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To cite this version:

Andrés Jaramillo Puentes. Rigid isotopy classification of real quintic rational plane curves. Algebraic Geometry [math.AG]. Université Pierre et Marie Curie - Paris VI, 2017. English. NNT : 2017PA066116 . tel-01624267

HAL Id: tel-01624267

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Université Pierre et Marie Curie

École doctorale de Sciences Mathématiques de Paris Centre

THÈSE DE DOCTORAT

Discipline: Mathématiques

presentée par

Andrés Jaramillo Puentes

Rigid isotopy classification of real quintic rational plane curves

---

dirigée par Ilia Itenberg

Soutenue le 28 septembre 2017 devant le jury composé de:

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M. Antoine Ducros	Professeur, UPMC	Examinateur
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A María Teresa, Avi y Juli.
Diego, Sofía y Davide.

Acknowledgements

I would like to thank my advisor, professor Ilia Itenberg, for his constant support during all this time, his insightful answers to even the most basic questions I came up with, the opportunity to participate at academic events and for correcting in all detail the long complicated proofs I presented (et pour tous les cafés aussi!). J'aimerais remercier chaleureusement professeur Erwan Brugallé, qui m'a toujours encouragé à travailler dans ma thèse et à participer activement aux activités du groupe de recherche (¡gracias también por los tragos!). Merci aussi pour avoir accepté d'être un membre du jury.

J'aimerais remercier énormement professeur Frédéric Mangolte d'avoir été un rapporteur de ma thèse (même si ça a impliqué lire le document pendant la periode de vacances) et pour ces remarques sur le document. I would like to thank professor Hannah Markwig for having reviewed my thesis in such a short period of time, for her comments on this dissertation and for having accepted being a member of the jury. I would like to thank cordially indeed professors Antoine Ducros, Alexander Oancea and Alexander Polishchuk for being members of the jury.

Un agradecimiento muy especial a Florent "F" Schaffhauser, por haberme apoyado durante todos estos años, por haberme motivado a seguir este camino y por haberme brindado su amistad.

I would like to thank Benoit Bertrand, Frédéric Bihan, Alex Degtyarev, Penka Georgieva, Lucia López de Medrano, Gregory Mikhalkin, Kristin Shaw and Johannes Raw for their several seminars, mini-courses and discussions that helped me understand many ideas of real algebraic, tropical and toric geometry (most of the time separately). And of course I would like to thank my real tropical toric anabelian comrades Cristhian Garay López, Johannes Josi, Matilde Manzerolli and Arthur Reneaudineau for all the unadventurous adventures in and out of the math world. A posthumous thank goes to professor Jean Jacques Risler, a great and funny man who always made us feel welcome.

I would like to thank professor Melissa Liu for her patient explanations about Mirror Symmetry, professor Martijn Kool for discussing with me details about real DT-invariants and professor Johannes Huisman for sharing ideas on equivariant homotopy of real algebraic varities.

J'aimerais remercier les membres de l'Institut de Mathématiques de Jussieu - Paris Rive Gauche, en particulier les membres de l'equipe d'Analyse Algebrique, notamment à Nicolas Bergeron, Grégory Ginot, Antonin Guilloux, Julien Marché, Maxime Wolff; mes co-organisateurs du Séminaire de Thésards de l'IMJ-PRG: Eric Balandraud, Julie Desjardins, Catherine Gille et Anne Giralt; mes cobureaux: Thibaud, Rafael (y por extensión Sami, y por no extensión Silvain), Cuong, Martin et Yining; les doctorants de l'IMJ-PRG, specialement à Amiel, Arthur, Jesua, Joaquin, Juliette, Ildar, Karam, Louis, Lucas, Macarena, Malick, Marc, Nicolina, Viet... pour l'ambiance relaxée et motivante, pour les groupes de travail, pour les pains au chocolat du seminaire des lapins, pour les cafés, les thés, pour les déjeuners des fois en parlant de maths, des fois de la politique du pays, pour

les soirées, les annifs, les nouvel ans. Merci d'avoir partagé avec moi.

Je remercie la Fondation de Sciences Mathématiques de Paris, spécialement Jean Dolbeault et Sylvie Delabrière pour leur magnifique travail avec les programmes Paris Graduated School of Mathematics and Domaine d'intérêt majeur | Région Île-de-France, auquels j'ai eu l'honneur d'appartenir.

Grazie a Dile e Matteo, Fabio, Giacomo, Enrica, Andrea. È sempre un piacere trovarci in giro. Gracias con todo mi corazón a Ana, por toda la motivación y ánimo que me diste. Un remerciement à Rémi et Vlado. Un agradecimiento especial a Ernesto.

Dehors le monde mathématique, j'aimerais remercier mes amis parisiens Oana, Jean Marie et Camille. A mis amigas Carolina, Elvira, Linda, Diana y Mariana, con quien se pasó bueno comiendo y comiendo. Alla mia professoressa di romano, Michelle, porterò la tua carbonara nel cuore. Je remercie énormement Laurent pour son soutien constant. J'aimerais remercier énormement Irène, sans son soutien mon séjour à Paris n'aurait jamais été si fantastique.

Quisiera agradecer a mi gran amigo Luis por todas las discusiones filosóficas sobre la vida y las salidas en París. And I would like to thank João and Noé for not having talked about philosophy with me. I would like to thank Martin, who came into my life in an unexpected way, for all the silly jokes and his friendship. A Josephin y a Miguelito, quienes han seguido compartiendo conmigo desde hace mucho tiempo y con quienes espero compartir mucho más.

Quisiera agradecer a mi familia, a mi mamá Maria Teresa, mi hermanita Avi, mi hermanito Juli, mis tías que siempre me tienen en sus oraciones. Finalmente, quisiera agradecer a my fish, my spear, my key, Diego Cachicornious, por su incondicional amistad. A Sofía, por haberme apoyado todos estos años y por su incondicional amor de familia. Y a Davide, a quien no existen palabras en ningún idioma que yo conozca para expresar cuán agradecido estoy por todo su apoyo y cariño.

Abstract

In order to study the rigid isotopy classes of nodal rational curves of degree 5 in $\mathbb{RP}^2$, we associate to every real rational quintic curve with a marked real nodal point a trigonal curve in the Hirzebruch surface Σ_3 and the corresponding nodal real dessin on $\mathbb{CP}^1/(z \mapsto \bar{z})$. The dessins are real versions, proposed by S. Orevkov [10], of Grothendieck's *dessins d'enfants*. The *dessins* are graphs embedded in a topological surface and endowed with a certain additional structure. We study the combinatorial properties and decompositions of dessins corresponding to real nodal trigonal curves $C \subset \Sigma$ in real ruled surfaces Σ. Uninodal dessins in any surface with non-empty boundary and nodal dessins in the disk can be decomposed in blocks corresponding to cubic dessins in the disk $\mathbf{D}^2$, which produces a classification of these dessins. The classification of dessins under consideration leads to a rigid isotopy classification of real rational quintics in $\mathbb{RP}^2$.

Résumé

Afin d'étudier les classes d'isotopie rigide des courbes rationnelles nodales de degré 5 dans $\mathbb{RP}^2$, nous associons à chaque quintique avec un point double réel marqué une courbe trigonale dans la surface de Hirzebruch Σ_3 et le dessin réel nodal correspondant dans $\mathbb{CP}^1/(z \mapsto \bar{z})$. Les dessins sont des versions réelles, proposées par S. Orevkov dans [10], des *dessins d'enfants* de Grothendieck. Un *dessin* est un graphe contenu dans une surface topologique, muni d'une certaine structure supplémentaire. Dans cette thèse, nous étudions les propriétés combinatoires et les décompositions des dessins correspondants aux courbes trigonales nodales $C \subset \Sigma$ dans les surfaces réglées réelles Σ. Les dessins uninodaux sur une surface à bord quelconque et les dessins nodaux sur le disque peuvent être décomposés en blocs correspondant aux dessins cubiques sur le disque $\mathbf{D}^2$, ce qui conduit à une classification de ces dessins. La classification des dessins considédérés mène à une classification à isotopie rigide près des courbes rationnelles nodales de degré 5 dans $\mathbb{RP}^2$.

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Introduction

> You do not have to be good.
> You do not have to walk on your knees
> for a hundred miles through the
> desert, repenting.
> You only have to let the soft animal of
> your body love what it loves.
>
> Mary Oliver, *Wild Geese*

The first part of Hilbert's 16th problem [7] asks for a topological classification of all possible pairs $(\mathbb{RP}^2, \mathbb{R}A)$, where $\mathbb{R}A$ is the real point set of a non-singular curve A of fixed degree d in the real projective plane $\mathbb{RP}^2$. The fact that any homeomorphism of $\mathbb{RP}^2$ is isotopic to the identity implies that two topological pairs $(\mathbb{RP}^2, \mathbb{R}A)$ and $(\mathbb{RP}^2, \mathbb{R}B)$ are homeomorphic if and only if $\mathbb{R}A$ is isotopic to $\mathbb{R}B$ as subsets of $\mathbb{RP}^2$. The moduli space $\mathbb{RP}^{\frac{d(d+3)}{2}}$ of real homogeneous polynomials of degree d in three variables (up to multiplication by a non-zero real number) has open strata formed by the non-singular curves (defined by polynomials with non-zero gradient in $\mathbb{C}^3 \setminus \{\bar{0}\}$) and a codimension one subset formed by the singular curves; the latter subset is an algebraic hypersurface called the *discriminant*. The discriminant divides $\mathbb{RP}^{\frac{d(d+3)}{2}}$ into connected components, called *chambers*. Two curves in the same chamber can be connected by a path that does not cross the discriminant, and therefore every point of the path corresponds to a non-singular curve. Such a path is called a *rigid isotopy*. A version of Hilbert's 16th problem asks for a description of the set of chambers of the moduli space $\mathbb{RP}^{\frac{d(d+3)}{2}}$, equivalently, for a classification up to rigid isotopy of non-singular curves of degree d in $\mathbb{RP}^2$. Nowadays, the rigid isotopy classification of non-singular curves in $\mathbb{RP}^2$ is known up to degree 6. More generally, given a class of complex algebraic varieties, one can ask for a description of open strata of the set of real varieties in this class. Classical examples are provided by non-singular algebraic surfaces in $\mathbb{RP}^3$ and non-singular real algebraic curves in various algebraic surfaces (for example, in $\mathbb{RP}^1 \times \mathbb{RP}^1$). In the case of rational curves of degree d in $\mathbb{RP}^2$, the open strata correspond to the rational curves having only nodal points. A rigid isotopy classification of real rational curves of degree 4 in $\mathbb{RP}^2$ can be found in [6].

In the case of curves of degree 5 in $\mathbb{RP}^2$, the rigid isotopy classification of non-singular curves and curves with one non-degenerate double point can be found in [9]. The purpose of this thesis is a rigid isotopy classification of real nodal rational curves of degree 5 in $\mathbb{RP}^2$. (The isotopy classification of nodal rational curves of degree 5 in $\mathbb{RP}^2$ was obtained in [8].) In the case where the studied curves have at least one real nodal point, we use a correspondence between real nodal rational curves of degree 5 with a marked real nodal point and toiles endowed with some extra information.

The above correspondence makes use of trigonal curves on the Hirzebruch surface Σ_3.

In this text, a trigonal curve is a complex curve C lying in a ruled surface such that the restriction to C of the projection provided by the ruling is a morphism of degree 3. We consider an auxiliary morphism of j-invariant type in order to associate to the curve C a dessin, a real version, proposed by S. Orevkov, of the dessins d'enfants introduced by A. Grothendieck. A toile is a particular type of dessin corresponding the to the curves under consideration.

In Chapter 1, we introduce the basic notions and discuss some properties of trigonal curves and the associated dessins in the complex and real case. We explain how the study of equivalence classes of dessins allows us to obtain information about the equivariant deformation classes of real trigonal curves.

In Chapter 2, we state combinatorial properties of nodal dessins. These combinatorial properties are valid in a more general setting than the one needed for the case of rational curves of degree 5. We present a systematical treatment of dessins in two cases, namely, dessins lying on an arbitrary compact surface with boundary and having only one nodal point, and dessins lying on the disk $\mathbb{D}^2$ and having an arbitrary number of nodal points.

In Chapter 3, we deal with a relation between plane curves and trigonal curves on the Hirzebruch surfaces and detail the case of plane cubic curves, which leads to the dessins that serve as building blocks for our constructions.

Chapter 4 contains an application to the case of real pointed quartic curves and explicits their rigid isotopy classification *via* dessins. Such classification was first obtained in [11].

Finally, in Chapter 5 we present a rigid isotopy classification of nodal rational curves of degree 5 in $\mathbb{RP}^2$.

Chapter 1

Trigonal curves and dessins

In this chapter we introduce trigonal curves and dessins, which are the principal tool we use in order to study the rigid isotopy classification of degree 5 real rational curves. The content of this chapter is based on the book [2] and the article [4].

1.1 Ruled surfaces and trigonal curves

1.1.1 Basic definitions

A compact complex surface Σ is a *(geometrically) ruled surface* over a curve B if Σ is endowed with a projection $\pi : \Sigma \longrightarrow B$ of fiber $\mathbb{CP}^1$ as well as a special section E of non-positive self-intersection.

Definition 1.1.1. A *trigonal curve* is a reduced curve C lying in a ruled surface Σ such that C contains neither the exceptional section E nor a fiber as component, and the restriction $\pi|_C : C \longrightarrow B$ is a degree 3 map.

A trigonal curve $C \subset \Sigma$ is *proper* if it does not intersect the exceptional section E.

A *singular fiber* of a trigonal curve $C \subset \Sigma$ is a fiber F of Σ intersecting $C \cup E$ geometrically in less than 4 points.

A fiber F is singular if C passes through $E \cap F$, or if C is tangent to F or if C has a singular point belonging to F (those cases are not exclusive). A singular fiber F is *proper* if C does not pass through $E \cap F$. Then, C is proper if and only if all its singular fibers are. We call a singular fiber F *simple* if either C is tangent to F or F contains a node, a cusp or an inflection point. The set $\{b \in B \mid F_b \text{ is a singular fiber }\}$ of points in the base having singular fibers is a discrete subset of the base B. We denote its complement by $B^\# = B^\#(C)$; it is a curve with punctures.

Certain trigonal curves can be seen as *pullbacks* of more simple ones using the following construction. If $\varphi : B' \longrightarrow B$ is a non-constant morphism between two compact curves and $C \subset \Sigma$ is a trigonal curve, then $\Sigma' = \Sigma \underset{B}{\times} B'$ is a ruled surface over B', $E' = \varphi^* E \subset \Sigma'$ is the exceptional section of Σ', $C' = \varphi^* C$ is the trigonal curve induced of C by φ.

1.1.2 Deformations

We are interested in the study of trigonal curves up to deformation. In the real case, we consider the curves up to equivariant deformation (with respect to the action of the complex conjugation, cf. 1.1.7).

In the Kodaira-Spencer sense, a *deformation* of the quintuple $(\pi\colon \Sigma \longrightarrow B, E, C)$ refers to an analytic space $X \longrightarrow S$ fibered over an marked open disk $S \ni o$ endowed with analytic subspaces $\mathcal{B}, \mathcal{E}, \mathcal{C} \subset X$ such that for every $s \in S$, the fiber X_s is diffeomorphic to Σ and the intersections $\mathcal{B}_s := X_s \cap \mathcal{B}$, $\mathcal{E}_s := X_s \cap \mathcal{E}$ and $\mathcal{C}_s := X_s \cap \mathcal{C}$ are diffeomorphic to B, E and C, respectively, and there exists a map $\pi_s\colon X_s \longrightarrow B_s$ making X_s a geometrically ruled surface over B_s with exceptional section E_s, such that the diagram in Figure 1.1 commutes and $(\pi_o\colon X_o \longrightarrow B_o, E_o, C_o) = (\pi\colon \Sigma \longrightarrow B, E, C)$.

$$\begin{tikzcd}
 & E_s \arrow[r, "\text{diff.}"] \arrow[d, hook] & E \arrow[d, hook] & \\
C_s \arrow[r, hook] \arrow[dr, "\pi_s|_{C_s}"'] & X_s \arrow[r, "\text{diff.}"] \arrow[d, "\pi_s"] & \Sigma \arrow[d, "\pi"'] & C \arrow[l, hook'] \arrow[dl, "\pi|_C"] \\
 & B_s \arrow[r, "\text{diff.}"] & B &
\end{tikzcd}$$

Figure 1.1: Commuting diagram of isomorphisms and diffeomorphisms of the fibers of a deformation.

Definition 1.1.2. An *elementary deformation* of a trigonal curve $C \subset \Sigma \longrightarrow B$ is a deformation of the quintuple $(\pi\colon \Sigma \longrightarrow B, E, C)$ in the Kodaira-Spencer sense.

An elementary deformation $X \longrightarrow S$ is *equisingular* if for every $s \in S$ there exists a neighborhood $U_s \subset S$ of s such that for every singular fiber F of C, there exists a neighborhood $V_{\pi(F)} \subset B$ of $\pi(F)$, where $\pi(F)$ is the only point with a singular fiber for every $t \in U_s$. An elementary deformation over D^2 is a *degeneration* or *perturbation* if the restriction to $D^2 \setminus \{0\}$ is equisingular and for a set of singular fibers F_i there exists a neighborhood $V_{\pi(F_i)} \subset B$ where there are no points with a singular fiber for every $t \neq 0$. In this case we say that C_t *degenerates* to C_0 or C_0 is *perturbed* to C_t, for $t \neq 0$.

1.1.3 Nagata transformations

One of our principal tools in this paper are the dessins, which we are going to associate to proper trigonal curves. A trigonal curve C intersects the exceptional section E in a finite number of points, since C does not contain E as component. We use the Nagata transformations in order to construct a proper trigonal curve out of a non-proper one.

Definition 1.1.3. A *Nagata transformation* is a fiberwise birational transformation $\Sigma \longrightarrow \Sigma'$ consisting of blowing up a point $p \in \Sigma$ and contracting the strict transform of the fiber $F_{\pi(p)}$ containing p. The transformation is called *positive* if $p \in E$, and *negative* otherwise.

The result of a positive Nagata transformation is a ruled surface Σ', with an exceptional divisor E' such that $-E'^2 = -E^2 - 1$. The trigonal curves C_1 and C_2 over the same base B are *Nagata equivalent* if there exists a sequence of Nagata transformations mapping one curve to the other by strict transforms. Since all the points at the intersection $C \cap E$ can be resolved, every trigonal curve C is Nagata equivalent to a proper trigonal curve C' over the same base, called a *proper model* of C.

1.1.4 Weierstraß equations

For a trigonal curve, the Weierstraß equations are an algebraic tool which allows us to study the behavior of the trigonal curve with respect to the zero section and the exceptional one. They give rise to an auxiliary morphism of j-invariant type, which plays an intermediary role between trigonal curves and dessins. Let $C \subset \Sigma \longrightarrow B$ be a proper trigonal curve. Mapping a point $b \in B$ of the base to the barycenter of the points in $C \cap F_B^0$ (weighted according to their multiplicity) defines a section $B \longrightarrow Z \subset \Sigma$ called the *zero section*; it is disjoint from the exceptional section E.

The surface Σ can be seen as the projectivization of a rank 2 vector bundle, which splits as a direct sum of two line bundles such that the zero section Z corresponds to the projectivization of $\mathcal{Y}$, one of the terms of this decomposition. In this context, the trigonal curve C can be described by a Weierstraß equation, which in suitable affine charts has the form

$$x^3 + g_2 x + g_3 = 0, \tag{1.1}$$

where g_2, g_3 are sections of $\mathcal{Y}^2$, $\mathcal{Y}^3$ respectively, and x is an affine coordinate such that $Z = \{x = 0\}$ and $E = \{x = \infty\}$. For this construction, we can identify $\Sigma \setminus B$ with the total space of $\mathcal{Y}$ and take x as a local trivialization of this bundle. Nonetheless, the sections g_2, g_3 are globally defined. The line bundle $\mathcal{Y}$ is determined by C. The sections g_2, g_3 are determined up to change of variable defined by

$$(g_2, g_3) \longrightarrow (s^2 g_2, s^3 g_3),\ s \in H^0(B, \mathcal{O}_B^*).$$

Hence, the singular fibers of the trigonal curve C correspond to the points where the equation (1.1) has multiple root, i.e., the zeros of the discriminant section

$$\Delta := -4g_2^3 - 27g_3^2 \in H^0(B, \mathcal{O}_B(\mathcal{Y}^6)). \tag{1.2}$$

Therefore, C being reduced is equivalent to Δ being identically zero. A Nagata transformation over a point $b \in B$ changes the line bundle $\mathcal{Y}$ to $\mathcal{Y} \otimes \mathcal{O}_B(b)$ and the sections g_2 and g_3 to $s^2 g_2$ and $s^3 g_3$, where $s \in H^0(B, \mathcal{O}_B)$ is any holomorphic function having a zero at b.

Definition 1.1.4. Let C be a non-singular trigonal curve with Weierstraß model determined by the sections g_2 and g_3 as in (1.1). The trigonal curve C is *almost generic* if every singular fiber corresponds to a simple root of the determinant section $\Delta = -4g_2^3 - 27g_3^2$ which is not a root of g_2 nor of g_3. The trigonal curve C is *generic* if it is almost generic and the sections g_2 and g_3 have only simple roots.

1.1.5 The j-invariant

The j-invariant describes the relative position of four points in the complex projective line $\mathbb{CP}^1$. We describe some properties of the j-invariant in order to use them in the description of the dessins.

Definition 1.1.5. Let z_1, z_2, z_3, $z_4 \in \mathbb{CP}^1$. The *j-invariant* of a set $\{z_1, z_2, z_3, z_4\}$ is given by

$$j(z_1, z_2, z_3, z_4) = \frac{4(\lambda^2 - \lambda + 1)^3}{27\lambda^2(\lambda - 1)^2}, \tag{1.3}$$

where λ is the *cross-ratio* of the quadruple (z_1, z_2, z_3, z_4) defined as

$$\lambda(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}.$$

The cross-ratio depends on the order of the points while the j-invariant does not. Since the cross-ratio λ is invariant under Möbius transformations, so is the j-invariant. When two points z_i, z_j coincide, the cross-ratio λ equals either 0, 1 or ∞, and the j-invariant equals ∞.

Let us consider a polynomial $z^3 + g_2 z + g_3$. We define the j-invariant $j(z_1, z_2, z_3)$ of its roots z_1, z_2, z_3 as $j(z_1, z_2, z_3, \infty)$. If $\Delta = -4g_2^3 - 27g_3^2$ is the discriminant of the polynomial, then

$$j(z_1, z_2, z_3, \infty) = \frac{-4g_2^3}{\Delta}.$$

A subset A of $\mathbb{CP}^1$ is real if A is invariant under the complex conjugation. We say that A has a nontrivial symmetry if there is a nontrivial permutation of its elements which extends to a linear map $z \longmapsto az + b$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$.

Lemma 1.1.6 ([2]). *The set $\{z_1, z_2, z_3\}$ of roots of the polynomial $z^3 + g_2 z + g_3$ has a nontrivial symmetry if and only if its j-invariant equals 0 (for an order 3 symmetry) or 1 (for an order 2 symmetry).*

Proposition 1.1.7 ([2]). *Assume that $j(z_1, z_2, z_3) \in \mathbb{R}$. Then, the following holds*

- *The j-invariant $j(z_1, z_2, z_3) < 1$ if and only if the points z_1, z_2, z_3 form an isosceles triangle. The special angle seen as a function of the j-invariant is a increasing monotone function. This angle tends to 0 when j tends to $-\infty$, equals $\frac{\pi}{3}$ at $j = 0$ and tends to $\frac{\pi}{2}$ when j approaches 1.*

- *The j-invariant $j(z_1, z_2, z_3) \geq 1$ if and only if the points z_1, z_2, z_3 are collinear.The ratio between the lengths of the smallest segment and the longest segment $\overline{z_l z_k}$ seen as a function of the j-invariant is a decreasing monotone function. This ratio equals 1 when j equals 1, and 0 when j approaches ∞.*

As a corollary, if the j-invariant of $\{z_1, z_2, z_3\}$ is not real, then the points form a triangle having side lengths pairwise different. Therefore, in this case, the points z_1, z_2, z_3 can be ordered according to the decreasing order of side lengths of the opposite edges. E.g., for $\{z_1, z_2, z_3\} = \{\frac{i}{2}, 0, 1\}$ the order is 0, $\frac{i}{2}$ and 1.

Proposition 1.1.8 ([2]). *If $j(z_1, z_2, z_3) \notin \mathbb{R}$, then the above order on the points z_1, z_2, z_3 is clockwise if $\operatorname{Im}(j(z_1, z_2, z_3)) > 0$ and anti-clockwise if $\operatorname{Im}(j(z_1, z_2, z_3)) < 0$.*

1.1.6 The j-invariant of a trigonal curve

Let C be a proper trigonal curve. We use the j-invariant defined for triples of complex numbers in order to define a meromorphic map j_C on the base curve B. The map j_C encodes the topology of the trigonal curve C. The map j_C is called the *j-invariant* of the curve C and provides a correspondence between trigonal curves and dessins.

Definition 1.1.9. For a proper trigonal curve C, we define its j-invariant j_C as the analytic continuation of the map

$$\begin{array}{ccc} B^{\#} & \longrightarrow & \mathbb{C} \\ b & \longmapsto & j\text{-invariant of } C \cap F_b^0 \subset F_b^0 \cong \mathbb{C}. \end{array}$$

We call the trigonal curve C *isotrivial* if its j-invariant is constant.

If a proper trigonal curve C is given by a Weierstraß equation of the form (1.1), then

$$j_C = -\frac{4g_2^3}{\Delta}\ , \text{ where } \Delta = -4g_2^3 - 27g_3^2. \tag{1.4}$$

For a sufficiently generic trigonal curve (i.e., if g_2 and g_3 have no common zeros) j_C is a degree $6d$ map, where $d = -E^2$. If $C = \varphi^* C'$ is the pullback by a morphism $\varphi\colon B \longrightarrow B'$ of another trigonal curve, then we have $j_C = j_{C'} \circ \varphi$.

The surface Σ_1 can be defined as the surface obtained by blowing up a point of the complex projective plane. Choose affine coordinates $(\bar{t}, \bar{x})$ in $\mathbb{CP}^2$ such that the blown-up point is $(0, \infty)$. The projection to the first coordinate $(\bar{t}, \bar{x})) \longmapsto \bar{t}$ defines a ruling of Σ_1 over $\mathbb{CP}^1$ with the exceptional section given by the exceptional divisor of the blow-up.

Definition 1.1.10. The *universal cubic* is the trigonal curve $\overline{C} \subset \Sigma_1$ given by the Weierstraß equation

$$\bar{x}^3 - 3\bar{t}(\bar{t}-1)\bar{x} + 2\bar{t}(\bar{t}-1)^2 = 0. \tag{1.5}$$

The j-invariant of the universal curve is the identity map of $\mathbb{CP}^1$.

Theorem 1.1.11 ([2])**.** *Up to Nagata equivalence, every non-isotrivial curve C is equivalent to $j_C^* \overline{C}$, i.e., C is induced from the universal curve $\overline{C}$ by its j-invariant j_C.*

Theorem 1.1.12 ([2])**.** *Let B be a compact curve and $j\colon B \longrightarrow \mathbb{CP}^1$ a non-constant meromorphic map. Up to Nagata equivalence, there exists a unique trigonal curve $C \subset \Sigma \longrightarrow B$ such that $j_C = j$.*

Following the proof of the theorem, $j_B \longrightarrow \mathbb{CP}^1$ leads to a unique *minimal* proper trigonal curve C_j, in the sense that any other trigonal curve with the same j-invariant can be obtained by positive Nagata transformations from C_j.

An equisingular deformation C_s, $s \in S$, of C leads to an analytic deformation of the couple (B_s, j_{C_s}).

Corollary 1.1.13 ([2])**.** *Let (B, j) be a couple, where B is a compact curve and $j\colon B \longrightarrow \mathbb{CP}^1$ is a non-constant meromorphic map. Then, any deformation of (B, j) results in a deformation of the minimal curve $C_j \subset \Sigma \longrightarrow B$ associated to j.*

The j-invariant of a generic trigonal curve $C \subset \Sigma \longrightarrow B$ has degree $\deg(j_C) = 6d$, where $d = -E^2$. A positive Nagata transformation increases d by one while leaving j_C invariant. The j-invariant of a generic trigonal curve C has a ramification index equal to 3, 2 or 1 at every point $b \in B$ such that $j_C(b)$ equals 0, 1 or ∞, respectively. We can assume, up to perturbation, that every critical value of j_C is simple. In this case we say that j_C has a *generic branching behavior.*

1.1.7 Real structures

We are mostly interested in real trigonal curves. A *real structure* on a complex variety X is an anti-holomorphic involution $c\colon X \longrightarrow X$. We define a *real variety* as a couple (X, c), where c is a real structure on a complex variety X. We denote by $X_{\mathbb{R}}$ the fixed point set of the involution c and we call $X_{\mathbb{R}}$ *the set of real points* of c.

Theorem 1.1.14 (Smith-Thom inequality [5])**.** *If (X, c) is a compact real variety, then*

$$b_*(X_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}) \leq b_*(X; \mathbb{Z}/2\mathbb{Z}), \tag{1.6}$$

where b_* denotes the total Betti number with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.1.15 (Smith congruence [5]). *If (X, c) is a compact real variety, then*

$$b_*(X_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}) \equiv b_*(X; \mathbb{Z}/2\mathbb{Z}) \mod 2. \tag{1.7}$$

Definition 1.1.16. A real variety (X, c) is *an $(M - a)$-variety* if

$$b_*(X; \mathbb{Z}/2\mathbb{Z}) - b_*(X_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}) = 2a.$$

We say that a real curve (X, c) is of *type* I if $\widetilde{X} \setminus \widetilde{X}_{\mathbb{R}}$ is disconnected, where $\widetilde{X}$ is the normalization of X.

1.1.8 Real trigonal curves

A geometrically ruled surface $\pi\colon \Sigma \longrightarrow B$ is *real* if there exist real structures $c_\Sigma\colon \Sigma \longrightarrow \Sigma$ and $c_B\colon B \longrightarrow B$ compatible with the projection π, i.e., such that $\pi \circ c_\Sigma = c_B \circ \pi$. We assume the exceptional section is *real* in the sense that it is invariant by conjugation, i.e., $c_\Sigma(E) = E$. Put $\pi_{\mathbb{R}} := \pi|_{\Sigma_{\mathbb{R}}}\colon \Sigma_{\mathbb{R}} \longrightarrow B_{\mathbb{R}}$. Since the exceptional section is real, the fixed point set of every fiber is not empty, implying that the real structure on the fiber is isomorphic to the standard complex conjugation on $\mathbb{CP}^1$. Hence all the fibers of $\pi_{\mathbb{R}}$ are isomorphic to $\mathbb{RP}^1$. Thus, the map $\pi_{\mathbb{R}}$ establishes a bijection between the connected components of the real part $\Sigma_{\mathbb{R}}$ of the surface Σ and the connected components of the real part $B_{\mathbb{R}}$ of the curve B. Every connected component of $\Sigma_{\mathbb{R}}$ is homeomorphic either to a torus or to a Klein bottle.

If $\Sigma = \mathbf{P}(1 \oplus \mathcal{Y})$, with $\mathcal{Y} \in \mathrm{Pic}(B)$, we put $\mathcal{Y}_i := \mathcal{Y}_{\mathbb{R}}|_{B_i}$ for every connected component B_i of $B_{\mathbb{R}}$. Hence $\Sigma_i := \Sigma_{\mathbb{R}}|_{B_i}$ is orientable if and only if $\mathcal{Y}_i$ is topologically trivial, i.e., its first Stiefel-Whitney class $w_1(\mathcal{Y}_i)$ is zero.

Definition 1.1.17. A *real trigonal curve* C is a trigonal curve contained in a real ruled surface $(\Sigma, c_\Sigma) \longrightarrow (B, c_B)$ such that C is c_Σ-invariant, i.e., $c_\Sigma(C) = C$.

If a real trigonal curve is proper, then $\mathcal{Y}$ is real as well as its j-invariant, seen as a morphism $j_C\colon (B, c_B) \longrightarrow (\mathbb{CP}^1, z \longmapsto \bar{z})$, where $z \longmapsto \bar{z}$ denotes the standard complex conjugation on $\mathbb{CP}^1$. In addition, the sections g_2 and g_3 can be chosen real.

Let us consider the restriction $\pi|_{C_{\mathbb{R}}}\colon C_{\mathbb{R}} \longrightarrow B_{\mathbb{R}}$. We put $C_i := \pi|_{C_{\mathbb{R}}}^{-1}(B_i)$ for every connected component B_i of $B_{\mathbb{R}}$. We say that B_i is *hyperbolic* if $\pi|_{C_i}\colon C_i \longrightarrow B_i$ has generically a fiber with three elements. The trigonal curve C is *hyperbolic* if its real part is non-empty and all the connected components of $B_{\mathbb{R}}$ are hyperbolic.

Definition 1.1.18. Let C be a non-singular generic real trigonal curve. A connected component of the set $\{b \in B \mid \mathrm{Card}(\pi|_{C_{\mathbb{R}}}^{-1}(b)) \geq 2\}$ is an *oval* if it is not a hyperbolic component and its preimage by $\pi|_{C_{\mathbb{R}}}$ is disconnected. Otherwise, the connected component is called a *zigzag*.

1.2 Dessins

The dessins d'enfants were introduced by A. Grothendieck (cf. [13]) in order to study the action of the absolute Galois group. We use a modified version of dessins d'enfants which was proposed by S. Orevkov [10].

1.2.1 Trichotomic graphs

Let S be a compact connected topological surface. A graph D on the surface S is a graph embedded into the surface and considered as a subset $D \subset S$. We denote by $\mathrm{Cut}(D)$ the *cut* of S along D, i.e., the disjoint union of the closure of connected components of $S \setminus D$.

Definition 1.2.1. A *trichotomic graph* on a compact surface S is an embedded finite directed graph $D \subset S$ decorated with the following additional structures (referred to as *colorings* of the edges and vertices of D, respectively):

- every edge of D is color solid, bold or dotted,
- every vertex of D is black ($\bullet$), white ($\circ$), cross ($\times$) or monochrome (the vertices of the first three types are called *essential*),

and satisfying the following conditions:

1. $\partial S \subset D$,
2. every essential vertex is incident to at least 2 edges,
3. every monochrome vertex is incident to at least 3 edges,
4. the orientations of the edges of D form an orientation of the boundary $\partial\,\mathrm{Cut}(D)$ which is compatible with an orientation on $\mathrm{Cut}(D)$,
5. all edges incident to a monochrome vertex are of the same color,
6. $\times$-vertices are incident to incoming dotted edges and outgoing solid edges,
7. $\bullet$-vertices are incident to incoming solid edges and outgoing bold edges,
8. $\circ$-vertices are incident to incoming bold edges and outgoing dotted edges.

Let $D \subset S$ a trichotomic graph. A *region* R is an element of of Cut(D). The boundary ∂R of R contains $n = 3k$ essential vertices. A region with n vertices on its boundary is called an *n-gonal region.* We denote by D_{solid}, D_{bold}, D_{dotted} the monochrome parts of D, i.e., the sets of vertices and edges of the specific color. On the set of vertices of a specific color, we define the relation $u \preceq v$ if there is a monochrome path from u to v, i.e., a path form entirely of edge of the same color. We call the graph D *admissible* if the relation $\preceq$ is a partial order, equivalently if there are no directed monochrome cycles.

Definition 1.2.2. A trichotomic graph D is a *dessin* if

1. D is admissible;
2. every trigonal region of D is homeomorphic to a disk.

The orientation of the graph D is determined by the pattern of colors of the vertices on the boundary of every region.

1.2.2 Complex and real dessins

Let S be an orientable surface. Every orientation of S induces a *chessboard coloring* of $\mathrm{Cut}(D)$, i.e., a function on $\mathrm{Cut}(D)$ determining if a region R endowed with the orientation set by D coincides with the orientation of S.

Definition 1.2.3. A *real trichotomic graph* on a real compact surface (S, c) is a trichotomic graph D on S which is invariant under the action of c. Explicitly, every vertex v of D has as image $c(v)$ a vertex of the same color; every edge e of D has as image $c(e)$ an edge of the same color.

Let D be a real trichotomic graph on (S, c). Put $\overline{S} := S/c$ as the quotient surface and put $\overline{D} \subset \overline{S}$ as the image of D by the quotient map $S \longrightarrow S/c$. The graph $\overline{D}$ is a well defined graph on the surface S/c.

In the inverse sense, let S be a compact surface, which can be non-orientable or can have non-empty boundary. Let $D \subset S$ be a trichotomic graph on S. Consider its complex double covering $\widetilde{S} \longrightarrow S$ (cf. [1] for details), which has a real structure given by the deck transformation, and put $\widetilde{D} \subset \widetilde{S}$ the inverse image of D. The graph $\widetilde{D}$ is a graph on $\widetilde{S}$ invariant by the deck transformation. We use these constructions in order to identify real trichotomic graphs on real surfaces with their images on the quotient surface.

