le \mathbb{P}\{\exists E\in\mathbb{R},\ \ell_1,\ell_2;\ L^{1/\alpha}\le\ell_1,\ell_2\le L,\ \mathbf{x}\in\mathbf{C}^{(n')}_L(\mathbf{u}'),\ \mathbf{y}\in\mathbf{C}^{(n')}_L(\mathbf{v}'),\\
&\quad\mathbf{C}^{(n')}_{\ell_1}(\mathbf{x})\times\mathbf{C}^{(n'')}_L(\mathbf{u}'')\subset\mathbf{C}^{(n)}_L(\mathbf{u}) \text{ and } \mathbf{C}^{(n')}_{\ell_2}(\mathbf{y})\times\mathbf{C}^{(n'')}_L(\mathbf{v}'')\subset\mathbf{C}^{(n)}_L(\mathbf{v}) \text{ are } E\text{-R}\} \qquad (3.13)\\
&\quad+\mathbb{P}\{\exists E\in\mathbb{R},\ \ell_1,\ell_2,\ L^{1/\alpha}\le\ell_1,\ell_2\le L,\ \mathbf{x}\in\mathbf{C}^{(n'')}_L(\mathbf{u}''),\ \mathbf{y}\in\mathbf{C}^{(n'')}(\mathbf{v}''),\\
&\quad\mathbf{C}^{(n')}_L(\mathbf{u}')\times\mathbf{C}^{(n'')}_{\ell_1}(\mathbf{x})\subset\mathbf{C}^{(n)}_L(\mathbf{u}) \text{ and } \mathbf{C}^{(n')}_L(\mathbf{v}')\times\mathbf{C}^{(n'')}_{\ell_2}(\mathbf{y})\subset\mathbf{C}^{(n)}_L(\mathbf{v}) \text{ are } E\text{-R}\} \qquad (3.14)\\
&\quad+\mathbb{P}\{\exists E\in\mathbb{R},\ \ell_1,\ell_2\colon\ L^{1/\alpha}\le\ell_1,\ell_2\le L\ \mathbf{x}\in\mathbf{C}^{(n')}_L(\mathbf{u}'),\ \mathbf{y}\in\mathbf{C}^{(n'')}_L(\mathbf{v}'')\colon\\
&\quad\mathbf{C}^{(n')}_{\ell_1}(\mathbf{x})\times\mathbf{C}^{(n'')}_L(\mathbf{u}'')\subset\mathbf{C}^{(n)}_L(\mathbf{u}) \text{ and } \mathbf{C}^{(n'}_L(\mathbf{v}')\times\mathbf{C}^{(n'')}_{\ell_2}(\mathbf{y})\subset\mathbf{C}^{(n)}_L(\mathbf{v}) \text{ are } E\text{-R}\} \qquad (3.15)\\
&\quad+\mathbb{P}\{\exists E\in\mathbb{R},\ \ell_1,\ell_2 :\ L^{1/\alpha}\le\ell_1,\ell_2\le L,\ \mathbf{x}\in\mathbf{C}^{(n'')}_L(\mathbf{u}''),\ \mathbf{y}\in\mathbf{C}^{(n')}_L(\mathbf{v}')\colon\\
&\quad\mathbf{C}^{(n')}_L(\mathbf{u}')\times\mathbf{C}^{(n'')}_{\ell_1}(\mathbf{x})\subset\mathbf{C}^{(n)}_L(\mathbf{u}) \text{ and } \mathbf{C}^{(n')}_{\ell_2}(\mathbf{y})\times\mathbf{C}^{(n'')}_L(\mathbf{v}'')\subset\mathbf{C}^{(n)}_L(\mathbf{v}) \text{ are } E\text{-R}\} \qquad (3.16)
\end{aligned}$$

Let us compute the first term in the RHS. The arguments for the other terms are similar. We estimate

$$\begin{aligned}
(3.13) &\le \{\exists E\in\mathbb{R},\ \operatorname{dist}(E,\sigma(\mathbf{H}_{\mathbf{C}^{(n')}_{\ell_1}(\mathbf{x})\times\mathbf{C}^{(n'')}_L(\mathbf{u}'')}))<\mathrm{e}^{-\ell_1^\beta},\\
&\qquad \operatorname{dist}(E,\sigma(\mathbf{H}_{\mathbf{C}^{(n')}_{\ell_2}(\mathbf{y})\times\mathbf{C}^{(n'')}_L(\mathbf{v}'')}))<\mathrm{e}^{-\ell_2^\beta}\}\\
&\le \{\operatorname{dist}(\sigma(\mathbf{H}_{\mathbf{C}^{(n')}_{\ell_1}(\mathbf{x})\times\mathbf{C}^{(n'')}_L(\mathbf{u}'')}),\sigma(\mathbf{H}_{\mathbf{C}^{(n')}_{\ell_2}(\mathbf{y})\times\mathbf{C}^{(n'')}_L(\mathbf{v}'')}))<\mathrm{e}^{-L^{\beta/\alpha}}\}\\
&\le (3L)^{2n+1)d}\cdot Const\cdot L^{-A/\alpha}.
\end{aligned}$$

Above, we used Theorem 2.8 and condition (**P1**). Counting the number of possible centers in the respective cubes of radius L and the number of possible values of ℓ_1,ℓ_2 we obtain:

$$\begin{aligned}
(3.13) &\le (3L)2nd\cdot L^2\cdot(3L)^{(2n+1)d}\cdot Const\cdot L^{-A/\alpha}\\
&\le \frac{1}{4}L^{-4^N p},
\end{aligned}$$

since $A>4^N p\alpha+9Nd$. Finally the sum of the four probability terms yields the assertion (A) of the Lemma.

(B)

$$
\begin{aligned}
&\mathbb{P}\{\exists E\in\mathbb{R},\ \mathbf{C}_L^{(n)}(\mathbf{u}) \text{ is not } E\text{-HNR and } \mathbf{C}^{(n)}(\mathbf{v}) \text{ is not } E\text{-CNR}\}\\
&\qquad\le \mathbb{P}\{\exists E\in\mathbb{R},\ \ell_1,\ell_2,\ L^{1/\alpha}\le \ell_1,\ell_2\le L,\ \mathbf{x}\in \mathbf{C}_L^{(n')}(\mathbf{u}'),\ \mathbf{y}\in\mathbf{C}_L^{(n)}(\mathbf{v}),\\
&\mathbf{C}_{\ell_1}^{(n')}(\mathbf{x})\times \mathbf{C}_L^{(n'')}(\mathbf{u}'')\subset \mathbf{C}_L^{(n)}(\mathbf{u}) \text{ and } \mathbf{C}_{\ell_2}^{(n)}(\mathbf{y})\subset \mathbf{C}_L^{(n)}(\mathbf{v}) \text{ are } E\text{-R}\} \qquad (3.17)\\
&\qquad + \mathbb{P}\{\exists E\in\mathbb{R},\ \ell_1,\ell_2,\ L^{1/\alpha}\le \ell_1,\ell_2\le L,\ \mathbf{x}\in \mathbf{C}_{\ell_1}^{(n'')}(\mathbf{u}'''),\ \mathbf{y}\in\mathbf{C}_L^{(n)}(\mathbf{v}):\\
&\qquad \mathbf{C}_L^{(n')}(\mathbf{u}')\times \mathbf{C}_{\ell_1}^{(n'')}(\mathbf{x})\subset \mathbf{C}(n)_L(\mathbf{u}) \text{ and } \mathbf{C}_{\ell_2}^{(n)}(\mathbf{y})\subset \mathbf{C}_L^{(n)}(\mathbf{v}) \text{ are } E\text{-R}\}. \quad (3.18)
\end{aligned}
$$

We compute the first term in the RHS, the calculation for the second term is similar. Following the same lines as in part (A), we have:

$$
\begin{aligned}
(3.17) &\le (3L)^{2nd}\cdot L^2\cdot (3L)^{2n+1)d}\cdot Const\cdot L^{-A/\alpha}\\
&\le \frac{1}{2}L^{-4^Np},
\end{aligned}
$$

since $A > 4^N p\alpha + 9Nd$. Therefore the sum of the two probability terms yields the assertion. □

Lemma 3.12 has its analog for the weakly interacting regime under assumption (**P2**). The following statement, where the main ideas of the proof do not change, is proved in a similar way as Lemma 3.12 using Theorem 2.10 instead of Theorem 2.8 as above.

Lemma 3.13. *Let $E\in I$ and $\mathbf{C}_{L_k}^{(n)}(\mathbf{u})$ be a PI cube. Assume that $\mathbf{C}_{L_k}^{(n)}(\mathbf{u})$ is (E,m)-NT and E-HNR. Then $\mathbf{C}_{L_k}^{(n)}(\mathbf{u})$ is (E,m)-NS.*

Proof. Consider a PI cube $\mathbf{C}_{L_k}^{(n)}(\mathbf{u}) = \mathbf{C}_{L_k}^{(n')}(\mathbf{u}')\times\mathbf{C}_{L_k}^{(n'')}(\mathbf{u}'')$. Let $\{\lambda_i,\varphi_i\}$ and $\{\mu_j,\phi_j\}$ be the eigenvalues and corresponding eigenvectors of $\mathbf{H}^{(n')}_{\mathbf{C}_{L_k}^{(n')}(\mathbf{u}')}$ and $\mathbf{H}^{(n'')}_{\mathbf{C}_{L_k}^{(n'')}(\mathbf{u}'')}$ respectively. Then we can choose the eigenvectors $\mathbf{\Psi}_{ij}$ and corresponding eigenvalues E_{ij} of $\mathbf{H}_{\mathbf{C}_{L_k}^{(n)}(\mathbf{u})}(\omega)$ as follows.

$$
\mathbf{\Psi}_{ij} = \varphi_i\otimes\phi_j, \qquad E_{ij} = \lambda_i+\mu_j.
$$

By the assumed E-HNR property of the cube $\mathbf{C}_{L_k}^{(n)}(\mathbf{u})$, for all eigenvalues λ_i one has $\mathbf{C}_L^{(n'')}(\mathbf{u}'')$ is $E-\lambda_i$-CNR. Next, by assumption of E-NT, $\mathbf{C}_{L_k}^{(n'')}(\mathbf{u}'')$ does not contain any pair of separable $(E-\lambda_i,m)$-S cubes of radius L_{k-1} therefore by Lemma 2.18 $M(\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{u}),E) < \kappa(n)+2$ and Lemma 2.20 implies that it is also

$(E-\lambda_i, m)$-NS, yielding

$$\max_{\{\lambda_i\}} \max_{\mathbf{v}'' \in \partial^- \mathbf{C}_{L_k}^{(n'')}(\mathbf{u}'')} \left|\mathbf{G}^{(n'')}(\mathbf{u}'', \mathbf{v}''; E-\lambda_i)\right| \leq \mathrm{e}^{-\gamma(m, L_k, n'')L_k}. \tag{3.19}$$

The same analysis for $\mathbf{C}_L^{(n')}(\mathbf{u}')$ also gives

$$\max_{\{\mu_j\}} \max_{\mathbf{v}' \in \partial^- \mathbf{C}_{L_k}^{(n')}(\mathbf{u}')} \left|\mathbf{G}^{(n')}(\mathbf{u}', \mathbf{v}'; E-\mu_j)\right| \leq \mathrm{e}^{-\gamma(m, L_k, n')L_k}. \tag{3.20}$$

For any $\mathbf{v} \in \partial \mathbf{C}_{L_k}^{(n)}(\mathbf{u})$, $|\mathbf{u}-\mathbf{v}| = L_k$, thus either $|\mathbf{v}'-\mathbf{u}'| = L_k$ or $|\mathbf{v}''-\mathbf{u}''| = L_k$. Consider first the latter case. Equation (3.19) applies and we get

$$\begin{aligned}
\left|\mathbf{G}^{(n)}(\mathbf{u}, \mathbf{v}; E)\right| &= \left|\sum_{i,j} \frac{\varphi_i(\mathbf{u}')\varphi_i(\mathbf{v}')\phi_j(\mathbf{u}'')\phi_j(\mathbf{v}'')}{E-\lambda_i-\mu_j}\right| \\
&\leq \sum_i |\varphi_i(\mathbf{u}')\varphi_i(\mathbf{v}')| \cdot \left|\mathbf{G}^{(n)}(\mathbf{u}'', \mathbf{v}''; E-\lambda_i)\right| \\
&\leq (2L_k+1)^{(n-1)d} \max_{\{\lambda_i\}} \max_{\mathbf{v}'' \in \partial \mathbf{C}_{L_k}^{(n)}(\mathbf{u}'')} \left|\mathbf{G}^{(n)}(\mathbf{u}'', \mathbf{v}''; E-\lambda_i)\right|, \quad (\text{since } \|\varphi\|_\infty \leq 1) \\
&\leq (2L_k+1)^{(n-1)d} \cdot \mathrm{e}^{-\gamma(m, L_k, n-1)L_k} \\
&= \mathrm{e}^{-[\gamma(m, L_k, n-1) - L_k^{-1}\ln(2L_k+1)^{(n-1)d}]L_k}.
\end{aligned}$$

But by Definition (2.33):

$$\gamma(m, L_k, n) = m(1+L_k^{-1/8})^{N-n+1},$$

with $m = (12Nd)L_0^{-1/2}$. Moreover, for $2 \leq n \leq N$,

$$\gamma(m, L_k, n-1) - \gamma(m, L_k, n) > L_k^{-1}\ln(2L_k+1)^{(n-1)d}.$$

Indeed, setting $C_1 = 12Nd$,

$$\begin{aligned}
\gamma(m, L_k, n-1) - \gamma(m, L_k, n) &= mL_k^{-1/8}(1+L_k^{-1/8})^{N-n+1} \\
&= C_1 L_0^{-1/2} L_k^{-1/8}(1+L_k^{-1/8})^{N-n+1} > C_1 L_k^{-5/8},
\end{aligned}$$

and for L_0 (hence L_k) sufficiently large,

$$L_k^{-1}\ln(2L_k+1)^{(n-1)d} \leq L_k^{-1}(n-1)d(3L_k)^{3/8} \leq C_1 L_k^{-5/8}.$$

Thus, $\mathbf{C}_{L_k}^{(n)}(\mathbf{u})$ is (E, m)-NS. Finally, the case $|\mathbf{u}'-\mathbf{v}'| = L_k$ is similar. □

Lemma 3.14. *Let $2 \leq n \leq N$ and assume property $(\mathbf{DS}.k, n', N)$ for any $1 \leq n' < n$. Then for any PI cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ one has*

$$\mathbb{P}\big\{\exists E \in I, \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } (E,m)\text{-}T\big\} \leq \frac{1}{2} L_{k+1}^{-4p\,4^{N-n}}. \tag{3.21}$$

Proof. Consider a PI cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) = \mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{y}') \times \mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{y}'')$. Define the events

$\mathrm{T}_{\mathbf{y}}$:={ There exists $E \in I$, such that $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is (E,m)-tunnelling},

$\mathrm{LT}_{\mathbf{y}}$:={ There exist $E \in I$ and $\mu_j \in \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{y}'')})$ such that $\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{y}')$ contains two separable $(E-\mu_j, m)$-singular cubes of size L_k},

$\mathrm{RT}_{\mathbf{y}}$:={ There exist $E \in I$ and $\lambda_i \in \sigma(\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{y}')})$ such that $\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{y}'')$ contains two separable $(E-\lambda_i, m)$-singular cubes of size L_k}.

By Definition 3.10, $\mathrm{T}_{\mathbf{y}} \subset \mathrm{LT}_{\mathbf{y}'} \cup \mathrm{RT}_{\mathbf{y}''}$.

Now, since $E \in I$ and $\mu_j \geq 0$ we have $E - \mu_j \leq E^*$. So for any j, $E - \mu_j \in I$. Further using property $(\mathbf{DS}.k, n', N)$ we have

$$\begin{aligned}\mathbb{P}\left\{\mathrm{LT}_{\mathbf{y}}\right\} &\leq \frac{|\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{y}')|^2}{2} |\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{y}'')| L_k^{-2p4^{N-n'}} \\ &\leq C(n,N,d) L_{k+1}^{-2p\frac{4^{N-(n-1)}}{\alpha}+3(n-1)d}.\end{aligned}$$

A similar argument also shows that

$$\mathbb{P}\left\{\mathrm{RT}_{\mathbf{y}}\right\} \leq C(n,N,d) L_{k+1}^{-2p\frac{4^{N-(n-1)}}{\alpha}+3(n-1)d},$$

so that

$$\mathbb{P}\left\{\exists E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } (E,m)\text{-T}\right\} \leq C(n,N,d) L_{k+1}^{-2p\frac{4^{N-(n-1)}}{\alpha}+3(n-1)d}.$$

The assertion follows by observing that $2p\,4^{N-(n-1)}/\alpha - 3(n-1)d > 4p\,4^{N-n}$ for $\alpha = 3/2$ provided L_0 is large enough and $p > 4\alpha N d = 6Nd$. □

Theorem 3.15. *Let $n = 2, \ldots, N$. There exists $L_1^* = L_1^*(N,d) > 0$ such that if $L_0 \geq L_1^*$ and if for $k \geq 0$ $(\mathbf{DS}.k, n', N)$ holds true for any $n' = 1, \ldots, n-1$, then $(\mathbf{DS}.k+1, n, N)$ holds true for any pair of separable PI cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$.*

Proof. Let $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ be two separable PI-cubes. Consider the events:

$$\begin{aligned}
\mathrm{B}_{k+1} &= \left\{\exists\, E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ and } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ are } (E,m)\text{-S}\right\},\\
\mathrm{R} &= \left\{\exists\, E \in I : \text{neither } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ nor } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } E\text{-HNR}\right\},\\
\mathrm{T}_{\mathbf{x}} &= \left\{\exists E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ is } (E,m)\text{-T}\right\},\\
\mathrm{T}_{\mathbf{y}} &= \left\{\exists E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } (E,m)\text{-T}\right\}.
\end{aligned}$$

If $\omega \in \mathrm{B}_{k+1} \setminus \mathrm{R}$, then $\forall E \in I$, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ or $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-HNR. If $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-HNR, then it must be (E,m)-T: otherwise it would have been (E,m)-NS by Lemma 3.13. Similarly, if $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ is E-HNR, then it must be (E,m)-T. This implies that

$$\mathrm{B}_{k+1} \subset \mathrm{R} \cup \mathrm{T}_{\mathbf{x}} \cup \mathrm{T}_{\mathbf{y}}.$$

