Stabilisation and simulation of fluid-structure interaction models

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THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le *09/12/2016* **par :**

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Stabilisation et simulation de modèles d'interaction fluide-structure

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À ma grand-mère

Remerciements

Mes premières pensées vont à mes directeurs de thèse. Tout d'abord, je tiens à remercier Jean-Pierre Raymond qui m'a fait découvrir ce sujet riche et passionnant, au carrefour de plusieurs domaines. Nous avons passé de nombreuses heures à discuter ainsi qu'à relire et corriger les innombrables versions de mes papiers. Je remercie aussi Denis Matignon pour sa disponibilité malgré un emploi du temps chargé. Sa grande culture mathématique et son souci de la précision m'ont toujours poussé à peaufiner mes raisonnements et ma rédaction. Je remercie enfin Michel Fournié qui nous a rejoints en cours de route. C'est sa disponibilité et l'intérêt qu'il a porté à ma thèse qui ont permis l'aboutissement de la partie numérique de ce travail.

Je remercie Bertrand Maury et Marius Tucsnak qui m'ont fait l'honneur d'accepter d'être les rapporteurs de cette thèse. Je voudrais de même adresser mes remerciements aux autres membres du jury : Franck Boyer, Miguel Fernández et Céline Grandmont.

Je remercie tous les doctorants et jeunes docteurs de l'équipe MIP que j'ai eu un réel plaisir à côtoyer durant ces trois années : Antoine, Marc, Lien, Guillaume, Laurent, Gilles, Jean-Jérôme, Sourav, Baptiste.

Toute ma reconnaissance va à mes parents qui m'ont toujours encouragé et poussé à aller plus loin. Je suis profondément redevable à mon grand frère Boursine. Son soutien matériel et ses précieux conseils ont été pour beaucoup dans la réussite de mes études supérieures. Je remercie aussi les autres membres de ma famille et mes amis.

Pour terminer, je voudrais remercier ma femme pour son soutien moral ininterrompu. Ma troisième année de thèse, connue comme étant la plus difficile, a été, par la naissance de ma petite fille Halima, la plus belle de ces trois années.

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Introduction

1 Contexte

Les modèles d'interaction-fluide permettent de décrire divers phénomènes physiques comme l'écoulement d'air autour d'une aile d'avion ou d'un pont, le déplacement d'un poisson dans l'eau, l'écoulement sanguin dans les artères, etc. Plus généralement, les modèles d'interaction fluide-structure décrivent l'écoulement d'un fluide en présence d'une structure élastique située à l'intérieur ou à la frontière du domaine fluide. On parle d'interaction car la force exercée par le fluide déforme la structure et cette déformation modifie l'écoulement du fluide.
Le contrôle de tels modèles est un enjeu important dans des domaines comme l'aéronautique, la biologie ou le génie civil car les conséquences résultantes de cette interaction peuvent être catastrophiques. Dans cette thèse, nous nous intéressons uniquement à la stabilisation par feedback autour d'états stationnaires instables. Pour définir cette notion de stabilisation, nous considérons le système suivant :

$$z' = F(z) + Bv \text{ dans } (0, \infty), \quad z(0) = z_0, \tag{1 .1}$$

où F est une application non-linéaire et non bornée dans un espace de Hilbert Z et $B \in \mathcal{L}(\mathcal{V}, Z)$ est l'opérateur de contrôle. L'espace de Hilbert $\mathcal{V}$ est l'espace des contrôles. Soit z_s une solution stationnaire du système non-linéaire, i.e. une fonction vérifiant

$$F(z_s) = 0.$$

Définition 1 .1. *Le système (1 .2) est stabilisable en boucle fermée, localement au voisinage de z_s, s'il existe une loi de contrôle $K \in \mathcal{L}(Z, \mathcal{V})$ et $r > 0$ tels que si*

$$\|z_0 - z_s\| \leq r,$$

alors la solution z du système en boucle fermée

$$z' = F(z) + BK(z - z_s) \text{ dans } (0, \infty), \quad z(0) = z_0, \tag{1 .2}$$

vérifie

$$\|z(t) - z_s\| \leq Ce^{-\tau t}\|z_0 - z_s\|, \quad \forall\, t \geq 0, \tag{1 .3}$$

où $\tau > 0$, $C > 0$ et $\|\cdot\|$ est une norme sur l'espace des conditions initiales.

Dans les problèmes étudiés ici la stabilisation locale (où sens de la définition 1 .1) du système (1 .2) se ramène à l'étude de la stabilisation du système linéarisé

$$z' = Az + Bv \text{ dans } (0, \infty), \quad z(0) = z_0, \tag{1 .4}$$

où $(A, D(A))$ est le générateur d'un semi-groupe continu sur Z. Le système (1 .4) est appelé système de contrôle. Nous le représenterons aussi par la paire (A, B).

Définition 1 .2. *Le système (1 .4) ou la paire* (A, B) *est stabilisable par feedback s'il existe une loi de contrôle* $K \in \mathcal{L}(Z, \mathcal{V})$ *tel que* $(A + BK, D(A + BK))$ *est le générateur d'un semi-groupe continu exponentiellement stable.*

Dans cette thèse, nous considérons des écoulements dans des domaines de type canal, avec ou sans obstacle, et en présence d'une structure située à la frontière du domaine occupé par le fluide. L'écoulement du fluide est décrit par les équations de Navier-Stokes incompressibles tandis que le déplacement de la structure vérifie une équation de poutre avec amortissement. Nous supposons de plus que l'on a des conditions aux limites mixtes pour le fluide. Une condition limite de type Dirichlet est donnée en entrée du canal. Cette partie de la frontière est notée Γ_i (l'indice i est choisi pour *inflow*). En sortie du canal, nous supposons que l'écoulement vérifie une condition limite de type Neumann et cette partie est notée Γ_n. Sur la partie supérieure et inférieure du canal, nous imposons soit des conditions de type Dirichlet (chapitre 1) soit des conditions de type Navier (chapitres 2, 3 et 4).

Le caractère bien posé de ce type de modèles a été étudié dans [15, 25] pour l'existence de solutions faibles et dans [8, 39, 26] pour l'existence de solutions fortes. La stabilisation autour d'une solution nulle, par des contrôles agissant uniquement sur la structure, a été traitée dans [51].

La simulation numérique de ces modèles a fait l'objet de nombreux travaux. Deux approches sont utilisées pour simuler ces systèmes. La première approche dite *monolithique* consiste à résoudre un système couplé en un seul bloc (voir par exemple [56, 30]). Dans l'autre approche dite *partitionnée*, on découple le système pour résoudre séparément les équations du fluide et celles de la structure puis les recoupler (voir par exemple [21]). L'approche *monolithique* a l'avantage d'être généralement stable contrairement à l'approche *partitionnée* qui peut donner lieu à des instabilités numériques si le couplage n'est pas bien pris en compte. Cependant, cette dernière est plus facile à mettre en œuvre car on bénéficie des outils déjà développés séparément pour le fluide et la structure.

Dans le chapitre 1, nous supposons que la partie supérieure de la frontière du canal est occupée par une structure élastique. Le but sera de stabiliser le système fluide-structure, autour d'une solution stationnaire instable de ce système, par soufflage-aspiration sur la partie inférieure de la frontière du canal.

Dans les chapitres 2, 3 et 4, un obstacle (une plaque épaisse) est située à l'entrée du canal. L'écoulement autour de cette plaque épaisse correspond à un écoulement dans une soufflerie autour d'un obstacle. Le domaine de calcul est ici tronqué (voir *Figure* 1). Dans ces trois chapitres, nous présentons des travaux réalisés dans le cadre du projet CARPE [1] du RTRA STAE (Sciences et Techniques de L'Aéronautique et de L'Espace) auquel participent l'IMFT, le LAAS, l'ISAE, l'ONERA et l'IMT. Le but de ce projet est d'étudier la stabilisation de l'écoulement autour d'une solution stationnaire instable par soufflage-aspiration sur les parois inférieure et supérieure, supposées fixes, de la plaque. Ici, nous considérons le cas où les parois horizontales de la plaque épaisse sont des structures élastiques. Nous étudierons la stabilisation du point de vue théorique (chapitres 2 et 3) et numérique (chapitres 2 et 4) du système couplé fluide-structure par déformation de paroi. Nous avons développé un code éléments finis à partir de la librairie getfem++ écrite en C++ pour simuler ces modèles et vérifier l'efficacité des contrôles construits.

2 Configuration du domaine fluide

Dans tous les chapitres, on désigne par Ω la configuration de référence du domaine fluide et par $\Omega_{\eta(t)}$ le domaine occupé par le fluide à l'instant $t \geq 0$, où $\eta(t)$ désigne le déplacement de la

[1] http ://www.fondation-stae.net/fr/actions/projets-soutenus/ ?pg=3

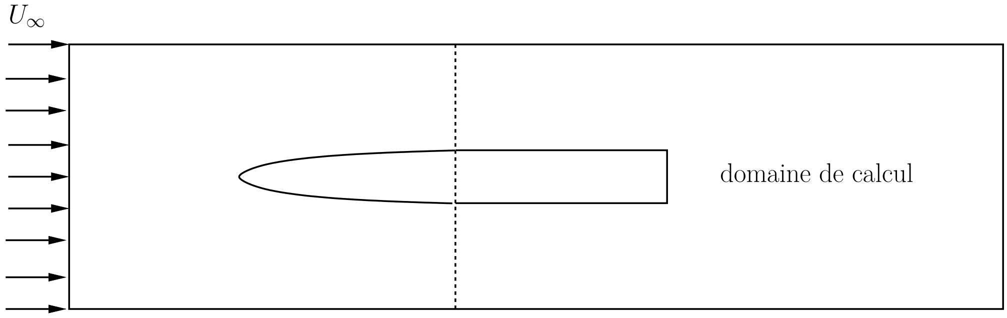
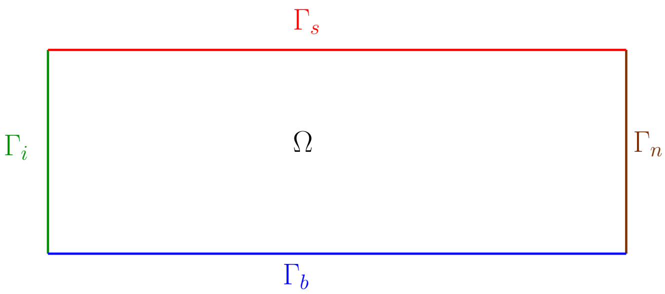
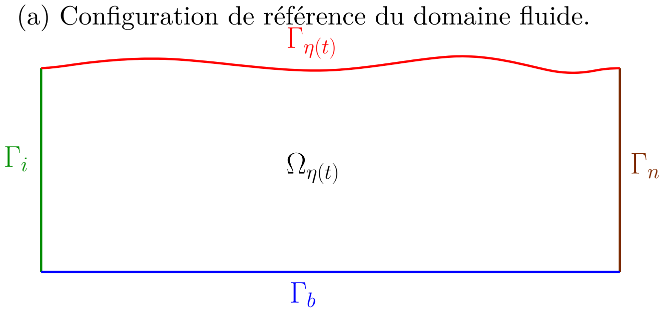


Figure 1 – Configuration dans le projet Carpe.



