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Variational and viscosity solutions of the Hamilton-Jacobi equation

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THÈSE DE DOCTORAT

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Préparée à l'Université Paris-Dauphine

Solutions variationnelles et solutions de viscosité
de l'équation de Hamilton-Jacobi

École Doctorale de Dauphine — ED 543

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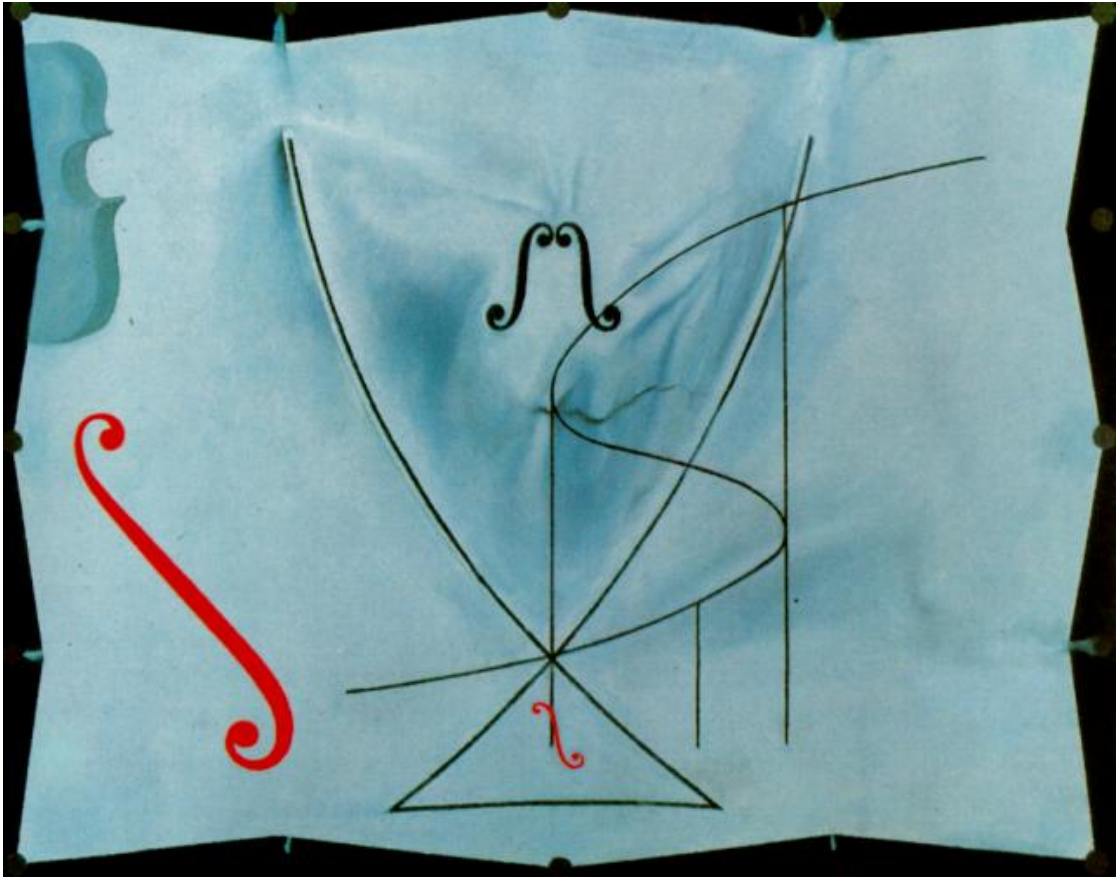
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*À Jacqueline et Arthur,
qui se seront manqués de peu.*



La queue d'aronde — Série des catastrophes, Salvador Dali, Mai 1983

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Introduction

L'équation de Hamilton-Jacobi

Dans cette thèse on étudie différents types de solutions faibles pour l'équation de Hamilton-Jacobi évolutive du premier ordre. Cette équation est donnée par un hamiltonien, c'est-à-dire une fonction $H : \mathbb{R} \times T^*\mathbb{R}^d \rightarrow \mathbb{R}$ que l'on supposera tout au long de cette thèse de classe \mathcal{C}^2 , et s'écrit ainsi :

$$\partial_t u(t, q) + H(t, q, \partial_q u(t, q)) = 0, \quad (\text{HJ})$$

où $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ est la fonction inconnue.

L'équation de Hamilton-Jacobi apparaît dans le cadre de la mécanique hamiltonienne comme l'équation vérifiée par l'action hamiltonienne d'un système. Elle connaît un nouvel essor depuis le milieu du siècle dernier, lorsque R. Bellman observe qu'elle est plus généralement l'équation vérifiée par la valeur optimale d'un problème d'optimisation en contrôle optimal. Sous cette forme, elle intervient dans de nombreux domaines d'applications, comme l'économie, le trafic routier ou encore le problème des tourtereaux¹.

On étudie le problème de Cauchy formé par cette équation et la donnée d'une condition initiale $u(0, \cdot) = u_0$, qu'on supposera au moins lipschitzienne. Même pour un hamiltonien et une donnée initiale lisses, ce problème de Cauchy n'admet pas forcément de solutions classiques en temps long, et différents types de solutions faibles ont ainsi été introduites pour donner un sens à l'équation pour des fonctions non différentiables. L'objet de cette thèse est de comparer deux de ces notions : d'un côté, les solutions de viscosité, définies par P.-L. Lions et M. G. Crandall, qui sont communément utilisées dans l'analyse des équations de Hamilton-Jacobi et plus largement dans l'étude d'équations aux dérivées partielles elliptiques, et de l'autre côté les solutions variationnelles, introduites dans le cadre de la géométrie symplectique par J.-C. Sikorav et M. Chaperon, qui sont plus directement en lien avec la dynamique hamiltonienne sous-jacente à l'équation.

S'il est établi (voir [Jou91]) que ces deux solutions coïncident dans le cas très physique d'un hamiltonien convexe par rapport à la variable impulsion, des exemples de solutions variationnelles ne vérifiant pas l'équation au sens de la viscosité sont également connus de longue date, voir par exemple [Che75], [Vit96], [BC11] et [Wei14].

Pour pouvoir comparer les deux notions, on se place dans des hypothèses de travail bien adaptées à la fois au cadre variationnel et aux solutions de viscosité, en prenant une donnée initiale lipschitzienne et un hamiltonien vérifiant l'hypothèse suivante.

Hypothèse. Il existe $C > 0$ tel que pour tout (t, q, p) dans $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\|\partial_{(q,p)}^2 H(t, q, p)\| < C, \quad \|\partial_{(q,p)} H(t, q, p)\| < C(1 + \|p\|), \quad |H(t, q, p)| < C(1 + \|p\|)^2, \quad (1)$$

où l'on note $\partial_{(q,p)} H$ et $\partial_{(q,p)}^2 H$ les dérivées spatiales de H de premier et second ordre.

¹Voir [GL15].

La majoration de la dérivée seconde de H est classique en dynamique hamiltonienne, puisqu'elle garantit que les trajectoires n'explosent pas en temps fini. La majoration de la dérivée première apparaît naturellement pour des problèmes de contrôle optimal.

Cette hypothèse de travail garantit un principe de propagation finie à la fois dans le cadre variationnel (voir l'annexe B de [CV08]) et pour les solutions de viscosité (voir [ABI99]), ce qui permet de travailler avec des hamiltoniens qui ne sont pas nécessairement à support compact.

La méthode des caractéristiques en dynamique hamiltonienne

La mécanique hamiltonienne associe à un hamiltonien le système d'équations suivant,

$$\begin{cases} \dot{q}(t) = \partial_p H(t, q(t), p(t)), \\ \dot{p}(t) = -\partial_q H(t, q(t), p(t)), \end{cases} \quad (\text{HS})$$

nommé *système hamiltonien*. On appelle *trajectoire hamiltonienne* une solution $(q(t), p(t))$ du système hamiltonien. Lorsque le hamiltonien est à dérivée seconde bornée, le système admet un flot complet, c'est-à-dire qu'il existe une famille de fonctions $\phi_s^t : T^*\mathbb{R}^d \rightarrow T^*\mathbb{R}^d$, définie pour tout $s \leq t$, telle que $t \mapsto (q(t), p(t)) = \phi_s^t(q, p)$ est l'unique trajectoire hamiltonienne vérifiant $(q(s), p(s)) = (q, p)$ au temps s : on dit que ϕ est le *flot hamiltonien* associé à H .

L'*action hamiltonienne* entre le temps s et t d'un chemin régulier $\gamma(t) = (q(t), p(t))$ dans l'espace cotangent $T^*\mathbb{R}^d$ est définie par

$$\mathcal{A}_s^t(\gamma) = \int_s^t p(\tau) \cdot \dot{q}(\tau) - H(\tau, q(\tau), p(\tau)) d\tau,$$

et le calcul des variations montre que si γ est un chemin qui est un point critique de l'action \mathcal{A}_s^t parmi les chemins à extrémités fixées, γ satisfait le système hamiltonien (HS).

La *méthode des caractéristiques* est une technique classique de résolution d'équations aux dérivées partielles. Adaptée au cadre de l'équation de Hamilton-Jacobi, elle garantit que si u est une solution \mathcal{C}^2 de l'équation de Hamilton-Jacobi sur le domaine $[0, T] \times \mathbb{R}^d$, et si u_s et u_t désignent la fonction u à s ou t fixé, le flot hamiltonien ϕ_s^t envoie le graphe de la différentielle du_s sur le graphe de la différentielle du_t pour tout $0 \leq s \leq t \leq T$. De plus, si ϕ_s^t envoie le point $(q_s, du_s(q_s))$ sur $(q_t, du_t(q_t))$, la différence de u entre les points (s, q_s) et (t, q_t) est donnée par l'action de la trajectoire hamiltonienne envoyant $(q_s, du_s(q_s))$ sur $(q_t, du_t(q_t))$. Autrement dit, si $\gamma(\tau) = \phi_s^\tau(q_s, du_s(q))$,

$$u(t, q_t) = u(s, q_s) + \mathcal{A}_s^t(\gamma).$$

Cette méthode donne aussi l'existence de solutions classiques lorsque la donnée initiale et le hamiltonien sont à dérivée seconde bornée, voir Proposition 1.3.

Solution géométrique et front d'onde associés au problème de Cauchy

Si u_0 est une donnée initiale lisse (au moins de classe \mathcal{C}^2), on note Γ_0 le graphe de la dérivée de u_0 et on appelle *solution géométrique au temps t* son évolution par le flot hamiltonien, $\phi_0^t(\Gamma_0)$. La méthode des caractéristiques implique que si u est une solution \mathcal{C}^2 de l'équation de Hamilton-Jacobi sur le domaine $[0, \tau] \times \mathbb{R}^d$, le graphe de la différentielle de u_t est égal à la solution géométrique au temps t pour tout t dans $[0, \tau]$: $(q, du_t(q)) \in \phi_0^t(\Gamma_0)$. En particulier, si la solution géométrique n'est plus un graphe pour un certain temps T , comme c'est le cas sur la figure 1.1, l'existence de solutions \mathcal{C}^2 sur le domaine $[0, T] \times \mathbb{R}^d$ est exclue.

Le *front d'onde* au temps t associé au problème de Cauchy pour une donnée initiale u_0 de classe \mathcal{C}^2 , noté $\mathcal{F}_0^t u_0$, est défini ainsi :

$$\mathcal{F}_0^t u_0 = \left\{ (q, u_0(q_0) + \mathcal{A}_0^t(\phi_0^\tau(q_0, du_0(q_0)))) \mid \begin{array}{l} t \geq 0, q \in \mathbb{R}^d, q_0 \in \mathbb{R}^d, \\ Q_0^t(q_0, du_0(q_0)) = q. \end{array} \right\} \quad (\text{F})$$

Au dessus de chaque point q , le front d'onde au temps t donne l'action hamiltonienne de chacune des trajectoires qui démarrent en un point du graphe de du_0 au temps 0, et arrivent au dessus du point q au temps t , à laquelle on ajoute la valeur de la donnée initiale pour la position de départ.

La méthode des caractéristiques garantit que si u est une solution \mathcal{C}^2 de l'équation de Hamilton-Jacobi sur le domaine $[0, \tau] \times \mathbb{R}^d$, le graphe de u_t est égal au front d'onde au temps t pour tout t dans $[0, \tau]$. Le front d'onde peut être vu comme une solution multivaluée au problème de Cauchy lorsqu'il n'est plus un graphe, comme c'est le cas sur la figure 1.2 à droite.

Enfin, la méthode des caractéristiques impliquent que la solution géométrique pour une solution classique donne point à point la dérivée du front d'onde associé. C'est toujours le cas lorsque la solution géométrique et le front d'onde ne sont plus des graphes, voir la figure 1.2 à droite.

Solutions de viscosité

En ajoutant un petit terme de viscosité à l'équation de Hamilton-Jacobi (HJ), on obtient une équation aux dérivées partielles parabolique :

$$\partial_t u^\varepsilon(t, q) + H(t, q, \partial_q u^\varepsilon(t, q)) = \varepsilon \Delta_q u^\varepsilon(t, q).$$

Une telle équation admet une unique solution u^ε , et la famille (u^ε) atteint une limite lorsque ε tend vers 0. Cette technique, appelée *méthode de la viscosité évanescence*, a été introduite initialement pour des équations quasi-linéaires, voir [Ole59b] et [Kru70].

P.-L. Lions et M. G. Crandall donnèrent en 1981 (voir [CL83]) une définition de solution de viscosité plus pratique à manipuler, qui s'inscrit dans la continuité des travaux de L. Evans (voir [Eva80]). Voici une version possible de cette définition :

Définition. Une fonction continue u est une *sous-solution* de viscosité de (HJ) en un point $(t, q) \in (0, \infty) \times \mathbb{R}^d$ si pour toute fonction \mathcal{C}^∞ $\phi : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ telle que $u - \phi$ atteint un maximum local (strict) en (t, q) ,

$$\partial_t \phi(t, q) + H(t, q, \partial_q \phi(t, q)) \leq 0.$$

Une fonction continue u est une *sursolution* de viscosité de (HJ) en un point $(t, q) \in (0, \infty) \times \mathbb{R}^d$ si pour toute fonction \mathcal{C}^∞ $\phi : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ telle que $u - \phi$ atteint un minimum local (strict) en (t, q) ,

$$\partial_t \phi(t, q) + H(t, q, \partial_q \phi(t, q)) \geq 0.$$

La fonction u est solution de viscosité au point (t, q) si elle est à la fois sous-solution et sursolution en ce point.

Cette définition implique entre autres qu'une solution différentiable de l'équation est solution de viscosité, et qu'une solution de viscosité résout l'équation au sens classique en tout point de différentiabilité.

Cette notion est rapidement apparue comme *la* bonne notion de solution généralisée pour l'équation de Hamilton-Jacobi (et d'autres), de par ses bonnes propriétés d'existence, d'unicité

et de stabilité dans de nombreux jeux d'hypothèses, incluant celui de cette thèse. La théorie des solutions de viscosité s'est alors vigoureusement développée, donnant naissance à une littérature à présent très vaste. On renvoie à [CIL92], [Bar94] ou [BCD97] pour des présentations générales et détaillées du sujet.

Caractérisation axiomatique

Dans le cadre du traitement d'images, [AGLM93] (Theorem 2) propose l'idée de caractériser les solutions de viscosité par le biais d'un opérateur satisfaisant un certain nombre d'axiomes, voir aussi [FS06] (Theorem 5.1) et [Bit01] (Theorem 3.1) pour une extension de ces résultats sous des hypothèses plus faibles. On utilise une caractérisation similaire dans cette thèse : on appelle *opérateur de viscosité* une famille d'opérateurs $(V_s^t)_{s \leq t}$ sur $\mathcal{C}^{0,1}(\mathbb{R}^d)$ (l'ensemble des fonctions lipschitziennes sur \mathbb{R}^d) vérifiant les propriétés suivantes :

- (i) Monotonie : si $u \leq v$ sur \mathbb{R}^d , $V_s^t u \leq V_s^t v$ sur \mathbb{R}^d pour tout $s \leq t$,
- (ii) Additivité : si $c \in \mathbb{R}$, $V_s^t(c + u) = c + V_s^t u$ pour tout u dans $\mathcal{C}^{0,1}(\mathbb{R}^d)$,
- (iii) Régularité : si $u \in \mathcal{C}^{0,1}(\mathbb{R}^d)$ et $\tau \leq T$, la famille de fonctions $\{q \mapsto V_\tau^t u(q), t \in [\tau, T]\}$ est équi-lipschitzienne et $(t, q) \mapsto V_\tau^t u(q)$ est localement lipschitzienne sur $(\tau, \infty) \times \mathbb{R}^d$,
- (iv) Compatibilité avec l'équation de Hamilton-Jacobi : si u est une solution \mathcal{C}^2 et lipschitzienne de l'équation de Hamilton-Jacobi, alors $V_s^t u_s = u_t$ pour tout $s \leq t$,
- (v) Propriété de Markov : $V_s^t = V_\tau^t \circ V_s^\tau$ pour tout $s \leq \tau \leq t$.

La proposition suivante, démontrée dans [Ber12] (Proposition 20), justifie cette appellation.

Proposition. *Soit H un hamiltonien \mathcal{C}^2 à dérivée spatiale seconde bornée et $V_s^t : \mathcal{C}^{0,1}(\mathbb{R}^d, \mathbb{R}) \rightarrow \mathcal{C}^{0,1}(\mathbb{R}^d, \mathbb{R})$ un opérateur de viscosité défini pour tout $0 \leq s \leq t$. Alors pour toute donnée initiale $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ lipschitzienne,*

$$u : (t, q) \mapsto V_0^t u_0(q)$$

est solution de viscosité de l'équation de Hamilton-Jacobi sur $(0, \infty) \times \mathbb{R}^d$.

Théorème 1. *Si H vérifie l'hypothèse (1), il existe un unique opérateur de viscosité V_s^t .*

L'unicité est la conséquence d'un résultat d'unicité plus fort établi par H. Ishii dans [Ish84] pour des solutions non bornées (Theorem 2.1 et Remark 2.2), voir aussi [CIL92]. On en donne une autre preuve dans l'annexe A, inspirée de [ABIL13], où l'on démontre une propriété de *vitesse de propagation finie* (Proposition A.1) en appliquant la *méthode de dédoublement des variables*, qui est une technique classique de l'analyse des solutions de viscosité.

L'existence d'un tel opérateur était déjà garantie dans notre contexte par les travaux de Crandall, Lions et Ishii (voir [CIL92]). Cette thèse en donne une autre preuve : on va déduire par un procédé itératif l'existence d'une solution de viscosité de l'existence de solutions variationnelles (voir le théorème 3).

Condition d'entropie d'Oleinik

En dimension 1, la théorie des solutions de viscosité de l'équation de Hamilton-Jacobi est la contrepartie de la théorie des solutions entropiques pour les lois de conservation scalaire : en effet si u résout l'équation de Hamilton-Jacobi, $p(t, q) = \partial_q u(t, q)$ résout l'équation

$$\partial_t p(t, q) + \partial_q (H(t, q, p(t, q))) = 0.$$

La condition d'entropie qui suit, introduite par O. Oleinik dans [Ole59a] pour des lois de conservation, donne un critère géométrique pour décider si une fonction est solution de viscosité en un point de singularité. Elle est démontrée en ces termes par exemple dans [Kos93] (Theorem 2.2), en application directe du Theorem 1.3 de [CEL84]. On l'énonce ici pour un hamiltonien intégrable, c'est-à-dire qui ne dépend que de p .

Par convention, on énonce la condition d'entropie d'Oleinik pour ce qu'on appellera *choc simple descendant*, en référence aux lois de conservation, et on appellera *condition d'entropie d'Oleinik inverse* la condition analogue pour un choc simple ascendant.

Définition (Condition d'entropie d'Oleinik). Soit $H : \mathbb{R} \rightarrow \mathbb{R}$ un hamiltonien de classe \mathcal{C}^2 . Si $(p_1, p_2) \in \mathbb{R}^2$, on dit que la *condition d'entropie d'Oleinik* est (strictement) vérifiée entre p_1 et p_2 si

$$H(\mu p_1 + (1 - \mu)p_2) \stackrel{(<)}{\leq} \mu H(p_1) + (1 - \mu)H(p_2) \quad \forall \mu \in (0, 1),$$

c'est-à-dire si et seulement si le graphe de H est situé sous la corde reliant $(p_1, H(p_1))$ à $(p_2, H(p_2))$.

On dit que la *condition de Lax* est (strictement) vérifiée entre p_1 et p_2 si

$$H'(p_1)(p_2 - p_1) \stackrel{(<)}{\leq} H(p_2) - H(p_1) \stackrel{(<)}{\leq} H'(p_2)(p_2 - p_1),$$

ce qui est automatiquement vérifiée si la condition d'entropie d'Oleinik est satisfaite.

Plus de détails sur ces conditions se trouvent dans l'Appendix F.

Proposition. Soit $u = \min(f_1, f_2)$ sur un voisinage ouvert U de (t, q) dans $\mathbb{R}^+ \times \mathbb{R}$, où f_1 et f_2 sont des solutions \mathcal{C}^1 de l'équation de Hamilton-Jacobi sur U . On note $p_1 = \partial_q f_1(t, q)$ et $p_2 = \partial_q f_2(t, q)$. Si $f_1(t, q) = f_2(t, q)$, alors u est solution de viscosité de l'équation de Hamilton-Jacobi au point (t, q) si et seulement si la condition d'entropie d'Oleinik est satisfaite entre p_1 et p_2 .

La condition d'entropie d'Oleinik est également valable pour des hamiltoniens non intégrables et en dimension supérieure, pour des chocs situés sur une hypersurface, voir [IK96]. Elle peut aussi être généralisée lorsque u est le minimum de plus de deux fonctions, voir [Ber13].

Solutions variationnelles

Le graphe sélecteur

La description de la solution géométrique et du front d'onde associés au problème de Cauchy motive la discussion qui suit : pour définir une solution univaluée à l'équation de Hamilton-Jacobi, on cherche à sélectionner une section continue du front d'onde.

Pour cela, on se place dans un cadre symplectique standard. On considère le fibré cotangent $\pi : T^*M \rightarrow M$ d'une variété riemannienne M complète et de dimension d . Si $q = (q_1, \dots, q_d)$ sont des coordonnées sur M , les coordonnées duales $p = (p_1, \dots, p_d)$ sur T_q^*M sont définies par $p_i(e_j) = \delta_{ij}$, où e_j désigne le j^e vecteur de la base canonique et δ_{ij} est le symbole de Kronecker. La variété T^*M est munie de la 1-forme de Liouville λ qui s'écrit $\lambda = pdq$ dans le système de coordonnées dual. La structure symplectique sur T^*M est donnée par la forme symplectique $\omega = d\lambda = dp \wedge dq$.

Une sous-variété \mathcal{L} de T^*M est dite *lagrangienne* si elle est de dimension d et si $i_{\mathcal{L}}^* \omega = 0$, où $i_{\mathcal{L}} : \mathcal{L} \rightarrow T^*M$ est l'inclusion. Une sous-variété lagrangienne est dite exacte si de plus $i_{\mathcal{L}}^* \lambda$

est exacte, c'est-à-dire s'il existe une fonction lisse $S : \mathcal{L} \rightarrow \mathbb{R}$ telle que $dS = i_{\mathcal{L}}^* \lambda$. Une telle fonction est appelée *primitive* de \mathcal{L} , et est déterminée à constante près. On appelle alors *front d'onde* pour \mathcal{L} l'ensemble défini (à constante près) par $\mathcal{W} = \{(\pi(x), S(x)), x \in \mathcal{L}\}$. La figure 1.2 présente deux exemples de lagrangiennes (en bas) avec leurs fronts d'onde associés (en haut).

Si \mathcal{L} est une sous-variété lagrangienne exacte, et \mathcal{W} est un front d'onde associé, on appelle *graphe sélecteur* une application lipschitzienne² $u : M \rightarrow \mathbb{R}$ dont le graphe est inclus dans \mathcal{W} . Dans les cas les plus favorables, une primitive de \mathcal{L} peut être définie en terme d'actions, et on utilise alors des *sélecteurs d'action* pour obtenir un graphe sélecteur. Ceux-ci peuvent être construits avec des familles génératrices (voir [Sik86], [Cha91]), via l'homologie de Floer (voir [Flo88], [Oh97]) ou encore par des techniques d'analyse microlocale des faisceaux (voir [Gui12]). Le lien entre les invariants obtenus avec les familles génératrices ou avec l'homologie de Floer est étudié dans [MO98], voir aussi [MVZ12].

La proposition suivante, dont la démonstration est donnée dans la version anglaise (voir Proposition 1.12), montre qu'un graphe sélecteur sélectionne à la fois une section continue du front d'onde et une section discontinue de la lagrangienne.

Proposition. *Si \mathcal{L} est une sous-variété lagrangienne exacte telle que $\pi|_{\mathcal{L}}$ est propre, \mathcal{W} un front d'onde associé, et $u : M \rightarrow \mathbb{R}$ est un graphe sélecteur, alors $(q, du(q)) \in \mathcal{L}$ pour presque tout q .*

Le concept de graphe sélecteur est utile pour aborder d'autres problèmes dynamiques, voir par exemple [PPS03], [Arn10] et [BdS12].

Définition axiomatique

On appelle *opérateur variationnel* une famille d'opérateurs $(R_s^t)_{s \leq t}$ sur $\mathcal{C}^{0,1}(\mathbb{R}^d)$ qui vérifie les propriétés de Monotonie, d'Additivité et de Régularité (i), (ii) et (iii) de l'opérateur de viscosité, ainsi que la propriété suivante.

- (iv') Propriété variationnelle : pour toute fonction u lipschitzienne et de classe \mathcal{C}^1 , pour tout Q dans \mathbb{R}^d et $s \leq t$, il existe (q, p) dans le graphe de du tels que $Q_s^t(q, p) = Q$ et

$$R_s^t u(Q) = u(q) + \mathcal{A}_s^t(\gamma),$$

où γ désigne la trajectoire hamiltonienne issue de (q, p) au temps s .

Cette propriété revient à demander, en termes de front d'onde (voir (F)), que le graphe de $R_0^t u_0$ soit inclus dans $\mathcal{F}_0^t u_0$.

L'unicité d'un tel opérateur variationnel n'est pas garantie a priori.

On appelle *solution variationnelle* du problème de Cauchy associé à la donnée initiale u_0 toute fonction donnée par un opérateur variationnel de la manière suivante : $u(t, q) = R_0^t u_0(q)$.

Observons que la propriété variationnelle implique la propriété de Compatibilité (iv), d'après la méthode des caractéristiques. Ainsi, si un opérateur variationnel vérifie la propriété de Markov (v), il satisfait tous les axiomes caractérisant l'opérateur de viscosité, et coïncide donc avec cet opérateur.

Explicitons le lien entre un opérateur variationnel et la notion de graphe sélecteur introduite dans le paragraphe précédent pour une donnée initiale u_0 de classe \mathcal{C}^2 .

²Si la lagrangienne est uniformément bornée en la fibre, toute application continue dont le graphe est inclus dans \mathcal{W} est en fait lipschitzienne.

La *suspension autonome* de H est le hamiltonien $K(t, s, q, p) = s + H(t, q, p)$ défini sur $T^*(\mathbb{R} \times \mathbb{R}^d)$, qu'on identifie à $T^*\mathbb{R} \times T^*\mathbb{R}^d$. On note son flot hamiltonien Φ . Le système hamiltonien pour K s'écrit

$$\begin{cases} \dot{t} = 1, & \dot{q} = \partial_p H(t, q, p), \\ \dot{s} = -\partial_t H(t, q, p), & \dot{p} = -\partial_q H(t, q, p), \end{cases}$$

et on identifie donc t à la variable temps du flot.

La sous-variété $\Gamma_0 = \{(0, -H(0, q_0, du_0(q_0)), q_0, du_0(q_0)), q_0 \in \mathbb{R}^d\}$ est définie de sorte à être contenue dans le niveau d'énergie nulle pour K . Comme le hamiltonien K est autonome, il est constant le long de ses trajectoires, et par conséquent

$$\Phi^t(\Gamma_0) = \{(t, -H(t, \phi_0^t(q_0, du_0(q_0))), \phi_0^t(q_0, du_0(q_0))), q_0 \in \mathbb{R}^d\}.$$

On appelle *solution géométrique suspendue* associée au problème de Cauchy la sous-variété lagrangienne $\mathcal{L} = \cup_{t \in \mathbb{R}} \Phi^t(\Gamma_0) \subset T^*(\mathbb{R} \times \mathbb{R}^d)$, et l'ensemble suivant est un front d'onde pour \mathcal{L} :

$$\mathcal{W} = \left\{ (t, q, u_0(q_0) + \mathcal{A}_0^t(\phi_0^t(q_0, du_0(q_0)))) \mid \begin{array}{l} t \in \mathbb{R}, q \in \mathbb{R}^d, q_0 \in \mathbb{R}^d, \\ Q_0^t(q_0, du_0(q_0)) = q. \end{array} \right\}$$

Les axiomes caractérisant un opérateur variationnel impliquent que la fonction $u : (t, q) \mapsto R_0^t u_0(q)$ est un sélecteur de graphe pour \mathcal{L} : elle est lipschitzienne d'après la propriété de régularité (iii), et son graphe est contenu dans le front d'onde d'après la propriété variationnelle (iv'). La proposition énoncée dans le paragraphe précédent indique alors que pour presque tout (t, q) , $(t, \partial_t u(t, q), q, \partial_q u(t, q))$ appartient à \mathcal{L} qui est dans le niveau d'énergie nulle de K .

En d'autres termes, si R_s^t est un opérateur variationnel et u_0 est une donnée initiale de classe \mathcal{C}^2 , on vient d'établir que $(t, q) \mapsto R_s^t u_0(q)$ résout presque partout l'équation de Hamilton-Jacobi.

Notons que ce résultat est plus faible que l'analogue pour les solutions de viscosité : on ne sait pas si l'équation est vérifiée sur tout le domaine de différentiabilité, ni si la conclusion reste valable pour une donnée initiale seulement lipschitzienne.

Existence d'un opérateur variationnel et estimées locales

Dans cette thèse, on présente la construction complète d'un opérateur variationnel, ce qui revient à construire un graphe sélecteur directement pour la solution géométrique suspendue \mathcal{L} et le front d'onde \mathcal{W} associés introduits dans le paragraphe précédent. Pour cela, on suit l'idée de J.-C. Sikorav (voir [Cha91]) consistant à sélectionner adéquatement les valeurs critiques d'une *famille génératrice* décrivant cette solution géométrique. On travaille avec la famille génératrice explicite construite par M. Chaperon à l'aide de la méthode des *géodésiques brisées* (voir [Cha84] et [Cha91]), dont les éléments critiques sont directement liés aux objets dynamiques du problème. On utilise un *sélecteur de valeur critique* σ défini de manière axiomatique (voir Proposition 2.7) pour des fonctions qui s'écrivent comme la somme d'une forme quadratique non dégénérée et d'une fonction lipschitzienne (ce qu'on appelle *quadratique à l'infini*). Il sera vérifié qu'un tel sélecteur existe : on peut le construire en prenant différents types de minmax, qui ne donnent pas forcément le même sélecteur (voir l'exemple de F. Laudenbach étudié dans [Wei13b]). On doit aussi contourner la difficulté relative au fait que la famille génératrice de Chaperon n'est pas a priori quadratique à l'infini, en modifiant le hamiltonien pour p grand de sorte à ce qu'il soit égal à une forme quadratique, sans omettre de vérifier que l'opérateur ainsi obtenu ne dépend pas du choix de la forme quadratique imposée à l'infini.

On note R_s^t l'opérateur construit par ce procédé, en gardant en tête que cet opérateur dépend du choix de sélecteur σ . Les dérivées explicites de la famille génératrice permettent alors d'établir les estimées énoncées ici :

Théorème 2. *Il existe un opérateur variationnel, noté \mathbf{R}_s^t , qui vérifie les estimées locales suivantes : pour toutes fonctions L -lipschitziennes u et v , pour tout $0 \leq s \leq s' \leq t' \leq t$,*

1. $\mathbf{R}_s^t u$ est lipschitzienne, avec $\text{Lip}(\mathbf{R}_s^t u) \leq e^{C(t-s)}(1+L) - 1$,
2. $\|\mathbf{R}_s^{t'} u - \mathbf{R}_s^t u\|_\infty \leq C e^{2C(t-s)}(1+L)^2 |t' - t|$,
3. $\|\mathbf{R}_{s'}^t u - \mathbf{R}_s^t u\|_\infty \leq C(1+L)^2 |s' - s|$,
4. $\forall Q \in \mathbb{R}^d, |\mathbf{R}_s^t u(Q) - \bar{R}_s^t v(Q)| \leq \|u - v\|_{\bar{B}(Q, (e^{C(t-s)} - 1)(1+L))}$,

où $\bar{B}(Q, r)$ désigne la boule fermée de centre Q et de rayon r et $\|u\|_K := \sup_K |u|$.

L'intérêt de ces estimées est qu'elles se comportent bien lorsqu'on itère l'opérateur variationnel. Elles interviennent ainsi de manière cruciale dans la démonstration du théorème 3, présenté dans le prochain paragraphe, où l'on obtient l'opérateur de viscosité par itération d'un opérateur variationnel.

Les mêmes techniques permettent aussi d'estimer la dépendance de l'opérateur \mathbf{R}_s^t par rapport au hamiltonien : si H_0 et H_1 sont des hamiltoniens de classe C^2 vérifiant l'hypothèse (1) pour C , u est L -lipschitzienne, Q est dans \mathbb{R}^d et $s \leq t$, alors

$$|\mathbf{R}_{s,H_1}^t u(Q) - \mathbf{R}_{s,H_0}^t u(Q)| \leq (t-s) \|H_1 - H_0\|_{\bar{V}},$$

où $\bar{V} = [s, t] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1+L)) \times \bar{B}(0, e^{C(t-s)}(1+L) - 1)$.

Les deux dernières estimées peuvent être reformulées en propriétés de monotonie locale : si H_0 et H_1 sont des hamiltoniens de classe C^2 vérifiant l'hypothèse (1) pour C , alors pour tout $s \leq t$, Q dans \mathbb{R}^d et u et v L -lipschitziennes,

- $\mathbf{R}_s^t u(Q) \leq \mathbf{R}_s^t v(Q)$ si $u \leq v$ sur $\bar{B}(Q, (e^{C(t-s)} - 1)(1+L))$,
- $\mathbf{R}_{s,H_1}^t u(Q) \leq \mathbf{R}_{s,H_0}^t u(Q)$ si $H_1 \geq H_0$
sur $[s, t] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1+L)) \times \bar{B}(0, e^{C(t-s)}(1+L) - 1)$.

Un procédé itératif

Les opérateurs variationnel et de viscosité ne coïncident pas forcément. Par contre, Q. Wei a établi dans [Wei14], pour des hamiltoniens à support compact, qu'on peut obtenir l'opérateur de viscosité comme limite d'une famille d'opérateurs obtenus en itérant un opérateur variationnel le long d'une subdivision en temps de plus en plus fine. Ceci rentre dans le cadre du procédé d'approximation proposé par Souganidis dans [Sou85] sous un jeu d'hypothèses un peu différent, en observant que l'opérateur variationnel remplit le rôle du *generator* utilisé dans l'article. On renvoie à [BS91] pour une présentation plus complète de ce type de schéma numérique, également valable pour des équations de Hamilton-Jacobi du deuxième ordre.

On fixe une suite de subdivisions de $[0, \infty]$ $((\tau_i^N)_{i \in \mathbb{N}})_{N \in \mathbb{N}}$ telle que pour tout N , $0 = \tau_0^N$, $\tau_i^N \xrightarrow{i \rightarrow \infty} \infty$ et $i \mapsto \tau_i^N$ est strictement croissante. On suppose que pour tout N , $i \mapsto \tau_{i+1}^N - \tau_i^N$ est bornée par une constante δ_N qui tend vers 0 quand N tend vers l'infini. Pour t dans \mathbb{R} , on note $i_N(t)$ le seul entier tel que $t \in [\tau_{i_N(t)}^N, \tau_{i_N(t)+1}^N)$. On définit l'opérateur itéré de rang N comme suit : si $0 \leq s \leq t$,

$$R_{s,N}^t u = R_{\tau_{i_N(t)}^N}^t R_{\tau_{i_N(t)-1}^N}^{\tau_{i_N(t)}^N} \cdots R_s^{\tau_{i_N(s)+1}^N} u,$$

pour R_s^t un opérateur variationnel vérifiant les estimées lipschitziennes du théorème 2.

Théorème 3 (Théorème de Wei). *Pour tout hamiltonien H vérifiant l'hypothèse (1), la suite d'opérateurs itérés $(R_{s,N}^t)$ converge simplement vers l'opérateur de viscosité V_s^t . De plus, pour toute fonction lipschitzienne u , la suite de fonctions $\{(s,t,Q) \mapsto R_{s,N}^t u(Q)\}_N$ converge uniformément vers $(s,t,Q) \mapsto V_s^t u(Q)$ sur les compacts de $\{0 \leq s \leq t\} \times \mathbb{R}^d$.*

Une part conséquente de cette thèse est consacrée à démontrer ce résultat sans hypothèse de compacité sur le support de H . Ce théorème prouve entre autres l'existence de l'opérateur de viscosité pour un hamiltonien vérifiant l'hypothèse (1).

Remarque. Ce théorème permet d'établir un critère pour décider au cas par cas si la solution variationnelle et la solution de viscosité associée à une donnée initiale fixée u coïncide ou non : si $R_\tau^t R_s^\tau u = R_s^t u$ pour tout $s \leq \tau \leq t$, l'opérateur itéré appliqué à u se réduit à $R_{s,N}^t u = R_s^t u$ et ne dépend donc pas de N , ce qui implique que $V_s^t u = R_s^t u$ pour tout $s \leq t$. L'hypothèse est moins forte que la propriété de Markov (v) puisqu'on ne vérifie celle-ci que pour une seule donnée initiale. Cette observation est due à M. Zavidovique.

Une conséquence intéressante de cette convergence et que les estimées obtenues pour l'opérateur variationnel se voient automatiquement transférées à l'opérateur de viscosité, voir Proposition 1.21. Les estimées obtenues ne sont pas surprenantes (ce sont finalement celles vérifiées par les solutions classiques), mais comme elles sont obtenues de manière dynamique, elles sont susceptibles d'améliorer les estimées obtenues en travaillant avec des techniques de viscosité.

Données initiales non lisses

Pour une donnée initiale de classe \mathcal{C}^2 à dérivée seconde bornée, la méthode des caractéristiques donne que le front d'onde est en petit temps le graphe d'une solution différentiable. La solution variationnelle coïncide alors avec cette solution différentiable, qui est également solution de viscosité. Pour observer une différence entre les deux types de solution dès que $t > 0$, on doit donc travailler avec des données initiales non lisses.

Extension aux données initiales lipschitziennes

La propriété variationnelle (iv') peut s'étendre aux données initiales lipschitziennes en choisissant là encore une notion de différentielle généralisée bien adaptée. Si $u : \mathbb{R}^d \rightarrow \mathbb{R}$ est lipschitzienne, on définit sa dérivée de Clarke en un point q , notée $\partial_u(q)$, comme l'enveloppe convexe de l'ensemble

$$\left\{ \lim_{n \rightarrow \infty} du(q_n), q_n \xrightarrow[n \rightarrow \infty]{} q, q_n \in \text{dom}(du) \right\}.$$

Cette dérivée est réduite au singleton $\{du(q)\}$ là où u est de classe \mathcal{C}^1 .

Si R_s^t est un opérateur variationnel, il vérifie alors la propriété variationnelle généralisée suivante : pour toute fonction u lipschitzienne, pour tout Q dans \mathbb{R}^d et $s \leq t$, il existe (q,p) dans le graphe de ∂u tels que $Q_s^t(q,p) = Q$ et

$$R_s^t u(Q) = u(q) + \mathcal{A}_s^t(\gamma),$$

où γ désigne la trajectoire hamiltonienne issue de (q,p) au temps s .

On définit alors le front d'onde généralisé au temps t :

$$\mathcal{F}_0^t u_0 = \left\{ (q, u_0(q_0) + \mathcal{A}_0^t(\phi_0^\tau(q_0, p_0))) \mid \begin{array}{l} t \geq 0, q \in \mathbb{R}^d, \\ p_0 \in \partial u_0(q_0), \\ Q_0^t(q_0, p_0) = q. \end{array} \right\}, \quad (\text{F}')$$

de sorte à ce que le graphe d'une solution variationnelle au temps t soit contenu dans $\mathcal{F}_0^t u_0$ même pour une donnée initiale seulement lipschitzienne.

Ce choix de différentielle généralisée n'est pas forcément optimal, voir Remark 1.23.

Caractérisation de la solution variationnelle en petit temps

On dit qu'une fonction $u : \mathbb{R}^d \rightarrow \mathbb{R}$ est B -semiconcave si la fonction $q \mapsto u(q) - \frac{B}{2}\|q\|^2$ est concave. Une fonction est semiconcave s'il existe une constante $B \in \mathbb{R}$ pour laquelle elle est B -semiconcave, et semiconvexe si son opposée est semiconcave.

Le théorème qui suit énonce que pour une donnée initiale semiconcave, la solution variationnelle est donnée en petit temps par la section minimale du front d'onde généralisé.

Théorème 4. *Si R_s^t est un opérateur variationnel et u_0 est une donnée initiale lipschitzienne et B -semiconcave, il existe une constante $T > 0$ ne dépendant que de B et C tel que pour tout (t, q) dans $[0, T] \times \mathbb{R}^d$,*

$$R_0^t u_0(q) = \inf \{S(q, S) \in \mathcal{F}_0^t u_0\} = \inf \left\{ u_0(q_0) + \mathcal{A}_0^t(\gamma) \left| \begin{array}{l} (q_0, p_0) \in \mathbb{R}^d \times \mathbb{R}^d, \\ p_0 \in \partial u_0(q_0), \\ Q_0^t(q_0, p_0) = q. \end{array} \right. \right\}, \quad (2)$$

où γ désigne la trajectoire hamiltonienne issue de (q_0, p_0) au temps 0.

De plus, si H est intégrable (c'est-à-dire ne dépend que de p), on peut prendre $T = 1/BC$.

En particulier, dans le domaine de validité de ce théorème, les estimées obtenues sur l'opérateur \mathbf{R} sont vérifiées par la solution variationnelle.

Illustrons ce théorème par un exemple en dimension 1 : si $u_0(q) = -|q|$ et si H est un hamiltonien intégrable dont le graphe est donné par la figure 1 à gauche, le front d'onde au temps t est représenté sur la figure 1 à droite, et sa section minimale, en gras, est le graphe de la solution variationnelle. Le même genre d'arguments donne un premier élément de comparaison

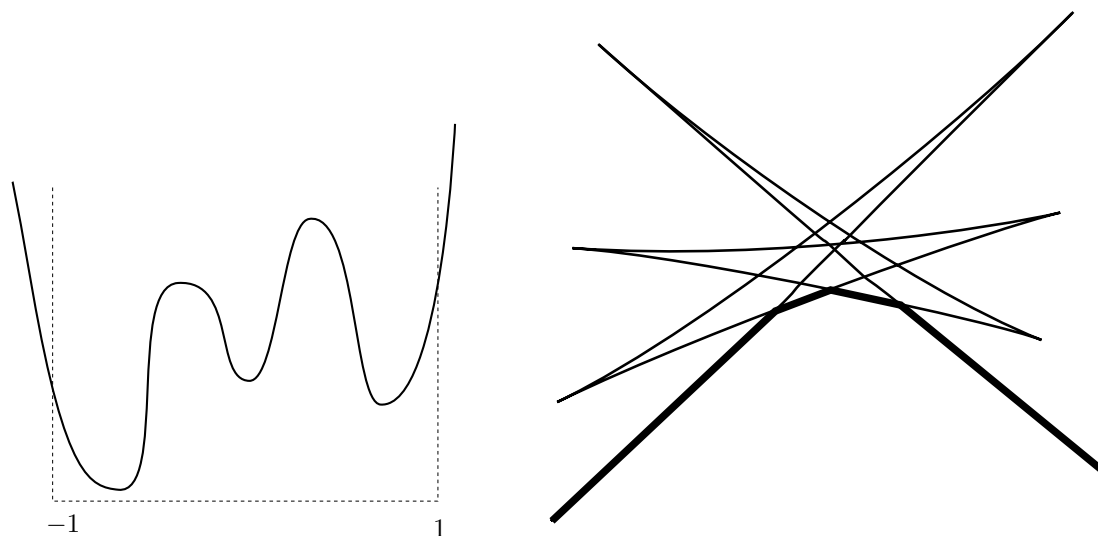


FIGURE 1 : À gauche : graphe de H . À droite : front d'onde $\mathcal{F}_0^t u_0$ pour $t > 0$ et sa section minimale en gras.

entre solution variationnelle et solution de viscosité pour une donnée initiale semiconcave. Ce résultat est dû à P. Bernard, voir [Ber13].

Proposition. *Si R_s^t est un opérateur variationnel et u_0 est une donnée initiale lipschitzienne et B -semiconcave, il existe $T > 0$ ne dépendant que de B et C tel que pour tout $0 \leq t \leq T$,*

$$V_0^t u_0 \leq R_0^t u_0.$$

De plus, si H est intégrable, on peut prendre $T = 1/BC$.

Liens entre les deux types de solution

Formules de Lax-Hopf dans le cas intégrable

On dit d'un hamiltonien qu'il est intégrable s'il ne dépend que de la variable impulsion p .

Sous des hypothèses de convexité portant sur le hamiltonien ou sur la donnée initiale, Lax [Lax57] puis Hopf [Hop65] ont proposé des formules duales décrivant des solutions généralisées pour l'équation de Hamilton-Jacobi sous la forme de problèmes d'optimisation.

Proposition (Formule de Lax). *Soit H un hamiltonien intégrable convexe à dérivée seconde bornée et u_0 une condition initiale lipschitzienne. Alors*

$$R_0^t u_0(q) = V_0^t u_0(q) = u_{Lax}(t, q) = \inf_{x \in \mathbb{R}^d} \sup_{p \in \mathbb{R}^d} u_0(x) + p \cdot (q - x) - tH(p).$$

Proposition (Formule de Hopf). *Soit H un hamiltonien intégrable à dérivée seconde bornée et u_0 une condition initiale lipschitzienne concave. Alors pour tout opérateur variationnel R_s^t ,*

$$R_0^t u_0(q) = V_0^t u_0(q) = u_{Hopf}(t, q) = \inf_{p \in \mathbb{R}^d} \sup_{x \in \mathbb{R}^d} u_0(x) + p \cdot (q - x) - tH(p).$$

Une référence possible pour la preuve de ces propositions côté viscosité est [BE84], où le hamiltonien est seulement supposé continu. La formule de Lax est démontrée en utilisant des méthodes de théorie du contrôle, alors que la formule de Hopf est obtenue par des techniques de théorie des jeux. La partie variationnelle de ces énoncés est prouvée dans [Ber13] pour la formule de Hopf, et est une conséquence du théorème de Joukovskaia que nous allons présenter dans le paragraphe suivant pour la formule de Lax.

Les formules de Lax-Hopf ont été abondamment étudiées dans [Lio82], [LR86], [Bar87], voir aussi [ABI99] et [Imb01] pour l'étude de ces formules pour des hamiltoniens ou conditions initiales pas nécessairement continus.

Lorsque le hamiltonien ou la donnée initiale s'écrit comme somme de fonctions convexe et concave, des estimées de type Lax-Hopf peuvent être construites pour borner la solution variationnelle ([BC11]) ou la solution de viscosité ([BF98]).

Semi-groupe de Lax-Oleinik dans le cas convexe

Le semi-groupe de Lax-Oleinik est la généralisation de la formule de Lax pour un hamiltonien convexe en p mais pas forcément intégrable. C'est un objet central de la théorie KAM faible conçue par J. Mather et A. Fathi, puisque les solutions KAM faibles de niveau 0 peuvent être vues comme les points fixes de cet opérateur, voir [Fat].

Si H est un hamiltonien strictement convexe par rapport à p , une fonction lagrangienne L définie sur le fibré tangent lui est associée par la transformation de Legendre :

$$L(t, q, v) = \sup_{p \in (\mathbb{R}^d)^*} p \cdot v - H(t, q, p).$$

Pour tout t, q, p , l'inégalité de Legendre suivante est vérifiée :

$$L(t, q, v) + H(t, q, p) \geq p \cdot v$$

et il y a égalité si et seulement si $p = \partial_v L(t, q, v)$, ou de manière équivalente $v = \partial_p H(t, q, p)$. En particulier, si $(q(\tau), p(\tau))$ est une trajectoire hamiltonienne, $\dot{q}(\tau) = \partial_p H(\tau, q(\tau), p(\tau))$ et

$$\int_s^t L(\tau, q(\tau), \dot{q}(\tau)) d\tau = \int_s^t p(\tau) \cdot \dot{q}(\tau) - H(\tau, q(\tau), p(\tau)) d\tau.$$

Autrement dit, l'action hamiltonienne d'une trajectoire hamiltonienne est égale à ce qu'on va appeler l'action lagrangienne de sa projection sur l'espace des positions.

Le semi-groupe de Lax-Oleinik $(T_s^t)_{s \leq t}$ peut être exprimé à l'aide de cette action lagrangienne : si u est une application lipschitzienne sur \mathbb{R}^d , on définit $T_s^t u$ par

$$T_s^t u(q) = \inf_c u(c(s)) + \int_s^t L(\tau, c(\tau), \dot{c}(\tau)) d\tau,$$

où l'infimum est pris sur l'ensemble des chemins lipschitziens $c : [s, t] \rightarrow \mathbb{R}^d$ tels que $c(t) = q$.

Proposition. *Si le hamiltonien H est uniformément strictement convexe en p , le semi-groupe de Lax-Oleinik est à la fois un opérateur variationnel et l'opérateur de viscosité.*

La propriété de Markov se lit directement sur la définition de T . Le théorème 5 démontre les autres propriétés. Dans la version anglaise de l'introduction, on propose une preuve didactique de la propriété variationnelle (iv'), voir Proposition 1.28, qui explicite par la méthode classique de calcul des variations le lien entre les points critiques de l'action lagrangienne et l'équation d'Euler-Lagrange (EL).

Le théorème suivant établit que l'opérateur variationnel construit dans cette thèse donne effectivement le semi-groupe de Lax-Oleinik pour un hamiltonien uniformément strictement convexe, et coïncide avec l'opérateur de viscosité dans le cas convexe. On suppose pour démontrer ce résultat que le sélecteur de valeur critique σ satisfait deux axiomes supplémentaires, énoncés dans la Proposition 4.4.

Théorème 5 (Théorème de Joukovskaia). *Si $p \mapsto H(t, q, p)$ est convexe pour tout (t, q) ou concave pour tout (t, q) , l'opérateur variationnel \mathbf{R}_s^t associé au sélecteur de valeur critique σ est l'opérateur de viscosité. En particulier, il coïncide avec le semi-groupe de Lax-Oleinik si H est uniformément strictement convexe par rapport à p .*

La deuxième partie de ce résultat a été prouvée par T. Joukovskaia dans le cas d'une variété compacte, voir [Jou91].

Ce théorème a été généralisé à des hamiltoniens de type convexe-concave, voir [Wei13a] et [BC11], mais seulement pour un hamiltonien et une donnée initiale à variables séparées, c'est-à-dire tels que

$$H(t, q, p) = H_1(t, q_1, p_1) + H_2(t, q_2, p_2) \text{ et } u_0(q) = u_1(q_1) + u_2(q_2)$$

où $d = d_1 + d_2$, (q_i, p_i) désignent les coordonnées dans $T^*\mathbb{R}^{d_i}$, H_1 (resp. H_2) est un hamiltonien sur $\mathbb{R} \times \mathbb{R}^{d_1}$ (resp. sur $\mathbb{R} \times \mathbb{R}^{d_2}$) convexe en p_1 (resp. concave en p_2), et u_1 et u_2 sont des fonctions lipschitziennes sur \mathbb{R}^{d_1} et \mathbb{R}^{d_2} .

Caractérisation des hamiltoniens intégrables tels que les deux notions coïncident

Le théorème de Joukovskaia donne une classe d'hamiltoniens pour lesquels les opérateurs variationnel et de viscosité coïncident. On donne dans cette thèse une réponse à la question réciproque, dans le cas intégrable.

Théorème 6. *Soit H est un hamiltonien intégrable (c'est-à-dire qui ne dépend que de p). Si l'opérateur de viscosité V_s^t est un opérateur variationnel, alors H est convexe ou concave.*

Pour montrer ce théorème, on réduit le problème à l'étude de deux situations élémentaires en dimension 1 et 2, énoncées dans les Proposition 5.6 et 6.6. L'exemple pertinent pour la dimension 1 était déjà bien connu : il apparaissait dans [Che75], voir également [IK96]. L'exemple clé pour la dimension 2, présenté dans le paragraphe §6.2, est a priori nouveau.

Étude de la propagation d'un choc simple en dimension 1

Afin de la comparer à la solution de viscosité, on présente une étude précise du comportement en petit temps de la solution variationnelle pour le problème de Cauchy associé à un hamiltonien intégrable sur \mathbb{R} et une donnée initiale semiconcave présentant un seul choc, c'est-à-dire un unique point de singularité avec changement de dérivée. On se place dans ce cadre parce qu'il suffit à démontrer la partie unidimensionnelle du théorème 6. Ce travail réunit et généralise de nombreuses observations faites par exemple dans [Lax57], [Che75], [IK96] et [Wei14].

On note \mathcal{E} l'ensemble des fonctions lipschitziennes f de classe \mathcal{C}^2 sur \mathbb{R} , à dérivée seconde bornée, qui vérifient $f(0) = f'(0) = 0$.

On étudie le problème de Cauchy donné par un hamiltonien intégrable $H(p)$ à dérivée seconde bornée et une donnée initiale de la forme

$$u_0(q) = \min(p_1 q, p_2 q) + f(q),$$

pour $p_1 < p_2$ et $f(q) = \begin{cases} f_1(q), & q \geq 0, \\ f_2(q), & q \leq 0, \end{cases}$ avec f_1 et f_2 des éléments de \mathcal{E} .

Les résultats suivants peuvent aussi servir pour une donnée initiale avec des chocs séparés, aussi longtemps que les singularités issus des chocs n'interagissent pas.

Comme u_0 est semiconcave, le théorème 4 nous autorise à parler de la solution variationnelle en petit temps, et les classifications qui suivent sont valables quel que soit l'opérateur variationnel R_s^t .

On note \widehat{H} l'enveloppe concave de H sur l'intervalle $[p_1, p_2]$. Le choc initial vérifie la condition d'entropie proposée par Oleinik si et seulement si \widehat{H} est une fonction affine, et dans ce cas $\widehat{H}' = \frac{H(p_2) - H(p_1)}{p_2 - p_1}$ est constante.

Si la condition d'entropie est strictement vérifiée, et si la constante \widehat{H}' est une valeur régulière de H' , on établit dans §5.3 la classification suivante :

$H'(p_1) = H'(p_2)(= \widehat{H}')$		$R = V$
$H'(p_1) < \widehat{H}' = H'(p_2)$ (resp. $H'(p_1) = \widehat{H}' < H'(p_2)$)	\nearrow si f est strictement convexe sur un $[0, \delta]$ (resp. sur un $[-\delta, 0]$) \searrow si f est concave sur un $[0, \delta]$ (resp. sur un $[-\delta, 0]$)	$R \neq V$ $R = V$
$H'(p_1) < \widehat{H}' < H'(p_2)$		$R = V$

où " $R = V$ " veut dire "il existe $\tau > 0$ tel que $(t, q) \mapsto R_0^t u_0(q)$ est solution de l'équation de Hamilton-Jacobi sur $(0, \tau] \times \mathbb{R}^d$ ", et " $R \neq V$ " veut dire "il existe $\tau > 0$ tel que pour tout $0 < t < \tau$, il existe un point q tel que $(t, q) \mapsto R_0^t u_0(q)$ nie l'équation de Hamilton-Jacobi au sens de viscosité au point (t, q) ".

La condition d'entropie est niée si et seulement si $\widehat{H}'(p_1) > \widehat{H}'(p_2)$. Dans ce cas, et si $\widehat{H}'(p_1)$ et $\widehat{H}'(p_2)$ sont des valeurs régulières de H' , on établit dans §5.4 la classification suivante :

$H'(p_1) = \widehat{H}'(p_1)$ et $\widehat{H}'(p_2) = H'(p_2)$		$R = V$
$H'(p_1) < \widehat{H}'(p_1)$ et $\widehat{H}'(p_2) = H'(p_2)$ (resp. $H'(p_1) = \widehat{H}'(p_1)$, $\widehat{H}'(p_2) < H'(p_2)$)	\nearrow f strictement convexe sur $[0, \delta]$ (resp. sur $[-\delta, 0]$) \searrow f concave sur $[0, \delta]$ (resp. sur $[-\delta, 0]$)	$R \neq V$ $R = V$
$H'(p_1) < \widehat{H}'(p_1)$ et $\widehat{H}'(p_2) < H'(p_2)$	\nearrow f strictement convexe sur $[0, \delta]$ OU sur $[-\delta, 0]$ \searrow f concave sur $[-\delta, \delta]$	$R \neq V$ $R = V$

Dans les deux énoncés, l'hypothèse portant sur les valeurs régulières de H' n'est utilisée que partiellement selon les cas. Il n'est par ailleurs pas exclu qu'on pourrait se passer d'une telle hypothèse en utilisant d'autres approches que la nôtre. Les résultats analogues pour une donnée initiale semiconvexe sont énoncées dans les Propositions 5.10 et 5.13.

La discussion est un peu plus subtile lorsque la condition d'entropie est vérifiée, mais pas strictement vérifiée : on développe dans §5.5 un exemple, appelé *Perestroïka*, où la coïncidence entre la solution variationnelle et la solution de viscosité dépend d'une comparaison numérique impliquant la valeur des dérivées du hamiltonien et de la donnée initiale.

Enfin, pour illustrer cette discussion, on présente dans §5.6 un exemple pour lequel il est possible de construire explicitement la solution de viscosité, qui est différente de la solution variationnelle, en suivant une idée d'O. Oleinik.

Organisation du mémoire

La version anglaise de cette introduction contient certaines preuves supplémentaires et quelques précisions techniques.

Dans le chapitre 2, on construit l'opérateur variationnel \mathbf{R} et on déduit de cette construction les différentes propriétés lipschitziennes de cet opérateur, afin de prouver le théorème 2. Pour cela, on commence par détailler la construction de la famille génératrice de Chaperon et ses propriétés (§2.1), ainsi que la notion de sélecteur de valeurs critiques, définie de manière axiomatique (§2.2).

On définit ensuite l'opérateur variationnel en appliquant le sélecteur à la famille génératrice. Pour cela, il faut rendre le hamiltonien quadratique à l'infini tout en s'assurant que le choix de forme à l'infini n'a pas d'incidence sur la définition de l'opérateur (§2.3). Enfin, on montre que l'opérateur obtenu est variationnel et vérifie les propriétés lipschitziennes voulues (§2.4).

Dans le chapitre 3, on démontre le théorème 3 de convergence de l'opérateur itéré. Pour cela, on donne des estimées uniformes sur l'opérateur itéré pour pouvoir appliquer le théorème d'Arzelà-Ascoli. La sous-suite obtenue converge vers l'opérateur de viscosité, et par unicité on obtient donc la convergence de toute la suite.

Dans le chapitre 4, on démontre le théorème 5 (dit de Joukovskaia). Pour ce faire, on décrit le semi-groupe de Lax-Oleinik à l'aide de la famille génératrice obtenue par la méthode des géodésiques brisées dans le cas convexe, et on fait le lien entre cette famille génératrice et celle obtenue dans le cas général.

Dans le chapitre 5, on étudie le problème de Cauchy associé à un hamiltonien intégrable et une donnée initiale semiconcave présentant un unique choc, en dimension 1. Après avoir détaillé certaines propriétés structurelles du front d'onde (§5.1), on prouve les deux résultats de classification annoncés dans cette introduction, pour un choc vérifiant strictement la condition d'entropie (§5.3) ou la niant (§5.4). On étudie dans §5.5 un exemple exclu de ces classifications, et dans §5.6 on construit explicitement les solutions variationnelle et de viscosité pour un couple commode de donnée initiale et d'hamiltonien.

Dans le chapitre 6, on démontre le théorème 6 caractérisant les hamiltoniens intégrables pour lesquels l'opérateur de viscosité est variationnel. Pour cela, on donne les outils de réduction permettant de découper le problème en un énoncé en dimension 1 contenu dans §5.3 et en un exemple explicite en dimension 2, présenté dans §6.2.

L'annexe A donne une preuve élémentaire de l'unicité des solutions lipschitziennes de viscosité sous l'hypothèse (1), en présentant un argument classique de dédoublement de variables. L'annexe B détaille la construction et les propriétés des familles génératrices du flot hamiltonien, à la fois dans le cas général (§B.1) et dans le cas convexe (§B.2). L'annexe C propose une construction fonctorielle d'un sélecteur de valeur critique. Les deux lemmes de déformation utilisés pour cela font l'objet de l'annexe D. L'annexe E se place dans le cadre d'une donnée initiale semiconcave : on y démontre le théorème 4. Enfin, l'annexe F énonce des considérations élémentaires sur la stabilité des conditions d'entropie et de Lax.

Chapter 1

Introduction

1.1 The Hamilton-Jacobi equation

The concern of this thesis is the study of the evolutive Hamilton-Jacobi equation

$$\partial_t u(t, q) + H(t, q, \partial_q u(t, q)) = 0, \quad (\text{HJ})$$

where $H : \mathbb{R} \times T^*\mathbb{R}^d \rightarrow \mathbb{R}$ is a C^2 Hamiltonian, and $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the unknown function.

This equation was first introduced in the Hamiltonian mechanics framework, in which it is naturally solved by a certain Hamiltonian action. In the last century, it has appeared to be central in optimal control theory, and matters therefore in various domains of applications: economy, traffic flows studies...

We study the Cauchy problem formed by the (HJ) equation associated with an initial condition $u(0, \cdot) = u_0$, which will be at least Lipschitz. This Cauchy problem does not admit classical solutions in large time even for smooth u_0 and H , and different types of weak solutions were then introduced. The viscosity solutions, defined by P.-L. Lions and M.G. Crandall (see [CL83]), are considered as the "good" notion of generalized solution, and take a large part in the analysis of optimal control problems. The variational solutions were introduced by J.-C. Sikorav and M. Chaperon (see [Cha91]) with the help of symplectic geometry tools such as the generating family of a Lagrangian submanifold, and are closely related to the Hamiltonian dynamics associated with the Cauchy problem.

T. Joukovskaia showed that the two solutions coincide for compactly supported fiberwise convex Hamiltonians (see [Jou91]), but this is not true in general. Examples where the solutions differ were proposed in [Che75], [Vit96], [BC11] and [Wei14]. The purpose of this thesis is to clarify whether and when the two types of solution coincide.

To do so, we work in a set of assumptions that suits both the viscosity and the variational framework, taking the initial condition u_0 Lipschitz and a C^2 Hamiltonian as follows:

Hypothesis 1.1. There is a $C > 0$ such that for each (t, q, p) in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\|\partial_{(q,p)}^2 H(t, q, p)\| < C, \quad \|\partial_{(q,p)} H(t, q, p)\| < C(1 + \|p\|), \quad |H(t, q, p)| < C(1 + \|p\|)^2,$$

where $\partial_{(q,p)} H$ and $\partial_{(q,p)}^2 H$ denote the first and second order spatial derivatives of H .

The bound on the second derivative is standard in Hamiltonian dynamics, since it implies that the Hamiltonian flow is complete. The bound on the first derivative is standard in optimal control theory.

This hypothesis implies a finite propagation speed principle in both viscosity and variational contexts, which allows to deal with non compactly supported Hamiltonians. We refer for example to [Bar94] for the viscosity side, where in particular the uniqueness of the viscosity operator (see also Proposition A.3) is studied, and to Appendix B of [CV08] for the existence of variational solutions for Hamiltonians satisfying this finite propagation speed principle.

The method of characteristics

The method of characteristics is a standard technique used to solve partial differential equation. Adapted to this situation, it gives the link between the Hamiltonian dynamics objects and the classical solution of the evolutive Hamilton-Jacobi equation.

Under Hypothesis 1.1, the *Hamiltonian system*

$$\begin{cases} \dot{q}(t) = \partial_p H(t, q(t), p(t)), \\ \dot{p}(t) = -\partial_q H(t, q(t), p(t)) \end{cases} \quad (\text{HS})$$

admits a complete *Hamiltonian flow* ϕ_s^t , meaning that $t \mapsto \phi_s^t(q, p)$ is the unique solution of (HS) with initial conditions $(q(s), p(s)) = (q, p)$. We denote by (Q_s^t, P_s^t) the coordinates of ϕ_s^t . We call a function $t \mapsto (q(t), p(t))$ solving the Hamiltonian system (HS) a *Hamiltonian trajectory*. The *Hamiltonian action* of a C^1 path $\gamma(t) = (q(t), p(t)) \in T^*\mathbb{R}^d$ is denoted by

$$\mathcal{A}_s^t(\gamma) = \int_s^t p(\tau) \cdot \dot{q}(\tau) - H(\tau, q(\tau), p(\tau)) d\tau.$$

The next lemmas state respectively the existence of *characteristics* for C^2 solutions of the Hamilton-Jacobi equation (HJ) and the existence of small time C^2 solutions for C^2 initial condition with bounded second derivative.