Therefore,

$$\begin{aligned}
\mathbb{P}\left\{\mathrm{B}_{k+1}\right\} &\le \mathbb{P}\left\{\mathrm{R}\right\} + \mathbb{P}\{\mathrm{T}_{\mathbf{x}}\} + \mathbb{P}\{\mathrm{T}_{\mathbf{y}}\}\\
&\le \mathbb{P}\left\{\mathrm{R}\right\} + \frac{1}{2} L_{k+1}^{-4p\,4^{N-n}} + \frac{1}{2} L_{k+1}^{-4p\,4^{N-n}},
\end{aligned}$$

where we used (3.21) to estimate $\mathbb{P}\{\mathrm{T}_{\mathbf{x}}\}$ and $\mathbb{P}\{\mathrm{T}_{\mathbf{y}}\}$. Also by Lemma 3.12, we have that $\mathbb{P}\left\{\mathrm{R}\right\} \le L_{k+1}^{-4^N p}$. Finally

$$\mathbb{P}\left\{\mathrm{B}_{k+1}\right\} \le L_{k+1}^{-4^N p} + L_{k+1}^{-4p4^{N-n}} < L_{k+1}^{-2p4^{N-n}}. \tag{3.22}$$

□

Lemma 3.16. *With the above notations, assume that* $(\mathbf{DS}.k-1, n', N)$ *holds true for all* $n' = 1, \dots, n-1$ *then*

$$\mathbb{P}\left\{M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), I) \ge \kappa(n) + 2\right\} \le \frac{3^{2nd}}{2} L_{k+1}^{2nd} \left(L_k^{-4^N p} + L_k^{-4p\,4^{N-n}}\right). \tag{3.23}$$

Proof. Suppose that $M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), I) \ge \kappa(n)+2$, then by Lemma 2.18, $M^{\mathrm{sep}}_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), I) \ge 2$, i.e., there are at least two separable (E,m)-singular PI cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j_1)})$, $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j_2)})$ inside $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$. The number of possible pairs of centers $\{\mathbf{u}^{(j_1)}, \mathbf{u}^{(j_2)}\}$ such that

$$\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j_1)}),\ \mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j_2)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$$

is bounded by $\frac{3^{2nd}}{2}L_{k+1}^{2nd}$. Then, setting

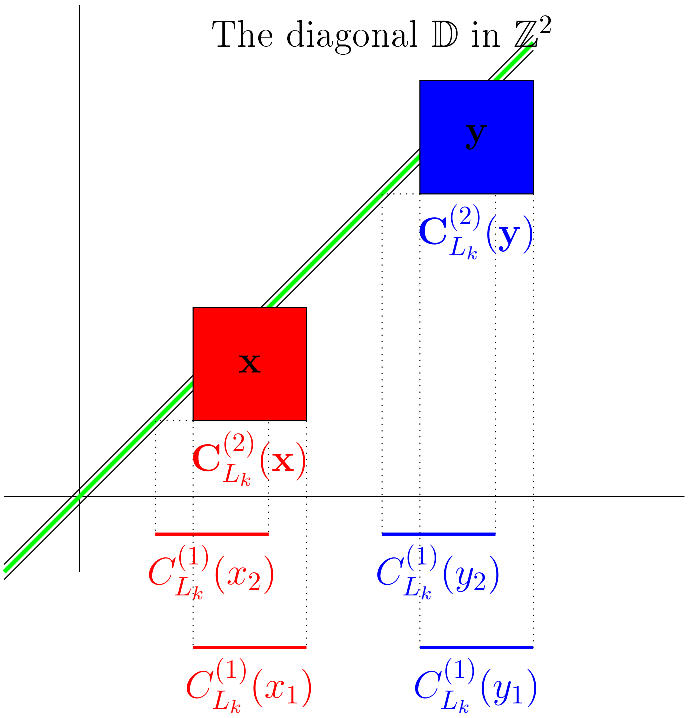
$$\mathrm{B}_k = \{\exists E \in I,\ \mathbf{C}_{L_k}^{(n)}(\mathbf{u}^{(j_1)}),\ \mathbf{C}_{L_k}^{(n)}(\mathbf{u}^{(j_2)}) \text{ are } (E,m)\text{-S}\},$$

we have

$$\mathbb{P}\left\{M_{\mathrm{PI}}^{\mathrm{sep}}(\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{u}), I) \geq 2\right\} \leq \frac{3^{2nd}}{2}L_{k+1}^{2nd} \times \mathbb{P}\left\{\mathrm{B}_k\right\}$$

with $\mathbb{P}\left\{\mathrm{B}_k\right\} \leq L_k^{-4^N p} + L_k^{-4p\,4^{N-n}}$ by (3.22). Here B_k is defined as in Theorem 3.15. $\square$

3.3.2 Pairs of FI cubes



Pairs of FI-cubes: for $d = 1$, $n = 2$, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.

Our aim now is to prove $(\mathbf{DS}.k+1, n, N)$ for a pair of separable fully interactive cubes $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{x})$ and $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{y})$. The main result of this subsection is Theorem 3.18. We need the following preliminary result.

Lemma 3.17. *Given $k \geq 0$, assume that property $(\mathbf{DS}.k, n, N)$ holds true for all pairs of separable FI cubes. Then for any $\ell \geq 1$,*

$$\mathbb{P}\left\{M_{\mathrm{FI}}(\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{u}), I) \geq 2\ell\right\} \leq C(n,N,d,\ell)L_k^{2\ell d n\alpha}L_k^{-2\ell p\,4^{N-n}}. \tag{3.24}$$

Proof. Suppose there exist 2ℓ pairwise separable, fully interactive cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$, $j = 1, \ldots, 2\ell$. Then, by Lemma 2.15, for any pair $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i-1)})$, $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i)})$, the corresponding random Hamiltonians $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i-1)})}$ and $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i)})}$ are independent, and so are their spectra and their Green functions. For $i = 1, \ldots, \ell$ we consider the events:

$$\mathrm{A}_i = \left\{ \exists\, E \in I : \mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i-1)}) \text{ and } \mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i)}) \text{ are } (E,m)\text{-S} \right\}.$$

Then by assumption $(\mathbf{DS}.k,n,N)$, we have, for $i = 1, \ldots, \ell$,

$$\mathbb{P}\left\{\mathrm{A}_i\right\} \le L_k^{-2p\,4^{N-n}}, \tag{3.25}$$

and, by independence of events $\mathrm{A}_1, \ldots, \mathrm{A}_\ell$,

$$\mathbb{P}\Big\{ \bigcap_{1\le i\le \ell} \mathrm{A}_i \Big\} = \prod_{i=1}^{\ell} \mathbb{P}(\mathrm{A}_i) \le \big(L_k^{-2p\,4^{N-n}}\big)^{\ell}. \tag{3.26}$$

To complete the proof, note that the total number of different families of 2ℓ cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$, $j = 1, \ldots, 2\ell$, is bounded by

$$\frac{1}{(2\ell)!} \left| \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}) \right|^{2\ell} \le C(n,N,\ell,d) L_k^{2\ell d n \alpha}. \qquad \square$$

Theorem 3.18. *Let $n = 2, \ldots, N$. There exists $L_2^* = L_2^*(N,d) > 0$ such that if $L_0 \ge L_2^*$ and if for $k \ge 0$*

(i) $(\mathbf{DS}.k-1,n',N)$ *for all $n' = 1, \ldots, n-1$ holds true,*

(ii) $(\mathbf{DS}.k,n,N)$ *holds true for all pairs of FI cubes,*

then $(\mathbf{DS}.k+1,n,N)$ holds true for any pair of separable FI cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$.

Above, we used the convention that $(\mathbf{DS}.-1,n,N)$ means no assumption.

Proof. Consider a pair of separable FI cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ and set $J = \kappa(n) + 5$. Define

$$\begin{aligned}
\mathrm{B}_{k+1} &= \left\{ \exists\, E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ and } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ are } (E,m)\text{-S} \right\}, \\
\Sigma &= \left\{ \exists\, E \in I : \text{neither } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ nor } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } E\text{-CNR} \right\}, \\
\mathrm{S}_{\mathbf{x}} &= \left\{ \exists\, E \in I : M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}); E) \ge J+1 \right\}, \\
\mathrm{S}_{\mathbf{y}} &= \left\{ \exists\, E \in I : M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}), E) \ge J+1 \right\}.
\end{aligned}$$

Let $\omega \in \mathrm{B}_{k+1}$. If $\omega \notin \Sigma \cup \mathrm{S}_{\mathbf{x}}$, then $\forall E \in I$ either $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ or $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-CNR and $M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) \leq J$. Thus the cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ cannot be E-CNR: indeed, by Lemma 2.20 it would be (E,m)-NS. So the cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-CNR and (E,m)-S. This implies again by Lemma 2.20 that

$$M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}), E) \geq J+1.$$

Therefore $\omega \in \mathrm{S}_{\mathbf{y}}$, so that $\mathrm{B}_{k+1} \subset \Sigma \cup \mathrm{S}_{\mathbf{x}} \cup \mathrm{S}_{\mathbf{y}}$, hence

$$\mathbb{P}\left\{\mathrm{B}_{k+1}\right\} \leq \mathbb{P}\{\Sigma\} + \mathbb{P}\{\mathrm{S}_{\mathbf{x}}\} + \mathbb{P}\{\mathrm{S}_{\mathbf{y}}\}$$

and $\mathbb{P}\left\{\Sigma\right\} \leq L_{k+1}^{-4^N p}$ by Corollary 2.9. Now let us estimate $\mathbb{P}\{\mathrm{S}_{\mathbf{x}}\}$ and similarly $\mathbb{P}\{\mathrm{S}_{\mathbf{y}}\}$. Since

$$M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) + M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) \geq M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E),$$

the inequality $M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) \geq \kappa(n)+6$, implies that either $M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) \geq \kappa(n)+2$, or $M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) \geq 4$. Therefore, by Lemma 3.16 and Lemma 3.17 (with $\ell = 2$),

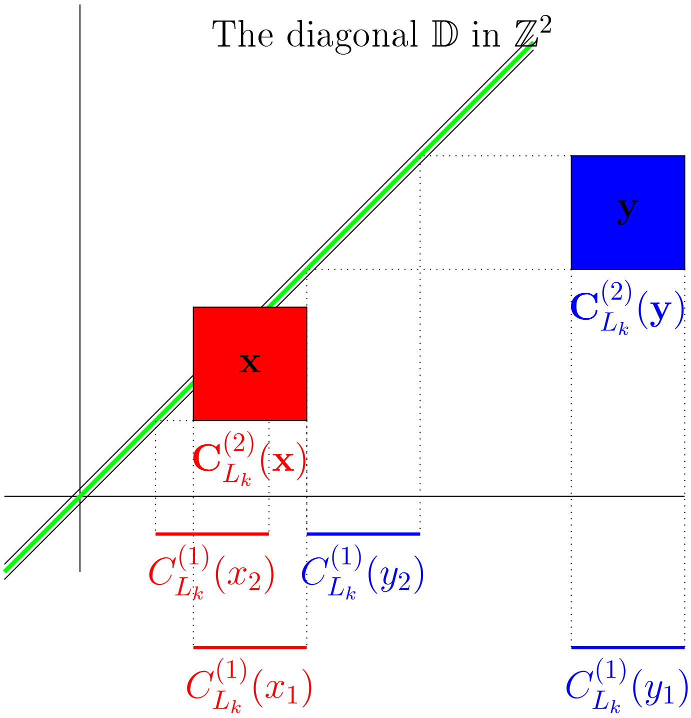
$$\begin{aligned}
\mathbb{P}\{\mathrm{S}_{\mathbf{x}}\} &\leq \mathbb{P}\left\{\exists\, E \in I : M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) \geq \kappa(n)+2\right\} \\
&\quad + \mathbb{P}\left\{\exists\, E \in I : M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) \geq 4\right\} \\
&\leq \frac{3^{2nd}}{2} L_{k+1}^{2nd}(L_k^{-4^N p} + L_k^{-4p\,4^{N-n}}) + C(n,N,d) L_{k+1}^{4dn - \frac{4p}{\alpha}4^{N-n}} \\
&\leq C(n,N,d)\left(L_{k+1}^{-\frac{4^N p}{\alpha}+2nd} + L_{k+1}^{-\frac{4p}{\alpha}4^{N-n}+2nd} + L_{k+1}^{-\frac{4p}{\alpha}4^{N-n}+4nd}\right) \\
&\leq \tilde{C}(n,N,d)\left(L_{k+1}^{-\frac{4p}{\alpha}4^{N-n}+4nd}\right) \qquad (\alpha = 3/2) \\
&\leq \frac{1}{4} L_{k+1}^{-2p\,4^{N-n}},
\end{aligned}$$

where we used $p > 4\alpha N d = 6Nd$. Finally

$$\mathbb{P}\left\{\mathrm{B}_{k+1}\right\} \leq L_{k+1}^{-4^N p} + \frac{1}{2} L_{k+1}^{-2p4^{N-n}} < L_{k+1}^{-2p4^{N-n}}.$$

$\square$

3.3.3 Mixed pairs of cubes



Mixed pairs of cubes: for $d = 1$, $n = 2$, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.

Now it remains only to derive $(\mathbf{DS}.k+1, n, N)$ in case (III), i.e., for pairs of n-particle cubes where one is PI while the other is FI.

Theorem 3.19. *Let $n = 2, \ldots, N$. There exists $L_3^* = L_3^*(N, d) > 0$ such that if $L_0 \geq L_3^*(N, d)$ and if for $k \geq 0$,*

(i) $(\mathbf{DS}.k-1, n', N)$ *holds true for all $n' = 1, \ldots, n-1$,*

(ii) $(\mathbf{DS}.k, n', N)$ *holds true for all $n' = 1, \ldots, n-1$ and*

(iii) $(\mathbf{DS}.k, n, N)$ *holds true for all pairs of FI cubes,*

then $(\mathbf{DS}.k+1, n, N)$ holds true for any pair of separable cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ where one is PI while the other is FI.

Proof. Consider a pair of separable n-particle cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ and suppose that $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ is PI while $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is FI. Set $J = \kappa(n) + 5$ and introduce the events

$$\begin{aligned}
\mathrm{B}_{k+1} &= \left\{\exists\, E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ and } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ are } (E,m)\text{-S}\right\},\\
\mathbf{X} &= \left\{\exists\, E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ is not } E\text{-HNR and } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is not } E\text{-CNR}\right\},\\
\mathrm{T}_{\mathbf{x}} &= \left\{\exists E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ is } (E,m)\text{-T}\right\},\\
\mathrm{S}_{\mathbf{y}} &= \left\{\exists\, E \in I : M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}), E) \geq J+1\right\}.
\end{aligned}$$

Let $\omega \in \mathrm{B}_{k+1} \setminus (\mathbf{X} \cup \mathrm{T}_{\mathbf{x}})$, then for all $E \in I$ either $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ is E-HNR or $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-CNR and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ is (E,m)-NT. The cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ cannot be E-HNR. Indeed, by Lemma 3.13 it would have been (E,m)-NS. Thus the cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-CNR, so by Lemma 2.20, $M(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}); E) \geq J+1$: otherwise $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ would be (E,m)-NS. Therefore $\omega \in \mathrm{S}_{\mathbf{y}}$. Consequently,

$$\mathrm{B}_{k+1} \subset \mathbf{X} \cup \mathrm{T}_{\mathbf{x}} \cup \mathrm{S}_{\mathbf{y}}.$$

Recall that the probabilities $\mathbb{P}\{\mathrm{T}_{\mathbf{x}}\}$ and $\mathbb{P}\{\mathrm{S}_{\mathbf{y}}\}$ have already been estimated in Sections 3.3.1 and 3.3.2. Also $\mathbb{P}\{\mathbf{X}\} \leq L_{k+1}^{-4^N p}$ by Lemma 3.12. We therefore obtain

$$\begin{aligned}
\mathbb{P}\{\mathrm{B}_{k+1}\} &\leq \mathbb{P}\{\mathbf{X}\} + \mathbb{P}\{\mathrm{T}_{\mathbf{x}}\} + \mathbb{P}\{\mathrm{S}_{\mathbf{y}}\}\\
&\leq L_{k+1}^{-4^N p} + \frac{1}{2} L_{k+1}^{-4p\,4^{N-n}} + \frac{1}{4} L_{k+1}^{-2p\,4^{N-n}} \leq L_{k+1}^{-2p\,4^{N-n}}.
\end{aligned}$$

□

3.3.4 Conclusion

Theorem 3.20. *Let $n = 1, \dots, N$ and $\mathbf{H}^{(n)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^{n} V(x_j, \omega) + \mathbf{U}$, where $\mathbf{U}$, V satisfy* (**I1**) *and* (**P1**) *respectively. Then for any $p > 6Nd$ there exist $m > 0$ and $E^* > E_0^{(N)}$ such that* (**DS**.k,n,N) *holds true for all $k \geq 0$ provided $L_0 \geq L_0^*(N,d,r-0) > 0$ for some large enough $L_0^*(N,d,r_0) > 0$.*

Proof. We show by induction on n that (**DS**.k,n,N) holds true for all $k \geq 0$. Notice that the parameters $m, E^* > 0$ are given by Theorem 3.6. For $n = 1$, property (**DS**.$k,1,N$) for all $k \geq 0$ is valid using Theorems 3.6 and 3.7. Now suppose that for all $n' = 1, \dots, n-1$, (**DS**.k,n',N) holds true for all $k \geq 0$, we aim to prove that (**DS**.k,n,N) holds true for all $k \geq 0$. For $k = 0$, (**DS**.$0,n,N$) is valid using Theorem 3.6. Next, suppose that (**DS**.k',n,N) holds true for all $k' < k$, then combining this last assumption with (**DS**.k,n',N) above, one can conclude that

(i) $(\mathbf{DS}.k, n, N)$ holds true for all $k \geq 0$ and for all pairs of PI cubes, using Theorem 3.15,

(ii) $(\mathbf{DS}.k, n, N)$ holds true for all $k \geq 0$ and for all pairs of FI cubes, using Theorem 3.18,

(iii) $(\mathbf{DS}.k, n, N)$ holds true for all $k \geq 0$ and for all mixed (PI + FI) pairs of cubes, using Theorem 3.19.

