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Exposants de Lyapunov d'opérateurs de Schrödinger ergodiques

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Résumé

L'objectif de cette thèse est de traiter de deux aspects différents de la théorie de l'exposant de Lyapunov de cocycles de Schrödinger définis par une dynamique ergodique.

Dans la première partie, on s'intéresse aux estimées de grandes déviations de type Bourgain & Goldstein pour des cocycles quasi-périodiques, puis pour ceux définis par le doublement de l'angle. Après avoir montré que seule une estimée par-dessus sur une bande complexe est nécessaire pour avoir la minoration requise pour l'estimée des grandes déviations, on donne une nouvelle preuve de cette majoration dans le cas d'une dynamique quasi-périodique. La preuve utilise des techniques de mouvement brownien en lien avec des fonctions sous-harmoniques. Ensuite on adapte la méthode au cas du doublement de l'angle pour lequel on établit des estimées de grandes déviations sur les branches inverses de cette dynamique.

Dans la deuxième partie sont étudiés des cocycles de Schrödinger dont le potentiel mêle des dynamiques quasi-périodique et aléatoire. On démontre que, dans un régime perturbatif, les développements asymptotiques de l'exposant de Lyapunov attaché à ces cocycles sont similaires à ceux démontrés dans le cas aléatoire par Figotin & Pastur et Sadel & Schulz-Baldes. L'analyse se fait en fonction du caractère diophantien ou résonant de l'énergie par rapport à la fréquence de la partie quasi-périodique du potentiel.

Mots clés Exposant de Lyapunov, cocycles de Schrödinger quasi-périodiques et aléatoires, opérateur de transfert, équation cohomologique, théorie ergodique, difféomorphismes aléatoires, mesure stationnaire, théorie du potentiel, opérateurs différentiels.

Abstract

In this thesis we are interested in the Lyapunov exponent of ergodic Schrödinger cocycles. These cocycles occur in the analysis of solutions to the Schrödinger equation where the potential is defined with ergodic dynamics. We study two distinct aspects of the theory of the Lyapunov exponent for different kinds of dynamics.

Firstly we focus on a large deviation theorem for quasi-periodic cocycles and then for potentials defined by the doubling map. We prove that estimates of Bourgain & Goldstein type are granted if an upper estimate involved in the theorem is true on a strip in the complex plane. Then we develop a new technique to prove this upper bound in the quasi-periodic setting, based on subharmonic arguments related to the Brownian motion and suggested by the work of Avila, Jitomirskaya & Sadel. We adapt afterwards the method and prove a large deviation theorem for the inverse branches of the doubling map.

In the second part, we establish an asymptotic development similar to the results of Figotin & Pastur and Sadel & Schulz-Baldes for the Lyapunov exponent of Schrödinger cocycles at small coupling with potentials that are a mixture of quasi-periodic and random. The analysis distinguishes whether the energy is diophantine or resonant with respect to the frequency of the quasi-periodic part of the potential.

Keywords Lyapunov exponent, quasi-periodic and random Schrödinger cocycles, transfer operator, cohomological equation, ergodic theory, random diffeomorphisms, stationary measure, potential theory, differential operators.

Contents

CHAPTER 1

Introduction

1.1 From the Schrödinger equation to Schrödinger cocycles

1.1.1 Background from Physics

The mathematical model of an electron under the action of a potential V is:

$$-\,\mathrm{i}\,\partial_t\psi = -\Delta\psi + \mathrm{V}\psi =: \mathrm{H}_{\mathrm{V}}\psi \tag{1.1}$$

where

- $\psi = \psi(x,t)$ is the wave function of the electron (up to some phase);
- $\mathrm{d}\mathbf{P} = |\psi(x,t)|^2 \,\mathrm{d}x$ is the probability density at position x, and time t of the particle;
- $\Delta = \sum_i \partial_i^2$ is the standard Laplacian operator;
- $\mathrm{V} = \mathrm{V}(x,t)$ is a scalar potential.

The Schrödinger operator is then $\mathrm{H} = \mathrm{H}_\mathrm{V} = -\Delta + \mathrm{V}$. Thanks to functional calculus and Stone's theorem (see [RS80]) the solutions of the time-dependent Schrödinger equation are

$$\psi(t) = \mathrm{e}^{\mathrm{i}t\mathrm{H}}\psi(0)$$

One major question is then to understand the dynamics of the wave function: typically one wants to know whether the wave propagates information in time or in average in time (*scattering or mean scattering*), or if it vanishes quickly around its initial position (*localization*).

Let us denote $\mathscr{H}_{\mathrm{pp}}, \mathscr{H}_{\mathrm{c}}$ and $\mathscr{H}_{\mathrm{ac}}$ the respective pure point, continuous and absolutely continuous part of the decomposition according to the respective components of the spectrum of H (see appendix B). Then the spectral theory of self-adjoint operators gives some results. In particular the R.A.G.E. theorem (see [RS79]) gives information about the dynamical behaviour of the solution.

Theorem 1.1 (Ruelle, Amrein, Georgescu, Enss.)

For H *a self adjoint operator on a Hilbert space* $\mathscr{H}$

- $\psi \in \mathscr{H}_{\mathrm{pp}} \Leftrightarrow \sup_{t\geqslant 0} \|\mathbf{1}_{\mathrm{B_R}^c}\, \mathrm{e}^{\pm \mathrm{i}t\mathrm{H}}\psi\| \xrightarrow[\mathrm{R}\to\infty]{} 0$ *(bound state)*
- $\psi \in \mathscr{H}_{\mathrm{c}} \Leftrightarrow \forall \mathrm{R} > 0 \quad \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \|\mathbf{1}_{\mathrm{B_R}}\mathrm{e}^{\pm \mathrm{i}t\mathrm{H}}\psi\|^2 \,\mathrm{d}t \xrightarrow[\mathrm{T}\to\infty]{} 0$ *(mean scattering state)*
- $\psi \in \mathscr{H}_{\mathrm{ac}} \Rightarrow \forall \mathrm{R} > 0 \quad \|\mathbf{1}_{\mathrm{B_R}}\mathrm{e}^{\pm \mathrm{i}t\mathrm{H}}\psi\|^2 \xrightarrow[t\to\infty]{} 0$ *(scattering state)*

One great motivation to study the dynamics of Schrödinger solutions is based on the work of Anderson ([And58] and see [Hun08] for a nice introduction) in the 50's. He noted the absence of diffusion of waves in a disordered medium and was the first to suggest the possibility of electron localization inside a semiconductor, provided the randomness caused by impurities in the crystal is sufficiently large. This strongly contrasts with the ideal model of a perfect crystal consisting of a periodic lattice generating a periodic potential under which a particle evolves. In the case of a periodic potential V, the operator H_V exhibits only purely continuous spectrum and so the crystal should always conduct charged particles.

Rigorous proofs were given in the 80's by several authors ([KS80, FS83, SW86, AM93]).

These authors studied an operator $\mathrm{H}_\omega = -\Delta + \mathrm{V}_{\text{per}} + \lambda \mathrm{V}_\omega$ for a parameter ω and a coupling constant $\lambda > 0$ and (sometimes under largeness assumptions on λ) proved some *localization* properties of the operator H_ω that relate to the initial conjecture of Anderson. Namely, one example among others is the *exponential localization* in the interval $[\,a\,;b\,]$ that requires that for almost every ω, the spectrum of H_ω is pure point in $[\,a\,;b\,]$ with associated eigenfunctions exponentially decaying.

Later, other types of Schrödinger operators that are also models to understand the impurities in a crystal have been studied: the quasi-periodic operators. In this case, the potential is given by the evaluation of a function on an orbit of the translation $x \mapsto x+\alpha$ on the torus $\mathbf{T}^d$. Those functions naturally appear when one takes the sum of two periodic potentials with immeasurable periods. They exhibit no true period but still seem to preserve a somehow regular pattern.

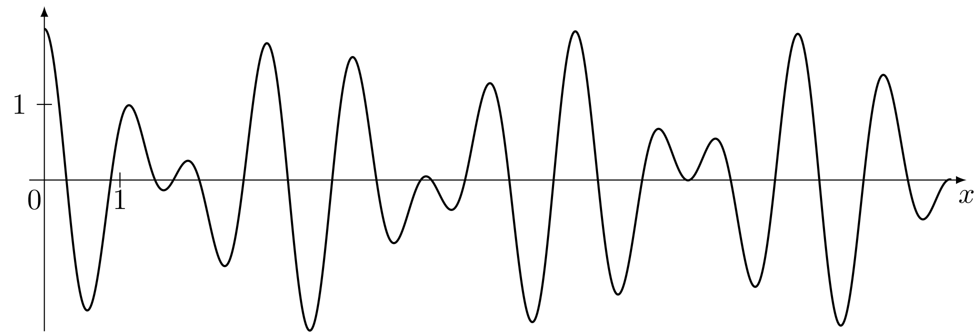

Figure 1.1: The quasi-periodic function $x \mapsto \cos(2\pi x) + \cos(\sqrt{2}\pi x)$

1.1.2 Dynamically defined Schrödinger operators and related cocycles

The basic idea is that for a function $\varphi \in \mathscr{C}^2(\mathbf{R})$ and $h \neq 0$ a Taylor development gives $\varphi''(x) = \frac{1}{h^2}\left(\varphi(x+h) + \varphi(x-h) - 2\varphi(x)\right) + \mathrm{O}(h)$. Then if one would like to understand a solution of the Schrödinger equation by discretizing it on a lattice of $\mathbf{R}^3$ they would need to study (neglecting the error terms) the equation $-\mathrm{i}h^2\partial_t\psi_n = (-\Delta_{\mathbf{Z}} + v_n)\,\psi_n$ for $\psi_n(\cdot) = \varphi(nh,\cdot)$ and $v_n = h^2\mathrm{V}(nh)$. This is how emerges the discrete operator $\mathrm{H}_v = -\widetilde{\Delta}_{\mathbf{Z}^d} + \mathcal{M}_v$ (see definition *B.2* in the appendix).

In the sequel, we will only focus on discrete one dimensional operators. Our setting is the following

- $d = 1$
- $(\Omega, \mathcal{F}, \mu)$ is a probability space

- $\mathrm{T} : \Omega \circlearrowleft$ is an ergodic map for μ called the *base dynamics*
- $v : \Omega \to \mathbf{R}$ is the map which we evaluate along the orbit of the dynamics T
- the *potential* is $(v_n(\omega))_n$ where $v_n(\omega) = v(\mathrm{T}^n\omega)$ for all n

Then the Schrödinger operator is defined by

$$\mathrm{H}_{v,\omega} : \begin{cases} \quad \ell^2 & \longrightarrow \ell^2 \\ (u_n)_n & \longmapsto (u_{n+1} + u_{n-1} + v_n(\omega)u_n)_n \end{cases}$$

Depending on whether T is invertible or not, the operator is defined on $\ell^2(\mathbf{Z})$ or $\ell^2(\mathbf{N})$. Here are some of the most studied types of operators depending on their dynamics.

The random operator

Such an operator is defined on $\ell^2(\mathbf{Z})$ when $\Omega = \mathbf{R}^{\otimes \mathbf{Z}}$ and $\mu = \mathbf{P}_0^{\otimes \mathbf{Z}}$ is some product probability space endowed with the usual product measure once a probability on $(\mathbf{R}, \mathrm{Bor}(\mathbf{R}))$ is given and where the potential is defined by the canonical projections $v_n(\omega) = \omega_n$ for some $\omega = (\omega_n)_{n\in\mathbf{Z}}$ in $\mathbf{R}^{\mathbf{Z}}$.

The *doubling map* operator

Such an operator is defined on $\ell^2(\mathbf{Z}_+)$ with the dynamics

$$\mathrm{T} : \mathbf{T} \ni x \longmapsto 2x \mod 1 \in \mathbf{T}$$

The potential is then $v_n = v(2^n\omega)$. After a conjugacy, it can be treated as a certain type of random operator for a dynamics given by the shift on $\mathbf{R}^{\otimes \mathbf{Z}}$ endowed by the Bernoulli measure.

The *quasi-periodic* Schrödinger operator

Here $\Omega = \mathbf{T}^d \simeq [\,0\,;1\,]^d / \sim$ (with $x \sim y \iff x - y \in \mathbf{Z}^d$) is the d-th dimensional torus endowed with $\mu = \mathrm{d}x$ the Haar-Lebesgue measure, and the potential is given by the evaluation along an orbit of the translation by α

$$\mathrm{T} : \mathbf{T}^d \ni \omega \longmapsto \omega + \alpha \in \mathbf{T}^d$$

of a function $v : \mathbf{T}^d \to \mathbf{R}$ regular enough (quite often analytic in the literature) that is $v_n(\omega) = v(\omega + n\alpha)$. Note that we identify maps on $\mathbf{T}$ with 1-periodic maps on $\mathbf{R}$, and may abusively write $\int_0^1 f$ instead of $\int_{\mathbf{T}} f$. Of course, if for example $d = 1$ and α is rational, the potential is periodic and everything is known about periodic operators ([CL12, RS78]) so we assume that α has rationally independent coordinates. In fact we will almost always work with more accurate arithmetics conditions on α and namely *diophantine conditions* which are very convenient because they express quantitative irrationality of α.

Our potential

In the first part, we are interested in quasi-periodic cocycles and then in cocycles given by the doubling map.

Then, in the second part, we study Schrödinger cocycles with a potential mixing both quasi-periodic and random components, namely

$$v_n(x,\omega) = f(x+n\alpha) + \mathrm{W}_n(\omega)$$

for f analytic or regular enough on $\mathbf{T}$ and $(\mathrm{W}_n)_n$ i.i.d. random variables. The idea is to obtain results for the mixed potentials since many results for each potential taken separately have been proved.

Remark 1.1

We carried our analysis for quasi-periodic potentials with only one frequency α. The analysis for $\alpha \in \mathbf{T}^d$ with $d \geqslant 2$ would actually be similar. The diophantine condition would be on the frequency vector $(\alpha_1, \dots, \alpha_d)$ and β. Yet, it would lead to more complicated computations for the cohomological equations we deal with throughout part II. This could especially be painful for the resonant energies (chapter 6) where the calculations get pretty technical.

1.1.3 Cocycles

It is important to understand the eigenvalue equations $\mathrm{H}\psi = \mathrm{E}\psi$ (see appendix B). These equations lead to the sequence recurrence equations

$$\forall n \in \mathbf{Z} \quad (\text{or } \mathbf{Z}_+) \qquad \psi_{n+1} = (\mathrm{E} - v_n(\omega))\psi_n - \psi_{n-1} \tag{1.2}$$

which can be written matricially

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} \mathrm{E} - v_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}$$

This is how emerges the definition of dynamically defined Schrödinger cocycles: the Schrödinger cocycle for the dynamics T at energy E and potential v is defined by

$$(\mathrm{T}, \mathrm{S}_\mathrm{E}) \colon \begin{cases} \Omega \times \mathbf{R}^2 \longrightarrow \Omega \times \mathbf{R}^2 \\ \quad (\omega, u) \ \longmapsto (\mathrm{T}\omega, \mathrm{S}_{\mathrm{E},v}(\omega)u) \end{cases}$$

where $\mathrm{S}_{\mathrm{E},v}(\cdot) = \begin{pmatrix} \mathrm{E} - v(\cdot) & -1 \\ 1 & 0 \end{pmatrix}$ is the elementary transfer matrix for the recurrent equation (1.2). Note that under iteration one gets $(\mathrm{T}, \mathrm{S}_\mathrm{E})^n = (\mathrm{T}^n, \mathrm{S}_\mathrm{E}^{(n)})$ where for $n \geqslant 0$ the iterates are given by the fibered matrix products

$$\mathrm{S}_{\mathrm{E}}^{(n)}(\omega) = \mathrm{S}_{\mathrm{E},v}(\mathrm{T}^{n-1}\omega)\dots\mathrm{S}_{\mathrm{E},v}(\omega)$$
$$\mathrm{S}_{\mathrm{E}}^{(-n)}(\omega) = \mathrm{S}_{\mathrm{E},v}(\mathrm{T}^{-n}\omega)^{-1}\dots\mathrm{S}_{\mathrm{E},v}(\mathrm{T}^{-1}\omega)^{-1} \quad \text{(when T is invertible)}$$

It is also of greater importance to remark that $\mathrm{S}_{\mathrm{E}}^{(n)} \in \mathrm{SL}_2(\mathbf{R})$ for all n.

1.2 The Lyapunov exponent

The Lyapunov exponent is defined as the averaged exponential rate of the norm of the transfer matrices:

Theorem 1.2 (definition of the LE, (Furstenberg-Kesten))

When (T,μ) *is ergodic the following limit exists, is non negative and defines* $\mathrm{L(E)} = \mathrm{L}(\mathrm{E}, v, \mathrm{T})$ *the Lyapunov exponent at energy* E

$$\lim_{n\to\pm\infty} \frac{1}{n}\int_\Omega \ln \|\mathrm{S}_{\mathrm{E}}^{(n)}(\omega)\| \,\mathrm{d}\mu(\omega)$$

Moreover one has the μ-a.s. *result of convergence*

$$\frac{1}{|n|} \ln \|\mathrm{S}_{\mathrm{E}}^{(n)}(\omega)\| \xrightarrow[n\to\pm\infty]{} \mathrm{L(E)}$$

The non negativity is straightforward as we choose the operator norm associated with the euclidean norm on $\mathbf{R}^2$ namely $\|\mathrm{A}\| = |||\mathrm{A}|||_2 = \sqrt{\max_{\lambda\in \mathrm{sp}\, {}^t\mathrm{AA}} |\lambda|}$ so that $\|\mathrm{A}\| = \|\mathrm{A}^{-1}\| \geqslant 1$ for all $\mathrm{A} \in \mathrm{SL}_2(\mathbf{R})$.

The convergence of the integral quantity comes from a result about subadditive sequences. The almost sure convergence is based on Kingman's subadditive ergodic theorem (see [Kin68, AB08]). So the Lyapunov exponent of a cocycle measures the exponential growth of the n-th transfer matrix $\mathrm{S}_{\mathrm{E}}^{(n)}$. In particular if there is such an exponential growth, the Lyapunov exponent gives its rate and measures the exponential rate of the solutions of $\mathrm{H}\psi = \mathrm{E}\psi$. It is clearly an asymptotic property and does not depend on any fixed first matrices $\mathrm{S}_0, \dots, \mathrm{S}_p$. Checking whether L(E) is positive or vanishes is a central question in the theory of Schrödinger operators and reveals to be sometimes quite challenging. We now explain why it is of major interest.

1.2.1 The relevance of studying the positivity of L(E)

Knowing if L(E) is positive or not is essential to get information about the dynamics of the solutions of $\mathrm{H}\psi = \mathrm{E}\psi$, as stated by Oseledet's theorem (see [AB08]). It states if the Lyapunov exponent is positive then almost surely we have a solution that is exponentially decaying in the future $(n \to +\infty)$ for an initial condition in an *stable* manifold E_ω^s and another one exponentially decaying in the past ($n \to -\infty$) for an initial condition in an *unstable* manifold E^u_ω, both at same rate $-\mathrm{L}(\mathrm{E})$. Those manifolds constitute a splitting of $\mathbf{R}^2$ and verify some *fibered* invariance w.r.t. the dynamics.

The importance to study the LE is also illustrated by some dictionary properties between spectral and dynamical properties of the Schrödinger operator. For example it is not difficult to prove that the exponent is positive outside the (almost sure) spectrum of the operator H_ω. There are other crucial results, and for example the theorem of Ishii-Pastur-Kotani that states that the absolutely continuous part of the spectrum $\sigma_{\mathrm{ac}}(\mathrm{H})$ is the essential closure* of the set of energies where $\mathrm{L}(\mathrm{E})$ vanishes. Hence, if $\mathrm{L}(\mathrm{E})$ is positive on an interval, then there is a. s. no a.c. spectrum (see appendix B) in it.

1.2.2 Some results of positivity

There are many results about the positivity of the Lyapunov exponent in different dynamical settings. We recall some famous ones.

In the general random setting

One major result is obtained by Fürstenberg in [Fur63]: for a large class of i.i.d. random matrices the exponent is positive.

Theorem 1.3 (Fürstenberg)

Let μ be a log*-integrable probability on* $\mathrm{SL}_2(\mathbf{R})$ *i.e.*

$$\textstyle\int_{\mathrm{SL}_2(\mathbf{R})} \log(\|\mathrm{M}\|)\,\mathrm{d}\mu(\mathrm{M}) < +\infty$$

and also $(\mathrm{Y}_n)_{n\in\mathbf{N}} \in \mathrm{SL}_2(\mathbf{R})^{\mathbf{N}}$ *a family of random i.i.d. matrices with same law* μ. *We denote* G_μ *the smallest closed subspace of* $\mathrm{SL}_2(\mathbf{R})$ *containing the support of* μ *and assume that it satisfies*

- G_μ *is not compact;*
- *there is no* G_μ *invariant* ($\mathrm{MX} = \mathrm{X}$ *for* $\mathrm{M} \in \mathrm{G}_\mu$) *finite subset* $\mathrm{X} \subset \mathbf{P}^1$ *except* $\varnothing$.

Then $\gamma(\mathrm{Y}) = \lim_{n\to+\infty} \frac{1}{n} \log(\|\mathrm{Y}_n \ldots \mathrm{Y}_0\|) > 0$ *where the limit exists a.s.*

Later, Aizenman and Molchanov [AM93] obtained more in a more general setting but at large coupling ($\lambda \gg 1$) or for so called *extreme energies* with some lower bound of type $\gtrsim \log|\lambda|$.

*The *essential closure* of a set A is $\overline{\mathrm{A}}^{\mathrm{ess}} = \{x \in \mathbf{R} \mid \forall\varepsilon > 0 \quad \mathrm{Leb}(\mathrm{A} \cap \,]\,x-\varepsilon\,;x+\varepsilon\,[\,) > 0\}$.

For the doubling map

Here the potential is $v_n(x) = \lambda v(2^n x)$ for an operator defined on $\ell^2(\mathbf{Z}_+)$. Chulaevsky and Spencer ([CS95]) proved positivity with some perturbative formula $\mathrm{L}_\lambda(\mathrm{E}) \sim c\lambda^2$ like Figotin-Pastur's, at small coupling $\lambda \ll 1$ and for energies in $]-2\,;2\,[$ away from the edges and the band center. Damanik and Killip ([DK05]) generalized the positivity (but without asymptotic formula) to $\lambda > 0$. We shall elaborate more on the Figotin-Pastur type formula in section 1.5. Note that Hermann's subharmonic trick (see [Her83]) that we mention in the next paragraph also works in this case for large trigonometric polynomials.

In the quasi-periodic setting

This has been a particularly active field since the 80's. A first grounding and inspiring result was obtained by Michael Herman ([Her83]) who used an analytic extension and an elegant subharmonic trick to establish that if $v(x) = \sum a_k \mathrm{e}^{2\mathrm{i}\pi kx}$ is a real trigonometric polynomial of degree N, then $\mathrm{L}(\mathrm{E}) \geqslant \ln|a_\mathrm{N}|$. This settled the basis for many other authors. For the *almost Mathieu* operator on $\mathbf{Z}$ with $v_n = 2\lambda\cos(2\pi(n\alpha+\omega))$ and $\alpha \notin \mathbf{Q}$, this implies positivity whenever $\lambda > 1$ with a lower bound $\gtrsim \log\lambda$. For $\mathrm{E} \in \sigma(\mathrm{H})$, Bourgain and Jitomyrskaya even refined the result and proved $\mathrm{L}(\mathrm{E}) = \max(0, \ln|\lambda|)$ (see [BJ02]). In the 90's, Sorets and Spencer ([SS91]) extended the result to the case of non constant real analytic potentials $\mathbf{T} \to \mathbf{R}$ at large coupling ($\lambda \gg 1$) and obtained the same lower bound. Later, Bourgain and Goldstein developed in [BG00] a method to prove this inequality for non constant real analytic potentials on $\mathbf{T}^d$ with $d > 1$ under some diophantine assumptions on α. It is based on a *large deviation estimate* which states that the n-th transfer matrices are close to their average for large n.

1.3 Large deviation estimates

1.3.1 Pre-existing results

Referring to these large deviations (LD) estimates means to evaluate the difference between some quantity and its average. It has become a standard part of the probability theory that deals with the asymptotic behaviour of remote tails of sequences of probability distributions.

In our setting of quasi-periodic Schrödinger cocycles and Lyapunov exponent we are interested in the estimation of

$$\delta_n = \mathrm{Leb}\left\{x \in \mathbf{T} \ :\ \left|\frac{1}{n}\ln\|\mathrm{S}_\mathrm{E}^{(n)}(x)\| - \frac{1}{n}\int_{\mathbf{T}^d}\ln\|\mathrm{S}_\mathrm{E}^{(n)}(x)\|\,\mathrm{d}x\right| \geqslant n^{-\gamma}\right\}$$

Under a *diophantine condition* $\alpha \in \mathrm{DC}(\kappa, \tau)$:

$$\forall n \in \mathbf{Z} \setminus \{0\} \qquad \|n\alpha\|_{\mathbf{Z}} =: \inf_{p \in \mathbf{Z}} |n\alpha - p| \geqslant \frac{\kappa}{|n|^{\tau}} \tag{1.3}$$

Bourgain and Goldstein proved in [BG00] the following estimate: for some $\gamma' > 0$ and all large enough n

$$\delta_n \lesssim \mathrm{e}^{-cn^{\gamma'}} \tag{1.4}$$

Then the authors used it to establish the positivity of the Lyapunov exponent for potentials on $\mathbf{T}^d$ using a central tool in the theory: the *avalanche principle*, developed by Goldstein & Schlag in [GS01]. This principle relates the expansion of a long product of matrices to the product of the individual expansions of the matrices.

It is also a hypothesis that is used to prove continuity of the Lyapunov exponent, Anderson localization, etc.

1.3.2 Our results

At first we take care of the quasi-periodic case. Here we use a method inspired by arguments of [AJS14] (especially the Brownian motion argument for subharmonic functions in section 2) to provide an alternative proof of the estimates in the one dimensional case like Bourgain and Goldstein did. In particular we use potential theory in the complex plane to establish that only an upper estimate on some complex strip is sufficient:

Proposition 1.1

Let us define $\varphi_n : \mathbf{T} \ni x \longmapsto \frac{1}{n} \ln \|\mathrm{S}_{\mathrm{E}}^{(n)}(x)\|$. For (1.4) to be true, it is enough that for all $n \geqslant 1$, the function φ_n admits a subharmonic extension on a fixed strip $\{z \in \mathbf{C} : |\Im\mathfrak{m}\, z| \leqslant \rho\}$ and that for some $b_2 > 0$ and all $0 < b_1 < 1$, for all $x \in \mathbf{T}$

$$\forall 0 < \varepsilon < \rho \qquad \varphi_n(x \pm \mathrm{i}\varepsilon) - \int_{\mathbf{T}} \varphi_n \lesssim \varepsilon \sup_{|\Im\mathfrak{m}\, z| \leqslant \rho} |\varphi_n| + \frac{1}{n^{b_1} \varepsilon^{b_2}} + \frac{1}{n^{1-b_1}} \tag{1.5}$$

Please note that, at this point, we do not need any arithmetic assumption on the frequency α, nor even the precise dynamics: (1.5) and the assumptions on φ_n in the previous proposition are enough to get (1.4).

We also provide a proof of (1.5) using the Brownian motion on $\mathbf{R}^2$:

Proposition 1.2

Under a diophantine condition $\alpha \in \mathrm{DC}(\kappa, \tau)$ (see (1.3)), the estimate (1.5) is true for an appropriate choice $b_2 = \tau + 2^+$ which denotes a quantity greater than $\tau + 2$.

And so for this proposition the diophantine condition (1.3) on α and the type of dynamics (quasi-periodic) are important.

We originally had in mind to use the technique of proof developed in the quasi-periodic setting later in the context of Schrödinger operators with potentials given by the doubling-map to prove positivity with bounds at large coupling and maybe even localization. Results already exist (like [DK05]) but we thought we could remove the condition on almost all initial conditions. The major technical difficulty is that, whereas the quasi-periodic dynamics can easily be extended to the upper half plane and preserves any analyticity strip $\{|\Im\mathfrak{m}\, z| \leqslant \delta\}$, the doubling map exponential growth prevents us to use a very crucial argument of boundedness and *pseudo invariance* for the dynamics of the complexified functions $\varphi_n(z)$. We then chose to look at the inverse branches of the dynamics with the idea that some information obtained in the past could pass on the future. We prove some deviation estimates on the inverse branches. These could lead to the initial goal with a finer analysis.

Theorem 1.4 (LDT for the inverse branches of the doubling-map)

Let us denote m_2 the doubling map on $\mathbf{T} \simeq [\,0\,;1\,]$. We set

$$\mathrm{T} : \varphi \longmapsto \frac{1}{2}(\varphi(\cdot/2) + \varphi(\cdot/2 + 1/2))$$

and $\psi_n = \mathrm{T}^n \varphi_n$ be the (combinatorial) average of φ_n over all the 2^n inverse images for ${m_2}^n$. Then ψ_n verifies an LDT like (1.4).

We recently found new arguments that, joint with the large deviation estimates on the inverse branches, could help to prove quantitative positivity of the Lyapunov exponent without energy exclusion (like in [Krü11]).

1.4 Asymptotic development for small perturbations: the Figotin-Pastur formula

1.4.1 Pre-existing results

Now we investigate some other major results concerning the exponent of Schrödinger cocycles obtained by several authors at small coupling $\varepsilon \mathrm{V}$ with $\varepsilon \ll 1$. It originally begins with the Figotin-Pastur proof of a formula due to Thouless ([Tho79]) in the random setting at small coupling and for energies in the spectrum of the free operator ($H_0 = -\Delta$) away from the band edges and the center: if $\mathbf{V}(v_\omega)$ denotes the variance of the random potential v_ω then

Theorem 1.5 (Figotin, Pastur [PF92], theorem (14.6))

Assume the potential is εv_ω for i.i.d. random variables v_ω bounded by 1 with zero expectation and fix an energy $E \in \,]-2;2[\, \smallsetminus \{0\}$. Then for $\varepsilon > 0$ small enough

$$L_{\varepsilon v_\omega}(E) = \frac{\mathbf{V}(v_\omega)}{2(4-E^2)}\varepsilon^2 + O\left(\frac{\varepsilon^3}{4-E^2}\right)$$

The formula was generalized by Chulaevsky and Spencer in [CS95] to other potentials like Markov automorphisms, hyperbolic toral automorphisms; it particularly applies to the doubling map $x \mapsto 2x \mod 1$ using some tools and tricks of [PF92], namely a change of coordinates called the *Prüfer phase variables* and the spectral density of the random process generated by the v_ω's.

1.4.2 Figotin-Pastur formulæ for quasi-periodic and random potential

In this part we analyze the model where the potential is given by

$$\forall n \in \mathbf{Z} \qquad v_n = \varepsilon\big(V(x+n\alpha) + W_n(\omega)\big) \tag{1.6}$$

for V regular enough on $\mathbf{T}$ (we will precise the condition when needed) and i.i.d. random variables W_n. More precisely we want to get a Figotin-Pastur type formula for the exponent when ε is small.

In chapter 4, we prove a similar expansion formula for diophantine α when the energy is *diophantine w.r.t. α*. That means $\beta = \frac{1}{\pi}\operatorname{Arc}\cos(E/2)$ is diophantine w.r.t. α which is

$$\exists \kappa > 0 \quad \exists \tau > 1 \qquad \forall n \in \mathbf{Z} \smallsetminus \{0\} \qquad \|n\alpha + \beta\|_{\mathbf{Z}} \geqslant \frac{\kappa}{|n|^\tau} \tag{1.7}$$

We write $DC_\alpha(\kappa, \tau)$ the subset of $\beta \in \mathbf{T}$ defined by (1.7). Let us also denote $\mathbf{V}(W_0)$ the variance of the i.i.d. random variables W_n.

Theorem 1.6

For a potential given by (1.6) and all energies $E \in DC_\alpha(\kappa, \tau)$ (see (1.7)), we have the following perturbative development for $\varepsilon \leqslant \varepsilon_0(V, \alpha)$ small enough and for a *quasi-periodic part of the potential* $\mathrm{V} \in \mathscr{C}^{4\tau+11}(\mathbf{T})$

$$\mathrm{L(E)} = \frac{\mathbf{V}(\mathrm{W}_0)}{2(4-\mathrm{E}^2)}\varepsilon^2 + \mathrm{O}_\tau\left(\frac{\varepsilon^3}{\kappa^3(4-\mathrm{E}^2)^3}\right)$$

That means we obtain a Figotin-Pastur type formula for a positive set of energies in the interval $]-2\,;2\,[$, the measure of which depends on the diophantine constant of β w.r.t. α. The exact same main order term given by the variance of the random part of the potential. Hence, if the potential is purely quasi-periodic, the lowest order of the development vanishes. This is coherent with the fact that, in the purely quasi-periodic setting, for ε small enough the set of energies for which the cocycle S is $\mathscr{C}^\omega$-reducible to a constant has full Lebesgue measure by Eliasson's theorem [Eli92] and so the Lyapunov exponent is expected to vanish for almost every energy in the spectrum.

We used some ideas of the work of Malicet who recovers the Figotin-Pastur's formula in [Mal12] using the theory of random diffeomorphisms of the torus $\mathbf{T}^1$ and their associated Lyapunov exponent. We extended some of his results to stationary measures of random diffeomorphisms of $\mathbf{T}^2$ that appear in our context of mixed quasi-periodic and random Schrödinger operators. Once one translates the Lyapunov exponent of the cocycle in terms of Lyapunov exponent of the associated random diffeomorphism, the key is to obtain a perturbative development of any of its stationary measures. As one can easily observe, in our perturbative setting, the same conjugacy made by Figotin and Pastur brings us to study a diffeomorphism that is close to the rotation $\mathrm{R}_{\alpha,\beta} : (x, y) \in \mathbf{T}^2 \mapsto (x + \alpha, y + \beta)$ mod $\mathbf{Z}^2$ where $\mathrm{E} = 2\cos\pi\beta$. In the diophantine case (1.7), the main order reveals to be, as expected, the Lebesgue measure which arises as the only obstruction to solve a cohomological equation for the transfer operator induced by the random diffeomorphism.

What has to be underlined here is that, contrary to theorem 1.5, we do not assume any zero expectation on the global potential (1.6), nor separately for the quasi-periodic part V, or the random part W_ω. However surprising this could be as one would think of insuring the condition (1.7) for all energies by changing a little bit the potential or the energy, we will explain (remark 4.7 in chapter 4) why this is not possible because of the contradiction the arithmetic condition leads to for the boundedness of the potential that we need for our estimates.

The next chapters are dedicated to the treatment of the resonant energies $\mathrm{E} = 2\cos(k\pi\alpha)$, for which the cohomological equation for the rotation $\mathrm{R}_{\alpha,\beta}$ exhibits too many obstructions that are not straightforward computable. In those chapters, we impose a vanishing condition on the random potential W_n in (1.6).

In chapter 5, we treat the first obstruction $k = 0$, that appears when we are at the band edge of the free spectrum. We extend the result of [SSB07] in the purely random case to prove a similar scaling diagram for the exponent which has a scale depending on the speed at which we approach the edge $\mathrm{E} = 2$. We even give (like in [SSB07]) a perturbative development outside the free spectrum.

Theorem 1.7

For $\frac{4}{5} < \eta < \frac{4}{3}$ we have the following asymptotic expansion

$$\mathrm{L}(2 - d\varepsilon^{\eta}) = \frac{\mathbf{V}(\mathrm{W}_\omega)}{8}\frac{\varepsilon^{2-\eta}}{d} + \mathrm{O}\left(d^{-6}(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{4-\frac{5}{2}\eta})\right)$$

For $\mathrm{E} = 2 - d\varepsilon^{4/3}$, the Lyapunov exponent admits the asymptotic development below

$$\mathrm{L}(2 - d\varepsilon^{4/3}) = \frac{\sigma^2\varepsilon^{2/3}}{4d}\left|\int_{\mathbf{T}}(\cos(2\pi y) + \cos^2(2\pi y))\rho(y)\,\mathrm{d}y\right| + \mathrm{O}(d^{-\frac{9}{2}}\varepsilon)$$

where ρ is some density defined by the L^1 normalization solution of an (explicit) first order differential equation.
Finally for $\frac{4}{7} < \eta < \frac{4}{3}$, the Lyapunov exponent can be expanded as follows

$$\mathrm{L}(2 + d\varepsilon^{\eta}) = \sqrt{d}\,\varepsilon^{\eta/2} + \mathrm{O}\left(d^{-3}\left(\varepsilon^{1-\frac{\eta}{4}}, \varepsilon^{\frac{9}{4}\eta-1}, \varepsilon^{\frac{4}{3}\eta}\right)\right)$$

Our technique is still based on the development of any stationary measure of the random diffeomorphism induced on $\mathbf{T}^2$ by the cocycle. So it differs from the one of [SSB07] that relies on estimates for the Birkhoff sums of the projective random dynamical system. The different scales give different types of dynamics for the diffeomorphism (elliptic, parabolic, hyperbolic). We however retrieve some of the issues and technical difficulties of [SSB07].

Finally, in chapter 6, we treat the more complicated case when the diffeomorphism cannot be expanded, up to some order of approximation, to a map with separate variables $(x, y) \mapsto (x+\alpha, g(y))$ that forces us to develop a method of conjugacy for differential operators that naturally appear in this model. The goal is to get rid of the dependency on x on the second factor in order to solve a simpler cohomological equation. We need in this case V to be real analytic. We obtain the following theorem:

Theorem 1.8

For $\alpha \in \mathrm{DC}(\kappa, \tau)$ and an α-resonant energy $\mathrm{E} = 2\cos(k\pi\alpha)$ with $k \neq 0$ for a potential $\varepsilon(\mathrm{V}(x) + \mathrm{W}_\omega)$ with $\delta_k = \mathbf{E}(\mathrm{W}) + \widehat{\mathrm{V}}_0 - \left|\widehat{\mathrm{V}}_k\right| > 0$, the following holds for sufficiently small ε:

$$\mathrm{L}(\mathrm{E}) = \varepsilon^2\int_{\mathbf{T}}\Phi(y)\,\rho(y)\,\mathrm{d}y + \mathrm{O}_{\kappa,k,\delta_k}(\varepsilon^3)$$

for an explicit density ρ and Φ an explicit positive function.

We can also deal with the *parabolic* case $\delta_k = 0$ that differs from the previous one because the first order method used for $\delta_k > 0$ cannot be applied and we have to develop a method of order 2 to solve the cohomological equation.

Theorem 1.9

For $\alpha \in \mathrm{DC}(\kappa, \tau)$ *and an* α*-resonant energy* $\mathrm{E} = 2\cos(k\pi\alpha)$ *with* $k \neq 0$ *for a potential* $\varepsilon(\mathrm{V}(x) + \mathrm{W}_\omega)$ *with* $\delta_k = \mathbf{E}(\mathrm{W}) + \widehat{\mathrm{V}}_0 - |\widehat{\mathrm{V}}_k| = 0$*, the following holds for sufficiently small* ε*:*

$$\mathrm{L}(\mathrm{E}) = \mathrm{A}_\varepsilon + \mathrm{O}_{\kappa,k,\delta_k}\left(\varepsilon^{5/3}\right)$$

where A_ε *either vanishes or is* $\asymp \varepsilon^{4/3}$*, or is* $\mathrm{O}(\varepsilon^{4/3})$*.*

Here the numerical factor seems out of reach because of the complexity of the computations. As in the parabolic case at the band edge of the free spectrum, we face a situation where it is unclear whether the lowest order of our development vanishes or not. However not entirely satisfactory this is, it still provides a perturbative formula. We shall elaborate more on this issue in remark 6.7. Note that these two last theorems highlight the difference of nature of the two resonant problems $k = 0$ and $k \neq 0$ and, within these two conditions depending on the condition on δ_k, there are substantial differences of behaviour. This can also be compared to the results of [SSB07] where the resonant case $k = 0$ is treated as we did for theorem 1.7. All resonances are then not similarly treatable.

Part I

Large deviation estimates

CHAPTER 2

Large deviation estimates for Schrödinger cocycles using Brownian motion

2.1 Setting and main results

Let us begin by making our setting explicit. Let $v \in \mathscr{C}^\omega(\mathbf{T}, \mathbf{R})$ be a real analytic function on the torus, commonly named the *potential*, $\mathrm{E} \in \mathbf{R}$ be the *energy* and $\alpha \in \mathbf{T} \setminus \mathbf{Q}/\mathbf{Z}$ the *frequency*. Then we define the elementary transfer matrix

$$\mathrm{S}_{\mathrm{E},v}(x) = \begin{pmatrix} \mathrm{E} - v(x) & -1 \\ 1 & 0 \end{pmatrix}$$

and the associated Schrödinger cocycle

$$(\alpha, \mathrm{S}) \colon \begin{cases} \mathbf{T} \times \mathbf{R}^2 \longrightarrow \mathrm{T} \times \mathbf{R}^2 \\ \quad (x, u) \ \longmapsto (x + \alpha, \mathrm{S}_{\mathrm{E},v}(x) u) \end{cases}$$

We denote $(\alpha, \mathrm{S}_\mathrm{E})^n = (n\alpha, \mathrm{S}_\mathrm{E}^{(n)})$ where

$$\forall n \in \mathbf{N} \qquad \mathrm{S}_\mathrm{E}^{(n)}(x) = \mathrm{S}_{\mathrm{E},v}(x + (n-1)\alpha) \cdots \mathrm{S}_{\mathrm{E},v}(x+\alpha)\, \mathrm{S}_{\mathrm{E},v}(x) = \prod_{k=n-1}^{0} \mathrm{S}_\mathrm{E}(x + k\alpha)$$

We keep the dependence on all the parameters E, v, α implicit when no confusion is possible. Since we consider the potential fixed and study the results when the energy varies, the dependence on E is more likely to be pointed out. Let us denote $\|\cdot\|$ the operator norm associated with the euclidean norm on $\mathbf{C}^2$. Let us remind that $\|\mathrm{A}\|$ is given by the singular values of the matrix A, namely the roots of the eigenvalues of $\mathrm{A}\,{}^t\mathrm{A}$. The quantity

$$\mathrm{L}_n = \mathrm{L}_n(\alpha, \mathrm{E}) = \frac{1}{n} \int_\mathbf{T} \ln \|\mathrm{S}_\mathrm{E}^{(n)}(x)\| \,\mathrm{d}x$$

defines a subadditive sequence bounded from below. It is then a general result that it converges as n goes to $\pm\infty$, which defines the *Lyapunov exponent* of the cocycle (α, S), one has

$$\mathrm{L}(\mathrm{E}) = \mathrm{L}(\mathrm{E}, \alpha, v) = \inf_{n \to \pm+\infty} \frac{1}{n} \int_\mathbf{T} \ln \|\mathrm{S}_\mathrm{E}^{(n)}(x)\| \,\mathrm{d}x$$

We set for all $x \in \mathbf{T}$

$$\varphi_n(x) = \frac{1}{n} \ln \|\mathrm{S}_\mathrm{E}^{(n)}(x)\| \tag{2.1}$$

so that

$$\mathrm{L}_n = \int_\mathbf{T} \varphi_n = \int_0^1 \varphi_n \tag{2.2}$$

where we abusively write $\int_\mathrm{I} \varphi_n$ for the integral on a interval $\mathrm{I} \subset \mathbf{R}$ of any lift of φ_n on $\mathbf{R}$, that is we identify a map on the torus with the associated 1-periodic function on $\mathbf{R}$.

Our goal is to obtain some estimates for the following measure

$$\mathrm{Leb}\left\{x \in \mathbf{T} \,:\, |\varphi_n(x) - \mathrm{L}_n| \geqslant n^{-\gamma}\right\}$$

for a suitable $\gamma > 0$, and more explicitly, to get an exponentially small measure like in the estimates of Bourgain and Goldstein in [BG00]. This means that the function φ_n is close to its integral for large n.

We need some arithmetic assumptions on α. Let us recall some classical definitions used in the theory of quasi-periodic Schrödinger cocycles.

Definition 2.1 (Diophantine class)

We write for $\kappa, \tau > 0$ that $\alpha \in \mathrm{DC}(\kappa, \tau)$ if

$$\forall n \in \mathbf{Z} \setminus \{0\} \qquad \|n\alpha\|_{\mathbf{T}} = d(n\alpha, \mathbf{Z}) \geqslant \frac{\kappa}{|n|^{\tau}} \tag{2.3}$$

In this case, we say that α is in some diophantine class *(or is diophantine) with* constant κ *and* exponent τ*. For all $\tau > 1$, the set $\mathrm{DC}(\tau) = \cup_{\kappa>0}\mathrm{DC}(\kappa, \tau) \subset \mathbf{T}$ has full Haar-Lebesgue measure.*

We prove indeed under these arithmetic conditions:

Theorem 2.1

Let us assume that α satisfies a diophantine condition $\mathrm{DC}(\kappa, \tau)$*. Then, for some constants $a, a' \in \,]\,0\,;1\,[$ depending on κ, τ we have the deviation estimate:*

$$\exists c > 0 \quad \forall n \gg 1 \qquad \mathrm{Leb}\left\{x \in [\,0\,;1\,] \,:\, |\varphi_n(x) - \mathrm{L}_n| \geqslant n^{-a}\right\} \leqslant c\,\mathrm{e}^{-n^{a'}}$$

Remark 2.1

We shall see in the proofs that a, a' depend on κ, τ but cannot be easily written as functions of these parameters.

2.2 Ideas and strategy of the proof

The function defined by (2.1) admits a subhmarmonic extension to a complex strip, so our result (theorem 2.1) actually deals with estimates of the difference between a subharmonic function in the complex plane and its average. It is then no surprise indeed to get estimates for subharmonic functions that one controls at the boundary of their domain. Actually, the map φ_n also satisfies some pseudo-invariance in α for large n which we should elaborate later on in lemma 2.1.

The relevance to complexify and exploit subharmonic properties could be said "classic" and particularly dates back to Herman's subharmonic trick ([Her83]), or a generalization of a Jensen formula by Sorets and Spencer ([SS91]), or also Bourgain and Goldstein ([BG00]). One main idea used by Bourgain and Goldstein (see eq. (1.11) in [BG00]) is the following: the Fourier coefficients of a subharmonic function behave like $\mathrm{O}(|n|^{-1})$ as $n \to +\infty$ (see also corollary 4.7 in [Bou04]).

Yet, what is remarkable here is that upper estimates for φ_n on a complex strip in the complex plane like proposition 2.1 actually give lower bounds without using the Fourier coefficients argument, nor even the type of dynamics used to define the potential of the Schrödinger operator. Our proof in the quasi-periodic setting relies on what we call *subharmonic properties of the Brownian motion*. This means the link between the value of a subharmonic function at z in a domain and the expectation of $\varphi(\mathrm{B}_\tau)$ for (B_t) a Brownian motion starting at z and exiting the domain at time τ.

It is interesting to the Brownian motion (BM) in this context because of its geometrical properties that give access to more flexibility in order to get and idea of and then prove the estimates. Note that these properties were also used on another form in section 2 of [AJS14]. So, although the result is not new, the new proof we give illustrates the workability of these BM techniques in this setting.

The scheme of the proof is the following.

(i) Firstly we notice in lemma 2.1 the *pseudo invariance in α for φ_n*: $\varphi_n(x+k\alpha) \simeq \varphi_n(x)$ for all x and $k \simeq \sqrt{n}$. This is a heuristic argument for φ_n to be close to the constant $\int_{\mathbf{T}} \varphi_n$ since $x \mapsto x+\alpha$ is ergodic for irrational α, and so for theorem 2.1 to be true.

(ii) Then we prove upper estimates for the subharmonic extension of φ_n in proposition 2.1 from which we deduce in proposition 2.2 that the upper estimate needed in theorem 2.1 is true for all x and not only outside a subset of exponentially small measure. We begin by recalling in subsection 2.4.1 the harmonic and subharmonic properties of the Brownian motion we will use later on. This leads us to compare the exit point of the Brownian motion from our strip of analytic continuation with the exit point from the upper half-plane. We prove in lemma 2.3 that they are as close as the imaginary part of the starting point of the BM. As we know the distribution (Poisson kernel) for the upper half-plane we then study convergence properties of this kernel in proposition 2.4 that also require some quantitative control of the speed of convergence for Birkhoff's sums over the irrational rotation $x \mapsto x+\alpha$ for diophantine α (lemma 2.4). Putting all the previous estimates together and with some optimization in the parameters gives the result.

(iii) The next step is to take care of the lower bound (proposition 2.5). We first present a *guiding principle* (subsection 2.5.1) that highlights how the subharmonic properties of φ_n require to estimate the probability the Brownian motion exits some domain (look at figure 2.3) limited by two boundaries consisting in a circle and in the set K_n (see (2.15)) of the *bad points*, the measure of which we want to control in theorem 2.1. Indeed we prove that this probability is of order $-1/(\ln \mathrm{Leb}\,\mathrm{K}_n)$.

After some reminder about potential theory that we will use, we construct an explicit map $\widetilde{\mathrm{G}}$ (see (2.28)) with the potential and the equilibrium measure of the compact set of the bad points. This maps is actually a kind of Green function for the domain. We prove (lemma 2.7) that it agrees almost everywhere with the exit probability from some domain for which one boundary is K_n. Then easy estimates for the map $\widetilde{\mathrm{G}}$ give the estimates for the exit probability in corollary 2.1.

(iv) Finally we apply the estimates to obtain a contradiction of type $\int_{\mathbf{T}} \varphi_n < \int_{\mathbf{T}} \varphi_n$ if theorem 2.1 is not true.

2.3 Elementary bounds for the growth of the cocycle

Since v is analytic and $\mathbf{T}$ is compact, one has, by submultiplicativity of the norm

$$\forall x \in \mathbf{T} \qquad \|\mathrm{S}_{\mathrm{E}}^{(n)}(x)\| \leqslant \mathrm{C}^n$$

where $\mathrm{C} = \sup_{x\in\mathbf{T}} \|\mathrm{S}_{\mathrm{E}}(x)\| < +\infty$. Let us recall that for every $\mathrm{A} \in \mathrm{SL}_2(\mathbf{C})$ one has $\|\mathrm{A}\| \geqslant 1$. Choosing the operator norm associated with the euclidean norm on $\mathbf{R}^2$ we can add that $\|\mathrm{A}\| = \|\mathrm{A}^{-1}\|$. It follows that φ_n is bounded by $\ln \mathrm{C}$. One can get finer results about the behaviour of the translates of φ_n:

Lemma 2.1

We have the following pseudo invariance property

$$\forall x \in \mathbf{T} \quad \forall n \in \mathbf{N} \smallsetminus \{0\} \quad \forall k \in \mathbf{Z} \qquad |\varphi_n(x+k\alpha) - \varphi_n(x)| \leqslant c\frac{|k|}{n} \tag{2.4}$$

where $c = 2\ln \sup_{x\in\mathbf{T}} \|\mathrm{S}_{\mathrm{E}}(x)\|$.

This expresses a kind of pseudo-periodicity in α for φ_n, if one thinks, for example, of k being of order $\sqrt{n}$.

Proof.

This result (of [BG00]) comes from the fact that

$$\mathrm{S}_{\mathrm{E}}^{(n)}(x+k\alpha) = \underbrace{\prod_{j=k+n-1}^{n} \mathrm{S}_{\mathrm{E}}(x+j\alpha)}_{\Pi_1} \cdot \mathrm{S}_{\mathrm{E}}^{(n)}(x) \cdot \underbrace{\prod_{i=0}^{k-1} \mathrm{S}_{\mathrm{E}}(x+i\alpha)^{-1}}_{\Pi_2}$$

So that, $$\|\mathrm{S}_{\mathrm{E}}^{(n)}(x+k\alpha)\| \leqslant \mathrm{C}^{2|k|}\|\mathrm{S}_{\mathrm{E}}^{(n)}(x)\|$$

Using $\mathrm{S}_{\mathrm{E}}^{(n)}(x) = {\Pi_1}^{-1}\mathrm{S}_{\mathrm{E}}^{(n)}(x+k\alpha){\Pi_2}^{-1}$ one gets the other required estimate $\|\mathrm{S}_{\mathrm{E}}^{(n)}(x)\| \leqslant \mathrm{C}^{2|k|}\|\mathrm{S}_{\mathrm{E}}^{(n)}(x+k\alpha)\|$. This implies

$$-2\,|k|\ln \mathrm{C} \leqslant \ln \frac{\|\mathrm{S}_{\mathrm{E}}^{(n)}(x+k\alpha)\|}{\|\mathrm{S}_{\mathrm{E}}^{(n)}(x)\|} \leqslant 2\,|k|\ln \mathrm{C}$$

which proves $$|\varphi_n(x+k\alpha) - \varphi_n(x)| \leqslant c\,|k|\,/n$$

as expected with $c = 2\ln \mathrm{C}$.

■

We have established the uniform (w.r.t x) estimate:

$$\varphi_n(x+k\alpha) = \varphi_n(x) + \varepsilon_{n,k} \qquad \text{with} \qquad |\varepsilon_{n,k}| \leqslant c\,|k|\,/n$$

2.4 Upper estimates

Let us fix the energy E. If b is some real number, we denote b^+ any quantity $b+\delta$ where $\delta > 0$ can eventually be small enough to fit in some criteria required on b. We will prove the following:

Proposition 2.1

Let us assume $\alpha \in \mathrm{DC}(\kappa,\tau)$ (see definition 2.3). For all integer $n \geqslant 1$, $x \in \mathbf{R}$ and all $0 < c < 1$, the following inequality holds for $0 < \varepsilon \leqslant \rho$

$$\varphi_n(x \pm \mathrm{i}\varepsilon) - \textstyle\int_0^1 \varphi_n \lesssim \dfrac{\varepsilon}{\rho}\sup_{|\Im\mathfrak{m}\, z| \leqslant \rho}|\varphi_n| + \dfrac{1}{n^c\,\varepsilon^{\tau+2^+}} + \dfrac{1}{n^{1-c}}$$

As $\int_0^1 \varphi_n = \mathrm{L}_n$ (remind (2.2)), choosing adequate b, c and $\varepsilon \asymp n^{-b}$, we then get what we want by subharmonicity of φ_n:

Proposition 2.2

For $\alpha \in \mathrm{DC}(\kappa,\tau)$, $a = \frac{1}{\tau+4^+} > 0$ and some $c' > 0$ and all $n \geqslant 1$

$$\forall x \in \mathbf{T} \qquad \varphi_n(x) \leqslant \mathrm{L}_n + c'n^{-a}$$

Remark 2.2

Proposition 2.2 states that the upper inequality required on the set in theorem (2.1) is fulfilled for all x and not only outside a set of x's of exponentially small measure.

2.4.1 Complex extension of φ_n and use of the Brownian motion

Since we consider real analytic potentials v, the map $\mathrm{S}_{\mathrm{E}}^{(n)}$ extends to an holomorphic function on a strip $\mathrm{S}_\rho = \{|\Im\mathfrak{m}\, z| \leqslant \rho\}$ in the complex plane that we denote $\mathrm{S}_{\mathrm{E}}^{(n)}(z) \in \mathrm{SL}_2(\mathbf{C})$. Hence, φ_n has a subharmonic and continuous extension to the strip S_ρ still written $\varphi_n(z)$. Let us fix $\mathcal{C}$ the open square in $\mathbf{R}^2$ of vertices $(0,0), (0,\rho), (1,\rho)$ and $(1,0)$. Moreover we denote its four edges $\mathcal{C}_{\mathrm{u}} = [\,\mathrm{i}\rho\,;\mathrm{i}\rho+1\,]$, $\mathcal{C}_{\mathrm{r}} = [\,1+\mathrm{i}\rho\,;1\,]$, $\mathcal{C}_{\mathrm{d}} = [\,0\,;1\,]$ and $\mathcal{C}_{\mathrm{l}} = [\,0\,;\mathrm{i}\rho\,]$. The subscripts u, r, d, and l respectively mean up, right, down and left.

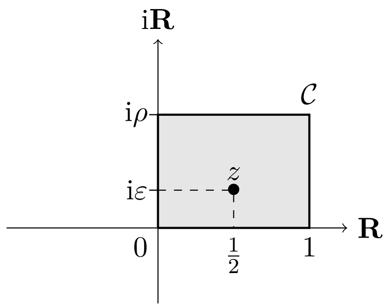

Figure 2.1: The domain $\mathcal{C}$

In order to prove proposition 2.1 we now want to get upper estimates for some one-periodic function that is continuous and subharmonic on the domain $\mathcal{C}$. The introduction of the Brownian motion (BM) is then useful if one recalls (see theorems 3.5 and 3.12 in [MP10]) that for any harmonic function ϕ on a bounded domain D, continuous on ∂D

$$\phi(z) = \mathbf{E}_z(\phi(\mathrm{B}_\tau)) \tag{2.5}$$

where $(\mathrm{B}_t)_{t\geqslant 0}$ is a Brownian motion starting in the domain D with

$$\tau = \tau_{\mathrm{D}} = \inf\{t \geqslant 0 \,:\, \mathrm{B}_t \in \partial\mathrm{D}\} \tag{2.6}$$

and $\mathbf{E}_z$ (resp. $\mathbf{P}_z$) denotes the conditional expectation (resp. probability) knowing that the Brownian motion starts at $z \in \mathrm{D}$, that is, the conditional expectation (resp. probability) knowing $\mathrm{B}_0 = z$. The domain D has to satisfy some conditions and for example the *Poincaré cone condition* (see definition 3.10 in [MP10]). Note that our domain $\mathrm{D} = \mathcal{C}$ is bounded and satisfies the Poincaré cone condition.

Moreover (theorem 8.5 in [MP10]) without any assumption on the domain D other than its boundedness, the map $\mathrm{D} \ni z \mapsto \mathbf{E}_z\phi(\mathrm{B}_\tau)$ is harmonic for continuous $\phi : \partial\mathrm{D} \to \mathbf{R}$. So the maximum principle ensures that if ϕ is only subharmonic on D and continuous on ∂D

$$\phi(z) \leqslant \mathbf{E}_z(\phi(\mathrm{B}_\tau)) \tag{2.7}$$

Indeed, the difference $z \mapsto \phi(z) - \mathbf{E}_z(\phi(\mathrm{B}_\tau))$ has non negative Laplacian and so is subharmonic, and also vanishes on ∂D.

To be completely rigorous, since in our model φ_n can only be extended to some strip $\{|\Im\mathfrak{m}\, z| \leqslant \rho\}$ we need to stop the Brownian motion when it escapes from the strip, otherwise $\varphi_n(\mathrm{B}_t)$ makes no sense.

2.4.2 The main idea: the closest exit is on the real axis

Let τ the exit time (see (2.6)) from the open square $\mathcal{C}$. The main idea is that, if we are close enough to the real axis, the Brownian motion will exit from $\mathcal{C}$ by it with great probability, and so everything goes like we were just considering when a Brownian motion escapes from the upper half-plane.

Hence, our goal is to obtain information about the distribution of the exit point B_τ of a Brownian motion starting at $x + \mathrm{i}\varepsilon$ for any $x \in [\,0\,;1\,]$. We prove that it is close to the law of the exit point B_T of a Brownian motion on $\mathbf{H}^+$ with exit time T from the upper half-plane, for which we know the distribution:

$$\forall\varphi \in \mathscr{C}_b^0(\mathbf{R}) \quad \forall z \in \mathbf{H}^+ \qquad \mathbf{E}_z(\varphi(\mathrm{B}_\mathrm{T})) = \int_\mathbf{R} \varphi(t)\,\mathrm{d}\mathbf{P}_z(t)$$

where $\mathrm{d}\mathbf{P}_z(t) = \mathrm{P}_z(t)\,\mathrm{d}t$ with P_z the Poisson kernel for the upper half-plane:

$$\mathrm{P}_{x+\mathrm{i}y}(t) = \frac{1}{\pi}\frac{y}{(x-t)^2 + y^2} \tag{2.8}$$

Indeed, the harmonic measure (see §4 of chapter 3 and theorem 3.43 in [MP10]) of the unit disk with base point 0 is the Lebesgue-Haar measure and there is a conformal mapping from the upper half-plane onto the disk that maps z to 0. This gives the result as the Brownian motion is invariant under conformal mapping (theorem 7.19 in [MP10] and see also theorem 2.33 in the same reference).

This allows to use the equality

$$\mathbf{E}_{x+\mathrm{i}\varepsilon}(\varphi(\mathrm{B}_\mathrm{T})) = \frac{1}{\pi}\int_\mathbf{R} \varphi(t)\,\frac{\varepsilon}{(x-t)^2 + \varepsilon^2}\,\mathrm{d}t \tag{2.9}$$

with our function φ_n to get the estimates we want, provided we can manage the error terms.

2.4.3 Estimates for the probability to leave the domain by the real axis

We now prove that the BM most likely exits from the domain $\mathcal{C}$ (see figure 2.1) on the real axis:

Proposition 2.3

The probability to leave $\mathcal{C}$ by another side than the real axis is ε-small: for some $c_1 > 0$

$$\forall \varepsilon > 0 \qquad \mathbf{P}_{\frac{1}{2}+\mathrm{i}\varepsilon}(\mathrm{B}_\tau \notin [\,0\,;1\,]) \leqslant c_1\,\varepsilon/\rho$$

Proof.

Let us denote $\mathrm{W} = \mathcal{C}_\mathrm{r} \cup \mathcal{C}_\mathrm{l}$ the vertical edges of $\mathcal{C}$ and

$$\mathscr{C}_i = \{\mathrm{B}_\tau \in \mathcal{C}_i\} \qquad i \in \{\mathrm{u}, \mathrm{d}, \mathrm{r}, \mathrm{l}\}$$

By harmonicity of $\psi : z \mapsto \Im\mathfrak{m}\, z$, one has with (2.5) that for $z \in \mathcal{C}$

$$\begin{aligned}\Im\mathfrak{m}\, z = \psi(z) &= \mathbf{E}_z(\psi(\mathrm{B}_\tau)) \\ &= 0 \times \mathbf{P}_z(\mathscr{C}_\mathrm{d}) + \rho \times \mathbf{P}_z(\mathscr{C}_\mathrm{u}) + \mathbf{E}_z(\Im\mathfrak{m}\,\mathrm{B}_\tau \mid \mathscr{C}_\mathrm{r})\mathbf{P}_z(\mathscr{C}_\mathrm{r}) + \mathbf{E}_z(\Im\mathfrak{m}\,\mathrm{B}_\tau \mid \mathscr{C}_\mathrm{l})\mathbf{P}_z(\mathscr{C}_\mathrm{l})\end{aligned}$$

Since all terms are non negative, one gets

$$\mathbf{P}_z(\mathrm{B}_\tau \in \mathcal{C}_\mathrm{u}) \leqslant \frac{\Im\mathfrak{m}\, z}{\rho} \tag{2.10}$$

So that the probability for B_t to exit $\mathcal{C}$ by the upper edge is small if z is close to the real axis.

Now we want to bound $\mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{W}) = \mathbf{E}\{\mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{W} \mid \tau)\}$. We can write (see definition 2.1 in [MP10])

$$\mathrm{B}_t = \left(\mathrm{B}_t^{(1)}, \mathrm{B}_t^{(2)}\right)$$

where the $(\mathrm{B}_t^{(i)})$'s are unidimensional and independent Brownian motions. Let T be the exit time of (the one-dimensional Brownian motion) $\mathrm{B}_t^{(1)}$ starting at 1/2:

$$\mathrm{T} = \inf\left\{t \geqslant 0 \;:\; \mathrm{B}_t^{(1)} \in \{0\,;1\} \mid \mathrm{B}_0^{(1)} = 1/2\right\}$$

So, $$\{\mathrm{B}_\tau \in \mathrm{W} \mid \mathrm{B}_0 = 1/2 + \mathrm{i}\varepsilon\} = \left\{\forall t < \mathrm{T} \;:\; 0 < \mathrm{B}_t^{(2)} < \rho \mid \mathrm{B}_0^{(2)} = \varepsilon\right\}$$

Thus, $$\mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{W}) = \mathbf{P}_\varepsilon\left(\forall t < \mathrm{T} \quad 0 < \mathrm{B}_t^{(2)} < \rho\right) \leqslant \mathbf{P}_\varepsilon\left(\forall t < \mathrm{T} \quad 0 < \mathrm{B}_t^{(2)}\right)$$

with $z = 1/2 + \mathrm{i}\varepsilon$. The translation invariance (which directly comes from the definition – see 1.1 in [MP10]–) of the Brownian motion implies

$$\mathrm{T} \overset{d}{=} \mathrm{T}' = \inf\{t \geqslant 0 \,:\, b_t = \pm 1/2 \mid b_0 = 0\}$$

for a 1-dimensional Brownian motion $(b_t)_{t\geqslant 0}$. So

$$\mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{W}) \leqslant \mathbf{E}\left\{\mathbf{P}_\varepsilon\left(\forall t < \mathrm{T}' \quad b_t > 0 \mid \mathrm{T}'\right)\right\}$$

By symmetry of the BM (which also comes from definition 1.1 in [MP10]) we get

$$\begin{aligned}
\mathbf{P}_\varepsilon\left(\forall t < \mathrm{T}' \quad b_t > 0 \mid \mathrm{T}'\right) &= \mathbf{P}_\varepsilon\Big(\inf_{[0\,;\mathrm{T}']} b_t > 0 \mid \mathrm{T}'\Big) \\
&= \mathbf{P}_0\Big(\inf_{[0\,;\mathrm{T}']} b_t > -\varepsilon \mid \mathrm{T}'\Big) \\
&= \mathbf{P}_0\Big(\sup_{[0\,;\mathrm{T}']} b_t < \varepsilon \mid \mathrm{T}'\Big) \\
\mathbf{P}_\varepsilon\left(\forall t < \mathrm{T}' \quad b_t > 0 \mid \mathrm{T}'\right) &= \mathbf{P}_0\left(|b_{\mathrm{T}'}| < \varepsilon \mid \mathrm{T}'\right)
\end{aligned}$$

where the last equality follows from the equality $\sup_{[0\,;t]} b_s \overset{d}{=} |b_t|$ knowing that $b_0 = 0$ (see theorem 2.18 in [MP10]). Finally, the scaling invariance (lemma 1.7 in[MP10]) ensures that

$$\mathbf{P}_\varepsilon\left(\forall t < \mathrm{T}' \quad b_t > 0 \mid \mathrm{T}'\right) = \mathbf{P}_0\left(|b_1| < \varepsilon/\sqrt{\mathrm{T}'} \mid \mathrm{T}'\right)$$

Hence, $$\mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{W}) \leqslant \mathbf{E}\left(\int_{-\varepsilon/\sqrt{\mathrm{T}'}}^{\varepsilon/\sqrt{\mathrm{T}'}} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x\right) \leqslant \frac{2\varepsilon}{\sqrt{2\pi}}\,\mathbf{E}\left(\frac{1}{\sqrt{\mathrm{T}'}}\right)$$

It remains to prove the following lemma:

Lemma 2.2

The random variable $1/\sqrt{\mathrm{T}'}$ has finite expectation.