Lemma 1.2. *If u is a C^2 solution of (HJ) on $[T_-, T_+] \times \mathbb{R}^d$ and $\gamma : \tau \mapsto (q(\tau), p(\tau))$ is a Hamiltonian trajectory satisfying $p(s) = \partial_q u(s, q(s))$ for some $s \in [T_-, T_+]$, then $p(t) = \partial_q u(t, q(t))$ for each $t \in [T_-, T_+]$ and*

$$u(t, q(t)) = u(s, q(s)) + \mathcal{A}_s^t(\gamma) \quad \forall t \in [T_-, T_+].$$

Proof. If $f(t)$ denotes the quantity $\partial_q u(t, q(t))$, one can show that both f and p solve the ODE $\dot{y}(t) = -\partial_q H(t, q(t), y(t))$ and $p(s) = f(s)$ implies that $p(t) = f(t)$ for each time $t \in [T_-, T_+]$. Then, differentiating the function $t \mapsto u(t, q(t))$ gives the result. \square

This implies in particular the uniqueness of C^2 solutions for the Cauchy problem. The following lemma, proved in Appendix B, states the existence of C^2 solutions for an initial condition with bounded second derivative, where the temporal bound of existence depends only on the bounds of the second derivatives.

Proposition 1.3. *If u_0 is a C^2 function with second derivative bounded by $B > 0$, there exists T depending only on C and B (for example $T < C^{-1} \ln \left(\frac{2+B}{1+B} \right)$, or $T < 1/BC$ in case of an integrable Hamiltonian, i.e. that depends only on p) such that $(t, q) \mapsto (t, Q_0^t(q, du_0(q)))$ is a C^1 -diffeomorphism on $[0, T] \times \mathbb{R}^d$. Then if $q^{t, Q}$ denotes the second coordinate of the inverse diffeomorphism and $\gamma^{t, Q}$ denotes the Hamiltonian trajectory issued from $(q(0), p(0)) = (q^{t, Q}, du_0(q^{t, Q}))$, the function*

$$u(t, Q) = u_0(q^{t, Q}) + \mathcal{A}_0^t(\gamma^{t, Q})$$

is a C^2 solution of the Cauchy problem on $[0, T] \times \mathbb{R}^d$.

Geometric solution and wavefront associated with the Cauchy problem

If u_0 is \mathcal{C}^1 , and Γ_0 is the graph of du_0 , we call the set $\phi_0^t \Gamma_0$ *geometric solution at time t* of the Cauchy problem associated with u_0 . Lemma 1.2 states that if u is a \mathcal{C}^2 solution on $[0, \tau] \times \mathbb{R}^d$, the geometric solution coincide with the graph of $\partial_q u_t$ above \mathbb{R}^d for each t in $[0, \tau]$. In particular, if $\phi_0^T \Gamma_0$ is not a graph for some time $T > 0$, as in Figure 1.1, the existence of classical solution on $[0, T] \times \mathbb{R}^d$ is not possible, hence the introduction of generalized solutions.

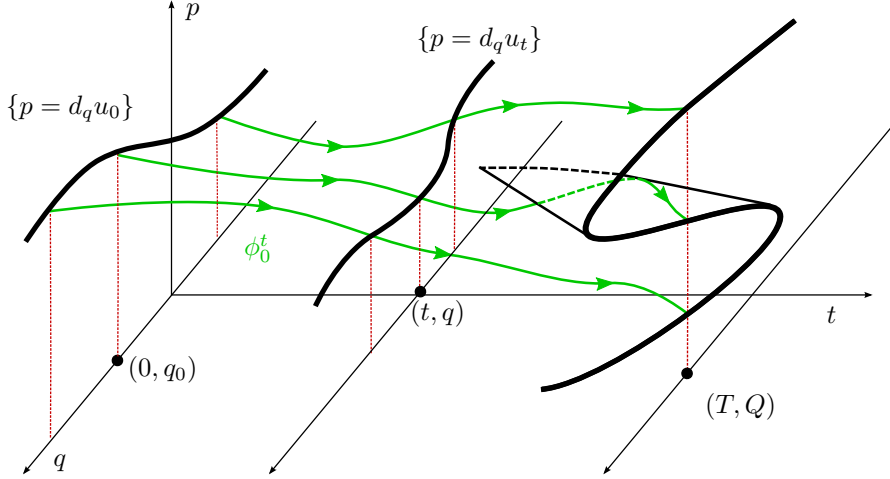


Figure 1.1: Geometric solution associated with a smooth initial condition u_0 .

The *wavefront at time t* associated with the Cauchy problem for u_0 is denoted by $\mathcal{F}_0^t u_0$ and defined by

$$\mathcal{F}_0^t u_0 = \left\{ (q, u_0(q_0) + \mathcal{A}_0^t(\phi_0^t(q_0, du_0(q_0)))) \mid \begin{array}{l} t \geq 0, q \in \mathbb{R}^d, q_0 \in \mathbb{R}^d, \\ Q_0^t(q_0, du_0(q_0)) = q. \end{array} \right\} \quad (\text{F})$$

Above each point q , the wavefront at time t gives the Hamiltonian action of every Hamiltonian trajectory issued from the graph of du_0 at time 0 and ending above q at time t , added to the value of u_0 at the initial endpoint of this trajectory.

Lemma 1.2 states that if u is a \mathcal{C}^2 solution on $[0, \tau] \times \mathbb{R}^d$, $\mathcal{F}_0^t u_0$ is the graph of u_t for each t in $[0, \tau]$. The wavefront can hence be viewed as a multivalued solution of the Cauchy problem when it is not a graph.

Lemma 1.2 implies that the geometric solution for a classical solution gives the slopes of the associated wavefront with respect to q . This is still true when the geometric solution and the wavefront are no longer graphs, see Figure 1.2.

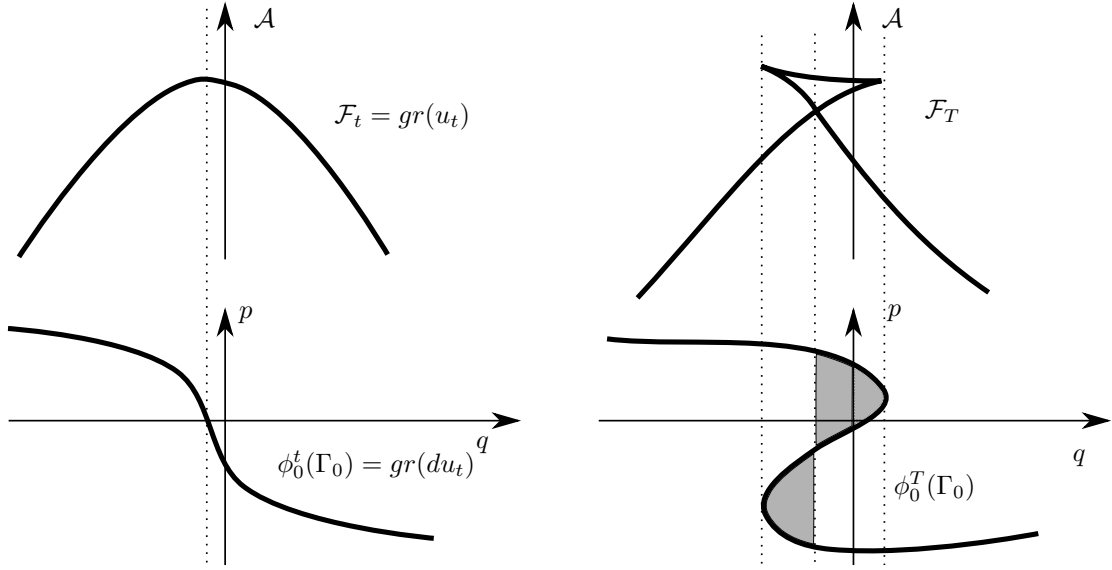


Figure 1.2: Geometric solution of Figure 1.1 and associated wavefront for time t (left) and T (right). The geometric solution is locally the derivative of the wavefront, and the two greyed domains delimited by the position of the intersection in the wavefront have hence the same area.

1.2 Viscosity solutions

Adding a small viscosity term to the evolutive (HJ) equation makes it parabolic:

$$\partial_t u^\varepsilon(t, q) + H(t, q, \partial_q u^\varepsilon(t, q)) = \varepsilon \Delta_q u^\varepsilon(t, q),$$

and the uniquely defined solution u^ε then admits a limit when $\varepsilon \rightarrow 0$. This is called the *vanishing viscosity method*, first introduced for quasilinear equations (see [Ole59b], [Kru70]).

P.-L. Lions and M. G. Crandall gave in 1981 a practical definition of viscosity solutions (see [CL83]), closely related to the work on the vanishing viscosity method for Hamilton-Jacobi equations made by L. Evans in [Eva80]. Here is a possible version of this definition:

Definition 1.4. A continuous function u is a *subsolution* of (HJ) on the set $(0, T) \times \mathbb{R}^d$ if for each \mathcal{C}^∞ function $\phi : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $u - \phi$ admits a (strict) local maximum at a point $(t, q) \in (0, T) \times \mathbb{R}^d$,

$$\partial_t \phi(t, q) + H(t, q, \partial_q \phi(t, q)) \leq 0.$$

It is a *supersolution* of (HJ) on the set $(0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$ if for each \mathcal{C}^∞ function $\phi : (0, T) \times \mathbb{R}^d$ such that $u - \phi$ admits a (strict) local minimum at a point $(t, q) \in (0, T) \times \mathbb{R}^d$,

$$\partial_t \phi(t, q) + H(t, q, \partial_q \phi(t, q)) \geq 0.$$

A viscosity solution is both a sub- and supersolution of (HJ).

This definition implies that classical solutions are in particular viscosity solutions, and that viscosity solutions are weak solutions, in the sense that

Proposition 1.5. *If u is differentiable at a point (t, q) and solves (HJ) in the viscosity sense at this point, then*

$$\partial_t u(t, q) + H(t, q, \partial_q u(t, q)) = 0.$$

Viscosity solutions appears to be a good notion of weak solutions: the existence and uniqueness are guaranteed, and it behaves well (*stability*) with respect to the Hamiltonian, all this being satisfied in various settings of assumptions on H and u_0 , including the one of this thesis. As a consequence, the theory of viscosity solution has been flourishing in the last decades, giving birth to a vast literature. We refer to [CIL92], [Bar94] or [BCD97] for overviews of the viscosity solutions theory.

Axiomatic characterization

In [AGLM93] (Theorem 2), an axiomatic description of the viscosity solutions is proposed, in the framework of multiscale analysis, see also [FS06] (Theorem 5.1) and [Bit01] (Theorem 3.1) for an extension of this result under weaker assumptions. In this thesis we will use a similar axiomatic characterization: a family of operators $(V_s^t)_{s \leq t}$ mapping $\mathcal{C}^{0,1}(\mathbb{R}^d)$ (the space of Lipschitz functions) into itself is called a *viscosity operator* if it satisfies the following conditions:

Hypotheses 1.6 (Viscosity operator).

- (i) Monotonicity: if $u \leq v$ are Lipschitz on \mathbb{R}^d , then $V_s^t u \leq V_s^t v$ on \mathbb{R}^d for each $s \leq t$,
- (ii) Additivity: if u is Lipschitz on \mathbb{R}^d and $c \in \mathbb{R}$, then $V_s^t(c + u) = c + V_s^t u$,
- (iii) Regularity: if u is Lipschitz, then for each $\tau \leq T$, $\{q \mapsto V_\tau^t u(q), t \in [\tau, T]\}$ is equi-Lipschitz and $(t, q) \mapsto V_\tau^t u(q)$ is locally Lipschitz on $(\tau, \infty) \times \mathbb{R}^d$,
- (iv) Compatibility with Hamilton-Jacobi equation: if u is a Lipschitz \mathcal{C}^2 solution of the Hamilton-Jacobi equation, then $V_s^t u_s = u_t$ for each $s \leq t$,
- (v) Markov property: $V_s^t = V_\tau^t \circ V_s^\tau$ for all $s \leq \tau \leq t$.

The following Remark allows to work by density for any operator satisfying the Monotonicity and Additivity properties:

Remark 1.7. If an operator V satisfies (i) and (ii), and u and v are two Lipschitz functions on \mathbb{R}^d with bounded difference, then

$$|V_s^t u - V_s^t v| \leq \|u - v\|_\infty.$$

The following proposition, proved in [Ber12] (Proposition 20), justifies the name of *viscosity operator*.

Proposition 1.8. *Let H be a \mathcal{C}^2 Hamiltonian with uniformly bounded second spatial derivative and $V_s^t : \mathcal{C}^{0,1}(\mathbb{R}^d, \mathbb{R}) \rightarrow \mathcal{C}^{0,1}(\mathbb{R}^d, \mathbb{R})$ be a viscosity operator defined for each $0 \leq s \leq t$. Then for each Lipschitz function $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$,*

$$u(t, q) = V_0^t u_0(q)$$

solves the Hamilton-Jacobi equation in the viscosity sense on $(0, \infty) \times \mathbb{R}^d$.

Theorem 1.9. *If H satisfies Hypothesis 1.1, there exists a unique viscosity operator V_s^t .*

The uniqueness is the consequence of a stronger uniqueness result for unbounded solutions stated by H. Ishii in [Ish84] (Theorem 2.1 with Remark 2.2), see also [CL87]. We give another proof in Appendix A, where we deduce the uniqueness of the viscosity solution (Consequence A.3) from a *finite speed of propagation* property (Proposition A.1) inspired from [ABIL13], using a standard technique for viscosity solutions called *doubling variables argument*.

The existence of the viscosity operator for our framework was already granted by the work of Crandall, Lions and Ishii (see [CIL92]) and it is proved again in this thesis, where we deduce the existence of a viscosity operator from the existence of a variational operator via a limiting process, see Theorem 1.19.

Note that since a Lipschitz function is almost everywhere differentiable, Proposition 1.5 implies that the viscosity solution solves the (HJ) equation almost everywhere.

Oleinik's entropy condition

In dimension 1, the theory of viscosity solutions of the (HJ) equation is the counterpart of the theory of entropy solutions for conservation laws: if $p(t, q) = \partial_q u(t, q)$ and u satisfies (HJ),

$$\partial_t p(t, q) + \partial_q (H(t, q, p(t, q))) = 0.$$

The following entropy condition, first proposed by O. Oleinik in [Ole59a] for conservation laws, gives a geometric criterion to decide if a function solves the (HJ) equation in the viscosity sense at a point of shock. It is proved for example in [Kos93] (Theorem 2.2) in the modern viscosity terms, as a direct application of Theorem 1.3 in [CEL84]. We give the statement for H integrable, *i.e.* which depends only on p .

Definition 1.10 (Oleinik's entropy condition). Let $H : \mathbb{R} \rightarrow \mathbb{R}$ be a \mathcal{C}^2 Hamiltonian. If $(p_1, p_2) \in \mathbb{R}^2$, we say that *Oleinik's entropy condition* is (strictly) satisfied between p_1 and p_2 if

$$H(\mu p_1 + (1 - \mu)p_2) \stackrel{(<)}{\leq} \mu H(p_1) + (1 - \mu)H(p_2) \quad \forall \mu \in (0, 1),$$

i.e. if and only if the graph of H lies (strictly) under the cord joining $(p_1, H(p_1))$ and $(p_2, H(p_2))$.

We say that the *Lax condition* is (strictly) satisfied if

$$H'(p_1)(p_2 - p_1) \stackrel{(<)}{\leq} H(p_2) - H(p_1) \stackrel{(<)}{\leq} H'(p_2)(p_2 - p_1),$$

which is implied by the entropy condition.

See Appendix F for more details on these conditions.

Proposition 1.11. *Let $u = \min(f_1, f_2)$ on an open neighbourhood U of (t, q) in $\mathbb{R}_+ \times \mathbb{R}$, with f_1 and f_2 \mathcal{C}^1 solutions on U of the Hamilton-Jacobi solution (HJ). Let p_1 and p_2 denote respectively $\partial_q f_1(t, q)$ and $\partial_q f_2(t, q)$. If $f_1(t, q) = f_2(t, q)$, then u is a viscosity solution at (t, q) if and only if the entropy condition is satisfied between p_1 and p_2 .*

Oleinik's entropy condition is also valid in higher dimensions (for shock along a smooth hypersurface), see Theorem 3.1 in [IK96], and can be generalized when u is the minimum of more than two functions, see [Ber13].

1.3 Variational solutions

Graph selector

In view of the geometric solution and the wavefront description, a way to define a meaningful singlevalued solution to the Cauchy problem is to select a continuous section of the wavefront.

Let us settle in a usual symplectic framework: we assume that M is a closed Riemannian d -manifold and look at its cotangent bundle $\pi : T^*M \rightarrow M$. If $q = (q_1, \dots, q_d)$ are the coordinates of a chart on M , the dual coordinates $p = (p_1, \dots, p_d) \in T_q^*M$ are defined by $p_i(e_j) = \delta_{ij}$, where e_j is the j^{th} vector of the canonical basis and $\delta_{i,j}$ is the Kronecker symbol. The manifold T^*M is endowed with the Liouville 1-form λ , which writes $\lambda = pdq$ in this dual chart. The symplectic structure on T^*M is given by the symplectic form $\omega = d\lambda = dp \wedge dq$ in the dual chart.

A submanifold \mathcal{L} of T^*M is called *Lagrangian* if it is d -dimensional and if $i_{\mathcal{L}}^*w = 0$, where $i_{\mathcal{L}} : \mathcal{L} \rightarrow T^*M$ is the inclusion. It is *exact* if $i_{\mathcal{L}}^*\lambda$ is exact, *i.e.* if there exists a smooth function $S : \mathcal{L} \rightarrow \mathbb{R}$ such that $dS = i_{\mathcal{L}}^*\lambda$. Such a function is called a primitive of \mathcal{L} , and is uniquely determined up to the addition of a constant. If \mathcal{L} is an exact Lagrangian submanifold, we call *wavefront* for \mathcal{L} a set of the form $\mathcal{W} = \{(\pi(x), S(x)), x \in \mathcal{L}\}$ for S a primitive of \mathcal{L} . Figure 1.2 right presents an example of Lagrangian submanifold (down) and associated wavefront (up).

If \mathcal{L} is an exact Lagrangian submanifold and \mathcal{W} is a wavefront for \mathcal{L} , we call *graph selector* a Lipschitz¹ function u whose graph is included in \mathcal{W} . Since a possible primitive S of the Lagrangian submanifold is given by an underlying action, the existence of a graph selector can be deduced under reasonable hypotheses from the existence of *action selectors*. These action selectors are obtained by using either generating family techniques (see [Cha91]), via Floer homology (see [Flo88] and [Oh97]) or lately by microlocal sheaf techniques (see [Gui12]). In [MO97], the link between the invariants constructed with generating families and via the Floer homology is studied, which leads to the conclusion that they give the same graph selector under a suitable normalization (see also [MVZ12]).

A graph selector provides simultaneously a continuous section of the wavefront and a discontinuous section of the Lagrangian submanifold:

Proposition 1.12 (Graph selector). *Let \mathcal{L} be an exact Lagrangian submanifold of T^*M such that $\pi|_{\mathcal{L}}$ is proper, \mathcal{W} be a wavefront for \mathcal{L} , and u be a graph selector. Then $(q, du(q)) \in \mathcal{L}$ for almost every q .*

The author was unable to locate the proof of this statement in the literature, yet it is close to Proposition 2.4 in [PPS03] and to Proposition II in [OV94], which both deal with the graph selector in terms of generating family. We present a proof improved by J.-C. Sikorav.

Proof. Let $S : \mathcal{L} \rightarrow \mathbb{R}$ be a primitive of \mathcal{L} and u be a graph selector of the associated wavefront. If x is in \mathcal{L} , we will denote by $p_x \in T_{\pi(x)}^*\mathcal{L}$ the second coordinate of $x = (\pi(x), p_x)$.

We are going to prove that if $q \in M$ is a regular value of $\pi|_{\mathcal{L}}$ and a point of differentiability of u , $(q, du(q))$ is in \mathcal{L} . Then combining Rademacher's theorem (on u) and Sard's theorem (on $\pi|_{\mathcal{L}}$) imply that the statement holds for almost every q .

Let us fix such a point q . We denote by \mathcal{L}_q the fiber $\pi|_{\mathcal{L}}^{-1}(\{q\})$, which is finite set since q is a regular value of the proper map $\pi|_{\mathcal{L}}$. We are going to prove that for all v in \mathbb{S}^{d-1} , there exists $x = (q, p) \in \mathcal{L}_q$ such that $du(q).v = p.v$.

Let $v \in \mathbb{S}^{d-1}$. We work in a local chart in the neighbourhood of $q \in M$: take a sequence q_n such that $\lim_{n \rightarrow \infty} \frac{q_n - q}{\|q_n - q\|} = v$. For all n , there exists x_n in \mathcal{L}_{q_n} such that $u(q_n) = S(x_n)$. Since

¹J.-C. Sikorav pointed out that a continuous function with graph is included in \mathcal{W} is automatically Lipschitz if \mathcal{L} is uniformly bounded in the fiber variable.

$\pi|_{\mathcal{L}}$ is proper, we may assume without loss of generality that x_n admits a limit x in \mathcal{L} . We again work in the local chart to write $x_n = x + x_n - x$, where $x_n - x$ is a sequence of $T_x\mathcal{L}$ converging to zero. We have on one hand

$$u(q_n) - u(q) = du(q) \cdot (q_n - q) + o(\|q_n - q\|) = \|q_n - q\| du(q) \cdot v + o(\|q_n - q\|)$$

and on the other hand

$$u(q_n) - u(q) = S(x_n) - S(x) = dS(x) \cdot (x_n - x) + o(\|x_n - x\|) = p_x d\pi(x) \cdot (x_n - x) + o(\|x_n - x\|).$$

Now, since $\pi(x_n) = q_n$ for each n , we have since $d\pi|_{\mathcal{L}}(x)$ is invertible

$$d\pi(x) \cdot (x_n - x) = q_n - q + o(\|q_n - q\|) = \|q_n - q\| v + o(\|q_n - q\|).$$

Putting these three equations together we get

$$\|q_n - q\| du(q) \cdot v = \|q_n - q\| p_x v + o(\|q_n - q\|),$$

and dividing by $\|q_n - q\|$ and letting n tend to $+\infty$ gives that $du(q) \cdot v = p_x \cdot v$.

Now we define $E_x = \{v \in \mathbb{S}^{d-1} \mid du(q) \cdot v = p_x \cdot v\}$. The previous result implies that $\{E_x\}_{x \in \mathcal{L}_q}$ is a finite cover of \mathbb{S}^{d-1} , hence $\{\text{Vect}(E_x)\}_{x \in \mathcal{L}_q}$ is a finite cover of \mathbb{R}^d made of vector subspaces: one of them is hence the whole space \mathbb{R}^d , and the corresponding $x \in \mathcal{L}_q$ hence satisfies $du(q) = p_x$. \square

The graph selector concept can also be used to address other dynamical questions, see [PPS03], [Arn10] and [BdS12].

Axiomatic definition

We will call a family of operators $(R_s^t)_{s \leq t}$ mapping $\mathcal{C}^{0,1}(\mathbb{R}^d)$ into itself a *variational operator* if it satisfies the monotonicity, additivity and regularity properties (i), (ii), (iii) of Hypotheses 1.6 and the following one:

- (iv') Variational property: for each Lipschitz \mathcal{C}^1 function u , Q in \mathbb{R}^d and $s \leq t$, there exists (q, p) such that $p = d_q u$, $Q_s^t(q, p) = Q$ and if γ denotes the Hamiltonian trajectory issued from $(q(s), p(s)) = (q, p)$,

$$R_s^t u(Q) = u(q) + \mathcal{A}_s^t(\gamma).$$

In terms of wavefront, we ask that the graph of $q \mapsto R_0^t u_0(q)$ is included in $\mathcal{F}_0^t u_0$, see (F).

The uniqueness of a variational operator is not guaranteed a priori.

We call *variational solution* to the Cauchy problem associated with u_0 a function given by a variational operator as follows: $u(t, q) = R_0^t u_0(q)$.

Remark 1.13. In view of the characteristics method, Variational property (iv') implies Compatibility property (iv).

The Markov property (v) of Hypotheses 1.6 appears then to be the crucial property for the discussion: if a variational operator satisfies this Markov property, it is the viscosity operator.

We follow [Vit96] to explicit the link between the variational operator and the graph selector introduced in the previous paragraph for a \mathcal{C}^2 initial condition u_0 . We define the autonomous suspension of H by $K(t, s, q, p) = s + H(t, q, p)$ on the cotangent $T^*(\mathbb{R} \times \mathbb{R}^d)$, identified with $T^*\mathbb{R} \times T^*\mathbb{R}^d$, and denote by Φ its Hamiltonian flow. The Hamiltonian system for K writes

$$\begin{cases} \dot{t} = 1, & \dot{q} = \partial_p H(t, q, p), \\ \dot{s} = -\partial_t H(t, q, p), & \dot{p} = -\partial_q H(t, q, p), \end{cases}$$

hence t can be taken as the time variable.

The submanifold $\Gamma_0 = \{(0, -H(0, q_0, du_0(q_0)), q_0, du_0(q_0)), q_0 \in \mathbb{R}^d\}$ is contained in the level set $K^{-1}(\{0\})$, and since K is autonomous, it is constant along its trajectories, and as a consequence $\Phi^t(\Gamma_0) = \{(t, -H(t, \phi_0^t(q_0, du_0(q_0))), \phi_0^t(q_0, du_0(q_0)), q_0 \in \mathbb{R}^d\}$. We call *suspended geometric solution* of the Cauchy problem the Lagrangian submanifold $\mathcal{L} = \cup_{t \in \mathbb{R}} \Phi^t(\Gamma_0)$, and the following set is a wavefront for \mathcal{L} :

$$\mathcal{W} = \left\{ (t, q, u_0(q_0) + \mathcal{A}_0^t(\phi_0^t(q_0, du_0(q_0)))) \mid \begin{array}{l} t \in \mathbb{R}, q \in \mathbb{R}^d, q_0 \in \mathbb{R}^d, \\ Q_0^t(q_0, du_0(q_0)) = q. \end{array} \right\}$$

The axioms required to be a variational operator implies that the function $u : (t, q) \mapsto R_0^t u_0(q)$ is a graph selector for \mathcal{L} : it is Lipschitz, and the variational property asks that its graph is contained in \mathcal{W} . Also, Proposition 1.12 states that for almost every (t, q) , $(t, \partial_t u(t, q), q, \partial_q u(t, q))$ belongs to $\mathcal{L} \subset K^{-1}(\{0\})$, which proves the following statement.

Proposition 1.14. *If u_0 is \mathcal{C}^2 and R_s^t is a variational operator, $(t, q) \mapsto R_0^t u_0(q)$ solves (HJ) in the classical sense for almost every (t, q) in $(0, \infty) \times \mathbb{R}^d$.*

This is a weaker equivalent of Proposition 1.5: we do not know in general, even for a \mathcal{C}^2 initial condition, if a variational solution u solves the equation on its domain of differentiability. We do not know either if $(t, q) \mapsto R_0^t u_0(q)$ solves the equation everywhere when u_0 is only Lipschitz.

Existence and local estimates of a variational operator

In this thesis we present a complete construction of the variational operator under Hypothesis 1.1, which comes down to build a graph selector directly for the suspended geometric solution \mathcal{L} and its wavefront \mathcal{W} , introduced in the previous paragraph. We follow the idea of J.-C. Sikorav (see [Sik86] or [Vit96]) consisting in selecting suitable critical values of a *generating family* describing this geometric solution. In order to get Lipschitz estimates for this operator, we work with the explicit generating family constructed by M. Chaperon via the *broken geodesics* method (see [Cha84] and [Cha91]), whose critical points and values are related to the Hamiltonian objects of the problem. We use a general *critical value selector* σ defined from an axiomatic point of view (see Proposition 2.7), for functions which differ by a Lipschitz function from a nondegenerate quadratic form. An obstacle is that the generating family of Chaperon is of this form only for Hamiltonians that are quadratic for large $\|p\|$, so we need to modify the Hamiltonian for large $\|p\|$ into a quadratic form Z to be able to use the critical value selector, and check that the choice of Z does not matter in the definition of the operator.

We denote by \mathbf{R}_s^t the obtained operator, keeping in mind that it depends a priori on the choice of a critical value selector σ . The explicit derivatives of the generating family allow to prove the estimates of the following statement.

Theorem 1.15. *If H satisfies Hypothesis 1.1 with constant C , there exists a variational operator, denoted by $(\mathbf{R}_s^t)_{s \leq t}$, such that for $0 \leq s \leq s' \leq t' \leq t$ and u and v two L -Lipschitz functions,*

1. $\mathbf{R}_s^t u$ is Lipschitz with $\text{Lip}(\mathbf{R}_s^t u) \leq e^{C(t-s)}(1+L) - 1$,
2. $\|\mathbf{R}_s^{t'} u - \mathbf{R}_s^t u\|_\infty \leq C e^{2C(t-s)}(1+L)^2 |t' - t|$,
3. $\|\mathbf{R}_{s'}^t u - \mathbf{R}_s^t u\|_\infty \leq C(1+L)^2 |s' - s|$,
4. $\forall Q \in \mathbb{R}^d, |\mathbf{R}_s^t u(Q) - \mathbf{R}_s^t v(Q)| \leq \|u - v\|_{\bar{B}(Q, (e^{C(t-s)} - 1)(1+L))}$,

where $\bar{B}(Q, r)$ denotes the closed ball of radius r centered in Q and $\|u\|_K := \sup_K |u|$.

The interest of these estimates is that they behave well with the iteration of the operator, and Theorem 1.15 allows then to prove Theorem 1.19 with no compactness assumptions on H .

Remark 1.16. The variational operator can also be constructed while omitting the third assumption $|H(t, q, p)| \leq C(1 + |p|)^2$ of Hypothesis 1.1. It is still Lipschitz and shares the Lipschitz constants of Theorem 1.15 except for the one associated with s and t :

$$\|\mathbf{R}_s^{t'} u - \mathbf{R}_s^t u\|_\infty \leq |t' - t| \|H\|_{\bar{B}(0, e^{C(t-s)}(1+L)-1)}, \quad |\mathbf{R}_{s'}^t u - \mathbf{R}_s^t u|_\infty \leq |s' - s| \|H\|_{\bar{B}(0, L)}.$$

These constants are a bit less practical to handle, but they do also well behave with the iteration of the operator, and would be enough to prove Theorem 1.19. The third assumption of Hypothesis 1.1 is hence merely cosmetic.

With the same method we are also able to quantify the dependence of the constructed operator \mathbf{R}_s^t with respect to the Hamiltonian:

Proposition 1.17. *Let H_0 and H_1 be two C^2 Hamiltonians satisfying Hypothesis 1.1 with constant C , u be a L -Lipschitz function, Q be in \mathbb{R}^d and $s \leq t$. Then*

$$|\mathbf{R}_{s, H_1}^t u(Q) - \mathbf{R}_{s, H_0}^t u(Q)| \leq (t - s) \|H_1 - H_0\|_{\bar{V}},$$

where $\bar{V} = [s, t] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1 + L)) \times \bar{B}(0, e^{C(t-s)}(1 + L) - 1)$.

An other formulation of the two last estimates is a localized version of the monotonicity of this variational operator with respect to the initial condition or to the Hamiltonian:

Proposition 1.18. *If H_0 and H_1 are two C^2 Hamiltonians satisfying Hypothesis 1.1 with constant C , then for $s \leq t$, Q in \mathbb{R}^d and u and v two L -Lipschitz functions,*

- $\mathbf{R}_s^t u(Q) \leq \mathbf{R}_s^t v(Q)$ if $u \leq v$ on the set $\bar{B}(Q, (e^{C(t-s)} - 1)(1 + L))$,
- $\mathbf{R}_{s, H_1}^t u(Q) \leq \mathbf{R}_{s, H_0}^t u(Q)$ if $H_1 \geq H_0$
on the set $[s, t] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1 + L)) \times \bar{B}(0, e^{C(t-s)}(1 + L) - 1)$.

An iterative procedure

If the variational and viscosity operators do not coincide in general, Q. Wei showed in [Wei14] that, for compactly supported Hamiltonians, it is possible to obtain the viscosity operator by iterating the variational operator along a subdivision of the time space and letting then the maximal step of this subsequence tend to 0. This result fits in the approximation scheme proposed by Souganidis in [Sou85] for a slightly different set of assumptions, where the variational operator acts like a *generator*. We also refer to [BS91] for a presentation of this approximation scheme method in a wider framework that includes second order Hamilton-Jacobi equations.

Let us fix a sequence of subdivisions of $[0, \infty)$ $((\tau_i^N)_{i \in \mathbb{N}})_{N \in \mathbb{N}}$ such that for all N , $0 = \tau_0^N$, $\tau_i^N \xrightarrow{i \rightarrow \infty} \infty$ and $i \mapsto \tau_i^N$ is increasing. Let us also assume that for all N , $i \mapsto \tau_{i+1}^N - \tau_i^N$ is bounded by a constant δ_N such that $\delta_N \rightarrow 0$ when $N \rightarrow \infty$. For t in \mathbb{R}_+ , we denote by $i_N(t)$ the unique integer such that t belongs to $[\tau_{i_N(t)}^N, \tau_{i_N(t)+1}^N)$. If u is Lipschitz on \mathbb{R}^d , and $0 \leq s \leq t$, let us define the iterated operator at rank N by

$$R_{s, N}^t u = R_{\tau_{i_N(t)}^N}^t R_{\tau_{i_N(t)-1}^N}^{\tau_{i_N(t)}^N} \cdots R_s^{\tau_{i_N(s)+1}^N} u,$$

where R_s^t is any variational operator satisfying the Lipschitz estimate of Theorem 1.15.

Theorem 1.19 (Wei's theorem). *For each Hamiltonian H satisfying Hypothesis 1.1, the sequence of iterated operators $(R_{s,N}^t)$ converges simply when $N \rightarrow \infty$ to the viscosity operator V_s^t . Furthermore, for each Lipschitz function u , $\{(s, t, Q) \mapsto R_{s,N}^t u(Q)\}_N$ converges uniformly towards $(s, t, Q) \mapsto V_s^t u(Q)$ on every compact subset of $\{0 \leq s \leq t\} \times \mathbb{R}^d$.*

A consequent part of this thesis is aimed at proving this theorem without compactness assumptions on H .

Remark 1.20. M. Zavidovique pointed out a consequence of this theorem that may be useful to check if the variational solution coincides with the viscosity solution for a given initial condition u : if $R_\tau^t R_s^\tau u = R_s^t u$ for all $s \leq \tau \leq t$, the iterated operator applied to u is given by $R_{s,N}^t u = R_s^t u$ and does not depend on N , hence $V_s^t u = R_s^t u$ for all $s \leq t$.

Theorem 1.19 implies amongst other things the existence of the viscosity operator, and the local uniform convergence allows to transfer Lipschitz estimates to the viscosity framework:

Proposition 1.21. *If H satisfies Hypothesis 1.1 with constant C , the viscosity operator $(V_s^t)_{s \leq t}$ satisfies the following estimates: for $0 \leq s \leq s' \leq t' \leq t$ and u and v two Lipschitz functions with Lipschitz constant L ,*

1. $V_s^t u$ is Lipschitz with $\text{Lip}(V_s^t u) \leq e^{C(t-s)}(1+L) - 1$,
2. $\|V_s^{t'} u - V_s^t u\|_\infty \leq C e^{2C(t-s)}(1+L)^2 |t' - t|$,
3. $\|V_{s'}^t u - V_s^t u\|_\infty \leq C(1+L)^2 |s' - s|$,
4. $\forall Q \in \mathbb{R}^d, |V_s^t u(Q) - V_{s'}^t v(Q)| \leq \|u - v\|_{\bar{B}(Q, (e^{C(t-s)} - 1)(1+L))}$.

Moreover, if H_0 and H_1 are two Hamiltonians satisfying Hypothesis 1.1 with constant C , u is a L -Lipschitz function, Q is in \mathbb{R}^d and $s \leq t$, the associated operators satisfy

$$|V_{s,H_1}^t u(Q) - V_{s,H_0}^t u(Q)| \leq (t-s) \|H_1 - H_0\|_{\bar{V}},$$

where $\bar{V} = [s, t] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1+L)) \times \bar{B}(0, e^{C(t-s)}(1+L) - 1)$.

Furthermore,

- $V_s^t u(Q) \leq V_s^t v(Q)$ if $u \leq v$ on the set $\bar{B}(Q, (e^{C(t-s)} - 1)(1+L))$,
- $V_{s,H_1}^t u(Q) \leq V_{s,H_0}^t u(Q)$ if $H_1 \geq H_0$
on the set $[s, t] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1+L)) \times \bar{B}(0, e^{C(t-s)}(1+L) - 1)$.

These estimates are not a priori very surprising since they are satisfied for classical solutions, but due to their dynamical origin they are likely to be sharper than the ones obtained using viscosity arguments. For example, the Lipschitz estimate with respect to u gives a better speed of propagation than the one obtained in Proposition A.2 with $e^{CT}(1+L) - 1$ as uniform Lipschitz constant.

Nonsmooth initial condition

For a \mathcal{C}^2 initial condition with bounded second derivative, the method of characteristics gives the existence of a \mathcal{C}^2 solution for small time, and implies also that the wavefront has a unique section for small time. In particular, viscosity and variational solutions coincide with the classical solution for small time for \mathcal{C}^2 initial condition with bounded second derivative. As a consequence, in order to find a difference between the two types of solution as soon as $t > 0$, we focus on nonsmooth initial conditions.

Extension to a Lipschitz initial condition

The variational property for cc^1 Lipschitz functions (iv') extends to all Lipschitz functions, using a suitable choice of generalized differential. If $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Lipschitz function, we will denote by $\partial u(q)$ the *Clarke derivative* of u at a point $q \in \mathbb{R}^d$, which is defined as the convex envelop of the set

$$\left\{ \lim_{n \rightarrow \infty} du(q_n), q_n \xrightarrow{n \rightarrow \infty} q, q_n \in \text{dom}(du) \right\}.$$

It is the singleton $\{du(q)\}$ if u is C^1 at q .

Proposition 1.22. *If R_s^t is a variational operator, for each Lipschitz function u_s , q in \mathbb{R}^d and $s \leq t$, there exists (q_s, p_s) such that $p_s \in \partial_{q_s} u_s$, $Q_s^t(q_s, p_s) = q$ and if γ denotes the Hamiltonian trajectory issued from $(q(s), p(s)) = (q_s, p_s)$,*

$$R_s^t u_s(q) = u_s(q_s) + \mathcal{A}_s^t(\gamma).$$

Proof. Remark 1.7 allows to work by density. The Lasry-Lions approximation for a Lipschitz function u gives a sequence of C^1 equi-Lipschitz functions u_n converging uniformly towards u_s and such that if $(q_n, du_n(q_n))$ admits a limit (q, p) , it lies necessarily in the graph of ∂u : $p \in \partial u(q)$. This statement can be found in [Ben92], combining Proposition 2, Théorème 3 and Remarque 4.

Let us fix $s \leq t$ and q . For each n , the variational property applies and gives a point (q_n, p_n) such that $p_n \in \partial_{q_n} u_n$, $Q_s^t(q_n, p_n) = q$ and if γ_n denotes the Hamiltonian trajectory issued from $(q(s), p(s)) = (q_n, p_n)$,

$$R_s^t u_n(q) = u_n(q_n) + \mathcal{A}_s^t(\gamma_n).$$

Since the family (u_n) is equi-Lipschitz, (p_n) is bounded by a constant L , and as a consequence (q_n) is bounded by Lemma 2.5. We can hence assume without loss of generality that the sequence (q_n, p_n) admits a limit (q_s, p_s) , which belongs to the graph of ∂u thanks to the choice of regularizing sequence. If γ denotes the Hamiltonian trajectory issued from $(q(s), p(s)) = (q_s, p_s)$, the continuity of the different objects concludes the argument:

$$R_s^t u(q) = \lim_{n \rightarrow \infty} R_s^t u_n(q) = \lim_{n \rightarrow \infty} u_n(q_n) + \mathcal{A}_s^t(\gamma_n) = u_s(q_s) + \mathcal{A}_s^t(\gamma).$$

□

If u_0 is a Lipschitz initial condition, the generalized wavefront at time t associated with the Cauchy problem for u_0 , denoted by $\mathcal{F}_0^t u_0$, is defined by

$$\mathcal{F}_0^t u_0 = \left\{ (q, u_0(q_0) + \mathcal{A}_0^t(\phi_0^\tau(q_0, p_0))) \mid \begin{array}{l} t \geq 0, q \in \mathbb{R}^d, \\ p_0 \in \partial u_0(q_0), \\ Q_0^t(q_0, p_0) = q. \end{array} \right\} \quad (\text{F}')$$

Proposition 1.22 implies that if u is a variational solution to the Cauchy problem associated with a Lipschitz initial condition u_0 , the graph of u_t is included in the wavefront $\mathcal{F}_0^t u_0$.

Remark 1.23. The Clarke derivative appears as the natural generalized differential for convex functions. And the extended Variational property states that it is a sufficient notion of weak differential for Lipschitz functions, since it is large enough to contain the initial endpoint of the characteristic giving the variational solution. However, we ran into an example in dimension 2 letting one think that the Clarke derivative may be too large for nonconvex and nonsmooth data, *i.e.* contains points irrelevant to the variational resolution for any Hamiltonian. Let $f(q_1, q_2) = \frac{q_1 q_2}{\sqrt{q_1^2 + q_2^2}}$, for which the set of limits of derivatives at $(0, 0)$ is the green astroid of

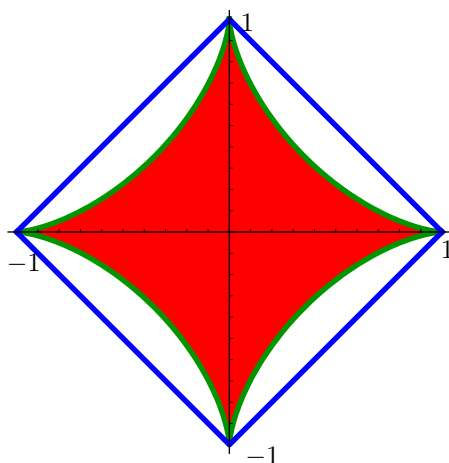


Figure 1.3: Relevant derivative (red) and Clarke derivative boundary (blue) for f .

Figure 1.3. The relevant derivative of this function seems to be the domain enclosed by this astroid, see Figure 1.3.

Indeed, on the following figure, we compare the wavefront associated with the initial condition f and the Hamiltonian $H(p_1, p_2) = p_1 p_2$ obtained either (left) with the Clarke derivative or (right) with the yellow derivative of Figure 1.3. The red part of the wavefront is issued from the singularity $(0, 0)$, whereas the blue part is issued from the domain of differentiability of f , where the differential is reduced to a point. The part of the Clarke derivative exterior to the astroid produces parts of the wavefront that cannot belong to a continuous section on one hand, and that breach the geometric structure of the wavefront on the other hand.

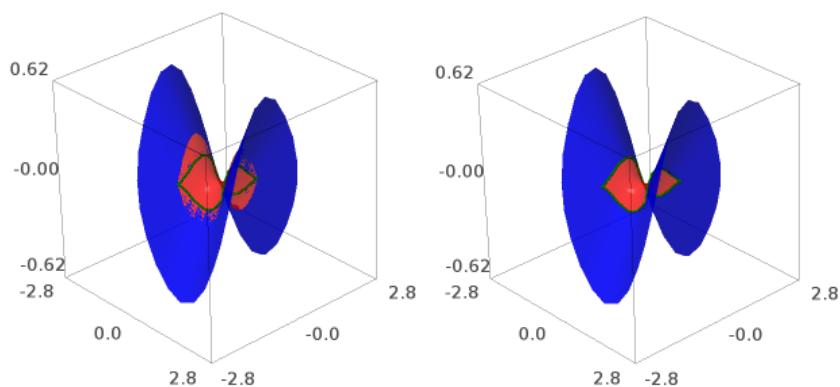


Figure 1.4: Left: with Clarke derivative. Right: with the candidate.

The relevant derivative of this example coincides with the homological generalized differential defined by N. Vichery in [Vic13]. A natural question is to prove that this homological generalized differential notion is also adapted to the variational resolution, and to decide if it is optimal.

Characterization of the variational solution for a semiconcave initial condition

A function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is B -semiconcave if $q \mapsto u(q) - \frac{B}{2}\|q\|^2$ is concave. The function u is said to be *semiconcave* if there exists B for which u is B -semiconcave. It is said to be *semiconvex* if $-u$ is semiconcave.

The following theorem states that if u_0 is a B -semiconcave function, the variational solution is given by the minimal section of the wavefront $\mathcal{F}_0^t u_0$ for small time.

Theorem 1.24. *Let H be a Hamiltonian satisfying Hypothesis 1.1 with constant C . If R_s^t is a variational operator and if u_0 is a Lipschitz B -semiconcave initial condition, then there exists $T > 0$ depending only on C and B such that for all (t, q) in $[0, T] \times \mathbb{R}^d$,*

$$R_0^t u_0(q) = \inf \{S|(q, S) \in \mathcal{F}_0^t u_0\} = \inf \left\{ u_0(q_0) + \mathcal{A}_0^t(\gamma) \left| \begin{array}{l} (q_0, p_0) \in \mathbb{R}^d \times \mathbb{R}^d, \\ p_0 \in \partial u_0(q_0), \\ Q_0^t(q_0, p_0) = q. \end{array} \right. \right\} \quad (1.1)$$

where γ denotes the Hamiltonian trajectory issued from $(q(0), p(0)) = (q_0, p_0)$.

Furthermore if H is integrable (i.e. depends only on p), we can choose $T = 1/BC$.

In particular, on the domain of validity of Theorem 1.24, the estimates (Theorem 1.15, Propositions 1.17, 1.18 and Addendum 2.26) satisfied by the variational operator \mathbf{R} hold.

Example. In dimension 1, if $u_0(q) = -|q|$ and if the Hamiltonian is integrable and has the shape of Figure 1.5 left, the wavefront at time t has the shape of Figure 1.5 right and its minimal section, thickened on the figure, is the graph of $R_0^t u_0$.

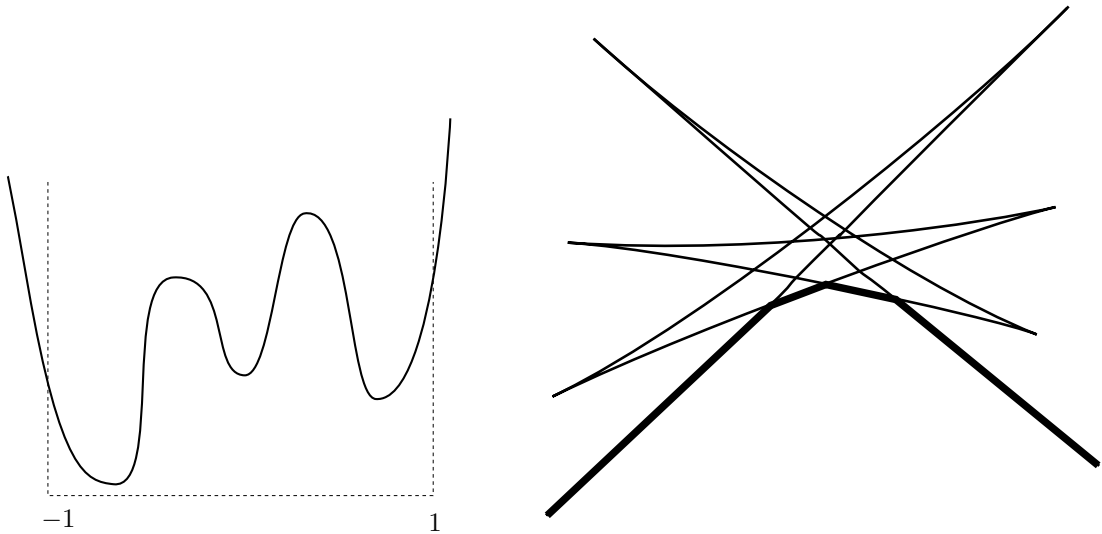


Figure 1.5: Left: graph of H . Right: minimal section of the wavefront $\mathcal{F}_0^t u_0$ at time $t > 0$.

A first element of comparison between viscosity and variational solutions is the following statement, which is proved jointly with Theorem 1.24. It is originally due to P. Bernard, see [Ber13].

Proposition 1.25. *If R_s^t is a variational operator and if u_0 is a Lipschitz B -semiconcave initial condition, then there exists $T > 0$ depending only on C and B such that if $0 \leq t \leq T$,*

$$V_0^t u_0 \leq R_0^t u_0.$$

Furthermore if H is integrable, we can choose $T = 1/BC$.

1.4 On the equality between viscosity and variational solutions

Lax-Hopf formulae in the integrable case

A Hamiltonian is said to be *integrable* if it depends only on the momentum variable p .

When convexity assumptions are made on the Hamiltonian or the initial condition, Lax [Lax57] and then Hopf [Hop65] introduced explicit and dual generalized solutions of the Cauchy problem under the form of an optimization problem.

Proposition 1.26 (Lax formula). *Let $H(p)$ be a convex integrable Hamiltonian with bounded second derivative, and u_0 be a Lipschitz initial condition. Then*

$$R_0^t u_0(q) = V_0^t u_0(q) = u_{Lax}(t, q) = \inf_{x \in \mathbb{R}^d} \sup_{p \in \mathbb{R}^d} u_0(x) + p \cdot (q - x) - tH(p).$$

Proposition 1.27 (Hopf formula). *Let $H(p)$ be an integrable Hamiltonian with bounded second derivative and u_0 be a concave Lipschitz initial condition. Then for any variational operator R_s^t ,*

$$R_0^t u_0(q) = V_0^t u_0(q) = u_{Hopf}(t, q) = \inf_{p \in \mathbb{R}^d} \sup_{x \in \mathbb{R}^d} u_0(x) + p \cdot (q - x) - tH(p).$$

A possible reference for the proof of these statements for viscosity solutions, with the Hamiltonian only supposed continuous, is [BE84], where the Lax formula (H convex) is proved using control theory methods, whereas the Hopf formula (u_0 concave) is obtained by game theory techniques. The variational part of the proposition is proved in [Ber13] for the Hopf formula (u_0 concave), and a consequence of the next Theorem 1.29 for the Lax formula (H convex).

The Lax-Hopf formulae were intensively studied in [Lio82], [LR86], [Bar87], see also [ABI99] and [Imb01] for the study in the case of merely lower semicontinuous initial data.

In the case where the Hamiltonian or the initial condition is the sum of a convex and a concave function, Lax-Hopf-type estimates can be constructed to bound the viscosity and the variational solution, see respectively [BF98] and [BC11].

Lax-Oleinik semigroup in the convex case

The Lax-Oleinik semigroup is a generalization of the function u_{Lax} when the Hamiltonian is convex but not integrable. It is a central object for the weak KAM theory, a subject pioneered by J. Mather and A. Fathi, since weak KAM solutions at level 0 can be defined as fixed points of this operator, see [Fat].

If H is strictly convex w.r.t. p , the Lagrangian function, defined on the tangent bundle, is the Legendre transform of H :

$$L(t, q, v) = \sup_{p \in (\mathbb{R}^d)^*} p \cdot v - H(t, q, p).$$

The Legendre inequality writes

$$L(t, q, v) + H(t, q, p) \geq p \cdot v$$

for all t, q, p and v , and is an equality if and only if $p = \partial_v L(t, q, v)$ or equivalently $v = \partial_p H(t, q, p)$. In particular, if $(q(\tau), p(\tau))$ is a Hamiltonian trajectory, $\dot{q}(\tau) = \partial_p H(\tau, q(\tau), p(\tau))$ and

$$\int_s^t L(\tau, q(\tau), \dot{q}(\tau)) d\tau = \int_s^t p(\tau) \cdot \dot{q}(\tau) - H(\tau, q(\tau), p(\tau)) d\tau.$$

In other words, the Hamiltonian action of a Hamiltonian trajectory coincides with the so-called Lagrangian action of its projection on the position space.

The Lax-Oleinik semigroup $(T_s^t)_{s \leq t}$ is usually expressed with this Lagrangian action: if u is a Lipschitz function on \mathbb{R}^d , then $T_s^t u$ is defined by

$$T_s^t u(q) = \inf_c u(c(s)) + \int_s^t L(\tau, c(\tau), \dot{c}(\tau)) d\tau, \quad (1.2)$$

where the infimum is taken over all the Lipschitz curves $c : [s, t] \rightarrow \mathbb{R}^d$ such that $c(t) = q$.

Proposition 1.28. *If the Hamiltonian H is uniformly strictly convex w.r.t. p , the Lax-Oleinik semigroup is both a variational and a viscosity operator.*

The Markov property is a straightforward consequence of the definition, and it is the only property we are going to use in the thesis. We give in a didactic purpose a proof of the Variational property (iv') in the case of an autonomous (*i.e.* that does not depend on t) Tonelli Hamiltonian.

Didactic proof. Let u be a \mathcal{C}^1 function on \mathbb{R}^d . Since the Hamiltonian is Tonelli, Tonelli's Theorem implies that the infimum defining the Lax-Oleinik semi-group is reached by a \mathcal{C}^1 curve c^* (see for example [Fat]). Let us denote by $\mathbb{L}(c) = u(c(s)) + \int_s^t L(c(\tau), \dot{c}(\tau)) d\tau$ the considered functional and apply the classical variational calculus technique. Since c^* minimizes \mathbb{L} , for small \mathcal{C}^1 variation curve h on \mathbb{R}^d we get at first order

$$\begin{aligned} 0 &\leq \mathbb{L}(c^* + h) - \mathbb{L}(c^*) \simeq du(c^*(s))h(s) + \int_s^t \partial_q L(c^*(\tau), \dot{c}^*(\tau))h(\tau) + \partial_v L(c^*(\tau), \dot{c}^*(\tau))\dot{h}(\tau) d\tau \\ &= du(c^*(s))h(s) + \underbrace{\int_s^t \left(\partial_q L(c^*(\tau), \dot{c}^*(\tau)) - \frac{d}{d\tau} (\partial_v L(c^*(\tau), \dot{c}^*(\tau))) \right) h(\tau) d\tau}_{(*)} + [\partial_v L(c^*, \dot{c}^*)h]_s^t. \end{aligned} \quad (1.3)$$

If the variation curve is taken with both endpoints fixed, *i.e.* $h(s) = h(t) = 0$, the only term remaining in (1.3) is the integral. Since this integral is positive for any small h , $(*)$ must cancel. In other words, a minimizer of the Lagrangian action with endpoints fixed solves the Euler-Lagrange equation:

$$\frac{d}{dt} (\partial_v L(c^*(t), \dot{c}^*(t))) = \partial_q L(c^*(t), \dot{c}^*(t)). \quad (\text{EL})$$

If the initial endpoint is free and the final endpoint is fixed (*i.e.* $h(t) = 0$), (1.3) gives, since the integral vanishes,

$$0 \leq (du(c^*(s)) - \partial_v L(s, c^*(s), \dot{c}^*(s))) h(s),$$

and this can be true for any small h only if $du(c^*(s)) = \partial_v L(s, c^*(s), \dot{c}^*(s))$.

As a consequence, setting $p^* = \partial_v L(c^*, \dot{c}^*)$, we get a Hamiltonian trajectory $\gamma^* = (c^*, p^*)$: on one hand, $p^* = \partial_v L(c^*, \dot{c}^*)$ is equivalent to $\dot{c}^* = \partial_p H(c^*, p^*)$, and on the other hand, the Euler-Lagrange equation (EL) gives that $\dot{p}^* = \partial_q L(c^*, \dot{c}^*)$ which is equal to $-\partial_q H(c^*, p^*)$, using the Legendre definition. Furthermore, $p^*(s) = \partial_v L(c^*(s), \dot{c}^*(s)) = du(c^*(s))$. Since the Hamiltonian action of the Hamiltonian trajectory $\gamma^* = (c^*, p^*)$ coincides with the Lagrangian action of the curve c^* , we have shown that the variational property is satisfied: $c^*(t) = q$, $\gamma^*(s)$ is in the graph of du , and

$$T_s^t u(q) = u(c^*(s)) + \mathcal{A}_s^t(\gamma^*).$$

□

The following theorem states that the variational operator construction of this thesis gives effectively the Lax-Oleinik semigroup for uniformly strictly convex Hamiltonian, and the viscosity operator in the convex case. We assume for this result that the critical value selector σ satisfies two additional assumptions, presented in Proposition 4.4.

Theorem 1.29 (Joukovskaia's theorem). *If $p \mapsto H(t, q, p)$ is convex for each (t, q) or concave for each (t, q) , the variational operator \mathbf{R}_s^t associated with the critical value selector σ is the viscosity operator. In particular, it coincides with the Lax-Oleinik semigroup if H is uniformly strictly convex w.r.t. p .*

The last part of this statement was proved by T. Joukovskaia in the case of a compact manifold, see [Jou91].

This theorem was generalized to convex-concave type Hamiltonians, see [Wei13a] and [BC11], but only when both the Hamiltonian and the initial condition are in the form of splitting variables:

$$H(t, q, p) = H_1(t, q_1, p_1) + H_2(t, q_2, p_2) \text{ and } u_0(q) = u_1(q_1) + u_2(q_2)$$

where $d = d_1 + d_2$, (q_i, p_i) denotes the variables in $T^*\mathbb{R}^{d_i}$, H_1 (resp. H_2) is a Hamiltonian on $\mathbb{R} \times \mathbb{R}^{d_1}$ (resp. on $\mathbb{R} \times \mathbb{R}^{d_2}$) convex in p_1 (resp. concave in p_2), and u_1 and u_2 are Lipschitz functions on \mathbb{R}^{d_1} and \mathbb{R}^{d_2} .

Characterization of the integrable Hamiltonians for which variational and viscosity operators coincide

Joukovskaia's theorem gives a class of Hamiltonians for which the variational and viscosity operators coincide. We give a first answer to the converse question for integrable Hamiltonians.

Theorem 1.30. *Let H be an integrable Hamiltonian (i.e. that depends only on p). If the viscosity operator V_s^t is a variational operator, H is convex or concave.*

To show this theorem, we reduce the problem to the study of two situations in dimension 1 and 2, namely Proposition 5.6 and Proposition 6.6. The example for the dimension 1 was already well studied: it appears already in [Che75], see also [IK96]. It is also contained in the one-dimensional work presented in this thesis (see next paragraph). The crucial example for the dimension 2 is a priori new, and presented in §6.2.

An overview of what may happen in dimension 1 for a simple shock

This thesis presents a thorough study of the short-term behaviour of the variational solution in dimension 1, for an integrable Hamiltonian and an initial condition presenting a single shock, in comparison with the viscosity solution. The results gathered and generalized here are essentially

well known, see for example [Lax57], [Che75], [IK96] and [Wei14]. The chosen framework is enough to prove (a part of) Theorem 1.30.

By *shock*, in the whole thesis, we mean a continuous singularity with a change of derivative.

We denote by \mathcal{E} the set of Lipschitz \mathcal{C}^2 functions f on \mathbb{R} , with bounded second derivative, such that $f(0) = f'(0) = 0$.

We study the Cauchy problem given by a general integrable Hamiltonian $H(p)$ with bounded second derivative bounded, and an initial condition of the form

$$u_0(q) = \min(p_1 q, p_2 q) + f(q),$$

where $p_1 < p_2$ and $f(q) = \begin{cases} f_1(q), & q \geq 0, \\ f_2(q), & q \leq 0, \end{cases}$ with f_1 and f_2 in \mathcal{E} .

The next results can also be of use for an initial condition with separated shocks in small time, *i.e.* as long as the singularities caused by the shocks do not interact.

Since u_0 is semiconcave, Theorem 1.24 implies that we can talk about *the* variational solution for small time, and the following classification holds for any variational operator R_s^t .

We denote by \widehat{H} the concave envelope of H on the set $[p_1, p_2]$. The slopes of the initial shock satisfies Oleinik's entropy condition (see Definition 1.10) if and only if \widehat{H} is affine, and in that case, $\widehat{H}' = \frac{H(p_2) - H(p_1)}{p_2 - p_1}$ is constant.

If Oleinik's entropy condition is strictly satisfied, and if \widehat{H}' is a regular value of H' , the following classification holds, see §5.3:

$H'(p_1) = H'(p_2) (= \widehat{H}')$		$R = V$
$H'(p_1) < \widehat{H}' = H'(p_2)$ (resp. $H'(p_1) = \widehat{H}' < H'(p_2)$)	\nearrow if f strictly convex on some $[0, \delta]$ (resp. on some $[-\delta, 0]$) \searrow if f concave on some $[0, \delta]$ (resp. on some $[-\delta, 0]$)	$R \neq V$ $R = V$
$H'(p_1) < \widehat{H}' < H'(p_2)$		$R = V$

where by " $R = V$ " we mean "there exists $\tau > 0$ such that $(t, q) \mapsto R_0^t u_0(q)$ solves the (HJ) equation in the viscosity sense on $(0, \tau] \times \mathbb{R}^d$ ", and by " $R \neq V$ " we mean "there exists $\tau > 0$ such that for all $0 < t < \tau$, there exists a point q such that $(t, q) \mapsto R_0^t u_0(q)$ does not satisfy the (HJ) equation in the viscosity sense at (t, q) ".

The entropy condition is violated if and only if $\widehat{H}'(p_1) > \widehat{H}'(p_2)$. In that case, and if $\widehat{H}'(p_1)$ and $\widehat{H}'(p_2)$ are regular values of H' , the following classification holds, see §5.4:

$H'(p_1) = \widehat{H}'(p_1)$ and $\widehat{H}'(p_2) = H'(p_2)$		$R = V$
$H'(p_1) < \widehat{H}'(p_1)$ and $\widehat{H}'(p_2) = H'(p_2)$ (resp. $H'(p_1) = \widehat{H}'(p_1), \widehat{H}'(p_2) < H'(p_2)$)	\nearrow if f strictly convex on some $[0, \delta]$ (resp. on some $[-\delta, 0]$) \searrow if f concave on some $[0, \delta]$ (resp. on some $[-\delta, 0]$)	$R \neq V$ $R = V$
$H'(p_1) < \widehat{H}'(p_1)$ and $\widehat{H}'(p_2) < H'(p_2)$	\nearrow if f strictly convex on some $[0, \delta]$ OR on some $[-\delta, 0]$ \searrow if f concave on some $[-\delta, \delta]$	$R \neq V$ $R = V$

For both results, the assumption requiring regular values is only used in some of the situations. Besides, it is not excluded that it could be removed using other techniques than ours. The analogous results for a semiconvex initial condition are stated in Propositions 5.10 and 5.13.

If the entropy condition is satisfied but not strictly, the situation is slightly more subtle. In §5.5, we develop such an example, called *Perestroika*, where the coincidence between viscosity and variational solutions depends on a numerical comparison involving the first and second derivatives of the Hamiltonian and the initial condition.

We also study an example where viscosity and variational solutions differ, see §5.6, for which we were able to build explicitly the viscosity solution, following an idea of O. Oleinik.

Organization of the thesis

The thesis is organized as follows: in Chapter 2 we build the variational operator and prove Theorem 1.15. We first describe the construction of Chaperon's generating family and its properties (§2.1) and introduce the notion of critical value selector and its properties (§2.2). Then, we address carefully the difficulty related to the behaviour of the Hamiltonian for large p in order to define the variational operator without compactness assumption (§2.3). We finally collect some properties of the variational and its Lipschitz estimates, proving Theorem 1.15 and Proposition 1.17 (§2.4).

In Chapter 3 we prove Theorem 1.19. We study the uniform Lipschitz estimates of the iterated operator $R_{s,N}^t$ (§3.1), and then show that the limit of any subsequence is the viscosity operator (§3.2). Chapter 3 can be read independently from Chapter 2, once the Lipschitz constants of Theorem 1.15 are granted.

In Chapter 4 we give a direct proof of Joukovskaia's theorem, while describing the Lax-Oleinik semigroup with the broken geodesics method (§4.1).

In Chapter 5 we study in dimension 1 the Cauchy problem associated with an integrable Hamiltonian and a semiconcave initial condition with one shock. After giving a few details on the wavefront structure (§5.1), we prove the two classification results announced in this introduction, for a shock strictly satisfying the entropy condition (§5.3) or denying it (§5.4). We study in §5.5 an example not included in the classification statements, and in §5.6 we build explicitly the (different !) variational and viscosity solutions for a convenient couple of initial condition and Hamiltonian.

In Chapter 6 we prove Theorem 1.30. To do so, we give reduction tools for integrable Hamiltonians that allow to split the problem into a statement in dimension 1 contained in §5.3 and an explicit example in dimension 2, studied in §6.2.

Appendix A is about viscosity solutions, and gives an elementary proof of the uniqueness for Lipschitz initial data and under Hypothesis 1.1, via a standard doubling variables argument. Appendix B details the construction and properties of Chaperon's generating families for the Hamiltonian flow, both in the general (§B.1) and in the convex case (§B.2). Appendix C proposes a functorial construction of a critical value selector as needed in the construction of the variational operator. It requires two deformation lemmas proved in Appendix D. In Appendix E we prove Theorem 1.24 and Proposition 1.25 for semiconcave initial conditions. Appendix F states elementary considerations on the Lax and entropy conditions.

Chapter 2

Building a variational operator

Dans ce chapitre, on construit l'opérateur variationnel \mathbf{R} et on déduit de cette construction les différentes propriétés lipschitziennes de cet opérateur. Pour cela, on commence par détailler la construction de la famille génératrice de Chaperon et ses propriétés. La Proposition 2.1 décrit les points et valeurs critiques de cette famille en termes hamiltoniens, et donne avec la Proposition 2.2 différentes dérivées en ces points critiques, dont on décrit la localisation dans la Proposition 2.4. La Proposition 2.3 donne la forme de cette famille génératrice pour un hamiltonien quadratique à l'infini. On introduit la notion de sélecteur de valeurs critiques dans la Proposition 2.7, et on rassemble certaines de ses propriétés, notamment la Consequence 2.12 qui exprime la localisation du sélecteur.