Hence Theorem 3.20 is proven. □

3.4 Almost sure lower spectral edge: proof of Theorem 3.1

The proof is based on the well-known Weyl argument. Assumption (**I1**) implies that the interaction $\mathbf{U}$ is non-negative and (**P1**) implies that the random potential is non-negative almost surely. So $\sigma(\mathbf{H}^{(n)}(\omega)) \subset [0, +\infty)$ almost surely. It remains to prove that $[0, 4nd] \subset \sigma(\mathbf{H}^{(n)}(\omega))$ *a.s.* Let $k, m \in \mathbb{N}^*$. Define $I_{k,m} = [0, \frac{1}{km}]$ and

$$\mathrm{B}_{k,m} = \left\{ \mathbf{x} \in \mathbb{Z}^{nd} : \min_{i \neq j} |x_i - x_j| > r_0 + 2km \right\}$$

where r_0 is the range of the interaction $\mathbf{U}$. For $k, m \in \mathbb{N}^*$, introduce the following sequence $\{\mathbf{x}^{\ell,k,m}\}_{\ell \in \mathbb{N}^*}$ defined by

$$\mathbf{x}^{\ell,k,m} = C_{k,m}(C_{k,m}\ell + 1, C_{k,m}\ell + 2, \cdots, C_{k,m}\ell + nd) \in \mathbb{Z}^{nd},$$

where $C_{k,m} = r_0 + 2km + Nd + 1$. Obviously $\mathbf{x}^{\ell,k,m} \in \mathrm{B}_{k,m}$ for any $\ell \in \mathbb{N}^*$. Using the identification $\mathbb{Z}^{nd} \cong (\mathbb{Z}^d)^n$, $\mathbf{x}^{\ell,k,m}$ write as $\mathbf{x}^{\ell,k,m} = C_{k,m}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ with $\mathbf{x}_i = ((i-1)d + 1 + C_{k,m}\ell, \ldots, id + C_{k,m}\ell)$, $i = 1, \ldots, n$.

We claim that for $\ell \neq \ell'$, $\Pi\mathbf{C}^{(n)}_{km}(\mathbf{x}^{\ell,k,m}) \cap \Pi\mathbf{C}^{(n)}_{km}(\mathbf{x}^{\ell',k,m}) = \emptyset$. Indeed

$$\begin{aligned} \operatorname{dist}\left(\Pi\mathbf{C}^{(n)}_{km}(\mathbf{x}^{\ell,k,m}), \Pi\mathbf{C}^{(n)}_{km}(\mathbf{x}^{\ell',k,m})\right) &= \min_{1 \leq i,j \leq n} \operatorname{dist}\left(\mathbf{C}^{(n)}_{km}(\mathbf{x}_i^{\ell,k,m}), \mathbf{C}^{(n)}_{km}(\mathbf{x}_j^{\ell',k,m})\right) \\ &\geq \min_{i,j} \left|\mathbf{x}_i^{\ell,k,m} - \mathbf{x}_j^{\ell',k,m}\right| - 2km. \end{aligned}$$

For any $i, j \in \{1, \ldots, n\}$,

$$\begin{aligned} \left|\mathbf{x}_i^{\ell,k,m} - \mathbf{x}_j^{\ell',k,m}\right| &= C_{k,m}|(i-j)d + C_{k,m}(\ell - \ell')| \\ &\geq C^2_{k,m}|\ell - \ell'| - C_{k,m}|i-j|d. \end{aligned}$$

Thus

$$\begin{aligned}\operatorname{dist}\left(\Pi\mathbf{C}^{(n)}_{km}(\mathbf{x}^{\ell,k,m}),\Pi\mathbf{C}^{(n)}_{km}(\mathbf{x}^{\ell',k,m})\right)&\geq C^2_{k,m}-C_{k,m}\max_{i,j}|i-j|d-2km\\&\geq C^2_{k,m}-C_{k,m}Nd-2km\\&=C_{k,m}(C_{k,m}-Nd)-2km>0.\end{aligned}$$

For any $\ell,k,m\in\mathbb{N}^*$, define

$$\Omega_{\ell,k,m}(\mathbf{x}^{\ell,k,m})=\left\{\omega\in\Omega:\forall j=1,\dots,n,\forall y_j\in C^{(1)}_{km}(x^{\ell,k,m}_j),V(y_j,\omega)\in I_{km}\right\}.$$

We have that $\mathbb{P}\left\{\Omega_{\ell,k,m}(\mathbf{x}^{\ell,k,m})\right\}=\mu(I_{km})^{n(2km+1)^d}$ and the latter quantity is >0, since $0\in\operatorname{supp}\mu$. So $\sum_{\ell\geq 1}\mathbb{P}\left\{\Omega_{\ell,k,m}(\mathbf{x}^{\ell,k,m})\right\}=+\infty$. Since $\Pi\mathbf{C}^{(n)}_{km}(\mathbf{x}^{\ell,k,m})\cap\Pi\mathbf{C}^{(n)}_{km}(\mathbf{x}^{\ell',k,m})=\emptyset$, the corresponding events $\Omega_{\ell,k,m}(\mathbf{x}^{\ell,k,m})$ and $\Omega_{\ell 4,k,m}(\mathbf{x}^{\ell',k,m})$ are independent. Set

$$\Omega_{k,m}=\left\{\omega:\Omega_{\ell,k,m}(\mathbf{x}^{\ell,k,m})\text{ occurs for infinitely many }\ell\geq 1\right\};$$

the Borel-Cantelli Lemma implies that $\mathbb{P}\left\{\Omega_{k,m}\right\}=1$. Note that if $\omega\in\Omega_{k,m}$, then $\exists\ell\geq 1$ such that $\omega\in\Omega^{\ell,k,m}(\mathbf{x}^{\ell,k,m})$. Define

$$\Omega_\infty=\bigcap_{k\geq 1}\bigcap_{m\geq 1}\Omega_{k,m}.$$

We have that $\mathbb{P}\left\{\Omega_\infty\right\}=1$. Now, let $\omega\in\Omega_\infty$ and recall that $[0,4nd]\subset\sigma(\mathbf{H}^{(n)}(\omega))$. Since $\omega\in\Omega_\infty$, we have that for any $k,m\geq 1$, $\exists\ell^*\geq 1$ such that $\omega\in\Omega_{\ell^*,k,m}(\mathbf{x}^{\ell^*,k,m})$. Recall that $[0,4nd]=\sigma(-\boldsymbol{\Delta})$. Let $E\in[0,4nd]$. By Theorem A.4, there exists a Weyl sequence ϕ^E_m for E and $-\boldsymbol{\Delta}$ with a bounded support, i.e., $\|\phi^E_m\|=1$, $\|((-\boldsymbol{\Delta})-E)\phi^E_m\|\to 0$ as $m\to\infty$ and $\operatorname{supp}\phi^E_m\subset\mathbf{C}^{(n)}_{k_Em}(\mathbf{0})$ for some $k_E\in\mathbb{N}^*$. The translated sequence $\widetilde{\phi}^E_m$ defined by $\widetilde{\phi}^E_m(\mathbf{x})=\phi^E_m(\mathbf{x}-\mathbf{x}^{\ell^*,k_E,m})$ is also a Weyl sequence for E and $(-\boldsymbol{\Delta})$, i.e,

$$\|((-\boldsymbol{\Delta})-E)\widetilde{\phi}^E_m\|\underset{m\to\infty}{\longrightarrow}0,\tag{3.27}$$

and $\operatorname{supp}\widetilde{\phi}^E_m\subset\mathbf{C}^{(n)}_{k_Em}(\mathbf{x}^{\ell^*,k_E,m})$. Further, for all $\mathbf{y}\in\mathbb{Z}^{nd}$ one has

$$\left|\mathbf{V}(\mathbf{y},\omega)\widetilde{\phi}^E_m(\mathbf{y})\right|\leq\frac{n}{k_Em}|\widetilde{\phi}^E_m(\mathbf{y})|.\tag{3.28}$$

Indeed, both sides of (3.28) vanish outside $\mathbf{C}^{(n)}_{k_Em}(\mathbf{x}^{\ell^*,k_E,m})$, while on $\mathbf{C}^{(n)}_{k_Em}(\mathbf{x}^{\ell^*,k_E,m})$ it follows directly from our choice of ω. Therefore,

$$\left\|\mathbf{V}(\omega)\widetilde{\phi}^E_m\right\|\leq\frac{n}{k_Em}\|\widetilde{\phi}^E_m\|,\tag{3.29}$$

while $\mathbf{U}\widetilde{\phi}_m^E = 0$. Because $\mathbf{U}$ vanishes on $\mathbf{C}^{(n)}_{k_E m}(\mathbf{x}^{\ell^*,k_E,m})$ since $\mathbf{x}^{\ell^*,k_E,m} \in B_{k_E m}$.

Collecting (3.27) and (3.29), we conclude that

$$\|(\mathbf{H}(\omega) - E)\widetilde{\phi}_m\| \leq \|((-\boldsymbol{\Delta}) - E)\widetilde{\phi}_m\| + \|\mathbf{V}\widetilde{\phi}_m\| \underset{m\to\infty}{\longrightarrow} 0.$$

Thus $\{\widetilde{\phi}_m^E, m \in \mathbb{N}^*\}$ is indeed a Weyl sequence for $\mathbf{H}^{(n)}(\omega)$ and E. Therefore $E \in \sigma(\mathbf{H}^{(n)}(\omega))$. Finally, since $\mathbb{P}\{\Omega_\infty\} = 1$, we conclude that $[0, 4nd] \subset \sigma(\mathbf{H}^{(n)}(\omega))$ almost surely. $\square$

3.5 Exponential localization at low energy: proof of Theorem 3.2

Now we will derive from the results of Theorem 3.20, spectral exponential localization in the low energy regime. We will use the fact ([Ber68, Kir08, Sim83]) that almost every energy E with respect to the spectral measure of $\mathbf{H}^{(N)}(\omega)$ is a generalized eigenvalue of $\mathbf{H}^{(N)}(\omega)$, i.e., there is a polynomially bounded solution of the equation $\mathbf{H}^{(N)}(\omega)\mathbf{\Psi} = E\mathbf{\Psi}$. It suffices therefore to prove that with probability one, the generalized eigenfunctions of $\mathbf{H}^{(N)}(\omega)$ decay exponentially fast at infinity. For a proof see Theorem A.7 Let $E \in [E_0^{(N)}, E^*]$ be a generalized eigenvalue of $\mathbf{H}^{(N)}(\omega)$. Following [vDK89, CS09a, FMSS85], we will prove that if the bound $(\mathbf{DS}.k, N)$ is satisfied for all $k \geq 0$, with some $m > 0$, then

$$\forall \tilde{\rho} \in (0,1): \quad \limsup_{|\mathbf{x}|\to\infty} \ln \frac{|\mathbf{\Psi}(\mathbf{x},\omega)|}{|\mathbf{x}|} \leq -\tilde{\rho} m.$$

Given $\mathbf{u} \in \mathbb{Z}^{Nd}$ and an integer $k \geq 0$, set, using the notations of Lemma 2.3,

$$R(\mathbf{u}) := \max_{\ell=1,\ldots,\kappa(N)} |\mathbf{u}-\mathbf{u}^{(\ell)}|, \quad b_k(\mathbf{u}) := 7N + R(\mathbf{u})L_k^{-1}, \quad M_k(\mathbf{u}) = \bigcup_{\ell=1}^{\kappa(N)} \mathbf{C}^{(N)}_{7NL_k}(\mathbf{u}^{(\ell)})$$

and define

$$A_{k+1}(\mathbf{u}) := \mathbf{C}^{(N)}_{bb_{k+1}L_{k+1}}(\mathbf{u}) \setminus \mathbf{C}^{(N)}_{b_k L_k}(\mathbf{u}),$$

where the parameter $b > 0$ will be chosen later. One can easily check that:

$$M_k(\mathbf{u}) \subset \mathbf{C}^{(N)}_{b_k L_k}(\mathbf{u}).$$

Hence, if $\mathbf{x} \in A_{k+1}(\mathbf{u})$, then the cubes $\mathbf{C}^{(N)}_{L_k}(\mathbf{x})$ and $\mathbf{C}^{(N)}_{L_k}(\mathbf{u})$ are separable by Lemma 2.3. Define the event

$$\Omega_k(\mathbf{u}) := \Big\{\exists E \in [E_0^{(N)}, E^*], \text{ and } \mathbf{x} \in A_{k+1}(\mathbf{u})\text{: } \mathbf{C}^{(N)}_{L_k}(\mathbf{x}) \text{ and } \mathbf{C}^{(N)}_{L_k}(\mathbf{u}) \text{ are } (E,m)\text{-S}\Big\}.$$

Property (**DS**.k, N) implies that

$$\begin{aligned}\mathbb{P}\left\{\Omega_k(\mathbf{u})\right\} &\leq \mathbb{P}\left\{\exists E \in (-\infty, E^*],\ \text{and } \mathbf{x} \in A_{k+1}(\mathbf{u})\text{: } \mathbf{C}^{(N)}_{L_k}(\mathbf{x}) \text{ and } \mathbf{C}^{(N)}_{L_k}(\mathbf{u}) \text{ are } (E,m)\text{-S}\right\}\\ &\leq (2bb_{k+1}L_{k+1}+1)^{Nd}L_k^{-2p}\\ &\leq (2bb_{k+1}+1)^{Nd}L_k^{-2p+\alpha Nd},\end{aligned}$$

since $p > (\alpha Nd + 1)/2$ (in fact $p > 6Nd$) we have $\sum_{k=0}^{\infty} \mathbb{P}\left\{\Omega_k(\mathbf{u})\right\} < \infty$. Thus, setting

$$\Omega_{<\infty} := \{\forall \mathbf{u} \in \mathbb{Z}^{Nd}, \Omega_k(\mathbf{u}) \text{ occurs finitely many times}\}.$$

By the Borel-Cantelli Lemma and the countability of $\mathbb{Z}^{Nd}$, we have that

$$\mathbb{P}\left\{\Omega_{<\infty}\right\} = 1.$$

Therefore, it suffices to pick $\omega \in \Omega_{<\infty}$ and prove the exponential decay of any nonzero generalised eigenfunction $\mathbf{\Psi}$ of $\mathbf{H}^{(N)}(\omega)$. Since $\mathbf{\Psi}$ is polynomially bounded, there exist $C, t \in (0,\infty)$ such that for all $\mathbf{x} \in \mathbb{Z}^{Nd}$

$$|\mathbf{\Psi}(\mathbf{x},\omega)| \leq C(|\mathbf{x}|)^t.$$

Since $\mathbf{\Psi}$ is not identically zero, there exists $\mathbf{u} \in \mathbb{Z}^{Nd}$ such that $\mathbf{\Psi}(\mathbf{u}) \neq 0$. Let us show that there is an integer $k_1 = k_1(\omega, E, \mathbf{u})$ such that $\forall k \geq k_1$, the cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is (E,m)-S. Indeed, given an integer $k \geq 0$, assume that $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is (E,m)-NS. Then by the geometric resolvent inequality for the eigenfunction (2.9), combined with the definition of an (E,m)-NS cube (cf. (2.32)), we have

$$\begin{aligned}|\mathbf{\Psi}(\mathbf{u})| &\leq C(N,d)L_k^{Nd-1}\mathrm{e}^{-mL_k}\cdot \max_{\mathbf{v}:|\mathbf{v}-\mathbf{u}|\leq L_k+1}|\mathbf{\Psi}(\mathbf{v})|\\ &\leq C'(N,d)L_k^{Nd-1}\mathrm{e}^{-mL_k}(1+|\mathbf{u}|+L_k)^t \underset{L_k\to\infty}{\longrightarrow} 0.\end{aligned}$$

This shows that if $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is (E,m)-NS for arbitrary large values of L_k (i.e., for an infinite number of values k), then $|\mathbf{\Psi}(\mathbf{u})| = 0$, in contradiction with the definition of the point $\mathbf{u}$. So there is an integer $k_1 = k_1(\omega, E, \mathbf{u}) < \infty$ such that $\forall k \geq k_1$ the cube $\mathbf{C}^{(N)}_{L_k}(\mathbf{u})$ is (E,m)-S. At the same time since $\omega \in \Omega_{<\infty}$ there exists $k_2 = k_2(\omega, \mathbf{u})$ such that if $k \geq k_2$, $\Omega_k(\mathbf{u})$ does not occurs. We conclude that $\forall k \geq \max\{k_1, k_2\}$ for all $\mathbf{x} \in A_{k+1}(\mathbf{u})$, $\mathbf{C}^{(N)}_{L_k}(\mathbf{x})$ is (E,m)-NS. Let $\rho \in (0,1)$ and choose b such that

$$b > \frac{1+\rho}{1-\rho},$$

so that obviously $\mathbf{C}^{(N)}_{\frac{b_kL_k}{1-\rho}}(\mathbf{u}) \subset \mathbf{C}^{(N)}_{\frac{bb_{k+1}L_{k+1}}{1+\rho}}(\mathbf{u})$. Define

$$\tilde{A}_{k+1}(\mathbf{u}) = \mathbf{C}^{(N)}_{\frac{bb_{k+1}L_{k+1}}{1+\rho}}(\mathbf{u}) \setminus \mathbf{C}^{(N)}_{\frac{b_kL_k}{1-\rho}}(\mathbf{u}) \subset A_{k+1}(\mathbf{u}).$$

Fix $\mathbf{x} \in \tilde{A}_{k+1}(\mathbf{u})$.