(a) Configuration de référence du domaine fluide.



(b) Configuration déformée du domaine fluide.

Figure 2 – Configurations du domaine fluide (chapitre 1)

structure. On notera par Γ_s (resp. $\Gamma_{\eta(t)}$) la frontière de Ω (resp. $\Omega_{\eta(t)}$) occupée par la structure.

2 .1 Configuration géométrique dans le chapitre 1

Dans ce chapitre, la configuration de référence du domaine fluide est

$$\Omega = (0, \ell) \times (0, 1),$$

où $\ell > 0$ est la longueur du domaine. La structure est localisée sur la frontière supérieure du domaine fluide. Nous avons donc

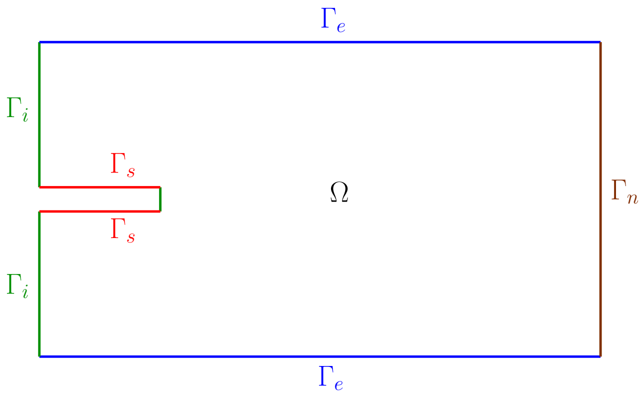
$$\Gamma_s = (0, \ell) \times \{1\}.$$

Les domaines Ω et $\Omega_{\eta(t)}$ sont représentés sur la *Figure* 2.

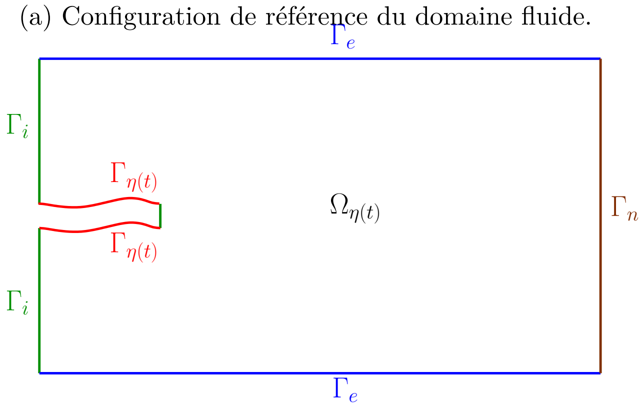
2 .2 Configuration géométrique dans les chapitres 2, 3 et 4

Dans ces trois chapitres, la configuration de référence du domaine fluide est

$$\Omega = (0, L) \times (-\ell, \ell) \setminus (0, \ell_s) \times (-e, +e),$$



(a) Configuration de référence du domaine fluide.



(b) Configuration déformée du domaine fluide.

FIGURE 3 – Configurations du domaine fluide (chapitres 2, 3 et 4)

où e, ℓ_s, L et ℓ sont strictement positifs. La structure est localisée sur les parois inférieure $\Gamma_s^- = (0, \ell_s) \times \{-e\}$ et supérieure $\Gamma_s^+ = (0, \ell_s) \times \{+e\}$ de la plaque épaisse. Nous avons donc

$$\Gamma_s = \Gamma_s^- \cup \Gamma_s^+.$$

Les domaines Ω et $\Omega_{\eta(t)}$ sont représentés sur la *Figure* 3.

3 Les équations du modèle fluide-structure

Dans les quatre chapitres de la thèse, la vitesse $u = (u_1, u_2)$ et la pression p du fluide vérifient les équations de Navier-Stokes incompressibles écrites dans la configuration déformée

$$\begin{aligned}
&u_t - \operatorname{div} \sigma(u,p) + (u \cdot \nabla)u = 0 \text{ dans } \bigcup_{t \in (0,\infty)} \{t\} \times \Omega_{\eta(t)}, \\
&\operatorname{div} u = 0 \text{ dans } \bigcup_{t \in (0,\infty)} \{t\} \times \Omega_{\eta(t)}, \\
&u(0) = u^0 \text{ dans } \Omega_{\eta(0)}.
\end{aligned}$$

Le tenseur des contraintes $\sigma(u,p)$ est défini par l'expression

$$\sigma(u,p) = 2\nu\varepsilon(u) - pI, \quad \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

où ν est le coefficient de viscosité du fluide.

Nous supposons que le déplacement η de la structure satisfait une équation d'Euler-Bernoulli avec amortissement écrite dans la configuration de référence (en variables Lagrangiennes)

$$\begin{aligned}
&\eta_{tt} - \beta\Delta\eta - \gamma\Delta\eta_t + \alpha\Delta^2\eta = \Phi[u,p] + f_b \text{ dans } (0,\infty)\times\Gamma_s,\\
&\eta(0) = \eta_1^0 \text{ dans } \Gamma_s, \quad \eta_t(0) = \eta_2^0 \text{ dans } \Gamma_s,\\
&\eta = 0 \ \text{ sur } \ (0,\infty)\times\partial\Gamma_s, \quad \eta_x = 0 \ \text{ sur } \ (0,\infty)\times\partial\Gamma_s.
\end{aligned}$$

Les coefficients $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$ sont respectivement la rigidité, l'étirement et l'amortissement de la structure. La quantité $\Phi[u,p]$ représente l'action du fluide sur la structure et elle est définie par

$$\Phi[u,p] = -\sigma(u,p)|_{\Gamma_{\eta(t)}}\, n_{\eta(t)} \cdot e_2,$$

où $n_{\eta(t)}$ est la normale unitaire à $\Gamma_{\eta(t)}$ extérieure à $\Omega_{\eta(t)}$ et $e_2 = (0,1)$. La quantité f_b représente l'ensemble des autres forces qui s'exercent sur la structure.
Nous allons compléter les équations du fluide par les conditions aux limites. A l'interface fluide-structure, nous avons l'égalité des vitesses du fluide et de déplacement de la structure

$$u = \eta_t e_2 \ \text{ sur } \bigcup_{t\in(0,\infty)} \{t\}\times\Gamma_{\eta(t)}.$$

Dans la configuration du chapitre 1, cette équation signifie

$$u(t,x,1+\eta(t,x)) = \eta_t(t,x) \ \text{ sur } \ (0,\infty)\times\Gamma_s.$$

Dans les 2, 3 et 4, il faut modifier cette condition en fonction de la configuration du domaine. On impose un certain profil en entrée Γ_i du domaine et on laisse la sortie Γ_n libre, i.e :

$$u = u_i \ \text{ sur } \ (0,\infty)\times\Gamma_i \quad \text{ et } \quad \sigma(u,p)n = 0 \ \text{ sur } \ (0,\infty)\times\Gamma_n,$$

où u_i est le profil imposé (profil de Poiseuille ou approximation d'une couche limite de Blasius par exemple). Dans le chapitre 1, on a une condition de Dirichlet homogène sur Γ_b

$$u = 0 \text{ sur } (0,\infty)\times\Gamma_b.$$

Dans les chapitres 2, 3 et 4, on impose des conditions de Navier sur Γ_e

$$u\cdot n = 0 \ \text{ sur } \ (0,\infty)\times\Gamma_e \quad \text{ et } \quad \varepsilon(u)n\cdot\tau = 0 \ \text{ sur } \ (0,\infty)\times\Gamma_e.$$

4 Quelques notations

Dans ce manuscrit, nous utiliserons les notations suivantes

$$\begin{aligned}
&Q_\eta^\infty = \textstyle\bigcup_{t\in(0,\infty)}\{t\}\times\Omega_{\eta(t)}, \quad \Gamma_\eta^\infty = \bigcup_{t\in(0,\infty)}\{t\}\times\Gamma_{\eta(t)},\\
&Q^\infty = (0,\infty)\times\Omega, \quad \Sigma_s^\infty = (0,\infty)\times\Gamma_s,\\
&\Sigma_i^\infty = (0,\infty)\times\Gamma_i, \quad \Sigma_e^\infty = (0,\infty)\times\Gamma_e,\\
&\Sigma_b^\infty = (0,\infty)\times\Gamma_b, \quad \Sigma_n^\infty = (0,\infty)\times\Gamma_n.
\end{aligned}$$

5 Objectifs de la thèse

Nous présentons tout d'abord les objectifs des chapitres 2, 3 et 4. Les objectifs du chapitre 1 seront présentés ensuite.