Pour des hamiltoniens quadratiques à l'infini, on définit directement l'opérateur variationnel en appliquant le sélecteur à la famille génératrice, et on établit que la valeur de l'opérateur ne dépend que des valeurs du hamiltonien sur une large bande de $\mathbb{R} \times T^*\mathbb{R}^d$ (voir Proposition 2.16). Pour étendre la construction à des hamiltoniens vérifiant seulement l'hypothèse de travail (1), on rend le hamiltonien quadratique à l'infini tout en s'assurant que le choix de forme à l'infini n'a pas d'incidence sur la définition de l'opérateur (voir Proposition 2.17 et Définition 2.18). Enfin, on montre que l'opérateur obtenu est variationnel (Propositions 2.21, 2.22 et 2.23), et on démontre les estimées locales du Theorem 1.15 et des Propositions 1.17 et 1.18.

In this chapter we present the complete construction of the variational operator, following the idea proposed by J.-C. Sikorav in [Sik90] and M. Chaperon in [Cha91]. We work with an explicit generating family of the geometric solution defined by Chaperon via the *broken geodesics* method (see [Cha84]). We gather its properties in the next paragraph, referring to Appendix B for some of the proofs. Then we apply on this generating family a *critical value selector*, which we handle only via a few axioms, see Proposition 2.7. The existence of a selector satisfying these axioms is proved in Appendix C. This selector can only be directly applied to generating families associated with Hamiltonians coinciding with a quadratic form at infinity, so we need to handle this difficulty by modifying the Hamiltonian for large p , see Proposition 2.17 and Definition 2.18. The rest of the chapter consists in verifying that the obtained operator is a variational operator, and that it satisfies the Lipschitz estimates of Theorem 1.15.

2.1 Chaperon's generating families

We first build a *generating family* of the Hamiltonian flow, following Chaperon's *broken geodesics* method introduced in [Cha84] and detailed in [Cha90], and then adapt it to the Cauchy problem. The results of this section are detailed and proved in Appendix B.

Under Hypothesis 1.1, it is possible to find a $\delta_1 > 0$ depending only on C (for example $\delta_1 = \frac{\ln(3/2)}{C}$) such that $\phi_s^t - \text{id}$ is $\frac{1}{2}$ -Lipschitz (see Proposition B.2), and as a consequence $(q, p) \mapsto (Q_s^t(q, p), p)$ is a \mathcal{C}^1 -diffeomorphism for each $|t - s| \leq \delta_1$, where (Q_s^t, P_s^t) denotes the components of the Hamiltonian flow ϕ_s^t .

For a Hamiltonian H satisfying Hypothesis 1.1 and $0 \leq t - s \leq \delta_1$, let $F_s^t : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ be the \mathcal{C}^1 function defined by

$$F_s^t(Q, p) = \int_s^t (P_s^\tau(q, p) - p) \cdot \partial_\tau Q_s^\tau(q, p) - H(\tau, \phi_s^\tau(q, p)) d\tau, \quad (2.1)$$

where q is the only point satisfying $Q_s^t(q, p) = Q$. The function F_s^t is called a *generating function* for the flow ϕ_s^t , meaning that

$$(Q, P) = \phi_s^t(q, p) \iff \begin{cases} \partial_p F_s^t(Q, p) &= q - Q, \\ \partial_Q F_s^t(Q, p) &= P - p, \end{cases}$$

which is proved in Proposition B.5.

Note that if $H(t, q, p) = H(p)$ is integrable, Hamiltonian trajectories have constant impulsion p and $F_s^t(Q, p) = -(t - s)H(p)$ does not depend on Q .

When $t - s$ is large, we choose a subdivision of the time interval with steps smaller than δ_1 and add intermediate coordinates along this trajectory. For each $s \leq t$ and (t_i) such that $t_0 = s \leq t_1 \leq \dots \leq t_{N+1} = t$ and $t_{i+1} - t_i \leq \delta_1$ for each i , let $G_s^t : \mathbb{R}^{2d(1+N)} \rightarrow \mathbb{R}$ be the function defined by

$$G_s^t(p_0, Q_0, p_1, Q_1, \dots, Q_{N-1}, p_N, Q_N) = \sum_{i=0}^N F_{t_i}^{t_{i+1}}(Q_i, p_i) + p_{i+1} \cdot (Q_{i+1} - Q_i) \quad (2.2)$$

where indices are taken modulo $N + 1$.

In Proposition B.7 we prove that G_s^t is a *generating function* for the flow ϕ_s^t , meaning that if $(Q, p) = (Q_N, p_0)$ and $\nu = (Q_0, p_1, \dots, Q_{N-1}, p_N)$,

$$(Q, P) = \phi_s^t(q, p) \iff \exists \nu \in \mathbb{R}^{2dN}, \begin{cases} \partial_p G_s^t(p, \nu, Q) &= q - Q, \\ \partial_Q G_s^t(p, \nu, Q) &= P - p, \\ \partial_\nu G_s^t(p, \nu, Q) &= 0, \end{cases}$$

and in this case $(Q_i, p_{i+1}) = \phi_s^{t_{i+1}}(q, p)$ for all $0 \leq i \leq N - 1$. Furthermore, if $Q = Q_s^t(q, p)$ and γ denotes the Hamiltonian trajectory issued from (q, p) ,

$$G_s^t(p, \nu, Q) = \mathcal{A}_s^t(\gamma) - p \cdot (Q - q) \quad (2.3)$$

for critical points ν of $\nu \mapsto G_s^t(p, \nu, Q)$.

This is called the *broken geodesics method*: G_s^t is actually the sum of the actions of the unique Hamiltonian trajectories γ_i such that $\gamma_i(t_i) = (\star, p_i)$ and $\gamma_i(t_{i+1}) = (Q_i, \star)$ and of boundary terms (of the form $p_{i+1} \cdot (q_{i+1} - Q_i)$) smartly arranged in order that taking critical values for G_s^t is equivalent to sew the pieces of trajectories γ_i at the intermediate points into a nonbroken geodesic on the whole time interval.

Note that if $H(t, q, p) = H(p)$, this function is quite simple:

$$G_s^t(p_0, Q_0, p_1, Q_1, \dots, p_{N-1}, Q_{N-1}, p_N, Q_N) = \sum_{i=0}^{N-1} -(t_{i+1} - t_i)H(p_i) + p_{i+1} \cdot (Q_{i+1} - Q_i). \quad (2.4)$$

Now let us use the generating family G_s^t of the flow to build what is called a *generating family* for the Cauchy problem associated with the Hamilton-Jacobi equation (HJ) and an initial condition u , using a composition formula proposed by Chekanov. If $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is Lipschitz and $s \leq t$, let us define $S_s^t u$ by

$$\begin{aligned} S_s^t u : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{2dN} &\rightarrow \mathbb{R} \\ (Q, \underbrace{q, p, \nu}_{\xi}) &\mapsto u(q) + G_s^t(p, \nu, Q) + p \cdot (Q - q). \end{aligned} \quad (2.5)$$

Proposition 2.1. *Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be a Lipschitz C^1 initial condition and $0 \leq t - s \leq T$. If Q is fixed in \mathbb{R}^d , $(q, p, Q_0, p_1, \dots, p_N)$ is a critical point of $S_s^t u(Q, \cdot)$ if and only if*

$$\begin{cases} p = du(q), \\ Q_s^t(q, p) = Q, \\ (Q_{i-1}, p_i) = \phi_s^{t_i}(q, p) \quad \forall 1 \leq i \leq N, \end{cases}$$

and in that case, $\partial_Q S_s^t u(Q, q, p, Q_0, \dots, p_N) = P_s^t(q, p)$.

Furthermore, the critical value of $S_s^t u(Q, \cdot)$ associated with a critical point (q, p, ν) is equal to $u(q) + \mathcal{A}_s^t(\gamma)$, where γ denotes the Hamiltonian trajectory $\tau \mapsto \phi_s^\tau(q, p)$.

Proof. The point (q, p, ν) is a critical point of $S_s^t u(Q, \cdot)$, if and only if

$$\begin{cases} 0 = \partial_q S_s^t u(Q, q, p, \nu) = du(q) - p, \\ 0 = \partial_p S_s^t u(Q, q, p, \nu) = \partial_p G_s^t(p, \nu, Q) + Q - q, \\ 0 = \partial_\nu S_s^t u(Q, q, p, \nu) = \partial_\nu G_s^t(p, \nu, Q). \end{cases}$$

Since G is a generating family of the flow, the two last lines implies that $Q_s^t(q, p) = Q$ and $\phi_s^{t_i}(q, p) = (Q_{i-1}, p_i)$, hence $P_s^t(q, p) = \partial_Q G_s^t u(p, \nu, Q) + p = \partial_Q S_s^t u(Q, \xi)$. The form of the critical values directly follows from the form of the critical values of G , see (2.3). \square

In other words, if Γ denotes the graph of $du \{(q, du(q)), q \in \mathbb{R}^d\}$, the generating family that we built describes the so-called *geometric solution* $\phi_s^t(\Gamma)$ as follows:

$$\phi_s^t(\Gamma) = \{(Q, \partial_q S_s^t u(Q, \xi)) \mid Q \in \mathbb{R}^d, \partial_\xi S_s^t u(Q, \xi) = 0\},$$

meaning that above each point Q , a point (Q, P) is in $\phi_s^t(\Gamma)$ if and only if there is a critical point ξ of $\xi \mapsto S_s^t u(Q, \xi)$ such that $P = \partial_Q S_s^t u(Q, \xi)$.

Let us state the values of the other derivatives of $S_s^t u$ at the points of interest:

Proposition 2.2. *Let u a C^1 L -Lipschitz function and Q in \mathbb{R}^d be fixed.*

1. *If $\xi = (q, p, \nu)$ is a critical point of $\xi \mapsto S_s^t u(Q, \xi)$, then*

$$\begin{cases} \partial_t S_s^t u(Q, \xi) &= -H(t, Q, P_s^t(q, p)), \\ \partial_s S_s^t u(Q, \xi) &= H(s, q, p). \end{cases}$$

2. If H_μ is a C^2 family of Hamiltonians satisfying Hypothesis 1.1 with constant C , the same subdivision can be chosen to build the associated generating families $S_{s,\mu}^t u$, and then $\mu \mapsto S_{s,\mu}^t u(Q, \xi)$ is C^1 and if $\xi = (q, p, \nu)$ is a critical point of $\xi \mapsto S_{s,\mu}^t u(Q, \xi)$,

$$\partial_\mu S_{s,\mu}^t u(Q, \xi) = - \int_s^t \partial_\mu H_\mu(\phi_s^\tau(q, p)) d\tau.$$

Proof. We obtain these derivatives using Proposition B.5 and B.6, and the fact that a critical point $\xi = (q, p, \nu)$ of the generating family $\xi \mapsto S_s^t u(Q, \xi)$ describes steps of a nonbroken Hamiltonian trajectory from (q, p) to $(Q, P_s^t(q, p))$ (Proposition 2.1). \square

Propositions 2.1 and 2.2 imply that if ξ is a critical point of $S_s^t u(Q, \cdot)$, the Hamilton-Jacobi equation is satisfied at this one point: $\partial_t S_s^t u(Q, \xi) + H(t, Q, \partial_Q S_s^t u(Q, \xi)) = 0$. In particular if $(t, Q) \mapsto \xi(t, Q)$ is a differentiable function giving for each (t, Q) a critical point of $S_s^t u(Q, \cdot)$, then $(t, Q) \mapsto S_s^t u(Q, \xi(t, Q))$ is a differentiable solution of the Cauchy problem. An idea to build a generalized solution is then to select adequately critical values of $S_s^t u(Q, \cdot)$, which we are going to do in the next paragraphs.

Until now, we only used the part of Hypothesis 1.1 stating that $\|\partial_{(q,p)}^2 H\|$ is uniformly bounded. The two next propositions requires the fact that $\|\partial_{(q,p)} H(t, q, p)\| \leq C(1 + \|p\|)$. The first one states that if H is nearly quadratic at infinity, so is $\xi \mapsto S_s^t u(Q, \xi)$, and the second one allows to localize the critical points of $S_s^t u$.

Proposition 2.3. *Let Z be a (possibly degenerate) quadratic form on \mathbb{R}^d . If both H and $(t, q, p) \mapsto Z(p)$ satisfy Hypothesis 1.1 with the same constant C , and $H(t, q, p) = Z(p)$ for all $\|p\| \geq R$, then $S_s^t u(Q, \xi) = \mathcal{Z}(\xi) + \ell(Q, \xi)$, where $\xi \mapsto \ell(Q, \xi)$ is a Lipschitz function with constant $\|Q\| + \text{Lip}(u) + 4(1 + R)$ and \mathcal{Z} is the nondegenerate quadratic form with associated matrix*

$$\frac{1}{2} \begin{pmatrix} 2\tau_0 Z & 0 & 0 & \cdots & 0 & -\text{Id} & \text{Id} & 0 & \cdots & 0 \\ 0 & 2\tau_1 Z & 0 & \cdots & 0 & 0 & -\text{Id} & \text{Id} & \ddots & \vdots \\ 0 & 0 & 2\tau_2 Z & \ddots & 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & -\text{Id} & \text{Id} \\ 0 & 0 & \cdots & 0 & 2\tau_N Z & 0 & 0 & \cdots & 0 & -\text{Id} \\ \hline -\text{Id} & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \text{Id} & -\text{Id} & 0 & \cdots & 0 & \vdots & & 0 & & \vdots \\ 0 & \text{Id} & \ddots & \ddots & \vdots & \vdots & & 0 & & \vdots \\ \vdots & \ddots & \ddots & -\text{Id} & 0 & 0 & & \cdots & & 0 \\ 0 & \cdots & 0 & \text{Id} & -\text{Id} & 0 & \cdots & \cdots & & 0 \end{pmatrix}$$

when written in the basis $(p, p_1, \dots, p_N, q, Q_0, \dots, Q_{N-1})$, where $\tau_i = t_{i+1} - t_i$.

Proof. Let us denote $\tilde{H}(t, q, p) = Z(p)$, and apply Proposition B.8, noticing that since $H = \tilde{H}$ for $\|p\| \geq R$, $\|d_{q,p}(H - \tilde{H})(t, q, p)\| \leq 2C(1 + \|p\|) \leq 2C(1 + R)$. It gives that a subdivision can be chosen for both H and \tilde{H} and that $\tilde{G}_s^t - G_s^t$ is then $4(1 + R)$ -Lipschitz.

For \tilde{H} , it directly follows from (2.4) that $\tilde{S}_s^t u(Q, q, p, \nu) = u(q) + \mathcal{Z}(\xi) + p_N \cdot Q$. The quadratic form \mathcal{Z} is nondegenerate as the associated matrix is invertible.

Since $\xi \mapsto \tilde{S}_s^t u(Q, \xi) - S_s^t u(Q, \xi) = \tilde{G}_s^t(Q, p, \nu) - G_s^t u(Q, p, \nu)$, it is $4(1 + R)$ -Lipschitz, which proves the point. \square

Proposition 2.4. *Let H be a Hamiltonian satisfying Hypothesis 1.1 with constant C , u be a C^1 L -Lipschitz function, $s < t$ and Q be in \mathbb{R}^d . If $\xi = (q, p, \nu)$ is a critical point of $\xi \mapsto S_s^t u(Q, \xi)$, then for all τ in $[s, t]$,*

$$\phi_s^\tau(q, p) \in B\left(Q, (e^{C(t-s)} - 1)(1 + L)\right) \times B\left(0, e^{C(t-s)}(1 + L) - 1\right),$$

where $B(x, r)$ denotes the open ball of radius r centered on x .

As a consequence, if H and \tilde{H} are two Hamiltonians satisfying Hypothesis 1.1 with constant C and coinciding on $[s, t] \times B(Q, (e^{C(t-s)} - 1)(1 + L)) \times B(0, e^{C(t-s)}(1 + L) - 1)$, the functions $\xi \mapsto S_{s,H}^t u(Q, \xi)$ and $\xi \mapsto S_{s,\tilde{H}}^t u(Q, \xi)$ have the same critical points and the same associated critical values.

Proof. We need to quantify the maximal distance covered by Hamiltonian trajectories. Hypothesis 1.1 gives an estimate which is uniform with respect to the initial position q :

Lemma 2.5. *If H satisfies Hypothesis 1.1 with constant C , then for each (q, p) , $s \leq t$,*

$$\|P_s^t(q, p) - p\| < (1 + \|p\|)(e^{C(t-s)} - 1), \quad \|Q_s^t(q, p) - q\| < (1 + \|p\|)(e^{C(t-s)} - 1).$$

In other words, $\phi_s^t(q, p)$ belongs to $B(q, (1 + \|p\|)(e^{C(t-s)} - 1)) \times B(p, (1 + \|p\|)(e^{C(t-s)} - 1))$.

Proof. The Hamiltonian system gives that $\|P_s^t(q, p) - p\| \leq \int_s^t \|\partial_q H(\tau, \phi_s^\tau(q, p))\| d\tau$ and using the hypothesis, we get

$$\|P_s^t(q, p) - p\| < C \int_s^t (1 + \|P_s^\tau(q, p)\|) d\tau \leq C \int_s^t (\|P_s^\tau(q, p) - p\| + 1 + \|p\|) d\tau. \quad (2.6)$$

Lemma B.3 applied to $f(t) = \|P_s^t(q, p) - p\|$ with $K = C(1 + \|p\|)$ gives the first estimate. Since $\|Q_s^t(q, p) - q\|$ is bounded by the same inequality (2.6), it is easy to check the second one. \square

Now, if $\xi = (q, p, \nu)$ is a critical point, Proposition 2.1 states that $p = du(q)$, whence $\|p\| \leq L$. Lemma 2.5 hence implies that for all $s \leq \tau \leq t$,

$$\|P_s^\tau(q, p)\| \leq \|p\| + (1 + \|p\|)(e^{C(\tau-s)} - 1) \leq e^{C(\tau-s)}(1 + L) - 1.$$

Now using Lemma 2.5 between τ and t gives, since $Q = Q_\tau^t(Q_s^\tau(q, p), P_s^\tau(q, p))$:

$$\|Q - Q_s^\tau(q, p)\| \leq (1 + \|P_s^\tau(q, p)\|)(e^{C(t-\tau)} - 1),$$

and since $1 + \|P_s^\tau(q, p)\| \leq e^{C(\tau-s)}(1 + L)$, we get

$$\|Q - Q_s^\tau(q, p)\| \leq (1 + L)(e^{C(t-s)} - e^{C(\tau-s)}) \leq (1 + L)(e^{C(t-s)} - 1).$$

To prove the second statement, let us recall that if $\tilde{\phi}_s^t = (\tilde{Q}_s^t, \tilde{P}_s^t)$ denotes the Hamiltonian flow for \tilde{H} , Proposition 2.1 states that $\xi = (q, p, Q_0, p_1, \dots, p_N)$ is a critical point of $\xi \mapsto S_{s,H}^t u(Q, \xi)$ (resp. of $\xi \mapsto S_{s,\tilde{H}}^t u(Q, \xi)$) if and only if

$$\begin{cases} p = du(q), \\ Q_s^t(q, p) = Q, \text{ (resp. } \tilde{Q}_s^t(q, p) = Q, \text{)} \\ (Q_{i-1}, p_i) = \phi_s^{t_i}(q, p) \text{ (resp. } (Q_{i-1}, p_i) = \tilde{\phi}_s^{t_i}(q, p)) \quad \forall 1 \leq i \leq N. \end{cases}$$

But if ξ is a critical point of $\xi \mapsto S_{s,H}^t u(Q, \xi)$, the previous work shows that the trajectory $\gamma(\tau) = \phi_s^\tau(q, p)$ stays in $B(Q, (e^{C(t-s)} - 1)(1 + L)) \times B(0, e^{C(t-s)}(1 + L) - 1)$. It is hence a Hamiltonian trajectory both for H and \tilde{H} and $\tilde{\phi}_s^\tau(q, p) = \phi_s^\tau(q, p)$ for all $s \leq \tau \leq t$, which hence shows that ξ is a critical point of $\xi \mapsto S_{s,\tilde{H}}^t u(Q, \xi)$. The associated critical value $u(q) + \mathcal{A}_s^t(\gamma)$ is also the same for H and \tilde{H} since γ stays in the set where H and \tilde{H} coincide. \square

Remark 2.6. If $H(p)$ is an integrable Hamiltonian satisfying Hypothesis 1.1 with constant C , then for each (q, p) , $s \leq t$, $P_s^t(q, p) = p$ and Lemma 2.5 may be improved:

$$\|Q_s^t(q, p) - q\| < C(t - s)(1 + \|p\|).$$

As a consequence, if u is a C^1 L -Lipschitz function, $s < t$ and Q is in \mathbb{R}^d , and $\xi = (q, p, \nu)$ is a critical point of $\xi \mapsto S_s^t u(Q, \xi)$, then for all τ in $[s, t]$,

$$\phi_s^\tau(q, p) \in B(Q, C(t - s)(1 + L)) \times B(0, L).$$

2.2 Critical value selector

Let us denote by \mathcal{Q}_m the set of functions on \mathbb{R}^m that can be written as the sum of a nondegenerate quadratic form and of a Lipschitz function.

Proposition 2.7. *There exists a function $\sigma : \bigcup_{m \in \mathbb{N}} \mathcal{Q}_m \rightarrow \mathbb{R}$ that satisfies:*

1. if f is C^1 , then $\sigma(f)$ is a critical value of f ,
2. if c is a real constant, then $\sigma(c + f) = c + \sigma(f)$,
3. if ϕ is a Lipschitz C^∞ -diffeomorphism of \mathbb{R}^m such that $f \circ \phi$ is in \mathcal{Q}_m , then

$$\sigma(f \circ \phi) = \sigma(f),$$

4. if $f_0 - f_1$ is Lipschitz and $f_0 \leq f_1$ on \mathbb{R}^d , then $\sigma(f_0) \leq \sigma(f_1)$,
5. if $(f_\mu)_{\mu \in [s, t]}$ is a C^1 family of \mathcal{Q}_m with $(\mathcal{Z} - f_\mu)_\mu$ equi-Lipschitz for some nondegenerate quadratic form \mathcal{Z} , then for all $\mu \neq \tilde{\mu} \in [s, t]$,

$$\min_{\mu \in [s, t]} \min_{x \in \text{Crit}(f_\mu)} \partial_\mu f_\mu(x) \leq \frac{\sigma(f_{\tilde{\mu}}) - \sigma(f_\mu)}{\tilde{\mu} - \mu} \leq \max_{\mu \in [s, t]} \max_{x \in \text{Crit}(f_\mu)} \partial_\mu f_\mu(x).$$

6. if $g(x, \eta) = f(x) + \mathcal{Z}(\eta)$ where f is in \mathcal{Q}_m and \mathcal{Z} is a nondegenerate quadratic form, then $\sigma(g) = \sigma(f)$.

We call such an object a critical value selector.

Such a critical value selector, named *minmax*, was introduced by Chaperon in 1991, see [Cha91]. Its construction and properties are detailed in Appendix C, which proves Proposition 2.7. The uniqueness of such a selector is not guaranteed, see [Wei14].

Remark 2.8. Additional assumptions, which are satisfied by the minmax, will be made on the critical value selector (see Proposition 4.4) in order to prove Joukovskaia's theorem. They are not needed to prove Theorems 1.15 and 1.19, so we choose not to require them until then.

Remark 2.9. Properties 2.7-(2), 2.7-(3) and 2.7-(6) coupled with Viterbo's uniqueness theorem on generating functions (see [Vit92] and [Thé99]) imply that the variational operator we are going to obtain does not depend on the choice of generating family. See Remark C.2 for more details. Property 2.7-(3) implies in particular that $\sigma(f \circ \tau) = \sigma(f)$ for each affine transformation τ of \mathbb{R}^d , which would be sufficient to prove Theorems 1.15 and 1.19.

Let us fix a critical value selector σ for the rest of the discussion. We gather here three consequences of the properties of the critical value selector.

Consequence 2.10. If f and g are two functions of \mathcal{Q}_m with difference bounded and Lipschitz on \mathbb{R}^m , then

$$|\sigma(f) - \sigma(g)| \leq \|f - g\|_\infty.$$

This is obtained by combining 2.7-(4) and 2.7-(2).

Consequence 2.11. If f is a coercive function of \mathcal{Q}_m , then $\sigma(f) = \min(f)$.

Proof. Since f is in \mathcal{Q}_m , there exist a nondegenerate quadratic form \mathcal{Z} and an L -Lipschitz function ℓ on \mathbb{R}^m such that $f = \mathcal{Z} + \ell$. Since f is coercive, it attains a global minimum at some point x_0 , and necessarily \mathcal{Z} is coercive, hence convex. Without loss of generality, we assume that $x_0 = 0$.

We are going to use the following regularization of the norm: for each $\varepsilon > 0$, the function $x \mapsto \|x\| + \varepsilon e^{-\|x\|/\varepsilon}$ is \mathcal{C}^1 , strictly convex, 1-Lipschitz and attains its global minimum ε at 0 which is its only critical point.

We have necessarily $\sigma(f) \geq \min(f) = f(0)$ (if f is \mathcal{C}^1 , this is true because $\sigma(f)$ is a critical value of f - see Proposition 2.7-(1) - and we get the result for a general f by continuity - see Consequence 2.10). Let us prove the other inequality. For each x ,

$$f(x) = \mathcal{Z}(x) + \ell(x) \leq \mathcal{Z}(x) + \ell(0) + L\|x\| \leq \mathcal{Z}(x) + \ell(0) + L(\|x\| + \varepsilon e^{-\|x\|/\varepsilon}).$$

The function $x \mapsto \mathcal{Z}(x) + \ell(0) + L(\|x\| + \varepsilon e^{-\|x\|/\varepsilon})$ is convex as a sum of convex functions and admits 0 as a critical point, hence its only critical value is $\ell(0) + \varepsilon$. Since the difference with f is $2L$ -Lipschitz, we may apply the Monotonicity property (Proposition 2.7-(4)) which gives $\sigma(f) \leq \ell(0) + \varepsilon = f(0) + \varepsilon$. Letting ε tend to 0 gives the wanted inequality. \square

Consequence 2.12. If $f_\mu = \mathcal{Z}_\mu + \ell_\mu$ is a \mathcal{C}^1 family of \mathcal{Q}_m with ℓ_μ equi-Lipschitz, such that the set of critical points f_μ does not depend on μ and such that $\mu \mapsto f_\mu$ is constant on this set, then $\mu \mapsto \sigma(f_\mu)$ is constant.

Proof. Let us take μ in some bounded set $[s, t]$. Since $\mu \mapsto \mathcal{Z}_\mu$ is \mathcal{C}^1 and \mathcal{Z}_μ is non degenerate for all μ , the index of \mathcal{Z}_μ does not depend on μ and for all μ there exists a linear isomorphism $\phi_\mu : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that $\mathcal{Z}_\mu \circ \phi_\mu = \mathcal{Z}_s$, and $\mu \mapsto \phi_\mu$ is \mathcal{C}^1 . Let us define $\tilde{f}_\mu = f_\mu \circ \phi_\mu = \mathcal{Z}_s + \ell_\mu \circ \phi_\mu$ and observe that \tilde{f}_μ satisfies the hypotheses of Proposition 2.7-(5): to do so, we only need to check that $\ell_\mu \circ \phi_\mu$ is equi-Lipschitz, which follows from the fact that ϕ_μ is equi-Lipschitz for μ in the compact set $[s, t]$.

Now, let us check that $\partial_\mu \tilde{f}_\mu(x) = 0$ for each critical point x of \tilde{f}_μ , so that both bounds of Proposition 2.7-(5) are zero. Since ϕ_μ is a \mathcal{C}^1 -diffeomorphism, x is a critical point of \tilde{f}_μ if and only if $\phi_\mu(x)$ is a critical point of f_μ , i.e. $df_\mu(\phi_\mu(x)) = 0$. Then since $\mu \mapsto f_\mu$ is constant on its critical points, $\partial_\mu f_\mu(\phi_\mu(x)) = 0$. As a consequence, $\partial_\mu \tilde{f}_\mu(x) = \partial_\mu f_\mu(\phi_\mu(x)) + \partial_\mu \phi_\mu(x) df_\mu(\phi_\mu(x)) = 0$ and $\mu \mapsto \sigma(\tilde{f}_\mu)$ is constant by Proposition 2.7-(5). Proposition 2.7-(3) ends the proof, stating that for all μ , $\sigma(\tilde{f}_\mu) = \sigma(f_\mu \circ \phi_\mu) = \sigma(f_\mu)$. \square

2.3 Definition of R_s^t

In this section, we will say that a Hamiltonian is *fiberwise compactly supported* if there exists a $R > 0$ such that $H(t, q, p) = 0$ for $\|p\| \geq R$. If $Z(p)$ is a quadratic form, we denote by \mathcal{H}_Z^C the set of \mathcal{C}^2 Hamiltonians H satisfying Hypothesis 1.1 with constant C and such that $H(t, q, p) - Z(p)$ is fiberwise compactly supported.

If Z is a (possibly degenerate) quadratic form, Proposition 2.3 proves that the generating family associated with a Hamiltonian in \mathcal{H}_Z^C differs by a Lipschitz function from a nondegenerate quadratic form. For Hamiltonians in \mathcal{H}_Z^C , we are then able to define the operator \mathbf{R}_s^t directly by applying the critical value selector σ on the generating family. The localization of the critical points of the generating family (Proposition 2.4) allows then to show that the value of the operator does only depend on the behaviour of H on a large enough strip $\mathbb{R} \times \mathbb{R}^d \times B(0, R)$.

For general Hamiltonians satisfying Hypothesis 1.1, the generating family is a priori not in any \mathcal{Q}_m , so we cannot select a critical value with the selector σ . To get around this difficulty, we modify the Hamiltonian outside a large enough strip into some $Z(p)$. It is remarkable that the choice of Z has no incidence on the value of the operator: we hence obtain exactly the same operator by making the Hamiltonian compactly supported with respect to p or by setting it on $\|p\|^2$, for example. To prove Theorems 1.15 and 1.19, we will simply use $Z = 0$, but when dealing with fiberwise convex Hamiltonians, for example to prove Theorem 1.29, the choice of a convex nondegenerate quadratic form will be more adequate.

Definition 2.13. If H is in \mathcal{H}_Z^C and $s \leq t$, let the operator (\mathbf{R}_s^t) be defined for Lipschitz functions u on \mathbb{R}^d by

$$\mathbf{R}_s^t u(Q) = \sigma(S_s^t u(Q, \cdot)) \quad \forall Q \in \mathbb{R}^d,$$

where $S_s^t u(Q, \cdot)$ is the function $\xi \mapsto S_s^t u(Q, \xi)$ and S is the generating family defined at (2.5). In particular, if u is \mathcal{C}^1 , $\mathbf{R}_s^t u(Q)$ is a critical value of $\xi \mapsto S_s^t u(Q, \xi)$.

This definition is possible since Proposition 2.3 states that $\xi \mapsto S_s^t u(Q, \xi)$ is in some \mathcal{Q}_m .

Proposition 2.14. *The operator \mathbf{R}_s^t does not depend on the choice of subdivision of $[s, t]$ in the definition of G , see (2.2).*

Proof. It is enough to consider two cases: either the subdivisions are identical with only one intermediate step t_i changing, or one subdivision is obtained from the other by adding artificially an intermediate step of length zero.

In the first case, we observe that if the subdivision is fixed except for one intermediate step t_i , the function $t_i \mapsto S_s^t u(Q, \xi)$ is \mathcal{C}^1 , hence uniformly continuous, and by Consequence 2.10 this implies that $t_i \mapsto \mathbf{R}_s^t u(Q)$ is continuous. But the set of critical values of $\xi \mapsto S_s^t u(Q, \xi)$ does not depend on t_i (see Proposition 2.1) and is discrete, hence $t_i \mapsto \mathbf{R}_s^t u(Q)$ must be a constant function.

In the second case, let us artificially add an intermediate step t_ι equal to t_i : the subdivision is now $s = t_0 \leq t_1 \leq \dots \leq t_{i-1} \leq t_\iota = t_i \leq \dots \leq t_{N+1} = t$ and the variables $(Q, p, Q_0, p_1, Q_1, \dots, Q_{i-1}, p_\iota, Q_\iota, p_i, \dots, p_N)$. We denote by G (resp. \tilde{G}) the family associated with the subdivision without (resp. with) t_ι , that takes variables $(Q, p, Q_0, \dots, Q_{i-1}, p_i, \dots, p_N)$ (resp. $(Q, p, Q_0, \dots, Q_{i-1}, p_\iota, Q_\iota, p_i, \dots, p_N)$).

Since $F_{t_i}^{t_i} = 0$ and $F_{t_{i-1}}^{t_i} = F_{t_{i-1}}^{t_\iota}$, we may observe that:

$$\tilde{G}(Q, \dots, Q_i, p_\iota, Q_\iota, p_{i+1}, \dots, p_N) = G(Q, \dots, Q_i, p_{i+1}, \dots, p_N) - (p_i - p_\iota) \cdot (Q_\iota - Q_{i-1}),$$

and the same holds for the associated families S and \tilde{S} :

$$\tilde{S}(Q, q, \dots, Q_i, p_\iota, Q_\iota, p_{i+1}, \dots, p_N) = S(Q, q, \dots, Q_i, p_{i+1}, \dots, p_N) - (p_i - p_\iota) \cdot (Q_\iota - Q_{i-1}).$$

The affine transformation mapping p_ι to $\tilde{p}_\iota = p_i - p_\iota$, Q_ι to $\tilde{Q}_\iota = Q_\iota - Q_{i-1}$ and keeping the other variables fixed preserves the value of the selector by property 2.7-(3) of σ . In these new coordinates, the family writes:

$$\tilde{S}(Q, q, \dots, Q_i, \tilde{p}_\iota, \tilde{Q}_\iota, p_{i+1}, \dots, p_N) = S(Q, q, \dots, Q_i, p_{i+1}, \dots, p_N) - \tilde{p}_\iota \cdot \tilde{Q}_\iota$$

and since $(\tilde{p}_\iota, \tilde{Q}_\iota) \mapsto -\tilde{p}_\iota \cdot \tilde{Q}_\iota$ is a nondegenerate quadratic function of $(\tilde{p}_\iota, \tilde{Q}_\iota)$, the invariance by stabilization 2.7-(6) for σ of the critical value selector concludes the proof. \square

The following basic continuity result for \mathbf{R}_s^t , which is improved in Theorem 1.15, is only there to allow to work with u of class C^1 and extend the results by density:

Proposition 2.15 (Weak contraction). *If H is in \mathcal{H}_Z^C and u and v are two Lipschitz functions such that $u - v$ is bounded, then $\mathbf{R}_s^t u - \mathbf{R}_s^t v$ is bounded by $\|u - v\|_\infty$.*

Proof. Let us fix s, t and Q , and note that the quantity $S_s^t u(Q, \xi) - S_s^t v(Q, \xi) = u(q) - v(q)$ is a Lipschitz and bounded function of ξ . The continuity of σ established in Consequence 2.10 gives that

$$\|\mathbf{R}_s^t u(Q) - \mathbf{R}_s^t v(Q)\| \leq \|S_s^t u(Q, \cdot) - S_s^t v(Q, \cdot)\|_\infty \leq \|u - v\|_\infty.$$

\square

The following proposition implies that the value of the operator depends only on the value of H on a large enough compact set:

Proposition 2.16. *Let Z and \tilde{Z} be two quadratic forms, and H (resp. \tilde{H}) be a Hamiltonian in \mathcal{H}_Z^C (resp. $\mathcal{H}_{\tilde{Z}}^C$). For each L -Lipschitz function u and $s \leq t$, if $H = \tilde{H}$ on $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(t-s)}(1+L) - 1)$, then $\mathbf{R}_{s,H}^t u = \mathbf{R}_{s,\tilde{H}}^t u$.*

Proof. Let us first assume that u a C^1 L -Lipschitz function and $s \leq t$. Let us define $H_\mu = \mu H + (1 - \mu)\tilde{H}$. Observe that H_μ is in $\mathcal{H}_{Z_\mu}^C$ where $Z_\mu = \mu Z + (1 - \mu)\tilde{Z}$ is a quadratic form, and that there exists $R > 0$ such that for all μ in $[0, 1]$, $H_\mu(t, q, p) = Z_\mu(p)$ if $\|p\| \geq R$.

Proposition 2.3 hence guarantees that for all μ , $S_{s,H_\mu}^t u(Q, \xi) = Z_\mu(\xi) + \ell_\mu(Q, \xi)$ where Z_μ is a nondegenerate quadratic form and $\xi \mapsto \ell_\mu(Q, \xi)$ is Lipschitz with constant $\text{Lip}(u) + \|Q\| + 4(1+R)$. Note that if Q is fixed, the family $\xi \mapsto \ell_\mu(Q, \xi)$ is hence equi-Lipschitz when μ is in $[0, 1]$.

As H_μ is constant on $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(t-s)}(1+L) - 1)$, the second part of Proposition 2.4 states that the set of critical points of $\xi \mapsto S_{s,H_\mu}^t u(Q, \xi)$ does not depend on μ , and neither do the associated critical values.

So if Q is fixed, the family of functions $f_\mu = S_{s,H_\mu}^t u(Q, \cdot)$ satisfies the conditions of Consequence 2.12, and hence $\mathbf{R}_{s,H_\mu}^t u(Q) = \sigma(f_\mu)$ does not depend on μ . As a consequence, $\mathbf{R}_{s,H}^t u = \mathbf{R}_{s,\tilde{H}}^t u$.

The result extends to every L -Lipschitz u thanks to Proposition 2.15 and the fact that u can be L^∞ -approximated by a C^1 L -Lipschitz function. \square

We now want to extend the definition to a Hamiltonian that is not quadratic at infinity, by modifying it outside some large enough strip $\mathbb{R} \times \mathbb{R}^d \times B(0, R)$ into some $Z(p)$. We cannot make sure that the modified Hamiltonian still satisfies Hypothesis 1.1 with the same constant C than H , so we have to be cautious since the width of the strip depends on C . Lemma 2.19 shows that the constant of the modified Hamiltonian can be arbitrarily close to C , and this independently from the width of the strip, which avoids any trouble.

Proposition 2.17. *Let H be a C^2 Hamiltonian satisfying Hypothesis 1.1 with constant C , u be a L -Lipschitz function and $s \leq t$. For all $\delta > 0$, and for each quadratic form Z such that $\|d^2 Z\| \leq C$, there exists a Hamiltonian $H_{\delta,Z}$ in $\mathcal{H}_Z^{C(1+\delta)}$ that coincides with H on $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(1+\delta)(t-s)}(1+L) - 1)$. Then, $\mathbf{R}_{s,H_{\delta,Z}}^t u$ does neither depend on the choice of $H_{\delta,Z}$, nor on the choice of Z , nor on $\delta > 0$.*

This proposition allows to define the variational operator for general Hamiltonians:

Definition 2.18. Let H be a \mathcal{C}^2 Hamiltonian satisfying Hypothesis 1.1 with constant C . For each L -Lipschitz function u and $s \leq t$, we define $\mathbf{R}_{s,H}^t u = \mathbf{R}_{s,H_\delta,Z}^t u$, where $\delta > 0$ and $H_{\delta,Z}$ is a Hamiltonian of $\mathcal{H}_Z^{C(1+\delta)}$ for some quadratic form Z such that $\|d^2Z\| \leq C$, which coincides with H on $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(1+\delta)(t-s)}(1+L) - 1)$.

Proof of Proposition 2.17. Let us show that for all $\delta > 0$, there exists H_δ in $\mathcal{H}_Z^{C(1+\delta)}$ coinciding with H on $\mathbb{R} \times \mathbb{R}^d \times B(0, R_\delta)$, where $R_\delta = e^{C(1+\delta)(t-s)}(1+L) - 1$. To do so, we use the following lemma:

Lemma 2.19. *If $R > 0$ and $\varepsilon > 0$, there exists a compactly supported \mathcal{C}^2 function $\varphi : \mathbb{R}_+ \rightarrow [0, 1]$, equal to 1 on $[0, R]$, such that for all $r \geq 0$,*

$$|\varphi'(r)| \leq \frac{\varepsilon}{6(1+r)}, \quad |\varphi''(r)| \leq \frac{\varepsilon}{6(1+r)^2} \quad \text{and} \quad \frac{|\varphi'(r)|}{r} \leq \frac{\varepsilon}{6(1+r)^2}.$$

For such a function φ , if H and \tilde{H} are two Hamiltonians satisfying Hypothesis 1.1 with constant C , the Hamiltonian $H_\varphi : (t, q, p) \mapsto \varphi(\|p\|)H(t, q, p) + (1 - \varphi(\|p\|))\tilde{H}(t, q, p)$ satisfies Hypothesis 1.1 with constant $C(1+\varepsilon)$, is equal to H on $\mathbb{R} \times \mathbb{R}^d \times B(0, R)$ and $H_\varphi - \tilde{H}$ is fiberwise compactly supported.

Proof. Take some $R' > \max(1, R)$ and let us define

$$\varphi(r) = \max\left(0, 1 - \frac{\varepsilon}{12} \max(0, \ln(1+r) - \ln(1+R'))\right).$$

If $r \leq R'$, $\varphi(r) = 1$. If $r \geq (1+R')e^{12/\varepsilon} - 1$, $\varphi(r) = 0$. For all $r \geq 0$, $0 \leq \varphi(r) \leq 1$.

The function φ is \mathcal{C}^∞ except at $r = R'$ or $r = (1+R')e^{12/\varepsilon} - 1$. Let us evaluate its derivatives on $(R', (1+R')e^{12/\varepsilon} - 1)$, where $f(r) = 1 - \frac{\varepsilon}{12} (\ln(1+r) - \ln(1+R'))$:

$$\varphi'(r) = \frac{-\varepsilon}{12(1+r)}, \quad \varphi''(r) = \frac{\varepsilon}{12(1+r)^2}.$$

Furthermore, as long as $r \geq R' > 1$, this implies that

$$|\varphi'(r)| = \frac{\varepsilon}{12(1+r)} \leq \frac{\varepsilon r}{6(1+r)^2}.$$

Hence the three wanted estimates are satisfied on $(R', (1+R')e^{12/\varepsilon} - 1)$. Since φ' and φ'' are zero if $r < R'$ or $r > (1+R')e^{12/\varepsilon} - 1$, it is possible to smooth φ by below at R' and by above at $(1+R')e^{12/\varepsilon} - 1$ without increasing the derivative bounds, keeping $\varphi = 1$ for $r \leq R$ and φ compactly supported.

Now if H and \tilde{H} are two Hamiltonians satisfying Hypothesis 1.1 with constant C , let us define H_φ by $H_\varphi(t, q, p) = \varphi(\|p\|)H(t, q, p) + (1 - \varphi(\|p\|))\tilde{H}(t, q, p)$. It is \mathcal{C}^2 , coincides with H on $\mathbb{R} \times \mathbb{R}^d \times B(0, R_\delta)$, and $H_\varphi(t, q, p) - \tilde{H}(t, q, p) = \varphi(\|p\|)(H(t, q, p) - \tilde{H}(t, q, p))$ is fiberwise compactly supported since $\varphi(r) = 0$ for r large enough.

In order to verify that H_φ satisfies Hypothesis 1.1 with constant $C(1+\varepsilon)$, let us bound the derivatives of $\phi(p) = \varphi(\|p\|)$:

$$\begin{aligned} \|d\phi(p)\| &= |\varphi'(\|p\|)| \leq \frac{\varepsilon}{6(1+\|p\|)}, \\ \|d^2\phi(p)\| &\leq \max\left(|\varphi''(\|p\|)|, \frac{|\varphi'(\|p\|)|}{\|p\|}\right) \leq \frac{\varepsilon}{6(1+\|p\|)^2}. \end{aligned}$$

Now, since both H and \tilde{H} satisfy $|H(t, q, p)| \leq C(1 + \|p\|)^2$ and $\phi(p) \in [0, 1]$ for all p ,

$$|H_\varphi(t, q, p)| \leq \phi(p)|H(t, q, p)| + (1 - \phi(p))|\tilde{H}(t, q, p)| \leq C(1 + \|p\|)^2,$$

Since H and \tilde{H} satisfies Hypothesis 1.1 with constant C , $H - \tilde{H}$ satisfies Hypothesis 1.1 with constant $2C$, and the following holds:

$$\begin{aligned} \|dH_\varphi\| &\leq \phi(p) \underbrace{\|dH\|}_{\leq C(1+\|p\|)} + (1 - \phi(p)) \underbrace{\|d\tilde{H}\|}_{\leq C(1+\|p\|)} + \underbrace{|d\phi(p)|}_{\leq \frac{\varepsilon}{\delta(1+\|p\|)}} \underbrace{|H - \tilde{H}|}_{\leq 2C(1+\|p\|)^2} \\ &\leq C(1 + \|p\|) + \frac{\varepsilon}{3}C(1 + \|p\|) \leq C(1 + \varepsilon)(1 + \|p\|), \\ \|d^2H_\varphi\| &\leq \phi\|d^2H\| + (1 - \phi)\|d^2\tilde{H}\| + 2\|d\phi\|\|dH - d\tilde{H}\| + \|d^2\phi\|\|H - \tilde{H}\| \\ &\leq \phi C + (1 - \phi)C + 2\frac{\varepsilon}{6(1 + \|p\|)} \cdot 2C(1 + \|p\|) + \frac{\varepsilon}{6(1 + \|p\|)^2} \cdot 2C(1 + \|p\|)^2 \\ &\leq C + 2\frac{\varepsilon}{3}C + \frac{\varepsilon}{3}C \leq C(1 + \varepsilon). \end{aligned}$$

□

To build $H_{\delta, Z}$ in $\mathcal{H}_Z^{C(1+\delta)}$ coinciding with H on $\mathbb{R} \times \mathbb{R}^d \times B(0, R_\delta)$, it is enough to apply Lemma 2.19 with $\tilde{H}(t, q, p) = Z(p)$, $\varepsilon = \delta$ and $R = R_\delta = e^{C(1+\delta)(t-s)}(1 + L) - 1$.

Let us now check that $\mathbf{R}_{s, H_{\delta, Z}}^t u$ is independent from the choice of $H_{\delta, Z}$ and Z : if $H_{\delta, Z}$ in $\mathcal{H}_Z^{C(1+\delta)}$ and $\tilde{H}_{\delta, \tilde{Z}}$ in $\mathcal{H}_{\tilde{Z}}^{C(1+\delta)}$ coincide on $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(1+\delta)(t-s)}(1 + L) - 1)$, Proposition 2.16 applies and $\mathbf{R}_{s, H_{\delta, Z}}^t u = \mathbf{R}_{s, \tilde{H}_{\delta, \tilde{Z}}}^t u$.

From now on, we may take $Z = 0$, hence the set \mathcal{H}_0^C is exactly the set of \mathcal{C}^2 fiberwise compactly supported Hamiltonians satisfying Hypothesis 1.1 with constant C . Let us prove the independence with respect to δ .

Let $s \leq t$ and u a L -Lipschitz function be fixed, and still denote by R_δ the radius given by $e^{C(1+\delta)(t-s)}(1 + L) - 1$, which is increasing with respect to δ . Take $\delta > \tilde{\delta} > 0$, and H_δ (resp. $H_{\tilde{\delta}}$) a Hamiltonian in $\mathcal{H}_0^{C(1+\delta)}$ (resp. $\mathcal{H}_0^{C(1+\tilde{\delta})}$) coinciding with H on $\mathbb{R} \times \mathbb{R}^d \times B(0, R_\delta)$ (resp. $\times B(0, R_{\tilde{\delta}})$), so that $\mathbf{R}_{s, H_\delta}^{t, \delta} u(Q) = \mathbf{R}_{s, H_\delta}^t u(Q)$ and $\mathbf{R}_{s, H_\delta}^{t, \tilde{\delta}} u(Q) = \mathbf{R}_{s, H_{\tilde{\delta}}}^t u(Q)$.

Lemma 2.19 applied with $R = R_\delta$, $\varepsilon = \tilde{\delta}$ and $\tilde{H} = 0$ gives a Hamiltonian H_φ in $\mathcal{H}_0^{C(1+\tilde{\delta})}$ coinciding with H (hence H_δ) on $\mathbb{R} \times \mathbb{R}^d \times B(0, R_\delta)$, and therefore since $B(0, R_{\tilde{\delta}}) \subset B(0, R_\delta)$, with $H_{\tilde{\delta}}$ on $\mathbb{R} \times \mathbb{R}^d \times B(0, R_{\tilde{\delta}})$. Proposition 2.16 gives on the one hand that $\mathbf{R}_{s, H_\delta}^t u = \mathbf{R}_{s, H_\varphi}^t u$, and on the other hand that $\mathbf{R}_{s, H_\varphi}^t u = \mathbf{R}_{s, H_{\tilde{\delta}}}^t u$, hence the result.

□

Addendum 2.20. *If H is uniformly strictly convex with respect to p (i.e. there exists $m > 0$ such that $\partial_p^2 H(t, q, p) \geq m \text{id}$ for all (t, q, p)) and Z is a strictly positive quadratic form such that $\frac{m}{2} \text{id} \leq Z \leq \frac{C}{2} \text{id}$, then the function $H_{\delta, Z}$ of Proposition 2.17 can be chosen uniformly strictly convex w.r.t. p .*

Proof. In the proof of Lemma 2.19, we assume that H and \tilde{H} are uniformly strictly convex with respect to p with a constant $m > 0$. Then following the construction of H_φ , we may estimate its second derivative with respect to p :

$$\begin{aligned} \partial_p^2 H_\varphi &\geq \phi \partial_p^2 H + (1 - \phi) \partial_p^2 \tilde{H} - \left(2\|d\phi\|\|\partial_p H - \partial_p \tilde{H}\| + \|d^2\phi\|\|H - \tilde{H}\| \right) \text{id} \\ &\geq (m - C\varepsilon) \text{id} \end{aligned}$$

using the estimates on the derivatives of φ , H and \tilde{H} . So, if $\varepsilon < m/C$, the obtained function is uniformly strictly convex. \square

2.4 Properties and Lipschitz estimates of \mathbf{R}_s^t .

Let us prove that $(\mathbf{R}_s^t)_{s \leq t}$ is a variational operator. Monotonicity and additivity properties are straightforward:

Proposition 2.21 (Monotonicity). *If $u \leq v$ are Lipschitz functions on \mathbb{R}^d , then for each $s \leq t$, $\mathbf{R}_s^t u \leq \mathbf{R}_s^t v$ on \mathbb{R}^d .*

Proof. Let L be a Lipschitz constant for both u and v , and fix $s \leq t$, $\delta > 0$. Let H_δ be a Hamiltonian in $\mathcal{H}_0^{C(1+\delta)}$ coinciding with H on $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(1+\delta)(t-s)}(1+L) - 1)$ as in Definition 2.18, so that $\mathbf{R}_{s,H}^t u(Q) = \mathbf{R}_{s,H_\delta}^t u(Q)$ and $\mathbf{R}_{s,H}^t v(Q) = \mathbf{R}_{s,H_\delta}^t v(Q)$.

Since $S_{s,H_\delta}^t v(Q, \xi) - S_{s,H_\delta}^t u(Q, \xi) = v(q) - u(q)$ is a non negative and Lipschitz function of ξ , the monotonicity 2.7-(4) of σ applies and $\mathbf{R}_{s,H_\delta}^t u(Q) \leq \mathbf{R}_{s,H_\delta}^t v(Q)$, thus

$$\mathbf{R}_{s,H}^t u(Q) \leq \mathbf{R}_{s,H}^t v(Q).$$

\square

Proposition 2.22 (Additivity). *If c is a real constant, then $\mathbf{R}_s^t(c + u) = c + \mathbf{R}_s^t u$ for each Lipschitz function u .*

Proof. The additivity property 2.7-(2) of σ and the form of $S_s^t u$ conclude, as in the previous proof. \square

Proposition 2.23 (Variational property). *For each \mathcal{C}^1 Lipschitz function u , Q in \mathbb{R}^d and $s \leq t$, there exists (q, p) such that $p = d_q u$, $Q_s^t(q, p) = Q$ and if γ denotes the Hamiltonian trajectory issued from $(q(s), p(s)) = (q, p)$,*

$$\mathbf{R}_s^t u(Q) = u(q) + \mathcal{A}_s^t(\gamma),$$

Proof. Let us fix u , $s \leq t$ and $\delta > 0$ and take as in Definition 2.18 a Hamiltonian H_δ in $\mathcal{H}_0^{C(1+\delta)}$ equal to H on $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(1+\delta)(t-s)}(1+L) - 1)$, such that $\mathbf{R}_{s,H}^t u(Q) = \mathbf{R}_{s,H_\delta}^t u(Q)$.

Since u is \mathcal{C}^1 , $\mathbf{R}_{s,H_\delta}^t u(Q)$ is a critical value of $\chi \mapsto S_{s,H_\delta}^t u(Q, \chi)$. Proposition 2.1, which describes the critical points and values of S , gives the existence of (q, p) such that $Q_{s,H_\delta}^t(q, p) = Q$ and $p = d_q u(q)$, and states that if $\gamma_\delta(\tau) = \phi_{s,H_\delta}^\tau(q, p)$ denotes the Hamiltonian trajectory issued from (q, p) for the Hamiltonian H_δ ,

$$\mathbf{R}_{s,H_\delta}^t u(Q) = u(q) + \mathcal{A}_{s,H_\delta}^t(\gamma_\delta).$$

Proposition 2.4, which localizes the critical points of S under Hypothesis 1.1, gives that $\gamma_\delta(\tau)$ belongs to the set $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(1+\delta)(t-s)}(1+L) - 1)$ for all τ in $[s, t]$.

Since H and H_δ coincide on that set for each time in $[s, t]$, γ_δ is also a Hamiltonian trajectory for H on $[s, t]$, the Hamiltonian action of γ_δ has the same expression for H and H_δ , and the conclusion holds: $Q = Q_{s,H_\delta}^t(q, p) = Q_{s,H}^t(q, p)$ and

$$\mathbf{R}_{s,H}^t u(Q) = \mathbf{R}_{s,H_\delta}^t u(Q) = u(q) + \mathcal{A}_{s,H}^t(\gamma_\delta).$$

\square

The next proposition proves the Lipschitz estimates of Theorem 1.15, which implies that \mathbf{R}_s^t satisfies the regularity property (iii) of Hypotheses 1.6.

Proof of Theorem 1.15. Suppose to begin with that u is \mathcal{C}^1 and that H is fiberwise compactly supported, meaning that there exists $R > 0$ such that $H(t, q, p) = 0$ for $\|p\| \geq R$. Under that assumption, in Proposition 2.3, the nondegenerate quadratic form \mathcal{Z} does not depend on s or t .

For each item of this proof, we are going to use Property 2.7-(5) on a suitable homotopy f_μ , the form of the derivatives of $S_s^t u$ given in Propositions 2.1 and 2.2 and the localization of the critical points of $S_s^t u$ described in Proposition 2.4.

1. Let us show that $\mathbf{R}_s^t u$ is Lipschitz with $\text{Lip}(\mathbf{R}_s^t u) \leq e^{C(t-s)}(1+L) - 1$. Let us fix Q and h in \mathbb{R}^d and define $f_\mu(\xi) = S_s^t u(Q + \mu h, \xi)$ for μ in $[0, 1]$. The aim is to estimate $|\mathbf{R}_s^t u(Q + h) - \mathbf{R}_s^t u(Q)| = |\sigma(f_1) - \sigma(f_0)|$.

Proposition 2.3 states that the family f_μ is of the form required in Property 2.7-(5), *i.e.* $f_\mu(\xi) = \mathcal{Z}(\xi) + \ell_\mu(\xi)$, where \mathcal{Z} is nondegenerate and the family ℓ_μ is equi-Lipschitz with constant $\text{Lip}(u) + \|Q\| + \|h\| + 4(1+R)$.

Let us then estimate $\partial_\mu f_\mu$:

$$\partial_\mu f_\mu(q, p, \nu) = h \cdot \partial_Q S_s^t(Q + \mu h, \xi).$$

If $\xi_\mu = (q_\mu, p_\mu, \nu_\mu)$ is a critical point of f_μ , Proposition 2.4 gives on one hand that $\|P_s^t(q_\mu, p_\mu)\| \leq e^{C(t-s)}(1+L) - 1$, and Proposition 2.1, on the other hand, gives that $\partial_Q S_s^t(Q + \mu h, \xi_\mu) = P_s^t(q_\mu, p_\mu)$.

To sum it up, we have just proved that $\|\partial_\mu f_\mu\| \leq \|h\|(e^{C(t-s)}(1+L) - 1)$ for each critical point of f_μ . This implies that $|\sigma(f_1) - \sigma(f_0)| \leq \|h\|(e^{C(t-s)}(1+L) - 1)$ by Property 2.7-(5) of the selector, hence the result.

2. Let us show that $\|\mathbf{R}_s^{t'} u - \mathbf{R}_s^t u\|_\infty \leq C e^{2C(t-s)}(1+L)^2 |t' - t|$. It is enough to prove the result for $|t - t'| < \delta_1/2$. We may therefore assume that (t_1, \dots, t_N) is a subdivision suitable both between s and t and between s and t' , since the choice of the subdivision does not change the value of the variational operator \mathbf{R} (see Proposition 2.14).

Let us fix Q , $t' < t$ and s and define $f_\mu(\xi) = S_s^\mu u(Q, \xi)$ for μ in $[t', t]$. The aim is to estimate $|\mathbf{R}_s^t u(Q) - \mathbf{R}_s^{t'} u(Q)| = |\sigma(f_t) - \sigma(f_{t'})|$.

By Proposition 2.3, the family f_μ is as required in Property 2.7-(5), thanks to the fact that the nondegenerate quadratic form \mathcal{Z} does not depend on t ($= \mu$).

If $\xi_\mu = (q_\mu, p_\mu, \nu_\mu)$ is a critical point of f_μ , Proposition 2.2-(1) gives on one hand that $\partial_\mu S_s^\mu(Q, \xi_\mu) = -H(\mu, Q, P_s^\mu(q_\mu, p_\mu))$ and Proposition 2.4 gives on the other hand that $\|P_s^\mu(q_\mu, p_\mu)\| \leq e^{C(\mu-s)}(1+L) - 1$.

By Hypothesis 1.1, we hence get that

$$|\partial_\mu S_s^\mu(Q, \xi_\mu)| \leq C(1 + \|P_s^\mu(q_\mu, p_\mu)\|)^2 \leq C e^{2C(\mu-s)}(1+L)^2.$$

To sum it up, we have just proved that $\|\partial_\mu f_\mu\| \leq C e^{2C(t-s)}(1+L)^2$ for each μ in $[t', t]$ and each critical point of f_μ . Property 2.7-(5) hence states that $\mu \mapsto \sigma(f_\mu)$ is Lipschitz with constant $C e^{2C(t-s)}(1+L)^2$ on $[t', t]$, hence the result.

3. Let us show that $\|\mathbf{R}_{s'}^t u - \mathbf{R}_s^t u\|_\infty \leq C(1+L)^2|s' - s|$. Again we may assume that $|s - s'|$ is small enough to choose a subdivision suitable both between s and t and between s' and t .

Let us fix Q , t and $s \leq s'$ and define $f_\mu(\xi) = S_\mu^t u(Q, \xi)$ for μ in $[s, s']$. The aim is to estimate $|\mathbf{R}_{s'}^t u(Q) - \mathbf{R}_s^t u(Q)| = |\sigma(f_{s'}) - \sigma(f_s)|$.

By Proposition 2.3, the family f_μ is, again, as required in Property 2.7-(5).

If $\xi_\mu = (q_\mu, p_\mu, \nu_\mu)$ is a critical point of f_μ , Proposition 2.2-(1) gives on one hand that $\partial_\mu S_\mu^t(Q, \xi_\mu) = H(\mu, q_\mu, p_\mu)$ and Proposition 2.1 on the other hand that $\|p_\mu\| \leq L$.

By Hypothesis 1.1, we hence get that

$$|\partial_\mu S_\mu^t(Q, \xi) \leq C(1+L)^2.$$

To sum it up, we have just proved that $\|\partial_\mu f_\mu\| \leq C(1+L)^2$ for each μ in $[s, s']$ and each critical point of f_μ , hence $\mu \mapsto \sigma(f_\mu)$ is Lipschitz with constant $C(1+L)^2$ on $[s, s']$ and the result holds.

4. Let us show that $\forall Q \in \mathbb{R}^d, |\mathbf{R}_s^t u(Q) - \mathbf{R}_s^t v(Q)| \leq \|u - v\|_{\bar{B}(Q, (e^{C(t-s)} - 1)(1+L))}$.

For Q fixed, let us again define $f_\mu = S_\mu^t((1-\mu)u + \mu v)(Q, \cdot)$ for μ in $[0, 1]$. The aim is to estimate $|\mathbf{R}_s^t v(Q) - \mathbf{R}_s^t u(Q)| = |\sigma(f_1) - \sigma(f_0)|$.

By Proposition 2.3, since $(1-\mu)u + \mu v$ is L -Lipschitz, the family f_μ is, again, as required in Property 2.7-(5). Let us then estimate $\partial_\mu f_\mu$:

$$\partial_\mu f_\mu(q, p, \nu) = v(q) - u(q).$$

If $\xi_\mu = (q_\mu, p_\mu, \nu_\mu)$ is a critical point of f_μ , Proposition 2.4 gives that q_μ belongs to $B(Q, (e^{C(t-s)} - 1)(1+L))$, so that $\|\partial_\mu f_\mu\| \leq \|u - v\|_{\bar{B}(Q, (e^{C(t-s)} - 1)(1+L))}$ for each critical point of f_μ , hence the result.

Remark 2.24. The proof of the alternative Proposition 1.18 is contained here: if $u \leq v$ on $B(Q, (e^{C(t-s)} - 1)(1+L))$, then $\partial_\mu f_\mu(q, p, \nu) = v(q) - u(q) \geq 0$ for each critical point of f_μ , hence $\mathbf{R}_s^t v(Q) - \mathbf{R}_s^t u(Q) = \sigma(f_1) - \sigma(f_0) \geq 0$.

If u is only Lipschitz with constant L , for all $\varepsilon > 0$ we may find a C^1 and L -Lipschitz function u_ε such that $\|u - u_\varepsilon\|_\infty \leq \varepsilon$, and then by weak contraction (Proposition 2.15) $\mathbf{R}_s^t u - \mathbf{R}_s^t u_\varepsilon$ is also bounded by ε for each $s \leq t$. Writing the previous results for u_ε and then letting ε tend to zero gives us the wanted estimates.

If H is not fiberwise compactly supported, let us fix L , T , and $\delta > 0$ and take a Hamiltonian H_δ in $\mathcal{H}_0^{C(1+\delta)}$ that coincides with H on $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(1+\delta)T}(1+L) - 1)$ as in Definition 2.18, so that if u is L -Lipschitz and $0 \leq s \leq t \leq T$, $\mathbf{R}_s^t u = \mathbf{R}_{s, H_\delta}^t u$.

The previous Lipschitz estimates, applied to $\mathbf{R}_{s, H_\delta}^t$, give that:

1. $\mathbf{R}_s^t u$ is Lipschitz with constant $\text{Lip}(\mathbf{R}_s^t u) \leq e^{C(1+\delta)(t-s)}(1+L) - 1$,
2. $\|\mathbf{R}_s^{t'} u - \mathbf{R}_s^t u\|_\infty \leq C(1+\delta)e^{2C(1+\delta)(t-s)}(1+L)^2|t' - t|$,
3. $\|\mathbf{R}_{s'}^t u(Q) - \mathbf{R}_s^t u(Q)\|_\infty \leq C(1+\delta)(1+L)^2|s' - s|$,
4. $|\mathbf{R}_s^t u(Q) - \mathbf{R}_s^t v(Q)| \leq \|u - v\|_{\bar{B}(Q, (e^{C(1+\delta)(t-s)} - 1)(1+L))}$,

and we conclude the proof by letting δ tend to 0. \square

Let us end this section with the analogous proof of Proposition 1.17, which describes the dependence of the constructed operator with respect to the Hamiltonian.

Proof of Proposition 1.17. Let H_0 and H_1 be two C^2 Hamiltonians satisfying Hypothesis 1.1 with constant C , u be a L -Lipschitz function, Q be in \mathbb{R}^d and $s \leq t$. We are going to show that

$$|\mathbf{R}_{s,H_1}^t u(Q) - \mathbf{R}_{s,H_0}^t u(Q)| \leq (t-s) \|H_1 - H_0\|_{\bar{V}},$$

where $\bar{V} = [s, t] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1+L)) \times \bar{B}(0, e^{C(t-s)}(1+L) - 1)$.

Let us first assume that u is a C^1 function, and that H_0 and H_1 are fiberwise compactly supported. Let us define $H_\mu = (1-\mu)H_0 + \mu H_1$ for μ in $[0, 1]$ and observe that H_μ is in \mathcal{H}_0^C , and that there exists a $R > 0$ such that $H_\mu(t, q, p) = 0$ for all $\|p\| \geq R$ and all μ in $[0, 1]$. Let us denote by $\phi_{s,\mu}^t = (Q_{s,\mu}^t, P_{s,\mu}^t)$ the Hamiltonian flow for H_μ .

Let us fix Q and h in \mathbb{R}^d and define $f_\mu(\xi) = S_{s,H_\mu}^t u(Q, \xi)$ for μ in $[0, 1]$. The aim is to estimate $|\mathbf{R}_{s,H_1}^t u(Q) - \mathbf{R}_{s,H_0}^t u(Q)| = |\sigma(f_1) - \sigma(f_0)|$.