1. Since $|\mathbf{x}-\mathbf{u}| > \frac{b_k L_k}{1-\rho}$,

$$\begin{aligned}\operatorname{dist}\Big(\mathbf{x}, \partial^+ \mathbf{C}^{(N)}_{b_k L_k}(\mathbf{u})\Big) &= |\mathbf{x}-\mathbf{u}| - b_k L_k \\ &> |\mathbf{x}-\mathbf{u}| - (1-\rho)|\mathbf{x}-\mathbf{u}| \\ &= \rho|\mathbf{x}-\mathbf{u}|.\end{aligned}$$

2. Since $|\mathbf{x}-\mathbf{u}| \le \frac{b b_{k+1} L_{k+1}}{1+\rho}$,

$$\begin{aligned}\operatorname{dist}\Big(\mathbf{x}, \partial^- \mathbf{C}^{(N)}_{b b_{k+1} L_{k+1}}(\mathbf{u})\Big) &= b b_{k+1} L_{k+1} - |\mathbf{x}-\mathbf{u}| \\ &\ge (1+\rho)|\mathbf{x}-\mathbf{u}| - |\mathbf{x}-\mathbf{u}| \\ &= \rho|\mathbf{x}-\mathbf{u}|.\end{aligned}$$

The interior boundary of the annulus $A_{k+1}(\mathbf{u})$ is given by $\partial^- A_{k+1}(\mathbf{u}) = \partial^- \mathbf{C}^{(N)}_{b b_{k+1} L_{k+1}}(\mathbf{u}) \cup \partial^+ \mathbf{C}^{(N)}_{b_k L_k}(\mathbf{u})$. Thus

$$\begin{aligned}\operatorname{dist}\Big(\mathbf{x}, \partial^- A_{k+1}(\mathbf{u})\Big) &= \min\Big[\operatorname{dist}\Big(\mathbf{x}, \partial^+ \mathbf{C}^{(N)}_{b_k L_k}(\mathbf{u})\Big), \operatorname{dist}\Big(\mathbf{x}, \partial^- \mathbf{C}^{(N)}_{b b_{k+1} L_{k+1}}(\mathbf{u})\Big)\Big] \\ &\ge \rho|\mathbf{x}-\mathbf{u}|.\end{aligned}$$

We have that if $|\mathbf{x}-\mathbf{u}| > b_0 L_0/(1-\rho)$ then there exists $k \ge 0$ such that $\mathbf{x} \in \tilde{A}_{k+1}(\mathbf{u})$.

Now, let $k \ge \max\{k_1, k_2\}$, so the cube $\mathbf{C}^{(N)}_{L_k}(\mathbf{x})$ must be (E, m)-NS. Hence by the geometric resolvent inequality for eigenfunctions (2.9), we get

$$|\mathbf{\Psi}(\mathbf{x})| \le C(N,d) L_k^{Nd-1} \mathrm{e}^{-m(1+L_k^{-1/8})L_k} |\mathbf{\Psi}(\mathbf{v}_1)| \quad \text{with } \mathbf{v}_1 \in \partial^+ \mathbf{C}^{(N)}_{L_k}(\mathbf{x}). \tag{3.30}$$

If $\mathbf{x} \in \tilde{A}_{k+1}(\mathbf{u})$ with $k \ge \max\{k_1, k_2\}$, we can iterate the bound (3.30) at least $(L_k+1)^{-1}\rho|\mathbf{x}-\mathbf{u}|$ times and, using the polynomial bound on $\mathbf{\Psi}$ to obtain

$$|\mathbf{\Psi}(\mathbf{x})| \le \Big[C(N,d) L_k^{Nd-1} \mathrm{e}^{-m(1+L_k^{-1/8})L_k}\Big]^{(L_k+1)^{-1}\rho|\mathbf{x}-\mathbf{u}|} C(1+|\mathbf{u}|+bL_{k+1})^t. \tag{3.31}$$

We can conclude that given ρ', $0 < \rho' < 1$, we can find $k_3 \ge \max\{k_1, k_2\}$ such that if $k \ge k_3$ then

$$|\mathbf{\Psi}(\mathbf{x})| \le \mathrm{e}^{-\rho\rho' m|\mathbf{x}-\mathbf{u}|},$$

if $|\mathbf{x}-\mathbf{u}| > \frac{b_{k_3} L_{k_3}}{1-\rho}$. Finally, we see that

$$\limsup_{|\mathbf{x}|\to\infty} \frac{1}{|\mathbf{x}|} \ln|\mathbf{\Psi}(\mathbf{x})| \le -\rho\rho' m. \quad \square$$

3.6 Dynamical localization at low energies: Proof of Theorem 3.3

Dynamical localization via multi-scale analysis was proven initially by Germinet and De Bièvre [GB98] and by Damanik and Stollmann (cf. [DS01, Sto01]). In our work, we use a different approach, originally developed by Germinet and Klein [GK01] in the framework of differential operators in $\mathbb{R}^d$. This will allows us to derive from the results of the multiparticle multi-scale analysis strong localization.

Before we turn to the proof of Theorem 3.3, we need to make the following observations. For every bounded set $\mathbf{K} \subset \mathbb{Z}^{Nd}$ figuring in Theorem 3.3 there exists $k_0 > 0$ such that $\mathbf{K} \subset \mathbf{C}^{(N)}_{L_{k_0}}(\mathbf{0})$. Now for $j \geq k_0$, set

$$\mathbf{M}_j(\mathbf{0}) = \mathbf{C}^{(N)}_{(7N+1)L_{j+1}}(\mathbf{0}) \setminus \mathbf{C}^{(N)}_{(7N+1)L_j}(\mathbf{0}).$$

Observe that for any $\mathbf{y} \in \mathbf{C}^{(N)}_{L_j}(\mathbf{0})$ we have $\mathbf{C}^{(N)}_{7NL_j}(\mathbf{y}) \subset \mathbf{C}^{(N)}_{(7N+1)L_j}(\mathbf{0})$. Hence, if $\mathbf{x} \in \mathbf{M}_j(\mathbf{0})$ and $\mathbf{y} \in \mathbf{C}^{(N)}_{L_j}(\mathbf{0})$, we have that $|\mathbf{y} - \mathbf{x}| > 7NL_j$. Since $\operatorname{diam}(\varPi\mathbf{y}) \leq 2L_j$, it follows that $|\mathbf{y} - \mathbf{x}| > \operatorname{diam}(\varPi\mathbf{y}) + 3NL_j$. Thus using assertion (B) of Lemma 2.3, the cubes $\mathbf{C}^{(N)}_{L_j}(\mathbf{x})$ and $\mathbf{C}^{(N)}_{L_j}(\mathbf{y})$ are separable. We need the following statement establishing the decay of the kernels in the Hilbert-Schmidt norm.

Lemma 3.21. *Under assumptions* (**I1**) *and* (**P1**). *Then there exists an integer $k_1 \geq 0$ such that for any bounded function $f : \mathbb{R} \to \mathbb{R}$ all $j \geq k_1$, $\mathbf{x} \in \mathbf{M}_j(\mathbf{0})$ and $\mathbf{y} \in \mathbf{C}^{(N)}_{L_j}(\mathbf{0})$*

$$\mathbb{E}\left[\sup_{\|f\|_\infty \leq 1} \|\delta_{\mathbf{x}} f(\mathbf{H})\mathbf{P}_I(\mathbf{H})\delta_{\mathbf{y}}\|^2_{HS}\right] \leq \mathrm{e}^{-mL_j/2} + L_j^{-2p}, \tag{3.32}$$

where $p > 6Nd$.

Proof. Define

$$B_j := \left\{\forall\lambda \in I, \text{either } \mathbf{C}^{(N)}_{L_j}(\mathbf{x}) \text{ or } \mathbf{C}^{(N)}_{L_j}(\mathbf{y}) \text{ is } (\lambda, m)\text{-NS}\right\}.$$

Consider a bounded function $f : \mathbb{R} \to \mathbb{R}$ and set $f_I = f\chi_I$, where χ_I is the indicator function of the interval I. We have by the definition of the Hilbert-Schmidt norm:

$$
\begin{aligned}
\|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|_{HS}^2 &= \sum_{\mathbf{z},\mathbf{v}\in\mathbb{Z}^{Nd}} |\langle \delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\delta_{\mathbf{z}}, \delta_{\mathbf{v}}\rangle|^2 \\
&= \sum_{\mathbf{z},\mathbf{v}\in\mathbb{Z}^{Nd}} |\langle f_I(\mathbf{H})\delta_{\mathbf{y}}\delta_{\mathbf{z}}, \delta_{\mathbf{x}}\delta_{\mathbf{v}}\rangle|^2 \\
&= |\langle f_I(\mathbf{H})\delta_{\mathbf{y}}, \delta_{\mathbf{x}}\rangle|^2 \\
&= |\langle \delta_{\mathbf{x}}, f_I(\mathbf{H})\delta_{\mathbf{y}}\rangle|^2 .
\end{aligned}
$$

If $\omega \in B_j$ then either $\mathbf{C}^{(N)}_{L_j}(\mathbf{x})$ or $\mathbf{C}^{(N)}_{L_j}(\mathbf{y})$ is (λ, m)-NS for all $\lambda \in I$. Since

$$
|\,\langle \delta_{\mathbf{x}}, f_I(\mathbf{H})\delta_{\mathbf{y}}\rangle\,| = |\,\langle \delta_{\mathbf{y}}, f_I(\mathbf{H})\delta_{\mathbf{x}}\rangle\,|,
$$

we can assume without loss of generality that $\mathbf{C}^{(N)}_{L_j}(\mathbf{x})$ is (λ, m)-NS. Now using Theorem A.7, we get

$$
\begin{aligned}
\|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|_{HS} &\le |\langle \delta_{\mathbf{x}}, f_I(\mathbf{H})\delta_{\mathbf{y}}\rangle| \\
&\le \int_I |f(\lambda)|\,|T_{\mathbf{x},\mathbf{y}}(\lambda)|\,d\rho(\lambda)
\end{aligned}
$$

and the function $\mathbf{\Psi} : \mathbf{x} \to T_{\mathbf{x},\mathbf{y}}(\lambda)$ is a polynomially bounded generalized eigenfunction for ρ a.e. λ. Next, the geometric resolvent inequality for generalised eigenfunctions (2.9) gives

$$
\begin{aligned}
|\mathbf{\Psi}(\mathbf{x})| &\le \left|\partial \mathbf{C}^{(N)}_{L_j}(\mathbf{x})\right| \mathrm{e}^{-mL_j} |\mathbf{\Psi}(\mathbf{x}')| \\
&\le C(N,d) L_j^{Nd-1} \mathrm{e}^{-mL_j} (1 + |\mathbf{x}| + L_j)^t \\
&\le C(N,d) L_j^{\alpha t + Nd - 1} \mathrm{e}^{-mL_j} < \mathrm{e}^{-mL_j/2},
\end{aligned}
$$

for $j \ge k_1$ with $k_1 \ge 0$ large enough. Yielding

$$
\begin{aligned}
\|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|_{HS} &\le \|f\|_\infty \rho(I) \mathrm{e}^{-mL_j/2} \\
&\le \|f\|_\infty \mathrm{e}^{-mL_j/2}.
\end{aligned}
$$

For $\omega \in B_j^c$, we have

$$
\|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|_{HS} = |\langle \delta_{\mathbf{x}}, f_I(\mathbf{H})\delta_{\mathbf{y}}\rangle| \le \|f\|_\infty.
$$

Finally, we can conclude that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{\|f\|_\infty \le 1} \|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|_{HS}^2\right] &\le \mathrm{e}^{-mL_j/2}\mathbb{P}\{B_j\} + \mathbb{P}\{B_j^c\} \\
&\le \mathrm{e}^{-mL_j/2} + L_j^{-2p}.
\end{aligned}
$$

Where we used $(\mathbf{DS}.k, N)$ to bound $\mathbb{P}\{B_j^c\}$. □

We are now ready to finish the proof of Theorem 3.3. Namely, let $\mathbf{K} \subset \mathbb{Z}^{Nd}$, $j \geq k_1$ as in Lemma 3.21 and $p > 6Nd$ the parameter appearing in the RHS of property ($\mathbf{DS}.k, N$). Set $s^* = \frac{2p}{\alpha} - Nd - 1$. For any $s \in (0, s^*)$ and any bounded function $f : \mathbb{R} \to \mathbb{R}$ we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{\|f\|_\infty \leq 1} \left\| |X|^{\frac{s}{2}} f_I(\mathbf{H}) \mathbf{1}_{\mathbf{K}} \right\|_{HS}^2\right] \\
&\leq \mathbb{E}\left[\sup_{\|f\|_\infty \leq 1} \left\| \mathbf{1}_{\mathbf{C}^{(N)}_{(7N+1)L_{k_1}}(\mathbf{0})} |X|^{\frac{s}{2}} f_I(\mathbf{H}) \mathbf{1}_{\mathbf{K}} \right\|_{HS}^2\right] \\
&\qquad + \sum_{j \geq k_1} C_1(N,d) L_{j+1}^s \sum_{\mathbf{x} \in \mathbf{M}_j(\mathbf{0})} \mathbb{E}\left[\sup_{\|f\|_\infty \leq 1} \|\delta_{\mathbf{x}} f_I(\mathbf{H}) \mathbf{1}_{\mathbf{K}}\|_{HS}^2\right] \\
&\leq \mathbb{E}\left[\sup_{\|f\|_\infty \leq 1} \sum_{\mathbf{x} \in \mathbb{Z}^{Nd}} \left\| \mathbf{1}_{\mathbf{C}^{(N)}_{(7N+1)L_{k_1}}(\mathbf{0})} |X|^{\frac{s}{2}} f_I(\mathbf{H}) \mathbf{1}_{\mathbf{K}} \delta_{\mathbf{x}} \right\|^2\right] \\
&\qquad + \sum_{j \geq k_1} C_1(N,d) L_{j+1}^s \sum_{\substack{\mathbf{x} \in \mathbf{M}_j(\mathbf{0}) \\ \mathbf{y} \in \mathbf{C}^{(N)}_{L_{k_1}}(\mathbf{0})}} \mathbb{E}\left[\sup_{\|f\|_\infty \leq 1} \|\delta_{\mathbf{x}} f_I(\mathbf{H}) \delta_{\mathbf{y}}\|_{HS}^2\right] \\
&\leq \mathbb{E}\left[\sup_{\|f\|_\infty \leq 1} \sum_{\mathbf{x} \in \mathbf{K}} \left\| \mathbf{1}_{\mathbf{C}^{(N)}_{(7N+1)L_{k_1}}(\mathbf{0})} |X|^{\frac{s}{2}} f_I(\mathbf{H}) \delta_{\mathbf{x}} \right\|^2\right] \\
&\qquad + C_2(N,d) L_{k_1}^{Nd} \sum_{j \geq k_1} L_j^{\alpha s + \alpha Nd} \left(\mathrm{e}^{-mL_j/2} + L_j^{-2p} \right) \\
&\leq C_3(N,d,|\mathbf{K}|) L_{k_1}^s + C_2(N,d) L_{k_1}^{Nd} \sum_{j \geq k_1} L_j^{\alpha s + \alpha Nd} \left(\mathrm{e}^{-\frac{mL_j}{2}} + L_j^{-2p} \right) \\
&< \infty \quad \square. \qquad\qquad\qquad\qquad \left(\text{since } s < \tfrac{2}{\alpha} p - Nd - 1\right)
\end{aligned}
$$

Chapter 4

Multi-particle localization for the weakly interacting system

For the analysis of localization properties of a multi-particle system with weak interaction, it is convenient to introduce the amplitude h of the interaction and represent the N-particle Hamiltonian of the system by:

$$\mathbf{H} := \mathbf{H}_h^{(N)}(\omega) = -\mathbf{\Delta} + \mathbf{V}(\omega) + h\mathbf{U}, \quad h \in \mathbb{R}, \tag{4.1}$$

where $\mathbf{V}(\omega) = \mathbf{V}(\mathbf{x}, \omega) = V(x_1, \omega) + \cdots + V(x_N, \omega)$ and h is a parameter measuring the strength of the non-random interaction $\mathbf{U}$. For $h = 0$, one has a system of N non-interacting quantum particles subject to a common random potential $V\colon \mathbb{Z}^d \times \Omega \to \mathbb{R}$. Technically, we will prove in this chapter stability of initial MSA bounds from the one-dimensional single-particle estimates to the multiparticle estimates under sufficiently weak interaction. This will allows us to perform multiparticle multiscale analysis for the weak interacting multiparticle Anderson model leading to the complete Anderson localization.

Since the notions of resonance and singularity have been defined in Chapters 2 and 3 with respect to a given random ensemble of Hamiltonian, with fixed parameters, it is convenient to change now the notations and to introduce explicitly the parameter h of $\mathbf{H}_h^{(N)}(\omega)$. For any $n = 1, \ldots, N$ we will say that

(i) a cube $\mathbf{C}_L^{(n)}(\mathbf{u})$ is (E, h)-resonant, if it is E-resonant with respect to $\mathbf{H}_h^{(n)}(\omega)$;

(ii) a cube $\mathbf{C}_L^{(n)}(\mathbf{u})$ is (E, m, h)-singular, if it is (E, m)-singular with respect to $\mathbf{H}_h^{(n)}(\omega)$.

4.1 Assumptions and the main results

4.1.1 Assumptions

(I2). Fix any $n = 1, \dots, N$. The potential of inter-particle interaction $\mathbf{U}$ is bounded and of the form

$$\mathbf{U}(\mathbf{x}) = \sum_{1 \leq i < j \leq n} \Phi(|x_i - x_j|), \quad \mathbf{x} = (x_1, \dots, x_n),$$

where $\phi : \mathbb{N} :\to \mathbb{R}$ is a compactly supported function such that

$$\exists r_0 \in \mathbb{N} : \operatorname{supp} \Phi \subset [0, r_0]. \tag{4.2}$$

To describe our assumptions on the random potential

$$V : \mathbb{Z}^d \times \Omega \to \mathbb{R},$$

we need some notations. Denote by F_V the distribution function on $\mathbb{R}$ of the i.i.d. random variables. Namely, let $E \in \mathbb{R}$

$$F_V(E) := \mathbb{P}\left\{V(0,\omega) \leq E\right\}$$

and let μ be the associated measure on $\mathbb{R}$, i.e,

$$\mu([a,b]) = F_V(b) - F_V(a).$$

Define the quantity

$$s(\mu, \varepsilon) = \sup_{a \in \mathbb{R}} \mu([a, a+\varepsilon]) = \sup_{a \in \mathbb{R}} (F_V(a+\varepsilon) - F_V(a)).$$

We also define

$$L^\infty((1+|x|)^{1+\kappa}) := \{f : \mathbb{R} \to \mathbb{R} | \sup_{x \in \mathbb{R}} |f(x)(1+|x|)^{1+\kappa}| < \infty\},$$

for some $\kappa \in (0,1)$.

(P2). The random potential $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ is i.i.d. and bounded, i.e., there exists $M \in (0,\infty)$ such that $\operatorname{supp} \mu \subset [-M, M]$. Furthermore, the probability distribution measure μ has a bounded density, i.e., $\mathbb{P}\left\{V(0,\omega) \in A\right\} = \mu(A) = \int_A \rho(\lambda) d\lambda$, and $\rho \in L^\infty((1+|x|)^{1+\kappa})$.