Proof.

For this purpose, let us decompose

$$\frac{1}{\sqrt{\mathrm{T}'}} = \frac{1}{\sqrt{\mathrm{T}'}}\mathbf{1}_{\mathrm{T}'\geqslant 1} + \frac{1}{\sqrt{\mathrm{T}'}}\mathbf{1}_{\mathrm{T}'<1}$$

When $\mathrm{T}' \geqslant 1$, we have $\dfrac{1}{\sqrt{\mathrm{T}'}} \leqslant 1$ and when $\mathrm{T}' < 1$, we have $\sqrt{\mathrm{T}'} > \mathrm{T}'$ so that

$$\mathbf{E}\left(\frac{1}{\sqrt{\mathrm{T}'}}\right) \leqslant 1 + \mathbf{E}\left(\frac{1}{\mathrm{T}'}\right)$$

It is known (see proposition 3.7 in [RY13]) that T' has Laplace transform

$$\forall \lambda \in \mathbf{R}_+ \qquad \mathcal{L}_{\mathrm{T}'}(\lambda) = \frac{1}{\mathrm{ch}\left(\frac{1}{2}\sqrt{2\lambda}\right)}$$

Since $\mathcal{L}_{T'}(x) \underset{x\to+\infty}{\sim} 2e^{-\sqrt{2x}/2}$ we can deduce that $\mathcal{L}_{T'} \in L^1(\mathbf{R}_+, \mathrm{Leb})$ so that

$$\mathbf{E}\left(\frac{1}{T'}\right) = \int_0^{+\infty} \mathcal{L}_{T'}(x)\,\mathrm{d}x < +\infty$$

■

Finally we get

$$\forall \varepsilon > 0 \qquad \mathbf{P}_{\frac{1}{2}+\mathrm{i}\varepsilon}(B_\tau \in W) \leqslant c_2\varepsilon \quad \text{with} \quad c_2 = \sqrt{\frac{\pi}{2}}\left(1 + \int_0^{+\infty} \frac{\mathrm{d}\lambda}{\mathrm{ch}\left(\sqrt{2\lambda}/2\right)}\right)$$

Adding the estimate (2.10) we obtain the proposition.

■

Remark 2.3

In fact we could prove a similar (uniform in x) estimate $\mathbf{P}_{x+\mathrm{i}\varepsilon}(B_\tau \notin [\,0\,;1\,]) \lesssim \varepsilon/\rho$ as long as $x \in [\,\delta\,;1-\delta'\,]$ with $0 < \delta, \delta' < 1/2$ is far enough from the vertical edges of $\mathcal{C}$.

2.4.4 Estimating the law of the exit point

Let us denote the exit time from the square $\mathcal{C}_x$ translated by the vector $(x - 1/2, 0)$

$$\tau = \inf\,\{t \geqslant 0 \;:\; B_t \in \mathcal{C}_x\}$$

We define the exit time from the upper half-plane

$$T = \inf\,\{t \geqslant 0 \;:\; B_t \in \mathbf{R}\}$$

We will prove that, knowing $B_0 = x + \mathrm{i}\varepsilon$, the distributions of B_T and B_τ are ε-close:

Lemma 2.3

For all real bounded continuous function φ on $S_\rho = \{0 \leqslant \Im\mathfrak{m}\, z \leqslant \rho\}$ one has for $\varepsilon \leqslant \rho$ and uniformly in x:

$$|\mathbf{E}_{x+\mathrm{i}\varepsilon}(\varphi(B_T)) - \mathbf{E}_{x+\mathrm{i}\varepsilon}(\varphi(B_\tau))| \leqslant 4c_1 \sup\nolimits_{S_\rho} |\varphi| \frac{\varepsilon}{\rho}$$

Proof.

Let us set $\|\varphi\|_0 = \sup_{S_\rho} |\varphi|$. It is clear from their definitions that

$$\{T \leqslant \tau\} = \{T = \tau\} = \{B_\tau \in [\,x - 1/2\,; x + 1/2\,]\}$$

For $z = x + \mathrm{i}\varepsilon$ and $I_x = [\,x - 1/2\,; x + 1/2\,]$ we have

$$\mathbf{E}_z\left[\varphi(B_\tau) \mid B_\tau \in I_x\right] = \mathbf{E}_z\left[\varphi(B_\tau) \mid T = \tau\right] = \mathbf{E}_z\left[\varphi(B_T) \mid T = \tau\right]$$

Then we obtain

$$\mathbf{E}_z(\varphi(\mathrm{B}_\tau)) = \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{T} \leqslant \tau\right] \mathbf{P}_z(\mathrm{T} \leqslant \tau) + \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{T} > \tau\right] \mathbf{P}_z(\mathrm{T} > \tau)$$

So that

$$\begin{aligned}\mathbf{E}_z(\varphi(\mathrm{B}_\mathrm{T})) - \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{B}_\tau \in \mathrm{I}_x\right] &= \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{T} = \tau\right] \left(\mathbf{P}_z(\mathrm{T} = \tau) - 1\right) \\ &\quad + \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{T} > \tau\right] \mathbf{P}_z(\mathrm{T} > \tau) \\ &= \left(\mathbf{E}_z\left[\varphi(\mathrm{B}_\mathrm{T}) \mid \mathrm{T} > \tau\right] - \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{T} > \tau\right]\right) \mathbf{P}_z(\mathrm{T} > \tau)\end{aligned}$$

Hence,

$$|\mathbf{E}_z(\varphi(\mathrm{B}_\mathrm{T})) - \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{B}_\tau \in \mathrm{I}_x\right]| \leqslant 2\|\varphi\|_0\, \mathbf{P}_z(\mathrm{T} > \tau) = 2\|\varphi\|_0\, \mathbf{P}_z(\mathrm{B}_\tau \notin \mathrm{I}_x)$$

The invariance of the Brownian motion by translation gives:

$$\begin{aligned}\mathbf{P}_z(\mathrm{B}_\tau \notin \mathrm{I}_x) &= 1 - \mathbf{P}_{1/2+\mathrm{i}\varepsilon+x-1/2}(\mathrm{B}_\tau - x + 1/2 \in [\,0\,;1\,]) \\ &= 1 - \mathbf{P}_{1/2+\mathrm{i}\varepsilon}(\mathrm{B}_\tau \in [\,0\,;1\,]) \\ \mathbf{P}_z(\mathrm{B}_\tau \notin \mathrm{I}_x) &= \mathbf{P}_{1/2+\mathrm{i}\varepsilon}(\mathrm{B}_\tau \notin [\,0\,;1\,]) \leqslant c_1\, \varepsilon/\rho \quad \text{(prop 2.3)}\end{aligned}$$

So that

$$|\mathbf{E}_z(\varphi(\mathrm{B}_\mathrm{T})) - \mathbf{E}_z\left\{\varphi(\mathrm{B}_\tau) \mid \mathrm{B}_\tau \in \mathrm{I}_x\right\}| \leqslant 2\|\varphi\|_0\, c_1\, \varepsilon/\rho$$

Finally let us write

$$\mathbf{E}_z(\varphi(\mathrm{B}_\tau)) = \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{B}_\tau \in \mathrm{I}_x\right] \mathbf{P}(\mathrm{B}_\tau \in \mathrm{I}_x) + \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{B}_\tau \notin \mathrm{I}_x\right] \mathbf{P}(\mathrm{B}_\tau \notin \mathrm{I}_x)$$

Consequently, $$|\mathbf{E}_z\left[\varphi(\mathrm{B}_\tau)\right] - \mathbf{E}_z\left[\varphi(\mathrm{B}_\tau) \mid \mathrm{B}_\tau \in \mathrm{I}_x\right]| \leqslant 2\|\varphi\|_0\, c_1\, \varepsilon$$

from which the lemma easily follows by the triangular inequality. ∎

We can deduce that, for any continuous bounded real function φ

$$\mathbf{E}_{x+\mathrm{i}\varepsilon}(\varphi(\mathrm{B}_\tau)) = \frac{1}{\pi}\int_{\mathbf{R}} \varphi(t)\, \frac{\varepsilon}{(x-t)^2+\varepsilon^2}\, \mathrm{d}t + \mathrm{O}(\|\varphi\|_0 \varepsilon)$$

Now we want to see that for n large enough:

$$\frac{1}{\pi}\int_{\mathbf{R}} \varphi_n(t)\, \frac{\varepsilon}{(x-t)^2+\varepsilon^2}\, \mathrm{d}t \simeq \int_0^1 \varphi_n = \mathrm{L}_n$$

2.4.5 Convergence properties of the Poisson kernel

Let us first recall some notations we will use throughout the manuscript. If f, g are real functions or sequences we write $f \lesssim g$ or $g \gtrsim f$ for an inequality $f(x) \leqslant cg(x)$ for all x where c is a positive constant that does not need to be taken account into. If both $f \lesssim g$ and $f \gtrsim g$ are true, we write $f \asymp g$. We will prove:

Proposition 2.4

For all integer $n \geqslant 1$ and all $\mathrm{K} \leqslant n$:

$$\left|\int_{\mathbf{R}} \varphi_n \, \mathrm{P}_{x+\mathrm{i}\varepsilon}(t)\,\mathrm{d}t - \int_0^1 \varphi_n(t)\,\mathrm{d}t\right| \lesssim \frac{\mathrm{K}}{n} + \frac{1}{\mathrm{K}\varepsilon^{\tau+2^+}}$$

where $\mathrm{P}_{x+\mathrm{i}\varepsilon}$ is the Poisson kernel defined by (2.8), $\tau > 1$ is the exponent in the diophantine condition of $\alpha \in \mathrm{DC}(\kappa,\tau)$ (see definition 2.3).

Let us write $\varphi = \varphi_n$. Let us recall that φ is a real one-periodic continuous function. That is almost all we will need for the upcoming computations. Then we can write

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbf{R}} \varphi(t)\,\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\,\mathrm{d}t &= \frac{1}{\pi}\sum_{k=-\infty}^{+\infty}\int_k^{k+1}\varphi(t)\,\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\,\mathrm{d}t \\
&= \frac{1}{\pi}\sum_{k=-\infty}^{+\infty}\int_0^1 \varphi(u+k)\,\frac{\varepsilon}{(x-u-k)^2+\varepsilon^2}\,\mathrm{d}u \\
&= \frac{1}{\pi}\sum_{k=-\infty}^{+\infty}\int_0^1 \varphi(u)\,\frac{\varepsilon}{(x-u-k)^2+\varepsilon^2}\,\mathrm{d}u
\end{aligned}$$

So, $$\frac{1}{\pi}\int_{\mathbf{R}} \varphi(t)\,\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\,\mathrm{d}t = \int_0^1 \varphi(u)\,p_{\varepsilon,x}(u)\,\mathrm{d}u \qquad \text{(Fubini-Tonnelli)}$$

where $p_{\varepsilon,x}$ is the Kernel defined as follows:

$$p_{\varepsilon,x}(u) = \frac{1}{\pi}\sum_{k=-\infty}^{+\infty}\frac{\varepsilon}{(x-u-k)^2+\varepsilon^2} \tag{2.11}$$

This defines a one-periodic positive continuous bounded function on the real axis satisfying

$$\int_0^1 p_{\varepsilon,x} = 1$$

Let us make a useful ergodic observation: for all $x \in \mathbf{R}$, by one-periodicity of φ, the unique ergodicity of the map $t \mod 1 \longmapsto t+\alpha \mod 1$ ensures that for all $f \in \mathscr{C}^0([\,0\,;1\,])$ and $t \in [\,0\,;1\,]$

$$\frac{1}{\mathrm{K}}\sum_{k=1}^{\mathrm{K}} f(t+k\alpha) \xrightarrow[\mathrm{K}\to+\infty]{} \int_0^1 f$$

which provides a way to estimate the difference

$$\int_{\mathbf{R}} \varphi(t)\,\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\,\mathrm{d}t - \int_0^1 \varphi = \int_0^1 \varphi\, p_{\varepsilon,x} - \int_0^1 \varphi$$

Indeed, the dominated convergence theorem permits to write

$$\mathrm{A}_{\mathrm{K}} := \int_0^1 \frac{1}{\mathrm{K}}\sum_{k=1}^{\mathrm{K}} \varphi(t+k\alpha)\,p_{\varepsilon,x}(t)\,\mathrm{d}t \xrightarrow[\mathrm{K}\to+\infty]{} \int_0^1 \left(\int_0^1 \varphi(t)\,\mathrm{d}t\right) p_{\varepsilon,x}(u)\,\mathrm{d}u = \int_0^1 \varphi$$

But also
$$\left|\mathrm{A_K} - \int_0^1 \varphi\, p_{\varepsilon,x}\right| = \left|\int_0^1 \frac{1}{\mathrm{K}} \sum_{k=0}^{\mathrm{K}-1} \varphi(t+k\alpha)\, p_{\varepsilon,x}(t)\,\mathrm{d}t - \int_0^1 \varphi(t)\, p_{\varepsilon,x}(t)\,\mathrm{d}t\right|$$
$$= \left|\int_0^1 \frac{1}{\mathrm{K}} \sum_{k=0}^{\mathrm{K}-1} \left[\varphi(t+k\alpha)\, p_{\varepsilon,x}(t) - \varphi(t)\, p_{\varepsilon,x}(t)\right]\right| \mathrm{d}t$$
$$= \int_0^1 p_{\varepsilon,x}(t) \frac{1}{\mathrm{K}} \sum_{k=0}^{\mathrm{K}-1} \underbrace{|\varphi(t+k\alpha) - \varphi(t)|}_{\leqslant c|k|/n}\,\mathrm{d}t$$
$$\left|\mathrm{A_K} - \int_0^1 \varphi\, p_{\varepsilon,x}\right| \leqslant \int_0^1 p_{\varepsilon,x}(t)\,\mathrm{d}t \times \frac{1}{\mathrm{K}} \sum_{k=0}^{\mathrm{K}-1} \frac{c\,|k|}{n}$$

Hence
$$\left|\mathrm{A_K} - \int_0^1 \varphi\, p_{\varepsilon,x}\right| \leqslant c\frac{\mathrm{K}}{2n} \tag{2.12}$$
Consequently, $\int_0^1 \varphi\, p_{\varepsilon,x}$ and $\int_0^1 \varphi$ are close if K is chosen appropriately, typically when $\mathrm{K} \sim \sqrt{n}$. We want to quantify the difference between these two quantities.

Using the linearity of the integral and the change of variable $u = t + k\alpha$ for each k one gets
$$\mathrm{A_K} = \frac{1}{\mathrm{K}} \sum_{k=1}^{\mathrm{K}} \int_{k\alpha}^{k\alpha+1} \varphi(u)\, p_{\varepsilon,x}(u-k\alpha)\,\mathrm{d}u$$
But since both φ and $p_{\varepsilon,x}$ are continuous and one-periodic, the following clearly holds
$$\int_{k\alpha}^{k\alpha+1} \varphi(u)\, p_{\varepsilon,x}(u-k\alpha)\,\mathrm{d}u = \int_0^1 \varphi(u)\, p_{\varepsilon,x}(u-k\alpha)\,\mathrm{d}u$$
So that:
$$\mathrm{A_K} - \int_0^1 \varphi = \int_0^1 \varphi(u) \left(\frac{1}{\mathrm{K}} \sum_{k=1}^{\mathrm{K}} p_{\varepsilon,x}(u-k\alpha) - 1\right) \mathrm{d}u \tag{2.13}$$
$$= \int_0^1 \varphi(u) \left(\frac{1}{\mathrm{K}} \sum_{k=1}^{\mathrm{K}} p_{\varepsilon,x}(u-k\alpha) - \int_0^1 p_{\varepsilon,x}\right) \mathrm{d}u$$

It remains to get an estimate for the speed of convergence of the Birkhoff averaged sums of $p_{\varepsilon,x}$ under the uniquely ergodic translation $x \mapsto x - \alpha$. Note that $-\alpha$ has the same diophantine properties as α.

2.4.6 Speed of convergence in Birkhoff's ergodic theorem

The unique ergodicity of the translation $x \mod 1 \mapsto x+\alpha \mod 1$ ensures that for any $f \in \mathscr{C}^0([\,0\,;1\,])$ and $x \in [\,0\,;1\,]$
$$\frac{1}{n} \sum_{k=1}^{n} f(x+k\alpha) \xrightarrow[n\to\infty]{} \int_0^1 f$$

We can give more precise statements concerning the speed of convergence of the Birkhoff averaged sums of the translation $x \mapsto x+\alpha$

$$\mathrm{B}_{\mathrm{K}}(f)(x) = \frac{1}{\mathrm{K}}\sum_{k=1}^{\mathrm{K}} f(x+k\alpha)$$

for a function $f \in \mathscr{C}^k([\,0\,;1\,])$ for some $k \geqslant 1$ to be specified in regard to the diophantine condition we will consider on α. Let us denote $\|\mathrm{F}\|_k$ denotes the $\mathscr{C}^k$-norm of a function F.

Lemma 2.4

Let us assume that α satisfies the diophantine condition $\mathrm{DC}(\kappa,\tau)$ (see (2.3)). For all integer $k > \tau+2$ and $f \in \mathscr{C}^k(\mathbf{T},\mathbf{R})$ we define the constant

$$c_f = c(k,f,\tau,\kappa) = \kappa^{-1}(2\pi)^{-k}\|f^{(k)}\|_0 \sum_{\ell\in\mathbf{Z}\smallsetminus\{0\}} |\ell|^{\tau-k}$$

Then for all integer $\mathrm{K} \geqslant 1$

$$\begin{aligned}\sup_{x\in[\,0\,;1\,]} \left|\mathrm{B}_{\mathrm{K}}(f)(x) - \textstyle\int_0^1 f\right| &\lesssim \frac{\|f^{(\tau+2^+)}\|_0}{\kappa\,\mathrm{K}}\\ &\leqslant \frac{c_f}{\mathrm{K}}\end{aligned}$$

Remark 2.4

Please notice that this is not the classical Denjoy-Koksma inequality (see [Her79]). Here we do not assume that K is the denominator of the convergents of α. We use regularity assumptions on f to prove the same result about the speed of convergence of the Birkhoff averages for all K.

Proof.

Expanding f in Fourier series on the form $f(x) = \sum_{\ell=-\infty}^{+\infty} \widehat{f}(\ell)\mathrm{e}^{2\mathrm{i}\pi\ell x}$, we can compute the following

$$\begin{aligned}\mathrm{B}_{\mathrm{K}}(x) := \mathrm{B}_{\mathrm{K}}(f)(x) &= \frac{1}{\mathrm{K}}\sum_{k=1}^{\mathrm{K}}\sum_{\ell\in\mathbf{Z}\smallsetminus\{0\}} \widehat{f}(\ell)\,\mathrm{e}^{2\mathrm{i}\pi\ell k\alpha}\mathrm{e}^{2\mathrm{i}\pi\ell x} + \widehat{f}(0)\\ &= \frac{1}{\mathrm{K}}\sum_{\ell\in\mathbf{Z}\smallsetminus\{0\}} \widehat{f}(\ell)\,\mathrm{e}^{2\mathrm{i}\pi\ell x}\,\frac{1-\mathrm{e}^{2\mathrm{i}\pi k\ell\alpha}}{1-\mathrm{e}^{2\mathrm{i}\pi\ell\alpha}} + \widehat{f}(0)\end{aligned}$$

with $$\left|\frac{1-\mathrm{e}^{2\mathrm{i}\pi k\ell\alpha}}{1-\mathrm{e}^{2\mathrm{i}\pi\ell\alpha}}\right| = \frac{|\sin(\pi\ell k\alpha)|}{|\sin(\pi\ell\alpha)|} \leqslant \frac{1}{|\sin(\pi\ell\alpha)|}$$

and using $$x = n_x + d(x,\mathbf{Z}) \qquad \text{with} \quad n_x \in \mathbf{Z}$$

we have $$|\sin(\pi\ell\alpha)| = |\sin(\pi\|\ell\alpha\|_{\mathbf{Z}})| \geqslant \|\ell\alpha\|_{\mathbf{Z}} \qquad \text{since } \|\ell\alpha\|_{\mathbf{Z}} \in [\,0\,;1/2\,[$$

Thus, $$\frac{|\sin(\pi\ell k\alpha)|}{|\sin(\pi\ell\alpha)|} \leqslant \kappa^{-1}\,|\ell|^{\tau}$$

Moreover, $$\forall k \in \mathbf{Z} \qquad \widehat{f}(\ell) = (2\mathrm{i}\pi\ell)^{-k}\,\widehat{f^{(k)}}(\ell)$$

so choosing $k > \tau+2$ and reminding that $\widehat{f}(0) = \int_0^1 f$, one obtains the expected inequality:

$$\left|\mathrm{B_K}(f) - \textstyle\int_0^1 f\right| \leqslant \frac{c_f}{\mathrm{K}} \qquad \text{with} \qquad c_f = \kappa^{-1}(2\pi)^{-k}\|f^{(k)}\|_\infty \sum_{\ell\in\mathbf{Z}\smallsetminus\{0\}} \frac{1}{|\ell|^{k-\tau}}$$

■

In order to use lemma 2.4 for the Birkhoff averages of $p_{\varepsilon,x}$ we need to estimate the $\mathscr{C}^j$-norms of $p_{\varepsilon,x}$ for $j > \tau+2$.

2.4.7 Estimates of the $\mathscr{C}^j$-norms of the kernel $p_{\varepsilon,x}$

We will prove the following:

Lemma 2.5

For all integer $j \in \mathbf{N}$ one has, uniformly in $x \in \mathbf{R}$,

$$\sup_{t\in[\,0\,;1\,]}\left|\frac{\mathrm{d}^j p_{\varepsilon,x}}{\mathrm{d}t^j}\right| \lesssim \frac{1}{\varepsilon^{j+1}}$$

Proof.

Let us recall the definition of $p_{\varepsilon,x}$:

$$p_{\varepsilon,x}(t) = \frac{1}{\pi}\sum_{k=-\infty}^{+\infty}\frac{\varepsilon}{\varepsilon^2+(t-x+k)^2}$$

It is clearly a one-periodic function. Then the Fourier coefficients of $p_{\varepsilon,x}$ are, thanks to the periodicity,

$$\widehat{p}_n = \frac{1}{\pi}\int_0^1 p_{\varepsilon,x}(t)\,\mathrm{e}^{-2\mathrm{i}\pi nt}\,\mathrm{d}t = \frac{1}{\pi}\int_{\mathbf{R}}\frac{\varepsilon}{(x-v)^2+\varepsilon^2}\,\mathrm{e}^{-2\mathrm{i}\pi nv}\,\mathrm{d}v = \mathrm{e}^{-2\mathrm{i}\pi nx}\int_{\mathbf{R}}\frac{\mathrm{e}^{-2\mathrm{i}\pi n\varepsilon z}}{z^2+1}\,\mathrm{d}z$$

which is the Fourier transform of the elementary Cauchy Kernel $\frac{1}{x^2+1}$ at $n\varepsilon$ and so

$$\widehat{p}_n = \mathrm{e}^{-2\mathrm{i}\pi nx}\mathrm{e}^{-2\pi|n|\varepsilon}$$

Since $p_{\varepsilon,x}$ is $\mathscr{C}^\infty$ we have

$$\|{p_{\varepsilon,x}}^{(j)}\|_0 \lesssim \sum_{n\in\mathbf{Z}}|n|^j\,|\widehat{p}_n| \lesssim \sum_{n\in\mathbf{Z}}|n|^j\,\mathrm{e}^{-2\pi|n|\varepsilon} = \mathrm{O}_{\varepsilon\to0}(\varepsilon^{-(j+1)})$$

where the last estimate is given by the j-th derivative of the geometric series $\sum_{n=0}^{+\infty} x^n = 1/(1-x)$ evaluated at $x = \mathrm{e}^{-2\pi\varepsilon} = 1-2\pi\varepsilon+\mathrm{O}(\varepsilon^2)$.

■

2.4.8 End of the proof of the upper estimate

Finally, the triangular inequality, the inequalities (2.13) and (2.12) lead with lemmas 2.4 and 2.5 to:

$$\left|\int_0^1 \varphi - \int_0^1 \varphi\, p_{\varepsilon,x}\right| \leqslant \left|\int_0^1 \varphi - \mathrm{A_K}\right| + \left|\mathrm{A_K} - \int_0^1 \varphi\, p_{\varepsilon,x}\right| \lesssim \frac{1}{\mathrm{K}\varepsilon^{\tau+2^+}} + \frac{\mathrm{K}}{n}$$

which proves proposition 2.4 and consequently proposition 2.1 for $\varphi_n(x+\mathrm{i}\varepsilon)$ with $\mathrm{K} = n^c$. The same inequality for $\varphi_n(x+\mathrm{i}\varepsilon)$ also holds with the same arguments because of the symmetry of the Brownian motion (so we just have to symmetrize the domains used in the previous proofs).

We can now finish the proof of proposition 2.2.

Proof.

The function φ_n is subharmonic on $\{z \in \mathbf{C} \;:\; -\varepsilon \leqslant \Im\mathfrak{m}\, z \leqslant \varepsilon\}$ so that the maximum principle ensures the following:

$$\forall x \in \mathbf{R} \qquad \varphi_n(x) \leqslant \varphi_n(x+\mathrm{i}\varepsilon)$$

It remains to recall that $\int_0^1 \varphi_n = \mathrm{L}_n$ and choose $\mathrm{K} = n^c$ with $0 < c < 1$ and $\varepsilon = n^{-b}$ with $b > 0$. Then for $1 > c > b(\tau + 2^+)$

$$\varphi_n(x) - \mathrm{L}_n \lesssim \frac{1}{n^{c-b(\tau+2^+)}} + \frac{1}{n^{1-c}} + \frac{1}{n^b}$$

Optimizing in b, c gives $c - b(\tau + 2^+) = 1 - c = b = (\tau + 4^+)^{-1}$ and so we get the estimation for all $n \geqslant 1$ with $a = a(\tau) = \frac{1}{\tau+4^+} \in \,]\,0\,;1\,[$:

$$\exists c' > 0 \qquad \forall n \geqslant 1 \quad \forall x \in \mathbf{R} \qquad \varphi_n(x) \leqslant \mathrm{L}_n + c'\, n^{-a}$$

■

2.5 Lower bounds

In this section we prove a result in terms of the measure of the *bad points*, namely the set of the x's satisfying $\varphi_n(x) - \mathrm{L}_n \lesssim n^{-a}$ which is the converse inequality required for the lower bound in theorem 2.1. Let us establish the following result:

Proposition 2.5

Suppose for some $\rho > 0$, some $b_2 > 0$ and all $0 < b_1 < 1$ we have for all integer $n \geqslant 1$ a continuous subharmonic function φ_n on $\mathrm{S}_\rho = \{|\Im\mathfrak{m}\, z| \leqslant \rho\}$ satisfying: for all $0 < \varepsilon \leqslant \rho$

$$\forall x \in \mathbf{R} \qquad \varphi_n(x \pm \mathrm{i}\varepsilon) - \int_0^1 \varphi_n \lesssim \varepsilon \sup_{|\Im\mathfrak{m}\, z| \leqslant \rho} |\varphi_n| + \frac{1}{n^{b_1}\varepsilon^{b_2}} + \frac{1}{n^{1-b_1}} \qquad (2.14)$$

Let us define for $n \geqslant 1$ and $\lambda > 0$ the following compact set

$$\mathrm{K}_n(\lambda) = \left\{ x \in [\,0\,;1\,] \,:\, \varphi_n(x) \leqslant \int_0^1 \varphi_n - n^{-\lambda} \right\} \tag{2.15}$$

Then for an appropriate choice of $\lambda, \sigma > 0$ (depending on the b_i's)

$$\forall n \gg 1 \qquad \mathrm{Leb}\,\mathrm{K}_n(\lambda) \lesssim \mathrm{e}^{-n^\sigma}$$

2.5.1 The guiding principle

Let us denote $\mathrm{K} := \mathrm{K}_n(\lambda) \subset [\,0\,;1\,]$ the set of the points where the inequality in theorem 2.1 is violated. We assume it is in some interval $[\,-r/2\,;r/2\,]$ of size $r \ll \rho$ and we consider D_{2r} the disc circling K and the domain $\mathscr{D} = \mathrm{D}_{2r} \setminus \mathrm{K}$ with boundary $\partial\mathrm{D}_{2r} \cup \mathrm{K}$.

Now we decompose the boundary of $\mathscr{D}$ and apply (2.7) with the subharmonic non negative map φ_n and τ the exit time from $\mathscr{D}$. One gets for all $x \in [\,-r/2\,;r/2\,]$:

$$\begin{aligned}
\varphi_n(x) &\leqslant \mathbf{E}_x \varphi_n(\mathrm{B}_\tau) \\
&\leqslant \mathbf{E}_x\left[\varphi_n(\mathrm{B}_\tau)\mathbf{1}_{\mathrm{B}_\tau \in \partial\mathrm{D}_{2r}}\right] + \mathbf{E}_x\left[\varphi_n(\mathrm{B}_\tau)\mathbf{1}_{\mathrm{B}_\tau \in \mathrm{K}}\right] \\
&\leqslant \left(\sup\nolimits_{\partial\mathrm{D}_{2r}} \varphi_n\right) \mathbf{E}_x\left(\mathbf{1}_{\mathrm{B}_\tau \in \mathrm{K}}\right) + \left(\sup\nolimits_{\mathrm{K}} \varphi_n\right) \mathbf{E}_x\left(\mathbf{1}_{\mathrm{B}_\tau \in \mathrm{K}}\right) \\
&\leqslant \left(\sup\nolimits_{\partial\mathrm{D}_{2r}} \varphi_n\right) \mathbf{P}_x\left(\mathrm{B}_\tau \in \partial\mathrm{D}_{2r}\right) + \left(\sup\nolimits_{\mathrm{K}} \varphi_n\right) \mathbf{P}_x\left(\mathrm{B}_\tau \in \mathrm{K}\right) \\
\varphi_n(x) &\leqslant \left(\sup\nolimits_{\partial\mathrm{D}_{2r}} \varphi_n\right)\left(1 - \mathbf{P}_x\left(\mathrm{B}_\tau \in \mathrm{K}\right)\right) + \left(\sup\nolimits_{\mathrm{K}} \varphi_n\right) \mathbf{P}_x\left(\mathrm{B}_\tau \in \mathrm{K}\right)
\end{aligned}$$

By subharmonicity of φ_n, the maximum principle gives $\sup_{\partial\mathrm{D}_{2r}} \varphi_n = \sup_{|\Im\mathfrak{m}\, z| = 2r} \varphi_n$ so that

$$\varphi_n(x) \leqslant (1 - p_x) \sup_{|\Im\mathfrak{m}\, z| = 2r} \varphi_n + p_x \sup_{\mathrm{K}} \varphi_n \tag{2.16}$$

where

$$p_x = \mathbf{P}_x(\mathrm{B}_\tau \in \mathrm{K}) \tag{2.17}$$

Remind that we want an exponential estimate on Leb K. In the end we will prove that (2.16) leads to a contradiction for $\int_{\mathbf{T}} \varphi_n$ if we assume that $\mathrm{Leb}\,\mathrm{K} \geqslant \mathrm{Ce}^{-n^\sigma}$.

Thus according to (2.16) we have to relate p_x to Leb K. Note that we can estimate $\sup_{|\Im\mathfrak{m}\, z| = 2r} \varphi_n$ with the hypothesis (2.14) where we have to choose the b_1, and $\varepsilon < \rho$. Of course $\sup_{\partial\mathrm{D}_{2r}} \varphi_n$ can also be estimated with the definition of K.

We will use potential theory to relate the probability that a planar Brownian motion exits from a domain by some compact border to the Lebesgue measure of this compact set. Actually potential theory relates this probability to the *capacity* of the compact set, that we relate to its Lebesgue measure.

Let us recall this well known and telling result for the exit time of the Brownian motion from a ring:

Proposition 2.6 (theorem 3.17 in [MP10])

Let $\mathrm{A} = \mathrm{A}_{r,\mathrm{R}} = \{z \in \mathbf{C} : r < |z| < \mathrm{R}\}$ *be the ring of radii* $0 < r < \mathrm{R}$ *(see figure 2.2). Then, if* $\tau = \inf\{t \geqslant 0 : \mathrm{B}_t \in \partial\mathrm{A}\}$ *denotes the exit time from* A *of a Brownian starting inside* A*:*

$$\mathbf{P}_z(|\mathrm{B}_\tau| = r) = \frac{\ln(\mathrm{R}/|z|)}{\ln(\mathrm{R}/r)}$$

This gives, as r goes to zero and other parameters kept fixed,

$$\mathbf{P}_z(|\mathrm{B}_\tau| = r) \underset{r\to 0}{\sim} \frac{\ln(\mathrm{R}/|z|)}{-\ln r}$$

with $\ln r$ being proportional to the logarithm of the Lebesgue measure of the disc $\mathrm{D}(0, r)$ which stands for the points we want the Brownian motion to get to. We want to prove some similar estimate for a different compact subset of points in a segment $\subset \mathbf{R}$, namely the set K of the points where the large deviation estimate is not verified.

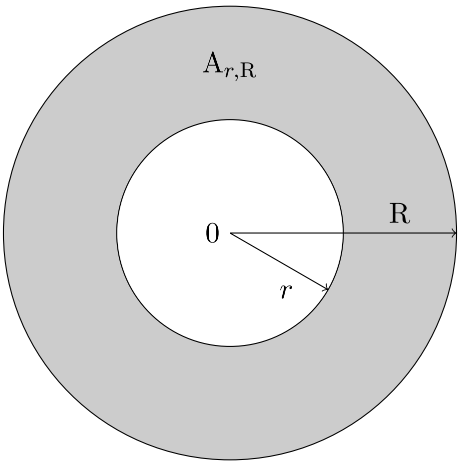

Figure 2.2: The ring $\mathrm{A}_{r,\mathrm{R}}$

2.5.2 Bounds for the probability to exit a domain in $\mathbf{R}^2$

We want an inequality such as

$$\mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{K}) \geqslant \frac{\mathrm{C}}{-\ln\rho} \tag{2.18}$$

where ρ is related to the (one dimensional) Lebesgue measure of K and τ the exit time from the domain limited by the disc outside and the compact centered at the origin as an inside boundary.

Note that an argument of inclusions of domains gives the following. Let K be a subset of $\mathbf{C}$ included in some disc $\mathrm{D}_{r<1}$ centered at the origin. Then, clearly, considering a Brownian motion (B_t) starting at $z \in \mathrm{A}_{r,r'}$ with exit time τ when it leaves the domain $\mathrm{D}_{r'} \setminus \mathrm{K}$, and (W_t) a Brownian motion in the domain $\mathrm{A}_{r,r'}$ with exit time θ

$$\mathbf{P}_z\left(\mathrm{B}_\tau \in \mathrm{K}\right) \leqslant \mathbf{P}_z\left(|\mathrm{W}_\theta| = r\right) = \frac{\ln(r'/\,|z|)}{\ln(r'/r)} \tag{2.19}$$

This of course is not enough for our purpose since the formula does not involve the Lebesgue measure of K but illustrates the type of arguments used in the proof. We now refine the arguments to get more accurate estimates.

Potential theory

Let us make some reminder about potential theory such as developed in [Ran95]. If μ is a finite Borelian measure on $\mathbf{C}$ with compact support K, one can define its potential function

$$p_\mu : \begin{cases} \mathbf{C} \longrightarrow [-\infty\,;+\infty\,[\\ z \longmapsto \displaystyle\int_{\mathrm{K}} \ln|z-w| \,\mathrm{d}\mu(w) \end{cases}$$

Since $\Delta \log|\cdot| \overset{\mathcal{D}'}{=} 2\pi\delta_0$, where δ_0 is the Dirac peak at 0, one has that p_μ is subharmonic on $\mathbf{C}$ and harmonic on $\mathbf{C} \setminus \operatorname{Supp}\mu$. The *energy* of such a measure is defined by the formula

$$\mathrm{I}(\mu) = \iint_{\mathrm{K}\times\mathrm{K}} \ln|z-w| \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w) = \int_{\mathrm{K}} p_\mu(z)\,\mathrm{d}\mu(z)$$

A subset $\mathrm{A} \subset \mathbf{C}$ is called *polar* if

$$\mathrm{I}(\mu) = -\infty \tag{2.20}$$

for every non zero Borelian measure μ for which $\operatorname{Supp}\mu$ is a compact subset of A. We say that a property holds *nearly everywhere on* $\mathrm{S} \subset \mathbf{C}$ if it holds everywhere on $\mathrm{S} \setminus \mathrm{E}$ for some Borel polar set $\mathrm{E} \subset \mathrm{S}$.

Remark 2.5

It is a fact that polar sets have two-dimensional Lebesgue measure zero (see chapter 3.2 in [Ran95]). So, a property holding nearly everywhere *also holds* Leb-*almost everywhere.*

Let us remark that if additionally K *is a compact set in* $\mathbf{R}$ *with* $\mathrm{Leb}_{\mathbf{R}}\,\mathrm{K} > 0$, *then the same set considered as a compact subset of* $\mathbf{R}^2$ *is not polar. Indeed, the* trace *measure* $\mu(\mathrm{O} \subset \mathbf{R}^2) := \mathrm{Leb}_{\mathbf{R}}(\mathrm{O} \cap \mathrm{K})$ *is a non zero Borel measure since* $\mathrm{Leb}_{\mathbf{R}}\,\mathrm{K} > 0$ *for which*

$$\mathrm{I}(\mu) = \iint_{\mathrm{K}\times\mathrm{K}} \ln|z-w|\;\mathrm{d}z\,\mathrm{d}w$$

is finite.

If K is a compact subset of $\mathbf{C}$, when looking at $\mathrm{Prob}(\mathrm{K})$, the set of all Borel probability measures on K, there always exists a measure ν verifying

$$\mathrm{I}(\nu) = \sup_{\mu\in\mathrm{Prob}(\mathrm{K})} \mathrm{I}(\mu) \tag{2.21}$$

that is called an *equilibrium measure* for K. For this measure, Frostman's theorem (see theorem 3.3.4 in [Ran95]) ensures that

- $p_\nu \geqslant \mathrm{I}(\nu)$ on $\mathbf{C}$;
- $p_\nu = \mathrm{I}(\nu)$ on $\mathrm{K} \smallsetminus \mathrm{P}$ where P is an F_σ polar subset of $\partial\mathrm{K}$.

Let us fix K a compact subset of $\mathbf{C}$ for which $\partial\mathrm{K}$ has positive one-dimensional Lebesgue measure and ν its equilibrium measure. We define

$$\mathrm{G}(z) = p_\nu(z) - \mathrm{I}(\nu) \tag{2.22}$$

By Frostman's theorem, G is non negative, harmonic on the open set $\mathbf{C} \smallsetminus \mathrm{K}$, subharmonic on $\mathbf{C}$ and $\mathrm{G}_{|\mathrm{K}\smallsetminus\mathrm{P}} = 0$. We shall see below that it is a kind of Green's function (see chapter 4.4 in [Ran95]) for a domain to be specified.

Notice that $\mathrm{I}(\nu) = \ln\operatorname{cap}\mathrm{K}$ is called the *logarithmic capacity* of the compact set K.

Proof of the inequality (2.18)

We denote D_a the open disc of center 0 and radius $a > 0$ in the complex plane. Let us now assume that $\mathrm{K} \subseteq [\,-r/2\,;r/2\,]$ with $r \ll 1$ is a subset of Lebesgue measure ρ of the real line and is included in the disc D_{2r} (see figure 2.3).

Let us denote

$$c = -\mathrm{I}(\nu) \tag{2.23}$$

Clearly for all $z \in \partial\mathrm{D}_{2r}$ and $w \in \mathrm{K}$

$$\tfrac{3}{2}r \leqslant |z-w| \leqslant \tfrac{5}{2}r$$

so that
$$\ln(\tfrac{3}{2}r) + c \leqslant \mathrm{G}(z) = p_\nu(z) + c \leqslant \ln(\tfrac{5}{2}r) + c$$

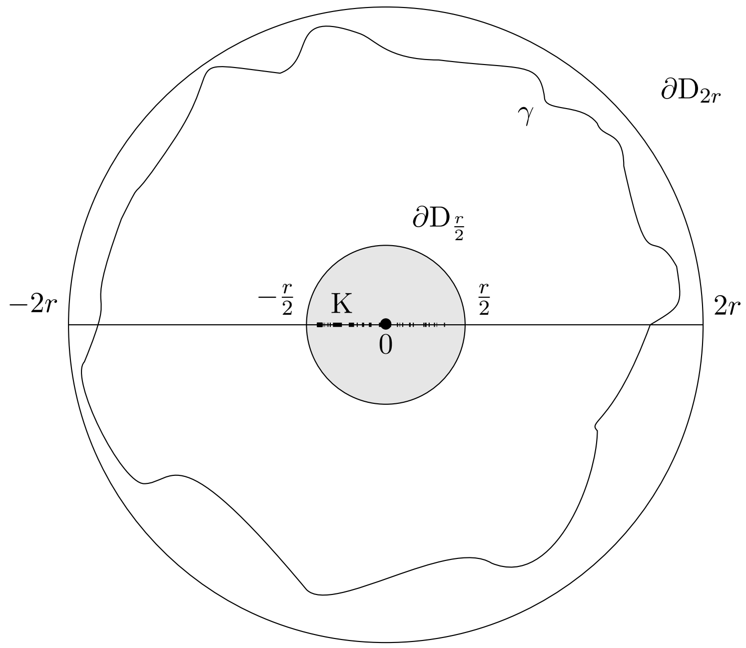

Figure 2.3: Our domains

The subharmonicity of G ensures

$$\forall z \in \overline{D_{2r}} \qquad G(z) \geqslant c + \ln\left(\frac{3}{2}r\right) \tag{2.24}$$

We similarly obtain

$$\forall z \in \overline{D_{r/2}} \qquad G(z) \leqslant \ln r + c \tag{2.25}$$

So let us define

$$b = \frac{1}{2}\left(2c\ln r + \ln(3r/2)\right) = c + \ln r + \ln(\sqrt{6}/2) \tag{2.26}$$

This chosen value of b (see (2.26)) and the maximum principle give that

$$D_{r/2} \subset \gamma \subset D_{2r} \tag{2.27}$$

Lemma 2.6

For b defined by (2.26), the set $\gamma = G^{-1}(\{b\})$ is a loop included in D_{2r} and encircling $D_{r/2}$ (with no intersection) and so encircling K.

Proof.

Since $K = \operatorname{Supp}\mu \subset \mathbf{R}$ we can write

$$G(x+\mathrm{i}y) = \int_K \ln|x+\mathrm{i}y-w|\,\mathrm{d}\nu(w) = \frac{1}{2}\int_K \ln((x-w)^2+y^2)\,\mathrm{d}\nu(w) = g(x,y)$$

and we compute for $x+\mathrm{i}y \notin \operatorname{Supp}\nu$

$$\frac{\partial g}{\partial x}(x,y) = \int_K \frac{x-w}{(x-w)^2+y^2}\,\mathrm{d}\nu(w) \qquad \text{and} \qquad \frac{\partial g}{\partial y}(x,y) = \int_K \frac{y}{(x-w)^2+y^2}\,\mathrm{d}\nu(w)$$

If $y=0$ then $x\notin K$ so that $\frac{\partial g}{\partial x}(x,y)\neq 0$. If $y\neq 0$ then $\frac{\partial g}{\partial y}(x,y)>0$ so that in both cases the implicit function theorem applies and ensures that γ is locally a curve.

We have $\gamma = G^{-1}(\{b\}) = {G_{\mathbf{C}\smallsetminus K}}^{-1}(\{b\})$ is closed since G is continuous on $\mathbf{C}\smallsetminus K$. Every connected component of γ is then a closed curve. Since all these are also bounded because they are included in D_{2r}, they are compact submanifolds of dimension one, and so loops included in D_{2r}.

It remains to see why there is just on such loop, and that it encircles K. Given a connected component ℓ of γ, the inclusion (2.27) leads to 2 possibilities:

(i) ℓ circles $D_{r/2}$ and so K, which is what we want;

(ii) ℓ does not encircles $D_{r/2}$: in this case the bounded domain d limited by ℓ is biholomorphic to the unit disc D, say via h. Since ℓ does not encircles $D_{r/2}$, the domain d does not contain any point of K and so $G\circ h^{-1}$ is a harmonic function on D which is constant on $\overline{D}$, and so is constant by the maximum principle. This implies G is constant on d, a contradiction.

We have proved that γ is a union of loops circling D_r but there is at most one such curve. Indeed two distinct curves cannot intersect so one is encircling the other which contradicts the maximum principle applied to the subharmonic function G. The lemma is proved. ■

Define

$$\widetilde{G} = \frac{1}{b}G = \frac{p_\nu(z)+c}{c+\ln r+\ln(6\sqrt{2})} \tag{2.28}$$

and the bounded domain D limited by the boundaries $\partial D = K\cup\gamma$, and where $\tau = \inf\{t\geqslant 0 \,:\, B_t\in\partial D\}$ is the exit time from D. Then $\widetilde{G}$ is bounded thanks to Frostman's theorem and it solves the problem

$$\begin{cases} \Delta\widetilde{G} = 0 \quad \text{on the open set } D\smallsetminus K \\ \widetilde{G}_{|K\smallsetminus P} = 0 \\ \widetilde{G}_{|\gamma} = 1 \end{cases} \tag{2.29}$$

Here is what relates potential theory as we described using equilibrium measure and (sub)harmonic functions with (sub)harmonic properties of the Brownian motion: we also have, thanks to theorem 3.8 in [MP10] with the measurable map $\varphi = \mathbf{1}_\gamma$, that the function $\wp : z\mapsto \mathbf{P}_z(B_\tau\in\gamma)$ solves the problem

$$\begin{cases} \Delta\wp = 0 \quad \text{on D} \\ \wp_{|\mathrm{K}} = 0 \\ \wp_{|\gamma} = 1 \end{cases}$$

So that one hopes for an argument of uniqueness to conclude that $\widetilde{\mathrm{G}} = \wp$.

Remark 2.6

Note that this is not a classical Dirichlet problem for $\widetilde{\mathrm{G}}$ since the function $\widetilde{\mathrm{G}}$ may be non zero on the polar Borel subset P. So uniqueness in the Dirichlet problem cannot be directly applied here since $\widetilde{\mathrm{G}}$ and $\wp$ may not coincide on $\partial\mathrm{D}$. We will use potential theory and (sub)harmonic techniques for the Brownian motion to prove that, nevertheless, $\widetilde{\mathrm{G}}$ and $\wp$ are equal nearly everywhere.

Lemma 2.7

The function $\widetilde{\mathrm{G}}$ defined by (2.28) satisfies for every $z \in \mathrm{D}\cup\gamma$ and nearly every $z \in \mathrm{K}$:

$$\widetilde{\mathrm{G}}(z) = \mathbf{P}_z\left(\mathrm{B}_\tau \in \gamma\right) = 1 - \mathbf{P}_z\left(\mathrm{B}_\tau \in \mathrm{K}\right)$$

where $(\mathrm{B}_t)_{t\geqslant 0}$ is any Brownian motion starting in the bounded domain D limited by the boundary $\partial\mathrm{D} = \mathrm{K}\cup\gamma$, and with $\tau = \inf\{t \geqslant 0 : \mathrm{B}_t \in \partial\mathrm{D}\}$.

Proof.

The map $\widetilde{\mathrm{G}}$ is harmonic and bounded on D. We claim that

$$n.e.\ \zeta \in \mathrm{K} \qquad \limsup_{z\to\zeta} \widetilde{\mathrm{G}}(z) = 0$$

This is a consequence of the fact that $\overline{\mathrm{D}} = \mathrm{D}\cup\mathrm{K}\cup\gamma$ is non polar (because $\mathrm{Leb}\,\overline{\mathrm{D}} > 0$), and so (look at definition 3.8.1 and theorem 3.8.6 in [Ran95]) it is *non-thin* at nearly every point of itself, which means that for some Borel polar set $\mathrm{B}_1 \subset \overline{\mathrm{D}}$

$$\forall\zeta \in \overline{\mathrm{D}} \setminus \mathrm{B}_1 \qquad \limsup_{z\to\zeta} \widetilde{\mathrm{G}}(z) = \widetilde{\mathrm{G}}(\zeta)$$

and also $$\forall z \in \mathrm{K} \setminus (\mathrm{B}_1 \cap \mathrm{K}) \qquad \limsup_{z\to\zeta} \widetilde{\mathrm{G}}(z) = \widetilde{\mathrm{G}}(\zeta)$$

and $\mathrm{B}' = \mathrm{B}_1 \cap \mathrm{K}$ is polar as a subset of a polar set. But, by Frostmann's theorem, for some Borel polar set $\mathrm{B} \subset \mathrm{K}$

$$\forall\zeta \in \mathrm{K} \setminus \mathrm{B} \qquad \widetilde{\mathrm{G}}(\zeta) = 0$$

Then $\mathrm{B}'' = \mathrm{B} \cap \mathrm{B}'$ is a Borel polar subset of K and

$$\forall \zeta \in \mathrm{K} \setminus \mathrm{B}'' \qquad \limsup_{z\to\zeta} \widetilde{\mathrm{G}}(z) = 0$$

The positivity of $\widetilde{\mathrm{G}}$ thus gives

$$\forall \zeta \in \mathrm{K} \setminus \mathrm{B}'' \qquad \liminf_{z\to\zeta} \widetilde{\mathrm{G}}(z) = 0$$

The continuity of $\widetilde{\mathrm{G}}$ on $\mathbf{C} \setminus \mathrm{K}$ (Frostmann) ensures that

$$\forall \zeta \in \gamma \qquad \widetilde{\mathrm{G}}(z) \xrightarrow[z\to\zeta]{} 1$$

which gives the lim sup and lim inf (everywhere) on γ. But the map $\wp : z \mapsto \mathbf{P}_z(\mathrm{B}_\tau \in \gamma)$ is positive and harmonic on D and so has the same lim sup and lim inf limits nearly everywhere on K and γ by non-thinness. As $\partial\mathrm{D} = \mathrm{K} \cup \gamma$ is non polar, we can apply the result of the *extended maximum principle* for subharmonic functions (theorem 3.6.9 in [Ran95]), which states that if D is domain in $\mathbf{C}$ with $\partial\mathrm{D}$ non-polar, and u a bounded above subharmonic function on D satisfying

$$\text{for nearly every } z \in \partial\mathrm{D} \qquad \limsup_{z\to\zeta} \widetilde{u} \leqslant 0$$

then $u \leqslant 0$ on D. Applying this to $\pm(\widetilde{\mathrm{G}} - \wp)$, we get $\widetilde{\mathrm{G}} = \wp$ on D. Considering the boundary conditions (2.29) on $\widetilde{\mathrm{G}}$ and $\wp$, we also conclude for the equality on γ and nearly everywhere on K.

■

Now we go back to our initial issue to estimate p_x (see (2.17)) in (2.16).

Corollary 2.1

Let K *be a compact set of* $\mathbf{R}$ *with* $\mathrm{Leb}_{\mathbf{R}} > 0$ *and* $\mathrm{K} \subset [-r/2\,;r/2]$. *Let* $\mathscr{D}$ *be the domain* $\mathrm{D}_{2r} \setminus \mathrm{K}$ *of boundary* $\mathrm{K} \cup \partial\mathrm{D}_{2r}$. *If* $\ln r - \ln \mathrm{Leb}\,\mathrm{K} \gg 1$, *the exit time from* $\mathscr{D}$ *verifies:*

(i) *uniformly for a.e.* $z \in [-r/2\,;r/2]$ *we have*

$$\mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{K}) \gtrsim \frac{1}{\ln r - \ln \mathrm{Leb}_{\mathbf{R}}\,\mathrm{K}}$$

(ii) *uniformly for* $z \in \mathrm{D}$ *with* $|\mathfrak{Im}\, z| \asymp r$

$$\mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{K}) \asymp \frac{1}{\ln r - \ln \mathrm{Leb}_{\mathbf{R}}\,\mathrm{K}}$$

Remark 2.7

Under the additional assumption $\ln \mathrm{Leb}\,\mathrm{K} \ll \ln r$, *then the estimates are*

$$\text{for a.e. } z \in [-r/2\,;r/2] \qquad \mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{K}) \gtrsim \frac{1}{-\ln \mathrm{Leb}\,\mathrm{K}}$$

$$\textit{for } z \in \mathrm{D} \textit{ with } |\Im\mathfrak{m}\, z| \asymp r \qquad \mathbf{P}_z(\mathrm{B}_\tau \in \mathrm{K}) \asymp \frac{1}{-\ln \mathrm{Leb}\,\mathrm{K}} \tag{2.30}$$

Proof.

An easy inclusion of domains argument (look at figure 2.3) gives for $z \in \mathrm{D}$:

$$\begin{aligned}\mathbf{P}(\text{going from } z \text{ to K before touching } \mathrm{D}_{2r}) &\geqslant \mathbf{P}(\text{going from } z \text{ to K before touching } \gamma)\\ &= 1 - \widetilde{\mathrm{G}}(z)\end{aligned}$$

where $c = -\ln \mathrm{cap}\,\mathrm{K}$ (see (2.23)) is the capacity of K, so we have the desired result if we remind definitions (2.26) and (2.28):

$$1 - \widetilde{\mathrm{G}}(z) = \frac{\ln r - p_\mu(z) + \ln(\sqrt{6}/2)}{\ln r - \ln \mathrm{cap}\,\mathrm{K} + \ln(\sqrt{6}/2)} = \frac{\ln r - \ln|z| - f(z) + \ln(\sqrt{6}/2)}{\ln r - \ln \mathrm{cap}\,\mathrm{K} + \ln(\sqrt{6}/2)} \tag{2.31}$$

So we find back the initial formula for two discs with same center (proposition 2.6), now corrected with the constant $\ln(\sqrt{6}/2)$, the logarithmic capacity of K (instead of the measure of the smaller disc), and

$$f(z) = \int_{\mathrm{K}} \log|1 - w/z| \,\mathrm{d}\mu(w) = p_\mu(z) - \ln|z|$$

being a function bounded away from zero because p_μ is bounded everywhere.

Additionally, (see chapter 5.2 in [Ran95]), as $\mathrm{K} \subset [\,-r/2\,; r/2\,]$ has diameter $d \leqslant r$, the following holds

$$\mathrm{cap}\,\mathrm{K} \leqslant d/2 \leqslant r/2$$

so that

$$\ln r - \ln \mathrm{cap}\,\mathrm{K} \geqslant 0$$

For all $z, w \in [\,-r/2\,; r/2\,]$ we have $|z - w| \leqslant r$. That gives $p_\mu(z) \leqslant \ln r$ and with (2.28)

$$\widetilde{\mathrm{G}}(z) \leqslant \frac{\ln r + c}{\ln r - \ln \mathrm{cap}\,\mathrm{K} + \ln(\sqrt{6}/2)}$$

$$1 - \widetilde{\mathrm{G}}(z) \geqslant \frac{\ln(\sqrt{6}/2)}{\ln r - \ln \mathrm{cap}\,\mathrm{K} + \ln(\sqrt{6}/2)}$$

But also (chapter 5.2 in [Ran95]), as $\mathrm{K} \subset \mathbf{R}$

$$\mathrm{cap}\,\mathrm{K} \geqslant \tfrac{1}{4}\,\mathrm{Leb}_{\mathbf{R}}\,\mathrm{K}$$

so

$$\frac{1}{b} = \frac{1}{\ln r - \ln \mathrm{cap}\,\mathrm{K} + \ln(\sqrt{6}/2)} \geqslant \frac{1}{\ln r - \ln \mathrm{Leb}\,\mathrm{K} + \ln(8\sqrt{6})}$$

and finally

$$1 - \widetilde{\mathrm{G}}(z) \geqslant \frac{\ln(\sqrt{6}/2)}{\ln r - \ln \mathrm{Leb}\,\mathrm{K} + \ln(8\sqrt{6})} \tag{2.32}$$

The point (i) then just comes from lemma 2.7 and (2.32).

For the bound from below we have for $z \in \overline{\mathrm{D}}_{\beta r} \setminus [\,-r/2\,;r/2\,]$ with $2 \geqslant \beta \geqslant 1/2$, $\mathfrak{Im}\, z \neq 0$ and $w \in \mathrm{K} \subset [\,-r/2\,;r/2\,]$:

$$(\mathfrak{Im}\, z)^2 \leqslant |z-w|^2 \leqslant r^2\left((\beta^2+1/4) - \beta\sqrt{1-\left(\frac{\mathfrak{Im}\, z}{\beta r}\right)^2}\right) \leqslant r^2(\beta-1/2)^2$$

such that

$$\ln|\mathfrak{Im}\, z| \leqslant p_\nu(z) \leqslant \ln r + \ln\left|\beta - \tfrac{1}{2}\right|$$

So
$$\frac{\ln(\sqrt{6}/2) - \ln|\beta-1/2|}{\ln r + c + \ln(\sqrt{6}/2)} \leqslant 1 - \widetilde{\mathrm{G}}(z) \leqslant \frac{\ln r - \ln|\mathfrak{Im}\, z| + \ln(\sqrt{6}/2)}{\ln r + c + \ln(\sqrt{6}/2)}$$

which is a bound of the required type for $2 \geqslant \beta \geqslant \frac{\sqrt{6}+1}{2}$ and $|\mathfrak{Im}\, z| \asymp \beta r$. This ends the proof with lemma 2.7. ■

Remark 2.8

In the case where $\mathrm{K} = [\,a\,;b\,]$ *is an interval one has*

$$\operatorname{cap}\mathrm{K} = \frac{b-a}{4} \qquad \textit{and} \qquad c = -\ln\rho + \mathrm{c}^{\mathrm{st}}$$

where ρ *is exactly the Lebesgue measure of* K.

2.5.3 The proof of the main theorem

We are now able to prove the main result (theorem 2.1). Let us first prove proposition 2.5.

Proof.

By proposition 2.2 we already have for a $c>0$ and $a = \frac{1}{\tau+4^+}$ and all $n \geqslant 1$

$$\forall x \in \mathbf{R} \qquad \varphi_n(x) \leqslant \mathrm{L}_n + c\,n^{-a}$$

so it is enough to prove an exponential estimate for the $x \in \mathrm{K}_n$ where (remind (2.15))

$$\mathrm{K}_n = \mathrm{K}_n(\lambda) = \left\{x \in [\,0\,;1\,] \;:\; \varphi_n(x) \leqslant \mathrm{L}_n - n^{-\lambda}\right\}$$

where $\lambda > 0$ is to be specified. Let us make an elementary remark: if $\mathrm{K}_n(\lambda)$ was an interval $\mathrm{I} \subset [\,0\,;1\,]$ of size $n^{-\beta}$ ($\beta > 0$) satisfying

$$\text{Leb a.e. } x \in \mathrm{I} \qquad \varphi_n(x) \leqslant \mathrm{L}_n - n^{-\lambda} \tag{2.33}$$

Then a simple integration split over the intervals I and I^c, which sizes are know, gives

$$\begin{aligned}\int_0^1 \varphi_n &= \int_{\mathrm{I}} \varphi_n + \int_{\mathrm{I}^c} \varphi_n \\ &\leqslant n^{-\beta}(\mathrm{L}_n - n^{-\lambda}) + (1-n^{-\beta})(\mathrm{L}_n + c\,n^{-a}) \\ \int_0^1 \varphi_n &\leqslant \mathrm{L}_n + c\,n^{-a} - n^{-\beta-\lambda} - c\,n^{-a-\beta}\end{aligned}$$

If $\lambda + \beta < a$ there exists $c' > 0$ satisfying for all n large enough

$$c'n^{-a} - n^{-\beta-\lambda} - c'n^{-a-\beta} \leqslant -c'n^{-\lambda-\beta}$$

And so for n large enough:

$$\int_0^1 \varphi_n \leqslant \int_0^1 \varphi_n - c'n^{-\lambda-\beta}$$

which is a contradiction. All boils down to having such an interval satisfying (2.33).

The problem is that K_n may not contain an interval e.g. if K_n is a Cantor set. In this case $\partial \mathrm{K}_n = \mathrm{K}_n$ and $\partial \mathrm{K}_n$ has positive measure and so is not polar (see remark 2.5). Here is how we deal with this technical difficulty. Fixing $\ell > 0$, for a continuous function f on the torus $\mathbf{T}$, moving along $\mathbf{T}$ one can find a closed interval $\mathrm{I} \subset \mathbf{T}$ of length ℓ such that

$$\frac{1}{|\mathrm{I}|}\int_{\mathrm{I}} f = \int_{\mathbf{T}} f$$

This implies the existence of an interval I such that, for any β *a priori* fixed,

$$\frac{|\mathrm{I} \cap \mathrm{K}_n|}{|\mathrm{I}|} = |\mathrm{K}_n| \quad \text{with} \quad |\mathrm{I}| = n^{-\beta} \tag{2.34}$$

Let us fix (λ, σ). Assume that for all $\mathrm{C} > 0$ there are infinitely many integer n verifying

$$\mathrm{Leb}(\mathrm{K}_n(\lambda)) \geqslant \mathrm{C} \exp\left(-n^{\sigma}\right) \tag{2.35}$$

Our goal is to choose an adequate (λ, σ) such that (2.35) cannot be true, and so we will get a $\mathrm{C} > 0$ such that $\mathrm{Leb}(\mathrm{K}_n(\lambda)) < \mathrm{C} \exp\left(-n^{\sigma}\right)$ for all n large enough.

We define the compact set $\mathrm{K} = \mathrm{I} \cap \mathrm{K}_n \subset \mathbf{R}$. Up to a translation, we can suppose that $\mathrm{I} = [-r/2\,; r/2]$ with $r = n^{-\beta}$ and so $\mathrm{K} \subset [-r/2\,; r/2]$. We consider D_{2r} the disc circling K and the domain $\mathscr{D} = \mathrm{D}_{2r} \smallsetminus \mathrm{K}$ with boundary $\partial \mathrm{D}_{2r} \cup \mathrm{K}$. Since

$$r = n^{-\beta} \ll \rho \tag{2.36}$$

the extension φ_n is well defined on $\mathrm{D}_{2r} \subset \mathrm{S}_\rho$. Now we use (2.16). Let us estimate $\sup_{\partial \mathrm{D}_{2r}} \varphi_n$ with (2.14) where we choose $\varepsilon = 2r = 2n^{-\beta}$. One gets for all $x \in [-r/2\,; r/2]$:

$$\varphi_n(x) \leqslant (1 - p_x)(\mathrm{L}_n + 2n^{-\beta}\|\varphi_n\|_0 + n^{\beta b_2 - b_1} + n^{b_2 - 1}) + p_x(\mathrm{L}_n - n^{-\lambda})$$

Then corollary 2.1 and (2.34) give, uniformly for a. e. $x \in \mathrm{I}$

$$p_x = \mathbf{P}_x(\mathrm{B}_\tau \in \mathrm{K}) \gtrsim \frac{1}{\ln r - \ln(\mathrm{Leb}\,\mathrm{I} \times \mathrm{Leb}\,\mathrm{K}_n)}$$

The assumptions (2.35), $\mathrm{Leb}\,\mathrm{I} = n^{-\beta}$ and $r = n^{-\beta}$ imply that

$$p_x \gtrsim \frac{1}{n^{\sigma}}$$

Summing up it remains, for almost every $x \in \mathrm{I}$

$$\varphi_n(x) - \mathrm{L}_n \lesssim n^{-\beta} + n^{\beta b_2 - b_1} + n^{b_2 - 1} - n^{-\lambda-\sigma} =: g(n)$$

We finally impose the following conditions on β, and on (λ, σ) in (2.35) and $0 < b_1 < 1$ in (2.14)

$$0 < \lambda + \sigma + \beta < a \qquad \lambda + \sigma < \min\,(\beta, 1 - b_2, b_1 - \beta b_2) \qquad \beta b_2 < b_1 < 1$$

This ensures that $g(n) \sim_{n\to+\infty} -n^{-(\lambda+\sigma)}$ so that the constant $\gamma = \lambda + \sigma$ verifies

$$\exists c''' > 0 \quad \forall n \gg 1 \qquad g(n) \leqslant -c''' n^{-\gamma}$$

Consequently we obtain for infinitely many n's

$$\mathrm{Leb\,a.\,e.}\ x \in \mathrm{I} \qquad \varphi_n(x) \leqslant \mathrm{L}_n - c'''\, n^{-\gamma}$$

with $\mathrm{Leb}\,\mathrm{I} = n^{-\beta}$ and $\gamma + \beta < a$. Now we conclude with the observation (2.33) made at the beginning of the proof. Hence there is a choice of λ, σ such that there is a $\mathrm{C} > 0$ satisfying $\mathrm{Leb}\,\mathrm{K}_n(\lambda) < \mathrm{C}\,\mathrm{e}^{-n^{\sigma}}$ for all n large enough.