Proposition 2.3 states that the homotopy f_μ is of the form required in the condition 2.7-(5): $f_\mu(\xi) = \mathcal{Z}(\xi) + \ell_\mu(\xi)$, where the family (ℓ_μ) is equi-Lipschitz with constant $\text{Lip}(u) + \|Q\| + 4(1+R)$.

Let $\xi = (q, p, \nu)$ be a critical point of f_μ . On the one hand, Proposition 2.4 gives that $\phi_{s,\mu}^\tau(q, p)$ is in $B(Q, (e^{C(t-s)} - 1)(1+L)) \times B(0, e^{C(t-s)}(1+L) - 1)$ for all $s \leq \tau \leq t$, since H_μ satisfies Hypothesis 1.1 with constant C . On the other hand, Proposition 2.2-(2) gives that

$$\partial_\mu f_\mu(\xi) = \partial_\mu S_{s,H_\mu}^t u(Q, q, p, \nu) = - \int_s^t \partial_\mu H_\mu(\tau, \phi_{s,\mu}^\tau(q, p)) d\tau.$$

Since $\partial_\mu H_\mu = H_1 - H_0$, we have just proved that $\|\partial_\mu f_\mu\| \leq (t-s) \|H_0 - H_1\|_V$ for each critical point of f_μ . This implies that $|\sigma(f_1) - \sigma(f_0)| \leq (t-s) \|H_0 - H_1\|_V$ by Property 2.7-(5) of the selector, hence the result.

Remark 2.25. The proof of the alternative Proposition 1.18 is contained here: if $H_0 \leq H_1$ on V , then $\partial_\mu f_\mu(\xi) = - \int_s^t (H_1 - H_0)(\tau, \phi_{s,\mu}^\tau(q, p)) \leq 0$ for each critical point of f_μ , hence $\mathbf{R}_{s,H_1}^t u(Q) - \mathbf{R}_{s,H_0}^t u(Q) = \sigma(f_1) - \sigma(f_0) \leq 0$.

If u is only Lipschitz with constant L , for all $\varepsilon > 0$ we may find a C^1 and L -Lipschitz function u_ε such that $\|u - u_\varepsilon\|_\infty \leq \varepsilon$, and then by continuity (Proposition 2.15) $\mathbf{R}_s^t u - \mathbf{R}_s^t u_\varepsilon$ is also bounded by ε for each $s \leq t$. Writing the previous results for u_ε and then letting ε tend to zero gives us the wanted estimates.

If H_0 and H_1 are not fiberwise compactly supported, take $\delta > 0$ and $H_{0,\delta}$ (resp. $H_{1,\delta}$) in $\mathcal{H}_0^{C(1+\delta)}$ coinciding with H_0 (resp. with H_1) on $\mathbb{R} \times \mathbb{R}^d \times B(0, e^{C(1+\delta)(t-s)}(1+L) - 1)$ as in Definition 2.18, so that $\mathbf{R}_{s,H_0}^t u = \mathbf{R}_{s,H_{0,\delta}}^t u$ and $\mathbf{R}_{s,H_1}^t u = \mathbf{R}_{s,H_{1,\delta}}^t u$. The previous work applied to $H_{0,\delta}$ and $H_{1,\delta}$ gives that

$$|\mathbf{R}_{s,H_1}^t u(Q) - \mathbf{R}_{s,H_0}^t u(Q)| = \left| \mathbf{R}_{s,H_{1,\delta}}^t u(Q) - \mathbf{R}_{s,H_{0,\delta}}^t u(Q) \right| \leq (t-s) \underbrace{\|H_{1,\delta} - H_{0,\delta}\|_{V_\delta}}_{=\|H_1 - H_0\|_{V_\delta}},$$

where $V_\delta = [s, t] \times B(Q, (e^{C(1+\delta)(t-s)} - 1)(1+L)) \times B(0, e^{C(1+\delta)(t-s)}(1+L) - 1)$. The result is then obtained by letting δ tend to 0. \square

Let us add here the considerably simpler Lipschitz estimates obtained for integrable Hamiltonians, using Remark 2.6 instead of Proposition 2.4 in the previous proofs.

Addendum 2.26. *If $H(p)$ (resp. $\tilde{H}(p)$) satisfies Hypothesis 1.1 with constant C , then for $0 \leq s \leq s' \leq t' \leq t$ and u and v two L -Lipschitz functions,*

1. $\mathbf{R}_s^t u$ is L -Lipschitz,
2. $\|\mathbf{R}_s^{t'} u - \mathbf{R}_s^t u\|_\infty \leq C(1+L)^2|t' - t|$,
3. $\|\mathbf{R}_{s'}^t u - \mathbf{R}_s^t u\|_\infty \leq C(1+L)^2|s' - s|$,
4. $\forall Q \in \mathbb{R}^d, |\mathbf{R}_s^t u(Q) - \mathbf{R}_s^t v(Q)| \leq \|u - v\|_{\bar{B}(Q, C(t-s)(1+L))}$,
5. $\|\mathbf{R}_{s, \tilde{H}}^t u - \mathbf{R}_{s, H}^t u\|_\infty \leq (t-s)\|\tilde{H} - H\|_{\bar{B}(0, L)}$.

where $\bar{B}(Q, r)$ denotes the closed ball of radius r centered in Q and $\|u\|_K := \sup_K |u|$.

Chapter 3

Iterating the variational operator

Ce chapitre est consacré à la démonstration du théorème de convergence de l'opérateur variationnel itéré, voir Theorem 1.19. On donne les estimées uniformes de l'opérateur itéré dans la Proposition 3.2, afin de pouvoir y appliquer le théorème d'Arzelà-Ascoli (Theorem 3.9). On montre alors que la valeur d'adhérence obtenue est nécessairement l'opérateur de viscosité (voir Proposition 3.10), ce qui permet de conclure la preuve.

A variational operator does a priori not satisfy the Markov property (v) of Hypotheses 1.6, and in that case it cannot coincide with the viscosity operator. Yet we may obtain the viscosity operator from the variational operator we have just constructed by iterating it along a subdivision of the time space and letting then the maximal step of the subdivision tend to zero. Doing so preserves the monotonicity, additivity, regularity and compatibility properties of the operator and the limit operator satisfies the Markov property, hence is the viscosity operator.

3.1 Iterated operator and uniform Lipschitz estimates

Let us recall the definition of the iterated operator. We fix a sequence of subdivisions of $[0, \infty)$ $((\tau_i^N)_{i \in \mathbb{N}})_{N \in \mathbb{N}}$ such that for all N , $0 = \tau_0^N$, $\tau_i^N \xrightarrow{i \rightarrow \infty} \infty$ and $i \mapsto \tau_i^N$ is increasing. Assume also that for all N , $i \mapsto \tau_{i+1}^N - \tau_i^N$ is bounded a constant δ_N such that δ_N tends to zero when N tends to the infinite.

Definition 3.1. Let N be fixed and omitted in the notations. For t in \mathbb{R}_+ , denote by $i(t)$ the unique integer such that t belongs to $[\tau_{i(t)}, \tau_{i(t)+1})$. Now, if u is a Lipschitz function on \mathbb{R}^d , and $0 \leq s \leq t$, let us define the iterated operator at rank N by

$$R_{s,N}^t u = R_{\tau_{i(t)}}^t R_{\tau_{i(t)-1}}^{\tau_{i(t)}} \cdots R_s^{\tau_{i(s)+1}} u,$$

where R_s^t is any variational operator satisfying the Lipschitz estimate of Theorem 1.15.

Let us now sum up the Lipschitz estimates of the iterated operator: note that thanks to the semigroup form of Lipschitz constants for the non iterated operator in Theorem 1.15, the new estimates do not depend on N .

Proposition 3.2. *Let $0 \leq s \leq s' \leq t' \leq t \leq T$ and u and v two L -Lipschitz functions. The Lipschitz constants for the iterated operator are:*

1. $\text{Lip}(R_{s,N}^t u) \leq e^{CT}(1+L) - 1$,
2. $\|R_{s,N}^{t'} u - R_{s,N}^t u\|_\infty \leq C e^{2CT}(1+L)^2 |t' - t|$,
3. $\|R_{s',N}^t u - R_{s,N}^t u\|_\infty \leq C(1+L)^2 |s' - s|$,
4. $\forall Q \in \mathbb{R}^d, \left| R_{s,N}^t u(Q) - R_{s,N}^t v(Q) \right| \leq \|u - v\|_{\bar{B}(Q, (e^{CT}-1)(1+L))}$.

Proof. This whole proof consists in exploiting the results of Theorem 1.15 while keeping the Lipschitz estimates independent of N .

1. Since $\text{Lip}(R_s^t u) \leq e^{C(t-s)}(1 + \text{Lip}(u)) - 1$ and $R_{s,N}^t u = R_{\tau_i(t)}^t (R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u)$:

$$\begin{aligned} \text{Lip}(R_{s,N}^t u) &\leq e^{C(t-\tau_i(t))}(1 + \text{Lip}(R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u)) - 1 \\ &\leq e^{C(t-\tau_i(t))} e^{C(\tau_i(t)-\tau_i(t)-1)} (1 + \text{Lip}(R_{\tau_i(t)-2}^{\tau_i(t)-1} \cdots R_s^{\tau_i(s)+1} u)) - 1 \\ &\leq e^{C(t-\tau_i(t)+\tau_i(t)-\cdots-s)} (1 + \text{Lip}(u)) - 1 \\ &\leq e^{CT}(1+L) - 1. \end{aligned}$$

2. Assume that $0 \leq s \leq t' \leq t \leq T$. It is enough to prove the result for $|t - t'| \leq \delta_N$, and in that case either $i(t) = i(t')$, or $i(t) = i(t') + 1$. If $i(t) = i(t')$, then

$$\begin{aligned} \|R_{s,N}^t u - R_{s,N}^{t'} u\|_\infty &= \|R_{\tau_i(t)}^t (R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u) - R_{\tau_i(t)}^{t'} (R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u)\|_\infty \\ &\leq C e^{2C(t-\tau_i(t))} (1 + \text{Lip}(R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u))^2 |t' - t|. \end{aligned}$$

Now since $1 + \text{Lip}(R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u) \leq e^{C(\tau_i(t)-s)}(1+L)$,

$$\|R_{s,N}^t u - R_{s,N}^{t'} u\|_\infty \leq C e^{2C(t-s)}(1+L)^2 |t - t'| \leq C e^{2CT}(1+L)^2 |t - t'|.$$

Else, assume that $i(t) = i(t') + 1$. Then

$$\|R_{s,N}^t u - R_{s,N}^{t'} u\|_\infty = \|R_{s,N}^t u - R_{s,N}^{\tau_i(t)} u + R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u - R_{\tau_i(t)-1}^{t'} \cdots R_s^{\tau_i(s)+1} u\|_\infty$$

and we may use the previous case to estimate both quantities:

$$\begin{aligned} \|R_{s,N}^t u - R_{s,N}^{t'} u\|_\infty &\leq C e^{2C(t-s)}(1+L)^2 |t - \tau_i(t)| + C e^{2C(t-s)}(1+L)^2 |\tau_i(t) - t'| \\ &\leq C e^{2C(t-s)}(1+L)^2 |t - t'| \leq C e^{2CT}(1+L)^2 |t - t'| \end{aligned}$$

since in that case $t' \leq \tau_i(t) \leq t$.

3. Again, it is enough to prove the result for $|s - s'| \leq \delta_N$. We freely use a consequence of the estimate proved in the next point:

$$\|R_{s,N}^t u - R_{s,N}^t v\|_\infty \leq \|u - v\|_\infty$$

If $i(s') = i(s)$,

$$\begin{aligned} \|R_{s,N}^t u - R_{s',N}^t u\|_\infty &= \|R_{\tau_i(s)+1,N}^t R_s^{\tau_i(s)+1} u - R_{\tau_i(s)+1,N}^t R_{s'}^{\tau_i(s)+1} u\|_\infty \\ &\leq \|R_{s'}^{\tau_i(s)+1} u - R_s^{\tau_i(s)+1} u\|_\infty \leq C(1+L)^2 |s - s'|. \end{aligned}$$

If $i(s') = i(s) + 1$,

$$\begin{aligned} \|R_{s',N}^t u - R_{s,N}^t u\|_\infty &\leq \|R_{s',N}^t u - R_{\tau_i(s'),N}^t u\|_\infty + \|R_{\tau_i(s'),N}^t u - R_{s,N}^t u\|_\infty \\ &\leq C(1+L)^2 ((s' - i(s')) + (i(s') - s)) \leq C(1+L)^2 |s - s'|. \end{aligned}$$

4. Let Q be fixed. Note that $R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u$ and $R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} v$ are both Lipschitz with constant $(e^{C(\tau_i(t)-s)}(1+L) - 1)$. Then

$$\begin{aligned} & |R_{s,N}^t u(Q) - R_{s,N}^t v(Q)| \\ &= |R_{\tau_i(t)}^t \left(R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u \right) (Q) - R_{\tau_i(t)}^t \left(R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} v \right) (Q)| \\ &\leq \|R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} u - R_{\tau_i(t)-1}^{\tau_i(t)} \cdots R_s^{\tau_i(s)+1} v\|_{\bar{B}(Q, (e^{C(\tau_i(t)-s)} - 1)e^{C(\tau_i(t)-s)}(1+L))}. \end{aligned}$$

Estimating the Lipschitz constant of $R_{\tau_i(t)-2}^{\tau_i(t)-1} \cdots R_s^{\tau_i(s)+1} u$ and $R_{\tau_i(t)-2}^{\tau_i(t)-1} \cdots R_s^{\tau_i(s)+1} v$ gives the next step:

$$\begin{aligned} & |R_{s,N}^t u(Q) - R_{s,N}^t v(Q)| \\ &\leq \|R_{\tau_i(t)-2}^{\tau_i(t)-1} \cdots R_s u - R_{\tau_i(t)-2}^{\tau_i(t)-1} \cdots R_s v\|_{\bar{B}(Q, (e^{C(t-s)} - e^{C(\tau_i(t)-1-s)})(1+L))} \\ &\leq \cdots \leq \|u - v\|_{\bar{B}(Q, (e^{C(t-s)} - 1)(1+L))}. \end{aligned}$$

□

Let us gather the Lipschitz dependence in s and t to obtain an estimation of how non-Markov the iterated operator is:

Proposition 3.3. *Take $0 \leq s \leq r \leq t \leq T$ and u L -Lipschitz. Then for all integer N ,*

$$\|R_{s,N}^t u - R_{r,N}^r R_{s,N}^r u\|_\infty \leq 2Ce^{2CT} (1+L)^2 \delta_N$$

where δ_N is the upper bound of $i \mapsto \tau_{i+1}^N - \tau_i^N$.

Proof. Let us first show that if $s \leq r \leq t$, then

$$\|R_s^t u - R_r^r R_s^r u\|_\infty \leq 2Ce^{2C(t-s)} (1 + \text{Lip}(u))^2 |r - s|$$

for each Lipschitz function u . Since $R_s^s u = u$, we might write

$$\begin{aligned} \|R_s^t u - R_r^r R_s^r u\|_\infty &\leq \|R_s^t u - R_r^t u\|_\infty + \|R_r^t R_s^s u - R_r^t R_s^r u\|_\infty \\ &\leq C(1 + \text{Lip}(u))^2 |r - s| + \|R_s^s u - R_s^r u\|_\infty \\ &\leq C(1 + \text{Lip}(u))^2 |r - s| + Ce^{2C(r-s)} (1 + \text{Lip}(u))^2 |r - s| \\ &\leq C(1 + e^{2C(t-s)}) (1 + \text{Lip}(u))^2 |r - s| \\ &\leq 2Ce^{2C(t-s)} (1 + \text{Lip}(u))^2 |r - s|. \end{aligned}$$

The second line is obtained by applying the Lipschitz estimates w.r.t. s and u of Theorem 1.15, the third line by applying the Lipschitz estimate w.r.t. t (same Theorem).

Now, let us fix N and estimate $\|R_{s,N}^t u - R_{r,N}^r R_{s,N}^r u\|_\infty$. The fourth point of Proposition 3.2 implies that

$$\begin{aligned} \|R_{s,N}^t u - R_{r,N}^r R_{s,N}^r u\|_\infty &\leq \|R_{s,N}^{\tau_i(r)+1} u - R_r^{\tau_i(r)+1} R_{s,N}^r u\|_\infty \\ &\leq \|R_{\tau_i(r)}^{\tau_i(r)+1} R_{s,N}^{\tau_i(r)} u - R_r^{\tau_i(r)+1} R_{\tau_i(r)}^{\tau_i(r)} R_{s,N}^{\tau_i(r)} u\|_\infty. \end{aligned}$$

Using the previous result gives that

$$\|R_{s,N}^t u - R_{r,N}^r R_{s,N}^r u\|_\infty \leq 2Ce^{2C(\tau_i(r)+1-\tau_i(r))} \left(1 + \text{Lip}(R_{s,N}^{\tau_i(r)} u)\right)^2 |r - \tau_i(r)|$$

and since $(1 + \text{Lip}(R_{s,N}^{\tau_i(r)} u))^2 \leq e^{2C(\tau_i(r)-s)} (1 + \text{Lip}(u))^2$, we get

$$\|R_{s,N}^t u - R_{r,N}^t R_{s,N}^r u\|_\infty \leq 2C e^{2C(\tau_i(r)+1-s)} (1 + \text{Lip}(u))^2 |r - \tau_i(r)|.$$

Then the result comes by using the definition of δ_N . \square

Let us add a word on the dependence with respect to H , extending Proposition 1.17:

Proposition 3.4. *Let H_0 and H_1 be two C^2 Hamiltonians satisfying Hypothesis 1.1 with constant C , u be a L -Lipschitz function, Q be in \mathbb{R}^d and $s \leq t$. Then*

$$|R_{s,H_1,N}^t u(Q) - R_{s,H_0,N}^t u(Q)| \leq (t-s) \|H_1 - H_0\|_{\bar{V}},$$

where $\bar{V} = [s, t] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1+L)) \times \bar{B}(0, e^{C(t-s)}(1+L) - 1)$.

Proof. To lighten the notation, let us prove that for the non iterated operator,

$$\begin{aligned} & |R_{\tau,H_1}^t R_{s,H_1}^r u(Q) - R_{\tau,H_0}^t R_{s,H_0}^r u(Q)| \\ & \leq (t-s) \|H_1 - H_0\|_{[s,t] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1+L)) \times \bar{B}(0, e^{C(t-s)}(1+L) - 1)}. \end{aligned}$$

The result is then obtained for the iterated operator by induction on the number of steps between s and t .

For both H_0 and H_1 , $1 + \text{Lip}(R_s^r u) \leq e^{C(\tau-s)}(1+L)$ by Theorem 1.15. Hence, on the one hand, Proposition 1.17 gives that

$$\begin{aligned} & |R_{\tau,H_1}^t R_{s,H_1}^r u(Q) - R_{\tau,H_0}^t R_{s,H_1}^r u(Q)| \\ & \leq (t-\tau) \|H_1 - H_0\|_{[\tau,t] \times \bar{B}(Q, (e^{C(t-\tau)} - 1)e^{C(\tau-s)}(1+L)) \times \bar{B}(0, e^{C(t-\tau)} e^{C(\tau-s)}(1+L) - 1)} \\ & \leq (t-\tau) \|H_1 - H_0\|_{\bar{V}}. \end{aligned}$$

On the other hand, using the Lipschitz estimate with respect to u of Theorem 1.15,

$$|R_{\tau,H_0}^t R_{s,H_1}^r u(Q) - R_{\tau,H_0}^t R_{s,H_0}^r u(Q)| \leq \|R_{s,H_1}^r u - R_{s,H_0}^r u\|_{\bar{B}(Q, (e^{C(t-s)} - 1)e^{C(\tau-s)}(1+L))}$$

Proposition 1.17 gives that for each q of $\bar{B}(Q, (e^{C(t-s)} - 1)e^{C(\tau-s)}(1+L))$,

$$|R_{s,H_1}^r u(q) - R_{s,H_0}^r u(q)| \leq (\tau-s) \|H_1 - H_0\|_{[s,\tau] \times \bar{B}(q, (e^{C(\tau-s)} - 1)(1+L)) \times \bar{B}(0, e^{C(\tau-s)}(1+L) - 1)},$$

and then summing up the radius of the balls gives

$$\begin{aligned} & |R_{\tau,H_0}^t R_{s,H_1}^r u(Q) - R_{\tau,H_0}^t R_{s,H_0}^r u(Q)| \\ & \leq (\tau-s) \|H_1 - H_0\|_{[s,\tau] \times \bar{B}(Q, (e^{C(t-s)} - 1)(1+L)) \times \bar{B}(0, e^{C(\tau-s)}(1+L) - 1)} \\ & \leq (\tau-s) \|H_1 - H_0\|_{\bar{V}}. \end{aligned}$$

Summing up the two estimates concludes the proof. \square

3.2 Convergence towards the viscosity operator

In this section we prove that the iterated operator sequence $(R_{s,N}^t)_N$ converges to a limit operator when the maximal step of the subdivision tends to 0. To do so, we first use a compactness argument to get a converging subsequence (Theorem 3.9), then show that the limit of such a subsequence is the viscosity operator (Proposition 3.10) and finally prove Theorem 1.19 with the uniqueness of this operator.

Definition 3.5. Let $\|\cdot\|_{Lip}$ be the norm on the sets of real-valued Lipschitz functions on \mathbb{R}^d given by

$$\|u\|_{Lip} = |u(0)| + \text{Lip}(u).$$

Definition 3.6. We denote by $\mathcal{L}^L(K)$ the set of Lipschitz functions on \mathbb{R}^d supported by the compact set K and with Lipschitz norm $\|\cdot\|_{Lip}$ bounded by the constant L :

$$\mathcal{L}^L(K) = \left\{ u \in C^{0,1}(\mathbb{R}^d, \mathbb{R}) \mid \begin{array}{l} \text{supp}(u) \subset K \\ \|u\|_{Lip} \leq L \end{array} \right\}$$

Proposition 3.7. *The set $\mathcal{L}^L(K)$ is a compact set for the uniform norm.*

Proof. The Arzelà-Ascoli theorem immediately gives that the closure of $\mathcal{L}^L(K)$ is compact. Then, it is easy to check that $\mathcal{L}^L(K)$ is closed. Hence, it is compact. \square

Proposition 3.8. *For each $T > 0$, $R > 0$, $L > 0$, the family $\{(s, t, Q, u) \mapsto R_{s,N}^t u(Q)\}_N$ is equi-Lipschitz on the set $\{0 \leq s \leq t \leq T\} \times \bar{B}(0, R) \times \mathcal{L}^L(\bar{B}(0, R))$.*

Proof. It is enough to observe that the Lipschitz constants obtained in Proposition 3.2 depend only on T , R , L , and that if u and v are compactly supported Lipschitz functions,

$$\|R_{s,N}^t u - R_{s,N}^t v\| \leq \|u - v\|_\infty.$$

\square

Theorem 3.9. *There exists a subsequence N_k such that for all $0 \leq s \leq t$, $Q \in \mathbb{R}^d$, u Lipschitz function on \mathbb{R}^d , $R_{s,N_k}^t u(Q)$ has a limit when k tends to ∞ , denoted $\bar{R}_s^t u(Q)$. Furthermore, the sequence of functions $\{(s, t, Q) \mapsto R_{s,N_k}^t u(Q)\}_k$ converges uniformly towards $(s, t, Q) \mapsto \bar{R}_s^t u(Q)$ on every compact subset of $\{0 \leq s \leq t\} \times \mathbb{R}^d$.*

Proof. The first step consists in applying Arzelà-Ascoli theorem with (s, t, Q, u) living in the compact set $\{0 \leq s \leq t \leq T\} \times \bar{B}(0, R) \times \mathcal{L}^L(\bar{B}(0, R))$, where T , R and L are fixed. The second step is to get a subsequence working for all T , R and L . The third step consists in extending the result to Lipschitz functions which are not compactly supported.

First step. Since Proposition 3.8 gives that $\{(s, t, Q, u) \mapsto R_{s,N}^t v(Q)\}_N$ is equi-Lipschitz on $\{0 \leq s \leq t \leq T\} \times \bar{B}(0, R) \times \mathcal{L}^L(\bar{B}(0, R + CT))$, it is enough to prove that it is uniformly bounded at one point - for example $(s, s, Q, 0)$ - to gather all the conditions required to apply Arzelà-Ascoli theorem.

$$|R_{s,N}^s 0(Q)| = |0(Q)| = 0,$$

hence, there exists a subsequence N_k (a priori depending on T , R and L) such that the sequence $\{(s, t, Q, u) \mapsto R_{s,N_k}^t u(Q)\}_k$ converges uniformly to a limit $(s, t, Q, u) \mapsto \bar{R}_s^t u(Q)$ on the compact set $\{0 \leq s \leq t \leq T\} \times \bar{B}(0, R) \times \mathcal{L}^L(\bar{B}(0, R))$.

Second step. In this paragraph we will describe a subsequence by the diagonal process. Note that the first step also applies on every subsequence of $(R_{s,N}^t)_N$.

Let $T_i = R_i = L_i = i$ for each integer i .

For $i = 1$, let ψ^1 be the subsequence given by the Arzelà-Ascoli theorem for the sequence $(R_{s,N}^t)_{N \in \mathbb{N}}$ and the constants $T_1 = L_1 = R_1 = 1$.

For $i > 1$, let ψ^i be the subsequence given by the Arzelà-Ascoli theorem for the sequence $(R_{s,\psi^{i-1}(N)}^t)_{N \in \mathbb{N}}$ and the constants $T_i = L_i = R_i = i$.

Now define the diagonal subsequence $N_k = \psi^k(k)$: for all k , $(N_i)_{i \geq k}$ is extracted from ψ^k .

For each T, R, L , there exists i such that $T \leq i$, $R \leq i$ and $L \leq i$. Since $R_{s,\psi^i(k)}^t$ converges on $\{0 \leq s \leq t \leq i\} \times \bar{B}(0, i) \times \mathcal{L}^i(\bar{B}(0, i))$, it converges on $\{0 \leq s \leq t \leq T\} \times \bar{B}(0, R) \times \mathcal{L}^L(\bar{B}(0, R))$, and so does R_{s,N_k}^t since N_k is a subsequence of $\psi^i(k)$. Hence we have constructed a subsequence that works for all L, R, T positive constants. If \mathcal{L}_c denotes the set of compactly supported Lipschitz functions,

$$\bigcup_{T,L,R} \{0 \leq s \leq t \leq T\} \times \bar{B}(0, R) \times \mathcal{L}^L(\bar{B}(0, R)) = \{0 \leq s \leq t\} \times \mathbb{R}^d \times \mathcal{L}_c,$$

and the subsequence we have constructed converges for all $s \leq t$, $Q \in \mathbb{R}^d$ and u compactly supported Lipschitz function.

Third step. Now take T and R two constants and u a Lipschitz function on \mathbb{R}^d , with Lipschitz constant L . For all $\bar{L} > L$, we build a compactly supported \bar{L} -Lipschitz function \bar{u} such that $\bar{u} = u$ on $\bar{B}(0, R + (e^{CT} - 1)(1 + \bar{L}))$: to do so, let us take a compactly supported \mathcal{C}^1 function $\phi : \mathbb{R}^+ \rightarrow [0, 1]$ such that

$$\begin{cases} \phi = 1 \text{ on } [0, R + (e^{CT} - 1)(1 + \bar{L})], \\ |\phi'(x)| \leq \frac{L' - L}{|u(0)| + Lx} \quad \forall x \geq 0, \end{cases}$$

and $\bar{u}(q) = \phi(\|q\|) \cdot u(q)$.

If u is \mathcal{C}^1 , so is \bar{u} , and since $\|d_q(\phi(\|q\|))\| = |\phi'(\|q\|)| \leq \frac{L' - L}{|u(0)| + L\|q\|}$, the differential of \bar{u} is bounded by \bar{L} :

$$\|d\bar{u}(q)\| \leq \underbrace{\|d_q(\phi(\|q\|))\|}_{\leq \bar{L} - L} \cdot |u(q)| + \underbrace{|\phi(q)|}_{\leq 1} \cdot \underbrace{\|du(q)\|}_{\leq L} \leq \bar{L}.$$

If u is not \mathcal{C}^1 , one can show that \bar{u} is \bar{L} -Lipschitz by applying the mean value theorem to ϕ .

For all Q in the ball $\bar{B}(0, R)$, since u and \bar{u} are \bar{L} -Lipschitz and coincide on the ball centered in Q of radius $(e^{CT} - 1)(1 + \bar{L})$, the Lipschitz property 3.2-(4) gives

$$R_{s,N}^t \bar{u}(Q) = R_{s,N}^t u(Q) \quad \forall N \in \mathbb{N}, \forall 0 \leq s \leq t \leq T.$$

Since \bar{u} is a compactly supported function, $\{(s, t, Q) \mapsto R_{s,N_k}^t \bar{u}(Q)\}_k$ uniformly converges on $\{0 \leq s \leq t \leq T\} \times \bar{B}(0, R)$, and thus the same holds for $\{(s, t, Q) \mapsto R_{s,N_k}^t u(Q)\}_k$. \square

Proposition 3.10. *The limit operator \bar{R}_s^t is the viscosity operator: $\bar{R}_s^t = V_s^t$.*

Proof. 1. Monotonicity property follows from the monotonicity of R_s^t , for $s \leq t$.

2. Same thing for the additivity property.

3. Regularity: since the convergence of $\{(s, t, Q) \mapsto R_{s,N_k}^t v(Q)\}_k$ is uniform on every compact subset of $\{0 \leq s \leq t\} \times \mathbb{R}^d$, and the family is equi-Lipschitz in time and space, the limit satisfies that $\{\bar{R}_\tau^t u, t \in [\tau, T]\}$ is uniformly Lipschitz for each $\tau \leq T$ and $(t, q) \mapsto \bar{R}_\tau^t u(q)$ is locally Lipschitz on $(\tau, \infty) \times \mathbb{R}^d$.

4. Compatibility with Hamilton-Jacobi equation: Remark 1.13 and Proposition 2.23 give the compatibility property for the operator R_s^t . Hence if u is a Lipschitz C^2 solution of the Hamilton-Jacobi equation, for all N :

$$R_{s,N}^t u_s = R_{\tau_i(t)}^t \cdots \underbrace{R_s^{\tau_i(s)+1} u_s}_{=u_{\tau_i(s)+1}} = R_{\tau_i(t)}^t u_{\tau_i(t)} = u_t,$$

and the limit satisfies $\bar{R}_s^t u_s = u_t$.

5. Markov property: take u Lipschitz, and $0 \leq s \leq \tau \leq t \leq T$. Let us show the equality $\bar{R}_\tau^t \circ \bar{R}_s^\tau u = \bar{R}_s^t u$. Let Q be fixed in \mathbb{R}^d .

Since $Q \mapsto \bar{R}_s^\tau u(Q)$ is Lipschitz, $(R_{\tau,N_k}^t \bar{R}_s^\tau u(Q))_k$ converges to $\bar{R}_\tau^t \bar{R}_s^\tau u(Q)$.

Let us first show that $R_{\tau,N_k}^t R_{s,N_k}^\tau u(Q)$ tends to $\bar{R}_\tau^t \bar{R}_s^\tau u(Q)$.

$$\begin{aligned} |R_{\tau,N_k}^t R_{s,N_k}^\tau u(Q) - \bar{R}_\tau^t \bar{R}_s^\tau u(Q)| &\leq |R_{\tau,N_k}^t R_{s,N_k}^\tau u(Q) - R_{\tau,N_k}^t \bar{R}_s^\tau u(Q)| \\ &\quad + \underbrace{|R_{\tau,N_k}^t \bar{R}_s^\tau u(Q) - \bar{R}_\tau^t \bar{R}_s^\tau u(Q)|}_{\rightarrow 0}. \end{aligned}$$

Now, the uniform Lipschitz estimates of property 3.2-(4) give

$$|R_{\tau,N_k}^t R_{s,N_k}^\tau u(Q) - R_{\tau,N_k}^t \bar{R}_s^\tau u(Q)| \leq \|R_{s,N_k}^\tau u - \bar{R}_s^\tau u\|_{\bar{B}(Q,r)}$$

for some radius r depending only on C, T, L ; as the convergence is uniform on every compact subset of \mathbb{R}^d , the right hand side tends to 0 when k tends to ∞ .

Now, since $\delta_{N_k} \xrightarrow[k \rightarrow \infty]{} 0$, Proposition 3.3 implies that $R_{\tau,N_k}^t R_{s,N_k}^\tau u(Q)$ and $R_{s,N_k}^t u(Q)$ have the same limit, hence the conclusion:

$$\bar{R}_s^t u(Q) = \bar{R}_\tau^t \bar{R}_s^\tau u(Q).$$

□

Consequence 3.11. We have proved, for every Hamiltonian satisfying Hypothesis 1.1, that the viscosity operator exists. In particular, for such a Hamiltonian and for a Lipschitz initial condition, there exists a viscosity solution of (HJ) on $(0, \infty) \times \mathbb{R}^d$ that coincides with the initial condition at time 0, see Proposition 1.8.

Proof of Theorem 1.19. Since every subsequence of $R_{s,N}^t u$ admits a subsequence uniformly converging to the viscosity solution $V_s^t u$ on every compact set, the whole family $(R_{s,N}^t u)_N$ converge to $V_s^t u$ by uniqueness of the viscosity solution. □

The local Lipschitz estimates on the viscosity operator V and the local monotonicity properties stated in Proposition 1.21 are directly deduced from this uniform convergence and the estimates on the variational operator R . In the integrable case, the iterated operator $R_{s,N}^t$ satisfies the same Lipschitz estimate than the variational operator R_s^t (see Addendum 2.26), whence the following result.

Addendum 3.12. *If $H(p)$ (resp. $\tilde{H}(p)$) satisfies Hypothesis 1.1 with constant C , then for $0 \leq s \leq s' \leq t' \leq t$ and u and v two L -Lipschitz functions,*

1. $V_s^t u$ is L -Lipschitz,
2. $\|V_s^{t'} u - V_s^t u\|_\infty \leq C(1+L)^2 |t' - t|$,
3. $\|V_{s'}^t u - V_s^t u\|_\infty \leq C(1+L)^2 |s' - s|$,
4. $\forall Q \in \mathbb{R}^d, |V_s^t u(Q) - V_s^t v(Q)| \leq \|u - v\|_{\bar{B}(Q, C(t-s)(1+L))}$,
5. $\|V_{s, \tilde{H}}^t u - V_{s, H}^t u\|_\infty \leq (t-s) \|\tilde{H} - H\|_{\bar{B}(0, L)}$.

where $\bar{B}(Q, r)$ denotes the closed ball of radius r centered in Q and $\|u\|_K := \sup_K |u|$.

Chapter 4

The convex case

Le but de ce chapitre est de vérifier que la construction de l'opérateur variationnel proposée dans cette thèse donne le semi-groupe de Lax-Oleinik dans le cas d'un hamiltonien uniformément strictement convexe en la fibre. Pour voir cela, on décrit le semi-groupe de Lax-Oleinik à l'aide de la famille génératrice obtenue par la méthode des géodésiques brisées dans le cas convexe, et on fait le lien entre cette famille génératrice et celle obtenue dans le cas général en utilisant les propriétés du sélecteur de valeur critique.

The purpose of this chapter is to prove Theorem 1.29, that states in particular that for strictly convex Hamiltonians, the variational operator constructed in this thesis coincides with the Lax-Oleinik semi-group. To do so, we give a description of the Lax-Oleinik semigroup in terms of broken geodesics, and discuss the link between the so-called *Lagrangian generating family* involved in this description and the generating family used for general Hamiltonians.

4.1 The Lax-Oleinik semigroup with broken geodesics

The Lax-Oleinik semigroup defined by the equation (1.2) in the introduction may also be written as a finite dimensional optimization problem. If H is strictly uniformly convex w.r.t. p and satisfies Hypothesis 1.1, we fix $\delta_2 > 0$ such that $(q, p) \mapsto (q, Q_s^t(q, p))$ is a C^1 -diffeomorphism for each $|t - s| \leq \delta_2$ (see Proposition B.9).

Proposition 4.1. *If $s = t_0 \leq t_1 \leq \dots \leq t_N = t$ is a subdivision such that $t_{i+1} - t_i < \delta_2$ for all i , then*

$$T_s^t u(Q) = \min_{q, Q_0, \dots, Q_{N-1}} \mathcal{A}_s^t u(Q, q, Q_0, \dots, Q_{N-1}),$$

with the Lagrangian generating family \mathcal{A} defined by

$$\mathcal{A}_s^t u(Q, q, Q_0, \dots, Q_{N-1}) = u(q) + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} L(\tau, Q_{t_i}^\tau(Q_{i-1}, p_i), \partial_\tau Q_{t_i}^\tau(Q_{i-1}, p_i)) d\tau$$

where p_i is uniquely defined by $Q_{t_i}^{t_{i+1}}(Q_{i-1}, p_i) = Q_i$ and while denoting $q = Q_{-1}$ and $Q = Q_N$.

A proof of this statement can be found in [Ber12], Lemma 48 and Proposition 49.

The two next propositions gather properties of the Lagrangian generating family \mathcal{A} .

Proposition 4.2. *If H is uniformly strictly convex w.r.t. p , for δ_2 small enough,*

$$\mathcal{A}_s^t u(Q, q, Q_0, \dots, Q_{N-1}) = \max_{p, p_1, \dots, p_N} S_s^t u(Q, q, p, Q_0, \dots, p_N).$$

Proof. This is a direct consequence of Proposition B.12, since by definition

$$\mathcal{A}_s^t u(Q, q, Q_0, \dots, Q_{N-1}) = u(q) + A_s^t(q, Q_0, \dots, Q)$$

and

$$S_s^t u(Q, q, p, Q_0, \dots, p_N) = u(q) + G_s^t(p, Q_0, \dots, p_N, Q) + p \cdot (Q - q),$$

with the notations of Appendix B. □

Proposition 4.3. *If H satisfies Hypothesis 1.1 with constant C , is uniformly strictly convex w.r.t. p and $H(t, q, p) = \frac{\|p\|^2}{2}$ outside of a band $\mathbb{R} \times \mathbb{R}^d \times B(0, R)$, then the function $(q, Q_0, \dots, Q_{N-1}) \mapsto \mathcal{A}_s^t u(Q, q, Q_0, \dots, Q_{N-1})$ is coercive and in some \mathcal{Q}_m .*

Proof. We are first going to prove the result for $H(t, q, p) = \frac{\|p\|^2}{2}$. In that case, $L(t, q, v) = \frac{\|v\|^2}{2}$ and $Q_{t_i}^{t_{i+1}}(Q_{i-1}, p_i) = Q_i$ if and only if $Q_i = Q_{i-1} + (t_{i+1} - t_i)p_i$. Thus

$$\begin{aligned} \mathcal{A}_s^t u(Q, q, Q_0, \dots, Q_{N-1}) &= u(q) + \sum_{i=0}^N \int_{t_i}^{t_{i+1}} L(\tau, Q_{t_i}^\tau(Q_{i-1}, p_i), \partial_\tau Q_{t_i}^\tau(Q_{i-1}, p_i)) d\tau \\ &= u(q) + \frac{1}{2} \sum_{i=0}^N \int_{t_i}^{t_{i+1}} \frac{\|Q_i - Q_{i-1}\|^2}{(t_{i+1} - t_i)^2} d\tau = u(q) + \frac{1}{2} \sum_{i=0}^N \frac{\|Q_i - Q_{i-1}\|^2}{t_{i+1} - t_i} \end{aligned}$$

always denoting $q = Q_{-1}$ and $Q = Q_N$. To see that the considered function is coercive and in some \mathcal{Q}_m , we may then use for example the affine diffeomorphism $(q, Q_0, \dots, Q_{N-1}) \mapsto \left(\frac{Q_0 - q}{\sqrt{t_1 - s}}, \frac{Q_1 - Q_0}{\sqrt{t_2 - t_1}}, \dots, \frac{Q - Q_{N-1}}{\sqrt{t - t_N}} \right)$.

Now, if $H(t, q, p) = \frac{\|p\|^2}{2}$ outside of a band $\mathbb{R} \times \mathbb{R}^d \times B(0, R)$, and if \tilde{H} denotes the quadratic form $\tilde{H}(p) = \frac{\|p\|^2}{2}$, H and \tilde{H} satisfy the hypotheses of Proposition B.14 with constants C and $K = C(1 + R)$, and thus $\tilde{A}_s^t u - \mathcal{A}_s^t u = \tilde{A}_s^t - \mathcal{A}_s^t$ is a Lipschitz function of (q, Q_0, \dots, Q_{N-1}) . The previous part hence proves that the function $(q, Q_0, \dots, Q_{N-1}) \mapsto \mathcal{A}_s^t u(Q, q, Q_0, \dots, Q_{N-1})$ is coercive and in some \mathcal{Q}_m . □

4.2 Proof of Joukovskaia's theorem

To prove that the variational operator \mathbf{R}_s^t constructed in Chapter 2 is the viscosity operator, it is enough to prove that it satisfies the Markov property (v), see Remark 1.13. In that purpose, we need the critical value selector to satisfy the two additional following properties - properties that are actually satisfied by the minmax constructed in Appendix C.

Proposition 4.4. *There exists a critical value selector $\sigma : \bigcup_{m \in \mathbb{N}} \mathcal{Q}_m \rightarrow \mathbb{R}$, as defined in Proposition 2.7, that satisfies:*

1. $\sigma(-f) = -\sigma(f)$,
2. if $f(x, y)$ is a \mathcal{C}^2 function of \mathcal{Q}_m such that $\partial_y^2 f \geq cid$ for some $c > 0$, and if g defined by $g(x) = \min_y f(x, y)$ is in some $\mathcal{Q}_{\tilde{m}}$, then $\sigma(g) = \sigma(f)$.

We assume σ to be such a critical value selector.

Proof of Theorem 1.29. First step. We assume that the Hamiltonian H is uniformly strictly convex w.r.t. p ($\partial_p^2 H \geq \text{mid}$), satisfies Hypothesis 1.1 with some constant C and coincides with the quadratic form $Z(p) = \|p\|^2$ outside of a band $\mathbb{R} \times \mathbb{R}^d \times B(0, R)$. Then the variational operator constructed in Chapter 2 is the Lax-Oleinik operator: $\mathbf{R}_s^t = T_s^t$.

To see this, we apply the last item to the function $f(x, y) = S_s^t u(Q, q, p, Q_0, \dots, p_N)$ where $x = (q, Q_0, Q_1, \dots, Q_{N-1})$ and $y = (p, \dots, p_N)$. Proposition B.11 gives that $y \mapsto f(x, y)$ is uniformly strictly concave, since $S_s^t u(Q, q, p, Q_0, \dots, p_N) = u(q) + G_s^t(p, Q_0, \dots, p_N, Q) + p \cdot (Q - q)$ and Proposition 4.2 that

$$g(x) = \max_y f(x, y) = u(q) + \sum_{i=0}^N \int_{t_i}^{t_{i+1}} L(\tau, Q_{t_i}^\tau(Q_{i-1}, p_i), \partial_\tau Q_{t_i}^\tau(Q_{i-1}, p_i)) d\tau.$$

and Proposition 4.3 that g is a coercive function of some $Q_{\tilde{m}}$. Since g is coercive, Consequence 2.11 states that $\sigma(g) = \min g$, so we have that

$$T_s^t u(Q) = \min g = \sigma(g) = \sigma(f) = \mathbf{R}_s^t u(Q).$$

Second step. We only assume that the Hamiltonian H is uniformly strictly convex w.r.t. p ($\partial_p^2 H \geq \text{mid}$) and satisfies Hypothesis 1.1 with some constant C . It does not a priori coincides with a quadratic form at infinity.

Let us prove the Markov property: we fix u , $s \leq \tau \leq t$ and Q and we are going to show that $\mathbf{R}_\tau^t \mathbf{R}_s^\tau u(Q) = \mathbf{R}_s^t u(Q)$. If Z denotes the quadratic form $Z(p) = \|p\|^2$, we may choose $\delta > 0$ and build as in Definition 2.18 a Hamiltonian H_δ in $\mathcal{H}_Z^{C(1+\delta)}$ such that both $\mathbf{R}_s^t u(Q) = \mathbf{R}_{s, H_\delta}^t u(Q)$ and $\mathbf{R}_{\tau, H_\delta}^t \mathbf{R}_{s, H_\delta}^\tau u(Q) = \mathbf{R}_\tau^t \mathbf{R}_s^\tau u(Q)$. Addendum 2.20 states that H_δ can moreover be constructed uniformly strictly convex w.r.t. p .

The previous work applies to H_δ , and hence

$$\mathbf{R}_\tau^t \mathbf{R}_s^\tau u(Q) = \mathbf{R}_{\tau, H_\delta}^t \mathbf{R}_{s, H_\delta}^\tau u(Q) = T_{\tau, H_\delta}^t T_{s, H_\delta}^\tau u(Q) = T_{s, H_\delta}^t u(Q) = \mathbf{R}_{s, H_\delta}^t u(Q) = \mathbf{R}_s^t u(Q)$$

since T_{s, H_δ}^t is a semi-group. We hence showed that \mathbf{R}_s^t satisfies the Markov property (v).

The uniqueness of the viscosity operator concludes: $\mathbf{R}_s^t = V_s^t = T_s^t$.

Third step. If H is convex with respect to p and satisfies Hypothesis 1.1 with constant C , $H_\varepsilon(t, q, p) = H(t, q, p) + \frac{1}{2}\varepsilon\|p\|^2$ is uniformly strictly convex w.r.t. p ($\partial_p^2 H_\varepsilon \geq \varepsilon \text{id}$) and satisfies Hypothesis 1.1 with constant $C + \varepsilon$.

Now for all $\varepsilon \leq 1$, the estimates of Propositions 1.17 and 1.21 give, for all $s \leq t$ and Lipschitz function u :

$$\begin{aligned} \|\mathbf{R}_{s, H_\varepsilon}^t u - \mathbf{R}_{s, H}^t u\|_\infty &\leq (t-s)\|H_\varepsilon - H\|_V, \\ \|V_{s, H_\varepsilon}^t u - V_{s, H}^t u\|_\infty &\leq (t-s)\|H_\varepsilon - H\|_{\bar{V}}, \end{aligned}$$

where $V = \mathbb{R} \times \mathbb{R}^d \times \bar{B}(0, e^{(C+1)(t-s)}(1 + \text{Lip}(u)) - 1)$. In other words,

$$\begin{aligned} \|\mathbf{R}_{s, H_\varepsilon}^t u - \mathbf{R}_{s, H}^t u\|_\infty &\leq \frac{1}{2}\varepsilon(t-s) \left(e^{(C+1)(t-s)}(1 + \text{Lip}(u)) - 1 \right)^2, \\ \|V_{s, H_\varepsilon}^t u - V_{s, H}^t u\|_\infty &\leq \frac{1}{2}\varepsilon(t-s) \left(e^{(C+1)(t-s)}(1 + \text{Lip}(u)) - 1 \right)^2. \end{aligned}$$

The second step applied to H_ε states that $\mathbf{R}_{s, H_\varepsilon}^t u = V_{s, H_\varepsilon}^t u$, and hence letting ε tend to zero gives the conclusion: $\mathbf{R}_{s, H}^t u = V_{s, H}^t u$.

The result is obtained analogously in the concave case, where the Lax-Oleinik semigroup is defined as a maximum, see Remark B.13. \square

Chapter 5

Overview of the integrable case in dimension 1

Dans ce chapitre, on étudie le problème de Cauchy associé à un hamiltonien intégrable et à une donnée semi-concave qui présente une seule singularité, en dimension 1. L'objectif principal est d'expliquer les classifications proposées dans le paragraphe §1.4 et de prouver la Proposition 5.6 qui sert à la démonstration du Theorem 1.30 caractérisant les hamiltoniens pour lesquels l'opérateur de viscosité est variationnel.

La solution variationnelle est donnée en petit temps par la section minimale du front d'onde d'après le Theorem 1.24. On commence donc par étudier la structure du front d'onde, dont le comportement en petit temps (voir Proposition 5.2) suggère d'étudier le problème de Cauchy associé à la linéarisée de la condition initiale. La formule de Hopf appliquée à ce cas (Proposition 5.4) éclaire le lien entre la section minimale du front d'onde et l'enveloppe concave du hamiltonien. Les Proposition 5.6 et 5.11, avec les Addenda 5.8 et 5.9, prouvent la classification lorsque la condition d'entropie est strictement vérifiée par la donnée initiale, et le Theorem 5.12 donne la classification lorsque la condition d'entropie n'est pas vérifiée.

Par ailleurs on étudie un exemple pour lequel la condition d'entropie est vérifiée de manière dégénérée, appelée la *Perestroïka*, pour laquelle une estimation plus fine sur les dérivées en jeu est nécessaire pour décider si les deux types de solutions coïncident ou non en petit temps, voir Proposition 5.14. Enfin, le paragraphe §5.6 présente un exemple pour lequel la solution variationnelle et la solution de viscosité, différentes, peuvent être explicitées et graphiquement représentées (voir Figure 5.13), ainsi que leurs caractéristiques (voir Figure 5.14).

In this chapter, $H : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 Hamiltonian with second derivative bounded by C . It satisfies Hypothesis 1.1. The Hamiltonian flow is given by $\phi_s^t(q, p) = (q + (t-s)H'(p), p)$ and the action of a Hamiltonian trajectory depends only on the (constant) impulsion along the trajectory: $\mathcal{A}_s^t(\gamma) = (t-s)(pH'(p) - H(p))$.

The aim of this chapter is to prove the classification results announced in §1.4 for an initial condition with only one shock. We first present some properties of the wavefront for such an initial condition, and for the linearized problem, that will be useful in the further discussion.

5.1 Wavefront structure for an initial condition with one shock

By *shock*, we mean a continuous singularity with a change of derivative.

We denote by \mathcal{E}_B the sets of Lipschitz C^2 functions f on \mathbb{R} , with second derivative bounded by B , such that $f(0) = f'(0) = 0$.

In this chapter we take $p_1 < p_2$ and $f(q) = \begin{cases} f_1(q), & q \geq 0, \\ f_2(q), & q \leq 0, \end{cases}$ with f_1 and f_2 in \mathcal{E}_B and assume that the initial condition is of the form

$$u_0(q) = \min(p_1q, p_2q) + f(q) = \begin{cases} p_1q + f_1(q), & q \geq 0, \\ p_2q + f_2(q), & q \leq 0. \end{cases}$$

We denote by $\mathcal{F}_t \subset \mathbb{R}^2$ the wavefront at time t fixed (see (F')). Since u_0 is differentiable on $\mathbb{R} \setminus \{0\}$, its Clarke derivative is a point outside zero and the segment $[p_1, p_2]$ at zero. The wavefront is hence the union of three pieces \mathcal{F}_t^ℓ , \mathcal{F}_t^r and \mathcal{F}_t^0 respectively issued from the left part, the right part, and the singularity of the initial condition. A first parametrization follows directly from the wavefront definition:

$$\begin{aligned} \mathcal{F}_t^\ell &: \begin{cases} q + tH'(u'_0(q)), \\ u_0(q) + tu'_0(q)H'(u'_0(q)) - tH(u'_0(q)), \end{cases} & q < 0, \\ \mathcal{F}_t^r &: \begin{cases} q + tH'(u'_0(q)), \\ u_0(q) + tu'_0(q)H'(u'_0(q)) - tH(u'_0(q)), \end{cases} & q > 0, \\ \mathcal{F}_t^0 &: \begin{cases} tH'(p), \\ t(pH'(p) - H(p)), \end{cases} & p \in [p_1, p_2]. \end{aligned}$$

This parametrization allows to evaluate the slopes and convexity of the wavefront.

Proposition 5.1. *1. Slopes on the wavefront. If $H''(p) \neq 0$ and $t > 0$, the slope of \mathcal{F}_t^0 at the point of parameter p is p . If $t < 1/BC$, the slope of \mathcal{F}_t^r at the point of parameter q is $u'_0(q)$.*

2. Convexity of the right arm. If u_0 is convex (resp. concave) on $[0, \delta]$, then for $t < 1/BC$, the portion of \mathcal{F}_t^r parametrized by $q \in (0, \delta]$ is convex (resp. concave).

Proof. 1. If $(x(u), y(u))$ is the parametrization of a curve, the slope at the point of parameter u is given by $y'(u)/x'(u)$ when $x'(u)$ is nonzero. For \mathcal{F}_t^0 , $x'(p) = tH''(p)$ and $y'(p) = pH'(p)$, which proves the statement. For \mathcal{F}_t^r , if $t < 1/BC$, $x'(q) = 1 + tu''_0(q)H'(u'_0(q)) > 0$ since u''_0 and H' are respectively bounded by B and C , and the statement results from $y'(q) = u'_0(q)x'(q)$.

2. The convexity of \mathcal{F}_t^r at a point of parameter q is given by the sign of $\frac{x'(q)y''(q) - x''(q)y'(q)}{x'(q)^3}$. For $t < 1/BC$, $x'(q) > 0$ and as $y'(q) = u'_0(q)x'(q)$,

$$\frac{x'(q)y''(q) - x''(q)y'(q)}{x'(q)^3} = \frac{x'(u''_0x' + u'_0x'') - x''u'_0x'}{x'^3} = \frac{u''_0(q)}{x'(q)},$$

which proves the statement. □

The fact that \mathcal{F}_t^0 depends homothetically on t suggests to look for each $t > 0$ at the homothetic reduction of the wavefront at time t , where both coordinates are divided by t . We call it *reduced wavefront*, denote it by $\tilde{\mathcal{F}}_t$, and it admits the following parametrizations:

$$\begin{aligned} \tilde{\mathcal{F}}_t^\ell &: \begin{cases} \tilde{q} + H'(u'_0(t\tilde{q})), \\ \frac{u_0(t\tilde{q})}{t} + u'_0(t\tilde{q})H'(u'_0(t\tilde{q})) - H(u'_0(t\tilde{q})), \end{cases} & \tilde{q} < 0, \\ \tilde{\mathcal{F}}_t^r &: \begin{cases} \tilde{q} + H'(u'_0(t\tilde{q})), \\ \frac{u_0(t\tilde{q})}{t} + u'_0(t\tilde{q})H'(u'_0(t\tilde{q})) - H(u'_0(t\tilde{q})), \end{cases} & \tilde{q} > 0, \\ \tilde{\mathcal{F}}_t^0 &: \begin{cases} H'(p), \\ pH'(p) - H(p), \end{cases} & p \in [p_1, p_2]. \end{aligned}$$

This reduced wavefront admits a non trivial limit when t tends to 0.

Proposition 5.2. *The reduced wavefront tends pointwise when t tends to 0 to the reduced wavefront associated with the linearized function of u_0 at zero, i.e. $\min(p_1q, p_2q)$:*

$$\begin{aligned} \tilde{\mathcal{F}}_t^\ell &: \begin{cases} \tilde{q} + H'(u'_0(t\tilde{q})), \\ \frac{u_0(t\tilde{q})}{t} + u'_0(t\tilde{q})H'(u'_0(t\tilde{q})) - H(u'_0(t\tilde{q})), \end{cases} & \tilde{q} < 0, \\ \tilde{\mathcal{F}}_t^r &: \begin{cases} \tilde{q} + H'(u'_0(t\tilde{q})), \\ \frac{u_0(t\tilde{q})}{t} + u'_0(t\tilde{q})H'(u'_0(t\tilde{q})) - H(u'_0(t\tilde{q})), \end{cases} & \tilde{q} > 0, \\ \tilde{\mathcal{F}}_t^0 &: \begin{cases} H'(p), \\ pH'(p) - H(p), \end{cases} & p \in [p_1, p_2], \\ \xrightarrow{t \rightarrow 0} \mathcal{F}^\ell &: \begin{cases} \tilde{q} + H'(p_2), \\ p_2\tilde{q} + p_2H'(p_2) - H(p_2), \end{cases} & \tilde{q} < 0, \\ \mathcal{F}^r &: \begin{cases} \tilde{q} + H'(p_1), \\ p_1\tilde{q} + p_1H'(p_1) - H(p_1), \end{cases} & \tilde{q} > 0, \\ \mathcal{F}^0 &: \begin{cases} H'(p), \\ pH'(p) - H(p), \end{cases} & p \in [p_1, p_2]. \end{aligned}$$

The parametrization of the limit shows explicitly that \mathcal{F}^r and \mathcal{F}^ℓ are two straight half-lines with respective slopes p_1 and p_2 . The convergence is illustrated in Figure 5.1.

The method of characteristics gives that the left and right arms are the graph of classical solutions of the (HJ) equation. More precisely, since $q \mapsto p_1q + f_1(q)$ and $q \mapsto p_2q + f_2(q)$ are \mathcal{C}^2 functions with second derivative bounded by B , and H'' is bounded by C :

Proposition 5.3. *There exists on $[0, 1/BC] \times \mathbb{R}^d$ a unique \mathcal{C}^2 solution of the (HJ) equation u_ℓ (resp. u_r) for the initial condition $q \mapsto p_1q + f_1(q)$ (resp. $q \mapsto p_2q + f_2(q)$). Then \mathcal{F}_t^ℓ (resp. \mathcal{F}_t^r) coincides with the graph of $u_\ell(t, \cdot)$ (resp. $u_r(t, \cdot)$) on $(-\infty, tH'(p_2))$ (resp. on $(tH'(p_1), \infty)$).*

5.2 Homogeneous initial condition

In view of Proposition 5.2, we study the case of the homogeneous concave initial condition $u_0(q) = \min(p_1q, p_2q)$, with $p_1 < p_2$. We still denote by \mathcal{F}_t^ℓ , \mathcal{F}_t^r and \mathcal{F}_t^0 the three pieces of wavefront respectively issued from the left part, the right part, and the singularity of the initial condition. The parametrization stated in Proposition 5.2 shows that \mathcal{F}_t^ℓ and \mathcal{F}_t^r are half-lines, and that the whole wavefront is homothetic with respect to t . We will hence use the notations $\mathcal{F}_t^r = t\mathcal{F}^r$, $\mathcal{F}_t^0 = t\mathcal{F}^0$ and $\mathcal{F}_t^\ell = t\mathcal{F}^\ell$ to keep in mind this fact.

We denote by \widehat{H} the concave envelope of H on the set $[p_1, p_2]$. It is a \mathcal{C}^1 function on $[p_1, p_2]$. Proposition 1.27 has a particularly simple counterpart in this framework, and explicits the link between the minimal section of the wavefront and the concave envelope of H .

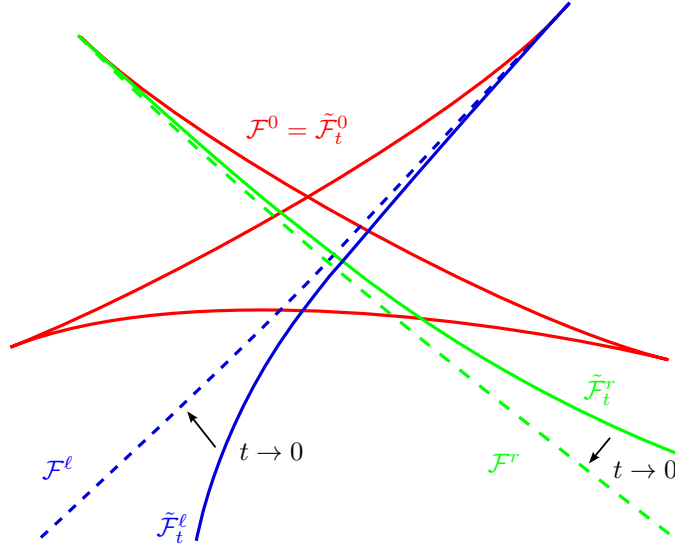


Figure 5.1: Asymptotic behaviour of the homothetically reduced wavefront

Proposition 5.4. For all $t \geq 0$,

$$V_0^t u_0(q) = R_0^t u_0(q) = \min_{p \in [p_1, p_2]} pq - tH(p) = \min_{p \in [p_1, p_2]} pq - t\widehat{H}(p).$$

As a consequence, the graph of $R_0^t u_0$ may be parametrized as follows:

$$\begin{cases} q, & q < t\widehat{H}'(p_2), \\ p_2 q - t\widehat{H}(p_2), & \\ q, & q > t\widehat{H}'(p_1), \\ p_1 q - t\widehat{H}(p_1), & \\ t\widehat{H}'(p), & p \in [p_1, p_2]. \\ t(p\widehat{H}'(p) - \widehat{H}(p)), & \end{cases}$$

The Hopf formula implies that $(t, q) \mapsto R_0^t u_0(q)$ is concave and positively 1-homogeneous, meaning that $R_0^{\lambda t} u_0(\lambda q) = \lambda R_0^t u_0(q)$ for all $\lambda > 0$.

Proof. Proposition 1.27 gives directly the two first equalities: since u_0 is concave,

$$R_0^t u_0(q) = V_0^t u_0(q) = u_{Hopf}(t, q) = \inf_{p \in \mathbb{R}^d} \sup_{x \in \mathbb{R}^d} u_0(x) + p \cdot (q - x) - tH(p).$$

and since $u_0(q) = \min(p_1 q, p_2 q)$, $\sup_{x \in \mathbb{R}^d} u_0(x) - px = \begin{cases} 0 & \text{if } p \in [p_1, p_2], \\ +\infty & \text{else.} \end{cases}$

The fact that $\min_{p \in [p_1, p_2]} pq - tH(p) = \min_{p \in [p_1, p_2]} pq - t\widehat{H}(p)$ is then a classical convex analysis result. In other words, for any C^2 Hamiltonian \tilde{H} with bounded second derivative that coincides with \widehat{H} on $[p_1, p_2]$, $R_{0, H}^t u_0 = R_{0, \tilde{H}}^t u_0$. Since $\tilde{H}' = \widehat{H}'$ is nonincreasing on $[p_1, p_2]$, the wavefront associated with \tilde{H} is a graph, whence the parametrization. \square

Figures 5.2 and 5.3 illustrate the situation. The parameters indicated on the wavefront correspond to the parametrization of \mathcal{F}^0 , and they give for each point the slope of \mathcal{F}^0 (see Proposition 5.1). The part of \mathcal{F}^0 that appears in the minimal section of the wavefront is parametrized by the set $\{p \in (p_1, p_2) | H(p) = \widehat{H}(p)\}$, which for Figure 5.2 is $[p_1^*, p_3] \cup [p_4, p_2)$. The segment $[p_3, p_4]$ parametrizes a stationary point for the parametrization given in Proposition 5.4, which gives the red shock in the graph of the variational solution.

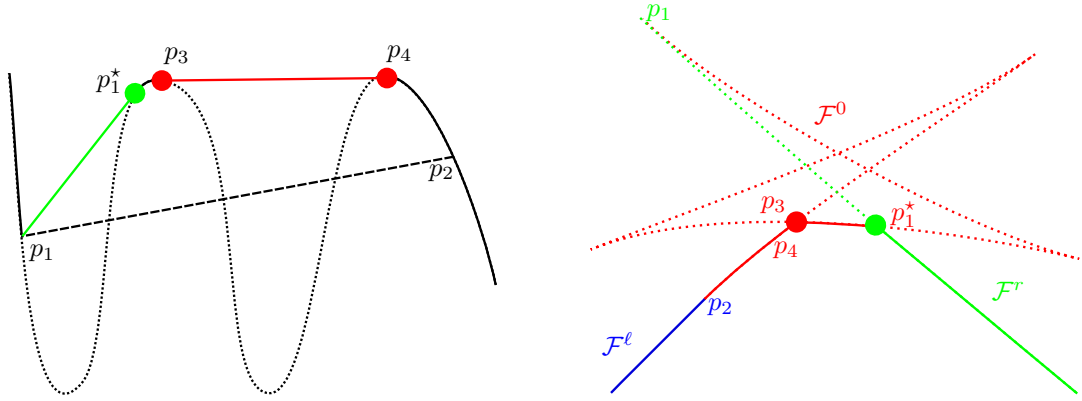


Figure 5.2: Concave envelope of H and minimal section of the wavefront.

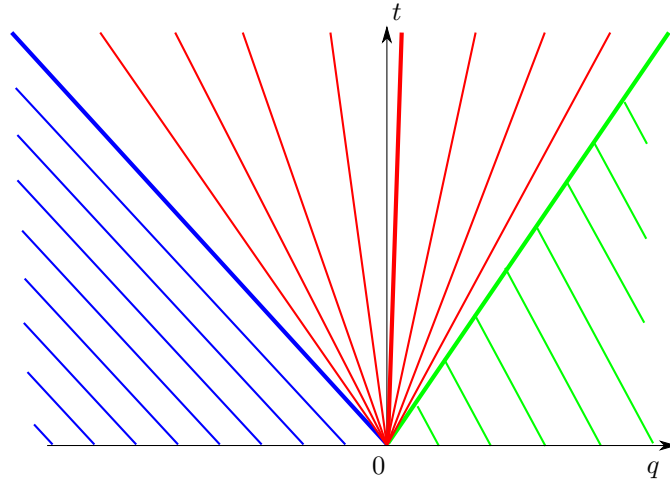


Figure 5.3: Characteristics representation for the variational solution of Figure 5.2. The thin lines are levels of $\partial_q u$, associated with the values p_2 (blue), p_1 (green), and any p in $[p_1^*, p_3] \cup (p_4, p_2)$ (red), and their slope is then equal to $1/H'(p)$. The thick blue line ($q = t\widehat{H}'(p_2) = tH'(p_2)$) represents the junction between \mathcal{F}^ℓ and \mathcal{F}^0 , the thick red line ($q = tH'(p_3) = tH'(p_4)$) represents the red shock of the inner front, and the thick green line ($q = t\widehat{H}'(p_1)$) represents the shock between \mathcal{F}^r and \mathcal{F}^0 .