Note that assumption (**P2**) above implies that μ is Hölder continuous of order 1.

Remark that, under the assumptions (**I2**) and (**P2**), we have that the spectrum of the one-dimensional mutiparticle Hamiltonian $\mathbf{H}_h^{(N)}(\omega)$ satisfies:

$$\sigma(\mathbf{H}^{(N)}(\omega)h) \subset [-N(4d+M) - |h|\|\mathbf{U}\|, N(4d+M) + |h|\|\mathbf{U}\|] \quad a.s.$$

Therefore it suffices for our purposes to show Anderson localization on

$$I = [-1 - N(4d+M) - |h|\|\mathbf{U}\|, N(4d+M) + |h|\|\mathbf{U}\| + 1].$$

4.1.2 The results

Theorem 4.1. *Under assumptions* (**I2**) *and* (**P2**), *there exists $h^* > 0$ such that for any $h \in (-h^*, h^*)$ the Hamiltonian $\mathbf{H}_h^{(N)}$, with interaction of amplitude $|h|$, exhibits complete Anderson localization,* i.e., *with $\mathbb{P}$-probability one,*

(i) *the spectrum of $\mathbf{H}^{(n)}$ is pure point,*

(ii) *the eigenfunctions $\mathbf{\Psi}_i(\mathbf{x},\omega)$ relative to eigenvalues $E_i(\omega) \in I$ are rapidly decaying at infinity: for each $\mathbf{\Psi}_j$ for all $\mathbf{x} \in \mathbb{Z}^{Nd}$ and some constants $a, c, C_i(\omega) > 0$*

$$|\mathbf{\Psi}_i(\mathbf{x},\omega)| \leq C_i(\omega) \mathrm{e}^{-a(\ln|\mathbf{x}|)^{1+c}}.$$

Denote by $\mathcal{B}_1$ the set of bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\|f\|_\infty \leq 1$. We now give our result on strong Hilbert-Schmidt dynamical localization of any order.

Theorem 4.2. *Under assumptions* (**I2**) *and* (**P2**), *there exists $h^* > 0$ such that for any $h \in (-h^*, h^*)$ any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$, any bounded region $\mathbf{K} \subset \mathbb{Z}^{Nd}$ and any $s > 0$ we have:*

$$\mathbb{E}\left[\sup_{f\in\mathcal{B}_1} \left\| |\mathbf{X}|^{\frac{s}{2}} f(\mathbf{H}^{(N)}(\omega)) \mathbf{P}_I(\mathbf{H}_h^{(N)}(\omega)) \mathbf{1}_\mathbf{K} \right\|_{HS}^2 \right] < \infty. \tag{4.3}$$

where $(|\mathbf{X}|\mathbf{\Psi})(\mathbf{x}) := |\mathbf{x}|\mathbf{\Psi}(\mathbf{x})$, $\mathbf{P}_I(\mathbf{H}^{(N)}(\omega))$ is the spectral projection of $\mathbf{H}^{(N)}(\omega)$ onto the interval I.

4.2 Fix-energy MSA bounds under weak interaction

In this Section we fix $d = 1$, i.e., we consider here the one-dimensional multi-particle random Hamiltonian $\mathbf{H}_h^{(N)}(\omega) = -\mathbf{\Delta} + \mathbf{V}(\mathbf{x},\omega) + h\mathbf{U}(\mathbf{x})$ acting in the Hilbert space $\ell^2(\mathbb{Z}^N)$. Now, we aim to prove stability of the MSA bounds from the single-particle lattice systems to multiparticle systems with sufficiently weak interaction in the interval

$$I = [-1 - N(4d + M)|h|\|\mathbf{U}\|, N(4d + M) + |h|\|\mathbf{U}\| + 1],$$

Consider the one-dimesional single-particle Hamiltonian

$$H^{(1)}(\omega) = -\Delta + V(\omega)$$

with a non-constant i.i.d. random potential $V : \mathbb{Z} \times \Omega \to \mathbb{R}$. Recall that in one-dimensional Anderson models one can prove strong estimates on the Green functions (cf. [CL90, KS80, vDK89]) and also on eigenfunction correlators.

It follows for example by Theorem IX (a) in [KS80] that for all L_0 large enough and any bounded interval $I \subset \mathbb{R}$ if,

$$\Upsilon_{C^{(1)}_{L_0}(u)}(x, y, I; \omega) := \sum_{E_j \in \sigma(H^{(1)}_{C^{(1)}_{L_0}(u)}) \cap I} |\psi_j(x)\psi_j(y)|,$$

There exists a constant $\widetilde{\mu}(I) > 0$ such that

$$\mathbb{E}\left[\Upsilon_{C^{(1)}_{L_0}(u)}(x, y, I; \omega)\right] \le \mathrm{e}^{-\widetilde{\mu}|x-y|}. \tag{4.4}$$

where $\{E_j, \psi_j\}_{j=1,\cdots,|C^{(1)}_{L_0}(u)|}$ are the eigenvalues and corresponding eigenfunctions of
$H^{(1)}_{C^{(1)}_{L_0}(u)}(\omega)$.

4.2.1 The initial MSA bound for the n-particle system without interaction

The main result of this subsection is Lemma 4.3 given below. The proof of Lemma 4.3 relies on an auxiliary statement formulated below, Lemma 4.4. We need to introduce first

$$\{(\lambda^{(i)}_{j_i}, \psi^{(i)}_{j_i}) : j_i = 1, \ldots, |C^{(1)}_{L_0}(u_i)|\},$$

the eigenvalues and the corresponding eigenfunctions of $H^{(1)}_{C^{(1)}_{L_0}(u_i)}(\omega)$, $i = 1, \ldots, n$. Then the eigenvalues $E_{j_1 \ldots j_n}$ of the non-interacting multiparticle random Hamiltonian
$\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega)$ are written as sum

$$E_{j_1 \ldots j_n} = \sum_{i=1}^{n} \lambda^{(i)}_{j_i} = \lambda^{(1)}_{j_1} + \cdots + \lambda^{(n)}_{j_n},$$

while the corresponding eigenfunctions $\mathbf{\Psi}_{j_1\dots j_n}$ can be chosen as tensor products

$$\mathbf{\Psi}_{j_1\dots j_n} = \psi^{(1)}_{j_1} \otimes \cdots \otimes \psi^{(n)}_{j_n}.$$

The eigenfunctions of finite volume Hamiltonian are assumed normalised.

Theorem 4.3. *Let $1 \le n \le N$ and $\widetilde{\mu}$ as in (4.4). Consider $m^* = \min(\frac{1}{2^N 12Nd}, 2^{-N-1}\widetilde{\mu})$. Then for all $E \in I$ and all $\mathbf{u} \in \mathbb{Z}^n$:*

$$\mathbb{P}\left\{ \mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \text{ is } (E, m^*, 0)\text{-}S \right\} \le L_0^{-2p^* 4^{N-n}}, \tag{4.5}$$

with L_0 large enough.

The proof of Lemma 4.3 relies on the following auxiliary statement.

Lemma 4.4. *Let be given $N \ge n \ge 2$, $m^* > 0$, a cube $\mathbf{C}^{(n)}_{L_0}(\mathbf{u})$, and $E \in \mathbb{R}$. Suppose that $\mathbf{C}^{(n)}_{L_0}(\mathbf{u})$ is E-NR, and for any operator $H^{(1)}_{C_{L_0}(u_i)}$, all its eigenfunctions ψ_j satisfy*

$$|\psi_j(u_i)\, \psi_j(u_i \pm L_0)| \le \mathrm{e}^{-2\gamma(m^*, L_0, n) L_0}. \tag{4.6}$$

Then $\mathbf{C}^{(n)}_{L_0}(\mathbf{u})$ is $(E, m^*, 0)$-NS, provided that $L_0 \ge L_*(m^*, N, d)$.

Proof. Fix any $\mathbf{y} \in \partial^- \mathbf{C}^{(n)}_{L_0}(\mathbf{u})$. There exists $i \in [1, n]$ such that $|u_i - y_i| = L_0$. Decompose the cube $\mathbf{C}^{(n)}_{L_0}(\mathbf{u})$ as follows: $\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) = \mathbf{C}^{(n-1)}_{L_0}(\mathbf{u}') \times \mathbf{C}^{(1)}_{L_0}(u_i)$, $\mathbf{u}' \in \mathbb{Z}^{(n-1)d}$. In a similar way, we factorize every eigenfunction, $\mathbf{\Psi}_k(\mathbf{u}) = \mathbf{\Psi}_{k'}(\mathbf{u}')\, \psi_i(u_i)$, and the respective eigenvalue, $E_k = E_{\neq i} + \lambda^{(i)}_{j_i}$. Now we have that

$$\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E) = \sum_{k'} \mathbf{\Psi}_{k'}(\mathbf{u}') \mathbf{\Psi}_{k'}(\mathbf{y}')\; G^{(1)}_{C^{(1)}_{L_0}(u_i)}(u_i, y_i; E - E_{\neq i}).$$

Here $\|\mathbf{\Psi}_{k'}\|_\infty \le 1$, since $\|\mathbf{\Psi}_k\|_2 = 1$, therefore,

$$\left| \mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E) \right| \le (2L_0 + 1)^{nd} \frac{\mathrm{e}^{-2\gamma(m^*, L_0, n) L_0}}{\mathrm{e}^{-L_0^\beta}} \le \mathrm{e}^{-\gamma(m^*, L_0, n) L_0},$$

for L large enough (depending on m^*, N, d). $\square$

Proof of Lemma 4.3. We will say that $\mathbf{H}^{(n)}_L(\mathbf{u})$ is m^*-localized if all conditions (4.6) for the eigenfunctions of all operators $H^{(1)}_{C^{(1)}_{L_0}(u_i)}$ are satisfied. Introduce the events

$$\mathcal{N} := \{ \mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega) \text{ is not } m^*\text{-localized } \},$$

$$\mathcal{R} := \{ \mathbf{C}^{(n)}_{\mathbf{C}^{(n)}_{L_0}}(\mathbf{u}) \text{ is } E\text{-R } \}.$$

Then by Lemma 4.4,

$$\begin{aligned}
\mathbb{P}\left\{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \text{ is } (E,m^*,0)\text{-S}\right\} &\le \mathbb{P}\left\{\mathcal{N}\right\} + \mathbb{P}\left\{\mathcal{R}\right\},\\
&\le \frac{\mathbb{E}\left[\Upsilon_{C^{(1)}_{L_0}(u_i)}(u_i, u_i \pm L_0, I;\omega)\right]}{\mathrm{e}^{-2\gamma(m^*,L_0,n)L_0}} + \mathbb{P}\left\{\mathcal{R}\right\},\\
&\le \mathrm{e}^{(-\widetilde{\mu}+2\gamma(m^*,L_0,n)L_0} + \mathrm{e}^{-L_0^{\beta'}} \le L_0^{-p^*\,4^{N-n}},
\end{aligned}$$

since $2\gamma(m^*,L_0,n) < 2^{N+1}m^* \le \widetilde{\mu}$.

□

4.2.2 The initial MSA bound for weakly interacting multi-particle systems

Now we derive the required initial estimate from its counterpart established for non-interacting systems.

Theorem 4.5. *Let $1 \le n \le N$, m^* Suppose that the Hamiltonians $\mathbf{H}^{(n)}_0(\omega)$ (without inter-particle interaction) fulfill the following condition: for all $E \in I$ and all $\mathbf{u} \in \mathbb{Z}^{nd}$*

$$\begin{aligned}
&\mathbb{P}\left\{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \textit{ is } (E,m^*,0)\textit{-S}\right\} \le L_0^{-2p^*4^{N-n}}, \qquad (4.7)\\
&p^* > 6Nd,
\end{aligned}$$

Then there exists $h^* > 0$ such that for all $h \in (-h^*, h^*)$ the Hamiltonian $\mathbf{H}^{(n)}_h(\omega)$, with interaction of amplitude $|h|$, satisfies a similar bound: there exist some $p > 6Nd, m > 0$ such that for all $E \in I$ and all $\mathbf{u} \in \mathbb{Z}^{nd}$

$$\mathbb{P}\left\{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \textit{ is } (E,m,h)\textit{-S}\right\} \le L_0^{-2p\,4^{N-n}}.$$

Proof. Note first that the assumption of Lemma 4.5 is proved in the statement of Lemma 4.3. Set

$$\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h}(E) = (\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h} - E)^{-1},\ h \in \mathbb{R}.$$

By definition, a cube $\mathbf{C}^{(n)}_{L_0}(\mathbf{u})$ is $(E,m^*,0)$-NS iff

$$\max_{\mathbf{y}\in\partial^-\mathbf{C}^{(n)}_{L_0}(\mathbf{u})} \left|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0}(\mathbf{u},\mathbf{y};E)\right| \le \mathrm{e}^{-m^*(1+L_0^{-1/8})^{N-n+1}L_0}, \qquad (4.8)$$

Therefore, there exists sufficiently small $\epsilon > 0$ such that

$$\max_{\mathbf{y}\in\partial^-\mathbf{C}^{(n)}_{L_0}(\mathbf{u})} \left|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0}(\mathbf{u},\mathbf{y};E)\right| \le \mathrm{e}^{-m(1+L_0^{-1/8})^{N-n}L_0} - \epsilon, \qquad (4.9)$$

where $m = m^*/2 > 0$. Since, by assumption, $p^* > 6Nd$, there exists $6Nd < p < p^*$ and $\tau > 0$ such that $L_0^{-2p4^{N-n}} - \tau > L_0^{-2p^*4^{N-n}}$. With such values p' and τ, inequality (4.7) with $p^* > 6Nd$ implies

$$\mathbb{P}\{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \text{ is } (E, m^*, 0)\text{-S}\} < L_0^{-2p4^{N-n}} - \tau. \tag{4.10}$$

Next, it follows from the second resolvent identity that

$$\|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0}(E) - \mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h}(E)\| \leq |h|\,\|\mathbf{U}\| \cdot \|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0}(E)\| \cdot \|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h}(E)\|. \tag{4.11}$$

By Lemma 2.8, applied to Hamiltonians $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0}$ and $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h}$, for any $\tau > 0$ there is $B(\tau) \in (0, +\infty)$ such that

$$\begin{aligned}
&\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0}(E)\| \geq B(\tau)\big\} \leq \frac{\tau}{4}\,,\\
&\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h}(E)\| \geq B(\tau)\big\} \leq \frac{\tau}{4}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0}(E) - \mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h}(E)\| \geq |h|\,\|\mathbf{U}\|B^2(\tau)\big\}\\
&\qquad \leq \mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0}(E)\| \geq B(\tau)\big\} + \mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h}(E)\| \geq B(\tau)\big\}\\
&\qquad \leq 2\,\frac{\tau}{4} = \frac{\tau}{2}\,.
\end{aligned}$$

Set $h^* := \frac{\epsilon}{2\|\mathbf{U}\|(B(\tau))^2} > 0$. We see that if $|h| \leq h^*$, then $|h| \times \|\mathbf{U}\| \times (B(\tau))^2 \leq \frac{\epsilon}{2}$. Hence,

$$\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0} - \mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h}\| \geq \frac{\epsilon}{2}\big\} \leq 2\,\frac{\tau}{4}\,. \tag{4.12}$$

Combining (4.9), (4.10), and (4.12), we obtain that for all $E \in I$

$$\begin{aligned}
&\mathbb{P}\big\{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \text{ is } (E, m, h)\text{-S}\big\}\\
&\quad \leq \mathbb{P}\big\{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \text{ is } (E, m^*, 0)\text{-S}\big\}\\
&\quad + \mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),0}(E) - \mathbf{G}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u}),h}(E)\| \geq \frac{\epsilon}{2}\big\}\\
&\quad \leq \big(L_0^{-2p'4^{N-n}} - \tau\big) + \frac{\tau}{2} < L_0^{-2p'4^{N-n}}.
\end{aligned}$$

□

4.2.3 Multiscale induction

We recall the following facts: For any $1 \leq n \leq N$, if $\mathbf{C}^{(n)}_L(\mathbf{u})$ and $\mathbf{C}^{(n)}_L(\mathbf{v})$ are two FI cubes with $|\mathbf{u} - \mathbf{v}| > 7NL_k$, then $\Pi\mathbf{C}^{(n)}_L(\mathbf{u}) \cap \Pi\mathbf{C}^{(n)}_L(\mathbf{v}) = \emptyset$.

Given an n-particle cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ and $E \in \mathbb{R}$,

— by $M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E)$ the maximal number of (not necessarily separable) *partially interactive* (E,m)-singular cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ with $|\mathbf{u}^{(j)} - \mathbf{u}^{(j')}| > 7NL_k$ for all $j \neq j'$;

— by $M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E)$ the maximal number of *fully interactive* (E,m)-singular cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ with $|\mathbf{u}^{(j)} - \mathbf{u}^{(j')}| > 7NL_k$ for all $j \neq j'$

Fixed energy bounds for PI cubes

Lemma 4.6. *Let $E \in \mathbb{R}$. Consider a PI cube with the canonical decomposition $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}) = \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \times \mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')$. Assume that*

(i) $\forall \mu_j \in \sigma(\mathbf{H}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')})$ *the cube* $\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')$ *is $(E-\mu_j, m)$-NS,*

(ii) $\forall \lambda_i \in \sigma(\mathbf{H}_{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')})$ *the cube* $\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')$ *is $(E-\lambda_i, m)$-NS.*

Then the cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is (E,m)-NS.