■

Then proposition 2.2 and proposition 2.5 easily imply theorem 2.1.

CHAPTER **3**

LDT for the inverse branches of the dynamics $x \mapsto 2x \mod 1$

3.1 Setting, main theorem and strategy of the proof

3.1.1 Model and notations

In this chapter, we are interested in Schrödinger operators with a potential given by the dynamics of the *doubling map*

$$m = m_2 \colon \begin{cases} \mathbf{T} \longrightarrow \mathbf{T} \\ x \longmapsto 2x \mod 1 \end{cases}$$

where $\mathbf{T}$ is $\mathbf{R}/\mathbf{Z} \simeq [\,0\,;1\,]/\sim$. This means that the potential in (1.2) is $v_n(x) = v(2^n x)$ where $v : \mathbf{T} \to \mathbf{R}$ is real analytic. Of course such an operator is defined on $\ell^2(\mathbf{Z}_+)$.

In this context, the Lyapunov exponent of the associated cocycle is $\mathrm{L}(\mathrm{E}) = \lim_{n\to+\infty} \mathrm{L}_n$ where

$$\mathrm{L}_n = \int_{\mathbf{T}} \varphi_n(x)\,\mathrm{d}x \tag{3.1}$$

$$\varphi_n(x) = \frac{1}{n} \log \|\mathrm{S}_{\mathrm{E},v}(2^{n-1}x)\ldots\mathrm{S}_{\mathrm{E},v}(2x)\,\mathrm{S}_{\mathrm{E},v}(x)\| \tag{3.2}$$

$$\forall x \in \mathbf{T} \qquad \mathrm{S}_{\mathrm{E},v}(x) = \begin{pmatrix} \mathrm{E} - v(x) & -1 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

Note that we still get, with the same proof, a pseudo invariance property similar to the one in lemma 2.1 of chapter 2: if $c = 2\log\sup_{x\in\mathbf{T}} \|\mathrm{S}_{\mathrm{E},v}(x)\|$ then

$$\forall x \in \mathbf{T} \qquad \left|\varphi_n(2^k x) - \varphi_n(x)\right| \leqslant c\frac{k}{n}$$

A natural idea is to reproduce the scheme of the proof made in chapter 2 which begins with the analytic extension of $\mathrm{S}_{\mathrm{E},v}(\cdot)$ and so the continuous subharmonic extension of φ_n. Unfortunately, it is clear that, contrary to the quasi-periodic dynamics $x \mapsto x + \alpha$ that preserves any complex strip $\mathrm{S}_\rho = \{|\Im\mathfrak{m}\, z| \leqslant \rho\}$, the *complexified* doubling map $z \mapsto 2z \mod (1,0)$ does **not** leave it invariant. Consequently, $\varphi_n(z)$ is only defined and uniformly bounded in n and $z \in \mathrm{S}_{\rho_n}$ with

$$\rho_n = 2^{-n}\rho \tag{3.4}$$

But this is an issue as all the estimates in proposition 2.1 involve the ratio ε/ρ which is now exponentially large. This is why we turn to the inverse branches of the doubling map.

3.1.2 Main result

We prove the following result of large deviation estimates:

Theorem 3.1

Let us denote $\psi_n := \mathrm{T}^n\varphi_n$ the average of φ_n over all inverse images for $m_2{}^n$ (see (3.2) for the precise definition of T*). Then for some $\nu, \lambda \in \,]\,0\,;1\,[$*

$$\mathrm{Leb}\,\{x \in \mathbf{T} \,:\, |\psi_n(x) - \textstyle\int_{\mathbf{T}} \varphi_n(t)\,\mathrm{d}t| \geqslant n^{-\nu}\} \lesssim \mathrm{e}^{-n^\lambda}$$

3.1.3 Strategy of the proof

(i) At first we prove in proposition 3.1 upper estimates for φ_n that are similar to those in the quasi-periodic case (proposition 2.1). The small difference here is that the dynamics is not quasi-periodic, so we have to use the mixing of the doubling map instead of the unique ergodic of irrational rotations. We establish a speed of mixing for the Cauchy Kernel $p_{\varepsilon,x}$ (see (2.11)) and any subharmonic map φ on a strip S_ρ. Here we use the Fourier coefficients argument of Bourgain that we mentioned in the section 2.2 of the previous chapter.

(ii) Then we explain why the size of the strip (3.4) is actually an issue to get estimates that do not degenerate as the dynamics of the doubling map is iterated.

(iii) Finally we define the transfer operator T and state in lemma 3.2 some of its properties that we will use throughout this chapter. This leads us to study the inverse branches of the doubling map. We prove these estimates with the same arguments as in the previous chapter because $\psi_n = \mathrm{T}^n\varphi_n$ is now defined on a strip of fixed size $\asymp 1$ as $n \to +\infty$.

3.2 Upper estimates for subharmonic functions on a strip with pseudo invariance

Using the same protocole as in the previous chapter we obtain:

Proposition 3.1

For any sequence φ_n of continuous subharmonic maps uniformly bounded in n on $\mathrm{S}_\rho = \{|\Im\mathfrak{m}\, z| \leqslant \rho\}$ that satisfy a pseudo invariance for T such as: for a $c_\varphi > 0$

$$\forall n, p \geqslant 1 \qquad \|\mathrm{T}^p\varphi_n - \varphi_n\|_{\mathscr{C}^0(\mathrm{S}_\rho)} \leqslant c_\varphi \frac{p}{n} \tag{3.5}$$

the following holds for all positive integers k, n and all $0 < \varepsilon \leqslant \rho$

$$\varphi_n(x \pm \mathrm{i}\varepsilon) - \int_0^1 \varphi_n \lesssim \frac{\varepsilon}{\rho}\|\varphi_n\|_{\mathscr{C}^0(\mathrm{S}_\rho)} + c_\varphi \frac{k}{n} + \frac{1}{\rho\, 2^k \sqrt{\varepsilon}}$$

Proof.

The proof is the same as in the previous chapter 2 and uses the Brownian motion so that we get the same results as in proposition 2.3, and lemma 2.3 concerning the law of the exit time of the domains $\mathbf{H}^+$ and $\mathcal{C}$ of a BM starting at $\frac{1}{2} + \mathrm{i}\varepsilon$.

The only difference in this context is the unique ergodicity argument of the irrational rotation $x \mapsto x + \alpha$ that cannot be used anymore to evaluate the quantity (see (2.9) in chapter 2)

$$\mathbf{E}_{x+\mathrm{i}\varepsilon}\Big(\varphi_n(\mathrm{B}_{\tau_{\mathbf{H}^+}})\Big) = \frac{\varepsilon}{\pi}\int_{\mathbf{R}} \frac{\varphi_n(t)}{(x-t)^2+\varepsilon^2}\,\mathrm{d}t = \int_0^1 \varphi_n(t)\,p_{\varepsilon,x}(t)\,\mathrm{d}t$$

where we remind that $p_{\varepsilon,x}$ is the kernel satisfying (see (2.11) in chapter 2)

$$p_{\varepsilon,x}(u) = \frac{1}{\pi}\sum_{k=-\infty}^{+\infty} \frac{\varepsilon}{(x-u-k)^2+\varepsilon^2} \qquad \text{and} \qquad \int_0^1 p_{\varepsilon,x}(u)\,\mathrm{d}u = 1 \tag{3.6}$$

which is a one-periodic positive continuous bounded function on the real axis.

The useful ergodic observation here is that $x \mapsto 2x \mod 1$ is mixing: for all $f \in \mathrm{L}^2([\,0\,;1\,])$:

$$\int_0^1 f(2^k t)\,g(t)\,\mathrm{d}t \xrightarrow[\mathrm{K}\to+\infty]{} \int_0^1 f \int_0^1 g$$

and so for $\varphi = \varphi_n$

$$\int_0^1 \mathrm{T}^k\varphi\,p_{\varepsilon,x} = \int_0^1 \varphi(u)\,p_{\varepsilon,x}(2^k u)\,\mathrm{d}u \xrightarrow[k\to+\infty]{} \int_0^1 \varphi(t)\,\mathrm{d}t \int_0^1 p_{\varepsilon,x}(u)\,\mathrm{d}u = \int_0^1 \varphi$$

which provides a way to estimate the difference

$$\int_{\mathbf{R}} \varphi(t)\,\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\,\mathrm{d}t - \int_0^1 \varphi(t)\,\mathrm{d}t = \int_0^1 \varphi(t)\,p_{\varepsilon,x}(t)\,\mathrm{d}t - \int_0^1 \varphi(t)\,\mathrm{d}t$$

if we are able to get some quantitative estimates on the speed of mixing for the kernel $p_{\varepsilon,x}$.

Let us define

$$\mathrm{A}_k = \int_0^1 \varphi(u)\,p_{\varepsilon,x}(2^k u)\,\mathrm{d}u \tag{3.7}$$

so that

$$\int_0^1 \varphi(u)\,p_{\varepsilon,x}(2^k u)\,\mathrm{d}u - \int_0^1 \varphi(t)\,\mathrm{d}t \int_0^1 p_{\varepsilon,x}(u)\,\mathrm{d}u = \mathrm{A}_k - \int_0^1 \varphi$$

With (3.5) one can estimate

$$\begin{aligned}\left|\mathrm{A}_k - \int_0^1 \varphi(t)\,p_{\varepsilon,x}(t)\,\mathrm{d}t\right| &= \left|\int_0^1 \mathrm{T}^k\varphi(t)\,p_{\varepsilon,x}(t)\,\mathrm{d}t - \int_0^1 \varphi(t)\,p_{\varepsilon,x}(t)\,\mathrm{d}t\right| \\ &= \left|\int_0^1 \Big[\mathrm{T}^k\varphi(t)\,p_{\varepsilon,x}(t) - \varphi(t)\,p_{\varepsilon,x}(t)\Big]\right|\,\mathrm{d}t \\ \left|\mathrm{A}_k - \int_0^1 \varphi(t)\,p_{\varepsilon,x}(t)\,\mathrm{d}t\right| &= \int_0^1 p_{\varepsilon,x}(t)\,\underbrace{\left|\mathrm{T}^k\varphi(t) - \varphi(t)\right|}_{\leqslant c|k|/n}\,\mathrm{d}t\end{aligned}$$

Hence

$$\left|\mathrm{A}_k - \int_0^1 \varphi(t)\,p_{\varepsilon,x}(t)\,\mathrm{d}t\right| \leqslant c\frac{k}{n} \tag{3.8}$$

So $\int_0^1 \varphi\,p_{\varepsilon,x}$ and $\int_0^1 \varphi$ are close if k is chosen appropriately, typically when $k \asymp n^d$ $(0 < d < 1)$. We want to quantify the difference between these two quantities.

Lemma 3.1 (Speed of mixing of $t \mapsto 2t$ for $p_{\varepsilon,x}$ and φ)

For any one-periodic map φ with bounded subharmonic extension to a strip S_ρ we have for all $k \geqslant 1$

$$\left|\int_0^1 \varphi(u)\, p_{\varepsilon,x}(2^k u)\,\mathrm{d}u - \int_0^1 \varphi(u)\,\mathrm{d}u \int_0^1 p_{\varepsilon,x}(u)\,\mathrm{d}u\right| \lesssim \frac{1}{\rho}\,\frac{1}{2^k}\,\frac{1}{\sqrt{\varepsilon}}$$

Proof.

For fixed (ε, x), we evaluate $(\widehat{p}_j)_{j\in\mathbf{Z}}$, the Fourier coefficients of $p_{\varepsilon,x}$, using the Fourier transform of the Cauchy kernel (remind (3.6)):

$$\mathcal{F}\left(t \mapsto \frac{1}{1+t^2}\right)(\omega) = \int_{\mathbf{R}} \frac{\mathrm{e}^{-2\mathrm{i}\pi\omega t}}{1+t^2}\,\mathrm{d}t = \pi\mathrm{e}^{-2\pi|\omega|}$$

This gives for all $j \in \mathbf{Z}$

$$\begin{aligned}
\widehat{p}_j &= \int_0^1 \frac{1}{\pi} \sum_{k=-\infty}^{+\infty} \frac{\varepsilon}{(x-t-k)^2+\varepsilon^2}\mathrm{e}^{-2\mathrm{i}\pi jt}\,\mathrm{d}t \\
&= \int_{\mathbf{R}} \frac{1}{\pi}\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\,\mathrm{e}^{-2\mathrm{i}\pi jt}\,\mathrm{d}t \\
\widehat{p}_j &= \mathrm{e}^{-2\mathrm{i}\pi jx}\,\mathrm{e}^{-2\pi|j|\varepsilon}
\end{aligned}$$

We also have for all $k \in \mathbf{N} \smallsetminus \{0\}$

$$\begin{aligned}
\mathrm{A}_k &= \sum_{j\in\mathbf{Z}} \widehat{\varphi}_j\, \overline{\widehat{p_\varepsilon(2^k\cdot)}_j} \\
&= \sum_{j\in 2^k\mathbf{Z}} \widehat{\varphi}_j\, \overline{\widehat{p_\varepsilon(2^k\cdot)}_j} \\
\mathrm{A}_k &= \sum_{j\in\mathbf{Z}} \widehat{\varphi}_{2^k j}\, \overline{\widehat{p}_j}
\end{aligned}$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbf{C}$. Corollary 4.7 in [Bou04] ensures that the Fourier coefficients of a subharmonic function on S_ρ decay as $\rho^{-1}|k|^{-1}$ as $|k| \to +\infty$ so that the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\left|\mathrm{A}_k - \widehat{\varphi}_0\,\overline{\widehat{p}_0}\right| &\lesssim \rho^{-1}\frac{1}{2^k}\frac{\pi}{\sqrt{3}}\left(2\sum_{j\geqslant 1}\mathrm{e}^{-4\pi j\varepsilon}\right)^{1/2} \\
&\lesssim \frac{1}{\rho}\frac{1}{2^k}\frac{1}{\sqrt{\varepsilon}}
\end{aligned} \tag{3.9}$$

This ends the proof since $\overline{\widehat{p}_0} = \widehat{p}_0 = \int_0^1 p_{\varepsilon,x}(t)\,\mathrm{d}t = 1$ (remind (3.6)). ∎

Finally one obtains the proof of proposition 3.1 thanks to the subharmonicity of φ and the triangular inequality with (3.8) and (3.9).

∎

3.3 The issue of the too small size of the strip

Of course we want to apply proposition 3.1 to our function φ_n but this goes wrong for the following reasons. Note that to control the term k/n one has to chose reasonably small growth of k such as $k \asymp n^d$ with $0 < d < 1$. As $\varepsilon \to 0$ we also need to ensure $\rho\, 2^k \varepsilon \to +\infty$, but at the same time we need $\varepsilon < \rho_n = \rho\, 2^{-n}$ (remind (3.4)) to satisfy (2.36) in the case we would adapt the proof made in chapter 2. This is not doable with the previous assumption on k. We will now focus on the inverse branches for which we can obtain an estimate because the size of the strip ρ will be fixed and so will not go to zero too fast as n goes to infinity.

3.4 Inverse branches: back to the future

To avoid the exponential growth of the doubling map we choose to consider its inverse branches. Indeed, m_2 admits two right inverses: if we use $\mathbf{T} = [\,0\,;1\,]/\sim$ and identify $x \mod 1$ with $\{x\}$ the fractional part these are

$$f_0 : x \longmapsto \frac{x}{2} \qquad \text{and} \qquad f_1 : x \longmapsto \frac{x+1}{2}$$

so that $x = f_0(2x)$ for $x \in \left[\,0\,; \frac{1}{2}\right]$ and $x = f_1(2x)$ for $x \in \left[\,\frac{1}{2}\,; 1\right]$

Remark 3.1

Note that in terms of shift if $x = \sum_{i=1}^{+\infty} \frac{x_i}{2^i} = (x_1, x_2, \dots)$ *we have*

$$\begin{aligned} f_0\left((x_i)_{i\geqslant 1}\right) &= (0,\, x_1, x_2 \dots) \\ f_1\left((x_i)_{i\geqslant 1}\right) &= (1,\, x_1, x_2 \dots) \end{aligned}$$

Splitting the integration over $[\,0\,;1/2\,]$ and $[\,1/2\,;1\,]$ gives:

Lemma 3.2

The following holds for all $\varphi \in \mathrm{L}^2(\mathbf{T})$

$$\int_{\mathbf{T}} \varphi(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbf{T}} (\varphi \circ f_0(x) + \varphi \circ f_1(x))\,\mathrm{d}x$$

which can be rephrased in terms of the transfer operator

$$\mathrm{T} \colon \begin{cases} \mathrm{L}^2(\mathbf{T}) \longrightarrow \mathrm{L}^2(\mathbf{T}) \\ \quad\varphi \quad\;\longmapsto \frac{1}{2}\left(\varphi \circ f_0 + \varphi \circ f_1\right) \end{cases}$$

This operator T *is Lebesgue invariant: we have* $\int_{\mathrm{T}} \varphi\,\mathrm{d}x = \int_{\mathrm{T}} \mathrm{T} \circ \varphi\,\mathrm{d}x$ *for all* $\varphi \in \mathrm{L}^2(\mathbf{T})$. *It also satisfies for all* $\varphi, \psi \in \mathrm{L}^2(\mathbf{T})$

$$\int_{\mathbf{T}} \varphi \circ \mathrm{T}(x)\, \psi(x)\, \mathrm{d}x = \int_{\mathbf{T}} \varphi(x)\, \psi(2x)\, \mathrm{d}x$$

From the previous lemma we deduce:

Proposition 3.2

Let us denote $\mathrm{S} = \mathrm{S}_{\mathrm{E},v}$*. For all integers* $k, n \geqslant 0$

$$\begin{aligned}
\mathrm{L}_n &= \frac{1}{n}\int_{\mathbf{T}} \log \|\mathrm{S}(2^n x)\dots \mathrm{S}(2x)\,\mathrm{S}(x)\|\,\mathrm{d}x \\
&= \frac{1}{2^k}\sum_{(i_1,\dots,i_k)\in\{0;1\}^k} \frac{1}{n}\int_{\mathbf{T}} \log \|\mathrm{S}(2^n f_{i_1}\circ\dots\circ f_{i_k}(x))\dots \mathrm{S}(f_{i_1}\circ\dots\circ f_{i_k}(x))\|\,\mathrm{d}x \\
\mathrm{L}_n &= \frac{1}{2^n}\sum_{(i_1,\dots,i_n)\in\{0;1\}^n} \frac{1}{n}\int_{\mathbf{T}} \log \|\mathrm{S}(x)\,\mathrm{S}(f_{i_1}(x))\dots \mathrm{S}(f_{i_1}\circ\dots\circ f_{i_n}(x))\|\,\mathrm{d}x
\end{aligned}$$

Proof.

This is just a k-times iteration of lemma 3.2 for the first equality and an easy change of variables for the second one.

∎

Remark 3.2

Note that for $k = n$*, we have* $2^n f_{i_1}\circ\dots\circ f_{i_n}(x) = x$ *so that for a fixed* $(i_1,\dots,i_n) \in \{0;1\}^n$ *we are just browsing through a branch of the tree of all inverses of* x *for the doubling map until an antecedent of order* n*.*

Let us establish a pseudo-invariance property with respect to the dynamics on the inverse branches:

Lemma 3.3

For all positive k, n *and* $z \in \mathrm{S}_{\rho_n}$

$$\left|\mathrm{T}^k\varphi_n(z) - \varphi_n(z)\right| \leqslant c\frac{k}{n}$$

where the estimate is uniform in n, z *for* $z \in \mathrm{S}_{\rho_n}$*.*

Proof.

It is enough to prove the case $k = 1$ since an easy induction argument then gives the result for all k thanks to the triangular inequality: if $\|\mathrm{T}^k\varphi_n - \varphi_n\|_0 \leqslant ck/n$ then for all x

$$\begin{aligned}\left|\mathrm{T}^{k+1}\varphi_n(x)-\varphi_n(x)\right| &\leqslant \left|\mathrm{T}(\mathrm{T}^k\varphi_n)(x)-\mathrm{T}^k\varphi_n(x)\right|+\left|\mathrm{T}^k\varphi_n(x)-\varphi_n(x)\right|\\ &\leqslant \frac{c}{n}+\frac{ck}{n}=c\frac{k+1}{n}\end{aligned}$$

We compute for $c=\max_{z\in \mathrm{S}_\rho}\log\|\mathrm{S}(z)\|$

$$\begin{aligned}n\,\varphi_n\circ f_0(x) &= \log\|\mathrm{S}(2^{n-1}x)\ldots\mathrm{S}(x)\,\mathrm{S}(\tfrac{x}{2})\|\\ &= \log\|\mathrm{S}(2^n x)^{-1}\,\mathrm{S}(2^n x)\,\mathrm{S}(2^{n-1}x)\ldots\mathrm{S}(x)\,\mathrm{S}(\tfrac{x}{2})\|\\ n\,\varphi_n\circ f_0(x) &\leqslant 2c+n\varphi_n(x)\end{aligned}$$

and

$$\begin{aligned}n\,\varphi_n\circ f_1(x) &= \log\|\mathrm{S}(2^{n-1}\tfrac{x+1}{2})\ldots\mathrm{S}(2\tfrac{x+1}{2})\,\mathrm{S}(\tfrac{x+1}{2})\|\\ &= \log\|\mathrm{S}(2^n x)^{-1}\,\mathrm{S}(2^n x)\,\mathrm{S}(2^{n-1}x)\ldots\mathrm{S}(x)\,\mathrm{S}(\tfrac{x+1}{2})\|\\ n\,\varphi_n\circ f_1(x) &\leqslant 2c+n\varphi_n(x)\end{aligned}$$

so

$$\mathrm{T}\varphi_n(x)-\varphi_n(x) \leqslant 2\tfrac{c}{n}$$

Then similarly:

$$\begin{aligned}n\,\varphi_n(x) &= \log\|\mathrm{S}(2^n x)\ldots\mathrm{S}(x)\,\mathrm{S}(\tfrac{x}{2})\|\\ &= \tfrac{1}{2}\log\|\mathrm{S}(2^n x)\,\mathrm{S}(2^{n-1}x)\ldots\mathrm{S}(x)\,\mathrm{S}(\tfrac{x}{2})\,\mathrm{S}(\tfrac{x}{2})^{-1}\|\\ &\qquad+\tfrac{1}{2}\log\|\mathrm{S}(2^n x)\,\mathrm{S}(2^{n-1}x)\ldots\mathrm{S}(x)\,\mathrm{S}(\tfrac{x+1}{2})\,\mathrm{S}(\tfrac{x+1}{2})^{-1}\|\\ n\,\varphi_n(x) &\leqslant 2c+\mathrm{T}\varphi_n(x)\end{aligned}$$

and

$$\begin{aligned}n\,\varphi_n\circ f_1(x) &= \log\|\mathrm{S}(2^{n-1}\tfrac{x+1}{2})\ldots\mathrm{S}(2\tfrac{x+1}{2})\,\mathrm{S}(\tfrac{x+1}{2})\|\\ &= \log\|\mathrm{S}(2^n x)^{-1}\,\mathrm{S}(2^n x)\,\mathrm{S}(2^{n-1}x)\ldots\mathrm{S}(x)\,\mathrm{S}(\tfrac{x+1}{2})\|\\ n\,\varphi_n\circ f_1(x) &\leqslant 2c+n\varphi_n(x)\end{aligned}$$

■

3.5 LDT for the inverse branches

3.5.1 Upper bounds

As we noticed in section 3.3, we cannot directly apply the result for φ_n as the estimate in proposition 3.1 degenerates as $n\to+\infty$ and $\varepsilon=\varepsilon(n)\to 0$. Let us however consider the inverse branches: by this we mean that we work with

$$\psi_n=\mathrm{T}^n\varphi_n \tag{3.10}$$

Remark that $\psi_n(x)$ is the average over all the 2^n inverse images of x for ${m_2}^n$. Indeed let us denote $f^{(i)}=f_{i_k}\circ\cdots\circ f_{i_2}\circ f_{i_1}$ for an index $i=(i_1,\ldots,i_k)\in\{0\,;1\}^k$ where f_0, f_1 are the previous left inverses of m_2. We compute with lemma 3.2

$$\begin{aligned}\mathrm{T}^2\varphi &= \frac{1}{2}\left(\tfrac{1}{2}\varphi\circ f_0\circ f_0+\tfrac{1}{2}\varphi\circ f_0\circ f_1+\tfrac{1}{2}\varphi\circ f_1\circ f_0+\tfrac{1}{2}\varphi\circ f_1\circ f_1\right)\\ &= \frac{1}{4}\sum_{i\in\{0;1\}^2}\varphi\circ f^{(i)}\end{aligned}$$

Then an easy induction gives

$$\mathrm{T}^n\varphi = \frac{1}{2^n}\sum_{i\in\{0;1\}^n} \varphi\circ f^{(i)}$$

As $\left\{f^{(i)}(x)\ :\ i\in\{0;1\}^n\right\}$ is the set of all inverses images of x for ${m_2}^n$, we obtain the claim about the average. Let us just give some details in order to interpret ψ in terms of the transfer matrices that are involved. We observe that for $i=(i_1,\dots,i_n)\in\{0\,;1\}^n$ and all $k\in[\![\,1\,;\,n\,]\!]$

$$2^{n-k}f^{(i)} = f_{i_k}\circ\cdots\circ f_{i_1}$$

And so

$$\varphi_n\circ f^{(i)}(x) = \frac{1}{n}\log\|\mathrm{S}(x)\,\mathrm{S}(f_{i_1}(x))\,\mathrm{S}(f_{i_2}\circ f_{i_1}(x))\dots\mathrm{S}(f^{(i)}(x))\|$$

This is the *log-norm* product of the transfer matrices browsing through a branch in the tree of all inverses images and so it explains that ψ_n is the average over the 2^n inverse branches of order n of the dynamics $x\mapsto 2x \mod 1$. Notice that this function ψ_n admits a well defined continuous subharmonic extension on $\{|\mathfrak{Im}\,z|\leqslant\rho\}$ a strip of size $\asymp_{n\to+\infty} 1$ (contrary to φ_n) because each branch is locally subharmonic as the log of an analytic map. This extension still is a one-periodic function as, when evaluating at $z+1$, one only permutes the 2^n inverse images of a point z for ${m_2}^n$ so that the averaged sum that defines ψ_n remains unchanged. Each $\varphi_n\circ f^{(i)}$ verifies the pseudo-invariance criteria and so does the average ψ_n. According to proposition 3.1, we then have the following estimate

Proposition 3.3

For $\varepsilon<\rho$ and all integers $k,n\geqslant 1$

$$\psi_n(x+\mathrm{i}\varepsilon)-\int_0^1\psi_n \lesssim \frac{\varepsilon}{\rho}+\frac{k}{n}+\frac{1}{\rho\,2^k\sqrt{\varepsilon}}$$

with a constant in $\lesssim$ that only depends on $\max_{z\in\mathrm{S}_\rho}\log\|\mathrm{S}_{\mathrm{E},v}(z)\|$.

Similarly to the work done in chapter 2, one could obtain an upper estimate on $\mathbf{T}$ for all x. Indeed, using subharmonicity of ψ_n, choosing $k\asymp n^a$ with $0<a<1$ and $\varepsilon\asymp 2^{-n^{a^-}}$ one would obtain

$$\psi_n(x)-\int_0^1\psi_n \leqslant \sup\nolimits_{x\in\mathbf{R}}\psi_n(x+\mathrm{i}\varepsilon)-\int_0^1\psi_n\lesssim n^{a-1}$$

Thanks to (3.10) and lemma 3.2 we have $\int_0^1\psi_n=\int_0^1\varphi_n$ so we have proved with $a'=1-a>0$ that

$$\forall x\in\mathbf{T}\qquad \mathrm{T}^n\varphi_n(x)-\int_0^1\varphi_n\lesssim\frac{1}{n^{a'}}$$

3.5.2 Lower bounds

We obtained in chapter 2 that the set $\mathrm{K}_n(\lambda)$ of the *bad points* (see (2.15)), where the lower estimate is violated, has exponentially small measure. The proof made in section 2.5.3 of chapter 2 totally adapts here because the arguments are independent from the dynamics but rely on the potential theory in the complex plane that we exposed in the previous chapter in section 2.2, and the guiding principle in section 2.5.1.

The only assumption required (see proposition 2.5) is an upper estimate like in proposition 3.3 which was proved in the previous section. This ensures theorem 3.1.

Part II

Figotin-Pastur formula for Schrödinger cocycles with quasi-periodic and random potential

CHAPTER 4

Figotin-Pastur formula for diophantine energies w.r.t. α

4.1 Introduction, main result and ideas of the proof

4.1.1 Setting and notations

Let us define the following potential

$$v(\omega, x) = \lambda\omega_0 + \mathrm{V}(x) = \lambda \mathrm{W}_0(\omega) + \mathrm{V}(x) \tag{4.1}$$

for $(\omega, x) \in \Omega \times \mathbf{T}$, where:

- $\Omega = \mathbf{R}^{\otimes \mathbf{Z}}$ is a product probability space endowed with $\mathbf{P} = \mathbf{P}_0{}^{\otimes \mathbf{Z}}$ for some $\mathbf{P}_0 \in \mathrm{Prob}(\mathbf{R})$; the only assumption we need on the probability is that we can use Fubini's theorem with $\mathrm{d}\mathbf{P}_0$ and $\mathrm{d}x$ the Haar-Lebesgue measure on $\mathbf{T}$. For example this is the case when $\mathrm{d}\mathbf{P}_0$ is absolutely continuous w.r.t. the Lebesgue measure on $\mathbf{R}$ with a compactly supported L^1 density;
- $\mathbf{T} \simeq [\,0\,;1\,]$ is the one-dimensional torus;
- $\mathrm{V} \in \mathscr{C}^p(\mathbf{T}, \mathbf{R})$ is a real potential (which will ultimately be needed as small as λ) the regularity of which will be specified when needed;
- $\mathrm{W}_0 : \Omega \ni (\omega_n)_{n\in\mathbf{Z}} \longmapsto \omega_0 \in \mathbf{R}$ is the random potential;
- λ is a positive real *coupling constant.*

Remark 4.1

Of course V *will be assumed small but we first carry out our analysis without specifying the smallness of* V *compared to* λ *until the very end of the chapter where we discuss this condition to establish the formula for the Lyapunov exponent.*

Let us define the dynamics on $\Omega \times \mathbf{T}$ as the direct product of the full-shift σ on Ω

$$\sigma(\omega_n)_{n\in\mathbf{Z}} = (\omega_{n+1})_{n\in\mathbf{Z}} \tag{4.2}$$

with the rotation $r_\alpha : x \mapsto x + \alpha \mod 1$ on the torus:

$$\mathrm{T} = \sigma \times r_\alpha \colon \begin{cases} \Omega \times \mathbf{T} \longrightarrow \Omega \times \mathbf{T} \\ (\omega, x) \longmapsto (\sigma\omega, x + \alpha) \end{cases} \tag{4.3}$$

The cocycle for an energy E is then defined on $\Omega \times \mathbf{T}$ by

$$(\mathrm{T}, \mathrm{S}_\mathrm{E}) \colon \begin{cases} \Omega \times \mathbf{T} \times \mathbf{R}^2 \longrightarrow \Omega \times \mathbf{T} \times \mathbf{R}^2 \\ (\omega, x, u) \longmapsto (\sigma\omega, x + \alpha, \mathrm{S}_{\mathrm{E},\omega,x} u) \end{cases} \tag{4.4}$$

where
$$\mathrm{S}_{\omega,x} = \mathrm{S}_{\mathrm{E},\omega,x} = \begin{pmatrix} \mathrm{E} - v(\omega, x) & -1 \\ 1 & 0 \end{pmatrix}$$

We keep the dependency on E implicit when no confusion is possible. The iterates of the cocycle are denoted by $(\mathrm{T}, \mathrm{S}_\mathrm{E})^n = (\mathrm{T}^n, \mathrm{S}_\mathrm{E}^{(n)})$ where

$$\forall n \in \mathbf{N} \qquad \mathrm{S}_\mathrm{E}^{(n)}(\omega, x) = \mathrm{S}_{\mathrm{E},\mathrm{T}^{n-1}(\omega,x)} \cdots \mathrm{S}_{\mathrm{E},\mathrm{T}(\omega,x)}\, \mathrm{S}_{\mathrm{E},\omega,x} = \prod_{k=n-1}^{0} \mathrm{S}_{\mathrm{E},\mathrm{T}^k(\omega,x)}$$

To compute the Lyapunov exponent we need an ergodic measure for T. The Lyapunov exponent in our context of mixed quasi-periodic and random Schrödinger cocycle is defined as follows:

Definition 4.1 (LE for quasi-periodic + random dynamics)

The Lyapunov exponent that we are interested in is:

$$\mathrm{L}(\mathrm{E}) = \lim_{n\to+\infty} \frac{1}{n} \int_{\Omega\times\mathbf{T}} \log \|\mathrm{S}_\mathrm{E}^{(n)}(\omega, x)\| \,\mathrm{d}\mathbf{P}(\omega)\,\mathrm{d}x = \lim_{n\to+\infty} \frac{1}{n}\, \mathbf{E} \int_{\mathbf{T}} \log \|\mathrm{S}_\mathrm{E}^{(n)}(\omega, x)\|\,\mathrm{d}x \tag{4.5}$$

4.1.2 Main theorem

In this chapter we prove that a Figotin-Pastur formula remains valid when the potential is a small mixture of quasi-periodic and random potentials, provided the quasi-periodic part is differentiable enough w.r.t. the diophantine condition.

Theorem 4.1 (Figotin-Pastur formula QP + random)

Assume the potential (4.1) *is given by* $v(\omega, x) = \varepsilon\,(\mathrm{V}(x) + \mathrm{W}_0(\omega))$. *For an energy* $\mathrm{E} = 2\cos(\pi\beta)$ *with* β *in some diophantine class* $\mathrm{DC}_\alpha(\kappa, \tau)$ *w.r.t.* α *(see* (1.7)*) then the Lyapunov exponent of the quasi-periodic and random cocycle* (4.5) *admits the following perturbative development of Figotin-Pastur type: for* $\mathrm{V} \in \mathscr{C}^{4\tau+11}(\mathbf{T})$ *and sufficiently small* ε

$$\mathrm{L}(\mathrm{E}) = \frac{\mathbf{V}(\mathrm{W}_0)}{2(4-\mathrm{E}^2)}\varepsilon^2 + \mathrm{O}\left(\frac{\varepsilon^3}{\kappa^3(4-\mathrm{E}^2)^3}\right)$$

4.1.3 Strategy of the proof

To prove the previous theorem, we use the notion of random diffeomorphism induced by the projective action of the cocycle (see (4.7)).

Firstly we establish a formula (proposition 4.1) that relates the Lyapunov exponent of the initial Schrödinger cocycle to the exponent of the induced random diffeomorphism on $\mathbf{T}^2$.

We obtain a development for any stationary measure of the random diffeomorphism obtained after an adequate conjugacy made in section 4.3.1. As this diffeomorphism is actually close to a rotation on $\mathbf{T}^2$, we show that our goal can be achieved if we are able to solve cohomological equations (4.25) for the rotation. These equations are of course more explicit and workable: under an arithmetic assumption on the frequencies of the rotation, the $\mathscr{C}^k$-norms of the solution can be controlled (see lemma 4.2).

An additional usefull conjugacy is exploited to get a simpler formula for the LE (proposition 4.3). Once all the error terms are properly analyzed (see (4.48)), we apply the development of a stationary measure to get an expansion of the Lyapunov exponent using explicit Taylor developments for the Lyapunov exponent of the random diffeomorphism: this is formula (4.50).

We end this chapter by commenting on the error term and the smallness required on V in the potential (4.1) and, ultimately, why despite the vanishing assumption on the expectation of the potential is not necessary to our work, one cannot perturb the potential to insure the arithmetic condition (4.28) needed on the energy for our approach (remark 4.7).

4.2 Random diffeomorphisms and Lyapunov exponent

4.2.1 Random diffeomorphisms

We shall now give the setting of random diffeomorphism we will use throughout the remainder of the manuscript. Let us remind that, given a probability space $(\Omega, \mathcal{T}, \mathbf{P})$, a random map f on a topological space $\mathcal{X}$ is a map

$$f\colon \begin{cases} \Omega \times \mathcal{X} \longrightarrow \mathcal{X} \\ (\omega, \chi) \longmapsto f_\omega(\chi) \end{cases}$$

such that for all $\chi \in \mathcal{X}$ the map $\omega \mapsto f_\omega(\chi)$ is measurable. The definition of a random diffeomorphism additionally requires that for almost every ω, the map f_ω is a diffeomorphism (see chapter I in [Kif12] and chapter 1 in [Mal12]).

In this context, the relevant notion of invariance is the notion of *stationary measure*. A probability measure μ on $\mathbf{T}^2$ is said stationary if it is invariant under the transfer operator $\mathscr{T}$ defined as follows

$$\forall \varphi \in \mathrm{L}^1(\mathbf{T}^2) \qquad \mathscr{T}\varphi = \mathbf{E}(\varphi \circ \mathrm{F}) \tag{4.6}$$

We recall that a stationary measure is said *ergodic* if it is an extremal point among all stationary measures.

There is a way to translate all what precedes about random diffeomorphisms into a classical dynamical setting. If $\widetilde{\Omega} = \Omega^{\mathbf{N}}$ and $\mathrm{F}(\omega, \chi) = (\sigma\omega, f_{\omega_0}(\chi))$ then

$$\mathrm{F}^n(\omega,\chi) = (\sigma^n\omega_0, f_{\omega_{n-1}}\circ\cdots\circ f_{\omega_0}(\chi))$$

so that the dynamics of F gives access to the random dynamics of f.

This relates the random diffeomorphism F to the dynamical system $\widetilde{\mathrm{F}}$ and so, a measure $\mu\in\mathrm{Prob}(\mathcal{X})$ is stationary for f if and only if $\mathbf{P}\times\mu$ is invariant for F, and the ergodicity of the measures are equivalent. The interesting thing is that we can use the ergodicity of $\mathbf{P}\times\mu$ in terms of the convergence of Birkhoff's averages given by standard results about dynamical systems. Stated in this context, the standard Birkhoff theorem ensures that if φ is any $\mathrm{L}^1(\mathcal{X},\mu)$ function then for μ-almost every $\mathrm{X}_0\in\mathcal{X}$ we have for $\mathbf{P}$ almost every ω

$$\frac{1}{n}\sum_{k=1}^{n}\varphi(\mathrm{F}^k_\omega(\mathrm{X}_0)) \xrightarrow[n\to\infty]{} \int_{\mathcal{X}}\varphi\,\mathrm{d}\mu$$

where $\mathrm{F}^k_\omega = \mathrm{F}_{\sigma^{k-1}\omega}\circ\cdots\circ\mathrm{F}_\omega$.

Let us see the random diffeomorphism that arises in our setting where $\mathcal{X}=\mathbf{T}^2$ and $\Omega=\mathbf{R}^{\mathbf{Z}}$ is already a product space.

Definition 4.2

Let $\mathrm{E}\in\mathbf{R}$ *be fixed. The elementary transfer matrix* $\mathrm{S}_{\mathrm{E},\omega,x}$ *induces a random diffeomorphism* $f_{\omega,x}=f_{\mathrm{E},\omega,x}$ *on the torus by its projective action*

$$\mathrm{e}^{\,\mathrm{i}\pi f_{\omega,x}(y)} = \pm\frac{\mathrm{S}_{\mathrm{E},\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}}{\|\mathrm{S}_{\mathrm{E},\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}\|} \tag{4.7}$$

where we write abusively $\mathrm{e}^{\,\mathrm{i}t}=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$ *identifying* $\mathbf{C}$ *and* $\mathbf{R}^2$.

We then work with

$$\widetilde{\mathrm{F}}\colon\begin{cases}\Omega\times\mathbf{T}^2 \longrightarrow \Omega\times\mathbf{T}^2\\ (\omega,(x,y))\longmapsto(\sigma\omega,\mathrm{F}^\omega(x,y))\end{cases}$$

where $\mathrm{F}^\omega(x,y)=(x+\alpha, f_{\omega,x}(y))$, so that we have for all $n\geqslant 1$

$$\widetilde{\mathrm{F}}^n(\omega,(x,y)) = \left(\sigma^n\omega, x+n\alpha, f^{(n)}_{\omega,x}(y)\right) \quad\text{with}\quad f^{(n)}_{\omega,x} = f_{\mathrm{T}^{n-1}(\omega,x)}\circ\cdots\circ f_{\sigma\omega,x+\alpha}\circ f_{\omega,x}$$

In order to deal with the whole 2-dimensional dynamics, we need to study the following random diffeomorphism of $\mathbf{T}^2$:

$$\mathrm{F}_\omega\colon\begin{cases}\mathbf{T}^2\longrightarrow\mathbf{T}^2\\ (x,y)\longmapsto(x+\alpha, f_{\omega,x}(y))\end{cases} \tag{4.8}$$

We now deduce an ergodic measure for the initial cocycle which is adapted to the notion of random diffeomorphism that we just developed.

Lemma 4.1

Let us fix μ an ergodic measure for the random diffeomorphism $\omega \mapsto \mathrm{F}_\omega$ and denote $\widetilde{\mu} = \pi_{1*}\mu$. Then $\widetilde{\mu} = \mathrm{Leb}_{\mathbf{T}}$ and so $\mathbf{P} \times \widetilde{\mu}$ is ergodic for $\mathrm{T} = \sigma \times r_\alpha$ on $\Omega \times \mathbf{T}$ whenever $\alpha \in \mathbf{T} \setminus \mathbf{Q}/\mathbf{Z}$.

Proof.

Such a stationary and ergodic for F_ω measure μ exists thanks to the compactness of $\mathbf{T}^2$, the Krylov-Bogolyubov theorem, and the existence of extremal points in compact convex sets. Then, for $\varphi \in \mathscr{C}^0(\mathbf{T})$, the unique ergodicity of $x \mapsto x + \alpha$ for all irrational α gives

$$\forall x \in \mathbf{T} \quad \forall n \geqslant 1 \qquad \mathrm{B}_n(\varphi)(x) = \frac{1}{n}\sum_{k=1}^{n}\varphi(x + k\alpha) \xrightarrow[n\to\infty]{} \int_{\mathbf{T}}\varphi(x)\,\mathrm{d}x$$

but also
$$\mathrm{B}_n(\varphi)(x) = \frac{1}{n}\sum_{k=1}^{n}\Phi(\sigma^k\omega, \mathrm{F}_\omega^k(x,y)) = \mathrm{B}_n(\Phi)(\omega, x, y)$$

where $\Phi(\omega, x, y) = \varphi(x)$ so the ergodicity of $\mathbf{P} \times \mu$ ensures that for $\mathbf{P}$-almost every $\omega \in \Omega$ and μ-almost every $(x, y) \in \mathbf{T}^2$

$$\mathrm{B}_n(f)(x) \xrightarrow[n\to\infty]{} \int_{\Omega\times\mathbf{T}^2}\Phi\,\mathrm{d}\mathbf{P}\,\mathrm{d}\mu = \int_{\mathbf{T}}\varphi\,\mathrm{d}\widetilde{\mu}$$

The previous convergence is valid for $\widetilde{\mu}$-a.e. x and so

$$\int_{\mathbf{T}}\varphi\,\mathrm{dLeb} = \int_{\mathbf{T}}\varphi\,\mathrm{d}\widetilde{\mu}$$

The ergodicity of $(\Omega \times \mathbf{T}, \mathrm{T}, \mathbf{P} \times \widetilde{\mu})$ comes from the weak-mixing of $(\Omega, \sigma, \mathbf{P})$ and the ergodicity of $(\mathbf{T}, r_\alpha, \mathrm{Leb}_{\mathbf{T}})$. ■

4.2.2 Link between the two notions of Lyapunov exponent

It is clear that $\mathbf{P}_0{}^{\otimes\mathbf{Z}} \times \mathrm{Leb}_{\mathbf{T}}$ is an ergodic measure for the cocycle for any probability measure $\mathbf{P}_0$ on $\mathbf{R}$. What is more interesting is its relevance to get estimates with the theory of random diffeomorphisms. Let us elaborate on the two different notions of LE.

Definition 4.3 (LE of a diffeomorphism)

The Lyapunov exponent of a random diffeomorphism $\mathrm{F} = \mathrm{F}_\omega$ of $\mathbf{T}^2$ for a stationary and ergodic measure μ for F is defined as:

$$\gamma = \gamma(\mathrm{F}, \mu) = \mathbf{E}\int_{\mathbf{T}^2}\ln\left|\det(\mathrm{DF}_\omega)_{(x,y)}\right|\,\mathrm{d}\mu(x, y)$$

We will prove the following:

Proposition 4.1 (LE of the cocycle vs LE of the random diffeo.)

The Lyapunov exponent of the cocycle (4.5) *and the one of the random diffeomorphism it induces on* $\mathbf{T}^2$ *as in* (4.7) *are related as follows:*

$$\mathrm{L(E)} = \frac{1}{2}\left|\gamma(\mathrm{F})\right| \tag{4.9}$$

Remark 4.2

Actually the equality $\mathrm{L(E)} = -\frac{1}{2}\gamma(\mathrm{F})$ *holds in many situations as we will conclude at the end of this chapter with the Figotin-Pastur formula we obtain.*

Proof.

We define on $\Omega \times \mathbf{T}^2$, $\Psi : (\omega, x, y) \mapsto \log \|\mathrm{S}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}\|$ so that for $k \in \mathbf{N}$

$$\Psi \circ \widetilde{\mathrm{F}}^k(\omega, x, y) = \log \|\mathrm{S}_{\sigma^k\omega, x+k\alpha}\mathrm{e}^{\,\mathrm{i}\pi f^{(k)}_{\omega,x}(y)}\|$$

But the definition

$$\mathrm{e}^{\,\mathrm{i}\pi f_{\omega,x}(y)} = \pm\frac{\mathrm{S}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}}{\|\mathrm{S}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}\|}$$

and an easy induction argument give:

$$\forall n \in \mathbf{N} \qquad \mathrm{e}^{\,\mathrm{i}\pi f^{(n)}_{\omega,x}(y)} = \pm\frac{\mathrm{S}^{(n)}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}}{\|\mathrm{S}^{(n)}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}\|}$$

which leads to

$$\Psi \circ \widetilde{\mathrm{F}}^k(\omega, x, y) = \log\frac{\|\mathrm{S}^{(k+1)}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}\|}{\|\mathrm{S}^{(k)}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}\|}$$

This ensures that Birkhoff's sums of Ψ under $\widetilde{\mathrm{F}}$ are telescopic. Moreover thanks to the ergodicity of $\mathbf{P} \times \widetilde{\mu}$, the Fürstenberg-Kesten theorem implies that for all (ω, x, y) in a set B_1 of $\mathbf{P} \times \widetilde{\mu}$ measure one

$$\frac{1}{n}\log\|\mathrm{S}^{(n)}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}\| \xrightarrow[n\to\infty]{} \int_{\Omega\times\mathbf{T}^2} \log\|\mathrm{S}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}\|\,\mathrm{d}\mathbf{P}(\omega)\,\mathrm{d}\mu(x,y) \tag{4.10}$$

Now using Oseledets' theorem (see [Led84, AB08]) there exists $\mathrm{B} \subset \Omega \times \mathbf{T}$ with $\mathbf{P} \times \widetilde{\mu}(\mathrm{B}) = 1$ and for all $(\omega, x) \in \mathrm{B}$ there is at most one so named *direction of contraction* $u^-_{\omega,x} \in \mathbf{T}$ such that

$$\forall(\omega, x) \in \mathrm{B} \qquad \frac{1}{n}\log\|\mathrm{S}^{(n)}_{\omega,x}\mathrm{e}^{\,\mathrm{i}\pi y}\| \xrightarrow[n\to\infty]{} \begin{cases} \mathrm{L(E)} & \forall y \in \mathbf{T} \setminus \left\{u^-_{\omega,x}\right\} \\ -\mathrm{L(E)} & \text{if } y = u^-_{\omega,x} \end{cases}$$

Consequently the set $\widetilde{\mathrm{B}} = \mathrm{B}_1 \cap (\mathrm{B} \times \mathbf{T})$ is of $\mathbf{P} \times \mu$ measure one and for (ω, x) in this set, the previous limit exists and is equal to the integral in (4.10). It is also equal either to L(E) or −L(E). Hence we can conclude that the sign is constant on $\widetilde{\mathrm{B}}$ and that, with the non negativity of L(E)

$$\mathrm{L(E)} = \left| \int_{\Omega \times \mathbf{T}^2} \log \|\mathrm{S}_{\omega,x} \mathrm{e}^{\,\mathrm{i}\pi y}\| \, \mathrm{d}\mathbf{P}(\omega) \, \mathrm{d}\mu(x, y) \right|$$

We finish by proving a link between the previous equality and the Lyapunov exponent of the random diffeomorphism F_ω.

Since $\mathrm{S}_{\omega,x} \in \mathrm{SL}_2(\mathbf{R})$ for all ω, x the area conservation formula gives

$$\|\mathrm{S}_{\omega,x} \mathrm{e}^{\,\mathrm{i}\pi y}\| \|\mathrm{S}_{\omega,x} \mathrm{e}^{\,\mathrm{i}\pi y'}\| \, |\sin \pi (f(y) - f(y'))| = |\sin \pi (y - y')|$$

and letting $y' \to y$ gives

$$1 = \|\mathrm{S}_{\omega,x} \mathrm{e}^{\,\mathrm{i}\pi y}\|^2 \, |f_{\omega,x}{}'(y)|$$

Let us remark that

$$(\mathrm{DF}_\omega)_{(x,y)} = \begin{pmatrix} 1 & 0 \\ \partial_x f_{\omega,x}(y) & f_{\omega,x}{}'(y) \end{pmatrix}$$

such that $f_{\omega,x}{}'(y) = \det(\mathrm{DF}_\omega)_{(x,y)}$ which yields to the announced formula.

■

4.3 Estimation of a stationary measure

4.3.1 The initial change of variables

The formula (4.9) obtained in the previous section brings us to get estimates for any stationary measure μ. Intuitively, if we suppose that the potential of the Schrödinger operator we initially considered is small, the random diffeomorphism F_ω is close to be the 2-dimensional rotation of $\mathbf{T}^2$:

$$\mathrm{R}_{\alpha,\beta} : (x, y) \mapsto (x + \alpha, y + \beta) \tag{4.11}$$

where

$$\beta = \beta(\mathrm{E}) = \frac{1}{\pi} \operatorname{Arc\,cos} (\mathrm{E}/2) \tag{4.12}$$

This is due to algebraic conjugacy, used by Figotin-Pastur and Chulaevsky and Spencer in [PF92, CS95], by the matrix

$$\mathrm{P}_\beta = \begin{pmatrix} 1 & -\cos \pi\beta \\ 0 & \sin \pi\beta \end{pmatrix} \tag{4.13}$$

This conjugacy gives a new cocycle which is close to the $\pi\beta$-rotation in $\mathbf{R}^2$ (remind that $v(\omega, x)$ is supposed to be small). The elementary matrix for the new cocycle is then

$$\mathcal{S}_{\omega,x} = \mathrm{P}_\beta \, \mathrm{S}_{\mathrm{E},\omega,x} \, \mathrm{P}_\beta^{-1} = \mathrm{R}_{\pi\beta} + \mathrm{N}_{\omega,x} \tag{4.14}$$

where

$$\mathrm{R}_{\pi\beta} = \begin{pmatrix} \cos \pi\beta & -\sin \pi\beta \\ \sin \pi\beta & \cos \pi\beta \end{pmatrix} \qquad \mathrm{N}_\beta = \begin{pmatrix} 1 & \cot \pi\beta \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{N}_{\omega,x} = -v(\omega, x)\mathrm{N}_\beta \tag{4.15}$$

As a consequence, both cocycles have the same Lyapunov exponent and the cocycle obtained after the conjugacy is a small perturbation of the rotation $\mathrm{R}_{\pi\beta}$ in $\mathbf{R}^2$, which explains why F_ω is close to the rotation $\mathrm{R}_{\alpha,\beta}$ of $\mathbf{T}^2$. This is an argument in favor of the closeness of μ to the Lebesgue measure on $\mathbf{T}^2$ under some arithmetic condition on (α, β). After some notations, we shall give precise estimates.

4.3.2 Functional & calculus notations

We denote $\mathscr{C}^k(\mathbf{T}^2)$ the space of real $\mathscr{C}^k$ functions on $\mathbf{T}^2$. Let φ be in $\mathscr{C}^k(\mathbf{T}^2)$ and j, j' be two natural integers with $j + j' \leqslant k$. We define

$$\|\varphi\|_{j,j'} = \sup_{\mathbf{T}^2} \left| \frac{\partial^{j+j'} \varphi}{\partial x^j \partial y^{j'}} \right| \tag{4.16}$$

And so is $\mathscr{C}^k(\mathbf{T}^2)$ equipped with the norm

$$\|\varphi\|_k = \max_{0 \leqslant j+j' \leqslant k} \|\varphi\|_{j,j'}$$

We also denote for a random $\mathscr{C}^k$ map $\omega \longmapsto \varphi_\omega$ (that is φ_ω is $\mathscr{C}^k$ for almost every ω):

$$|||\varphi|||_k = \mathbf{E}\left(\|\varphi\|_k^2\right)^{1/2} \tag{4.17}$$

so that with Jensen's and Cauchy-Schwarz inequalities

$$\mathbf{E}\,\|\varphi\|_k \asymp |||\varphi|||_k$$

Remind the definition of F_ω in (4.8). Let $\mathscr{T}$ be the operator on $\mathrm{L}^2(\mathbf{T}^2)$

$$\mathscr{T} : \varphi \longmapsto \mathbf{E}(\varphi \circ \mathrm{F}_\omega) \tag{4.18}$$

and $\mathscr{T}_0$ the composition operator on $\mathrm{L}^2(\mathbf{T}^2)$ for the translation $\mathrm{R}_{\alpha,\beta}$

$$\mathscr{T}_0 : \varphi \longmapsto \varphi \circ \mathrm{R}_{\alpha,\beta} \tag{4.19}$$

For $\varphi \in \mathrm{L}^2(\mathbf{T}^2, \mathbf{C})$ we denote $\widehat{\varphi}_{m,n}$ the usual Fourier coefficient by

$$\widehat{\varphi}_{m,n} = \int_{\mathbf{T}^2} \varphi(x,y)\,\mathrm{e}^{-2\mathrm{i}\pi(mx+ny)}\,\mathrm{d}x\,\mathrm{d}y$$

4.3.3 Cohomological equation

We want to prove that $\int_{\mathbf{T}^2}\psi\,\mathrm{d}\mu \approx \int_{\mathbf{T}^2}\psi\,\mathrm{d}x\,\mathrm{d}y$ or, equivalently, that $\int_{\mathbf{T}^2}(\psi - \widehat{\psi}_{0,0})\,\mathrm{d}\mu \approx 0$ where $\widehat{\psi}_{m,n}$ stands for the Fourier coefficient of ψ of index (m,n). As $\mathrm{F}_\omega \approx \mathrm{R}_{\alpha,\beta}$ and $\int_{\mathbf{T}^2}\mathbf{E}(\varphi\circ\mathrm{F})\,\mathrm{d}\mu = \int_{\mathbf{T}^2}\varphi\,\mathrm{d}\mu$ the result is guaranteed provided we can write

$$\psi - \int_{\mathbf{T}^2}\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \varphi - \varphi\circ\mathrm{R}_{\alpha,\beta} \tag{4.20}$$

Here we need an arithmetic conditions on the couple (α,β) in order to get quantitative estimates.

Definition 4.4 (Diophantine vector)

We say that $(\alpha,\beta)\in\mathbf{T}^2$ belongs to the diophantine classe $\mathrm{DC}(\kappa,\tau)$ if

$$\forall (m,n)\in\mathbf{Z}^2\smallsetminus\{(0,0)\} \qquad \|m\alpha+n\beta\|_{\mathbf{Z}} \geqslant \frac{\kappa}{(|m|+|n|)^\tau} \tag{4.21}$$

For fixed α and $\tau>1$, the set of β's satisfying (4.21) for some $\kappa>0$ has full Lebesgue measure. If (4.21) is only satisfied for $n=\pm1$ then we are in the case $\beta\in\mathrm{DC}_\alpha(\kappa,\tau)$ of (1.7).

A particular case when (4.21) is not satisfied for which we say that β is resonant *for α (or α-*resonant*) is*

$$\exists k\in\mathbf{Z} \qquad \beta = k\alpha \tag{4.22}$$

Now we can prove the following result for the cohomological equation:

Lemma 4.2 (Cohomological operator, diophantine case)

Suppose (α,β) satisfies (4.21) for $\kappa=\kappa_{\alpha,\beta}$. Then the linear operator $\mathrm{U}=\mathrm{U}_{\alpha,\beta}$ defined as follows:

$$\mathrm{U}_{\alpha,\beta}\colon \begin{cases} \mathscr{C}^{j+2\tau+4}(\mathbf{T}^2) \longrightarrow \mathscr{C}^j(\mathbf{T}^2) \\ \qquad \psi \qquad\longmapsto \varphi \text{ the solution of } \varphi - \varphi\circ\mathrm{R}_{\alpha,\beta} = \psi - \int_{\mathbf{T}^2}\psi\,\mathrm{d}x\,\mathrm{d}y \end{cases}$$

which is explicitly given by (see (4.26)):

$$\mathrm{U}_{\alpha,\beta}\psi(x,y) = \sum_{(m,n)\neq(0,0)} \frac{\widehat{\psi}_{m,n}}{1-\mathrm{e}^{2\mathrm{i}\pi(m\alpha+n\beta)}}\,\mathrm{e}^{2\mathrm{i}\pi(mx+ny)} \tag{4.23}$$

is continuous and has norm

$$|||\mathrm{U}_{\alpha,\beta}||| \leqslant (72\pi^{2\tau}\kappa_{\alpha,\beta})^{-1} =: c_{\alpha,\beta} \lesssim 1/\kappa_{\alpha,\beta} \tag{4.24}$$

Proof.

We are then interested in solving the previous cohomological equation (4.20). Using Fourier series this is equivalent to

$$\forall (m,n) \in \mathbf{Z}^2 \smallsetminus \{(0,0)\} \qquad (1-\mathrm{e}^{2\mathrm{i}\pi(m\alpha+n\beta)})\widehat{\varphi}_{m,n} = \widehat{\psi}_{m,n} \tag{4.25}$$

The minimal condition to solve equation (4.25) is that (α,β) is rationally independent which means that the quantity $m\alpha+n\beta$ only vanishes for $(m,n)=(0,0)$. But we also need some estimates on the regularity of the solution:

$$\varphi = \sum_{(m,n)\neq(0,0)} \frac{\widehat{\psi}_{m,n}}{1-\mathrm{e}^{2\mathrm{i}\pi(m\alpha+n\beta)}} e_{m,n} \qquad \text{where} \qquad e_{m,n} : \mathbf{T}^2 \ni (x,y) \longmapsto \mathrm{e}^{2\mathrm{i}\pi(mx+ny)} \tag{4.26}$$

Clearly, for all integers k, k', ℓ, ℓ',

$$\frac{\partial^{k+\ell}\varphi}{\partial x^k \partial y^\ell} = \sum_{(m,n)\neq(0,0)} \frac{\widehat{\psi}_{m,n}}{1-\mathrm{e}^{2\mathrm{i}\pi(m\alpha+n\beta)}} (2\mathrm{i}\pi)^{k+\ell} m^k n^\ell e_{m,n}$$

and also $$\widehat{\psi}_{m,n} = \frac{1}{(2\mathrm{i}\pi m)^{k'}} \widehat{\partial_x{}^{k'}\psi}_{m,n} = \frac{1}{(2\mathrm{i}\pi n)^{\ell'}} \widehat{\partial_y{}^{\ell'}\psi}_{m,n}$$

so that $$\left|\widehat{\psi}_{m,n}\right| \leqslant \frac{1}{(2\pi)^{k+\ell}\,|m|^k\,|n|^\ell} \left\| \frac{\partial^{k+\ell}\psi}{\partial x^k \partial y^\ell} \right\|_0$$

We set $d_{m,n} = \left|\mathrm{e}^{2\mathrm{i}\pi(m\alpha+n\beta)} - 1\right| = \left|-2\mathrm{i}\sin\pi(m\alpha+n\beta)\mathrm{e}^{\mathrm{i}\pi(m\alpha+n\beta)}\right|$. Then

$$\begin{aligned} d_{m,n} &= 2\,|\sin\pi(m\alpha+n\beta)| \\ &= 2\,|\sin\pi\|m\alpha+n\beta\|_{\mathbf{Z}}| \\ &\geqslant 2\times\frac{2}{\pi}\pi\|m\alpha+n\beta\|_{\mathbf{Z}} \\ d_{m,n} &\geqslant 4\|m\alpha+n\beta\|_{\mathbf{Z}} \end{aligned}$$

Under the assumption (4.21) we get for all integers k, k', ℓ, ℓ'

$$\left\| \frac{\partial^{k+\ell}\varphi}{\partial x^k \partial y^\ell} \right\|_0 \leqslant \frac{1}{\kappa} \sum_{(m,n)\neq(0,0)} \frac{(2\pi)^{k+\ell}}{(2\pi)^{k'+\ell'}} \frac{|m|^k\,|n|^\ell\,(|m|+|n|)^\tau}{|m|^{k'}\,|n|^{\ell'}} \|\psi\|_{k',\ell'}$$

But convexity arguments and basic calculus show that for all $\tau > 0$ there is $c_\tau > 0$ such that for all real x, y one has $(|x|+|y|)^\tau \leqslant c_\tau(|x|^\tau + |y|^\tau)$. Indeed $c_\tau = 1$ if $\tau < 1$ and $c_\tau = 2^{\tau-1}$ if $\tau \geqslant 1$. Thus we have

$$\left\| \frac{\partial^{k+\ell}\varphi}{\partial x^k \partial y^\ell} \right\|_0 \leqslant \frac{1}{\kappa} \frac{1}{(2\pi)^{k'+\ell'-k-\ell}} \sum_{(m,n)\neq(0,0)} \left(\frac{1}{|m|^{k'-k-\tau}\,|n|^{\ell'-\ell}} + \frac{1}{|n|^{\ell'-\ell-\tau}\,|m|^{k'-k}} \right) \|\psi\|_{k',\ell'}$$

the sum which is involved converges if and only if $k' > k+\tau+1$ and $\ell' > \ell+\tau+1$. This means we can solve the cohomological equation (4.25), and so (4.20), with a loss of regularity strictly greater than $\tau+1$ in each direction. For example we can take $k' = k+\tau+2$ and $\ell' = \ell+\tau+2$ and get

$$\|\varphi\|_{k,\ell} \leqslant \frac{1}{72\pi^{2\tau}} \frac{1}{\kappa} \|\psi\|_{k+\tau+2,\ell+\tau+2}$$

this gives

$$\|\varphi\|_j \leqslant c_{\alpha,\beta} \|\psi\|_{j+2\tau+4}$$

So there is a global loss of $2\tau+4$ derivatives. The lemma is proved. ■

Remark 4.3

We shall say a few words on the dual operator of U *in* $\mathrm{L}^2(\mathbf{T}^2,\mathbf{R})$ *that we will need in the sequel. The operator* U *has a dense domain as it is well defined and continuous on* $\mathscr{C}^\infty(\mathbf{T}^2)$. *Solving for all* $\varphi,\psi \in \mathscr{C}^\infty(\mathbf{T}^2,\mathbf{R})$

$$\int_{\mathbf{T}^2} \overline{\mathrm{U}\psi}\cdot\varphi \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbf{T}^2} \overline{\psi}\cdot \mathrm{U}^*\varphi\,\mathrm{d}x\,\mathrm{d}y$$

gives the expression of U^**:*

$$\widehat{\mathrm{U}^*\varphi}_{0,0} = 0 \qquad \textit{and} \qquad \forall (m,n)\neq(0,0) \quad \widehat{\mathrm{U}^*\varphi}_{m,n} = \frac{\widehat{\varphi}_{m,n}}{1-\mathrm{e}^{-2\mathrm{i}\pi(m\alpha+n\beta)}} \tag{4.27}$$

this proves that $\mathrm{U}^* : \mathscr{C}^{j+2\tau+4}(\mathbf{T}^2) \to \mathscr{C}^j(\mathbf{T}^2)$ *has same domain and same norm as* U.

Remark 4.4

Let us point out that, for our purpose of computing a formula for the Lyapunov exponent, we will actually need a weaker diophantine condition on (α,β). *Indeed, the functions that arise in the computation of the LE will be trigonometric polynomials in the variable* y *of degree at most 2 of the form* $c_+(x)\mathrm{e}^{2\mathrm{i}\pi y} + c_-(x)\mathrm{e}^{-2\mathrm{i}\pi y}$. *Hence we will only require that* $\beta \in \mathrm{DC}_\alpha(\kappa,\tau)$ *which means*

$$\forall m \in \mathbf{Z}\setminus\{0\} \qquad |m\alpha+\beta| \geqslant \frac{\kappa}{|m|^\tau} \tag{4.28}$$

Also note that under the condition $\beta \in \mathrm{DC}_\alpha(\kappa,\tau)$ *the operator* U *is bounded on the space of trigonometric polynomials of degre 2 in* y *and still verifies* (4.24).

Now we give the key proposition to get a Figotin-Pastur formula.