5 .1 Objectifs des chapitres 2, 3 et 4

Dans le chapitre 3, nous étudions la stabilisation du système fluide-structure suivant

$$
\begin{aligned}
& u_t - \operatorname{div}\sigma(u,p) + (u\cdot\nabla)u = 0 \text{ dans } Q_\eta^\infty,\\
& \operatorname{div} u = 0 \text{ dans } Q_\eta^\infty, \quad u = \eta_t e_2 \text{ on } \Sigma_\eta^\infty, \quad u = g_s + h \text{ sur } \Sigma_i^\infty,\\
& u\cdot n = 0 \text{ sur } \Sigma_e^\infty, \quad \varepsilon(u)n\cdot\tau = 0 \text{ sur } \Sigma_e^\infty, \quad \sigma(u,p)n = 0 \text{ sur } \Sigma_n^\infty,\\
& \eta_{tt} - \beta\Delta\eta - \gamma\Delta\eta_t + \alpha\Delta^2\eta = -\sigma(u,p)|_{\Gamma_{\eta(t)}} n_{\eta(t)}\cdot e_2 - p_s|_{\Gamma_s} e_2\cdot n + f \text{ sur } \Sigma_s^\infty,\\
& \eta = 0 \text{ sur } (0,\infty)\times\partial\Gamma_s, \quad \eta_x = 0 \text{ sur } (0,\infty)\times\partial\Gamma_s,\\
& u(0) = u^0 \text{ dans } \Omega_{\eta_1^0}, \quad \eta(0) = \eta_1^0 \text{ sur } \Gamma_s, \quad \eta_t(0) = \eta_2^0 \text{ sur } \Gamma_s,
\end{aligned}
\tag{5 .1}
$$

où (u_s,p_s) est une solution des équations de Navier-Stokes stationnaires

$$
\begin{aligned}
& -\operatorname{div}\sigma(u_s,p_s) + (u_s\cdot\nabla)u_s = 0, \operatorname{div} u_s = 0 \text{ dans } \Omega,\\
& \operatorname{div} u_s = 0 \text{ dans } \Omega, \quad u_s = 0 \text{ sur } \Gamma_s, \quad u_s = g_s \text{ sur } \Gamma_i,\\
& u_s\cdot n = 0 \text{ sur } \Gamma_e, \quad \varepsilon(u_s)n\cdot\tau = 0 \text{ sur } \Gamma_e,\\
& \sigma(u_s,p_s)n = 0 \text{ sur } \Gamma_n.
\end{aligned}
\tag{5 .2}
$$

La variable de contrôle f est de dimension finie n_c :

$$
f(t,x,y) = \sum_{i=1}^{n_c} f_i(t) w_i(x,y),
$$

(voir Section 3.7 pour le choix des fonctions w_i). Remarquons que si $f = 0$ et $h = 0$, alors $(u_s, p_s, 0)$ est solution stationnaire du système (5 .1). Dans le chapitre 2, nous étudions un problème similaire dans lequel l'équation d'Euler-Bernoulli pour la structure est remplacée par un système de dimension finie. Pour cela, nous recherchons le déplacement η de la structure sous la forme

$$
\eta(t,x) = \sum_{i=1}^{n_c} \eta^i(t) w_i(x),
$$

(voir Section 2.7 .3 sur la construction de la famille $(w_i)_{1\le i\le n_c}$). Dans ce cas, le système fluide-structure s'écrit

$$
\begin{aligned}
& u_t - \operatorname{div}\sigma(u,p) + (u\cdot\nabla)u = 0 \text{ dans } Q_\eta^\infty,\\
& \operatorname{div} u = 0 \text{ dans } Q_\eta^\infty, \quad u = \eta_t e_2 \text{ sur } \Sigma_\eta^\infty, \quad u = g_s + h \text{ sur } (0,\infty)\times\Gamma_i,\\
& u\cdot n = 0 \text{ sur } (0,\infty)\times\Gamma_e, \quad \varepsilon(u)n\cdot\tau = 0 \text{ sur } (0,\infty)\times\Gamma_e,\\
& \sigma(u,p)n = 0 \text{ sur } (0,\infty)\times\Gamma_n, \quad u(0) = u_0 \text{ sur } \Omega_{\eta_1^0},\\
& \boldsymbol{\eta}_{tt} + \alpha\boldsymbol{\eta} + \gamma\boldsymbol{\eta}_t = f,\\
& \boldsymbol{\eta}(0) = \boldsymbol{\eta}_1^0, \quad \boldsymbol{\eta}_t(0) = \boldsymbol{\eta}_2^0,
\end{aligned}
\tag{5 .3}
$$

où $\boldsymbol{\eta} = [\eta^1, \cdots, \eta^{n_c}]$ est le vecteur des déplacements. Remarquons que, pour simplifier le problème, nous n'avons pas rajouté dans l'équation de la structure le terme correspondant à l'action du fluide. Nous vérifions que $(u_s, p_s, 0)$ est aussi une solution stationnaire du système (5 .3) avec $h = 0$ et $f = 0$. On sait que si le nombre de Reynolds dépasse une certaine valeur critique, la solution stationnaire $(u_s, p_s, 0)$ des systèmes (5 .1) et (5 .3) est instable. On se placera dans ce cas en choisissant le nombre de Reynolds $Rey = \frac{eu_m}{\nu} = 200$ où u_m est la vitesse maximale à

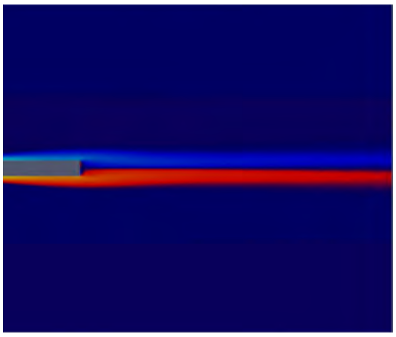


FIGURE 4 – Vorticité de l'écoulement stationnaire à Reynolds 200.

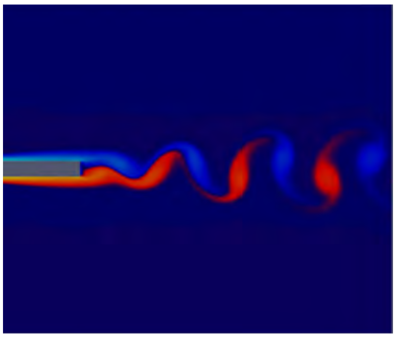


FIGURE 5 – Vorticité de l'écoulement perturbé.

l'entrée, e l'épaisseur de la plaque et ν est la viscosité du fluide. Dans la *Figure* 4, nous avons représenté la vorticité de l'écoulement stationnaire pour ce nombre de Reynolds.

En présence d'une perturbation de la condition d'entrée ($h \neq 0$), on observe des tourbillons de von Karman à l'arrière de la plaque (voir *Figure* 5). Notre objectif est de déterminer une loi de contrôle permettant de stabiliser localement les systèmes (5 .1) et (5 .3) autour de la solution stationnaire $(u_s, p_s, 0)$, et d'en déduire une loi de contrôle permettant de stabiliser les systèmes semi-discrétisés associés et satisfaisant les conditions suivantes :

1. L'algorithme de calcul de la loi de contrôle du système discrétisé doit être facile à mettre en œuvre numériquement. En particulier, nous devons éviter d'avoir à inverser des matrices pleines.

2. Le système projeté semi-discrétisé utilisé pour calculer le contrôle doit être de petite dimension, identique à la dimension du système projeté pour le modèle continu.

3. Même si l'analyse d'erreur de discrétisation n'est pas faite dans cette thèse, la stratégie mise en œuvre doit nous permettre de faire une étude numérique de convergence. Dans les calculs numériques menés au cours de cette thèse, une étude de la convergence de la partie droite du spectre des opérateurs linéarisés et des valeurs propres associées a été faite, et un compromis acceptable a été retenu entre précision et temps de calcul.

4. La stabilisabilité des systèmes de contrôle, quand elle ne peut pas être totalement garantie pour le problème de dimension infinie (dû au fait que le système n'est pas linéarisé autour de zéro et que uniquement le déplacement normal de la structure est pris en compte) doit pouvoir être étudiée pour le problème semi-discrétisé.

Une telle stratégie, prenant en compte les quatre conditions listées ci-dessus, a déjà été mise en œuvre pour la stabilisation des équations de Navier-Stokes dans [1]. Nous allons voir que sa mise en œuvre pour des problèmes fluide-structure, tels que ceux étudiés dans cette thèse, soulève des difficultés nouvelles. Cela nous a conduit à étudier deux stratégies différentes pour les modèles des chapitres 2 et 3. Nous avons souhaité étudier le modèle du chapitre 2, car étant plus simple que celui du chapitre 3, nous pensions que, d'un point de vue théorique et numérique, la stabilisation du système (5 .3) serait plus simple et plus efficace que celle du système (5 .1). L'étude de la stabilisation de ces deux systèmes consiste en différentes étapes listées ci-dessous.

Étape 1 : *Réécriture en domaine fixe et linéarisation*

On réécrit les systèmes (5 .1) et (5 .3) dans la configuration de référence, en effectuant un changement de variables. Puis, nous linéarisons les systèmes obtenus autour de la solution stationnaire $(u_s, p_s, 0)$.

Étape 2 : *Réécriture du système linéarisé sous la forme d'un système de contrôle* $(\mathcal{A} + \omega I, \mathcal{B})$

Nous montrons que le système linéarisé se décompose en un système de contrôle couplé à une (chapitre 2) ou deux (chapitre 3) équations algébriques. Nous prouvons que le générateur infinitésimal $\mathcal{A}$ du système linéarisé est le générateur d'un semi-groupe analytique à résolvante compacte. Cela nous permet de ramener l'étude de la stabilisabilité du système de contrôle à un problème de continuation unique pour les fonctions propres du système adjoint.

Étape 3 : *Construction de la loi de contrôle*

Pour construire la loi de contrôle, nous écrivons le système de contrôle projeté sur le sous espace instable. La loi de contrôle du système projeté (de dimension finie) s'obtient par résolution d'une équation de Riccati de même dimension que celle de l'espace instable.

Étape 4 : *Stabilisation du système non-linéaire*

La loi de contrôle construite dans l'étape 3 permet de stabiliser le système linéarisé de dimension finie, puis le système linéarisé non-homogène. Nous démontrons enfin que cette loi stabilise localement le système non-linéaire initial.

Ce programme a été mise en œuvre dans les chapitres 2 et 3, mais nous avons suivi une approche différente au niveau de l'étape 2. La réécriture des systèmes linéarisés associés à (5 .1) et (5 .3) sous la forme d'un système de contrôle repose sur l'élimination de la pression pour obtenir une équation d'évolution. L'élimination de la pression peut s'obtenir de deux manières différentes. Une manière consiste à utiliser le projecteur de Leray pour les équations de Navier-Stokes, et à exprimer la pression en fonction de la vitesse de l'écoulement et du déplacement de la structure dans l'équation de la structure. Cette approche a été utilisée dans [51] et c'est elle que nous mettons en œuvre dans le chapitre 3. Une deuxième manière d'éliminer la pression consiste à utiliser un projecteur pour le système linéarisé qui prend en compte les différents couplages. C'est l'approche suivie dans [5] et nous l'avons utilisée dans le chapitre 2. Pour le système fluide-structure étudié dans [5], le projecteur permettant d'éliminer la pression est un projecteur orthogonal. Dans notre cas, la linéarisation étant effectuée au voisinage d'une solution $(u_s, p_s, 0)$ non nulle, nous n'avons pas les mêmes couplages fluide-structure pour les systèmes direct et adjoint. Ainsi, nous sommes conduits à introduire un projecteur oblique Π. L'approche du chapitre 2 n'aurait pas pu être faite pour le modèle du chapitre 3, car l'analyticité du semi-groupe $(e^{t\mathcal{A}})_{t\geq 0}$ dans le chapitre 2 repose essentiellement sur le fait que l'équation de la structure est de dimension finie. L'approche consistant à utiliser le projecteur de Leray peut s'appliquer au cas d'une équation de structure de dimension finie ou infinie. Cependant, il est plus difficile de montrer que la loi de contrôle ne dépend pas du projecteur. Dans le cas où le projecteur conserve les couplages fluide-structure, les calculs sont beaucoup plus simples (à l'exception du calcul de l'adjoint de l'opérateur du système linéarisé, dont la caractérisation s'avère délicate).