Proof. By Definition of the canonical decomposition, the subconfigurations $\mathbf{u}'$ and $\mathbf{u}''$ are non-interacting so that $\mathbf{U}(\mathbf{u}) = \mathbf{U}(\mathbf{u}') + \mathbf{U}(\mathbf{u}'')$ and $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u})}$ reads as

$$\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u})} = \mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}.$$

Therefore its eigenvalues are the sums $E_{ij} = \lambda_i + \mu_j$ where $\{\lambda_i\} = \sigma(\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')})$ and $\{\mu_j\} = \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')})$. Eigenvectors of $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u})}$ can be chosen in the form $\mathbf{\Psi}_{ij} = \varphi_i \otimes \phi_j$, where $\{\varphi_i\}$ are eigenvectors of $\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{Lk}(\mathbf{u}')}$ and $\{\phi_j\}$ are eigenvectors of $\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}$. We have

$$\begin{aligned}
\mathbf{G}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E) &= \sum_{\lambda_i} \sum_{\mu_j} \frac{\varphi_i(\mathbf{u}')\varphi_i(\mathbf{y}')\phi_j(\mathbf{u}'')\phi_j(\mathbf{y}'')}{E - \lambda_i - \mu_j} && (4.13)\\
&= \sum_{\lambda_i} \varphi_i(\mathbf{u}')\varphi_i(\mathbf{y}')\mathbf{G}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}(\mathbf{u}'', \mathbf{y}''; E - \lambda_i) && (4.14)\\
&= \sum_{\mu_j} \phi_j(\mathbf{u}'')\phi_j(\mathbf{y}'')\mathbf{G}_{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')}(\mathbf{u}', \mathbf{y}'; E - \mu_j). && (4.15)
\end{aligned}$$

For any $\mathbf{y} \in \partial^- \mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ either $|\mathbf{u}' - \mathbf{y}'| = L_k$ or $|\mathbf{u}'' - \mathbf{y}''| = L_k$. In the former case using (4.15) combined with assumption (i) we have

$$|\mathbf{G}_{\mathbf{C}^{(n)}_{L_k}}(\mathbf{u}, \mathbf{y}; E)| \leq |\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')| \times \mathrm{e}^{-\gamma(m,L_k,n-1)L_k},$$

while in the latter case, we have by (4.14) and (ii),

$$|\mathbf{G}_{\mathbf{C}^{(n)}_{L_k}}(\mathbf{u}, \mathbf{y}; E)| \leq |\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')| \times \mathrm{e}^{-\gamma(m,L_k,n-1)L_k}.$$

Now it is easy to see that, for $2 \leq n \leq N$

$$\gamma(m, L_k, n-1) - L_k^{-1} \ln(2L_k + 1)^{(n-1)d} > \gamma(m, L_k, n),$$

if L_0 is sufficiently large. This implies the required result. $\square$

Lemma 4.7. *Let $2 \leq n \leq N$. Consider $\theta \in (0, 1/3)$, and $0 < m \leq 1/(2^{N+1}12Nd)$. Assume that for all $E \in I$, $1 \leq n' < n$, $\mathbf{u}' \in \mathbb{Z}^{n'd}$ and $p > \frac{6Nd}{1-3\theta}$*

$$\mathbb{P}\left\{ \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \text{ is } (E, m, h)\text{-}S \right\} \leq L_k^{-2p4^{N-n'}(1+\theta)^k},$$

then for all $E \in I$ and any PI cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}) \subset \mathbb{Z}^{nd}$

$$\mathbb{P}\left\{ \mathbf{C}^{(n)}_{L_k}(\mathbf{u}) \text{ is } (E, m, h)\text{-}S \right\} \leq L_k^{-6p4^{N-n}(1+\theta)^k}.$$

Proof. Consider the canonical decomposition $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}) = \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \times \mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')$ of the cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$. It follows from the definition of a PI cube that $\Pi\mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \cap \Pi\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'') = \emptyset$. By virtue of Lemma 4.6

$$\begin{aligned}
&\mathbb{P}\left\{ \mathbf{C}^{(n)}_{L_k}(\mathbf{u}) \text{ is } (E, m, h)\text{-S} \right\} \\
&\leq \mathbb{P}\left\{ \exists \mu_j \in \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}) : \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \text{ is } (E - \mu_j, m, h)\text{-S} \right\} \\
&+ \mathbb{P}\left\{ \exists \lambda_i \in \sigma(\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')}) : \mathbf{C}^{(n'')}_{L_k}(\mathbf{u}') \text{ is } (E - \lambda_i, m, h)\text{-S} \right\}.
\end{aligned}$$

Let us focus on the first term in the RHS. The second term will be bound with similar arguments.

$$\begin{aligned}
&\mathbb{P}\left\{ \exists \mu_j \in \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}) : \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \text{ is } (E - \mu_j, m, h)\text{-S} \right\} \\
&= \mathbb{E}\left[\mathbb{P}\left\{ \exists \mu_j \in \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}) : \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \text{ is } (E - \mu_j, m, h)\text{-S} | \mathfrak{B}'' \right\} \right],
\end{aligned}$$

where $\mathfrak{B}''$ is the sigma-algebra generated by the values of the random potential in $\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')$. Now for $E \in I$, assume first that $E-\mu_j < -1-N(4d+M)-|h|\|\mathbf{U}\|$, then since $\lambda_i \geq -N(4d+M)-|h|\|\mathbf{U}\|$, we have that

$$\begin{aligned}\operatorname{dist}\left[E-\mu_j, \sigma(\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')})\right] &= \min_{i=1,\dots,|\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')|}(\lambda_i-(E-\mu_j))\\ &\geq -N(4d+M)-|h|\|\mathbf{U}\|-(E-\mu_j) > 1,\end{aligned}$$

next if $E-\mu_j > N(4d+M)+|h|\|\mathbf{U}\|+1$, then since $\lambda_i \leq N(4d+M)+|h|\|\mathbf{U}\|$, we also have that:

$$\begin{aligned}\operatorname{dist}\left[E-\mu_j, \sigma(\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')})\right] &= \min_{i=1,\dots,|\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')|}((E-\mu_j)-\lambda_i)\\ &\geq (E-\mu_j)-N(4d+M)+|h|\|\mathbf{U}\| > 1.\end{aligned}$$

In either case, the Combes-Thomas estimate (Theorem 2.12) implies that

$$\begin{aligned}\max_{\mathbf{y}'\in\partial^-\mathbf{C}^{(n')}_{L_0}(\mathbf{u}')}|\mathbf{G}^{(n')}_{\mathbf{C}^{(n')}_{L_0}(\mathbf{u}')}(\mathbf{u}',\mathbf{y}';E-\mu_j)| &\leq 2\mathrm{e}^{-\frac{1}{12Nd}|\mathbf{u}'-\mathbf{y}'|}\\ &< \mathrm{e}^{-m(1+L_0^{-1/8})^{N-n'+1}L_0} = \mathrm{e}^{-\gamma(m,L_0,n')L_0}.\end{aligned}$$

Thus, $\mathbf{C}^{(n')}_{L_0}(\mathbf{u}')$ must be $(E-\mu_j,m,0)$-NS provided $m \leq 1/(2^{N+1}12Nd)$. Since the random potential is i.i.d. and $\Pi\mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \cap \Pi\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'') = \emptyset$ one has

$$\begin{aligned}&\mathbb{P}\left\{\exists\mu_j \in \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}) : \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \text{ is } (E-\mu_j,m,h)\text{-S}|\mathfrak{B}''\right\}\\ &\leq \sup_{E''\in I}\mathbb{P}\left\{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \text{ is } (E'',m,h)\text{-S}\right\} \quad \text{a.s.}\end{aligned}$$

Hence by hypothesis

$$\sup_{E''\in I}\mathbb{P}\left\{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \text{ is } (E'',m,h)\text{-S}\right\} \leq L_k^{-2p4^{N-n'}(1+\theta)^k}.$$

We finally obtain:

$$\begin{aligned}&\mathbb{P}\left\{\exists\mu_j \in \sigma(\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}) : \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \text{ is } (E-\mu_j,m,h)\text{-S}\right\}\\ &\leq C(n,N,d)L_k^{nd-2p\cdot 4\cdot 4^{N-n}(1+\theta)^k}\\ &\leq L_k^{-6p4^{N-n}(1+\theta)^k},\end{aligned}$$

since $p > \frac{6Nd}{1-3\theta}$. Similarly

$$\mathbb{P}\left\{\exists\lambda_i \in \sigma(\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')}) : \mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'') \text{ is } (E-\lambda_i, m, h)\text{-S}\right\} \le L_k^{-6p4^{N-n}(1+\theta)^k}.$$

$\square$

Fixed energy scale induction

Let us set for $k \ge 0$

$$\begin{aligned}
\mathrm{P_k} &= \sup_{\mathbf{u}\in\mathbb{Z}^{nd}} \mathbb{P}\left\{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}) \text{ is } (E,m,h)\text{-S}\right\}.\\
\mathrm{Q_{k+1}} &= \sup_{\mathbf{u}\in\mathbb{Z}^{nd}} \mathbb{P}\left\{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}) \text{ is } (E,h)\text{-R}\right\}.\\
\mathrm{S_{k+1}} &= \sup_{\mathbf{u}\in\mathbb{Z}^{nd}} \mathbb{P}\left\{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}) \text{ contains a PI } (E,m,h)\text{-S cube of size } L_k\right\}.
\end{aligned}$$

Theorem 4.8. *Assume that condition* (**P2**) *holds true. Assume that for all $k \ge 0$ and $E \in I$ the following conditions are satisfied:*

(i) *for all $\mathbf{u}' \in \mathbb{Z}^{n'd}$, $1 \le n' < n$,*

$$\mathbb{P}\left\{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}) \textit{ is } (E,m,h)\textit{-S}\right\} \le L_k^{-2p\,4^{N-n'}(1+\theta)^k},$$

(ii) *for all $\mathbf{u} \in \mathbb{Z}^{nd}$*

$$\mathbb{P}\left\{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}) \textit{ is } (E,m,h)\textit{-S}\right\} \le L_k^{-2p\,4^{N-n}(1+\theta)^k}.$$

Then for all $E \in I$ and all $\mathbf{u} \in \mathbb{Z}^{nd}$

$$\mathbb{P}\left\{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}) \textit{ is } (E,m,h)\textit{-S}\right\} \le L_{k+1}^{-2p\,4^{N-n}\,(1+\theta)^{k+1}}$$

Proof. First observe that assumption (i) implies by Lemma 4.7 that

$$\begin{aligned}
\mathrm{S_{k+1}} &\le C(N,d) L_{k+1}^{nd} L_k^{-6p\,4^{N-n}(1+\theta)^k}\\
&\le C(n,d) L_{k+1}^{nd-\frac{6}{\alpha(1+\theta)}\,p\,4^{N-n}(1+\theta)^{k+1}}\\
&\le \frac{1}{4} L_{k+1}^{-2p\,4^{N-n}(1+\theta)^{k+1}},
\end{aligned}$$

since $p > \frac{6Nd}{1-3\theta}$. Next, by lemma 2.20, if $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ is (E,m,g,h)-S then, it is (E,g,h)-R or $M_{\mathrm{PI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E) + M_{\mathrm{FI}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E) \ge n^n + 6 > 2$. There are

$< \frac{3^{2nd}}{2} L_{k+1}^{2nd}$ pairs of L_k-cubes with centers at distantce $> 7NL_k$ in $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$. Thus,

$$\mathrm{P_{k+1}} \leq \frac{3^{\mathrm{2nd}}}{2} \mathrm{L_{k+1}^{2nd} P_k^2} + \mathrm{Q_{k+1}} + \mathrm{S_{k+1}}.$$

Let us assess the first term in the RHS:

$$\begin{aligned}\frac{3^{2nd}}{2} L_{k+1}^{2nd} \mathrm{P_k^2} &\leq C(n,d) L_{k+1}^{2nd - \frac{4}{\alpha(1+\theta)} p\, 4^{N-n}(1+\theta)^{k+1}} \\ &\leq \frac{1}{4} L_{k+1}^{-2p\, 4^{N-n}(1+\theta)^{k+1}},\end{aligned}$$

since $p > \frac{6Nd}{1-3\theta}$ and $0 < \theta < 1/3$. On the other hand using corollary 2.10, we have that

$$\mathrm{Q_{k+1}} \leq \mathrm{e}^{-\mathrm{L}_{k+1}^{\beta'}} < \frac{1}{2} \mathrm{L_{k+1}^{-2p\, 4^{N-n}(1+\theta)^{k+1}}},$$

since

$$\begin{aligned}-\ln(1/2) + 2p4^{N-n}(1+\theta)^{k+1} \ln L_k &\leq \ln 2 + 2p2^{k+1}\alpha^{k+1} \ln L_0 \\ &\leq C(p,n,N) 2^{k+1} \alpha^{k+1} \ln L_0 \\ &\leq L_0^{\beta' \alpha^{k+1}} \leq L_{k+1}^{\beta'}.\end{aligned}$$

Indeed

$$\alpha^{k+1} \geq \frac{\ln C(n,N,p)}{\beta' \ln L_0} + (k+1) \frac{\ln(2\alpha)}{\beta' \ln L_0} + \frac{\ln \ln L_0}{\beta' \ln L_0},$$

if $L_0 \geq L(n,N,p)$ for some large enough $L(n,N,p) > 0$. Now we can conclude that

$$\mathrm{S_{k+1}} \leq \mathrm{L_{k+1}^{-2p\, 4^{N-n}(1+\theta)^{k+1}}},$$

as required. $\square$

Conclusion

Taking into account the above results, we finally get by induction on n that for all $E \in I$ and all $\mathbf{u} \in \mathbb{Z}^{nd}$

$$\mathbb{P}\left\{ \mathbf{C}^{(n)}_{L_k}(\mathbf{u}) \text{ is } (E,m,h)\text{-S} \right\} \leq L_k^{-2p\, 4^{N-n}(1+\theta)^k} \quad \text{for all } k \geq 0. \tag{4.16}$$

4.3 Variable-energy MSA bounds for the weakly interacting system

To derive the variable energy bound from its fixed energy counterpart, we follow the same strategy as in [Chu12]. Let I be a bounded interval, i.e., $|I| < \infty$.

Given an integer $L > 0$ and points $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$ set:

$$\mathrm{F}_{\mathbf{x},\mathbf{y}}(E) = |\mathbf{G}_{\mathbf{C}_L^{(N)}(\mathbf{x})}(\mathbf{x}, \mathbf{y}; E)|, \quad \mathrm{F}_{\mathbf{x}}(E) = \max_{\mathbf{y} \in \partial \mathbf{C}_L^{(N)}(\mathbf{x})} \mathrm{F}_{\mathbf{x},\mathbf{y}}(E).$$

Let $a > 0$, also introduce

$$\mathsf{E}_{\mathbf{x},\mathbf{y}}(a) = \{E \in I : \mathrm{F}_{\mathbf{x},\mathbf{y}}(E) \geq a\} \quad \mathsf{E}_{\mathbf{x}}(a) = \{E \in I : \mathrm{F}_{\mathbf{x}}(E) \geq a\}.$$

Theorem 4.9. *Let $L \in \mathbb{N}^*$, $\mathbf{x} \in \mathbb{Z}^{Nd}$, $\mathbf{y} \in \partial^- \mathbf{C}_L^{(N)}(\mathbf{x})$. Let $\{\lambda_j\}_{j=1,\dots,|\mathbf{C}_L^{(N)}(\mathbf{x})|}$ be the eigenvalues of $\mathbf{H}^{(N)}_{\mathbf{C}_L^{(N)}(\mathbf{x})}(\omega)$ and $I \subset \mathbb{R}$ a bounded interval. Let be given numbers $a, b, c, \mathcal{P}_L > 0$ such that*

$$b \leq \min\{|\mathbf{C}_L^{(N)}(\mathbf{x})|^{-1} a c^2, c\} \tag{4.17}$$

and for all $E \in I$

$$\mathbb{P}\{\mathrm{F}_{\mathbf{x}}(E) \geq a\} \leq \mathcal{P}_L. \tag{4.18}$$

Then there is an event $\mathcal{B}_{\mathbf{x}}(b)$ with $\mathbb{P}\{\mathcal{B}_{\mathbf{x}}(b)\} \leq b^{-1}|I|\mathcal{P}_L$ such that for all $\omega \notin \mathcal{B}_{\mathbf{x}}(b)$, the set

$$\mathsf{E}_{\mathbf{x}}(2a) = \mathsf{E}_{\mathbf{x}}(2a, \omega) = \{E \in I : \mathrm{F}_{\mathbf{x}}(E) \geq 2a\}$$

is contained in a union of intervals $\bigcup_{j=1,\dots,|\mathbf{C}_L^{(N)}(\mathbf{x})|} I_j$, $I_j = \{E \in I : |E - \lambda_j| \leq 2c\}$, $\lambda_j \in I$.

Proof. Set for $b > 0$

$$\mathcal{B}_{\mathbf{x}}(b) = \{\omega \in \Omega : \mathrm{mes}(\mathsf{E}_{\mathbf{x}}(a)) > b\}$$

By Chebychev's inequality and Fubini's Theorem combined with (4.18) we have

$$\begin{aligned} \mathbb{P}\{\mathcal{B}_{\mathbf{x}}(b)\} &\leq b^{-1} \mathbb{E}\left[\, \mathrm{mes}(\mathsf{E}_{\mathbf{x}}(a)) \,\right] \\ &= b^{-1} \int_I dE \, \mathbb{E}\left[\, \mathbf{1}_{\{\mathrm{F}_{\mathbf{x}}(E) \geq a\}} \,\right] \leq b^{-1}|I|\mathcal{P}_L. \end{aligned}$$

Let $\omega \notin \mathcal{B}_{\mathbf{x}}(b)$, so that $\mathrm{mes}(\mathsf{E}_{\mathbf{x}}(a, \omega)) \leq b$. Consider the function

$$f : E \to \mathbf{G}_{\mathbf{C}_L^{(N)}(\mathbf{x})}(\mathbf{x}, \mathbf{y}; E) = \sum_j \frac{\mathbf{\Psi}_j(\mathbf{x}) \mathbf{\Psi}_j(\mathbf{y})}{\lambda_j - E}$$

and set

$$\mathsf{R}(2c) = \{\lambda \in \mathbb{R} : \min_j |\lambda_j - \lambda| \geq 2c\}$$
$$\mathsf{R}(c) = \{\lambda \in \mathbb{R} : \min_j |\lambda_j - \lambda| \geq c\}.$$

Observe that for $E \in \mathsf{R}(c)$, $|f'(E)| \leq c^{-2}|\mathbf{C}_L^{(N)}(\mathbf{x})|$. Now, we show by contraposition that with $\omega \notin \mathcal{B}_{\mathbf{x},\mathbf{y}}(b)$

$$\{E \in I : |\mathbf{G}_{\mathbf{C}_L^{(N)}(\mathbf{x})}(\mathbf{x}, \mathbf{y}; E)| \geq 2a\} \cap \mathsf{R}(2c) = \emptyset.$$

Assume otherwise and consider any point λ^* in the non empty set figuring in the above LHS and let $J = \{E' \in I : |E' - \lambda^*| \leq b\} \subset \mathsf{R}(c)$, $b \leq c$. Then for any $E \in J$, one has by (4.17)

$$|f(E)| \geq |f(\lambda^*)| - |E - \lambda^*| \sup_{E' \in J} |f'(E')| > 2a - |\mathbf{C}_L^{(N)}(\mathbf{x})| c^{-2} \cdot b \geq a$$

so $J \subset \mathsf{E}_{\mathbf{x},\mathbf{y}}(a)$ and $|\mathsf{E}_{\mathbf{x},\mathbf{y}}(a)| \geq \mathrm{mes}(J) = 2b > b$ contrary to the choice of ω. We therefore get the assertion, since the set $\mathsf{R}(c)$ is independent of $\mathbf{y}$. □

Taking into account the results of the fixed energy MSA established in the previous subsection (cf. (4.16)), we have $\mathcal{P}_{L_k} \leq L_k^{-2p(1+\theta)^k}$ with $p > \frac{6Nd}{1-3\theta}$, $0 < \theta < 1/3$. A straightforward calculation shows that, with $L = L_k$, $k \geq 0$, the condition (4.17) is satisfied with the following numbers: $a = a(L_k)$, $b = b(L_k)$, $c = c(L_k)$

$$a(L_k) = L_k^{-\frac{p_k}{5}}, \quad b(L_k) = L_k^{-\frac{4p_k}{5}}, \quad c(L_k) = 3^{\frac{Nd}{2}} L_k^{-\frac{p_k}{5}},$$

provided that $p_k \geq 5Nd$. The latter inequality follows from our assumption that $p > 6Nd/(1-3\theta)$, since $p_k = p(1+\theta)^k$. These values will be used below.