Lemma 1.2.4 ([2])**.** *Let D be a c-invariant trichotomic graph on a real surface $(\widetilde{S}, c)$. Then, every region R of D is disjoint from its image $c(R)$.*

Proposition 1.2.5 ([2])**.** *Let S be a compact surface. Given a trichotomic graph $D \subset S$, then its oriented double covering $\widetilde{D} \subset \widetilde{S}$ is a real trichotomic graph. Moreover, $\widetilde{D} \subset \widetilde{S}$ is a dessin if and only if so is $D \subset S$. Conversely, if (S, c) is a real compact surface and $D \subset S$ is a real trichotomic graph, then its image $\overline{D}$ in the quotient $\overline{S} := S \longrightarrow S/c$ is a trichotomic graph. Moreover, $\overline{D} \subset \overline{S}$ is a dessin if and only if so is $D \subset S$.*

Definition 1.2.6. Let D be a dessin on a compact surface S. Let us denote by $\mathrm{Ver}(D)$ the set of vertices of D. For a vertex $v \in \mathrm{Ver}(D)$, we define the *index* $\mathrm{Ind}(v)$ of v as half of the number of incident edges of $\widetilde{v}$, where $\widetilde{v}$ is a preimage of v by the double complex cover of S as in Proposition 1.2.5.

A vertex $v \in \mathrm{Ver}(D)$ is *singular* if

- v is black and $\mathrm{Ind}(v) \not\equiv 0 \mod 3$,
- or v is white and $\mathrm{Ind}(v) \not\equiv 0 \mod 2$,
- or v has color $\times$ and $\mathrm{Ind}(v) \geq 2$.

We denote by $\mathrm{Sing}(D)$ the set of singular vertices of D. A dessin is *non-singular* if none of its vertices is singular.

Definition 1.2.7. Let B be a complex curve and let $j\colon B \longrightarrow \mathbb{CP}^1$ a non-constant meromorphic function, in other words, a ramified covering of the complex projective line. The dessin $D := \mathrm{Dssn}(j)$ associated to j is the graph given by the following construction:

- as a set, the dessin D coincides with $j^{-1}(\mathbb{RP}^1)$, where $\mathbb{RP}^1$ is the fixed point set of the standard complex conjugation in $\mathbb{CP}^1$;
- black vertices ($\bullet$) are the inverse images of 0;
- white vertices ($\circ$) are the inverse images of 1;

- vertices of color $\times$ are the inverse images of ∞;
- monochrome vertices are the critical points of j with critical value in $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$;
- solid edges are the inverse images of the interval $[\infty, 0]$;
- bold edges are the inverse images of the interval $[0, 1]$;
- dotted edges are the inverse images of the interval $[1, \infty]$;
- orientation on edges is induced from an orientation of $\mathbb{RP}^1$.

Lemma 1.2.8 ([2]). *Let S be an oriented connected closed surface. Let $j\colon S \longrightarrow \mathbb{CP}^1$ a ramified covering map. The trichotomic graph $D = \mathrm{Dssn}(j) \subset S$ is a dessin. Moreover, if j is real with respect to an orientation-reversing involution $c\colon S \longrightarrow S$, then D is c-invariant.*

Let (S, c) be a compact real surface. If $j\colon (S, c) \longrightarrow (\mathbb{CP}^1, z \longrightarrow \bar{z})$ is a real map, we define $\mathrm{Dssn}_c(j) := \mathrm{Dssn}(j)/c \subset S/c$.

Theorem 1.2.9 ([2]). *Let S be an oriented connected closed surface (and let $c\colon S \longrightarrow S$ a orientation-reversing involution). A (real) trichotomic graph $D \subset S$ is a (real) dessin if and only if $D = \mathrm{Dssn}(j)$ for a (real) ramified covering $j\colon S \longrightarrow \mathbb{CP}^1$.*

Moreover, j is unique up to homotopy in the class of (real) ramified coverings with dessin D.

The last theorem together with the Riemann existence theorem provides the next corollaries, for the complex and real settings.

Corollary 1.2.10 ([2]). *Let $D \subset S$ be a dessin on a compact close orientable surface S. Then there exists a complex structure on S and a holomorphic map $j\colon S \longrightarrow \mathbb{CP}^1$ such that $\mathrm{Dssn}(j) = D$. Moreover, this structure is unique up to deformation of the complex structure on S and the map j in the Kodaira-Spencer sense.*

Corollary 1.2.11 ([2]). *Let $D \subset S$ be a dessin on a compact surface S. Then there exists a complex structure on its double cover $\widetilde{S}$ and a holomorphic map $j\colon \widetilde{S} \longrightarrow \mathbb{CP}^1$ such that j is real with respect to the real structure c of $\widetilde{S}$ and $\mathrm{Dssn}_c(j) = D$. Moreover, this structure is unique up to equivariant deformation of the complex structure on S and the map j in the Kodaira-Spencer sense.*

1.2.3 Deformations of dessins

In this section we describe the notions of deformations which allow us to associate classes of non-isotrivial trigonal curves and classes of dessins, up to deformations and equivalences that we explicit.

Definition 1.2.12. A *deformation of coverings* is a homotopy $S \times [0, 1] \longrightarrow \mathbb{CP}^1$ within the class of (equivariant) ramified coverings. The deformation is *simple* if it preserves the multiplicity of the inverse images of 0, 1, ∞ and of the other real critical values.

Any deformation is locally simple except for a finite number of values $t \in I = [0, 1]$.

Proposition 1.2.13 ([2]). *Let $j_0, j_1\colon S \longrightarrow \mathbb{CP}^1$ be (c-equivariant) ramified coverings. They can be connected by a simple (equivariant) deformation if and only their dessins $D(j_0)$ and $D(j_1)$ are isotopic (respectively, $D_c(j_0)$ and $D_c(j_1)$).*

Definition 1.2.14. A deformation $j_t\colon S \longrightarrow \mathbb{CP}^1$ of ramified coverings is *equisingular* if the union of the supports

$$\bigcup_{t\in[0,1]} \operatorname{supp}\{(j_t^*(0) \mod 3) + (j_t^*(1) \mod 2) + j_t^*(\infty)\}$$

considered as a subset of $S \times [0,1]$ is an isotopy. Here * denotes the divisorial pullback of a map $\varphi : S \longrightarrow S'$ at a point $s' \in S'$:

$$\varphi^*(s') = \sum_{s\in\varphi^{-1}(s')} r_s s,$$

where r_s if the ramification index of φ at $s \in S$.

A dessin $D_1 \subset S$ is called a *perturbation* of a dessin $D_0 \subset S$, and D_0 is called a *degeneration* of D_1, if for every vertex $v \in \operatorname{Ver}(D_0)$ there exists a small neighboring disk $U_v \subset S$ such that $D_0 \cap U_v$ only has edges incident to v and $D_1 \cap U_v$ contains essential vertices of at most one color.

Theorem 1.2.15 ([2])**.** *Let $D_0 \subset S$ be a dessin, and let D_1 be an admissible perturbation. Then there exists a map $j_t\colon S \longrightarrow \mathbb{CP}^1$ such that*

1. *$D_0 = \operatorname{Dssn}(j_0)$ and $D_1 = \operatorname{Dssn}(j_1)$;*
2. *$j_t|_{S\setminus\bigcup_v U_v} = j_{t'}|_{S\setminus\bigcup_v U_v}$ for every t, t';*
3. *the deformation restricted to $S \times (0,1]$ is simple.*

Corollary 1.2.16 ([2])**.** *Let S be a complex compact curve, $j\colon S \longrightarrow \mathbb{CP}^1$ a non-constant holomorphic map, and let $\operatorname{Dssn}_c(j) = D_0, D_1, \ldots, D_n$ be a chain of dessins in S such that for $i = 1, \ldots, n$ either D_i is a perturbation of D_{i-1}, or D_i is a degeneration of D_{i-1}, or D_i is isotopic to D_{i-1}. Then there exists a piecewise-analytic deformation $j_t\colon S_t \longrightarrow \mathbb{CP}^1$, $t \in [0,1]$, of $j_0 = j$ such that $\operatorname{Dssn}(j_1) = D_n$.*

Corollary 1.2.17 ([2])**.** *Let (S,c) be a real compact curve, $j\colon (S,c) \longrightarrow (\mathbb{CP}^1, z \longmapsto \bar{z})$ be a real non-constant holomorphic map, and let $\operatorname{Dssn}_c(j) = D_0, D_1, \ldots, D_n$ be a chain of real dessins in (S,c) such that for $i = 1, \ldots, n$ either D_i is a equivariant perturbation of D_{i-1}, or D_i is a equivariant degeneration of D_{i-1}, or D_i is equivariantly isotopic to D_{i-1}. Then there is a piecewise-analytic real deformation $j_t\colon (S_t, c_t) \longrightarrow (\mathbb{CP}^1, \bar{\cdot})$, $t \in [0,1]$, of $j_0 = j$ such that $\operatorname{Dssn}_c(j_1) = D_n$.*

Due to Theorem 1.2.15, the deformation j_t given by Corollaries 1.2.16 and 1.2.17 is equisingular in the sense of Definition 1.2.14 if and only if all perturbations and degenerations of the dessins on the chain $D_0, D_1, \ldots, D_n$ are equisingular.

1.3 Trigonal curves and their dessins

In this section we describe an equivalence between dessins.

1.3.1 Correspondence theorems

Let $C \subset \Sigma \longrightarrow B$ be a non-isotrivial proper trigonal curve. We associate to C the dessin corresponding to its j-invariant $\mathrm{Dssn}(C) := \mathrm{Dssn}(j_C) \subset B$. In the case when C is a real trigonal curve we associate to C the real dessin corresponding to its j-invariant, $\mathrm{Dssn}_c(C) := \mathrm{Dssn}(j_C) \subset B/c_B$, where c_B is the real structure of the base curve B.

So far, we have focused on one direction of the correspondences: we start with a trigonal curve C, consider its j-invariant and construct the dessin associated to it. Now, we study the opposite direction. Let us consider a dessin D on a topological orientable closed surface S. By Corollary 1.2.10, there exist a complex structure B on S and a holomorphic map $j_D \colon B \longrightarrow \mathbb{CP}^1$ such that $\mathrm{Dssn}(j_D) = D$. By Theorem 1.1.12 and Corollary 1.1.13 there exists a trigonal curve C having j_D as j-invariant; such a curve is unique up to deformation in the class of trigonal curves with fixed dessin. Moreover, due to Corollary 1.2.16, any sequence of isotopies, perturbations and degenerations of dessins gives rise to a piecewise-analytic deformation of trigonal curves, which is singular if and only if all perturbations and degenerations are.

In the real framework, let (S, c) a compact close oriented topological surface endowed with a orientation-reversing involution. Let D be a real dessin on (S, c). By Corollary 1.2.11, there exists a real structure (B, c_B) on (S, c) and a real map $j_D \colon (B, c_B) \longrightarrow (\mathbb{CP}^1, z \longmapsto \bar{z})$ such that $\mathrm{Dssn}_c(j_D) = D$. By Theorem 1.1.12, Corollary 1.1.13 and the remarks made in Section 1.1.8, there exists a real trigonal curve C having j_D as j-invariant; such a curve is unique up to equivariant deformation in the class of real trigonal curves with fixed dessin. Furthermore, due to Corollary 1.2.17, any sequence of isotopies, perturbations and degenerations of dessins gives rise to a piecewise-analytic equivariant deformation of real trigonal curves, which is equisingular if and only if all perturbations and degenerations are.

Definition 1.3.1. A dessin is *reduced* if

- for every v $\bullet$-vertex one has $\operatorname{Ind} v \leq 3$,
- for every v $\circ$-vertex one has $\operatorname{Ind} v \leq 2$,
- every monochrome vertex is real and has index 2.

A reduced dessin is *generic* if all its $\bullet$-vertices and $\circ$-vertices are non-singular and all its $\times$-vertices have index 1.

Any dessin admits an equisingular perturbation to a reduced dessin. The vertices with excessive index (i.e., index greater than 3 for $\bullet$-vertices or than 2 for $\circ$-vertices) can be reduced by introducing new vertices of the same color.

In order to define an equivalence relation of dessins, we introduce *elementary moves*. Consider two reduced dessins D, $D' \subset S$ such that they coincide outside a closed disk $V \subset S$. If V does not intersect ∂S and the graphs $D \cap V$ and $D' \cap V$ are as shown in Figure 1.2(a), then we say that performing a *monochrome modification* on the edges intersecting V produces D' from D, or *vice versa*. This is the first type of *elementary moves*. Otherwise, the boundary component inside V is shown in light gray. In this setting, if the graphs $D \cap V$ and $D' \cap V$ are as shown in one of the subfigures in Figure 1.2, we say that performing an *elementary move* of the corresponding type on $D \cap V$ produces D' from D, or *vice versa*.

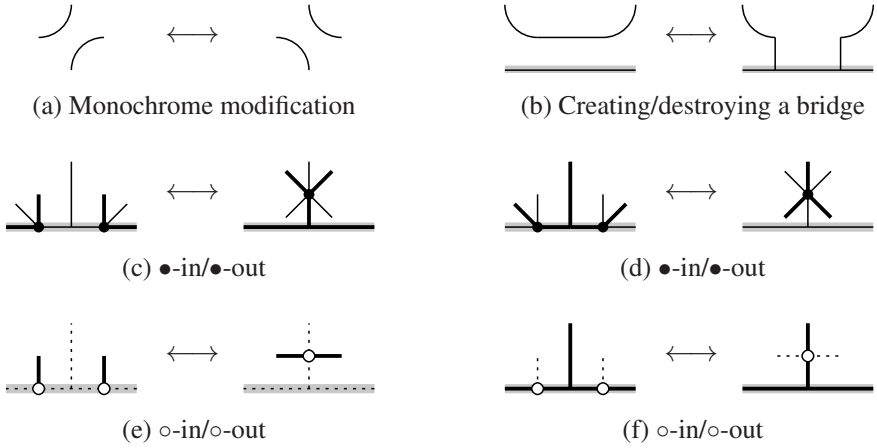


Figure 1.2: Elementary moves.

Definition 1.3.2. Two reduced dessins D, $D' \subset S$ are *elementary equivalent* if, after a (preserving orientation, in the complex case) homeomorphism of the underlying surface S they can be connected by a sequence of isotopies and elementary moves between dessins, as described in Figure 1.2.

This definition is meant so that two reduced dessins are elementary equivalent if and only if they can be connected up to homeomorphism by a sequence of isotopies, equisingular perturbations and degenerations.

The following theorems establish the equivalences between the deformation classes of trigonal curves we are interested in and elementary equivalence classes of certain dessins. We use these links to obtain different classifications of curves *via* the combinatorial study of dessins.

Theorem 1.3.3 ([2])**.** *There is a one-to-one correspondence between the set of equisingular deformation classes of non-isotrivial proper trigonal curves $C \subset \Sigma \longrightarrow B$ with $\widetilde{A}$ type singular fibers only and the set of elementary equivalence classes of reduced dessins $D \subset B$.*

Theorem 1.3.4 ([2])**.** *There is a one-to-one correspondence between the set of equivariant equisingular deformation classes of non-isotrivial proper real trigonal curves $C \subset \Sigma \longrightarrow (B, c)$ with $\widetilde{A}$ type singular fibers only and the set of elementary equivalence classes of reduced real dessins $D \subset B/c$.*

Theorem 1.3.5 ([2])**.** *There is a one-to-one correspondence between the set of equivariant equisingular deformation classes of almost generic real trigonal curves $C \subset \Sigma \longrightarrow (B, c)$ and the set of elementary equivalence classes of generic real dessins $D \subset B/c$.*

This correspondences can be extended to trigonal curves with more general singular fibers (see [2]).

Definition 1.3.6. Let $C \subset \Sigma \longrightarrow B$ be a proper trigonal curve. We define the *degree* of the curve C as $\deg(C) := -3E^2$ where E is the exceptional section of Σ. For a dessin D, we define its *degree* as $\deg(D) = \deg(C)$ where C is a minimal proper trigonal curve such that $\mathrm{Dssn}(C) = D$.

Figure 1.3: Straightening/creating a zigzag.

1.3.2 Real generic curves

Let C be a generic real trigonal curve and let $D := \mathrm{Dssn}_c(C)$ be a generic dessin. The *real part* of $D \subset S$ is the intersection $D \cap \partial S$. For a specific color $* \in \{\text{solid}, \text{bold}, \text{dotted}\}$, D_* is the subgraph of the corresponding color and its adjacent vertices. The components of $D_* \cap \partial S$ are either components of ∂S, called *monochrome components* of D, or segments, called *maximal monochrome segments* of D. We call these monochrome components or segments *even* or *odd* according to the parity of the number of $\circ$-vertices they contain.

Moreover, we refer to the dotted monochrome components as *hyperbolic components*. A dotted segment without $\times$-vertices of even index is referred to as an *oval* if it is even, or as a *zigzag* if it is odd.

Definition 1.3.7. Let $D \subset S$ be a real dessin. Let us assume there is a subset of S in which D has a configuration of vertices and edges as in Figure 1.3. Replacing the corresponding configuration with the alternative one defines another dessin $D' \subset S$. We say that D' is obtained from D by *straightening/creating* a zigzag.

Two dessins D, D' are *weakly equivalent* if there exists a finite sequence of dessins $D = D_0, D_1, \ldots, D_n = D'$ such that D_{i+1} is either elementary equivalent to D_i, or D_{i+1} is obtained from D_i by straightening/creating a zigzag.

Let us remark that if D' is obtained from D by straightening/creating a zigzag and $\widetilde{D}, \widetilde{D'} \subset \widetilde{S}$ are the liftings of D and D' in $\widetilde{S}$, the double complex of S, then $\widetilde{D}$ and $\widetilde{D'}$ are elementary equivalent as complex dessins. However, D and D' are not elementary equivalent, since the number of zigzags of a real dessin is an invariant on the equivalence class of real dessins.

1.3.3 Type of a dessin

Let $C \subset \Sigma \longrightarrow B$ be a real trigonal curve over a base curve of type 1. We define C_{Im} as the closure of the set $\pi^{-1}|_C(B_{\mathbb{R}}) \setminus C_{\mathbb{R}}$ and let $B_{\mathrm{Im}} = \pi(C_{\mathrm{Im}})$. By definition C_{Im} is invariant with respect to the real structure of C. Moreover, $C_{\mathrm{Im}} = \emptyset$ if and only if C is a hyperbolic trigonal curve.

Lemma 1.3.8 ([4])**.** *A trigonal curve is of type* I *if and only if the homology class* $[C_{\mathrm{Im}}] \in H_1(C; \mathbb{Z}/2\mathbb{Z})$ *is zero.*

In view of the last lemma, we can represent a trigonal curve of type I as the union of two orientable surfaces C_+ and C_-, intersecting at their boundaries $C_+ \cap C_- = \partial C_+ = \partial C_-$. Both surfaces, C_+ and C_-, are invariant under the real structure of C. We define

$$m_\pm : \begin{array}{ccc} B & \longrightarrow & \mathbb{Z} \\ b & \longmapsto & \mathrm{Card}(p|_C^{-1}(b) \cap C_\pm) - \chi_{B_{\mathrm{Im}}}(b), \end{array}$$

where $\chi_{B_{\mathrm{Im}}}$ is the characteristic function of the set B_{Im}. These maps are locally constant over $B^{\#}$, and since $B^{\#}$ is connected, the maps are actually constant. Moreover, $m_+ + m_- = 3$, so we choose the surfaces $C_\pm$ such that $m_+|_{B^{\#}} \equiv 1$ and $m_-|_{B^{\#}} \equiv 2$.

We can label each region $R \in \mathrm{Cut}(D)$ where $D = \mathrm{Dssn}(C)$ according to the label on C_+. Given any point $b \in R$ on the interior, the vertices of the triangle $\pi|_C^{-1}(b) \subset F_b$ are labeled by 1, 2, 3, according to the increasing of the opposite side of the triangle. We label the region R by the label of the point $\pi|_C^{-1}(b) \cap C_+$.

We can also label the interior edges according to the adjacent regions in the following way:

- every solid edge can be of type 1 (i.e., both adjacent regions are of type 1) or type $\overline{1}$ (i.e., one region of type 2 and one of type 3);
- every bold edge can be of type 3 or type $\overline{3}$;
- every dotted edge can be of type 1, 2 or 3.

We use the same rule for the real edges of D. Note that there are no real solid edges of type 1 nor real bold edges of type 3 (otherwise the morphism $C_+ \longrightarrow B$ would have two layers over the regions of D adjacent to the edge).

Theorem 1.3.9 ([4]). *A generic non-hyperbolic curve C is of type* I *if and only if the regions of D admit a labeling which satisfies the conditions described above.*

As a consequence, we can assign to every non-singular vertex of a dessin associated to a type I trigonal curve a label according to its incident edges.

- The interior $\times$-vertices are of type $(1, 1)$.
- The interior $\bullet$-vertices are of type $(1, \overline{3}, 1, \overline{3}, 1, \overline{3})$.
- The interior $\circ$-vertices can be of type $(3, 3, 3, 3)$ or type $(\overline{3}, 1, \overline{3}, 2)$.
- The real $\times$-vertices are of type $(2, 2)$ or $(3, 3)$.
- The real $\bullet$-vertices are of type $(2, \overline{3}, 1, 1)$ or $(3, 3, \overline{1}, 2)$.
- The real $\circ$-vertices can be of type $(3, 3, 3)$ or $(1, \overline{3}, 2)$ in case of dotted real incident edges, or can be of type $(1, 1, 1)$ or $(2, 2, 2)$ in case of bold real incident edges.

Corollary 1.3.10 ([4]). *Let D be a dessin associated to a type* I *trigonal curve. Then, the real $\times$-vertices of D limiting an oval can be of type* 3 *(i.e., of type* $(3, 3)$*) and so all the edges of the oval are of type* 3*, or they can be of type* 2 *(i.e., of type* $(2, 2)$*) and the edges in between alternate on type* 1 *and* 2*.*

Similarly, the real $\times$-vertices limiting a zigzag are of type 3 as well as the edges in between. A hyperbolic component is of *type* 3, 2, 1 if all its edges are of type 3, 2, 1, respectively, or of *type* $\overline{3}$ if the type of its edges alternates between 1 and 2.

Lemma 1.3.11 ([4]). *Let D be a dessin associated to a type* I *trigonal curve. The odd hyperbolic components of D are of type* 3*. The even hyperbolic components can be of type* 3*,* 2*,* 1 *or* $\overline{3}$*.*

The even hyperbolic components of type 1 and 2 are the ones without essential vertices.

Definition 1.3.12 ([4]). A real dessin is *unramified* if all its $\times$-vertices are real.

Lemma 1.3.13 ([4]). *Let D be an unramified dessin of type* I*. Then D does not have solid nor dotted edges of type* 1*.*

Proof. Since solid or dotted edges must finish (after passing by monochrome vertices) in a $\times$-vertex, which can not have type 1 since D is of type I and does not have solid edges of type 1. $\square$

Lemma 1.3.14 ([4])**.** *Let D be an unramified dessin of type* I*. Every $\bullet$-vertex $v \in \mathrm{Vert}(D)$ is real. Moreover, the incident edges of v are of type $(3, 3, \bar{1}, 2)$. Its essential neighbor vertices are either $\circ$ or $\times$-vertices.*

Proof. If the dessin D has an interior $\bullet$-vertex, it must have type $(1, \bar{3}, 1, \bar{3}, 1, \bar{3})$ having a solid edge of type 1, contradicting the previous lemma. Moreover, if it has a $\bullet$-vertex as neighbor, after an elementary move of type $\bullet$-in (up to destruction of bridges, cf 1.2) in order to obtain an interior $\bullet$-vertex. $\square$

In the following corollaries we consider D as an unramified dessin of type I.

Corollary 1.3.15 ([4])**.** *The dessin D does not have interior bold edges of type $\bar{3}$ (i.e., all the interior bold edges are of type 3).*

Corollary 1.3.16 ([4])**.** *Every real $\circ$-vertex with real bold edges is of type $(2, 2, 2)$. All the other $\circ$-vertex are of type 3 (i.e., either it is real of type $(3, 3, 3)$ or interior of type $(3, 3, 3, 3)$).*

Corollary 1.3.17 ([4])**.** *The dessin D does not have ovals of type 3 (i.e., every oval is of type 2).*

Lemma 1.3.18 ([4])**.** *Let v be a $\circ$-vertex of type 2. Then v is real with $\bullet$-vertices as essential neighbors.*

Proof. According to Corollary 1.3.16, the vertex v is real and its adjacent real edges are bold. If one of those connects v to a $\circ$-vertex (possibly, with v itself), then this segment has a monochrome vertex having an interior bold edge of type $\bar{3}$, contradicting Corollary 1.3.15. $\square$

Chapter 2

Nodal trigonal curves and their dessins

Within the moduli space of trigonal curves of fixed degree, generic trigonal curves are smooth and the discriminant (i.e., the set of singular trigonal curves, cf. [5]) has a stratification which we can describe by means of dessins. Singular proper trigonal curves have singular dessins and the singular points are represented by singular vertices. A generic singular trigonal curve C has exactly one singular point, which is a non-degenerate double point (*node*). Moreover, if C is a proper trigonal curve, then the double point on it is represented by a $\times$-vertex of index 2 on its dessin. In addition, if C has a real structure, the double point is real and so is its corresponding vertex, leading to the cases where the $\times$-vertex of index 2 has dotted real edges (representing the intersection of two real branches) or has solid real edges (representing one isolated real point, which is the intersection of two complex conjugated branches).

Definition 2.0.1. Let $D \subset S$ be a dessin on a compact surface S. A *nodal vertex* (*node*) of D is a $\times$-vertex of index 2. The dessin D is called *nodal* if all its singular vertices are nodal vertices; it is called *uninodal* if it has exactly one singular vertex which is a node. We call a *toile* a non-hyperbolic real nodal dessin on $(\mathbb{CP}^1, z \longmapsto \bar{z})$.

Since a real dessin on $(\mathbb{CP}^1, z \longmapsto \bar{z})$ descends to the quotient, we represent toiles on the disk.

In a real dessin, there are two types of real nodal vertices, namely, vertices having either real solid edges and interior dotted edges, or dotted real edges and interior solid edges. We call *isolated nodes* of a dessin D thoses $\times$-vertices of index 2 corresponding to the former case and *non-isolated nodes* those corresponding to the latter.

Definition 2.0.2. Let $D \subset S$ be a real dessin. A *bridge* of D is an edge e contained in a connected component of the boundary ∂S having more than two vertices, such that e connects two monochrome vertices. The dessin D is called *bridge-free* if it has no bridges. The dessin D is called *peripheral* if it has no inner vertices other than $\times$-vertices.

For non-singular dessins, combinatorial statements analogous to Lemma 2.0.3 and Proposition 2.1.1 are proved in [3].

Lemma 2.0.3. *A nodal dessin D is elementary equivalent to a bridge-free dessin D' having the same number of inner essential vertices and real essential vertices.*

Proof. Let D be a dessin on S. Let e be a bridge of D lying on a connected component of ∂S. Let u and v be the endpoints of e. Since e is a bridge, there exists at least one

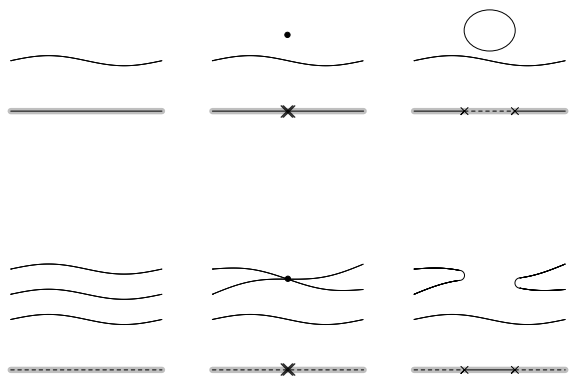


Figure 2.1: Perturbations of a real $\times$-vertex of index 2.

real vertex $w \neq u$ of D adjacent to v. If w is an essential vertex, destroying the bridge is an admissible elementary move. Otherwise, w is monochrome of the same type as u and v and the edge connecting v and w is another bridge e' of D. The fact that every region of the dessin contains on its boundary essential vertices implies that after destroying the bridge e the regions of the new graph have an oriented boundary with essential vertices. Therefore the resulting graph is a dessin and destroying that bridge is admissible. All the elementary moves used to construct the elementary equivalent dessin D' from D are destructions of bridges. Since destroying bridges does not change the nature of essential vertices, D and D' have the same amount of inner essential vertices and real essential vertices. $\square$

A real singular $\times$-vertex v of index 2 in a dessin can be perturbed within the class of real dessins on the same surface in two different ways. Locally, when v is isolated, the real part of the corresponding real trigonal curve has an isolated point as connected component of its real part, which can be perturbed to a topological circle or disappears, when v becomes two real $\times$-vertices of index 1 or one pair of complex conjugated interior $\times$-vertices of index 1, respectively. When v is non-isolated, the real part of the corresponding real trigonal curve has a double point, which can be perturbed to two real branches without ramification or leaving a segment of the third branch being one-to-one with respect to the projection π while being three-to-one after two vertical tangents (see Figure 2.1).

Definition 2.0.4. Given a dessin D, a subgraph $\Gamma \subset D$ is a *cut* if it consists of a single interior edge connecting two real monochrome vertices. An *axe* is an interior edge of a dessin connecting a $\times$-vertex of index 2 and a real monochrome vertex.

Let us consider the image of an axe T under the perturbation of the $\times$-vertex v of index 2 into $\times$-vertices of index 1. If v becomes a pair of interior conjugated vertices having an interior solid edge e, the axe T becomes an interior edge, which together with e form a cut of the new dessin. If v becomes two real $\times$, the image of T has a monochrome vertex of the color of T which determines a cut.

Now, let us consider a dessin D lying on a surface S having a cut or an axe T. Assume that T divides S and consider the connected components S_1 and S_2 of $S \setminus T$. Then, we can define two dessins D_1, D_2, each lying on the compact surface $\overline{S_i} \subset S$, respectively for

$i = 1, 2$, and determined by $D_i := (D \cap S_i) \cup \{T\}$. If $S \setminus T$ is connected, we define the surface $S' = (S \setminus T) \sqcup T_1 \sqcup T_2/\varphi_1, \varphi_2$, where $\varphi_i : T_i \longrightarrow S$ is the inclusion of one copy T_i of T into S, and the dessin $D' := (D \setminus T) \sqcup T_1 \sqcup T_2/\varphi_1, \varphi_2$.

By these means, a dessin having a cut or an axe determines either two other dessins of smaller degree or a dessin lying on a surface with a smaller fundamental group. Moreover, in the case of an axe, the resulting dessins have one singular vertex less. Considering the inverse process, we call D the *gluing* of D_1 and D_2 along T or the *gluing* of D' with itself along T_1 and T_2.

Definition 2.0.5. Let $C \subset \Sigma$ be a nodal proper trigonal curve with nodes $n_1, n_2, \dots n_l$. Consider a Weierstraß model of C determined by sections g_2 and g_3. We say that the trigonal curve C is *almost generic* if every singular fiber different from the fiber at $\pi(n_i)$ corresponds to a simple root of the determinant section $\Delta = -4g_2^3 - 27g_3^2$ which is not a root of g_2 nor of g_3. The nodal trigonal curve C is *generic* if it is almost generic and the sections g_2 and g_3 only have simple roots.

2.1 Uninodal dessins

Proposition 2.1.1. *A non-hyperbolic uninodal dessin is elementary equivalent to a peripheral one.*

Proof. We proceed by contradiction: let us assume that D is a non-hyperbolic uninodal dessin, which is not elementary equivalent to a peripheral one. Let us choose a dessin D' within the class of elementary equivalence of D having a minimal number of inner essential vertices. By Corollary 2.0.3, we can assume that D' is a bridge-free dessin.

We call an *inner chain* a chain of vertices $v_0, v_1, \dots, v_k$ within D' such that every edge $[v_i, v_{i+1}]$, $0 \leq i < k$ and every vertex v_i, $0 \leq i < k$ are inner. Let us assume that there is an inner chain $v_0, v_1, \dots, v_k$ of minimal length starting at a vertex v_0, which is either white or black, arriving to a non-hyperbolic real component.

We pick the inner chain in a way that either the ending vertex v_k is monochrome or every other inner chain of minimal length from a black or white vertex to the boundary ends in an essential vertex. This assumption and the fact that D' is a bridge-free dessin assure the monochrome modifications we make are admissible. We study the possible configurations of the chosen inner chain in order to do elementary modifications decreasing the number of inner essential vertices.

Case 0: v_k is a monochrome vertex and v_{k-1} is either black or white. Thus, the number of inner essential vertices is reduced after an elementary move of type $\bullet$-out or $\circ$-out.

Case 1.1: v_k is a black vertex and v_{k-1} is a $\times$-vertex. Then $k \geq 2$ and v_{k-2} is white. Up to symmetry, the configuration corresponds to Figure 2.2a. Hence, the creation of a bold bridge brings us to Case 0.

Case 1.2.1: v_k is black and v_{k-1} is white. We consider the region R such that its boundary contains the edge $[v_{k-1}, v_k]$ and the other inner edge of v_k. After one monochrome modification R can be reduced to a triangle. If the region R contains an inner $\times$-vertex (see Figure 2.2b), the creation of a bold bridge brings us to Case 0.

Case 1.2.2: v_k is black and v_{k-1} is white, and the above-mentioned region R contains the nodal vertex n. By the minimality condition of the chain, the vertex n is an elliptic nodal vertex (since R is a triangle, and v_{k-1} is not connected to a monochrome vertex) and the inner solid edge of v_k connects it to a vertex u, at the boundary beside n. We look at the solid real edge of v_k, which connects v_k with a vertex v, either an index 1 $\times$-vertex

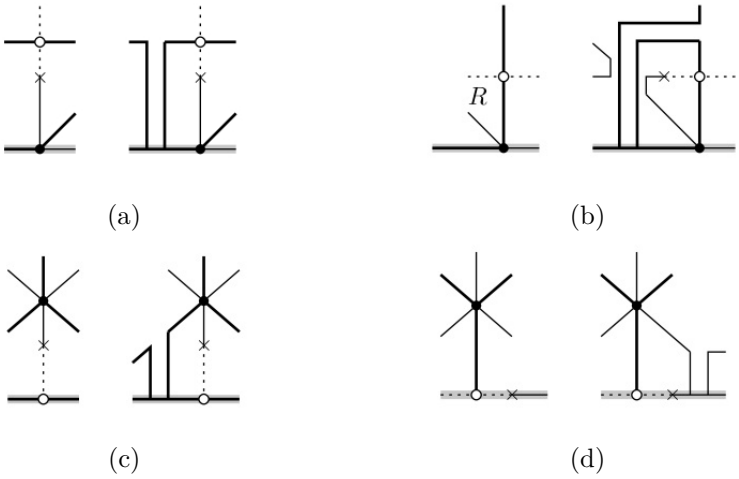


Figure 2.2

(since D' is a uninodal dessin) or a monochrome vertex. When the vertex v is a $\times$-vertex, there is a real dotted edge incident to v where the construction of a dotted bridge brings us to Case 0.

When the vertex v is a monochrome vertex, let e be its inner solid edge. This edge connects the boundary with either another monochrome vertex or with an inner $\times$-vertex. If e is adjacent to another monochrome vertex w, since D' is a uninodal bridge-free dessin, the vertex w belongs to a solid segment ending in two real $\times$-vertices, each defining a real dotted edge. Thus, creating a bridge with one of the inner dotted edge of v_{k-1} at the real dotted edge in the same region brings us to Case 0. Otherwise, the edge e connects v to an inner $\times$-vertex w. A monochrome modification defines an edge connecting w to v_{k-1}. Since D' is a bridge-free dessin, v has two black neighbors. An elementary move of type $\bullet$-in at v, followed by an elementary move of type $\circ$-out at u, connects v_{k-1} to the boundary at a monochrome vertex as in Case 0.

Case 2.1: v_k is white and v_{k-1} is a $\times$-vertex. Hence, $k \geq 2$ and v_{k-2} is black. Therefore, creating a bold bridge brings us to Case 0 (see Figure 2.2c).

Case 2.2: v_k is white and v_{k-1} is black. We look at the neighboring vertices of v_k. If there is a $\times$-vertex of index 1, creating a solid bridge at the incident solid edge brings us to Case 0 (see Figure 2.2d). Otherwise, let a be one of its monochrome neighbor such that the region R containing the edges $[v_k, v_{k-1}]$ and $[v_k, v]$, as in the left part of Figure 2.3a, does not contain the nodal vertex. Let w be the white neighboring vertex of a other than v_k. We replace the original chain by the chain $v_0, v_1, \ldots, v_{k-1}, w$. Since the real component is non-hyperbolic, after a finite number of iterations, the vertex v_k will have a $\times$-vertex of index 1 as neighbor.

Case 3: v_k is monochrome and v_{k-1} is a $\times$-vertex. Then $k \geq 2$ and v_{k-2} is either white or black. We can perform a monochrome modification in order to connect v_{k-2} with one of the real neighbors of v_k and bring us to Case 1.2 (see Figure 2.3b) or to Case 2.2 (see Figure 2.3c).

Case 4: v_k is the nodal vertex. Let R be one of the regions containing the edge $[v_{k-1}, v_k]$. Let e be the edge in the boundary of R, incident to v_{k-1} and different from $[v_{k-1}, v_k]$. If v_{k-1} is black, the edge e is bold and connects v_{k-1} with a white vertex v. If v is an inner white vertex, the creation of a dotted bridge, followed by an elementary move

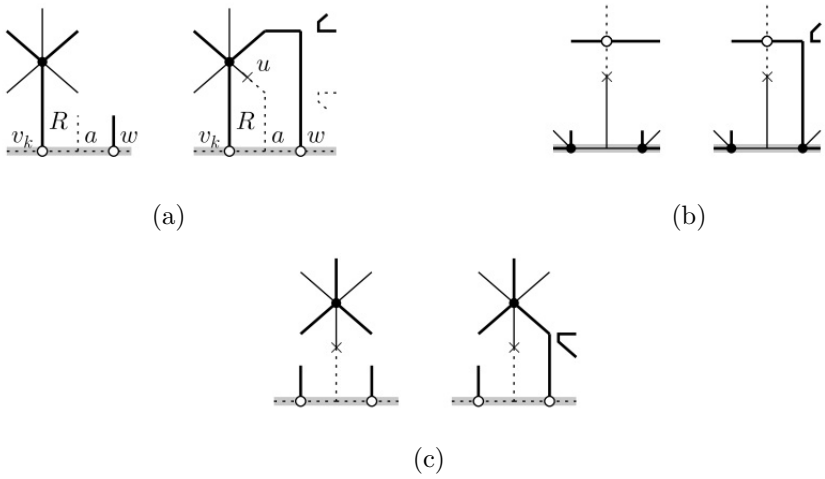


Figure 2.3

of type ∘-out reduces the number of inner essential vertices (see Figure 2.4a). Otherwise, v is a real white vertex. We consider instead the chain $v_0, v_1, \ldots, v_{k-1}, v$ as in Case 2.