Mise en œuvre des ces approches pour les systèmes semi-discrétisés

Les deux approches mentionnées ci-dessus ont été adaptées aux systèmes semi-discrétisés associés (chapitre 4 et Sections 2.8 à 2.11 du chapitre 2). Les différentes étapes à mettre en œuvre pour les systèmes discrétisés sont les suivantes.

Étape 5 : *Approximations semi-discrètes*

La discrétisation des systèmes linéarisés de dimension finie par la méthode des éléments finis conduit à des systèmes semi-discrétisés, de type algébrico-différentiels de dimension finie. Dans ces systèmes, la matrice de rigidité A_L est du type

$$\begin{bmatrix} A & \widehat{A}_{v\eta\theta} \\ A_{v\eta\theta}^T & 0 \end{bmatrix}.$$

La matrice A correspond à l'équation différentielle du système, la matrice $A_{v\eta\theta}$ correspond à l'équation algébrique du système, et la matrice $\widehat{A}_{v\eta\theta}$ correspond à la prise en compte des multiplicateurs de Lagrange dans le système différentiel. Le système n'est pas standard dans la mesure où $\widehat{A}_{v\eta\theta} \neq A_{v\eta\theta}$.

Étape 6 : *Réécriture des systèmes discrétisés*

Grâce à l'approximation du projecteur de Leray et du projecteur oblique Π, nous découplons les systèmes discrétisés en un système de contrôle $(\mathbf{A}, \mathbf{B})$ couplé à des équations algébriques.

Étape 7 : *Problèmes aux valeurs propres et système projeté*

La construction de la loi de contrôle permettant de stabiliser le système de contrôle $(\mathbf{A}, \mathbf{B})$ repose la décomposition spectrale de l'opérateur $\mathbf{A}$. Malheureusement, il est difficile de calculer numériquement $\mathbf{A}$ car cela nécessite d'inverser des matrices de grande taille. Pour surmonter cette difficulté, nous montrons que le problème aux valeurs propres associé à $\mathbf{A}$ est équivalent au celui associé à A_L, et ce dernier est facile à résoudre numériquement.

Étape 8 : *Loi de contrôle*

D'abord, on projette le système de contrôle sur le sous espace instable. Grâce aux liens entre les problèmes aux valeurs propres associés à $\mathbf{A}$ et A_L, nous prouvons que la détermination du système projeté sur le sous espace instable ne dépend pas du projecteur utilisé pour éliminer la pression. La loi de contrôle du système projeté est obtenue en résolvant une équation de Riccati de petite dimension. A partir de cette loi, nous construisons une loi de contrôle toujours indépendante du projecteur et permettant de stabiliser le système linéarisé discrétisé. Grâce à des tests numériques, nous montrons que la loi de contrôle ainsi construite permet de stabiliser le système non-linéaire discrétisé.

L'approche du chapitre 2 consistant à utiliser une projection oblique permet de lever la difficulté liée au fait que $\widehat{A}_{v\eta\theta} \neq A_{v\eta\theta}$. L'approche du chapitre 3, moins adaptée à ce type de problème et à la justification de l'étape 4, s'avère beaucoup plus délicate. En effet, nous devons montrer que la construction du système projeté ne fait intervenir ni le projecteur de Leray, ni le projecteur oblique (car non calculable numériquement) mais fait intervenir uniquement des produits matriciels faciles à mettre en œuvre.

5 .2 Objectifs du chapitre 1

Nous présentons maintenant le chapitre 1 dans lequel nous avons étudié (uniquement d'un point d'un point de vue théorique) la stabilisation d'un modèle d'interaction fluide-structure par un

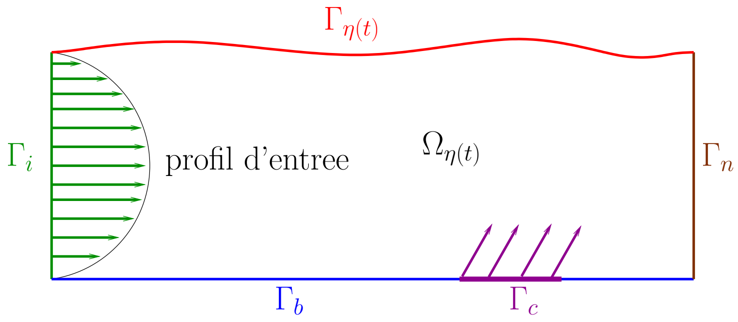


FIGURE 6 – Configuration déformée du domaine fluide.

contrôle agissant sur les conditions limites du fluide (voir *Figure* 6). La configuration géométrique retenue ici est simple (voir *Figure* 2). Mais la stratégie développée peut être appliquée pour des géométries plus complexes telles celles étudiées numériquement dans [63]. Les applications potentielles des résultats obtenus de ce chapitre sont donc également importantes. Le système considéré ici est le suivant

$$\begin{aligned}
&u_t - \operatorname{div}\sigma(u,p) + (u\cdot\nabla)u = 0 \text{ dans } Q_\eta^\infty,\\
&\operatorname{div} u = 0 \text{ dans } Q_\eta^\infty, \quad u = \eta_t e_2 \text{ sur } \Sigma_\eta^\infty,\\
&u = g \text{ sur } \Sigma_b^\infty, \quad u = u_i \text{ sur } \Sigma_i^\infty, \quad \sigma(u,p)n = 0 \text{ sur } \Sigma_n^\infty,\\
&\eta_{tt} - \beta\Delta\eta - \gamma\Delta\eta_t + \alpha\Delta^2\eta = -\sigma(u,p)|_{\Gamma_{\eta(t)}} n_{\eta(t)}\cdot e_2 - p_s \text{ sur } \Sigma_s^\infty,\\
&\eta = 0 \text{ sur } (0,\infty)\times\partial\Gamma_s, \quad \eta_x = 0 \text{ sur } (0,\infty)\times\partial\Gamma_s,\\
&u(0) = u^0 \text{ dans } \Omega_{\eta_1^0}, \quad \eta(0) = \eta_1^0 \text{ sur } \Gamma_s, \quad \eta_t(0) = \eta_2^0 \text{ sur } \Gamma_s.
\end{aligned} \tag{5 .4}$$

La variable de contrôle g est sous la forme

$$g(t,x,y) = \sum_{i=1}^{n_c} g_i(t) w_i(x,y),$$

où les fonctions w_i sont localisées sur une zone de contrôle $\Gamma_c \subset \Gamma_b$ (voir Section 1.6 .4 pour les détails sur leur construction). La stabilisation du système couplé au voisinage de zéro, par un contrôle agissant uniquement sur l'équation de la structure élastique, a été étudié dans [51]. Notre problème semble très proche de celui traité dans [51] dans le cas de conditions de Dirichlet. Le fait de considérer la stabilisation, par un contrôle frontière localisé ailleurs que sur la structure, autour d'une solution stationnaire non nulle, et avec des conditions mixtes Dirichlet/Neumann pour les équations de Navier-Stokes, introduit des difficultés supplémentaires. Tout d'abord, à cause des conditions mixtes, les solutions des équations du fluide sont moins régulières. En particulier, cela rend plus difficile la caractérisation de la pression en fonction de la vitesse du fluide et du déplacement de la structure. Une autre difficulté est que la réécriture du système couplé sous la forme d'un système de contrôle fait apparaître la dérivée du contrôle dans les équations de la structure. Nous la surmontons en considérant un système étendu avec le contrôle comme l'une des variables de ce système.

6 Résultats obtenus

Dans le chapitre 1, nous avons montré la stabilisation locale du système (5 .4) autour d'une solution stationnaire instable $(u_s, p_s, 0)$ par des contrôles de dimension finie. De plus le vecteur de contrôle $\mathbf{g} = (g_1, \cdots, g_{n_c})$ est obtenu en résolvant l'équation différentielle ordinaire

$$\mathbf{g}'(t) = \Lambda \mathbf{g}(t) + \mathcal{K}_\omega(e^{\omega t}(Pu \circ \mathcal{T}_{\eta(t)}^{-1} - u_s, \eta, \eta_t),$$

où $\omega > 0$ est le taux de décroissance exponentielle recherché, P est le projecteur de Leray, $\mathcal{T}_{\eta(t)}$ est changement de variable qui transforme le domaine $\Omega_{\eta(t)}$ en Ω (voir (1.3 .2)). L'opérateur $\mathcal{K}_\omega$ est une loi de contrôle obtenue en résolvant une équation de Riccati.

Dans la première partie du chapitre 2, nous avons trouvé des lois de contrôles permettant de stabiliser localement par feedback le système couplé (5 .3) autour de la solution stationnaire $(u_s, p_s, 0)$. Les lois de contrôle sont obtenues en résolvant des équations de Riccati de petite dimension. De plus, nous montrons que les contrôles calculés par feedback ne font pas intervenir le projecteur Π utilisé pour réécrire le système linéarisé en un système de contrôle. Dans la deuxième partie du chapitre, nous avons étudié la stabilisation d'une semi-discrétisation par éléments finis du système (5 .3). En utilisant une approche similaire à celle du modèle continu, nous déterminons des lois de contrôles, faciles à calculer numériquement, capables de stabiliser le système semi-discrète linéarisé. Pour finir, grâce à des tests numériques, nous montrons que ces lois de contrôles permettent de stabiliser le système discrétisé associé à (5 .3) autour de la solution stationnaire.

Dans le chapitre 3, nous obtenons des résultats de stabilisation similaires à ceux du chapitre 2, mais dans ce chapitre la structure vérifie une équation d'Euler-Bernoulli avec amortissement.

Dans le chapitre 4, nous avons montré la stabilisation par feedback de la linéarisation d'un système semi-discrétisé associé (5 .1) autour d'une solution stationnaire instable en utilisant une approche similaire à celle du chapitre 3. Les lois de contrôle sont obtenues en résolvant des équations de Riccati de petite dimension. Grâce à des tests numériques, nous avons montré que ces lois de contrôle permettent de stabiliser une discrétisation du système (5 .1) autour d'une solution stationnaire instable.