Theorem 4.10. *Assume that assumption* (**P2**) *holds true, then for any pairs of separable cubes* $\mathbf{C}_{L_k}^{(N)}(\mathbf{x})$, $\mathbf{C}_{L_k}^{(N)}(\mathbf{y})$, $k \geq 0$, *one has*

$$\mathbb{P}\left\{\exists E \in I : \min\{\mathrm{F}_{\mathbf{x}}(E), \mathrm{F}_{\mathbf{y}}(E)\} \geq 2L_k^{-\frac{p_k}{5}}\right\} \leq C(|I|, N, d)\, L_k^{-\frac{p_k}{5}+(2N+1)d}$$

Proof. With the above choice of a, b, c, introduce the events

$$\mathsf{E}_{\mathbf{x}} = \{E \in I : \mathrm{F}_{\mathbf{x}}(E) \geq 2a(L_k)\}, \quad \mathsf{E}_{\mathbf{y}} = \{E \in I : \mathrm{F}_{\mathbf{y}}(E) \geq 2a(L_k)\}$$

as in Theorem 4.9. Further, introduce the events $\mathcal{B}_{\mathbf{x}}(b)$, $b = b(L_k)$ relative to the cube $\mathbf{C}_{L_k}^{(N)}(\mathbf{x})$ in the same way as in the proof of Theorem 4.9, and $\mathcal{B}_{\mathbf{y}}(b)$ relative to the cube $\mathbf{C}_{L_k}^{(N)}(\mathbf{y})$. Set $\mathcal{B}(b) = \mathcal{B}_{\mathbf{x}}(b) \cup \mathcal{B}_{\mathbf{y}}(b)$. For any $\omega \notin \mathcal{B}(b)$,

$$\mathrm{dist}(E, \sigma(\mathbf{H}^{(N)}_{\mathbf{C}_{L_k}^{(N)}(\mathbf{x})})) \leq 2c(L_k), \ \mathrm{dist}(E, \sigma(\mathbf{H}^{(N)}_{\mathbf{C}_{L_k}^{(N)}(\mathbf{y})})) \leq 2c(L_k).$$

Thus applying Theorem 2.8 (B), we obtain:

$$
\begin{aligned}
\mathbb{P}\left\{\mathsf{E}_{\mathbf{x}} \cap \mathsf{E}_{\mathbf{y}} \neq \emptyset\right\} &\leq \mathbb{P}\left\{\mathcal{B}(b)\right\} + \mathbb{P}\left\{\operatorname{dist}(\sigma(\mathbf{H}^{(N)}_{\mathbf{C}^{(N)}_{L_k}(\mathbf{x})}), \sigma(\mathbf{H}^{(N)}_{\mathbf{C}^{(N)}_{L_k}(\mathbf{y})})) \leq 4c(L_k)\right\} \\
&\leq |I| L_k^{-\frac{p_k}{6}} + C(N,d)\, L_k^{(2N+1)d}(c(L_k)) \\
&\leq C(|I|, N, d)\, L_k^{-\frac{p_k}{5}+(2N+1)d}.
\end{aligned}
$$

□

4.4 Proof of Theorem 4.1

Given $\mathbf{u} \in \mathbb{Z}^{Nd}$ and an integer $k \geq 0$, set, using the notations of Lemma 2.3,

$$
R(\mathbf{u}) := \max_{\ell=1,\dots,\kappa(N)} |\mathbf{u}-\mathbf{u}^{(\ell)}|, \quad b_k(\mathbf{u}) := 7N + R(\mathbf{u}) L_k^{-1}, \quad M_k(\mathbf{u}) = \bigcup_{\ell=1}^{\kappa(N)} \mathbf{C}^{(N)}_{7NL_k}(\mathbf{u}^{(\ell)})
$$

and define

$$
A_{k+1}(\mathbf{u}) := \mathbf{C}^{(N)}_{bb_{k+1}L_{k+1}}(\mathbf{u}) \setminus \mathbf{C}^{(N)}_{b_k L_k}(\mathbf{u}),
$$

where the parameter b is to be chosen later. One can easily check that:

$$
M_k(\mathbf{u}) \subset \mathbf{C}^{(N)}_{b_k L_k}(\mathbf{u}).
$$

Moreover, if $\mathbf{x} \in A_{k+1}(\mathbf{u})$, then cubes $\mathbf{C}^{(N)}_{L_k}(\mathbf{x})$ and $\mathbf{C}^{(N)}_{L_k}(\mathbf{u})$ are separable by Lemma 2.3. Define the event

$$
\Omega_k(\mathbf{u}) := \left\{\exists E \in I, \text{ and } \mathbf{x} \in A_{k+1}(\mathbf{u})\colon \min\{\mathrm{F}_{\mathbf{x}}(E), \mathrm{F}_{\mathbf{u}}(E)\} \geq 2L_k^{-\frac{p_k}{5}}\right\}.
$$

Theorem 4.10 implies that

$$
\begin{aligned}
\mathbb{P}\left\{\Omega_k\right\} &\leq (2bb_{k+1}L_{k+1}+1)^{Nd} C(|I|, N, d)\, L_k^{-\frac{p_k}{5}+(2N+1)d} \\
&\leq (2bb_{k+1}+1)^{Nd} L_k^{-\frac{p(1+\theta)^k}{5}+\alpha Nd},
\end{aligned}
$$

The series $\sum_{k=0}^{\infty} \mathbb{P}\left\{\Omega_k(\mathbf{u})\right\} < \infty$ converges. Thus By Borel-Cantelli Lemma,

$$
\mathbb{P}\left\{\Omega_{<\infty}\right\} := \mathbb{P}\left\{\forall \mathbf{u} \in \mathbb{Z}^{Nd}, \Omega_k(\mathbf{u}) \text{ occurs finitely many times}\right\} = 1.
$$

Therefore, it suffices to pick the potential $\{V(y,\omega), y \in \mathbb{Z}^d\}$ with $\omega \in \Omega_{<\infty}$ and prove the exponential decay of any nonzero generalised eigenfunction $\mathbf{\Psi}$ of $\mathbf{H}$. Since $\mathbf{\Psi}$ is polynomially bounded, there exist $C, t \in (0,\infty)$ such that for all $\mathbf{x} \in \mathbb{Z}^{Nd}$

$$
|\mathbf{\Psi}(\mathbf{x},\omega)| \leq C(|\mathbf{x}|)^t.
$$

Since $\mathbf{\Psi}$ is not identically zero, there exists $\mathbf{u}\in\mathbb{Z}^{Nd}$ such that $\mathbf{\Psi}(\mathbf{u})\neq 0$. Let us show that there is an integer $k_1=k_1(\omega,E,\mathbf{u})$ such that $\forall k\geq k_1$, $\mathrm{F}_{\mathbf{u}}(E)\geq 2L_k^{-\frac{p_k}{5}}$. Indeed, given an integer $k\geq 0$, assume that $\mathrm{F}_{\mathbf{u}}(E)<2L_k^{-\frac{p_k}{5}}$. Then by the geometric resolvent inequality for the eigenfunction implies that

$$\begin{aligned}|\mathbf{\Psi}(\mathbf{u})| &\leq C(N,d)L_k^{Nd-1}L_k^{-\frac{p_k}{5}}\cdot\max_{\mathbf{v}:|\mathbf{v}-\mathbf{u}|\leq L_k+1}|\mathbf{\Psi}(\mathbf{v})|\\ &\leq C'(N,d)L_k^{-\frac{p_k}{5}+Nd-1}(1+|\mathbf{u}|+L_k)^t\underset{L_k\to\infty}{\longrightarrow}0.\end{aligned}$$

This shows that if $\mathrm{F}_{\mathbf{u}}(E)<2L_k^{-\frac{p_k}{5}}$ for arbitrary large values of L_k (i.e., for an infinite number of values k), then $|\mathbf{\Psi}(\mathbf{u})|=0$, in contradiction with the definition of the point $\mathbf{u}$. So there is an integer $k_1=k_1(\omega,E,\mathbf{u})<\infty$ such that $\forall k\geq k_1$ $\mathrm{F}_{\mathbf{u}}(E)\geq 2L_k^{-\frac{p_k}{5}}$. At the same time since $\omega\in\Omega_\infty$ there exists $k_2=k_2(\omega,\mathbf{u})$ such that if $k\geq k_2$, $\Omega_k(\mathbf{u})$ does not occurs. We conclude that $\forall k\geq\max\{k_1,k_2\}$ for all $\mathbf{x}\in A_{k+1}(\mathbf{u})$, $\mathrm{F}_{\mathbf{x}}(E)<2L_k^{-\frac{p_k}{5}}$. Let $\rho\in(0,1)$ and choose b such that

$$b>\frac{1+\rho}{1-\rho}.$$

introduce the annuli

$$\widetilde{A}_{k+1}=\mathbf{C}^{(N)}_{c'_kL_{k+1}}(\mathbf{u})\setminus\mathbf{C}^{(N)}_{c''_kL_k}(\mathbf{u}),$$

with

$$c'_k=\frac{bb_{k+1}}{1+\rho},\quad c''_k=\frac{b_{k+1}}{1-\rho}.$$

Now observe that, if $|\mathbf{x}-\mathbf{u}|>b_0L_0/(1-\rho)$, then there exists $k\geq\max\{k_1,k_2\}$ such that $\mathbf{x}\in\widetilde{A}_{k+1}(\mathbf{u})$. Thus, the cube $\mathbf{C}^{(N)}_{L_k}(\mathbf{x})$ must satisfied the bound

$$\max_{\mathbf{v}\in\partial^-\mathbf{C}^{(N)}_{L_k}(\mathbf{u})}|\mathbf{G}_{\mathbf{C}^{(N)}_{L_k}(\mathbf{u})}(\mathbf{u},\mathbf{v};E)|\leq 2\,L_k^{-\frac{p}{5}(1+\theta)^k}\leq C\mathrm{e}^{-a\,(\ln L_k)^{1+c}},\tag{4.19}$$

for some constants $a,c,C>0$. The bound (4.19) implies that

$$|\mathbf{\Psi}(\mathbf{x})|\leq C\cdot\mathrm{e}^{-a(\ln L_k)^{1+c}}\cdot C(|\mathbf{u}|)\cdot L_k^{Const}\leq C(|\mathbf{u}|)\cdot\mathrm{e}^{-a'(\ln L_k)^{1+c}},\tag{4.20}$$

where $a'\geq a/2>0$ for large values of k (hence, L_k). Next, since $\mathbf{x}\in\widetilde{A}_{k+1}$, we have that

$$\ln|\mathbf{x}-\mathbf{u}|\leq\ln(c'_kL_{k+1})=\ln(c'_kL_k^\alpha)=c'''_k+\alpha\ln L_k,$$

which implies that

$$\ln L_k\geq\frac{\ln|\mathbf{x}-\mathbf{u}|-c'''_k}{\alpha}\geq\frac{1}{2}\ln|\mathbf{x}-\mathbf{u}|,$$

for large values of k. Therefore, the bound (4.20) implies that

$$|\mathbf{\Psi}(\mathbf{x})| \le C(|\mathbf{u}|)\cdot \mathrm{e}^{-a'\cdot(\frac{1}{2}\ln|\mathbf{x}-\mathbf{u}|)^{1+c}} \le C(|\mathbf{u}|)\cdot \mathrm{e}^{-a''(\ln|\mathbf{x}-\mathbf{u}|)^{1+c}},$$

with $a'' = a'/2 > 0 \quad \square$.

4.5 Proof of Theorem 4.2

Before we turn to the proof of Theorem 4.2, we need to make the following observations. For every bounded set $\mathbf{K} \subset \mathbb{Z}^{Nd}$ figuring in Theorem 4.2 there exists $k_0 > 0$ such that $\mathbf{K} \subset \mathbf{C}^{(N)}_{L_{k_0}}(\mathbf{0})$. Now for $j \ge k_0$, set

$$\mathbf{M}_j(\mathbf{0}) = \mathbf{C}^{(N)}_{(7N+1)L_{j+1}}(\mathbf{0}) \setminus \mathbf{C}^{(N)}_{(7N+1)L_j}(\mathbf{0}).$$

Observe that for any $\mathbf{y} \in \mathbf{C}^{(N)}_{L_j}(\mathbf{0})$, $\mathbf{C}^{(N)}_{7NL_j}(\mathbf{y}) \subset \mathbf{C}^{(N)}_{(7N+1)L_j}(\mathbf{0})$. Then, if $\mathbf{x} \in \mathbf{M}_j(\mathbf{0})$ and $\mathbf{y} \in \mathbf{C}^{(N)}_{L_j}(\mathbf{0})$, we have that $|\mathbf{y}-\mathbf{x}| > 7NL_j$. Since $\mathrm{diam}(\Pi\mathbf{y}) \le 2L_j$, it follows that $|\mathbf{y}-\mathbf{x}| > \mathrm{diam}(\Pi\mathbf{y}) + 3NL_j$. Thus using assertion (B) of Lemma 2.3, the cubes $\mathbf{C}^{(N)}_{L_j}(\mathbf{x})$ and $\mathbf{C}^{(N)}_{L_j}(\mathbf{y})$ are separable. We need the following statement establishing the decay of the kernels in the Hilbert-Schmidt norm.

Lemma 4.11. *Consider a random Schrödinger operator* $\mathbf{H}$ *sastisfying assumptions of Theorem 4.2. There exists* $k_1 \ge 0$ *such that for any bounded Borel function* $f : \mathbb{R} \to \mathbb{R}$ *all* $k \ge k_1$, $\mathbf{x} \in \mathbf{M}_j(\mathbf{0})$ *and* $\mathbf{y} \in \mathbf{C}^{(N)}_{L_j}(\mathbf{0})$*:*

$$\mathbb{E}\left[\sup_{\|f\|_\infty \le 1} \|\delta_{\mathbf{x}} f(\mathbf{H})\mathbf{P}_I(\mathbf{H})\delta_{\mathbf{y}}\|^2_{HS}\right] \le \mathrm{e}^{-a(\ln L_j)^{1+c}} + C(|I|,N,d)L_j^{-\frac{p(1+\theta)^j}{5}+(2N+1)d}, \tag{4.21}$$

where $a, c \in (0,\infty)$, $\theta \in (0,1/3)$ *and* $p > 6Nd/(1-3\theta)$.