Finally, if v_{k-1} is a white vertex, the edge e must connect v_{k-1} to a black vertex v. If v is an inner black vertex, the creation of a solid bridge, followed by an elementary move of type •-out reduces the number of inner essential vertices (see Figure 2.4b). Otherwise, v is a real black vertex. We consider instead the chain $v_0, v_1, \ldots, v_{k-1}, v$ as in Case 1.

In the case when all black or white inner vertices have no inner chain to a non-hyperbolic component, let us consider a black or white inner vertex v_0, and let $v_0, v_1, \ldots, v_k$ be a minimal inner chain connecting v_0 to a hyperbolic component. By the aforementioned considerations, the vertex v_0 must be black. In this setting, every hyperbolic component is connected to a non-hyperbolic component through a chain $C = \{u_0, u_1, \ldots, u\}$ free of inner black or white vertices. Up to monochrome modifications, the vertex v_0 belongs to a region whose boundary contains the inner chain C. Thus, the vertices v_0 and u belong to the same region R.

Case 5.1: the vertex u is monochrome. If it is solid or bold, the creation of a bridge on the real solid edge incident to u in the region R with an inner solid edge incident to v_0 brings us to Case 0. If the vertex u is dotted, it has a real neighboring ×-vertex in the region R determining a solid segment where the creation of a solid bridge with an inner solid edge incident to v_0 brings us to Case 0.

Case 5.2: the vertex u is nodal. Since the chain C has no inner black vertices, u has real solid edges. Thus, the creation of a solid bridge on the real solid edge incident to u in the region R with an inner solid edge incident to v_0 brings us to Case 0.

Case 5.3: the vertex u is white or black. If the vertex u is white, it has real bold edges. Thus, the creation of a bold bridge on the real bold edge incident to u in the region R with an inner bold edge incident to v_0 brings us to Case 0. Otherwise, the vertex u is black. If in the region R a bold real edge is incident to u, the creation of a bold bridge with an inner bold edge incident to v_0 brings us to Case 0. Otherwise, a solid monochrome modification between the inner solid edges incident to u and v_0 in the region R brings us to a configuration where the creation of a bold bridge on the real bold edge incident to u with an inner bold edge incident to v_0 brings us to Case 0. □

Proposition 2.1.2. *A hyperbolic uninodal dessin is elementary equivalent to a dessin without inner white vertices.*

(a) (b)

Figure 2.4

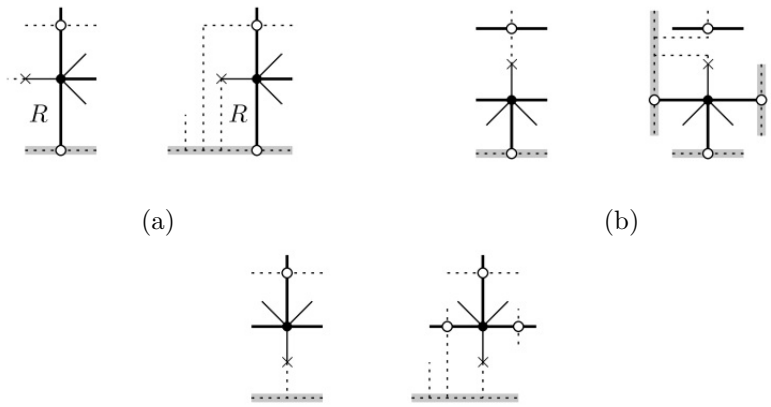
Proof. By Corollary 2.0.3, we can suppose that the dessin is bridge-free. Let $v_0, \ldots, v_k$ be a minimal length chain from an inner white vertex v_0 to a real vertex v_k. The vertex v_0 is the only white inner vertex in the chain. Since every inner black or $\times$-vertex is connected to a white vertex, the length k is at most three.

Since the dessin is hyperbolic, if $k = 1$, then v_k is a monochrome vertex. An elementary move of type $\circ$-out reduces the number of inner white vertices.

If $k = 2$, then v_1 is black and v_2 is white or a nodal vertex. In the case when v_2 is white, we consider the solid edge e incident to v_1 between the edges $[v_0, v_1]$ and $[v_1, v_2]$.

Let v be the $\times$-vertex of the edge e. If v is an inner $\times$-vertex, a monochrome modification and the creation of a dotted bridge followed by an elementary move of type $\circ$-out reduces the number of inner white vertices (see Figure 2.5a). Otherwise, the vertex v is a nodal vertex and the creation of a dotted bridge followed by an elementary move of type $\circ$-out reduces the number of inner white vertices (see Figure 2.4a). When the vertex v_2 is nodal, we look at the white vertices connected to v_1. If the black vertex v_1 is connected to a real white vertex v, we consider the chain v_0, v_1, v instead.

If $k = 3$, the colors of the vertices v_1, v_2, v_3 are either $\times$, black and white, or black, $\times$ and monochrome, respectively. Due to the length minimality of the chain, in the former case every white vertex connected to v_2 is real, and the number of inner white vertices can be reduced by the creation of a dotted bridge followed by an elementary move of type $\circ$-out (see Figure 2.5b); in the latter case every white vertex connected to v_1 is inner, thus any of the white vertices connected to v_1 within the region containing the edge v_2, v_3 becomes real after the creation of a dotted bridge and an elementary move of type $\circ$-out (see Figure 2.5c), reducing the number of inner white vertices. $\square$



(a) (b)

(c)

Figure 2.5

Proposition 2.1.3. *A non-hyperbolic uninodal real dessin D on a surface S of degree higher than 3 is weakly equivalent either to the gluing of dessins of smaller degree, or to the gluing of two real edges of a dessin D'. In both cases, the gluing corresponds to either a dotted cut or an axe.*

Proof. Let D_1 be a dessin within the weak equivalence class of D, maximal with respect to the number of zigzags (equivalently, minimal with respect to the number of inner $\times$-vertices). The dessin D_1 is non-hyperbolic; therefore, by Proposition 2.1.1, the dessin D_1 is elementary equivalent to a peripheral dessin D_0, which we can assume bridge-free, by Corollary 2.0.3.

A *dotted cut* is an inner dotted edge c connecting two monochrome vertices. If the dessin D_0 has a cut or an axe c, then cutting through the edge c either defines two dessins D_1 and D_2 (in the case when $S \setminus c$ is disconnected), or it defines one dessin D' on a surface S' (then D_0 is the image by the gluing of two different edges in S'). Let us assume D_0 has no dotted cut nor an axe. Let T be the nodal vertex of D_0, let e be the inner edge incident to T and let $\mathcal{S}$ be the monochrome segment containing T. The edge e connects T with a real vertex v, which is either black (if e is solid) or white (if e is dotted).

Case 1: the edge e is dotted and the vertex v is white. Let us consider one region R adjacent to e. In the case when the edge e divides the surface S, let us assume that R is on the connected component with a maximal number of black vertices. Let u be the real neighbor vertex of v in R.

Case 1.1: the vertex u is monochrome. Let us assume that u has a real white neighbor vertex v' different than v. The vertex v' has an inner dotted edge e'. If the edge e' connects v' with a monochrome vertex w, an elementary move of type $\circ$-in at u followed by an elementary move of type $\circ$-out at w produces a dotted axe. Otherwise, the edge e connects v' with an inner $\times$-vertex w.

If the vertex u defines a bold cut, since the dessin is bridge-free, an elementary move of type $\circ$-in at u followed by an elementary move of type $\circ$-out along the cut brings us to a configuration where, up to monochrome modifications, the vertex w can be set up in order to create a zigzag, contradicting the maximality of the number of zigzags of D_0.

Otherwise, the vertex u is connected to a real black vertex w'. If the vertex w' has a real neighbor $\times$-vertex w'', we can create a dotted axe, either by creating a dotted bridge with the edge e if w'' belongs to R or by creating a dotted bridge with the edge e' beside w'' and proceeding as v' is connected to a monochrome vertex, if the vertex e' is contained on the region containing w''. If the vertex w' is neighbor to a monochrome vertex w'', an elementary move of type $\bullet$-in at w'' followed by an elementary move of type $\bullet$-out at u bring us to a setting around w where we can create a zigzag.

If the vertex w' is connected to the nodal vertex T, up to a solid monochrome modification, the vertices w and w' are connected. In this case, an elementary move of type $\circ$-in, followed by the creation of a bold bridge beside v' and an elementary move of type $\circ$-out bring us to a setting around w where we can create a zigzag.

One special case is when v belongs to a monochrome boundary component, having a bold monochrome vertex u with inner edge e'. If the vertex u is connected by e' to a black vertex w neighbor to a real $\times$-vertex determining a dotted segment where the creation of a bridge with e produces an axe. If the vertex u is connected by e' to a monochrome vertex w or to a black vertex neighbor to a solid monochrome vertex w', then an elementary of type $\bullet$-in at w or w' followed by an elementary move of type $\bullet$-out at u brings us to the configuration where v has to black real neighbors u_1, u_2, they three being the only essential vertices of that connected component of the boundary.

If one of the black vertices u_1 or u_2 is connected to an inner $\times$-vertex w, a dotted monochrome modification connects w with v and the creation of a bold bridge brings us to a setting around w where we can create a zigzag.

If one of the black vertices u_1 or u_2 is connected to a solid monochrome vertex w not in the segment $\mathcal{S}$, its real neighbor on the region R is a $\times$-vertex since the dessin is bridge-free, determining a dotted segment where the creation of a bridge with e produces an axe.

In the remaining case, both black vertices u_1 and u_2 are connected to solid monochrome vertices w_1 and w_2, respectively, neighbors of the nodal vertex T. Since the dessin is bridge-free, each w_i $(i = 1, 2)$ is neighbor to a non-singular $\times$-vertex w_i', respectively. If a vertex u_i $(i = 1, 2)$ is connected to a bold monochrome vertex, an elementary move of type $\circ$-in at it followed by the creation of a dotted bridge beside w_i' (if needed) and an elementary move of type $\bullet$-out set the vertex u_i connecting to a white vertex w_i'' neighbor to w_i', respectively. If a vertex u_i $(i = 1, 2)$ is connected to a white vertex and the vertex w_i' is connected to a dotted monochrome vertex, the creation of a dotted bridge after a bold monochrome modification produces a cut. In any case, up to a bold monochrome modification, we can assume each vertex u_i connected to a white vertex w_i'' neighbor to w_i', respectively.

Let u' be the solid monochrome vertex neighbor to u_1 and u_2. If the vertex u' is connected to an inner $\times$-vertex, then the creation of dotted bridges (if needed) at the dotted segments determined by w_i'' produces a cut. Otherwise, the vertex u' determined a solid cut, with monochrome vertex u'' having as neighbor $\times$-vertices w_i'''. If one of the $\times$-vertices w_1''' or w_2''' is neighbor to a monochrome vertex, the creation of a dotted bridge beside w_i'' produces a cut. Otherwise, each w_i''' is neighbor to a white vertex v_i. It can happen that $v_i = w_i''$ for one $i = 1, 2$ but not for both since the degree is greater than 3. Let i_1 be an index for which $v_{i_1} \neq w_{i_1}''$ and let $i_0 \in \{1, 2\} \setminus \{i_1\}$. If the vertex v_{i_0} is neighbor to a dotted monochrome vertex, the creation of a bridge beside w_{i_0}'' produces a cut. Otherwise, v_{i_0} is neighbor to a $\times$-vertex determining a zigzag. We do a bold monochrome modification so u_{i_0} is connected to v_{i_0}. Then, an elementary move of type $\bullet$-in at u' followed by an elementary move of type $\bullet$-out through the solid cut and the creation of a bridge bring us to a setting around v_{i_0} where we destroy the zigzag and the remaining bridge. The resulting configuration has an inner $\times$-vertex having a dotted edge. The creation of a dotted bridge beside v_{i_1} followed by the creation of a bridge beside w_{i_1}'' produces a cut.

Otherwise, the vertex u is connected by e' to a black vertex w neighbor of T. In this setting, we look at the inner solid edge f of w. If the edge f connects w with a monochrome vertex w', which has a $\times$-vertex w'' neighbor in the region determined by e and f since the dessin is bridge-free, then the creation of a bridge with e on the dotted segment determined by w'' produces an axe. If the edge f connects w with an inner $\times$-vertex w', we look at the real neighbor vertex w'' of w. If the vertex w'' is white, up to a monochrome modification it is connected to w' and the creation of a bold bridge bring us to a setting around w' where we can create a zigzag. If the vertex w'' is monochrome, then an elementary move of type $\bullet$-in at w'', followed by an elementary move of type $\bullet$-out and a dotted monochrome modification to connect v with w' bring us to a setting around w' where we can create a zigzag.

Case 1.2: the vertex u is black. Let f to be the inner solid edge of u and let w to be the vertex connected to u by f. If the vertex w is monochrome and neighbor to an index 1 $\times$-vertex on the region R, it determines a dotted segment in which the creation of a bridge with e produces an axe. If the vertex w is monochrome and neighbor to the nodal

vertex T, it has a real neighbor $\times$-vertex w'. Then, we ask for the real neighbor vertex w'' of u. If the vertex w'' is monochrome, an elementary move of type $\bullet$-in at w'' followed by an elementary move of type $\bullet$-out at w and the destruction of a possible bridge, repeated a finite number of times, bring us to Case 1.1 or a considered case.

If the vertices w' and w'' are index 1 $\times$-vertices, we consider the bold inner edge f' of u. If the edge f' connects u to a monochrome vertex within a monochrome component with on essential vertex u', then the creation dotted bridges with the inner edge of u beside w' and w'' produces a dotted cut. It can not happen that the vertices w' and w'' are real neighbors of the same dotted monochrome vertex since the degree of the dessin is greater than 3. If the edge f' connects u to a monochrome vertex w''' having two real neighbors, then an elementary move of type $\circ$-in at w''' followed by the creation of a dotted bridge beside w', an elementary move of type $\circ$-out and the creation of a bridge beside w'' produce a dotted cut.

If the edge f' connects u to a white vertex u', then the vertex u' cannot be a real neighbor of w' and w'' since the degree is greater than 3. If the vertex w'' has a real neighbor monochrome vertex, then the creation of a dotted bridge beside u' produces a dotted cut. Otherwise we can assume the vertex u' is a real neighbor of w'', up to a monochrome modification. If one of the vertices u' and w' has a real neighbor monochrome vertex, then the creation of a dotted bridge produces a dotted cut. If in the region determined by the edges f and f' there is a dotted inner edge, up to the creation of bridges beside the vertices u' and w' there exists a dotted cut. If the vertex u' has a real neighbor $\times$-vertex connected to a solid cut, an elementary move of type $\bullet$-in followed by an elementary move of type $\bullet$-out bring us to the case where the vertex u' has a real neighbor $\times$-vertex connected to a real black vertex b. Constructing a solid bridge with the edge f beside the vertex b results in a solid cut and allows us to destroy the zigzag determined by u'. An elementary move of type $\bullet$-in followed by an elementary move of type $\bullet$-out along the cut, elementary moves of type $\circ$-in and $\circ$-out and the creation of a zigzag bring us to a configuration where we can restart the algorithm. Since the degree is greater than 3 and the dotted edge of the nodal vertex T is dividing, we do not obtain a cyclic argument.

If the vertex w' is equal to the nodal vertex, then the creation of a dotted bridge with the edge e beside the vertex u' results in the creation of a dotted axe.

If the vertex w'' is equal to the nodal vertex, we look at the inner bold edge f' of u. If the edge f' connects the vertex u to a white vertex belonging to a dotted segment where the creation of a dotted bridge with the edge e creates an axe. If the edge f' has a monochrome vertex, then there is a white vertex b in the region determined by the edges f and f'. If the inner dotted edge of the vertex b connects it to a monochrome vertex, then we can create an axe by constructing a bridge with the edge e by either eluding the monochrome boundary component containing b or by performing elementary moves of type $\circ$-in and $\circ$-out on the dotted segment. If the inner dotted edge of the vertex b connects it to a $\times$-vertex vertex, monochrome modifications to join it to the vertices u and v and the creation of a bold bridge allow us to create a zigzag, which implies that the dessin D_0 has not a maximal number of zigzags.

Case 2: the edge e is solid and the vertex v is black.

Case 2.1: the vertex v has a real bold edge connecting to a white vertex u, having an inner dotted edge f. The edge f cannot connect u to an inner $\times$-vertex, since a monochromatic modification would allows us to create a zigzag. Since the dessin is peripheral, the edge f connects u to a monochrome vertex w. If the vertex w is not a real neighbor of the nodal vertex T, the creation of a bridge with the edge f beside T produces a dotted

cut. Otherwise, the vertex w is a real neighbor of the vertex T. The vertex u has a real neighbor vertex v'. If the vertex v' is black, its inner solid edge e' must be incident to a monochrome vertex neighbor to two real $\times$-vertices w' and w''. Let us assume the vertex w' belongs to the region R' containing the edges f and e'. Since the dessin is uninodal, the $\times$-vertex w' determines a dotted segment, at which the creation of a bridge produces a dotted cut unless the vertices w and w' are connected.

In the case where the vertices w and w' are real neighbors, let f' be the inner bold edge incident to v', connecting it to a real vertex u'. Let us assume the vertex u' is white. If the region determined by the edges e' and f' contains an inner dotted edge, up to the creation of bridges beside the vertices w'' and u' we can obtain a dotted cut. Otherwise, up to a monochrome modification the region R' is triangular. Let w''' be the real neighbor vertex of v'. If the vertex w''' is a $\times$-vertex followed by a dotted monochrome vertex, the creation of a bridge beside the vertex u' produces a dotted cut.

It can occur the vertex u' is neighbor to the vertices w'' and w'''. We will analyze this configuration as Case 2.3.

If the case when the vertex w''' is a $\times$-vertex followed by a white vertex, we can assume that w''' and u' are connected. If the region determined by the edges e' and f' contains an inner dotted edge, up to the creation of bridges beside the vertices w'' and u' we can obtain a dotted cut. Otherwise, we look at the sequence of vertices on the boundary after w''' and u'. If the vertex u' is followed by a $\times$-vertex and a solid monochrome vertex, it must be followed by a $\times$-vertex b since the dessin is bridge-free and uninodal. Up to a monochrome modification, we can construct a bridge with the edge f beside the vertex b producing a cut. If the vertex u is followed by a $\times$-vertex and a black vertex b, having an inner solid edge connecting to a monochrome vertex m. The destruction of the zigzag at u, elementary moves of type $\bullet$-in and $\bullet$-out, and the construction of a bridge bring us to a configuration where the creation of a bridge beside m with the edge e produces an axe. If the vertex b is connected by its inner solid edge to an inner $\times$-vertex m, after monochrome modifications we can construct bridges with the inner dotted edge of m beside w''' and the real white neighbor of w''' in order to obtain a cut.

In the case when the vertex w''' is monochrome, we perform elementary moves of type $\bullet$-in at w''' and $\bullet$-out along e', destroying the possible residual bridge. This corresponds to a configuration already considered.

We proceed in the same way if the edge f' connects the vertex v' to a monochrome vertex m and v' has a monochrome solid neighbor.

Otherwise v' has a real $\times$-vertex neighbor w''' determining a dotted segment. Then, either the vertex m has two real white vertices as neighbors, an elementary move of type $\circ$-in at m followed by an elementary move of type $\circ$-out (up to the creation of a bridge beside w''') and the creation of a bridge beside w'' produces a dotted cut; or the vertex m belongs to a monochrome component, having a white vertex with inner dotted edge, with which the creation of two dotted bridges beside w'' and w''' produces a cut.

Finally, when the vertex v' is monochrome, an elementary move of type $\circ$-in at v' followed by an elementary move of type $\circ$-out at w and the destruction of a possible residual bold bridge bring us to a configuration to which we iterate this process. Due to the finite number of vertices this process must stop.

Case 2.2: the vertex v has a real bold edge connecting to a monochrome vertex u, having an inner bold edge f connecting u to a vertex w. When the vertex w is monochrome, an elementary move of type $\bullet$-in at u followed by an elementary move of type $\bullet$-out at w bring us to Case 2.1.

If the vertex w is white and it is neighbor to a monochrome or simple $\times$-vertex in

the region determined by the edges e and f, then the creation of a dotted bridge beside T produces a cut or a solid bridge beside the $\times$-vertex produces an axe, respectively. Otherwise, w is a real neighbor of T. Since the dessin is bridge-free, the vertex u has a real neighbor $v' \neq v$ having an inner bold edge f'. If the edge f' connects v' to a monochrome vertex, an elementary move of type $\bullet$-in at u followed by an elementary move of type $\bullet$-out along f' produces a solid axe.

Let us assume the edge f' connects v' to a white vertex w'. Let e' be the solid inner edge incident to v'. If the edge e' connects v' to an inner $\times$-vertex w'' having an inner dotted edge g, then the creation of dotted bridges (if needed) beside them with g produces a dotted cut, unless the vertices w and w' are real neighbors to a monochrome vertex incident to g, at which an elementary move of type $\circ$-in followed by an elementary move of type $\circ$-out at u brings us to a configuration where we can construct a zigzag determined by w''.

If the edge e' connects v' to a monochrome vertex w'', an elementary move of type $\circ$-in at u followed by an elementary move of type $\circ$-out at w'' brings us to a configuration where we can iterate the process. If it cycles to this configuration, we look at the sequence of real neighbor vertices of w'' in the region determined by the edges f' and e'. If the vertex w'' if followed by a $\times$-vertex and a monochrome vertex, or by a $\times$-vertex, a white vertex w''' and a monochrome vertex, then the creation of a dotted bridge beside w' or the monochrome modification to connect the vertices v' and w''' followed by the creation of a dotted bridge produce a cut. This move is admissible since otherwise w'' and w''' would be the same and the degree of the dessin would be 3. If the vertex w'' is followed by a zigzag, we do the same consideration in this paragraph with respect to the vertex v'.

Finally, in the case where the vertices v' and w'' are followed by zigzags with white vertices w_1 and w_2, let us assume v' to be connected to w_1, the white vertex determining its adjacent zigzag. Let g be the inner bold edge incident to w_2 and let m be its other vertex. If the vertex m is monochrome, the creation of bold bridge with the edge f' beside m allows us to proceed as when the edge f' connected v' to a monochrome vertex. If the vertex m is black, being connected to an inner solid edge incident to an inner $\times$-vertex m', then up to a monochrome modification between f' and g, the vertex m' belongs to the same region that w_1 and w_2. Thus, the creation of dotted bridges with the inner edge incident to m' beside w_1 and w_2 produces a cut.

If the vertex m is black being connected to a monochrome solid vertex m', which up to a monochrome modification between f' and g belongs to the same region as w_1 and w_2, then a monochrome modification between the solid inner edges of v' and m brings us to a configuration already considered.

Case 2.3: the edge e does not divide the dessin into two parts, one of these consisting of two triangles.

Case 2.3.1: the vertex v has a bold inner incident edge e' connecting v to a white vertex v'. If within the region determined by the edges e and e' the vertex T has a monochrome real neighbor, having an inner dotted edge f, we can create a bridge beside v' in order to obtain a cut. Otherwise, the vertex T has a white real neighbor, which up to a monochrome modification is the vertex v'. Let us assume the vertex v' has a real neighbor vertex u which is monochrome.

We ask for the color of the real neighbor vertex w of v. If w is a $\times$-vertex, then the creation of a dotted bridge with the inner edge f of u beside w forms a cut, as in Figure 2.6a. Otherwise w is a solid monochrome vertex, either defining a solid cut where w is connected to a segment bounded by $\times$-vertices, which allows us to create a bridge

with f producing a cut, or w is connected to an inner $\times$-vertex, and after a triangulation, the configuration of the dessin is as in Figure 2.6b.

The vertex w has a black neighbor v'' with an inner solid edge g incident to a vertex w'. If the vertex w' is monochrome then an elementary move of type $\bullet$-in on w followed by an elementary move of type $\bullet$-out set us as in Figure 2.6c. In this case, an elementary move of type $\circ$-in followed by an elementary move of type $\circ$-out allows us to create a zigzag as in Figure 2.6d.

Otherwise, the vertex w' is an inner $\times$-vertex connected to a monochrome vertex w'', which can be on the segment S or on a different segment. In the former case, the creation of a dotted bridge on S with the inner dotted edge of w' produces a dotted cut. In the latter case, w'' has two white real neighbor vertices and, up to the creation of a bold bridge, an elementary move of type $\circ$-in on w'' followed by an elementary move of type $\circ$-out on the bold segment of v' allow us to create a zigzag, contradicting the maximality of the number of zigzag of the dessin (see Figure 2.6e).

In the case where the vertex w' is an inner $\times$-vertex connected to a white vertex, up to the creation of a bold bridge, an elementary move of type $\bullet$-in, followed by an elementary move of type $\bullet$-out produces a solid axe, as in Figure 2.6f.

Finally, the vertex w' can be a solid monochrome vertex. Since the dessin is bridge-free, the vertex w' has two real neighboring $\times$-vertices. The creation of a dotted bridge with the edge f beside the $\times$-vertex in its region produces a dotted cut.

Let us assume now that the vertex u is a $\times$-vertex. By the assumption on the present Case, it cannot be connected to the vertex v. Let w be the real neighbor vertex of v on the region containing e'. If the vertex w is monochrome, having an inner solid edge f, the creation of a bridge with f beside u, followed by elementary moves of type $\bullet$-in at w, $\bullet$-out along f and the destruction of a possible residual bold bridge bring us to a configuration where we can iterate this process, considering that the new edge e does divide the dessin, with one connected component being formed by two triangles.

In the case when the vertex w is a $\times$-vertex, neighbor of a dotted monochrome vertex, this one has an inner dotted edge, with which the creation of a bridge beside u produces a cut. Instead, if the vertex w is a $\times$-vertex, neighbor of a white vertex w', then the vertex w' has an inner bold edge f connecting it to a black vertex w''. If the inner solid edge f' of w'' belong to the same region of e', a bold monochromatic modification between e' and f changes the region so the creation of a solid bridge with e beside w'' produces an axe. Otherwise, let w''' be the incident vertex of f' other than w''. If the vertex w''' is a $\times$-vertex, it has an inner dotted edge, with which the the creation of bridges beside the vertices u and w'' after the bold monochrome modification between the edges e' and f produces a cut. If the vertex w''' is monochrome, the creation of a solid bridge with the edge e beside w''' produces a cut.

Case 2.3.2: the vertex v has a bold inner incident edge e' connecting v to a monochrome vertex v'. Assume that the vertex v' has two real neighboring white vertices. Then, the region containing the edges e and e' has an inner dotted edge f which, up to the creation of a bridge, is incident to a monochrome vertex neighbor to the nodal vertex T. An elementary move of type $\circ$-in on v' followed by an elementary move of type $\circ$-out on the monochrome of the edge f brings us to Case 2.3.1, as seen in the see Figure 2.6a.

If the vertex v has a bold inner incident edge e' connecting v to a monochrome vertex v' belonging to a bold monochrome component with only one white vertex b having an inner dotted edge f. Then we look at the real neighbor w of v by a solid edge. Either the vertex w is a $\times$-vertex and the construction of bridges (if necessary) with the edge f beside w and T produces a cut, or the vertex w is monochrome and an elementary move of

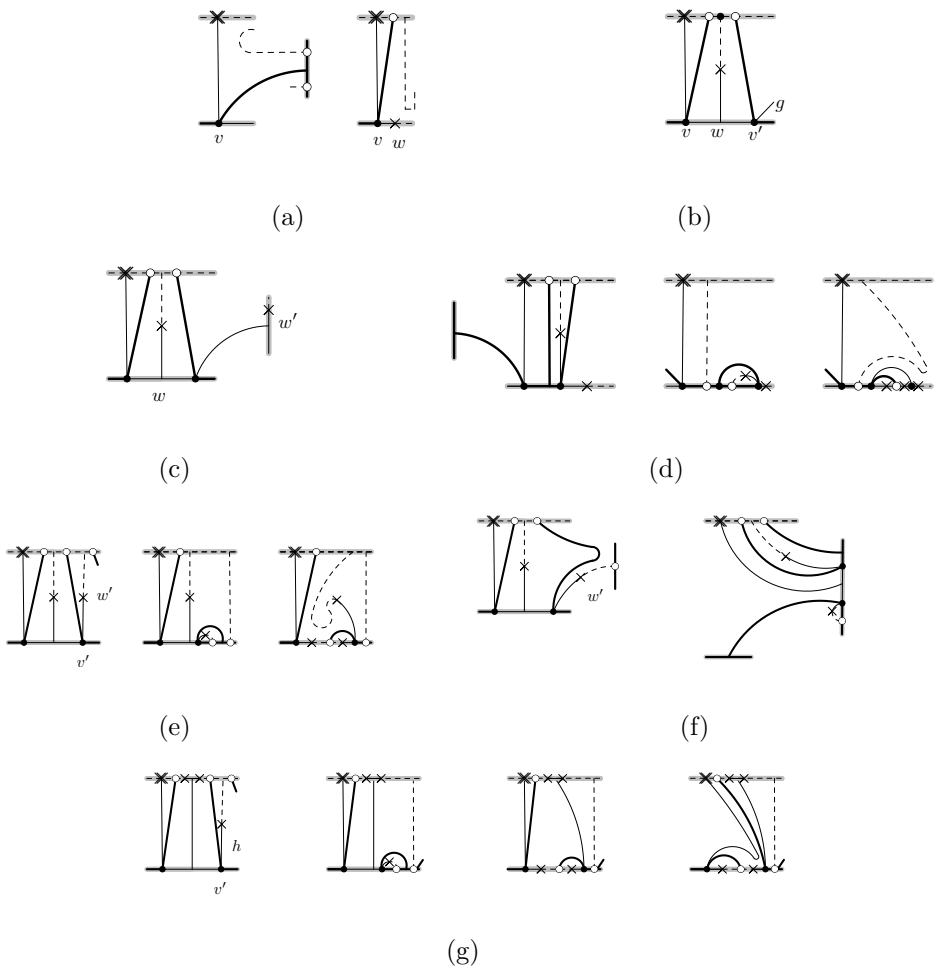


Figure 2.6

type •-in at w followed by an elementary move of type •-out at v' bring us to Case 2.1. □

Proposition 2.1.4. *A hyperbolic uninodal real dessin D of degree higher than 3 is weakly equivalent either to the gluing of dessins of smaller degree, or to the gluing of two real edges of a dessin D'. In both cases, the gluing corresponds to a dotted cut.*

Proof. Let D be a hyperbolic uninodal dessin and let T be the nodal vertex of D. Due to Corollary 2.0.3 and Proposition 2.1.2 we can assume that the dessin D is bridge-free and has no inner white vertex. The vertex T is connected to an inner black vertex b, which is connected to inner ×-vertices v_1 and v_2, each connected to a monochrome vertex u_1 and u_2, respectively.

Case 1: the vertices u_1 and u_2 have a common real neighboring white vertex. Since the degree is greater than 3 at least one of the vertices u_1 or u_2 has no common real neighboring vertex with T. Let us assume u_1 has no common real neighboring vertex with T. An elementary move of type ∘-in at u_1 produces an inner white vertex w which up to a monochrome modification is connected *via* two different inner bold edges to b. In this setting, the creation of a dotted bridge near T with an inner dotted edge of w followed by an elementary move of type ∘-out sending w next to the vertex T produces a dotted cut in a dessin without inner white vertices.

Case 2: the vertices u_1 and u_2 have no common real neighboring vertex.

Case 2.1: at least one of the vertices u_1 or u_2 has two real white neighboring vertices. Let us assume that u_1 has two real white neighboring vertices. An elementary move of type $\circ$-in at u_1 produces an inner white vertex w which up to a monochrome modification is connected *via* two different inner bold edges to b. In this setting, the creation of a dotted bridge near u_2 with an inner dotted edge of w followed by an elementary move of type $\circ$-out sending w next to the vertex u_2 produces a dotted cut in a dessin without inner white vertices.

Case 2.2: every vertex u_i, $i = 1, 2$, belongs to a component with only one essential vertex. Let w be the white vertex in the real component of u_1. Up to a monochrome modification the vertex w is connected to b such that b, v_1, w form a triangle. In this setting, the vertices v_1 and u_2 belong to the same region. Creating a dotted bridge near u_2 with the inner dotted edge of v_1 produces a dotted cut. □

2.2 Toiles

Definition 2.2.1. Given a real dessin D and a vertex $v \in \mathrm{Ver}(D)$, we call the *depth* of v the minimal number n such that there exists an undirected inner chain $v_0, \ldots, v_n$ in D from $v_0 = v$ to a real vertex v_n and we denote the depth of v by $\mathrm{dp}(v)$. The depth of a dessin D is defined as the maximum of the depth of the black and white vertices of D and it is denoted by $\mathrm{dp}(D)$.

Definition 2.2.2. A *generalized cut* of a dessin D is an inner undirected chain formed entirely of inner edges of the same color, either dotted or solid, connecting two distinct real nodal or monochrome vertices.

Analogously to a cut, cutting a dessin $D \subset S$ by a generalized cut produces two dessins of lower degree or a dessin of the same degree in a surface with a simpler topology, depending on whether the inner chain divides or not the surface S.

Proposition 2.2.3. *Let $D \subset \mathbb{D}^2$ be a toile of degree greater than 3. Then, there exists a toile D' weakly equivalent to D such that either D' has depth 1 or D' has a generalized cut.*

Proof. Within the class of elementary equivalence of our initial toile D, let us choose a toile D_0 having minimal depth. Due to Corollary 2.0.3 we can choose D_0 bridge-free. If $\mathrm{dp}(D_0) \geq 2$, then there is a vertex $v_0 \in \mathrm{Ver}(D_0)$ having $\mathrm{dp}(v_0) \geq 2$, hence there exists an undirected inner chain $v_0, v_1, \ldots, v_n$, with $n = \mathrm{dp}(v_0)$, of minimal size connecting v_0 to the boundary of D^2. By definition, $\mathrm{dp}(v_{n-2}) = 2$. Put $u = v_{n-2}, v = v_{n-1}, w = v_n, e = [u, v], f = [v, w]$. We study the possible configurations of the chain u, v, w in order to show that, unless there exists a generalized cut, we can decreases the depth of every vertex having depth at least 2, contradicting minimality assumption. We assume that the toile D_0 does not have dotted cuts, since otherwise the Proposition follows trivially.

Case 0: whenever v is black or white and connected to a monochrome vertex, an elementary move of type $\bullet$-out or $\circ$-out, respectively, reduces the depth of u. For simplicity, from now on we assume that v is only connected to essential vertices.

Case 1.1: the vertex u is white and the vertex v is black.

Case 1.1.1: the vertex w is white. Let g be the solid edge adjacent to v sharing a region with the edges e and f. Let w_1 be the real neighbor vertex to w in the region R determined by f and g. If w_1 is a simple $\times$-vertex, it determines a solid segment where

the construction of a bridge with the edge g bring us to Case 0. If w_1 is a nodal $\times$-vertex, up to a monochrome modification it is connected to v, and then the creation of a bridge with the inner dotted edge of u beside w_1 decreases the depth of u. Otherwise, w_1 is monochrome. If it is connected to an inner simple $\times$-vertex w_2, up to a monochrome modification the vertex w_2 is connected to v, and as before, the creation of a bridge with the inner dotted edge of u beside w_1 decreases the depth of u. If w_1 is connected to an inner nodal $\times$-vertex w_2, up to a monochrome modifications the vertex w_2 is connected to u and v. Since the dessin is a bridge-free toile, w_1 has a white real neighbor vertex $w_3 \neq w$.

If w_3 is connected to a monochrome vertex by an inner edge, then in the region determined by the vertices w_1 and w_3 there is a black vertex w_4 and up to monochrome modifications the vertices u and w_2 are connected to w_4, reducing the depth of u. If w_3 is connected to a real black vertex w_4, then there are two cases: in the region determined by the vertices w_1 and w_3, the vertex w_4 is adjacent to a inner or real solid edge h. When the edge h is inner, up to a monochrome modification, the vertices v and w_4 are connected, then the creation of a bridge with a inner bold edge of u beside w_4 bring us to Case 0.

When the edge h is real, connecting w_4 with a $\times$-vertex w_5, then the creation of a dotted bridge with the edge adjacent to w_1 beside w_5 or a monochrome modification connecting w_1 and w_5, respectively if it is a simple or nodal $\times$-vertex, produces a generalized cut. When the edge h connects w_4 with a monochrome vertexw_5, sine the toile is bridge free, there exist a real black vertex $w_6 \neq w_4$ connected to w_5. Monochrome modifications connect w_2 with w_5 and u with w_6 reducing the depth of u. Finally, if the vertex w_3 is connected to an inner black vertex w_4, which up to monochrome modifications is connected to u and w_2, we consider the real neighbor vertex $w_5 \neq w_1$. If w_5 is a simple $\times$-vertex, it determines a solid segment where the creation of a bridge with the inner solid edge of v in the region bring us to Case 0. If w_5 is a nodal $\times$-vertex, up to a monochrome modification it is connected to v. Let $w_6 \neq u, w$ be the vertex connected to v by a bold edge. If w_6 is an inner white vertex, the creation of a dotted bridge followed by an elementary move of type $\circ$-out lead us to the next consideration. When the vertex w_6 is a real white vertex, let us defined $w' := w_6$ and consider instead the chain u, v, w' and the considerations made in this algorithm. If the algorithm cycles back to this configuration and we denote by w'_i the vertices on the second iteration, the edges $[w_1, w_2], [w_2, u], [u, w'_2], [w'_2, w'_1]$ form a generalized cut.