4.3.4 Estimates for a stationary measure

Remark 4.5

From now on we denote $\mathrm{O(Q)}$ *a quantity which is bounded by* $c\mathrm{Q}$ *where* c *is a constant depending at most on* (α, β)*, namely some power of the constant* $c_{\alpha,\beta} \asymp \kappa^{-1}$ *mentioned in the previous paragraph (see (4.24)). We shall remind this fact in the final formula to elaborate on the error term in the perturbative development we obtain.*

To state our estimate we need to define

Definition 4.5 (2-dim. adjoint)

For a function f *on* $\mathbf{T}^2 \to \mathbf{R}^2$ *written* $f(x, y) = (f_1(x, y), f_2(x, y))$ *we define the action of the adjoint* U^*

$$\mathrm{U}^* f := (\mathrm{U}^* f_1, \mathrm{U}^* f_2) \tag{4.29}$$

where U^* *is defined in (4.27) for maps* $\mathbf{T}^2 \to \mathbf{R}$.

Proposition 4.2

We have the following development for any stationary measure μ*:*

(i) *For all* $\psi \in \mathscr{C}^{2\tau+5}(\mathbf{T}^2)$

$$\int_{\mathbf{T}^2} \psi \, \mathrm{d}\mu = \int_{\mathbf{T}^2} \psi \, \mathrm{d}x \, \mathrm{d}y + \mathrm{O}(\epsilon \|\psi\|_{2\tau+5})$$

(ii) *Now if* $\psi \in \mathscr{C}^{4\tau+10}(\mathbf{T}^2)$ *we have more precisely*

$$\int_{\mathbf{T}^2} \psi \, \mathrm{d}\mu = \int_{\mathbf{T}^2} \psi \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbf{T}^2} (\mathrm{U}^* \widetilde{\Xi}) \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}y + \mathrm{O}\left(\epsilon^2 \|\psi\|_{4\tau+10}\right)$$

with the following notations: (remember (4.17))

$$\begin{cases} \Xi = \mathrm{F}_\omega - \mathrm{R}_{\alpha,\beta} \\ \widetilde{\Xi} = \mathbf{E}(\Xi \circ \mathrm{R}_{-\alpha,-\beta}) \\ \epsilon = |||\Xi|||_{2\tau+5} \end{cases} \tag{4.30}$$

Proof.

(i) A Taylor expansion to order 0 gives

$$\varphi \circ \mathrm{F}_\omega = \varphi \circ \mathrm{R}_{\alpha,\beta} + \mathrm{O}\left(\|\Xi\|_0 \|\varphi\|_1\right)$$

so that with the definitions in (4.18), (4.19), taking expectation gives

$$\mathscr{T}\varphi = \mathscr{T}_0\varphi + \mathrm{O}\left(|||\Xi|||_0 \|\varphi\|_1\right) = \mathscr{T}_0\varphi + \mathrm{O}\left(\epsilon\|\varphi\|_1\right)$$

The stationarity of the measure μ leads to

$$\int_{\mathbf{T}^2} \varphi \,\mathrm{d}\mu = \int_{\mathbf{T}^2} \mathscr{T}\varphi \,\mathrm{d}\mu = \int_{\mathbf{T}^2} \mathscr{T}_0\varphi \,\mathrm{d}\mu + \mathrm{O}\left(\epsilon\|\varphi\|_1\right)$$

We apply this equation to $\varphi = \mathrm{U}\psi$ and get with (4.24)

$$\int_{\mathbf{T}^2} \psi \,\mathrm{d}\mu - \int_{\mathbf{T}^2} \psi \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbf{T}^2} (\psi - \textstyle\int_{\mathbf{T}^2} \psi \,\mathrm{d}x\,\mathrm{d}y)\,\mathrm{d}\mu = \mathrm{O}\left(\epsilon\|\psi\|_{2\tau+5}\right) \tag{4.31}$$

(ii) A first order Taylor expansion yields

$$\varphi \circ \mathrm{F}_\omega = \varphi \circ \mathrm{R}_{\alpha,\beta} + \Xi \cdot \nabla\varphi \circ \mathrm{R}_{\alpha,\beta} + \mathrm{O}\left(\|\varphi\|_2 \|\Xi\|_0^2\right)$$

then
$$\begin{aligned}\mathscr{T}\varphi &= \mathscr{T}_0\varphi + \mathbf{E}\left(\Xi \cdot \nabla\varphi \circ \mathrm{R}_{\alpha,\beta}\right) + \mathrm{O}\left(\epsilon^2\|\varphi\|_2\right)\\ &= \mathscr{T}_0\varphi + (\mathbf{E}\,\Xi) \cdot \nabla\varphi \circ \mathrm{R}_{\alpha,\beta} + \mathrm{O}\left(\epsilon^2\|\varphi\|_2\right)\end{aligned}$$

Integrating the previous relation w.r.t μ and using the first estimation (4.31) leads to

$$\begin{aligned}\int_{\mathbf{T}^2} \varphi - \mathscr{T}_0\varphi \,\mathrm{d}\mu &= \int_{\mathbf{T}^2} \mathbf{E}(\Xi) \cdot \nabla\varphi \circ \mathrm{R}_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}\left(\epsilon\|\,\mathbf{E}(\Xi) \cdot \nabla\varphi \circ \mathrm{R}_{\alpha,\beta})\|_{2\tau+5}\right) + \mathrm{O}\left(\epsilon^2\right)\|\varphi\|_2\\ &= \int_{\mathbf{T}^2} \mathbf{E}(\Xi \circ \mathrm{R}_{-\alpha-\beta}) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}\left(\epsilon^2\|\varphi\|_{2\tau+6}\right) + \mathrm{O}\left(\epsilon^2\|\varphi\|_2\right)\\ \int_{\mathbf{T}^2} \varphi - \mathscr{T}_0\varphi \,\mathrm{d}\mu &= \int_{\mathbf{T}^2} \widetilde{\Xi} \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}\left(\epsilon^2\|\varphi\|_{2\tau+6}\right)\end{aligned}$$

once again for $\varphi = \mathrm{U}\psi$ we obtain

$$\int_{\mathbf{T}^2} \varphi - \mathscr{T}_0\varphi \,\mathrm{d}\mu = \int_{\mathbf{T}^2} \psi \,\mathrm{d}\mu - \int_{\mathbf{T}^2} \psi \,\mathrm{d}x\,\mathrm{d}y$$

and with (4.24):
$$\|\varphi\|_{2\tau+6} = \mathrm{O}\left(\|\psi\|_{4\tau+10}\right)$$

An easy computation gives for $\psi : \mathbf{T}^2 \to \mathbf{R}$

$$\nabla(\mathrm{U}\psi) = (\mathrm{U}(\nabla\psi)_1, \mathrm{U}(\nabla\psi)_2) \tag{4.32}$$

We can then compute with (4.32) and (4.29)

$$\begin{aligned}
\int_{\mathbf{T}^2} \widetilde{\Xi} \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}y &= \int_{\mathbf{T}^2} \widetilde{\Xi} \cdot (\mathrm{U}(\nabla\psi)_1, \mathrm{U}(\nabla\psi)_2) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbf{T}^2} \widetilde{\Xi}_1.\mathrm{U}(\nabla\psi)_1 + \widetilde{\Xi}_2.\mathrm{U}(\nabla\psi)_2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbf{T}^2} \mathrm{U}^*(\widetilde{\Xi}_1).(\nabla\psi)_1 + \mathrm{U}^*(\widetilde{\Xi}_2).(\nabla\psi)_2 \, \mathrm{d}x \, \mathrm{d}y \\
\int_{\mathbf{T}^2} \widetilde{\Xi} \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}y &= \int_{\mathbf{T}^2} \mathrm{U}^*(\widetilde{\Xi}) \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}y
\end{aligned}$$

which ensures the result we announced.

∎

4.4 Proof of the formula for a.e. rotation and energy

The key proposition is the following:

Proposition 4.3

Let us denote ξ_ω (resp. $\widetilde{\xi}$) the second coordinate of Ξ_ω (resp. $\widetilde{\Xi}$) (see (4.30)):

$$\xi_\omega(x,y) = f_{\omega,x}(y) - (y+\beta) \tag{4.33}$$

$$\widetilde{\xi}(x,y) = \mathbf{E}\,\xi(x-\alpha, y-\beta) = \mathbf{E}\left(f_{\omega,x-\alpha}(y-\beta)\right) - (y-\beta) \tag{4.34}$$

We also set (remind definition (4.17) of $|||\cdot|||$):

$$\epsilon = |||\Xi|||_{4\tau+11} = |||\xi_\omega|||_{4\tau+11} \tag{4.35}$$

Then we have

$$\begin{aligned}
\gamma(\mathrm{F},\mu) &= -\tfrac{1}{2}\,\mathbf{E}\int_{\mathbf{T}^2} \mathrm{tr}^{\,2}\left(\mathrm{D}\Xi + \mathrm{DU}^*(\widetilde{\Xi}) - \mathrm{DU}^*(\widetilde{\Xi}) \circ \mathrm{R}_{\alpha,\beta}\right) \mathrm{d}x \, \mathrm{d}y + \mathrm{O}\left(\epsilon^3\right) \\
&= -\tfrac{1}{2}\,\mathbf{E}\int_{\mathbf{T}^2} \left(\partial_y \xi_\omega(x,y) + \partial_y \mathrm{U}^*\widetilde{\xi}(x,y) - \partial_y \mathrm{U}^*\widetilde{\xi}(x+\alpha, y+\beta)\right)^2 \mathrm{d}x \, \mathrm{d}y \\
&\qquad + \mathrm{O}\left(\epsilon^3\right)
\end{aligned}$$

Proof.

Let us define

$$g\colon \begin{cases} \mathbf{T}^2 \longrightarrow \mathbf{T}^2 \\ (x,y) \longmapsto \mathrm{U}^*\widetilde{\Xi} = \left(0, (\mathrm{U}^*\widetilde{\xi})(x,y)\right) \end{cases} \qquad \text{and} \qquad \mathrm{G} = \mathrm{Id}_{\mathbf{T}^2} - g \tag{4.36}$$

Lemma 4.3

The following properties hold:

(i) *the map* G *is a* $\mathscr{C}^1$*-diffeomorphism of* $\mathbf{T}^2$ *for* $\|g\|_1 \leqslant 1/4$*;*

(ii) $\|g\|_k \lesssim \mathbf{E}\,\|\xi\|_{k+2\tau+4} \lesssim |||\xi|||_{k+2\tau+4}$

(iii) $\|\mathrm{G}^{-1} - \mathrm{Id}\|_0 \leqslant \|g\|_0$

(iv) $(\mathrm{DG}^{-1})_{\mathrm{X}} = \mathrm{I}_2 + \mathrm{D}g_{\mathrm{G}^{-1}(\mathrm{X})} + \mathrm{O}\left(\|g\|_1^2\right)$

Proof.

(i) The mean value theorem ensures that

$$\|\mathrm{G}(\mathrm{X}) - \mathrm{G}(\mathrm{X}')\| \geqslant \frac{3}{4}\|\mathrm{X} - \mathrm{X}'\|$$

Hence G is injective. Let us denote $\widetilde{\mathrm{G}}$ any lift of G on $\mathbf{R}^2$. Then $\mathrm{DG} = \mathrm{D}\widetilde{\mathrm{G}} = \mathrm{I}_2 - \mathrm{D}g$ so that G is invertible everywhere for $\|g\|_1 < 1$ and the previous estimate yields

$$\|\widetilde{\mathrm{G}}(x,y)\| \xrightarrow[\|x,y\|\to+\infty]{} +\infty$$

Applying Hadamard-Lévy's theorem gives that $\widetilde{\mathrm{G}}$ is a $\mathscr{C}^1$ diffeomorphism of $\mathbf{R}^2$ and we can conclude for G.

(ii) This comes from $\|\,\mathbf{E}(\xi)\|_k \leqslant \mathbf{E}(\|\xi\|_k)$ and $\|\mathrm{U}\| = \|\mathrm{U}^*\|$ (see remark 4.3).

(iii) This is an easy consequence of $\mathrm{X} = \mathrm{G}(\mathrm{G}^{-1}(\mathrm{X})) = \mathrm{G}^{-1}(\mathrm{X}) - g(\mathrm{G}^{-1}(\mathrm{X}))$.

(iv) With $\mathrm{DG} = \mathrm{I}_2 - \mathrm{D}g$, $\mathrm{D}(\mathrm{G}^{-1})_{\mathrm{X}} = (\mathrm{DG}_{\mathrm{G}^{-1}(\mathrm{X})})^{-1}$, and $(\mathrm{I}_2 - \mathrm{A})^{-1} = \sum_{n=0}^{+\infty} \mathrm{A}^n$ for $\|\mathrm{A}\| < 1$ we can conclude.

■

Now we use G to conjugate a stationary measure and get nicer estimates that more explicitly exhibits the negativity of $\gamma(\mathrm{F},\mu)$. Let us define $\mu_1 = \mathrm{G}_*\mu$. Then one has

$$\begin{aligned}
\int_{\mathbf{T}^2} \varphi\,\mathrm{d}\mu_1 &= \int_{\mathbf{T}^2} \varphi \circ \mathrm{G}\,\mathrm{d}\mu \\
&= \int_{\mathbf{T}^2} \varphi(x,y) - \nabla\varphi(x,y)\cdot g(x,y) + \mathrm{O}\left(\|g\|_0^2\,\|\varphi\|_2\right)\,\mathrm{d}\mu(x,y) \\
&= \int_{\mathbf{T}^2} \varphi - \nabla\varphi\cdot g\,\mathrm{d}\mu + \mathrm{O}\left(\epsilon^2\,\|\varphi\|_2\right) \\
&= \int_{\mathbf{T}^2} \varphi\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbf{T}^2} g\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}\left(\epsilon^2\|\varphi\|_{4\tau+10}\right) \\
&\qquad - \int_{\mathbf{T}^2} g\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}\left(\epsilon\|g\cdot\nabla\varphi\|_{2\tau+5}\right)
\end{aligned}$$

Finally

$$\int_{\mathbf{T}^2} \varphi\,\mathrm{d}\mu_1 = \int_{\mathbf{T}^2} \varphi\,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}\left(\epsilon^2\|\varphi\|_{4\tau+10}\right) \tag{4.37}$$

Let us define

$$\mathrm{F}_1^\omega = \mathrm{G}\circ\mathrm{F}_\omega\circ\mathrm{G}^{-1} \tag{4.38}$$

The stationarity of μ leads to

$$\gamma(\mathrm{F}, \mu) = \gamma(\mathrm{F}_1, \mu_1) \tag{4.39}$$

Indeed

$$\begin{aligned}
\gamma(\mathrm{F}_1, \mu_1) &= \mathbf{E} \int_{\mathbf{T}^2} \ln \det(\mathrm{DG}_{\mathrm{F}\circ\mathrm{G}^{-1}(\mathrm{G}(\mathrm{X}))} .\mathrm{DF}_{\mathrm{G}^{-1}(\mathrm{G}(\mathrm{X}))} .(\mathrm{DG}_{\mathrm{G}^{-1}(\mathrm{G}(\mathrm{X}))})^{-1})\, \mathrm{d}\mu(\mathrm{X}) \\
&= \mathbf{E} \int_{\mathbf{T}^2} \ln \det(\mathrm{DG} \circ \mathrm{F})\, \mathrm{d}\mu + \mathbf{E} \int_{\mathbf{T}^2} \ln \det \mathrm{DF}\, \mathrm{d}\mu - \mathbf{E} \int_{\mathbf{T}^2} \ln \det \mathrm{DG}\, \mathrm{d}\mu \\
&= \mathbf{E} \int_{\mathbf{T}^2} \ln \det \mathrm{DF}\, \mathrm{d}\mu \\
\gamma(\mathrm{F}_1, \mu_1) &= \gamma(\mathrm{F}, \mu)
\end{aligned}$$

Let us recall that for square matrices $\mathrm{A}, \mathrm{H}, \mathrm{K}$:

$$\mathrm{D}^2 \det{}_{\mathrm{A}}(\mathrm{H}, \mathrm{K}) = \det \mathrm{A} \left[\mathrm{tr}\, (\mathrm{A}^{-1}\mathrm{H})\, \mathrm{tr}\, (\mathrm{A}^{-1}\mathrm{K}) - \mathrm{tr}\, \left(\mathrm{A}^{-1}\mathrm{H}\mathrm{A}^{-1}\mathrm{K}\right)\right]$$

which ensures $\quad \det(\mathrm{I} + \mathrm{H}) = 1 + \mathrm{tr}\ \mathrm{H} + \frac{1}{2} \left(\mathrm{tr}^{\,2}(\mathrm{H}) - \mathrm{tr}\, (\mathrm{H}^2)\right) + \underset{\mathrm{H}\to 0}{\mathrm{O}}(\mathrm{H}^3)$

Let us set with (4.38)

$$\Xi_1 = \Xi_1^\omega := \mathrm{F}_1^\omega - \mathrm{R}_{\alpha,\beta} \tag{4.40}$$

A second order Taylor expansion and estimate (4.37) then give

$$\begin{aligned}
\gamma(\mathrm{F}_1, \mu_1) &= \mathbf{E} \int_{\mathbf{T}^2} \ln(\mathrm{I}_2 + \mathrm{D}\Xi_1)\, \mathrm{d}\mu_1 \\
&= \mathbf{E} \int_{\mathbf{T}^2} \Big(\mathrm{tr}\, (\mathrm{D}\Xi_1) - \tfrac{1}{2}\, \mathrm{tr}\, (\mathrm{D}\Xi_1{}^2) + \mathrm{O}\left(\|\mathrm{D}\Xi_1\|_0^3\right) \Big)\, \mathrm{d}\mu_1 \\
\gamma(\mathrm{F}_1, \mu_1) &= \mathbf{E} \int_{\mathbf{T}^2} \Big(\mathrm{tr}\, (\mathrm{D}\Xi_1) - \tfrac{1}{2}\, \mathrm{tr}\, (\mathrm{D}\Xi_1{}^2)\Big)\, \mathrm{d}x\, \mathrm{d}y + \mathrm{O}\left(\epsilon^2\, \mathbf{E}\, \|\, \mathrm{tr}\, (\mathrm{D}\Xi_1)\|_{4\tau+10}\right) + \mathrm{O}\left(|\!|\!|\mathrm{D}\Xi_1|\!|\!|_0^3\right)
\end{aligned}$$

And so we deduce

$$\gamma(\mathrm{F}_1, \mu_1) = \mathbf{E} \int_{\mathbf{T}^2} \Big(\mathrm{tr}\, (\mathrm{D}\Xi_1) - \tfrac{1}{2}\, \mathrm{tr}\, (\mathrm{D}\Xi_1{}^2)\Big)\, \mathrm{d}x\, \mathrm{d}y + \mathrm{O}\left(\epsilon^3\right) \tag{4.41}$$

Writing $\mathrm{tr}\ \mathrm{D}\Xi_1 = \partial_x(\Xi_1)_1 + \partial_y(\Xi_1)_2$ one gets for all ω

$$\int_{\mathbf{T}^2} \mathrm{tr}\, (\mathrm{D}\Xi_1^\omega)\, \mathrm{d}x\, \mathrm{d}y = 0$$

We shall now give an estimate on $\mathrm{D}\Xi_1$ with Ξ and g. Equations (4.30) and (4.38) ensure that

$$\begin{aligned}
\mathrm{DF}_1 &= \mathrm{DG}_{\mathrm{F}\circ\mathrm{G}^{-1}} \mathrm{DF}_{\mathrm{G}^{-1}} \mathrm{D}(\mathrm{G}^{-1}) \qquad (4.42)\\
&= (\mathrm{I}_2 - \mathrm{D}g)_{\mathrm{F}\circ\mathrm{G}^{-1}} (\mathrm{I}_2 + \mathrm{D}\Xi)_{\mathrm{G}^{-1}} \mathrm{D}(\mathrm{G}^{-1})
\end{aligned}$$

Then (4.36), (4.40) and lemma 4.3 give

$$\begin{cases} \mathrm{D}\Xi_{\mathrm{G}^{-1}(\mathrm{X})} = \mathrm{D}\Xi_{\mathrm{X}} + \mathrm{O}\left(\|\Xi\|_2\|g\|_0\right) = \mathrm{D}\Xi_{\mathrm{X}} + \mathrm{O}\left(\|\xi\|^2_{2\tau+4}\right) \\ \mathrm{D}g_{\mathrm{G}^{-1}(\mathrm{X})} = \mathrm{D}g_{\mathrm{X}} + \mathrm{O}\left(\|g\|_0\|g\|_1\right) = \mathrm{D}g_{\mathrm{X}} + \mathrm{O}\left(\|\xi\|^2_{2\tau+5}\right) \\ \mathrm{F}(\mathrm{G}^{-1}(\mathrm{X})) = \mathrm{R}_{\alpha,\beta}\circ\mathrm{G}^{-1}(\mathrm{X}) + \Xi(\mathrm{G}^{-1}(\mathrm{X})) = \mathrm{R}_{\alpha,\beta}(\mathrm{X}) + \mathrm{O}\left(\|g\|_0\right) = \mathrm{R}_{\alpha,\beta}(\mathrm{X}) + \mathrm{O}\left(\|\xi\|_{2\tau+4}\right) \\ \mathrm{D}g_{\mathrm{F}\circ\mathrm{G}^{-1}} = \mathrm{D}g\circ\mathrm{R}_{\alpha,\beta} + \mathrm{O}(\|g\|_0\|g\|_2) = \mathrm{D}g\circ\mathrm{R}_{\alpha,\beta} + \mathrm{O}(\|\xi\|^2_{2\tau+6}) \end{cases}$$

so that in (4.42)

$$\begin{aligned}(\mathrm{DF}_1)_{\mathrm{X}} &= \left(\mathrm{I}_2 - \mathrm{D}g_{\mathrm{F}\circ\mathrm{G}^{-1}(\mathrm{X})}\right)\left(\mathrm{I}_2 + \mathrm{D}\Xi_{\mathrm{G}^{-1}(\mathrm{X})}\right)\left(\mathrm{I}_2 + \mathrm{D}g_{\mathrm{G}^{-1}(\mathrm{X})} + \mathrm{O}\left(\|g\|_1^2\right)\right) \\ &= \mathrm{I}_2 + \mathrm{D}\Xi_{\mathrm{G}^{-1}(\mathrm{X})} + \mathrm{D}g_{\mathrm{G}^{-1}(\mathrm{X})} - \mathrm{D}g_{\mathrm{F}\circ\mathrm{G}^{-1}(\mathrm{X})} + \mathrm{O}\left(\|\xi\|^2_{2\tau+6}\right) \\ (\mathrm{DF}_1)_{\mathrm{X}} &= \mathrm{I}_2 + \mathrm{D}\Xi + \mathrm{D}g - \mathrm{D}g\circ\mathrm{R}_{\alpha,\beta} + \mathrm{O}(\|\xi\|^2_{2\tau+6})\end{aligned}$$

We finally obtain with these estimates and definition (4.40):

$$\begin{aligned}\mathrm{D}\Xi_1 &= \mathrm{D}\Xi + \mathrm{D}g - \mathrm{D}g\circ\mathrm{R}_{\alpha,\beta} + \mathrm{O}(\|\xi\|^2_{2\tau+6}) \\ (\mathrm{D}\Xi_1)^2 &= (\mathrm{D}\Xi + \mathrm{D}g - \mathrm{D}g\circ\mathrm{R}_{\alpha,\beta})^2 + \mathrm{O}(\|\xi\|^3_{2\tau+6})\end{aligned} \tag{4.43}$$

It remains to compute, using (4.30), (4.33) and (4.36)

$$\mathrm{D}\Xi + \mathrm{D}g - \mathrm{D}g\circ\mathrm{R}_{\alpha,\beta} = \begin{pmatrix} 0 & 0 \\ * & \partial_y(\xi + \mathrm{U}^*\widetilde{\xi} - \mathrm{U}^*\widetilde{\xi}\circ\mathrm{R}_{\alpha,\beta}) \end{pmatrix}$$

This ends the proof of the theorem with (4.39), (4.41) and (4.43).

■

Let us now give an explicit formula for $\xi = \xi_\omega$ (recall (4.33)). Let us recall the definitions (4.7), (4.14) and (4.15). Then one can compute

$$\mathrm{e}^{\mathrm{i}\pi\xi} = \frac{1 + \mathrm{N}_{\omega,x}(\mathrm{e}^{\mathrm{i}\pi y}).\mathrm{e}^{-\mathrm{i}\pi(y+\beta)}}{\left|1 + \mathrm{N}_{\omega,x}(\mathrm{e}^{\mathrm{i}\pi y}).\mathrm{e}^{-\mathrm{i}\pi(y+\beta)}\right|}$$

as a consequence

$$\mathrm{i}\pi\xi = \log\frac{1 + \mathrm{N}_{\omega,x}(\mathrm{e}^{\mathrm{i}\pi y}).\mathrm{e}^{-\mathrm{i}\pi(y+\beta)}}{\left|1 + \mathrm{N}_{\omega,x}(\mathrm{e}^{\mathrm{i}\pi y}).\mathrm{e}^{-\mathrm{i}\pi(y+\beta)}\right|}$$

The Taylor-Young expansion of the complex logarithm to the first order $\log(1+z) = z + \underset{z\to 0}{\mathrm{O}}(z^2)$ gives

$$\begin{aligned}\mathrm{i}\pi\xi &= z - \tfrac{1}{2}\left(2\,\mathfrak{Re}\,z\right) + \mathrm{O}(z^2) \\ &= \mathrm{i}\,\mathfrak{Im}\,z + \mathrm{O}(z^2)\end{aligned}$$

where (see definition (4.15))

$$z = \mathrm{N}_{\omega,x}(\mathrm{e}^{\mathrm{i}\pi y}).\mathrm{e}^{-\mathrm{i}\pi(y+\beta)} = -\frac{v(\omega,x)}{\sin\pi\beta}\,\sin(\pi(\beta+y))\,\mathrm{e}^{-\mathrm{i}\pi(y+\beta)} \tag{4.44}$$

This leads to $\xi_\omega = \xi_\omega^{(1)} + \xi_\omega^{(2)}$ with

$$\begin{cases} \xi_\omega^{(1)}(x,y) = \frac{1}{\pi}\,\mathfrak{Im}\, z = \dfrac{\sin^2(\pi(y+\beta))}{\sin \pi\beta}\, v(\omega,x) \\ \qquad \xi_\omega^{(2)} = \xi_\omega - \xi_\omega^{(1)} \end{cases} \tag{4.45}$$

- $\xi_\omega^{(1)}$ is an explicit trigonometric polynomial in y of order 1 such that $|||\xi^{(1)}|||_0 = \mathrm{O}(\|\mathrm{N}_{\omega,x}\|)$;
- $\xi_\omega^{(2)}$ satisfies $\xi_\omega^{(2)}(x,y) = \dfrac{1}{\sin^2 \pi\beta}\, v^2(\omega,x)\, a_\beta(y)$ for some analytic map a_β uniformly bounded in β.

Remark 4.6

Note that we could obtain a development of ξ_ω at any order if we expanded the complex logarithm sufficiently far. For example to the second order we have $\log(1+z) = z - \frac{1}{2}z^2 + \underset{z\to 0}{\mathrm{O}}(z^3)$ and so we would get

$$\mathrm{i}\pi\xi = \mathrm{i}\,\mathfrak{Im}\, z - \mathrm{i}\,\mathfrak{Re}\, z\; \mathfrak{Im}\, z + \mathrm{O}(z^3)$$

which would give $\xi_\omega = \xi_\omega^{(1)} + \xi_\omega^{(2)} + \xi_\omega^{(3)}$ with

$$\begin{cases} \xi_\omega^{(1)}(x,y) = \frac{1}{\pi}\,\mathfrak{Im}\, z \\ \xi_\omega^{(2)}(x,y) = -\frac{1}{\pi}\,\mathfrak{Im}\, z\; \mathfrak{Re}\, z \\ \qquad \xi_\omega^{(3)} = \xi_\omega - \xi_\omega^{(1)} - \xi_\omega^{(2)} \end{cases}$$

- *$\xi_\omega^{(1)}$ is a trigonometric polynomial in y of order 1 such that $|||\xi^{(1)}|||_0 = \mathrm{O}(\|\mathrm{N}_{\omega,x}\|)$*
- *$\xi_\omega^{(2)}$ is a trigonometric polynomial in y of order 2 with $|||\xi^{(2)}|||_0 = \mathrm{O}(\|\mathrm{N}_{\omega,x}\|^2)$*
- *$\xi_\omega^{(3)} = \xi_\omega - \xi_\omega^{(1)} - \xi_\omega^{(2)}$ satisfies $|||\xi^{(3)}|||_0 = \mathrm{O}(\|\mathrm{N}_{\omega,x}\|^3)$*

However, this makes the error term degenerate since $\mathrm{N}_{\omega,x}$ has a factor $1/\sin(\pi\beta) \asymp (\mathrm{E}^2-4)^{-1/2}$ (see (4.15)): the more we want a precise expansion of ξ, the more the error term degenerates as a positive power of $(\mathrm{E}^2-4)^{-1}$. Besides, the first order will be sufficient for our estimates

The definitions and estimates of (4.45) coupled with proposition 4.3 give for $\widetilde{\xi}^{(1)} = \mathbf{E}\left(\xi_\omega^{(1)} \circ \mathrm{R}_{-\alpha,-\beta}\right)$:

$$\gamma(\mathrm{F},\mu) = -\tfrac{1}{2}\mathbf{E}\int_{\mathbf{T}^2} \left[\partial_y \xi_\omega^{(1)}(x,y) + \partial_y \mathrm{U}^*\widetilde{\xi}^{(1)}(x,y) - \partial_y \mathrm{U}^*\widetilde{\xi}^{(1)}(x+\alpha,y+\beta)\right]^2 \mathrm{d}x\,\mathrm{d}y \tag{4.46}$$

up to an error term (recall that $\epsilon = |||\Xi|||_{4\tau+11} = |||\xi_\omega|||_{4\tau+11}$ and $\xi = \xi^{(1)} + \xi^{(2)}$ with (4.30) and (4.45))

$$\begin{aligned}\mathcal{O} &= \mathrm{O}\left(\epsilon^3, \mathbf{E}\left(\|\xi^{(1)}\|_{2\tau+5}\,\|\xi^{(2)}\|_{2\tau+5}\right), \mathbf{E}\,\|\xi^{(2)}\|^2_{2\tau+5}\right) \\ &= \mathrm{O}\left(\mathbf{E}\,\|\xi\|^3_{4\tau+11}, \mathbf{E}\left(\|\xi^{(1)}\|_{2\tau+5}\,\|\xi^{(2)}\|_{2\tau+5}\right), \mathbf{E}\,\|\xi^{(2)}\|^2_{2\tau+5}\right)\end{aligned} \tag{4.47}$$

Let us analyze this error term with (4.1) and (4.45):

$$\begin{aligned}\|\xi^{(1)}\|_{2\tau+5} &= \mathrm{O}\left(\frac{1}{|\sin\pi\beta|}(\lambda\,|\mathrm{W}_0| + \|\mathrm{V}\|_{2\tau+5})\right) = \mathrm{O}\left(\frac{\lambda\,|\mathrm{W}_0| + \|\mathrm{V}\|_{2\tau+5})}{(4-\mathrm{E}^2)^{1/2}}\right) \\ \|\xi^{(2)}\|_{2\tau+5} &= \mathrm{O}\left(\frac{1}{|\sin\pi\beta|^2}(\lambda^2\,|\mathrm{W}_0|^2 + \|\mathrm{V}\|^2_{2\tau+5})\right) = \mathrm{O}\left(\frac{\lambda^2\,|\mathrm{W}_0|^2 + \|\mathrm{V}\|^2_{2\tau+5})}{4-\mathrm{E}^2}\right) \\ \|\xi\|_{4\tau+11} &= \mathrm{O}\left(\frac{1}{|\sin\pi\beta|^2}(\lambda\,|\mathrm{W}_0| + \|\mathrm{V}\|_{4\tau+11})\right) = \mathrm{O}\left(\frac{\lambda\,|\mathrm{W}_0| + \|\mathrm{V}\|_{4\tau+11})}{4-\mathrm{E}^2}\right)\end{aligned}$$

Recall remark 4.5: the estimates obtained for $\xi, \xi^{(1)}, \xi^{(2)}$ actually hide some positive power of κ^{-1}. Thus

$$\mathcal{O} = \mathrm{O}\left(\kappa^{-3}\frac{\lambda^3\,|\mathrm{W}_0|^3 + \|\mathrm{V}\|^3_{4\tau+11})}{(4-\mathrm{E}^2)^3}\right) \tag{4.48}$$

Let us finish with the computation of a simpler expression of the formula for the LE. Using $\mathfrak{Im}(z_1\overline{z_2}) = \mathfrak{Im}\,z_1\,\mathfrak{Re}\,z_2 - \mathfrak{Im}\,z_2\,\mathfrak{Re}\,z_1$, de Moivre's formulæ we get in definition (4.45):

$$\xi^{(1)}_\omega(x,y) = \frac{\sin^2(\pi(y+\beta))}{\sin\pi\beta}\,v(\omega,x) = -\,\mathfrak{Re}\left(\frac{v(\omega,x)}{2\pi\sin\pi\beta}\,\mathrm{e}^{\,2\mathrm{i}\pi(y+\beta)}\right) + c_\beta(x) \tag{4.49}$$

with
$$c_\beta(x) = \frac{v(\omega,x)}{2\pi\sin\pi\beta}$$

Note that we actually have to calculate the integral of a function that is a trigonometric polynomial of small degree as explained in remark 4.4. The diophantine condition on β w.r.t. α is then enough to be able to use (4.46) and get a development.

Since the formula (4.46) only requires derivatives along y, the quantity c_β does not contribute to $\gamma(\mathrm{F},\mu)$. We recall that $\widetilde{\xi} = \mathbf{E}\,\xi_\omega(\cdot-\alpha,\cdot-\beta)$. Expanding the potential $v(\omega,x)$ (see (4.1)) in Fourier series on the form $v(\omega,x) = \sum_{m=-\infty}^{+\infty}\widehat{(v^\omega)}_m\,\mathrm{e}^{\,2\mathrm{i}\pi mx}$ we obtain thanks to (4.27) and (4.49)

$$\mathrm{U}^*\widetilde{\xi}^{(1)}(x,y) = -\tfrac{1}{2\pi\sin\pi\beta}\,\mathfrak{Re}\sum_{m=-\infty}^{+\infty}\frac{\mathrm{e}^{\,-2\mathrm{i}\pi(m\alpha+\beta)}}{1-\mathrm{e}^{\,-2\mathrm{i}\pi(m\alpha+\beta)}}\,\mathbf{E}(\widehat{v^\omega}_m)\,\mathrm{e}^{\,2\mathrm{i}\pi\beta}\,\mathrm{e}^{\,2\mathrm{i}\pi(mx+y)}$$

so that

$$\begin{aligned}\partial_y\mathrm{U}^*\widetilde{\xi}^{(1)}(x,y) &= -\tfrac{1}{\sin\pi\beta}\,\mathfrak{Re}\,\mathrm{i}\sum_{m=-\infty}^{+\infty}\frac{\mathrm{e}^{\,-2\mathrm{i}\pi(m\alpha+\beta)}}{1-\mathrm{e}^{\,-2\mathrm{i}\pi(m\alpha+\beta)}}\,\mathbf{E}(\widehat{v^\omega}_m)\,\mathrm{e}^{\,2\mathrm{i}\pi\beta}\,\mathrm{e}^{\,2\mathrm{i}\pi(mx+y)} \\ \partial_y\mathrm{U}^*\widetilde{\xi}^{(1)}(x+\alpha,y+\beta) &= -\tfrac{1}{\sin\pi\beta}\,\mathfrak{Re}\,\mathrm{i}\sum_{m=-\infty}^{+\infty}\frac{1}{1-\mathrm{e}^{\,-2\mathrm{i}\pi(m\alpha+\beta)}}\,\mathbf{E}(\widehat{v^\omega}_m)\,\mathrm{e}^{\,2\mathrm{i}\pi\beta}\,\mathrm{e}^{\,2\mathrm{i}\pi(mx+y)}\end{aligned}$$

Denoting

$$\Delta(x,y) = \partial_y \mathrm{U}^* \widetilde{\xi}^{(1)}(x,y) - \partial_y \mathrm{U}^* \widetilde{\xi}^{(1)}(x+\alpha, y+\beta)$$

we get the following expression

$$\begin{aligned}\Delta(x,y) &= \tfrac{1}{\sin \pi\beta}\, \mathfrak{Re}\left(\mathrm{i} \sum_{m=-\infty}^{+\infty} \mathbf{E}(\widehat{v^{\omega}}_m)\, \mathrm{e}^{2\mathrm{i}\pi\beta}\, \mathrm{e}^{2\mathrm{i}\pi(mx+y))} \right) \\ &= \mathfrak{Re}\left[\mathrm{i} \frac{\mathbf{E}(v(\omega,x))}{\sin \pi\beta}\, \mathrm{e}^{2\mathrm{i}\pi(y+\beta)} \right]\end{aligned}$$

Then with (4.49) we also have

$$\partial_y \xi^{(1)}(x,y) = \mathfrak{Re}\left[\mathrm{i} \frac{v(\omega,x)}{\sin \pi\beta}\, \mathrm{e}^{2\mathrm{i}\pi(y+\beta)} \right]$$

which, using definition (4.1), leads to

$$\begin{aligned}\mathrm{I}_\omega(x,y) :&= \partial_y \xi^{(1)}_\omega(x,y) + \partial_y \mathrm{U}^* \widetilde{\xi}^{(1)}(x,y) - \partial_y \mathrm{U}^* \widetilde{\xi}^{(1)}(x+\alpha, y+\beta) \\ &= -\,\mathfrak{Im}\left[\frac{v(\omega,x) - \mathbf{E}(v(\omega,x))}{\sin \pi\beta} \mathrm{e}^{2\mathrm{i}\pi(y+\beta)} \right] \\ \mathrm{I}_\omega(x,y)^2 &= \tfrac{1}{2\sin^2 \pi\beta} \left| v(\omega,x) - \mathbf{E}(v(\omega,x)) \right|^2 - \tfrac{1}{2\sin^2 \pi\beta} \cos(4\pi(y+\beta)) \left| v(\omega,x) - \mathbf{E}(v(\omega,x)) \right|^2 \\ &= \tfrac{1}{2\sin^2 \pi\beta} \left| \mathrm{W}_0 - \mathbf{E}(\mathrm{W}_0) \right|^2 - \tfrac{1}{2\sin^2 \pi\beta} \cos(4\pi(y+\beta)) \left| \mathrm{W}_0 - \mathbf{E}(\mathrm{W}_0) \right|^2\end{aligned}$$

The quantity $\cos(4\pi(y+\beta)) \left| \mathrm{W}_0 - \mathbf{E}(\mathrm{W}_0) \right|^2$ vanishes after integration in x, y and taking expectation because it is a trigonometric polynomial in y with no constant term. So it remains in (4.46)

$$\gamma(\mathrm{F},\mu) = -\frac{1}{4\sin^2 \pi\beta} \int_{\mathbf{T}^2} \mathbf{V}(v(\omega,x))\, \mathrm{d}x + \mathcal{O}$$

In our setting $v(x,\omega) = \mathrm{V}(x) + \lambda \mathrm{W}_0(\omega)$ and $\beta = \operatorname{Arc\,cos}(\mathrm{E}/2)$, so we compute

$$\mathbf{V}(v(\omega,x)) = \lambda^2\, \mathbf{V}(\mathrm{W}_0) \qquad \text{and} \qquad \sin^2 \pi\beta = \frac{4 - \mathrm{E}^2}{4}$$

Reminding (see (4.9)) that $\mathrm{L}(\mathrm{E}) = \frac{1}{2} \left| \gamma(\mathrm{F},\mu) \right|$ gives the sign in this formula and the expansion:

$$\mathrm{L}(\mathrm{E}) = -\tfrac{1}{2}\gamma(\mathrm{F},\mu) = \frac{\mathbf{V}(\mathrm{W}_0)}{2(4 - \mathrm{E}^2)} \lambda^2 + \mathcal{O} \tag{4.50}$$

We shall say a few words on $\mathcal{O}$ in (4.5) and on the size of the quasi-periodic part $\mathrm{V}(x)$ of the potential in (4.1) compared to the coupling constant λ for the random part of the potential. The formulæ (4.50) and (4.48) prove a Figotin-Pastur formula for V in $\mathscr{C}^{4\tau+11}(\mathbf{T})$ whose norm is at least as small as λ for all $\beta = \operatorname{Arc\,cos}(\mathrm{E}/2)$ which is diophantine with respect to α. That means for a positive set of energies in $]-2\,;2\,[$ the measure of which depends on the diophantine constant of β w.r.t. α. Notice that we also have the result when $v(\omega,x) = \varepsilon \mathrm{W}_0(\omega) + \nu \mathrm{V}(x)$ where ν is a coupling constant smaller than ε, and it clearly applies when $\nu \ll \varepsilon$.

In particular, if the random and quasi-periodic parts of the potential are of comparably same small size ε, then we obtain the proof of theorem 4.1.

Remark 4.7

Notice that the previous theorem does not require any assumption on the vanishing expectation of the potential $v(\omega, x) = \varepsilon\left(\mathrm{V}(x) + \mathrm{W}_\omega\right)$. *This may seem strange for two reasons. The first is that Figotin-Pastur's formula ((14.57) in [PF92]) has been established for random potentials with zero expectation. So setting* $\mathrm{V} = 0$ *in our context should recover their result. The second and more relevant question here is that one would then think of moving away from non-diophantine energies (i.e. those for which* $\beta = \frac{1}{\pi}\operatorname{Arc}\cos\left(\mathrm{E}/2\right)$ *does not belong to any* $\mathrm{DC}_\alpha(\kappa, \tau)$ *for any* κ, τ *– see (1.7)) to diophantine energies to get the result for all energies. Let us explain why this is* **not** *doable with our analysis. Indeed, suppose that* β *in (4.12) is not diophantine w.r.t.* α. *In order to apply our technique based on the resolution of the cohomological equation (4.20) for the rotation* $\mathrm{R}_{\alpha,\beta}$, *we need to move to a diophantine energy. So one would fix* Δv *some constant which they would perturb the potential* $v(\omega, x)$ *with:* $v(\omega, x) = \epsilon(\mathrm{V}(x) + \mathrm{W}_0 + \Delta v)$ *and such that the energy is now* $\widetilde{\mathrm{E}} = \mathrm{E} - \epsilon\Delta v = 2\cos(\pi(\beta + \epsilon\Delta\beta))$ *with* $\Delta\beta \asymp \Delta v$ *and* $\beta + \epsilon\Delta\beta$ *in some class* $\mathrm{DC}_\alpha(\kappa, \tau)$*:*

$$\forall n \neq 0 \qquad |(\beta + \epsilon\Delta\beta) + n\alpha| \geqslant \frac{\kappa}{|n|^\tau}$$

then for $n = 1$ *we get by triangular inequality* $\Delta\beta \gtrsim \epsilon^{-1}\kappa$ *and so* $\Delta v \gtrsim \kappa\epsilon^{-1}$ *which cannot be since we need the potential to be uniformly bounded otherwise the error term in (4.48) degenerates. So, even if none of the vanishing expectation hypotheses* $\int_{\mathbf{T}} f(x)\,\mathrm{d}x = 0$, *or* $\mathbf{E}(\mathrm{W}_0) = 0$ *or* $\int_{\mathbf{T}} f(x)\,\mathrm{d}x + \mathbf{E}(\mathrm{W}_0) = 0$ *(resp. for* x *or* ω *or even in the global variable* (x, ω)*) is needed for our result, one cannot move the potential to apply the diophantine regime strategy to all energies.*

CHAPTER 5

Scaling diagram for energies at the band edge of the free operator

5.1 The problem, its difficulties and ideas to solve it

5.1.1 The point of view

In this chapter we want to study the case when the energy E is close to ± 2, both by superior and inferior values. Such a problem is interesting because the *free operator*, which means when $\varepsilon = 0$, *i.e.* when the potential is identically zero and $\mathrm{E} = 2$, has spectrum $\Sigma(\mathrm{H}_{\varepsilon=0}) = [-2\,;2]$. This also corresponds to the case $\beta(\mathrm{E}) = \mathrm{Arc}\cos(\mathrm{E}/2) \approx 0$, a particular case of *resonant* β with respect to α (see (4.22)) for which we cannot solve the cohomological equation as in chapter 4, and where the error terms in the previous chapter degenerate because they are positive powers of $\frac{1}{4-\mathrm{E}^2}$ (see (4.48)). As we will see, instead of being close to an elliptic regime with good estimates, we are now close to a parabolic dynamics that we conjugate close to the identity with good estimates.

The setting is the following: the elementary transfer matrix for the Schrödinger cocycle is

$$\mathrm{S}_{\mathrm{E},\omega,\varepsilon}(x) = \begin{pmatrix} \mathrm{E} + \varepsilon(\mathrm{V}(x) + \mathrm{W}_\omega) & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{where} \qquad \mathrm{E} = 2 + \sigma d\varepsilon^\eta \tag{5.1}$$

where σ is a sign $\pm$ that reflects whether the energy is inside or outside the free spectrum, and $d > 0$ is some parameter. We assume that $\mathbf{E}(\mathrm{W}_\omega) = 0$ and $\mathbf{V}(\mathrm{W}_\omega) > 0$, and $\int_{\mathbf{T}} \mathrm{V}(x)\,\mathrm{d}x = 0$. This means that we perturb the free operator with a quasi-periodic and random potential and then approach the band edge of the spectrum at a speed $\asymp \varepsilon^\eta$.

A final important remark is that we will also obtain estimates for positive d, that is we are able to obtain a formula for the Lyapunov exponent outside the spectrum when $\mathrm{E} = 2 + d\varepsilon^\eta$, in the hyperbolic regime of the free operator perturbed by a potential of size ε . So all boils down to study a perturbation of size $\varepsilon^{\min(\eta,1)}$ of the *free operator*

$$\mathrm{S}_{\mathrm{E}=2} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.2}$$

Note that we have now completely lost control of the $1/\sqrt{4-\mathrm{E}^2}$ term. The cocycle is, as in the previous chapter (see (4.4)), defined on $\Omega \times \mathbf{T} \times \mathbf{R}^2$ by

$$(\mathrm{T},\mathrm{S})\colon \begin{cases} \Omega \times \mathbf{T} \times \mathbf{R}^2 \longrightarrow \Omega \times \mathbf{T} \times \mathbf{R}^2 \\ (\omega, x, u) \quad \longmapsto (\sigma\omega, x+\alpha, \mathrm{S}_{\mathrm{E},\omega,\varepsilon}(x)u) \end{cases}$$

Let us mention that we use the notation with ε as an index and x as a variable for the cocycle to highlight the dependency w.r.t. x that we want to cancel up to a certain order of approximation as we will explain in the strategy below.

5.1.2 Strategy of the proof

We first get rid of the dependency on x which brings us back to the situation of a perturbation of the identity in order to deal with an easier cohomological equation. A perturbatively method is developed to get an asymptotic expansion of any stationary measure of the random diffeomorphism of $\mathbf{T}^2$ induced by the cocycle. As seen in the previous chapter, such an estimate for a stationary measure ultimately allows to compute an asymptotic formula for the Lyapunov exponent.

The principle is based on the following elementary remark. If we are looking for an estimation of $\int_{\mathbf{T}^2} \psi \, \mathrm{d}\mu$ for a stationary measure μ, and if we can write ψ as a coboundary *i.e.* $\psi = \mathbf{E}(\varphi \circ \mathrm{F}_\omega) - \varphi$, then the definition of stationarity (remind (4.6)) gives $\int_{\mathbf{T}^2} \psi \, \mathrm{d}\mu = 0$. Obviously every function ψ cannot be a coboundary and we rather want to solve, for a given map $\psi : \mathbf{T}^2 \to \mathbf{R}$ regular enough, a cohomological equation (in φ) with obstruction terms

$$\psi = \mathbf{E} \left(\varphi \circ \mathrm{F}_\omega\right) - \varphi + \mathrm{Obs}(\psi) \tag{5.3}$$

The estimate for a stationary measure $\mu = \mu_\varepsilon$ would then be

$$\int_{\mathbf{T}^2} \psi \, \mathrm{d}\mu = \int_{\mathbf{T}^2} \mathrm{Obs}(\psi) \, \mathrm{d}\mu$$

Of course we are looking for obstructions that are as explicit as possible and especially concerning their order w.r.t. ε. Note that in the previous chapter (proposition 4.2) we found the obstruction to be of the following type

$$\mathrm{Obs}(\psi) = \int_{\mathbf{T}^2} \psi \, \mathrm{d}x \, \mathrm{d}y + \varepsilon \Lambda_\psi + \mathrm{O}(\varepsilon^2 \|\psi\|)$$

where $\psi \mapsto \Lambda_\psi$ is a linear map. So our obstruction were "simply" constants (and not functions). This means that we were able to compute an asymptotic development of μ_ε as $\varepsilon \to 0$ for functions with enough regularity, and where the main order term was the Lebesgue measure on $\mathbf{T}^2$.

We shall prove similar estimates but with obstructions that can involve a density with respect to the Lebesgue measure at the first order or Dirac peaks, depending on the dynamical regime of the mean diffeomorphism.

5.2 Reduction of the problem: getting rid of the dependency on x

We will use elementary ideas of the well developed theory of the reducibility of cocycles close to constants, that is conjugating quasi-periodic cocycles to constant cocycles. This theory has initially been developed by Moser and Pöschel ([MP84]), and then improved and extended by Eliasson and Kuksin ([EK09]), and also Avila and Krikorian ([AK06]).

Lemma 5.1

Let us set

$$m_\eta = \min(1+\eta, 2) \tag{5.4}$$

Assume $\mathrm{V} \in \mathscr{C}^p(\mathbf{T}, \mathbf{R})$ *with* $p \in \mathbf{N} \cup \{\infty, \omega\}$ *and* $\alpha \in \mathrm{DC}(\kappa, \tau)$ *(remind definition (2.3)). Then there exists* $\mathrm{Y} \in \mathscr{C}^{p-3\tau}(\mathbf{T}, \mathfrak{sl}_2(\mathbf{R}))$ *such that* $\mathrm{B}(\cdot) = \mathrm{e}^{\varepsilon \mathrm{Y}(\cdot)}$ *satisfies for all* $x \in \mathbf{T}$

$$\mathrm{B}(x+\alpha)\, \mathrm{S}_{\mathrm{E},\omega,\varepsilon}(x)\, \mathrm{B}(x)^{-1} = \mathrm{S}_{\omega,\varepsilon} + \mathrm{O}_{\omega,x}(\varepsilon^{m_\eta})$$

$$\mathrm{S}_{\omega,\varepsilon} = \begin{pmatrix} \mathrm{E} + \varepsilon(\widehat{\mathrm{V}}_0 + \mathrm{W}_\omega) & -1 \\ 1 & 0 \end{pmatrix} \tag{5.5}$$

$$\mathrm{O}_{\omega,x}(\varepsilon^{m_\eta}) = \varepsilon^{m_\eta} \mathrm{Z}_\omega(x)$$

for some matrix $\mathrm{Z}_\omega(x)$ *which is* $\mathscr{C}^{p-3\tau}$ *w.r.t* x.

Proof.

Let us define $\mathrm{S}_0 = \mathrm{S}_{\mathrm{E}=2}$ (recall (5.2) and (5.1)) and decompose

$$\mathrm{S}_{\mathrm{E},\omega,\varepsilon}(x) = \mathrm{S}_0 + \varepsilon \mathrm{L}_{\omega,x} + \varepsilon^\eta \sigma d^{-1} \mathrm{M}$$

with

$$\mathrm{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{L}_{\omega,x} = (\mathrm{V}(x) + \mathrm{W}_\omega)\mathrm{M} \tag{5.6}$$

The Taylor formula with Lagrange remainder gives

$$\mathrm{B}(x+\alpha) = \mathrm{e}^{\varepsilon \mathrm{Y}(x+\alpha)} = \mathrm{I}_2 + \varepsilon \mathrm{Y}(x+\alpha) + \varepsilon^2 \mathrm{D}(x)$$
$$\mathrm{B}(x)^{-1} = \mathrm{e}^{-\varepsilon \mathrm{Y}(x)} = \mathrm{I}_2 - \varepsilon \mathrm{Y}(x) + \varepsilon^2 \mathrm{E}(x)$$

so that

$$\mathrm{B}(x+\alpha)\, \mathrm{S}_{\mathrm{E},\omega,\varepsilon}(x)\, \mathrm{B}(x)^{-1} = \mathrm{S}_0 + \varepsilon\big(\mathrm{Y}(x+\alpha)\mathrm{S}_0 - \mathrm{S}_0 \mathrm{Y}(x)\big) + \begin{pmatrix} \varepsilon(\mathrm{V}(x) + \mathrm{W}_\omega) + \sigma d \varepsilon^\eta & 0 \\ 0 & 0 \end{pmatrix}$$
$$+ \varepsilon^{m_\eta} \mathrm{Z}_\omega(x)$$

with $\mathrm{Z}_\omega(x) \in \mathbf{R}_2[\mathrm{L}_{\omega,x}, \mathrm{M}, \mathrm{Y}(x), \mathrm{Y}(x+\alpha), \mathrm{D}(x), \mathrm{E}(x)]$. It is then enough that Y satisfies

$$\mathrm{Y}(x+\alpha)\mathrm{S}_0 - \mathrm{S}_0 \mathrm{Y}(x) = \begin{pmatrix} \widehat{\mathrm{V}}_0 - \mathrm{V}(x) & 0 \\ 0 & 0 \end{pmatrix}$$

or, equivalently, $\mathrm{Y}(x+\alpha) - \mathrm{S_0}\mathrm{Y}(x){\mathrm{S_0}}^{-1} = \mathrm{M}(x)$ where

$$\mathrm{M}(x) = \begin{pmatrix} \widehat{\mathrm{V}}_0 - \mathrm{V}(x) & 0 \\ 0 & 0 \end{pmatrix} {\mathrm{S_0}}^{-1}$$

Denoting, for $\mathrm{A} \in \mathrm{GL_2}(\mathbf{R})$, the map $\mathrm{Ad_A} : \mathrm{M} \in \mathfrak{M}_2(\mathbf{R}) \longmapsto \mathrm{AMA^{-1}}$ we then want to solve

$$\mathrm{Y}(x+\alpha) - \mathrm{Ad_{S_0}}\, \mathrm{Y}(x) = \mathrm{M}(x) \tag{5.7}$$

Using Fourier series this is equivalent to

$$\forall n \in \mathbf{Z} \qquad \left(\mathrm{e}^{2\mathrm{i}\pi n\alpha}\mathrm{Id} - \mathrm{Ad_{S_0}}\right) \widehat{\mathrm{Y}}_n = -\widehat{\mathrm{V}}_n(1-\delta_{n,0}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} {\mathrm{S_0}}^{-1} \tag{5.8}$$

We obviously need information about the inverse of the linear operator $\mathrm{e}^{2\mathrm{i}\pi n\alpha}\mathrm{Id} - \mathrm{Ad_{S_0}}$ with some estimates on the $\mathscr{C}^k$-regularity of the solution. Let us recall the following facts from basic linear algebra about the operator $\mathrm{Ad_A}$ on $\mathfrak{sl}_2(\mathbf{R})$ (look also at appendice A.1)

Lemma 5.2

For $\mathrm{A} \in \mathrm{SL_2}(\mathbf{R})$ *let us consider the operator*

$$\mathrm{Ad_A} : \begin{cases} \mathfrak{sl}_2(\mathbf{R}) \longrightarrow \mathfrak{sl}_2(\mathbf{R}) \\ \quad \mathrm{X} \quad \longmapsto \mathrm{AXA^{-1}} \end{cases} \in \mathrm{Aut}(\mathfrak{sl}_2(\mathbf{R}))$$

The spectrum (eigenvalues) of $\mathrm{Ad_A}$ *consists of* $\left\{1, \mathrm{e}^{\mathrm{i}\beta}, \mathrm{e}^{-\mathrm{i}\beta}\right\}$ *where* $\beta^2 = \det \mathrm{B}$ *and* $\mathrm{B} \in \mathfrak{sl}_2(\mathbf{R})$ *satisfies* $\mathrm{A} = \mathrm{e}^{\mathrm{B}}$. *More precisely*

- $\beta \in \mathbf{R} \smallsetminus \{0\}$ *if* A *is elliptic* ($\mathrm{tr}\ \mathrm{A} < 2$) *;*
- $\beta \in \mathrm{i}\mathbf{R} \smallsetminus \{0\}$ *if* A *is hyperbolic* ($\mathrm{tr}\ \mathrm{A} > 2$) *;*
- $\beta = 0$ *if* A *is parabolic* ($\mathrm{tr}\ \mathrm{A} = \pm 2$)*.*

Let us denote $\widetilde{\beta} = \beta/(2\pi)$. *Then for* $n \in \mathbf{Z}$ *the operator* $\mathrm{L}_n = \mathrm{e}^{2\mathrm{i}\pi n\alpha}\mathrm{Id} - \mathrm{Ad_{S_0}}$ *is invertible for* $\mathrm{e}^{2\mathrm{i}\pi n\alpha} \notin \left\{1, \mathrm{e}^{\mathrm{i}\beta}, \mathrm{e}^{-\mathrm{i}\beta}\right\}$ *that is* $\pm\widetilde{\beta} + n\alpha \notin \mathbf{Z}$ *and* $n\alpha \notin \mathbf{Z}$. *Under these conditions we denote the solution* $\mathrm{Y} = {\mathrm{L}_n}^{-1}(\mathrm{F})$ *and one has the following explicit formula:*

(i) *in the diagonalizable case* ($\beta \neq 0$)*: if* X_λ *is the projection of* X *on the* $\mathrm{e}^{\mathrm{i}\lambda}$*-eigenspace for* $\mathrm{Ad_A}$ *we have*

$$\forall \lambda \in \{0, \beta, -\beta\} \qquad \mathrm{Y}_\lambda = \frac{\mathrm{F}_\lambda}{\mathrm{e}^{2\mathrm{i}\pi(n\alpha+\lambda)} - 1}$$

(ii) *in the non diagonalizable case ($\beta = 0$) we can write the Jordan normal form of* A

$$\mathrm{A} = \mathrm{P}\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}\mathrm{P}^{-1}$$

When A *has spectrum* $\{1\}$ *(+-sign case) one gets* $\mathrm{Y} = \left(\begin{smallmatrix} a_{\mathrm{Y}} & b_{\mathrm{Y}} \\ c_{\mathrm{Y}} & -a_{\mathrm{Y}} \end{smallmatrix}\right)$ *where*

$$\begin{cases} c_{\mathrm{Y}} = \dfrac{c_{\widetilde{\mathrm{F}}}}{\gamma_n} \\ a_{\mathrm{Y}} = \dfrac{a_{\widetilde{\mathrm{F}}} - c_{\mathrm{Y}}}{\gamma_n} = \dfrac{\gamma_n\, a_{\widetilde{\mathrm{F}}} - c_{\widetilde{\mathrm{F}}}}{{\gamma_n}^2} \\ b_{\mathrm{Y}} = \dfrac{b_{\widetilde{\mathrm{F}}} + 2a_{\mathrm{Y}} + c_{\mathrm{Y}}}{\gamma_n} = \dfrac{{\gamma_n}^2\, b_{\widetilde{\mathrm{F}}} + 2\gamma_n\, a_{\widetilde{\mathrm{F}}} + (1-\gamma_n)c_{\widetilde{\mathrm{F}}}}{{\gamma_n}^3} \\ \widetilde{\mathrm{F}} = \mathrm{P}^{-1}\mathrm{F}\mathrm{P} \qquad \textit{and} \qquad \gamma_n = \mathrm{e}^{2\mathrm{i}\pi n\alpha} - 1 \end{cases}$$

If A *has spectrum* $\{-1\}$ *the formula are analogous.*

Let us use these results in our context. We impose the condition $\widehat{\mathrm{Y}}_0 = 0$ such that we only have to solve $(\mathrm{e}^{2\mathrm{i}\pi n\alpha}\mathrm{Id} - \mathrm{Ad}_{\mathrm{S}_0})\,\widehat{\mathrm{Y}}_n = -\widehat{\mathrm{V}}_n\mathrm{M}$ where $\mathrm{M} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\mathrm{S}_0{}^{-1}$ for $n \in \mathbf{Z} \smallsetminus \{0\}$. In our case, A is not diagonalizable and has spectrum $\{1\}$ so we only need an arithmetic condition on α* to get nice estimates on the solution. Namely, if $\alpha \in \mathrm{DC}(\kappa, \tau)$ satisfies the associated diophantine condition (2.3) one gets the following $\mathscr{C}^k$ or $\mathscr{C}^\omega_h$ (which stands for the analytic norm on a strip $\{|\mathfrak{Im}\, z| \leqslant h\}$) estimates

$$\begin{cases} \|\mathrm{Y}\|_{\mathscr{C}^k} \lesssim \dfrac{1}{\kappa^3}\|\mathrm{F}\|_{\mathscr{C}^{k+3\tau}} \\ \|\mathrm{Y}\|_{h'<h} \lesssim \dfrac{\kappa^{-3}}{(h-h')^{3\tau+1}}\|\mathrm{F}\|_h \end{cases}$$

Thus we can find a $\mathrm{Y}(\cdot)$ as required. ■

5.3 Random diffeomorphisms induced on T^2 by the cocycle

5.3.1 Influence on the Lyapunov exponents

We have proved in the previous section that the cocycle $\mathrm{S}_{\mathrm{E},\omega,\varepsilon}(x)$ can be conjugated to the perturbation of $\left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (see lemma (5.1)): with a cocycle conjugacy we obtained

$$\mathrm{C}(x+\alpha)\,\mathrm{S}_{\mathrm{E},\omega,\varepsilon}(x)\,\mathrm{C}(x)^{-1} = \widetilde{\mathrm{S}}_{\omega,\varepsilon} := \mathrm{S}_{\omega,\varepsilon} + \mathrm{O}_{\omega,x}(\varepsilon^{m_\eta})$$

1 has multiplicity at least 1 for $\dim\{\mathrm{X} \in \mathfrak{M}_2(\mathbf{R}) \mid [\mathrm{A},\mathrm{M}] = 0\} \geqslant 2$ and if $[\mathrm{A},\mathrm{M}] = 0$ then $[\mathrm{A}, \mathrm{M} - (\mathrm{tr}\ \mathrm{M})\mathrm{I}_2] = 0$.

*whereas in general one needs $\widetilde{\beta}$ to be diophantine with respect to α in order to have estimates of the solution.

This shows that the cocycle $(\alpha, \omega, \mathrm{S}_{\mathrm{E},\omega,\varepsilon}(x))$ is conjugated to $(\alpha, \omega, \widetilde{\mathrm{S}}_{\omega,\varepsilon})$ such that their Lyapunov exponents are equal. We then have to look for a stationary measure for the random diffeomorphism of $\mathbf{T}^2$ induced by $\mathrm{S}_{\omega,\varepsilon}$ which we will algebraically conjugate to a perturbation of the identity with no dependency on x up to a term of order m_η. The error term given by $\varepsilon^{m_\eta}\mathrm{Z}_\omega(x)$ (see (5.5)) will be taken into account for the estimate of the Lyapunov exponent in the end. It will actually be of higher order than the other error term that arises from our analysis to come term, so that it will not explicitly appear in the estimates.

5.3.2 Computation of the diffeomorphism

Any $\mathrm{SL}_2(\mathbf{R})$ cocycle S induces a diffeomorphism f_{S} of $\mathbf{T}$ by its projective action (see definition (4.7)):

$$\mathrm{e}^{\,\mathrm{i}\pi f_{\mathrm{S}}(y)} = \pm\frac{\mathrm{S}(\mathrm{e}^{\,\mathrm{i}\pi y})}{\|\mathrm{S}(\mathrm{e}^{\,\mathrm{i}\pi y})\|}$$

Since the cocycle is a perturbation of the identity of the form $\mathrm{S} = \mathrm{I}_2 + \mathrm{N}$ where N is supposed to be small, here the adequate sign for the projective action is $+$. Let us set

$$z = \mathrm{N}(\mathrm{e}^{\,\mathrm{i}\pi y}).\mathrm{e}^{\,-\mathrm{i}\pi y} \tag{5.9}$$

which is small by assumption, and compute

$$\begin{aligned}
\mathrm{e}^{\,2\mathrm{i}\pi f_{\mathrm{S}}(y)} &= \frac{\mathrm{S}(\mathrm{e}^{\,\mathrm{i}\pi y})}{\overline{\mathrm{S}(\mathrm{e}^{\,\mathrm{i}\pi y})}} = \frac{\mathrm{e}^{\,\mathrm{i}\pi y} + \mathrm{N}(\mathrm{e}^{\,\mathrm{i}\pi y})}{\mathrm{e}^{\,-\mathrm{i}\pi y} + \overline{\mathrm{N}(\mathrm{e}^{\,\mathrm{i}\pi y})}} = \mathrm{e}^{\,2\mathrm{i}\pi y}\frac{1+z}{1+\overline{z}} \\
&= \mathrm{e}^{\,2\mathrm{i}\pi y}\,(1+z)\,(1-\overline{z}+\overline{z}^2+\mathrm{O}(z^3)) \\
\mathrm{e}^{\,2\mathrm{i}\pi(f_{\mathrm{S}}(y)-y)} &= 1 + 2\mathrm{i}\,\mathfrak{Im}(z) + \overline{z}^2 - |z|^2 + \mathrm{O}(z^3)
\end{aligned}$$

A second order Taylor expansion of the complex logarithm near 1 then leads to

$$f_{\mathrm{S}}(y) = y + \frac{1}{\pi}\,\mathfrak{Im}\,z - \frac{1}{\pi}\,\mathfrak{Im}(z)\,\mathfrak{Re}(z) + \mathrm{O}(\mathrm{N}^3) \tag{5.10}$$

Let us sum up all the important notations we will use later on. The random diffeomorphism of $\mathbf{T}^2$ for which we have to find a stationary measure is:

$$\mathrm{F}_\omega(x,y) = (x+\alpha, y + v_\omega(y)) \tag{5.11}$$

where the random perturbation $v_\omega(y) = \sum_{j=1}^{\mathrm{J}-1}\varepsilon^{\eta_j} f_j + \mathrm{O}(\varepsilon^{\eta_{\mathrm{J}}})$ is, up to an error term, a linear combination of analytic functions f_j with coefficients that are increasing powers of ε.