The parametrization given in Proposition 5.4 implies the following statements.

- Proposition 5.5.** 1. *If the entropy condition is satisfied between p_1 and p_2 , \widehat{H}' is a constant equal to $\frac{H(p_2)-H(p_1)}{p_2-p_1}$, and then $R_0^t u_0$ is affine on $(t\widehat{H}', \infty)$ (resp. on $(-\infty, t\widehat{H}')$) with derivative p_1 (resp. p_2).*
2. *If the entropy condition is strictly denied, $\widehat{H}'(p_1) > \widehat{H}'(p_2)$, and then $R_0^t u_0$ is affine on $(t\widehat{H}'(p_1), \infty)$ (resp. on $(-\infty, \widehat{H}'(p_2))$) with derivative p_1 (resp. p_2). On the non trivial interval $[t\widehat{H}'(p_2), \widehat{H}'(p_1)]$, $R_0^t u_0$ is given by a so-called rarefaction wave issued from the singularity.*

5.3 Entropy condition strictly satisfied by the initial shock

We give here an elementary example where the variational and viscosity solutions do not coincide, which is a step towards the one-dimensional case of Theorem 6.1 (see §6). With the vocabulary of Definition 1.10 and Appendix F, we work on a specific case where the entropy condition is strictly satisfied between the derivatives at 0 of the initial condition, and the Lax condition is strictly satisfied on one side, and an equality on the other side, see Figure 5.4.

Recall that \mathcal{E}_B is the set of Lipschitz C^2 functions on \mathbb{R} , with second derivative bounded by B , such that $f(0) = f'(0) = 0$.

Proposition 5.6. *Let $H : \mathbb{R} \rightarrow \mathbb{R}$ be a C^2 Hamiltonian with bounded second derivative, and $p_1 < p_2$ be such that the entropy condition is strictly satisfied on $[p_1, p_2]$, $H''(p_2) < 0$, and moreover that*

$$H'(p_1) < \frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2).$$

Assume that $u_0(q) = \begin{cases} p_1 q + f_1(q), & q \geq 0, \\ p_2 q + f_2(q), & q \leq 0, \end{cases}$, where f_1 and f_2 are in \mathcal{E}_B and f_1 is strictly convex on \mathbb{R}_+ . Then, for every t small enough, the variational solution $(t, q) \mapsto R_0^t u_0(q)$ is not a viscosity solution.

We are going to show that under the assumptions of the proposition, the variational solution presents a shock between \mathcal{F}_t^0 and \mathcal{F}_t^r which denies Oleinik's entropy condition (see Definition 1.10) when t is small enough. Figure 5.4 presents an example of the situation. Note that the left arm of the initial condition matters only by its derivative at 0.

Lemma 5.7. *Under the assumptions of Proposition 5.6, there exists $\tau > 0$ such that the wavefront \mathcal{F}_t has a unique continuous section if $0 < t < \tau$, presenting a shock between \mathcal{F}_t^0 and \mathcal{F}_t^r .*

Proof. It is equivalent to prove the result for the reduced wavefront $\tilde{\mathcal{F}}_t$, where both coordinates are divided by t . Proposition 5.2 gives that this reduced wavefront tends when $t \rightarrow 0$ to the reduced wavefront associated with the linearized initial condition $\min(p_1 q, p_2 q)$.

$$\begin{aligned} \tilde{\mathcal{F}}_t^\ell &\xrightarrow[t \rightarrow 0]{} \mathcal{F}^\ell : \begin{cases} q + H'(p_2), \\ p_2 q + p_2 H'(p_2) - H(p_2), \end{cases} & q < 0, \\ \tilde{\mathcal{F}}_t^r &\xrightarrow[t \rightarrow 0]{} \mathcal{F}^r : \begin{cases} q + H'(p_1), \\ p_1 q + p_1 H'(p_1) - H(p_1), \end{cases} & q > 0, \\ \tilde{\mathcal{F}}_t^0 &= \mathcal{F}^0 : \begin{cases} H'(p), \\ p H'(p) - H(p), \end{cases} & p \in [p_1, p_2]. \end{aligned}$$

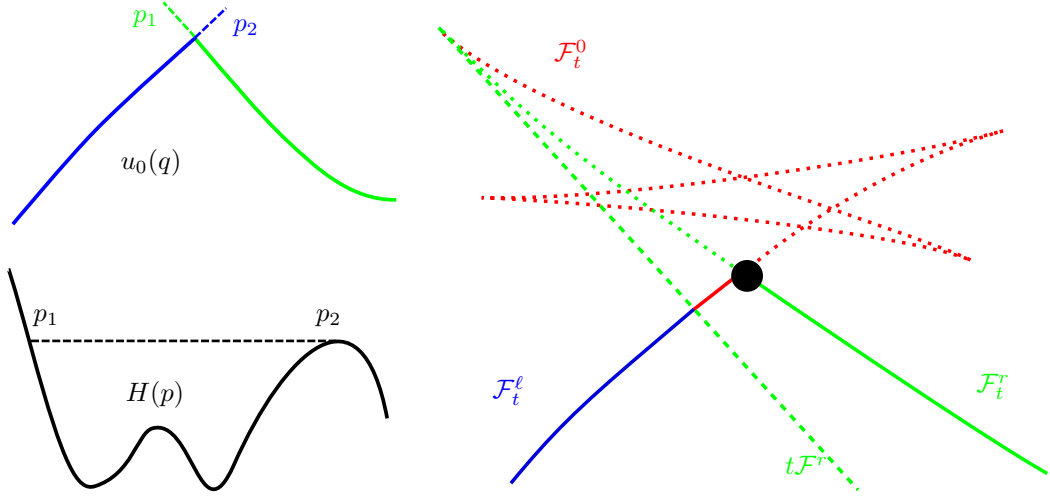


Figure 5.4: The variational solution, given by the minimal section of the wavefront, does not solve the (HJ) equation in the viscosity sense at the dot. The dashed green half line is the right piece of wavefront $t\mathcal{F}^r$ associated with the linearized function $\min(p_1q, p_2q)$.

Proposition 5.5 states that, since the entropy condition is satisfied, the minimal section of the limit front is affine on both components of $\mathbb{R} \setminus \left\{ \frac{H(p_2) - H(p_1)}{p_2 - p_1} \right\}$, with left slope p_2 and right slope p_1 . We denote by (Q, S) the point of shock of this minimal section and check that it is attained exactly once on \mathcal{F}^0 , belongs to \mathcal{F}^r and not to \mathcal{F}^ℓ .

It is attained on \mathcal{F}^r for the parameter $q = \frac{H(p_2) - H(p_1)}{p_2 - p_1} - H'(p_1)$ which is positive given the Lax strict inequality. The Lax equality $H'(p_2) = \frac{H(p_2) - H(p_1)}{p_2 - p_1}$ proves that it is not attained on \mathcal{F}^ℓ , but on \mathcal{F}^0 for the parameter $p = p_2$. It is not a double point of \mathcal{F}^0 , or else the entropy condition would not be strictly satisfied.

Since $H''(p_2) < 0$, there exists $\eta > 0$ such that $H'' < 0$ on $[p_2 - \eta, p_2]$, and the piece of \mathcal{F}^0 parametrized by $p \in (p_2 - \eta, p_2]$, denoted $\mathcal{F}_{(p_2 - \eta, p_2]}^0$, is immersed. Since \mathcal{F}^0 is compact, we may assume up to taking a smaller η that $\mathcal{F}_{(p_2 - \eta, p_2]}^0$ does not contain any double point either.

Now, by Proposition 5.2, the families of \mathcal{C}^1 curves $(\tilde{\mathcal{F}}_t^r)_{t \geq 0}$ and $(\tilde{\mathcal{F}}_t^\ell \cup \mathcal{F}_{(p_2 - \eta, p_2]}^0)_{t \geq 0}$ are continuous, when extended respectively to \mathcal{F}^r and $\mathcal{F}^\ell \cup \mathcal{F}_{(p_2 - \eta, p_2]}^0$ for $t = 0$. The intersection $\mathcal{F}^r \cap (\mathcal{F}^\ell \cup \mathcal{F}_{(p_2 - \eta, p_2]}^0)$, which is exactly the point $(0, 0)$, is transverse, and hence there exists $\tau > 0$ such that for all $t < \tau$, the intersection $\tilde{\mathcal{F}}_t^r \cap (\tilde{\mathcal{F}}_t^\ell \cup \mathcal{F}_{(p_2 - \eta, p_2]}^0)$ is exactly a point.

Proposition 5.1 states that since f_1 is strictly convex on \mathbb{R}_+ , \mathcal{F}_t^r and hence $\tilde{\mathcal{F}}_t^r$ are convex curves for all $t > 0$. Looking at the slope for a parameter $q \rightarrow 0$ shows that $\tilde{\mathcal{F}}_t^r$ admits \mathcal{F}^r as a tangent at its endpoint, and is hence positioned above \mathcal{F}^r . As a consequence, the intersection between $\tilde{\mathcal{F}}_t^r$ and $\mathcal{F}_t^\ell \cup \mathcal{F}_{(p_2 - \eta, p_2]}^0$ is necessarily an intersection between $\tilde{\mathcal{F}}_t^r$ and \mathcal{F}^0 . \square

Proof of Proposition 5.6. For all t , the graph of the variational solution is included in the wavefront \mathcal{F}_t . Lemma 5.7 states that \mathcal{F}_t has a unique continuous section for $t \leq \tau$, which implies that the variational solution, which is continuous, is given by this section. Lemma 5.7 states also that this section presents a shock between \mathcal{F}_0^t and \mathcal{F}_t^r .

Let us prove that the Lax condition is violated at this shock. A fortiori, Oleinik's entropy

condition is violated, which by Proposition 1.11 will imply that it is not a viscosity solution. For all t in $(0, \tau)$, the shock is given by parameters (q_t, p_t) , such that $q_t > 0$, $p_t \in [p_1, p_2]$ and

$$\begin{cases} q_t + tH'(u'_0(q_t)) &= tH'(p_t), \\ u_0(q_t) + tu'_0(q_t)H'(u'_0(q_t)) - tH(u'_0(q_t)) &= tp_tH'(p_t) - tH(p_t). \end{cases}$$

Injecting the first equation multiplied by $u'_0(q_t)$ into the second gives, after reorganization:

$$t(H(p_t) - H(u'_0(q_t)) - (p_t - u'_0(q_t))H'(p_t)) = q_t u'_0(q_t) - u_0(q_t).$$

The linear part of u_0 cancels in the right hand side, which equals $q_t f'_1(q_t) - f_1(q_t)$. The strict convexity of f_1 implies that $f'_1(h) > f_1(h)/h$ for all $h > 0$, hence the right hand side is strictly positive for $t > 0$, and as a consequence, for t in $(0, \tau)$,

$$H(p_t) - H(u'_0(q_t)) > (p_t - u'_0(q_t))H'(p_t). \quad (5.1)$$

By Proposition 5.1, the slopes at the shock are $u'_0(q_t)$ and p_t . This inequality hence proves that the variational solution breaches the Lax condition, hence Oleinik's entropy condition (see Definition 1.10), and consequently does not solve (HJ) in the viscosity sense at the intersection between \mathcal{F}_t^r and \mathcal{F}_t^0 for all t in $(0, \tau)$ by Proposition 1.11. \square

Addendum 5.8. *The conclusion of Proposition 5.6 still holds if f_1 is only strictly convex on some $[0, \delta]$.*

Proof. It is enough to prove that the shock of the previous proof is attained in \mathcal{F}_t^r at a parameter in $(0, \delta]$ for t small enough. If L denote the Lipschitz constant of u_0 , we denote by $A = \sup_{[-L, L]} |H'|$. The projection of the wavefront \mathcal{F}_t^0 on its first coordinate is contained in the ball $B(0, tA)$. The first coordinate of the shock is hence also bounded by tA , and if the shock belongs to \mathcal{F}_t^r , the parameter giving the shock is then bounded by $2tA$. In particular, if $t < \delta/2A$, the function f_1 is strictly convex on the domain parametrizing the part of the front preceding the shock, and the proofs of Lemma 5.7 and Proposition 5.6 both hold. \square

We now deal with what happens to Proposition 5.6 when the initial condition is concave on \mathbb{R}^+ .

Addendum 5.9. *Let $H : \mathbb{R} \rightarrow \mathbb{R}$ be a C^2 Hamiltonian with bounded second derivative, and $p_1 < p_2$ be such that the entropy condition is strictly satisfied on $[p_1, p_2]$, $H''(p_2) < 0$, and*

$$H'(p_1) < \frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2).$$

Assume that $u_0(q) = \begin{cases} p_1 q + f_1(q), & q \geq 0, \\ p_2 q + f_2(q), & q \leq 0, \end{cases}$, where f_1 and f_2 are elements of \mathcal{E}_B and f_1 is concave on \mathbb{R}^+ . Then the variational solution $(t, q) \mapsto R_0^t u_0(q)$ solves the Hamilton-Jacobi equation (HJ) in the viscosity sense for all t small enough.

Proof. The analogous of Lemma 5.7 when f_1 is concave is that \mathcal{F}_t has a unique continuous section, presenting a shock on $\mathcal{F}_t^r \cap \overline{\mathcal{F}_t^\ell}$. For all t in $(0, \tau)$, the shock is then given by parameters (q_t^r, q_t^ℓ) , such that $q_t^r > 0$, $q_t^\ell \leq 0$ and

$$\begin{cases} q_t^r + tH'(u'_0(q_t^r)) &= q_t^\ell + tH'(u'_0(q_t^\ell)), \\ u_0(q_t^r) + tu'_0(q_t^r)H'(u'_0(q_t^r)) - tH(u'_0(q_t^r)) &= u_0(q_t^\ell) + tu'_0(q_t^\ell)H'(u'_0(q_t^\ell)) - tH(u'_0(q_t^\ell)), \end{cases}$$

and the slopes at the shock are then $u'_0(q_t^r)$ and $u'_0(q_t^\ell)$ by Proposition 5.1.

Injecting the first equation multiplied by $u'_0(q_t^r)$ into the second gives, after reorganization:

$$t(H(u'_0(q_t^\ell)) - H(u'_0(q_t^r)) - (u'_0(q_t^\ell) - u'_0(q_t^r))H'(u'_0(q_t^\ell))) = (q_t^r - q_t^\ell)u'_0(q_t^r) - u_0(q_t^r) + u_0(q_t^\ell). \quad (5.2)$$

Note that since $q_t^r > 0$ and $q_t^\ell \leq 0$, if $A = \sup_{[-\text{Lip}(u_0), \text{Lip}(u_0)]} |H'|$, the first equation gives

$$-tA \leq q_t^r + tH'(u'_0(q_t^r)) = q_t^\ell + tH'(u'_0(q_t^\ell)) \leq tA,$$

and as a consequence $|q_t^\ell| \leq 2tA$ as well as $|q_t^r| \leq 2tA$ are arbitrarily small when t is small.

Since u_0 is concave on \mathbb{R}^+ and its left derivative p_2 is strictly smaller than its right derivative p_1 at zero, there exists $\delta > 0$ such that for all $q_- \in (-\delta, 0]$ and $q_+ > 0$,

$$u'_0(q_+) < \frac{u_0(q_+) - u_0(q_-)}{q_+ - q_-}.$$

Since q_t^ℓ is in $(-\delta, 0]$ for t small enough, the equation (5.2) hence gives the following Lax inequality:

$$\frac{H(u'_0(q_t^\ell)) - H(u'_0(q_t^r))}{(u'_0(q_t^\ell) - u'_0(q_t^r))} < H'(u'_0(q_t^\ell))$$

and we can apply Proposition F.4: the entropy condition is strictly satisfied on $[p_1, p_2]$, the Lax condition is strict at p_1 and an equality at p_2 (with $H''(p_2) < 0$), so there exists $\varepsilon > 0$ such that for $(\tilde{p}_1, \tilde{p}_2)$ in $[p_1 - \varepsilon, p_1 + \varepsilon] \times [p_2 - \varepsilon, p_2 + \varepsilon]$, if the Lax condition is satisfied on $[\tilde{p}_1, \tilde{p}_2]$, so is the entropy condition. Since q_t^ℓ and q_t^r are arbitrarily small for t small, $(u'_0(q_t^\ell), u'_0(q_t^r))$ is in $[p_1 - \varepsilon, p_1 + \varepsilon] \times [p_2 - \varepsilon, p_2 + \varepsilon]$ for t small enough, hence the entropy condition is satisfied by the shock of the variational solution. As \mathcal{F}_t^r and \mathcal{F}_t^ℓ are the graphs of classical solutions of (HJ) (see Proposition 5.3), Proposition 1.11 applies, and the variational solution solves then the Hamilton-Jacobi equation for small time. \square

We state Proposition 5.6 analogous result for a semiconvex initial condition, see Figure 5.5.

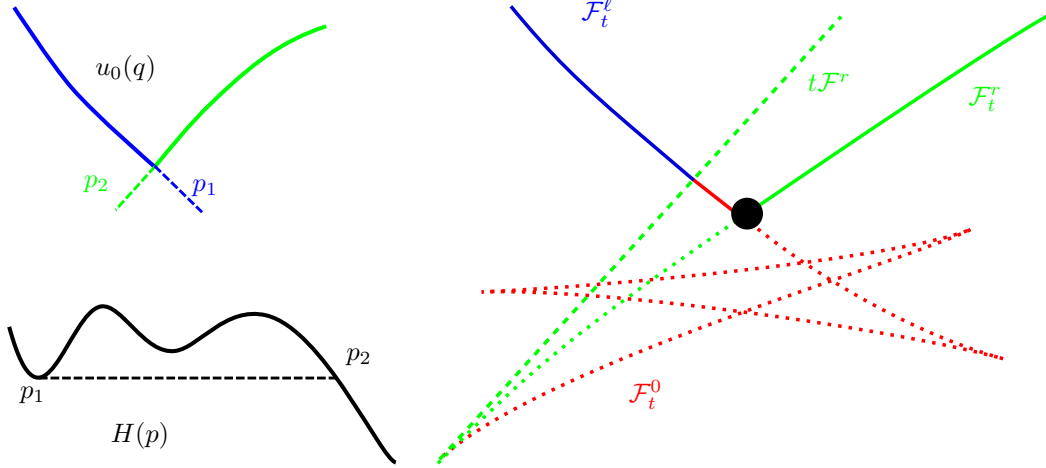


Figure 5.5: The variational solution, given by the maximal section of the wavefront, does not solve the (HJ) equation in the viscosity sense at the dot.

Proposition 5.10. *Let us assume that H is such that $p_1 < p_2$ be such that the reverse entropy condition (i.e. the Hamiltonian lies above the cord) is strictly satisfied on $[p_1, p_2]$, $H''(p_1) > 0$, and*

$$H'(p_1) = \frac{H(p_2) - H(p_1)}{p_2 - p_1} > H'(p_2).$$

Assume that $u_0(q) = \begin{cases} p_2q + f_1(q), & q \geq 0, \\ p_1q + f_2(q), & q \leq 0, \end{cases}$, where f_1 and f_2 are in \mathcal{E}_B and f_1 is strictly concave on \mathbb{R}_+ . Then for every t small enough, the variational solution $(t, q) \mapsto R_0^t u_0(q)$ is not a viscosity solution.

The next proposition states that if the entropy condition is strictly satisfied, and the Lax condition is either strict, or an equality on both sides, then the variational and viscosity solutions coincide for a small time.

Proposition 5.11. *If $p_1 < p_2$ be such that the entropy condition is strictly satisfied on $[p_1, p_2]$, and either*

$$H'(p_1) < \frac{H(p_2) - H(p_1)}{p_2 - p_1} < H'(p_2)$$

or

$$H'(p_1) = \frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2),$$

in which case we assume that $H''(p_1) < 0$ and $H''(p_2) < 0$.

Assume that $u_0(q) = \begin{cases} p_1q + f_1(q), & q \geq 0, \\ p_2q + f_2(q), & q \leq 0, \end{cases}$ with f_1 and f_2 in \mathcal{E}_B . Then the variational solution $(t, q) \mapsto R_0^t u_0(q)$ solves the Hamilton-Jacobi equation (HJ) in the viscosity sense for all t small enough.

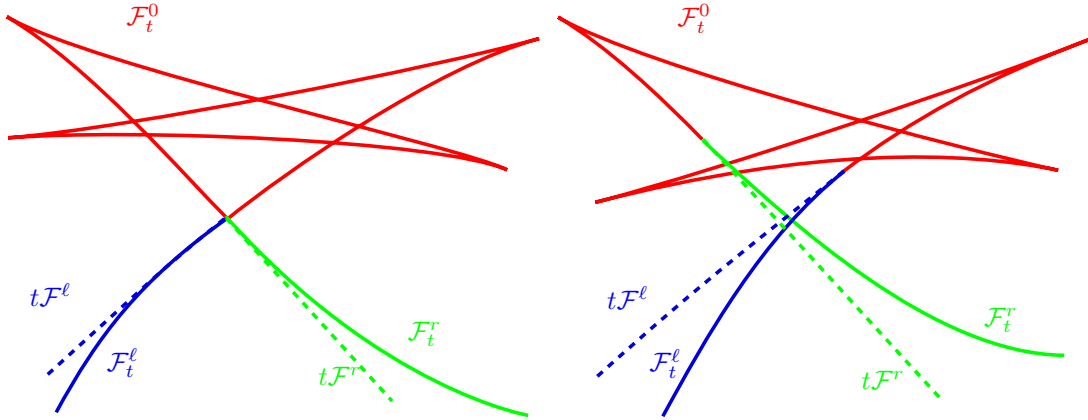


Figure 5.6: Left: example of wavefront when the entropy condition is strictly satisfied and the Lax condition is a double equality. Right: example of wavefront when the entropy condition and the Lax condition are strictly satisfied.

Proof. As in the previous proof, Proposition 5.5 states, since the entropy condition is satisfied, that the minimal section of the limit front is affine on $\mathbb{R} \setminus \left\{ \frac{H(p_2) - H(p_1)}{p_2 - p_1} \right\}$, with left slope p_2 and right slope p_1 .

In the case of the Lax equality $H'(p_1) = \frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2)$, the shock presented is attained at both endpoints of \mathcal{F}^0 by the parameters $p = p_1$ and $p = p_2$ and by no other parameter in (p_1, p_2) , or else the entropy condition would not be strictly satisfied, see Figure 5.6 left. The shock does not belong to \mathcal{F}^r or \mathcal{F}^ℓ . Since $H''(p_1)$ and $H''(p_2)$ are non zero, the double point parametrized by p_1 and p_2 in \mathcal{F}^0 is regular, and \mathcal{F}^0 is hence the union of two \mathcal{C}^1 curves on a neighbourhood of the shock. As a consequence, for small $t > 0$, the intersections $\mathcal{F}_t^r \cap \mathcal{F}_t^0$ and \mathcal{F}_t^ℓ are still empty, and the structure of the wavefront is preserved. Since the slopes of \mathcal{F}_t^r and \mathcal{F}_t^ℓ are respectively p_1 and p_2 at their endpoints, the entropy condition is still satisfied at the new point of shock. As \mathcal{F}_t^r and \mathcal{F}_t^ℓ are the graphs of classical solutions of (HJ) (see Proposition 5.3), Proposition 1.11 applies, and the variational solution solves then the Hamilton-Jacobi equation for small time.

In the case of the Lax strict inequality $H'(p_1) < \frac{H(p_2) - H(p_1)}{p_2 - p_1} < H'(p_2)$, the shock belongs to $\mathcal{F}^r \cap \mathcal{F}^\ell$ and does not belong to \mathcal{F}^0 , or else the entropy condition would be denied, see Figure 5.6 right. For small $t > 0$, since the intersection is transverse, the structure of the wavefront is preserved, and the shock between \mathcal{F}_t^r and \mathcal{F}_t^ℓ presents slopes close to p_1 and p_2 . By Proposition F.3, since the Lax condition is strictly satisfied between p_1 and p_2 , the entropy condition is satisfied for slopes close enough to p_1 and p_2 . Hence Proposition 1.11, which applies since \mathcal{F}_t^r and \mathcal{F}_t^ℓ are the graphs of classical solutions of (HJ) (see Proposition 5.3), concludes that the variational solution solves then the Hamilton-Jacobi equation for small time. \square

5.4 Entropy condition violated by the initial shock

Theorem 5.12. *Let $u_0(q) = \min(p_1q, p_2q) + f(q)$, where $p_1 < p_2$ and $f(q) = \begin{cases} f_1(q), & q \geq 0, \\ f_2(q), & q \leq 0, \end{cases}$ with f_1 and f_2 in \mathcal{E}_B . Let us assume that $\widehat{H}'(p_1) > \widehat{H}'(p_2)$ (i.e. the entropy condition is initially violated), and that $\widehat{H}'(p_1)$ and $\widehat{H}'(p_2)$ are regular values of H' . The following classification holds:*

$H'(p_1) = \widehat{H}'(p_1)$ and $\widehat{H}'(p_2) = H'(p_2)$		$R = V$
$H'(p_1) < \widehat{H}'(p_1)$ and $\widehat{H}'(p_2) = H'(p_2)$ (resp. $H'(p_1) = \widehat{H}'(p_1)$, $\widehat{H}'(p_2) < H'(p_2)$)	\nearrow if f strictly convex on some $[0, \delta]$ (resp. on some $[-\delta, 0]$) \searrow if f concave on some $[0, \delta]$ (resp. on some $[-\delta, 0]$)	$R \neq V$ $R = V$
$H'(p_1) < \widehat{H}'(p_1)$ and $\widehat{H}'(p_2) < H'(p_2)$	\nearrow if f strictly convex on some $[0, \delta]$ OR on some $[-\delta, 0]$ \searrow if f concave on some $[-\delta, \delta]$	$R \neq V$ $R = V$

where by " $R = V$ " we mean "there exists $\tau > 0$ such that $(t, q) \mapsto R_0^t u_0(q)$ solves the (HJ) equation in the viscosity sense on $(0, \tau] \times \mathbb{R}^d$ ", and by " $R \neq V$ " we mean "there exists $\tau > 0$ such that for all $0 < t < \tau$, there exists a point q such that $(t, q) \mapsto R_0^t u_0(q)$ does not satisfy the (HJ) equation in the viscosity sense at (t, q) ".

Proof. For the linearized initial condition $\bar{u}_0(q) = \min(p_1q, p_2q)$, Proposition 5.5 states that when the entropy condition is denied between p_1 and p_2 , the pieces of wavefront issued from the regular parts of the initial condition are strictly separated by a rarefaction wave in the minimal section of the limit front. In other words, the potential shock on the minimal section of the wavefront

involving \mathcal{F}^r does not involve \mathcal{F}^ℓ , and vice versa. By Proposition 5.2, this is still the case for the wavefront \mathcal{F}_t if t is small enough. This is why the behaviour of f on \mathbb{R}_- and \mathbb{R}_+ may be looked at independently, and proving the following two points is enough to get the whole classification.

1. If $H'(p_1) = \widehat{H}'(p_1)$ and $H'(p_2) = \widehat{H}'(p_2)$, the reduced wavefront associated with the linearized function $\min(p_1q, p_2q)$ (see Proposition 5.2) presents no intersection between \mathcal{F}^r (resp. \mathcal{F}^ℓ) and \mathcal{F}^0 : this is for example given by the parametrization of the minimal section (Proposition 5.4), which if $H'(p_1) = \widehat{H}'(p_1)$ and $H'(p_2) = \widehat{H}'(p_2)$ implies that both arms are entirely included in the minimal section. Figure 5.7 presents a particularly pathological example included in this set of assumptions, where the left junction is attained by multiple parameters in \mathcal{F}^0 .

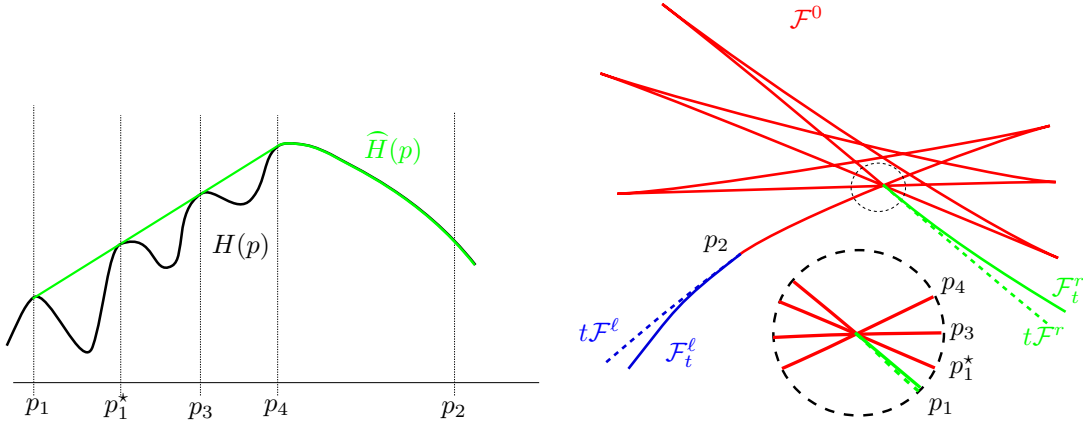


Figure 5.7: Graph of a Hamiltonian and its concave envelope (left) satisfying the assumption $H'(p_1) = \widehat{H}'(p_1)$ and $H'(p_2) = \widehat{H}'(p_2)$ and giving a wavefront (right) with a multiple shock.

Let us denote by (Q^0, S^0) the point of junction between \mathcal{F}^r and \mathcal{F}^0 , that belongs to \mathcal{F}^0 for the parameter p_1 . Let us denote by \mathcal{P} the set of parameters of $[p_1, p_2]$ for which (Q^0, S^0) is attained. In particular the first coordinate of the parametrization gives that $H'(p) = H'(p_1)$ for all p in \mathcal{P} .

Here is the moment where we use the regular value assumption. Since $H'(p_1)$ is a regular value of H' , the set \mathcal{P} is finite: it is a closed set by continuity of the parametrization of \mathcal{F}^0 , so if it contains an accumulation point $p_\infty = \lim p_n$, since $H'(p_n)$ is constant, $H''(p_\infty) = 0$ which is excluded since $H'(p_\infty) = H'(p_1)$ is a regular value of H' .

On a neighbourhood of (Q^0, S^0) , the reduced wavefront for the linearized function \bar{u}_0 is hence the union of $\#\mathcal{P}$ \mathcal{C}^1 curves crossing only at (Q^0, S^0) with slopes taking the values of \mathcal{P} , see Figure 5.7. In particular, when t is small, the structure of the (reduced) wavefront is preserved by transversality, the position of the shock does not depend on the behaviour of f , and the slope of the right arm at its endpoint is still p_1 .

This shock (which can be a simple junction) hence satisfies the entropy condition, since it has the same slopes than the shock for the linearized function, for which the variational solution coincides with the viscosity solution, by Proposition 5.4.

To put it in a nutshell, if u_r and u_ℓ denotes the \mathcal{C}^2 solutions respectively associated with

$q \mapsto p_1q + f_1(q)$ and $q \mapsto p_2q + f_2(q)$ as in Proposition 5.3, for small t ,

$$R_0^t u_0(q) = \begin{cases} u_\ell(t, q) & \text{if } q < tH'(p_2), \\ R_0^t \bar{u}_0(q) & \text{if } q \in (tH'(p_2), tH'(p_1)), \\ u_r(t, q) & \text{if } q > tH'(p_1). \end{cases}$$

Since $R_0^t \bar{u}_0 = V_0^t \bar{u}_0$ (see Proposition 5.4), u_ℓ and u_r are classical solutions, and the entropy condition is satisfied at the potential shocks, $(t, q) \mapsto R_0^t u_0(q)$ solves the (HJ) equation for all q in \mathbb{R} and small $t > 0$.

2. If $H'(p_1) < \widehat{H}'(p_1)$ and $\widehat{H}'(p_2) = H'(p_2)$, Proposition 5.5 implies that the minimal section of the wavefront for the linearized initial condition \bar{u}_0 contains a shock in $\mathcal{F}^r \cap \mathcal{F}^0$, and no shock between \mathcal{F}^ℓ and \mathcal{F}^0 (the left arm is entirely included in the minimal section of the wavefront).

The left junction (which can be a multiple point of \mathcal{F}^0) between \mathcal{F}^ℓ and \mathcal{F}^0 is studied as in the previous point, and in particular $R_0^t u_0(q) = u_\ell(t, q)$ if $q < tH'(p_2)$, where u_ℓ is the \mathcal{C}^2 solution associated with $q \mapsto p_2q + f(q)$.

Let us then focus on the shock in $\mathcal{F}^r \cap \mathcal{F}^0$, that we denote (Q^0, S^0) . We denote by \mathcal{P} the set of parameters of $[p_1, p_2]$ for which (Q^0, S^0) is attained in \mathcal{F}^0 . The assumption implies that $\mathcal{P} \subset (p_1, p_2)$. Again, the first coordinate of the parametrization gives that $H'(p) = \widehat{H}'(p_1)$ for all p in \mathcal{P} , and the fact that $\widehat{H}'(p_1)$ is a regular value of H' implies that the set \mathcal{P} is finite as previously. On a neighbourhood of (Q^0, S^0) , \mathcal{F}^0 is hence the union of $\#\mathcal{P}$ \mathcal{C}^1 curves crossing only at (Q^0, S^0) with slopes taking the values of $\mathcal{P} \subset (p_1, p_2)$, see Figure 5.8, and for each of this curve, the intersection with \mathcal{F}_r is hence transverse.

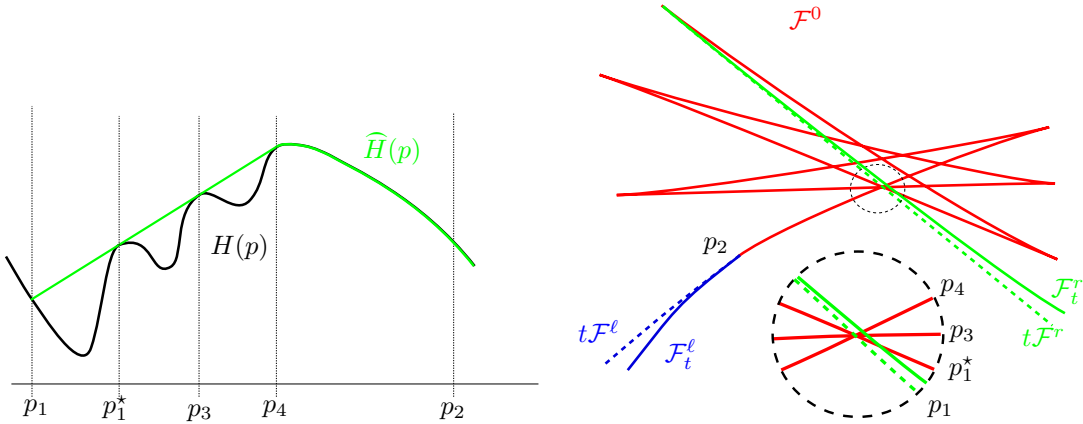


Figure 5.8: Graph of a Hamiltonian and its concave envelope (left) satisfying the assumption $H'(p_1) < \widehat{H}'(p_1)$ and $H'(p_2) = \widehat{H}'(p_2)$ and giving a wavefront (right) with a multiple shock.

Proposition 5.2 shows that the reduced wavefront \mathcal{F}_t^r tends to \mathcal{F}_r when t tends to 0, and by transversality, there exists $\tau > 0$ such that the minimal section of the wavefront presents a shock between \mathcal{F}_r^t and $t\mathcal{F}^0 = \mathcal{F}_t^0$ in the neighbourhood of $t(Q^0, S^0)$ for all t in $(0, \tau)$, that we will denote by (Q^t, S^t) .

To put it in a nutshell, if u_r and u_ℓ denotes the \mathcal{C}^2 solutions respectively associated with

$q \mapsto p_1q + f_1(q)$ and $q \mapsto p_2q + f_2(q)$ as in Proposition 5.3, for small t ,

$$R_0^t u_0(q) = \begin{cases} u_\ell(t, q) & \text{if } q < tH'(p_2), \\ R_0^t \bar{u}_0(q) & \text{if } q \in (tH'(p_2), Q^t) \\ u_r(t, q) & \text{if } q > Q^t. \end{cases}$$

- Assume that f is strictly convex on \mathbb{R}_+ . The shock (Q^t, S^t) is given by parameters (q_t, p_t) , such that $q_t > 0$, $p_t \in [p_1, p_2]$ and

$$\begin{cases} q_t + tH'(u'_0(q_t)) & = tH'(p_t), \\ u_0(q_t) + tu'_0(q_t)H'(u'_0(q_t)) - tH(u'_0(q_t)) & = tp_tH'(p_t) - tH(p_t), \end{cases}$$

and the slopes at the shock are $u'_0(q_t)$ and p_t by Proposition 5.1. We prove as for Proposition 5.6 that the Lax condition is denied between the slopes of this shock, see (5.1), and as a consequence the variational solution does not satisfy the (HJ) equation in the viscosity sense at the intersection between \mathcal{F}_t^r and \mathcal{F}_t^0 for all t in $(0, \tau)$.

- Assume that f is concave on \mathbb{R}_+ . Since the left junction satisfies the entropy condition as in the previous argument, it is enough to prove that there exists $\tau > 0$ such that $(t, q) \mapsto R_0^t u_0$ is a viscosity solution on the set $\{(t, q), 0 < t < \tau, q > tH'(p_2)\}$.

Let \tilde{f} be a \mathcal{C}^2 concave function of \mathbb{R} , with bounded second derivative by B , that coincides with f on \mathbb{R}_+ . We define $\tilde{u}_0(q) = \min(p_1q, p_2q) + \tilde{f}(q)$, which is concave, and denote by \tilde{u}_r and \tilde{u}_ℓ the \mathcal{C}^2 solutions associated with $q \mapsto p_1q + \tilde{f}(q)$ and $q \mapsto p_2q + \tilde{f}(q)$ as in Proposition 5.3. Since f and \tilde{f} coincide on $[0, \infty)$, $u_r(t, q) = \tilde{u}_r(t, q)$ for all $q \geq tH'(p_1)$ as a consequence of Proposition 5.3. Since \tilde{u}_0 has the same linearized function \bar{u}_0 than u_0 , the previous work applied to \tilde{u}_0 gives in particular that

$$R_0^t \tilde{u}_0(q) = \begin{cases} R_0^t \bar{u}_0(q) & \text{if } q \in (tH'(p_2), Q^t) \\ \tilde{u}_r(t, q) = u_r(t, q) & \text{if } q > Q^t. \end{cases}$$

since for t small enough, Q^t is close to $q^0 = t\widehat{H}'(p_1) > tH'(p_1)$. In other words, there exists $\tau > 0$ such that for $0 < t < \tau$, $R_0^t u_0$ coincides with $R_0^t \tilde{u}_0$ on $(tH'(p_2), \infty)$, which solves (HJ) in the viscosity sense on its domain since \tilde{u}_0 is concave (see Proposition 1.27). We hence proved that $(t, q) \mapsto R_0^t u_0$ is a viscosity solution on the set $\{(t, q), 0 < t < \tau, q > tH'(p_2)\}$, hence on the whole $(0, \tau) \times \mathbb{R}$.

We get the result for f locally convex or concave using the arguments of Addendum 5.8.

□

We state the analogous statement for semiconvex initial conditions and the convex envelope of H . Let \widetilde{H} denote the largest convex function on $[p_1, p_2]$ which is smaller than H on this set.

Proposition 5.13. *Let $u_0(q) = \max(p_1q, p_2q) + f(q)$, where $p_1 < p_2$ and $f(q) = \begin{cases} f_1(q), & q \geq 0, \\ f_2(q), & q \leq 0, \end{cases}$ with f_1 and f_2 in \mathcal{E}_B . Note that p_2 (resp. p_1) is now the right (resp. left) derivative of u_0 at zero. If $\widetilde{H}'(p_1) < \widetilde{H}'(p_2)$ are regular values of H' , the following classification holds:*

$H'(p_1) = \widetilde{H}'(p_1)$ and $\widetilde{H}'(p_2) = H'(p_2)$		$R=V$
$H'(p_1) = \widetilde{H}'(p_1)$ and $\widetilde{H}'(p_2) > H'(p_2)$ (resp. $H'(p_1) > \widetilde{H}'(p_1)$, $\widetilde{H}'(p_2) = H'(p_2)$)	\nearrow if f strictly concave on some $[0, \delta]$ (resp. on some $[-\delta, 0]$) \searrow if f convex on some $[0, \delta]$ (resp. on some $[-\delta, 0]$)	$R \neq V$ $R=V$
$H'(p_1) > \widetilde{H}'(p_1)$ and $\widetilde{H}'(p_2) > H'(p_2)$	\nearrow if f strictly concave on some $[0, \delta]$ OR on some $[-\delta, 0]$ \searrow if f convex on some $[-\delta, \delta]$	$R \neq V$ $R=V$

5.5 *Perestroika*: entropy condition satisfied, but not strictly, by the initial shock

In this part, let us take H as in Figure 5.9, *i.e.* such that the concave envelope of H between p_1 and p_2 coincides with H at a unique point of (p_1, p_2) , denoted by p_0 . For example, take $H(p) = p^4 - p^2$, $p_1 = -1$, $p_0 = 0$ and $p_2 = 1$, and any $u_0(q) = \min(p_1q, p_2q) + f(q)$ as in the previous paragraphs.

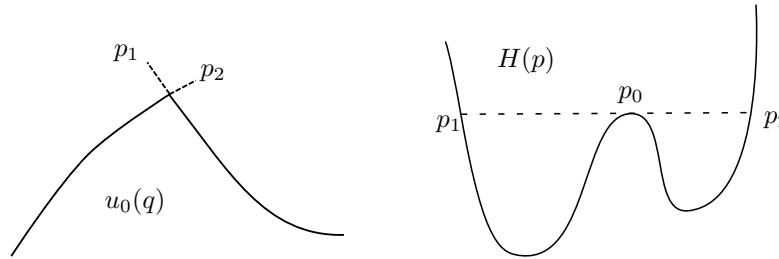


Figure 5.9: Graphs of u_0 and H for the considered situation.

We are going to show that depending on the local behaviour of u_0 at p_1 and p_2 , one of the three situations of Figure 5.10 may appear.

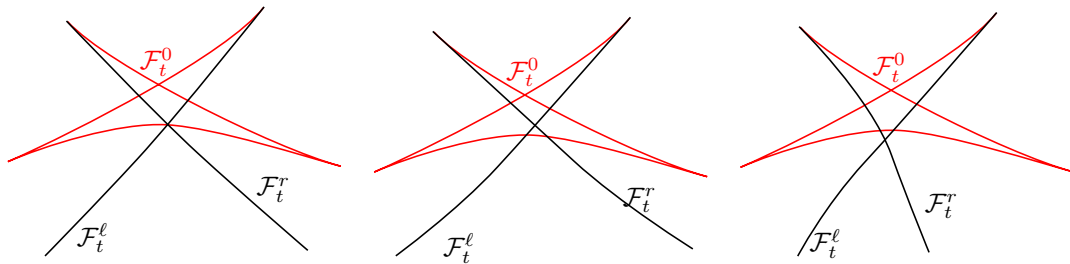


Figure 5.10: Possible evolutions of a triple shock.

We will assume for simplicity that $H(p_1) = H(p_0) = H(p_2) = 0$, hence $H'(p_0) = 0$. We denote by s_1 (resp. s_2) the right (resp. left) second derivative of u_0 at zero, and we assume to

avoid additional degenerate effects that $H'(p_1) < 0 < H'(p_2)$ and that $H''(p_0)$, s_1 and s_2 are non zero.

Proposition 5.14. *If $\frac{H'(p_1)^2 s_1}{p_0 - p_1} > \frac{H'(p_2)^2 s_2}{p_0 - p_2}$, the variational solution presents two shocks (Figure 5.10 middle) and is not a viscosity solution for small time.*

If $\frac{H'(p_1)^2 s_1}{p_0 - p_1} < \frac{H'(p_2)^2 s_2}{p_0 - p_2}$, the variational solution presents one shock (Figure 5.10 right) and is a viscosity solution for small time.

Proof. The study of the wavefront for the linearized function $\bar{u}_0(q) = \min(p_1 q, p_2 q)$ gives that the limit triple shock is attained exactly once in \mathcal{F}^0 by the parameter p_0 . The intersections $\mathcal{F}^r \cap \mathcal{F}^0$ and $\mathcal{F}^\ell \cap \mathcal{F}^0$ are transverse, hence they are preserved for small t for the reduced wavefront. We denote by $Q_r(t)$ (resp. $Q_\ell(t)$) the position of the shock issued of the triple shock between $\tilde{\mathcal{F}}_t^r$ (resp. $\tilde{\mathcal{F}}_t^\ell$) and $\tilde{\mathcal{F}}_t^0$.

Let us prove that there exist \mathcal{C}^1 parameter functions $(q_r(t), p_r(t))$ and $(q_\ell(t), p_\ell(t))$ such that $q_r(t) > 0$, $q_\ell(t) < 0$, $p_r(t)$ and $p_\ell(t)$ are in (p_1, p_2) , $p_r(0) = p_\ell(0) = p_0$ and

$$\begin{cases} Q_r(t) := q_r(t) + H'(u'_0(tq_r(t))) = H'(p_r(t)), \\ \frac{u_0(tq_r(t))}{t} + u'_0(tq_r(t))H'(u'_0(tq_r(t))) - H(u'_0(tq_r(t))) = p_r(t)H'(p_r(t)) - H(p_r(t)), \end{cases}$$

$$\begin{cases} Q_\ell(t) := q_\ell(t) + H'(u'_0(tq_\ell(t))) = H'(p_\ell(t)), \\ \frac{u_0(tq_\ell(t))}{t} + u'_0(tq_\ell(t))H'(u'_0(tq_\ell(t))) - H(u'_0(tq_\ell(t))) = p_\ell(t)H'(p_\ell(t)) - H(p_\ell(t)). \end{cases}$$

We define the \mathcal{C}^1 function $F : \mathbb{R}_+ \times \mathbb{R}_+ \times (p_1, p_2) \rightarrow \mathbb{R}^2$ by

$$F(t, q, p) = \begin{pmatrix} q + H'(u'_0(tq)) - H'(p) \\ \frac{u_0(tq)}{t} - H(u'_0(tq)) + u'_0(tq)H'(u'_0(tq)) - pH'(p) + H(p) \end{pmatrix}$$

where $\frac{u_0(tq)}{t}$ is \mathcal{C}^1 -continuously extended to $p_1 q$ when $t = 0$.

Here are the derivatives of F at time $t = 0$:

$$(\partial_q F(0, q, p), \partial_p F(0, q, p)) = \begin{pmatrix} 1 & -H''(p) \\ p_1 & -pH''(p) \end{pmatrix}$$

The implicit function theorem hence applies at the point $(q(0), p(0)) = (-H'(p_1), p_0)$ since $H''(p_0) \neq 0$ and $p_1 < p_0$, giving the first parameter function $(q_r(t), p_r(t))$. We obtain the other parameter function similarly.

To decide if the situation of Figure 5.10 right or middle happens, it is enough to check in which order the shock appears, *i.e.* to compare $Q_\ell(t)$ and $Q_r(t)$. The first lines of the systems yield $Q_r(0) = Q_\ell(0) = H'(p_0) = 0$, hence $q_r(0) = -H'(p_1)$ (resp. $q_\ell(0) = -H'(p_2)$). To compare $Q_r(t)$ and $Q_\ell(t)$ for small time, we then write the derivative w.r.t. t of both systems in order to get $Q'_r(0)$ and $Q'_\ell(0)$.

We compute

$$\partial_t \frac{u_0(tq_r(t))}{t} = \frac{-u_0(tq_r(t)) + t(q_r(t) + tq'_r(t))u'_0(tq_r(t))}{t^2} \xrightarrow{t \rightarrow 0} q'_r(0)p_1 + \frac{q_r(0)^2 s_1}{2}.$$

When $t = 0$, the systems of derivatives are

$$\begin{cases} Q'_r(0) := q'_r(0) + q_r(0)s_1 H''(p_1) = p'_r(0)H''(p_0), \\ q'_r(0)p_1 + \frac{q_r(0)^2 s_1}{2} + q_r(0)s_1 p_1 H''(p_1) = p'_r(0)p_0 H''(p_0), \end{cases}$$

$$\begin{cases} Q'_\ell(0) := q'_\ell(0) + q_\ell(0)s_2H''(p_2) = p'_\ell(0)H''(p_0), \\ q'_\ell(0)p_2 + \frac{q_\ell(0)^2s_2}{2} + q_\ell(0)s_2p_2H''(p_2) = p'_\ell(0)p_0H''(p_0), \end{cases}$$

and combining both lines of each system gives

$$\begin{cases} p_1Q'_r(0) + \frac{q_r(0)^2s_1}{2} = p_0Q'_r(0), \\ p_2Q'_\ell(0) + \frac{q_\ell(0)^2s_2}{2} = p_0Q'_\ell(0). \end{cases}$$

As a consequence, as $q_r(0) = -H'(p_1)$ and $q_\ell(0) = -H'(p_2)$,

$$Q'_r(0) > Q'_\ell(0) \iff \frac{H'(p_1)^2s_1}{p_0 - p_1} > \frac{H'(p_2)^2s_2}{p_0 - p_2}$$

and in that case the minimal section of the wavefront present for small time two shocks as in Figure 5.10 middle. Note that in that case, necessarily s_1 or s_2 is positive, *i.e.* u_0 is strictly convex on some $[-\delta, 0]$ or on some $[0, \delta]$, and then one can show as in the proof of Theorem 5.12 that the shock on the convex side denies Oleinik's entropy condition for small time t .

Conversely,

$$Q'_r(0) < Q'_\ell(0) \iff \frac{H'(p_1)^2s_1}{p_0 - p_1} < \frac{H'(p_2)^2s_2}{p_0 - p_2}$$

and in that case the minimal section of the wavefront presents for small time only one shock, between $\tilde{\mathcal{F}}_t^r$ and $\tilde{\mathcal{F}}_t^\ell$, as in Figure 5.10 right. Let us now prove that under the assumption $\frac{H'(p_1)^2s_1}{p_0 - p_1} < \frac{H'(p_2)^2s_2}{p_0 - p_2}$, this shock satisfies the entropy condition. There exist \mathcal{C}^1 parameter functions $(q_r(t), q_\ell(t))$ giving this intersection, *i.e.* such that $q_r(t) > 0$, $q_\ell(t) < 0$,

$$\begin{cases} Q(t) := q_r(t) + H'(u'_0(tq_r(t))) = q_\ell(t) + H'(u'_0(tq_\ell(t))), \\ \frac{u_0(tq_r(t))}{t} + u'_0(tq_r(t))H'(u'_0(tq_r(t))) - H(u'_0(tq_r(t))) \\ = \frac{u_0(tq_\ell(t))}{t} + u'_0(tq_\ell(t))H'(u'_0(tq_\ell(t))) - H(u'_0(tq_\ell(t))). \end{cases}$$

When t is zero, both $H(u'_0(tq_\ell(t)))$ and $H(u'_0(tq_r(t)))$ vanish, which implies combining both lines of the system that

$$Q(0) = q_r(0) + H'(p_1) = q_\ell(0) + H'(p_2) = 0.$$

To prove that the shock satisfies the entropy condition for t small enough, it is enough to check that the assumption implies at the first order in t the following strict inequality between the slopes of the cords joining the slopes of the shock and p_0 :

$$\frac{H(u'_0(tq_\ell(t))) - H(p_0)}{u'_0(tq_\ell(t)) - p_0} > \frac{H(u'_0(tq_r(t))) - H(p_0)}{u'_0(tq_r(t)) - p_0}.$$

Since $H(p_1) = H(p_0) = H(p_2)$ are zero, the right hand side (resp. left hand side) is equivalent to $t \frac{H'(p_1)q_r(0)s_1}{p_1 - p_0}$ (resp. $t \frac{H'(p_2)q_\ell(0)s_2}{p_2 - p_0}$), and using the fact that $q_r(0) + H'(p_1)$ and $q_\ell(0) + H'(p_2)$ are both zero, we get the wanted strict inequality for small t :

$$\frac{H(u'_0(tq_\ell(t))) - H(p_0)}{u'_0(tq_\ell(t)) - p_0} \underset{t \rightarrow 0}{\sim} t \frac{H'(p_2)^2s_2}{p_0 - p_2} > t \frac{H'(p_1)^2s_1}{p_0 - p_1} \underset{t \rightarrow 0}{\sim} \frac{H(u'_0(tq_r(t))) - H(p_0)}{u'_0(tq_r(t)) - p_0}.$$

We have hence proved that for small t , if $\frac{H'(p_2)^2s_2}{p_0 - p_2} > \frac{H'(p_1)^2s_1}{p_0 - p_1}$, the case of Figure 5.10 right happens and the variational solution is a viscosity solution. \square

5.6 An explicit example where the solutions differ

Following an idea of N. Vichery, we take piecewise quadratic Hamiltonian and initial condition in order to be able to compute explicitly the viscosity and variational solutions of the Cauchy problem. Let $\delta > 0$ be small, and take

$$u_0(q) = \begin{cases} q & \text{if } q < 0, \\ -q + q^2/2 & \text{if } 0 < q < 1, \\ -1/2 & \text{if } q > 1 \end{cases}, \quad H(p) = \begin{cases} p + p^2 & \text{if } p < 0, \\ p - p^2 & \text{if } p > \delta, \end{cases}$$

where H is extended to a \mathcal{C}^2 Hamiltonian on \mathbb{R} so that H'' has exactly one zero in $(0, \delta)$.

Proposition 5.15. *For $t > 0$ small enough, the variational solution is given by*

$$R_0^t u_0(q) = \begin{cases} q & \text{if } q < -t, \\ f_0(t, q) & \text{if } q \in [-t, c(t)], \\ f_1(t, q) & \text{if } q \in [c(t), 1+t], \\ -1/2 & \text{if } q \geq 1+t \end{cases}$$

where

$$f_0(t, q) = -t \left(\frac{q-t}{2t} \right)^2, \quad f_1(t, q) = \frac{-2q + (q-t)^2}{2(1+2t)} \quad \text{and} \quad c(t) = \frac{3t + 4t^2 - 2t\sqrt{2+4t}}{1+4t}.$$

The function f_0 (resp. f_1) is a classical solution of the (HJ) equation associated with the Hamiltonian $H_+(p) = p - p^2$ (resp. with the Hamiltonian $H_-(p) = p + p^2$) on $(0, \infty) \times \mathbb{R}$ (resp. on $[0, \infty) \times \mathbb{R}$) and the function $c(t)$, which is defined by the equation $f_0(t, c(t)) = f_1(t, c(t))$, is called the *variational shock*.

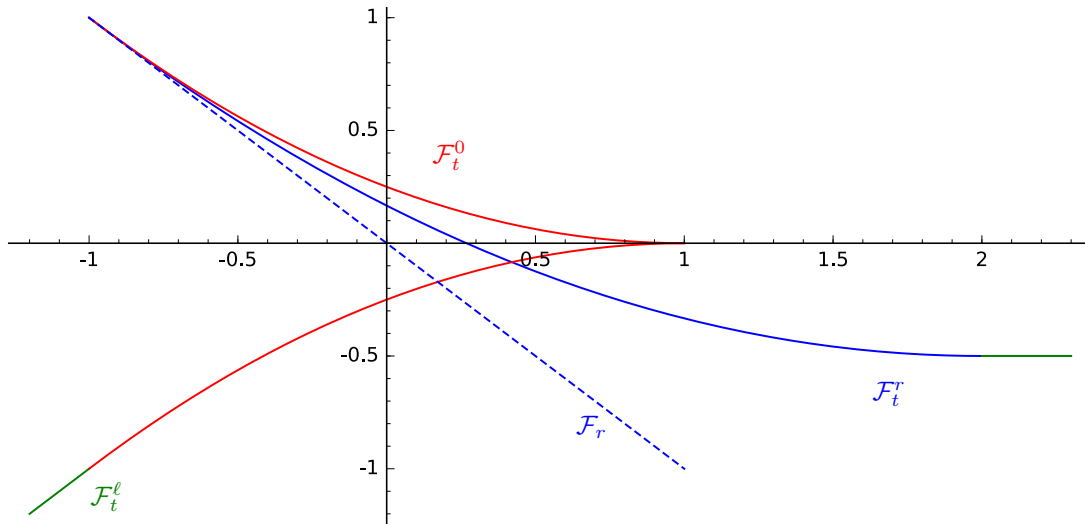


Figure 5.11: Wavefront at time $t = 1$. The blue dashed half line represents the limit of the wavefront when t tends to 0. The green half lines are the affine part of the wavefront.

Proof. The wavefront associated with the Cauchy problem is presented on Figure 5.11, and admits the following parametrization:

$$\begin{aligned} \mathcal{F}_t^\ell &: \begin{cases} q_0 + tH'(u'_0(q_0)), \\ q_0 + tu'_0(q_0)H'(u'_0(q_0)) - tH(u'_0(q_0)), \end{cases} & q_0 < 0, \\ \mathcal{F}_t^r &: \begin{cases} q_0 + tH'(u'_0(q_0)), \\ u_0(q_0) + tu'_0(q_0)H'(u'_0(q_0)) - tH(u'_0(q_0)), \end{cases} & q_0 > 0, \\ \mathcal{F}_t^0 &: \begin{cases} tH'(p), \\ t(pH'(p) - H(p)), \end{cases} & p \in [-1, 1]. \end{aligned}$$

In particular, since $u'_0(q_0) = 1$ for $q_0 < 0$, $H(1) = 0$ and $H'(1) = -1$, \mathcal{F}_t^ℓ is the graph of the identity restricted to $(-\infty, -t)$. If $0 < q_0 < 1$, $u'_0(q_0) = -1 + q_0 < 0$, where $H(p) = p + p^2$ and hence the part of \mathcal{F}_t^r parametrized by q_0 in $(0, 1)$ is:

$$\begin{cases} q_0 + t(1 + 2(-1 + q_0)), \\ -q_0 + q_0^2/2 + t(-1 + q_0)^2, \end{cases}$$

which is a graph over the interval $(-t, 1 + t)$. If $q = q_0 + t(1 + 2(-1 + q_0))$, $q_0 = \frac{q-t}{1+2t}$ and simplifying the second term gives the function $f_1(t, q)$. Hence $\mathcal{F}_t^r \cap (-t, 1 + t) \times \mathbb{R}$ is the graph of $q \mapsto f_1(t, q)$ over the interval $(-t, 1 + t)$. If $q_0 \geq 1$, $u'_0(q_0) = 0$ and $H'(0) = H(0) = 0$, hence the part of \mathcal{F}_t^r parametrized by q_0 in $[1, \infty)$ is the horizontal half line $\{(q_0, -1/2), q_0 \geq 1\}$.

The part of \mathcal{F}_t^0 parametrized by p in $(\delta, 1)$ is:

$$\begin{cases} t(1 - 2p), \\ -tp^2, \end{cases}$$

which is on the set $(-t, t(1 - 2\delta))$ the graph of the function $q \mapsto f_0(t, q)$: if $q = t(1 - 2p)$, $p = -(q - t)/2t$ and the second term has the wanted form.

Resolving straightforward the equation $f_0(t, c(t)) = f_1(t, c(t))$ gives the value of $c(t)$. So, if $c(t)$ belongs to $(-t, t(1 - 2\delta))$ and to $(-t, 1 + t)$, the (unique) continuous section of the wavefront, which gives the variational solution, is as stated in the proposition. This is the case for small t , since for small δ

$$\frac{c(t)}{t} \xrightarrow{t \rightarrow 0} 3 - 2\sqrt{2} \in (0, 1 - 2\delta).$$

□

We denote by $p^* = \sqrt{2} - 1$ the positive parameter for which $H'(p^*) = \frac{H(p^*) - H(-1)}{p^* + 1}$, which is the point of contact between H and its concave envelope of H on $[-1, 1]$, see Figure 5.12.

Comparing u_0 to its linearized function at 0 gives already a large domain on which viscosity and variational solutions coincide.

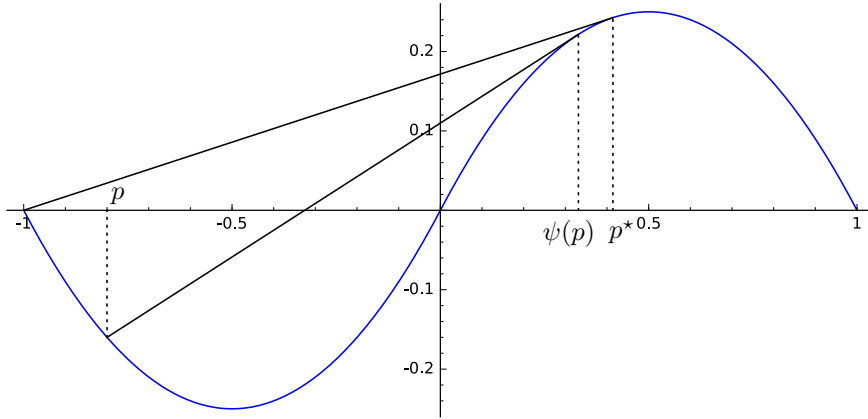
Proposition 5.16. *If $t > 0$ is small enough and $q \leq tH'(p^*)$, $R_0^t u_0(q) = V_0^t u_0(q)$.*

Proof. We denote by $\bar{u}_0 : q \mapsto -|q|$ the linearized function of u_0 at 0, which is smaller than u_0 on \mathbb{R} . The continuous sections of the wavefronts associated with u_0 and \bar{u}_0 are the same for $q \leq tH'(p^*)$, see Figure 5.11, where the wavefront for \bar{u}_0 is the one with the dashed right arm. As a consequence, $R_0^t \bar{u}_0(q) = R_0^t u_0(q)$ when $q \leq tH'(p^*)$.

Now, since $\bar{u}_0 \leq u_0$, the monotonicity of the viscosity operator implies $V_0^t \bar{u}_0 \leq V_0^t u_0$. Since \bar{u}_0 is convex, by Proposition 1.27, $R_0^t \bar{u}_0 = V_0^t \bar{u}_0$. And Proposition 1.25 gives that $V_0^t u_0 \leq R_0^t u_0$ for small t . Hence for $q \leq tH'(p^*)$,

$$R_0^t u_0(q) = R_0^t \bar{u}_0(q) = V_0^t \bar{u}_0(q) \leq V_0^t u_0(q) \leq R_0^t u_0(q).$$

□

Figure 5.12: Graph of H and tangents defining p^* and ψ .

Let $\psi(p) = -p \cdot p^*$, which is defined, see Figure 5.12, such that for all $p < -\frac{\delta}{p^*}$,

$$H'(\psi(p)) = \frac{H(\psi(p)) - H(p)}{\psi(p) - p}.$$

To build the viscosity solution, we first identify the viscosity shock by solving an ODE, following an idea of O. Oleinik explained in [Che75], and then build the viscosity solution by following the characteristics tangentially issued from this line of shock.

Proposition 5.17. *The shock of the viscosity solution, called viscosity shock, is given for small t by the Cauchy problem*

$$\begin{cases} x'(t) = H'(\psi(\partial_q f_1(t, x(t)))) \\ x(0) = H'(p^*), \end{cases}$$

and equal to $x(t) = 1 + t - (1 + 2t)p^*$. We denote by $p(t)$ the quantity $\psi(\partial_q f_1(t, x(t)))$.

The viscosity solution coincides with the variational solution, i.e. $V_0^t u_0(q) = R_0^t u_0(q)$, for all q in $\mathbb{R} \setminus (tH'(p^*), x(t))$, and if $q \in (tH'(p^*), x(t))$, there exists a unique $0 < \tau < t$ such that $q = Q_\tau^t(x(\tau), p(\tau))$, and then the viscosity solution is given by

$$V_0^t u_0(q) = f_1(\tau, x(\tau)) + \mathcal{A}_\tau^t(\gamma),$$

where γ is the Hamiltonian trajectory issued from $(x(\tau), p(\tau))$ at time τ .