Case 1.1.2: the vertex w is nodal. If w has a real white neighboring vertex w', we consider the chain u, v, w' as in Case 1.1.1. Otherwise w has a real monochrome neighboring vertex w_1. If w_1 is connected to an inner white vertex w_2, an elementary move of type $\circ$-out with it at w_2 creates real white neighboring vertex of w and we consider the previous case. If w_1 is connected real white vertex w_2, then the creation of a bold bridge with an inner bold edge of v beside w_2 followed by an elementary move of type $\bullet$-out bring us to a configuration in which u is connected to a real vertex, so its depth is reduced.

Case 1.2: the vertex u is white and the vertex v is a simple $\times$-vertex.

Case 1.2.1: the vertex w is black. The vertex w has a real solid bold edge in the same region as u. Then, the creation of a bridge with an inner bold edge of u beside w followed by an elementary move of type $\circ$-out transfers the vertex u to the boundary in a bridge-free toile.

Case 1.2.2: the vertex w is monochrome. Since the toile is bridge-free, w has a real black neighboring vertex which is connected to u after a monochrome modification reducing the depth of u.

Case 1.3: the vertex u is white and the vertex v is a nodal $\times$-vertex.

Case 1.3.1: the vertex w is a monochrome dotted vertex. Let w_1 be a vertex connected to v by a solid edge. If w_1 is real, then either its bold inner edge can be connected to u after a monochrome modification or its bold real edge allows us to create a bridge with the inner bold edge of u and then an elementary move of type $\circ$-out transfers the vertex u to the boundary in a bridge-free toile. If w_1 is a monochrome vertex, since the toile is bridge-free, then it has has a real black neighboring vertex w_2. A monochrome modification connects u to w_2 decreasing the depth of u in a bridge-free toile. Otherwise, the vertex w_1 is an inner black vertex. Since the the toile is bridge-free, there exist a white vertex w_2 real neighbor of w in the same region as w_1. A monochrome modification connects w_1 to w_2. We consider instead the chain u, w_1, w_2 as in Case 1.1.1.

Case 1.3.2: the vertex w is a monochrome solid vertex. This configuration was considered in Case 1.3.1.

Case 1.3.3: the vertex w is white. Let w_1 be a vertex connected to v by a solid edge. If w_1 is a real vertex, the considerations made on the Case 1.3.1 apply. Otherwise, w_1 is an inner black vertex. Up to monochrome modifications The creation of a bold bridge beside w with an inner bold edge adjacent to w_1 followed by an elementary move of type $\bullet$-out decreases the depth of u in a bridge-free toile.

Case 1.3.4: the vertex w is black. This case correspond to the configuration in Case 1.3.1 when w_1 is a real vertex.

Case 2.1: the vertex u is black and the vertex v is white.

Case 2.1.1: the vertex w is black. Let w_1 be a vertex connected to v by a dotted edge. If w_1 is a real nodal $\times$-vertex, then the creation of a bridge beside w_1 with an inner solid edge of u in the region followed by an elementary move of type $\bullet$-out transfers the vertex u to the boundary in a bridge-free toile. If w_1 is an inner simple $\times$-vertex, then we consider the inner solid edge g adjacent to w. If g and w_1 belong to the same region, then we can assume g is adjacent to w_1 up to a monochrome modification. In this setting, the creation of a bridge with the edge e beside w followed by an elementary move of type $\bullet$-out transfers the vertex u to the boundary in a bridge-free toile. If g and w_1 do not share any region, then we can assume w_1 is connected to u up to a monochrome modification. In this setting the creation of a bridge with the inner solid edge connecting w_1 and u beside w followed by an elementary move of type $\bullet$-out transfers the vertex u to the boundary in a bridge-free toile. Lastly, if w_1 is a nodal $\times$-vertex, up to monochrome modifications it is connected to u and w. Let $w_2 \neq v$ be the vertex connected to w_2 by a dotted edge. If w_2 is a real white vertex, then the creation of a bridge with an inner bold edge adjacent to u beside w_2 followed by an elementary move of type $\circ$-out transfers the vertex u to the boundary in a bridge-free toile. If w_2 is an inner white vertex, up to a monochrome modification it is connected to u, then the creation of a bridge with an inner bold edge adjacent to w_2 beside w followed by elementary moves of type $\circ$-out at w_2 and $\bullet$-out at u transfers the vertex u to the boundary in a bridge-free toile. If w_2 is a monochrome vertex, it has two real white neighboring vertices, then an elementary move of type $\circ$-in at w_2 brings us to the previous consideration.

Case 2.1.2: the vertex w is a nodal $\times$-vertex. This case corresponds to the configuration in Case 2.1.1 when w_1 is a real nodal $\times$-vertex.

Case 2.2: the vertex u is black and the vertex v is a simple $\times$-vertex.

Case 2.2.1: the vertex w is monochrome. Since the toile is bridge-free, the vertex w has two different real white neighboring vertices. A monochrome modification connects u to one of those vertices decreasing the depth of u.

Case 2.2.2: the vertex w is white. In this setting, the creation of a bold bridge beside w with an inner bold edge adjacent to v followed by an elementary move of type $\bullet$-out transfers the vertex u to the boundary in a bridge-free toile.

Case 2.3: the vertex u is black and the vertex v is a nodal $\times$-vertex.

Case 2.3.1: the vertex w is monochrome dotted. The vertex u is connected to an inner white vertex w_1 on the same region as the edge f. The creation of a dotted bridge beside w with an inner dotted edge of w_1 followed by an elementary move of type $\circ$-out decreasing the depth of u.

Case 2.3.2: the vertex w is white. In this setting, the creation of a bold bridge beside w with an inner bold edge adjacent to v follow by an elementary move of type $\bullet$-out transfers the vertex u to the boundary in a bridge-free toile.

Case 2.3.3: the vertex w is monochrome solid. Let w_1 be a vertex connected to v by a dotted edge. If w_1 is a real monochrome vertex or a white vertex, we consider instead the chain u, v, w_1 as in Case 2.3.1 and Case 2.3.2, respectively. If w_1 is an inner white vertex, let w_2 be a real black vertex neighbor of w sharing a region with w_1. Up to a monochrome modification w_1 is connected to w_2. We consider instead the chain u, w_1, w_2 as in Case 2.1.1.

Case 2.3.4: the vertex w is black. Let w_1 be a vertex connected to v by a dotted edge. If w_1 is a real monochrome vertex or a white vertex, we consider instead the chain u, v, w_1 as in Case 2.3.1 and Case 2.3.2, respectively. If w_1 is an inner white vertex, then up to a monochrome modification w_1 is connected to w. We consider instead the chain u, w_1, w as in Case 2.1.1.

Case 3.1: the vertex u is a simple $\times$-vertex and the vertex v is white.

Case 3.1.1: the vertex w is black. Let g be the solid edge adjacent to w sharing a region with the vertex u. If g is an inner edge, up to monochrome modification between g and the solid edge adjacent to u decreases the depth of u. If g is a real edge, let w_1 be the vertex connected to u by a solid edge. Since the depth of u is two, the vertex w_1 is an inner black vertex. The creation of a bridge on g with the edge $[u, w_1]$ followed by an elementary move of type $\bullet$-out at w_1 decreases the depth of u in a bridge-free toile.

Case 3.1.2: the vertex w is a nodal $\times$-vertex. Let w_1 be a real neighbor vertex of w. If w_1 is a monochrome vertex determining a solid cut, let w_2 be the monochrome vertex connected to w_1 through the cut and let w_3 be the black vertex neighbor to w_2 sharing a region with u. A monochrome modification between the inner solid edges adjacent to u and w_2 connects these two vertices, reducing the depth of u. If w_1 is a monochrome vertex connected to a real black vertex w_2, then in the region determined by v and w_2 the vertex w_2 either has an inner bold edge and a monochrome modification allows us to consider instead the chain u, v, w_2 as in Case 3.1.1 or it has a real bold edge where the creation of a bridge with the inner bold edge adjacent to v in the region followed by an elementary move of type $\circ$-out at v decreases the depth of u in a bridge-free toile. If w_1 is black, up to a monochrome modification v and w_1 are connected and we consider instead the chain u, v, w_1 as in Case 3.1.1. Lastly, if w_1 is a monochrome vertex connected to an inner black vertex w_2, an elementary move of type $\bullet$-out at w_2 bring us to the previous consideration.

Case 3.2: the vertex u is a simple $\times$-vertex and the vertex v is black.

Case 3.2.1: the vertex w is white. Since $\mathrm{dp}(u) = 2$, the vertex u is connected to an inner white vertex w_1. If the vertices u and w share a region, the creation of a dotted bridge beside w with the dotted edge adjacent to u followed by an elementary move of type $\circ$-out at w_2 decreases the depth of u in a bridge-free toile. If the vertices u and w do not share a region, let $w_2, w_2' \neq u$ be the vertices connected to v by a solid edge.

We can assume up to monochrome modifications that v and w_1 are connected by two different bold edges. If one of the vertices w_2 or w_2' is an inner simple $\times$-vertex, up to a monochrome modifications it is connected to w_1, and then the creation of a bridge beside w with the inner dotted edge adjacent to w_1 followed by an elementary move of type $\circ$-out at w_1 decreases the depth of u in a bridge-free toile. Otherwise, the vertices w_2 and w_2' are nodal $\times$-vertices. We can assume up to one monochrome modification that w_2 and w_1 are connected. If both w_2 and w_2' are real, the creation of two bridges with an inner dotted edge adjacent to w_1, one beside w_2 and one beside w_2', produces a dotted cut. If both w_2 and w_2' are inner, the creation of two bridges beside w, one in every side, with inner dotted edges adjacent to w_2 and w_2' respectively, followed by the creation of an inner monochrome vertex between the inner dotted edges of w_2 and w_2' sharing a region with w_1, create a generalized cut. It between the vertices w_2 and w_2' one is real and one is inner, then the creation of dotted bridges and an inner monochrome vertex as in the previous considerations allow us to create a generalized cut.

Case 3.2.2: the vertex w is a nodal $\times$-vertex. Since $\mathrm{dp}(u) = 2$, the vertex u it is connected to an inner white vertex w_1, which we can assume connected to v by two different bold edges up to monochrome modification. In this setting, the creation of a bridge beside w with the inner dotted edge adjacent to w_1 followed by an elementary move of type $\circ$-out decreases the depth of u in a bridge-free toile.

Case 4.1: the vertex u is a nodal $\times$-vertex and the vertex v is white.

Case 4.1.1: the vertex w is black. If it was an inner solid edge g sharing a region with u, then the vertices u and w can be connected by a monochrome modification between the edges e and g decreasing the depth of u. Otherwise, the vertex w has a real solid edge sharing a region with u. Since $\mathrm{dp}(u) = 2$, the vertex u is connected to an inner black vertex w_1. In this setting, the creation of a bridge beside w with a solid inner edge adjacent to w_1 followed by an elementary move of type $\bullet$-out at w_1 decreases the depth of u in a bridge-free toile.

Case 4.1.2: the vertex w is a nodal $\times$-vertex. Since $\mathrm{dp}(u) = 2$, the vertex u is connected to an inner black vertex w_1, which up to a monochrome modification it is connected to v. Then, the creation of a bridge beside w with a solid inner edge adjacent to w_1 followed by an elementary move of type $\bullet$-out at w_1 decreases the depth of u in a bridge-free toile.

Case 4.2: the vertex u is a nodal $\times$-vertex and the vertex v is black.

Case 4.2.1: the vertex w is a nodal $\times$-vertex. Since $\mathrm{dp}(u) = 2$, the vertex u is connected to an inner white vertex w_1 sharing a region with w, which we can assume connected to v up to a monochrome modification. In this setting, the creation of a bridge beside w with an inner dotted edge adjacent to w_1 followed by an elementary move of type $\circ$-out at w_1 decreases the depth of u in a bridge-free toile.

Case 4.2.2: the vertex w is white. Since $\mathrm{dp}(u) = 2$, the vertex u is connected to an inner white vertex w_1. Then, if the vertices u and w belong to the same region, we can choose w_1 belonging to the same region as them. Then, the creation of a bridge beside w with an inner dotted edge adjacent to w_1 followed by an elementary move of type $\circ$-out at w_1 decreases the depth of u in a bridge-free toile. Otherwise, let $w_2 \neq u$ be a vertex connected to v by a solid edge sharing a region with w_1. If w_2 is a real nodal $\times$-vertex, we consider instead the chain u, v, w_2 as in Case 4.2.1. If w_2 is an inner simple $\times$-vertex, up to monochrome modification it is connected to w_1, and then, the creation of a bridge beside w with an inner dotted edge adjacent to w_1 followed by an elementary move of type $\circ$-out at w_1 decreases the depth of u in a bridge-free toile. Finally, if w_2 is an inner nodal $\times$-vertex, we consider instead the white vertex $w_1' \neq w_1$ connected to u and the

$\times$-vertex $w_2' \neq u, w_2$ connected to v. If the aforementioned consideration cycle to this configuration, then the creation of two bridges beside w, one at every side, with inner dotted edges adjacent to w_2 and w_2', respectively, produces a generalized cut. $\square$

Corollary 2.2.4. *Let D be as in Proposition 2.2.3. If there exists a toile D' weakly equivalent to D with depth 1, then D' can be chosen bridge-free.*

Proposition 2.2.5. *Let D be a toile of degree at least 6 and depth at most 1. Then, there exists a toile D' weakly equivalent to D such that D' has a generalized cut. Moreover, if D has isolated real nodal $\times$-vertices, the generalized cut is dotted or a solid axe.*

Proof. Let D_0 be a dessin within the weak equivalence class of D, maximal with respect to the number of zigzags. Due to Proposition 2.2.3 and Corollary 2.2.4, we can choose D_0 within the weak equivalence class of D_0 such that $\mathrm{dp}(D_0) \leq 1$ and D_0 is bridge-free. For simplicity we assume that there are no black or white inner vertices connected to monochrome vertices. We assume that there are no dotted cuts, since otherwise the Proposition follows trivially. If there is a bold cut $[w, w']$ in a bridge-free toile, assuming the vertex w has black neighbor real vertices, then an elementary move of type $\bullet$-in followed by an elementary move of type $\bullet$-out at the vertices w and w', respectively, eliminates the bold cut. This way we can assume there are no bold cuts on the dessin without breaking the bridge-free property nor changing its depth. Let us start by the case when D_0 has singular vertices on the boundary of the disk $\mathbb{D}^2$. Let v be a real nodal $\times$-vertex.

Case 1: the vertex v is isolated, being connected to a white vertex u. If the vertex u is real, the edge $e := [v, u]$ is dividing. Let R be the region containing e on the connected component of $D_0 \setminus e$ with a maximal number of white vertices. Let w be the real neighbor vertex of u in the region R. Let S be the bold segment containing u.

Case 1.1: the vertex w is monochrome. If w is connected to a real black vertex w_1, then there are two cases: in the region R, the vertex w_1 is adjacent to an inner or real solid edge f.

If the edge f is inner, let w_2 be the real vertex connected to w_1 by a solid real edge (see Figure 2.7a). If w_2 is a $\times$-vertex, let $w_3 \neq u$ be a white real vertex connected to w. If w_2 is a simple $\times$-vertex, then, the creation of a bridge with the inner dotted edge adjacent to w_3 beside w_2, followed by elementary moves of type $\circ$-in at w and $\circ$-out at the bridge, produces a dotted axe in a bridge-free dessin (see Figure 2.7b). If w_2 is a nodal $\times$-vertex, up to a monochrome modification it is connected to w_3. An elementary move of type $\circ$-out at w produces a generalized cut (see Figure 2.7c). If w_2 is a monochrome vertex, then an elementary move of type $\bullet$-in at w_2 followed by an elementary move of type $\bullet$-out at w bring us to a configuration we study within the Case 1.2.

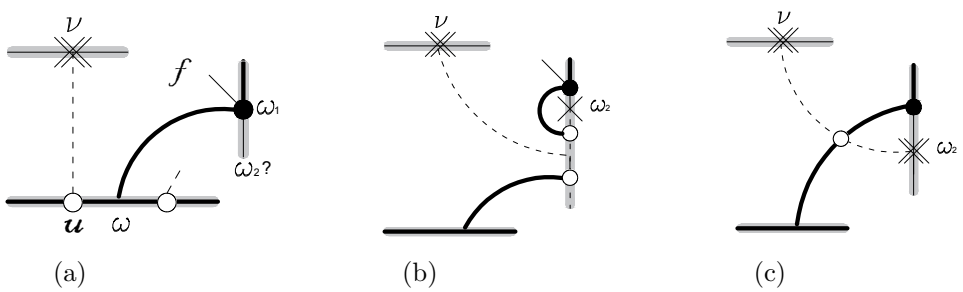


Figure 2.7

If the edge f is real, let w_2 be the real vertex connected to w_1 by the edge f (see Figure 2.8a). If w_2 is a simple $\times$-vertex, it determines a real dotted segment where the creation of a bridge with the edge e produces a dotted axe (see Figure 2.8b). If w_2 is a monochrome vertex, then an elementary move of type $\bullet$-in at w_2 followed by an elementary move of type $\bullet$-out at w bring us to a configuration we study within the Case 1.2. In the case when w_2 is a nodal $\times$-vertex different from v, and in this case the creation of an inner monochrome vertex with the edge e and the inner dotted edge adjacent to w_2 produces a generalized cut (see Figure 2.8c).

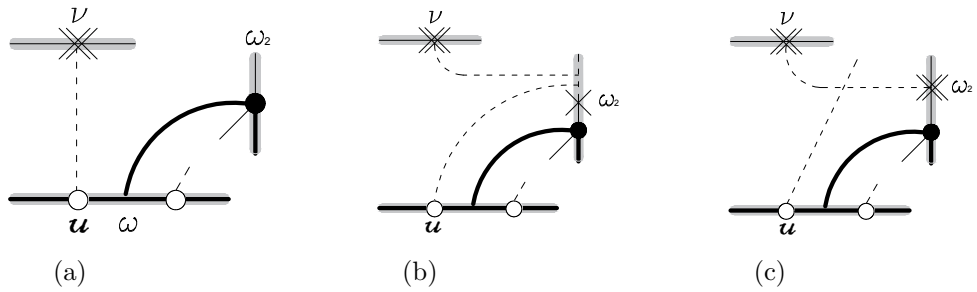


Figure 2.8

A special case is when $w_2 = v$. Let $w_3 \neq u$ be a white real vertex connected to w and let w_4 be the vertex connected to w_3 by an inner dotted edge. If w_4 is a monochrome vertex or a real nodal $\times$-vertex, an elementary move of type $\circ$-in produces a generalized cut (see Figure 2.9a and Figure 2.9b). If w_4 is an inner simple $\times$-vertex, it is connected to w_1 up to a monochrome modification. An elementary move of type $\circ$-in at w followed by the creation of a bold bridge beside w_1 and an elementary move of type $\circ$-out bring us to a configuration where we can create a zigzag, contradicting the maximality assumption (see Figure 2.9c).

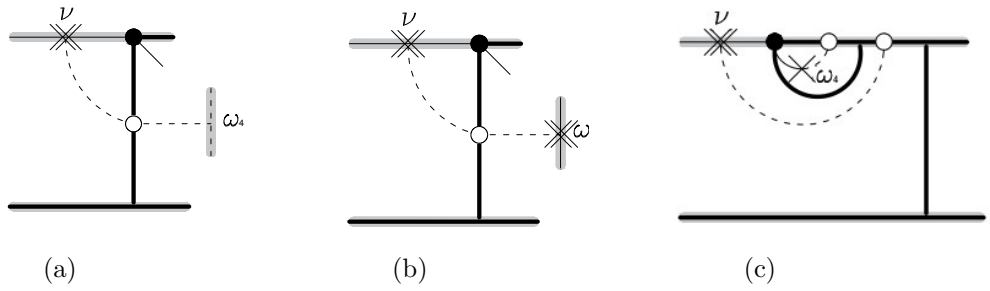


Figure 2.9

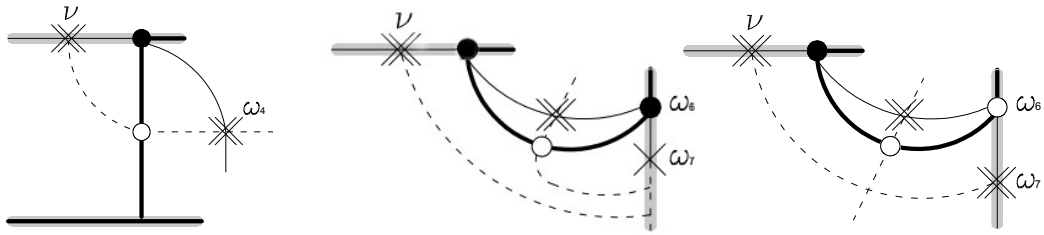
If w_4 is an inner nodal $\times$-vertex, up to a monochrome modification it is connected to w_1. Let $w_5 \neq w_3$ be a vertex connected to w_4 by a dotted edge. If w_5 is a monochrome vertex, then an elementary move of type $\circ$-in at w creates an inner white vertex w' such that the chain v, w', w_4, w_5 is a generalized cut (see Figure 2.10a).

If w_5 is a white inner vertex, the creation of a bold bridge beside w_1 with a bold edge of w_5 followed by an elementary move of type $\circ$-out allows us to consider w_5 as a real white vertex.

If w_5 is a white real vertex, let $w_6 \neq w_1$ be a vertex connected to w_4 by a solid edge. When w_6 is a real black vertex, there are two cases: in the region R' determined by

w_4, w_5 and w_6 the bold edge g adjacent to w_6 is either real or inner. We perform an elementary move of type $\circ$-in at w producing a white inner vertex w', and destroy the potential residual bold bridge.

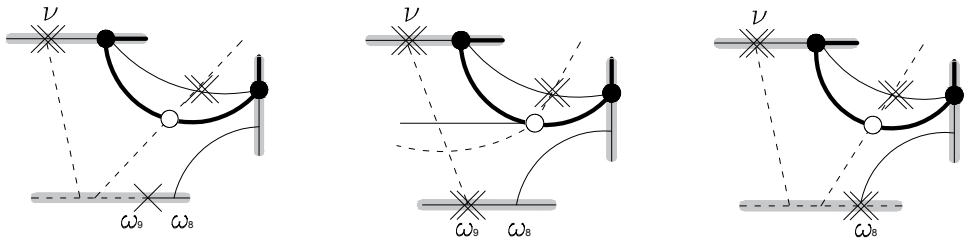
If g is a real bold edge, let w_7 be the vertex connected to w_6 by a real solid edge. Up to a monochrome modification the vertex w' is connected to w_6. If w_7 is a simple $\times$-vertex, the creation of a bridge beside w_7 with an inner dotted edge incident to w' produces a generalized cut (see Figure 2.10b). If w_7 is a nodal $\times$-vertex, the creation of an inner monochrome vertex with the edge e and the inner dotted edge adjacent to w_7 produces a generalized cut (see Figure 2.10c). If w_7 is a monochrome vertex, let w_8 be the vertex connected to w_7 by an inner solid edge.



(a) (b) (c)

Figure 2.10

If w_8 is a monochrome vertex, let w_9 be the vertex connected to w_8 in the region determined by w_6, w_7 and w_8. If w_9 is a simple $\times$-vertex, the creation of a bridge beside w_9 with the edge e produces an axe (see Figure 2.11a). Otherwise w_9 is a nodal $\times$-vertex, the creation of an inner monochrome vertex with the edge e and the inner dotted edge adjacent to w_9 produces a generalized cut (see Figure 2.11b). If w_8 is a real nodal $\times$-vertex, the creation of a bridge beside it with the edge e produces an axe (see Figure 2.11c).

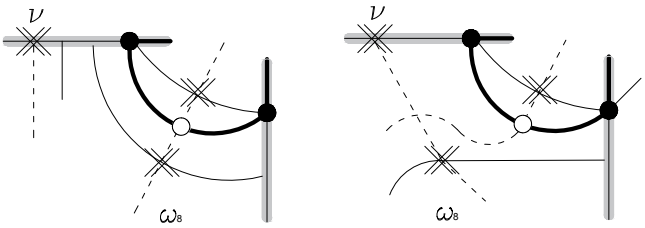


(a) (b) (c)

Figure 2.11

If w_8 is an inner nodal $\times$-vertex, we make monochrome modification in order to connect w_8 to w'. Then, the creation of a bride h beside w_1 with an inner solid edge adjacent to w_8 produces a solid generalized cut and cutting by it produces two different toiles (see Figure 2.12a). Let D_0' be the resulting toile containing v. The toile D_0' is a toile of degree strictly greater than 3 since there are no nodal cubic toiles having two isolated nodes (cf. Section 3.3). Since $\mathrm{dp}(D_0') \leq 1$, we can restart the algorithm with the toile D_0' and it does not cycle back to this consideration. A dotted generalized cut in D' having w_8 as an end can be extended to a generalized cut in D_0 by deleting the solid bridge h and creating an inner monochrome dotted vertex with the edge e and the inner dotted edge of w_8 (see

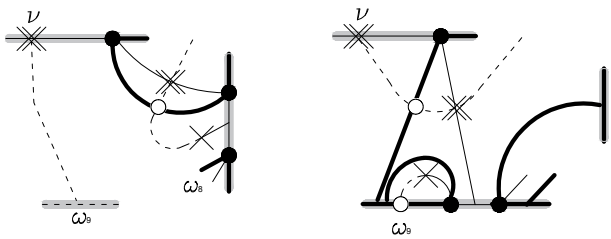
Figure 2.12b).



(a) (b)

Figure 2.12

If w_8 is a simple $\times$-vertex, let w_9 be the vertex connected to it by a dotted edge. If w_9 is a monochrome vertex, a monochrome modification between the inner dotted edge adjacent to w_9 and the edge e produces an axe (see Figure 2.13a). If w_9 is a real white vertex, then an elementary move of type $\bullet$-in at w_7 followed by an elementary move of type $\bullet$-out beside w_9 allows us to create a zigzag with w_8, contradicting the maximality assumption (see Figure 2.13b). Lastly, if w_9 is an inner white vertex, the creation of a bridge in the segment S followed by an elementary move of type $\circ$-out allows us to consider w_9 as a real white vertex.



(a) (b)

Figure 2.13

If g is an inner bold edge, let w_7 be the vertex connected to w_6 by an inner bold edge. If w_7 is a real white vertex, the creation of a bridge beside it with an inner dotted edge adjacent to w_4 produces a generalized cut (see Figure 2.14a). If w_7 is a monochrome vertex, an elementary move of type $\circ$-in allows us to consider it as an inner white vertex. If w_7 is an inner white vertex, up to a monochrome modification it is connected to w_4. Let w_8 be the vertex connected to w_6 by a real solid edge. If w_8 is a simple $\times$-vertex, the creation of a dotted bridge beside it with an inner dotted edge adjacent to w_4 produces a generalized cut (see Figure 2.14b). If w_8 is a nodal $\times$-vertex, up to a monochrome modification it is connected to w_7 determining a generalized cut (see Figure 2.14c). If w_8 is a monochrome vertex, the creation of a bold bridge beside w_1 with an inner bold edge adjacent to w_7 followed by an elementary move of type $\circ$-out at w_7, an elementary move of type $\bullet$-in at w_8 and an elementary move of type $\bullet$-out at the bridge bring us to the configuration when the edge g was a real bold edge.

In the case when w_6 is a monochrome vertex, an elementary move of type $\bullet$-in at w_6 produces an inner black vertex w_6'. We destroy any possible remaining bridge. Then, the creation of a bridge beside w_5 with an inner bold edge adjacent to w_6' followed by an

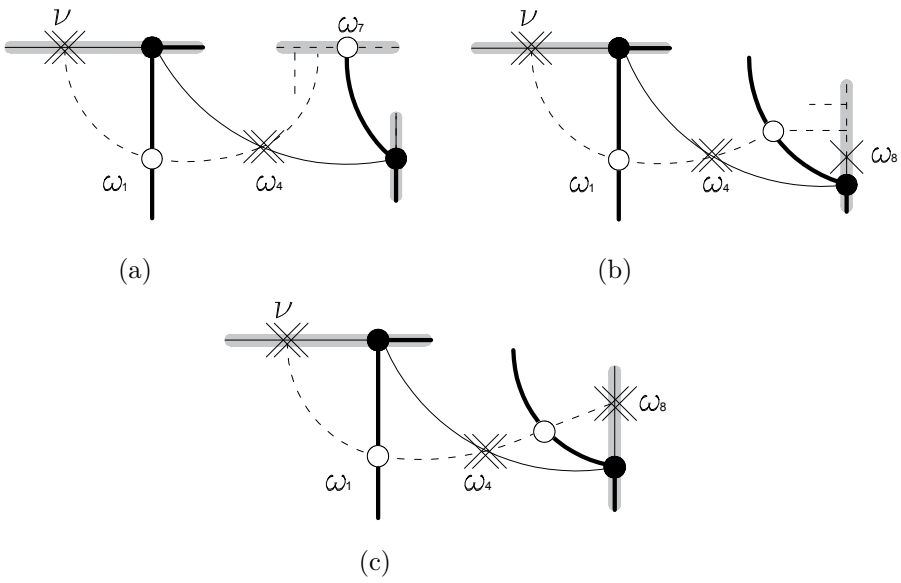


Figure 2.14

elementary move of type •-out at w_6' bring us to the previous consideration when w_6 was a black vertex. The same applies to the case when w_6 is an inner black vertex.

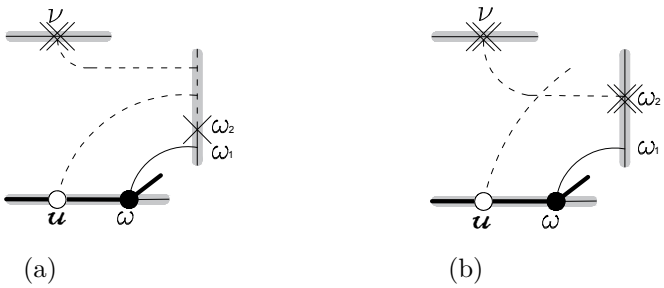


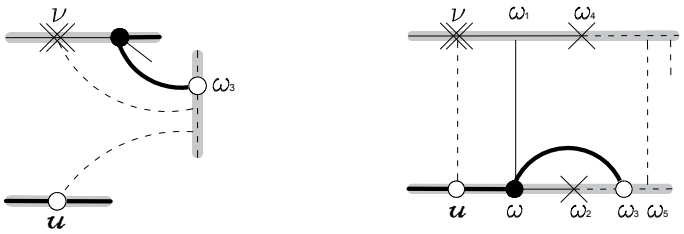
Figure 2.15

Case 1.2: the vertex w is black. Let w_1 be the vertex connected to w by an inner solid edge. If w_1 is a monochrome vertex in a different solid segment that the one containing v, let w_2 be the vertex connected to w_1 on the region R. If w_2 is a simple $\times$-vertex, the creation of a bridge beside it with the edge e produces a generalized cut (see Figure 2.15a). Otherwise, the vertex w_2 is a nodal $\times$-vertex and the creation of an inner dotted monochrome vertex with the inner edge of w_2 and the edge e produces a generalized cut (see Figure 2.15b). If w_1 is a monochrome vertex connected to v, let w_2 be the vertex connected to w by a real solid edge. If w_2 is a monochrome vertex, we do an elementary move of type •-in at w_2 followed by an elementary move •-out at w_1. In the case when there is no resulting bold bridge, this configuration has been consider in the Case 1.1. Otherwise, we destroy the bold bridge and let w_2' be the real black vertex connected to v and let w_3 be the vertex connected to it by an inner bold edge. If w_3 is an inner white vertex, the creation of a bridge beside u followed by an elementary move of type ∘-out sets the configuration as when there was no bold bridge. If w_3 is a monochrome vertex, an elementary move of type ∘-in allows us to consider it as an inner white vertex. If w_3 is a real white vertex, the

creation of a bridge beside it with the edge e produces an axe (see Figure 2.16a).

In the case when the vertex w_2 is a simple $\times$-vertex, let w_3 be the vertex connected to w_2 by a dotted real edge. If w_3 is a white vertex, it is connected to w up to a monochrome modification. Let $w_4 \neq v$ be the real neighbor vertex of w_1.

When the vertex w_4 is a simple $\times$-vertex, if it is connected to w_3, then the toile would be a cubic. Hence the vertices w_3 and w_4 are not neighbors. Let $w_5 \neq w_2$ be the real neighbor vertex of w_3.



(a) (b)

Figure 2.16

When w_5 is a monochrome vertex, the creation of a bridge with its inner dotted edge beside w_4 produces a cut (see Figure 2.16b). If w_5 is a simple $\times$-vertex having a black neighboring vertex, then we create a bridge beside w_5 with the inner solid edge adjacent to w creating a solid cut, we destroy the correspondent zigzag, make an elementary move of type $\bullet$-it followed by an elementary move of type $\bullet$-out, make an elementary move of type $\circ$-in with u and w_5 in order to create a bold bridge and perform an elementary move of type $\circ$-out. Then, the creation of a zigzag brings us to a configuration considered in Case 1.1 without breaking the maximality assumption (see Figure 2.17).

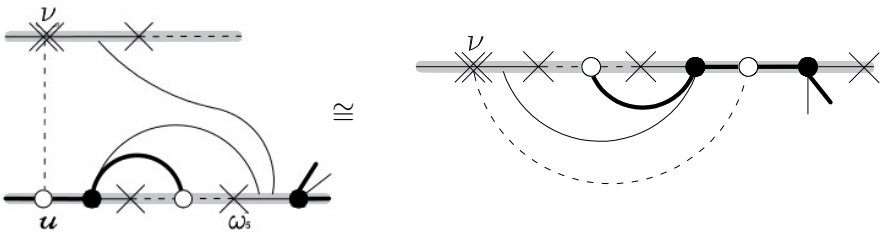


Figure 2.17

If w_5 is a simple $\times$-vertex having a solid neighboring monochrome vertex w_6, we perform a monochrome modification to connect w_6 with w. Let $w_7 \neq w_5$ be the real neighboring vertex of w_6. If w_7 is a simple $\times$-vertex, the creation of a dotted bridge beside with the edge e produces an axe (see Figure 2.18a). If w_7 is a nodal $\times$-vertex, the creation of an inner monochrome vertex between the edge e and the inner dotted edge adjacent to w_7 produces a generalized cut (see Figure 2.18b).

If w_5 is a nodal $\times$-vertex, a monochrome modification connect it to w and then, the creation of a bridge beside w_5 with the edge e produces an axe (see Figure 2.18c).

If w_4 is a nodal $\times$-vertex, the creation of a dotted bridge beside w_3 with the inner edge adjacent to w_4 produces an axe (see Figure 2.19a).

If w_3 is a monochrome vertex, let w_4 be the vertex connected to w by an inner bold edge. If w_4 is a real white vertex, then the creation of a bridge beside it with the inner

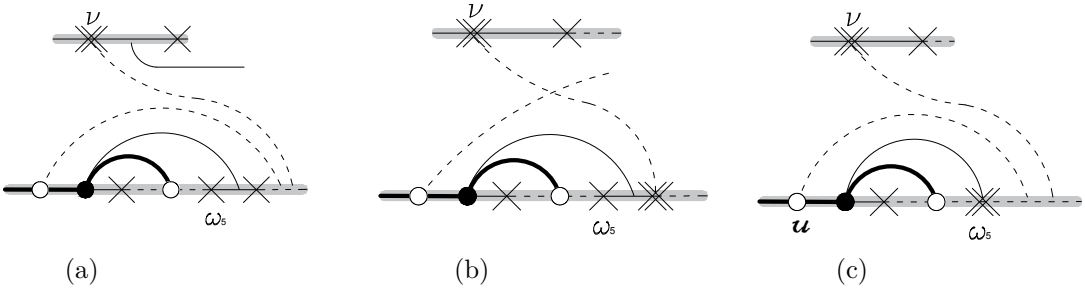


Figure 2.18

dotted edge adjacent to w_3 produces a cut (see Figure 2.19b). If w_4 is a monochrome vertex, an elementary move of type $\circ$-in allows us to consider it as an inner white vertex. If w_4 is an inner white vertex, up to a monochrome modification it is connected to w_3 and then an elementary move of type $\circ$-out bring us to the configuration where w_3 was a white vertex.