Remark 5.1

Please note that the $\mathrm{O}(\varepsilon^a)$ *involved are actually functions of the variables* x, y *multiplied by* ε^a.

5.3.3 The argument of the time-one map of vector fields on $\mathbf{T}$

Let us mention another way to compute the random diffeomorphism $f_{\omega,x}$ using tools from Lie algebras. We define the following functions on $\mathbf{T}$:

$$c(y) = \frac{1}{\pi}\cos(2\pi y) \qquad s(y) = \frac{1}{\pi}\sin(2\pi y) \qquad \text{and} \qquad \iota(y) = \frac{1}{\pi}$$

If $\{a, b\} = [a\partial, b\partial] = (ab' - a'b)\partial$ is the Lie bracket of the vector fields $\mathrm{X} = a\partial$ and $\mathrm{Y} = b\partial$ on $\mathbf{T}$ then it is straightforward that

$$\{c, s\} = 2\iota \qquad \{\iota, s\} = 2c \qquad \text{and} \qquad \{\iota, s\} = -2s$$

so that $\mathcal{X} = \mathrm{Vect}\,\{\iota\partial, c\partial, s\partial\}$ endowed with $[\cdot,\cdot]$ is a 3-dimensional Lie algebra of vector fields on $\mathbf{T}$. The following matrices

$$\mathcal{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathcal{S} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathcal{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

form a basis of the Lie algebra $\mathfrak{sl}_2(\mathbf{R})$ endowed with the classical Lie bracket for matrices $[\mathrm{M}, \mathrm{N}] = \mathrm{MN} - \mathrm{NM}$ and satisfy the same brackets relations as c, s, ι (with self-explanatory notations). As a result, the following linear mapping is a Lie algebra isomorphism

$$c \longleftrightarrow \mathcal{C} \qquad s \longleftrightarrow \mathcal{S} \qquad \text{and} \qquad \iota \longleftrightarrow \mathcal{I}$$

We shall explain why the following diagram commutes

$$\begin{tikzcd}
\mathrm{B} \in \mathfrak{sl}_2(\mathbf{R}) \arrow[r] \arrow[d, "\exp\text{ in }\mathfrak{M}_n(\mathbf{R})"'] & \chi_{\mathrm{B}} \in \mathcal{X} \arrow[d, "\Phi^1\text{(time-one)}"] \\
\mathrm{A} = \mathrm{e}^{\mathrm{B}} \in \mathrm{SL}_2(\mathbf{R}) \arrow[r, "\rho"'] & f_{\mathrm{A}} \backslash \Phi^1_{\mathrm{B}}
\end{tikzcd}$$

Proposition 5.1

If Φ^1_{X} *denotes the time-one map of the vector field* X *on* $\mathbf{T}$ *and* χ_{B} *the vector field associated to the matrix* $\mathrm{B} \in \mathfrak{sl}_2(\mathbf{R})$ *through the previous dictionary, then*

$$f_{\mathrm{e}^{\mathrm{B}}} = \Phi^1_{\chi_{\mathrm{B}}}$$

In terms of Lie algebras: the exponential map in $\mathfrak{sl}_2(\mathbf{R})$ *corresponds to the exponential map in* $\mathcal{X}$.

Proof. ______________

The previous map ρ used in the above diagram is actually an $\mathrm{SL}_2(\mathbf{R})$-action on $\mathbf{T}$

$$\rho\colon \begin{cases} \mathrm{SL}_2(\mathbf{R}) \longrightarrow \mathrm{Diff}(\mathbf{T}) \\ \quad \mathrm{A} \qquad \longmapsto f_{\mathrm{A}} \end{cases}$$

We write $\cdot$ for the action, which means $\mathrm{B}\cdot y = f_{\mathrm{B}}(y)$. Set $g_{\mathrm{B}}(t) = \mathrm{e}^{t\mathrm{B}}$ the usual matrix exponentiation and $y(t) = g_{\mathrm{B}}(t)\cdot y$. We then have $y(t+h) = g_{\mathrm{B}}(t+h)\cdot y = g_{\mathrm{B}}(t+h)g_{\mathrm{B}}(t)^{-1}\cdot y(t) = g_{\mathrm{B}}(h)\cdot y(t)$

We can also compute, as done in the previous subsection,

$$\mathrm{e}^{2\mathrm{i}\pi g_{\mathrm{B}}(h)\cdot y} = \frac{(\mathrm{I}+h\mathrm{B}+\mathrm{O}(h^2))(\mathrm{e}^{\mathrm{i}\pi y})}{\overline{(\mathrm{I}+h\mathrm{B}+\mathrm{O}(h^2))(\mathrm{e}^{\mathrm{i}\pi y})}}$$

which gives $$g_{\mathrm{G}}(h)\cdot y - y = h\frac{1}{\pi}\,\Im\mathfrak{m}\left(\mathrm{B}(\mathrm{e}^{\mathrm{i}\pi y}).\mathrm{e}^{-\mathrm{i}\pi y}\right) + \mathrm{O}(h^2)$$

Setting $\mathrm{B} = \left(\begin{smallmatrix} a & b \\ c & -a\end{smallmatrix}\right)$ and letting $h\to 0$ in the ratio $\frac{y(t+h)-y(t)}{h}$ leads to

$$y'(t) = \chi(y(t)) \qquad \text{where} \qquad \begin{aligned}\chi(y) &= \frac{2}{\pi}\,\Im\mathfrak{m}\left(\mathrm{B}(\mathrm{e}^{\mathrm{i}\pi y}).\mathrm{e}^{-\mathrm{i}\pi y}\right) \\ &= (c-b)\frac{1}{\pi} + (b+c)\frac{\cos(2\pi y)}{\pi} - 2a\frac{\sin(2\pi y)}{\pi}\end{aligned}$$

This exactly means that $\chi = \chi_{\mathrm{B}}$ for the previous dictionary. We have just proved

$$f_{\mathrm{e}^{\mathrm{B}}}(y) = y(1) = \Phi^1_{\chi_{\mathrm{B}}}(y)$$

as expected. ■

It only remains to give an expression of the time-one map. We have $\Phi^1_{\mathrm{X}} = \mathrm{e}^{\mathrm{X}\partial}$ for $\mathrm{X}\in\mathcal{X}$, in the sense that for all (real) analytic f

$$f\circ\Phi^1_{\mathrm{X}} = \sum_{n=0}^{+\infty}\frac{(\mathrm{X}\partial)^n}{n!}f \qquad \text{where} \qquad \left((\mathrm{X}\partial)f\right)(y) = \mathrm{X}(y).f'(y) \qquad (y\in\mathbf{T}) \tag{5.12}$$

This follows from the differentiation of any analytic observation of the flow $t\mapsto f(\Phi^t(y))$:

$$\frac{\mathrm{d}}{\mathrm{d}t}f(\Phi^t(y)) = \mathrm{X}(\Phi^t(y)).f'(\Phi^t(y))$$

and so $$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\Phi^t(y)) = \mathrm{X}(y).f'(y)$$

an easy induction gives

$$\left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}\right|_{t=0} f(\Phi^t(y)) = \left((\mathrm{X}\partial)^n f\right)(y)$$

The Cauchy-Kowalevska theorem ensures that $t\mapsto f\circ\Phi^t(y)$ is analytic, so the claim (5.12) is proved since we just computed its Taylor coefficients at 0.

5.3.4 Transfer operator and cohomological equation

We recall the definition of the transfer operator (see (5.11) for F_ω)

$$\mathrm{T} : \varphi \longmapsto \mathbf{E}(\varphi \circ \mathrm{F}_\omega) \tag{5.13}$$

In order to find the appropriated obstruction, let us try to solve $\psi = \mathrm{T}\varphi - \varphi$ to understand where this obstruction comes from.

Using Taylor expansions, one can compute the action of the transfer operator for $\varphi \in \mathscr{C}^{p+1}(\mathbf{T}^2)$:

$$\begin{aligned}\mathrm{T}\varphi(x,y) = \mathbf{E}\Bigg(&\varphi(x+\alpha,y) + \sum_{k=1}^{p} \tfrac{1}{k!}\varphi_y^{(k)}(x+\alpha,y)\, v_\omega(y)^k \\ &+ \tfrac{1}{p!} v_\omega(y)^{(p+1)} \int_0^1 (1-s)^p\, \varphi_y^{(p+1)}(x+\alpha, y + s v_\omega(y))\,\mathrm{d}s\Bigg) \\ = \mathbf{E}\Bigg(&\varphi(x+\alpha,y) + \varphi'_y(x+\alpha,y)\, v_\omega(y) + v_\omega(y)^2 \int_0^1 (1-s)\, \varphi''_y(x+\alpha, y + s\, v_\omega(y))\,\mathrm{d}s\Bigg)\end{aligned}$$

The second equality holds in the particular case $p = 2$. As we will see, for our purpose the expansion for $p = 2, 3$ will be enough.

We use Fourier series in x and write $f_n(y) = \widehat{f}_n(y) = \int_{\mathbf{T}} f(x,y)\mathrm{e}^{-2\mathrm{i}\pi n x}\,\mathrm{d}x$. The former equation (5.13) is then equivalent to the fact that for all $n \in \mathbf{Z}$, the following equation is satisfied in $\mathrm{L}^2(\mathbf{T})$

$$\widehat{\psi}_n = \mathrm{T}_n(\widehat{\varphi}_n) \tag{5.14}$$

where we set (the integer p can be chosen as large as needed for the computations as long as φ is $\mathscr{C}^{p+1}(\mathbf{T}^2)$)

$$\begin{aligned}\mathrm{T}_n\varphi(y) = {}& \gamma_n\varphi(y) + \alpha_n \sum_{k=1}^{p} \frac{1}{k!}\varphi_y^{(k)}(y)\ \mathbf{E}(v_\omega(y)^k) \\ &+ \frac{\alpha_n}{p!}\ \mathbf{E}\left(v_\omega(y)^{p+1} \int_0^1 (1-s)^p\, \varphi_y^{(p+1)}(y + s v_\omega(y))\,\mathrm{d}s\right) \\ = {}& \gamma_n\varphi(y) + \alpha_n \sum_{k=1}^{p} \overline{v}_k(y)\, \varphi_y^{(k)}(y) + \frac{\alpha_n}{p!}\ \mathbf{E}\left(v_\omega(y)^{p+1} \int_0^1 (1-s)^p\, \varphi_y^{(p+1)}(y + s v_\omega(y))\,\mathrm{d}s\right)\end{aligned} \tag{5.15}$$

and for integers $k \geqslant 1$ and $n \in \mathbf{Z}$

$$\overline{v}_k = \frac{1}{k!}\mathbf{E}\left({v_\omega}^k\right) \qquad \alpha_n = \mathrm{e}^{2\mathrm{i}\pi n\alpha} \qquad \gamma_n = \alpha_n - 1 \tag{5.16}$$

We remind that we assume a diophantine condition (2.3) on α. Let us point out the fundamental difference between equations (5.14): for $n \neq 0$ the equation has a non vanishing derivative of order 0 whereas $\gamma_0 = 0$, so we have a *pseudo* differential equation of order 2, meaning that ${\varphi_0}'$ satisfies a first order differential equation. Moreover, ${\varphi_0}'$ multiplied by a small function $\overline{v}_\varepsilon$ that passes on the error term. We will compute the $\overline{v}_k$'s in each case we study.

5.4 Energies inside the free spectrum ($\sigma = -1$) with $\eta < \frac{4}{3}$

The main theorem is the following:

Theorem 5.1 (L.E. inside the free spectrum)

For $\frac{4}{5} < \eta < \frac{4}{3}$, the following asymptotic expansion holds for ε small enough:

$$\mathrm{L}(2 - d\varepsilon^\eta) = \tfrac{1}{8}\mathbf{V}(\mathrm{W}_\omega)\frac{\varepsilon^{2-\eta}}{d} + \mathrm{O}\left(d^{-6}(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{4-\frac{5}{2}\eta})\right)$$

5.4.1 Conjugacy close to an elliptic matrix

For an energy $\mathrm{E} = 2 - d\varepsilon^\eta$, we are in presence of a QPR (quasi-periodic and random) perturbation of the following elliptic matrix of $\mathrm{SL}_2(\mathbf{R})$

$$\mathrm{A}_\varepsilon := \begin{pmatrix} 2 - d\varepsilon^\eta & -1 \\ 1 & 0 \end{pmatrix}$$

It has eigenvalues $\mathrm{e}^{\pm \mathrm{i}\theta}$ which are the roots of the polynomial $\mathrm{X}^2 - (2 - d\varepsilon^\eta)\mathrm{X} + 1$ with $\theta = \theta_\varepsilon \in [\,0\,; \pi\,[$. Thus $\mathrm{Q} := \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ 1 & 0 \end{smallmatrix}\right)$ diagonalizes A_ε to R_θ the rotation matrix by the angle θ. We also compute

$$\mathrm{Q}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathrm{Q} = \begin{pmatrix} 0 & 0 \\ \cot\theta & 1 \end{pmatrix}$$

And with $2\cos\theta = 2 - d\varepsilon^\eta$ we get

$$\cot\theta = \frac{1}{\sqrt{d}\,\varepsilon^{\eta/2}} - \frac{13}{24}\sqrt{d}\,\varepsilon^{\eta/2} + \mathrm{O}(d^{3/2}\varepsilon^{3\eta/2}) \tag{5.17}$$

Using $$\mathrm{R}_\theta = \exp\theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathrm{I}_2 + \theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \tfrac{1}{2}\theta^2\,\mathrm{I}_2 + \mathrm{O}(d^{3/2}\varepsilon^{3\eta/2})$$

we obtain $$\begin{aligned} \mathrm{C}(x+\alpha)\mathrm{S}_{\omega,\varepsilon,x}\mathrm{C}(x)^{-1} &= \mathrm{R}_\theta + \varepsilon\mathrm{W}_\omega\begin{pmatrix} 0 & 0 \\ \cot\theta & 1 \end{pmatrix} + \mathrm{O}(\varepsilon^{m_\eta - \eta/2}/\sqrt{d}) \\ &= \mathrm{I}_2 + \mathrm{N}_{\omega,\varepsilon} \end{aligned}$$

where $\mathrm{C}(x) = \mathrm{Q}^{-1}\mathrm{B}(x)$ and

$$\mathrm{N}_{\omega,\varepsilon} = \sqrt{d}\,\varepsilon^{\eta/2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\varepsilon^{1-\eta/2}}{\sqrt{d}}\mathrm{W}_\omega\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}d\,\varepsilon^\eta\mathrm{I}_2 + \mathrm{W}_\omega\mathrm{O}(\varepsilon) + \mathcal{O} \tag{5.18}$$

with

$$\mathcal{O} = \mathrm{O}\left(\varepsilon^{\frac{3}{2}\eta} d^{3/2}, \sqrt{d}\varepsilon^{1+\frac{\eta}{2}}, \varepsilon^{m_\eta - \frac{\eta}{2}}/\sqrt{d}\right) = \mathrm{O}\left(d^{-\frac{1}{2}}(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{2-\frac{\eta}{2}})\right) \tag{5.19}$$

Note that we write $\mathrm{W}_\omega \mathrm{O}(q)$ for a quantity that is proportional to W_ω times a quantity q which is independent of ω. The main error term in $\mathcal{O}$ depends on the position of η relatively to 1. But we already need the condition $\eta < 2$ for this development (5.18) to be properly perturbative.

5.4.2 The diffeomorphism in the elliptic regime

In this elliptic regime we have in (5.9)

$$z = \mathrm{i}\sqrt{d}\varepsilon^{\eta/2} + \mathrm{W}_\omega \cos(\pi y) \mathrm{i}\, \mathrm{e}^{-\mathrm{i}\pi y} \frac{\varepsilon^{1-\eta/2}}{\sqrt{d}} - \tfrac{1}{2} d\varepsilon^\eta + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathcal{O}$$

where $\mathcal{O}$ is defined by (5.19). So we obtain

$$\begin{cases} \mathfrak{Im}\, z = \sqrt{d}\varepsilon^{\eta/2} + \mathrm{W}_\omega \cos^2(\pi y) \dfrac{\varepsilon^{1-\eta/2}}{\sqrt{d}} + \mathcal{O} \\ \mathfrak{Re}\, z = \mathrm{W}_\omega \dfrac{\sin(2\pi y)}{2} \dfrac{\varepsilon^{1-\eta/2}}{\sqrt{d}} - \tfrac{1}{2} d\varepsilon^\eta + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathcal{O} \end{cases}$$

Then $$\mathfrak{Im}(z)\, \mathfrak{Re}(z) = \tfrac{1}{2} {\mathrm{W}_\omega}^2 \cos^2(\pi y)\, \sin(2\pi y) \frac{\varepsilon^{2-\eta}}{d} + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \tfrac{1}{d}\mathcal{O}$$

Finally we get with formula (5.10)

Definition 5.1 (Random diffeomorphism, inner energies case)

In the case $\mathrm{E} = 2 - d\varepsilon^\eta$ *the random diffeomorphism is*

$$\mathrm{F} = \mathrm{F}_{\omega,\varepsilon} : \mathbf{T}^2 \ni (x, y) \longmapsto (x + \alpha, y + v_\omega(y))$$

with the following random part

$$\begin{aligned} v_\omega(y) = \frac{1}{\pi}\sqrt{d}\,\varepsilon^{\eta/2} - \frac{1}{2\pi} {\mathrm{W}_\omega}^2 \cos^2(\pi y)\, \sin(2\pi y) \frac{\varepsilon^{2-\eta}}{d} \\ + \frac{1}{\pi} \mathrm{W}_\omega \cos^2(\pi y) \frac{\varepsilon^{1-\eta/2}}{\sqrt{d}} + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathcal{O}_2 \end{aligned}$$

where $$\mathcal{O}_2 = \mathrm{O}\left(d^{-\frac{3}{2}}\left(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{3(1-\frac{\eta}{2})}\right)\right)$$

Notice that, for the averaged dynamics, the lowest order term is $\sqrt{d}\varepsilon^{\eta/2}$ which does not vanish when $\eta/2 < 2 - \eta$: that is the condition $\eta < \frac{4}{3}$ as in [SSB07]. This hypothesis more likely ensures an elliptic behaviour of our dynamics which is close to the (fixed) rotation $y \mapsto y + \frac{1}{\pi}\sqrt{d}\varepsilon^{\eta/2}$. For $\eta > \frac{4}{3}$, the diffeomorphism rather exhibits a hyperbolic behaviour with two fixed points. The case $\eta = \frac{4}{3}$ will be called the *parabolic* case; it is treated in the next section.

5.4.3 Transfer operator and cohomological equation for $\sigma = -1$

We compute

$$
\begin{aligned}
{v_\omega}^2 &= \frac{1}{\pi^2} d\varepsilon^\eta + \frac{1}{\pi^2}\cos^4(\pi\cdot)\frac{\varepsilon^{2-\eta}}{d}{\mathrm{W}_\omega}^2 + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathrm{W}_\omega \mathrm{O}(\varepsilon^{1+\eta/2}) \\
&\qquad + \begin{cases} \mathrm{O}(d^{-2}(\varepsilon^{2\eta}, \varepsilon^{2-\eta/2})) & \text{if } \eta \leqslant 1 \\ \mathrm{O}(d^{-2}\varepsilon^{2-\eta/2}) & \text{if } \eta \geqslant 1 \end{cases}
\end{aligned}
$$

$$
\overline{v}_2 = \frac{1}{2}\mathbf{E}({v_\omega}^2) = \frac{1}{2\pi^2} d\varepsilon^\eta + \frac{1}{2\pi^2}\cos^4(\pi\cdot) d^{-1}\varepsilon^{2-\eta}\,\mathbf{V}(\mathrm{W}) + \begin{cases} \mathrm{O}\left(d^{-2}(\varepsilon^{2\eta}, \varepsilon^{2-\frac{\eta}{2}})\right) & \text{if } \eta \leqslant 1 \\ \mathrm{O}(d^{-2}\varepsilon^{2-\frac{\eta}{2}}) & \text{if } \eta \geqslant 1 \end{cases} \tag{5.20}
$$

Let us state precise results for the equations (5.14).

Lemma 5.3

The equations given by the Fourier transform of the cohomological equation (5.14) can be solved as follows:

(i) *for $n \neq 0$, we can find $\varphi_n(\cdot)$ such that*

$$
\mathrm{T}_n\varphi_n = \psi_n + \varepsilon_n \qquad \textit{with} \qquad |\varepsilon_n| \lesssim \frac{\|\psi_n\|_4}{|\gamma_n|^2}\varepsilon^{\min\left(2-\frac{\eta}{2}, \frac{3}{2}\eta\right)}
$$

(ii) *for $n = 0$ we can solve*

$$
\psi_0 - \lambda_\psi = \mathrm{T}_0\varphi + \Delta_\psi
$$

with the following quantities:

$$
\sigma^2 = \mathbf{V}(\mathrm{W}_\omega) = \mathbf{E}({\mathrm{W}_\omega}^2) \tag{5.21}
$$

$$
\begin{aligned}
\lambda_\psi &= \widehat{\psi}_{0,0} = \int_{\mathbf{T}^2}\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
\Delta_\psi &= \frac{1}{\pi}{\psi_0}'\sqrt{d}\varepsilon^{\eta/2} + \frac{1}{2}\sigma^2\left(\frac{1}{\pi}\cos^4(\pi\cdot)\,{\psi_0}' - \cos^2(\pi\cdot)\,\sin(2\pi\cdot)(\psi_0 - \lambda_\psi)\right) d^{-\frac{3}{2}}\varepsilon^{2-\frac{3}{2}\eta} \\
&\qquad + \mathrm{O}\left(\|\psi_0\|_2\, d^{-3}(\varepsilon^\eta, \varepsilon^{3-2\eta})\right) \\
&= \mathrm{O}\left(\|\psi_0\|_2\, d^{-3}(\varepsilon^{\frac{\eta}{2}}, \varepsilon^{2-\frac{3}{2}\eta})\right)
\end{aligned} \tag{5.22}
$$

Proof. ________________

(i) Let us begin with $n \neq 0$. Then at its lowest order in ε the equation is $\psi_n = \gamma_n \varphi_n$. Let us set $\varphi_{n,0} = \frac{1}{\gamma_n}\psi_n$. We evaluate the error made with this approximation:

$$\mathrm{T}_n \varphi_{n,0} = \psi_n + \varepsilon_{n,1}$$

with
$$\begin{aligned}\varepsilon_{n,1} &= \alpha_n \, \mathbf{E}\left(v_\omega \int_0^1 {\varphi_{n,0}}'(\cdot + s v_\omega)\,\mathrm{d}s\right)\\ &= \frac{\alpha_n}{\gamma_n}\, \mathbf{E}\left(v_\omega \int_0^1 {\psi_n}'(\cdot + s v_\omega)\,\mathrm{d}s\right)\\ \varepsilon_{n,1} &= \mathrm{O}\left({\gamma_n}^{-1} \varepsilon^{\min(\frac{\eta}{2}, 2-\eta)}\right)\end{aligned}$$

Obviously we want to continue this procedure and write $-\varepsilon_{n,1} = \mathrm{T}_n \varphi_{n,1}$. Again, looking at the lowest terms in this equation, we set

$$\varphi_{n,1} = -\frac{\varepsilon_{n,1}}{\gamma_n} = -\frac{\alpha_n}{{\gamma_n}^2}\,\overline{v}_1\, {\psi_n}' - \frac{\alpha_n}{{\gamma_n}^2}\,\overline{v}_2\, {\psi_n}'' + \mathrm{O}\left({\gamma_n}^{-2}\,\overline{v}_3\right)$$

The error made is then

$$\mathrm{T}_n \varphi_{n,1} = -\varepsilon_{n,1} + \varepsilon_{n,2}$$

with
$$\begin{aligned}\varepsilon_{n,2} &= \alpha_n \, \mathbf{E}\left(v_\omega \int_0^1 {\varphi_{n,1}}'(\cdot + s v_\omega)\,\mathrm{d}s\right)\\ &= \alpha_n {\varphi_{n,1}}'\,\overline{v}_1 + \mathrm{O}\left({\varphi_{n,1}}''\,\overline{v}_2\right)\end{aligned}$$

and
$$\begin{aligned}{\varphi_{n,1}}' &= -\frac{\alpha_n}{{\gamma_n}^2}\, \mathbf{E}\left({v_\omega}' \int_0^1 {\psi_n}'(\cdot + s v_\omega)\,\mathrm{d}s + v_\omega^2 \int_0^1 {\psi_n}''(\cdot + s v_\omega)\,\mathrm{d}s\right)\\ &= -\frac{\alpha_n}{{\gamma_n}^2}\,{\psi_n}'\,\mathbf{E}\left({v_\omega}'\right) + \mathrm{O}\left(\left(\overline{v}_2 + \mathbf{E}\,|{v_\omega}' v_\omega|\right){\gamma_n}^{-2}\,\|{\psi_n}''\|_0\right)\\ &= \mathrm{O}\left({\gamma_n}^{-2}\, d^{-\frac{3}{2}}\|{\psi_n}''\|_0(\varepsilon^{2-\eta}, \varepsilon^{3\eta/2}, \varepsilon^{3(1-\eta/2)}, \varepsilon^{\eta})\right)\\ {\varphi_{n,1}}' &= \mathrm{O}\left({\gamma_n}^{-2}\, d^{-\frac{3}{2}}\|{\psi_n}''\|_0(\varepsilon^{2-\eta}, \varepsilon^{\eta})\right)\end{aligned}$$

Finally
$$\varepsilon_{n,2} = \mathrm{O}\left({\gamma_n}^{-2}\, d^{-\frac{3}{2}}\|{\psi_n}''\|_0(\varepsilon^{3\eta/2}, \varepsilon^{2-\eta/2}, \varepsilon^{4-2\eta})\right)$$

Now $\varphi_n := \varphi_{n,0} + \varphi_{n,1}$ solves

$$\mathrm{T}_n \varphi_n = \psi_n + \varepsilon_{n,2}$$

with an error $\varepsilon_{n,2}$ which is of order ${\varepsilon_{n,1}}^2$.

Remark 5.2

It is possible to go on with this procedure and solve $-\varepsilon_{n,2} = \mathrm{T}_n \varphi_{n,2}$ which would give a better approximation. What we lose of course is some derivatives, since at the k-th step we would get a factor ${\gamma_n}^{-k}$, which implies *some loss of the derivatives in order to control the convergence of the $\mathscr{C}^k$ norms of the solution. Anyway we limit ourselves to the second step, as it will be sufficient considering the obstructions that occur for the Fourier coefficient φ_0.*

(ii) We only solve the equation at its lowest order (provided $\eta > 4/3$)

$$\psi_0 = \left(\tfrac{1}{\pi}\sqrt{d}\varepsilon^{\eta/2}\right)\,{\varphi_0}'$$

This is an antiderivative equation but we are interested in one-periodic solution so that we need $\int_{\mathbf{T}} \psi_0 \,\mathrm{d}y = 0$. The obstruction is then $\lambda_\psi = \int_{\mathbf{T}} \psi_0 \,\mathrm{d}y = \int_{\mathbf{T}^2} \psi \,\mathrm{d}x\,\mathrm{d}y$ so that we rather solve

$$\frac{\pi}{\sqrt{d}\,\varepsilon^{\frac{\eta}{2}}}\,(\psi_0 - \lambda_\psi) = {\varphi_0}'$$

That gives

$$\varphi_0(y) = \frac{\pi}{\sqrt{d}\varepsilon^{\frac{\eta}{2}}}\int_0^y (\psi_0 - \lambda_\psi) \tag{5.23}$$

such that for any $k \geqslant 1$

$$\|\varphi_0\|_k \lesssim \frac{1}{\sqrt{d}\varepsilon^{\eta/2}}\|\psi_0\|_{k-1} \tag{5.24}$$

Then we evaluate the error made by computing

$$\begin{aligned}
\mathrm{T}_0\varphi_0 &= {\varphi_0}'\,\overline{v}_1 + {\varphi_0}''\,\overline{v}_2 + \mathbf{E}\left({v_\omega}^3\int_0^1 (1-s)\,{\varphi_0}'''(\cdot + s v_\omega)\,\mathrm{d}s\right)\\
&= \psi_0 - \lambda_\psi - \tfrac{1}{2\pi}\sigma^2\cos^2(\pi\cdot)\,\sin(2\pi\cdot)\,{\varphi_0}' d^{-1}\,\varepsilon^{2-\eta} + \mathrm{O}\left(\|{\varphi_0}'\|_0\, d^{-1}(\varepsilon^{3\eta/2}, \varepsilon^{3(1-\eta/2)})\right)\\
&\qquad + {\varphi_0}''\,\overline{v}_2 + \mathrm{O}(\|\varphi_0\|_3\,\mathbf{E}(|{v_\omega}^3|))
\end{aligned}$$

With the definitions of φ_0 and $\overline{v}_2$ (remind (5.23) and (5.16)) what precedes leads to

$$\begin{aligned}
\mathrm{T}_0\varphi_0 &= \psi_0 - \lambda_\psi - \sigma^2\cos^2(\pi\cdot)\,\sin(2\pi\cdot)(\psi - \lambda_\psi)d^{-\frac{3}{2}}\varepsilon^{2-3\eta/2} + \mathrm{O}\left(\|\psi_0\|_0\, d^{-\frac{3}{2}}(\varepsilon^{\eta}, \varepsilon^{3-2\eta})\right)\\
&\qquad + \pi{\psi_0}'\,\overline{v}_2\varepsilon^{-\frac{\eta}{2}}d^{-\frac{1}{2}} + \mathrm{O}\left(\|\psi_0\|_2\,\mathbf{E}(|{v_\omega}^3|)\varepsilon^{-\eta/2}\right)\\
&= \psi_0 - \lambda_\psi + \tfrac{1}{\pi}{\psi_0}'\sqrt{d}\varepsilon^{\eta/2} + \tfrac{1}{2}\sigma^2\left(\tfrac{1}{\pi}\cos^4(\pi\cdot)\,{\psi_0}' - \cos^2(\pi\cdot)\,\sin(2\pi\cdot)(\psi_0 - \lambda_\psi)\right)d^{-\frac{3}{2}}\,\varepsilon^{2-3\eta/2}\\
&\qquad + \mathrm{O}\left(\|\psi_0\|_2\, d^{-3}\,(\varepsilon^{\eta}, \varepsilon^{2-\eta}, \varepsilon^{3-2\eta})\right)
\end{aligned}$$

$$\mathrm{T}_0\varphi_0 = \psi_0 - \lambda_\psi + \Delta_\psi$$

where Δ_ψ is the quantity defined by (5.22) in the proposition.

■

Consequently we can solve the cohomological equation with obstructions (5.3) as follows:

Corollary 5.1

Given a $\psi \in \mathscr{C}^p(\mathbf{T}^2)$ with $p > \max(4, 2+2\tau)$, we can construct φ which satisfies

$$\psi = \lambda_\psi + \nu_\psi d^{-\frac{3}{2}}\varepsilon^{2-\frac{3}{2}\eta} + \mathbf{E}(\varphi \circ \mathrm{F}_\omega) - \varphi + \mathrm{O}\left(\|\psi\|_{0,\max(4,2\tau+2^+)} d^{-6}\varepsilon^{\min(\eta,4-3\eta)}\right)$$

where $$\nu_\psi = \sigma^2 \int_{\mathbf{T}^2} \psi(x,y)\, \cos^2(\pi y)\, \sin(2\pi y)\, \mathrm{d}x\, \mathrm{d}y$$

Proof.

Remind our notations $\|\varphi\|_{j,j'}$ introduced in chapter 4 with (4.16). We set $\varphi(x,y) = \sum_{n\in\mathbf{Z}} \varphi_n(y)\mathrm{e}^{2\mathrm{i}\pi n x}$. Then

$$\begin{aligned}
\mathrm{T}\varphi - \varphi &= \sum_{n\in\mathbf{Z}\setminus\{0\}} \psi_n e_n + \sum_{n\in\mathbf{Z}\setminus\{0\}} \varepsilon_n e_n + \psi_0 - \lambda_\psi - \Delta_\psi \\
&= \psi - \lambda_\psi + \sum_{n\in\mathbf{Z}\setminus\{0\}} \varepsilon_n e_n - \Delta_\psi
\end{aligned}$$

Thanks to the diophantine condition (2.3) one has

$$\begin{aligned}
\sum_{n\in\mathbf{Z}\setminus\{0\}} |\varepsilon_n| &\lesssim \frac{1}{|\kappa|}\|\psi\|_{0,k+1} \sum_{n\in\mathbf{Z}\setminus\{0\}} |n|^{\tau-k}\, d^{-3}\varepsilon^{\min(\frac{\eta}{2},2-\eta)} \\
&\lesssim \frac{1}{\kappa^2}\|\psi\|_{\tau+1^+}\, d^{-3}\varepsilon^{\min(\frac{\eta}{2},2-\eta)}
\end{aligned}$$

The estimate (5.22) on Δ_ψ ensures that φ verifies

$$\psi - \lambda_\psi = \mathrm{T}\varphi - \varphi + \mathrm{O}\left(\|\psi\|_{0,\max(2,\tau+1^+)}\, d^{-3}(\varepsilon^{\eta/2}, \varepsilon^{2-\frac{3}{2}\eta})\right)$$

We apply the same procedure to $\Delta = \Delta_\psi - \sum_{n\in\mathbf{Z}\setminus\{0\}} \varepsilon_n e_n$ and construct $\widetilde{\varphi}$ such that

$$\begin{aligned}
\Delta - \lambda_\Delta = \Delta - \lambda_{\Delta_\psi} &= \mathrm{T}\widetilde{\varphi} - \widetilde{\varphi} + \mathrm{O}\left(\|\Delta\|_{0,\max(2,\tau+1^+)} d^{-3}(\varepsilon^{\eta/2}, \varepsilon^{2-\frac{3}{2}\eta})\right) \\
&= \mathrm{T}\widetilde{\varphi} - \widetilde{\varphi} + \mathrm{O}\left(\|\psi\|_{0,2\max(2,\tau+1^+)} d^{-3}(\varepsilon^{\eta}, \varepsilon^{4-3\eta})\right)
\end{aligned}$$

Now $\phi := \varphi + \widetilde{\varphi}$ obviously satisfies

$$\psi - \lambda_\psi - \lambda_{\Delta_\psi} = \mathbf{E}(\phi \circ \mathrm{F}_\omega) - \phi + \mathrm{O}\left(\|\psi\|_{0,\max(4,2\tau+2^+)}\, d^{-6}(\varepsilon^{\eta}, \varepsilon^{4-3\eta})\right)$$

It remains to give an estimate of λ_{Δ_ψ}. With its expression, it is straightforward that

$$\begin{aligned}
\lambda_{\Delta_\psi} &= \int_{\mathbf{T}^2} \Delta_\psi \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbf{T}} \left(\frac{\sqrt{d}}{\pi} \psi_0{}' \varepsilon^{\eta/2} + \tfrac{1}{2}\sigma^2 \, \left(\tfrac{1}{\pi} \cos^4(\pi y) \, \psi_0{}'(y) - \cos^2(\pi y) \sin(2\pi y)(\psi_0(y) - \lambda_\psi) \right) d^{-\frac{3}{2}} \varepsilon^{2-3\eta/2} \right) \mathrm{d}y \\
&\qquad + \mathrm{O}\left(\|\psi_0\|_2 \, d^{-3} (\varepsilon^\eta, \varepsilon^{2-\eta}, \varepsilon^{3-2\eta}) \right) \\
&= \frac{1}{2}\sigma^2 \left(\int_{\mathbf{T}} \frac{1}{\pi} \cos^4(\pi y) \, \psi_0{}'(y) \, \mathrm{d}y - \int_{\mathbf{T}} \cos^2(\pi y) \, \sin(2\pi y) \, \psi_0(y) \, \mathrm{d}y \right) d^{-\frac{3}{2}} \varepsilon^{2-\frac{3}{2}\eta} \\
&\qquad + \mathrm{O}\left(\|\psi_0\|_2 \, d^{-3} (\varepsilon^\eta, \varepsilon^{2-\eta}, \varepsilon^{3-2\eta}) \right) \\
\lambda_{\Delta_\psi} &= \left(\int_{\mathbf{T}} \cos^2(\pi y) \, \sin(2\pi y) \, \psi_0(y) \, \mathrm{d}y \right) \frac{\sigma^2}{2d^{3/2}} \varepsilon^{2-\frac{3}{2}\eta} + \mathrm{O}\left(\|\psi_0\|_2 \, d^{-3} (\varepsilon^\eta, \varepsilon^{3-2\eta}) \right)
\end{aligned}$$

This ends the proof. ■

5.4.4 Stationary measure and LE in the elliptic regime

We are now in a position to state the result for any stationary measure of our random diffeomorphism $\mathrm{F}_\omega : (x, y) \longmapsto (x + \alpha, y + v_\omega(y))$ (remind definition 5.1).

Proposition 5.2

In the elliptic regime $\eta < \frac{4}{3}$, any stationary measure μ_ε for F_ω satisfies for any ψ differentiable enough

$$\int_{\mathbf{T}^2} \psi \, \mathrm{d}\mu = \int_{\mathbf{T}^2} \psi \, \mathrm{d}x \, \mathrm{d}y + \left(\int_{\mathbf{T}^2} \psi'_y(x, y) \, p(y) \, \mathrm{d}x \, \mathrm{d}y \right) \frac{\sigma^2 \varepsilon^{2-\frac{3}{2}\eta}}{d^{3/2}} + \mathcal{O}$$

with
$$\begin{cases} p(y) = \cos^2(\pi y) \, \sin(2\pi y) = \frac{1}{2} \sin(2\pi y) + \frac{1}{4} \sin(4\pi y) \\ \mathcal{O} = \mathrm{O}\left(\|\psi\|_{0,\max(4, 2\tau+2^+)} \, d^{-6} (\varepsilon^\eta, \varepsilon^{4-3\eta}) \right) \end{cases}$$

Note that this is a proper asymptotic development to the first order if $2 - \frac{3}{2}\eta < \eta$ which is the condition $\eta > \frac{4}{5}$. Otherwise we only have the 0-th order which is the Lebesgue measure of course.

Proof.

Let us prove theorem 5.1. We use the formula $\mathrm{L(E)} = \frac{1}{2} |\gamma(\mathrm{F})|$ established in chapter 4 (see (4.9)). To use it we need to compute

$$\begin{aligned}
v_\omega{}' = -\mathrm{W}_\omega \sin(2\pi \cdot) \frac{\varepsilon^{1-\eta/2}}{\sqrt{d}} + \mathrm{W}_\omega \mathrm{O}(\varepsilon) - \tfrac{1}{2} \mathrm{W}_\omega{}^2 \left(\cos(2\pi \cdot) + \cos(4\pi \cdot) \right) \frac{\varepsilon^{2-\eta}}{d} \\
+ \mathrm{O}\left(d^{-\frac{3}{2}} (\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{3(1-\frac{\eta}{2})}) \right)
\end{aligned}$$

and also

$$
\begin{aligned}
(v_\omega{}')^2 &= \mathrm{W}_\omega{}^2 \sin^2(2\pi\cdot)\frac{\varepsilon^{2-\eta}}{d} + \mathrm{O}\left(d^{-\frac{9}{2}}(\varepsilon^{2-\frac{\eta}{2}}, \varepsilon^{1+\eta}, \varepsilon^{4-2\eta})\right) \\
|v_\omega{}'|^3 &= \mathrm{O}\left(d^{-\frac{9}{2}}(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{3(1-\frac{\eta}{2})}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbf{E}(v_\omega{}') &= -\frac{1}{2}\sigma^2\left(\cos(2\pi\cdot) + \cos(4\pi\cdot)\right)\frac{\varepsilon^{2-\eta}}{d} + \mathrm{O}\left(d^{-\frac{3}{2}}(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{3(1-\frac{\eta}{2})})\right) \\
\mathbf{E}\left[(v_\omega{}')^2\right] &= \sigma^2 \sin^2(2\pi\cdot)\frac{\varepsilon^{2-\eta}}{d} + \mathrm{O}\left(d^{-\frac{9}{2}}(\varepsilon^{2-\frac{\eta}{2}}, \varepsilon^{1+\eta}, \varepsilon^{4-2\eta})\right)
\end{aligned}
$$

Consequently, with proposition (5.2)

$$
\begin{aligned}
\int_{\mathbf{T}^2} \mathbf{E}(v_\omega{}')\,\mathrm{d}\mu &= \mathrm{O}\left(d^{-6-\frac{3}{2}}(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{4-\frac{5}{2}\eta})\right) \\
\int_{\mathbf{T}^2} \mathbf{E}\left[(v_\omega{}')^2)\right]\,\mathrm{d}\mu &= \sigma^2\frac{\varepsilon^{2-\eta}}{2d} + \mathrm{O}\left(d^{-\frac{9}{2}}(\varepsilon^{2-\frac{\eta}{2}}, \varepsilon^{1+\eta}, \varepsilon^{4-2\eta})\right) \\
\int_{\mathbf{T}^2} \mathrm{O}\left(|v_\omega{}'|^3\right)\,\mathrm{d}\mu &= \mathrm{O}\left(d^{-\frac{9}{2}}(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{3(1-\frac{\eta}{2})}\right)
\end{aligned}
$$

Now we can compute with definition 4.3 and proposition (5.2),

$$
\begin{aligned}
\gamma(\mathrm{F}) &= \int_{\mathbf{T}^2} \mathbf{E}\ln|1 + v_\omega{}'|\,\mathrm{d}\mu \\
&= \int_{\mathbf{T}^2} \mathbf{E}\left(v_\omega{}' - \tfrac{1}{2}v_\omega{}'^2 + \mathrm{O}\left(|v_\omega{}'|^3\right)\right)\,\mathrm{d}\mu \\
\gamma(\mathrm{F}) &= -\frac{\sigma^2}{4d}\varepsilon^{2-\eta} + \mathrm{O}\left(d^{-\frac{9}{2}}(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{4-\frac{5}{2}\eta})\right)
\end{aligned}
$$

Finally

$$
-\frac{1}{2}\gamma(\mathrm{F}) = \frac{\sigma^2}{8d}\varepsilon^{2-\eta} + \mathrm{O}\left(d^{-\frac{9}{2}}(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{4-\frac{5}{2}\eta})\right)
$$

Therefore can conclude for the announced formula of the Lyapunov exponent in theorem 5.1 with (4.9). This formula gives a proper development as long as $2 - \eta < \frac{3}{2}\eta$ that is $\eta > \frac{4}{5}$ and we find back the condition on η that arises in [SSB07]. Note that the term coming from the conjugacy of order ε^{m_η} is included in the error term we have written (remind (5.4) and section 5.3).

■

5.5 The parabolic case $\mathrm{E} = 2 - d\varepsilon^{4/3}$

We will prove the following:

Theorem 5.2 (LE, parabolic case)

The Lyapunov exponent admits the asymptotic expansion below:

$$\mathrm{L}(2 - d\varepsilon^{\frac{4}{3}}) = \frac{\sigma^2\varepsilon^{2/3}}{4d}\left|\int_{\mathbf{T}}\big(\cos(2\pi y) + \cos^2(2\pi y)\big)\rho(y)\,\mathrm{d}y\right| + \mathrm{O}(d^{-\frac{9}{2}}\varepsilon)$$

where ρ is some density defined by the L^1 normalization solution of an explicit first-order differential equation.

5.5.1 The diffeomorphism in the parabolic regime

All the computations we need have been made in the previous section (see definition (5.1)) to conclude that, when $\eta = \frac{4}{3}$, *i.e.* $\frac{\eta}{2} = 2 - \eta$:

Definition 5.2 (Random diffeomorphism for $\mathrm{E} = 2 - d\varepsilon^{4/3}$)

For an energy $\mathrm{E} = 2 - d\varepsilon^{4/3}$ the random diffeomorphism is

$$\mathrm{F} = \mathrm{F}_{\omega,\varepsilon} : \mathbf{T}^2 \ni (x,y) \longmapsto (x+\alpha, y + v_\omega(y))$$

with the following random part:

$$v_\omega(y) = \tfrac{1}{\pi}\mathrm{W}_\omega\cos^2(\pi y)\frac{\varepsilon^{1/3}}{\sqrt{d}} + \left(\tfrac{1}{\pi}d^{\frac{3}{2}} - \tfrac{1}{2\pi}{\mathrm{W}_\omega}^2\cos^2(\pi y)\,\sin(2\pi y)\right)\frac{\varepsilon^{2/3}}{d} + \mathrm{O}\left(d^{-\frac{3}{2}}\varepsilon\right)$$

5.5.2 Transfer operator and cohomological equation in the parabolic case

Let us state what we obtain in the parabolic case.

Proposition 5.3 (Parabolical cohomological equation)

For a given $\psi \in \mathscr{C}^{3+\tau^+}(\mathbf{T}^2)$, we are able to solve the approximated cohomological equation

$$\psi - \lambda_\psi = \mathrm{T}\varphi - \varphi + \mathrm{O}\left(d^{-\frac{7}{2}}\|\psi\|_{0,\max(2,\tau+1^+)}\varepsilon^{1/3}\right)$$

where the obstruction λ_ψ is defined by

$$\lambda_\psi = \int_{\mathbf{T}^2}\psi(x,y)\,\rho(y)\,\mathrm{d}x\,\mathrm{d}y$$

with ρ a density which is the L^1-normalized solution of an explicit first-order differential equation $p\,f' + (p' - q)\,f = 1$.

Here are some quantities we need in the sequel:

$$\begin{cases} q = \dfrac{d^{3/2}}{\pi} - \dfrac{\sigma^2}{2\pi}\cos^2(\pi\cdot)\,\sin(2\pi\cdot) \\ p = \dfrac{\sigma^2}{2\pi^2}\cos^4(\pi\cdot) \end{cases} \tag{5.25}$$

So we have

$$\begin{aligned} \mathbf{E}\left(v_\omega\right) &= q\frac{\varepsilon^{2/3}}{d} + \mathrm{O}\left(d^{-3/2}\varepsilon\right) \\ {v_\omega}^2 &= \tfrac{1}{\pi^2}{\mathrm{W}_\omega}^2\cos^4(\pi\cdot)\frac{\varepsilon^{2/3}}{d} + \mathrm{O}\left(d^{-3}\varepsilon\right) \\ \overline{v}_2 &= \tfrac{1}{2}\,\mathbf{E}\left({v_\omega}^2\right) = p\frac{\varepsilon^{2/3}}{d} + \mathrm{O}\left(d^{-3}\varepsilon\right) \end{aligned}$$

This is what we obtain for the operators T_n (equations (5.15)):

Lemma 5.4

- *for* $n = 0$ *we can find* φ_0 *such that*

$$\mathrm{T}_0\varphi_0 - \varphi_0 = \psi_0 - \lambda_\psi + \mathrm{O}\left(\varepsilon^{1/3}d^{-\frac{7}{2}}\|\psi_0\|_2\right)$$

with
$$\lambda_\psi = \int_{\mathbf{T}} \psi_0(y)\,\rho(y)\,\mathrm{d}y$$

where ρ *is the* L^1*-normalized solution of the one-periodic solution of the differential equation*

$$p\,f' + (p' - q)\,f = 1$$

- *for* $n \neq 0$ *the quantity* $\varphi_n = \frac{1}{\gamma_n}\psi_n$ *verifies*

$$\mathrm{T}_n\varphi_n = \psi_n + \mathrm{O}\left(\tfrac{1}{|\gamma_n|}\|\psi_n\|_1 d^{-\frac{3}{2}}\varepsilon^{1/3}\right)$$

Proof.

The equation for T_0 truncated at its lowest order gives the following differential equation

$$\psi_0 = \left(q\,{\varphi_0}' + p\,{\varphi_0}''\right)\frac{\varepsilon^{2/3}}{d}$$

Note that p admits $\frac{1}{2}$ as its only zero on $[\,0\,;1\,]$ which is of order 4, and in the meantime $q(\frac{1}{2}) > 0$. We also compute

$$\left(\frac{p}{q}\right)^{(4)}\left(\tfrac{1}{2}\right) = \pi d^{-1/2}\frac{\sigma^2}{48d\pi^2}\frac{\mathrm{d}^4}{\mathrm{d}y^4}\cos^4(\pi\cdot)_{y=\frac{1}{2}} = \tfrac{1}{2}d^{-3/2}\sigma^2\pi^3 > 0$$

Hence we are in situation (iv) of Proposition 3 in [SSB07]. That means that the differential equation

$$p\,\Phi' + q\,\Phi = \Psi \tag{5.26}$$

admits $\mathscr{C}^\infty$ solutions on $[\,0\,;1\,]$. These are unique for $y > \frac{1}{2}$ and have one free parameter for $y < \frac{1}{2}$. They are given by Φ_- on $\left[\,0\,;\frac{1}{2}\,\right[$ and Φ_+ on $\left]\,\frac{1}{2}\,;1\,\right]$ where

$$\begin{cases} \Phi_-(y) \;=\; \left(\Phi(0) + \displaystyle\int_0^y \frac{\Psi}{p}\,\mathrm{e}^{\,w_-}\right)\,\mathrm{e}^{\,-w_-(y)} \\[2ex] \Phi_+(y) \;=\; \mathrm{e}^{\,-w_+(y)}\displaystyle\int_{\frac{1}{2}}^y \frac{\Psi}{p}\,\mathrm{e}^{\,w_+} \\[2ex] w_\pm(s) \;=\; \displaystyle\int_{\frac{1\mp 1}{2}}^s \frac{q}{p} \end{cases}$$

We are of course looking for one-periodic solutions with zero average on $\mathbf{T}$ since the solution should be the one-periodic antiderivative of a solution of a first-order differential equation. The periodicity condition imposes the value of the free parameter with $\Phi(0) = \Phi_+(1)$. For the zero average condition we need to say a few more words:

Lemma 5.5

Let p, q and f be one-periodic smooth functions such that p, q satisfy the conditions of Proposition 3 (iv) in [SSB07]. Then the equation

$$p(y)\,u' + q(y)\,u = f$$

admits a 1-periodic solution with zero average on $\mathbf{T}$ iff

$$\int_{\mathbf{T}} f(y)\,\theta(y)\,\mathrm{d}y = 0$$

where $\theta = \theta_{p,q}$ is the 1-periodic solution of

$$p\,\theta' + (p' - q)\,\theta = 1$$

Moreover, the solution $\theta_{p,q}$ does not vanish on $\mathbf{T}$.

Proof.

The existence and uniqueness of one-periodic solutions of the two differential equations involved directly comes from the item (iv) of Proposition 3 in [SSB07] mentioned above. Concerning the criterion of zero average let us define the differential operator of order 2

$$\mathrm{A}\varphi = p\,\varphi'' + q\,\varphi'$$

We can compute its adjoint A^* on $\mathrm{L}^2(\mathbf{T})$

$$\int_{\mathbf{T}} (p(y)\,\varphi''(y) + q(y)\,\varphi'(y))\,\psi(y)\,\mathrm{d}y = \int_{\mathbf{T}} \varphi(y)\,((p\psi)''(y) - (q\,\psi)'(y))\,\mathrm{d}y$$

so that $$\mathrm{A}^* = \partial^2(p\cdot) - \partial(q\,\cdot)$$

It is a well know fact that $(\mathrm{Im\ A})^{\perp} = \mathrm{Ker}\,(\mathrm{A}^*)$ and obviously

$$\mathrm{A}^*\psi = 0 \quad \Longleftrightarrow \quad p\,\psi' + (p' - q)\,\psi = \mathrm{c}^{\mathrm{st}}$$

This proves that Ker A^* is spanned by θ the 1-periodic solution of $p\,\theta' + (p' - q)\,\theta = 1$. Now we write for φ a solution of $p\,u' + q\,u = f$

$$\int_{\mathbf{T}} \varphi = \int_{\mathbf{T}} \varphi\,(p\,\theta' + (p' - q)\,\theta) = -\int_{\mathbf{T}} \theta\;(p\,\varphi' + q\,\varphi) = -\int_{\mathbf{T}} \theta\, f$$

which gives the equivalence stated in the lemma.

The non-vanishing condition is a consequence of the resolvent formula, once one takes in account the singularities and chooses properly the relevant antiderivatives for $y > \frac{1}{2}$ or $y < \frac{1}{2}$, and because in this case $p(y) = \cos^4(\pi y) \geqslant 0$ and $\Psi = 1 > 0$. ■

Hence, in order to get a solution with zero average we rather solve

$$\psi_0 - \lambda_\psi = (q\,\varphi_0{}' + p\,\varphi_0{}'')\,\frac{\varepsilon^{2/3}}{d}$$

and so we must have

$$\int_{\mathbf{T}} (\psi - \lambda_\psi)\;\theta\,\mathrm{d}y = 0$$

This imposes the obstruction as follows:

$$\lambda_\psi = \frac{\displaystyle\int_{\mathbf{T}} \psi_0(y)\,\theta(y)\,\mathrm{d}y}{\displaystyle\int_{\mathbf{T}} \theta(y)\,\mathrm{d}y} =: \int_{\mathbf{T}} \psi_0(y)\,\rho(y)\,\mathrm{d}y \tag{5.27}$$

where ρ is the density defined by:

$$\rho = \theta/\widehat{\theta}_0$$

Note that θ does not depend on ε so that, with respect to ε we have $\lambda_\psi = \mathrm{O}\,(\|\psi_0\|_0)$.

Let us compute the error made with φ_0 constructed this way:

$$\begin{aligned} \mathrm{T}_0\varphi_0 &= \psi_0 - \lambda_\psi + \mathrm{O}\left(\varphi_0{}'\,d^{-3/2}\varepsilon\right) + \mathrm{O}\left(\varphi_0{}''\,d^3\varepsilon\right) + \mathrm{O}\left(\varphi_0{}'''\,d^3\varepsilon\right) \\ &= \psi_0 - \lambda_\psi + \mathrm{O}\left(\|\psi_0\|_2\,d^{-\frac{7}{2}}\varepsilon^{1/3}\right) \end{aligned}$$

For the other coefficients φ_n with $n \neq 0$ we use the first step of the iterative method which gives:

$$\varphi_n = \frac{\psi_n}{\gamma_n}$$

That is enough to ensure

$$\begin{aligned} \mathrm{T}_n\varphi_n &= \psi_n + \mathrm{O}\left(\|\varphi_n\|\,\mathbf{E}\,|v_\omega|\right) \\ &= \psi_n + \mathrm{O}\left(\tfrac{1}{|\gamma_n|}\|\psi_n\|_1 d^{-\frac{3}{2}}\varepsilon^{1/3}\right) \end{aligned}$$

■

Now proposition 5.3 can be proved. Indeed, the map $\varphi(x,y) = \sum_{n=-\infty}^{+\infty} \varphi_n(y) \mathrm{e}^{2i\pi n x}$ meets all the criteria as it has been constructed for this purpose and we estimated all the error terms required.

5.5.3 Formula for the exponent in the parabolic regime

In order to prove theorem 5.2 we need to compute with proposition 5.5

$$\begin{aligned}
\mathbf{E}\left(v_\omega{}'\right) &= \mathrm{W}_\omega \mathrm{O}\left(\varepsilon^{1/3}\right) - \tfrac{1}{2}\mathrm{W}_\omega{}^2\left(\cos(\pi\cdot) + \cos(4\pi\cdot)\right) d^{-1}\varepsilon^{2/3} + \mathrm{O}(d^{-\frac{3}{2}}\varepsilon) \\
\int_{\mathbf{T}^2} \mathbf{E}\left(v_\omega{}'\right) \mathrm{d}\mu_\varepsilon &= -\tfrac{1}{2}\sigma^2\left(\cos(\pi\cdot) + \cos(4\pi\cdot)\right) d^{-1}\varepsilon^{2/3} + \mathrm{O}(d^{-\frac{3}{2}}\varepsilon) \\
\mathbf{E}\left(v_\omega{}'^2\right) &= \mathrm{W}_\omega{}^2 \sin^2(2\pi\cdot) d^{-1}\varepsilon^{2/3} + \mathrm{O}(d^{-\frac{9}{2}}\varepsilon) \\
\int_{\mathbf{T}^2} \mathbf{E}\left(v_\omega{}'^2\right) \mathrm{d}\mu_\varepsilon &= \sigma^2 \sin^2(2\pi\cdot) d^{-1}\varepsilon^{2/3} + \mathrm{O}(d^{-\frac{9}{2}}\varepsilon)
\end{aligned}$$

So that with proposition 5.5

$$\begin{aligned}
\gamma(\mathrm{F}) &= \int_{\mathbf{T}^2} \mathbf{E} \ln\left|1 + v_\omega{}'\right| \mathrm{d}\mu_\varepsilon \\
&= \left(-\tfrac{1}{2}\sigma^2 \int_{\mathbf{T}} \left(\cos(2\pi y) + \cos(4\pi y) + \sin^2(2\pi y)\right) \rho(y)\,\mathrm{d}y\right) d^{-1}\varepsilon^{\frac{2}{3}} + \mathrm{O}\left(d^{-\frac{9}{2}}\varepsilon\right) \\
\gamma(\mathrm{F}) &= \left(-\frac{1}{2d}\sigma^2 \int_{\mathbf{T}} \left(\cos(2\pi y) + \cos^2(2\pi y)\right) \rho(y)\,\mathrm{d}y\right) \varepsilon^{\frac{2}{3}} + \mathrm{O}\left(d^{-\frac{9}{2}}\varepsilon\right)
\end{aligned}$$

And with $\mathrm{L}(\mathrm{E}) = \frac{1}{2}\left|\gamma(\mathrm{F})\right|$ (remind (4.9)) we obtain

$$\mathrm{L}(2 - d\varepsilon^{4/3}) = \frac{\sigma^2\varepsilon^{2/3}}{4d} \left|\int_{\mathbf{T}} \left(\cos(2\pi y) + \cos^2(2\pi y)\right) \rho(y)\,\mathrm{d}y\right| + \mathrm{O}\left(d^{-\frac{9}{2}}\varepsilon\right) \tag{5.28}$$

As before, in this regime the term coming from the conjugacy of order $\varepsilon^{m_\eta} = \varepsilon^2$ can be included in the error term we have written (remind (5.4) and section 5.3).

Remark 5.3

Note that the non-cancellation of the given integral in (5.28) is unclear, like in [SSB07].

5.6 The case $\sigma = 1$: energies outside the free spectrum

The main theorem is the following:

Theorem 5.3 (LE outside the free spectrum)

For $\frac{4}{7} < \eta < \frac{4}{3}$, the Lyapunov exponent admits the following asymptotics for ε small enough:

$$\mathrm{L}(2 + d\varepsilon^{\eta}) = \sqrt{d}\varepsilon^{\eta/2} + \mathrm{O}\left(d^{-3}\left(\varepsilon^{1-\frac{\eta}{4}}, \varepsilon^{\frac{9}{4}\eta-1}, \varepsilon^{\frac{4}{3}\eta}\right)\right)$$

5.6.1 Conjugacy close to an hyperbolic cocycle

When the energy is $\mathrm{E} = 2 + d\varepsilon^{\eta}$, we are facing a QPR perturbation of the following hyperbolic matrix of $\mathrm{SL}_2(\mathbf{R})$:

$$\mathrm{A}_{\varepsilon} := \begin{pmatrix} 2 + d\varepsilon^{\eta} & -1 \\ 1 & 0 \end{pmatrix}$$

It has eigenvalues λ, λ^{-1} which are the roots of $\mathrm{X}^2 - (2 + d\varepsilon^{\eta})\mathrm{X} + 1 = 0$ with $\lambda = \lambda_{\varepsilon} \in \mathbf{R}_+$ and so $\mathrm{Q} := \left(\begin{smallmatrix} \frac{1}{\lambda} & \lambda \\ 1 & 1 \end{smallmatrix}\right)$ diagonalizes A_{ε} to $\Lambda = \mathrm{Diag}\,(\lambda^{-1}, \lambda)$. We also compute

$$\widetilde{\mathrm{Q}} := \mathrm{Q}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathrm{Q} = \frac{1}{\lambda^2 - 1} \begin{pmatrix} -1 & -\lambda^2 \\ 1 & \lambda^2 \end{pmatrix} \tag{5.29}$$

From the quadratic equation we get $\lambda = 1 + u_{\varepsilon}$ where

$$\begin{aligned} u_{\varepsilon} &= \tfrac{1}{2} d\varepsilon^{\eta} + \tfrac{1}{2}\sqrt{4d\varepsilon^{\eta} + d^2\varepsilon^{2\eta}} \\ &= \sqrt{d}\varepsilon^{\frac{\eta}{2}} + \tfrac{1}{2} d\varepsilon^{\eta} + \mathrm{O}(d^{\frac{3}{2}}\varepsilon^{\frac{3}{2}\eta}) \end{aligned}$$

So that the other eigenvalue is

$$\begin{aligned} \lambda^{-1} &= \tfrac{1}{1+u_{\varepsilon}} = 1 - u_{\varepsilon} + \tfrac{1}{2}{u_{\varepsilon}}^2 + \mathrm{O}({u_{\varepsilon}}^3) \\ &= 1 - \sqrt{d}\varepsilon^{\frac{\eta}{2}} + \tfrac{1}{2} d\varepsilon^{\eta} + \mathrm{O}(d^{\frac{3}{2}}\varepsilon^{\frac{3}{2}}) \end{aligned}$$

This gives the development

$$\Lambda = \mathrm{Diag}\,(\lambda^{-1}, \lambda) = \mathrm{I}_2 + \sqrt{d}\varepsilon^{\frac{\eta}{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \tfrac{1}{2} d\varepsilon^{\eta}\mathrm{I}_2 + \mathrm{O}(d^{\frac{3}{2}}\varepsilon^{\frac{3}{2}\eta})$$

and in (5.29) we have

$$\begin{aligned} \widetilde{\mathrm{Q}} &= \frac{1}{2u_{\varepsilon} + {u_{\varepsilon}}^2} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2u_{\varepsilon} + {u_{\varepsilon}}^2} \begin{pmatrix} -(2u_{\varepsilon} + {u_{\varepsilon}}^2) & 0 \\ 2u_{\varepsilon} + {u_{\varepsilon}}^2 & 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{d}\varepsilon^{\eta/2}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \mathrm{O}\left(d^{-\frac{1}{2}}\varepsilon^{\eta/2}\right) \end{aligned}$$

Therefore

$$\mathrm{C}(x+\alpha)\,\mathrm{S}_{\omega,\varepsilon,x}\,\mathrm{C}(x)^{-1} = \Lambda + \varepsilon \mathrm{W}_\omega \widetilde{\mathrm{Q}} + \mathrm{O}\left(\frac{1}{\sqrt{d}}\varepsilon^{m_\eta - \frac{\eta}{2}}\right) \tag{5.30}$$
$$= \mathrm{I}_2 + \mathrm{N}_{\omega,\varepsilon}$$

where $\mathrm{C}(x) = \mathrm{Q}^{-1}\mathrm{B}(x)$ and

$$\mathrm{N}_{\omega,\varepsilon} = \sqrt{d}\,\varepsilon^{\eta/2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} + \frac{\varepsilon^{1-\frac{\eta}{2}}}{2\sqrt{d}}\mathrm{W}_\omega\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix} + \tfrac{1}{2}d\varepsilon^\eta \mathrm{I}_2 + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathcal{O}$$

$$\mathcal{O} = \mathrm{O}\left(d^{-1/2}\left(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{2-\frac{\eta}{2}}\right)\right) \tag{5.31}$$

Let us now compute the random diffeomorphism. We have

$$\begin{aligned} z &= \mathrm{N}_{\omega,\varepsilon}(\mathrm{e}^{\mathrm{i}\pi y}).\mathrm{e}^{-\mathrm{i}\pi y} \\ &= -\mathrm{e}^{-2\mathrm{i}\pi y}\sqrt{d}\varepsilon^{\frac{\eta}{2}} + \tfrac{d}{2}\varepsilon^\eta + \tfrac{1}{2}\mathrm{W}_\omega\left(\cos(\pi y) + \sin(\pi y)\right)(-1+\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\pi y}\,\frac{\varepsilon^{1-\frac{\eta}{2}}}{\sqrt{d}} \\ &\qquad + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathrm{O}\left(d^{-1/2}\left(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{2-\frac{\eta}{2}}\right)\right) \end{aligned}$$

So equation (5.31) gives

$$\mathfrak{Im}\, z = \sin(2\pi y)\sqrt{d}\varepsilon^{\frac{\eta}{2}} + \tfrac{\mathrm{W}_\omega}{2}\left(\cos(\pi y) + \sin(\pi y)\right)^2 \frac{\varepsilon^{1-\frac{\eta}{2}}}{\sqrt{d}} + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathcal{O}$$

$$\mathfrak{Re}\, z = -\cos(2\pi y)\sqrt{d}\varepsilon^{\frac{\eta}{2}} + \tfrac{d}{2}\varepsilon^\eta - \tfrac{\mathrm{W}_\omega}{2}\left(\cos^2(\pi y) - \sin^2(\pi y)\right)\frac{\varepsilon^{1-\frac{\eta}{2}}}{\sqrt{d}} + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathcal{O}$$

$$\mathfrak{Im}\, z\ \mathfrak{Re}\, z = -\sin(4\pi y)\tfrac{d}{2}\varepsilon^\eta - \tfrac{\mathrm{W}_\omega^2}{4}\left(1 + \sin(2\pi y)\right)\frac{\varepsilon^{2-\eta}}{d} + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathrm{O}\left(d^{-1}\left(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{2-\frac{\eta}{2}}\right)\right)$$

Finally, thanks to (5.10), outside the free spectrum the diffeomorphism to study is $(x+\alpha, y + v_\omega(y))$ where

$$\begin{aligned} v_\omega &= \frac{\sin(2\pi\cdot)}{\pi}\sqrt{d}\varepsilon^{\eta/2} + \frac{\sin(4\pi\cdot)}{2\pi}d\varepsilon^\eta + {\mathrm{W}_\omega}^2\frac{(1+\sin(2\pi\cdot))\,\cos(2\pi\cdot)}{4\pi}\frac{\varepsilon^{2-\eta}}{d} \\ &\qquad + \mathrm{W}_\omega \frac{1+\sin(2\pi\cdot)}{2\pi}\frac{\varepsilon^{1-\eta/2}}{\sqrt{d}} + \mathrm{W}_\omega \mathrm{O}(\varepsilon) + \mathrm{O}\left(d^{-1}\left(\varepsilon^{\frac{3}{2}\eta}, \varepsilon^{2-\eta/2}\right)\right) \end{aligned}$$

Please notice that, whether $\eta_1 = \frac{\eta}{2}$ is greater or smaller than $\eta_2 = 2 - \eta$ gives a different dynamical behaviour for the averaged diffeomorphism. Indeed, when $\eta_1 < \eta_2$, that is $\eta < \frac{4}{3}$, the development to the first order of f_ω is

$$y + \frac{\sin(2\pi y)}{\pi}\sqrt{d}\varepsilon^{\frac{\eta}{2}}$$

whose fixed points are: 0 that is repulsive and $\frac{1}{2}$ that is attractive. This indicates a hyperbolic dynamics and so any stationary measure should rather be close to a convex combination of the Dirac peaks at 0 and $\frac{1}{2}$. Actually we will show that only the Dirac peak at $\frac{1}{2}$ contributes to the stationary measure.