Rappelons que l'analyse conduite au chapitre 2 et celle conduite au chapitre 3 diffèrent au niveau de l'étape 2. Les méthodes de projection sont différentes. Les projecteurs introduits au chapitre 2 sont mieux adaptés aux systèmes couplés fluide-structure mais leur utilisation (pour l'analyse de la stabilisabilité) semble limitée au cas d'un structure décrite par une équation de dimension finie. Nous retrouvons cette différence pour les systèmes semi-discrétisés dans l'étape 6 (réécriture des systèmes linéarisés sous la forme de systèmes de contrôle). C'est la raison pour laquelle les calculs menés au chapitre 4 sont souvent plus complexes que ceux du chapitre 2.

Chapter 1

Feedback stabilization of a 2D fluid-structure model by a Dirichlet boundary control

1.1 Introduction

In this chapter we are interested in a system coupling a two dimensional fluid flow with the evolution of a beam localized at the boundary of the geometrical domain occupied by the fluid.

The fluid flow is described by the Navier-Stokes equations while the displacement of the structure satisfies a damped beam equation.

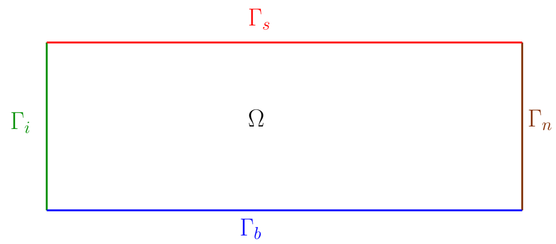
Stabilizing such systems in a neighborhood of an unstable stationary solution is a challenging problem. Depending on the applications we want to deal with, it is interesting to look for a stabilizing control acting either only in the structure equation or only in the fluid equation. The case of a control acting in the structure equation has been recently considered in [51]. Here we would like to study the local stabilization of such systems by a boundary control acting only in the fluid equation.

The procedure we follow in the chapter is a classical one, already used in [51], consisting in looking for a feedback control stabilizing a linearized system, that we next apply to the nonlinear system.

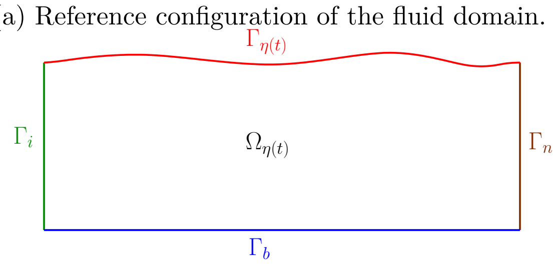
We have to emphasize, that in the considered model, the domain occupied by the fluid depends on the displacement of the beam. Therefore the main nonlinearity in this model comes from the fact that the equations of the fluid are written in Eulerian variables (therefore in the deformed configuration), while the equation of the structure is written in Lagrangian variables (i.e. in the reference configuration).

Thus before linearization, we have to rewrite this coupled system in the reference configuration. This is performed by using a change of variables. There are mainly two ways to do that. One way consists in writing the Navier-Stokes equations in Lagrangian variables, in that case the change of variables is associated with the fluid flow. The other way consists in using a change of variables associated with the displacement of the structure. The best choice usually corresponds to the change of variables associated with the most regular unknowns. In our case the displacement of the structure will be very regular. That is why it is more convenient to use a change of variables associated to the displacement of the beam. Let us now describe precisely the equations of this coupled system. The reference configuration of the fluid is

$$\Omega = (0, \ell) \times (0, 1),$$



(a) Reference configuration of the fluid domain.



(b) deformed configuration of the fluid domain.

Figure 1.1 – Configurations of the fluid domain.

where $\ell > 0$. The beam is located on $\Gamma_s = (0, \ell) \times \{1\}$. We also define the sets

$$\Gamma_b = (0, \ell) \times \{0\}, \quad \Gamma_i = \{0\} \times (0, 1), \quad \Gamma_n = \{1\} \times (0, \ell) \quad \text{and} \quad \Gamma_d = \Gamma_s \cup \Gamma_b \cup \Gamma_i.$$

Let η be the displacement of the beam from the reference configuration Γ_s. We denote by $\Omega_{\eta(t)}$ the domain occupied by the fluid at time t and by $\Gamma_{\eta(t)}$ the boundary of $\Omega_{\eta(t)}$ occupied by the beam (see *Figure 1*). We have

$$\Omega_{\eta(t)} = \{(x, y) \mid 0 < x < \ell,\, 0 < y < 1+\eta(t, x)\} \quad \text{and} \quad \Gamma_{\eta(t)} = \{(x, y) \mid 0 < x < \ell,\, y = 1+\eta(t, x)\}.$$

We use the notations

$$\begin{aligned}
&Q_\eta^\infty = \bigcup_{t\in(0,\infty)} \Big(\{t\} \times \Omega_{\eta(t)}\Big), \quad \Sigma_\eta^\infty = \bigcup_{t\in(0,\infty)} \Big(\{t\} \times \Gamma_{\eta(t)}\Big),\\
&Q^\infty = (0, \infty) \times \Omega, \quad \Sigma_s^\infty = (0, \infty) \times \Gamma_s, \quad \Sigma_b^\infty = (0, \infty) \times \Gamma_b,\\
&\Sigma_i^\infty = (0, \infty) \times \Gamma_i, \quad \Sigma_n^\infty = (0, \infty) \times \Gamma_n.
\end{aligned}$$

We also use the notations

$$e_1 = (1, 0) \quad \text{and} \quad e_2 = (0, 1).$$

The fluid flow is governed by the Navier-Stokes equations

$$\begin{aligned}
&u_t - \operatorname{div} \sigma(u, p) + (u \cdot \nabla)u = 0 \text{ in } Q_\eta^\infty,\\
&\operatorname{div} u = 0 \text{ in } Q_\eta^\infty, \quad u(t, x, 1 + \eta(t, x)) = \eta_t(t, x)e_2 \text{ on } \Sigma_s^\infty, \quad u = g \text{ on } \Sigma_b^\infty,\\
&u = u_i \text{ on } \Sigma_i^\infty, \quad \sigma(u, p)n = 0 \text{ on } \Sigma_n^\infty,\\
&u(0) = u^0 \text{ in } \Omega_{\eta_1^0},
\end{aligned} \tag{1.1 .1}$$

where ν is the fluid viscosity and

$$\sigma(u, p) = 2\,\nu\varepsilon(u) - pI, \quad \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

The inflow boundary condition u_i is of the form $u_i = g_s + h$, where $g_s \in H_0^2(\Gamma_i; \mathbb{R}^2)$ is stationary data (such as a Poiseuille profile) and h is a time-dependent perturbation of the inflow boundary condition g_s. The outflow condition on Γ_n is free, giving rise to a Neumann boundary condition. Let Γ_c be a subset of Γ_b. We assume that g is of finite dimension n_c

$$g(t,x,y)=\sum_{i=1}^{n_c} g_i(t)w_i(x,y).$$

The functions w_i, localized in the control zone $\Gamma_c\subset\Gamma_b$, are defined in (1.6 .18). The control variable is the vector

$$\mathbf{g}:=(g_1,\cdots,g_{n_c}).$$

The displacement η of the beam satisfies a damped beam equation

$$\begin{aligned}
&\eta_{tt}-\beta\Delta\eta-\gamma\Delta\eta_t+\alpha\Delta^2\eta=-\sigma(u,p)|_{\Gamma_{\eta(t)}}n_{\eta(t)}\cdot e_2+f_s \text{ on } \Sigma_s^\infty,\\
&\eta=0 \text{ on } (0,\infty)\times\partial\Gamma_s,\quad \eta_x=0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&\eta(0)=\eta_1^0 \text{ on } \Gamma_s,\quad \eta_t(0)=\eta_2^0 \text{ on } \Gamma_s,
\end{aligned}\tag{1.1 .2}$$

where $\alpha>0$, $\beta\geq 0$ and $\gamma>0$ are the rigidity, stretching and damping coefficients of the beam. The damping term $-\gamma\Delta\eta_t$ makes analytic the semigroup associated to the equation of the structure (see [17] and [16]). That is essential to obtain the main result of this chapter. The vector $n_{\eta(t)}$ is the unit normal to $\Gamma_{\eta(t)}$ exterior to $\Omega_{\eta(t)}$ and $\partial\Gamma_s$ stands for the boundary of Γ_s. The stationary force f_s is used to counteract the effect of a steady flow on the structure.
Let us notice that systems (1.1 .2) and (1.1 .1) are coupled through the equations $(1.1.2)_1$ and $(1.1.1)_2$. The equation $u=\eta_t e_2$ corresponds to the equality of the fluid velocity and the displacement velocity of the structure, while the first term in the right-hand side of $(1.1.2)_1$ represents the forces exerted by the fluid on the structure.
Let (u_s,p_s) be a solution of the stationary nonlinear system

$$\begin{aligned}
&-\operatorname{div}\sigma(u_s,p_s)+(u_s\cdot\nabla)u_s=0 \text{ in } \Omega,\\
&\operatorname{div}u_s=0 \text{ in } \Omega,\quad u_s=0 \text{ on } \Gamma_s,\quad u_s=0 \text{ on } \Gamma_b,\\
&u_s=g_s \text{ on } \Gamma_i,\quad \sigma(u_s,p_s)n=0 \text{ on } \Gamma_n,
\end{aligned}\tag{1.1 .3}$$