The last equality, while being implicit, allows though to plot the graph of the viscosity solution at time t , as a curve parametrized by τ in $(0, t)$, see Figure 5.13 up, where the viscosity solution is presented in black. We obtain a difference between the graphs of the viscosity and the variational solutions which is barely observable yet non zero, see Figure 5.13 down. The obtained viscosity solution is smaller than the variational one, in agreement with Proposition 1.25. On Figure 5.14 we present the characteristics and shock for the variational solution (up) or for the viscosity solution (down). The characteristics are lines along which $\partial_q u$ is constant, when u is differentiable. On the upper figure, the viscosity shock is represented by the dashed black curve, very close to the variational shock in thickened red, while the green thickened line represents the left C^1 junction. On the bottom figure, the green and red thickened lines represent the C^1

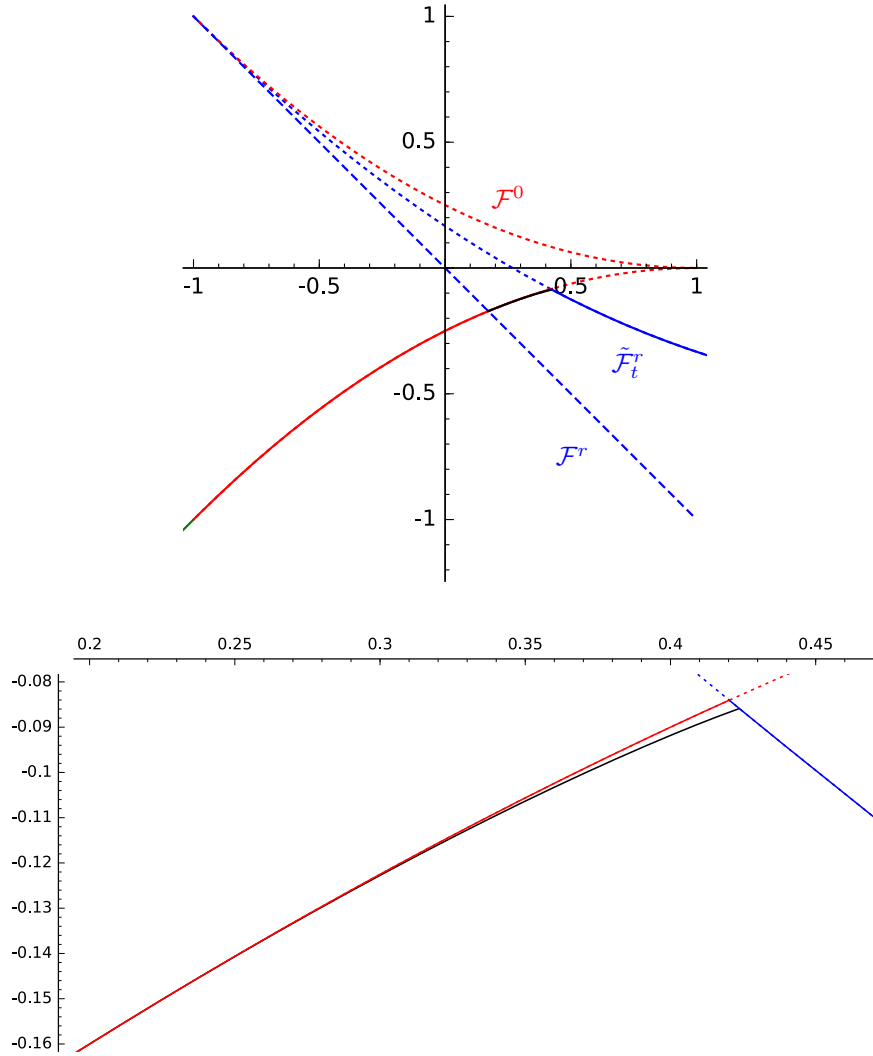


Figure 5.13: Up: Wavefront at time $t = 1$ and graph of the viscosity solution in black. Down: zoom on the area where viscosity and variational solutions differ.

junctions, and the black curve the viscosity shock. Note that in the area between the red junction and the black shock, the characteristics are tangentially issued from the viscosity shock, whereas for the variational solution, they are issued from the origin, "forgotten" for a certain time by the variational solution and then arise in the solution after the variational shock.

Proof of Proposition 5.17. Let us check that $x(t) = 1 + t - (1 + 2t)^{p^*}$ solves the considered Cauchy problem. The initial condition is clearly satisfied. Note that $\partial_q f_1(t, q) = \frac{q-1-t}{1+2t}$, and as a consequence $\partial_q f_1(t, x(t)) = -(1 + 2t)^{p^*-1}$. Hence $x'(t) = 1 + 2p^*(1 + 2t)^{p^*-1} = H'(\psi(\partial_q f_1(t, x(t))))$ as long as $\psi(\partial_q f_1(t, x(t))) > \delta$, where $H(p) = p - p^2$. Since $\partial_q f_1(0, x(0)) = -1$, we have $\psi(\partial_q f_1(0, x(0))) = p^* > \delta$ and the condition $\psi(\partial_q f_1(t, x(t))) > \delta$ is still satisfied for small t .

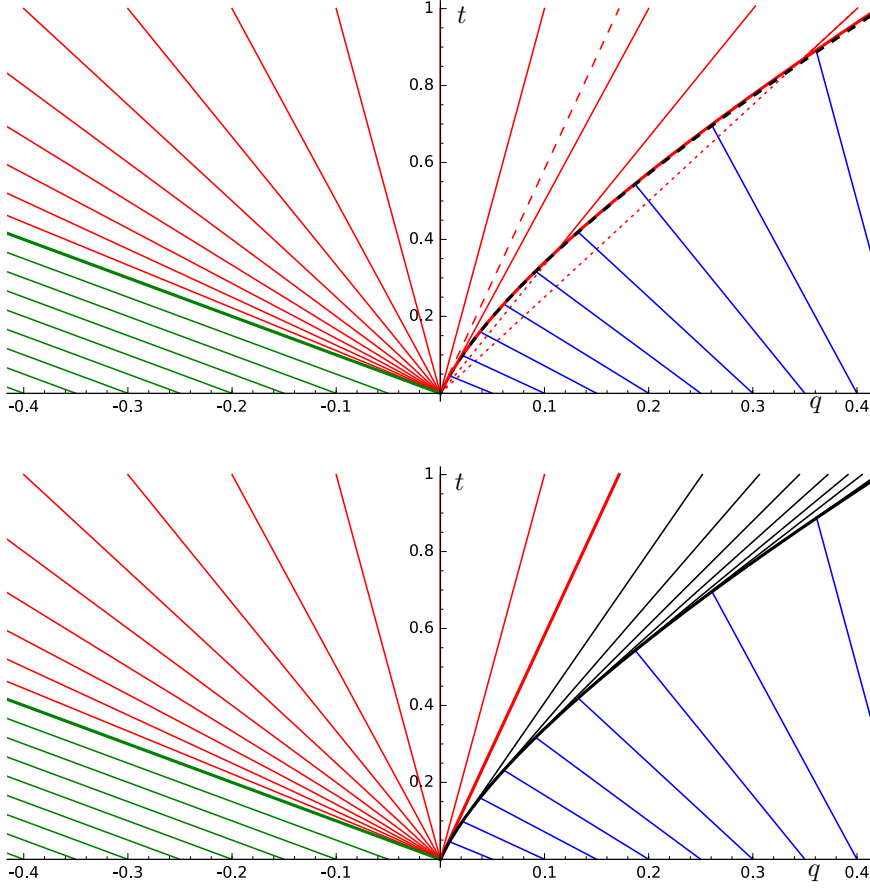


Figure 5.14: Characteristics and shock for the variational solution (up) and the viscosity solution (down). Up, the dashed black viscosity shock is presented in comparison with the variational shock (red).

Now, let us verify that for all q in $(tH'(p^*), x(t))$, there exists a unique τ in $(0, t)$ such that $q = x(\tau) + (t - \tau)H'(p(\tau)) = x(\tau) + (t - \tau)x'(\tau)$. If $t > 0$ is fixed, the function defined by $q_t(\tau) = x(\tau) + (t - \tau)x'(\tau)$ satisfies $q_t(0) = tH'(p^*)$, $q_t(t) = x(t)$ and for all $\tau < t$,

$$q_t'(\tau) = (t - \tau)x''(\tau) = -4(t - \tau)p^*(p^* - 1)(1 + 2t)^{p^* - 2} > 0.$$

The implicit function theorem, applied to the equation $q = q_t(\tau)$, states that the mapping $(t, q) \mapsto \tau(t, q)$ is \mathcal{C}^1 on the set $\{(t, q), t > 0, q \in (tH'(p^*), x(t))\}$, and it is continuously extended at the boundaries by $\tau(t, x(t)) = t$ and $\tau(t, tH'(p^*)) = 0$. Since $\partial_t q_t(\tau) = x'(\tau) = H'(p(\tau))$ and $q_t'(\tau) = (t - \tau)x''(\tau) = (t - \tau)p'(\tau)H''(p(\tau))$, differentiating the equation $q = q_t(\tau(t, q))$ with respect to q and t gives that

$$1 = \partial_q \tau (t - \tau)p'(\tau)H''(p(\tau)), \quad 0 = \partial_t \tau p'(\tau)H''(p(\tau)) + H'(p(\tau)). \quad (5.3)$$

We define for $q \in (tH'(p^*), x(t))$

$$f_2(t, q) = f_1(\tau(t, q), x(\tau(t, q))) + (t - \tau)(p(\tau(t, q))H'(p(\tau(t, q))) - H(p(\tau(t, q)))).$$

Let us show that this function is a classical solution of the (HJ) equation for $t > 0$ small enough and $q \in (tH'(p^*), x(t))$. We denote by

$$g(t, \tau) = f_1(\tau, x(\tau)) + (t - \tau)(p(\tau)H'(p(\tau)) - H(p(\tau))).$$

Using the fact that f_1 solves the (HJ) equation at the point of interest, and that

$$x'(\tau) = H'(p(\tau)) = \frac{H(p(\tau)) - H(\partial_q f_1(\tau, x(\tau)))}{p(\tau) - \partial_q f_1(\tau, x(\tau))},$$

one can show that $\partial_\tau g(t, \tau) = (t - \tau)p(\tau)p'(\tau)H''(p(\tau))$.

Now, using (5.3), we differentiate $f_2(t, q) = g(t, \tau(t, q))$ to get

$$\partial_q f_2(t, q) = \partial_t \tau(t, q) \partial_\tau g(t, \tau(t, q)) = \partial_t \tau(t - \tau)p(\tau)p'(\tau)H''(p(\tau)) = p(\tau),$$

and

$$\begin{aligned} \partial_t f_2(t, q) &= \partial_t \tau(t, q) \partial_\tau g(t, \tau(t, q)) + \partial_t g(t, \tau(t, q)) \\ &= \partial_t \tau(t, q)(t - \tau)p(\tau)p'(\tau)H''(p(\tau)) + p(\tau)H'(p(\tau)) - H(p(\tau)) = -H(p(\tau)). \end{aligned}$$

Now, let us check that

$$V_0^t u_0(q) = \begin{cases} q & \text{if } q < -t, \\ f_0(t, q) & \text{if } q \in [-t, tH'(p^*)], \\ f_2(t, q) & \text{if } q \in (tH'(p^*), x(t)), \\ f_1(t, q) & \text{if } q \in (x(t), 1 + t), \\ -1/2 & \text{if } q \geq 1 + t \end{cases}$$

Since $(t, q) \mapsto q$, $(t, q) \mapsto -1/2$, f_0 , f_1 and f_2 are \mathcal{C}^2 solutions on their domain of definition, we only have to look at the junctions. Since $\partial_q f_0(t, -t) = 1$, the junction at $q = -t$ is \mathcal{C}^1 and the equation is satisfied in the viscosity sense at $(t, -t)$ for all $t > 0$ small enough. Since $f_1(t, 1 + t) = -1/2$ and $\partial_q f_1(t, 1 + t) = 0$, the junction at $q = 1 + t$ is \mathcal{C}^1 and the equation is satisfied in the viscosity sense at $(t, 1 + t)$ for all $t > 0$ small enough. When $q = tH'(p^*)$, we also have that

$$\partial_q f_2(t, tH'(p^*)) = p(\tau(t, tH'(p^*))) = p(0) = p^* = \partial_q f_0(t, tH'(p^*)),$$

and again the junction is \mathcal{C}^1 .

Remark 5.18. One can even show that this junction is \mathcal{C}^2 :

$$\partial_q^2 f_2(t, tH'(p^*)) = \frac{1}{tH''(p^*)} = -\frac{1}{2t} = \partial_q^2 f_0(t, tH'(p^*)),$$

which explains that the variational and viscosity solutions are barely distinguishable at $tH'(p^*)$, see Figure 5.13.

If $q = x(t)$,

$$\partial_q f_2(t, x(t)) = p(\tau(t, x(t))) = p(t) = \psi(\partial_q f_1(t, x(t))) > \partial_q f_1(t, x(t)).$$

By definition of ψ , the Lax condition is then satisfied for this shock, and since H has a unique point of inflexion on $[-1, 1]$ this implies that the Oleinik's entropy condition is satisfied. By Proposition 1.11, the equation is then satisfied at $(t, x(t))$ for all $t > 0$ small enough, and the uniqueness of the viscosity solution gives the conclusion. \square

Chapter 6

Variational and viscosity operators differ for non convex non concave integrable Hamiltonians

Le but de ce chapitre est de montrer que les hamiltoniens intégrables pour lesquels l'opérateur de viscosité est un opérateur variationnel sont convexes ou concaves. Plus précisément, on construit pour tout hamiltonien intégrable ni convexe ni concave une donnée initiale pour laquelle la solution variationnelle n'est pas solution de viscosité en petit temps. On réduit le problème à l'étude de situations élémentaires en dimension 1 et 2 en caractérisant les fonctions ni convexes ni concaves sur \mathbb{R}^n (voir Proposition 6.2). L'exemple clé pour la dimension 2 (voir Proposition 6.6) est détaillé dans le paragraphe §6.2, alors que l'élément de dimension 1 nécessaire à la preuve a été établie dans le chapitre précédent (Proposition 5.6).

The aim of this chapter is to prove the following contrapositive statement of Theorem 1.30:

Theorem 6.1. *If $p \mapsto H(p)$ is a neither convex nor concave integrable Hamiltonian satisfying Hypothesis 1.1, and if R_s^t is a variational operator, there exists a Lipschitz initial condition u_0 such that $(t, q) \mapsto R_0^t u_0(q)$ does not solve (HJ) in the viscosity sense at some point (t, q) in $(0, \infty) \times \mathbb{R}^d$.*

6.1 Reduction

To prove Theorem 6.1, we are going to reduce the problem to the dimension 1 or 2 with the help of the three following propositions. The first one proposes a characterization of neither convex nor concave functions of \mathbb{R}^n .

Proposition 6.2. *A \mathcal{C}^2 function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is neither convex nor concave if and only if there exists a straight line along which it is neither convex nor concave, or there exists x in \mathbb{R}^n such that the Hessian $\mathcal{H}f(x)$ admits both (strictly) positive and negative eigenvalues.*

Proof. Since a \mathcal{C}^2 function is convex (resp. concave) if and only if its Hessian admits only non negative (resp. non positive) eigenvalues, it is enough to prove the following statement: if f is a non convex and non concave \mathcal{C}^2 function with $\mathcal{H}f(x) \in S_n^+(\mathbb{R}) \cup S_n^-(\mathbb{R})$ for all x , there exists a straight line along which f is neither concave nor convex.

Under the assumptions of this statement, the sets $U_1 = \{x \in \mathbb{R}^n | \mathcal{H}f(x) \in S_n^-(\mathbb{R}) \setminus \{0\}\}$ and $U_2 = \{x \in \mathbb{R}^n | \mathcal{H}f(x) \in S_n^+(\mathbb{R}) \setminus \{0\}\}$ are open and non empty: if U_1 is empty, f is necessarily convex. If x_1 is in U_1 , $\mathcal{H}f(x_1)$ admits a strictly negative eigenvalue. Hence for x close enough to x_1 , $\mathcal{H}f(x)$ admits a strictly positive eigenvalue and since $\mathcal{H}f(x) \in S_n^+(\mathbb{R}) \cup S_n^-(\mathbb{R})$ by hypothesis, necessarily $\mathcal{H}f(x)$ is in U_1 . We are going to apply the following lemma to the continuous function $A = \mathcal{H}f$ and the sets U_1 and U_2 .

Lemma 6.3. *If $A : \mathbb{R}^n \rightarrow M_n(\mathbb{R})$ is a continuous function and U_1 and U_2 are two disjoint open sets on which A does not vanish, there exists $(x_1, x_2) \in U_1 \times U_2$ such that*

$$x_1 - x_2 \notin \text{Ker}A(x_1) \cup \text{Ker}A(x_2).$$

Now, let us take (x_1, x_2) in $U_1 \times U_2$ such that $x_1 - x_2 \notin \text{Ker}\mathcal{H}f(x_1) \cup \text{Ker}\mathcal{H}f(x_2)$ and define $g(t) = f(tx_1 + (1-t)x_2)$. To show that the \mathcal{C}^2 function g is neither concave nor convex, we evaluate its second derivative:

$$g''(t) = \mathcal{H}f(tx_1 + (1-t)x_2)(x_1 - x_2) \cdot (x_1 - x_2).$$

If A is in $S_n^+(\mathbb{R}) \cup S_n^-(\mathbb{R})$, $Ax \cdot x = 0$ if and only if $Ax = 0$. Since $\mathcal{H}f(x_1)$ (resp. $\mathcal{H}f(x_2)$) is in $S_n^-(\mathbb{R})$ (resp. $S_n^+(\mathbb{R})$), and $x_1 - x_2 \notin \text{Ker}\mathcal{H}f(x_1) \cup \text{Ker}\mathcal{H}f(x_2)$, we obtain on one hand $g''(1) = \mathcal{H}f(x_1)(x_1 - x_2) \cdot (x_1 - x_2) < 0$ since $x_1 - x_2$ is not in $\text{Ker}\mathcal{H}f(x_1)$, and on the other hand $g''(0) = \mathcal{H}f(x_2)(x_1 - x_2) \cdot (x_1 - x_2) > 0$ since $x_1 - x_2$ is not in $\text{Ker}\mathcal{H}f(x_2)$. Thus, g is neither concave nor convex. \square

The following proof was improved by J.-C. Sikorav.

Proof of Lemma 6.3. For each $x_1^\circ \in U_1$, since $A(x_1^\circ)$ is a nonzero matrix, there exists x_2° in the open set U_2 such that $A(x_1^\circ)(x_1^\circ - x_2^\circ) \neq 0$. Since $(x_1, x_2) \mapsto A(x_1)(x_1 - x_2)$ is continuous, we may assume up to a diminution of U_1 and U_2 that $A(x_1)(x_1 - x_2) \neq 0$ for all $(x_1, x_2) \in U_1 \times U_2$.

Now let us fix x_2° in U_2 . Again, since $A(x_2^\circ)$ is nonzero, there exists x_1° in the open set U_1 such that $A(x_2^\circ)(x_1^\circ - x_2^\circ) \neq 0$, and the previous argument gives that $A(x_1^\circ)(x_1^\circ - x_2^\circ) \neq 0$, hence the conclusion. \square

The two next propositions deals with the behaviour of the variational and viscosity operators when reducing or transforming the Hamiltonian.

Proposition 6.4 (Affine transformations). *Let $H : \mathbb{R}^d \rightarrow \mathbb{R}$ be an integrable Hamiltonian, A be an invertible matrix of size d , b and n be vectors of \mathbb{R}^d , α a real value and λ a non zero real value, and define $K(p) = \frac{1}{\lambda}H(Ap + b) + p \cdot n + \alpha$. If $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is \mathcal{C}^1 , we define $v(t, q) = u(\lambda t, {}^tAq + \lambda tn) + b \cdot q + \alpha \lambda t$, and then*

$$\partial_t u(\lambda t, {}^tAq + \lambda tn) + K(\partial_q u(\lambda t, {}^tAq + \lambda tn)) = 0 \iff \partial_t v(t, q) + H(\partial_q v(t, q)) = 0.$$

If H is \mathcal{C}^2 with second derivative bounded by C , and u_0 is a Lipschitz B -semiconcave initial condition, we define $v_0(q) = u_0({}^tAq) + b \cdot q$. Then

$$V_{0,H}^t v_0(q) = V_{0,K}^{\lambda t} u_0({}^tAq + \lambda tn) + b \cdot q + \alpha \lambda t$$

for all (t, q) and

$$R_{0,H}^t v_0(q) = R_{0,K}^{\lambda t} u_0({}^tAq + \lambda tn) + b \cdot q + \alpha \lambda t$$

as long as $t < 1/||A||^2 BC$, since $||d^2 K|| \leq C||A||^2/\lambda$ and v_0 is $B||A||^2$ -semiconcave.

Proposition 6.5 (Reduction). *Let $(p_1, p_2) \mapsto H(p_1, p_2)$ be a C^2 Hamiltonian with second derivative bounded by C , with $(p_1, p_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Let us fix p_2 in \mathbb{R}^{d_2} and define $K(p_1) = H(p_1, p_2)$. If $u : \mathbb{R} \times \mathbb{R}^{d_1} \rightarrow \mathbb{R}$ is C^1 and $v(t, q_1, q_2) = u(t, q_1) + p_2 \cdot q_2$,*

$$\partial_t u(t, q_1) + K(\partial_{q_1} u(t, q_1)) = 0 \iff \partial_t v(t, q_1, q_2) + H(\partial_{q_1} v(t, q_1, q_2), \partial_{q_2} v(t, q_1, q_2)) = 0.$$

If u_0 is a Lipschitz B -semiconcave function on \mathbb{R}^{d_1} , and $v_0(q_1, q_2) = u_0(q_1) + p_2 \cdot q_2$, then

$$V_{0,H}^t v_0(q_1, q_2) = V_{0,K}^t u_0(q_1) + p_2 \cdot q_2$$

for all (t, q_1, q_2) and

$$R_{0,H}^t v_0(q_1, q_2) = R_{0,K}^t u_0(q_1) + p_2 \cdot q_2,$$

as long as $t < 1/BC$, since $\|d^2 K\| \leq C$ and v_0 is B -semiconcave.

Propositions 6.4 and 6.5 are proved in the same way. The first equivalence is a straightforward calculation, the viscosity equality is obtained by applying the same transformation or reduction on the test functions (see Definition 1.4), and the variational equality is obtained for small time by applying Theorem 1.24 with the domain of validity given for integrable Hamiltonians, which is the same for (K, u_0) and (H, v_0) .

6.2 Proof of Theorem 6.1 in the case of a quadratic saddle Hamiltonian

The aim of this section is to prove the following counterpart of Theorem 6.1 in the case of a quadratic saddle Hamiltonian on \mathbb{R}^2 .

Proposition 6.6. *If $H(p_1, p_2) = p_1 p_2$ is defined on \mathbb{R}^2 , For all $L > 0$, there exists a L -Lipschitz, L -semiconcave initial condition u_0 such that for all $t < 1/2L$, $R_{0,H_0}^t u_0 \neq V_{0,H_0}^t u_0$.*

Let $a < b$ and $u(q_1, q_2) = \min(a(q_1^2 - q_2), b(q_1^2 - q_2))$.

In a first time we are going to look at the wavefront for u , show that it admits a unique continuous section for all t , determine the function giving this section, and exhibit when $a > 0$ a point where this function is not a subsolution of (HJ), and a fortiori is not a viscosity solution. After that we will replace u by a Lipschitz function without modifying the wavefront in the neighbourhood of the point of interest. We still denote by $R_0^t u$ and call *variational solution* the unique continuous function whose graph is contained in the wavefront associated with u at time t even if u is not globally Lipschitz.

Lemma 6.7. *If $q_1 \leq -(b + a/2)t$,*

$$R_0^t u(t, q_1, q_2) = \min(a((q_1 + at)^2 - q_2), b((q_1 + bt)^2 - q_2)).$$

Proof. We are going to show that the unique continuous section of the wavefront is the graph of the continuous function defined piecewise by Figure 6.1, where the blue and black piece of parabola is parametrized by $q_2 = q_1^2 + 2t(a+b)q_1 + \frac{a^3-b^3}{a-b}t^2$, $q_1 \leq -(b+a/2)t$.

Let us fix $t > 0$ and give a parametrization of the wavefront at time t , denoted \mathcal{F}_t . The derivative of u is equal to $a \begin{pmatrix} 2q_1 \\ -1 \end{pmatrix}$ if $q_1^2 > q_2$ and to $b \begin{pmatrix} 2q_1 \\ -1 \end{pmatrix}$ if $q_1^2 < q_2$. The Clarke derivative of u is then a point outside the parabola $\{q_1^2 = q_2\}$, and $\partial u(q, q^2) = \left\{ p \begin{pmatrix} 2q \\ -1 \end{pmatrix}, p \in [a, b] \right\}$ for all q in \mathbb{R} .

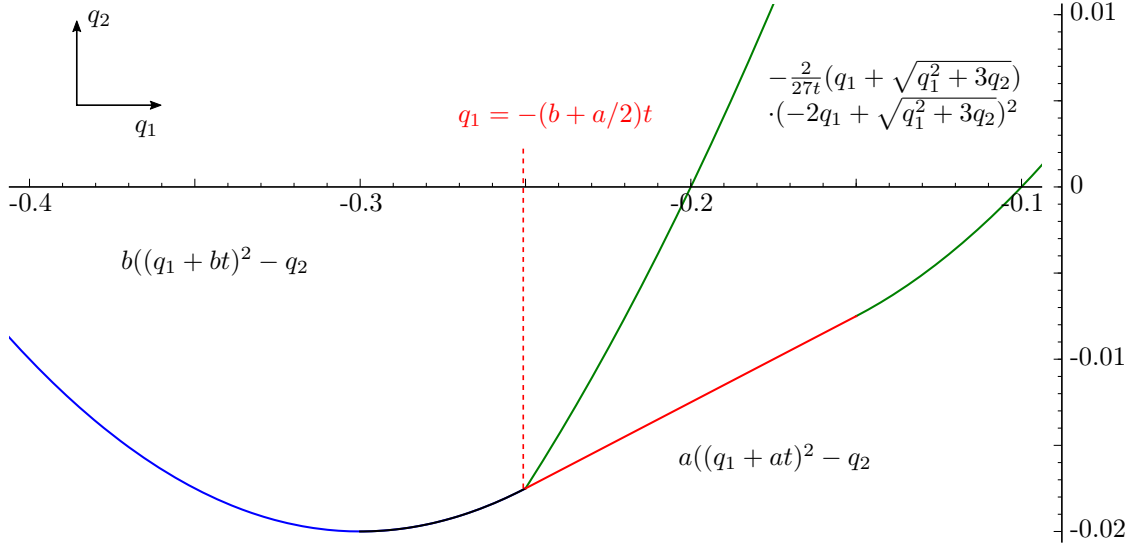


Figure 6.1: Variational solution at time t , here for $a = 1$, $b = 2$ and $t = 1/10$.

The Hamiltonian flow of H writes $\phi_0^t(q_1, q_2, p_1, p_2) = (q_1 + tp_2, q_2 + tp_1, p_1, p_2)$ and the action of a Hamiltonian trajectory depends only on the (constant) impulsions along the trajectory: $\mathcal{A}_0^t(\gamma) = t(p \cdot \nabla H(p) - H(p)) = tH(p) = (t - s)p_1 p_2$.

A parametrization of the wavefront is then:

$$\begin{aligned} \mathcal{F}_t^a &: \begin{cases} q_1 - at, \\ q_2 + 2atq_1, \\ a(q_1^2 - q_2) - 2a^2tq_1, \end{cases} & q_1^2 > q_2, \\ \mathcal{F}_t^b &: \begin{cases} q_1 - bt, \\ q_2 + 2btq_1, \\ b(q_1^2 - q_2) - 2b^2tq_1, \end{cases} & q_1^2 < q_2, \\ \mathcal{F}_t^0 &: \begin{cases} q - pt, \\ q^2 + 2ptq, \\ -2p^2tq, \end{cases} & (p, q) \in [a, b] \times \mathbb{R}. \end{aligned}$$

The two pieces of wavefront issued from the non singular part of u can be written directly as graphs of \mathcal{C}^1 solutions of the (HJ) equation:

$$\begin{aligned} (Q_1, Q_2, S) \in \mathcal{F}_t^a &\Leftrightarrow \begin{cases} S = a((Q_1 + ta)^2 - Q_2), \\ Q_1^2 + 4atQ_1 + 3a^2t^2 > Q_2 \end{cases} \\ (Q_1, Q_2, S) \in \mathcal{F}_t^b &\Leftrightarrow \begin{cases} S = b((Q_1 + tb)^2 - Q_2), \\ Q_1^2 + 4btQ_1 + 3b^2t^2 < Q_2 \end{cases} \end{aligned}$$

We define $u_a(t, Q_1, Q_2) = a((Q_1 + ta)^2 - Q_2)$ and $u_b(t, Q_1, Q_2) = b((Q_1 + tb)^2 - Q_2)$.

The piece of wavefront issued from the singularity is a \mathcal{C}^1 2-submanifold with two one-dimensional boundaries given by $p = a$ and $p = b$, projecting respectively on the parabolae $\mathcal{P}_a : Q_1^2 + 4atQ_1 + 3a^2t^2 = Q_2$ and $\mathcal{P}_b : Q_1^2 + 4btQ_1 + 3b^2t^2 = Q_2$, and a one-dimensional compact fold given by the parameters $(p, -tp/2)$ for p in $[a, b]$.

Both parts of this wavefront may be seen as the graph of C^1 solutions of (HJ):

$$(Q_1, Q_2, S) \in \mathcal{F}_t^0 \Leftrightarrow \begin{cases} S = -\frac{2}{27t} \left(-2Q_1 + \sqrt{Q_1^2 + 3Q_2}\right)^2 \left(Q_1 + \sqrt{Q_1^2 + 3Q_2}\right), \\ Q_1^2 + 3Q_2 > 0, \\ -2Q_1 + \sqrt{Q_1^2 + 3Q_2} \in [3ta, 3tb], \end{cases} \\ \text{or} \\ \begin{cases} S = -\frac{2}{27t} \left(-2Q_1 - \sqrt{Q_1^2 + 3Q_2}\right)^2 \left(Q_1 - \sqrt{Q_1^2 + 3Q_2}\right), \\ Q_1^2 + 3Q_2 > 0, \\ -2Q_1 - \sqrt{Q_1^2 + 3Q_2} \in [3ta, 3tb]. \end{cases}$$

To see this, we eliminate the q variable in the system of equations $\begin{cases} Q_1 = q - pt, \\ Q_2 = q^2 + 2ptq \end{cases}$ which leads to $p = \frac{1}{3t} \left(-2Q_1 \pm \sqrt{Q_1^2 + 3Q_2}\right)$, and then write that $S = -2tp^2q$ with $q = Q_1 + tp$.

We define

$$u_s(t, Q_1, Q_2) = -\frac{2}{27t} (Q_1 + \sqrt{Q_1^2 + 3Q_2}) (-2Q_1 + \sqrt{Q_1^2 + 3Q_2})^2, \\ \tilde{u}_s(t, Q_1, Q_2) = -\frac{2}{27t} (Q_1 - \sqrt{Q_1^2 + 3Q_2}) (-2Q_1 - \sqrt{Q_1^2 + 3Q_2})^2.$$

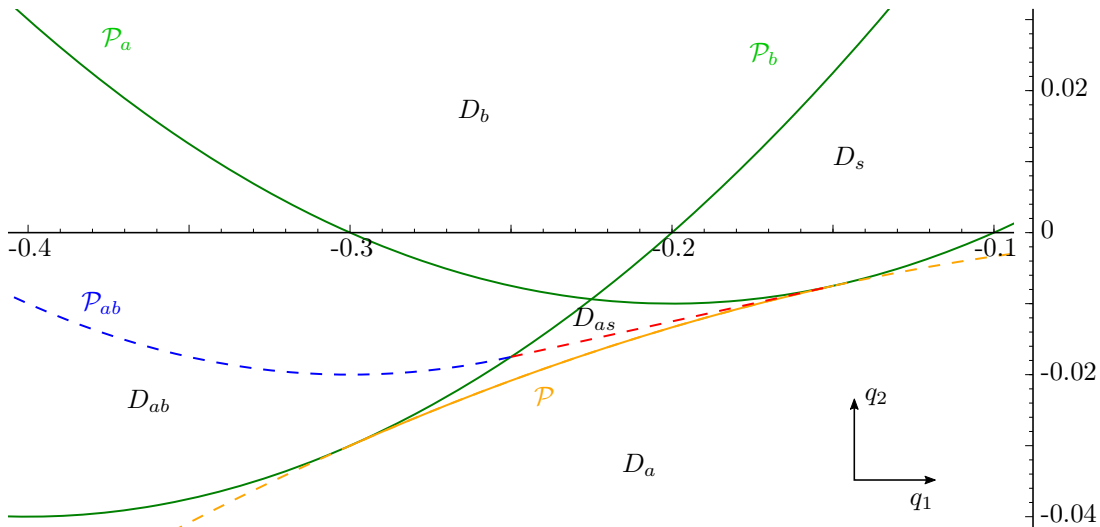


Figure 6.2: Projection of the wavefront on \mathbb{R}^2 , for $a = 1$, $b = 2$ and $t = 1/10$.

The projection of \mathcal{F}_t on \mathbb{R}^2 is described on Figure 6.2, where five domains are defined by the parabolae \mathcal{P}_a , \mathcal{P}_b and $\mathcal{P} : Q_1^2 + 3Q_2 = 0$. The projection is onto on D_a , D_b and D_s , and the variational solution is hence given respectively by u_a , u_b and u_s on these sets. On D_{ab} , the variational solution is given by $\min(u_a, u_b, \tilde{u}_s)$, but one can show that \tilde{u}_s is greater than both u_a and u_b on this set, and as a consequence the variational solution is given by $\min(u_a, u_b)$. On D_{as} , the variational solution is given by $\min(u_a, u_s, \tilde{u}_s)$, but one can show that \tilde{u}_s is greater than both u_a and u_s on this set, and as a consequence the variational solution is given by $\min(u_a, u_s)$.

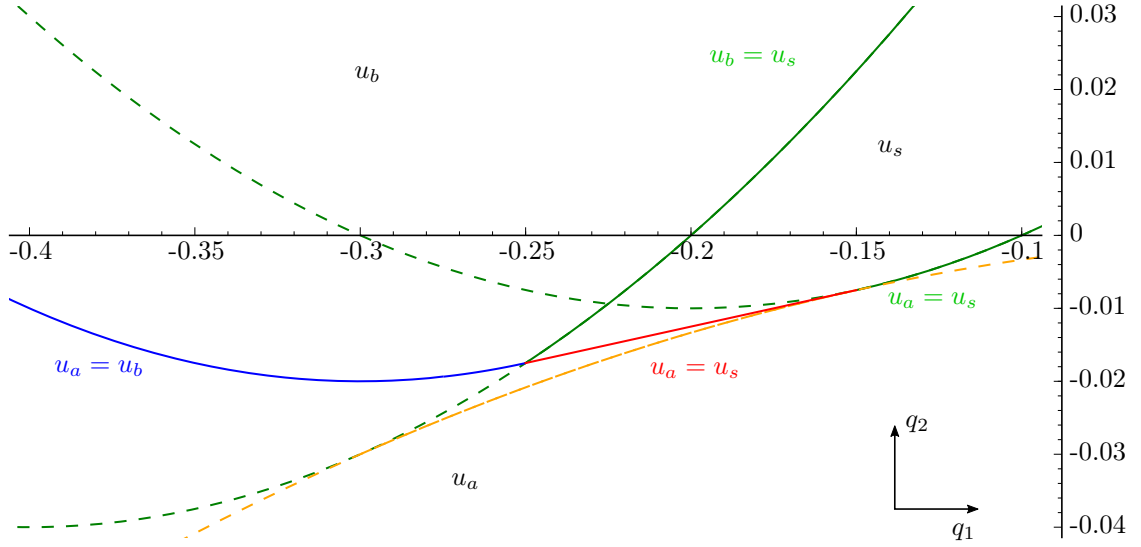


Figure 6.3: Functions giving the continuous section of the wavefront, for $a = 1$, $b = 2$ and $t = 1/10$.

Resolving $u_a(t, Q_1, Q_2) = u_b(t, Q_1, Q_2)$ gives the equation of a fourth parabola, namely $\mathcal{P}_{ab} : Q_2 = Q_1^2 + 2(a+b)tQ_1 + t^2(a^2 + ab + b^2)$, presented in blue on Figures 6.2 and 6.3, and the first coordinate of the point of intersection between \mathcal{P}_{ab} and \mathcal{P}_b is $-(b + a/2)t$.

As a consequence, the lemma holds. \square

Proof of Proposition 6.6. Let $a < b$ and $u(q_1, q_2) = \min(a(q_1^2 - q_2), b(q_1^2 - q_2))$. Lemma 6.7 gives the value of $R_0^t u(t, q_1, q_2)$ for $q_1 \leq -(b + a/2)t$. Let us prove that this variational solution denies the Hamilton-Jacobi equation at the point (t, q_1, q_2) if

$$q_2 = q_1^2 + 2(a+b)tq_1 + t^2(a^2 + ab + b^2) \text{ and } -(a+b)t < q_1 < -(b+a/2)t.$$

This corresponds to the black piece of parabola on Figure 6.1, which exists only if $a > 0$.

Remark 6.8. The red piece of curve on Figures 6.1 and 6.3 represents a shock (*i.e.* with change of derivative) between \mathcal{F}_t^a and \mathcal{F}_t^0 . One can show that the variational solution also denies the Hamilton-Jacobi equation in the viscosity sense along this shock. The variational solution is C^1 , hence viscosity, along the green curves, and shocks on the blue piece of parabola satisfy Oleinik's entropy condition. In other words, the variational solution satisfies the Hamilton-Jacobi equation in the viscosity sense everywhere except for the black and red curves presented on Figure 6.1.

Let us exhibit a test function denying the viscosity equation: we define the mean function $\phi = \frac{1}{2}(u_a + u_b)$ which is C^1 , larger than $\min(u_a, u_b)$ on a neighbourhood of (t, q_1, q_2) and equal to it at (t, q_1, q_2) since $u_a(t, q_1, q_2) = u_b(t, q_1, q_2)$, so that $u - \phi$ attains a local maximum at (t, q_1, q_2) . The derivatives of ϕ are given by

$$\begin{aligned} \partial_t \phi(t, q_1, q_2) &= a^2(q_1 + at) + b^2(q_1 + bt), \\ \partial_{q_1} \phi(t, q_1, q_2) &= a(q_1 + at) + b(q_1 + bt), \\ \partial_{q_2} \phi(t, q_1, q_2) &= -\frac{1}{2}(a + b). \end{aligned}$$

We compute

$$\begin{aligned} & \partial_t \phi(t, q_1, q_2) + H(\partial_q \phi(t, q_1, q_2)) \\ &= a^2(q_1 + at) + b^2(q_1 + bt) - \frac{1}{2}(a+b)(a(q_1 + at) + b(q_1 + bt)) \\ &= \frac{1}{2}(a-b)^2(at + bt + q_1) > 0 \end{aligned}$$

when $q_1 > -(a+b)t$, and as a consequence the variational solution is not a viscosity subsolution at the point (t, q_1, q_2) .

We end the proof modifying u outside a ball: note that u is $2b$ -Lipschitz and $2b$ -semiconcave on the ball $B((0, 0), 1)$. Let u_0 be a $3b$ -Lipschitz, $3b$ -semiconcave function on \mathbb{R}^2 coinciding with u on this ball.

For $t < 2/b$, the black piece of parabola \mathcal{P}_{ab} where the (HJ) equation is not satisfied in the viscosity sense is contained in the open ball $B(0, 2bt) \subset \mathbb{R}^2$: this can be seen on its characterization, $\{q_2 = q_1^2 + 2(a+b)tq_1 + t^2(a^2 + ab + b^2), q_1 \in [-(a+b)t, -(b+a/2)t]\}$, noticing that the minimum and maximum values of q_2 are negative, respectively attained for $q_1 = -(a+b)t$ and $q_1 = -(b+a/2)t$.

Let us prove that for $t < 1/5b$ (which is smaller than $2/b$), the wavefront $\mathcal{F}_t u_0$ associated with u_0 coincides with the wavefront associated with u above the ball $B(0, 2bt)$. To do so, it is enough to prove that the trajectories giving the wavefront above $B(0, 2bt)$ are issued from the domain $B(0, 1)$ where u and u_0 coincide.

If Q is in $B(0, 2bt)$ and (Q, S) is in $\mathcal{F}_t u_0$, there exists (q, p) with $p \in \partial u_0(q)$ such that $Q = q + t\nabla H(p)$. Since u_0 is $3b$ -Lipschitz and $\|\nabla H(p)\| = \|p\|$, this implies that $\|Q - q\| \leq 3bt$, and as a consequence $q \in B(0, 5bt) \subset B(0, 1)$.

If (Q, S) is in $\mathcal{F}_t u$, there exists (q, p) with $p \in \partial u_0(q)$ such that $Q = q + t\nabla H(p)$. The explicit expression of the Clarke derivative of u_0 on \mathbb{R}^2 gives that $\|p\| \leq bt(1 + 2\|q\|)$ and as a consequence, since $\|Q - q\| \leq t\|p\|$,

$$\|q\|(1 - 2bt) \leq \|Q\| + bt.$$

Now, since $\|Q\| \leq 2bt$ and $\frac{3x}{1-2x} \leq 1$ for $x \leq 1/5$, q belongs to $B(0, 1)$ when $t < 1/5b$.

In particular, the continuous sections of the wavefronts are the same above $B(0, 2bt)$, which contains the black piece of parabola \mathcal{P}_{ab} for $t < 1/5b$. Since $R_0^t u_0$ is given by the (unique) continuous section, $R_0^t u_0(q) = R_0^t u(q)$ and for $q \in (-(a+b)t, -(b+a/2)t)$.

Finally, choosing $b = L/3$ and for example $a = L/6$ (but any $0 < a < b$ would work), we showed that for all $L > 0$, there exists a L -Lipschitz, L -semiconcave function u_0 such that the variational solution $R_0^t u_0$ denies the (HJ) equation in the viscosity sense on the set $\{(t, q_1, q_2), t \in (0, 1/2L), q_1 \in (-(a+b)t, -(b+a/2)t), q_2 = q_1^2 + 2(a+b)tq_1 + t^2 \frac{a^3 - b^3}{a-b}\}$. \square

6.3 Proof of Theorem 6.1

Proof of Theorem 6.1. If H is a neither convex nor concave integrable Hamiltonian Proposition 6.2 states that there is either a straight line along which H is neither convex nor concave, or a point p_0 such that the Hessian matrix $\mathcal{H}H(p_0)$ has both a strictly positive and a strictly negative eigenvalue.

In the first case, applying an affine transformation on the vector space we may assume without loss of generality (see Proposition 6.4) that $p \in \mathbb{R} \mapsto H(p, 0, \dots, 0)$ is neither convex nor concave,

and we denote by $\bar{H}(p) = H(p, 0, \dots, 0)$ the reduced Hamiltonian. Since $\bar{H} : \mathbb{R} \rightarrow \mathbb{R}$ is neither convex nor concave, there exist in particular p_1 and p_2 such that $\bar{H}''(p_1) > 0$ and $\bar{H}''(p_2) < 0$, and we may assume without loss of generality that $p_1 < p_2$, using Proposition 6.4 with $A = -\text{id}$.

The graph of \bar{H} cannot be equal to the straight line joining $(p_1, \bar{H}(p_1))$ and $(p_2, \bar{H}(p_2))$, or the second derivatives at p_1 and p_2 would be zero. So, either there exists a point of the graph strictly above this line (*i.e.* Oleinik's entropy condition is denied, in terms of Definition 1.10), or a point of the graph strictly under this line (*i.e.* the reverse entropy condition is denied).

If the entropy condition is satisfied, we are going to use the following Lemma, proved at the end of this paragraph, in order to apply Proposition 5.6 to \bar{H} .

Lemma 6.9. *We define*

$$p_2^* = \inf \left\{ p \in (p_1, p_2), \frac{\bar{H}(p) - \bar{H}(p_1)}{p - p_1} = \sup_{\tilde{p} \in (p_1, p_2)} \frac{\bar{H}(\tilde{p}) - \bar{H}(p_1)}{\tilde{p} - p_1} \right\}.$$

If $\bar{H}''(p_1) > 0$ and the entropy condition is denied between p_1 and p_2 , then both bounds are attained, hence p_2^ is in (p_1, p_2) and $\bar{H}'(p_1) < \bar{H}'(p_2^*) = \frac{\bar{H}(p_2^*) - \bar{H}(p_1)}{p_2^* - p_1}$.*

Furthermore, if p_2 is fixed, $p_1 \mapsto \bar{H}'(p_2^)$ is increasing in a neighbourhood of p_1 .*

Since $p_1 \mapsto \bar{H}'(p_2^*)$ is (strictly) increasing in a neighbourhood of p_1 , Sard's theorem gives that we may assume without loss of generality that $\bar{H}'(p_2^*)$ is a regular value of \bar{H}' , up to a perturbation of p_1 within the open set $\{\bar{H}'' > 0\}$. Summing all this with the result of Lemma 6.9, we can check that \bar{H} satisfies the assumptions of Proposition 5.6 between p_1 and p_2^* : $\bar{H}''(p_1) > 0$, $\bar{H}''(p_2^*) \neq 0$, $\bar{H}'(p_2^*) = \frac{\bar{H}(p_2^*) - \bar{H}(p_1)}{p_2^* - p_1}$, and $\frac{\bar{H}(p) - \bar{H}(p_1)}{p - p_1} < \frac{\bar{H}(p_2^*) - \bar{H}(p_1)}{p_2^* - p_1}$ for all p in (p_1, p_2^*) , hence the entropy condition is strictly satisfied between p_1 and p_2^* .

Proposition 5.6 applied to \bar{H} between p_1 and p_2^* gives then a Lipschitz semiconcave initial condition u_0 , with right and left derivatives at 0 respectively equal to p_1 and p_2^* , such that the variational solution denies the (HJ) equation associated with \bar{H} for all t small enough.

If the reverse entropy condition is denied, we define

$$p_1^* = \sup \left\{ p \in (p_1, p_2) \mid \frac{\bar{H}(p_2) - \bar{H}(p)}{p_2 - p} = \inf_{\tilde{p} \in [p_1, p_2]} \frac{\bar{H}(p_2) - \bar{H}(\tilde{p})}{p_2 - \tilde{p}} \right\}$$

and work as previously to show that Proposition 5.10 can be applied to \bar{H} between p_1^* and p_2 . Applying Proposition 6.4 again, we finally get a Lipschitz semiconvex initial condition u_0 , with left and right derivatives at 0 respectively equal to p_1^* and p_2 , such that the variational solution denies the (HJ) equation associated with \bar{H} in the viscosity sense for all t small enough.

With Proposition 6.5, we return to H , and get from u_0 on \mathbb{R} a Lipschitz initial condition $v_0 : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, either semiconvex or semiconcave, for which $R_{0,H}^t v_0 \neq V_{0,H}^t v_0$ for all $t < T$.

In the second case, we may assume that the point of interest is a (strict) saddle point at 0: if p_0 denotes the point for which $\mathcal{H}H(p_0)$ has both a positive and a negative eigenvalue, take $\tilde{H}(p) = H(p_0 - p) + p \cdot \nabla H(p_0) - H(p_0)$ and apply Proposition 6.4.

Then, up to another linear transformation on the vector space, the Hamiltonian may even be taken as

$$H(p_1, p_2, \dots, p_d) = p_1 p_2 + K(p_1, p_2, \dots, p_d),$$

where K is a C^2 Hamiltonian vanishing at 0 to the second order, *i.e.* such that $K(0, \dots, 0) = 0$, $\partial_{p_1, 2} K(0, \dots, 0) = 0$ and $\partial_{(p_1, p_2)}^2 K(0, \dots, 0) = 0$. We denote by \bar{H} (resp. \bar{K}) the reduced Hamiltonians such that

$$\bar{H}(p_1, p_2) = H(p_1, p_2, 0, \dots, 0) = p_1 p_2 + \bar{K}(p_1, p_2).$$

We still denote by C a bound of the second derivatives of H and \bar{H} . It is necessarily larger than 2.

Now, we define

$$\bar{H}_\varepsilon(p_1, p_2) = \frac{1}{\varepsilon^2} \bar{H}(\varepsilon p_1, \varepsilon p_2) = p_1 p_2 + \frac{1}{\varepsilon^2} \bar{K}(\varepsilon p_1, \varepsilon p_2)$$

and

$$\bar{H}_0(p_1, p_2) = p_1 p_2.$$

The second derivative of \bar{H}_ε is also bounded by C .

In Proposition 6.6 (see Section 6.2) we built an initial condition such that the variational solution for \bar{H}_0 is not a viscosity solution. We fix $L > 0$ and take u_0 as in Proposition 6.6: for all $0 < t < 1/2L$, there exists a point q_t such that $R_{0, \bar{H}_0}^t u_0(q_t) \neq V_{0, \bar{H}_0}^t u_0(q_t)$. Let us now fix t in $(0, 1/LC) \subset (0, 1/2L)$.

Since u_0 is L -semiconcave, Theorem 1.24 states that $R_{0, \bar{H}_0}^t u_0 = \mathbf{R}_{0, \bar{H}_0}^t u_0$ as long as $t < 1/2L$ and $R_{0, \bar{H}_\varepsilon}^t u_0 = \mathbf{R}_{0, \bar{H}_\varepsilon}^t u_0$ as long as $t < 1/LC (< 1/2L)$.

Addenda 2.26 and 3.12 then give that

$$\|R_{0, \bar{H}_\varepsilon}^t u_0(q_t) - R_{0, \bar{H}_0}^t u_0(q_t)\| \leq t \sup_{\|p\| \leq L} \frac{1}{\varepsilon^2} \bar{K}(\varepsilon p)$$

and

$$\|V_{0, \bar{H}_\varepsilon}^t u_0(q_t) - V_{0, \bar{H}_0}^t u_0(q_t)\| \leq t \sup_{\|p\| \leq L} \frac{1}{\varepsilon^2} \bar{K}(\varepsilon p).$$

Since \bar{K} is zero until second order at 0, $\frac{1}{\varepsilon^2} \bar{K}(\varepsilon p) = o(\|p\|^2)$ and $\sup_{\|p\| \leq L} \frac{1}{\varepsilon^2} \bar{K}(\varepsilon p)$ tends to 0 when ε tends to 0. Thus, there exists $\varepsilon > 0$ (depending on t) such that

$$\sup_{\|p\| \leq L} \frac{1}{\varepsilon^2} \bar{K}(\varepsilon p) < \frac{1}{3t} \left| R_{0, \bar{H}_0}^t u_0(q_t) - V_{0, \bar{H}_0}^t u_0(q_t) \right|.$$

For such an ε , necessarily $R_{0, \bar{H}_\varepsilon}^t u_0(q_t) \neq V_{0, \bar{H}_\varepsilon}^t u_0(q_t)$.

Let us go back to \bar{H} , using Proposition 6.4 with $\lambda = \varepsilon^2$, $A = \varepsilon \text{id}$ and n , b and α equal to zero. Defining $v_0(q) = u_0(\varepsilon q)$, we get

$$V_{0, \bar{H}}^{t/\varepsilon^2} v_0(q_t/\varepsilon) = V_{0, \bar{H}_\varepsilon}^t u_0(q_t)$$

and

$$R_{0, \bar{H}}^{t/\varepsilon^2} v_0(q_t/\varepsilon) = R_{0, \bar{H}_\varepsilon}^t u_0(q_t)$$

since $\frac{t}{\varepsilon^2} < \frac{1}{\varepsilon^2 LC}$, and as a consequence

$$V_{0, \bar{H}}^{t/\varepsilon^2} v_0(q_t/\varepsilon) \neq R_{0, \bar{H}}^{t/\varepsilon^2} v_0(q_t/\varepsilon).$$

Note that since v_0 is $\varepsilon^2 L$ -semiconcave, t/ε^2 belongs to the domain of validity of Theorem 1.24 which is here $(0, 1/\varepsilon^2 LC)$. To finish the proof, we use Proposition 6.5 to get a initial condition suiting the non reduced Hamiltonian H as in the first case. \square

Proof of Lemma 6.9. The function $f : p \mapsto \frac{H(p) - H(p_1)}{p - p_1}$ may be extended continuously at p_1 by $H'(p_1)$, hence it reaches a maximum on $[p_1, p_2]$, denoted M . It cannot be attained at p_1 , or else the Taylor expansion of $\frac{H(p) - H(p_1)}{p - p_1} \leq H'(p_1)$ gives that $H''(p_1) \leq 0$, which is excluded, and

as a consequence $M > H'(p_1)$. It cannot be attained at p_2 because $\frac{H(p)-H(p_1)}{p-p_1} \leq \frac{H(p_2)-H(p_1)}{p_2-p_1}$ for all p in $[p_1, p_2)$ if and only if the entropy condition is satisfied between p_1 and p_2 , which is excluded. We hence proved that the supremum is attained on (p_1, p_2) . The infimum hence exists and belongs to $[p_1, p_2)$. By continuity of the function f , $f(p_2^*) = M$, which implies that $p_2^* > p_1$ since $f(p_1) = H'(p_1) < M$, and the infimum is a minimum. Since p_2^* is in (p_1, p_2) and maximises f , it is a critical point of f and hence $H'(p_2^*) = \frac{H(p_2^*)-H(p_1)}{p_2^*-p_1}$. As a consequence $H'(p_1) < H'(p_2^*) = M$.

For $\varepsilon > 0$ small enough, $p_1 + \varepsilon < p_2^*$, $H''(p_1 + \varepsilon) > 0$ and the entropy condition is denied between $p_1 + \varepsilon$ and p_2 . We denote by $p_{2,\varepsilon}^*$ the quantity associated with $p_1 + \varepsilon$ and p_2 as previously and show that $H'(p_{2,\varepsilon}^*) > H'(p_2^*)$ to prove the statement.

On one hand, by definition of p_2^* , the entropy condition is strictly satisfied between p_1 and p_2^* , and in particular since $p_1 + \varepsilon$ is in (p_1, p_2^*) ,

$$\frac{H(p_2^*) - H(p_1 + \varepsilon)}{p_2^* - (p_1 + \varepsilon)} > \frac{H(p_2^*) - H(p_1)}{p_2^* - p_1} = H'(p_2^*).$$

On the other hand, the previous work applied to $p_{2,\varepsilon}^*$ gives that

$$H'(p_{2,\varepsilon}^*) = \max_{p \in (p_1 + \varepsilon, p_2]} \frac{H(p) - H(p_1 + \varepsilon)}{p - (p_1 + \varepsilon)} \geq \frac{H(p_2^*) - H(p_1 + \varepsilon)}{p_2^* - (p_1 + \varepsilon)},$$

and the two inequalities combined give that $H'(p_{2,\varepsilon}^*) > H'(p_2^*)$. □

Appendix A

Uniqueness of the viscosity solution: a doubling variables argument

The uniqueness of the viscosity operator for H satisfying Hypothesis 1.1 is a consequence of a stronger uniqueness result for unbounded solutions stated by H. Ishii in [Ish84] (Theorem 2.1 with Remark 2.2), see also [CL87]. It is also a consequence of the following finite speed of propagation argument proposed by G. Barles in [Bar94] (Theorem 5.3). We write the proof here for the sake of completeness, adopting his arguments and notations, and using only the second estimate of Hypothesis 1.1.

Proposition A.1 (Finite speed of propagation). *If H satisfies $\|\partial_{q,p}H\| \leq C(1 + \|p\|)$ for some $C > 0$, and u and v are respectively sub- and supersolutions of (HJ) on $[0, T] \times \mathbb{R}^d$ which are L -Lipschitz uniformly in time with respect to the space variable, then:*

$$u(0, \cdot) \leq v(0, \cdot) \text{ on } B(0, R) \implies u \leq v \text{ on } [0, T] \times B(0, R - C(1 + 2L)T)$$

as long as R is strictly larger than $C(1 + 2L)T$.

Consequence A.2. If u and v are two viscosity solutions of (HJ) which are L -Lipschitz with respect to q on $[0, T] \times \mathbb{R}^d$, then for each t in $[0, T]$:

$$|u(t, q) - v(t, q)| \leq \|u(0, \cdot) - v(0, \cdot)\|_{\bar{B}(q, C(1+2L)t)}$$

Proof. We apply Proposition A.1 with $R = C(1 + 2L)t + \delta$ to the subsolution u and the supersolution $v + \|u(0, \cdot) - v(0, \cdot)\|_{\bar{B}(q, R)}$, use the symmetry and let δ tend to 0. \square

Consequence A.3. If u and v are both viscosity solutions on $[0, T] \times \mathbb{R}^d$ that satisfy $u(0, \cdot) = v(0, \cdot)$ on \mathbb{R}^d and are Lipschitz uniformly in time, they coincide on $[0, T] \times \mathbb{R}^d$. In particular, there exists at most one viscosity operator.

Lemma A.4. *If u is a continuous function of $(0, T] \times \mathbb{R}^d$ and also a subsolution of (HJ) on $(0, T] \times \mathbb{R}^d$, then it is a subsolution on $(0, T] \times \mathbb{R}^d$, meaning that if $u - \phi$ attains a strict maximum on $(0, T] \times \mathbb{R}^d$ at some (T, q_0) , the derivatives of ϕ satisfy the required inequality.*

Proof. Take $\phi \in C^\infty$ on $(0, T] \times \mathbb{R}^d$ such that $u - \phi$ attains its strict maximum at some (T, q_0) . Let us consider the functions $(t, q) \mapsto u(t, q) - \phi(t, q) - \frac{\eta}{T-t}$ for small $\eta > 0$. Since $u - \phi$ attains a strict maximum at (T, q_0) , there exists a sequence (t_η, q_η) in $(0, T) \times \mathbb{R}^d$ of local maximal points of $u - \phi - \frac{\eta}{T-t}$ such that (t_η, q_η) tends to (T, q_0) when $\eta \rightarrow 0$.

Since u is a subsolution on $(0, T) \times \mathbb{R}^d$, this implies that:

$$\partial_t \left(\phi(t, q) + \frac{\eta}{T-t} \right) + H \left(t_\eta, q_\eta, \partial_q \left(\phi(t, q) + \frac{\eta}{T-t} \right) \right) \leq 0$$

hence

$$\partial_t \phi(t_\eta, q_\eta) + \frac{\eta}{(T-t_\eta)^2} + H(t_\eta, q_\eta, \partial_q \phi(t_\eta, q_\eta)) \leq 0.$$

The positive term $\frac{\eta}{(T-t_\eta)^2}$ may be dropped, and then the continuity of ϕ gives that:

$$\partial_t \phi(T, q_0) + H(T, q_0, \partial_q \phi(T, q_0)) \leq 0.$$

□

Lemma A.5. *If the assumptions of Proposition A.1 are satisfied, the function $w = u - v$ is a subsolution on $(0, T] \times \mathbb{R}^d$ of*

$$\partial_t w - C(1 + 2L)\|\partial_q w\| = 0.$$

Proof. Let us assume that ϕ is a C^∞ function such that $w - \phi$ attains a strict local maximum at a point (t_0, q_0) in $(0, T) \times \mathbb{R}^d$. The aim is to show that

$$\partial_t \phi(t_0, q_0) \leq C(1 + 2L)\|\partial_q \phi(t_0, q_0)\|.$$

Here is where the variables are doubled: let us define the function

$$\Psi_{\varepsilon, \alpha} : (t, q, s, p) \mapsto u(t, q) - v(s, p) - \frac{\|q - p\|^2}{\varepsilon^2} - \frac{|t - s|^2}{\alpha^2} - \phi(t, q).$$

In particular $\Psi_{\varepsilon, \alpha}(t_0, q_0, t_0, q_0) = w(t_0, q_0) - \phi(t_0, q_0)$ is the local maximum of $w - \phi$ for all $\varepsilon > 0$ and $\alpha > 0$.

Take $r > 0$ such that the maximum of $w - \phi$ on $\bar{B}((t_0, q_0), r)$ is attained only at (t_0, q_0) . Then $\Psi_{\varepsilon, \alpha}$ attains a maximum on the compact set $\bar{B}((t_0, q_0), r) \times \bar{B}((t_0, q_0), r)$, and we denote by $(\bar{t}, \bar{q}, \bar{s}, \bar{p})$ a point reaching this maximum, without forgetting that these quantities depend on ε and α .

Lemma A.6. *The point $(\bar{t}, \bar{q}, \bar{s}, \bar{p})$ satisfies:*

1. $(\bar{t}, \bar{q}), (\bar{s}, \bar{p}) \rightarrow (t_0, q_0)$ when $\varepsilon, \alpha \rightarrow 0$,
2. $\frac{\|\bar{q} - \bar{p}\|}{\varepsilon^2} \leq L$.

Proof. 1. Since $(\bar{t}, \bar{q}, \bar{s}, \bar{p})$ belongs to the compact set $\bar{B}((t_0, q_0), r) \times \bar{B}((t_0, q_0), r)$, accumulation points (t, q, s, p) exist when ε and α tend to zero. These accumulation points must satisfy $(t, q) = (s, p)$: else, the value of $\Psi_{\varepsilon, \alpha}(\bar{t}, \bar{q}, \bar{s}, \bar{p})$ explodes towards $-\infty$ while it is supposed to remain larger than $\Psi_{\varepsilon, \alpha}(t_0, q_0, t_0, q_0)$ which is the maximum of $w - \phi$ and does not therefore depend on ε and α .

Now, let us denote by $(t, q) \in \bar{B}((t_0, q_0), r)$ an accumulation point of both (\bar{t}, \bar{q}) and (\bar{s}, \bar{p}) . Since $\Psi_{\varepsilon, \alpha}(\bar{t}, \bar{q}, \bar{s}, \bar{p}) \geq \Psi_{\varepsilon, \alpha}(t_0, q_0, t_0, q_0) = w(t_0, q_0) - \phi(t_0, q_0)$, we also have using the sign of $-\frac{\|\bar{q} - \bar{p}\|^2}{\varepsilon^2} - \frac{|\bar{t} - \bar{s}|^2}{\alpha^2}$ that

$$u(\bar{t}, \bar{q}) - v(\bar{s}, \bar{p}) - \phi(\bar{t}, \bar{q}) \geq w(t_0, q_0) - \phi(t_0, q_0).$$

Hence if ε and α tend to zero,

$$w(t, q) - \phi(t, q) \geq w(t_0, q_0) - \phi(t_0, q_0),$$

and the fact that (t_0, q_0) is the only point of $\bar{B}((t_0, q_0), r)$ where the maximum is attained concludes the proof.

2. Since $(\bar{t}, \bar{q}, \bar{s}, \bar{q})$ is in the set $\bar{B}((t_0, q_0), r) \times \bar{B}((t_0, q_0), r)$,

$$\Psi_{\varepsilon, \alpha}(\bar{t}, \bar{q}, \bar{s}, \bar{q}) \leq \Psi_{\varepsilon, \alpha}(\bar{t}, \bar{q}, \bar{s}, \bar{p})$$

hence

$$u(\bar{t}, \bar{q}) - v(\bar{s}, \bar{q}) - \frac{|\bar{t} - \bar{s}|^2}{\alpha^2} - \phi(\bar{t}, \bar{q}) \leq u(\bar{t}, \bar{q}) - v(\bar{s}, \bar{p}) - \frac{\|\bar{q} - \bar{p}\|^2}{\varepsilon^2} - \frac{|\bar{t} - \bar{s}|^2}{\alpha^2} - \phi(\bar{t}, \bar{q})$$

and since v is L -Lipschitz,

$$\frac{\|\bar{q} - \bar{p}\|^2}{\varepsilon^2} \leq v(\bar{s}, \bar{q}) - v(\bar{s}, \bar{p}) \leq L\|\bar{q} - \bar{p}\|.$$

□

Now, since $(\bar{t}, \bar{q}, \bar{s}, \bar{p})$ converge to (t_0, q_0, t_0, q_0) , it is in $B((t_0, q_0), r) \times B((t_0, q_0), r)$ for ε and α small enough, and the fact that it maximizes $\Psi_{\varepsilon, \alpha}$ tells us that:

- (\bar{t}, \bar{q}) is a maximum point of

$$(t, q) \mapsto u(t, q) - \underbrace{\left(\phi(t, q) + v(\bar{s}, \bar{p}) + \frac{\|q - \bar{p}\|^2}{\varepsilon^2} + \frac{|t - \bar{s}|^2}{\alpha^2} \right)}_{=\phi_1(t, q)},$$

and since u is a subsolution, the derivatives of ϕ_1 satisfy

$$\partial_t \phi_1(\bar{t}, \bar{q}) + H(\bar{t}, \bar{q}, \partial_q \phi_1(\bar{t}, \bar{q})) \leq 0,$$

hence

$$\partial_t \phi(\bar{t}, \bar{q}) + 2 \cdot \frac{\bar{t} - \bar{s}}{\varepsilon^2} + H\left(\bar{t}, \bar{q}, \partial_q \phi(\bar{t}, \bar{q}) + 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) \leq 0.$$

Note also that since u is L -Lipschitz with respect to q , the q -derivative of ϕ_1 at a point of maximum of $u - \phi$ is necessarily bounded by L , hence:

$$\|\partial_q \phi(\bar{t}, \bar{q}) + 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\| \leq L. \quad (\text{A.1})$$

- (\bar{s}, \bar{p}) is a minimum point of

$$(s, p) \mapsto v(s, p) - \underbrace{\left(u(\bar{t}, \bar{q}) - \phi(\bar{t}, \bar{q}) - \frac{\|\bar{q} - p\|^2}{\varepsilon^2} - \frac{|\bar{t} - s|^2}{\alpha^2} \right)}_{=\phi_2(s, p)},$$

and since v is a supersolution, the derivatives of ϕ_2 satisfy

$$\partial_s \phi_1(\bar{s}, \bar{p}) + H(\bar{s}, \bar{p}, \partial_p \phi_1(\bar{s}, \bar{p})) \leq 0,$$

hence

$$2 \cdot \frac{\bar{t} - \bar{s}}{\varepsilon^2} + H\left(\bar{s}, \bar{p}, 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) \geq 0.$$

Combining the two previous points gives that

$$\begin{aligned} \partial_t \phi(\bar{t}, \bar{q}) &\leq H\left(\bar{s}, \bar{p}, 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) - H\left(\bar{t}, \bar{q}, \partial_q \phi(\bar{t}, \bar{q}) + 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) \\ &\leq H\left(\bar{s}, \bar{p}, 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) - H\left(\bar{t}, \bar{q}, 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) \\ &\quad + \underbrace{H\left(\bar{t}, \bar{q}, 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) - H\left(\bar{t}, \bar{q}, \partial_q \phi(\bar{t}, \bar{q}) + 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right)}_{\leq C(1+2L)\|\partial_q \phi(\bar{t}, \bar{q})\|}. \end{aligned}$$

Let us explain the last point: the estimate (A.1) and the second result of Lemma A.6 state that both $\partial_q \phi(\bar{t}, \bar{q}) + 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}$ and $2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}$ are bounded by $2L$. The assumption made on $\|\partial_{p,q} H\|$ implies that $\partial_p H$ is bounded by $C(1+2L)$ on the set $[0, T] \times \mathbb{R}^d \times \bar{B}(0, 2L)$, and hence

$$\left| H\left(\bar{t}, \bar{q}, 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) - H\left(\bar{t}, \bar{q}, \partial_q \phi(\bar{t}, \bar{q}) + 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) \right| \leq C(1+2L)\|\partial_q \phi(\bar{t}, \bar{q})\|.$$

Lemma A.6 implies that the quantity $H\left(\bar{s}, \bar{p}, 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right) - H\left(\bar{t}, \bar{q}, 2 \cdot \frac{\bar{q} - \bar{p}}{\varepsilon^2}\right)$ tends to 0 when ε and α tend to 0. To finish, since (\bar{t}, \bar{q}) tends to (t_0, q_0) :

$$\partial_t \phi(t_0, q_0) \leq C(1+2L)\|\partial_q \phi(t_0, q_0)\|.$$

We then extend the subsolution property to $\{T\} \times \mathbb{R}^d$ with Lemma A.4. \square

Proof of Proposition A.1. Take $R > C(1+2L)T$ and let us denote by M the maximum of w on the set $[0, T] \times \bar{B}(0, R)$. We are going to prove that for all $\delta > 0$ such that $R > \delta + C(1+2L)T$, $w(t, q) \leq \delta t$ on the set $[0, T] \times B(0, R - C(1+2L)T - \delta)$, using a comparison with an ad hoc smooth solution of $\partial_t w - C(1+2L)\|\partial_q w\| = 0$.