Now if $\eta_1 > \eta_2$ that is $\eta > \frac{4}{3}$ then the first-order of f_ω is

$$y \longmapsto y + \sigma^2 \frac{(1+\sin(2\pi y))\,\cos(2\pi y)}{4\pi} \frac{\varepsilon^{2-\frac{\eta}{2}}}{d}$$

whose fixed points are: $\frac{1}{4}$ which is simple and attractive, whereas $\frac{3}{4}$ is of order 3 and neither attractive or repulsive. In this situation it is unclear if a hyperbolical behavior still could ensure Dirac peaks, and what combination of those two peaks it would give. So we will call *hyperbolic* the situation when $\eta < \frac{4}{3}$. Let us make an additional basis change by

$$\mathrm{P}_{\varepsilon,\delta} = \begin{pmatrix} \varepsilon^\delta & 0 \\ 0 & 1 \end{pmatrix} \tag{5.32}$$

with $\delta = 1 - \frac{3}{4}$ is positive in this setting. This is the same conjugacy made by [SSB07] to obtain what they call a *second-order anomaly*. The reason for this change of basis is that it seems more doable to deal with a second-order anomaly than with the hyperbolic first-order one. Their analysis of first-order anomalies basically works well for elliptic anomalies.

From our point of view, the technique used before to solve the cohomological equation to its lowest order does not work well anymore since the function p multiplied by φ' vanishes at $\frac{1}{2}$, so that taking an antiderivative is no longer easily doable. One should first substract functions that cancel the singularity of ψ/p at the zeros of p. It seems less technical to solve the equation up to the second order of perturbation which makes a *pseudo* second-order differential equation arise. By this, we mean that we need to solve $p\varphi' + q\varphi'' = \psi$ which requires in fact to solve a first-order differential equation and then take an anti-derivative of its solution, with the properties of periodicity and vanishing integral.

With an additional conjugacy in (5.30) by $\mathrm{P}_{\varepsilon,\delta}$ (defined by (5.32)), we obtain the transfer matrix $\mathrm{I} + \widetilde{\mathrm{N}}$ where

$$\widetilde{\mathrm{N}} = \frac{\mathrm{W}_\omega}{2} \frac{\varepsilon^{\frac{\eta}{4}}}{\sqrt{d}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \sqrt{d}\varepsilon^{\frac{\eta}{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{W}_\omega \mathrm{O}\left(d^{-1/2}\varepsilon^{1-\frac{\eta}{2}}\right) + \mathrm{W}_\omega \mathrm{O}\left(d^{-1/2}\varepsilon^{\frac{3}{4}\eta}\right)$$
$$+ \mathrm{O}\left(d^{-1/2}\left(\varepsilon^{\frac{9}{4}\eta-1}, \varepsilon^{1+\frac{\eta}{4}}\right)\right) \tag{5.33}$$

The random diffeomorphism can now be expanded.

5.6.2 The random diffeomorphism in the hyperbolic regime

In this case we compute $z = \widetilde{\mathrm{N}}(\mathrm{e}^{\mathrm{i}\pi y}).\mathrm{e}^{-\mathrm{i}\pi y}$ for the new cocycle given by (5.33)

$$z = \tfrac{1}{2}\mathrm{W}_\omega \cos(\pi y)\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi y}\, d^{-\frac{1}{2}}\varepsilon^{\eta/4} - \mathrm{e}^{-2\mathrm{i}\pi y}\sqrt{d}\varepsilon^{\eta/2} + \mathrm{W}_\omega \mathrm{O}\left(d^{-\frac{1}{2}}\varepsilon^{1-\eta/2}\right) + \mathrm{W}_\omega \mathrm{O}\left(d^{-\frac{1}{2}}\varepsilon^{\frac{3}{4}\eta}\right)$$
$$+\mathrm{O}\left(d^{-\frac{1}{2}}\left(\varepsilon^{\frac{9}{4}\eta-1}, \varepsilon^{1+\eta/4}\right)\right)$$

$$\Im\mathfrak{m}\, z = \tfrac{1}{2}\mathrm{W}_\omega \cos^2(\pi y) d^{-\frac{1}{2}}\varepsilon^{\eta/4} + \sin(2\pi y)\sqrt{d}\varepsilon^{\eta/2} + \mathrm{W}_\omega \mathrm{O} + \mathrm{O}\left(d^{-\frac{1}{2}}\left(\varepsilon^{\frac{9}{4}\eta-1}, \varepsilon^{1+\eta/4}\right)\right)$$

$$\Re\mathfrak{e}\, z = \mathrm{W}_\omega \sin(2\pi y)\tfrac{1}{4}\, d^{-\frac{1}{2}}\varepsilon^{\eta/4} - \cos(2\pi y)\sqrt{d}\varepsilon^{\eta/2} + \mathrm{W}_\omega \mathrm{O} + \mathrm{O}\left(d^{-\frac{1}{2}}\left(\varepsilon^{\frac{9}{4}\eta-1}, \varepsilon^{1+\eta/4}\right)\right)$$

$$\Im\mathfrak{m}\, z\,\Re\mathfrak{e}\, z = \tfrac{1}{8}{\mathrm{W}_\omega}^2 \cos^2(\pi y)\, \sin(2\pi y)\, d^{-1}\varepsilon^{\eta/2} + \cos(2\pi y)\sqrt{d}\varepsilon^{\eta/2}$$
$$+\mathrm{W}_\omega \mathrm{O} + \mathrm{O}\left(d^{-\frac{1}{2}}\left(\varepsilon^{\frac{9}{4}\eta-1}, \varepsilon^{1+\eta/4}\right)\right)$$

With (5.10), we have in this hyperbolic regime:

Definition 5.3 (Random diffeomorphism, hyperbolic case.)

For an energy $\mathrm{E} = 2 + d\varepsilon^{4/3}$ *the random diffeomorphism is*

$$\mathrm{F} = \mathrm{F}_{\omega,\varepsilon} : \mathbf{T}^2 \ni (x, y) \longmapsto (x + \alpha, y + v_\omega(y))$$

where

$$v_\omega = \mathrm{W}_\omega \frac{\cos^2(\pi\cdot)}{2\pi}\frac{\varepsilon^{\eta/4}}{\sqrt{d}} + \frac{\sin(2\pi\cdot)}{\pi}\left(d^{\frac{3}{2}} - \tfrac{1}{8}{\mathrm{W}_\omega}^2\cos^2(\pi\cdot)\right)\frac{\varepsilon^{\eta/2}}{d} + \mathrm{W}_\omega \mathrm{O}\left(\varepsilon^{\frac{3}{4}\eta}\right)$$
$$+\mathrm{O}\left(d^{-\frac{1}{2}}\varepsilon^{1-\eta/4}, d^{-1}\varepsilon^{\frac{9}{4}\eta-1}\right)$$

5.6.3 Transfer operator and cohomological equation, hyperbolic case

This is the cohomological equation (5.15) we solve:

Proposition 5.4

For ψ regular enough we can construct ϕ such that

$$\psi - \widehat{\psi}_0(\tfrac{1}{2}) = \mathrm{T}\phi - \phi + \mathrm{O}\left(\|\psi_0\|_2\, d^{-2}\left(\varepsilon^{\frac{\eta}{4}}, \varepsilon^{1-\frac{3}{4}\eta}, \varepsilon^{\frac{7}{4}\eta-1}\right)\right)$$

Proof.

The expansion of v_ω in definition 5.3 leads to

$${v_\omega}^2 = {\mathrm{W}_\omega}^2\frac{\cos^4(\pi\cdot)}{4\pi^2}\frac{\varepsilon^{\eta/2}}{d} + \mathrm{O}\left(d^{-2}\left(\varepsilon^{\frac{3}{4}\eta}, \varepsilon^{\frac{5}{2}\eta-1}\right)\right)$$

Taking expectations gives:

$$\overline{v}_1 = \mathbf{E}\left(v_\omega\right) = \sin(2\pi\cdot)\left(d^{\frac{3}{2}} - \tfrac{1}{8}\sigma^2\cos^2(\pi\cdot)\right)\frac{\varepsilon^{\eta/2}}{d} + \mathrm{O}\left(d^{-1}\left(\varepsilon^{1-\eta/4}, \varepsilon^{\frac{9}{4}\eta-1}\right)\right)$$

$$2\overline{v}_2 = \mathbf{E}\left({v_\omega}^2\right) = \sigma^2\frac{\cos^4(\pi\cdot)}{4\pi^2}\frac{\varepsilon^{\eta/2}}{d} + \mathrm{O}\left(d^{-2}\left(\varepsilon^{\frac{5}{2}\eta-1}, \varepsilon\right)\right)$$

These are proper perturbative developments when

$$\tfrac{\eta}{2} < \min\left(1, \tfrac{5}{2}\eta - 1, \tfrac{9}{4}\eta - 1, 1 - \tfrac{\eta}{4}\right)$$

which is the condition $\frac{4}{3} > \eta > \frac{4}{7}$. Under these conditions we want to solve the cohomological equation at its lowest order. As we already explained in the previous section, the equations $(\mathcal{E}_n)$ do not provide any difficulty and our iterative method gives a solution up to a reasonable order that we can always improve if we pay the price of the loss of derivatives due to the diophantine condition.

For the coefficient φ_0 we then have to solve for (5.15):

$$\psi_0 - \mathrm{Obs}(\psi) = \frac{1}{\pi}\sin(2\pi\cdot)\left(d^{\frac{3}{2}} - \frac{1}{8}\sigma^2\cos^2(\pi\cdot)\right)\frac{\varepsilon^{\eta/2}}{d}\,{\varphi_0}' + \frac{1}{8\pi^2}\sigma^2\cos^4(\pi\cdot)\frac{\varepsilon^{\eta/2}}{d}\,{\varphi_0}'' \quad (5.34)$$

which is again of the form (5.26) for which where we are looking for a one-periodic solution $\Phi = \varepsilon^{\eta/2}d^{-1}{\varphi_0}'$ satisfying $\int_{\mathbf{T}} \Phi \,\mathrm{d}y = 0$, and Ψ as well as all its derivatives are of order 1. The function p admits one zero of order 4: $y_0 = \frac{1}{2}$. It is also a zero of order 1 of q. This type of differential equation has been analyzed in Proposition 3 in [SSB07]. Let us use it in our setting. We need to compute

$$\left(\frac{p}{q}\right)^{(3)}(y_0) = c\frac{\mathrm{d}^3}{\mathrm{d}y^3}_{y=\frac{1}{2}}\frac{\cos^4(\pi y)}{\sin(2\pi y)} = \frac{c}{d^{3/2}}\frac{1}{4!}\frac{\frac{\mathrm{d}^4}{\mathrm{d}y^4}_{y=\frac{1}{2}}\cos^4(\pi y)}{\frac{\mathrm{d}}{\mathrm{d}y}_{y=\frac{1}{2}}\sin(2\pi y)} = -\frac{c}{2\pi d^{3/2}} < 0$$

where $c = c(\sigma^2, \pi) > 0$ is a constant.

We resolve on $[\,0\,;1\,]$ then look for periodicity conditions. If $\Psi(1/2) = 0$ then there is a two parameters family of $\mathscr{C}^\infty$ solutions, namely given by Φ_- on $\left[\,0\,;\frac{1}{2}\right[$ and Φ_+ on $\left]\,\frac{1}{2}\,;1\right]$ where

$$\begin{cases} \Phi_-(y) = \left(\Phi(0) + \displaystyle\int_0^y \frac{\Psi}{p}\,\mathrm{e}^{w_-}\right)\mathrm{e}^{-w_-(y)} \\ \Phi_+(y) = \left(\Phi(1) + \displaystyle\int_1^y \frac{\Psi}{p}\,\mathrm{e}^{w_+}\right)\mathrm{e}^{-w_+(y)} \\ w_\pm(s) = \displaystyle\int_{\frac{1\mp 1}{2}}^s \frac{q}{p} \end{cases}$$

The continuity, and more generally the $\mathscr{C}^k$-properties, are only to be checked at $y = \frac{1}{2}$ and easily result from de l'Hôpital's rule. Notice that the values of $\Phi^{(k)}(\frac{1}{2})$ are imposed by the differential equation (5.26). For example any $\mathscr{C}^1$ solution should satisfy

$$\Phi(\tfrac{1}{2}) = \lim_{y\to\frac{1}{2}}\frac{\Psi(y)}{q(y)}$$

which exists provided $\Psi(\frac{1}{2}) = 0$ since $\frac{1}{2}$ is a zero of q of order one. Taking derivatives in the differential equation gives the other values of $\Phi^{(k)}$ at $\frac{1}{2}$ for any $\mathscr{C}^{k+1}$ solution.

We then require $\Phi(0) = \Phi(1)$ and

$$\Phi(0)\int_{\mathbf{T}}\mathrm{e}^{-w_-} + \Phi(1)\int_{\mathbf{T}}\mathrm{e}^{-w_+} = -\int_{\mathbf{T}}\left(\mathrm{e}^{-w_-(y)}\int_0^y\frac{\Psi}{p}\,\mathrm{e}^{w_-} + \mathrm{e}^{-w_+(y)}\int_1^y\frac{\Psi}{p}\,\mathrm{e}^{w_+}\right)\mathrm{d}y$$

As $\int_{\mathbf{T}}(\mathrm{e}^{-w_-} + \mathrm{e}^{-w_+}) > 0$, this system has a unique solution and our differential equation admits a unique one-periodic solution with zero average provided $\Psi(\frac{1}{2}) = 0$. This gives rise to the obstruction $\mathrm{Obs}(\psi) = \psi_0(\frac{1}{2})$ in (5.34) and so we solve

$$\psi_0 - \psi(\tfrac{1}{2}) = \tfrac{1}{\pi}\sin(2\pi\cdot)\left(d^{\frac{3}{2}} - \tfrac{1}{8}\sigma^2\cos^2(\pi\cdot)\right)\frac{\varepsilon^{\eta/2}}{d}\,{\varphi_0}' + \tfrac{1}{8\pi^2}\sigma^2\cos^4(\pi\cdot)\frac{\varepsilon^{\eta/2}}{d}\,{\varphi_0}''$$

Considering the solution $\Phi = \frac{\varepsilon^{\eta/2}}{d}{\varphi_0}'$ and integrating it shows that for all $k \in \mathbf{N}$

$$\|\varphi_0\|_k \lesssim \|\psi_0\|_{\max(0,k-1)} d\varepsilon^{-\eta/2} \tag{5.35}$$

Let us estimate the error made for $\mathrm{T}_0\varphi_0$ with the following computations

$$\begin{aligned}\mathrm{T}_0\varphi_0 &= {\varphi_0}'\,\overline{v}_1 + {\varphi_0}''\,\overline{v}_2 + \mathbf{E}\left({v_\omega}^3\int_0^1 (1-s)^2\,{\varphi_0}'''(\cdot + sv_\omega)\,\mathrm{d}s\right)\\ &= \psi_0 - \psi_0(\tfrac{1}{2}) + \Delta_\psi\\ \Delta_\psi &=: {\varphi_0}'\mathrm{O}\left(d^{-1}(\varepsilon^{\frac{9}{4}\eta-1}, \varepsilon^{1-\eta/4})\right) + {\varphi_0}''\mathrm{O}\left(d^{-2}\left(\varepsilon^{\frac{5}{2}\eta-1}, \varepsilon\right)\right) + \mathrm{O}\left(\|\varphi_0\|_3\,\mathbf{E}(|v_\omega|^3)\right)\end{aligned}$$

The $\mathscr{C}^k$-estimates of φ_0 (see (5.35)) and the presence of the term ${v_\omega}^3$ ensure

$$\begin{aligned}\Delta_\psi &= \mathrm{O}\left(\|\psi_0\|_0(\varepsilon^{\frac{7}{4}\eta-1}, \varepsilon^{1-\frac{3}{4}\eta})\right) + \mathrm{O}\left(\|\psi_0\|_1\,d^{-1}\left(\varepsilon^{2\eta-1}, \varepsilon^{1-\eta/2}\right)\right) + \mathrm{O}\left(\|\psi_0\|_2\;\mathbf{E}(|v_\omega|^3)\varepsilon^{-\eta/2}\right)\\ &= \mathrm{O}\left(\|\psi_0\|_0\,(\varepsilon^{\frac{7}{4}\eta-1}, \varepsilon^{1-\frac{3}{4}\eta})\right) + \mathrm{O}\left(\|\psi_0\|_1\,d^{-1}\left(\varepsilon^{2\eta-1}, \varepsilon^{1-\eta/2}\right)\right) + \mathrm{O}\left(\|\psi_0\|_2\,d^{-2}\varepsilon^{\frac{3}{4}\eta}\varepsilon^{-\eta/2}\right)\\ \Delta_\psi &= \mathrm{O}\left(\|\psi_0\|_2\,d^{-2}\left(\varepsilon^{\eta/4}, \varepsilon^{1-\frac{3}{4}\eta}, \varepsilon^{\frac{7}{4}\eta-1}\right)\right)\end{aligned}$$

The lowest order of the error term therefore depends on whether $\eta \in \left]\,\frac{4}{7}\,;\frac{4}{3}\,\right[$ is larger or smaller than $\frac{4}{5}$ and $\frac{2}{3}$.

Let us deal now with the coefficients φ_n for $n \neq 0$. We use the same procedure as in the previous section. The first step is enough here: if we set

$$\varphi_n = \frac{\psi_n}{\gamma_n}$$

then $$\mathrm{T}_n\varphi_n = \psi_n + \varepsilon_n$$

where $$\varepsilon_n = \mathbf{E}\left(v_\omega\int_0^1 {\varphi_0}'(\cdot + sv_\omega)\,\mathrm{d}s\right) = \mathrm{O}\left(\|\psi_0\|_0 d^{-1}\varepsilon^{\eta/4}\right)$$

As done previously, we set $\varphi(x,y) = \sum\limits_{n\in\mathbf{Z}} \varphi_n(y)\mathrm{e}^{2\mathrm{i}\pi nx}$. It satisfies

$$\mathrm{T}\varphi - \varphi = \psi_0 - \psi_0(\tfrac{1}{2}) + \mathrm{O}\left(\|\psi\|_{0,\max(2,\tau+1^+)}\,d^{-2}\left(\varepsilon^{\eta/4}, \varepsilon^{1-\frac{3}{4}\eta}, \varepsilon^{\frac{7}{4}\eta-1}\right)\right)$$

This ends the proof of proposition 5.4.

■

Integration with respect to μ_ε gives the following expansion of any stationary measure:

Proposition 5.5

For $\psi \in \mathscr{C}^{3+\tau^+}(\mathbf{T}^2)$ *and any stationary measure* μ_ε *for* F_ω *we have*

$$\int_{\mathbf{T}^2} \psi \,\mathrm{d}\mu_\varepsilon = \widehat{\psi}_0(\tfrac{1}{2}) + \mathrm{O}\left(\|\psi\|_{0,\max(2,\tau+1^+)}\, d^{-2}\left(\varepsilon^{\frac{\eta}{4}}, \varepsilon^{1-\frac{3}{4}\eta}, \varepsilon^{\frac{7}{4}\eta-1}\right)\right)$$

with

$$\widehat{\psi}_0(y) = \int_{\mathbf{T}} \psi(x,y)\,\mathrm{d}x$$

5.6.4 Formula for the Lyapunov exponent in the hyperbolic case

We now prove theorem 5.3.

Proof.

With definition 5.3 one gets

$$\mathbf{E}\left(v_\omega{}'\right) = 2\cos(2\pi\cdot)\sqrt{d}\varepsilon^{\eta/2} + \mathrm{o}\left(y-\tfrac{1}{2}\right) d^{-1}\varepsilon^{\eta/2} + \mathrm{O}(d^{-\frac{1}{2}}\varepsilon^{1-\frac{\eta}{4}}, d^{-1}\varepsilon^{\frac{9}{4}\eta-1})$$

$$\int_{\mathbf{T}^2} \mathbf{E}\left(v_\omega{}'\right)\mathrm{d}\mu_\varepsilon = -2\sqrt{d}\varepsilon^{\eta/2} + \mathrm{O}(d^{-3}\left(\varepsilon^{\frac{3}{4}\eta}, \varepsilon^{1-\frac{\eta}{4}}, \varepsilon^{\frac{9}{4}\eta-1}\right))$$

$$\mathbf{E}\left(v_\omega{}'^2\right) = \mathrm{o}(y-\tfrac{1}{2})d^{-1}\varepsilon^{\eta/2} + \mathrm{O}\left(d^{-2}\left(\varepsilon^{\frac{3}{4}\eta}, \varepsilon^{\frac{5}{2}\eta-1}\right)\right)$$

$$\int_{\mathbf{T}^2} \mathbf{E}\left(v_\omega{}'^2\right)\mathrm{d}\mu_\varepsilon = \mathrm{O}\left(d^{-4}\left(\varepsilon^{\frac{3}{4}\eta}, \varepsilon^{1-\frac{\eta}{4}}, \varepsilon^{\frac{9}{4}\eta-1}\right)\right)$$

So that proposition 5.5 leads to

$$\begin{aligned}\gamma(\mathrm{F}) &= \int_{\mathbf{T}^2} \mathbf{E}\ln|1+v_\omega{}'|\,\mathrm{d}\mu_\varepsilon \\ &= -2\sqrt{d}\varepsilon^{\eta/2} + \mathrm{O}\left(d^{-5}\left(\varepsilon^{\frac{3}{4}\eta}, \varepsilon^{1-\frac{\eta}{4}}, \varepsilon^{\frac{9}{4}\eta-1}\right)\right)\end{aligned}$$

The formula $\mathrm{L}(\mathrm{E}) = \frac{1}{2}|\gamma(\mathrm{F})|$ ensures

$$\mathrm{L}(2+d\varepsilon^\eta) = \sqrt{d}\varepsilon^{\eta/2} + \mathrm{O}\left(d^{-5}\left(\varepsilon^{\frac{3}{4}\eta}, \varepsilon^{1-\frac{\eta}{4}}, \varepsilon^{\frac{9}{4}\eta-1}\right)\right)$$

Note that the formula is properly perturbative provided

$$\frac{\eta}{2} < \min\left(\frac{3}{4}\eta, 1-\frac{\eta}{4}, \frac{9}{4}\eta - 1\right)$$

which are the conditions $\eta < \frac{4}{3}$ (already required), and also $\eta > \frac{4}{7}$. We actually obtain a little wider range for which the formula established in [SSB07] holds as they need $\eta > \frac{4}{5}$. The error term of order ε^{m_η} due to the initial conjugacy (see (5.4) and section 5.3) is also included in the error term that we wrote.

∎

CHAPTER 6

Resonant energies

6.1 The model and its issues

For the purpose of a quite complete energy regime analysis, we now investigate the case of a Schrödinger cocycle with mixed quasi-periodic and random potential (as previously:

$v(\omega,x)=\mathrm{V}(x)+\mathrm{W}_\omega)$

$$\mathrm{S}_{\mathrm{E},\omega,x}=\begin{pmatrix}\mathrm{E}-\varepsilon v(\omega,x) & -1\\ 1 & 0\end{pmatrix} \tag{6.1}$$

when the energy E is supposed to be neither close to 2 nor diophantine with respect to α, but resonant for α which means

$$\beta=\beta(\mathrm{E})=\frac{1}{\pi}\operatorname{Arc}\cos\left(\mathrm{E}/2\right)=k\alpha \mod 1 \qquad (k\in\mathbf{Z}) \tag{6.2}$$

This obviously prevents the use of the techniques developed in chapter 4 since the cohomological equation associated with the transfer operator to solve would have no solution. Namely, the Fourier coefficients $(\widehat{\varphi}_{k,m})_{m\in\mathbf{Z}}$ of the cohomological solution φ would not necessary be defined as they should satisfy

$$\left(\mathrm{e}^{2\mathrm{i}\pi(m\alpha+n\beta)}-1\right)\widehat{\varphi}_{m,n}\approx\widehat{\psi}_{m,n}$$

This tends to indicate that the techniques associated to the study of a transfer operator close to a rotation on $\mathbf{T}^2$ are not relevant anymore. A better idea is rather to go back to the case where the random diffeomorphism induced by the cocycle is a perturbation of the identity function, as in chapter 5. Also we assume $k\neq 0$ in (6.2) because we already treated the energies at the band-edge of the free spectrum in the previous chapter.

6.2 Outline of the proof

The proof is divided as follows:

(i) At first we apply the conjugacy already mentioned in chapter 4 to conjugate our cocycle to a perturbation of the identity.

(ii) Then we compute the random diffeomorphism given by the conjugated cocycle which is now a perturbation of $(x,y)\mapsto(x+\alpha,y)$ on $\mathbf{T}^2$ with a dependance on x on the second factor. The Fourier transform in x leads to more complicated equations to solve for the coefficients with rather look driven by the action of a linear operator that is diagonal at its lowest order. The context of our work is then the setting of matricial differential operators with exponential decay w.r.t the diagonal (look at definition 6.2).

(iii) Thus we conjugate of the initial operator to a diagonal operator up to some error terms (proposition 6.4). We need to prove estimates for converging sequences in $\mathrm{L}^2(\mathbf{Z},\mathbf{T})$ and matricial differential operators acting upon those (proposition 6.2).

(iv) Once the equations for the diagonal operator are solved, we can back to the initial problem solve it (see section 6.8). Two regimes where the computations remain feasible occur, depending on a condition on $\mathbf{E}(\mathrm{W}_0)+\widehat{\mathrm{V}}_0-\left|\widehat{\mathrm{V}}_k\right|$ where k is defined by (6.2).

(v) We treat those two cases. We will see that the hardest Fourier equation to solve is actually the one for the coefficient of index 0. This will lead us to study solutions of first-order differential equations with some parameter (lemma 6.6). The computations get quite technical but give an exploitable perturbative formula in the end.

6.3 Reduction of the problem to a perturbation of the identity

We will first conjugate our cocycle to a perturbation of the identity. The conjugacy by $\mathrm{P}_\beta = \left(\begin{smallmatrix} 1 & -\cos\pi\beta \\ 0 & \sin\pi\beta \end{smallmatrix}\right)$ (see 4.13) (algebraically) conjugates our transfer matrix $\mathrm{S}_{\mathrm{E},\omega,x}$ to

$$\mathrm{R}_{\pi\beta} - \frac{\varepsilon v(\omega,x)}{\sin\pi\beta}\mathrm{N}_\beta = \begin{pmatrix} \cos\pi\beta & -\sin\pi\beta \\ \sin\pi\beta & \cos\pi\beta \end{pmatrix} - \frac{\varepsilon v(\omega,x)}{\sin\pi\beta}\begin{pmatrix} \sin\pi\beta & \cos\pi\beta \\ 0 & 0 \end{pmatrix}$$

Now we add a cocycle conjugacy of $(\alpha, \mathrm{S}_{\mathrm{E},\omega,x})$ by $(0, \mathrm{R}_{-\pi k x})$: if $\mathrm{C}(x) = \mathrm{R}_{-k\pi x}\mathrm{P}_\beta$ then

$$\begin{aligned}\widetilde{\mathrm{S}}_{\omega,x} := \mathrm{C}(x+\alpha)\mathrm{S}_{\mathrm{E},\omega,x}\mathrm{C}(x)^{-1} &= \mathrm{R}_{-\pi k(x+\alpha)}\mathrm{R}_{\pi\beta}\mathrm{R}_{\pi k x} - \frac{v(\omega,x)}{\sin\pi\beta}\mathrm{R}_{-k\pi(x+\alpha)}\mathrm{N}_\beta\mathrm{R}_{\pi k x} \\ &= \mathrm{R}_{\pi(\beta-k\alpha)} - \frac{v(\omega,x)}{\sin\pi\beta}\mathrm{R}_{-k\pi(x+\alpha)}\mathrm{N}_\beta\mathrm{R}_{\pi k x}\end{aligned}$$

As $\beta = k\alpha \mod 1$ we have $\mathrm{R}_{\pi(\beta-k\alpha)} = \sigma \mathrm{I}_2$ where this sign σ involved in the expression depends on the evenness of the integer $p = \beta - k\alpha$:

$$\sigma = (-1)^{\beta-k\alpha}$$

We can also compute $\mathrm{R}_{-k\pi(x+\alpha)}\,\mathrm{N}_\beta\,\mathrm{R}_{\pi k x} = \sigma\mathrm{Q}_{x,\alpha}$ where

$$\begin{aligned}\mathrm{Q}_{x,\alpha} &= \sigma\begin{pmatrix} \cos(k\pi(x+\alpha))\sin(\pi k(x+\alpha)) & \cos(k\pi(x+\alpha))\cos(\pi k(x+\alpha)) \\ -\sin(k\pi(x+\alpha))\sin(\pi k(x+\alpha)) & -\sin(k\pi(x+\alpha))\cos(\pi k(x+\alpha)) \end{pmatrix} \\ &= \sigma\begin{pmatrix} \frac{1}{2}\sin(2k\pi(x+\alpha)) & \cos^2(k\pi(x+\alpha)) \\ -\sin^2(k\pi(x+\alpha)) & -\frac{1}{2}\sin(2k\pi(x+\alpha)) \end{pmatrix}\end{aligned}$$

As a result, after this conjugacy we have to work with:

$$\widetilde{\mathrm{S}}_{\omega,x} = \sigma\left(\mathrm{I}_2 - \varepsilon\frac{v(\omega,x)}{\sin\pi\beta}\mathrm{Q}_{x,\alpha}\right) = \sigma\left(\mathrm{I}_2 - \varepsilon\frac{v(\omega,x)}{|\sin\pi k\alpha|}\mathrm{Q}_{x,\alpha}\right) \tag{6.3}$$

Therefore, the projective action induced by the cocycle does not depend on the sign σ and, like in previous chapter 5, is a perturbation of the identity map of $\mathbf{T}$ that we shall precisely compute afterwards. The main difference, which will directly lead to a more complex analysis of the transfer operator, is that the potential now cannot get rid of its dependency on the variable x.

6.4 The random diffeomorphism

In this section we compute the random diffeomorphism given by the cocycle $\widetilde{\mathrm{S}}_{\omega,x}$ (see 6.3) obtained after the conjugacy made in the previous section. Namely we show that the diffeomorphism is of the following type:

$$\mathrm{F}_\omega(x,y) = (x+\alpha, y+v_\omega(x,y))$$

with $v_\omega(x,y) = \varepsilon v_{1,\omega}(x,y) + \varepsilon^2 v_{2,\omega}(x,y) + \mathrm{O}(\varepsilon^3)$. We use the method explained in the previous chapters (see 5.10). Let us denote $a = k\pi(x+\alpha)$ and also

$$\widetilde{\mathrm{Q}} = \begin{pmatrix} \sin a\cos a & \cos^2 a \\ -\sin^2 a & -\sin a\cos a \end{pmatrix}$$

It is straightforward that for $\widetilde{y} = \pi y$ we have $z = \mathrm{Q}_{x,\alpha}(\mathrm{e}^{\mathrm{i}\widetilde{y}}).\mathrm{e}^{-\mathrm{i}\widetilde{y}} = \sin(a+\widetilde{y})\mathrm{e}^{-\mathrm{i}(a+\widetilde{y})}$ and so with the computations made previously chapter (see 5.10)

$$\begin{aligned} z &= -\frac{\varepsilon v(\omega,x)}{\sin\pi\beta}\,\sin\left(a+\widetilde{y}\right)\,\mathrm{e}^{-\mathrm{i}(a+\widetilde{y})} \\ \mathfrak{Im}\, z &= \frac{\varepsilon v(\omega,x)}{\sin\pi\beta}\sin^2(a+\widetilde{y}) \\ \mathfrak{Re}\, z &= -\frac{\varepsilon v(\omega,x)}{\sin\pi\beta}\,\sin(a+\widetilde{y})\cos(a+\widetilde{y}) \\ &= -\frac{\varepsilon v(\omega,x)}{2\sin\pi\beta}\,\sin(2a+2\widetilde{y}) \end{aligned}$$

This gives the diffeomorphism to study:

Definition 6.1 (Random diffeomorphism, resonant case)

The random diffeomorphism in the resonant case is $(x+\alpha, y+v_\omega(x,y))$ *with*

$$v_\omega(x,y) = \varepsilon v_{1,\omega}(x,y) + \varepsilon^2 v_{2,\omega}(x,y) + \mathrm{O}(\varepsilon^3)$$

where we set (recall that $v(\omega,x)$ *is the initial potential of the Schrödinger operator in* (6.1)*):*

$$\begin{cases} v_{1,\omega}(x,y) = \dfrac{1}{\pi}\dfrac{\sin^2\left(\pi\left(k(x+\alpha)+y\right)\right)}{|\sin\pi k\alpha|}\,v(\omega,x) \\[2ex] v_{2,\omega}(x,y) = \dfrac{1}{2\pi}\dfrac{\sin^2\left(\pi\left(k(x+\alpha)+y\right)\right)\sin\left(2\pi\left(k(x+\alpha)+y\right)\right)}{\sin^2\pi k\alpha}\,v(\omega,x)^2 \\[2ex] \qquad\qquad\quad = \frac{1}{2}\,v_{1,\omega}(x,y)\,\partial_y v_{1,\omega}(x,y) \\[1ex] v_{2,\omega}(x,y) = \frac{1}{4}\,\partial_y({v_{1,\omega}}^2(x,y)) \end{cases}$$

6.5 Transfer operator and cohomological equation

We still want to solve cohomological equations of the following type

$$\psi = \mathbf{E}\left(\varphi \circ \mathrm{F}_\omega\right) - \varphi + \mathrm{Obs}(\psi) + \mathrm{O}(\varepsilon^\eta \|\psi\|_k) \tag{6.4}$$

for some $\eta > 0$ and $k \in \mathbf{N}$ to be specified later, with obstructions that are as simple as possible and namely constants (linear forms in ψ). The cohomological equation is now obtained when one tries to solve the equation below:

$$\begin{aligned}
\psi(x,y) &= -\varphi(x,y) + \varphi(x+\alpha,y) + \varphi'_y(x+\alpha,y)\,\mathbf{E}(v_\omega(x,y)) + \tfrac{1}{2}\varphi''_y(x+\alpha,y)\,\mathbf{E}(v_\omega(x,y)^2) \\
&\qquad\qquad + \mathrm{O}(\varepsilon^3\|\varphi\|_3) \\
&= -\varphi(x,y) + \varphi(x+\alpha,y) + \varepsilon\varphi'_y(x+\alpha,y)\,\mathbf{E}(v_{1,\omega}(x,y)) \\
&\qquad + \varepsilon^2\left(\varphi'_y(x+\alpha,y)\ \mathbf{E}(v_{2,\omega}(x,y)) + \tfrac{1}{2}\varphi''_y(x+\alpha,y)\ \mathbf{E}(v_{1,\omega}(x,y)^2)\right) \\
&\qquad + \mathrm{O}(\varepsilon^3\|\varphi\|_3) \\
\psi(x,y) &= \varphi(x+\alpha,y) - \varphi(x,y) + \varepsilon v_1(x,y)\,\varphi'_y(x+\alpha,y) \\
&\qquad + \varepsilon^2\left(v_2(x,y)\,\varphi'_y(x+\alpha,y) + w(x,y)\,\varphi''_y(x+\alpha,y)\right) + \mathrm{O}(\varepsilon^3\|\varphi\|_3)
\end{aligned}$$

where we use the following notations

$$v_i(x,y) = \mathbf{E}\left(v_{i,\omega}(x,y)\right) \qquad \text{and} \qquad w(x,y) = \frac{1}{2}\,\mathbf{E}\left({v_{1,\omega}}^2(x,y)\right) \tag{6.5}$$

Hence $$v_2 = \tfrac{1}{4}\,\mathbf{E}\left(\partial_y({v_{1,\omega}}^2)\right) = \tfrac{1}{4}\partial_y\,\mathbf{E}\left({v_{1,\omega}}^2\right) = \tfrac{1}{2}\partial_y w$$

The cohomological equation is now an "α-schifted" PDE:

$$\psi = \varphi(\cdot+\alpha,\cdot) - \varphi + \varepsilon v_1\,\varphi'_y(\cdot+\alpha,\cdot) + \varepsilon^2\left(v_2\,\varphi'_y(\cdot+\alpha,\cdot) + w\,\varphi''_y(\cdot+\alpha,\cdot)\right) \tag{6.6}$$

with an error term of order $\mathrm{O}(\varepsilon^3\|\varphi\|_3)$ which is precisely

$$\mathbf{E}\left(v_\omega(x,y)^3\int_0^1(1-s)^2\,\varphi_y^{(3)}(x+\alpha,y+sv_\omega(x,y))\,\mathrm{d}s\right) \tag{6.7}$$

Let us use again Fourier series in x. As we noted, the dependency on x in the potentials v_i and w ((6.5)) make the computations more complicated. Let us recall that if f, g are in $\mathrm{L}^2(\mathbf{T}^2)$ then

$$(fg)_n(y) = \int_{\mathbf{T}^2} f(x,y)\,g(x,y)\mathrm{e}^{-2\mathrm{i}\pi nx}\,\mathrm{d}x = \sum_{k\in\mathbf{Z}} f_k(y)\,g_{n-k}(y)$$

Thanks to the previous computations (6.6), we see that we have to find, for all $n \in \mathbf{Z}$, a function φ_n satisfying, up to a term of order $\mathrm{O}(\varepsilon^3\|\varphi\|_3)$,

$$\psi_n(y) = \gamma_n\varphi_n(y) + \varepsilon \sum_{m\in\mathbf{Z}} \mathrm{e}^{2\mathrm{i}\pi m\alpha}\varphi_m{}'(y)\, v_{1,n-m}(y) \\ + \varepsilon^2 \sum_{m\in\mathbf{Z}} \mathrm{e}^{2\mathrm{i}\pi m\alpha}\left(\varphi_m{}'(y)\, v_{2,n-m}(y) + \varphi_m{}''(y)\, w_{n-m}(y)\right) \tag{6.8}$$

The equations are not independent like in the previous chapter as each one requires all the φ_n's: we are rather faced with an operator action since, up to terms of order ε^3,

$$\psi_n = \gamma_n\varphi_n + \varepsilon \sum_{m\in\mathbf{Z}} \mathrm{L}_{n,m}\varphi_m + \varepsilon^2 \sum_{m\in\mathbf{Z}} \mathrm{N}_{n,m}\varphi_m \tag{6.9}$$

which can be written in an operator theory formalism

$$(\psi_n)_{n\in\mathbf{Z}} = \left[\Gamma + \varepsilon\mathrm{L} + \varepsilon^2\mathrm{N}\right](\varphi_n)_{n\in\mathbf{Z}}$$

for operators $\Gamma, \mathrm{L}, \mathrm{N}$ defined by their coefficients ($\mathrm{Op}_{i,j} = \langle \mathrm{Op}\, e_i | e_j\rangle$) as follows:

$$\Gamma_{i,j} = \gamma_i\delta_{i,j} \qquad \mathrm{L}_{i,j} = \mathrm{e}^{2\mathrm{i}\pi j\alpha} v_{1,i-j}\,\partial_y \qquad \text{and} \qquad \mathrm{N}_{i,j} = \mathrm{e}^{2\mathrm{i}\pi j\alpha}\left(v_{2,i-j}\,\partial_y + w_{i-y}\,\partial_y^2\right) \tag{6.10}$$

Note that our operator

$$\mathrm{A} := \Gamma + \varepsilon\mathrm{L} + \varepsilon^2\mathrm{N} + \varepsilon^3\mathrm{M} \tag{6.11}$$

is, up to the third order in ε, the diagonal operator Γ of multiplication by $(\gamma_n)_n$ perturbed by differential operators of respective orders 1 and 2. The idea is to conjugate the operator A to get a simpler equation $\Psi = \mathrm{D}\Phi$ to be solved perturbatively. Namely, we diagonalize A up to the right order of approximation needed (in our case it is 3). First let us elaborate on our context of matricial operators.

6.6 Matricial operators on $\mathrm{L}^2(\mathbf{Z},\mathbf{T})$

The operator A acts on $\mathscr{H} = \mathrm{L}^2(\mathbf{Z},\mathbf{T}) = \left\{ (\psi_n(\cdot))_{n\in\mathbf{Z}} \mid \sum_{n\in\mathbf{Z}} |\psi_n|^2 \in \mathrm{L}^1(\mathbf{T})\right\}$ which is a Hilbert space endowed with the following inner product:

$$\langle\psi|\varphi\rangle := \int_{\mathbf{T}^2} \left(\sum_{n\in\mathbf{Z}} \psi_n(y)\,\mathrm{e}^{2\mathrm{i}\pi nx}\right)\left(\sum_{m\in\mathbf{Z}} \overline{\varphi_m}(y)\,\mathrm{e}^{-2\mathrm{i}\pi mx}\right)\mathrm{d}x\,\mathrm{d}y$$

whose associated norm is

$$\|(\psi_n)\|_{\mathscr{H}} = \sum_{n\in\mathbf{Z}} \int_{\mathbf{T}} |\psi_n(y)|^2\,\mathrm{d}y$$

It is clear that $\mathrm{L}^2(\mathbf{Z},\mathbf{T})$ is unitarily isomorphic to $\mathrm{L}^2(\mathbf{T}^2)$ with

$$\mathcal{U}\colon \begin{cases} \mathrm{L}^2(\mathbf{Z},\mathbf{T}) & \longrightarrow \mathrm{L}^2(\mathbf{T}^2) \\ (\psi_n(\cdot))_{n\in\mathbf{Z}} & \longmapsto \left((x,y)\mapsto \sum_{n\in\mathbf{Z}} \psi_n(y)\,\mathrm{e}^{2\mathrm{i}\pi n x}\right) \end{cases} \tag{6.12}$$

which admits the Fourier transform in x as an inverse

$$\mathcal{U}^{-1} = \mathcal{F}_x\colon \begin{cases} \mathrm{L}^2(\mathbf{T}^2) \longrightarrow \mathrm{L}^2(\mathbf{Z},\mathbf{T}) \\ \quad \psi \quad \longmapsto \left(\int_{\mathbf{T}} \psi(x,\cdot)\,\mathrm{e}^{-2\mathrm{i}\pi n x}\,\mathrm{d}x\right)_{n\in\mathbf{Z}} \end{cases}$$

That means

$$\|\mathcal{U}\psi\|_{\mathrm{L}^2}^2 = \sum_{m,n\in\mathbf{Z}} |\mathcal{U}\psi_{m,n}|^2 = \|\psi\|_{\mathscr{H}}^2 = \sum_{n\in\mathbf{Z}} \int_{\mathbf{T}} |\psi_n(y)|^2\,\mathrm{d}y$$

In order to give proper $\mathscr{C}^j$-estimates for any stationary measure in the end, we need to exploit the exponential decay of the matrix coefficients in (6.10) w.r.t their distance to the diagonal. Indeed they are Fourier transform in x of analytic functions provided one assumes the analyticity of the quasi-periodic part V in the potential (see (6.1)). First let us introduce relevant norms in our context of operators and sequences of functions.

Definition 6.2 (Matrix differential operators with exponential decay)

Let $\mathcal{O}_{\sigma,h}^{\mathrm{M}}$ *be the set of matrix differential operators* $\mathrm{Q} = (\mathrm{Q}_{i,j})_{(i,j)\in\mathbf{Z}^2}$ *such that* $\mathrm{Q}_{i,j} = \sum_{\ell=0}^{\mathrm{M}} q_{i,j,\ell}(\cdot)\,\partial_y^\ell$ *with* $q_{i,j,\ell}\in\mathscr{C}_h^\omega(\mathbf{T})$ *(analytic on the strip* $\{|\mathfrak{Im}\,z|\leqslant h\}$ *and*

$$\sum_{\substack{(i,j)\in\mathbf{Z}^2\\ 0\leqslant\ell\leqslant\mathrm{M}}} \mathrm{e}^{\sigma|i-j|}\|q_{i,j,\ell}(\cdot)\|_h < +\infty \tag{6.13}$$

For $\mathrm{Q}\in\mathcal{O}_{\sigma,h}^{\mathrm{M}}$ *let us define*

$$\|\mathrm{Q}\|_{\sigma,h} := \sum_{i,j,\ell} \mathrm{e}^{\sigma|i-j|}\|q_{i,j,\ell}(\cdot)\|_h < +\infty \tag{6.14}$$

We denote $\mathcal{O}_{\sigma,h}^{\mathrm{M},+}$ *the operators* Q *for which* $q_{i,j,0}=0$ *for all* i,j *that is when the differential operators have valuation* $\geqslant 1$ *(i.e. contain no term without derivative). And finally* $\mathcal{O}_{\sigma,h}, \mathcal{O}_{\sigma,h}^{+}$ *are the union of* $\mathcal{O}_{\sigma,h}^{\mathrm{M}}$ *(resp.* $\mathcal{O}_{\sigma,h}^{\mathrm{M},+}$*). over all* $\mathrm{M}\in\mathbf{N}$ *(resp.* $\mathrm{M}\geqslant 1$*)*

Now that these definitions have been settled, we can state the following lemme about the composition of matricial differential operators:

Lemma 6.1

If $\mathrm{Q}_1 \in \mathcal{O}_{\sigma,h}^{\mathrm{M}_1}$ *and* $\mathrm{Q}_2 \in \mathcal{O}_{\sigma,h}^{\mathrm{M}_2}$ *then* $\mathrm{Q}_1 \circ \mathrm{Q}_2 \in \mathcal{O}_{\sigma,h}^{\mathrm{M}_1+\mathrm{M}_2}$ *and we have for the norm:*

$$\|\mathrm{Q}_1 \circ \mathrm{Q}_2\|_{\sigma,h} \leqslant \frac{c(\mathrm{M}_1,\mathrm{M}_2)}{h^{\mathrm{M}_1+\mathrm{M}_2}} \|\mathrm{Q}_1\|_{\sigma,h} \|\mathrm{Q}_2\|_{\sigma,h}$$

Proof.

Let us begin the proof for operators of order one $\mathrm{Q}_{i,j} = q_{i,j}\partial$ and $\mathrm{R}_{i,j} = r_{i,j}\partial$. Then

$$\begin{aligned}
(\mathrm{QR})_{i,j} &= \sum_{\ell\in\mathbf{Z}} \mathrm{Q}_{i,\ell}\mathrm{R}_{\ell,j} \\
&= \sum_{\ell\in\mathbf{Z}} q_{i,\ell}\, r'_{\ell,j}\, \partial + q_{i,\ell}\, r_{\ell,j}\, \partial^2 \\
(\mathrm{QR})_{i,j} &= \Big(\sum_{\ell\in\mathbf{Z}} q_{i,\ell}\, {r_{\ell,j}}'\Big)\, \partial + \Big(\sum_{\ell\in\mathbf{Z}} q_{i,\ell}\, r_{\ell,j}\Big)\, \partial^2
\end{aligned}$$

So we are looking for the convergence and an upper bound of

$$\mathrm{S} = \sum_{(i,j)\in\mathbf{Z}^2} \mathrm{e}^{\sigma|i-j|} \left(\Big\| \sum_{\ell\in\mathbf{Z}} q_{i,\ell}\, {r_{\ell,j}}' \Big\|_h + \Big\| \sum_{\ell\in\mathbf{Z}} q_{i,\ell}\, r_{\ell,j} \Big\|_h \right)$$

The following inequalities, that come from the triangular inequality and Cauchy's estimates, will be useful:

$$\begin{aligned}
\Big\| \sum_{\ell\in\mathbf{Z}} q_{i,\ell}\, r_{\ell,j} \Big\|_h &\leqslant \sum_{\ell\in\mathbf{Z}} \|q_{i,\ell}\|_h\, \|r_{\ell,j}\|_h \qquad (6.15)\\
\|{r_{\ell,j}}'\|_h &\leqslant \frac{1}{h}\|r_{\ell,j}\|_h \\
\mathrm{e}^{\sigma|i-j|} &\leqslant \mathrm{e}^{\sigma|i-\ell|}\mathrm{e}^{\sigma|\ell-j|}
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathrm{S} &\leqslant \sum_{(i,j,\ell)\in\mathbf{Z}^3} \mathrm{e}^{\sigma|i-\ell|}\mathrm{e}^{\sigma|\ell-j|} \left(\|q_{i,\ell}\|_h\, \|{r_{\ell,j}}'\|_h + \|q_{i,\ell}\|_h\, \|r_{\ell,j}\|_h\right) \\
&\leqslant (1+\tfrac{1}{h}) \sum_{(j,\ell)\in\mathbf{Z}^2} \mathrm{e}^{\sigma|\ell-j|}\|r_{\ell,j}\|_h \sum_{i\in\mathbf{Z}} \mathrm{e}^{\sigma|i-\ell|}\|q_{i,\ell}\|_h \\
&\leqslant (1+\tfrac{1}{h}) \sum_{(j,\ell)\in\mathbf{Z}^2} \mathrm{e}^{\sigma|\ell-j|}\|r_{\ell,j}\|_h \sum_{(i,\ell)\in\mathbf{Z}^2} \mathrm{e}^{\sigma|i-\ell|}\|q_{i,\ell}\|_h \\
\mathrm{S} &\leqslant (1+\tfrac{1}{h})\|\mathrm{R}\|_{\sigma,h}\, \|\mathrm{Q}\|_{\sigma,h}
\end{aligned}$$

Now for general $\mathrm{Q}_{i,j} = \sum_{m=0}^{\mathrm{M_Q}} q_{i,j,m}\, \partial^m$ and $\mathrm{R}_{i,j} = \sum_{m'=0}^{\mathrm{M_R}} r_{i,j,m'}\, \partial^{m'}$ we have

$$\mathrm{Q}_{i,\ell}\mathrm{R}_{\ell,j} = \sum_{m=0}^{\mathrm{M_Q}}\sum_{m'=0}^{\mathrm{M_R}} q_{i,\ell,m}\,\partial^m(r_{\ell,j,m'}\,\partial^{m'})$$
$$= \sum_{m=0}^{\mathrm{M_Q}}\sum_{m'=0}^{\mathrm{M_R}} q_{i,\ell,m}\sum_{p=0}^{m}\binom{m}{p} r_{\ell,j,m'}{}^{(m-p)}\,\partial^{m'+p}$$
$$\mathrm{Q}_{i,\ell}\mathrm{R}_{\ell,j} = \sum_{\nu=0}^{\mathrm{M_Q+M_R}}\left(\sum_{m=0}^{\mathrm{M_Q}}\sum_{m'=0}^{\mathrm{M_R}} q_{i,\ell,m}\mathbf{1}_{\nu-m\leqslant m'\leqslant\nu}\binom{m}{\nu-m'} r_{\ell,j,m'}{}^{(m+m'-\nu)}\right)\partial^\nu$$

So that $\mathrm{S}_{i,j} := (\mathrm{QR})_{i,j} = \sum_{\nu=0}^{\mathrm{M_Q+M_R}} \mathrm{S}_{i,j,\nu}\partial^\nu$ where:

$$\mathrm{S}_{i,j,\nu} = \sum_{k\in\mathbf{Z}}\sum_{m=0}^{\mathrm{M_Q}}\sum_{m'=0}^{\mathrm{M_R}} q_{i,\ell,m}\mathbf{1}_{\nu-m\leqslant m'\leqslant\nu}\binom{m}{\nu-m'} r_{\ell,j,m'}{}^{(m+m'-\nu)}$$

With the same previous estimates (6.15) and using that $m+m'-\nu\leqslant\nu\leqslant\mathrm{M_Q+M_R}$ one gets:

$$\|\mathrm{Q}\circ\mathrm{R}\|_{\sigma,h} = \sum_{\substack{(i,j)\in\mathbf{Z}^2\\ 0\leqslant\nu\leqslant\mathrm{M_Q+M_R}}}\|\mathrm{S}_{i,j,\nu}\|\mathrm{e}^{\sigma|i-j|} \leqslant \frac{1}{h^{\mathrm{M_Q+M_R}}}c(\mathrm{M_Q},\mathrm{M_R})\|\mathrm{Q}\|_{\sigma,h}\,\|\mathrm{R}\|_{\sigma,h}$$

■

Let us take a look at the behaviour of the action of such operators on *analytic* sequences of $\mathrm{L}^2(\mathbf{Z},\mathbf{T})$.

Definition 6.3 (Norms on $\mathrm{L}^2(\mathbf{Z},\mathbf{T})$)

Let us define for $r\in\mathbf{N}$

$$\mathcal{E}_{\sigma,r} := \left\{(\varphi_n)_{n\in\mathbf{Z}} \quad \varphi_n\in\mathscr{C}^r(\mathbf{T}) \;\middle|\; \sum_{n\in\mathbf{Z}}\mathrm{e}^{\sigma|n|}\|\varphi_n\|_r < +\infty\right\} \tag{6.16}$$

We also set $\varphi = (\varphi_n)\in\mathcal{E}_{\sigma,r}$ *the following norm:*

$$\|\varphi\|_{\sigma,r} := \sum_{n\in\mathbf{Z}}\mathrm{e}^{\sigma|n|}\|\varphi_n\|_r \tag{6.17}$$

Remark 6.1

Note that for $\psi(x,y) = \sum_{n\in\mathbf{Z}}\psi_n(y)\mathrm{e}^{2\mathrm{i}\pi nx}$, *we have* $\|\psi\|_{\sigma,r} := \|\psi_n\|_{\sigma,r}$ *is the* $\mathscr{C}^\omega\times\mathscr{C}^j$*-norm of* ψ, *meaning that it is the analytic norm on the first variable, and the* $\mathscr{C}^j$*-norm on the second one.*

Here is some result about the compatibility between the norms of the operators and those of the sequences:

Lemma 6.2

For $\mathrm{Q} \in \mathcal{O}^{\mathrm{M}}_{\sigma,h}$ *and* $\varphi \in \mathcal{E}_{\sigma,r}$ *we have* $\mathrm{Q}\varphi \in \mathcal{E}_{\sigma,r-\mathrm{M}}$ *and*

$$\|\mathrm{Q}\varphi\|_{\sigma,r-\mathrm{M}} \lesssim \frac{1}{h^{r-\mathrm{M}}}\|\mathrm{Q}\|_{\sigma,h}\|\varphi\|_r \tag{6.18}$$

Proof.

Let us fix $\mathrm{Q} \in \mathcal{O}^{\mathrm{M}}_{\sigma,h}$ and $\varphi \in \mathcal{E}_{\sigma,r}$. For $\mathrm{Q}_{i,j} = q_{i,j,\ell}\partial^\ell$. Leibniz's derivation formula

$$\begin{aligned}(\mathrm{Q}\varphi)_n &= \sum_{j\in\mathbf{Z}} q_{n,j,\ell}\,{\varphi_j}^{(\ell)}\\ {(\mathrm{Q}\varphi)_n}^{(r-\mathrm{M})} &= \sum_{j\in\mathbf{Z}}\sum_{p=0}^{r-\mathrm{M}}\tbinom{r-\mathrm{M}}{p}{q_{n,j,\ell}}^{(r-\mathrm{M}-p)}\,{\varphi_j}^{(\ell+p)}\end{aligned}$$

$$\begin{aligned}\sum_{n\in\mathbf{Z}} \mathrm{e}^{\sigma|n|}\|(\mathrm{Q}\varphi)_n^{(r-\mathrm{M})}\|_0 &\leqslant \sum_{(n,j)\in\mathbf{Z}^2}\sum_{p=0}^{r-\mathrm{M}}\tbinom{r-\mathrm{M}}{p}\,\|{q_{n,j,\ell}}^{(r-\mathrm{M}-p)}\|_h\,\|\varphi_j\|_{\ell+p}\,\mathrm{e}^{\sigma|n-j|}\,\mathrm{e}^{\sigma|j|}\\ &\leqslant \sum_{j\in\mathbf{Z}}\sum_{p=0}^{r-\mathrm{M}}\tbinom{r-\mathrm{M}}{p}\,\mathrm{e}^{\sigma|j|}\,\|\varphi_j\|_{\ell+p}\,\tfrac{1}{h^{r-\mathrm{M}-p}}\sum_{n\in\mathbf{Z}}\|q_{n,j,\ell}\|_h\,\mathrm{e}^{\sigma|n-j|}\\ \sum_{n\in\mathbf{Z}} \mathrm{e}^{\sigma|n|}\,\|(\mathrm{Q}\varphi)_n^{(r-\mathrm{M})}\|_0 &\leqslant \tfrac{2^{r-\mathrm{M}}}{h^{r-\mathrm{M}}}\sum_{j\in\mathbf{Z}}\mathrm{e}^{\sigma|j|}\,\|\varphi_j\|_r\,\|\mathrm{Q}\|_{\sigma,h} = \tfrac{2^{r-\mathrm{M}}}{h^{r-\mathrm{M}}}\|\varphi\|_{\sigma,r}\,\|\mathrm{Q}\|_{\sigma,h}\end{aligned}$$

And so for $\mathrm{Q}_{i,j} = \sum\limits_{k=0}^{\mathrm{M}} q_{i,j,\ell}\,\partial^\ell$ we obtain $\mathrm{Q}\varphi \in \mathcal{E}_{\sigma,r-\mathrm{M}}$ with

$$\|\mathrm{Q}\varphi\|_{\sigma,r-\mathrm{M}} \leqslant \mathrm{M}\tfrac{2^{r-\mathrm{M}}}{h^{r-\mathrm{M}}}\,\|\varphi\|_{\sigma,r}\,\|\mathrm{Q}\|_{\sigma,h}$$

∎

For our purpose, we will only use operators with bounded degree M, and sequences with bounded $\mathscr{C}^j$-regularity in y.

6.7 Conjugacy of operators on $\mathrm{L}^2(\mathbf{Z},\mathbf{T})$

So if we set $\Psi = (\psi_n(\cdot))_{n\in\mathbf{Z}}$ and $\Phi = (\varphi_n(\cdot))_{n\in\mathbf{Z}}$ we have to solve the matricial equation

$$\Psi = (\Gamma + \varepsilon\mathrm{L} + \varepsilon^2\mathrm{N} + \mathrm{O}(\varepsilon^3))\Phi$$

As the lowest order term in A is Γ which is already a diagonal operator, we seek a conjugacy close to the identity and our aim is then to find operators $\mathrm{Y}_1, \mathrm{Y}_2, \dots$ such that

$$\mathrm{A}\Big(\mathrm{Id} + \varepsilon\mathrm{Y}_1 + \varepsilon^2\mathrm{Y}_2 + \dots\Big) = \Big(\mathrm{Id} + \varepsilon\mathrm{Y}_1 + \varepsilon^2\mathrm{Y}_2 + \dots\Big)\Big(\mathrm{D}_0 + \varepsilon\mathrm{D}_1 + \varepsilon^2\mathrm{D}_2 + \dots\Big) \tag{6.19}$$

where the D_i's are diagonal operators. Since invertible operators form an open set, this will be enough for the conjugacy we want because $\mathrm{Id} + \varepsilon\mathrm{Y}_1 + \varepsilon^2\mathrm{Y}_2 + \dots + \varepsilon^n\mathrm{Y}_n$ will then be invertible for ε small enough provided the Y_j's are bounded. The order of the perturbation n will be chosen depending on the order of the Taylor development used for the cohomological equation. We identify the same order terms in the development in ε that these equations give rise to. With $\mathrm{A} = \Gamma + \varepsilon\mathrm{L} + \varepsilon^2\mathrm{N} + \dots$, the system obtained from (6.19) is

$$\begin{cases} \Gamma = \mathrm{D}_0 \\ \mathrm{L} + \Gamma\mathrm{Y}_1 = \mathrm{D}_1 + \mathrm{Y}_1\mathrm{D}_0 \\ \mathrm{N} + \Gamma\mathrm{Y}_2 + \mathrm{L}\mathrm{Y}_1 = \mathrm{D}_2 + \mathrm{Y}_1\mathrm{D}_1 + \mathrm{Y}_2\mathrm{D}_0 \\ \vdots \end{cases}$$

To the 2nd order (and so that concerns the 3 first equations), this is equivalent to

$$\begin{cases} \mathrm{D}_0 = \Gamma \\ [\Gamma, \mathrm{Y}_1] = \mathrm{D}_1 - \mathrm{L} \\ [\Gamma, \mathrm{Y}_2] = \mathrm{D}_2 + \mathrm{Y}_1\mathrm{D}_1 - \mathrm{L}\mathrm{Y}_1 - \mathrm{N} \end{cases} \tag{6.20}$$

Let us analyze the equations of "bracket type" $[\Gamma, \mathrm{Y}] = \mathrm{F}$ given by (6.20). Remind that $\alpha_n = \mathrm{e}^{2\mathrm{i}\pi n\alpha}$ and $\gamma_n = \alpha_n - 1$, and also that we assumed a diophantine condition $\mathrm{DC}(\kappa, \tau)$ on α (see 2.3). This will provide quantitative estimates for the solutions of the brackets equations.

Lemma 6.3 (Bracket equations)

The equations

$$[\Gamma, \mathrm{Y}] = \mathrm{F} \tag{6.21}$$

can be solved formally for F *having zero diagonal part i.e.* $\mathrm{Diag}\,\mathrm{F} = 0$ *(that is* $\mathrm{F}_{i,i} = 0$ *for all* $i \in \mathbf{Z}$*) with*

$$\forall j \neq \ell \qquad \mathrm{Y}_{j,\ell} = \frac{1}{\gamma_j - \gamma_\ell}\mathrm{F}_{j,\ell} \tag{6.22}$$

Moreover, the solution Y *is unique up to its diagonal part* $\mathrm{Diag}\,\mathrm{Y}$ *which can be chosen arbitrarily.*

In the case where $\mathrm{F} \in \mathcal{O}^{\mathrm{M}}_{\sigma,h}$ *then the solution verifies* $\mathrm{Y} \in \mathcal{O}_{\sigma',h}$ *for* $\sigma' < \sigma$ *and*

$$\|\mathrm{Y}\|_{\sigma',h} \lesssim \frac{\kappa^{-1}}{(\sigma - \sigma')^{\tau+1}}\|\mathrm{F}\|_{\sigma,h} \tag{6.23}$$

where κ, τ *are the constant and the exponent in* (2.3).

Proof.

We can compute the coefficients of $[\Gamma, \mathrm{Y}]$:

$$[\Gamma, \mathrm{Y}]_{j,\ell} = (\gamma_j - \gamma_\ell)\, \mathrm{Y}_{j,\ell}$$

so that the bracket equation (6.21) is equivalent to $(\gamma_j - \gamma_\ell)\, \mathrm{Y}_{j,\ell} = \mathrm{F}_{j,\ell}$. Of course we have $\gamma_j \neq \gamma_\ell$ for $j \neq \ell$ thanks to the irrationality of α. So the formal solution would be $\mathrm{Y} = \mathrm{Y}_\mathrm{F}$ with coefficients

$$\forall j \neq \ell \qquad \mathrm{Y}_{j,\ell} = \frac{1}{\gamma_j - \gamma_\ell} \mathrm{F}_{j,\ell}$$

and one can choose anything for $\mathrm{Y}_{j,j}$ under the assumption that $\operatorname{Diag} \mathrm{F} = 0$.