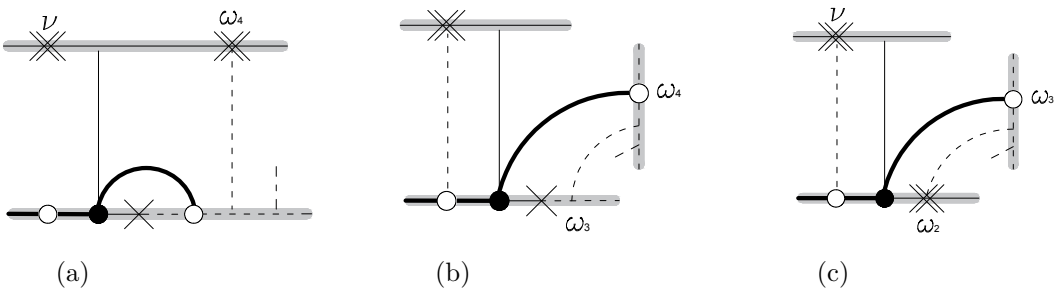
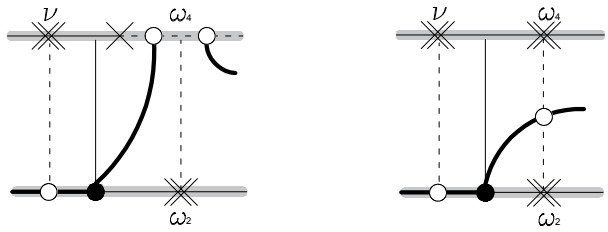


Figure 2.19

Otherwise w_2 is a nodal $\times$-vertex. Let w_3 be the vertex connected to w by an inner bold edge. If w_3 is a real white vertex, the creation of a bridge with the inner dotted edge adjacent to w_2 beside w_3 produces an axe (see Figure 2.19c). If w_3 is a monochrome vertex, an elementary move of type $\circ$-in allows us to consider it as an inner white vertex. Lastly, if w_3 is an inner white vertex, up to a monochrome modification it is connected to w_2. Let $w_4 \neq v$ be the vertex connected to w_1 by a real edge. If w_4 is a simple $\times$-vertex, up to the creation of a dotted bridge beside it with an inner dotted edge of w_3, we can perform an elementary move of type $\circ$-out producing an axe (see Figure 2.20a). If w_4 is a nodal $\times$-vertex, up to a monochrome modification it is connected to w_3 producing a generalized cut (see Figure 2.20b).

If w_1 is an inner simple $\times$-vertex, we can performs a monochrome modification so it is connected to u in order to create a zigzag, but this contradicts the maximality assumption.

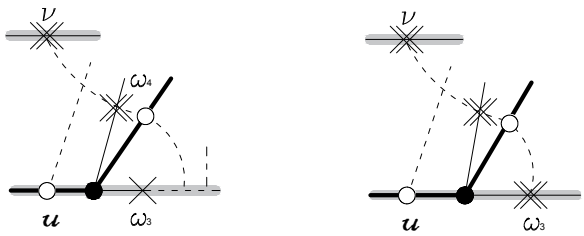
Otherwise, the vertex w_1 is an inner nodal $\times$-vertex. Let $w_2 \notin R$ be the vertex connected to w_1 by a dotted edge. If w_2 is an inner white vertex, it is connected to w up to a monochrome modification. Let w_3 be the vertex connected to w by a real solid edge. If w_3 is a simple $\times$-vertex, the creation of a bridge beside w_3 with an inner dotted edge adjacent to w_2 and a monochrome vertex with the edges e and an inner dotted edge adjacent to w_1 produce a generalized cut (see Figure 2.21a). If w_3 is a nodal $\times$-vertex, up to a monochrome modification it is connected to w_2. Then, the creation of an inner monochrome vertex with the edges e and an inner dotted edge adjacent to w_1 produce



(a) (b)

Figure 2.20

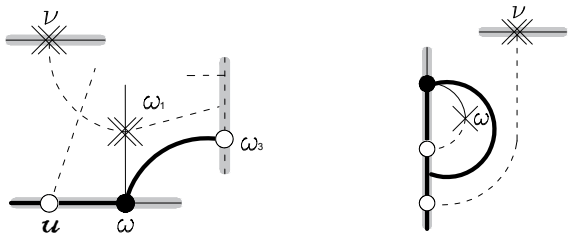
a generalized cut (see Figure 2.21b). Lastly, if w_3 is a monochrome vertex, we create a bridge beside v with the edge $[w, w_1]$ and we perform an elementary move of type $\bullet$-in at w_3 followed by an elementary move of type $\bullet$-out at the bridge, bringing us to a configuration considered in Case 1.1.



(a) (b)

Figure 2.21

If w_2 is a real white vertex, let w_3 be the vertex connected to w by a bold inner edge. If w_3 is a monochrome vertex, an elementary move of type $\circ$-in allows us to consider it as an inner white vertex. If w_3 is an inner white vertex, a monochrome modification connects it to w_1 and that corresponds to the configuration where w_2 was an inner white vertex. Finally, if w_3 is a real white vertex, the creation of a bridge beside it with an inner dotted edge adjacent to w_1 and a monochrome vertex with the edges e and an inner dotted edge adjacent to w_1 produce a generalized cut (see Figure 2.22a).



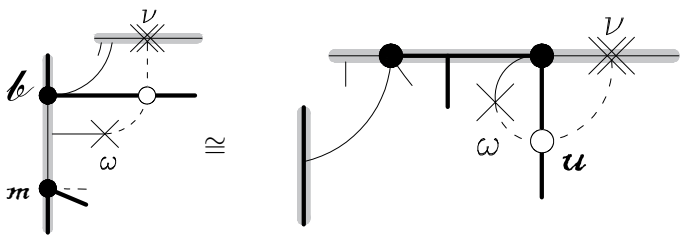
(a) (b)

Figure 2.22

Case 1.3: the vertex u is an inner white vertex. If u is connected to another nodal $\times$-vertex or a monochrome dotted vertex, it determines a generalized cut. Otherwise u is connected to an inner $\times$-vertex w. Let us assume that u is an inner simple $\times$-vertex.

If u is connected to a monochrome bold vertex, an elementary move of type $\circ$-out bring us to the previous cases. If u is connected to an inner black vertex, the creation of a bridge beside v with an inner solid edge adjacent to the black vertex and an elementary move of type $\bullet$-out allow us to consider u connected to a real black vertex. Thus, the vertex u must be connected to a black real vertex b. Let R to be the region determined by u, v and b, and let $f \neq [u, b]$ be the edge adjacent to b in R. If f is a real edge, then up to a monochrome modification the vertex b and w are connected by an inner solid edge, and then, the creation of a bold bridge beside b with an inner edge adjacent to u followed by an elementary move of type $\circ$-out allows us to create a zigzag, contradicting the maximality assumption (see Figure 2.22b).

If f is an inner edge, let m be the vertex connected to b by a real solid edge. If m is a $\times$-vertex, then either the creation of a bridge beside m with the edge $[u, w]$ or a monochrome modification connecting u and m, if m is simple or nodal, respectively, produces a generalized cut. If m is a monochrome vertex, it is connected to w and then, an elementary move of type $\bullet$-in at m followed by the creation of a solid bridge beside v with the edge f and an elementary move of type $\bullet$-out at the bridge allows us to create a zigzag, contradicting the maximality assumption (see Figure 2.23a).



(a) (b)

Figure 2.23

The missing case is when the vertex u is connected to an inner nodal $\times$-vertex w. Due to the aforementioned considerations, if any of the vertices u or w share a region with a real $\times$-vertex, the creation of a dotted bridge or an inner monochrome vertex produce a generalized cut. If the vertex u is connected to black vertices having neighboring real solid monochrome vertices, then the creation of solid bridges beside v followed by a elementary moves of type $\bullet$-in and $\bullet$-out bring us to the configuration where the vertex v has two neighboring black vertices b_1 and b_2, which are connected to u and w up to monochrome modifications. Let S be the bold segments containing b_1. If the segment S has no white vertices, it does have a monochrome vertex connected to a real white vertex determining a dotted segment where the construction of a bridge with the edge $[u, w]$ produces a generalized cut (see Figure 2.24a). If the segment S has exactly one white vertex w_1, it has two neighboring black vertices b_1 and b_1'. Up to monochrome modifications, the vertices b_1 and b_1' are connected to w and u. Then, if b_1' has a neighboring real $\times$-vertex, then either the creation of a bridge beside it with the edge $[v, u]$ or the creation of an inner monochrome vertex with the edge $[v, u]$ and the inner dotted edge adjacent to the nodal $\times$-vertex produces a generalized cut (see Figures 2.24b and 2.24c). Otherwise, the vertex b_1' is connected to a solid monochrome vertex m, in which case, the creation of a bridge beside v with the inner solid edge adjacent to m produces a solid cut where one of the resulting dessins is a cubic of type I^{***} (a cubic with an inner nodal $\times$-vertex and an isolated real nodal $\times$-vertex) (see Figure 2.24d). If the segment S has at least two white

vertices, by means of elementary moves of type ∘-in and ∘-out, we can transfer pairs of white vertices from the segment S to the bold segment containing the vertex b_2 bring us to the case when S has none or one white vertex.

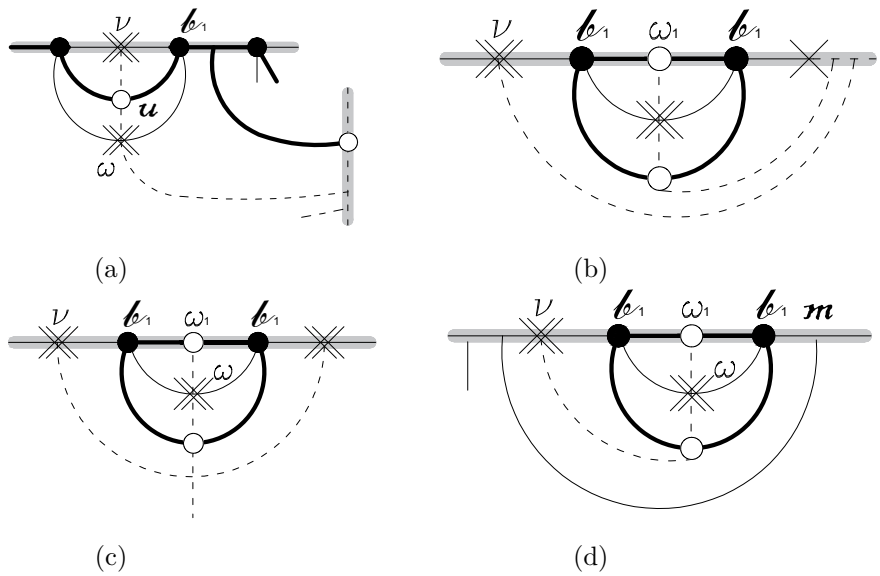


Figure 2.24

Case 2: the vertex v is non-isolated, being connected to a black vertex u. Due to Case 1, we can assume that there are no isolated real nodal ×-vertices. If the vertex u is real, the edge $e := [v, u]$ is dividing. Let w be vertex connected to u by a bold real edge. Let R be the region determined by v, u and w.

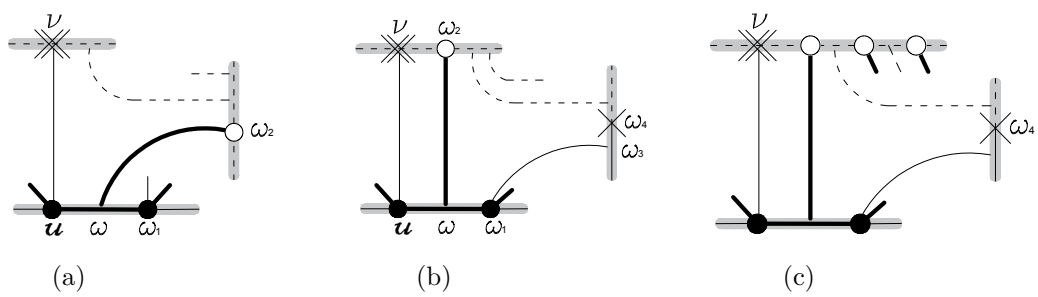


Figure 2.25

Case 2.1: the vertex w is a monochrome vertex. Since the toile is bridge-free, the vertex w is connected to a real black vertex $w_1 \neq u$. Let w_2 be the vertex connected to w by an inner bold edge. If w_2 is a monochrome vertex, an elementary move of type ∘-in followed by the possible creation of a bridge beside v and an elementary move of type ∘-out allows us to consider w_2 as a real white vertex. In the case when the vertex w_2 is a real white vertex, if it is not a neighboring vertex of v and the region R contains an inner dotted edge, up to the creation of bridges beside v and w_2 we can produce a dotted cut (see Figure 2.25a).

Otherwise, we can assume the vertices w_2 and v to be real neighboring vertices. Let w_3 be the vertex connected to w_1 by an inner solid edge. If w_3 is a monochrome vertex,

it has two real simple $\times$-vertices since the toile is bridge-free and there are no isolated nodal $\times$-vertices. Let w_4 be the simple $\times$-vertex connected to w_3 sharing a region with the vertices w and w_1. Let $w_4' \neq w_4$ be the other simple $\times$-vertex connected to w_3. All the following considerations apply to the case when w_3 is a nodal $\times$-vertex, in that case, the calls to w_4 and w_4' all refer to w_3. If w_4 is connected to a dotted monochrome vertex w_5, then, the creation of a dotted bridge beside w_2 with the inner edge adjacent to w_5 produces a cut (see Figure 2.25b).

If w_4 is connected to a real white vertex w_5 having a real neighboring monochrome vertex w_6, by applying a monochrome modification we can connect w_5 to w and then create a bridge beside v with the inner edge adjacent to w_6 in order to produce a dotted cut (see Figure 2.25c).

Otherwise, the vertex w_4 is connected to a real white vertex w_5 having a real neighboring $\times$-vertex $w_6 \neq w_4$. If w_6 is a simple $\times$-vertex, a monochrome modification connecting w_5 to w and the creation of a bridge beside w_6 with the edge e produces a solid axe (see Figure 2.26a).

If w_6 is a nodal $\times$-vertex, monochrome modifications connecting w_5 to w and w_6 to u bring us to a configuration equivalent to consider the vertices w_4 and w_2 as real neighbors. In the case where the vertices w_4 and w_2 are real neighbors, let us consider w_5 the vertex connected to w_1 by an inner bold edge. If w_5 is a monochrome vertex, an elementary move of type $\circ$-in at w_5 followed by the creation of a bridge beside w_4' and an elementary move of type $\circ$-out allows to consider the vertex w_5 as a white real vertex. The same applies if w_5 is an inner white vertex. Let us assume the vertex w_5 is a white real vertex. If w_4' has a monochrome dotted neighboring vertex, then the creation of a bridge with the corresponding inner edge beside w_5 produces a dotted cut (see Figure 2.26b). Otherwise, up to a monochrome modification the vertices w_4' and w_5 are neighbors.

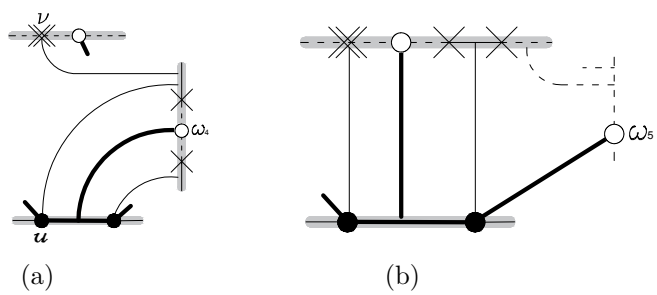


Figure 2.26

Let w_6 be the vertex connected to w_1 by a real solid edge. If w_6 is monochrome, it is connected to a real black vertex $w_1' \neq w_1$. Let w_7 be the vertex connected to w_6 by an inner solid edge. If w_7 is a monochrome vertex, it determines a solid cut.

If w_7 is a real nodal $\times$-vertex, it determines an axe. If w_7 is an inner $\times$-vertex, let w_8 be the vertex connected to u by an inner bold edge. If w_8 is a monochrome vertex, an elementary move of type $\circ$-in at it allows us to consider w_8 as an inner white vertex. If w_8 is an inner white vertex, up to the creation of a dotted bridge is it connected to a monochrome vertex beside v and then an elementary move of type $\circ$-out bring us to the case when w_8 is a real white vertex. If w_8 is a real white vertex, we perform an elementary move of type $\bullet$-in at w and the destruction of the resulting solid bridge. If w_7 is a simple $\times$-vertex, up to the creation of bridges beside the vertices w_5 and w_8 with the inner dotted edge adjacent to w_7 we construct a cut (see Figure 2.27a). In the case when

w_7 is an inner nodal $\times$-vertex, up to the construction of bridges beside the vertices w_5 and w_8 with the inner dotted edges adjacent to w_7, respectively, produces a generalized cut (see Figure 2.27b).

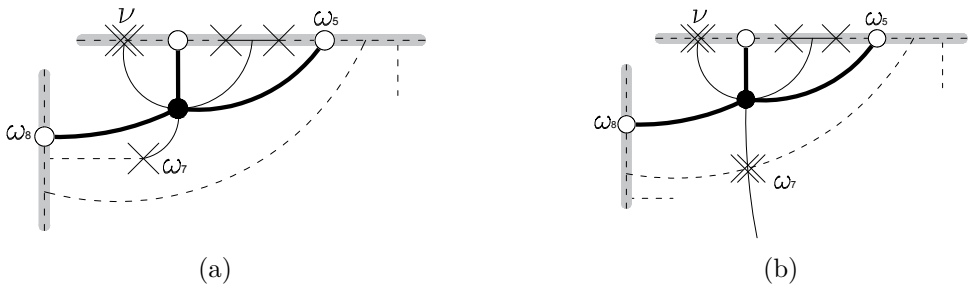


Figure 2.27

If w_6 is a real simple $\times$-vertex, let $w_7 \neq w_4'$ be the vertex connected to w_5. If w_7 is a monochrome vertex, the creation of a bridge with the inner dotted edge adjacent to it beside w_6 produces a cut (see Figure 2.28a). If w_7 is a nodal $\times$-vertex, the the creation of a bridge with the inner solid edge adjacent to it beside w_6 produces an axe (see Figure 2.28b). In the case when $w_7 \neq w_6$ is a simple $\times$-vertex, let w_8 be the vertex connected to w_6 by a real dotted edge. If w_8 is a monochrome vertex, the creation of a bridge with the inner dotted edge adjacent to it beside w_5 produces a cut (see Figure 2.28c).

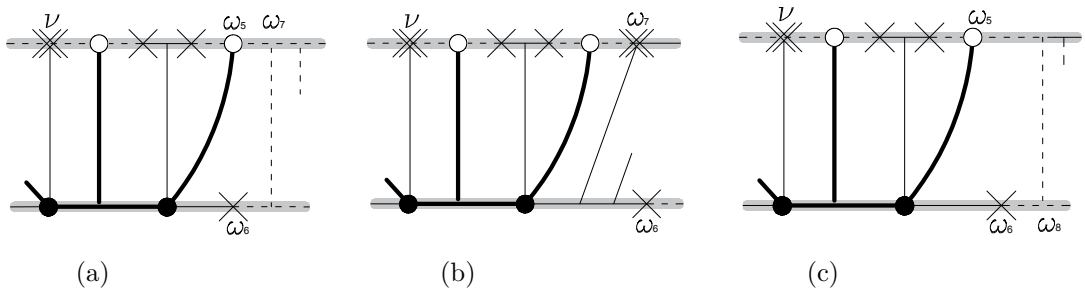


Figure 2.28

If w_8 is a real white vertex, it is connected to w_1 after a monochrome modification. Let $w_9 \neq w_6$ be the vertex connected to w_8 by a real dotted edge. If w_9 is a monochrome vertex, the creation of a bridge with the inner dotted edge adjacent to w_9 beside w_5 produce a cut (see Figure 2.29a). If w_9 is a nodal $\times$-vertex, the creation of an inner solid monochrome vertex with the inner edge adjacent to w_9 and the edge $[w_1, w_3]$ produces a solid generalized cut (see Figure 2.29b). If w_9 is a simple $\times$-vertex, the creation of a bridge beside w_9 with the edge $[w_1, w_3]$ produces a cut (see Figure 2.29c).

A special case is when the vertices w_7 and w_6 are equal. In this setting, we can consider the vertex w_3 as a nodal $\times$-vertex and made the aforementioned considerations applied to the vertices w_3 and w_1 as v and u, respectively. Since the degree of the toile is greater than 3, the process cannot cycle back to this case (see Figure 2.30a).

Case 2.2: the vertex w is a white vertex. Let w_1 be the vertex connected to w by an inner dotted edge and let $w_2 \neq u$ be the vertex connected to w by a real bold edge. If w_1 is a monochrome vertex and it is not a real neighboring vertex of v, then the creation of a dotted bridge beside the vertex v produces a cut (see Figure 2.30b). Otherwise, let

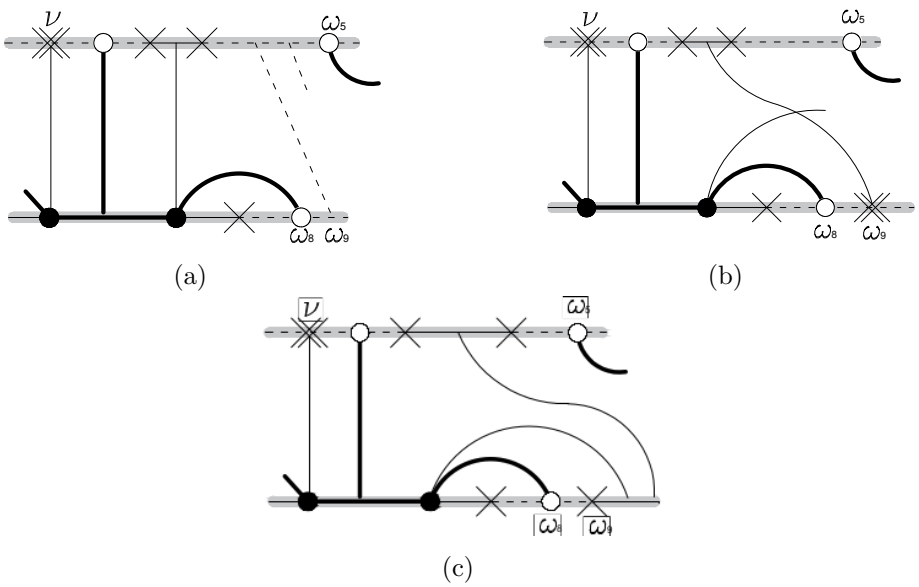


Figure 2.29

us assume the vertex w_1 to be monochrome and a real neighboring vertex of v. If w_2 is a monochrome vertex, an elementary move of type $\circ$-in at it followed by an elementary move of type $\circ$-out at w_1 and the destruction of a possible resulting bold bridge brings us either to Case 2.1 or to a different configuration within this case. From now on, we assume the vertex w_2 to be black. Let $w_3 \neq v$ be the vertex connected to w_1 by a real dotted edge.

If w_3 is a simple $\times$-vertex followed by a monochrome vertex w_4, which up to a monochrome modification is connected to w_2, let $w_3' \neq w_3$ be the simple $\times$-vertex connected to w_4 by a real solid edge. The following considerations apply when the vertex w_3 is a nodal $\times$-vertex, in which case we treat the vertices w_3' and w_4 equal to w_3. By symmetry of this configuration and since the degree is greater than 3, we can assume the dividing edge $[w_2, w_4]$ does not divide the toile resulting in a graph with only two triangular regions. Let w_5 be the vertex connected to w_2 by an inner bold edge. If w_5 is a monochrome vertex or an inner white vertex, up to an elementary move of type $\circ$-out at it, the creation of a dotted bridge beside w_3' and an elementary move of type $\circ$-out allow us to consider it as a real white vertex. If w_5 is a real white vertex and the vertex w_3' has a real neighboring dotted monochrome vertex, the creation of a bridge with its inner dotted edge beside

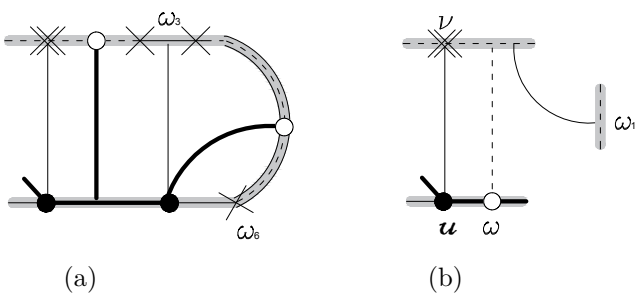


Figure 2.30

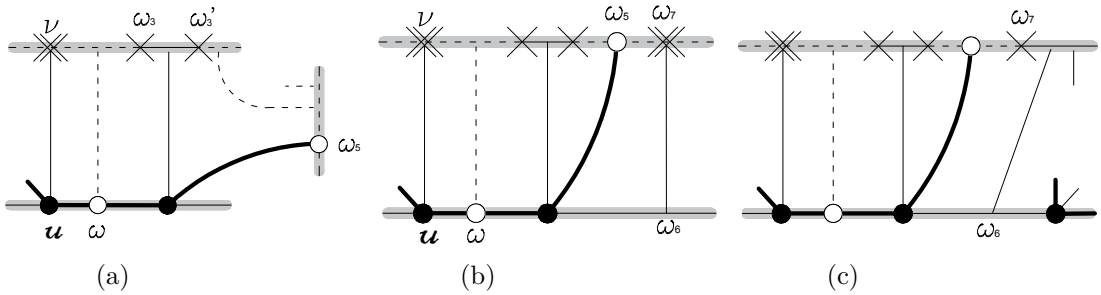


Figure 2.31

w_5 produces a cut (see Figure 2.31a). Otherwise, up to a monochrome modification the vertices w_3' and w_5 are neighbors. Let w_6 be the vertex connected to w_2 by a solid real edge.

If w_6 is a monochrome vertex, let $w_7 \neq w_3'$ be the vertex connected to w_5 by a real dotted edge. If w_7 is a real nodal $\times$-vertex, up to a monochrome modification it is connected to w_6 defining an axe (see Figure 2.31b). If w_7 is a real simple $\times$-vertex, the creation of a solid bridge beside w_7 with the inner solid edge adjacent to w_6 defines a solid cut (see Figure 2.31c).

Otherwise, the vertex w_7 is monochrome. Let $w_5' \neq w_5$ be the vertex connected to w_7 by a real dotted edge and let $w_2' \neq w_2$ be the vertex connected to w_6 by a real solid edge. Let w_8 be the vertex connected to w_6 by an inner solid edge, let w_9 be the vertex connected to w_2' by a real bold edge and let w_{10} be the vertex connected to w_2' by an inner solid edge (see Figure 2.32a).

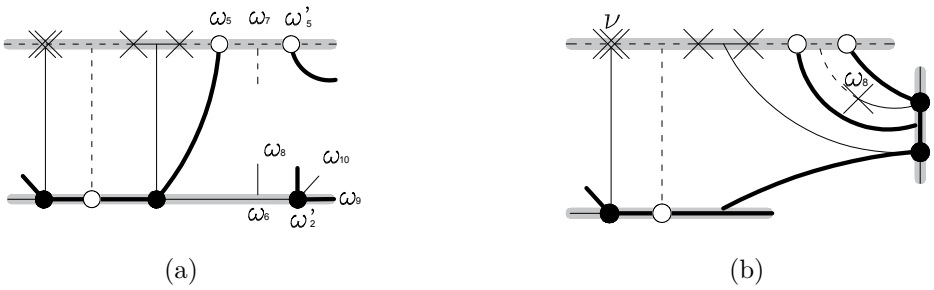
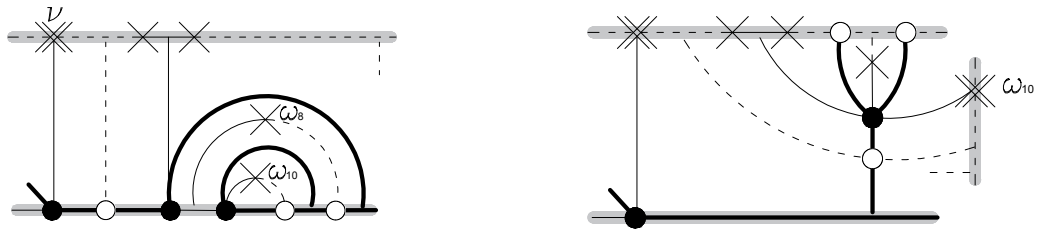


Figure 2.32

If the vertex w_8 is monochrome, it determines a cut.

If the vertex w_8 is an inner simple $\times$-vertex, it is connected to w_7 up to a monochrome modification. In this setting, the vertices w_5' and w_2' are connected up to a monochrome modification. If w_{10} is a monochrome vertex, an elementary move of type $\bullet$-in at w_6 followed by an elementary move of type $\bullet$-out at w_{10} and an elementary move of type $\circ$-in at w_7 followed by an elementary move of type $\circ$-out allow us to create a zigzag, contradicting the maximality assumption (see Figure 2.32b). If w_{10} is an inner simple $\times$-vertex, an elementary move of type $\circ$-in at w_7 produces an inner white vertex w' which is connected to w_{10} up to a monochrome modification. Then, up to the creation of a bold bridge beside w_2' with an inner bold edge adjacent to w', an elementary move of type $\circ$-out allows us to create a zigzag, contradicting the maximality assumption (see Figure 2.33a). If w_{10} is a real nodal $\times$-vertex, we consider the vertex w_9. If w_9 is a monochrome vertex,

the edge $[w_{10}, w_2']$ corresponds to the configuration in the Case 2.1. Otherwise, the vertex w_9 is a real white vertex, then, up to the creation of a dotted bridge beside w_{10} with the inner dotted edge adjacent to w_9, an elementary move of type $\bullet$-in at w_6 followed by an elementary move of type $\circ$-in produces a generalized cut (see Figure 2.33b).

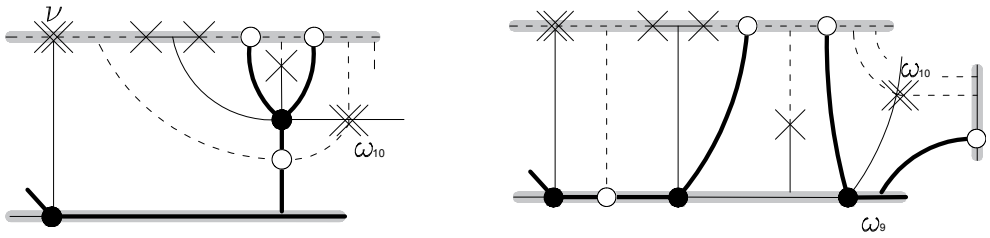


(a) (b)

Figure 2.33

Otherwise w_{10} is an inner nodal $\times$-vertex. If w_9 is a real white vertex, up to a monochrome modification it is connected to w_{10}. Up to the creation of a bridge beside w_5' with an inner dotted edge adjacent to w_{10}, an elementary move of type $\bullet$-in at w_6 followed by an elementary move of type $\circ$-in at the resulting bold monochrome vertex produces a generalized cut (see Figure 2.34a).

If w_9 is a monochrome vertex, it is connected to a real white vertex w_{11}. Then, the creation of bridges beside the vertices w_5' and w_{11} with the corresponding inner dotted edges adjacent to w_{10} produces a generalized cut (see Figure 2.34b).



(a) (b)

Figure 2.34

If w_8 is an inner nodal $\times$-vertex, up to a monochrome modification it is connected to w_7. Let $w_{11} \neq w_7$ be the vertex connected to w_8 by an inner dotted edge. If w_{11} is a monochrome vertex, it determines a generalized cut. If w_{11} is a real white vertex, up to the creation of a bold bridge beside w_{11} with the inner bold edge adjacent to w_2' an elementary move of type $\bullet$-in at w_6 followed by an elementary move of type $\bullet$-out at the bridge and the destruction of a possible resulting bridge produce a generalized cut (see Figure 2.35a).

If w_{11} is an inner white vertex, it is connected to w_2' up to a monochrome modification. Let $w_{12} \neq w_8$ be the vertex connected to w_{11} by an inner dotted edge. If w_{12} is a monochrome vertex, it determines a generalized cut. If w_{12} is an inner simple $\times$-vertex, up to a monochrome modification it is connected to w_2' and up to the creation of a bold bride beside w_2' and an elementary move of type $\circ$-out at the vertex w_{11} we can create a zigzag, contradicting the maximality assumption. If w_{12} is an inner nodal $\times$-vertex, up to

a monochrome modification is it connected to w_2'.

If w_9 is a monochrome vertex, it is connected to a real white vertex w_{11}. Then, the creation of a bridge beside w_{11} with an inner dotted edge adjacent to w_{10} produces a generalized cut (see Figure 2.35b).

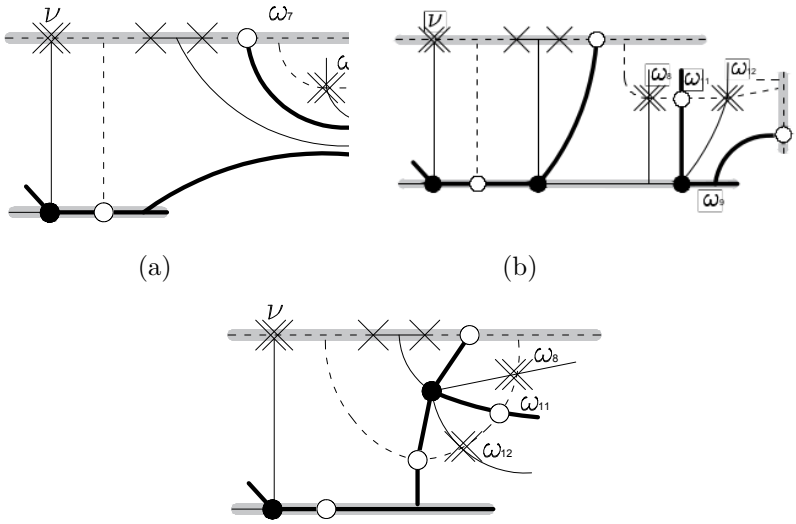


Figure 2.35

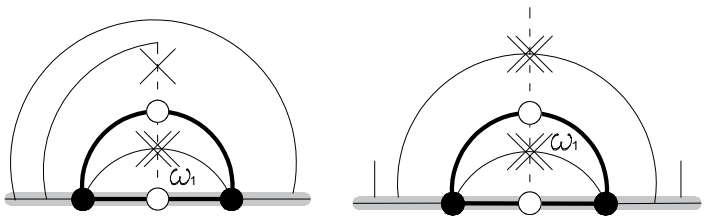
If w_9 is a real white vertex, up to a monochrome modification it is connected to w_{10}. Then, an elementary move of type $\bullet$-in at w_6 followed by an elementary move of type $\circ$-in at the resulting bold monochrome vertex produces a generalized cut (see Figure 2.35c).

If w_6 is a simple $\times$-vertex, let $w_7 \neq w_3'$ be the vertex connected to w_5 by a real dotted edge. If w_7 is a monochrome vertex or a real nodal $\times$-vertex, the creation of a bridge beside w_6 with the inner edge adjacent to w_7 produces a cut or an axe, respectively. If w_7 is a simple $\times$-vertex, let w_8 be the vertex connected to w_6 by a real dotted edge. If w_8 is a monochrome vertex, the creation of a bridge beside w_7 with the inner edge adjacent to w_8 produces a cut. If w_8 is a real white vertex, let w_9 be the vertex connected to w_8 by a real dotted edge. We perform a monochrome modification connecting w_2 to w_8. If w_9 is a monochrome vertex or a nodal $\times$-vertex, then either the creation of a bridge beside w_5 with the inner edge adjacent to w_9 or the creation of an inner solid monochrome vertex with the edge $[w_2, w_4]$ and the inner edge adjacent to w_9 produces a generalized cut. If w_9 is a simple $\times$-vertex, the creation of a bridge beside it with the edge $[w_2, w_4]$ produces a generalized cut.

In the case when w_3 is a simple $\times$-vertex followed by a black vertex w_4, let w_5 be the vertex connected to w_4 by an inner bold edge. If w_5 is a monochrome vertex, an elementary move of type $\circ$-in at it allows us to consider w_5 as an inner white vertex. If w_5 is a real white vertex, the creation of a bridge beside it with the edge $[w, w_1]$ produces a cut. If w_5 is an inner white vertex, the creation of a bridge beside w_3 with an inner dotted edge adjacent to w_3 followed by an elementary move of type $\circ$-out brings us to a configuration already considered.

If w_1 is an inner simple $\times$-vertex, a monochrome modification connecting it to the vertex u allows us to create a zigzag, contradicting the maximality assumption.

In the case when the vertex w_1 is an inner nodal $\times$-vertex, it is connected to w_2 up

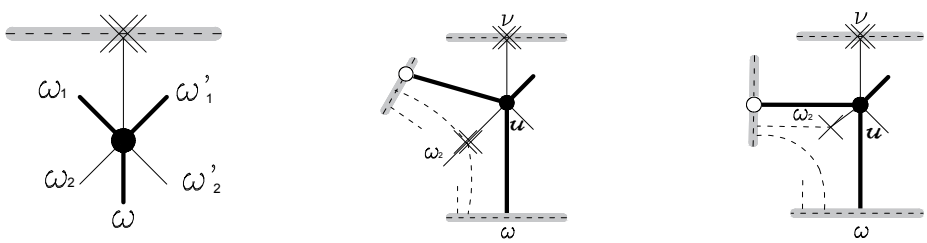


(a) (b)

Figure 2.36

to a monochrome modification. Let $w_3 \neq w$ be the vertex connected to w_1 by an inner dotted edge. If w_3 is a monochrome vertex, an elementary move of type $\circ$-in allows us to consider it as an inner white vertex. If w_3 is an inner white vertex, it is connected to w_2 up to a monochrome modification. Let w_4 be the vertex connected to w_2 by a real solid edge. If w_4 is a simple $\times$-vertex, the creation of a dotted bridge beside it with an inner dotted edge adjacent to w_3 produces a generalized cut. If w_4 is a monochrome vertex, let w_5 be the vertex connected to it by an inner solid edge. If w_5 is a real vertex, it determines a cut or an axe. If w_5 is an inner $\times$-vertex, it is connected to w_3 up to a monochrome modification. Then, we perform monochrome modifications connecting u to the vertices w_1 and w_3. In this setting, the creation of a bridge beside u with an inner solid edge adjacent to w_5 produces a generalized cut (see Figures 2.36a and 2.36b for when w_5 is simple or nodal, respectively). Otherwise, the vertex w_3 is a real white vertex. Let w_4 be the vertex connected to w_2 by an inner bold edge. If w_4 is a real white vertex, the creation of bridge beside it with the edge $[w_1, w_3]$ and a bridge beside v with the edge $[w, w_1]$ produces a generalized cut. If w_4 is a monochrome vertex, an elementary move of type $\circ$-in allows us to consider it as an inner white vertex. Lastly, if w_4 is an inner white vertex, a monochrome modification connecting it to w_2 brings us to a configuration already considered.