(see Assumption 1.4 .1). We choose $f_s=-p_s|_{\Gamma_s}$ which implies that $(u_s,p_s,0)$ is a stationary solution of system (1.1 .2)-(1.1 .1).
For any exponentially decay rate $-\omega<0$ and for given h, u^0, η_1^0 and η_2^0, such that h, u^0-u_s, η_1^0 and η_2^0 are small enough in appropriate functional spaces, we would like to find $\mathbf{g}$ so that the solution to the control system converges to the stationary solution $(u_s,p_s,0)$, in a sense we will define later (see Theorem 1.3 .1).
First we rewrite system (1.1 .2)-(1.1 .1) in the reference configuration, and we linearize it around the stationary solution $(u_s,p_s,0,0)$. The most important step is to find the feedback law able to stabilize the linearized system

$$\begin{aligned}
&v_t-\operatorname{div}\sigma(v,p)+(u_s\cdot\nabla)v+(v\cdot\nabla)u_s-A_1\eta_1-A_2\eta_2-\omega v=0 \text{ in } Q^\infty,\\
&\operatorname{div}v=A_3\eta_1 \text{ in } Q^\infty,\quad v=\eta_2 e_2 \text{ on } \Sigma_s^\infty,\quad v=\textstyle\sum_{i=1}^{n_c} g_i w_i \text{ on } \Sigma_b^\infty,\\
&v=0 \text{ on } \Sigma_i^\infty,\quad \sigma(v,p)n=0 \text{ on } \Sigma_n^\infty,\\
&\eta_{1,t}-\eta_2-\omega\eta_1=0 \text{ on } \Sigma_s^\infty,\\
&\eta_{2,t}-\beta\Delta\eta_1-\gamma\Delta\eta_2+\alpha\Delta^2\eta_1-A_4\eta_1-\omega\eta_2=\gamma_s p \text{ on } \Sigma_s^\infty,\\
&\eta_1=0 \text{ on } (0,\infty)\times\partial\Gamma_s,\quad \eta_{1,x}=0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&v(0)=v^0 \text{ in } \Omega,\quad \eta_1(0)=\eta_1^0 \text{ on } \Gamma_s,\quad \eta_2(0)=\eta_2^0 \text{ on } \Gamma_s,
\end{aligned}\tag{1.1 .4}$$

where γ_s is the trace operator on Γ_s; A_1, A_2, A_3 and A_4 are linear differential operators (see equation (1.3 .5)). In order to study the stabilizability of the system, we are going to rewrite it as a classical control problem. For that, we have to eliminate the pressure in the fluid and structure equations. The pressure is eliminated from the Oseen equations with the Leray projector P. Indeed, thanks to P, we can split the Oseen equations into two equations, an evolution equation satisfied by Pv and an algebraic equation verified by $(I-P)v$. More precisely, we can prove that

$$\begin{aligned}
&Pv_t = (A+\omega I)Pv + (\lambda_0 I - A)PL(\eta_2 e_2 \chi_{\Gamma_s} + \sum_{i=1}^{n_c} g_i w_i \chi_{\Gamma_b}, A_3\eta_1) + PA_1\eta_1 + PA_2\eta_2, \quad Pv(0) = v_0,\\
&(I-P)v = \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1,
\end{aligned}$$

where the Oseen operator A and the lifting operator L are defined in (1.4 .7) and (1.4 .8) respectively; N_s and N_c are defined in (1.5 .3); N_{div} is defined in Section 1.4 .4. Next to eliminate the pressure in the structure equations, we have to express it in terms of v, η_1 and η_2. Formally, we can derive an elliptic equation for the pressure p from the Oseen equation of system (1.1 .4). Unfortunately, due to the Dirichlet/Neumann mixed boundary conditions, we have not enough regularity on the velocity v for the well-posedness of the equation. However, by using a transposition method, we are able to express the pressure in terms of v, η_1 and η_2

$$\begin{aligned}
p = &-N_s(\eta_{2,t} - \omega\eta_2) - N_{\mathrm{div}} A_3(\eta_{1,t} - \omega\eta_1) + N_p A_1 \eta_1 + N_p A_2 \eta_2\\
&+ N_v(Pv + \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1) - N_c(\mathbf{g}_t - \omega \mathbf{g}),
\end{aligned}$$

where N_p and N_v are defined in Section 1.4 .4. Finally, we can rewrite system (1.1 .4) in the form

$$\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = (\mathcal{A}+\omega I)\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}_1 \mathbf{g} + \mathcal{B}_2(\mathbf{g}_t - \omega \mathbf{g}), \quad \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix}(0) = \begin{pmatrix} Pv^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix},\\
&(I-P)v = \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1,\\
&p = -N_s(\eta_{2,t} - \omega\eta_2) - N_{\mathrm{div}} A_3(\eta_{1,t} - \omega\eta_1) + N_p A_1 \eta_1 + N_p A_2 \eta_2\\
&\quad + N_v(Pv + \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1) - N_c(\mathbf{g}_t - \omega \mathbf{g}).
\end{aligned}$$

The operators $\mathcal{A}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined in Section 1.5 . The time derivative $\mathbf{g}_t$ of the control function $\mathbf{g}$ appears in the above equation. That is why we write an extended system by considering $\mathbf{g}$ as a new state variable and by introducing a new control variable $\mathbf{f} := \mathbf{g}_t - \Lambda\mathbf{g} - \omega\mathbf{g}$, where Λ is a diagonal matrix such that the spectra of $\mathcal{A}$ and Λ are disjoint. Thus, the extended system corresponding to (1.1 .4) is

$$\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = (\mathcal{A}_e+\omega I)\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} + \mathcal{B}_e \mathbf{f}, \quad \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix}(0) = \begin{pmatrix} Pv^0 \\ \eta_1^0 \\ \eta_2^0 \\ \mathbf{0} \end{pmatrix},\\
&(I-P)v = \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1,\\
&p = -N_s(\eta_{2,t} - \omega\eta_2) - N_{\mathrm{div}} A_3(\eta_{1,t} - \omega\eta_1) + N_p A_1 \eta_1 + N_p A_2 \eta_2\\
&\quad + N_v(Pv + \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1) - N_c(\mathbf{g}_t - \omega \mathbf{g}),
\end{aligned}$$

where $\mathcal{A}_e$ and $\mathcal{B}_e$ are defined in (1.6 .1) and (1.6 .3) respectively. Thus, we are reduced to find a feedback law $\mathcal{K}_\omega$ able to stabilize the pair $(\mathcal{A}_e + \omega I, \mathcal{B}_e)$. For that, we first prove that for an appropriate choice of the family of functions $(w_i)_{1\leq i\leq n_c}$ the pair $(\mathcal{A}_e+\omega I,\mathcal{B}_e)$ is stabilizable. We deduce that the algebraic Riccati equation

$$\begin{aligned}
&\mathcal{P}_\omega\in\mathcal{L}(Z_e),\quad \mathcal{P}_\omega=\mathcal{P}_\omega^*>0,\\
&\mathcal{P}_\omega(\mathcal{A}_e+\omega I)+(\mathcal{A}_e+\omega I)^*\mathcal{P}_\omega-\mathcal{P}_\omega\mathcal{B}_e\mathcal{B}_e^*\mathcal{P}_\omega=0,
\end{aligned}$$

where Z_e is the state space, admits a unique solution $\mathcal{P}_\omega$ and that the operator $\mathcal{K}_\omega=-\mathcal{B}_e^*\mathcal{P}_\omega$ provides a stabilizing feedback for $(\mathcal{A}_e+\omega I,\mathcal{B}_e)$. Since the unstable subspace for $\mathcal{A}_e+\omega I$ is of finite dimension, we can show that the solution to the Riccati equation can be deduced from the solution to a finite dimensional Riccati equation. In theorem 1.7 .1, we prove that the feedback law $\mathcal{K}_\omega$ is able to stabilize the nonhomogeneous linearized system associated to (1.1 .4). Next, thanks to a fixed point method, we prove that it is also able to locally stabilize the nonlinear system in the fixed domain (see Theorem 1.8 .2). Finally, we prove the feedback stabilisation of nonlinear system (1.1 .2)-(1.1 .1) (see Theorem 1.3 .1).

1.2 Functional setting

We define the classical Sobolev spaces in two dimensions

$$\mathbf{L}^2(\Omega)=L^2(\Omega;\mathbb{R}^2)\quad\text{and}\quad\mathbf{H}^s(\Omega)=H^s(\Omega;\mathbb{R}^2),\quad\forall\, s>0.$$

We also define the functional spaces

$$\begin{aligned}
&\mathbf{V}^0_{n,\Gamma_d}(\Omega)=\{v\in\mathbf{L}^2(\Omega)\mid \operatorname{div}v=0,\ v\cdot n=0 \text{ on } \Gamma_d\},\\
&\mathbf{V}(\Omega)=\{v\in\mathbf{L}^2(\Omega)\mid \operatorname{div}v\in\mathbf{L}^2(\Omega)\},\\
&\mathbf{H}^s_{\Gamma_d}(\Omega)=\{v\in\mathbf{H}^s(\Omega)\mid v=0 \text{ on } \Gamma_d\},\\
&\mathbf{V}^s_{n,\Gamma_d}(\Omega)=\mathbf{H}^s(\Omega)\cap\mathbf{V}^0_{n,\Gamma_d}(\Omega),\\
&\mathbf{V}^s_{\Gamma_d}(\Omega)=\mathbf{H}^s_{\Gamma_d}(\Omega)\cap\mathbf{V}(\Omega),\\
&H=H^1(\Omega)\times(H^3(\Gamma_s)\cap H^2_0(\Gamma_s))\times H^1_0(\Gamma_s),\\
&Z=\mathbf{V}^0_{n,\Gamma_d}(\Omega)\times H^2_0(\Gamma_s)\times L^2(\Gamma_s),\\
&Z_e=\mathbf{V}^0_{n,\Gamma_d}(\Omega)\times H^2_0(\Gamma_s)\times L^2(\Gamma_s)\times\mathbb{R}^{n_c},
\end{aligned}$$

for all $s>\frac{1}{2}$. For $w^0\in\mathbf{L}^2(\Omega)$ given, we define the space

$$\begin{aligned}
H_{ic}(w^0)=\{&(v^0,\eta_1^0,\eta_2^0)\in H\mid (Pv^0-Pw^0,\eta_1^0,\eta_2^0,\mathbf{0})\in[D(\mathcal{A}_e),Z_e]_{\frac{1}{2}},\ \operatorname{div}v^0=\operatorname{div}w^0+A_3\eta_1^0 \text{ in } \Omega,\\
&v^0=\eta_2^0e_2 \text{ on } \Gamma_s,\ v^0=0 \text{ on } \Gamma_i,\ v^0=0 \text{ on } \Gamma_b\},
\end{aligned}\tag{1.2 .1}$$

equipped with the norm

$$\|(v^0,\eta_1^0,\eta_2^0)\|_{ic}=\|(v^0,\eta_1^0,\eta_2^0)\|_H+\|(Pv^0-Pw^0,\eta_1^0,\eta_2^0,\mathbf{0})\|_{[D(\mathcal{A}_e),Z_e]_{\frac{1}{2}}}.$$