Next for $\mathrm{F}_{j,\ell} = \sum\limits_{p=0}^{\mathrm{M}} f_{j,\ell,p}(\cdot)\partial^p$ one has, if Y denotes the solution,

$$\begin{aligned}
\sum_{j,\ell \in \mathbf{Z}} \mathrm{e}^{\sigma'|j-\ell|} \|\mathrm{Y}_{j,\ell}\|_h &\leqslant \sum_{p=0}^{\mathrm{M}} \sum_{(j,\ell)\in\mathbf{Z}^2} \mathrm{e}^{-(\sigma-\sigma')|j-\ell|} \frac{|j-\ell|^\tau}{\kappa} \mathrm{e}^{\sigma|j-\ell|} \|f_{j,\ell,p}\|_h \\
&\lesssim \kappa^{-1} \|\mathrm{Q}\|_{\sigma,h} \sum_{(j,\ell)\in\mathbf{Z}^2} \mathrm{e}^{-(\sigma-\sigma')|j-\ell|} |j-\ell|^\tau \\
\sum_{j,\ell \in \mathbf{Z}} \mathrm{e}^{\sigma'|j-\ell|} \|\mathrm{Y}_{j,\ell}\|_h &\lesssim \frac{\kappa^{-1}}{(\sigma-\sigma')^{\tau+1}} \|\mathrm{Q}\|_{\sigma,h}
\end{aligned}$$

■

These estimates for the sequences and the operators help us to prove the following diagonalization result.

Lemma 6.4

For σ, h such that v_1, v_2, w admit an analytic extension to the strip $\{|\mathfrak{Im}\, z| \leqslant h\}$ with the convergence conditions $(v_{1,n}), (v_{2,n}), (w_n) \in \mathcal{E}_{\sigma,r}$ we can find $\mathrm{Y}_1 \in \mathcal{O}^{1,+}_{\sigma,h}, \mathrm{Y}_2 \in \mathcal{O}^{2,+}_{\sigma,h}$ matricial differential operators and $\mathrm{D}_0 = \Gamma$, $\mathrm{D}_1 \in \mathcal{O}^{1,+}_{\sigma,h}$ and $\mathrm{D}_2 \in \mathcal{O}^{2,+}_{\sigma,h}$ diagonal matrix operators, all explicit and defined with $\Gamma, \mathrm{L}, \mathrm{N}$, such that

$$\mathrm{AC} = \mathrm{CD} + \varepsilon^3 \mathrm{R} \tag{6.24}$$

where

$$\begin{aligned}
\mathrm{D} &= \mathrm{D}_0 + \varepsilon \mathrm{D}_1 + \varepsilon^2 \mathrm{D}_2 \\
\mathrm{C} &= \mathrm{Id} + \varepsilon \mathrm{Y}_1 + \varepsilon^2 \mathrm{Y}_2 \\
\mathrm{R} &= \mathrm{LY}_2 + \mathrm{NY}_1 - \mathrm{Y}_1\mathrm{D}_2 - \mathrm{Y}_2\mathrm{D}_1 + \varepsilon(\mathrm{NY}_2 - \varepsilon \mathrm{Y}_2\mathrm{D}_2)
\end{aligned} \tag{6.25}$$

and

$$\|\mathrm{R}\|_{\sigma',h} \lesssim \frac{\kappa^{-3}}{(\sigma-\sigma')^{3\tau+3}} \tag{6.26}$$

Remark 6.2

Please notice that, by definition, $\mathrm{R} = \mathrm{T} + \varepsilon\mathrm{F}$ *where* T *is a differential operator of order* 3 *and* $\mathrm{F} \in \mathcal{O}^{4,+}_{\sigma,h}$ *is of order* 4.

Proof.

Let us denote $\partial = \partial_y$ the operator of differentation along the 2nd variable y. We apply lemma 6.3 to equations (6.20) and obtain

$$\mathrm{D}_1 = \mathrm{Diag}\ \mathrm{L}$$

which means

$$(\mathrm{D}_1)_{j,\ell} = \delta_{j,\ell}\,\alpha_j\,v_{1,0}(\cdot)\,\partial_y$$

Since $\gamma_j - \gamma_\ell = \alpha_j - \alpha_\ell$, we get the following coefficients for Y_1:

$$\forall j \neq \ell \qquad (\mathrm{Y}_1)_{j,\ell} = -\frac{\alpha_k}{\gamma_j - \gamma_\ell} v_{1,j-\ell}(\cdot)\,\partial_y = -\frac{v_{1,j-\ell}}{\gamma_{j-\ell}}\,\partial_y \tag{6.27}$$

We also choose $\mathrm{Diag}\ \mathrm{Y}_1 = (0)_{j\in\mathbf{Z}}$.

Applying the procedure one more time gives

$$\begin{aligned}\mathrm{D}_2 &= \mathrm{Diag}\ \mathrm{N} + \mathrm{Diag}\,(\mathrm{L}\mathrm{Y}_1) - \mathrm{Diag}\,(\mathrm{Y}_1\mathrm{D}_1)\\ &= \mathrm{Diag}\ \mathrm{N} + \mathrm{Diag}\,(\mathrm{L}\mathrm{Y}_1)\end{aligned}$$

As v_1 is a real function we have $v_{1,-m} = \overline{v_{1,m}}$ and we deduce

$$\begin{aligned}(\mathrm{D}_2)_{j,j} &= \alpha_j\left(v_{2,0}\,\partial + w_0\,\partial^2\right) + \sum_{\ell\in\mathbf{Z}\smallsetminus\{j\}} \frac{\alpha_{j+\ell}}{\gamma_j - \gamma_\ell}\left(v_{1,j-\ell}\,(v_{1,\ell-j})'\,\partial + v_{1,j-\ell}\,v_{1,\ell-j}\,\partial^2\right)\\ &= \alpha_j\left(v_{2,0}\,\partial + w_0\,\partial^2\right) + \alpha_j\sum_{\ell\in\mathbf{Z}\smallsetminus\{j\}} \frac{v_{1,j-\ell}}{\gamma_{j-\ell}}\,\overline{v_{1,j-\ell}'}\,\partial + \alpha_j\sum_{\ell\in\mathbf{Z}\smallsetminus\{j\}} \frac{v_{1,j-\ell}}{\gamma_{j-\ell}}\,\overline{v_{1,j-\ell}}\,\partial^2\end{aligned}$$

$$(\mathrm{D}_2)_{j,j} = \alpha_j\left(v_{2,0}\,\partial + w_0\,\partial^2\right) + \alpha_j\sum_{m\in\mathbf{Z}\smallsetminus\{0\}} \frac{v_{1,m}}{\gamma_m}\,\overline{v_{1,m}'}\,\partial + \alpha_j\sum_{m\in\mathbf{Z}\smallsetminus\{0\}} \frac{v_{1,m}}{\gamma_m}\,\overline{v_{1,m}}\,\partial^2 \tag{6.28}$$

We can say a little more on the sums involved above by remarking that

$$\sum_{p\in\mathbf{Z}} a_p(y)\,\overline{b_p(y)} = \int_{\mathbf{T}} a(x,y)\,\overline{b(x,y)}\,\mathrm{d}x$$

where

$$a(x,y) = \mathcal{U}\,(a_p)_{p\in\mathbf{Z}} = \sum_{p\in\mathbf{Z}} a_p(y)\,\mathrm{e}^{2\mathrm{i}\pi p x}$$

Hence, if we let v_γ be the function defined by

$$v_\gamma(x,y) = \sum_{p\in\mathbf{Z}\smallsetminus\{0\}} \frac{v_{1,p}(y)}{\gamma_p}\,\mathrm{e}^{2\mathrm{i}\pi p x}$$

it is straightforward that $\overline{v_\gamma(x,y)} = v_\gamma(x,y)$ since $\overline{v_{1,p}(y)} = v_{1,-p}(y)$ (v_1 takes real values) and also $\overline{\gamma_p} = \gamma_{-p}$.

Remark 6.3

Let us note that v_γ actually solves the following cohomological equation:

$$\begin{aligned} v_\gamma(x+\alpha,y) - v_\gamma(x,y) &= \sum_{p\in\mathbf{Z}\smallsetminus\{0\}} \frac{\mathrm{e}^{2\mathrm{i}\pi p\alpha}-1}{\gamma_p} v_{1,p}(y)\,\mathrm{e}^{2\mathrm{i}\pi p x} \\ &= \sum_{p\in\mathbf{Z}\smallsetminus\{0\}} v_{1,p}(y)\,\mathrm{e}^{2\mathrm{i}\pi p x} = v_1(x,y) - \textstyle\int_{\mathbf{T}} v_1(x,y)\,\mathrm{d}x \end{aligned}$$

Then we deduce the following equalities

$$\begin{aligned} u(y) &:= \sum_{m\in\mathbf{Z}\smallsetminus\{0\}} \frac{v_{1,m}(y)}{\gamma_m}\overline{v_{1,m}{}'(y)} = \int_{\mathbf{T}} v_\gamma(x,y)\,\overline{\partial_y v_1(x,y)}\,\mathrm{d}x = \int_{\mathbf{T}} v_\gamma(x,y)\,\partial_y v_1(x,y)\,\mathrm{d}x \\ s(y) &:= \sum_{m\in\mathbf{Z}\smallsetminus\{0\}} \frac{v_{1,m}(y)}{\gamma_m}\overline{v_{1,m}(y)} = \int_{\mathbf{T}} v_\gamma(x,y)\,\overline{v_1(x,y)}\,\mathrm{d}x = \int_{\mathbf{T}} v_\gamma(x,y)\,v_1(x,y)\,\mathrm{d}x \end{aligned} \tag{6.29}$$

But we can also compute

$$\begin{aligned} s(\cdot) &= \sum_{m\in\mathbf{Z}\smallsetminus\{0\}} \frac{v_{1,m}}{\gamma_m}\overline{v_{1,m}} \\ &= \sum_{m\in\mathbf{Z}\smallsetminus\{0\}} \frac{|v_{1,m}|^2}{\gamma_m} = \sum_{m>0} |v_{1,m}|^2\left(\frac{1}{\gamma_m}+\frac{1}{\gamma_{-m}}\right) \\ &= \sum_{m>0} |v_{1,m}|^2\,\frac{2\,\mathfrak{Re}\,\gamma_m}{|\gamma_m|^2} = \sum_{m>0}|v_{1,m}|^2\,\frac{2(\cos 2\pi m\alpha - 1)}{|\mathrm{e}^{2\mathrm{i}\pi m\alpha}-1|^2} \\ &= -\sum_{m>0}|v_{1,m}|^2 = -\frac{1}{2}\sum_{m\neq 0}|v_{1,m}|^2 \\ s(\cdot) &= -\frac{1}{2}\int_{\mathbf{T}} {v_1}^2(x,\cdot)\,\mathrm{d}x + \frac{1}{2}{v_{1,0}}^2 \end{aligned}$$

Finally,

$$s(y) = -\frac{1}{2}\int_{\mathbf{T}} {v_1}^2(x,y)\,\mathrm{d}x + \frac{1}{2}\left(\int_{\mathbf{T}} v_1(x,y)\,\mathrm{d}x\right)^2 \tag{6.30}$$

So that $s(y)$ is non positive by Jensen's inequality, and vanishes if and only if $x\mapsto v_1(x,y)$ is constant a.e. We obtain thanks to (6.28):

$$(\mathrm{D}_2)_{j,j} = \alpha_j r(\cdot)\,\partial + \alpha_j t(\cdot)\,\partial^2 \tag{6.31}$$

where

$$r(y) = v_{2,0}(y) + u(y) = \frac{1}{2}{w_0}'(y) + \int_{\mathbf{T}} v_\gamma(x,y)\,\partial_y v_1(x,y)\,\mathrm{d}x$$
$$t(y) = w_0(y) + s(y) = w_0(y) + \int_{\mathbf{T}} v_\gamma(x,y)\,v_1(x,y)\,\mathrm{d}x \tag{6.32}$$

Let us give another useful expression of t. By definition,

$$w_0(y) = \frac{1}{2}\int_{\mathbf{T}} w(x,y)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbf{T}} \mathbf{E}\left(v_{1,\omega}(x,y)^2\right)\,\mathrm{d}x$$

Hence

$$\begin{aligned}
t(y) &= \frac{1}{2}\int_{\mathbf{T}} \mathbf{E}\left(v_{1,\omega}(x,y)^2\right)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbf{T}} {v_1}^2(x,y)\,\mathrm{d}x + \frac{1}{2}\left(\int_{\mathbf{T}} v_1(x,y)\,\mathrm{d}x\right)^2\\
&= \frac{1}{2}\int_{\mathbf{T}} \mathbf{E}\left(v_{1,\omega}(x,y)^2\right)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbf{T}} \left(\mathbf{E}\,v_{1,\omega}(x,y)\right)^2\,\mathrm{d}x + \frac{1}{2}\left(\int_{\mathbf{T}} \mathbf{E}\,v_{1,\omega}(x,y)\,\mathrm{d}x\right)^2\\
&= \frac{1}{2}\int_{\mathbf{T}} \mathbf{V}(v_{1,\omega}(x,y))\,\mathrm{d}x + \frac{1}{2}\left(\int_{\mathbf{T}} \mathbf{E}\,v_{1,\omega}(x,y)\,\mathrm{d}x\right)^2\\
&= \frac{1}{2\pi\,|\sin \pi k\alpha|}\,\mathbf{V}(\mathrm{W})\int_{\mathbf{T}} \sin^2\left(\pi\left(k(x+\alpha)+y\right)\right)\mathrm{d}x + \frac{1}{2}\left(\int_{\mathbf{T}} \mathbf{E}\,v_{1,\omega}(x,y)\,\mathrm{d}x\right)^2\\
t(y) &= \frac{1}{4\pi\,|\sin \pi k\alpha|}\,\mathbf{V}(\mathrm{W}) + \frac{1}{2}\left(\int_{\mathbf{T}} \mathbf{E}\,v_{1,\omega}(x,y)\,\mathrm{d}x\right)^2
\end{aligned}$$

And finally

$$t(y) = \frac{1}{4\pi\,|\sin \pi k\alpha|}\,\mathbf{V}(\mathrm{W}) + \frac{1}{2}{v_{1,0}}^2(y) \tag{6.33}$$

So that the positivity of t can be granted under some assumption on the largeness of $\mathbf{V}(\mathrm{W})$.

Let us finish by making Y_2 explicit. Let us set $\Delta_{i,j} := \gamma_i - \gamma_j$ for $i \neq j$. The resolution of the bracket equations then gives Y_2 with coefficients: for $i \neq j$

$$\begin{aligned}
(\mathrm{Y}_2)_{i,j} &= \frac{1}{\Delta_{i,j}}\left((\mathrm{Y}_1\mathrm{D}_1)_{i,j} - \mathrm{N}_{i,j} - (\mathrm{L}\mathrm{Y}_1)_{i,j}\right)\\
&= \frac{1}{\Delta_{i,j}}\left((\mathrm{Y}_1)_{i,j}\mathrm{D}_{j,j} - \mathrm{N}_{i,j} - \sum_{p\in\mathbf{Z}} \mathrm{L}_{i,p}(\mathrm{Y}_1)_{p,j}\right)\\
&= \frac{1}{\gamma_i-\gamma_j}\left(\frac{\mathrm{L}_{i,j}}{\gamma_j-\gamma_j}\mathrm{L}_{j,j} - \mathrm{N}_{i,j} - \sum_{p\in\mathbf{Z}\setminus\{j\}} \mathrm{L}_{i,p}\frac{\mathrm{L}_{p,j}}{\gamma_j-\gamma_p}\right)\\
(\mathrm{Y}_2)_{i,j} &= \frac{-{\alpha_j}^2}{(\Delta_{i,j})^2}\left(v_{1,i-j}(v_{1,0})'\,\partial + v_{1,i-j}\,v_{1,0}\,\partial^2\right) - \frac{\alpha_j}{\Delta_{i,j}}\left(v_{2,i-j}\,\partial + w_{i-j}\,\partial^2\right)\\
&\quad - \frac{1}{\Delta_{i,j}}\sum_{p\in\mathbf{Z}\setminus\{j\}} \frac{\alpha_{p+j}}{\Delta_{j,p}}\left(v_{1,i-p}\,(v_{1,p-j})'\,\partial + v_{1,i-j}\,v_{p-j}\,\partial^2\right)
\end{aligned}$$

Furthermore for $\mathrm{D} = \mathrm{D}_0 + \varepsilon \mathrm{D}_1 + \varepsilon^2 \mathrm{D}_2$

$$\begin{aligned}
\mathrm{AC} &= \left(\Gamma + \varepsilon \mathrm{L} + \varepsilon^2 \mathrm{N}\right)\left(\mathrm{Id} + \varepsilon \mathrm{Y}_1 + \varepsilon^2 \mathrm{Y}_2\right) \\
&= \Gamma + \varepsilon\left(\mathrm{L} + \Gamma \mathrm{Y}_1\right) + \varepsilon^2\left(\mathrm{N} + \mathrm{L}\mathrm{Y}_1 + \Gamma \mathrm{Y}_2\right) + \varepsilon^3 \mathrm{L}\mathrm{Y}_2 + \mathrm{N}\mathrm{Y}_1 + \varepsilon^4 \mathrm{N}\mathrm{Y}_2 \\
\mathrm{CD} &= \mathrm{D}_0 + \varepsilon\left(\mathrm{D}_1 + \mathrm{Y}_1\mathrm{D}_0\right) + \varepsilon^2\left(\mathrm{D}_2 + \mathrm{Y}_1\mathrm{D}_1 + \mathrm{Y}_2\mathrm{D}_0\right) + \varepsilon^3 \mathrm{Y}_1\mathrm{D}_2 + \mathrm{Y}_2\mathrm{D}_1 + \varepsilon^4 \mathrm{Y}_2\mathrm{D}_2
\end{aligned}$$

and so, as stated in (6.24), $\qquad \mathrm{AC} = \mathrm{CD} + \varepsilon^3 \mathrm{R}$

The estimate of $\|\mathrm{R}\|_{\sigma,\widetilde{h}}$ comes from the very definition of R, lemmas 6.3 and 6.2 successively applied for Y_1 and Y_2. ■

Corollary 6.1

For any $\mathrm{S} \in \mathbf{C}[\mathrm{Y}_1, \mathrm{Y}_2]$ *where* $\mathrm{Y}_1, \mathrm{Y}_2$ *are defined in lemma* 6.4 *with positive valuation and any* $\delta_m = (\delta_{n,m})_n \in \mathbf{Z}$ *we have*

$$\mathrm{S}\delta_m = 0$$

Proof.

This is straightforward if one remarks that $\mathrm{Y}_1, \mathrm{Y}_2$ have valuation $\geqslant 1$ whereas δ_m only contains constant coefficients. ■

All the ingredients required to solve the cohomological equation with obstruction have now been obtained.

6.8 Solution to the cohomological equation

We want to solve $\psi = \mathrm{A}\varphi + \lambda_\psi \delta_0$. The conjugacy we made reduces our problem to solve the following diagonal coefficient equations:

$$\psi = (\psi_p) = \mathrm{D}\varphi = \left(\mathrm{D}_0 + \varepsilon \mathrm{D}_1 + \varepsilon^2 \mathrm{D}_2\right)(\varphi_p)$$

Indeed, suppose we can find a solution $\Phi = \Phi_\psi$ to the equation

$$\psi - \lambda_\psi \delta_0 = \mathrm{D}\Phi_\psi + r_\psi \tag{6.34}$$

where λ_ψ is an obstruction and r_ψ is some error term. We obtain with (6.24), (6.34) and corollary 6.1:

$$\begin{aligned}
\mathrm{C}(\psi - \lambda\delta_0) &= \mathrm{CD}\Phi + \mathrm{C}r_\psi \\
&= \mathrm{AC}\Phi - \varepsilon^3 \mathrm{R}\Phi + \mathrm{C}r_\psi \\
\psi + \varepsilon \mathrm{Y}_1\psi + \varepsilon^2 \mathrm{Y}_2\psi - \lambda_\psi &= \mathrm{AC}\Phi - \varepsilon^3 \mathrm{R}\Phi + \mathrm{C}r_\psi
\end{aligned}$$

Still using corollary 6.1 it remains that

$$\varphi = \varphi_\psi := \mathrm{C}\Phi_\psi \tag{6.35}$$

solves the equation

$$\begin{aligned}\psi - \lambda_\psi \delta_0 &= \mathrm{A}\varphi - \varepsilon^3 \mathrm{R}\Phi + \mathrm{C} r_\psi - \varepsilon \mathrm{Y}_1 \psi - \varepsilon^2 \mathrm{Y}_2 \psi \\ &= \mathrm{A}\varphi + \Delta_\psi \end{aligned}\tag{6.36}$$

where

$$\begin{aligned}\Delta_\psi &= \mathrm{C} r_\psi - \varepsilon^3 \mathrm{R}\Phi_\psi - \varepsilon \mathrm{Y}_1 \psi - \varepsilon^2 \mathrm{Y}_2 \psi \\ &= r_\psi + \varepsilon \mathrm{Y}_1 r_\psi + \varepsilon^2 \mathrm{Y}_2 r_\psi - \varepsilon^3 \mathrm{R}\Phi_\psi - \varepsilon \mathrm{Y}_1 \psi - \varepsilon^2 \mathrm{Y}_2 \psi \end{aligned}\tag{6.37}$$

But (6.18) and remark 6.2 ensure that

$$\begin{aligned}\|\Delta_\psi\|_{\sigma,r} \lesssim\ & \|r_\psi\|_{\sigma,r} + \varepsilon \|r_\psi\|_{\sigma,r+1} + \varepsilon^2 \|r_\psi\|_{\sigma,r+2} \\ &+ \varepsilon^3 \|\Phi_\psi\|_{\sigma,r+3} + \varepsilon^4 \|\Phi_\psi\|_{\sigma,r+4} \\ &+ \varepsilon \|\psi\|_{\sigma,1} + \varepsilon^2 \|\psi\|_{\sigma,2} \end{aligned}\tag{6.38}$$

Assume that we have the following estimates for some functions $m, n : \mathbf{N} \to \mathbf{R}$

$$\|\Phi_\psi\|_{\sigma,r} \lesssim \varepsilon^{m(r)} \|\psi_0\|_{r-1} + \|\psi\|_{\sigma,r+4} \tag{6.39}$$

$$\|r_\psi\|_{\sigma,r} \lesssim \varepsilon^{n(r)} \|\psi_0\|_{r} + \varepsilon^2 \|\psi\|_{\sigma,r+4} \tag{6.40}$$

Then it remains in (6.38)

$$\begin{aligned}\|\Delta_\psi\|_{\sigma,r} \lesssim\ & \varepsilon^{n(r)} \|\psi_0\|_r + \varepsilon^{1+n(r+1)} \|\psi_0\|_{r+1} + \varepsilon^{2+n(r+2)} \|\psi_0\|_{r+2} \\ &+ \varepsilon^{3+m(r+3)} \|\psi_0\|_{r+2} + \varepsilon^{4+m(r+4)} \|\psi_0\|_{r+3} \\ &+ \varepsilon^2 \|\psi\|_{\sigma,r+4} + \varepsilon^3 \|\psi\|_{\sigma,r+5} + \varepsilon^4 \|\psi\|_{\sigma,r+6} \\ &+ \varepsilon^3 \|\psi\|_{\sigma,r+7} + \varepsilon^4 \|\psi\|_{\sigma,r+8} \\ &+ \varepsilon \|\psi\|_{\sigma,1} + \varepsilon^2 \|\psi\|_{\sigma,2} \end{aligned}\tag{6.41}$$

So, up to the error terms in (6.41) whose orders depend on ψ and will be more carefully analyzed throughout the remainder, the equation $\psi - \lambda_\psi = \mathrm{A}\varphi$ is equivalent to

$$\psi - \lambda_\psi = (\mathrm{D}_0 + \varepsilon \mathrm{D}_1 + \varepsilon^2 \mathrm{D}_2)\varphi$$

Like in chapter 5, this takes the form of an infinite countable system of differential equations

$$(\mathrm{ED}_n) \qquad \psi_n = \mathrm{T}_n \varphi_n$$

as follows:

$$\begin{aligned}\psi_n &= \gamma_n \varphi_n + \varepsilon \alpha_n v_{1,0}(\cdot)\, {\varphi_n}' + \varepsilon^2 \alpha_n r(\cdot)\, {\varphi_n}' + \varepsilon^2 \alpha_n t(\cdot)\, {\varphi_n}'' \\ &= \gamma_n \varphi_n + \varepsilon a_n(\cdot)\, {\varphi_n}' + \varepsilon^2 b_n(\cdot)\, {\varphi_n}''\end{aligned} \tag{6.42}$$

where r, t (see equation (6.32)) have already been specified and studied, and

$$a_n = \alpha_n(v_{1,0} + \varepsilon r) \qquad \text{and} \qquad b_n = \alpha_n t \tag{6.43}$$

Remark 6.4

Definition 6.1 *and Kolmogorov inequalities ensure that*

$$\begin{aligned}\left|r^{(p)}\right| &\lesssim \left|{v_{2,0}}^{(p)}\right| + \frac{1}{\kappa}\sum_{\ell \neq 0} \left|{v_{1,\ell}}^{(p)}\right| \left|{v_{1,-\ell}}^{(p+1)}\right| |\ell|^{\tau} \\ &\lesssim \frac{1}{k^4 \sin^2(k\pi\alpha)} + \frac{1}{\kappa}\int_{\mathbf{T}} \left|\partial_y^p v_1(x,y)\right|^2 \mathrm{d}x . \int_{\mathbf{T}} \left|\partial_x^{1+\tau^+} \partial^{p+1} v_1(x,y)\right|^2 \mathrm{d}x \\ &\lesssim \frac{1}{\kappa} \frac{|k|^{1+\tau^+}}{\sin^2(k\pi\alpha)} \|\mathrm{V}\|_{p+1} \\ \|r\|_p &\lesssim \frac{1}{\kappa^3} |k|^{3+\tau^+} \|\mathrm{V}\|_{p+1}\end{aligned}$$

And also the identity $t(y) = {v_{1,0}}^2(y) + \dfrac{\mathrm{c}^{\mathrm{st}}}{|\sin(k\pi\alpha)|}$ *leads to*

$$\|t\|_p \lesssim \|v_{1,0}\|_p^2 \lesssim \frac{1}{\sin^2(k\pi\alpha)} \lesssim \frac{|k|^{2\tau}}{\kappa^2}$$

where we remind that V *is the quasi-periodic part of the potential of the Schrödinger operator in* (6.1).

In our setting, the degree of all the operators involved is bounded by 4. Moreover we are interested in $\mathscr{C}^j$-estimates of $\mathcal{U}\varphi$ (for $\mathcal{U}$ see (6.12)) the function induced on $\mathbf{T}^2$ by the sequence obtained after solving the diagonal equations. Lemmas 6.3 and 6.2 give us those estimates. If we look back at the initial cohomological equation (6.6), we can see that we solved it with an additional error term which is $\mathrm{O}(\varepsilon^3 \mathcal{U}\widetilde{\varphi})$. Then we have solved (6.4) with $\phi = \mathcal{U}\widetilde{\varphi}$ provided we can estimate $\|\phi\|_j$ w.r.t. ε.

Let us now investigate the equation $\widetilde{\psi} = \mathrm{D}\widetilde{\varphi}$ and estimate its attached error terms in two different cases, depending on the positivity or the vanishing of the following quantity

$$\delta_k := \mathbf{E}(\mathrm{W}) + \widehat{\mathrm{V}}_0 - |\widehat{\mathrm{V}}_k| \tag{6.44}$$

6.9 The case $\delta_k > 0$

6.9.1 The diagonal cohomological equation

When $\delta_k = \mathbf{E}(\mathrm{W}) + \widehat{\mathrm{V}}_0 - |\widehat{\mathrm{V}}_k| > 0$ we obtain the following result:

Proposition 6.1

Given a $\psi = (\psi_n)_{n\in\mathbf{Z}} \in \mathcal{E}_{\sigma,4}$, we can construct $\varphi = (\varphi_n)$ satisfying the equation

$$\psi_n - \delta_{n,0}\left(\lambda_\psi + \varepsilon\lambda_{f_{r,t}(\psi-\lambda_\psi)}\right) = \left((\mathrm{D}_0 + \varepsilon\mathrm{D}_1 + \varepsilon^2\mathrm{D}_2)\varphi\right)_n + \varepsilon_n$$

with obstruction

$$\lambda_\psi = \frac{1}{\sqrt{(\widehat{\mathrm{V}}_0 + \mathbf{E}(\mathrm{W}))^2 - |\widehat{\mathrm{V}}_k|^2}} \int_{\mathbf{T}} \frac{\psi_0(s)}{\delta_k + 2|\widehat{\mathrm{V}}_k|\sin^2(\pi s + \pi k\alpha - \frac{1}{2}\theta_k)}\,\mathrm{d}s \qquad (6.45)$$

and error $\qquad \forall n \neq 0 \qquad \|\varepsilon_n\|_j \lesssim \varepsilon^2 \|\psi_n\|_{j+4}\,|n|^{2\tau}\,\frac{|k|^{6\tau}}{\kappa^8}$

$$|\varepsilon_0| \leqslant \frac{k^{3+\tau^+}}{\kappa^3}\frac{\|\psi_0\|_3}{\delta_k{}^4}\varepsilon^2$$

With respect to ε, we have $\varphi_n = \mathrm{O}(1)$ except for the case $n = 0$ where $\varphi_0 = \frac{1}{\varepsilon}\check{\varphi}_0$ and $\|\check{\varphi}_0\|_j = \mathrm{O}(1)$ for all integer j.

Proof.

For $n \neq 0$ we have $\gamma_n \neq 0$ so the iterative technique developed in the proof of lemma 5.3 (chapter 5) still gives a solution with similar estimates: we solve the equation with a perturbative scheme and try to improve the order of the error term. We first set $\varphi_{n,0} = \frac{1}{\gamma_n}\psi_n$ and compute the error made compared to an exact solution:

$$\gamma_n\varphi_{n,0} + \varepsilon a_n\,\varphi_{n,0}{}' + \varepsilon^2 b_n\,\varphi_{n,0}{}'' = \psi_n + \varepsilon_{n,1}$$

where $\qquad \varepsilon_{n,1} = \varepsilon a_n\,\varphi_{n,0}{}' + \varepsilon^2 b_n\,\varphi_{n,0}{}'' = \varepsilon\frac{a_n}{\gamma_n}\,\psi_n{}' + \varepsilon^2\frac{b_n}{\gamma_n}\,\psi_n{}''$

so that under the diophantine assumption $\alpha \in \mathrm{DC}(\kappa,\tau)$ (see 2.3) and the estimations of $\|a_n\|, \|b_n\|$ (see (6.43) and remark 6.4) we obtain

$$\|\varepsilon_{n,1}\|_j \lesssim \|\psi_n\|_{j+2}\frac{|n|^\tau}{\kappa^2}\varepsilon$$

We now want to write for the error term: $\varepsilon_{n,1} = -\mathrm{T}_n\varphi_{n,1}$. Always considering the lowest order we set

$$\varphi_{n,1} = -\frac{\varepsilon_{n,1}}{\gamma_n} = -\frac{\varepsilon a_n\,{\psi_n}' + \varepsilon^2 b_n\,{\psi_n}''}{{\gamma_n}^2}$$

so that $$\|\varphi_{n,1}\|_j = \mathrm{O}\left(\frac{\|\psi_n\|_{j+2}}{\kappa\,|{\gamma_n}^2|}\varepsilon\right) = \mathrm{O}\left(\frac{\|\psi_n\|_{j+2}}{\kappa^3}\,|n|^{2\tau}\,\varepsilon\right)$$

and $$\varphi_n = \varphi_{n,0} + \varphi_{n,1}$$

The evaluating of the error follows

$$\begin{aligned}\gamma_n\varphi_n + \varepsilon a_n\,{\varphi_n}' + \varepsilon^2 b_n\,{\varphi_n}'' &= \psi_n + \varepsilon_{n,1} + \gamma_n\varphi_{n,1} + \varepsilon a_n\,{\varphi_{n,1}}' + \varepsilon^2 b_n\,{\varphi_{n,1}}'' \\ &= \psi_n + \varepsilon_{n,1} - \varepsilon_{n,1} + \varepsilon_{n,2}\end{aligned}$$

with $$\begin{aligned}\varepsilon_{n,2} &= \varepsilon a_n\,{\varphi_{n,1}}' + \varepsilon^2 b_n\,{\varphi_{n,1}}'' \\ &= -\varepsilon a_n\left(\frac{\varepsilon a_n\,{\psi_n}' + \varepsilon^2 b_n\,{\psi_n}''}{{\gamma_n}^2}\right)' - \varepsilon^2 b_n\left(\frac{\varepsilon a_n\,{\psi_n}' + \varepsilon^2 b_n\,{\psi_n}''}{{\gamma_n}^2}\right)''\end{aligned}$$

so $$\begin{aligned}\|\varepsilon_{n,2}\|_j &\lesssim \varepsilon^2\frac{\|\psi_n\|_{j+4}}{|{\gamma_n}^2|}\left(\|a_n\|_2^2 + \|b_n\|_2^2\right) \\ &\lesssim \varepsilon^2\|\psi_n\|_{j+4}\,|n|^{2\tau}\,\frac{|k|^{2\tau}}{\kappa^4\sin^4(\pi k\alpha)} \\ \|\varepsilon_{n,2}\|_j &\lesssim \varepsilon^2\|\psi_n\|_{j+4}\,|n|^{2\tau}\,\frac{|k|^{6\tau}}{\kappa^8}\end{aligned}$$

We could go on with this procedure and obtain an error of order $\mathrm{O}(\varepsilon^3)$ but this is enough for our purpose.

For the case $n = 0$ we have to handle the following differential equation of *pseudo order* 2

$$\psi_0 - \lambda_\psi = \varepsilon a_0\,{\varphi_0}' + \varepsilon^2 b_0\,{\varphi_0}''$$

By pseudo order 2 we mean that we have to solve a first-order differential equation for ${\varphi_0}'$ and then take an antiderivative of this solution. The fulfillment of the periodicity conditions on φ_0 and ${\varphi_0}'$ give rise to the obstruction λ_ψ (see lemma 5.5 in chapter 5).

We already proved that $b_0 = t$ is positive with bounds depending on the variance $\mathbf{V}(\mathrm{W})$ (see (6.33)). So that one can divide by $\varepsilon^2 b_0$ to actually solve a first-order differential equation

$$\Phi' + \tfrac{1}{\varepsilon}a = \Psi$$

with Ψ of order ε^{-2}. Periodic solutions to those equations have $\mathscr{C}^j$-norms $\lesssim \varepsilon\|\Psi\|_j = \mathrm{O}(\varepsilon^{-1})$ when a does not vanish (see appendix A.2). Remind that $a_0 = v_{1,0} + \mathrm{O}(\varepsilon)$ (see (6.43)). Let us analyze $v_{1,0}(y) = \fint_{\mathbf{T}} v_1(x,y)\,\mathrm{d}x$. We recall that we computed in section 6.4

$$v_1(x,y) = \mathbf{E}\left(v_{1,\omega}(x,y)\right) = \frac{1}{\pi}\mathbf{E}(v(\omega,x))\frac{\sin^2\left(\pi\left(k(x+\alpha)+y\right)\right)}{|\sin \pi k\alpha|} \tag{6.46}$$
$$= \frac{1}{2\pi\,|\sin \pi k\alpha|}\big(\mathrm{V}(x)+\mathbf{E}(\mathrm{W}_\omega)\big)\big(1-\cos\left(2\pi kx+2\pi k\alpha+2\pi y\right)\big)$$

After integration w.r.t x we obtain

$$v_{1,0}(y) = (\widehat{v}_1)_0(y) = \frac{1}{2\pi\,|\sin \pi k\alpha|}\left(\widehat{\mathrm{V}}_0+\mathbf{E}(\mathrm{W}_\omega)-|\widehat{\mathrm{V}}_k|\cos(2\pi y+2\pi k\alpha-\theta_k)\right) \tag{6.47}$$
$$= \frac{1}{2\pi\,|\sin \pi k\alpha|}\left(\delta_k+2\left|\widehat{\mathrm{V}}_k\right|\sin^2(\pi y+\pi k\alpha-\tfrac{1}{2}\theta_k)\right)$$

where $\widehat{\mathrm{V}}_k = \int_{\mathbf{T}} \mathrm{V}(x)\mathrm{e}^{-2\mathrm{i}\pi kx}\,\mathrm{d}x$ stands for the k-th Fourier coefficient of $\mathrm{V}(\cdot)$ and $\theta_k = \operatorname{Arg}\widehat{\mathrm{V}}_k$ its argument. In the current context, the fact that v_1 does not vanish is granted under the condition $\delta_k > 0$ (remind (6.44)).

We could also use this method to solve the entire equation and get the obstruction of the form $\lambda_\psi = \int_{\mathbf{T}} \psi(y)\,\rho(y)\,\mathrm{d}y$ where ρ is the L^1 normalization of the solution of

$$f' - \frac{a}{\varepsilon}f = 1$$

But this would make the computation of the obstruction complicated as one would have to estimate integrals with respect to a density which is not explicit. So we rather use our perturbative technique that we call *method of order* 1 which will extract the ε orders out of the integrals w.r.t. ρ: we solve at the lowest order

$$\psi_0 - \lambda_\psi = \varepsilon\, v_{1,0}\,{\varphi_0}'$$

And consequently we set

$$\phi_0(y) = \frac{1}{\varepsilon}2\pi\,|\sin \pi k\alpha| \int_0^y \frac{\psi_0(s)-\lambda_\psi}{\delta_k+2|\widehat{\mathrm{V}}_k|\sin^2(\pi(s+k\alpha)-\frac{1}{2}\theta_k)}\,\mathrm{d}s$$

which gives the obstruction

$$\lambda_\psi = \frac{\displaystyle\int_{\mathbf{T}} \frac{\psi(s)}{\delta_k+2\left|\widehat{\mathrm{V}}_k\right|\sin^2(\pi(s+k\alpha)-\frac{1}{2}\theta_k)}\,\mathrm{d}s}{\displaystyle\int_{\mathbf{T}} \frac{\mathrm{d}s}{\delta_k+2\left|\widehat{\mathrm{V}}_k\right|\sin^2(\pi(s+k\alpha)-\frac{1}{2}\theta_k)}} = \frac{\displaystyle\int_{\mathbf{T}} \frac{\psi(s)}{\delta_k+2\left|\widehat{\mathrm{V}}_k\right|\sin^2(\pi(s+k\alpha)-\frac{1}{2}\theta_k)}\,\mathrm{d}s}{\displaystyle\int_{\mathbf{T}} \frac{\mathrm{d}s}{\delta_k+2\left|\widehat{\mathrm{V}}_k\right|\sin^2(\pi(s+\pi k\alpha)-\frac{1}{2}\theta_k)}}$$
$$= \int_{\mathbf{T}} \psi(s)\,\rho(s)\,\mathrm{d}s$$

where $\rho(\cdot) = \frac{f(\cdot)}{\int_{\mathbf{T}} f(u)\,\mathrm{d}u}$ is a density with

$$f(s) = \frac{1}{\delta_k+2|\widehat{\mathrm{V}}_k|\sin^2(\pi(s+\pi k\alpha)-\frac{1}{2}\theta_k)}$$
$$\int_{\mathbf{T}} f(u)\,\mathrm{d}u = \left(\left(\widehat{\mathrm{V}}_0+\mathbf{E}(\mathrm{W})\right)^2-|\widehat{\mathrm{V}}_k|^2\right)^{-\frac{1}{2}}$$

The following $\mathscr{C}^p$-estimates of ϕ_0 are straightforward

$$\|\phi_0\|_0 \lesssim \frac{|\sin(k\pi\alpha)|}{\sqrt{\left(\widehat{\mathrm{V}}_0 + \mathbf{E}(\mathrm{W})\right)^2 - |\widehat{\mathrm{V}}_k|^2}} \frac{\|\psi_0\|_0}{\varepsilon}$$

$$\|\phi_0\|_{p\geqslant 1} \lesssim \frac{|\sin(k\pi\alpha)|}{{\delta_k}^{p+1}} \frac{\|\psi_0\|_{p-1}}{\varepsilon}$$

Let us evaluate the error made with this method

$$\varepsilon a_0\, \phi_0{}' + \varepsilon^2 b_0\, \phi_0{}'' = \psi_0 - \lambda_\psi + \epsilon_1$$

$$\epsilon_1 = \epsilon_1(\psi) = \varepsilon r\, \phi_0{}' + \varepsilon^2 t\, \phi_0{}'' = \varepsilon r\, \frac{\psi_0 - \lambda_\psi}{v_{1,0}} + \varepsilon t \left(\frac{\psi_0 - \lambda_\psi}{v_{1,0}}\right)'$$

One can compute

$$\lambda_{\epsilon_1} = \varepsilon \int_{\mathbf{T}} r(y)\, \rho(y)\, \frac{\psi_0 - \lambda_\psi}{v_{1,0}} + t(y)\, \rho(y) \left(\frac{\psi_0 - \lambda_\psi}{v_{1,0}}\right)' \mathrm{d}y$$

$$= \varepsilon \lambda_{r\, \frac{\psi - \lambda_\psi}{v_{1,0}}} + \varepsilon \lambda_{t \left(\frac{\psi - \lambda_\psi}{v_{1,0}}\right)'}$$

and also

$$\begin{aligned}
\lambda_{\epsilon_1} &= \varepsilon \int_{\mathbf{T}} \left(r\, \rho - (t\, \rho)'\right) \frac{\psi_0 - \lambda_\psi}{v_{1,0}}\, \mathrm{d}y \\
&= \varepsilon \int_{\mathbf{T}} \left(r\, \rho - (t\, \rho)'\right) (\psi_0 - \lambda_\psi)\, \rho \left(\textstyle\int_{\mathbf{T}} \mathrm{d}t / v_{1,0}(t)\right) \mathrm{d}y \\
\lambda_{\epsilon_1} &= \varepsilon \int_{\mathbf{T}} f_{r,t}\, (\psi_0 - \lambda_\psi)\, \rho\, \mathrm{d}y
\end{aligned}$$

$$\text{where} \qquad f_{r,t} := \left(\textstyle\int_{\mathbf{T}} \mathrm{d}s / v_{1,0}(s)\right) \left(r\, \rho - (t\, \rho)'\right) \tag{6.48}$$

As a consequence

$$\lambda_{\epsilon_1}(\psi) = \varepsilon \lambda_{r \frac{\psi - \lambda_\psi}{v_{1,0}}} + \varepsilon \lambda_{t \left(\frac{\psi - \lambda_\psi}{v_{1,0}}\right)'} = \varepsilon \lambda_{f_{r,t}\psi} - \varepsilon \lambda_\psi \lambda_{f_{r,t}} \tag{6.49}$$

Then we want to write $-\epsilon_1 = \mathrm{T}_0 \phi_1$ and considering the lowest order we set

$$\phi_1(y) = -\frac{1}{\varepsilon} \int_0^y (\epsilon_1 - \lambda_{\epsilon_1}) \rho$$

The error made with $\varphi_0 := \phi_0 + \phi_1$ is

$$\begin{aligned}
\varepsilon a_0\, \varphi_0{}' + \varepsilon^2 b_0\, \varphi_0{}'' &= \psi_0 - \lambda_\psi + \epsilon_1 + \varepsilon a_0\, \phi_1{}' + \varepsilon^2 b_0\, \phi_1{}'' \\
&= \psi_0 - \lambda_\psi + \epsilon_1 - \epsilon_1 + \lambda_{\epsilon_1} + \epsilon_2
\end{aligned}$$

with $$\epsilon_2 = \varepsilon^2 r\, \phi_1{}' + \varepsilon^2 t\, \phi_1{}''$$

Such that, with the estimates of remark (6.4), we deduce

$$\|\epsilon_2\|_j \lesssim \frac{k^{3+\tau^+}}{\kappa^3}\frac{\|\psi_0\|_{j+3}}{\delta_k{}^4}\varepsilon^2$$

Finally the r_ψ in (6.34) is then the sequence $(\varepsilon_n)_{n\in\mathbf{Z}}$ and verifies, for σ such that $\psi_n \in \mathcal{E}_{\sigma,s+4}$ and $\sigma' < \sigma$,

$$\|\varepsilon_n\|_{\sigma',s} \leqslant \varepsilon^2 k^{6\tau} \frac{\kappa^{-8}}{\delta_k{}^4(\sigma-\sigma')^{2\tau+1}}\|\psi_n\|_{\sigma,s+4}$$

Furthermore, (remind (6.12))

$$\varepsilon^3\|\mathrm{R}\widetilde{\varphi}\|_{\sigma',0} \lesssim \varepsilon^3\|\widetilde{\varphi}\|_{\sigma,4} \lesssim \varepsilon^2\|\psi_n\|_{\sigma,3} \lesssim \varepsilon^2\|\psi = \mathcal{U}(\psi_n)\|_{\sigma,3}$$

Now we are in possession of all the error terms needed in (6.36) to write down a formula for a stationary measure.

Remark 6.5

If $\psi = \psi(y)$, *and so* $\psi_n = 0$ *for* $n \neq 0$, *or if* $\psi_n \neq 0$ *for at most a finite number of integers* n *then we have* $\|\psi_n\|_{\sigma,3} \lesssim \|\psi\|_3$ *and* $\|\psi_n\|_{\sigma,s+4} \lesssim \|\psi\|_{s+4}$. *These cases would in particular occur if we chose the quasi-periodic part of the potential* $\mathrm{V}(x)$ *in* (6.1) *to be a trigonometric polynomial.*

■

6.9.2 Development of a stationary measure

We can now conclude for a stationary measure:

Proposition 6.2

In the resonant case $\mathrm{E} = 2\cos(k\pi\alpha)$ *with* $k \neq 0$ *and under the condition*

$$\delta_k = \mathbf{E}(\mathrm{W}) + \widehat{\mathrm{V}}_0 - |\widehat{\mathrm{V}}_k| > 0$$

we can estimate the integral of $\psi \in \mathcal{E}_{\sigma,8}$ *(remind definition (6.3)) w.r.t. any stationary measure* μ_ε *for* $\mathrm{F}_\omega : (x,y) \mapsto (x+\alpha, y + v_\omega(x,y))$*:*

$$\int_{\mathbf{T}} \psi \,\mathrm{d}\mu_\varepsilon = \lambda_\psi + \varepsilon\nu_\psi + \mathrm{O}_{\kappa,k,\tau}\left(\varepsilon^2 \|\psi\|_{\sigma,8}\right)$$

where

$$\lambda_\psi = \int_{\mathbf{T}^2} \psi(x,y)\,\rho(y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\nu_\psi = \lambda_{v_\gamma\,\partial_y\psi} - \lambda_{r\,\frac{\psi-\lambda_\psi}{v_{1,0}}} - \lambda_{t\left(\frac{\psi-\lambda_\psi}{v_{1,0}}\right)'} = \lambda_{v_\gamma\,\partial_y\psi} - \lambda_{f_{r,t}\,\psi} + \lambda_\psi\lambda_{f_{r,t}}$$

for an explicit density ρ *which is the* L^1*-normalization of* $1/v_{1,0}$*:*

$$f(\cdot) = \left(\delta_k + 2|\widehat{\mathrm{V}}_k|\sin^2(\pi\cdot + \pi k\alpha - \tfrac{1}{2}\theta_k)\right)^{-1}$$

Proof.

In proposition 6.1, we have solved for any given ψ_n (remind (6.48))

$$\psi_n - \left(\lambda_\psi + \varepsilon\lambda_{f_{r,t}(\psi-\lambda_\psi)}\right)\delta_{n,0} = \left((\mathrm{D}_0 + \varepsilon\mathrm{D}_1 + \varepsilon^2\mathrm{D}_2)\widetilde{\varphi}\right)_n + \varepsilon_n$$

This gives (6.36) with $r_\psi = \sum_{n\in\mathbf{Z}} \varepsilon_n \mathrm{e}^{2\mathrm{i}\pi n\cdot}$ and $\Delta_\psi = -\varepsilon\mathrm{Y}_1\psi + \mathrm{O}(\varepsilon^2)$. Let us write (6.36) for $\psi = \Delta_\psi$

$$\Delta_\psi - \left(\lambda_{\Delta_\psi} + \varepsilon\lambda_{f_{r,t}(\Delta_\psi - \lambda_{\Delta_\psi})}\right)\delta_0 = \mathrm{A}\varphi_{\Delta_\psi} + \Delta_{\Delta_\psi}$$

Thanks to (6.37) we get $\lambda_{\Delta_\psi} = -\varepsilon\lambda_{\mathrm{Y}_1\psi} + \mathrm{O}(\varepsilon^2)$.

Note that if $\psi_n = \widehat{\Psi}_n$ are the Fourier coefficients of a function Ψ on $\mathbf{T}^2$ (which is the case for the Lyapunov exponent we compute), then

$$\begin{aligned}(\mathrm{Y}_1\psi)_0 &= -\sum_{n\in\mathbf{Z}\setminus\{0\}} \frac{v_{1,-n}}{\gamma_{-n}}\,{\psi_n}' = -\int_{\mathbf{T}} v_\gamma(x,\cdot)\,\partial_y\mathcal{U}\psi(x,\cdot)\,\mathrm{d}x \\ &= -\int_{\mathbf{T}} v_\gamma(x,\cdot)\,\partial_y\Psi(x,\cdot)\,\mathrm{d}x \end{aligned} \tag{6.50}$$

Hence

$$\lambda_{\mathrm{Y}_1\psi} = -\lambda_{v_\gamma\,\partial_y\psi} \tag{6.51}$$

With equation (6.36) and the development of the cohomological equation this proves that $\phi = \mathcal{U}(\varphi_\psi + \varphi_{\Delta_\psi})$ (remember the definition of $\mathcal{U}$ in (6.12)) verifies

$$\mathrm{T}\phi - \phi = \psi - \lambda_\psi - \varepsilon\lambda_{v_\gamma\,\partial_y\psi} + \varepsilon\lambda_{f_{r,t}(\psi-\lambda_\psi)} + \widetilde{\Delta}_\psi$$

where $\widetilde{\Delta}_\psi = \mathrm{O}(\varepsilon^2 \|\psi\|_{\sigma,8})$ since $\widetilde{\varphi}_0 = \mathrm{O}(\varepsilon^{-1})$ (see proposition 6.1). ■

Remark 6.6

The error term $\mathrm{O}(\varepsilon^2\|\psi\|_4)$ *is proportional to*

$$\varepsilon^2\|\psi\|_4 \frac{\mathrm{P}_\tau(|k|)}{\mathrm{Q}(\kappa)\,\mathrm{R}(\delta_k)}$$

where $\mathrm{P}, \mathrm{Q}, \mathrm{R}$ *are polynomials of valuation* $\geqslant 1$.

6.9.3 Formula for the exponent

We deduce for the Lyapunov exponent the following formula:

Theorem 6.1

Assume $\alpha \in \mathrm{DC}(\kappa, \tau)$. *For an* α*-resonant energy* $\mathrm{E} = 2\cos(k\pi\alpha)$ *with* $k \neq 0$ *and under the assumption* $\delta_k = \widehat{\mathrm{V}}_0 + \mathbf{E}(\mathrm{W}) - |\widehat{\mathrm{V}}_k| > 0$, *the following development holds for* ε *small enough:*

$$\mathrm{L}(\mathrm{E}) = \lambda_\Phi \varepsilon^2 + \mathrm{O}\left(\varepsilon^3 k^{\mathrm{A}} \kappa^{-\mathrm{B}} {\delta_k}^{-\mathrm{C}}\right)$$

where $\mathrm{A}, \mathrm{B}, \mathrm{C}$ *are integers,* $\psi \mapsto \lambda_\psi$ *is defined by* (6.45) *and* Φ *is the positive function*

$$\Phi = \frac{\mathbf{V}(\mathrm{W})}{8\pi\,|\sin(k\pi\alpha)|}\left(\frac{{v_{1,0}}'}{v_{1,0}}\right)^2 + \frac{1}{4}\int_{\mathbf{T}} \mathbf{E}\left[(\partial_y v_{1,\omega})^2\right] + \left[\mathbf{E}\,\partial_y v_{1,\omega}\right]^2 \,\mathrm{d}x$$

and $$v_{1,0}(y) = \frac{1}{2\pi\,|\sin \pi k\alpha|}\left(\delta_k + 2\left|\widehat{\mathrm{V}}_k\right| \sin^2(\pi y + \pi k\alpha - \tfrac{1}{2}\theta_k)\right)$$

Proof.

Let us use formula (4.9) established in chapter 4 to compute $\gamma(\mathrm{F})$ (remember (4.3)). Of course, as expected, the lowest order term of the stationary measure vanishes when one computes the Lyapunov exponent: indeed we have to integrate ${v_{1,0}}'/v_{1,0}$ w.r.t the Lebesgue measure. So the first order of the development is crucial to get a non trivial formula for the exponent. We need the following quantities that we compute using definition (6.1):

$$\partial_y v_\omega(x,y) = \varepsilon\partial_y v_{1,\omega}(x,y) + \varepsilon^2\partial_y v_{2,\omega}(x,y) + \mathrm{O}(\varepsilon^3)$$

$$\partial_y v_{1,\omega}(x,y) = \frac{\sin\left(2\pi(k(x+\alpha)+y)\right)}{|\sin(\pi k\alpha)|} v(\omega,x)$$

$$(\partial_y v_{1,\omega}(x,y))^2 = \frac{\sin^2\left(2\pi(k(x+\alpha)+y)\right)}{\sin^2(\pi k\alpha)} v(\omega,x)^2$$

$$\partial_y v_{2,\omega}(x,y) = \frac{\sin^2\left(2\pi(k(x+\alpha)+y)\right) + 2\sin^2\left(\pi(k(x+\alpha)+y)\right)\cos\left(2\pi(k(x+\alpha)+y)\right)}{2\sin^2(\pi k\alpha)} v(\omega,x)^2$$

Hence we obtain with a Taylor development

$$\mathbf{E}\ln|1+\partial_y v_\omega(x,y)| = \varepsilon l_1(x,y) + \varepsilon^2 l_2(x,y) + \mathrm{O}(\varepsilon^3) \tag{6.52}$$

where

$$l_1(x,y) = \mathbf{E}(\partial_y v_{1,\omega}(x,y)) = \partial_y v_1(x,y) = \frac{\sin\left(2\pi(k(x+\alpha)+y)\right)}{|\sin(\pi k\alpha)|}(\mathrm{V}(x)+\mathbf{E}(\mathrm{W}))$$

$$\begin{aligned} l_2(x,y) &= \partial_y v_2(x,y) - \tfrac{1}{2}\,\mathbf{E}\left[(\partial_y v_{1,\omega}(x,y))^2\right] = \tfrac{1}{2}\partial_y^2\,\mathbf{E}({v_{1,\omega}}^2(x,y)) - \tfrac{1}{2}\,\mathbf{E}\left[(\partial_y v_{1,\omega}(x,y))^2\right] \\ &= \frac{\sin^2\left(\pi(k(x+\alpha)+y)\right)\cos\left(2\pi(k(x+\alpha)+y)\right)}{\sin^2(\pi k\alpha)}\left(\mathrm{V}(x)^2 + \mathbf{E}^2(\mathrm{W}) + 2\mathrm{V}(x)\,\mathbf{E}(\mathrm{W})\right) \end{aligned}$$

We already get

$$\int_{\mathbf{T}} l_1(x,y)\,\mathrm{d}x = {v_{1,0}}'(y)$$

so that

$$\lambda_{l_1} \propto \int_{\mathbf{T}} {v_{1,0}}'/v_{1,0} = 0$$

and

$$\int_{\mathbf{T}^2} l_1\,\mathrm{d}\mu_\varepsilon = 0 + \varepsilon\nu_{l_1} + \mathrm{O}(\varepsilon^2)$$

Then proposition 6.2 ensures that

$$\begin{aligned} \nu_{l_1} &= \nu_{{v_{1,0}}'} \\ &= \lambda_{v_\gamma\,\partial_y l_1} - \lambda_{r\,\frac{{v_{1,0}}'}{v_{1,0}}} - \lambda_{t\left(\frac{{v_{1,0}}'}{v_{1,0}}\right)'} \\ \nu_{l_1} &= \lambda_{v_\gamma\,\partial_y^2 v_1} - \lambda_{r\,\frac{{v_{1,0}}'}{v_{1,0}}} - \lambda_{t\left(\frac{{v_{1,0}}'}{v_{1,0}}\right)'} \end{aligned} \tag{6.53}$$

Remind that (see equations (6.32))

$$\begin{aligned} r &= v_{2,0} + u = v_{2,0} + \int_{\mathbf{T}} v_\gamma(x,\cdot)\,\partial_y v_1(x,\cdot)\,\mathrm{d}x \\ &= \tfrac{1}{2}{w_0}'(y) + \int_{\mathbf{T}} v_\gamma(x,\cdot)\,\partial_y v_1(x,\cdot)\,\mathrm{d}x \\ t &= w_0 + \int_{\mathbf{T}} v_\gamma(x,\cdot)\,v_1(x,\cdot)\,\mathrm{d}x = \frac{1}{4\pi\,|\sin\pi k\alpha|}\,\mathbf{V}(\mathrm{W}) + \tfrac{1}{2}{v_{1,0}}^2 \end{aligned}$$

Let us take a closer look at

$$\begin{aligned}\left(\textstyle\int_{\mathbf{T}} \frac{\mathrm{d}y}{v_{1,0}(y)}\right) \lambda_{r\, v_{1,0}'/v_{1,0}} &= \int_{\mathbf{T}} \frac{r\, v_{1,0}'}{{v_{1,0}}^2} \\ &= \int_{\mathbf{T}} \frac{r'}{v_{1,0}} \\ \lambda_{r\, v_{1,0}'/v_{1,0}} &= \lambda_{r'}\end{aligned} \tag{6.54}$$

Next we compute

$$r' = \frac{1}{2} w_0'' + \int_{\mathbf{T}} \partial_y v_\gamma(x,\cdot)\, \partial_y v_1(x,\cdot)\, \mathrm{d}x + \int_{\mathbf{T}} v_\gamma(x,\cdot)\, \partial_y^2 v_1(x,\cdot)\, \mathrm{d}x \tag{6.55}$$

and also

$$\begin{aligned}\int_{\mathbf{T}} \partial_y v_\gamma(x,\cdot)\, \partial_y v_1(x,\cdot)\, \mathrm{d}x &= \sum_{m\in\mathbf{Z}\smallsetminus\{0\}} \frac{{v_{1,m}}'}{\gamma_m}\, \overline{{v_{1,m}}'} \\ &= \sum_{m\in\mathbf{Z}\smallsetminus\{0\}} \frac{|{v_{1,m}}'|^2}{\gamma_m} \\ &= \sum_{m>0} |{v_{1,m}}'|^2 \left({\gamma_m}^{-1} + \overline{{\gamma_m}^{-1}}\right) \\ &= -\sum_{m>0} |{v_{1,m}}'|^2 = -\frac{1}{2}\sum_{m\neq 0} |{v_{1,m}}'|^2 \\ \int_{\mathbf{T}} \partial_y v_\gamma(x,\cdot)\, \partial_y v_1(x,\cdot)\, \mathrm{d}x &= \tfrac{1}{2}({v_{1,0}}')^2 - \tfrac{1}{2}\int_{\mathbf{T}} \left[\partial_y v_1(x,y)\right]^2\, \mathrm{d}x\end{aligned}$$

Consequently we have

$$\int_{\mathbf{T}} \partial_y v_\gamma(x,\cdot)\, \partial_y v_1(x,\cdot)\, \mathrm{d}x = \frac{1}{2}({v_{1,0}}')^2 - \frac{1}{2}\int_{\mathbf{T}} \left[\mathbf{E}\, \partial_y v_{1,\omega}(x,y)\right]^2\, \mathrm{d}x \tag{6.56}$$

Note that with (6.55) and (6.56) we get

$$\begin{aligned}\int_{\mathbf{T}} l_2(x,\cdot)\, \mathrm{d}x &= \tfrac{1}{2} w_0'' + \tfrac{1}{2}\int_{\mathbf{T}} \mathbf{E}\left[(\partial_y v_{1,\omega}(x,\cdot))^2\right]\, \mathrm{d}x \\ &= r' - \int_{\mathbf{T}} \partial_y v_\gamma\, \partial_y v_1\, \mathrm{d}x - \int_{\mathbf{T}} v_\gamma\, \partial_y^2 v_1\, \mathrm{d}x + \tfrac{1}{2}\int_{\mathbf{T}} \mathbf{E}\left[(\partial_y v_{1,\omega}(x,\cdot))^2\right]\, \mathrm{d}x \\ \int_{\mathbf{T}} l_2(x,\cdot)\, \mathrm{d}x &= r' - \tfrac{1}{2}({v_{1,0}}')^2 - \int_{\mathbf{T}} v_\gamma\, \partial_y^2 v_1\, \mathrm{d}x + \tfrac{1}{2}\int_{\mathbf{T}} \mathbf{E}\left[(\partial_y v_{1,\omega}(x,\cdot))^2\right]\, \mathrm{d}x\end{aligned}$$

With (6.53), this ensures that we obtained so far:

$$\lambda_{l_2} + \nu_{l_1} = \tfrac{1}{2}\lambda_\Upsilon - \lambda_{t\left(\frac{v_{1,0}'}{v_{1,0}}\right)'}$$

where, thanks to Jensen's inequality, Υ is the positive quantity below:

$$\begin{aligned}\Upsilon(y) &= \int_{\mathbf{T}} \mathbf{E}\left[(\partial_y v_{1,\omega}(x,y))^2\right] \mathrm{d}x - ({v_{1,0}}')^2 + \int_{\mathbf{T}} \left[\mathbf{E}\,\partial_y v_{1,\omega}(x,y)\right]^2 \mathrm{d}x \\ &= \int_{\mathbf{T}} \mathbf{E}\left[(\partial_y v_{1,\omega}(x,y))^2\right] \mathrm{d}x - \left(\int_{\mathbf{T}} \mathbf{E}\,\partial_y v_{1,\omega}\,\mathrm{d}x\right)^2 + \int_{\mathbf{T}} \left[\mathbf{E}\,\partial_y v_{1,\omega}(x,y)\right]^2 \mathrm{d}x \\ \Upsilon(y) &= \int_{\mathbf{T}} \mathbf{V}_{\omega}(\partial_y v_{1,\omega})\,\mathrm{d}x + \mathbf{V}_{x}(\mathbf{E}\,\partial_y v_{1,\omega}) + \int_{\mathbf{T}} \left[\mathbf{E}\,\partial_y v_{1,\omega}(x,y)\right]^2 \mathrm{d}x\end{aligned}$$

In what just precedes, we denoted $\mathbf{V}_x$ the variance with respect to x, meaning that for a numeric function f

$$\mathbf{V}_x f = \int_{\mathbf{T}} f^2(x)\,\mathrm{d}x - \left(\int_{\mathbf{T}} f(x)\,\mathrm{d}x\right)^2$$

To continue the computations we set $c = \left(\int_{\mathbf{T}} \mathrm{d}y/v_{1,0}\right)^{-1}$. An integration by parts shows that

$$\begin{aligned}\lambda_{t({v_{1,0}}'/v_{1,0})'} &= \frac{c\,\mathbf{V}(\mathrm{W})}{4\pi\,|\sin(k\pi\alpha)|}\int_{\mathbf{T}}\left(\frac{{v_{1,0}}'}{v_{1,0}}\right)'\frac{1}{v_{1,0}}\,\mathrm{d}y + \frac{c}{2}\int_{\mathbf{T}}\left(\frac{{v_{1,0}}'}{v_{1,0}}\right)' v_{1,0}\,\mathrm{d}y \\ &= -\frac{c\,\mathbf{V}(\mathrm{W})}{4\pi\,|\sin(k\pi\alpha)|}\int_{\mathbf{T}}\frac{({v_{1,0}}')^2}{{v_{1,0}}^3}\,\mathrm{d}y - \frac{c}{2}\int_{\mathbf{T}}\frac{({v_{1,0}}')^2}{v_{1,0}}\,\mathrm{d}y \\ \lambda_{t({v_{1,0}}'/v_{1,0})'} &= -\frac{\mathbf{V}(\mathrm{W})}{4\pi\,|\sin(k\pi\alpha)|}\lambda_{({v_{1,0}}'/v_{1,0})^2} - \tfrac{1}{2}\lambda_{({v_{1,0}}')^2}\end{aligned}$$

As a result, the term $\frac{1}{2}({v_{1,0}}')^2$ vanishes in the final expression of $\gamma(\mathrm{F})$ (see (4.3)). This gives

$$\gamma(\mathrm{F}) = \lambda_{\Phi}\varepsilon^2 + \mathrm{O}(\varepsilon^3)$$

where Φ is the positive function given in the statement of theorem 6.1.

To make the coefficients explicit, let us add that for all $a > 0$

$$\begin{aligned}\int_0^1 \frac{1}{a+\sin^2(\pi y)}\,\mathrm{d}y &= \frac{1}{\sqrt{a(a+1)}} \\ \int_0^1 \frac{\sin^2 2\pi y}{a+\sin^2(\pi y)}\,\mathrm{d}y &= \left(\sqrt{a+1}-\sqrt{a}\right)^2 \\ \int_0^1 \frac{\sin^2 2\pi y}{\left(a+\sin^2(\pi y)\right)^3}\,\mathrm{d}y &= \frac{1}{2a(a+1)\sqrt{a(a+1)}}\end{aligned}$$

From which we deduce for $a_k = \frac{\delta_k}{2|\widehat{\mathrm{V}}_k|}$

$$\begin{aligned}
\lambda_{(v_{1,0}'/v_{1,0})^2} &= \frac{\pi^2}{\int_{\mathbf{T}} \frac{\mathrm{d}y}{a_k+\sin^2(\pi y)}} \int_{\mathbf{T}} \frac{\sin^2(2\pi u)}{(a_k+\sin^2\pi u)^3}\,\mathrm{d}u \\
&= \frac{\pi^2}{2a_k(a_k+1)} = \frac{2\pi^2|\widehat{\mathrm{V}}_k|^2}{\delta_k(\delta_k+2|\widehat{\mathrm{V}}_k|)} \\
\lambda_{(v_{1,0}'/v_{1,0})^2} &= \frac{2\pi^2|\widehat{\mathrm{V}}_k|^2}{\delta_k(\widehat{\mathrm{V}}_0+|\widehat{\mathrm{V}}_k|+\mathbf{E}(\mathrm{W}))}
\end{aligned}$$

The previous computations and (4.9) give the formula in theorems 1.8 and 6.1 for the Lyapunov exponent.