Case 2.3: the vertex u is an inner black vertex. If it is connected to a real monochrome vertex, an elementary move of type $\bullet$-out either produces an axe or bring us to a considered case. Let w_1, w_1' be the vertices connected to u by an inner bold edge sharing a region with v. If w_1 or w_1' is an inner white vertex, the creation of a bridge beside v with the inner dotted edge adjacent to it followed by an elementary move allows us to consider it as a real white vertex. Let w_2, $w_2' \neq v$ be the $\times$-vertices connected to u by an inner solid edge. Let $w \neq w_1, w_1'$ be the white vertex connected to u (see Figure 2.37a).



(a) (b) (c)

Figure 2.37

In the case when every vertex connected to u is a real vertex, since we assumed the dessin do not have cuts and do have degree greater than 3, it is not possible that every region containing u is triangular. Let R be one region containing u which is not triangular. Put v' and w' the $\times$-vertex and white vertex connected to u in R, respectively. Let w_3 be the neighboring vertex to w' in R. If w_3 is a monochrome vertex, the creation of a bridge beside v' with the inner edge adjacent to w_3 produces a cut. If w_3 is a nodal $\times$-vertex, the creation of an inner solid monochrome vertex with the inner edge adjacent to it and the edge $[v', u]$ produces a generalized cut. If w_3 is a simple $\times$-vertex, the creation of a bridge beside it with the edge $[v', u]$ produces an axe. Due to these considerations, we can assume the regions determined by v, u, w_1 and v, u, w_1' are triangular.

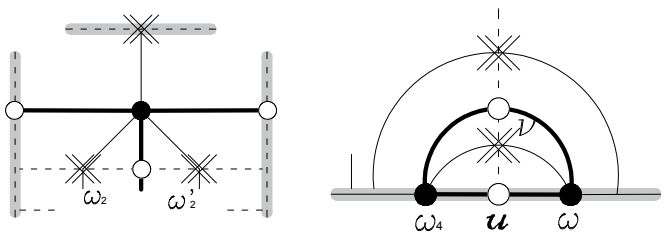
In the case when the vertex w is a real white vertex and w_2 is an inner $\times$-vertex, we have two different cases. If w_2 is an inner nodal $\times$-vertex, the creation of bridges beside w_1 and w with the inner dotted edges adjacent to w_2 produces a generalized cut (see Figure 2.37b). If w_2 is a simple $\times$-vertex, the creation of bridges beside w_1 and w with the inner dotted edge adjacent to w_2 produces a cut (see Figure 2.37c), unless this is an inadmissible elementary move, i.e., unless the vertices w_1, w and w_2 are connected to the same monochrome vertex.

In this setting, if w_2' it a real nodal $\times$-vertex, one of the regions determined by w_1', u, w_2' or w, u, w_2' is not triangular and the aforementioned considerations apply. Otherwise, the vertex w_2' is an inner $\times$-vertex. If w_2' is a nodal $\times$-vertex, the creation of bridges beside w_1' and w with the inner dotted edges adjacent to w_2 produces a generalized cut. If w_2' is a simple $\times$-vertex, the creation of bridges beside w_1' and w with the inner dotted edge adjacent to w_2' produces a cut, and since the degree is greater than 3, this is an admissible elementary move.

If w is an inner white vertex and at least one the vertices w_2 and w_2' is a real nodal $\times$-vertex, the creation of a bridge beside the real $\times$-vertex with an inner dotted edge adjacent to w and an elementary move of type $\circ$-out bring us to the configuration where w is a real vertex.

If w is an inner white vertex and at least one the vertices w_2 and w_2' is an inner simple $\times$-vertex, up to monochrome modification we can assume the vertex w is connected to the $\times$-vertex in which case, the creation of a bridge beside w_1 or w_1' with the inner dotted edge adjacent to the simple $\times$-vertex and an elementary move of type $\circ$-out bring us to the configuration where w is a real vertex.

Finally, if the vertex w is an inner white vertex and the vertices w_2 and w_2' are inner nodal $\times$-vertices, which up to monochrome modifications are connected to w, then the creation of dotted bridges beside w_1 and w_1' produces a cut (see Figure 2.38a).



(a) (b)

Figure 2.38

Case 3: let v be an inner nodal $\times$-vertex. Due to previous considerations we can assume

there are no real nodal $\times$-vertices. If v is connected to two different dotted monochrome vertices or solid vertices, it defines a generalized cut. Since $\mathrm{dp}(D) \leq 1$, the vertex v is connected to a real vertex u. Let us assume that u is a white vertex. If v is connected to inner black vertices, up to the creation of bold bridges beside u and elementary moves of type $\bullet$-out we can assume v is not connected to inner black vertices. Let w be a real vertex connected to u.

Case 3.1: the vertex w is a black vertex. Up to a monochrome modification it is connected to the vertex v. Let w_1 be the vertex connected to w by an inner bold edge. If w_1 is a monochrome vertex, let w_2 be the vertex connected to w by a real solid edge. If w_2 is a monochrome vertex, let w_3 be the vertex connected to w_2 by an inner solid edge. If w_3 is a monochrome vertex, it determines a cut. If w_3 is an inner simple $\times$-vertex, up to a monochrome modification it is connected to a real white vertex neighboring w_1. Then, an elementary move of type $\bullet$-in at w_2 followed by an elementary move of type $\bullet$-out at w_1 allows us to construct a zigzag, contradicting the maximality assumption.

If w_3 is an inner nodal $\times$-vertex, let $w_4 \neq w$ be the vertex connected to the vertex u by a real bold edge. If w_4 is a black vertex, up to a monochrome modification it is connected to v. We perform an elementary move of type $\circ$-in at w_1 producing an inner white vertex w' which up to monochrome modifications is connected to the vertices v, w_3 and w_4. Then, the creation of a bridge beside w_4 with an inner solid edge adjacent to w_3 produces a generalized cut (see Figure 2.38b).

If w_4 is a monochrome vertex, it is connected to a real black vertex w_5 by an inner bold edge and to a white vertex $u' \neq u$ by a real bold edge. If w_5 is connected to a solid monochrome vertex, an elementary move of type $\bullet$-in at it followed by an elementary move of type $\bullet$-out at w_4 bring us to the consideration where w_4 was a black vertex. Otherwise, the vertex w_5 is connected to a simple $\times$-vertex w_6. We perform an elementary move of type $\circ$-in at w_1. If w_6 shares a region with the vertex v, the creation of a bridge beside it with the edge $[v, u]$ connects v to a monochrome vertex by a chain of inner dotted edges (see Figure 2.39a). If w_6 does not share a region with the vertex v, the creation of a bridge beside it with the inner dotted edge adjacent to u', and elementary moves of type $\circ$-in at w_4 and w_1 connects v to a monochrome vertex by a chain of inner dotted edges (see Figure 2.39b).

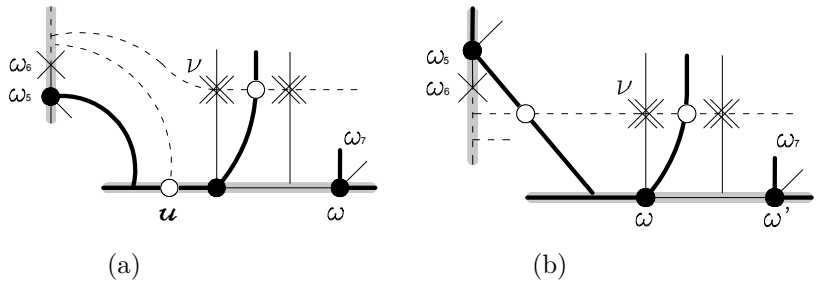


Figure 2.39

Let $w' \neq w$ be the vertex connected to w_2 by a real solid edge and let w_7 be the vertex connected to w' by an inner bold edge. If w_7 is a real white vertex, the creation of a bridge beside it with an inner dotted edge adjacent to w_3 produces a generalized cut. If w_7 is a monochrome vertex, an elementary move of type $\circ$-in allows us to consider it as an inner white vertex. If w_7 is an inner white vertex, up to a monochrome modification it is connected to w_3. Let w_8 be the vertex connected to w' by an inner solid edge. If w_8

is an inner simple $\times$-vertex, it is connected to w_7 up to a monochrome modification, then the creation of a bridge beside u' with an inner bold edge adjacent to w_7 followed by an elementary move of type $\circ$-out allows us to create a zigzag, contradicting the maximality assumption. If w_8 is a monochrome vertex, it is connected to a simple $\times$-vertex sharing a region with w_7. Then, the creation of a bridge beside this simple $\times$-vertex with a dotted edge adjacent to w_7 produces a generalized cut (see Figure 2.40a). If w_8 is a real nodal $\times$-vertex, the creation of a bridge beside it with a dotted edge adjacent to w_7 produces a generalized cut (see Figure 2.40b). If w_8 is an inner nodal $\times$-vertex, it is connected to w_7 up to a monochrome modification. Let w_9 be the vertex connected to w' by a real bold edge. If w_9 is a monochrome vertex, it is connected to a real white vertex determining a dotted segment, where the creation of a bridge with an inner dotted edge adjacent to w_8 produces a generalized cut (see Figure 2.40c). If w_9 is a white vertex, it is connected to w_8 up to a monochrome modification. We iterate the considerations starting with w_8 as the vertex v, producing a generalized cut.

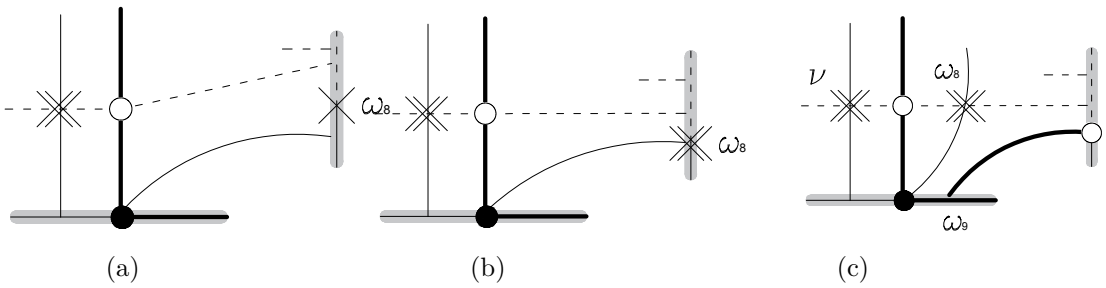
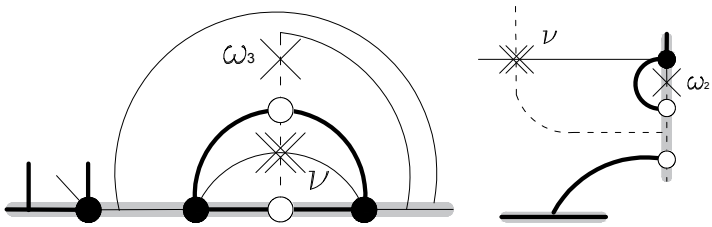


Figure 2.40

If w_2 is a simple $\times$-vertex, up to the creation of a bridge beside it with the inner dotted edge of a real white vertex connected to w_1, an elementary move of type $\circ$-in at w_1 followed by an elementary move of type $\circ$-out beside w_2 allow as to consider w_1 as a real white vertex. If w_1 is a real white vertex, up to a monochrome modification or the creation of a dotted bridge, the vertex v is connected to a monochrome vertex beside w_1. Then, we consider the other real vertex connected to u. If w_1 is an inner white vertex, it is connected to v up to a monochrome modification. Let w_2 the vertex connected to w by a real solid edge. If w_2 is a simple $\times$-vertex, the creation of a bridge beside it with an inner dotted edge adjacent to w_1 followed by an elementary move of type $\circ$-out at the bridge bring us to the configuration when w_1 is a real white vertex. If w_2 is a monochrome vertex, let $w' \neq w$ the black vertex connected to w_2 by a solid real edge and let w_3 be the vertex connected to w_2 by an inner solid edge. If w_3 is a monochrome vertex, it determines a cut. If w_3 is an inner nodal $\times$-vertex, up to a monochrome modification it is connected to w_1, corresponding to a previous configuration already considered.

If w_3 is an inner simple $\times$-vertex, up to monochrome modifications it is connected to w_1 and the vertex u' is connected to the vertices w_1 and v. Let S be the bold segment containing w'. If the segment S contains no white vertices, an elementary move of type $\bullet$-in followed by the creation of a bold bridge beside u and an elementary move of type $\bullet$-out produces a generalized cut (see Figure 2.41a). If the segment S containing exactly one white vertex w_4, which is connected to a black vertex $u'' \neq u'$. Up to monochrome modification the vertex w_4 is connected to the vertex v and the vertex u'' is connected to v and w_1 corresponding to a configuration already considered. Up to elementary moves of type $\circ$-in at monochrome vertex in the bold segment S containing u' followed by elementary

moves of type ∘-out at the bold segment containing u we reduce to the case when the segment S contains one or none white vertices.



(a) (b)

Figure 2.41

Case 3.2: the vertex w is a monochrome vertex. Let w_1 be the real black vertex connected to w by an inner bold edge. Let f be the solid real edge adjacent to w_1 and let w_2 be the vertex connected to w_1 by f. If f does not share a region with the vertex v, up to a monochrome modification v and w_1 are connected. If w_2 is a monochrome vertex, an elementary move of type •-in at w_2 followed by an elementary move of type •-out at w brings us to the previous case. If w_2 is a simple ×-vertex, up to the creation of a bridge beside it with the inner dotted edge adjacent to a neighboring white vertex of w, an elementary move of type ∘-in at w followed by an elementary move of type ∘-out beside w_2 connects the vertex v to a monochrome vertex (see Figure 2.41a). Then, we consider the other vertex connected to v by an inner dotted edge.

Otherwise, the edge f shares a region with the vertex v. If w_2 is a monochrome vertex, up to a monochrome modification it is connected to v. Then, an elementary move of type •-in at w_2 followed by an elementary move of type •-out at w brings us to Case 3.1. Lastly, if w_2 is a simple ×-vertex, the creation of a bridge beside it with the edge $[v, u]$ connects v with a monochrome vertex (see Figure 2.41b). Then, we consider the other real vertex connected to u.

Case 3.3: the vertex u is a real black vertex. Let f be the inner bold edge adjacent to u. Let w be an inner white vertex connected to v. If the edge f does not share a region with the vertex w, up to the creation of a bridge beside u with an inner bold edge adjacent to w, an elementary move of type ∘-out bring us to the previous case. If the edge f shares a region with the vertex w, up to a monochrome modification it does connects the vertex w to the vertex u. Let w_1 be the vertex connected to u by a real solid edge. If w_1 is a monochrome vertex, it is connected to a black vertex $u' \neq u$. Let w_2 be the vertex connected to w_1 by an inner solid edge. If w_2 is a monochrome vertex, it defines a solid cut. If w_2 is an inner simple ×-vertex, up to monochrome modifications the vertex w_2 is connected to w and the vertex u' is connected to the vertices w and v, bringing us to a configuration already considered. If w_2 is an inner nodal ×-vertex, up to a monochrome modification it is connected to the vertex w corresponding to a configuration considered in Case 3.1. If w_1 is a simple ×-vertex, up to the creation of a bridge beside w_1 with an inner dotted edge adjacent to w, an elementary move of type ∘-out connects v to a monochrome vertex.

□

Chapter 3

Real plane curves and Hirzebruch surfaces

3.1 Hirzebruch surfaces

From now on we consider ruled surfaces with the Riemann sphere $B \cong \mathbb{CP}^1$ as base curve. The Hirzebruch surfaces Σ_n are toric surfaces geometrically ruled over the Riemann sphere, determined up to isomorphism of complex surfaces by a parameter $n \in \mathbb{N}$. They are minimal except for Σ_1, which is isomorphic to $\mathbb{CP}^2$ blown up in a point. The surface Σ_n is defined by the local charts $U_0 \times \mathbb{CP}^1$ and $U_1 \times \mathbb{CP}^1$ where $U_0 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_0 \neq 0\}$, $U_1 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\}$ glued along $\mathbb{C}^*$ via the map

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{CP}^1 & \longrightarrow & \mathbb{C}^* \times \mathbb{CP}^1 \\ (z, w) & \longmapsto & \left(\frac{1}{z}, \frac{w}{z^n}\right) \end{array}.$$

The exceptional section is the section at infinity S_∞ such that $S_\infty^2 = -n\,(n \geq 0)$. We denote its homology class by $[E]$. The second homology group of the Hirzebruch surfaces $H_2(\Sigma_n, \mathbb{Z}/2\mathbb{Z})$ is generated by the homology classes $[Z]$ and $[f]$ of the null section S_0, denoted by Z, and of a generic fiber f, respectively. The intersection form is determined by the Gram matrix

$$\begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the base $\{[Z], [f]\}$. The homology class of the exceptional section is given by

$$[E] = [Z] - n[f].$$

Performing a positive Nagata transformation on Σ_n results in a geometrically ruled surface isomorphic to Σ_{n+1} (since the exceptional divisor decreases its self-intersection by one). Likewise, a negative Nagata transformation on Σ_n $(n > 0)$ results in a geometrically ruled surface isomorphic to Σ_{n-1}.

Setting $(\mathbb{C})^2 \subset \Sigma$, the trigonal curve C can be described by a polynomial in two variables $f(z, w) = q_0(z)w^3 + q_1(z)w^2 + q_2(z)w + q_3(z)$ in which q_0 determines the intersection with the exceptional fiber. If the trigonal curve C is proper, then q_0 must be constant. We can suppose q_0 equal to 1. Up to affine transformations of $\mathbb{C}$, we can set the sum of the roots of $f(z, \cdot)$ equal to 0, resulting in the Weiertraß equation

$$w^3 + q_2(z)w + q_3(z).$$

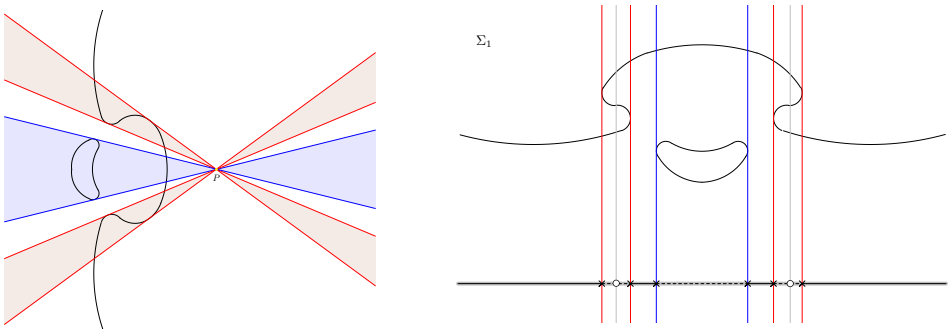


Figure 3.1: A real plane cubic with two zigzags and one oval, and its pencil of lines seen in Σ_1.

Since C is a trigonal curve, then $[C] = 3[Z] + 0[f]$. Therefore, the intersection product $[C] \cdot [Z] = 3n$ equals the degree of $q_3(z)$. Since this explicit description must be invariant by the coordinate change $(z, w) \longmapsto (\frac{1}{z}, \frac{w}{z^n})$, the degree of q_2 must be $2n$. Hence, the j-invariant

$$j = \frac{-4q_2^3}{\Delta}, \ \Delta = -4q_2^3 - 27q_3^2$$

is a rational function of degree $6n$ if the curve is generic (i.e., the polynomials q_2 and q_3 have no zeros in common).

3.2 Relation with plane curves

Let $A \subset \mathbb{CP}^2$ be a reduced algebraic curve with a distinguished point of multiplicity $\deg(A) - 3$ such that A does not have linear components passing through P. The blow-up of $\mathbb{CP}^2$ at P is isomorphic to Σ_1. The strict transform of A is a trigonal curve $C_A := \widetilde{A} \subset \Sigma_1$, called the *trigonal model* of the curve A. A *minimal proper model* of A consists of a proper model of C_A and markings corresponding to the images of the improper fibers of C_A by the Nagata transformations.

3.3 Real algebraic plane curves of degree 3

Let $A \subset \mathbb{CP}^2$ be a real smooth cubic. Let $p \in \mathbb{CP}^2$ a real point which does not belong to A and let $B \cong \mathbb{CP}^1 \subset \mathbb{CP}^2$ be a real line which does not pass through p. The pencil of lines passing through p can be parametrized by B, mapping every line $L \ni p$ to $L \cap B$. The blow-up of $\mathbb{CP}^2$ at p is isomorphic to a real geometrically ruled surface over B. The strict transform of A is a real proper trigonal curve $C \subset \Sigma_1$ (since it is proper, it is already a minimal proper model for A). Since the real structures are naturally compatible, we associate to C its real dessin $\mathrm{Dssn}(C)_c$ on the quotient of $B \cong \mathbb{CP}^1$ by the complex conjugation. Up to elementary equivalence, all the possible dessins are shown on Figure 3.2. They are named by either their type (cf. Definition 1.3.9) and the number of zigzags they possess (in the non-hyperbolic case, cf. Definition 1.1.18) or by H in the case of the *hyperbolic cubic*. Up to weak equivalence there are only three classes of cubics, namely the ones of type I, type II and the hyperbolic one, corresponding to the rigid isotopy classification of couples (A, p) of real cubic curves and one additional point of the plane outside A.

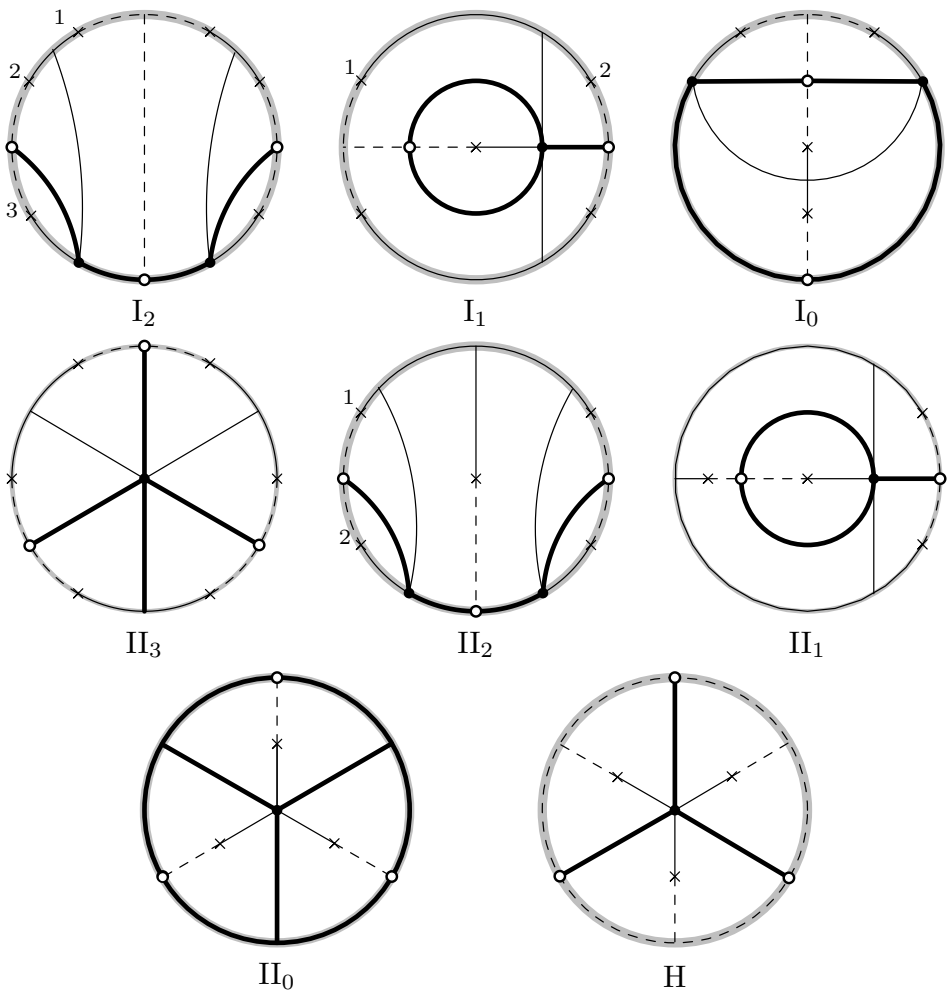


Figure 3.2: Classes of cubic dessins up to elementary equivalence.

3.3.1 Positive and negative inner nodal $\times$-vertices

Motivated by Proposition 5.1.10 we make the following definition.

Definition 3.3.1. Let v be an inner nodal $\times$-vertex of a type I dessin. We say that v is *positive* if all the regions containing v are labeled 1. Otherwise, we say that v is *negative*.

Given a real nodal curve X, we denote by c the number of nodal points in $\mathbb{C}X \setminus \mathbb{R}X$. If X is a plane or trigonal curve, let $\widehat{X}$ be the normalization of a type I perturbation of X and let $\widehat{X}_+$ and $\widehat{X}_-$ the connected components of $\mathbb{C}\widehat{X} \setminus \mathbb{R}\widehat{X}$. We put $\sigma(X) = \#(X_+ \cap X_-)$, where $X_\pm \subset \mathbb{C}X$ is the image of $\widehat{X}_\pm$ under the normalization map $\mathbb{C}\widehat{X} \longrightarrow \mathbb{C}X$.

Let X be a type I curve in $\mathbb{RP}^2$ of odd degree. We denote by $\Lambda^+(X)$ (resp., $\Lambda^-(X)$) the number of positive (resp., negative) oval and by $\Pi^+(X)$ (resp., $\Pi^-(X)$) the number of positive (resp., negative) injective pair of a type I perturbation of X.

Theorem 3.3.2 (Rokhlin's complex orientation formula for nodal plane curves in $\mathbb{RP}^2$ of odd degree [8])**.** *Let X be a nodal type* I *plane curve of degree $d = 2k+1$. Then*

$$\Lambda^+(X) - \Lambda^-(X) + 2(\Pi^+(X) - \Pi^-(X)) = l - k(k+1) + \sigma(X),$$

where l is the number of ovals in a type I *perturbation of X.*

We say that an elliptic nodal point of a type I plane curve X is *positive* (resp., *negative*) if its corresponding oval in the type I perturbation of X is positive (respectively, negative). In the same manner, we say that an isolated nodal $\times$-vertex of a type I dessin D is *positive* (resp., *negative*) if its corresponding oval in the type I perturbation of D is positive (resp., negative).

Definition 3.3.3. Given a dessin D, we denote by $c(D)$ twice the number of the inner nodal $\times$-vertices of D and by $\sigma(D)$ twice the number of the negative inner nodal $\times$-vertices of D.

Proposition 3.3.4. *If $X \subset \Sigma$ is a type* I *real trigonal curve with associated real dessin D, then*

$$\sigma(X) = \sigma(D).$$

Proof. Every pair $n, \bar{n}$ of complex conjugated nodal points of X corresponds to an inner nodal $\times$-vertex v_n, and vice versa. By definition, the number $\sigma(X)$ counts the nodal points of the curve X lying in $\mathbb{C}X \setminus \mathbb{R}X$ such that they belong to both the images of the halves $\mathbb{C}\widehat{X}_\pm$ of the normalization of X on X, each imposing an obstruction for X to be perturbed to a type I non-singular curve. In the case when the nodal point n does not contribute to $\sigma(X)$, it has a local type I perturbation and so does its corresponding vertex v_n. Since every inner simple $\times$-vertex of type I is labeled by 1, so are all the regions containing v_n and hence, it is positive. □

Corollary 3.3.5. *Let C be a nodal rational curve of degree* 5 *whose type* I *perturbation has l ovals. Assume that C has an elliptic nodal point $p \in \mathbb{R}C$. Then, the type* I *perturbation of the marked toile (D_C, v_{T_1}, v_{T_2}) associated to (C, p) has $l-1$ ovals and $c(D_C) = c(C)+2$.*

Proof. The elliptic nodal point p is perturbed to an oval in the type I perturbation of C, and every other oval different from the one arisen from p appears in the type I perturbation of D_C. Every inner nodal $\times$-vertex of D_C represents either a pair of complex conjugated nodal points of the curve C or the pair of complex conjugated tangent lines of C at p. □

Recall that for a proper non-isotrivial trigonal curve $X \subset \Sigma_n \longrightarrow \mathbb{CP}^1$, a real white vertex v in the associated toile corresponds to an intersection of the curve X with the zero section $Z \subset \Sigma_n$. When n is even, the real point set $\mathbb{R}\Sigma_n$ is homeomorphic to a torus and the long component of X together with the exceptional divisor divide $\mathbb{R}\Sigma_n$ into two connected components. When n is odd, the real point set $\mathbb{R}\Sigma_n$ is homeomorphic to a Klein bottle, and the dividing property does not hold. Assume that n is odd, the curve X is of type I, the real point set $\mathbb{R}X$ is non-singular and has a zigzag z delimitated by vertices u_1 and u_2 in D_X.

Denote by $s = z \setminus \{u_1, u_2\}$ the interior of the zigzag. We say that a simple $\times$-vertex in D_X is *on the same side* as the zigzag z if the number of white real vertices in the connected components of $\partial D_X \setminus (s \cup \{v\})$ is even. We say that an oval is *on the same side* as the zigzag z if its delimiting vertices are on the same side of the zigzag z. We denote by $\Lambda_z^+(X)$ (resp., $\Lambda_z^-(X)$) the number of positive (resp., negative) ovals on the same side as the zigzag z.

Theorem 3.3.6. *Let X be a type* I *nodal proper trigonal curve in the Hirzebruch surface $\Sigma_n \xrightarrow{\pi} \mathbb{CP}^1$, $n = 2k+1$, and let D_X be its associated toile. Assume that the real point set $\mathbb{R}X$ has a zigzag z. Then,*

$$2(\Lambda_z^-(X) - \Lambda_z^+(X)) + l + \sigma(X) = 2n + 2,$$

where l is the number of connected components in the real point set of the type I *perturbation of X. Furthermore, we have*

$$\Lambda^-(X) - \Lambda^+(X) + (l-1) + \sigma(X) = 2n.$$

Proof. The proof follows the standard scheme of the proof of Rokhlin's formulas of complex orientations. Considering instead of the curve X its type I perturbation, we can assume that the real point set $\mathbb{R}X$ of X is non-singular. Since X is a type I curve, the set $\mathbb{C}X \setminus \mathbb{R}X$ has two connected components. Choose one of the connected components of $\mathbb{C}X \setminus \mathbb{R}X$ and denote by $\mathbb{C}X_+ \subset \mathbb{C}X$ its closure. The orientation of $\mathbb{C}X_+$ induces an orientation of $\mathbb{R}X$. Then, the long component L of $\mathbb{R}X$ produces *via* π an orientation of $\mathbb{RP}^1$. Let $\mathbb{C}E_+$ be the closed hemisphere of the exceptional section $E \simeq \mathbb{CP}^1$ such that the induced orientation of $\mathbb{R}E$ produces *via* π the orientation of $\mathbb{RP}^1$ opposite to the one produced by the long component of $\mathbb{R}X$. Pick a white vertex v belonging to the zigzag z and consider the corresponding fiber $F = \pi^{-1}(v)$. Let $\mathbb{C}F_+$ be the closed hemisphere of $F \simeq \mathbb{CP}^1$ such that $L \cup \mathbb{R}E \cup \mathbb{R}F$ endowed with the described orientations of L and $\mathbb{R}E$ has a type I perturbation with a unique connected component. Note that if we equip $\mathbb{R}F$ with the opposite orientation, the set $L \cup \mathbb{R}E \cup \mathbb{R}F$ has a type I perturbation with three connected components.

Let Y_+ be the type I perturbation of $\mathbb{C}X_+ \cup \mathbb{C}E_+ \cup \mathbb{C}F_+$. Then, the boundary ∂Y_+ is a collection I of contractible circles in $\mathbb{R}\Sigma_n$. Put

$$w_+ = [Y_+ \cup \bigcup_{i \in I} U_i] \in H_2(\Sigma_n),$$
$$w_- = [Y_- \cup \bigcup_{i \in I} \overline{U}_i] \in H_2(\Sigma_n),$$

where $U_i \subset \mathbb{R}\Sigma_n$, $i \in I$, are the disks whose oriented boundaries coincide with the connected components of ∂Y_+, the set Y_- is $\mathrm{conj}(Y_+)$, and $\overline{U_i}$ is the disk U_i with the opposite orientation.

In order to calculate $w_+ \cdot w_-$, let us notice that $w_+ + w_- = [X] + [E] + [F]$ and $\mathrm{conj}_*(w_+) = -w_-$. Since conj_* acts in $H_2(\Sigma_n)$ as multiplication by -1, we have $w_+ + w_- = w_+ - \mathrm{conj}_*(w_+) = 2w_+$. Hence, $w_+ = w_- = \frac{1}{2}([X] + [E] + [F])$.

Since $X \subset \Sigma_n$ is a proper trigonal curve, we have $[X] = 3[E] + 3n[F]$. Then,

$$\begin{aligned} w_+ \cdot w_- &= \left(\frac{[X] + [E] + [F]}{2}\right)^2 \\ &= \left(\frac{4[E] + (3n+1)[F]}{2}\right)^2 \\ &= -4n + 2(3n+1) \\ &= 2n + 2. \end{aligned}$$

On the other hand, we can calculate the product $w_+ \cdot w_-$ in a geometrical way. We proceed using the arguments proposed in [12]. Choose a smooth tangent vector field V on $\mathbb{R}\Sigma_n$ such that the vector field has only nondegenerate singular points, the singular points are outside of ∂Y_+, and on ∂Y_+ the field is tangent to ∂Y_+ and directed according to the orientation induced from Y_+. Extend smoothly the vector field to Y_+ such that it is supported on a tubular neighborhood of ∂Y_+. Shift ∂Y_+ inside Y_+ along $\varepsilon\sqrt{-1}V$, where ε is a sufficiently small positive real number, and extend this shift to a shift of the disks D_i, $i \in I$, along $\varepsilon\sqrt{-1}V$. Denote by $\widetilde{Y}$ the result of the shift of $Y_+ \cup \bigcup_{i\in I} D_i$. By continuity of the shift, we have that $[\widetilde{Y}] = w_+$. Then, we can calculate $w_+ \cdot w_-$ as the intersection of the cycles $\widetilde{Y}$ and $Y_- \cup \bigcup_{i\in I} \overline{D}_i$, which intersect at the singular points of V and the complex nodes of X that contribute to $\sigma(X)$. At a singular point x they are smooth transversal two-dimensional submanifolds, each taken with multiplicity $-i_{\mathbb{R}A}(x)$. The local intersection number at x is equal to $(i_{\mathbb{R}A}(x))^2$ multiplied by the local intersection number of the submanifolds supporting the cycles. The latter is equal to the index of the vector field V at x multiplied by -1 due to the fact that multiplication by $\sqrt{-1}$ induces an isomorphism between the tangent and the normal fibrations of $\mathbb{R}A$ in $\mathbb{C}A$ reversing orientation. Calculating the local intersection numbers we have that every disk U_i, $i \in I$, contributes 1. For every positive (resp., negative) injective pair the contribution is -2 (resp., 2). The cardinality of I is the number l of connected components of $\mathbb{R}X$. The number of positive (resp., negative) injective pairs in ∂Y_+ correspond to the number $\Lambda_z^+(X)$ (resp., $\Lambda_z^-(X)$) of positive (resp., negative) ovals of X on the same side as the zigzag z. Therefore,

$$2(\Lambda_z^-(X) - \Lambda_z^+(X)) + l + \sigma(X) = 2n + 2. \tag{3.1}$$

Now, let us consider the other closed hemisphere $\mathbb{C}F_-$ of F. Let Z_+ be the type I perturbation of $\mathbb{C}X_+ \cup \mathbb{C}E_+ \cup \mathbb{C}F_-$. Then, the boundary ∂Z_+ is a collection J of contractible circles in $\mathbb{R}\Sigma_n$. Put

$$\begin{aligned} u_+ &= [Z_+ \cup \bigcup_{i\in J} U_i'] \in H_2(\Sigma_n), \\ u_- &= [J_- \cup \bigcup_{i\in J} \overline{U}_i'] \in H_2(\Sigma_n), \end{aligned}$$

where $U_i' \subset \mathbb{R}\Sigma_n$, $i \in J$, are the disks whose oriented boundaries coincide with the connected components of ∂Z_+, the set Z_- is $\mathrm{conj}(Z_+)$, and $\overline{U_i'}$ is the disk U_i' with the opposite orientation.

In order to calculate the homological product $u_+ \cdot u_-$ we proceed as before. We have that $u_+ + u_- = [X] + [E] + [F]$ and $\mathrm{conj}_*(w_+) = -w_-$, which lead us to the fact that $u_+ = u_- = \frac{1}{2}([X] + [E] + [F])$. Therefore, we have $u_+ \cdot u_- = w_+ \cdot w_- = 2n + 2$.