The operator P is the Leray projector (see Section 1.4 .2) and the operators $\mathcal{A}_e$ and A_3 are defined in (1.6 .1) and (1.3 .5) respectively. Now, we are going to introduce weighted Sobolev spaces as in [43]. We denote by $\mathcal{C}=(x_j,z_j)_{1\leq j\leq 4}$ the set of the corners of the domain Ω. For all $\delta,s\geq 0$ and for $n=1$ or $n=2$, we introduce the norms

$$\|v\|^2_{H^s_\delta(\Omega;\mathbb{R}^n)}=\sum_{|\alpha|\leq 2}\int_\Omega\prod_{j=1}^4\left(|x-x_j|^2+|z-z_j|^2\right)^\delta|\partial_\alpha v|^2,$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ denotes a two-index, $|\alpha| = \alpha_1 + \alpha_2$ is its length, ∂_α denotes the corresponding partial differential operator. We denote by $H^s_\delta(\Omega; \mathbb{R}^n)$ the closure of $C^\infty(\overline{\Omega}; \mathbb{R}^n)$ in the norm $\|\cdot\|_{H^s_\delta(\Omega;\mathbb{R}^n)}$. We define the weighted Sobolev spaces in two dimensions

$$\mathbf{L}^2_\delta(\Omega) = H^0_\delta(\Omega; \mathbb{R}^2) \quad \text{and} \quad \mathbf{H}^s_\delta(\Omega) = H^s_\delta(\Omega; \mathbb{R}^2), \quad \forall\, s > 0.$$

We introduce the spaces

$$\begin{aligned}
&H^{2,1}(Q^\infty) = L^2(0,\infty; \mathbf{H}^2(\Omega)) \cap H^1(0,\infty; \mathbf{L}^2(\Omega)),\\
&H^{2,1}_\delta(Q^\infty) = L^2(0,\infty; \mathbf{H}^2_\delta(\Omega)) \cap H^1(0,\infty; \mathbf{L}^2(\Omega)),\\
&H^{4,2}(\Sigma^\infty_s) = L^2(0,\infty; H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \cap H^2(0,\infty; L^2(\Gamma_s)),\\
&H^{2,1}(\Sigma^\infty_s) = L^2(0,\infty; H^2_0(\Gamma_s)) \cap H^1(0,\infty; L^2(\Gamma_s)),\\
&H^{2,1}(\Sigma^\infty_d) = L^2(0,\infty; \mathbf{H}^2_0(\Gamma_d)) \cap H^1(0,\infty; \mathbf{L}^2(\Gamma_d)),\\
&Y = L^2(0,\infty; \mathbf{L}^2(\Omega)) \times H^{2,1}(Q^\infty) \times L^2(0,\infty; L^2(\Gamma_s)),\\
&X_\delta = H^{2,1}_\delta(Q^\infty) \times L^2(0,\infty; H^1_\delta(\Omega)) \times H^{4,2}(\Sigma^\infty_s) \times H^{2,1}(\Sigma^\infty_s),\\
&X_{\delta,e} = H^{2,1}_\delta(Q^\infty) \times L^2(0,\infty; H^1_\delta(\Omega)) \times H^{4,2}(\Sigma^\infty_s) \times H^{2,1}(\Sigma^\infty_s) \times H^1(0,\infty; \mathbb{R}^{n_c}).
\end{aligned}$$

1.3 System in the reference configuration and main results

First, we are going to define the spaces of solutions for the coupled system (1.1 .2)-(1.1 .1). For that, we introduce the mappings

$$\mathcal{T}_\eta : (t,x,y) \quad \longmapsto \quad (t,x,z) = \left(t, x, \frac{y}{1+\eta(t,x)}\right), \tag{1.3 .1}$$

and

$$\mathcal{T}_{\eta(t)} : (x,y) \quad \longmapsto \quad (x,z) = \left(x, \frac{y}{1+\eta(t,x)}\right). \tag{1.3 .2}$$

We define the mapping $\mathcal{T}_{\eta^0_1}$. For all t in $(0,\infty)$, $\mathcal{T}_{\eta(t)}$ transforms $\Omega_{\eta(t)}$ into Ω. Let η belong to $H^{4,2}(\Sigma^\infty_s)$. We say that u belongs to the space $H^{2,1}_\delta(Q^\infty_\eta)$ if

- for all $t > 0$, the mapping $\mathcal{T}_{\eta(t)}$ is a $\mathcal{C}^1$-diffeomorphism from $\Omega_{\eta(t)}$ into Ω,
- $u \circ \mathcal{T}^{-1}_\eta$ belongs to $H^{2,1}_\delta(Q^\infty)$.

Let $\delta_0 > 0$ define in Theorem 1.4 .1. The main result of the chapter is the following:

Theorem 1.3 .1. *Let ω be positive. We assume that 1.4 .1 and 1.6 .1 are satisfied. Then, there exists $r > 0$ such that, for all $(u^0, \eta^0_1, \eta^0_2, h) \in \mathbf{H}^1(\Omega_{\eta^0_1}) \times H^3(\Gamma_s) \cap H^2_0(\Gamma_s) \times H^1_0(\Gamma_s) \times H^1_0(0,\infty; H^2_0(\Gamma_i))$ satisfying*

$$(u^0 \circ \mathcal{T}^{-1}_{\eta^0_1} - u_s, \eta^0_1, \eta^0_2) \in H_{ic}(\mathcal{F}_{\text{div}}[u^0 \circ \mathcal{T}^{-1}_{\eta^0_1} - u_s, \eta^0_1]),$$

and

$$\|(u^0 \circ \mathcal{T}^{-1}_{\eta^0_1} - u_s, \eta^0_1, \eta^0_2)\|_{ic} + \|e^{\omega t} h\|_{H^1_0(0,\infty;H^2_0(\Gamma_i))} \leq r,$$

we can find $\mathbf{g} \in H^1(0,\infty; \mathbb{R}^{n_c})$ for which there exists a unique solution (u,p,η) in

$$H^{2,1}_{\delta_0}(Q^\infty_\eta) \times L^2(0,\infty; H^1(\Omega_{\eta(t)})) \times H^{4,2}(\Sigma^\infty_s),$$

to the nonlinear system (1.1 .2)-(1.1 .1). Moreover, there exist $\varepsilon > 0$ and $C > 0$ such that

$$\left\|(u \circ \mathcal{T}_{\eta(t)}^{-1} - u_s, \eta(t), \eta_t(t))\right\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon}(\Omega)\times H^3(\Gamma_s)\times H^1(\Gamma_s)} \leq Ce^{-\omega t},$$

for all $t > 0$.

Remark 1.3 .1. *Actually, we are going to look for $\mathbf{g}$ as the solution to an additional equation*

$$\mathbf{g}_t = \Lambda \mathbf{g} + \mathbf{f}, \quad \mathbf{g}(0) = \mathbf{0},$$

where $\mathbf{f}$ is a control vector which can be found in feedback form (see Section 1.6).

In order to rewrite system (1.1 .2)-(1.1 .1) in the reference configuration, we make the changes of unknowns

$$\begin{aligned}
&\hat{u}(t,x,z) := e^{\omega t}[u(t, \mathcal{T}_{\eta(t)}^{-1}(x,z)) - u_s(x,z)], \quad \hat{p}(t,x,z) := e^{\omega t}[p(t, \mathcal{T}_{\eta(t)}^{-1}(x,z)) - p_s(x,z)],\\
&\hat{\eta}_1(t,x) := e^{\omega t}\eta(t,x), \quad \hat{\eta}_2(t,x) := e^{\omega t}\eta_t(t,x).
\end{aligned}$$

The system satisfied by $(\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2)$ is

$$\begin{aligned}
&\hat{u}_t - \operatorname{div}\sigma(\hat{u},\hat{p}) + (u_s\cdot\nabla)\hat{u} + (\hat{u}\cdot\nabla)u_s - A_1\hat{\eta}_1 - A_2\hat{\eta}_2 - \omega\hat{u} = e^{-\omega t}\mathcal{F}_f[\hat{u},\hat{p},\hat{\eta}_1,\hat{\eta}_2] \text{ in } Q^\infty,\\
&\operatorname{div}\hat{u} = e^{-\omega t}\operatorname{div}\mathcal{F}_{\text{div}}[\hat{u},\hat{\eta}_1] + A_3\hat{\eta}_1 \text{ in } Q^\infty, \quad \hat{u} = \hat{\eta}_2 e_2 \text{ on } \Sigma_s^\infty, \quad \hat{u} = \textstyle\sum_{i=1}^{n_c}\hat{g}_i w_i \text{ on } \Sigma_b^\infty,\\
&\hat{u} = \hat{h} \text{ on } \Sigma_i^\infty, \quad \sigma(\hat{u},\hat{p})n = 0 \text{ on } \Sigma_n^\infty,\\
&\hat{\eta}_{1,t} - \hat{\eta}_2 - \omega\hat{\eta}_1 = 0 \text{ on } \Sigma_s^\infty,\\
&\hat{\eta}_{2,t} - \beta\Delta\hat{\eta}_1 - \gamma\Delta\hat{\eta}_2 + \alpha\Delta^2\hat{\eta}_1 - A_4\hat{\eta}_1 - \omega\hat{\eta}_2 = \gamma_s\hat{p} + e^{-\omega t}\mathcal{F}_s[\hat{u},\hat{\eta}_1] \text{ on } \Sigma_s^\infty,\\
&\hat{\eta}_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \hat{\eta}_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&\hat{u}(0) = \hat{u}^0 \text{ in } \Omega, \quad \hat{\eta}_1(0) = \hat{\eta}_1^0 \text{ on } \Gamma_s, \quad \hat{\eta}_2(0) = \hat{\eta}_2^0 \text{ on } \Gamma_s,
\end{aligned} \tag{1.3 .3}$$

where γ_s is the trace operator on Γ_s and

$$\hat{u}^0 = u^0 \circ \mathcal{T}_{\eta_1^0}^{-1} - u_s, \quad \hat{\boldsymbol{g}}(t) = (\hat{g}_i(t))_{1\leq i\leq n_c} := e^{\omega t}\mathbf{g}(t), \quad \hat{h}(t,x,z) := e^{\omega t}h(t,x,z).$$

The nonlinear terms $\mathcal{F}_f$ and $\mathcal{F}_s$ are given in Appendix 1.B, and the nonlinear term $\mathcal{F}_{\text{div}}[\hat{u},\hat{\eta}_1]$ is defined by

$$\mathcal{F}_{\text{div}}[\hat{u},\hat{\eta}_1] = -\hat{\eta}_1\hat{u}_1 e_1 + z\hat{\eta}_{1,x}\hat{u}_1 e_2. \tag{1.3 .4}$$