Proof. Define

$$B_j := \left\{\forall\lambda \in I, \text{ either } \mathrm{F}_{\mathbf{x}}(\lambda) < 2L_j^{-\frac{p(1+\theta)^j}{5}} \text{ or } \mathrm{F}_{\mathbf{y}}(\lambda) < 2L_j^{-\frac{p(1+\theta)^j}{5}}\right\}.$$

Consider a bounded Borel function $f : \mathbb{R} \to \mathbb{C}$ and set $f_I = f\chi_I$, where χ_I is the indicator function of the interval I. We have:

$$\begin{aligned}\|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|^2_{HS} &= \sum_{\mathbf{z},\mathbf{v}\in\mathbb{Z}^{Nd}} |\langle \delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\delta_{\mathbf{z}}, \delta_{\mathbf{v}}\rangle|^2 \\ &= \sum_{\mathbf{z},\mathbf{v}\in\mathbb{Z}^{Nd}} |\langle f_I(\mathbf{H})\delta_{\mathbf{y}}\delta_{\mathbf{z}}, \delta_{\mathbf{x}}\delta_{\mathbf{v}}\rangle|^2 \\ &= |\langle f_I(\mathbf{H})\delta_{\mathbf{y}}, \delta_{\mathbf{x}}\rangle|^2 \\ &= |\langle \delta_{\mathbf{x}}, f_I(\mathbf{H})\delta_{\mathbf{y}}\rangle|^2.\end{aligned}$$

For $\omega \in B_j$, since

$$|\langle \delta_{\mathbf{x}}, f_I(\mathbf{H})\delta_{\mathbf{y}} \rangle| = |\left\langle \delta_{\mathbf{y}}, \overline{f_I}(\mathbf{H})\delta_{\mathbf{x}} \right\rangle|,$$

we can assume without loss of generality that $\mathrm{F}_{\mathbf{x}}(\lambda) < 2L_j^{-\frac{p(1+\theta)^j}{5}}$. Now using Theorem A.7, we get

$$\begin{aligned}
\|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|_2 &\le |\langle \delta_{\mathbf{x}}, f_I(\mathbf{H})\delta_{\mathbf{y}}\rangle| \\
&\le \int_I |f(\lambda)|\, |T_{\mathbf{x},\mathbf{y}}(\lambda)|\, d\rho(\lambda)
\end{aligned}$$

and the function $\mathbf{\Psi} : \mathbf{x} \to T_{\mathbf{x},\mathbf{y}}(\lambda)$ is polynomially bounded for ρ a.e. λ. Next, the geometric resolvent inequality for eigenfunctions gives

$$\begin{aligned}
|\mathbf{\Psi}(\mathbf{x})| &\le \left|\partial \mathbf{C}^{(N)}_{L_j}(\mathbf{x})\right| \cdot 2L_j^{-\frac{p(1+\theta)^j}{5}} \cdot |\mathbf{\Psi}(\mathbf{x}')| \\
&\le C(N,d) L_j^{Nd-1} L_j^{-\frac{p(1+\theta)^j}{5}} (1 + |\mathbf{x}| + L_j)^t \\
&\le C(N,d) L_j^{-\frac{p(1+\theta)^j}{5} + \alpha t + Nd - 1} < \mathrm{e}^{-a(\ln L_j)^{1+c}},
\end{aligned}$$

where $a, c > 0$ are some constants and provided $j \ge k_1$ for some $k_1 \ge 0$ large enough. yielding

$$\begin{aligned}
\|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|_{HS} &\le \|f\|_\infty \rho(I) \mathrm{e}^{-a(\ln L_j)^{1+c}} \\
&\le \|f\|_\infty \mathrm{e}^{-a(\ln L_j)^{1+c}}.
\end{aligned}$$

For $\omega \in B_j^c$, we have

$$\|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|_{HS} = |\langle \delta_{\mathbf{x}}, f_I(\mathbf{H})\delta_{\mathbf{y}}\rangle| \le \|f\|_\infty.$$

Finally, since by Theorem 4.10 $\mathbb{P}\left\{B_j^c\right\} \le C(|I|,N,d) L_j^{-\frac{p(1+\theta)^j}{5} + (2N+1)d}$, we can conclude that

$$\begin{aligned}
\mathbb{E}\left[\sup_{\|f\|_\infty \le 1} \|\delta_{\mathbf{x}} f_I(\mathbf{H})\delta_{\mathbf{y}}\|^2_{HS}\right] &\le \mathrm{e}^{-mL_j/2}\mathbb{P}\left\{B_j\right\} + \mathbb{P}\left\{B_j^c\right\} \\
&\le \mathrm{e}^{-a(\ln L_j)^{1+c}} + C(|I|,N,d) L_j^{-\frac{p(1+\theta)^j}{5} + (2N+1)d}.
\end{aligned}$$

$\square$

Now we finish the proof Theorem 4.2. Let $\mathbf{K} \subset \mathbb{Z}^{Nd}$ with $|\mathbf{K}| < \infty$, $j \ge k_1$ as in Lemma 4.11 and $p > 6Nd/(1-3\theta)$ for any $\theta \in (0, 1/3)$. Pick any $s > 0$ and any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$. Then we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{\|f\|_\infty \le 1} \left\| |X|^{\frac{s}{2}} f_I(\mathbf{H}) \mathbf{1}_{\mathbf{K}} \right\|_{HS}^2\right] \\
&\le \mathbb{E}\left[\sup_{\|f\|_\infty \le 1} \left\| \mathbf{1}_{\mathbf{C}^{(N)}_{(7N+1)L_{k_1}}(\mathbf{0})} |X|^{\frac{s}{2}} f_I(\mathbf{H}) \mathbf{1}_{\mathbf{K}} \right\|_{HS}^2\right] \\
&\qquad + \sum_{j \ge k_1} C_1(N,d) L_{j+1}^s \sum_{\mathbf{x} \in \mathbf{M}_j(\mathbf{0})} \mathbb{E}\left[\sup_{\|f\|_\infty \le 1} \|\delta_{\mathbf{x}} f_I(\mathbf{H}) \mathbf{1}_{\mathbf{K}}\|_{HS}^2\right] \\
&\le \mathbb{E}\left[\sup_{\|f\|_\infty \le 1} \sum_{\mathbf{x} \in \mathbb{Z}^{Nd}} \left\| \mathbf{1}_{\mathbf{C}^{(N)}_{(7N+1)L_{k_1}}(\mathbf{0})} |X|^{\frac{s}{2}} f_I(\mathbf{H}) \mathbf{1}_{\mathbf{K}} \delta_{\mathbf{x}} \right\|^2\right] \\
&\qquad + \sum_{j \ge k_1} C_1(N,d) L_{j+1}^s \sum_{\substack{\mathbf{x} \in \mathbf{M}_j(\mathbf{0}) \\ \mathbf{y} \in \mathbf{C}^{(N)}_{L_{k_1}}(\mathbf{0})}} \mathbb{E}\left[\sup_{\|f\|_\infty \le 1} \|\delta_{\mathbf{x}} f_I(\mathbf{H}) \delta_{\mathbf{y}}\|_{HS}^2\right] \\
&\le \mathbb{E}\left[\sup_{\|f\|_\infty \le 1} \sum_{\mathbf{x} \in \mathbf{K}} \left\| \mathbf{1}_{\mathbf{C}^{(N)}_{(7N+1)L_{k_1}}(\mathbf{0})} |X|^{\frac{s}{2}} f_I(\mathbf{H}) \delta_{\mathbf{x}} \right\|^2\right] \\
&\qquad + C_2(N,d) L_{k_1}^{Nd} \sum_{j \ge k_1} L_j^{\alpha s + \alpha N d} \left(\mathrm{e}^{-a(\ln L_j)^{1+c}} + C(|I|,N,d) L_j^{-\frac{p(1+\theta)^j}{5} + (2N+1)d} \right) \\
&\le C_3(N,d,|\mathbf{K}|) L_{k_1}^s + C_2'(|I|,N,d) L_{k_1}^{Nd} \sum_{j \ge k_1} L_j^{\alpha s + \alpha N d} \left(\mathrm{e}^{-a(\ln L_j)^{1+c}} + L_j^{-\frac{p(1+\theta)^j}{5} + (2N+1)d} \right) \\
&< \infty
\end{aligned}
$$

$\square$

Appendix A

Auxiliary results

A.1 Linear operators in Hilbert Spaces

The following statements can be found e.g. in the book [RS78, Sto01].

A.1.1 Basic definitions and properties

Let $\mathcal{H}$ be a complex Hilbert space with the scalar product $\langle , \rangle$ linear in the second argument. A linear operator in $\mathcal{H}$ is a map $T : D(T) \to \mathcal{H}$ where $D(T)$ is the domain of T.

Definition A.1. A linear operator T in $\mathcal{H}$ is

1. densely defined if the closure $\overline{D(T)} = \mathcal{H}$,
2. invertible if there exists an operator S in $\mathcal{H}$ such that $ST = TS = I$, where I is the identity operator of $\mathcal{H}$.
3. bounded if there exists $M < \infty$ such that $\|T\psi\| \leq M\|\psi\|$ for all $\psi \in \mathcal{H}$. In this case, we set

$$\|T\| = \sup_{\|\psi\|\neq 0} \frac{\|T\psi\|}{\|\psi\|} = \sup_{\|\psi\|\leq 1} \|T\psi\| = \sup_{\|\psi\|=1} \|T\psi\|.$$

Definition A.2. Let T be a linear operator in $\mathcal{H}$.

1. The resolvent set of T is defined by

$$\rho(T) = \{z \in \mathbb{R} \,\Big|\, T - z : D(T) \to \mathcal{H} \text{ is invertible and } (T-z)^{-1} \in B(\mathcal{H})\}.$$

2. The spectrum of T is denoted by $\sigma(T)$ and is the complementary of its resolvent set, i.e, $\sigma(T) = \mathbb{R} \setminus \rho(T)$.
3. We denote by $V_p(T)$ the set of eigenvalues of T. Clearly $V_p(T) \subset \sigma(T)$.

4. The discrete spectrum of T is the set $\sigma_{dis}(T)$ of all its isolated eigenvalues of finite multiplicity. The essential spectrum $\sigma_{ess}(T)$, is the complementary of the discrete spectrum, i.e., $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_{dis}(T)$.

For $z \in \rho(T)$, the operator $(T-z)^{-1}$ is called the resolvent of T and satisfies the following properties known as resolvent identities.

Proposition A.1. *Consider two linear operators S, T in $\mathcal{H}$.*

1. For all $z, z' \in \rho(T)$, we have

$$
\begin{aligned}
(T-z)^{-1} - (T-z')^{-1} &= (z-z')(T-z)^{-1}(T-z)^{-1} && \text{(A.1)}\\
&= (z-z')(T-z')^{-1}(T-z)^{-1}. && \text{(A.2)}
\end{aligned}
$$

2. For all $z \in \rho(T) \cap \rho(S)$ we have

$$
(T-z)^{-1} - (S-z)^{-1} = (T-z)^{-1}(S-T)(S-z)^{-1}. \tag{A.3}
$$

A.1.2 Self-adjoint operators

Let $T \in B(\mathcal{H})$. The adjoint T^* of T is defined by

$$
D(T^*) = \{\psi \in \mathcal{H} \,\Big|\, \exists \psi^* \in \mathcal{H} \text{ with } \forall \phi \in D(T), \langle T\phi, \psi\rangle = \langle \phi, \psi^*\rangle\},
$$

$$
T^*\psi = \psi^*.
$$

Definition A.3. A linear operator T in $\mathcal{H}$ is

— Hilbert-Schmidt if for any orthonormal basis $\{e_i\}$, we have $\sum_i \|Te_i\|^2 < \infty$. In this case the Hilbert-Schmidt-norm $\|T\|_{HS}^2 = \sum_i \|Te_i\|^2$ is independent of the basis.

— symmetric if it is densely defined and

$$
\langle T\phi, \psi\rangle = \langle \phi, T\psi\rangle, \quad \forall \phi, \psi \in D(T),
$$

— self-adjoint if it is densely defined and $T^* = T$.

We denote $S(\mathcal{H})$ the set of self-adjoint operators in $\mathcal{H}$.

Theorem A.4 (Weyl criterion [RS80])**.** *Let T be a self-adjoint operator and $\lambda \in \mathbb{R}$. Then the following assertions are equivalent*

(i) $\lambda \in \sigma(T)$

(ii) there exists a normalised sequence $\psi_n \in D(T)$ of bounded support such that $\|(T-\lambda)\psi_n\| \underset{n\to\infty}{\longrightarrow} 0$. Such a sequence is called a Weyl sequence for T and λ.

Let

$$\mu_0(T) = \inf \{\langle \phi, T\phi \rangle : \phi \in D(T), \|\phi\| = 1\},$$

and

$$\mu_k(T) = \sup_{\psi_1,\dots,\psi_k \in \mathcal{H}} \inf \{\langle \phi, T\phi \rangle : \phi \in D(T), \|\phi\| = 1, \phi \perp \psi_1, \dots, \psi_k\}$$

for $k \geq 1$. The operator T is bounded from below if $\mu_0(T) > -\infty$. $\mu_0(T)$ is then the infimum of the spectrum.

Let T be a self-adjoint operator with purely discrete spectrum, i.e., $\sigma(T) = \sigma_{dis}(T)$. Then, we can order its eignevalues as follows

$$E_0(T) \leq E_1(T) \leq E_2(T) \leq \cdots$$

where each eigenvalue is repeated according to its multiplicity. Theorem A.5 below establish a relation between the E_k and μ_k.

Theorem A.5 (The min-max principle [RS78]). *If T is a self-adjoint operator that is bounded from below and has purely discrete spectrum, then*

$$E_k(T) = \mu_k(T) \quad \forall k \geq 0.$$

A.2 Results from the single-particle theory

Consider a single-particle cube $C_L^{(1)}(u) \subset \mathbb{Z}^d$ and define $E_{0,L}^{(1)}(\omega) = \inf \sigma(H^{(1)}(\omega)|_{C_L^{(1)}(u)})$, the single-particle result at low energy is the following statement which we do not proof. Indeed a complete proof of the statement is given in the article by Kirsch [Kir08].

Theorem A.6 ([Kir08]). *Assume that the random potential V satisfies assumption* (**P1**). *Then, for any $C > 0$ there exists arbitrary large $L_0(C) > 0$ and $C_1, c > 0$ such that for any cube $C_{L_0}^{(1)}(\mathbf{u})$ the lowest eigenvalue $E_{0,L}^{(1)}(\omega)$ of $H_{C_{L_0}^{(1)}(\mathbf{u})}^{(1)}(\omega)$ satisfies*

$$\mathbb{P}\big\{E_{0,L}^{(1)}(\omega) \leq 2CL_0^{-1/2}\big\} \leq C_1 L_0^d \mathrm{e}^{-cL_0^{1/4}}. \tag{A.4}$$

A.3 Generalized eigenfunction expansion

Let $I \subset \mathbb{R}$ and denote by $\nu(I) = P_I(H)$ the projection valued measure associated to H. Set

$$\nu_{n,m}(I) = \langle \delta_n, \nu(I)\delta_m \rangle$$

Consider a sequence $\{\alpha_n\}_{n\in\mathbb{Z}^D}$ with $\alpha_n > 0$, $\sum \alpha_n = 1$ and define

$$\rho(I) = \sum_{n\in\mathbb{Z}^D} \alpha_n \nu_{n,n}(I) \tag{A.5}$$

ρ is called a *spectral measure* with total mass $\rho(\mathbb{R}) = 1$.

Theorem A.7. *Let ρ be a spectral measure for $H = \Delta + W(x)$ acting in $\ell^2(\mathbb{Z}^D)$. Then there exists measurable functions $T_{n,m}$ such that*

$$\langle \delta_n, f(H)\delta_m \rangle = \int f(\lambda) T_{n,m}(\lambda) d\rho(\lambda) \tag{A.6}$$

for any bounded measurable function f. Furthermore, the functions $\mathbf{\Psi}(n) := T_{n,m}(\lambda)$ for any fixed $m \in \mathbb{Z}^D$ satisfy

$$H\mathbf{\Psi} = \lambda\mathbf{\Psi} \quad \textit{for } \rho \textit{ a.e. } \lambda.$$

and are polynomially bounded i.e

$$|\mathbf{\Psi}(n)| \leq C(1 + |n|)^t, \quad \textit{for some } C > 0 \textit{ and } t > 0.$$

Proof. The Cauchy-Schwarz inequality implies that

$$|\nu_{n,m}(I)| \leq \nu_{n,n}(I)^{1/2} \nu_{m,m}(I)^{1/2}.$$

Hence, the $\nu_{n,m}$ are absolutely continuous w.r.t. ρ, i.e.,

$$\rho(J) = 0 \Longrightarrow \nu_{n,m}(J) = 0.$$

Thus by the Radon-Nikodym Theorem there exist measurable functions $T_{n,m}$ (densities) such that

$$\nu_{n,m}(I) = \int_I T_{n,m}(\lambda) d\rho(\lambda). \tag{A.7}$$

$T_{n,m}$ are defined up to sets of ρ-measure zero and since $\nu_{n,m} \geq 0$ the functions $T_{n,n}$ are nonnegative ρ-almost surely (ρ-a.s.). Moreover

$$\begin{aligned}
\rho(I) &= \sum_{n\in\mathbb{Z}^D} \alpha_n \nu_{n,n}(I) \\
&= \sum_{n\in\mathbb{Z}^D} \alpha_n \int_I T_{n,n}(\lambda) d\rho(\lambda) \\
&= \int_I \sum_{n\in\mathbb{Z}^D} \alpha_n T_{n,n}(\lambda) d\rho(\lambda).
\end{aligned}$$

Hence, $\sum \alpha_n T_{n,n}(\lambda) = 1$ (ρ-a.s.). Consequently,

$$T_{n,n}(\lambda) \leq \frac{1}{\alpha_n}.$$

Therefore

$$\begin{aligned}\left|\int_I T_{n,m}(\lambda)d\rho(\lambda)\right| &= |\nu_{n,m}(I)| \\ &\leq \nu_{n,n}(I)^{1/2}\nu_{m,m}(I)^{1/2} \\ &\left(\int_I T_{n,n}(\lambda)d\rho(\lambda)\right)^{1/2}\left(\int_I T_{n,m}(\lambda)d\rho(\lambda)\right)^{1/2} \\ &\leq \alpha_n^{-\frac{1}{2}}\alpha_m^{-\frac{1}{2}}\rho(I).\end{aligned}$$

Thus

$$|T_{n,m}| \leq \alpha_n^{-\frac{1}{2}}\alpha_m^{-\frac{1}{2}}. \tag{A.8}$$

It follows from A.7 that for any bounded measurable function f:

$$\langle \delta_n, f(H)\delta_m\rangle = \int f(\lambda)T_{n,m}(\lambda)d\rho(\lambda).$$

In particular for $f(\lambda) = \lambda g(\lambda)$ (g of compact support).

$$\begin{aligned}&\int \lambda g(\lambda)T_{n,m}(\lambda)d\rho(\lambda) \\ &\quad= \langle \delta_n, Hg(H)\delta_m\rangle \\ &\quad= \langle H\delta_n, g(H)\delta_m\rangle \\ &\quad= \sum_{|e|=1}\left(-\langle \delta_{n+e}, g(H)\delta_m\rangle\right) + (V(n)+2D)\langle \delta_n, g(H)\delta_m\rangle \\ &\quad= \sum_{|e|=1}\left(-\int g(\lambda)T_{n+e,m}(\lambda)d\rho(\lambda)\right) + \int g(\lambda)(V(n)+2D)T_{n,m}(\lambda)d\rho(\lambda) \\ &\quad= \int g(\lambda)H^{(n)}T_{n,m}(\lambda)d\rho(\lambda),\end{aligned}$$

where $H_nT_{n,m}(\lambda)$ is the operator H applied to the function $n \to T_{n,m}(\lambda)$. Thus

$$\int g(\lambda)\lambda T_{n,m}(\lambda)d\rho(\lambda) = \int g(\lambda)H_nT_{n,m}(\lambda)d\rho(\lambda),$$

for any bounded measurable function g with compact support. It follows that for ρ-almost all λ and for any fixed $m \in \mathbb{Z}^D$, the function $\psi(n) = T_{n,m}(\lambda)$ is a solution of $H\psi = \lambda\psi$. By (A.8) the function ψ satisfies

$$|\psi(n)| \leq C_0\alpha_n^{-1/2}.$$

Recall that the sequence α_n has only to fulfill $\alpha_n > 0$ and $\sum \alpha_n = 1$. We may now choose $\alpha_n = c(1 + |n|_\infty)^{-\beta}$ for an arbitrary $\beta > D$ yielding

$$|\psi(n)| \leq C(1 + |n|_\infty)^{\frac{D}{2}+\varepsilon}$$

for an $\varepsilon > 0$. The assertion of Theorem A.7 then follows. $\square$

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