The product $u_+ \cdot u_-$ can be calculated geometrically exactly in the same way as above. The cardinality of J is $l+2$. The number of positive (resp., negative) injective pairs in ∂Z_+ correspond to the number $\Lambda^+(X) - \Lambda_z^+(X)$ (resp., $\Lambda^-(X) - \Lambda_z^-(X)$) of positive (resp., negative) ovals of X which are not on the same side as the zigzag z. Therefore,

$$2[(\Lambda^-(X) - \Lambda_z^+(X)) - (\Lambda^-(X) - \Lambda_z^-(X))] + l + 2 + \sigma(X) = 2n + 2. \tag{3.2}$$

Hence, from Equations (3.1) and (3.2) we obtain that

$$\Lambda^-(X) - \Lambda^+(X) + (l - 1) + \sigma(X) = 2n. \quad \square$$

Chapter 4

Real pointed quartic curves

Let A be a curve of degree 4 in $\mathbb{RP}^2$, and let $p \in \mathbb{R}A$ be a point. Consider the geometrically ruled surface $\Sigma_1 \cong \mathrm{Bl}_p(\mathbb{CP}^2) \longrightarrow \mathbb{CP}^1$ which is the blow-up at the point p of the complex projective plane. Assume that $p \in A$ is smooth. Then, the strict transform of $\mathbb{C}A \subset \mathbb{CP}^2$ is a trigonal curve $C \subset \Sigma_1$. The intersection of the exceptional divisor E with the strict transform C consists of one point. Geometrically, it represents the tangent line to A at p. Performing the Nagata transformation on the intersection point $q \in E \cap C \subset \Sigma_1$ transforms C into a proper trigonal curve $\widetilde{C} \subset \Sigma_2$. Since the fiber of $C \subset \Sigma_1 \longrightarrow \mathbb{CP}^1$ passing through q intersects C in two other points, the trigonal curve $\widetilde{C}$ has a node. We mark this node in order to obtain a minimal proper model for (A, p).

We are interested in the moduli space of real pointed quartic curves. Therefore, we impose genericity conditions on the couple (A, p), namely we assume the curve A to be smooth and we assume the point p to be a point of A such that the tangent line of A at p intersects A in two other distinct points. Equivalently, the tangent line of A at p is not a bitangent of A and p is not an inflection point of A. This condition is based on the genericity condition for the Weierstraß coefficients sections presented in Definition 2.0.5 within trigonal curves with a fixed marked nodal point.

Proposition 4.0.1 (Classification of degree 6 uninodal toiles)**.** *The weak equivalence class of a degree 6 uninodal toile is determined by its number of ovals, its type (w.r.t. complex orientations) and the nature of the segment containing the only singular $\times$-vertex, as shown in Tables 4.1 to 4.5.*

Proof. Due to Proposition 2.1.3, a uninodal toile of degree 6 is the gluing of two cubics *via* a dotted cut or an axe. Let v be the nodal $\times$-vertex. If v is a non-isolated nodal $\times$-vertex, let S be the dotted segment containing v and let S_1, S_2 be the connected components of $S \setminus \{v\}$. Up to weak equivalence, the set of parities of the numbers of white vertices in S_1, S_2 is an invariant of the weak equivalence class. Enumerating all weak equivalence classes we obtain that every class is determined by its number of ovals, its type (w.r.t. complex orientations) and the set of parities of the numbers of white vertices in S_1, S_2 as shown in the Tables 4.1 to 4.5. □

Theorem 4.0.2 (Classification of generic real pointed quartic curves)**.** *There is a one-to-one correspondence between the weak equivalence classes of degree 6 uninodal toiles and the chambers of generic real pointed quartic curves in their moduli space, given by the figures in Tables 4.1 to 4.5.*

Proof. This is a consequence of Theorem 1.3.4 and the fact that creating/destroying a zigzag are deformations that remain within the rigid isotopy class. Every weak equivalence

class found in Proposition 4.0.1 corresponds to a rigid isotopy class of generic real pointed quartic curves. □

Table 4.1: $b_0(\mathbb{R}C) = 1$.

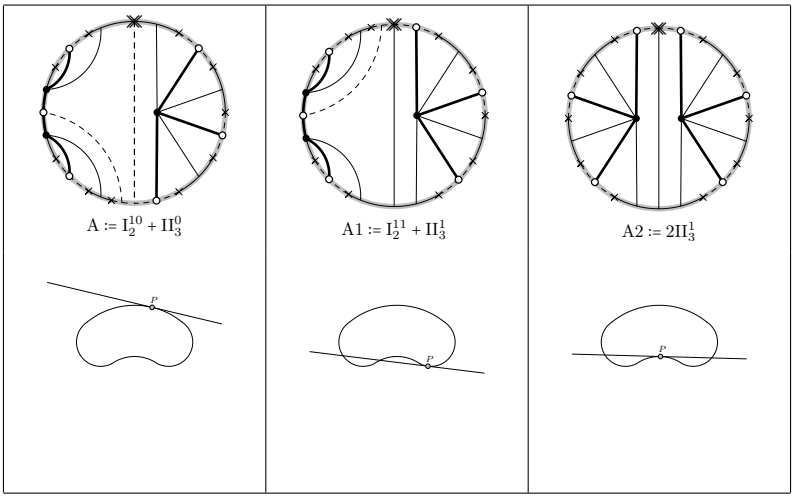


Table 4.2: $b_0(\mathbb{R}C) = 2$, type II.

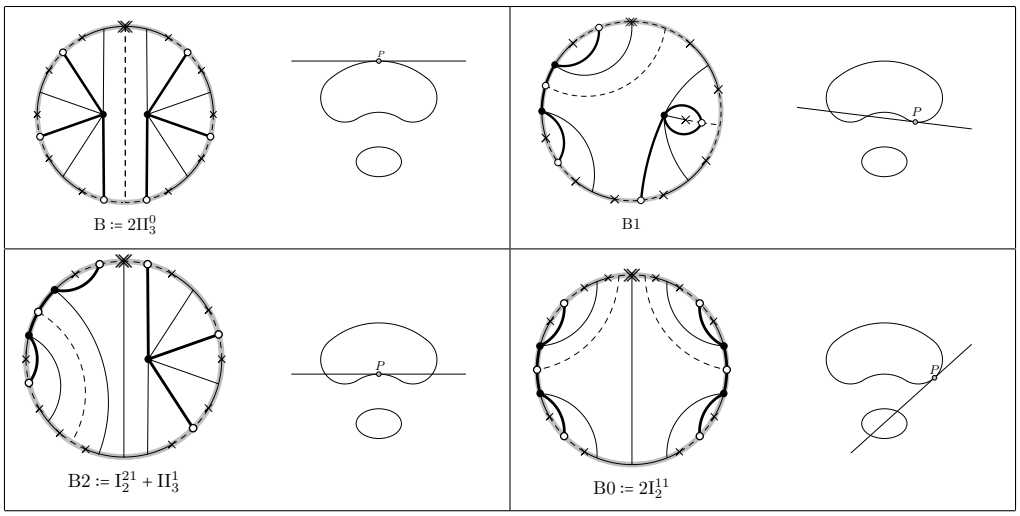


Table 4.3: $b_0(\mathbb{R}C) = 3$.

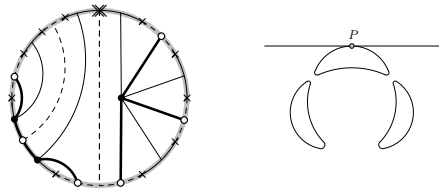
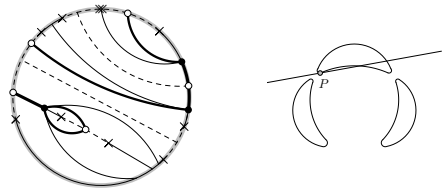
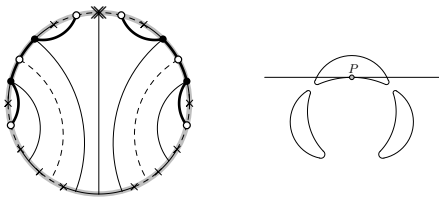
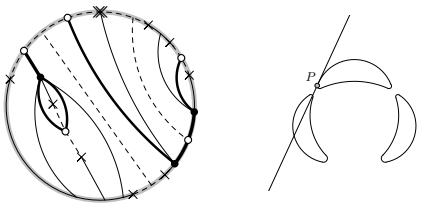
 C := $\mathrm{I}_2^{20} + \mathrm{II}_3^0$	 C1
 C2 := $2\mathrm{I}_2^{21}$	 C0

Table 4.4: $b_0(\mathbb{R}C) = 2$, type I.

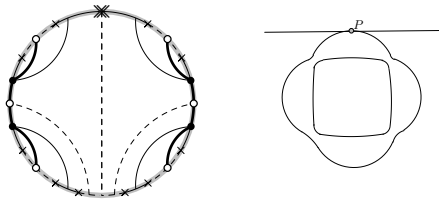
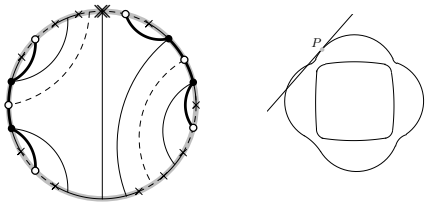
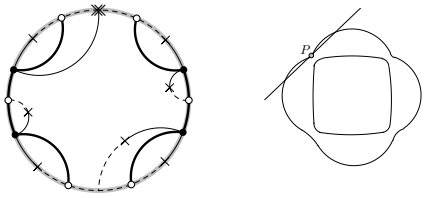
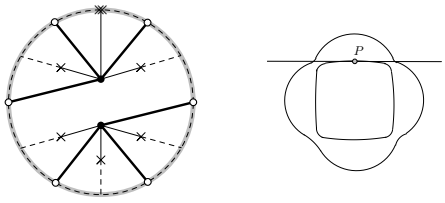
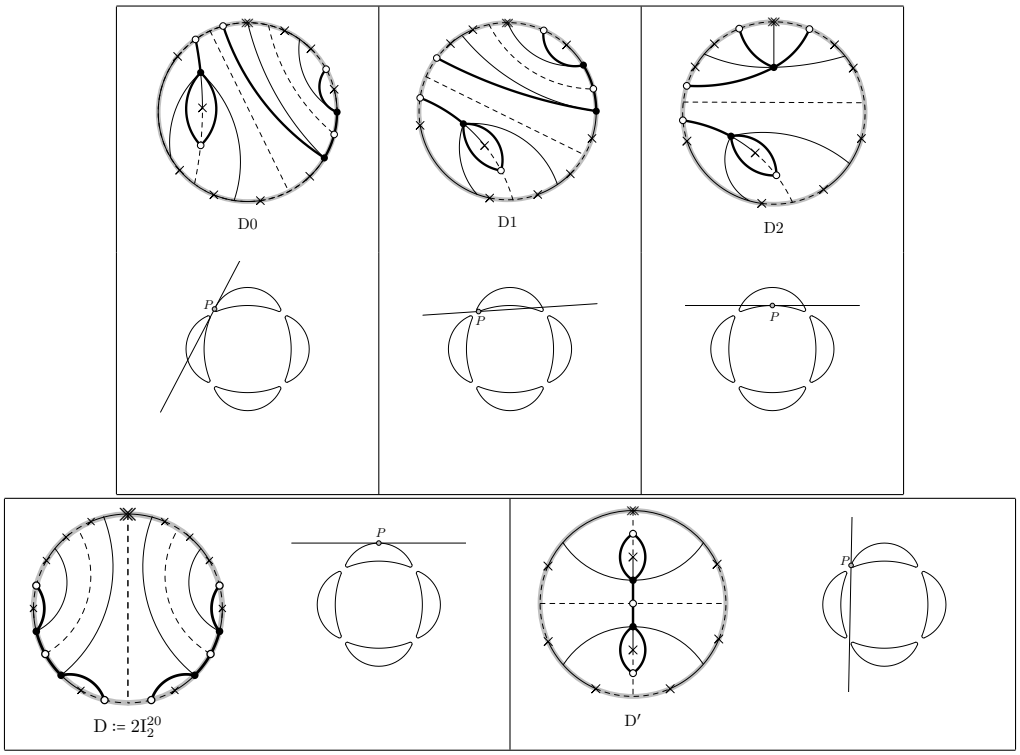
 BH := $2\mathrm{I}_2^{10}$	 BH1 := $\mathrm{I}_2^{11} + \mathrm{I}_2^{21}$
 BH2	 H2

Table 4.5: $b_0(\mathbb{R}C) = 4$.



Chapter 5

Real quintic rational curves

Let X be a real rational plane curve of degree 5. A generic rational plane curve has only nodal singular points. A double point of X is called *elliptic* (resp., *hyperbolic*) if it is given in local coordinates (x, y) by the equation $x^2 + y^2 = 0$ (resp., $x^2 - y^2 = 0$). A double point of X is called *imaginary* if it belongs to $\mathbb{C}X \setminus \mathbb{R}X$. Since X is real, the imaginary nodal points come in pairs of complex conjugated points. Topologically, the real point set $\mathbb{R}X \subset \mathbb{R}P^2$ of a generic real rational curve X is the disjoint union of a circle generically immersed in $\mathbb{RP}^2$ and a finite set of elliptic nodes (cf. [8]).

Let us denote by e the number of elliptic nodal points and by h the number of hyperbolic nodal points of the curve X. Then, we have $b_0(\mathbb{R}X) = 1 + e$ and $b_1(\mathbb{R}X) = 1 + h$. For the complex point set of a generic rational curve, the Betti numbers depend only on the degree of the curve, namely $b_0(\mathbb{C}X) = 1 = b_2(\mathbb{C}X)$ and $b_1(\mathbb{C}X) = \frac{(d-1)(d-2)}{2}$. Since X is of degree 5, applying the Smith-Thom theorem we obtain that $b_*(\mathbb{R}X) = 2 + e + h \leq b_*(\mathbb{C}X) = 2 + 6$. By the Smith congruence, the number of real nodal points of the curve X is an even number less or equal than 6.

Assume that the curve X has at least one real nodal point $p \in \mathbb{R}X$. Such a pair is called a *marked real rational plane curve*. Since the point p has multiplicity $\deg(X) - 3$, after blowing up the point $p \in \mathbb{CP}^2$, the pencil of lines passing through p determines a trigonal curve $C_1 \subset \Sigma_1$. The curve C_1 intersects the exceptional divisor in two different points T_1 and T_2, each one corresponding to a tangent line of the curve X at the point p in one of the crossing branches. If the curve is generic, each tangent line T_i intersects $X \setminus \{p\}$, the complement of the point p, in two different points. Then, applying positive Nagata transformations at the points T_i produce a proper trigonal curve $C_X \subset \Sigma_3$ with two additional nodal points n_{T_1} and n_{T_2}.

If the nodal point $p \in \mathbb{R}X$ is hyperbolic, the tangent lines to the curve X at the point p are real, and they transform into real nodes. Since the point p is nodal, a real tangent T_i intersects the curve X at the point p with multiplicity 3. If the line $\mathbb{R}T_i$ intersects $\mathbb{R}X \setminus \{p\}$ in two different real points, the corresponding nodal point $n_{T_i} \in C_X$ is a hyperbolic nodal point. Otherwise, the line T_i intersects $\mathbb{C}X$ at two non-real points and the corresponding nodal point n_{T_i} is elliptic.

If the base nodal point $p \in \mathbb{R}X$ is elliptic, the tangent lines to the curve X at the point p are a couple of complex conjugated lines T and $\overline{T}$, each one intersecting $\mathbb{C}X \setminus \mathbb{R}X$ in two different points. Hence, the corresponding nodal points n_T and $n_{\overline{T}}$ are a pair of complex conjugated nodal points and their associated vertex v_T is an inner nodal $\times$-vertex.

Definition 5.0.1. Given a marked real rational plane curve (X, p) of degree 5, we call the *marked* dessin of (X, p) the toile associated to the real proper trigonal curve $C_X \subset \Sigma_3$

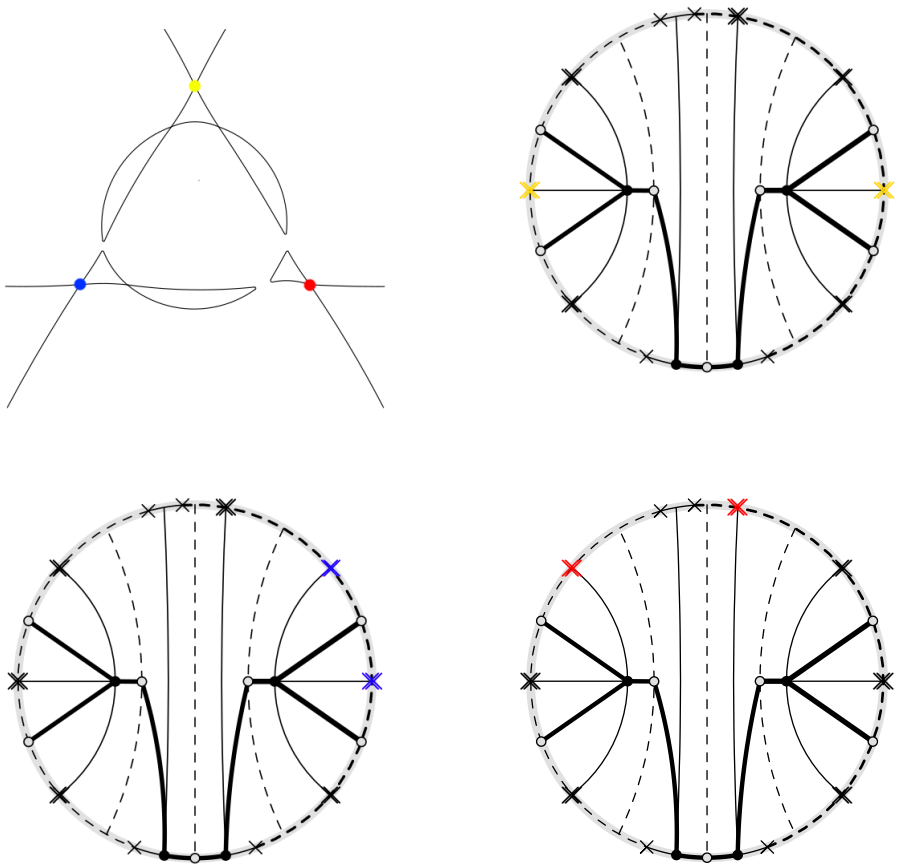


Figure 5.1: Example of a real nodal rational curve with three different markings and their associated marked dessin.

endowed with two nodal $\times$-vertices v_{T_1}, v_{T_2} corresponding to the nodal points n_{T_1} and n_{T_2}, respectively.

On the other hand, given a marked degree 9 toile (D, v_{T_1}, v_{T_2}), performing negative Nagata transformations of the associated trigonal curve $C_X \subset \Sigma_3$ at the nodes corresponding to the vertices v_{T_1} and v_{T_2} followed by the blow-down of the exceptional section of Σ_1, gives rise to a real degree 5 plane curve X_D endowed with a nodal point $p \in \mathbb{R}X_D$.

Definition 5.0.2. Let $D \subset S$ be a real dessin. Let us assume there is a subset of S in which D has a configuration of vertices and edges as in Figure 5.2. Replacing this configuration with the alternative one defines another dessin $D' \subset S$. We say that D' is obtained from D by *passing a nodal point through an inflexion point.* Two dessins D, D' are *very weakly equivalent* if there exists a finite sequence of dessins $D = D_0, D_1, \ldots, D_n = D'$ such that D_{i+1} is either weakly equivalent to D_i or it is obtained from D_i by passing a nodal point through an inflexion point.

Definition 5.0.3. Let $D \subset S$ be a real dessin and let v_T a marked nodal $\times$-vertex of D. Assume there is a neighborhood of v_T in which D has a configuration of vertices and edges

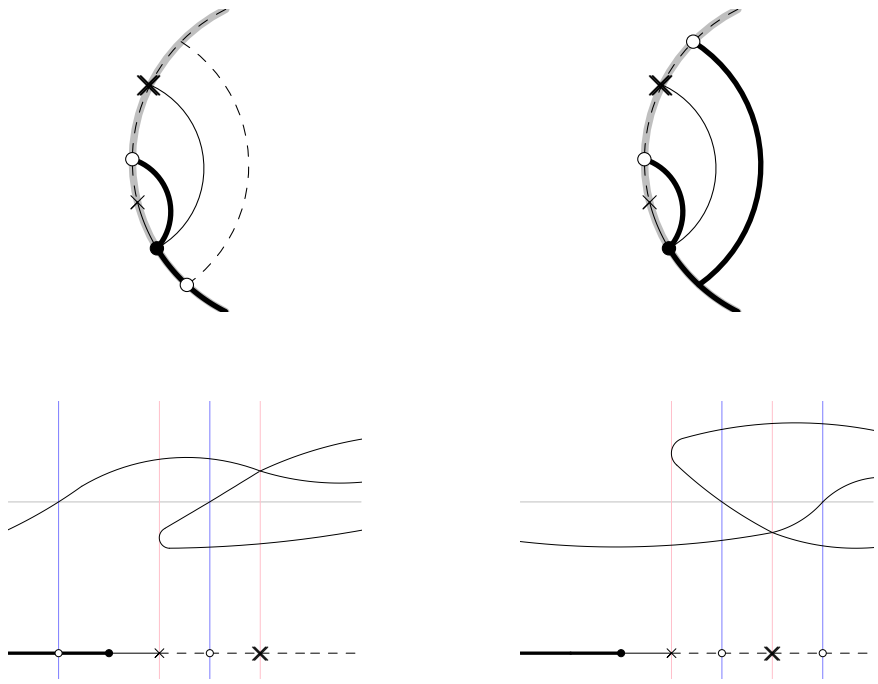


Figure 5.2: Passing a nodal point through an inflexion point.

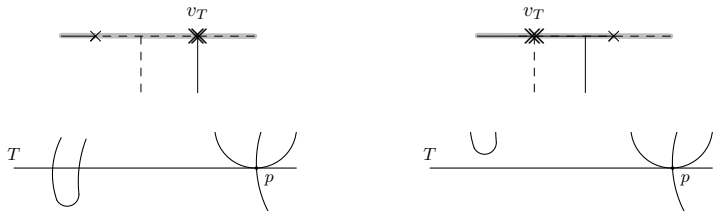
as one in Figure 5.3. Then, replacing this configuration with the corresponding alternative one defines another dessin $D' \subset S$. We say that D' is obtained from D by *passing a singular fiber through a tangent line*. Two marked dessins (D, v_{T_1}, v_{T_2}), (D', v'_{T_1}, v'_{T_2}) are *equivalent* if there exists a finite sequence of dessins $D = D_0, D_1, \ldots, D_n = D'$ such that D_{i+1} is either very weakly equivalent to D_i or it is obtained from D_i by passing a singular fiber through a tangent line.

In Figure 5.3 we have the possible equisingular deformations of the curve when the tangent line T is a bitangent of X or when the tangent line T intersects one additional nodal point (hyperbolic or elliptic). When the base point $p \in X$ is an inflection point of one branch of $\mathbb{R}X$, passing a marked nodal point through the inflexion point p defines a relation corresponding to the equisingular perturbations of the curve, as shown in Figure 5.4.

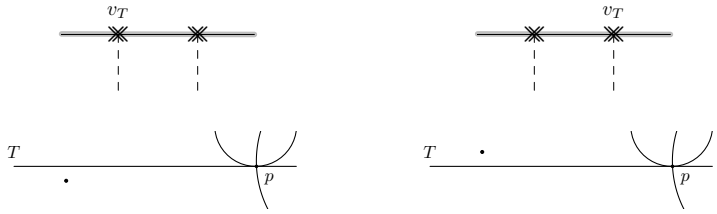
Theorem 5.0.4. *There is a one-to-one correspondence between the set of rigid isotopy classes of marked real nodal rational degree* 5 *curves and the set of equivalence classes of rational degree* 9 *marked toiles.*

Proof. This is a consequence of Theorem 1.3.4 and the fact that passing a nodal point through an inflexion point and passing a singular fiber through a tangent line are equivariant equisingular deformations that do not change the rigid isotopy class. □

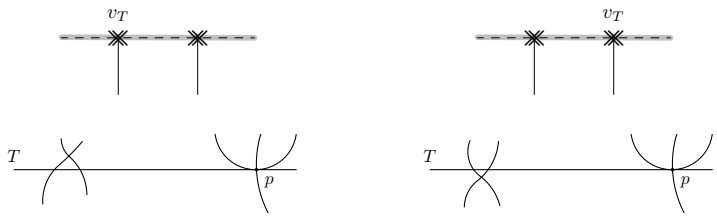
Definition 5.0.5. Given a nodal type I dessin D, we call the type I *perturbation* of D the dessin obtained by perturbing every non-marked isolated nodal $\times$-vertex into an oval and every non-isolated nodal vertex either into an inner simple $\times$-vertex if its labeling is 1 or into two real simple $\times$-vertices if its label is $\overline{1}$.



(a) Passing a simple tangent through the tangent line T.



(b) Passing an elliptic nodal point of the curve through the tangent line T.



(c) Passing a hyperbolic nodal point of the curve through the tangent line T.

Figure 5.3: Passing a singular fiber through a tangent line.

For an irreducible type I real nodal plane curve X, its type I perturbation consists of a curve having a non-singular real part $\mathbb{R}X$ where every elliptic node has been perturbed to an oval and every hyperbolic node has been perturbed according to a complex orientation of the crossing branches. Perturbing the real point set of the curve in any other way does not produce a type I curve. Moreover, if the curve X is an M-curve, its type I perturbation is non-singular.

5.1 Maximally perturbable curves

Definition 5.1.1. A real algebraic curve C in $\mathbb{R}P^2$ is *maximally perturbable* if there is a real perturbation C_0 of C such that C_0 is an M-curve.

Recall that an M-quintic has six ovals in a convex position: they form a hexagon. If we fix a point p at the interior of an oval, the pencil of lines passing through p induces an order on the ovals. Varying the fix point among the ovals, we can assign to every oval two neighboring ones, defining a reversible cyclic order on them. If a type I maximally perturbable curve C has an elliptic nodal point, the oval produced by it in a type I

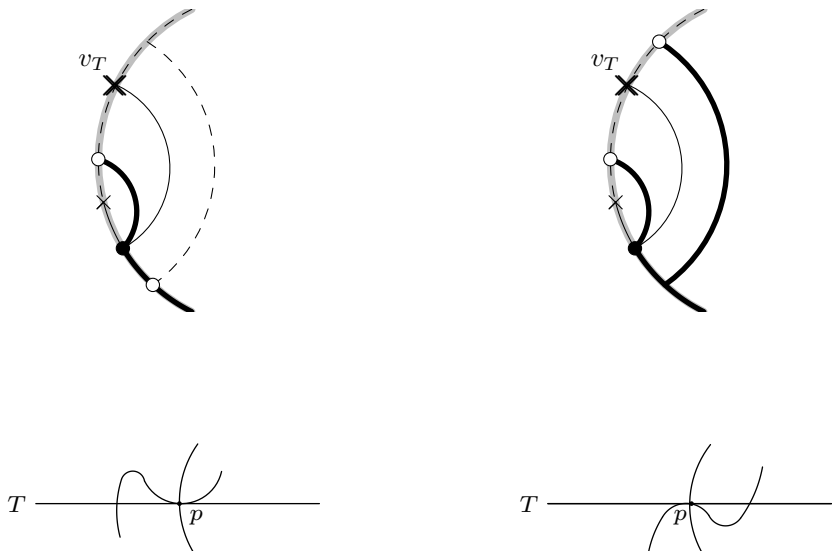


Figure 5.4: Passing a marked nodal point through an inflexion point.

perturbation of C must respect its relative position with respect to its neighboring ovals, and so does the elliptic nodal point of the original curve (see Figure 5.5).

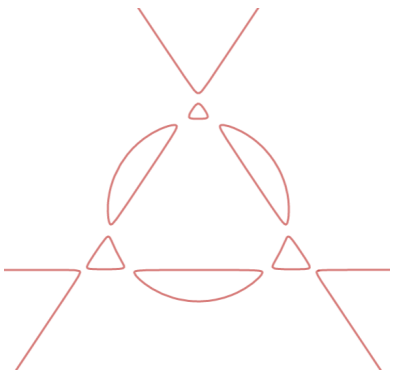


Figure 5.5: The rigid isotopy class of non-singular M-quintic curves.

Lemma 5.1.2. *If a real nodal rational plane curve is maximally perturbable, then it is maximal.*

Proof. Let C be a real nodal rational plane curve of degre d. In the type I perturbation $\widetilde{C}$ of the curve C every oval comes either from an elliptic nodal point or from an oval attached by a hyperbolic nodal point to a chain of ovals, which in the case when d is odd, is subsequently attached to the pseudoline. Therefore, the number of ovals l in $\widetilde{C}$ is less than of equal than the sum $e+h+1$. Since $\widetilde{C}$ is maximal, it has $\frac{(d-1)(d-2)}{2}+1$ connected components of the real point set. Thus, the total Betti number

$$b_*(\mathbb{R}C) = 2+e+h \geq 2+\frac{(d-1)(d-2)}{2} = b_*(\mathbb{C}C).$$

By the Smith-Thom inequality these quantities are in fact equal and the curve C is maximal. □

Corollary 5.1.3. *Let C be a maximal nodal rational curve of degree 5. Assume C has a hyperbolic nodal point $p \in \mathbb{R}C$. Then, the corresponding proper trigonal curve $\overline{C} \subset \Sigma_3$ is maximal.*

Proof. Since C is a maximal nodal rational curve of degree 5, we have that $e + h = 6$. Let e' and h' the numbers of elliptic and hyperbolic nodal points of the curve $\mathbb{R}\overline{C}$. Due to the fact that the base point p is hyperbolic, the nodal points n_{T_1} and n_{T_2} corresponding to the tangent lines of C at p are real. The base point p disappears on the strict transform, hence $e' + h' = e + (h - 1) + 2 = 7$. Since $\overline{C}$ is obtained from C by birational transformations, it is rational. Thus, the total Betti number of the real point set $b_*(\mathbb{R}C) = 2 + e' + h' = 9$. On the other hand, since a non-singular trigonal curve lying on Σ_3 has genus 7, the total Betti number of the complex point set $b_*(\mathbb{C}C)$ is equal to 9. □

Lemma 5.1.4. *Let C be a maximally perturbable real nodal rational curve of degree 5. Assume that C has a hyperbolic nodal point $p \in \mathbb{R}C$ belonging to the pseudoline of a type* I *perturbation of C. Then, the type* I *perturbation of the marked toile (D_C, v_{T_1}, v_{T_2}) associated to (C, p) is a non-singular maximal toile.*

Proof. Let us consider the curve C_0 obtained by a type I perturbation of every nodal point of C except for the base point p. The curve C_0 has five ovals and such ovals appears in the type I perturbation of the marked toile (D_C, v_{T_1}, v_{T_2}) associated to (C, p). If one of the tangent T_i produces an isolated nodal $\times$-vertex v_{T_i}, it is perturbed to an oval. Since the number of ovals of a type I curve has the same parity as a maximal curve, the number of ovals of (D_C, v_{T_1}, v_{T_2}) is seven. Otherwise the tangent lines T_1 and T_2 produce non-isolated nodal $\times$-vertices. Since the original trigonal curve $\overline{C} \subset \Sigma_3$ is rational, the hyperbolic nodes n_{T_1} and n_{T_2} belong to the long component. Topologically, the long component is a circle with two simple crossings, whose type I perturbation has 3 components. Therefore, the type I perturbation of the marked toile (D_C, v_{T_1}, v_{T_2}) has one long component and seven ovals. Since all singularities of the original curve C are real and the base point p is real, the toile D_C has no inner singular vertices. □

Definition 5.1.5. Given a type I marked dessin (D, v_1, v_2), we call an oval $\mathcal{O}$ of its type I perturbation a *true oval* if none of the vertices linking $\mathcal{O}$ to the pseudoline or other ovals is marked.

Corollary 5.1.6. *Let C be a nodal real rational curve of degree 5 whose type* I *perturbation has l ovals. Assume that C has a hyperbolic nodal point $p \in \mathbb{R}C$ belonging to the pseudoline of the type* I *perturbation of C. Then, the type* I *perturbation of the marked toile (D_C, v_{T_1}, v_{T_2}) associated to (C, p) has $l + 1$ ovals, among which $l - 1$ are true ovals.*

Proof. As in the proof of Lemma 5.1.4, there are $l - 1$ ovals in the type I perturbation of the marked dessin D_C arising from ovals of the the curve C_0 obtained by realizing a type I perturbation of every nodal point of C except for the base point p. The remaining two ovals come either from an isolated nodal point v_i or from a non-isolated nodal point v_i representing the self-intersection of the long component. □

Due to the previous lemma, we use the classification of smooth trigonal M-curves. We state a version of the theorem suitable for the content of this work. The theorem can be found in full statement in [3].

Theorem 5.1.7 ([3]). *Any smooth real trigonal M-curve $X \subset \Sigma_3$ has a corresponding dessin $D \subset \mathbb{D}^2$ which has a canonical decomposition as the gluing of three cubic dessins of type* I *along dotted generalized cuts corresponding to zigzags on the cubics.*

The previous theorem states that any non-singular maximal toile D of degree 9 is decomposed as the gluing of three type I cubics A, B and A' along edges belonging to a zigzag. If the decomposition takes the form $A - B - A'$, where $D_1 - D_2$ indicates that the dessin D_1 is glued to the dessin D_2, then the cubic B is the cubic I_2 and the cubics A and A' are either I_1 or I_2.

Lemma 5.1.8. *Let (D_C, v_{T_1}, v_{T_2}) be the marked toile associated to a maximally perturbable nodal rational curve C of degree 5 with a hyperbolic nodal point $p \in \mathbb{R}C$ belonging to the pseudoline. Then, the marked toile (D_C, v_{T_1}, v_{T_2}) is equivalent to a marked toile (D', v_1', v_2') whose type* I *perturbation is the gluing of cubics $\mathrm{I}_1 - \mathrm{I}_2 - \mathrm{I}_1$.*

Proof. Due to Lemma 5.1.4, the type I perturbation of the toile D_C is $A - \mathrm{I}_2 - A'$ where A and A' are either I_1 or I_2. If the cubic A equals I_2, let Z be the zigzag other than the one where the gluing takes place. If the zigzag Z comes from a zigzag belonging to the toile D_C, then the destruction of the zigzag Z followed by an elementary move of type $\bullet$-in produces a marked toile whose type I perturbation has a cubic I_1 in the place of A. If the zigzag Z comes from perturbing a non-isolated nodal $\times$-vertex v, then passing a nodal point through an inflexion point followed by an elementary move of type $\bullet$-in produces a marked toile whose type I perturbation has a cubic I_1 in the place of A. A symmetric argument applied to the cubic A' proves the statement. □

Henceforth, let v_e and v_h be the numbers of isolated and non-isolated marked nodal $\times$-vertices of a marked dessin (D, v_1, v_2).

Proposition 5.1.9. *Let C be a maximally perturbable nodal rational curve of degree 5 in $\mathbb{RP}^2$ such that the curve C has at least one hyperbolic nodal point. Then, the rigid isotopy class of C is determined by its isotopy class and the relative position of its elliptic nodes. Namely, the curve C belongs to one of the rigid isotopy classes represented in Figures 5.6 to 5.11.*

Proof. Since C is a maximally perturbable nodal rational curve, by Lemma 5.1.2 it is maximal, hence $e + h = 6$. Since C is a rational curve with hyperbolic nodal points, we can choose $p \in \mathbb{R}C$ a hyperbolic nodal point of C belonging to the pseudoline. Then, by Lemma 5.1.8, the marked toile (D_C, v_{T_1}, v_{T_2}) is equivalent to a marked toile (D', v_1', v_2') whose type I perturbation is the gluing of cubics $\mathrm{I}_1 - \mathrm{I}_2 - \mathrm{I}_1$, with associated trigonal curve C_M. In the real point set of the trigonal curve $\mathbb{R}C_M$ there is a set of ovals and possible hyperbolic nodes that can be created indicated from the dessin. We search to enumerate all equivalence classes of marked dessin (D', v_1', v_2'). For that, let us choose $e+v_e$ ovals of the curve C_M to contract, producing that amount of isolated nodal $\times$-vertices among which v_e are marked. Then, all the remaining ovals must be attached among them or to the long component in a way that there is only one irreducible component with a number of non-isolated $\times$-vertices equals to $h + v_h - 1$, among which v_h are marked. Then, enumerating all equivalence classes of marked dessins satisfying the aforementioned restrictions and realizing the birational transformations in order to recover the associated curves $\mathbb{R}C \subset \mathbb{RP}^2$ lead to the plane curves shown on Figures 5.6 to 5.11. □

Proposition 5.1.10. *Let C be a maximally perturbable nodal rational curve of degree 5 in $\mathbb{RP}^2$ such that the curve C has no hyperbolic nodal points. Then, there is only one rigid isotopy class of such a curve, represented in Figure 5.12a.*

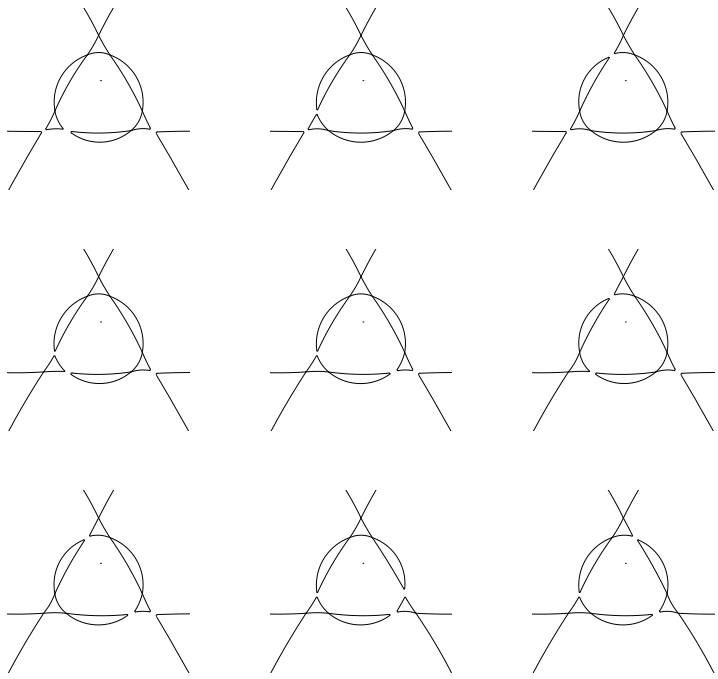


Figure 5.6: Rigid isotopy classes of maximally perturbable nodal rational curves of degree 5 in $\mathbb{RP}^2$ without isolated nodes.