For such a $\delta > 0$, it is possible to find a smooth and increasing function $\chi_\delta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\chi_\delta(r) = 0$ if $r \leq R - \delta$ and $\chi_\delta(r) = M$ if $r \geq R$. Then

$$\phi_\delta : (t, q) \mapsto \chi_\delta(\|q\|) + C(1+2L)t$$

is a smooth solution of $\partial_t w - C(1+2L)\|\partial_q w\| = 0$ on $[0, T] \times \bar{B}(0, R)$. Let us then show that the function $(t, q) \mapsto w(t, q) - \phi_\delta(t, q) - \delta t$ on $[0, T] \times \bar{B}(0, R)$ is non positive.

The maximum of this function cannot be attained at a point (t, q) of $(0, T] \times B(0, R)$, or else the fact that w is a subsolution on $(0, T] \times B(0, R)$ (Lemma A.5) gives that:

$$\partial_t \phi_\delta(t, q) + \delta - C(1+2L)\|\partial_q \phi_\delta(t, q)\| \leq 0.$$

Since ϕ_δ solves the equation in the classical way and δ is positive, this is impossible.

So, either the maximum is attained at a point $(0, q)$, or at a point (t, q) with $\|q\| = R$.

In the first case, the maximum is of the form $w(0, q) - \phi_\delta(\|q\|)$ and is hence non positive since $u \leq v$ on $\{0\} \times \mathbb{R}^d$ and ϕ_δ is non negative.

In the second case, $\phi_\delta(t, q) = M$ and the maximum is of the form $w(t, q) - M - \delta t$. Since w is smaller than M on $[0, T] \times \bar{B}(0, R)$, the maximum is non positive.

Hence, for each (t, q) in $[0, T] \times B(0, R)$,

$$w(t, q) \leq \phi_\delta(t, q) + \delta t.$$

Since $\phi_\delta(t, q)$ is zero on $[0, T] \times B(0, R - C(1+2L)T - \delta)$, on this set we have:

$$w(t, q) \leq \delta t.$$

Letting δ tend to zero gives that $w = u - v \leq 0$ on $[0, T] \times B(0, R - C(1+2L)T)$. \square

Appendix B

Generating families of the Hamiltonian flow

All the results and proofs of this appendix are inspired from [Cha90]. We write them down here only to explicit the time derivatives of the generating families (see Proposition B.5) and the Lipschitz constant in Proposition B.8.

We first state a useful basic technical lemma.

Lemma B.1. *If $u, v : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are C^1 and such that $\text{Lip}(u) < 1$ and $\text{Lip}(v) < 1$, then $f = \text{id} - u$ and $g = \text{id} - v$ are C^1 -diffeomorphisms of \mathbb{R}^n . If $f - g$ is bounded, then $f^{-1} - g^{-1}$ is bounded by $\frac{\|f-g\|_\infty}{1-\text{Lip}(u)}$.*

Proof. Let us first proof that f is a C^1 -diffeomorphism of \mathbb{R}^n . It is clearly C^1 , and is a local diffeomorphism since $\|df\| = \|\text{id} - du\| \geq 1 - \text{Lip}(u) > 0$. To see that it is invertible, we observe that $f(q) = \theta$ can be rewritten as a fixed point problem $q = u(q) + \theta$, where the map $q \mapsto u(q) + \theta$ is contracting.

Now, if $f - g$ is bounded, so is $u - v$, with $\|u - v\|_\infty = \|f - g\|_\infty$. Let us denote $x = f^{-1}(z)$ and $y = g^{-1}(z)$ for some z in \mathbb{R}^n . Then $x = u(x) + z$ and $y = v(y) + z$ and

$$|x - y| \leq |u(x) - v(y)| \leq |u(x) - u(y)| + |u(y) - v(y)| \leq \text{Lip}(u)|x - y| + \|u - v\|_\infty,$$

whence $|x - y| \leq \frac{\|f-g\|_\infty}{1-\text{Lip}(u)}$. □

Let us now state two Grönwall-type estimates on Hamiltonian flows:

Proposition B.2. *Let H and \tilde{H} be two C^2 Hamiltonians on $\mathbb{R} \times T^*\mathbb{R}^d$ such that $\|\partial_{q,p}^2 H\|$ and $\|\partial_{q,p}^2 \tilde{H}\|$ are uniformly bounded by a constant C and $\|\partial_{q,p} H - \partial_{q,p} \tilde{H}\|$ is uniformly bounded by a constant K . Then, if ϕ and $\tilde{\phi}$ denote the Hamiltonian flows respectively associated with H and \tilde{H} , we have for all $s \leq t$:*

1. $\|\phi_s^t - \tilde{\phi}_s^t\| \leq \frac{K}{C}(e^{C(t-s)} - 1)$,
2. $\|d\phi_s^t - \text{id}\| \leq e^{C(t-s)} - 1$.

In particular if $t - s \leq \delta_1 = \frac{\ln(3/2)}{C}$, $\phi_s^t - \text{id}$ is $\frac{1}{2}$ -Lipschitz.

Lemma B.3 (Grönwall's lemma, elementary version). *If $t \mapsto f(t)$ is a continuous non negative function such that $f(s) = 0$ and $f(t) \leq \int_s^t (Cf(u) + K) du$, then $f(t) \leq \frac{K}{C}(e^{C(t-s)} - 1)$.*

Proof. Observe that the assumed inequality can be written

$$\partial_t \left(e^{-C(t-s)} \int_s^t f(u) du \right) \leq e^{-C(t-s)} K(t-s),$$

and integrating this between s and t we get

$$\int_s^t f(u) du \leq K e^{C(t-s)} \int_s^t e^{-C(u-s)} (u-s) du = \frac{K}{C^2} (e^{C(t-s)} - C(t-s) - 1).$$

Reinjecting this into $f(t) \leq \int_s^t C f(u) + K du$ gives the result. \square

Proof of Proposition B.2. Let us define $\Gamma_t(q, p) = (\partial_p H(t, q, p), -\partial_q H(t, q, p))$, so that the Hamiltonian system (HS) can be rewritten $\partial_t \phi_s^t = \Gamma_t(\phi_s^t)$, and $\tilde{\Gamma}$ associated similarly to \tilde{H} .

1. Since $\|\partial_{q,p} H - \partial_{q,p} \tilde{H}\| \leq K$, $\|\Gamma_u - \tilde{\Gamma}_u\| \leq K$ for all u and since Γ_u is C -Lipschitz,

$$\begin{aligned} \|\phi_s^t - \tilde{\phi}_s^t\| &= \left\| \int_s^t \Gamma_u(\phi_s^u) - \tilde{\Gamma}_u(\tilde{\phi}_s^u) du \right\| \\ &\leq \int_s^t \|\Gamma_u(\phi_s^u) - \Gamma_u(\tilde{\phi}_s^u)\| + \|\Gamma_u(\tilde{\phi}_s^u) - \tilde{\Gamma}_u(\tilde{\phi}_s^u)\| du \\ &\leq \int_s^t C \|\phi_s^u - \tilde{\phi}_s^u\| + K du. \end{aligned}$$

So, $f(t) = \|\phi_s^t - \tilde{\phi}_s^t\|$ satisfies the conditions of Lemma B.3 and hence

$$\|\phi_s^t - \tilde{\phi}_s^t\| \leq \frac{K}{C} (e^{C(t-s)} - 1).$$

2. Since $\|\partial_{q,p}^2 H\| \leq C$, $d\Gamma_t$ is bounded by C and hence

$$\|\partial_t d\phi_s^t\| = \|d(\Gamma_t(\phi_s^t))\| = \|d\Gamma_t(\phi_s^t) \circ d\phi_s^t\| \leq C \|d\phi_s^t\|.$$

which implies that $\|d\phi_s^t - \text{id}\| \leq \int_s^t C (\|d\phi_s^t - \text{id}\| + 1) du$.

Since $d\phi_s^s = \text{id}$, $f(t) = \|d\phi_s^t - \text{id}\|$ satisfies the conditions of Lemma B.3 with $K = C$, and hence

$$\|d\phi_s^t - \text{id}\| \leq e^{C(t-s)} - 1.$$

\square

If $\gamma = (q, p)$ is a path on $T^*\mathbb{R}^d$, its Hamiltonian action is given by

$$\mathcal{A}_s^t(\gamma) = \int_s^t p(\tau) \cdot \partial_\tau q(\tau) - H(\tau, \gamma(\tau)) d\tau.$$

We give here a simple element of calculus of variations, giving for a parametrized family of Hamiltonian trajectories the link between the dependence of the Hamiltonian action with respect to the parameter and the behaviour of the family at the endpoints. It is useful to prove Proposition 1.3, and also to understand the construction of the generating family of the flow in the next paragraph.

Lemma B.4. *If $\gamma_u = (q_u, p_u) : \mathbb{R} \rightarrow T^*\mathbb{R}^d$ is a \mathcal{C}^1 family of Hamiltonian trajectories,*

$$\partial_u \mathcal{A}_s^t(\gamma_u) = p_u(t) \cdot \partial_u q_u(t) - p_u(s) \cdot \partial_u q_u(s).$$

Proof. We recall the Hamiltonian system satisfied by the Hamiltonian trajectory γ_u :

$$\begin{cases} \partial_\tau q_u(\tau) = \partial_p H(t, q_u(\tau), p_u(\tau)), \\ \partial_\tau p_u(\tau) = -\partial_q H(t, q_u(\tau), p_u(\tau)). \end{cases}$$

As a consequence,

$$\begin{aligned} \partial_u \mathcal{A}_s^t(\gamma_u) &= \partial_u \int_s^t p_u(\tau) \cdot \partial_\tau q_u(\tau) - H(\tau, q_u(\tau), p_u(\tau)) d\tau \\ &= \int_s^t \partial_u p_u(\tau) \cdot \partial_\tau q_u(\tau) + p_u(\tau) \cdot \partial_u \partial_\tau q_u(\tau) \\ &\quad - \partial_q H(\tau, q_u(\tau), p_u(\tau)) \cdot \partial_u q_u(\tau) - \partial_p H(\tau, q_u(\tau), p_u(\tau)) \cdot \partial_u p_u(\tau) d\tau \\ &= \int_s^t p_u(\tau) \cdot \partial_u \partial_\tau q_u(\tau) + \partial_\tau p_u(\tau) \cdot \partial_u q_u(\tau) d\tau = [p_u \cdot \partial_u q_u]_s^t. \end{aligned}$$

□

Proof of Proposition 1.3. Take $T < C^{-1} \ln \left(\frac{2+B}{1+B} \right)$. We first show that $g : (t, q) \mapsto (t, Q_0^t(q, du_0(q)))$ is a \mathcal{C}^1 -diffeomorphism on $[0, T] \times \mathbb{R}^d$. Since $\|d^2 H\| \leq C$, the Hamiltonian system (HS) implies that $\|d\phi_0^t - \text{id}\| \leq \exp(tC) - 1$ (see Proposition B.2).

The Jacobian matrix of g is $\begin{pmatrix} 1 & \partial_t Q_0^t(q, du_0(q)) \\ 0 & \partial_q Q_0^t(q, du_0(q)) + \partial_p Q_0^t(q, du_0(q)) d^2 u_0(q) \end{pmatrix}$, and keeping in mind that $\|\partial_q Q_0^t - \text{id}\| \leq \|d\phi_0^t - \text{id}\|$ and $\|\partial_p Q_0^t\| \leq \|d\phi_0^t - \text{id}\|$, the estimation on the flow gives for $t \leq T < C^{-1} \ln \left(\frac{2+B}{1+B} \right)$,

$$\|\partial_q Q + \partial_p Q d^2 u_0 - \text{id}\| \leq \|d\phi_0^t - \text{id}\| (1 + \|d^2 u_0\|) \leq (\exp(tC) - 1)(1 + B) < 1.$$

If H depends only on p , $Q_0^t(q, p) = q + tdH(p)$, hence $\partial_q Q_0^t = \text{id}$ and it is enough to check when $\|\partial_p Q_0^t(q, p)\| \|d^2 u_0\| < 1$. Since $\|\partial_p Q_0^t(q, p)\| = \|td^2 H(p)\| \leq tC$, this is true for all $0 \leq t \leq T$ as soon as $T < 1/BC$.

On the one hand, we then have that the Jacobian matrix of g is invertible, and g is a \mathcal{C}^1 local diffeomorphism. On the other hand, we have showed that the function $g(q) = Q_0^t(q, du_0(q))$ satisfies $\text{Lip}(g - \text{id}) < 1$, and it is then invertible by Lemma B.1, and therefore g is invertible, hence a global \mathcal{C}^1 -diffeomorphism on $[0, T] \times \mathbb{R}^d$.

If $(t, Q) \mapsto q^{t, Q}$ denotes the \mathcal{C}^1 second coordinate of g^{-1} , and $\gamma_{t, Q} = (q_{t, Q}, p_{t, Q})$ is the Hamiltonian trajectory issued from $(q^{t, Q}, du_0(q^{t, Q}))$ at time 0, u is defined as follows:

$$u(t, Q) = u_0(q_{t, Q}(0)) + \mathcal{A}_0^t \gamma_{t, Q}. \tag{B.1}$$

Differentiating $q_{t, Q}(t) = Q$ w.r.t. Q and t gives, denoting carefully by τ the time variable of $\tau \mapsto \gamma_{t, Q}(\tau)$, that $\partial_Q q_{t, Q}(t) = \text{id}$ and that $\partial_t q_{t, Q}(t) + \partial_\tau q_{t, Q}(t) = 0$.

We differentiate (B.1) with respect to Q , using Lemma B.4:

$$\partial_Q u(t, Q) = du_0(q_{t, Q}(0)) \cdot \partial_Q q_{t, Q}(0) + p_{t, Q}(t) \cdot \partial_Q q_{t, Q}(t) - p_{t, Q}(0) \cdot \partial_Q q_{t, Q}(0) = p_{t, Q}(t)$$

since $\partial_Q q_{t, Q}(t) = \text{id}$ and $p_{t, Q}(0) = du_0(q_{t, Q}(0))$.

We differentiate (B.1) with respect to t , without forgetting to differentiate the upper bound of the integral:

$$\begin{aligned} \partial_t u(t, Q) &= du_0(q_{t,Q}(0)) \cdot \partial_t q_{t,Q}(0) + p_{t,Q}(t) \partial_\tau q_{t,Q}(t) - H(t, q_{t,Q}(t), p_{t,Q}(t)) \\ &\quad + p_{t,Q}(t) \cdot \partial_t q_{t,Q}(t) - p_{t,Q}(0) \cdot \partial_t q_{t,Q}(0) = -H(t, Q, p_{t,Q}(t)) \end{aligned}$$

using the fact that $p_{t,Q}(0) = du_0(q_{t,Q}(0))$, $Q = q_{t,Q}(t)$ and $\partial_t q_{t,Q}(t) + \partial_\tau q_{t,Q}(t) = 0$.

Thus, we have proved that $\partial_t u(t, Q) = -H(t, Q, \partial_Q u(t, Q))$, and u is a C^2 solution since these derivatives are C^1 . \square

B.1 Generating family in the general case

As a consequence of Lemma B.1 and Proposition B.2, if we choose a $\delta \leq \delta_1 = \frac{\ln(3/2)}{C}$, the map $g_s^t : (q, p) \mapsto (Q_s^t(q, p), p)$ is a C^1 -diffeomorphism for all $0 \leq t - s \leq \delta$, since we have $\text{Lip}(g_s^t - \text{id}) \leq \text{Lip}(\phi_s^t - \text{id}) \leq 1/2$. If $0 \leq t - s \leq \delta$, let $F_s^t : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ be the function defined by

$$F_s^t(Q, p) = \int_s^t (P_s^\tau(q, p) - p) \cdot \partial_\tau Q_s^\tau(q, p) - H(\tau, \phi_s^\tau(q, p)) d\tau, \quad (\text{B.2})$$

where q is the only point satisfying $Q_s^t(q, p) = Q$, i.e. the first coordinate of $(g_s^t)^{-1}(Q, p)$. In other terms, if $\gamma(\tau) = (q(\tau), p(\tau))$ is the unique Hamiltonian trajectory such that $(q(t), p(s)) = (Q, p)$,

$$F_s^t(Q, p) = p \cdot (q(s) - Q) + \mathcal{A}_s^t(\gamma) = p \cdot (q(s) - Q) + \int_s^t p(\tau) \cdot \partial_\tau q(\tau) - H(\tau, \gamma(\tau)) d\tau. \quad (\text{B.3})$$

Proposition B.5. *The family of functions $(F_s^t)_{s \leq t \leq s + \delta}$ is C^1 with respect to s, t, Q, p and its derivatives are given by*

$$\begin{cases} \partial_p F_s^t(Q, p) = q - Q, & \partial_t F_s^t(Q, p) = -H(t, Q, P), \\ \partial_Q F_s^t(Q, p) = P - p, & \partial_s F_s^t(Q, p) = H(s, q, p), \end{cases}$$

where P and q are uniquely defined by $(Q, P) = \phi_s^t(q, p)$. In particular,

$$(Q, P) = \phi_s^t(q, p) \iff \begin{cases} \partial_p F_s^t(Q, p) = q - Q, \\ \partial_Q F_s^t(Q, p) = P - p. \end{cases}$$

Furthermore, if $Q = Q_s^t(q, p)$ and γ denotes the Hamiltonian trajectory issued from (q, p) ,

$$F_s^t(Q, p) = \mathcal{A}_s^t(\gamma) - p \cdot (Q - q).$$

The generating family is constructed by adding boundary terms to the Hamiltonian action of a Hamiltonian trajectory depending on parameters.

Proof of Proposition B.5. Let us differentiate F with respect to s, t, Q and p . The rest of the proposition is a straightforward consequence of the form of the derivatives of F . In terms of Lemma B.4, let us denote by $u = (s, t, Q, p)$ and by $\gamma_u = (q_u, p_u)$ the unique Hamiltonian trajectory such that $p_u(s) = p$ and $q_u(t) = Q$. Let us gather the derivatives of q_u at the endpoints in view of applying Lemma B.4: we differentiate $q_u(t) = Q$ with respect to s, t, Q and p , while denoting by τ the time variable of the trajectory γ_u :

$$\partial_s q_u(t) = 0, \quad \partial_t q_u(t) + \partial_\tau q_u(t) = 0, \quad \partial_Q q_u(t) = \text{id}, \quad \partial_p q_u(t) = 0. \quad (\text{B.4})$$

The equation (B.3) defining F may now be written as:

$$F_s^t(Q, p) = p \cdot (q_u(s) - Q) + \mathcal{A}_s^t(\gamma_u).$$

Lemma B.4 gives the dependence of $\mathcal{A}_s^t(\gamma_u)$ with respect to u . We differentiate this expression with respect to s , t , Q and p , cautiously denoting by τ the time variable of the trajectory $\gamma_u = (q_u, p_u)$, and taking into account the term $p \cdot (q_u(s) - Q)$ and the boundaries of the integral defining the action:

$$\begin{aligned} \partial_s F_s^t(Q, p) &= p \cdot (\partial_s q_u(s) + \partial_\tau q_u(s)) - (p_u(s) \cdot \partial_\tau q_u(s) - H(s, q_u(s), p_u(s))) + [p_u \cdot \partial_s q_u]_s^t \\ &= H(s, q_u(s), p_u(s)) + (p - p_u(s)) \cdot (\partial_s q_u(s) + \partial_\tau q_u(s)) + p_u(t) \cdot \partial_s q_u(t) \\ &= H(s, q, p), \end{aligned}$$

$$\begin{aligned} \partial_t F_s^t(Q, p) &= p \cdot \partial_t q_u(s) + (p_u(t) \cdot \partial_\tau q_u(t) - H(t, q_u(t), p_u(t))) + [p_u \cdot \partial_t q_u]_s^t \\ &= (p - p_u(s)) \cdot \partial_t q_u(s) + p_u(t) \cdot (\partial_\tau q_u(t) + \partial_t q_u(t)) - H(t, q_u(t), p_u(t)) \\ &= -H(t, Q, P), \end{aligned}$$

$$\begin{aligned} \partial_Q F_s^t(Q, p) &= p \cdot \partial_Q q_u(s) - p + [p_u \cdot \partial_Q q_u]_s^t \\ &= (p - p_u(s)) \cdot \partial_Q q_u(s) - p + p_u(t) \cdot \partial_Q q_u(t) = -p + P, \end{aligned}$$

$$\begin{aligned} \partial_p F_s^t(Q, p) &= p \cdot \partial_p q_u(s) + q_u(s) - Q + [p_u \cdot \partial_p q_u]_s^t \\ &= (p - p_u(s)) \cdot \partial_p q_u(s) + q_u(s) - Q + p_u(t) \cdot \partial_p q_u(t) = q - Q \end{aligned}$$

if we denote by $(P, q) = (p_u(t), q_u(s))$, using (B.4) and $(p_u(s), q_u(t)) = (p, Q)$. \square

Proposition B.6. *If H_μ is a \mathcal{C}^2 family of Hamiltonians such that $\|\partial_{q,p}^2 H_\mu\|$ is bounded by C , let us denote by $F_{s,\mu}^t$ associated with H_μ as previously for $t - s \leq \delta$. Then*

$$\partial_\mu F_{s,\mu}^t(Q, p) = - \int_s^t \partial_\mu H_\mu(\tau, \gamma_\mu(\tau)) \, d\tau$$

where $\gamma_\mu = (q_\mu, p_\mu)$ is the unique Hamiltonian trajectory for H_μ with $q_\mu(t) = Q$ and $p_\mu(s) = p$.

Proof. Let us fix Q , p , s and t , and take γ_μ as in the statement. By definition (B.3),

$$F_{s,\mu}^t(Q, p) = p \cdot (q_\mu(s) - Q) + \mathcal{A}_{s,H_\mu}^t(\gamma_\mu)$$

and thus differentiating w.r.t. μ gives the following, using Lemma B.4:

$$\partial_\mu F_{s,\mu}^t(Q, p) = p \cdot \partial_\mu q_\mu(s) + [p_\mu \cdot \partial_\mu q_\mu]_s^t - \int_s^t \partial_\mu H_\mu(\tau, \gamma_\mu(\tau)) \, d\tau.$$

Now, since $q_\mu(t) = Q$ for all μ , $\partial_\mu q_\mu(t) = 0$, and since $p = p_\mu(s)$, the two first terms of the right hand side cancel, hence the conclusion. \square

When $t - s$ is large, we choose a subdivision of the time interval with steps smaller than δ and add intermediate coordinates along this trajectory. For each $s \leq t$ and (t_i) such that $t_0 = s \leq t_1 \leq \dots \leq t_{N+1} = t$ and $t_{i+1} - t_i \leq \delta$ for each i , let $G_s^t : \mathbb{R}^{2d(1+N)} \rightarrow \mathbb{R}$ be the function defined by

$$G_s^t(p_0, Q_0, p_1, Q_1, \dots, p_{N-1}, Q_{N-1}, p_N, Q_N) = \sum_{i=0}^N F_{t_i}^{t_{i+1}}(Q_i, p_i) + p_{i+1} \cdot (Q_{i+1} - Q_i)$$

where indices are taken modulo $N + 1$.

Proposition B.7. *The family of functions $(G_s^t)_{s \leq t}$ is C^1 with respect to s, t, t_i, Q_i and p_i , and its derivatives are given by*

$$\begin{cases} \partial_{p_i} G_s^t(p_0, \dots, Q_N) = \partial_p F_{t_i}^{t_{i+1}}(Q_i, p_i) + Q_i - Q_{i-1} = q_i - Q_{i-1}, \\ \partial_{Q_i} G_s^t(p_0, \dots, Q_N) = \partial_Q F_{t_i}^{t_{i+1}}(Q_i, p_i) + p_i - p_{i+1} = P_i - p_{i+1}, \end{cases}$$

where P_i and q_i are uniquely defined by $(Q_i, P_i) = \phi_{t_i}^{t_{i+1}}(q_i, p_i)$ and indices are taken modulo $N+1$.

It is hence a generating family for the flow ϕ , meaning that if we denote $(Q, p) = (Q_N, p_0)$ and $\nu = (Q_0, p_1, \dots, Q_{N-1}, p_N)$,

$$(Q, P) = \phi_s^t(q, p) \iff \exists \nu \in \mathbb{R}^{2dN}, \begin{cases} \partial_p G_s^t(p, \nu, Q) = q - Q, \\ \partial_Q G_s^t(p, \nu, Q) = P - p, \\ \partial_\nu G_s^t(p, \nu, Q) = 0, \end{cases}$$

and in this case $(Q_i, p_{i+1}) = \phi_s^{t_{i+1}}(q, p)$ for all $0 \leq i \leq N-1$. Furthermore, if $Q = Q_s^t(q, p)$ and γ denotes the Hamiltonian trajectory issued from (q, p) ,

$$G_s^t(p, \nu, Q) = \mathcal{A}_s^t(\gamma) - p \cdot (Q - q)$$

if $\partial_\nu G_s^t(p, \nu, Q) = 0$.

Proof. The derivatives of G , which are directly obtained from the ones of F , give that, if p and Q are fixed,

$$\begin{cases} \partial_p G_s^t(p, \nu, Q) = q - Q, \\ \partial_Q G_s^t(p, \nu, Q) = P - p, \\ \partial_\nu G_s^t(p, \nu, Q) = 0, \end{cases} \iff \begin{cases} q = q_0, \\ P_N = P, \\ (Q_i, p_{i+1}) = \phi_{t_i}^{t_{i+1}}(Q_{i-1}, p_i) \quad \forall 0 \leq i \leq N-1. \end{cases}$$

If this is satisfied, ν describes a non broken Hamiltonian geodesic, $(Q_i, p_{i+1}) = \phi_s^{t_{i+1}}(q, p)$ and $(Q, P) = \phi_s^t(q, p)$. If $(Q, P) = \phi_s^t(q, p)$, then ν is given by $\phi_s^{t_i}(q, p)$ and the right hand system holds.

The critical value of $\nu \mapsto G_s^t(p, \nu, Q)$ is obtained by summing up the result obtained for F in Proposition B.5. \square

The last statement compares the generating families of flows related to Hamiltonians with Lipschitz difference.

Proposition B.8. *Let H and \tilde{H} be two C^2 Hamiltonians on $\mathbb{R} \times T^*\mathbb{R}^d$ such that $\|\partial_{q,p}^2 H\|$ and $\|\partial_{q,p}^2 \tilde{H}\|$ are uniformly bounded by a constant C and $\|\partial_{q,p} H - \partial_{q,p} \tilde{H}\|$ is uniformly bounded by a constant K . We can find a $\delta > 0$ suiting both \tilde{H} and H and build \tilde{G}_s^t and G_s^t with the same subdivision (t_i) , and then $\tilde{G}_s^t - G_s^t$ is Lipschitz with constant $4\frac{K}{C}(e^{C(t-s)} - 1)$ and also with constant $2\frac{K}{C}$.*

Proof. Let $\delta \leq \delta_1 = \frac{\ln(3/2)}{C}$ so that both $\phi_s^t - \text{id}$ and $\tilde{\phi}_s^t - \text{id}$ are $\frac{1}{2}$ -Lipschitz if $0 \leq t - s \leq \delta$, see Proposition B.2, and in that case $g_s^t : (q, p) \mapsto (Q_s^t(q, p), p)$ satisfies also $\text{Lip}(g_s^t - \text{id}) \leq 1/2$.

Proposition B.2 states that $\|\tilde{\phi}_{t_i}^{t_{i+1}} - \phi_{t_i}^{t_{i+1}}\|_\infty \leq \frac{K}{C}(e^{C(t_{i+1}-t_i)} - 1)$ under the assumptions made on H and \tilde{H} . We are hence going to check that for all i , $\|\partial_{Q_i} \tilde{G}_s^t - \partial_{Q_i} G_s^t\|$ and $\|\partial_{p_i} \tilde{G}_s^t - \partial_{p_i} G_s^t\|$ are both bounded by $4\|\tilde{\phi}_{t_i}^{t_{i+1}} - \phi_{t_i}^{t_{i+1}}\|_\infty$ in order to get the wanted Lipschitz constants. Proposition B.7 states that $\|\partial_{Q_i} \tilde{G}_s^t - \partial_{Q_i} G_s^t\| = \|\tilde{P}_i - P_i\|$ and $\|\partial_{p_i} \tilde{G}_s^t - \partial_{p_i} G_s^t\| = \|\tilde{q}_i - q_i\|$, where P_i and q_i

(resp. \tilde{P}_i and \tilde{q}_i) are uniquely defined by $(Q_i, P_i) = \phi_{t_i}^{t_{i+1}}(q_i, p_i)$ (resp. $(Q_i, \tilde{P}_i) = \tilde{\phi}_{t_i}^{t_{i+1}}(\tilde{q}_i, p_i)$). Since $(q_i, p_i) = (g_{t_i}^{t_{i+1}})^{-1}(Q_i, p_i)$ and $(\tilde{q}_i, p_i) = (\tilde{g}_{t_i}^{t_{i+1}})^{-1}(Q_i, p_i)$, Lemma B.1 gives

$$\|\tilde{q}_i - q_i\| \leq \|(\tilde{g}_{t_i}^{t_{i+1}})^{-1} - (g_{t_i}^{t_{i+1}})^{-1}\|_\infty \leq \frac{\|\tilde{g}_{t_i}^{t_{i+1}} - g_{t_i}^{t_{i+1}}\|_\infty}{1 - \text{Lip}(g_s^t - \text{id})} \leq 2\|\tilde{\phi}_{t_i}^{t_{i+1}} - \phi_{t_i}^{t_{i+1}}\|_\infty$$

since $\text{Lip}(g_{t_i}^{t_{i+1}} - \text{id}) \leq 1/2$. Now,

$$\begin{aligned} \|\tilde{P}_i - P_i\| &\leq \|\tilde{\phi}_{t_i}^{t_{i+1}}(\tilde{q}_i, p_i) - \phi_{t_i}^{t_{i+1}}(q_i, p_i)\| \\ &\leq \|\tilde{\phi}_{t_i}^{t_{i+1}}(\tilde{q}_i, p_i) - \phi_{t_i}^{t_{i+1}}(\tilde{q}_i, p_i)\| + \text{Lip}(\phi_{t_i}^{t_{i+1}})\|\tilde{q}_i - q_i\| \\ &\leq \|\tilde{\phi}_{t_i}^{t_{i+1}} - \phi_{t_i}^{t_{i+1}}\|_\infty + \text{Lip}(\phi_{t_i}^{t_{i+1}})2\|\tilde{\phi}_{t_i}^{t_{i+1}} - \phi_{t_i}^{t_{i+1}}\|_\infty \\ &\leq 4\|\tilde{\phi}_{t_i}^{t_{i+1}} - \phi_{t_i}^{t_{i+1}}\|_\infty \end{aligned}$$

since $\phi_{t_i}^{t_{i+1}}$ is $\frac{3}{2}$ -Lipschitz.

Since $t_{i+1} - t_i$ is smaller than $t - s$ and than δ for all i , we have proved that $\|d\tilde{G}_s^t - dG_s^t\|$ is bounded by $4\frac{K}{C}(e^{C\delta} - 1) \leq 2\frac{K}{C}$ and by $4\frac{K}{C}(e^{C(t-s)} - 1)$. \square

B.2 Generating family in the convex case

In this section we assume that the Hamiltonian H satisfies Hypothesis 1.1 with constant C , and that there exists $m > 0$ such that for each (t, q, p) , $\partial_p^2 H(t, q, p) \geq m \text{id}$ in the sense of quadratic forms.

Proposition B.9. *The following holds in the sense of quadratic forms:*

$$\partial_p Q_s^t \geq m(t-s)\text{id} - 2\left(e^{C(t-s)} - 1 - C(t-s)\right)\text{id}.$$

In particular there exists $\delta_2 > 0$ depending only on C and m such that if $|t-s| \leq \delta_2$,

$$\partial_p Q_s^t \geq \frac{m}{2}(t-s)\text{id}$$

which implies that the function $p \mapsto Q_s^t(q, p)$ is $\frac{m(t-s)}{2}$ -monotone, meaning that

$$(Q_s^t(q, \tilde{p}) - Q_s^t(q, p)) \cdot (\tilde{p} - p) \geq \frac{m}{2}(t-s)\|\tilde{p} - p\|^2.$$

In particular, if $|t-s| \leq \delta_2$, $(q, p) \mapsto (q, Q_s^t(q, p))$ is a C^1 -diffeomorphism.

Remark B.10. For A a not necessarily symmetric matrix, we say that $A \geq \text{cid}$ in the sense of quadratic forms if $Ax \cdot x \geq c\|x\|^2$ for all x . If $\|A\| \leq a$, then in particular $-\text{aid} \leq A \leq \text{aid}$.

Proof. Let us recall the variational equation

$$\partial_p \dot{Q}_s^t = \partial_p^2 H \partial_p P_s^t + \partial_{q,p}^2 H \partial_p Q_s^t$$

that we write under the form

$$\partial_p \dot{Q}_s^t - \partial_p^2 H = \partial_p^2 H (\partial_p P_s^t - \text{id}) + \partial_{q,p}^2 H \partial_p Q_s^t.$$

Lemma B.2 gives that $\|\partial_q Q_s^t - \text{id}\|$, $\|\partial_p Q_s^t\|$ and $\|\partial_p P_s^t - \text{id}\|$ are smaller than $e^{C(t-s)} - 1$. Adding the estimate on $\partial^2 H$, we get

$$\|\partial_p \dot{Q}_s^t - \partial_p^2 H\| \leq 2C(e^{C(t-s)} - 1),$$

which implies that

$$\partial_p \dot{Q}_s^t \geq \partial_p^2 H - 2C(e^{C(t-s)} - 1)\text{id} \geq (m - 2C(e^{C(t-s)} - 1))\text{id}$$

in the sense of quadratic forms, see Remark B.10. Integrating the result between s and t we obtain

$$\partial_p Q_s^t \geq m(t-s)\text{id} - 2(e^{C(t-s)} - 1 - C(t-s))\text{id}.$$

Since the second term of the right hand side is second order, there exists a constant $\delta_2 > 0$ depending only on C and m such that if $|t-s| \leq \delta_2$,

$$\partial_p Q_s^t \geq \frac{m}{2}(t-s)\text{id},$$

which means that for all z ,

$$\partial_p Q_s^t(q, p)z \cdot z \geq \frac{m}{2}(t-s)\|z\|^2.$$

Applying this to $z = \tilde{p} - p$ we get

$$\begin{aligned} (Q_s^t(q, \tilde{p}) - Q_s^t(q, p)) \cdot (\tilde{p} - p) &= \int_0^1 \partial_p Q_s^t(q, p + \tau(\tilde{p} - p))(\tilde{p} - p) d\tau \cdot (\tilde{p} - p) \\ &= \int_0^1 \partial_p Q_s^t(q, p + \tau(\tilde{p} - p))(\tilde{p} - p) \cdot (\tilde{p} - p) d\tau \\ &\geq \int_0^1 \frac{m}{2}(t-s)\|\tilde{p} - p\|^2 d\tau \geq \frac{m}{2}(t-s)\|\tilde{p} - p\|^2. \end{aligned}$$

We have proved that the function $p \mapsto Q_s^t(q, p)$ is $\frac{m(t-s)}{2}$ -monotone. It is then a classical result that $p \mapsto Q_s^t(q, p)$ is a global \mathcal{C}^1 -diffeomorphism (see for example Proposition 51 of [Ber12]), and therefore $(q, p) \mapsto (q, Q_s^t(q, p))$ is also a global \mathcal{C}^1 -diffeomorphism. \square

Proposition B.11. *There exists $\delta_3 > 0$ depending only on C and m such that if G_s^t is constructed with a maximal step smaller than δ_3 , $(p_0, p_1, \dots, p_N) \mapsto G_s^t(p_0, Q_0, p_1, Q_1, \dots, Q_{N-1}, p_N, Q_N)$ is uniformly strictly concave.*

Proof. Let us denote by g the function $(p_0, p_1, \dots, p_N) \mapsto G_s^t(p_0, Q_0, p_1, Q_1, \dots, Q_{N-1}, p_N, Q_N)$. Proposition B.7 gives that $\partial_{p_i} G_s^t(p_0, \dots, Q_N) = q_i - Q_{i-1}$, where q_i is the only point such that $Q_i = Q_{t_i}^{t_{i+1}}(q_i, p_i)$. On one hand, we get that if $i \neq j$, $\partial_{p_i p_j}^2 G_s^t$ is zero. On the other hand, $\partial_{p_i}^2 G_s^t = \partial_{p_i} q_i$.

Differentiating $Q_{t_i}^{t_{i+1}}(q_i, p_i) = Q_i$ w.r.t. p_i gives

$$\partial_q Q_{t_i}^{t_{i+1}}(q_i, p_i) \partial_{p_i} q_i + \partial_p Q_{t_i}^{t_{i+1}}(q_i, p_i) = 0,$$

so we have

$$\partial_{p_i}^2 G_s^t = -(\partial_q Q_{t_i}^{t_{i+1}})^{-1} \partial_p Q_{t_i}^{t_{i+1}}.$$

Lemma B.2 gives that $\|\partial_p Q_{t_i}^{t_{i+1}}\| \leq e^{C(t_{i+1}-t_i)} - 1$ and $\|\partial_q Q_{t_i}^{t_{i+1}} - \text{id}\| \leq e^{C(t_{i+1}-t_i)} - 1$, and hence $\partial_q Q_{t_i}^{t_{i+1}}$ is invertible as long as $e^{C(t_{i+1}-t_i)} < 2$ and satisfies

$$\left\| (\partial_q Q_{t_i}^{t_{i+1}})^{-1} - \text{id} \right\| \leq \frac{e^{C(t_{i+1}-t_i)} - 1}{2 - e^{C(t_{i+1}-t_i)}}. \quad (\text{B.5})$$

Using (B.5) and the estimate of Proposition B.9 we get

$$\begin{aligned} \partial_{p_i}^2 G_s^t &= -((\partial_q Q_{t_i}^{t_{i+1}})^{-1} - \text{id}) \partial_p Q_{t_i}^{t_{i+1}} - \partial_p Q_{t_i}^{t_{i+1}} \\ &\leq \frac{e^{C(t_{i+1}-t_i)} - 1}{2 - e^{C(t_{i+1}-t_i)}} (e^{C(t_{i+1}-t_i)} - 1) \text{id} \\ &\quad - m(t_{i+1} - t_i) \text{id} + 2 \left(e^{C(t_{i+1}-t_i)} - 1 - C(t_{i+1} - t_i) \right) \text{id}. \end{aligned}$$

Since the only first order term is $-m(t_{i+1} - t_i) \text{id}$, there exists a $\delta_3 > 0$ depending only on C and m such that if $t_{i+1} - t_i \leq \delta_3$,

$$\partial_{p_i}^2 G_s^t \leq -\frac{m}{2}(t_{i+1} - t_i) \text{id}.$$

If $\delta \leq \delta_3$, then $d^2 g$, which is a blockwise diagonal matrix, is smaller than $-\frac{m\delta}{2} \text{id}$ and g is hence uniformly strictly concave. \square

When the Hamiltonian H is strictly convex w.r.t. p , the Lagrangian function on the tangent bundle is associated as follows:

$$L(t, q, v) = \sup_{p \in (\mathbb{R}^d)^*} p \cdot v - H(t, q, p).$$

Assume that $\delta < \min(\delta_1, \delta_2, \delta_3)$, and let h_i be the inverse function of $(q, p) \mapsto (q, Q_{t_i}^{t_{i+1}}(q, p))$ (see Proposition B.9). We define

$$A_s^t(q, Q_0, \dots, Q_{N-1}, Q) = \sum_{i=0}^N \int_{t_i}^{t_{i+1}} L(\tau, Q_{t_i}^\tau(h_i(Q_{i-1}, Q_i)), \partial_\tau Q_{t_i}^\tau(h_i(Q_{i-1}, Q_i))) d\tau$$

with the notations $q = Q_{-1}$ and $Q = Q_N$.

Proposition B.12. *The so-called Lagrangian generating family A is C^1 and satisfies :*

1.

$$A_s^t(q, Q_0, \dots, Q_{N-1}, Q) = \max_{(p_0, \dots, p_N)} G_s^t(p_0, Q_0, \dots, Q_{N-1}, p_N, Q) + p_0 \cdot (Q - q).$$

2.

$$\begin{cases} \partial_{Q_i} A_s^t(q, Q_0, \dots, Q_{N-1}, Q) = P_i - p_{i+1} & \forall i = 0 \dots N-1, \\ \partial_q A_s^t(q, Q_0, \dots, Q_{N-1}, Q) = -p_0, \\ \partial_Q A_s^t(q, Q_0, \dots, Q_{N-1}, Q) = P_N, \end{cases}$$

where P_i and p_i are uniquely defined by $(Q_i, P_i) = \phi_{t_i}^{t_{i+1}}(Q_{i-1}, p_i)$.

This function is indeed a *generating family* for the flow, in the sense that if $v = (Q_0, \dots, Q_{N-1})$, the graph of the flow ϕ_s^t is the set

$$\{((q, -\partial_q A_s^t(q, v, Q)), (Q, \partial_Q A_s^t(p, v, Q))) \mid \partial_v A_s^t(p, v, Q) = 0\}.$$

Proof. 1. The function $(p_0, p_1, \dots, p_N) \mapsto G_s^t(p_0, Q_0, p_1, Q_1, \dots, Q_{N-1}, p_N, Q) + p_0 \cdot (Q - q)$ is uniformly strictly concave by Proposition B.11, and its maximum is hence attained by a unique point.

For i from 1 to N , this is a consequence of the derivative of G_s^t given in Proposition B.7: $\partial_{p_i} G_s^t(p_0, Q_0, p_1, Q_1, \dots, Q_{N-1}, p_N, Q) = q_i - Q_{i-1} = 0$ if and only if $Q_{t_i}^{t_i+1}(Q_{i-1}, p_i) = Q_i$. For $i = 0$, the derivative with respect to p_0 is $q_0 - q$ where q_0 is the only point such that $Q_s^{t_1}(q_0, p_0) = Q_0$, and consequently $\partial_{p_0} (G_s^t + p_0 \cdot (Q - q)) = 0$ if and only if $Q_s^{t_1}(q, p_0) = Q_0$.

The maximum is hence uniquely attained by the \mathcal{C}^1 function

$$\mathbf{p} : (q, Q_0 \dots, Q) \mapsto (h_0^2(q, Q_0), h_1^2(Q_0, Q_1), \dots, h_N^2(Q_{N-1}, Q)),$$

where h_i^2 denotes the second coordinate of h_i . In other terms, its coordinates satisfy $Q_{t_i}^{t_i+1}(Q_{i-1}, \mathbf{p}_i) = Q_i$ for all i from 0 to N , with the notations $q = Q_{-1}$ and $Q = Q_N$.

By definition of the Lagrangian, if $(q(t), p(t))$ is a Hamiltonian trajectory associated with H , then

$$L(t, q(t), \dot{q}(t)) = p(t) \cdot \dot{q}(t) - H(t, q(t), p(t)).$$

In particular the function F defined in (B.3) can be written in Lagrangian terms:

$$F_s^t(Q, p) = p \cdot (q - Q) + \int_s^t L(\tau, Q_s^\tau(q, p), \partial_\tau Q_s^\tau(q, p)) d\tau.$$

where q is the only point such that $Q_s^t(q, p) = Q$, and the function G is hence the following:

$$\begin{aligned} G_s^t(p_0, Q_0, p_1, Q_1, \dots, Q_{N-1}, p_N, Q_N) &= \sum_{i=0}^N F_{t_i}^{t_i+1}(Q_i, p_i) + p_{i+1} \cdot (Q_{i+1} - Q_i) \\ &= \sum_{i=0}^N \int_{t_i}^{t_{i+1}} L(\tau, Q_{t_i}^\tau(q_i, p_i), \partial_\tau Q_{t_i}^\tau(q_i, p_i)) d\tau + p_i \cdot (q_i - Q_i) + p_{i+1} \cdot (Q_{i+1} - Q_i) \\ &= \sum_{i=0}^N \int_{t_i}^{t_{i+1}} L(\tau, Q_{t_i}^\tau(q_i, p_i), \partial_\tau Q_{t_i}^\tau(q_i, p_i)) d\tau + p_i \cdot (q_i - Q_{i-1}), \end{aligned}$$

where q_i is the only point such that $Q_{t_i}^{t_i+1}(q_i, p_i) = Q_i$.

Now, if $(\mathbf{p}_0, \dots, \mathbf{p}_N)$ is the critical point, we have on one hand that $q_i = Q_{i-1}$ and on the other hand that $Q_{t_i}^{t_i+1}(q_i, \mathbf{p}_i) = Q_i$ if and only if $(q_i, \mathbf{p}_i) = h_i(q_i, Q_i)$, hence the result.

2. Since $A_s^t(q, \dots, Q) = G_s^t(Q_0, \dots, Q, \mathbf{p}(q, \dots, Q)) + \mathbf{p}_0(q, \dots, Q) \cdot (Q - q)$ while reorganising the variables, we have for all i from -1 to N

$$\begin{aligned} \partial_{Q_i} A_s^t(q, \dots, Q) &= \partial_{Q_i} (G_s^t(Q_0, \dots, Q, \mathbf{p}(q, \dots, Q)) + \mathbf{p}_0(q, \dots, Q) \cdot (Q - q)) \\ &\quad + \underbrace{\partial_{\mathbf{p}} (G_s^t(Q_0, \dots, Q, \mathbf{p}(q, \dots, Q)) + \mathbf{p}_0(q, \dots, Q) \cdot (Q - q))}_{=0} \partial_{Q_i} \mathbf{p} \end{aligned}$$

since $\mathbf{p}(q, \dots, Q)$ is the critical point. The result is then a straightforward consequence of Proposition B.7 and of the second point. \square

Let us state what happens in the case of a uniformly strictly concave Hamiltonian.

Remark B.13. If H is uniformly strictly concave (which means that $-H$ is uniformly strictly convex), Proposition B.9 analogous statement is that $-Q_s^t$ is $m(t-s)/2$ monotone, which implies the twist property: $(q, p) \mapsto (q, Q_s^t(q, p))$ is a C^1 -diffeomorphism for $|t-s| \leq \delta_2$. Proposition B.11 analogous statement is that $-G_s^t$ is strictly concave with respect to its p variable for $|t-s| \leq \delta_3$. The Lagrangian is now defined by

$$L(t, q, v) = \inf_{p \in (\mathbb{R}^d)^*} p \cdot v - H(t, q, p),$$

and the analogous statement of Proposition B.12 is that

$$A_s^t(q, Q_0, \dots, Q_{N-1}, Q) = \min_{(p_0, p_1, \dots, p_N)} G_s^t(p_0, Q_0, \dots, Q_{N-1}, p_N, Q) + p_0 \cdot (Q - q),$$

where A is defined as in the convex case. Finally, the next Proposition holds in both convex and concave cases.

Proposition B.14. *Let H and \tilde{H} be two C^2 Hamiltonians on $\mathbb{R} \times T^*\mathbb{R}^d$ such that*

- $\partial_{q,p}^2 H$ and $\partial_{q,p}^2 \tilde{H}$ are uniformly bounded by a constant C ,
- $\partial_p^2 H \geq \text{mid}$, $\partial_p^2 \tilde{H} \geq \text{mid}$ (or $\leq -\text{mid}$ in the concave case),
- $\partial_{q,p} H - \partial_{q,p} \tilde{H}$ is uniformly bounded by a constant K .

We fix a subdivision $s \leq t_0 \leq \dots \leq t_{N+1} = t$ such that $0 < t_{i+1} - t_i < \delta$, with δ smaller than δ_1 , δ_2 and δ_3 , and build the Lagrangian generating families A_s^t and \tilde{A}_s^t as previously, respectively for H and \tilde{H} . Then the difference $\tilde{A}_s^t - A_s^t$ is Lipschitz.

Proof. We denote by $\tilde{\cdot}$ the objects defined for \tilde{H} instead of H . Given the form of the derivatives of A_s^t obtained in Proposition B.12, it is enough to prove that $\tilde{p}_i - p_i$ and $\tilde{P}_i - P_i$ are bounded uniformly with respect to (q, \dots, Q) for all i , where P_i and p_i (resp. \tilde{P}_i and \tilde{p}_i) are uniquely defined by $(Q_i, P_i) = \phi_{t_i}^{t_{i+1}}(Q_{i-1}, p_i)$ (resp. $(Q_i, \tilde{P}_i) = \tilde{\phi}_{t_i}^{t_{i+1}}(Q_{i-1}, \tilde{p}_i)$).

Proposition B.9 states that $p \mapsto Q_{t_i}^{t_{i+1}}(q, p)$ is $\frac{m(t_{i+1}-t_i)}{2}$ -monotone, meaning that for all p and \tilde{p}

$$(Q_{t_i}^{t_{i+1}}(q, \tilde{p}) - Q_{t_i}^{t_{i+1}}(q, p)) \cdot (\tilde{p} - p) \geq \frac{m}{2}(t_{i+1} - t_i) \|\tilde{p} - p\|^2.$$

Applying the Cauchy-Schwarz inequality and dividing by $\|\tilde{p} - p\|$ we get

$$\|\tilde{p} - p\| \leq \frac{2}{m(t_{i+1} - t_i)} \left\| Q_{t_i}^{t_{i+1}}(q, \tilde{p}) - Q_{t_i}^{t_{i+1}}(q, p) \right\|.$$

Take $q = Q_{i-1}$, $\tilde{p} = \tilde{p}_i$ and $p = p_i$. Since $Q_{t_i}^{t_{i+1}}(Q_{i-1}, p_i) = \tilde{Q}_{t_i}^{t_{i+1}}(Q_{i-1}, \tilde{p}_i)$, we have

$$\|\tilde{p}_i - p_i\| \leq \frac{2}{m(t_{i+1} - t_i)} \left\| Q_{t_i}^{t_{i+1}}(Q_{i-1}, \tilde{p}_i) - \tilde{Q}_{t_i}^{t_{i+1}}(Q_{i-1}, \tilde{p}_i) \right\| \leq \frac{2}{m\mu} \left\| \phi_{t_i}^{t_{i+1}} - \tilde{\phi}_{t_i}^{t_{i+1}} \right\|_{\infty}$$

where μ denotes the minimum of $t_{i+1} - t_i$. The first estimate of Proposition B.2 gives:

$$\|\tilde{p}_i - p_i\| \leq \frac{2}{m\mu} \frac{K}{C} (e^{C\delta} - 1).$$

Finally, since $P_i = P_{t_i}^{t_{i+1}}(Q_{i-1}, p_i)$ and $\tilde{P}_i = \tilde{P}_{t_i}^{t_{i+1}}(Q_{i-1}, \tilde{p}_i)$,

$$\begin{aligned} \|\tilde{P}_i - P_i\| &\leq \left\| \phi_{t_i}^{t_{i+1}} - \tilde{\phi}_{t_i}^{t_{i+1}} \right\|_{\infty} + \left\| P_{t_i}^{t_{i+1}}(Q_{i-1}, p_i) - P_{t_i}^{t_{i+1}}(Q_{i-1}, \tilde{p}_i) \right\| \\ &\leq \left\| \phi_{t_i}^{t_{i+1}} - \tilde{\phi}_{t_i}^{t_{i+1}} \right\|_{\infty} + \text{Lip}(\phi_{t_i}^{t_{i+1}}) \|\tilde{p}_i - p_i\| \end{aligned}$$

is uniformly bounded since $\phi_{t_i}^{t_{i+1}}$ is $\frac{3}{2}$ -Lipschitz (see Proposition B.2). \square

Appendix C

Minmax: a critical value selector

We denote by \mathcal{Q}_m the set of functions on \mathbb{R}^m that can be written as the sum of a nondegenerate quadratic function and of a Lipschitz function. The aim of this appendix is to build a function $\sigma : \bigcup_{m \in \mathbb{N}} \mathcal{Q}_m \rightarrow \mathbb{R}$, named *minmax*, satisfying:

1. if f is \mathcal{C}^1 , then $\sigma(f)$ is a critical value of f ,
2. if c is a real constant, then $\sigma(c + f) = c + \sigma(f)$,
3. if ϕ is a Lipschitz \mathcal{C}^∞ -diffeomorphism on \mathbb{R}^m such that $f \circ \phi$ is in \mathcal{Q}_m , then

$$\sigma(f \circ \phi) = \sigma(f),$$

4. if $f_0 - f_1$ is Lipschitz and $f_0 \leq f_1$ on \mathbb{R}^d , then $\sigma(f_0) \leq \sigma(f_1)$,
5. if $(f_\mu)_{\mu \in [s,t]}$ is a \mathcal{C}^1 family of \mathcal{Q}_m with $(\mathcal{Z} - f_\mu)_\mu$ equi-Lipschitz for some nondegenerate quadratic form \mathcal{Z} , then for all $\mu \neq \tilde{\mu} \in [s, t]$,

$$\min_{\mu \in [s,t]} \min_{x \in \text{Crit}(f_\mu)} \partial_\mu f_\mu(x) \leq \frac{\sigma(f_{\tilde{\mu}}) - \sigma(f_\mu)}{\tilde{\mu} - \mu} \leq \max_{\mu \in [s,t]} \max_{x \in \text{Crit}(f_\mu)} \partial_\mu f_\mu(x).$$

6. $\sigma(-f) = -\sigma(f)$,
7. if $f(x, y)$ is a \mathcal{C}^2 function of \mathcal{Q}_m such that $\partial_y^2 f \geq \text{cid}$ for a $c > 0$, and if $g(x) = \min_y f(x, y)$ is in some $\mathcal{Q}_{\tilde{m}}$, then $\sigma(g) = \sigma(f)$.

For smooth functions, (1), (3) and (2) are proved in Proposition C.8, (4) is implied by Proposition C.11, and (6) and (7) are proved respectively in Propositions C.13 and C.15. They are extended to non smooth functions in Propositions C.17 and C.18, and (5) is proved in Proposition C.19.

Consequences C.1. These properties imply the following consequences:

1. If f and g are two functions of \mathcal{Q}_m with difference bounded and Lipschitz on \mathbb{R}^m , then $|\sigma(f) - \sigma(g)| \leq \|f - g\|_\infty$. This is a consequence of properties (2) and (4).
2. If $g(x, \eta) = f(x) + \mathcal{Z}(\eta)$ where \mathcal{Z} is a nondegenerate quadratic form and f is in \mathcal{Q}_m , then $\sigma(g) = \sigma(f)$. This is a consequence of properties (6) and (7) for smooth functions, which may be extended by continuity thanks to the previous point.

3. If $f_\mu = \mathcal{Z}_\mu + \ell_\mu$ is a \mathcal{C}^1 family of \mathcal{Q}_m with ℓ_μ equi-Lipschitz, such that the set of critical points f_μ does not depend on μ and such that $\mu \mapsto f_\mu$ is constant on this set, then $\mu \mapsto \sigma(f_\mu)$ is constant. This is a consequence of properties (3) and (5).
4. If f is bounded below, then $\sigma(f) = \min(f)$. This is a consequence of properties (1) and (4).

Consequences C.1-(3) and C.1-(4) are proved in the main corpus, see respectively Consequences 2.12 and 2.11.

The construction of such a critical value selector proves Propositions 2.7 and 4.4.

We will use two deformation lemmas proved in Appendix D, and we refer to [Wei13b] for a survey of minmax related subtleties, including an example due to F. Laudenbach where the minmax is not uniquely defined.

Remark C.2. In this thesis we describe the geometric solution associated with the considered Cauchy problem with a particular generating family proposed by Chaperon. In a more general setting, Viterbo's uniqueness theorem on generating functions state that if S and \tilde{S} are two generating functions quadratic at infinity describing a same Lagrangian submanifold which is Hamiltonianly isotopic to the zero section, they may be obtained one from another via a combination of the three following transformations:

- Addition of a constant: $\tilde{S} = S + c$ for some $c \in \mathbb{R}$,
- Diffeomorphism operation: $\tilde{S} = S \circ \phi$ for some fiber \mathcal{C}^∞ -diffeomorphism ϕ ,
- Stabilization: $\tilde{S}(x, \xi, \nu) = S(x, \xi) + \mathcal{Z}(\nu)$ for a nondegenerate quadratic form \mathcal{Z} .

The proof of D. Theret in [Th 99] puts forward the fact that the diffeomorphism ϕ may be chosen affine outside a compact set - in particular such a diffeomorphism is Lipschitz and if f is in \mathcal{Q}_m , so does $f \circ \phi$. Hence, the invariance of the minmax by additivity (property (2)), by diffeomorphism action (property (3)) and by stabilization (property C.1-(2)) gives that the minmax behave well when applied to generating functions. Up to adding a constant, it is the same for generating functions describing the same Lagrangian submanifold.

C.1 Definition of the minmax for smooth functions

Let us denote by \mathcal{Q}_m^∞ the set of \mathcal{C}^∞ functions of \mathcal{Q}_m . The critical points and values of \mathcal{C}^1 functions of \mathcal{Q}_m are bounded:

Proposition C.3. *If \mathcal{Z} is a nondegenerate quadratic form and ℓ is a \mathcal{C}^1 Lipschitz function with constant L , then the set of critical points of the function $f = \mathcal{Z} + \ell$ is closed and contained in the ball $\bar{B}(0, L/m)$ where $m = \inf_{\|x\|=1} \|d\mathcal{Z}(x)\|$. The set of critical values of f is hence closed and bounded.*

Notation C.4. For f a function and c a real number, let $f^c = \{x \in \mathbb{R}^m \mid f(x) \leq c\}$ be the sublevel set of f associated with the value c . Note that $f^c \subset f^{c'}$ if $c \leq c'$.

Definition C.5. Let f be a function of \mathcal{Q}_m^∞ and a be a real constant. Since the critical values of f are bounded, we can find $c \geq |a|$ greater than any critical values of f in modulus. For $a \leq c$, let i_a^c be the canonical injection

$$(f^a, f^{-c}) \hookrightarrow (f^c, f^{-c}).$$

It induces a morphism $i_a^{c^*}$ in relative cohomology:

$$H^\bullet(f^c, f^{-c}) \xrightarrow{i_a^{c^*}} H^\bullet(f^a, f^{-c}).$$

We assume that the cohomology is calculated with coefficients in a field, which allows to choose a simplified definition.

Let the *minmax* of f be defined by

$$\sigma(f) = \inf \{a \in \mathbb{R} \mid i_a^{c^*} \neq 0\} = \sup \{a \in \mathbb{R} \mid i_a^{c^*} = 0\}.$$

This definition does not depend on the choice of c when c is large enough.

Proof. The fact that $\sigma(f)$ does not depend on the choice of c when it is large enough is a consequence of the following lemma:

Lemma C.6. *If $c_1 \geq |a|$ and $c_2 \geq a$ are two real constants greater than any critical values of f in modulus, $i_a^{c_1^*}$ and $i_a^{c_2^*}$ are conjugate in cohomology. Therefore they are simultaneously zero or non-zero.*

Proof. Suppose $c_2 > c_1$. If $a = -c_1$, let us check that $i_a^{c_1^*} = i_a^{c_2^*} = 0$:

$$H^\bullet(f^{c_1}, f^{-c_1}) \xrightarrow{i_a^{c_1^*}} \underbrace{H^\bullet(f^{-c_1}, f^{-c_1})}_{=\{0\}}$$

and therefore $i_{-c_1}^{c_1^*} = 0$. We can prove that $i_{-c_1}^{c_2^*} = 0$ in the same way:

$$H^\bullet(f^{c_2}, f^{-c_2}) \xrightarrow{i_a^{c_2^*}} \underbrace{H^\bullet(f^{-c_1}, f^{-c_2})}_{=\{0\}}$$

where the nullity of $H^\bullet(f^{-c_1}, f^{-c_2})$ is guaranteed by the retraction constructed in Lemma D.1.

Now, if $a > -c_1$, there is an $\varepsilon > 0$ such that $-c_1 + \varepsilon \leq a$, and f has no critical value in $[-c_2 - \varepsilon, -c_1 + \varepsilon]$ or in $[c_1 - \varepsilon, c_2 + \varepsilon]$. Since $-c_1 + \varepsilon \leq a \leq c_1$, Deformation lemma D.1 gives two homotopy equivalences Φ_+ and Φ_- such that:

$$\begin{cases} \Phi_+(f^{c_2}) = f^{c_1} \\ \Phi_+(f^a) = f^a \end{cases} \quad \text{and} \quad \begin{cases} \Phi_-(f^{-c_1}) = f^{-c_2} \\ \Phi_-(f^a) = f^a. \end{cases}$$

The homotopy equivalences give isomorphisms in cohomology, and the following diagram commutes:

$$\begin{array}{ccc} H^\bullet(f^{c_1}, f^{-c_1}) & \xrightarrow{i_a^{c_1^*}} & H^\bullet(f^a, f^{-c_1}) \\ \wr \downarrow (\Phi_+^*)^{-1} & & \wr \downarrow (\Phi_+^*)^{-1} \\ H^\bullet(f^{c_2}, f^{-c_1}) & \circlearrowleft & H^\bullet(f^a, f^{-c_1}) \\ \wr \downarrow \Phi_-^* & & \wr \downarrow \Phi_-^* \\ H^\bullet(f^{c_2}, f^{-c_2}) & \xrightarrow{i_a^{c_2^*}} & H^\bullet(f^a, f^{-c_2}) \end{array}$$

which proves that $i_a^{c_1^*}$ and $i_a^{c_2^*}$ are conjugate in cohomology. \square

Let us now fix c large enough and prove that $\inf \{a \in \mathbb{R} \mid i_a^{c^*} \neq 0\} = \sup \{a \in \mathbb{R} \mid i_a^{c^*} = 0\}$. To do so, we are going to prove that any element of the set $\{a \in \mathbb{R} \mid i_a^{c^*} \neq 0\}$ is bigger than any element

of its complement set $\{a \in \mathbb{R} \mid i_a^{c^*} = 0\}$. Let a be such that $i_a^{c^*} \neq 0$ and b be such that $i_b^{c^*} = 0$. Assume that $b > a$. The following diagram commutes:

$$\begin{array}{ccc} (f^a, f^{-c}) & \xrightarrow{i} & (f^b, f^{-c}) \\ & \searrow i_a^c \circlearrowleft & \downarrow i_b^c \\ & & (f^c, f^{-c}) \end{array}$$

where i denotes the canonical injection from (f^a, f^{-c}) to (f^b, f^{-c}) . It induces a commutative diagram in cohomology:

$$\begin{array}{ccc} H^\bullet(f^a, f^{-c}) & \xleftarrow{i^*} & H^\bullet(f^b, f^{-c}) \\ & \nwarrow i_a^{c^*} \circlearrowleft & \uparrow i_b^{c^*} \\ & & H^\bullet(f^c, f^{-c}) \end{array}$$

Since $i_b^{c^*}$ is zero, $i_a^{c^*}$ is necessarily zero which is excluded. We have proved that $a \geq b$ (and then $a > b$ since $i_a^{c^*} \neq i_b^{c^*}$), and consequently:

$$\inf \{a \in \mathbb{R} \mid i_a^{c^*} \neq 0\} = \sup \{a \in \mathbb{R} \mid i_a^{c^*} = 0\}.$$

□

Theorem C.7. *The minmax $\sigma(f)$ is a critical value of f .*

Proof. Suppose that $\sigma(f)$ is not a critical value of f . Then, since the set of critical values of f is closed (see Proposition C.3), there is a $\varepsilon > 0$ such that f has no critical value in $[\sigma(f) - \varepsilon, \sigma(f) + \varepsilon]$. Since $\sigma(f)$ is finite, by definition, there exist a and b such that $\sigma(f) - \varepsilon < a \leq \sigma(f) \leq b < \sigma(f) + \varepsilon$, $i_a^* = 0$ and $i_b^* \neq 0$. Taking c strictly bigger than $|a|$, $|b|$ and any critical value of f , Proposition C.6 states that $i_a^{c^*} = 0$ and $i_b^{c^*} \neq 0$.

One can find an $\varepsilon' > 0$ such that $[a - \varepsilon', b + \varepsilon'] \subset [\sigma(f) - \varepsilon, \sigma(f) + \varepsilon]$ and $b + \varepsilon' \leq c$, so that $[a - \varepsilon', b + \varepsilon']$ does not contain any critical point of f , and Deformation lemma D.1 builds a continuous function Φ such that $\Phi(f^b, f^{-c}) = (f^a, f^{-c})$ and also $\Phi(f^c, f^{-c}) = (f^c, f^{-c})$ since $b + \varepsilon' \leq c$. Since Φ is a homotopy equivalence, it defines an isomorphism in cohomology. The following diagram should then commute:

$$\begin{array}{ccc} H^\bullet(f^c, f^{-c}) & \xrightarrow{i_a^{c^*=0}} & H^\bullet(f^a, f^{-c}) \\ \wr \downarrow \Phi^* & \circlearrowleft & \wr \downarrow \Phi^* \\ H^\bullet(f^c, f^{-c}) & \xrightarrow{i_b^{c^* \neq 0}} & H^\bullet(f^b, f^{-c}) \end{array}$$

which is impossible. Hence, $\sigma(f)$ is necessarily a critical value of f .