$\blacksquare$

6.10 The case $\delta_k = 0$

In this section we deal with the *parabolic* case $\delta_k = 0$. Let us state what we solve for the equations (6.42) that we recall:

$$\begin{aligned}
\psi_n &= \gamma_n\varphi_n + \varepsilon\alpha_n v_{1,0}(\cdot)\,\varphi_n' + \varepsilon^2\alpha_n r(\cdot)\,\varphi_n' + \varepsilon^2\alpha_n t(\cdot)\,\varphi_n'' \\
&= \gamma_n\varphi_n + \varepsilon a_n(\cdot)\,\varphi_n' + \varepsilon^2 b_n(\cdot)\,\varphi_n''
\end{aligned}$$

Lemma 6.5

We obtain the following for equations (6.42):

(i) *for $n \neq 0$, we can find $\varphi_n(y)$ such that*

$$(\mathrm{D}\varphi)_n = \gamma_n\varphi_n + \varepsilon\alpha_n v_{1,0}(\cdot)\,\varphi_n' + \varepsilon^2\alpha_n r(\cdot)\,\varphi_n' + \varepsilon^2\alpha_n t(\cdot)\,\varphi_n'' = \psi_n(y) + \varepsilon_n$$

with $$\|\varepsilon_n\|_j \lesssim \varepsilon^2\|\psi_n\|_{j+4}\,|n|^{2\tau}\,|k|^{6\tau}\kappa^{-8}$$

(ii) *for $n = 0$ the differential equation (ED_0) is solvable up to obstructions: we can solve*

$$(\mathrm{D}\varphi)_0 = \psi_0 - \lambda_\psi + \varepsilon_0$$

where $\varepsilon_0 = \varepsilon^2 r\varphi_0'$ with

$$\|\varepsilon_0\|_j \lesssim \varepsilon^{\frac{1-j}{3}}\|\psi_0\|_j$$

where $$\lambda_\psi = \lambda_\psi(\varepsilon) = \int_{\mathbf{T}^2} \psi(x,y)\,\rho_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y$$

and $\rho_\varepsilon = \dfrac{\theta_\varepsilon/t}{\int_{\mathbf{T}}\theta_\varepsilon/t}$ is a density given by the following explicit solvable differential equation on θ_ε: $\theta_\varepsilon' = \frac{v_{1,0}}{t(y)}\,\theta_\varepsilon + 1$ with

$$t(\eta) = \frac{\mathbf{V}(\mathrm{W})}{4\pi\,|\sin\pi k\alpha|} + \tfrac{1}{2}{v_{1,0}}^2(y) \qquad \textit{and} \qquad v_{1,0} = \frac{1}{\varepsilon}\frac{|\widehat{\mathrm{V}}_k|}{\pi\,|\sin\pi k\alpha|}\sin^2\pi(y-c_k)$$

Moreover the following $\mathscr{C}^j$-estimates of Y_ψ hold:

$$\|{\varphi_0}^{(j)}\|_0 \lesssim \varepsilon^{-\frac{j+4}{3}}\|\psi_0\|_{j-1}$$

In the setting of section 6.8 we have found Φ_ψ and r_ψ that verify

$$\begin{aligned}\psi - \lambda_\psi\delta_0 &= \mathrm{D}\Phi_\psi + r_\psi\\ \|\Phi_\psi\|_{\sigma,j} &\lesssim \varepsilon^{-\frac{j+4}{3}}\|\psi_0\|_{j-1} + \|\psi\|_{\sigma,j+4} &(6.57)\\ \|r_\psi\|_{\sigma,j} &\lesssim \varepsilon^{\frac{1-j}{3}}\|\psi_0\|_j + \varepsilon^2\|\psi\|_{\sigma,j+4} &(6.58)\end{aligned}$$

Proof.

For the first item, the technique used in the previous section in the proof of proposition 6.1 works as well.

For $n=0$ we want to find a 1-periodic solution to

$$\psi_0 = \varepsilon v_{1,0}(\cdot)\,{\varphi_0}' + \varepsilon^2 r(\cdot)\,{\varphi_0}' + \varepsilon^2 t(\cdot)\,{\varphi_0}'' \tag{6.59}$$

Remind that (see (6.32))

$$\begin{aligned} v_{1,0}(y) &= \frac{1}{2\pi\,|\sin\pi k\alpha|}\left(\delta_k + 2|\widehat{\mathrm{V}}_k|\sin^2(\pi(y+k\alpha)-\theta_k)\right)\\ &= \frac{|\widehat{\mathrm{V}}_k|}{\pi\,|\sin\pi k\alpha|}\sin^2(\pi(y-c_k))\\ t(y) &= \frac{\mathbf{V}(\mathrm{W})}{4\pi\,|\sin\pi k\alpha|} + \tfrac{1}{2}{v_{1,0}}^2(y)\end{aligned}$$

So the previous first-order method does not work anymore since $v_{1,0}$ vanishes at $c_k = \frac{1}{2\pi}\theta_k - k\alpha$ and we do not really have information on $r(y)$ (see (6.32)). Hence we solve the differential equation without taking the quantity $\varepsilon^2 r\,{\varphi_0}'$ into account. It will be treated as an error term. Namely, we resolve:

$${\varphi_0}''(y) - \frac{1}{\varepsilon}a(y)\,{\varphi_0}'(y) = \frac{1}{\varepsilon^2}\frac{\psi_0(y)}{t(y)} \tag{6.60}$$

where

$$a(y) = -\frac{v_{1,0}(y)}{t(y)}$$

With $\Phi = {\varphi_0}'$, this equation is of the following type:

$$\Phi' = \frac{a}{\delta}\,\Phi + \frac{1}{\delta^2}\,f$$

Let us first mention that if u is a solution of $(\mathcal{E}_\delta)$ then $v(y) = u(-y + c_k)$ satisfies

$$\begin{aligned} v'(y) &= -\frac{a(-y+c_k)}{\delta}\, v(y) - \frac{f(-y+c_k)}{\delta^2} \\ &= \frac{1}{\delta}\frac{\sin^2 \pi y}{\widetilde{t}(y)}\, v(y) + \frac{\widetilde{f}}{\delta^2} \end{aligned}$$

where $\widetilde{t}$ and $\widetilde{f}$ are affine translates of t, f having same positive bounds or $\mathscr{C}^j$-properties as t (resp. f).

Lemma 6.6 ("Parabolic" diff. equation with small parameter)

Let us consider the differential equation with (small) parameter $\delta > 0$

$$(\mathcal{E}_\delta) \qquad u' = \frac{1}{\delta} a(y)\, u + f(y) \tag{6.61}$$

with 1-periodic functions a, f. We assume that the map a is non negative and only vanishes at 0 with $a(x) \sim_{x\to 0} x^2$. Let u_δ be the 1-periodic solution of (6.61). The following estimates for general f and also $f = \delta^{-2}\psi$ hold:

$$\|u_\delta\|_0 \lesssim \delta^{1/3}\|f\|_0 = \delta^{1/3-2}\|\psi\|_0$$

$$\|u_\delta^{(j)}\|_0 \lesssim \delta^{1/3}\left(\|f^{(j)}\|_0 + \delta^{-1/3}\|f^{(j-1)}\|_0 + \cdots + \delta^{-j/3}\|f\|_0\right) \qquad (j \in \mathbf{N})$$

$$\|u_\delta\|_2 \lesssim \delta^{-1/3-2}\|\psi\|_2$$

In order to get a one-periodic solution of the initial differential equation (6.60) of order 2, we are brought back to find a solution of the previous equation (6.61) with zero average. As explained in lemma *A.2* of appendix A.2, this can be ensured by substracting the obstruction $\lambda_\psi = \int_{\mathbf{T}^2} \psi(x,y)\, \rho_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y$ to ψ_0. We deduce from the previous lemma the following corollary rephrased in our setting:

Corollary 6.2

On can find a one-periodic function $\varphi_0(\cdot)$ satisfying the differential equation

$$\varphi_0{}''(y) + \frac{a(y)}{\varepsilon}\,\varphi_0{}'(y) = \frac{1}{\varepsilon^2}\frac{\psi_0(y) - \lambda_\psi}{t(y)}$$

with the respective estimates for $j \geqslant 1$

$$\|\varphi_0\|_j \lesssim \varepsilon^{-(j+4)/3} \left\|\frac{\psi_0}{w}\right\|_{j-1} \lesssim \varepsilon^{-(j+4)/3}\, \|\psi_0\|_{j-1}$$

For the error term in lemma 6.5, note that

$$\begin{aligned} \mathrm{D}\varphi_0 &= \varepsilon v_{1,0}(\cdot)\,{\varphi_0}' + \varepsilon^2 r(\cdot)\,{\varphi_0}' + \varepsilon^2 t(\cdot)\,{\varphi_0}'' \\ &= \psi_0 - \lambda_\psi + \varepsilon^2 r(\cdot)\,{\varphi_0}' \end{aligned}$$

Such that with corollary 6.2 we have $\varepsilon_0 = \varepsilon^2 r(\cdot)\,{\varphi_0}'$, and so

$$\|\varepsilon_0\|_j = \mathrm{O}\left(\varepsilon^{\frac{1-j}{3}}\|\psi_0\|_j\right) \tag{6.62}$$

Proof. ______________

(of lemma 6.6)

As explained in appendix *A*.1, one-periodic solutions do exist. This is what we will prove first:

Lemma 6.7

The following estimates hold for all $t \in [\,0\,;1\,]$:

$$|u_\delta(t)| \lesssim \delta^{1/3}\|f\|_0 \qquad\qquad |u_\delta(t)| \lesssim \frac{\delta}{t^2}\|f\|_0$$

The second estimate is uniform in (δ, t) provided $t, 1-t \gtrsim \delta^{\frac{1}{2}-}$.

Proof. ______________

The resolvent formula gives indeed for fixed δ:

$$u(x) = \left(u(t) + \int_t^x f(s)\,\mathrm{e}^{-\mathrm{A}_{\delta,t}(s)}\,\mathrm{d}s\right)\mathrm{e}^{\mathrm{A}_{\delta,t}(x)}$$

where $u(t)$ is such that the solution is one-periodic and

$$\mathrm{A}_{\delta,t}(y) = \mathrm{A}_\delta(y) - \mathrm{A}_\delta(t) = \frac{1}{\delta}\int_t^y a$$

Note that with our hypotheses on a (see in lemma 6.6) we have for some $c > 0$:

$$\forall y \in [\,-3/4\,;3/4\,] \qquad a(y) \geqslant c\,y^2$$

Let us prove the estimates for $t \in [\,-1/2\,;1/2\,]$ and then conclude by periodicity.

It is straightforward that for all t

$$\mathrm{A}_{\delta,t}(x) \xrightarrow[x\to+\infty]{} +\infty$$

As our solution u_δ is periodic and hence bounded, this ensures the convergence and the equality below

$$u_\delta(t) = -\int_t^{+\infty} f(s)\,\mathrm{e}^{-\frac{1}{\delta}\int_t^s a}\,\mathrm{d}s = -\int_0^{+\infty} f(s)\,\mathrm{e}^{-\frac{1}{\delta}\int_t^{t+s} a}\,\mathrm{d}s$$

This equality and the inequality $|u_\delta(t)| \leqslant \|f\|_0\,|v_\delta(t)|$ prove that, in order to get $\mathscr{C}^0$-estimates for u_δ, it is enough to obtain estimates for

$$v_\delta(t) := \int_0^{+\infty} \exp\left(-\frac{1}{\delta}\int_t^{t+s} a(\cdot)\right)\,\mathrm{d}s$$

Note that $-v_\delta$ is the one periodic solution of

$$\mathrm{V}' = \frac{a}{\delta}\,\mathrm{V} + 1$$

We have

$$\int_1^{+\infty} \mathrm{e}^{-\frac{1}{\delta}\int_t^{t+s} a(\cdot)}\,\mathrm{d}s \leqslant \int_1^{+\infty} \mathrm{e}^{-\frac{1}{\delta}\int_t^{t+[s]} a(\cdot)}\,\mathrm{d}s \leqslant \sum_{k=1}^{+\infty} \mathrm{e}^{-\frac{k}{\delta}\widehat{a}_0} \lesssim \mathrm{e}^{-\frac{1}{\delta}\widehat{a}_0}$$

Then for all $t \in \left[-\frac{1}{2}\,;\frac{1}{2}\right]$ and $u \in \left[\frac{1}{4}\,;1\right]$ we have $\int_t^{t+u} a \geqslant c_2 > 0$ uniformly in t, u so that

$$\int_{1/4}^{1} \mathrm{e}^{-\frac{1}{\delta}\int_t^{t+s} a(\cdot)}\,\mathrm{d}s \lesssim \mathrm{e}^{-\frac{c_2}{\delta}}$$

Finally

$$v_\delta(t) = \int_0^{\frac{1}{4}} \mathrm{e}^{-\frac{1}{\delta}\int_t^{t+s} a(\cdot)}\,\mathrm{d}s + \mathrm{O}(\mathrm{e}^{-c'/\delta}) =: \mathrm{I}(t) + \mathrm{O}(\mathrm{e}^{-c'/\delta})$$

With the assumption on a we get

$$|\mathrm{I}(t)| \leqslant \int_0^{+\infty} \mathrm{e}^{-4c\frac{(t+s)^3-t^3}{\delta}}\,\mathrm{d}s = \int_0^{+\infty} \mathrm{e}^{-4c\frac{s^3+3s^2t+3st^2}{\delta}}\,\mathrm{d}s$$

The following elementary estimates will be useful: for all $x \geqslant 0$ and all $t \in \mathbf{R}$

$$\begin{aligned} s^3 + 3s^2t + 3st^2 &\geqslant \tfrac{1}{4}s^3 \\ s^3 + 3s^2t + 3st^2 &\geqslant \tfrac{3}{4}t^2 s \end{aligned}$$

It is sufficient to check that for all t and $s \geqslant 0$

$$\begin{aligned} &\left(\tfrac{1}{3} - \tfrac{1}{12}\right)s^2 + st + t^2 \geqslant 0 \\ &\tfrac{1}{3}s^2 + st + t^2\left(1 - \tfrac{1}{4}\right) \geqslant 0 \end{aligned}$$

But these inequations are true because both discriminants are zero. Then the condition $s \geqslant 0$ permits to conclude. These lower bounds lead to

$$|v_\delta(t)| \lesssim \delta^{1/3}c^{-1/3}\int_{\mathbf{R}} \mathrm{e}^{-v^3}\,\mathrm{d}v \lesssim \delta^{1/3}c^{-1/3}$$

$$|v_\delta(t)| \lesssim \frac{\delta}{t^2}c^{-1}\int_{\mathbf{R}} \mathrm{e}^{-v}\,\mathrm{d}v = \frac{\delta}{t^2}c^{-1}$$

■

Consequently:

Corollary 6.3

Let u_δ be the unique one-periodic solution of $w' = \frac{a}{\delta}w + f$. It satisfies the following estimates:

$$\|u_\delta\|_0 \lesssim \delta^{1/3}\|f\|_0$$

$$\|\tfrac{a}{\delta}\,u\|_0 \lesssim \|f\|_0$$

$$\|\left(\tfrac{a}{\delta}\right)'\,u\|_0 \lesssim \delta^{-1/3}\|f\|_0$$

Proof.

The assumption on a ensures that $a'(y) \sim_{y\to 0} 2cy$. Therefore,

- if $0 \leqslant t \leqslant \delta^{1/3}$

$$\left|\frac{a'(t)}{\delta}\, u(t)\right| \lesssim \frac{\delta^{1/3}}{\delta}\delta^{1/3}\|f\|_0 \lesssim \delta^{-1/3}\|f\|_0$$

- if $\delta^{1/3} \leqslant t \ll 1$

$$\left|\frac{a'(t)}{\delta} u(t)\right| \lesssim \frac{t}{\delta}\frac{\delta}{t^2}\|f\|_0 \lesssim \delta^{-1/3}\|f\|_0$$

■

Now we go for the $\mathscr{C}^j$-estimates using the preliminary result about the $\mathscr{C}^0$-estimate.

Lemma 6.8

The function ${u_\delta}^{(j)}$ is the only one-periodic solution of $w' + \frac{a}{\delta}\, w = \mathrm{F}_j$ where F_j is recursively defined and has the form

$$\mathrm{F}_j = f^{(j)} + c_{j,0}\left(\tfrac{a}{\delta}\right)^{(j-1)} u_\delta + \cdots + c_{j,j-2}\left(\tfrac{a}{\delta}\right)'' {u_\delta}^{(j-2)} + c_{j,j-1}\left(\tfrac{a}{\delta}\right)' {u_\delta}^{(j-1)}$$

Proof.

Let us give an inductive argument.

The hypothesis is true for $j = 0$ by definition of the differential equation we solved.

Then if one assume the hypothesis to be true for j:

$$\frac{\mathrm{d}}{\mathrm{d}y}{u_\delta}^{(j+1)} - \frac{a}{\delta}\,{u_\delta}^{(j)} = \left(\frac{a}{\delta}\right)' {u_\delta}^{(j)} + {\mathrm{F}_j}' =: \mathrm{F}_{j+1}$$

The Leibniz formula ensures that ${\mathrm{F}_j}'$ is the sum of $f^{(j+1)}$ and a linear combination of the $\left(\frac{a}{\delta}\right)^{(i)} {u_\delta}^{(j+1-i)}$ with $1 \leqslant i \leqslant j+1$.

■

We finish the proof of proposition 6.6 by proving inductively that

Lemma 6.9

For all $k \in \mathbf{N}$ these estimates hold:

$$\|\mathrm{F}_j\|_0 \lesssim \|f^{(j)}\|_0 + \delta^{-1/3}\|f^{(j-1)}\| + \cdots + \delta^{-j/3}\|f\|_0$$

$$\|{u_\delta}^{(j)}\|_0 \lesssim \delta^{1/3}\left(\|f^{(j)}\|_0 + \delta^{-1/3}\|f^{(j-1)}\| + \cdots + \delta^{-j/3}\|f\|_0\right)$$

Proof.

Let us assume that the lemma is true for $j \geqslant 0$. Then lemma 6.8 shows that

$$\begin{aligned}
\|\mathrm{F}_{j+1}\|_0 &\lesssim \|f^{(j+1)}\|_0 + \sum_{\substack{i+p=j+1\\ i\geqslant 1}} \|\tfrac{a^{(i)}}{\delta}\, u_\delta{}^{(p)}\|_0 \\
&\lesssim \|f^{(j+1)}\|_0 + \sum_{\substack{i+p=j+1\\ i\geqslant 2}} \|\tfrac{a^{(i)}}{\delta}\, u_\delta{}^{(p)}\|_0 + \|\tfrac{a'}{\delta}\, u_\delta{}^{(j)}\|_0 \\
\|\mathrm{F}_{j+1}\|_0 &\lesssim \|f^{(j+1)}\|_0 + \tfrac{1}{\delta}\sum_{p=0}^{j-1}\|u_\delta{}^{(p)}\|_0 + \delta^{-1/3}\|\mathrm{F}_j\|_0
\end{aligned}$$

where we used corollary 6.3 and the boundedness of $a^{(i)}$ for $i \geqslant 2$. With lemma 6.8 we obtain

$$\left\| \frac{a'}{\delta} u_\delta^{(j)} \right\| \lesssim \delta^{-1/3}\|\mathrm{F}_j\|_0$$

The induction hypothesis at rank j ensures that

$$\begin{aligned}
\|\mathrm{F}_{j+1}\|_0 \lesssim \|f^{(j+1)}\|_0 + \tfrac{1}{\delta}\Big(&\delta^{1/3}\|f\|_0 + \\
&+\|f\|_0 + \delta^{1/3}\|f'\|_0 \\
&\vdots \qquad\qquad \ddots \\
&+\delta^{-(j-2)/3}\|f\|_0 + \cdots + \delta^{1/3}\|f^{(j-1)}\|_0\Big) \\
+\delta^{-1/3}&\left(\|f^{(j)}\|_0 + \cdots + \delta^{-j/3}\|f\|_0\right)
\end{aligned}$$

$$\begin{aligned}
&\lesssim \|f^{jk+1)}\|_0 + \tfrac{1}{\delta}\left(\delta^{-(j-2)/3}\|f\|_0 + \delta^{-(j-3)/3}\|f'\|_0 + \cdots + \delta^{1/3}\|f^{(j-1)}\|_0\right) \\
&\qquad +\delta^{-1/3}\left(\|f^{(j)}\|_0 + \cdots + \delta^{-j/3}\|f\|_0\right)
\end{aligned}$$

$$\|\mathrm{F}_{j+1}\|_0 \lesssim \|f^{(j+1)}\|_0 + \delta^{-(j+1)/3}\|f\|_0 + \delta^{-j}\|f'\|_0 + \cdots + \delta^{-1/3}\|f^{(j)}\|_0$$

which is the formula expected at the rank $j+1$. Using corollary 6.8 we get

$$\|u_\delta^{(j+1)}\|_0 \lesssim \delta^{1/3}\left(\|f^{(j+1)}\|_0 + \delta^{-1/3}\|f^{(j)}\|_0 + \cdots + \delta^{-(j+1)/3}\|f\|_0\right)$$

This ends the proof of the induction.

■

In our context we have $f = \psi/\varepsilon^2$ so we obtain the second claim of proposition 6.6. ■

This ends the proof of the lemma 6.5. ■

6.10.1 Development of a stationary measure when $\delta_k = 0$

Let us state the result for any stationary measure.

Proposition 6.3

For an α-resonant energy $\mathrm{E} = 2\cos(k\pi\alpha)$ with $k \neq 0$ and under the condition $\delta_k = \mathbf{E}(\mathrm{W}) + \widehat{\mathrm{V}}_0 - |\widehat{\mathrm{V}}_k| = 0$, the integral of $\psi \in \mathcal{E}_{\sigma,8}$ w.r.t. any stationary measure μ_ε for $\mathrm{F}_\omega : (x,y) \mapsto (x+\alpha, y + v_\omega(x,y))$ can be expanded as follows:

$$\int_{\mathbf{T}} \psi \,\mathrm{d}\mu_\varepsilon = \lambda_\psi + \mathrm{O}_{\kappa,k}\left(\varepsilon^{1/3}\|\psi\|_4\right) + \mathrm{O}\left(\varepsilon^{2/3}\|\psi\|_4\right)$$

with $\lambda_g = \lambda_g(\varepsilon) = \int_{\mathbf{T}^2} g(x,y)\,\rho_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y$ *for a density* ρ_ε *which is the* L^1*-normalization of* θ/t *for* θ *satisfying*

$$\theta' = \frac{v_{1,0}}{\varepsilon\, t}\,\theta + 1$$

Proof.

Lemma 6.5 ensures (6.36) *i.e.*

$$\psi - \lambda_\psi \delta_0 = \mathrm{A}\varphi_\psi + \Delta_\psi$$

where

$$\|\varphi_\psi\|_{\sigma,j} \lesssim \varepsilon^{\frac{1-j}{3}} \|\psi_0\|_{j-1} + \varepsilon^2 \|\psi\|_{\sigma,j+4} \tag{6.63}$$

and we also have (6.39) with

$$m(j) = -\frac{j+4}{3} \qquad \text{and} \qquad n(j) = \frac{1-j}{3} \tag{6.64}$$

The lowest order term w.r.t ε in Δ_ψ (see (6.37)) is actually $\mathrm{C}r_\psi$ because of the equality $(r_\psi)_0 = \varepsilon^2 r\,{\varphi_0}'$. Moreover the following $\mathscr{C}^0$-estimate holds:

$$\|\mathrm{C}r_\psi\|_{\sigma,0} \lesssim \varepsilon^{\frac{1}{3}} \|\psi\|_{\sigma,4} \tag{6.65}$$

The next order term is $\varepsilon^3 \mathrm{R}\Phi_\psi$ with

$$\|\varepsilon^3 \mathrm{R}\Phi_\psi\|_{\sigma,0} \lesssim \varepsilon^{\frac{2}{3}} \|\psi\|_{\sigma,4} \tag{6.66}$$

Hence (6.65) and (6.66) lead to

$$\|\Delta_\psi\|_{\sigma,0} \lesssim \varepsilon^{\frac{1}{3}} \|\psi\|_{\sigma,4} \tag{6.67}$$

The other terms in (6.37) have orders $\mathrm{O}(\varepsilon^{1^+})$. With (6.7) and lemma 6.6, equation (6.36) shows that $\phi = \mathcal{U}\varphi$ (remember (6.12)) verifies

$$\mathrm{T}\phi - \phi = \psi - \lambda_\psi + \mathrm{O}(\varepsilon^3 \|\phi\|_3) + \mathrm{O}(\|\mathcal{U}\Delta_\psi\|_0) \tag{6.68}$$

Also (6.57) gives the $\mathscr{C}^3$-estimate for ϕ:

$$\|\phi\|_3 \lesssim \varepsilon^{-\frac{7}{3}} \|\psi\|_4 \tag{6.69}$$

Integrating with respect to any stationary probability measure μ_ε, the estimates (6.63) and (6.67) end the proof. ∎

Let us finally expand the Lyapunov exponent.

6.10.2 Asymptotic development for the Lyapunov exponent

Theorem 6.2

Assume $\alpha \in \mathrm{DC}(\kappa, \tau)$. For an α-resonant energy $\mathrm{E} = 2\cos(k\pi\alpha)$ with $k \neq 0$ and a potential $\varepsilon(\mathrm{V}(x) + \mathrm{W}_\omega)$ with $\delta_k = \mathbf{E}(\mathrm{W}) + \widehat{\mathrm{V}}_0 - |\widehat{\mathrm{V}}_k| = 0$ we have for sufficiently small $\varepsilon > 0$:

$$\mathrm{L}(\mathrm{E}) = \mathrm{A}_\varepsilon + \mathrm{O}_{\kappa,k,\delta_k}\left(\varepsilon^{5/3}\right)$$

where $\mathrm{A}_\varepsilon = -\frac{1}{2}\varepsilon\lambda_{v_{1,0}'} + \mathrm{O}(\varepsilon^{4/3})$ is a quantity that either vanishes, or is $\asymp \varepsilon^{4/3}$, or is a $\mathrm{O}(\varepsilon^{4/3})$. The constant λ_g is defined by $\lambda_g(\varepsilon) = \int_{\mathbf{T}^2} g(x,y)\,\rho_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y$ for a density ρ_ε which is the L^1-normalization of θ/t where θ satisfies:

$$\theta' = \frac{v_{1,0}}{\varepsilon\, t}\theta + 1$$

Proof.

We already computed in the previous section (see (6.52))

$$\mathbf{E}\ln|1 + \partial_y v_\omega(x,y)| = \varepsilon l_1(x,y) + \varepsilon^2 l_2(x,y) + \mathrm{O}(\varepsilon^3)$$

with

$$l_1(x,y) = \mathbf{E}(\partial_y v_{1,\omega}(x,y)) = \partial_y v_1(x,y) = \frac{\sin\left(2\pi(k(x+\alpha)+y)\right)}{|\sin(\pi k\alpha)|}(\mathrm{V}(x) + \mathbf{E}(\mathrm{W}))$$

$$\begin{aligned} l_2(x,y) &= \partial_y v_2(x,y) - \tfrac{1}{2}\,\mathbf{E}\left[(\partial_y v_{1,\omega}(x,y))^2\right] = \tfrac{1}{2}\partial_y^2\,\mathbf{E}({v_{1,\omega}}^2(x,y)) - \tfrac{1}{2}\,\mathbf{E}\left[(\partial_y v_{1,\omega}(x,y))^2\right] \\ &= \frac{\sin^2\left(\pi(k(x+\alpha)+y)\right)\cos\left(2\pi(k(x+\alpha)+y)\right)}{\sin^2(\pi k\alpha)}(\mathrm{V}(x)^2 + \mathbf{E}^2(\mathrm{W}) + 2\mathrm{V}(x)\,\mathbf{E}(\mathrm{W})) \end{aligned}$$

This gives (recall (6.47))

$$\int_{\mathbf{T}} l_1(x,y)\,\mathrm{d}x = {v_{1,0}}'(y)$$

Consequently with the expansion of an integral w.r.t. μ stated in proposition 6.3 we obtain the asymptotic expansion below:

$$\gamma(\mathrm{F}) = \varepsilon\lambda_{v_{1,0}'} + \mathrm{O}(\varepsilon^{4/3}) + \mathrm{O}(\varepsilon^{5/3}) \tag{6.70}$$

We need some preciser estimates for $\int_{\mathbf{T}}\theta_\varepsilon$ and integrals of ${v_{1,0}}'$ versus θ_ε. Let us prove the following:

Proposition 6.4

Let θ_ε be the solution of

$$\theta' = \frac{1}{\varepsilon} \frac{|\widehat{\mathrm{V}}_k|}{\pi\,|\sin \pi k\alpha|} \frac{\sin^2 \pi(y - c_k)}{t(y)}\,\theta + 1$$

Then θ_ε is negative and $\int_{\mathbf{T}} \sin(2\pi(y-c_k))\,\theta_\varepsilon(y)\,\mathrm{d}y > 0$ with the following estimates:

$$\int_{\mathbf{T}} \theta_\varepsilon(y)\,\mathrm{d}y \asymp -\varepsilon^{2/3} \qquad \textit{and} \qquad \int_{\mathbf{T}} \sin(2\pi(y-c_k))\,\theta_\varepsilon(y)\,\mathrm{d}y \asymp \varepsilon$$

Proof.

Let us denote $a(y) = \frac{1}{\varepsilon}\frac{|\widehat{\mathrm{V}}_k|}{\pi|\sin \pi k\alpha|}\frac{\sin^2 \pi(y-c_k)}{t(y)}$. The negativity of θ is a consequence of the resolvent formula

$$\theta_\varepsilon(t) = -\int_0^{+\infty} \mathrm{e}^{-\frac{1}{\varepsilon}\int_t^{t+s} a}\,\mathrm{d}s$$

We proved that t is positive with bounds such that, up to translation by c_k, we can limit ourselves to get an estimate for $\int_{\mathbf{T}} \sin(2\pi y)\,\theta(y)\,\mathrm{d}y$ where θ is the solution of $z' = \frac{1}{\varepsilon} a\,z + 1$ for $a(y) = a_k \sin^2(\pi y)$.

Let us first prove the positivity of the integral with an argument of asymmetry of the solution θ_ε. The following actually holds

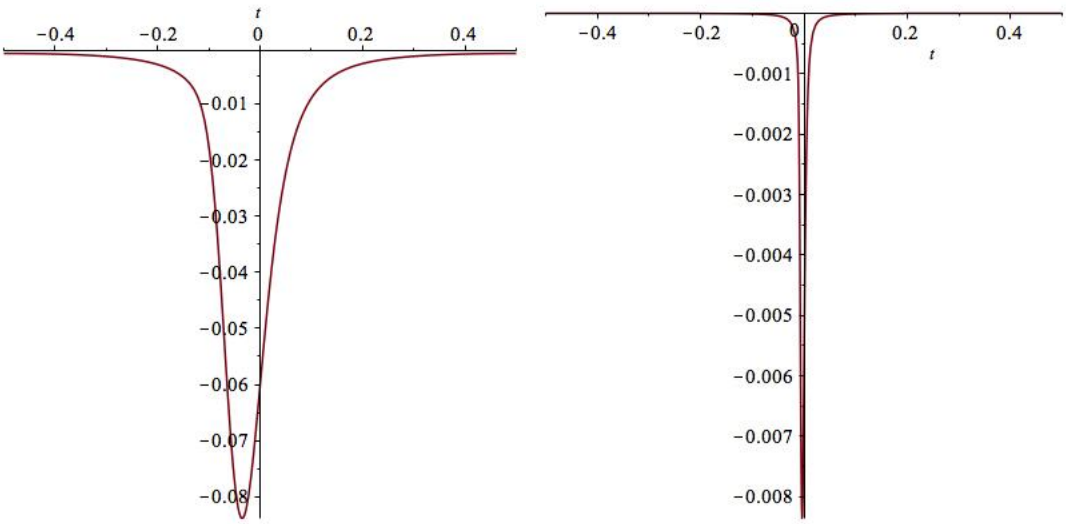

Figure 6.1: The graphs of $\theta_{10^{-3}}$ and $\theta_{10^{-6}}$ for $c_{k=0}$.

$$\forall y \in \,]-1/2\,;1/2\,[\qquad \theta(y) - \theta(-y) > 0$$

Indeed let us denote $\mathrm{F}(x) = \displaystyle\int_0^x \sin^2(\pi\cdot) = \frac{1}{2\pi}(2\pi x - \sin(2\pi x))$. Then

$$\begin{aligned}\theta(-y) &= -\int_0^{+\infty} \mathrm{e}^{-\frac{1}{\varepsilon}(\mathrm{F}(-y+s)-\mathrm{F}(-y))}\,\mathrm{d}s\\ &= -\int_0^{+\infty} \mathrm{e}^{-\frac{1}{\varepsilon}(\mathrm{F}(y)-\mathrm{F}(y-s))}\,\mathrm{d}s\\ \theta(y)-\theta(-y) &= \int_0^{+\infty} \mathrm{e}^{\frac{1}{\varepsilon}(\mathrm{F}(y-s)-\mathrm{F}(y))} - \mathrm{e}^{\frac{1}{\varepsilon}(\mathrm{F}(y)-\mathrm{F}(y+s))}\,\mathrm{d}s\end{aligned}$$

Now we compute for $y \in \,]-1/2\,;1/2\,[$

$$\mathrm{F}(y-s) - 2\mathrm{F}(y) + \mathrm{F}(y+s) = \tfrac{2}{\pi}\sin(2\pi y)\,\sin^2(\pi s) \geqslant 0$$

And we conclude for the integral defined by $\theta(y)-\theta(-y)$. Finally

$$\int_{-1/2}^{1/2} \sin(2\pi y)\,\theta(y)\,\mathrm{d}y = \int_0^{1/2} \sin(2\pi y)(\theta(y)-\theta(-y))\,\mathrm{d}y$$

so that the positivity of the integral is ensured. The arguments given in the proof of lemma 6.6 show that $\theta_\varepsilon(t) = \Theta(\mathrm{I}(t)) + \mathrm{O}(\mathrm{e}^{-c/\varepsilon})$ where

$$\mathrm{I}(t) = \int_0^{1/4} \mathrm{e}^{-\frac{1}{\varepsilon}(u^3+3u^2t+3ut^2)}\,\mathrm{d}u$$

This leads to the estimate for $\displaystyle\int_0^1 \sin(2\pi y)\,\theta_\varepsilon(y)\,\mathrm{d}y$. Then we compute

$$\int_{-\varepsilon^{1/3}}^{\varepsilon^{1/3}} \theta_\varepsilon \sim 2\theta(0)\varepsilon^{1/3} \sim -\varepsilon^{2/3}$$

and also

$$\left|\int_{\left[-1/2\,;-\varepsilon^{1/3}\right]\cup\left[\varepsilon^{1/3}\,;1/2\right]} \theta_\varepsilon\right| \lesssim \varepsilon\int_{\varepsilon^{1/3}}^{1/2} \mathrm{d}t/t^2 = \mathrm{O}(\varepsilon^{2/3})$$

so that

$$\left|\int_{-1/2}^{1/2} \theta_\varepsilon\right| \asymp \varepsilon^{2/3}$$

Given the sign of θ_ε, we can finally conclude that

$$\int_0^1 \theta_\varepsilon \asymp -\varepsilon^{2/3}$$

■

With (6.70) and the estimate from proposition 6.4 we obtain

$$\lambda_{v'_{1,0}} \asymp -\varepsilon^{1/3}$$

Hence (6.70) and (4.9) give a perturbative development of the following type:

$$\mathrm{L}(\mathrm{E}) = -\tfrac{1}{2}\varepsilon\lambda_{v_{1,0}'} + \mathrm{O}(\varepsilon^{4/3}) + \mathrm{O}(\varepsilon^{5/3})$$

with either $\mathrm{A}_\varepsilon \asymp \varepsilon^{4/3}$ or A_ε vanishes which is unclear, like in the parabolic case of chapter 5. Thus theorem 6.2 is proved.

■

Remark 6.7

The previous formula is not entirely satisfactory. Indeed it is not clear whether the development gives a positive lowest order term. One would think of optimizing the result about the asymptotic development of a stationary measure in proposition 6.3*. As we already explained in its proof, the lowest order term is given by* r_ψ *(see* (6.57)*). Its estimate is not good enough. One way of improving the estimates could be to refine the error term for* φ_0 *in lemma* 6.5*. The idea is to "extract" the possible* $\varepsilon^{1/3}$ *order term out of* $(r_\psi)_0 = \varepsilon^2 r\,\varphi_0{}'$.

We could reiterate the argument and solve (recall the definition of r *in* (6.32)*)*

$$\mathrm{D}\phi_0 = \varepsilon^2 r\,\varphi_0{}' - \lambda_{\varepsilon^2 r\,\varphi_0{}'} + \varepsilon^2 r\,\phi_0{}'$$

This would give a new solution for the 0*-th Fourier coefficient* $\varphi_0 + \phi_0$ *and a new* r_ψ *in lemma* 6.5 *with*

$$\|r_\psi\|_{\sigma,j} \lesssim \varepsilon^{\frac{3-2j}{3}} \|\psi\|_{\sigma,j-1}$$

which has better $\mathscr{C}^0$*-estimates. However this makes the* $\mathscr{C}^j$*-norms of* Φ_ψ *degenerate too rapidly. Indeed*

$$\|\Phi_\psi\|_{\sigma,j} \lesssim \varepsilon^{-\frac{2j+3}{3}} \|\psi\|_{\sigma,j+4}$$

which is now a problem for the estimate of $\varepsilon^3\|\phi\|_3$ *in* (6.68)*.*

The problem actually comes from the presence of the error term $(r_\psi)_0$ *which itself emerged because we did not solve the entire differential equation* (6.59)*, and set the term* $\varepsilon^2 r\,\varphi_0{}'$ *aside before estimating it thanks to the* $\mathscr{C}^1$*-estimate of* φ_0*. Note that we did not solve the differential equation with this term since it does not fill in the context of lemma* 6.6*. Indeed the function* a *is of type* $\sin^2(\pi y + c) + \varepsilon r$ *which can vanish. So a first attempt could be to establish a lemma for the* $\mathscr{C}^j$*-estimates of solutions of "slightly" hyperbolic differential equations, that is when* r *takes non positive values on a small region where it is negative, and behaves like* $y^2 - b^2$ *near* 0 *up to translation. If* r *is positive, then we are in the easier case of an "elliptic" differential equation, meaning that* a *is positive with bounds (which is treated in lemma A.*1*). Yet, the exact computation of* r *in* (6.32) *is quite hard and does not seem to be exploitable.*

CHAPTER 7

Conclusion

We studied two aspects of the Lyapunov exponent. Firstly we focused on a large deviation theorem (LDT) for the transfer matrices of the Schrödinger equation with potential either quasi-periodic or defined with the doubling map. The second aspect is an asymptotic expansion of the Lyapunov exponent of Schrödinger cocycles defined with mixed quasi-periodic and random potentials in a small coupling regime.

In the first part, we used subharmonic techniques and potential theory to prove that such an LDT is granted on the condition that only an upper estimate involved in the two-sided inequality is true on a complex strip. These techniques also helped us to find back the result of [BG00] in the quasi-periodic setting and permitted us to obtain a large deviation theorem for the inverse branches of the doubling map. These should be a good starting point to prove estimates like Bourgain-Goldstein's of type $\mathrm{L(E)} \gtrsim \log\lambda$ at large coupling using the avalanche principle and without any exclusion of parameter like in [Krü11]. Indeed, our result indicates that in measure, there are a very good proportion of the inverse branches that also verifies the LDT. Now we would like to get combinatorial results on the inverse images of a given x at some generation to propagate the estimates to its predecessors by taking sequences of blocks of adequate size chosen such that this procedure finally gives a lower bound for the Lyapunov exponent.

The second aspect concerns formulæ for the asymptotic development of the Lyapunov exponent in the case of mixed quasi-periodic and random potentials, both equally small. The crucial parameter in the analysis is the diophantine or resonant property of the energy w.r.t. the quasi-periodic frequency α. We recover developments similar to those of several authors in the random setting. Yet, one would have preferred a formula that gives a development in terms of some coupling constants λ, μ respectively for the quasi-periodic part and the random part of the potential: $v(\omega, x) = \mu \mathrm{W}_\omega + \lambda \mathrm{V}(x)$ for λ, ν small but with different regimes: $\lambda \approx \nu, \lambda \gg \mu$ or $\lambda \ll \nu$. Further analysis based on our result might prove to be successful, in particular looking for conjugacies (like in chapters 5 and 6) more accurately adapted to this extraction of a coupling in both λ and ν.

Finally, one could also hope to complete the analysis of the resonant case, at first with computations of the exact orders in the perturbative formula for the condition $\delta_k = 0$ to say whether the lowest order obtained vanishes or not. Also we want to treat the situation where the term of lowest degree in v_ω the random part of the diffeomorphism $(x, y) \mapsto (x+\alpha, y+\varepsilon v_{1,\omega}(x, y)+\dots)$ has two distinct zeros. One way to begin with this issue could be to find obstructions that are no longer constants like we did in part II, but rather maps f_ψ such that $\psi - f_\psi$ vanishes like $\int_{\mathbf{T}} \mathbf{E}\left(v_{1,\omega}(x, y)\right) \mathrm{d}x$. We expect a kind of hyperbolic behaviour to appear, like in chapter 5 for the energies outside the free spectrum. That should provide help with the evaluation of the integrals of the obstructions that appear and compute the Lyapunov exponent.

APPENDIX **A**

Adjoint operator and parametric differential equations

A.1 Spectrum of the adjoint operator

We want to solve $(\mathrm{e}^{2\mathrm{i}\pi n\alpha}\mathrm{Id} - \mathrm{Ad}_{\mathrm{A}})\mathrm{Y} = \mathrm{F} \in \mathfrak{sl}_2(\mathbf{C})$ for $\mathrm{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We compute

$$\mathrm{Ad}_{\mathrm{A}} . \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

So that we have to solve the system

$$\begin{cases} -a\gamma_n + c = a_{\mathrm{F}} \\ a\gamma_n - c = -a_{\mathrm{F}} \\ -2a - c - \gamma_n b = b_{\mathrm{F}} \\ c - \alpha_n c = c_{\mathrm{F}} \end{cases}$$

We successively obtain

$$\begin{cases} c = -c_{\mathrm{F}}/\gamma_n \\ a = \dfrac{c - a_{\mathrm{F}}}{\gamma_n} = \dfrac{-a_{\mathrm{F}}\gamma_n - c_{\mathrm{F}}}{\gamma_n{}^2} \\ b = \dfrac{b_{\mathrm{F}} + 2a + c}{\gamma_n} \end{cases}$$

So we can conclude for lemma 5.2.

A.2 Periodic differential equations with parameter

Lemma *A*.1 (Differential equation with small parameter)

Let us consider the differential equation with (small) parameter $\delta > 0$

$$u' = \frac{1}{\delta} a(y)\, u + f(y) \tag{A.1}$$

for 1-periodic $\mathscr{C}^1$ functions a and f. Assume a is positive and bounded below such as $|a| > a_* > 0$. Let us denote u_δ the unique 1-periodic solution of (A.1). The following estimates hold for all $k \in \mathbf{N}$:

$$\|u_\delta\|_k \lesssim \frac{\delta}{a_*{}^{k+1}} \|f\|_k$$

Proof.

Let us quickly explain why periodic solution do exist. We define

$$\mathrm{A}_\delta(y) = \frac{1}{\delta} \int_0^y a(s)\,\mathrm{d}s$$

Then the resolvent formula gives for $y \in [\,0\,;1\,]$

$$u(y) = u(0)\,\mathrm{e}^{-\mathrm{A}_\delta(y)} + \int_0^y f(s)\,\mathrm{e}^{\mathrm{A}_\delta(s) - \mathrm{A}_\delta(y)}\,\mathrm{d}s$$

Since a and f are 1-periodic, the function $y \mapsto u(1+y)$ is still a solution of the differential equation (A.1). By the Cauchy-Lipschitz theorem this proves that the 1-periodic solution u_δ satisfies $u_\delta(0) = u_\delta(1)$ and we necessarily have such a solution provided $\int_{\mathbf{T}} a(s)\,\mathrm{d}s \neq 0$. In this case

$$u_\delta(0) = \frac{1}{1 - \mathrm{e}^{-\mathrm{A}_\delta(1)}} \int_0^1 f(s)\,\mathrm{e}^{\mathrm{A}_\delta(s) - \mathrm{A}_\delta(1)}\,\mathrm{d}s$$

For the estimates of the first item we use the differential equation $u' = \dfrac{a}{\delta}\, u + f$. Since u is periodic and $\mathscr{C}^\infty$, u admits maxima and minima. If u is extremal at t we have $u'(t) = 0$ and hence $a(t)\,u(t) = -\delta f(t)$. The assumptions on a ensures

$$|u(t)| \leqslant \frac{1}{a_*} \delta \|f\|_0$$

which gives $\|u\|_0 \leqslant \delta/a_* \|f\|_0$.

To prove the other $\mathscr{C}^k$-estimates we just repeat the procedure using the differential equation (A.1) to get equations for the derivatives of u of higher order:

$$u'' = \frac{a}{\delta}\, u' + \frac{a'}{\delta}\, u + f'$$

And so for extremal points t of u' one gets

$$u'(t) = -\frac{1}{a(t)}\,(a'(t)\, u(t) + \delta f'(t))$$

The previous estimate gives

$$|u'(t)| \leqslant \frac{\delta}{a_*}\left(\|a'\|_0 \frac{\|f\|_0}{a_*} + \|f\|_1\right)$$

Thus
$$\|u'\|_0 \lesssim \frac{\delta}{{a_*}^2}\|f\|_1$$

The same method ensures for all integer k:

$$\|u^{(k)}\|_0 \lesssim \frac{\delta}{{a_*}^{k+1}}\|f\|_k$$

■

Lemma *A.2*

For one-periodic continuous functions a and f, the equation $u' + a(y)\,u = f$ admits a 1-periodic solution with zero average on $\mathbf{T}$ iff

$$\int_{\mathbf{T}} f(y)\,\theta(y)\,\mathrm{d}y = 0$$

where $\theta = \theta_a$ is the 1-periodic solution of

$$\theta' - a\,\theta = 1$$

Proof.

Let us define the differential operator of order 2

$$\mathrm{A}\varphi = \varphi'' + a\,\varphi'$$

We can compute its adjoint A^* on $\mathrm{L}^2(\mathbf{T})$

$$\int_{\mathbf{T}} (\varphi''(y) + a(y)\,\varphi'(y))\,\psi(y)\,\mathrm{d}y = \int_{\mathbf{T}} \varphi(y)\,(\psi''(y) - (a\,\psi)'(y))\,\mathrm{d}y$$

so that
$$\mathrm{A}^* = \partial^2 - \partial(a\,\cdot)$$

It is a well know fact that $(\mathrm{Im}\ \mathrm{A})^\perp = \mathrm{Ker}\,(\mathrm{A}^*)$ and obviously

$$\mathrm{A}^*\psi = 0 \quad\Longleftrightarrow\quad \psi' - a\,\psi = \mathrm{c}^{\mathrm{st}}$$

This proves that Ker A^* is spanned by θ the 1-periodic solution of $\theta' - a\,\theta = 1$. If φ is a solution of $u' + a(y)\,u = f$ we can write

$$\int_{\mathbf{T}} \varphi = \int_{\mathbf{T}} \varphi\,(\theta' - a\,\theta) = -\int_{\mathbf{T}} \theta\,(\varphi' + a\,\varphi) = -\int_{\mathbf{T}} \theta\, f$$

which gives the equivalence stated previously. ∎

APPENDIX **B**

Spectral theory

Let us recall some basic notions and results of spectral theory that are implicitly used in this manuscript.

B.1 Spectral theory of operators on Hilbert spaces

B.1.1 Definitions, examples

If $\mathscr{H}$ is a Hilbert space and $\mathcal{D}$ a dense subspace of $\mathscr{H}$ we say that a linear map $\mathrm{A} \in \mathcal{L}(\mathcal{D}, \mathscr{H})$ is *bounded* when

$$\|\mathrm{A}\| := \sup_{u\in\mathcal{D}\setminus\{0\}} \frac{\|\mathrm{A}u\|}{\|u\|} < +\infty$$

If the above condition holds, there is a unique extension of A to $\mathscr{H}$. We denote $\mathcal{B}(\mathscr{H})$ the set of all bounded operators on $\mathscr{H}$.

B.1.2 Crucial examples

Definition *B*.1 (Laplacian, shorted Laplacian (discrete case))

Let us define the following Laplacian operators:

- $\widetilde{\Delta}_d\colon \begin{cases} \ell^2(\mathbf{Z}^d) & \longrightarrow \ell^2(\mathbf{Z}^d) \\ (u_n)_{n\in\mathbf{Z}^d} & \longmapsto \left(\sum_{\|k\|_1=1} u_{n+k}\right)_{n\in\mathbf{Z}^d} \end{cases}$
- $\Delta = \widetilde{\Delta} - 2d\,\mathrm{Id}$

Then $$\|\widetilde{\Delta}_d\| = 2d < +\infty$$

The Schrödinger operator also involves the following operator:

Definition *B*.2 (Multiplication operator)

Let us define the multiplication operators either in the continuous or discrete case

- $\mathcal{M}_\varphi\colon \begin{cases} \mathrm{L}^2 \longrightarrow \mathrm{L}^2 \\ f \longmapsto f\varphi \end{cases}$
- $\mathcal{M}_v\colon \begin{cases} \ell^2 \longrightarrow \ell^2 \\ (u_n) \longmapsto (u_n v_n) \end{cases}$

Then their boundedness is submitted to the following conditions

$$\begin{aligned} \mathcal{M}_\varphi \in \mathcal{B}(\mathrm{L}^2) \quad &\Longleftrightarrow \varphi \in \mathrm{L}^\infty \\ \mathcal{M}_v \in \mathcal{B}(\ell^2) \quad &\Longleftrightarrow v \in \ell^\infty \end{aligned}$$

Remark *B*.1

In the whole manuscript we only consider bounded operators H *(the potentials of the Schrödinger operators are bounded).*

B.1.3 Spectrum of an operator

The notion of spectrum generalizes the notion of eigenvalues of linear maps in finite dimension.

Definition $B.3$ (Resolvent set, spectrum)

- $\rho(\mathrm{H}) = \{\lambda \in \mathbf{C} \ : \ \mathrm{H} - \lambda$ *is bijective with bounded inverse*$\}$ *is called the* resolvent set*;*
- *the resolvent of* H *is*

$$\mathrm{R_H}(\lambda) = (\mathrm{H} - \lambda)^{-1}$$

  Note that $\lambda \longmapsto \mathrm{R_H}$ *is holomorphic;*
- *we define* $\sigma(\mathrm{H}) = \mathbf{C} \setminus \rho(\mathrm{H})$ *the* spectrum *of* H*. It is a compact subset of* $\mathbf{C}$ *(for bounded* H*) which is invariant by conjugacy:*

$$\sigma(\mathrm{UHU}^{-1}) = \sigma(\mathrm{H})$$

We obviously have $\{\text{eigenvalues of } \ \mathrm{H}\} \subsetneq \sigma(\mathrm{H})$ and there are many ways to belong to the spectrum:

- $\sigma_{\text{point}}(\mathrm{H}) = \{\lambda \in \mathbf{C} \ : \ \mathrm{H} - \lambda \text{ non injective}\} = \{\text{eigenvalues of H}\}$
- $\sigma_{\text{continuous}}(\mathrm{H}) = \{\lambda \in \mathbf{C} \ : \ \mathrm{A} - \lambda \text{ injective with dense range}\}$
- $\sigma_{\text{residual}}(\mathrm{H}) = \{\lambda \in \mathbf{C} \ : \ \mathrm{A} - \lambda \text{ injective with non dense range}\}$

B.2 Adjoint, self adjointness

Definition $B.4$

The adjoint of a bounded operator H *is the linear operator defined by*

$$\forall \varphi, \psi \in \mathscr{H} \qquad \langle \mathrm{H}\varphi | \psi \rangle = \langle \varphi | \mathrm{H}^* \psi \rangle$$

The operator H *is said* self adjoint *(s.a.) if* $\mathrm{H}^* = \mathrm{H}$*. Such an operator has real spectrum:* $\sigma(\mathrm{H}) \subset \mathbf{R}$.

For our work, the following examples matter as they naturally arise from the Schrödinger equation.

Examples *B*.1

- *The Laplacians* Δ *and* $\widetilde{\Delta}$ *are s.a. and*

$$\sigma(\Delta) = [\,-4d\,;0\,] \qquad \text{and} \qquad \sigma(\widetilde{\Delta}) = [\,-2d\,;2d\,]$$

- *The multiplication operator* $\mathcal{M}_{\varphi\in \mathrm{L}^\infty}$ *is s.a. iff* $\overline{\varphi} \overset{\mathrm{L}^2(\mu)}{=} \varphi$ *and in that case*

$$\sigma(\mathcal{M}_\varphi) = \text{ess.}\operatorname{Im}\varphi$$

where$\text{ess.}\operatorname{Im}\varphi := \{z \in \mathbf{C} \;:\; \forall\varepsilon > 0 \quad \mu\,(x : |f(x) - z| < \varepsilon) > 0\}$

is the essential range *of* φ.

Actually the Fourier transform conjugates $\widetilde{\Delta}$ *to* $\mathcal{M}_{2d\cos(2\pi\|\cdot\|_1)}$ *so the first item is just a consequence of the second.*

B.3 Spectral theorem and spectral types

B.3.1 Herglotz maps and representation

Definition *B*.5 (Herglotz's functions)

We write $\mathrm{F} : \mathbf{H}^+ \to \mathbf{H}^+ \in \mathrm{Herg}$ *to design a function* F *that is holomorphic and verifies that* $z \mapsto \mathfrak{Im}(z)\,\mathrm{F}(z)$ *is bounded on* $\mathbf{H}^+$.

Proposition *B*.1 (Herglotz representation)

If $\mathrm{F} \in \mathrm{Herg}$ *then there exists a unique positive measure* μ_F *on* $(\mathbf{R}, \mathrm{Bor}(\mathbf{R}))$ *s.t.* F *is the Borel transform of* μ_F*:*

$$\mathrm{F}(z) = \int_\mathbf{R} \frac{\mathrm{d}\mu_\mathrm{F}(t)}{t - z}$$

B.3.2 Spectral theorem

The maps $\mathrm{F}_x : z \mapsto \langle x|\mathrm{R}_\mathrm{A}(z)x\rangle$ are Herglotz for any A s.a. Hence one obtains spectral measures μ_x and then $\mu_{x,y}$ by polarization. So we can integrate with respect to those measures and get

Theorem B.1 (Spectral thm - functional calculus version)

Let A *be s.a. on* $\mathscr{H}$*. Then there exists a unique continuous morphism of* C^**-unitary algebras with norm* 1 *denoted* $\Phi_{\mathrm{A}} : \mathrm{Bor}_b(\mathbf{R}, \mathbf{C}) \ni f \longmapsto f(\mathrm{A}) \in \mathcal{B}(\mathscr{H})$*. It verifies*

$$\mathrm{A}\varphi = \lambda\varphi \quad \Longrightarrow \quad f(\mathrm{A})\varphi = f(\lambda)\varphi$$

$$\langle x | f(\mathrm{A}) y \rangle = \int_{\mathbf{R}} f(t) \, \mathrm{d}\mu_{x,y}(t)$$

This permits to define $f(\mathrm{A})$ *for suitable functions* f *and in particular for* $\mathrm{E} \in \mathbf{R}$ *and* π_{F} *the projection on a subspace* F*:*

$$\mathbf{1}_{\{\mathrm{E}\}}(\mathrm{A}) = \pi_{\mathrm{Ker}\,(\mathrm{A}-\mathrm{E})}$$

Thanks to the spectral theorem we can divide the spectrum into relevant components. This decomposition is based on the Radon-Nikodym decomposition of measures.

Theorem B.2 (Radon-Nikodym)

Let ν *be any positive measure on* $(\mathbf{R}^d, \mathrm{Bor}(\mathbf{R}^d))$*. Then we can write*

$$\nu = \nu_{\mathrm{pp}} + \nu_{\mathrm{sc}} + f \,\mathrm{Leb} \text{ where}$$

- ν_{pp} *is a countable sum of atoms named the* pure point part*;*
- $\nu_{\mathrm{pp}} + \nu_{\mathrm{sc}} =: \nu_{\mathrm{s}} \perp \mathrm{Leb}$ *is the singular part of* ν*;*
- $f \,\mathrm{Leb}$ *with* $f \in \mathrm{L}^1(\mathbf{R}^d)$ *is the absolutely continuous part (w.r.t. the Lebesgue measure);*
- $\nu_{\mathrm{c}} = f \,\mathrm{Leb} + \nu_{\mathrm{sc}}$ *is the continuous part (without atom).*

Then one can define the spectral types of an operator:

Definition B.6 (Spectral types)

$$\begin{aligned} \textit{We denote} \quad \mathscr{H}_{\mathrm{p}} &= \mathrm{Vect}\,\{\textit{eigenvalues of } \mathrm{A}\} \\ \mathscr{H}_{\mathrm{pp}} &= \overline{\mathscr{H}_{\mathrm{p}}} \\ \mathscr{H}_{\bullet} &= \{x \in \mathscr{H} \mid \mu_x \textit{ of type} \bullet\} \quad \textit{for} \quad \bullet \in \{\mathrm{c}, \mathrm{ac}, \mathrm{sc}\} \end{aligned}$$

Those types lead to the following decomposition of the Hilbert space:

Proposition *B.2* (Spectral decomposition)

The different parts of $\mathscr{H}$ satisfy:

(i) $\mathscr{H}_{\mathrm{pp}} = \{x \in \mathscr{H} \mid \mu_x \text{ is pure point}\}$

(ii) $\mathscr{H} = \mathscr{H}_{\mathrm{pp}} \oplus \mathscr{H}_{\mathrm{sc}} \oplus \mathscr{H}_{\mathrm{ac}} \qquad \text{and} \qquad \mathscr{H}_{\mathrm{pp}} = {\mathscr{H}_{\mathrm{c}}}^{\perp}$

(iii) $\mathscr{H}_\bullet$ *is closed and "stable" under* A*:*

$$\left[\mathrm{A}_{|\mathscr{H}_\bullet}, \pi_{\mathscr{H}_\bullet}\right] = 0$$

Let $\sigma_\bullet(\mathrm{A}) = \sigma(\mathrm{A}_{|\mathscr{H}_\bullet})$. *Then*

$$\sigma_{\mathrm{pp}}(\mathrm{A}) = \overline{\sigma_{\mathrm{p}}(\mathrm{A})} = \overline{\{\text{e.v.}\}} \quad \text{and} \quad \sigma(\mathrm{A}) = \sigma_{\mathrm{pp}}(\mathrm{A}) \cup \sigma_{\mathrm{ac}}(\mathrm{A}) \cup \sigma_{\mathrm{sc}}(\mathrm{A})$$

B.4 Ergodic operators

B.4.1 Basic ergodic theory definitions and results

Let $(\Omega, \mathcal{F}, m)$ be a finite measured space. We say that a measurable map $\mathrm{T} : \Omega \to \Omega$ is *ergodic* if for all measurable set A one has $m(\mathrm{T}^{-1}\mathrm{A}) = m(\mathrm{A})$ and the only T-invariant subsets are trivial modulo μ :

$$\mathrm{T}^{-1}\mathrm{A} = \mathrm{A} \quad \Longrightarrow \quad \mu(\mathrm{A}) \in \{0\,;1\}$$

This is equivalent to ask for

$$\forall f \in \mathrm{L}^p \qquad f \circ \mathrm{T} = f \quad \Longrightarrow \quad f = \mathrm{c}^{\mathrm{st}} \quad m - \text{a.e.}$$

The main theorem in this setting is:

Theorem *B.3* (Birkhoff's ergodic theorem)

If $(\Omega, \mathcal{T}, m, \mathrm{T})$ is ergodic then for all $f \in \mathrm{L}^1(\Omega)$ its average in time coincides with its spatial one for almost every starting point:

$$\frac{1}{n}\sum_{k=0}^{n} f \circ \mathrm{T}^k(x) \xrightarrow[n\to\infty]{m(x)-\text{a.e.}} \frac{1}{m(\Omega)}\int_\Omega f \,\mathrm{d}m$$

B.4.2 Ergodic families of operators

Those families naturally arise when the potential of the Schrödinger operator contain some parameter.

Definition *B.7* (Ergodic s.a. operator)

A measurable map $\mathrm{H} : \omega \in \Omega \longmapsto \mathrm{H}_\omega \in \mathcal{SA}(\mathscr{H})$ *is said to be an* ergodic self-adjoint operator *when*

$$\forall i \in \mathrm{I} \quad \forall \omega \in \Omega \qquad \mathrm{H}_{\mathrm{T}_i(\omega)} = \mathrm{U}_i{}^* \, \mathrm{H}_\omega \, \mathrm{U}_i$$

where $(\mathrm{T}_i)_{i \in \mathrm{I}}$ *is a family of ergodic maps on* $(\Omega, \mathcal{F}, m)$ *and* $(\mathrm{U}_i)_{i \in \mathrm{I}}$ *a family of unitary operators on* $\mathscr{H}$ *which requires* $\mathrm{U}_i{}^* = \mathrm{U}_i{}^{-1}$ *for all* i*.*

In our model of quasi-periodic and random operators, we have such an ergodic operator thanks to ergodicity of $(\cdot + \alpha, \sigma)$ where $\sigma(\omega_n)_n = (\omega_{n+1})_n$.

The most significant result about ergodic operators is the following independence of the different spectra w.r.t. the parameter.

Theorem *B.4* (Pastur, Ishii, Kotani)

The spectrum of an ergodic s.a. operator is m*-a.e. constant.*

Remark *B.2*

The same result also holds for the components of the spectrum $\sigma_{\mathrm{pp}}, \sigma_{\mathrm{ac}}, \sigma_{\mathrm{c}}$ *but not the eigenvalues* σ_{p}*.*

B.5 Eigenvalue equations

Let us recall why eigenvalue equations $\mathrm{H}\psi = \mathrm{E}\psi$ are important. Another formulation of the spectral theorem is the following (see [RS79])

Theorem *B.5* (Spectral thm - mult. operator version)

Let $\mathrm{H} \in \mathcal{B}(\mathscr{H})$ *be s.a.. Then there exists on* $\mathbf{R}$ *finite Borelian measures* $(\mu_k)_{0 \leqslant k \leqslant r}$ *with* $r \in \overline{\mathbf{N}}$ *satisfying* $\mu_1 \gg \mu_2 \gg \cdots \gg \mu_i \gg \mu_{i+1} \ldots$ *and a unitary operator* $\mathrm{U} \in \mathcal{U}(\mathscr{H}, \bigoplus_{k=1}^r \mathrm{L}^2(\mathbf{R}, \mu_k))$ *s.t.*

$$\mathrm{A} = \mathrm{U}^{-1} \mathcal{M}_{\mathrm{Id}} \mathrm{U} \qquad \textit{and} \qquad \sigma(\mathrm{H}) = \operatorname{Supp} \mu_1$$

- r is called the multiplicity of H
- $\mu_{\mathrm{H}} := \mu_1$ is **the** spectral measure of H

Remark $B.3$

For $\mathrm{H} = \mathrm{H}_v \in \mathcal{B}(\ell^2(\mathbf{Z}^d))$, *we have* $r = 2$, $\mu_{\mathrm{H}} = \frac{1}{2}(\mu_{0,0} + \mu_{1,1})$ *in the sense of the previous spectral measure associated to* δ_0 *and* δ_1.

Now we relate the spectral measures to the eigenvalue equations.

Definition $B.8$ (Spectral meas., generalized e.v.)

The generalized *spectrum (made of generalized eigenvalues) is defined as follows:*

$$\sigma_{\mathrm{gen}}(\mathrm{H}) = \left\{ \mathrm{E} \in \mathbf{R} \mid \exists \psi \in \mathbf{R}^{\mathbf{Z}^d} \setminus \{0\} \text{ with polynomial growth s.t. } \quad \mathrm{H}\psi = \mathrm{E}\psi \right\}$$

That is we require

- $\forall n \in \mathbf{Z}^d \qquad -\Delta\psi(n) + v_n\psi(n) = \mathrm{E}\psi(n)$
- $\exists a, c > 0 \quad \forall n \in \mathbf{Z}^d \qquad |\psi(n)| \leqslant c(1 + \|n\|)^a$

Note that $\psi \notin \ell^2(\mathbf{Z}^d)$ *a priori*. The main result is the following:

Proposition $B.3$

- $\sigma_{\mathrm{gen}}(\mathrm{H}) \subset \sigma(\mathrm{H})$
- $\overline{\sigma_{\mathrm{gen}}(\mathrm{H})} = \sigma(\mathrm{H})$
- $\mu_{\mathrm{H}}\left(\sigma(\mathrm{H}) \setminus \sigma_{\mathrm{gen}}(\mathrm{H})\right) = 0$
- (Berezansky) μ_{H}-*almost every* $\mathrm{E} \in \mathbf{R}$ *is a generalized eigenvalue.*

These polynomially bounded solutions are often used in the literature to prove results about *Anderson localization* (see [BS00, BG00]. Those are formal solution verifying a bound that is not "too bad" which is used to ultimately prove the exponential decay of formal solutions of the eigenvalue equations.

List of Figures

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