Proof. Since C is a maximally perturbable nodal rational curve, by Lemma 5.1.2 it is maximal, hence $e = 6$. Let $p \in \mathbb{R}C$ an elliptic nodal point of the curve C and let (D_C, v_1, v_2) be the marked toile associated to (C, p). Since the base point p is elliptic, the nodes n_T and $n_{\overline{T}}$ corresponding to the tangent lines of C at p are a pair or complex conjugated nodal points. Therefore, the markings v_1 and v_2 are an inner nodal $\times$-vertices of D_C. The toile D_C has 5 isolated nodal $\times$-vertices corresponding to the elliptic nodal points different from p. Due to Proposition 2.2.5, since the toile D_C has only isolated nodal $\times$-vertices it can be decomposed as the gluing of three cubic toiles by dotted generalized cut or by a solid axe. Since the toile D_C is of type I, so are the glued cubic and the corresponding generalized cuts must be coherent with the labeling of the regions. Then, after enumerating the equivalence classes of marked toiles with the aforementioned restrictions, it turns out that there are two equivalence classes having representatives D_+ and D_-, respectively (see Figure 5.12a). The toile D_+ has a type I perturbation having 3 negative ovals and 2 positive ovals. Moreover, the toile D_+ has a type I non-singular perturbation. The toile D_- has a type I perturbation having 2 negative ovals and 3 positive ovals. Therefore, we can conclude that the toile D_+ (resp., D_-) corresponds to a pair (C, p) such that the elliptic nodal point p can be perturbed to a positive (resp., negative) oval. Hence, there is only one class of rigid isotopy of the curve C. □

Theorem 5.1.11 (Rigid isotopy classification of maximally perturbable real quintic rational curves)**.** *If C be a maximally perturbable real nodal rational curve of degree 5 in $\mathbb{RP}^2$, then the rigid isotopy class of C is determined by its isotopy class and the position of its isolated nodes.*

Proof. The case when $h \geq 1$ is considered in Proposition 5.1.9 and the case when $h = 0$ is considered in Proposition 5.1.10. □

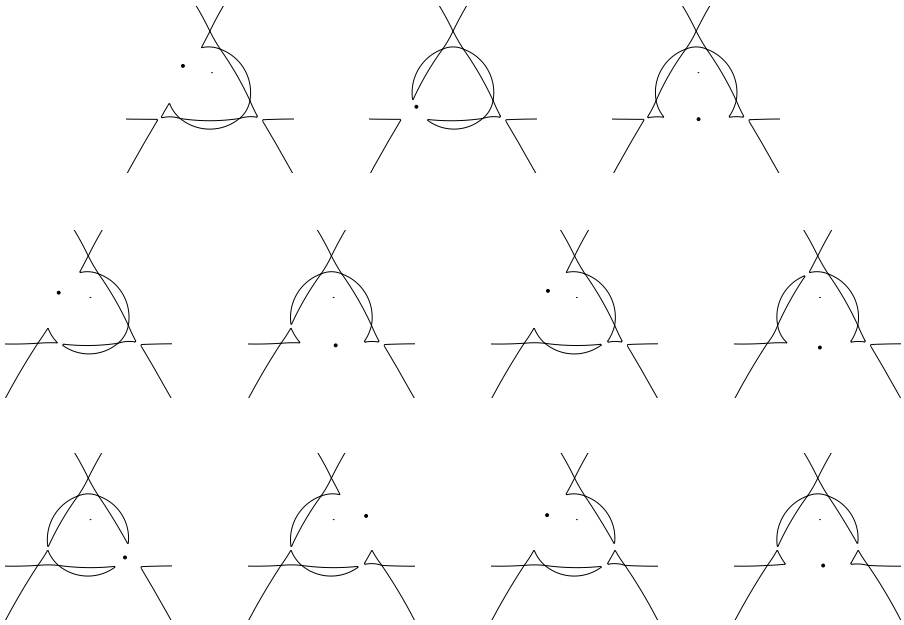


Figure 5.7: Rigid isotopy classes of maximally perturbable nodal rational curves of degree 5 in $\mathbb{RP}^2$ with exactly one isolated node.

5.2 $(M-2)$-perturbable curves

Definition 5.2.1. A real type I nodal plane curve C of degree d is $(M-s)$-*perturbable* if its type I perturbation has $\frac{(d-1)(d-2)}{2}+1-s$ connected components of the real point set.

Recall that an $(M-2)$-quintic C_0 has four ovals, which may or not be in a convex position, depending on whether $\sigma(C_0)=2$ or $\sigma(C_0)=0$, respectively. In the case when $\sigma(C_0)=2$, the curve C_0 has four ovals forming a quadrangle. As before, if we fix a point p at the interior of an oval, the pencil of lines passing for p induces an order on the ovals. Varying the fix point, we can assign to every oval two neighboring ones. In the case when $\sigma(C_0)=0$, the curve C_0 has four ovals forming Dynkin diagram D_4. If we fix a point p at the interior of the central oval and an orientation of the central oval, the pencil of lines passing through p induces a cyclic order on the three exterior ovals depending on the orientation given to the central oval. In both cases, if a type I $(M-2)$-perturbable curve C has an elliptic nodal point, the oval produce by it in a type I perturbation of C must respect its relative position with respect to its neighboring ovals, and so does the elliptic nodal point of the original curve (see Figure 5.13).

Lemma 5.2.2. *Let C be a nodal rational plane M-curve of degree 5. If C is $(M-2)$-perturbable, then $h \geq 3$.*

Proof. For a nodal rational plane M-curve of degree 5, we have that $e+h=6$. Since C is $(M-2)$-perturbable, its type I perturbation has 4 ovals and a pseudoline component. If $e \geq 4$, since each elliptic nodal point gives rise to an oval, then $e=4$. If $h=1$ or 2, then the real point set $\mathbb{R}C$ has a circle with h self-intersections and its I perturbation produces h extra ovals, contradicting the fact that C is $(M-2)$-perturbable. □

Theorem 5.2.3. *Let C be a nodal rational plane M-curve of degree 5. If C is $(M-2)$-perturbable, its rigid isotopy class is determined by its isotopy class.*

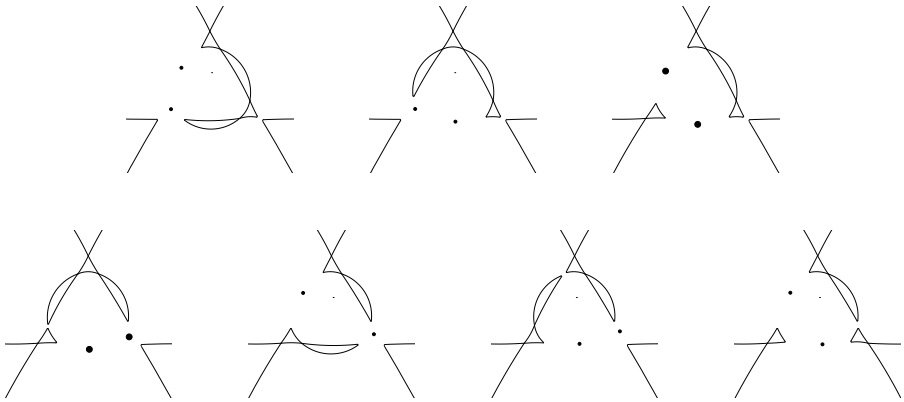


Figure 5.8: Rigid isotopy classes of maximally perturbable nodal rational curves of degree 5 in $\mathbb{RP}^2$ with exactly two isolated nodes.

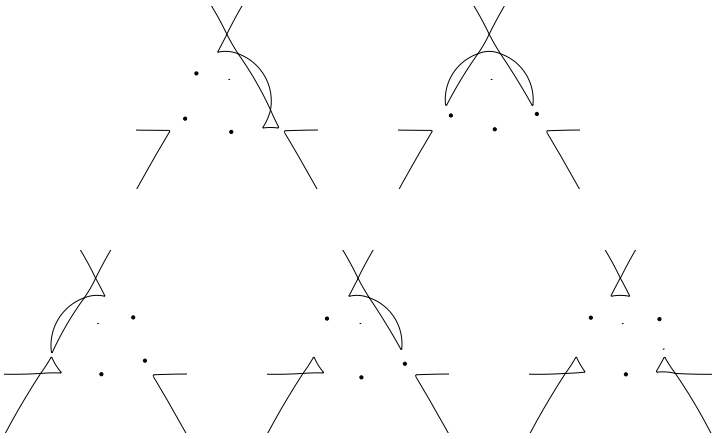


Figure 5.9: Rigid isotopy classes of maximally perturbable nodal rational curves of degree 5 in $\mathbb{RP}^2$ with exactly three isolated nodes.

Proof. Pick a hyperbolic nodal point $p \in \mathbb{R}C$ belonging to the pseudoline. The corresponding marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 7 real nodal $\times$-vertices, among which two are markings. By Corollary 5.1.6, the marked toile (D_C, v_1, v_2) has a type I perturbation with 3 true ovals. Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned restrictions and realizing the birational transformations in order to recover the associated curves $\mathbb{R}C \subset \mathbb{RP}^2$ lead to the plane curves shown on Figures 5.14 and 5.15. $\square$

Given a real type I nodal plane curve C of odd degree, denote by h_p the number of hyperbolic nodal points connecting the pseudoline to an oval in a type I perturbation of C.

Proposition 5.2.4. *Let C be a nodal rational plane $(M-2)$-curve of degree 5. If C is $(M-2)$-perturbable and $\sigma(C) = 2$, the rigid isotopy class of C is determined by its isotopy class and the position of its elliptic nodal points.*

Proof. Since C is a nodal rational plane $(M-2)$-curve of degree 5, we have $h + e = 4$. Let us begin with the case when $h \geq 1$. Then, the marked toile (D_C, v_1, v_2) associated

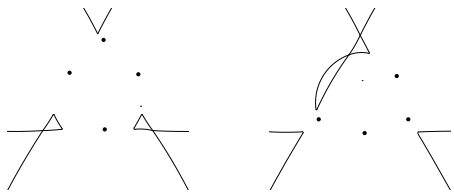


Figure 5.10: Rigid isotopy classes of maximally perturbable nodal rational curves of degree 5 in $\mathbb{RP}^2$ with exactly four isolated nodes.

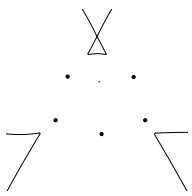


Figure 5.11: The rigid isotopy class of nodal rational maximally perturbable curves of degree 5 in $\mathbb{RP}^2$ with exactly five isolated nodes.

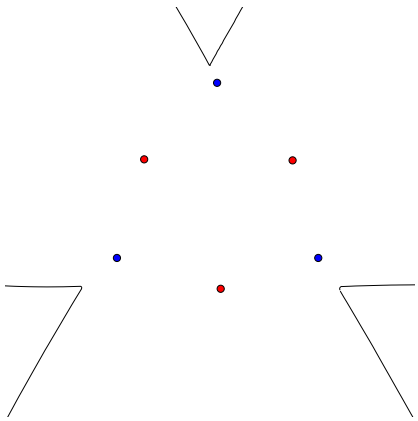
to (C, p) is a type I toile with 5 real nodal $\times$-vertices, among which two are markings, and a negative inner nodal $\times$-vertex. By Corollary 5.1.6, the marked toile (D_C, v_1, v_2) has a type I perturbation with 3 true ovals. Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned restrictions and realizing the birational transformations in order to recover the associated curves $\mathbb{R}C \subset \mathbb{RP}^2$ lead to the plane curves shown on Figure 5.16.

Remark that in the case when $h_p = 2$, the two negative ovals on a type I perturbation of the curve are opposite in the quadrangle formed by the ovals. Let n_1 and n_2 be the hyperbolic nodal points adding up to h_p and let L be the line passing through n_1 and n_2. There is a unique connected component of $L \setminus \{n_1, n_2\}$ that do not intersect C. Define this connected component as the segment $\overline{n_1 n_2}$. Analogously, there is a unique segment S of the pseudoline between n_1 and n_2 that do not intersect the line L. Then, the segments $\overline{n_1 n_2}$ and S bound a disk in $\mathbb{RP}^2$ that contains a nodal point q of C that can be distinguished. This construction is valid for any curve in the rigid isotopy class of C.

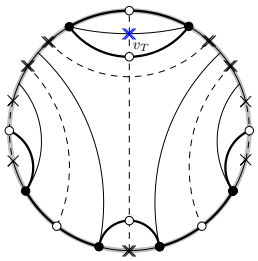
In the case when $h = 0$, by Rokhlin's complex orientation formula for nodal curves, the curve C has exactly 2 positive elliptic nodal points and 2 negative elliptic nodal points. Pick a negative elliptic nodal point $p \in \mathbb{R}C$ as a base point. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 3 real isolated nodal $\times$-vertices, among which exactly 1 is negative and the remaining 2 are positive, and two inner nodal $\times$-vertices, among which one is the nodal $\times$-vertex $v_1 = v_2$ and the remaining one is a negative inner nodal $\times$-vertex. Then, enumerating all possible toiles satisfying these conditions *via* the decompositions given in Proposition 2.2.5, there is only one equivalence class of marked dessins satisfying these restrictions. □

Proposition 5.2.5. *Let C be a nodal rational plane $(M-2)$-curve of degree 5. If C is $(M-2)$-perturbable, $\sigma(C) = 0$ and $h_p = 1$, then the rigid isotopy class of C is determined by its isotopy class.*

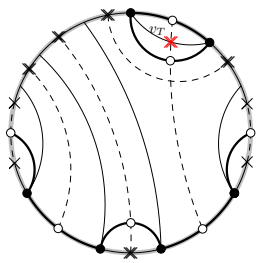
Proof. Since C is a nodal rational plane $(M-2)$-curve of degree 5, we have $h + e = 4$.



(a) The rigid isotopy class of maximally perturbable nodal rational curves of degree 5 in $\mathbb{RP}^2$ with exactly six isolated nodes.



(b) The marked dessin D_-, associated to a negative elliptic nodal point as the base point.



(c) The marked dessin D_+, associated to a positive elliptic nodal point as the base point.

Let $p \in \mathbb{R}C$ be the hyperbolic nodal point connecting the pseudoline to an oval in a type I perturbation of C. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 5 real nodal $\times$-vertices, among which two are markings, and with a positive inner nodal $\times$-vertex. By Corollary 5.1.6, the marked toile (D_C, v_1, v_2) has a type I perturbation with 3 true ovals. Since $h_p = 1$ for the curve C, the toile D_C has no non-marked non-isolated $\times$-vertices connecting the long component to an oval in the type I perturbation of D_C. Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned restrictions and realizing the birational transformations in order to recover the associated curves $\mathbb{R}C \subset \mathbb{RP}^2$ lead to the plane curves shown in Figure 5.17. □

Proposition 5.2.6. *Let C be a nodal rational plane $(M-2)$-curve of degree 5. If C is $(M-2)$-perturbable, $\sigma(C) = 0$ and $h_p = 2$, then the rigid isotopy class of C is determined by its isotopy class, the position of its elliptic nodal points and a cyclic order induced on the negative ovals of its type* I *perturbation.*

Proof. Pick a hyperbolic nodal point $p \in \mathbb{R}C$ connecting the pseudoline to an oval in a type I perturbation of C. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 5 real nodal $\times$-vertices, among which two are markings, and with a

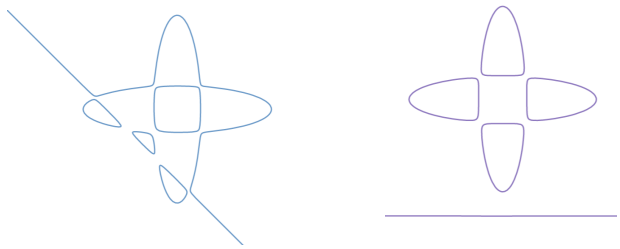


Figure 5.13: $(M-2)$-quintic curves.

positive inner nodal $\times$-vertex. By Corollary 5.1.6, the marked toile (D_C, v_1, v_2) has a type I perturbation with 3 true ovals. Since $h_p = 2$ for the curve C, the toile D_C has exactly one non-marked non-isolated $\times$-vertex connecting the long component to an oval in the type I perturbation of D_C. Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned and realizing the birational transformations in order to recover the associated curves $\mathbb{R}C \subset \mathbb{RP}^2$ lead to the plane curves shown in Figure 5.18.

In this setting, there are exactly two rigid isotopy classes per isotopy class. In order to distinguish them, we use the line L passing by the complex conjugated pair of nodal points of C. This line intersects the curve C in a unique real point p_0. Let $p' \neq p$ be the other hyperbolic nodal point connecting the pseudoline to an oval in a type I perturbation of C, and let C_0 be the type I perturbation of C. Then, the triple (p, p', p_0) induces an orientation on the pseudoline of the curve C that can be extended to a complex orientation of the whole curve, determining a cyclic order of the ovals of a type I perturbation of C. This orientation is a rigid isotopy invariant of the marked curve (C, p). Then, every isotopy class endowed with the orientation given by (p, p', p_0) determines a unique rigid isotopy class. □

Proposition 5.2.7. *Let C be a nodal rational plane $(M-2)$-curve of degree 5. If C is $(M-2)$-perturbable, $\sigma(C) = 0$ and $h_p = 3$, then the rigid isotopy class of C is determined by its isotopy class.*

Proof. Pick a hyperbolic nodal point $p \in \mathbb{R}C$ connecting the pseudoline to an oval in a type I perturbation of C. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 5 real nodal $\times$-vertices, among which two are markings, and with a positive inner nodal $\times$-vertex. By Corollary 5.1.6, the marked toile (D_C, v_1, v_2) has a type I perturbation with 3 true ovals. Since $h_p = 3$ for the curve C, the toile D_C has exactly 2 non-marked non-isolated $\times$-vertices connecting the long component to an oval in the type I perturbation of D_C. Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned restrictions and realizing the birational transformations in order to recover the associated curves $\mathbb{R}C \subset \mathbb{RP}^2$ lead to the plane curves shown in Figure 5.19. □

Theorem 5.2.8. *Let C be a nodal rational plane $(M-2)$-curve of degree 5. If C is $(M-2)$-perturbable and $\sigma(C) = 0$, its rigid isotopy class is determined by its isotopy class and the position of its elliptic nodal points.*

Proof. It remains to consider curves with $h = 0$, and hence $e = 4$. By Rokhlin's complex orientation formula for nodal curves, the curve C has exactly 1 positive elliptic nodal point and 3 negative elliptic nodal points. Pick the positive elliptic nodal point $p \in \mathbb{R}C$

Figure 5.14: type I perturbations of rigid isotopy classes of $(M-2)$-perturbable nodal rational plane M-curves. (Part I).

of C as a base point. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 3 real isolated nodal $\times$-vertices, which are all negative, and two inner nodal $\times$-vertices, among which one is the nodal $\times$-vertex $v_1 = v_2$ and the remaining one is a positive inner nodal $\times$-vertex. Then, enumerating all equivalence classes of marked toiles leads to the fact that there is only one equivalence class of marked dessins satisfying these restrictions. □

5.3 $(M-4)$-perturbable curves

Lemma 5.3.1. *Let C be a nodal rational plane curve of degree 5. If C is $(M-4)$-perturbable and $\sigma(C) = 0$, then $h \geq 1$.*

Proof. The fact that $\sigma(C) = 0$ implies that the curve C can be perturbed to a non-singular type I curve C'. The curve C' being a non-singular $(M-4)$ curve of type I has a nest of two ovals. If $h = 0$, then every nodal point of C is elliptic, producing a simple oval in C' and no nest can be produced in this manner. □

Theorem 5.3.2. *Let C be a nodal rational plane M-curve of degree 5. If C is $(M-4)$-perturbable, its rigid isotopy class is determined by its isotopy class.*

Proof. Pick a hyperbolic nodal point $p \in \mathbb{R}C$ belonging to the pseudoline. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 7 real nodal $\times$-vertices. By Corollary 5.1.6, the marked toile (D_C, v_1, v_2) has a type I perturbation with 1 true oval.

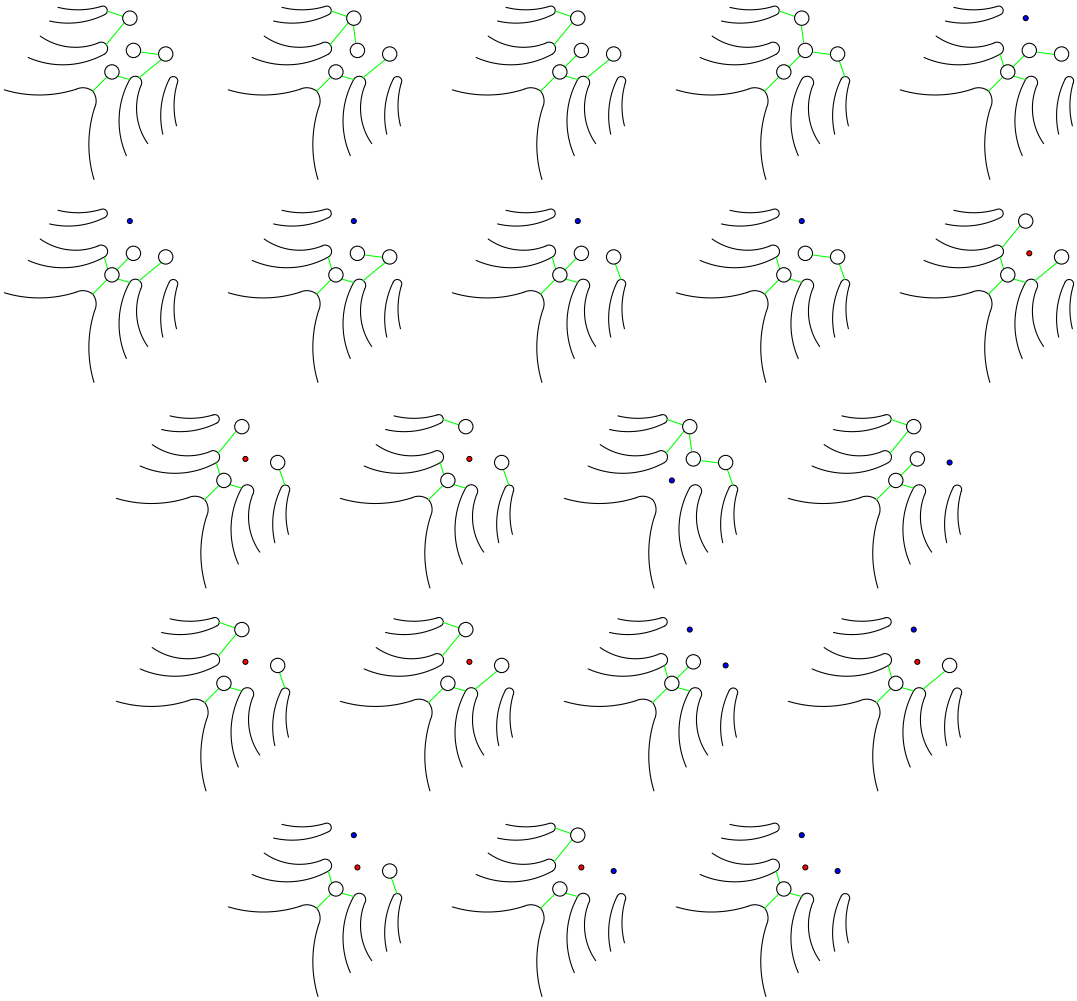


Figure 5.15: type I perturbations of rigid isotopy classes of $(M-2)$-perturbable nodal rational plane M-curves. (Part II).

Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned restrictions leads to the fact that every isotopy class in Figure 5.21 corresponds to exactly one rigid isotopy class. □

Lemma 5.3.3. *Let C be a nodal rational plane $(M-2)$-curve of degree 5. If C is $(M-4)$-perturbable, then $h \geq 1$.*

Proof. If $h = 0$, the type I perturbation of C has two ovals arising from elliptic nodes, in which case $h + e = 2$, contradicting the fact that for a nodal rational plane curve $(M-2)$-curve of degree 5 we have $h + e = 4$. □

Theorem 5.3.4. *Let C be a nodal rational plane $(M-2)$-curve of degree 5. If C is $(M-4)$-perturbable, its rigid isotopy class is determined by its isotopy class and the number $\sigma(C)$.*

Proof. Pick a hyperbolic nodal point $p \in \mathbb{R}C$ belonging to the pseudoline. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 5 real nodal $\times$-vertices, among which two are markings, and with an inner nodal $\times$-vertex. By Corollary 5.1.6, the

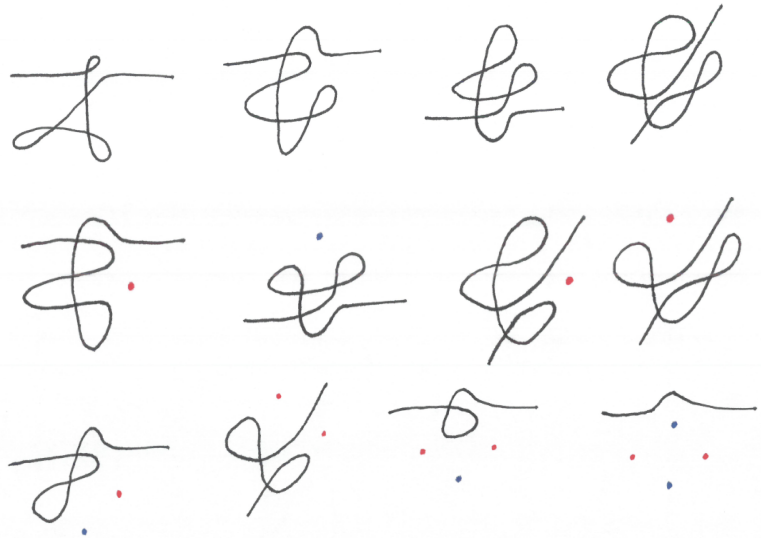


Figure 5.16: The rigid isotopy class of $(M-2)$-perturbable nodal rational plane $(M-2)$-curves with $\sigma = 2$.

Figure 5.17: Isotopy class of $(M-2)$-perturbable nodal rational plane $(M-2)$-curves of degree 5 with $h_p = 1$.

marked toile (D_C, v_1, v_2) has a type I perturbation with 1 true oval. Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned restrictions leads leads to the fact that every isotopy class in Figure 5.22 corresponds to exactly one rigid isotopy class. □

Theorem 5.3.5. *Let C be a nodal rational plane $(M-4)$-curve of degree 5. If C is $(M-4)$-perturbable, its rigid isotopy class is determined by its isotopy class and the sign of its elliptic nodal points.*

Proof. Let us consider all the possible values that $\sigma(C)$ can take.

Case 1: let us start with the case when $\sigma(C) = 0$. By Lemma 5.3.1, we have $h \geq 1$. Pick a hyperbolic nodal point $p \in \mathbb{R}C$ belonging to the pseudoline. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 3 real nodal $\times$-vertices, among which two are markings, and with 2 positive inner nodal $\times$-vertices. By Corollary 5.1.6, the marked toile (D_C, v_1, v_2) has a type I perturbation with 1 true oval. Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned restrictions leads leads to the fact that every isotopy class in Figure 5.23a corresponds to exactly one rigid isotopy class.

Case 2: In the case when $\sigma(C) = 2$, let us suppose $h \geq 1$. Pick a hyperbolic nodal point $p \in \mathbb{R}C$ belonging to the pseudoline. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 3 real nodal $\times$-vertices, among which two are markings, and with 1 positive inner nodal $\times$-vertex and 1 negative inner nodal $\times$-vertex. By Corollary 5.1.6, the marked toile (D_C, v_1, v_2) has a type I perturbation with 1 true oval.

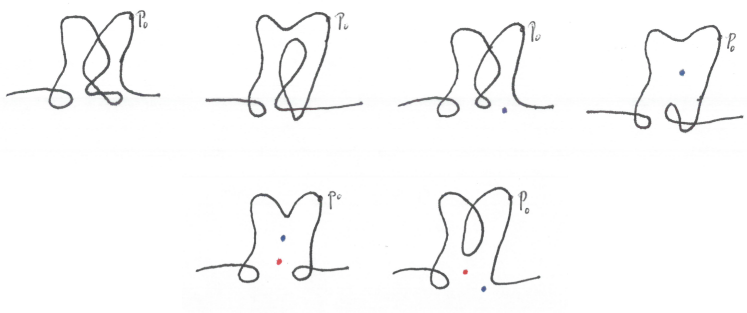


Figure 5.18: Isotopy class of $(M-2)$-perturbable nodal rational plane $(M-2)$-curves of degree 5 with $h_p = 2$.

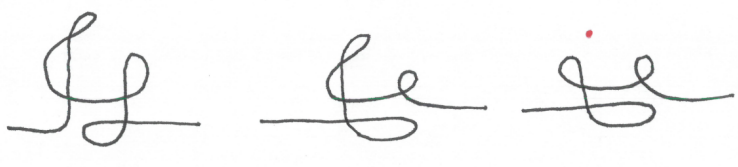


Figure 5.19: Isotopy class of $(M-2)$-perturbable nodal rational plane $(M-2)$-curves of degree 5 with $h_p = 3$.

Otherwise, we have $h = 0$, and consequently $e = 2$. By Rokhlin's complex orientation formula for nodal curves the curve C has exactly 2 negative elliptic nodal points. Let us pick $p \in \mathbb{R}C$ a negative elliptic nodal point of C as a base point. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 1 real isolated nodal $\times$-vertex, which is negative, and 3 inner nodal $\times$-vertices, among which one is the nodal $\times$-vertex $v_1 = v_2$ and the remaining two are one positive and one negative inner nodal $\times$-vertices. Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned restrictions leads leads to the fact that every isotopy class in Figure 5.23b corresponds to exactly one rigid isotopy class.

Case 3: In the case when $\sigma(C) = 4$, let us suppose $h \geq 1$. Pick a hyperbolic nodal point $p \in \mathbb{R}C$ belonging to the pseudoline. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 3 real nodal $\times$-vertices, among which two are markings, and with 2 negative inner nodal $\times$-vertices. By Corollary 5.1.6, the marked toile (D_C, v_1, v_2) has a type I perturbation with 1 true oval.

Otherwise we have that $h = 0$, and consequently $e = 2$. By Rokhlin's complex orientation formula for nodal curves the curve C has exactly 1 positive elliptic nodal point and1 negative elliptic nodal point. Let us pick $p \in \mathbb{R}C$ the positive elliptic nodal point of C as a base point. Then, the marked toile (D_C, v_1, v_2) associated to (C, p) is a type I toile with 1 real isolated nodal $\times$-vertex, which is negative, and 3 inner nodal $\times$-vertices, among which one is the nodal $\times$-vertex $v_1 = v_2$ and the remaining two are negative inner nodal $\times$-vertices. Then, enumerating all equivalence classes of marked toiles satisfying the aforementioned restrictions leads leads to the fact that every isotopy class in Figure 5.23c corresponds to exactly one rigid isotopy class. $\square$

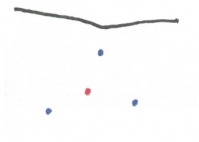


Figure 5.20: The isotopy class of $(M-2)$-perturbable nodal rational plane $(M-2)$-curves of degree 5 with $h = 0$.

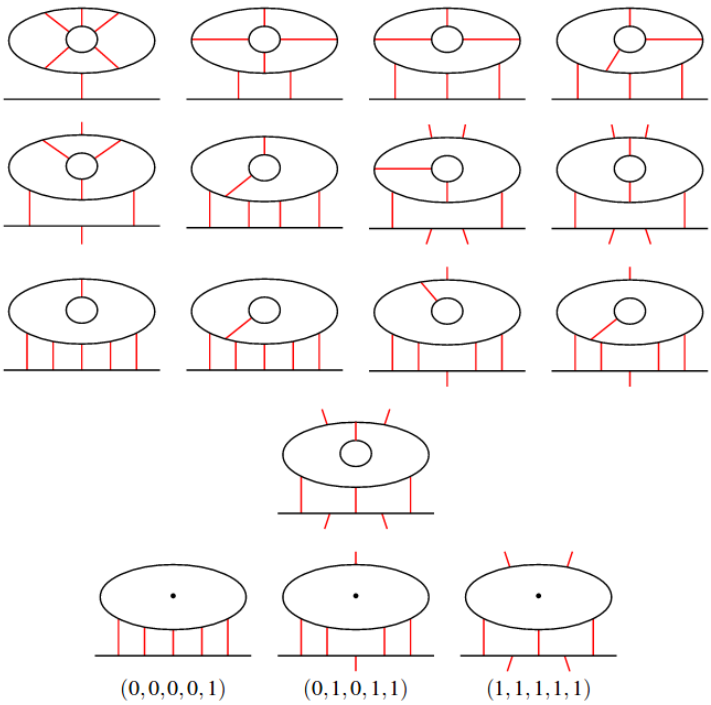


Figure 5.21: Isotopy classes of $(M-4)$-perturbable nodal rational plane M-curves of degree 5. Taken from [8].

5.4 $(M-6)$-perturbable curves

Theorem 5.4.1. *There is a unique rigid isotopy class of rational $(M-6)$-curves of degree 5 in $\mathbb{RP}^2$.*

This rigid isotopy class is shown in Figure 5.24.

Proof. Let C_0 be a rational $(M-6)$-curve of degree 5 in $\mathbb{RP}^2$. We have $h = e = 0$, and therefore $c(C_0) = 6$. Up to a small perturbation of the coefficients of C_0, we can assume that the six nodal points $p_1, p_2, \ldots, p_6$ of C_0 are in general position, i.e., they do not belong to a conic. Let us consider the linear system formed by the curves of degree 5 with singularities at $p_1, p_2, \ldots, p_6$. This is a plane in the space of quintic curves, which has dimension 20. In this linear system, the curves which are more degenerated than C_0 form a codimension 2 subset. Indeed, the condition for a curve C to belong to a codimension 1 stratum of singular curves of this linear system corresponds to the property of having a real nodal point. In this case, such a curve C is a reducible curve whose components are a rational cubic and a conic with empty real point set, which contradicts the fact that

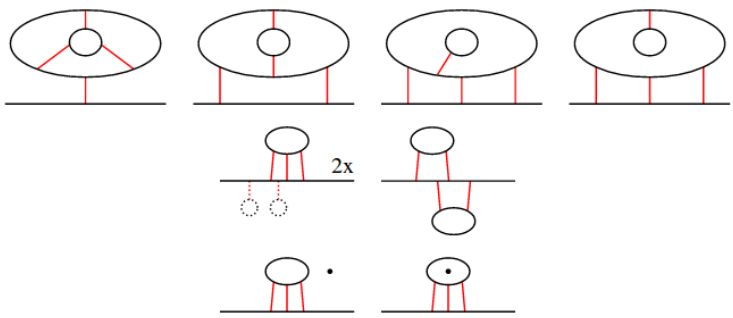
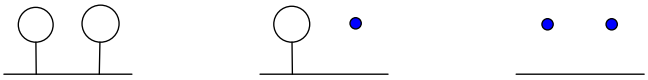


Figure 5.22: Isotopy classes of $(M-4)$-perturbable nodal rational plane $(M-2)$-curves of degree 5. Taken from [8].

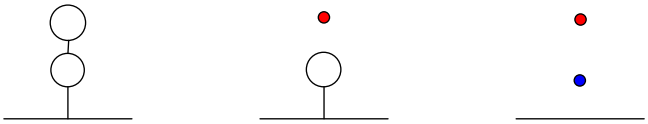
$p_1, p_2, \ldots, p_6$ do not belong to a conic. Hence, in this linear system, every degeneration is path-connected to C_0.

On the other hand, among the degenerations, there is a reducible curve C_0' formed by a real line and two complex conjugated conics such that the real line passes through the pair of complex conjugated points p_5 and p_6, one conic passes through the points $p_1, \ldots, p_5$, and the complex conjugated conic passes through the points $p_1, \ldots, p_4, p_6$. Thus, the statement of the theorem follows from the fact that any two real reducible curves of this kind (each of the curves being formed by a real line and two complex conjugated conics without real intersection points) can be connected within the class of such real reducible curves. $\square$

(a) Rigid isotopy classes of nodal rational $(M-4)$-perturbable curves of degree 5 in $\mathbb{RP}^2$ with $\sigma = 0$.



(b) Rigid isotopy classes of nodal rational $(M-4)$-perturbable curves of degree 5 in $\mathbb{RP}^2$ with $\sigma = 2$.



(c) Rigid isotopy classes of nodal rational $(M-4)$-perturbable curves of degree 5 in $\mathbb{RP}^2$ with $\sigma = 4$.

Figure 5.24: The rigid isotopy class of nodal rational $(M-6)$-perturbable curves of degree 5 in $\mathbb{RP}^2$.

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