□

C.2 Minmax properties for smooth functions

Proposition C.8. *Let f be in \mathcal{Q}_m^∞ . Then the minmax satisfies:*

1. $\sigma(f)$ is a critical value of f ,
2. if c is a real number, $\sigma(c + f) = c + \sigma(f)$,
3. if ϕ is a Lipschitz C^∞ -diffeomorphism on \mathbb{R}^m such that $f \circ \phi$ is in \mathcal{Q}_m , then

$$\sigma(f \circ \phi) = \sigma(f).$$

Proof. 1. has already been proved (see Theorem C.7).

2. If $b > 0$ is a real number, $g = b + f$ is in \mathcal{Q}_m^∞ . For all $c \in \mathbb{R}$, $f^c = g^{c+b}$. Choose c big enough so that $c - 2b$ is strictly greater than $|a|$ and than the critical values of f . Take a in \mathbb{R} and let us show that $i_a^{c,f^*} \neq 0 \iff i_{a+b}^{c-b,g^*} \neq 0$. There is an $\varepsilon > 0$ such that f has no critical value of f in $[c + \varepsilon, c - 2b - \varepsilon]$. Now take the homotopy equivalence constructed in Lemma D.1 and satisfying:

$$\begin{cases} \Phi(f^c) = f^{c-2b} \\ \Phi(f^u) = f^u \quad \forall u \leq c - 2b. \end{cases}$$

This gives the following commutative diagram, since a and $-c$ are smaller than $c - 2b$:

$$\begin{array}{ccc} H^\bullet(f^c, f^{-c}) & \xrightarrow{i_a^{c,f^*}} & H^\bullet(f^a, f^{-c}) \\ \wr \uparrow \Phi^* & & \wr \uparrow \Phi^* \\ H^\bullet(f^{c-2b}, f^{-c}) & \circlearrowleft & H^\bullet(f^a, f^{-c}) \\ \parallel & & \parallel \\ H^\bullet(g^{c-b}, g^{-c+b}) & \xrightarrow{i_{a+b}^{(c-b),g^*}} & H^\bullet(g^{a+b}, g^{-c+b}) \end{array}$$

which proves that $i_a^{c,f^*} = 0 \iff i_{a+b}^{(c-b),g^*} = 0$. But since the critical values of g are the critical values of f added to the constant b , $c - b$ is greater than any critical value of g in modulus since $c - 2b$ is greater in modulus than the critical values of f . Lemma C.6 states that the nullity of i_a^{c,f^*} (resp. i_a^{c,g^*}) does not depend on c large enough, hence:

$$\sigma(f) = \inf \{a \in \mathbb{R} \mid i_a^{c,f^*} \neq 0\} = \inf \{a \in \mathbb{R} \mid i_{a+b}^{(c-b),g^*} \neq 0\} = \sigma(g) - b.$$

3. Let ϕ be a Lipschitz C^∞ -diffeomorphism of \mathbb{R}^m such that $g = f \circ \phi$ is in \mathcal{Q}_m^∞ . Note that f and g have the same critical values. Take a in \mathbb{R} and $c \geq |a|$ greater than any critical value of f (hence g).

For all $u \in \mathbb{R}$, $f^u = \phi(g^u)$. Since ϕ is a C^∞ -diffeomorphism mapping the pair $(g^{u'}, g^u)$ to $(f^{u'}, f^u)$ for all real numbers $u < u'$, ϕ gives an isomorphism in cohomology. The following diagram commutes:

$$\begin{array}{ccc} H^\bullet(f^c, f^{-c}) & \xrightarrow{i_a^{c,f^*}} & H^\bullet(f^a, f^{-c}) \\ \wr \downarrow \phi^* & \circlearrowleft & \wr \downarrow \phi^* \\ H^\bullet(g^c, g^{-c}) & \xrightarrow{i_a^{c,g^*}} & H^\bullet(g^a, g^{-c}) \end{array}$$

which shows that $i_a^{c,f^*} \neq 0 \iff i_a^{c,g^*} \neq 0$, hence $\sigma(f) = \sigma(g)$. □

Now let us focus on the monotonicity of the minmax.

Definition C.9. If f_0 and f_1 are two functions of \mathcal{Q}_m^∞ with Lipschitz difference, let us consider the homotopy $f_t = (1-t)f_0 + tf_1$ between f_0 and f_1 and denote by \mathcal{C}_{f_0, f_1} the set of critical points $\mathcal{C}_{f_0, f_1} = \{x \in \mathbb{R}^m \mid \exists t \in [0, 1], df_t(x) = 0\}$.

Proposition C.10. *Under these assumptions, the set \mathcal{C}_{f_0, f_1} is compact.*

Proof. Let us denote by $f_0 = \mathcal{Z} + \ell_0$ and $f_1 = \mathcal{Z} + \ell_1$. If L is a Lipschitz constant suiting both ℓ_0 and ℓ_1 , note that $\ell_0 + t(\ell_1 - \ell_0)$ is also L -Lipschitz. The critical points of f_t are hence in the ball $\bar{B}(0, L/m)$ by Proposition C.3, and \mathcal{C}_{f_0, f_1} is a bounded set.

Let (x_n) be a converging sequence of \mathcal{C}_{f_0, f_1} and denote by x its limit. By definition of \mathcal{C}_{f_0, f_1} , there is a sequence $(t_n) \in [0, 1]$ such that $df_{t_n}(x_n) = 0$ for all n . Since (t_n) is bounded, it is possible to find a subsequence of t_n converging to some $t \in [0, 1]$. Since $(t, x) \mapsto f_t(x)$ is \mathcal{C}^1 , $df_t(x)$ is zero, and \mathcal{C}_{f_0, f_1} is closed. \square

Proposition C.11. *Let f_0 and f_1 be two functions of \mathcal{Q}_m^∞ with Lipschitz difference. If U is a set containing \mathcal{C}_{f_0, f_1} and $f_0 \geq f_1$ on U , then $\sigma(f_0) \geq \sigma(f_1)$. In particular if $f_0 \geq f_1$ on \mathcal{C}_{f_0, f_1} (or if $f_0 \geq f_1$ on \mathbb{R}^m), then $\sigma(f_0) \geq \sigma(f_1)$.*

Consequence C.12. If f_0 and f_1 are two functions of \mathcal{Q}_m^∞ with Lipschitz difference:

$$\inf_U (f_0 - f_1) \leq \inf_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1) \leq \sigma(f_0) - \sigma(f_1) \leq \sup_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1) \leq \sup_U (f_0 - f_1).$$

for each set U containing the set \mathcal{C}_{f_0, f_1} . In particular if $f_0 - f_1$ is Lipschitz and bounded on \mathbb{R}^m , then $|\sigma(f_0) - \sigma(f_1)| \leq \|f_0 - f_1\|_\infty$.

Proof. Since $f_1 + \inf_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1) \leq f_0 \leq f_1 + \sup_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1)$ on \mathcal{C}_{f_0, f_1} and the three functions are in \mathcal{Q}_m^∞ with Lipschitz difference, Proposition (C.11) gives

$$\sigma(f_1 + \inf_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1)) \leq \sigma(f_0) \leq \sigma(f_1 + \sup_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1)).$$

The additivity (C.8-2) then concludes:

$$\inf_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1) \leq \sigma(f_0) - \sigma(f_1) \leq \sup_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1).$$

\square

Proof. Let us first prove Proposition C.11 in the case of an open and bounded set U . Take a in \mathbb{R} and $C = \max_{t \in [0, 1]} \sup_U |f_t|$, and choose a c bigger than C and $|a|$. Note that c is bigger in modulus than the critical values of f_0 and f_1 (which are contained in U). Lemma D.2 gives a \mathcal{C}^1 -diffeomorphism $\Psi : (f_0^c, f_0^{-c}) \rightarrow (f_1^c, f_1^{-c})$, sending the pair (f_0^a, f_0^{-c}) into the pair (f_1^a, f_1^{-c}) (since $\Psi(f_0^a) \subset f_1^a$ and $\Psi(f_0^{-c}) = f_1^{-c}$). This results in the following commutative diagram:

$$\begin{array}{ccc} H^\bullet(f_1^c, f_1^{-c}) & \xrightarrow{i_a^{c, f_1^*}} & H^\bullet(f_1^a, f_1^{-c}) \\ \wr \downarrow \Psi^* & \circlearrowleft & \downarrow \Psi^* \\ H^\bullet(f_0^c, f_0^{-c}) & \xrightarrow{i_a^{c, f_0^*}} & H^\bullet(f_0^a, f_0^{-c}) \end{array}$$

Hence, if i_a^{c, f_1^*} is zero, since the left arrow is one-to-one, i_a^{c, f_0^*} is necessarily zero. This proves that $\{a \in \mathbb{R} | i_a^{c, f_0^*} \neq 0\} \subset \{a \in \mathbb{R} | i_a^{c, f_1^*} \neq 0\}$ and then $\sigma(f_1) \leq \sigma(f_0)$.

Now, if U is not open anymore, but bounded, it is contained for all $\delta > 0$ in the open and bounded set $U_\delta = \{x \in \mathbb{R}^d | d(x, U) < \delta\}$. Furthermore since $f_0 \geq f_1$ on U and since U_δ is bounded, we have by continuity of f_0 and f_1 that $f_0 \geq f_1 + w(\delta)$ on U_δ with $w(\delta) \rightarrow 0$ when $\delta \rightarrow 0$. The previous work states that $\sigma(f_0) \geq \sigma(f_1 + w(\delta)) = \sigma(f_1) + w(\delta)$ by additivity of the minmax, and letting δ tend to 0 finishes the proof.

Finally, we get rid of the boundness assumption by observing that since \mathcal{C}_{f_0, f_1} is compact (Proposition C.10), we may always replace U by the intersection of U with a ball large enough to contain \mathcal{C}_{f_0, f_1} , which ends the proof. \square

Proposition C.13. *If the cohomology is calculated with coefficients in a field, $\sigma(-f) = -\sigma(f)$ for each function f of \mathcal{Q}_m^∞ .*

Proof. If f is in \mathcal{Q}_m^∞ with an associate nondegenerate form \mathcal{Z} of index λ , take c bigger in modulus than the critical values of f . The homology calculation for the quadratic form gives that $H_k(f^c, f^{-c}) = 0$ if $k \neq \lambda$ and $H_\lambda(f^c, f^{-c})$ is one dimensional. In particular, if the homology is calculated with coefficients in a field, the homology morphism $i_{a^\star}^c : H_\bullet(f^a, f^{-c}) \rightarrow H_\bullet(f^c, f^{-c})$ induced by i_a^c is non zero if and only if it is one-to-one. Since $i_{a^\star}^c$ is the transposition of $i_a^{c^\star}$, they are simultaneously non zero.

Alexander duality gives the following commutative diagram, with exact columns:

$$\begin{array}{ccc} H_\bullet(f^a, f^{-c}) & \simeq & H^\bullet(\mathbb{R}^m \setminus f^{-c}, \mathbb{R}^m \setminus f^a) \\ i_{a^\star}^c \downarrow & \circlearrowleft & \downarrow \\ H_\bullet(f^c, f^{-c}) & \simeq & H^\bullet(\mathbb{R}^m \setminus f^{-c}, \mathbb{R}^m \setminus f^c) \\ \downarrow & \circlearrowleft & \downarrow \\ H_\bullet(f^c, f^a) & \simeq & H^\bullet(\mathbb{R}^m \setminus f^a, \mathbb{R}^m \setminus f^c) \end{array}$$

If a is not a critical value of f , for $\varepsilon > 0$ small enough $\mathbb{R}^m \setminus f^a = \{-f < -a\}$ retracts on $-f^{-a-\varepsilon}$ via the homotopy equivalence constructed in Lemma D.1, just as $-f^{-a}$. The same can be done for c and $-c$, and composing the cohomology induced isomorphisms we get an isomorphism Φ^\star , completing the previous diagram as follows:

$$\begin{array}{ccccc} H_\bullet(f^a, f^{-c}) & \simeq & H^\bullet(\mathbb{R}^m \setminus f^{-c}, \mathbb{R}^m \setminus f^a) & & \\ i_{a^\star}^c \downarrow & \circlearrowleft & \downarrow & & \\ H_\bullet(f^c, f^{-c}) & \simeq & H^\bullet(\mathbb{R}^m \setminus f^{-c}, \mathbb{R}^m \setminus f^c) & \xrightarrow{\Phi^\star} & H^\bullet((-f)^c, (-f)^{-c}) \\ \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow (i_{c, -f}^{-a})^\star \\ H_\bullet(f^c, f^a) & \simeq & H^\bullet(\mathbb{R}^m \setminus f^a, \mathbb{R}^m \setminus f^c) & \xrightarrow[\Phi^\star]{} & H^\bullet((-f)^{-a}, (-f)^{-c}) \end{array}$$

If a is larger than $\sigma(f)$, $i_a^{c^\star}$ is non zero, hence $i_{a^\star}^c$ is non zero and it is then one-to-one. Since the first column is exact, this implies that $(i_{c, -f}^{-a})^\star$ is zero, hence $-a \leq \sigma(-f)$. This being true for each a larger than $\sigma(g)$, it comes that $-\sigma(f) \leq \sigma(-f)$.

If a is smaller than $\sigma(f)$, $i_a^{c^\star}$, hence $i_{a^\star}^c$, are zero and it follows that $(i_{c, -f}^{-a})^\star$ is non zero, hence $-a \geq \sigma(-f)$. As before this implies that $-\sigma(f) \geq \sigma(-f)$, and the result holds. \square

Remark C.14. The proof of Proposition C.13 is the only place where we need to work with coefficients in a field.

Proposition C.15. *If $f : (x, y) \in \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$ is a function of \mathcal{Q}_{d+k}^∞ such that $\partial_y^2 f \geq \text{cid}$ for some $c > 0$, and if $g(x) = \min_y f(x, y)$ is in \mathcal{Q}_d , then $\sigma(g) = \sigma(f)$.*

Proof. If $\partial_y^2 f \geq \text{cid}$, $y \mapsto f(x, y)$ attains for each x a strict minimum at a point $y(x)$ and $x \mapsto y(x)$ is \mathcal{C}^1 by implicit differentiation of $\partial_y f(x, y(x)) = 0$. Note that $g(x) = f(x, y(x))$ and f have the same critical values and choose c larger in modulus than these critical values.

We denote by \tilde{g}^a the set $\{(x, y(x)) | g(x) \leq a\}$. It is the restriction of the graph of $x \mapsto y(x)$ on g^a . Hence $\Psi : x \mapsto (x, y(x))$, which is a \mathcal{C}^1 -diffeomorphism from \mathbb{R}^d to the graph of $x \mapsto y(x)$, maps for all a g^a on \tilde{g}^a , and it induces an isomorphism in relative cohomology.

For all a in \mathbb{R} , the sublevel set f^a retracts to \tilde{g}^a via $\Phi_t(x, y) = (x, (1-t)y + ty(x))$ which is a deformation retraction. One can indeed check, using the convexity of $y \mapsto f(x, y)$ and the fact that $y(x)$ is the minimum of this function, that:

$$\begin{cases} \Phi_0 = \text{id}, \\ \Phi_1(f^a) \subset \tilde{g}^a, \\ \Phi_t(f^a) \subset f^a \quad \forall t \in [0, 1], \\ \Phi_t = \text{id} \text{ on } \tilde{g}^a. \end{cases}$$

Since this retraction does not depend on a , the following diagram commutes:

$$\begin{array}{ccc}
H^\bullet(f^c, f^{-c}) & \xrightarrow{i_a^{c,f^*}} & H^\bullet(f^a, f^{-c}) \\
\wr \uparrow \Phi_1^* & & \wr \uparrow \Phi_1^* \\
H^\bullet(\tilde{g}^c, \tilde{g}^{-c}) & \circlearrowleft & H^\bullet(\tilde{g}^a, \tilde{g}^{-c}) \\
\wr \uparrow \Psi^{-1*} & & \wr \uparrow \Psi^{-1*} \\
H^\bullet(g^c, g^{-c}) & \xrightarrow{i_a^{c,g^*}} & H^\bullet(g^a, g^{-c})
\end{array}$$

Hence i_a^{c,g^*} and i_a^{c,f^*} are simultaneously nonzero and therefore $\sigma(g) = \sigma(f)$. \square

C.3 Extension to non-smooth functions

From now on the aim is to extend by continuity the definition and properties of the minmax to non-smooth functions.

Definition C.16. If f is in \mathcal{Q}_m , there exists by definition a nondegenerate quadratic form \mathcal{Z} and a Lipschitz function ℓ such that $f = \mathcal{Z} + \ell$. Since ℓ is Lipschitz, there exists an equi-Lipschitz sequence (ℓ_n) of \mathcal{C}^∞ functions such that ℓ_n converge uniformly towards ℓ . Then the minmax of $f = \mathcal{Z} + \ell$ is defined by

$$\sigma(f) = \lim_{n \rightarrow \infty} \sigma(\mathcal{Z} + \ell_n).$$

This does not depend on the choice of (ℓ_n) .

Proof. Let us show that the limit exists, and that it does not depend on the choice of the sequence (ℓ_n) .

- Let $\varepsilon > 0$ be fixed. Since ℓ_n converges uniformly, it is a Cauchy sequence and there is a $N > 0$ such that:

$$\|\ell_n - \ell_m\|_\infty \leq \varepsilon \quad \forall n, m \geq N.$$

Then, since $\mathcal{Z} + \ell_n$ and $\mathcal{Z} + \ell_m$ are in \mathcal{Q}_m^∞ with Lipschitz and bounded difference, Consequence C.12 gives:

$$|\sigma(\mathcal{Z} + \ell_n) - \sigma(\mathcal{Z} + \ell_m)| \leq \|\ell_n - \ell_m\|_\infty \leq \varepsilon \quad \forall n, m \geq N$$

and $(\sigma(\mathcal{Z} + \ell_n))$ is a Cauchy sequence in \mathbb{R} , hence has a limit denoted $\sigma(f)$.

- Let (ℓ_n) and $(\tilde{\ell}_n)$ be two equi-Lipschitz sequences of \mathcal{C}^∞ functions, and assume that ℓ_n and $\tilde{\ell}_n$ admit the same uniform limit ℓ . Let us show that $\sigma(\mathcal{Z} + \ell_n)$ and $\sigma(\mathcal{Z} + \tilde{\ell}_n)$ tend to the same limit.

Let $\varepsilon > 0$. Since ℓ_n and $\tilde{\ell}_n$ have the same limit, there is a $N > 0$ such that:

$$\|\ell_n - \tilde{\ell}_n\|_\infty \leq \varepsilon \quad \forall n \geq N.$$

Then, since $\mathcal{Z} + \ell_n$ and $\mathcal{Z} + \tilde{\ell}_n$ are in \mathcal{Q}_m^∞ with Lipschitz and bounded difference, Consequence C.12 gives:

$$|\sigma(\mathcal{Z} + \ell_n) - \sigma(\mathcal{Z} + \tilde{\ell}_n)| \leq \varepsilon \quad \forall n \geq N.$$

Letting n tend to ∞ shows that the limit does not depend on the choice of the sequence (ℓ_n) . \square

Let us gather the properties satisfied for continuous functions of \mathcal{Q}_m :

Proposition C.17. *If f is in \mathcal{Q}_m , the properties of the smooth minmax still hold:*

1. if c is a real constant, then $\sigma(c + f) = c + \sigma(f)$,
2. if $f_0 \leq f_1$ on \mathbb{R}^m and if $f_1 - f_0$ is Lipschitz, then $\sigma(f_0) \leq \sigma(f_1)$,
3. if ϕ is a Lipschitz \mathcal{C}^∞ -diffeomorphism on \mathbb{R}^m such that $f \circ \phi$ is in \mathcal{Q}_m , then

$$\sigma(f \circ \phi) = \sigma(f),$$

4. $\sigma(-f) = -\sigma(f)$.

Proof. 1. It is enough to notice that if $\mathcal{Z} + \ell_n$ converges to f as in the definition, then $\mathcal{Z} + \ell_n + c$ converges to $f + c$. Then, $\sigma(\mathcal{Z} + c + \ell_n) = c + \sigma(\mathcal{Z} + \ell_n)$ by the additivity property (C.8-2), and the statement holds when n tends to ∞ .

2. If $f^0 \leq f^1$ are in \mathcal{Q}_m and if their difference is Lipschitz, then there exist two sequences of equi-Lipschitz \mathcal{C}^∞ functions (ℓ_n^0) and (ℓ_n^1) such that $\mathcal{Z} + \ell_n^0$ (resp. $\mathcal{Z} + \ell_n^1$) converges uniformly to f^0 (resp. f^1) with $\ell_n^0 \leq \ell_n^1$ for n big enough. Then, Proposition C.11 states that $\sigma(\mathcal{Z} + \ell_n^0) \leq \sigma(\mathcal{Z} + \ell_n^1)$ for n big enough, and the statement holds when n tends to ∞ .

3. Since $\|(\mathcal{Z} + \ell_n) \circ \phi - f \circ \phi\|_\infty \leq \|\mathcal{Z} + \ell_n - f\|_\infty$, if $(\mathcal{Z} + \ell_n)$ converges uniformly to $f = \mathcal{Z} + \ell$, then $(\mathcal{Z} + \ell_n) \circ \phi$ converges uniformly to $f \circ \phi$. Moreover, since ϕ is Lipschitz, $\ell_n \circ \phi$ and $\ell \circ \phi$ are (equi-)Lipschitz. Now since $f \circ \phi = \mathcal{Z} \circ \phi + \ell \circ \phi$ is in \mathcal{Q}_m and $\ell \circ \phi$ is Lipschitz, $\mathcal{Z} \circ \phi$ is in \mathcal{Q}_m^∞ (as \mathcal{Z} is \mathcal{C}^∞) and the sequence $((\mathcal{Z} + \ell_n) \circ \phi)$ is still in \mathcal{Q}_m^∞ .

Thus, $((\mathcal{Z} + \ell_n) \circ \phi)$ is a sequence converging uniformly to $f \circ \phi$, as required in the definition. Since Property (C.8-3) states that $\sigma((\mathcal{Z} + \ell_n) \circ \phi) = \sigma(\mathcal{Z} + \ell_n)$ for all n , the statement holds when n tends to ∞ .

4. This is a direct consequence of Proposition C.13. □

Proposition C.18. *The properties involving critical elements hold for \mathcal{C}^1 functions of \mathcal{Q}_m :*

1. If $f \in \mathcal{Q}_m$ is \mathcal{C}^1 , then $\sigma(f)$ is a critical value of f .
2. If $f_0, f_1 \in \mathcal{Q}_m$ are \mathcal{C}^1 with Lipschitz difference, and \mathcal{C}_{f_0, f_1} is the set of critical points of the homotopy $f_t = (1 - t)f_0 + tf_1$, then

$$\inf_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1) \leq \sigma(f_0) - \sigma(f_1) \leq \sup_{\mathcal{C}_{f_0, f_1}} (f_0 - f_1).$$

Proof. 1. If $f = \mathcal{Z} + \ell$ is \mathcal{C}^1 , then ℓ is \mathcal{C}^1 and there exists an equi-Lipschitz sequence (ℓ_n) of \mathcal{C}^∞ functions such that ℓ_n uniformly converges towards ℓ and $d\ell_n$ converge uniformly towards $d\ell$, hence $\mathcal{Z} + \ell_n$ (resp. $d\mathcal{Z} + d\ell_n$) uniformly converges towards f (resp. df).

For all n , $\sigma(\mathcal{Z} + \ell_n)$ is a critical value of $\mathcal{Z} + \ell_n$, hence there exists x_n in \mathbb{R}^m such that $d\mathcal{Z}(x_n) + \ell_n(x_n) = 0$ and $\sigma(\mathcal{Z} + \ell_n) = (\mathcal{Z} + \ell_n)(x_n)$.

Since the sequence (ℓ_n) is equi-Lipschitz, the sequence (x_n) is contained in the closed ball $\bar{B}(L/m)$ where L denotes a Lipschitz constant suiting all ℓ_n and $m = \inf_{\|x\|=1} \|d\mathcal{Z}(x)\|$, see Proposition C.3.

Hence x_n admits a subsequence converging to some x in \mathbb{R}^m . On the one hand, since $d(\mathcal{Z} + \ell_n)$ converges uniformly towards df and $d(\mathcal{Z} + \ell_n)(x_n) = 0$, x is a critical point of f . On the other hand, since $\mathcal{Z} + \ell_n$ converges uniformly towards f and $(\mathcal{Z} + \ell_n)(x_n) = \sigma(\mathcal{Z} + \ell_n)$, $f(x) = \sigma(f)$. Thus $\sigma(f)$ is a critical value of f .

2. Take f^0 and f^1 in \mathcal{Q}_m , \mathcal{C}^1 and with Lipschitz difference. There exists an equi-Lipschitz sequence (ℓ_n^0) of \mathcal{C}^∞ functions such that $\mathcal{Z} + \ell_n^0$ (resp. $d\mathcal{Z} + d\ell_n^0$) converges uniformly to f^0 (resp. df^0). Note that if t is in $[0, 1]$, the sequence $(\ell_n^t) = (\ell_n^0 + t(f^1 - f^0))$ is equi-Lipschitz uniformly with respect to t , and $f_n^t = \mathcal{Z} + \ell_n^t$ converges uniformly to $f^t = (1-t)f^0 + tf^1$, and the derivative sequence (df_n^t) converges uniformly to df^t .

For all n , Consequence C.12 states that:

$$\inf_{\mathcal{C}_{f_n^0, f_n^1}} (f_n^1 - f_n^0) \leq \sigma(f_n^1) - \sigma(f_n^0) \leq \sup_{\mathcal{C}_{f_n^0, f_n^1}} (f_n^1 - f_n^0).$$

Let us focus on the second inequality. Since $\mathcal{C}_{f_n^0, f_n^1}$ is compact (Proposition C.10), the supremum is attained at some x_n in $\mathcal{C}_{f_n^0, f_n^1}$. By definition of $\mathcal{C}_{f_n^0, f_n^1}$, there exists a sequence (t_n) of $[0, 1]$ such that x_n is a critical point of $f_n^{t_n}$.

Now, since the sequence (ℓ_n^t) is equi-Lipschitz uniformly with respect to t , there exists a ball $B(0, R)$, where R depends only on the Lipschitz constants and on \mathcal{Z} , containing $\mathcal{C}_{f_n^0, f_n^1}$ for all n . The sequence (t_n, x_n) is hence bounded and we may assume it converges to some (t, x) . Since $df_n^{t_n}$ converges uniformly towards df^t , the fact that x_n is a critical point of $f_n^{t_n}$ implies that x is a critical point of f^t , hence x is in \mathcal{C}_{f^0, f^1} .

But then letting n tend to ∞ in

$$\sigma(f_n^1) - \sigma(f_n^0) \leq \sup_{\mathcal{C}_{f_n^0, f_n^1}} (f_n^1 - f_n^0) = f_n^1(x_n) - f_n^0(x_n)$$

gives that

$$\sigma(f_n^1) - \sigma(f_n^0) \leq f^1(x) - f^0(x) \leq \sup_{\mathcal{C}_{f^0, f^1}} (f^1 - f^0),$$

using first the uniform convergence of $f_n^1 - f_n^0$ towards $f^1 - f^0$ and then the fact that x is in \mathcal{C}_{f^0, f^1} . □

The next proposition is the improved version of Proposition C.18-(2) that we require in the definition of a critical value selector, see Definition 2.7.

Proposition C.19. *Let $(f_t)_{t \in [0, 1]}$ be a \mathcal{C}^1 homotopy of \mathcal{Q}_m such that there exists a nondegenerate quadratic function \mathcal{Z} and an equi-Lipschitz family of \mathcal{C}^1 functions $(\ell_t)_{t \in [0, 1]}$ with $f_t = \mathcal{Z} + \ell_t$. Then for all $s \neq t$ in $[0, 1]$*

$$\min_{t \in [0, 1]} \min_{x \in \text{Crit}(f_t)} \partial_t f_t(x) \leq \frac{\sigma(f_t) - \sigma(f_s)}{t - s} \leq \max_{t \in [0, 1]} \max_{x \in \text{Crit}(f_t)} \partial_t f_t(x).$$

Let $(f_t)_{t \in [0, 1]}$ be as in the proposition. Note that if $m = \inf_{\|x\|=1} \|d\mathcal{Z}(x)\|$, the critical points of f_t are contained in the compact set $C = \bar{B}(0, L/m)$. The set $\{(t, x), t \in [0, 1], \partial_x f_t(x) = 0\}$ is also compact: it is contained in the bounded set $[0, 1] \times C$ and is closed by continuity of $\partial_x f$ w.r.t. t and x . Both quantities $\min_{t \in [0, 1]} \min_{x \in \text{Crit}(f_t)} \partial_t f_t(x)$ and $\max_{t \in [0, 1]} \max_{x \in \text{Crit}(f_t)} \partial_t f_t(x)$ are hence attained, and we denote them respectively by a and b .

Lemma C.20. *For all $\varepsilon > 0$, there exists $\alpha > 0$ such that for all t in $[0, 1]$, $\|\partial_x f_t(x)\| \leq \alpha$ implies $a - \varepsilon \leq \partial_t f_t(x) \leq b + \varepsilon$.*

Proof. Assume that there exists an $\varepsilon > 0$ and a sequence (t_n, x_n) such that $\|\partial_x f_{t_n}(x_n)\| \leq 1/n$ and $\partial_t f_{t_n}(x_n) \notin (a + \varepsilon, b + \varepsilon)$. Since $f_{t_n} = \mathcal{Z} + \ell_{t_n}$, $\|\partial_x f_{t_n}(x_n)\| \geq m\|x_n\| - L$ and the sequence x_n is necessarily bounded. Since t_n is in $[0, 1]$, there exists a subsequence of (t_n, x_n) converging to some (t, x) . The continuity of df gives then a contradiction at the point (t, x) . \square

Proof of Proposition C.19. Let us define

$$w(\delta) = \sup_{x \in C, |t-s| \leq \delta} \{\partial_t f_s(x) - \partial_t f_t(x), \partial_x f_s(x) - \partial_x f_t(x)\}.$$

The continuity of df and the compactity of C grants that $w(\delta) \rightarrow 0$ when $\delta \rightarrow 0$.

Let us fix $\varepsilon > 0$ and prove that $(a - 2\varepsilon)(t - s) \leq \sigma(f_t) - \sigma(f_s) \leq (b + 2\varepsilon)(t - s)$ for all $s \leq t$ in $[0, 1]$. Take α as in Lemma C.20 and $\delta > 0$ such that both $w(\delta) < \varepsilon$ and $w(\delta) < \alpha$. We first show the result for $t - s \leq \delta$, and it is immediately extended to large $t - s$ by iteration.

For all x in \mathbb{R}^d , we have

$$(t - s) \inf_{\tau \in [s, t]} \partial_t f_\tau(x) \leq f_t(x) - f_s(x) \leq (t - s) \sup_{\tau \in [s, t]} \partial_t f_\tau(x).$$

Now if \mathcal{C}_{f_s, f_t} denotes the set of critical points of the functions $g_u = (1 - u)f_s + uf_t$ for u in $[0, 1]$, on the one hand, one has that $\mathcal{C}_{f_s, f_t} \subset C = \bar{B}(0, L/m)$, while on the other hand Proposition C.18-(2) states that:

$$\inf_{\mathcal{C}_{f_s, f_t}} (f_t - f_s) \leq \sigma(f_t) - \sigma(f_s) \leq \sup_{\mathcal{C}_{f_s, f_t}} (f_t - f_s),$$

which implies

$$(t - s) \inf_{\substack{\tau \in [s, t] \\ x \in \mathcal{C}_{f_s, f_t}}} \partial_t f_\tau(x) \leq \sigma(f_t) - \sigma(f_s) \leq (t - s) \sup_{\substack{\tau \in [s, t] \\ x \in \mathcal{C}_{f_s, f_t}}} \partial_t f_\tau(x).$$

Since \mathcal{C}_{f_s, f_t} and $[s, t]$ are compact, the right hand side supremum is attained for some τ and x , where x is the critical point of a function $g_u = (1 - u)f_s + uf_t$, and consequently satisfies $\partial_x f_s(x) = u(\partial_x f_s(x) - \partial_x f_t(x))$. Since x is in C and u is in $[0, 1]$, we get $\|\partial_x f_s(x)\| \leq w(|t-s|) \leq \alpha$ by definition of w and δ . Lemma C.20 then implies that $\partial_t f_s(x) \leq b + \varepsilon$.

Now let us estimate $\partial_t f_\tau(x)$: since x is in C and $w(\delta) \leq \varepsilon$,

$$\partial_t f_\tau(x) \leq \partial_t f_s(x) + w(|\tau - s|) \leq b + 2\varepsilon.$$

Putting it altogether we get that for all $\varepsilon > 0$,

$$\sigma(f_t) - \sigma(f_s) \leq f_t(x) - f_s(x) \leq (t - s)\partial_t f_\tau(x) \leq (t - s)(b + 2\varepsilon)$$

for $t - s \leq \delta$, and hence for all t and s . The same work for the left hand side infimum gives that for all $\varepsilon > 0$,

$$(t - s)(a - 2\varepsilon) \leq \sigma(f_t) - \sigma(f_s) \leq (t - s)(b + 2\varepsilon),$$

and letting ε tend to 0 gives the wanted estimate. \square

Appendix D

Deformation lemmas

D.1 Global deformation of sublevel sets

We still work with functions of \mathcal{Q}_m^∞ , i.e. with functions that can be written as the sum of a nondegenerate quadratic function and of a \mathcal{C}^∞ Lipschitz function.

Lemma D.1 (Strong deformation retraction). *Let f be a function of \mathcal{Q}_m^∞ . Take $\varepsilon > 0$ and $a < b$ in \mathbb{R} . If $[a - \varepsilon, b + \varepsilon]$ does not contain any critical value of f , then there is a strong deformation retraction mapping f^b to f^a , i.e. a continuous function $\Phi : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that*

$$\begin{cases} \Phi_0 = \text{id}_{\mathbb{R}^m}, \\ \Phi_1(f^b) \subset f^a, \\ \Phi_t|_{f^a} = \text{id}_{f^a} \quad \forall t \in [0, 1] \\ \Phi_t(f^c) \subset f^c \quad \forall t \in [0, 1], \forall c \in \mathbb{R} \end{cases}$$

satisfying the additional requirement $\Phi_t(f^c) = f^c$ for all $t \in [0, 1]$ and $c > b + \varepsilon$.

Proof. First step. We build a continuous function $\Psi : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that

$$\begin{cases} \Psi_0 = \text{id}_{\mathbb{R}^m}, \\ \Psi_1(f^b) \subset f^a, \\ \Psi_t(f^c) \subset f^c \quad \forall t \in [0, 1], \forall c \in \mathbb{R} \\ \Psi_t(f^c) = f^c, \quad \forall t \in [0, 1], \forall c > b + \varepsilon, \end{cases} \quad (\text{D.1})$$

without requiring that Ψ_t is the identity on f^a for all t .

Let X be the locally Lipschitz vector field defined for x in \mathbb{R}^m by

$$X(x) = \begin{cases} \nabla f(x) & \text{si } \|\nabla f(x)\| \leq 1 \\ \frac{\nabla f(x)}{\|\nabla f(x)\|} & \text{si } \|\nabla f(x)\| > 1 \end{cases}$$

Let us take a \mathcal{C}^∞ function $\phi : \mathbb{R} \rightarrow [0, 1]$ satisfying $\phi = 1$ on $(-\infty, b]$ and $\phi = 0$ on $[b + \varepsilon, \infty)$, and consider the following vector field:

$$Y(x) = \phi(f(x))X(x),$$

defined such that $Y = X$ on f^b and $Y(x) = 0$ if $f(x) \geq b + \varepsilon$.

Let us denote by $\Psi_t(x)$ the flow associated with $-Y$ as follows:

$$\begin{cases} \partial_t \Psi_t(x) = -Y(\Psi_t(x)) \\ \Psi_0(x) = x. \end{cases}$$

As $\|Y\|$ is locally Lipschitz and bounded by the constant 1, Ψ is defined on $\mathbb{R}^+ \times \mathbb{R}^m$ and Ψ_t is a homeomorphism of \mathbb{R}^m for all t . Let us check that $t \mapsto f(\Psi_t(x))$ is non-increasing:

$$\partial_t (f(\Psi_t(x))) = - \underbrace{\phi(f(\Psi_t(x)))}_{\geq 0} \underbrace{X(\Psi_t(x)) \cdot \nabla f(\Psi_t(x))}_{\geq \min\{\|\nabla f(\Psi_t(x))\|^2, \|\nabla f(\Psi_t(x))\|\}} \leq 0.$$

In particular, $\Psi_t(f^c) \subset f^c$ for all $t \geq 0$, and $c \in \mathbb{R}$.

Let us prove that $\Psi_t(f^c) = f^c$ for all $c > b + \varepsilon$. It is enough to prove that $f^c \subset \Psi_t(f^c)$ since the other inclusion is true for all c . Since $Y = 0$ on $\mathbb{R}^m \setminus f^{b+\varepsilon}$, $\Psi_t|_{\mathbb{R}^m \setminus f^c} = \text{id}_{\mathbb{R}^m \setminus f^c}$ for all $c > b + \varepsilon$. Then, if $x \in f^c$, there is a $y \in \mathbb{R}^m$ such that $\Psi_t(y) = x$ (since Ψ_t is onto), and y cannot be in $\mathbb{R}^m \setminus f^c$ since $x \in f^c$. Hence, x belongs to $\Psi_t(f^c)$.

The aim is now to find a $T > 0$ such that $\Psi_T(f^b) \subset f^a$.

Let us prove that there is a real constant $M_0 > 0$ such that:

$$\|df(x)\| \geq M_0 \quad \forall x \in f^b \setminus f^{a-\varepsilon}.$$

Suppose that (x_n) is a sequence of $f^b \setminus f^{a-\varepsilon}$ such that $df(x_n) \rightarrow 0$. Since $f = \mathcal{Z} + \ell$ with \mathcal{Z} nondegenerate quadratic and ℓ Lipschitz, (x_n) is hence bounded and admits a converging subsequence; let x be the limit. Since f and df are continuous, $df(x) = 0$ and $f(x)$ belongs to $[a - \varepsilon, b]$. As this is excluded, the existence of M_0 is proved.

Let x be in f^b . If $t \geq 0$, $\Psi_t(x)$ is in f^b too. Hence, we have $\|\nabla f(\Psi_t(x))\| \geq M_0$ as long as $f(\Psi_t(x)) > a - \varepsilon$, and the estimation of $\frac{d}{dt} f(\Psi_t(x))$ can be improved:

$$\begin{aligned} \partial_t (f(\Psi_t(x))) &\leq - \underbrace{\phi(f(\Psi_t(x)))}_{=1 \text{ since } \Psi_t(x) \in f^b} \min\{\|\nabla f(\Psi_t(x))\|^2, \|\nabla f(\Psi_t(x))\|\} \\ &\leq - \min\{M_0^2, M_0\} < 0. \end{aligned}$$

Let $K = \min(M_0, M_0^2) > 0$. As long as $f(\Psi_t(x)) > a - \varepsilon$, the previous calculation gives:

$$f(\Psi_t(x)) \leq \underbrace{f(\Psi_0(x))}_{=f(x) \leq b} - Kt \leq b - Kt.$$

Let $T = \frac{b-a}{K}$. Assume that for all $t \in [0, T]$, $f(\Psi_t(x)) > a$. The previous calculation shows that $f(\Psi_T(x)) \leq b - KT = a$, which is absurd. Hence there exists $t \in [0, T]$ such that $f(\Psi_t(x)) \leq a$ and then since $t \mapsto f(\Psi_t(x))$ is non increasing, $\Psi_T(x) \subset f^a$.

Up to a time rescaling sending T to 1 ($\tilde{\Psi}_t(x) = \Psi_{t/T}(x)$), we have just constructed a deformation retraction satisfying (D.1).

Second step. Let us now build the strong deformation retraction. For all x in \mathbb{R}^m , let $\tau(x)$ be defined by

$$\tau(x) = \inf \{t \in [0, 1] | \Psi_t(x) \in f^a\}.$$

It is a continuous function on \mathbb{R}^m . If $\Psi_t(x)$ stays out of f^a for all t in $[0, 1]$ (this is the case for all x in $\mathbb{R}^m \setminus f^{b+\varepsilon}$), then $\tau(x)$ is by convention equal to 1. Since $t \mapsto f(\Psi_t(x))$ is non-increasing, if $\Psi_t(x)$ is not in f^a , $t \leq \tau(x)$.

Let us define the mapping Φ :

$$\begin{aligned} \Phi : [0, 1] \times \mathbb{R}^m &\rightarrow \mathbb{R}^m \\ (t, x) &\mapsto \Phi_t(x) = \Psi_{\min(t, \tau(x))}(x) \end{aligned}$$

so that in particular $\Phi_0 = \Psi_0$ and $\Phi_1(x) = \Psi_{\tau(x)}(x)$. The continuity of τ and Ψ implies the continuity of Φ . Let us check that Φ is as required in the Lemma:

- $\Phi_0 = \Psi_0 = \text{id}_{\mathbb{R}^m}$,
- for all x in f^b , $\Psi_1(x)$ is in f^a , and as a consequence $\Phi_1(x) = \Psi_{\tau(x)}(x)$ is in f^a ,
- since $\tau = 0$ on f^a , $\Phi_t = \Psi_0 = \text{id}$ on f^a ,
- for all t in $[0, 1]$, $\Phi_t(f^c) \subset \cup_{u \in [0, 1]} \Psi_u(f^c) \subset f^c$.
- let us fix t in $[0, 1]$ and show that if $c > b + \varepsilon$, $f^c \subset \Phi_t(f^c)$. Since $f^c \subset \Psi_t(f^c)$ for such a c , for all x in f^c there exists y in f^c such that $\Psi_t(y) = x$. If x is not in f^a , $\tau(y) \geq t$, and hence $x = \Psi_t(y) = \Phi_t(y)$ is in $\Phi_t(f^c)$. If x is in f^a , since Φ_t is the identity on f^a , $x = \Phi_t(x)$ is in $\Phi_t(f^a) \subset \Phi_t(f^c)$.

□

D.2 Sending sublevel sets to sublevel sets

Lemma D.2 (Deformation of big sublevel sets of \mathcal{Q}_m^∞ functions with Lipschitz difference). *Let ℓ_0 and ℓ_1 be two C^∞ Lipschitz functions, \mathcal{Z} be a nondegenerate quadratic form on \mathbb{R}^m , and define $f_t = \mathcal{Z} + \ell_0 + t(\ell_1 - \ell_0)$ the homotopy between $f_0 = \mathcal{Z} + \ell_0$ and $f_1 = \mathcal{Z} + \ell_1$. Let U be an open and bounded set of \mathbb{R}^m containing $\mathcal{C} = \{x \in \mathbb{R}^m \mid \exists t \in [0, 1], df_t(x) = 0\}$. There exists a C^∞ -diffeomorphism Ψ of \mathbb{R}^m such that:*

$$\Psi(f_0^c) = f_1^c \quad \begin{aligned} &\forall c > \max_{t \in [0, 1]} \sup_U f_t, \\ &\forall c < \min_{t \in [0, 1]} \inf_U f_t. \end{aligned}$$

Moreover, if $f_0 \geq f_1$ on U , Ψ can be constructed so that $\Psi(f_0^a) \subset f_1^a$ for all $a \in \mathbb{R}$.

Proof. Since \mathcal{C} is compact (see Proposition C.10), there exists an open set Ω containing \mathcal{C} such that $\bar{\Omega} \cap U^c$ is empty (Ω is an open set which is "strictly included" in the open set U). Let X_t be the vector field defined on $\mathbb{R}^m \setminus \Omega$ by

$$X_t(x) = -\partial_t(f_t(x)) \frac{\nabla f_t(x)}{\|\nabla f_t(x)\|^2} \text{ for } t \in [0, 1].$$

Lemma D.3. *If $\gamma(t)$ is a trajectory for the vector field X_t , that is if $\gamma(t)$ stays in $\mathbb{R}^m \setminus \Omega$ and $\dot{\gamma}(t) = X_t(\gamma(t))$, then $f_t(\gamma(t))$ does not depend on t .*

Proof. This is proved by the following calculation:

$$\partial_t(f_t(\gamma(t))) = \underbrace{\dot{\gamma}(t) \cdot \nabla f_t(\gamma(t))}_{= -\partial_t f_t(\gamma(t))} + \partial_t f_t(\gamma(t)) = 0. \quad \square$$

Since $\bar{\Omega}$ and $\mathbb{R}^m \setminus U$ are closed and disjoint, it is possible to find $g : \mathbb{R}^m \rightarrow [0, 1]$ smooth such that $\begin{cases} g = 0 & \text{on } \bar{\Omega} \\ g = 1 & \text{on } \mathbb{R}^m \setminus U \end{cases}$. Let us define $Y_t(x) = g(x)X_t(x)$. The vector field Y is well-defined, \mathcal{C}^∞ on \mathbb{R}^m . It satisfies:

$$\begin{cases} Y_t = X_t & \text{on } \mathbb{R}^m \setminus U \\ Y_t = 0 & \text{on } \Omega. \end{cases}$$

Lemma D.4. *The vector field Y is bounded.*

Proof. If $m = \inf_{\|x\|=1} \|dZ(x)\|$, we get that $\|\nabla f_t(x)\| \geq m\|x\| - L$ for all x in \mathbb{R}^d . As a consequence, if $\|x\| \geq 2L/m$,

$$\|Y_t(x)\| \leq \frac{|\partial_t f_t(x)|}{\|\nabla f_t(x)\|} \leq \frac{L}{m(2L/m) - L} \leq 1.$$

Now, define

$$M = \sup_{\substack{t \in [0, 1] \\ \|x\| \leq 2L/m}} \|Y_t(x)\|.$$

Then Y is bounded by $\max(1, M)$ on \mathbb{R}^m . \square

The flow ψ of Y is hence defined on $\mathbb{R} \times \mathbb{R}^m$; it is the \mathcal{C}^∞ solution of the Cauchy problem:

$$\begin{cases} \partial_t \psi(t, x) = Y_t(\psi(t, x)) \\ \psi(0, x) = x. \end{cases}$$

Let Ψ be the \mathcal{C}^∞ -diffeomorphism mapping x to $\psi(1, x)$.

Let us denote by C_+ (resp. C_-) the quantity $\max_{t \in [0, 1]} \sup_U f_t$ (resp. $\min_{t \in [0, 1]} \inf_U f_t$), and prove that for $c \in \mathbb{R} \setminus [C_-, C_+]$, $\Psi(\{f_0 = c\}) = \{f_1 = c\}$. Take x and y such that $\Psi(x) = y$, and denote by $\gamma(t)$ the trajectory $t \mapsto \psi(t, x)$. Since Y and X coincide on $\mathbb{R}^m \setminus U \subset \mathbb{R}^m \setminus \Omega$, Lemma D.3 states that as long as $\gamma(t)$ is in $\mathbb{R}^m \setminus U$, $f_t(\gamma(t))$ is constant. By definition of C_+ and C_- , $\{f_t = c\}$ is included in $\mathbb{R}^m \setminus U$ for all t in $[0, 1]$, and as a consequence

$$\exists t \in [0, 1], f_t(\gamma(t)) = c \implies \forall t \in [0, 1], f_t(\gamma(t)) = c.$$

This means that $f_0(x) = c$ if and only if $f_1(y) = c$, and since Ψ is one-to-one we hence proved that $\Psi(\{f_0 = c\}) = \{f_1 = c\}$.

As a consequence, we obtain applying the previous work to a suitable union of levelsets that $\Psi(\{f_0 \leq c\}) = \{f_1 \leq c\}$ for $c < C_-$, and that $\Psi(\{f_0 > c\}) = \{f_1 > c\}$ for $c > C_+$. Since Ψ is one-to-one, this implies $\Psi(\{f_0 \leq c\}) = \{f_1 \leq c\}$ for $c > C_+$.

Finally, assume that $f_0 \geq f_1$ on U . Let us again estimate the evolution of $f_t(\gamma(t))$ for a trajectory $\dot{\gamma}(t) = Y_t(\gamma(t))$:

$$\begin{aligned} \partial_t f_t(\gamma(t)) &= \dot{\gamma}(t) \cdot \nabla f_t(\gamma(t)) + \partial_t f_t(\gamma(t)) \\ &= g(\gamma(t))(f_0 - f_1)(\gamma(t)) + (f_1 - f_0)(\gamma(t)) \\ &= (1 - g(\gamma(t)))(f_1 - f_0)(\gamma(t)) = \begin{cases} = 0 & \text{if } \gamma(t) \in \mathbb{R}^m \setminus U \\ \leq 0 & \text{if } \gamma(t) \in U. \end{cases} \end{aligned}$$

since $g = 1$ on $\mathbb{R}^m \setminus U$, $1 - g \geq 0$ and $f_1 \leq f_0$ on U . Now, for $a \in \mathbb{R}$, let $x \in f_0^a$. Since $t \mapsto f_t(\psi(t, x))$ is non-increasing, $f_1(\Psi(x)) \leq f_0(x) \leq a$ and we have proved that $\Psi(f_0^a) \subset f_1^a$. \square

Appendix E

Semiconcave initial condition

In this appendix we prove Theorem 1.24 and Proposition 1.25. Both proofs require only the monotonicity of the variational operator and Proposition 1.22, as well as the following lemma due to P. Bernard (see [Ber13]).

Lemma E.1. *If u is a Lipschitz and B -semiconcave function on \mathbb{R}^d , there exists a family F of \mathcal{C}^2 equi-Lipschitz functions with second derivatives bounded by B such that:*

- $u(q) = \min_{f \in F} f(q)$ for all q ,
- for each q in \mathbb{R}^d and p in $\partial u(q)$, there exists f in F such that $\begin{cases} f(q) = u(q), \\ df(q) = p. \end{cases}$

Proof. Since u is semiconcave, there exists a real constant B such that $q \mapsto u(q) - \frac{B}{2}\|q\|^2$ is concave, and as a consequence for all q_0 and q in \mathbb{R}^d , if p is in $\partial u(q_0)$,

$$u(q) \leq u(q_0) + p \cdot (q - q_0) + \frac{B}{2}\|q - q_0\|^2. \quad (\text{E.1})$$

We take L to be a Lipschitz constant for u . We are going to build a family of $6L$ -Lipschitz functions with second derivative bounded by B checking the wanted conditions.

Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous non-increasing function equal to B on $[0, 4L/B]$ and to 0 on $[5L/B, \infty)$. Let Ψ be the primitive of ψ such that $\Psi(0) = 0$. Note that $\Psi(r) \in [0, 5L]$ for each r in \mathbb{R}_+ . Let then φ be the primitive of Ψ such that $\varphi(0) = 0$. The function φ is $5L$ -Lipschitz, convex, and it satisfies $0 \leq \varphi'' \leq B$. Note also that

$$\varphi(r) \geq \min(Br^2/2, 2Lr). \quad (\text{E.2})$$

Let us consider the family F formed by the \mathcal{C}^2 functions

$$q \mapsto u(q_0) + p \cdot (q - q_0) + \varphi(\|q - q_0\|)$$

for $q_0 \in \mathbb{R}^d$ and $p \in \partial u(q_0)$. Since we have $\|p\| \leq L$, these functions are $6L$ -Lipschitz. Their second derivative is bounded by B , since both ϕ'' and $r \mapsto |\phi'(r)|/r$ are bounded by B . The derivative of $q \mapsto u(q_0) + p \cdot (q - q_0) + \varphi(\|q - q_0\|)$ at q_0 is p . The last thing to prove is that $u(q) = \min_{f \in F} f(q)$ for all $q \in \mathbb{R}^d$. Since $\|p\| \leq L$,

$$u(q) \leq u(q_0) + L\|q - q_0\| \leq u(q_0) + p \cdot (q - q_0) + 2L\|q - q_0\|, \quad (\text{E.3})$$

and putting (E.1), (E.2) and (E.3) together proves that $u(q) \leq u(q_0) + p \cdot (q - q_0) + \varphi(\|q - q_0\|)$. \square

Proof. Let us now prove Theorem 1.24. Proposition 1.22 gives that

$$R_0^t u_0(q) \geq \inf \left\{ u_0(q_0) + \mathcal{A}_0^t(\gamma) \left| \begin{array}{l} (q_0, p_0) \in \mathbb{R}^d \times \mathbb{R}^d, \\ p_0 \in \partial u_0(q_0), \\ Q_0^t(q_0, p_0) = q. \end{array} \right. \right\}.$$

If u is L -Lipschitz and B -semiconcave, take $T = (2M(1+B))^{-1}$ or $T = 1/BC$ if H is integrable. Let us fix definitively $q, q_0, p_0 \in \partial u_0(q_0)$ and $0 \leq t \leq T$ such that $Q_0^t(q_0, p_0) = q$ and show that $R_0^t u_0(q) \leq u_0(q_0) + \mathcal{A}_0^t(\gamma)$ where γ is the Hamiltonian trajectory issued from (q_0, p_0) .

Lemma E.1 gives a \mathcal{C}^2 function f_0 of F such that $f_0(q_0) = u_0(q_0)$ and $df_0(q_0) = p_0$. Since this function is \mathcal{C}^2 with second derivative bounded by B , the method of characteristics gives that q_0 is the only point such that $Q_0^t(q_0, df_0(q_0)) = q$, and the variational resolution for initial condition f_0 can only be

$$R_0^t f_0(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma).$$

But by definition of F , f_0 is larger than u_0 on \mathbb{R}^d , and the monotonicity of the variational operator brings the conclusion:

$$R_0^t u_0(q) \leq R_0^t f_0(q) = f_0(q_0) + \mathcal{A}_0^t(\gamma) = u_0(q_0) + \mathcal{A}_0^t(\gamma).$$

□

Proof. We now prove Proposition 1.25. If t and q are fixed, Proposition 1.22 gives the existence of (q_0, p_0) in $gr(\partial u_0)$ such that $Q_0^t(q_0, p_0) = q$ and that $R_0^t u_0(q) = u_0(q_0) + \mathcal{A}_0^t(\gamma)$ where γ is the Hamiltonian trajectory issued from (q_0, p_0) .

Lemma E.1 gives a \mathcal{C}^2 function f_0 of F such that $f_0(q_0) = u_0(q_0)$ and $df_0(q_0) = p_0$. The method of characteristics states that there exists a unique \mathcal{C}^2 solution of the (HJ) equation with initial condition f_0 , which satisfies in particular $f(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma)$.

Since a \mathcal{C}^1 solution is a viscosity solution, the uniqueness of viscosity solutions hence gives that $V_0^t f = f(t, \cdot)$ for all $t > 0$, and in particular

$$V_0^t f_0(q) = f(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma).$$

But by definition of F , f_0 is larger than u_0 on \mathbb{R}^d , and the monotonicity of the viscosity operator V_0^t brings the conclusion:

$$V_0^t u_0(q) \leq V_0^t f_0(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma) = R_0^t u_0(q).$$

□

Appendix F

Lax condition and entropy condition

Definition F.1. If $p_- < p_+$, the *entropy condition* between p_- and p_+ is said to be (strictly) satisfied if the graph of H lies (strictly) under the line joining the points $(p_-, H(p_-))$ and $(p_+, H(p_+))$, or equivalently if for all $p \in (p_-, p_+)$,

$$\frac{H(p) - H(p_-)}{p - p_-} \underset{(<)}{<} \frac{H(p_+) - H(p_-)}{p_+ - p_-},$$

or equivalently if for all $p \in (p_-, p_+)$,

$$\frac{H(p_+) - H(p)}{p_+ - p} \underset{(>)}{>} \frac{H(p_+) - H(p_-)}{p_+ - p_-}.$$

If $p_- < p_+$, the *Lax condition* between p_- and p_+ is said to be (strictly) satisfied if

$$H'(p_-) \underset{(<)}{<} \frac{H(p_+) - H(p_-)}{p_+ - p_-} \underset{(<)}{<} H'(p_+)$$

The aim of this appendix is to state two results of stability for the entropy condition, whether the Lax condition is strict or not.

Lemma F.2. *If the entropy condition is strictly satisfied between p_- and p_+ then for all $\delta > 0$, there exists $0 < \varepsilon < \delta$ such that for all $|\varepsilon_1|, |\varepsilon_2| < \varepsilon$, and $p \in [p_- + \delta, p_+ - \delta]$, the point $(p, H(p))$ lies under the line joining $(p_- + \varepsilon_1, H(p_- + \varepsilon_1))$ and $(p_+ - \varepsilon_2, H(p_+ - \varepsilon_2))$*

Proof. Let $\delta > 0$ and assume that there exists no such ε . Then for all $n \in \mathbb{N}$, there exists $p_n \in [p_- + \delta, p_+ - \delta]$ and $(p_n^-, p_n^+) \in [p_- - 1/n, p_- + 1/n] \times [p_+ - 1/n, p_+ + 1/n]$ such that $(p_n, H(p_n))$ lies above the line joining $(p_n^-, H(p_n^-))$ and $(p_n^+, H(p_n^+))$. Since p_n is in the fixed compact $[p_- + \delta, p_+ - \delta]$, we may extract to find a contradiction to the strict entropy condition. \square

Proposition F.3. *Let H and $p_- < p_+$ be such that the entropy condition and the Lax condition between p_- and p_+ are strictly satisfied. Then there exists $\varepsilon > 0$ such that for all $(\tilde{p}_-, \tilde{p}_+)$ in $[p_- - \varepsilon, p_- + \varepsilon] \times [p_+ - \varepsilon, p_+ + \varepsilon]$, the entropy condition between \tilde{p}_- and \tilde{p}_+ is satisfied.*

Proof. Without loss of generality, we may assume that $H(p_-) = H(p_+) = 0$. The strict Lax condition then writes $H'(p_-) < 0 < H'(p_+)$.

Since the Lax condition is strictly satisfied between p_- and p_+ , there exists $\varepsilon > 0$ such that for all $(\tilde{p}_-, \tilde{p}_+)$ in $[p_- - \varepsilon, p_- + \varepsilon] \times [p_+ - \varepsilon, p_+ + \varepsilon]$, the Lax condition between \tilde{p}_- and \tilde{p}_+ is satisfied (by continuity of the quantities involved).

Let $\frac{1}{2} > \delta > 0$ be such that

$$\begin{cases} |H(p) - (p - p_-)H'(p_-)| \leq \frac{-H'(p_-)}{2}|p - p_-| & \forall p \in [p_- - \delta, p_- + \delta] \\ \text{and} \\ |H(p) - (p - p_+)H'(p_+)| \leq \frac{H'(p_+)}{2}|p_+ - p| & \forall p \in [p_+ - \delta, p_+ + \delta]. \end{cases}$$

Lemma F.2 gives an $0 < \varepsilon < \delta$ such that for all $|\varepsilon_1|, |\varepsilon_2| < \varepsilon$, and $p \in [p_- + \delta, p_+ - \delta]$, the point $(p, H(p))$ lies under the line joining $(p_- + \varepsilon_1, H(p_- + \varepsilon_1))$ and $(p_+ - \varepsilon_2, H(p_+ - \varepsilon_2))$.

Hence take $(\tilde{p}_-, \tilde{p}_+)$ in $[p_- - \varepsilon, p_- + \varepsilon] \times [p_+ - \varepsilon, p_+ + \varepsilon]$, and then no p in $[p_- + \delta, p_+ - \delta]$ denies the entropy condition.

If $p \in [\tilde{p}_-, p_- + \delta]$, let us prove that

$$\frac{H(p) - H(\tilde{p}_-)}{p - \tilde{p}_-} \leq \frac{H(\tilde{p}_+) - H(\tilde{p}_-)}{\tilde{p}_+ - \tilde{p}_-}.$$

Because of the definition of δ ,

$$\frac{H(p) - H(\tilde{p}_-)}{p - \tilde{p}_-} \leq H'(p_-) - \delta H'(p_-) < \frac{1}{2}H'(p_-) < 0.$$

Since \tilde{p}_- and \tilde{p}_+ are δ close to p_- and p_+ ,

$$\frac{H(\tilde{p}_+) - H(\tilde{p}_-)}{\tilde{p}_+ - \tilde{p}_-} \geq \frac{-2\delta H'(p_+) + 2\delta H'(p_-)}{p_+ - p_- + 2\delta}.$$

Since this last term is arbitrarily small when δ tends to 0, for δ chosen small enough, the wanted inequality holds. The same work applies for $p \in [p_+ - \delta, \tilde{p}_+]$, which closes the discussion. \square

Proposition F.4. *Let H and $p_- < p_+$ be such that the entropy condition between p_- and p_+ is strictly satisfied and the Lax condition is satisfied but not strictly at p_- . Assume further that $H''(p_-) < 0$. Then there exists $\varepsilon > 0$ such that for all $(\tilde{p}_-, \tilde{p}_+)$ in $[p_- - \varepsilon, p_- + \varepsilon] \times [p_+ - \varepsilon, p_+ + \varepsilon]$, the Lax condition between \tilde{p}_- and \tilde{p}_+ implies the entropy condition between \tilde{p}_- and \tilde{p}_+ .*

Proof. Without loss of generality, we may assume that $H(p_-) = H(p_+) = 0$. We are going to prove the case when $H'(p_-) = 0 < H'(p_+)$.

Since $H''(p_-)$, there exists $\delta > 0$ such that H is concave on $[p_- - \delta, p_- + \delta]$ and for all p in $[p_+ - \delta, p_+ + \delta]$,

$$|H(p) - (p - p_+)H'(p_+)| \leq \frac{H'(p_+)}{2}|p_+ - p|.$$

Lemma F.2 gives an $0 < \varepsilon < \delta$ such that for all $|\varepsilon_1|, |\varepsilon_2| < \varepsilon$, and $p \in [p_- + \delta, p_+ - \delta]$, the point $(p, H(p))$ lies under the line joining $(p_- + \varepsilon_1, H(p_- + \varepsilon_1))$ and $(p_+ - \varepsilon_2, H(p_+ - \varepsilon_2))$. Hence take $(\tilde{p}_-, \tilde{p}_+)$ in $[p_- - \varepsilon, p_- + \varepsilon] \times [p_+ - \varepsilon, p_+ + \varepsilon]$, and then no p in $[p_- + \delta, p_+ - \delta]$ denies the entropy condition. If $p \in [p_+ - \delta, \tilde{p}_+]$, the same argument than in the previous case shows that p does not deny the entropy condition between \tilde{p}_- and \tilde{p}_+ .

Let now p be in $[\tilde{p}_-, p_- + \delta]$: since H is concave on $[p_- - \delta, p_- + \delta]$, we have

$$H'(\tilde{p}_-) \geq \frac{H(p) - H(\tilde{p}_-)}{p - \tilde{p}_-}$$

and hence, if the Lax condition is satisfied between \tilde{p}_- and \tilde{p}_+ ,

$$\frac{H(\tilde{p}_+) - H(\tilde{p}_-)}{\tilde{p}_+ - \tilde{p}_-} \geq H'(\tilde{p}_-) \geq \frac{H(p) - H(\tilde{p}_-)}{p - \tilde{p}_-}$$

and p does not violate the entropy condition. \square

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Résumé

On étudie l'équation de Hamilton-Jacobi évolutive du premier ordre, couplée avec une donnée initiale lipschitzienne. Le but est de comparer les solutions de viscosité et les solutions variationnelles pour cette équation, deux notions de solutions faibles qui coïncident en dynamique hamiltonienne convexe.

Pour travailler dans un cadre pertinent pour les deux types de solutions, on doit d'abord construire une solution variationnelle sans hypothèse de compacité sur la variété ou le hamiltonien étudiés. On retrace dans ce cas la construction historique des solutions variationnelles, en détaillant les propriétés de la famille génératrice obtenue par la méthode des géodésiques brisées. Il en découle des estimées permettant d'obtenir la solution de viscosité à partir de la solution variationnelle par un procédé d'itération.

Après avoir vérifié que la solution variationnelle construite coïncide effectivement avec la solution de viscosité pour un Hamiltonien convexe, on caractérise les Hamiltoniens intégrables pour lesquels cette propriété persiste, en étudiant attentivement des exemples élémentaires en dimension 1 et 2.

Mots Clés

Équation de Hamilton-Jacobi, dynamique hamiltonienne non convexe, solutions de viscosité, solutions variationnelles, fronts d'onde, familles génératrices, sélecteur minmax.

Abstract

We study the first order Hamilton-Jacobi equation associated with a Lipschitz initial condition. The purpose of this thesis is to compare two notions of weak solutions for this equation, namely the viscosity solution and the variational solution, that are known to coincide in convex Hamiltonian dynamics.

In order to work in a relevant framework for both notions, we first need to build a variational solution without compactness assumption on the manifold or the Hamiltonian. To do so, we follow the historical construction, detailing properties of the generating family obtained via the broken geodesics method. Local estimates allow to prove that the viscosity solution can be obtained from the variational solution via an iterative process.

We then check that this construction gives effectively the viscosity solution for a convex Hamiltonian, and characterize the integrable Hamiltonians for which this property persists by carefully studying elementary examples in dimension 1 and 2.

Keywords

Hamilton-Jacobi equation, nonconvex Hamiltonian dynamics, viscosity solutions, variational solutions, wavefronts, generating families, minmax selector.