The linear differential operators A_1, A_2, A_3 and A_4 are defined by

$$\begin{aligned}
A_1\hat{\eta}_1 \;&:=\; \hat{\eta}_1\left(p_{s,z}e_2 + u_{s,2}u_{s,z} - 2\nu u_{s,zz} + \nu u_{s,1,xx}e_1 + \nu u_{s,1,xz}e_2\right) - \nu\hat{\eta}_{1,x}\left(u_{s,1,z}e_2 - u_{s,1,x}e_1\right)\\
&\quad + z\hat{\eta}_{1,x}\left(p_{s,z}e_1 - 2\nu u_{s,xz} + u_{s,1}u_{s,z} - \nu u_{s,1,zz}e_2 - \nu u_{s,1,xz}e_1\right) - z\nu\hat{\eta}_{1,xx}\left(u_{s,z} + u_{s,1,z}e_1\right)\\
A_2\hat{\eta}_2 \;&:=\; z\hat{\eta}_2 u_{s,z},\\
A_3\hat{\eta}_1 \;&:=\; -\hat{\eta}_1 u_{s,1,x} + z\hat{\eta}_{1,x}u_{s,1,z},\\
A_4\hat{\eta}_1 \;&:=\; \nu\left(2\hat{\eta}_1 u_{s,2,z} - \hat{\eta}_{1,x}u_{s,1,z}\right).
\end{aligned} \tag{1.3 .5}$$

1.4 The Oseen system

The goal of this section is to study the nonhomogeneous Oseen system

$$\begin{aligned} &\lambda_0 w - \operatorname{div}\sigma(w,\pi) + (u_s\cdot\nabla)w + (w\cdot\nabla)u_s = F \text{ in } \Omega,\\ &\operatorname{div} w = h \text{ in } \Omega, \quad w = g \text{ on } \Gamma_d, \quad \sigma(w,\pi)n = 0 \text{ on } \Gamma_n, \end{aligned} \tag{1.4 .1}$$

where $F \in \mathbf{L}^2(\Omega)$, $h \in H^1(\Omega)$ and $g \in \mathbf{H}_0^{\frac{3}{2}}(\Gamma_d)$. We choose $\lambda_0 > 0$ in such a way that the coercivity inequality

$$\lambda_0 \int_\Omega v\cdot v + \int_\Omega \varepsilon(v):\varepsilon(v) + \int_\Omega [(u_s\cdot\nabla)v + (v\cdot\nabla)u_s]\cdot v \geq \frac{\nu}{2}\|v\|^2_{\mathbf{V}^1_{\Gamma_d}(\Omega)}, \quad \forall\, v \in V^1_{\Gamma_d}(\Omega), \tag{1.4 .2}$$

is satisfied.

1.4 .1 Regularity of the solutions to the Oseen system

First, we consider the Stokes system

$$\begin{aligned} &-\operatorname{div}\sigma(v,p) = F, \quad \operatorname{div} v = 0 \text{ in } \Omega,\\ &v = 0 \text{ on } \Gamma_d, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n, \end{aligned} \tag{1.4 .3}$$

Theorem 1.4 .1. *Let F belong to $\mathbf{L}^2(\Omega)$. Then, there exists $0 < \delta_0 < \frac{1}{2}$ such that the solution $(v,p) \in \mathbf{V}^1_{\Gamma_d}(\Omega) \times L^2(\Omega)$ to equation (1.4 .3) belongs $(v,p) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ with the estimate*

$$\|v\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|p\|_{H^1_{\delta_0}(\Omega)} \leq C\|F\|_{\mathbf{L}^2(\Omega)}.$$

Proof. See ([44], Theorem 2.5). □

We can assume the following result for the stationary Navier-Stokes equations (see [44]).

Assumption 1.4 .1. *System (1.1 .3) admits a solution $(u_s,p_s) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$.*

Lemma 1.4 .1. *There exists $\varepsilon_0 \in (0,1/2)$ such that $\mathbf{H}^2_{\delta_0}(\Omega) \subset \mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega)$ and $H^1_{\delta_0}(\Omega) \subset H^{\frac{1}{2}+\varepsilon_0}(\Omega)$.*

Proof. It follows from ([43], Lemma 6.2.1). □

Theorem 1.4 .2. *For all $(F,h,g) \in \mathbf{L}^2(\Omega)\times H^1(\Omega)\times \mathbf{H}_0^{\frac{3}{2}}(\Gamma_d)$, equation (1.4 .1) admits a unique solution $(w,\pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ with the estimate*

$$\|w\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|\pi\|_{H^1_{\delta_0}(\Omega)} \leq C\left(\|F\|_{\mathbf{L}^2(\Omega)} + \|h\|_{H^1(\Omega)} + \|g\|_{\mathbf{H}_0^{\frac{3}{2}}(\Gamma_d)}\right).$$

Proof. We look for the solution (w,π) in the form $(w,\pi) = (w_0,\pi_0) + (w_1,\pi_1)$ where (w_0,π_0) is the solution of the equation

$$\begin{aligned} &-\operatorname{div}\sigma(w_0,\pi_0) = F, \quad \operatorname{div} w_0 = h \text{ in } \Omega,\\ &w_0 = g \text{ on } \Gamma_d, \quad \sigma(w_0,\pi_0)n = 0 \text{ on } \Gamma_n, \end{aligned} \tag{1.4 .4}$$

and (w_1,π_1) satisfies

$$\begin{aligned}&\lambda_0 w_1-\operatorname{div}\sigma(w_1,\pi_1)+(u_s\cdot\nabla)w_1+(w_1\cdot\nabla)u_s=-\lambda_0 w_0-(u_s\cdot\nabla)w_0-(w_0\cdot\nabla)u_s \text{ in } \Omega,\\&\operatorname{div} w_1=0 \text{ in } \Omega,\quad w=0 \text{ on } \Gamma_d,\quad \sigma(w_1,\pi_1)n=0 \text{ on } \Gamma_n.\end{aligned} \tag{1.4 .5}$$

From ([43], Theorem 9.4.5), equation (1.4 .4) admits a unique solution (w_0,π_0) in $\mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ and

$$\|w_0\|_{\mathbf{H}^2_{\delta_0}(\Omega)}+\|\pi_0\|_{H^1_{\delta_0}(\Omega)}\le C\left(\|F\|_{\mathbf{L}^2(\Omega)}+\|h\|_{H^1(\Omega)}+\|g\|_{\mathbf{H}_0^{\frac{3}{2}}(\Gamma_d)}\right).$$

Since $u_s\in\mathbf{H}^2_{\delta_0}(\Omega)$, $\lambda_0 w_0+(u_s\cdot\nabla)w_0+(w_0\cdot\nabla)u_s$ belongs to $\mathbf{L}^2(\Omega)$ and

$$\|\lambda_0 w_0+(u_s\cdot\nabla)w_0+(w_0\cdot\nabla)u_s\|_{\mathbf{L}^2(\Omega)}\le C\left(\|F\|_{\mathbf{L}^2(\Omega)}+\|h\|_{H^1(\Omega)}+\|g\|_{\mathbf{H}_0^{\frac{3}{2}}(\Gamma_d)}\right).$$

Thus, for λ_0 large enough, from Lax-Milgram Theorem it follows that there exists a unique solution $(w_1,\pi_1)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)$ to equation (1.4 .5) with the estimate

$$\|w_1\|_{\mathbf{H}^1(\Omega)}+\|\pi_1\|_{L^2(\Omega)}\le C\left(\|F\|_{\mathbf{L}^2(\Omega)}+\|h\|_{H^1(\Omega)}+\|g\|_{\mathbf{H}_0^{\frac{3}{2}}(\Gamma_d)}\right).$$

We deduce that $\lambda_0 w_1+(u_s\cdot\nabla)w_1+(w_1\cdot\nabla)u_s$ belongs to $\mathbf{L}^2(\Omega)$ and

$$\|\lambda_0 w_1+(u_s\cdot\nabla)w_1+(w_1\cdot\nabla)u_s\|_{\mathbf{L}^2(\Omega)}\le C\left(\|F\|_{\mathbf{L}^2(\Omega)}+\|h\|_{H^1(\Omega)}+\|g\|_{\mathbf{H}_0^{\frac{3}{2}}(\Gamma_d)}\right).$$

Thus, thanks to ([43], Theorem 9.4.5), (w_1,π_1) belongs to $\mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ and

$$\|w_1\|_{\mathbf{H}^2_{\delta_0}(\Omega)}+\|\pi_1\|_{H^1_{\delta_0}(\Omega)}\le C\left(\|F\|_{\mathbf{L}^2(\Omega)}+\|h\|_{H^1(\Omega)}+\|g\|_{\mathbf{H}_0^{\frac{3}{2}}(\Gamma_d)}\right).$$

□

1.4 .2 The Oseen operator

We introduce P the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$. The operator P is called the Leray projector. Moreover, for all v in $\mathbf{L}^2(\Omega)$, we have

$$Pv=v-\nabla q_1-\nabla q_2,$$

where q_1 and q_2 are the solutions of the equations

$$\Delta q_1=\operatorname{div} v \text{ in } \Omega,\quad q_1=0 \text{ on } \partial\Omega,$$

and

$$\Delta q_2=0,\quad \frac{\partial q_2}{\partial n}=(v-\nabla q_1)n \text{ on } \Gamma_d,\quad q_2=0 \text{ on } \Gamma_n.$$

Proposition 1.4 .3. *If v belongs to $\mathbf{H}^2_{\delta_0}(\Omega)$ then Pv and $(I-P)v$ belong to $\mathbf{H}^1(\Omega)$. Moreover, we have*

$$(I-P)v=\nabla q,$$

where q is the solution of the equation

$$\Delta q=\operatorname{div} v \ \text{ in } \Omega,\quad \frac{\partial q}{\partial n}=v\cdot n \ \text{ on } \Gamma_0,\quad q=0 \ \text{ on } \Gamma_n. \tag{1.4 .6}$$

Proof. Equation (1.4 .6) admits a unique solution $q \in H^2(\Omega)$. Thus, we can easily show that $Pv = v - \nabla q \in \mathbf{H}^1(\Omega) \cap \mathbf{V}^0_{n,\Gamma_0}(\Omega)$ and $(I - P)v = \nabla q \in \mathbf{H}^1(\Omega)$.

□

Now, we introduce the Oseen operator $(A, D(A))$ defined by

$$\begin{aligned} &D(A) = \{v \in \mathbf{V}^1_{\Gamma_d}(\Omega) \mid \exists p \in H^{\frac{1}{2}+\varepsilon_0}(\Omega) \text{ s.t } \operatorname{div}\sigma(v,p) \in \mathbf{L}^2(\Omega),\ \sigma(v,p)n = 0 \text{ on } \Gamma_n\},\\ &\text{and}\\ &Av = P\operatorname{div}\sigma(v,p) - P[(u_s \cdot \nabla)v + (v \cdot \nabla)u_s]. \end{aligned} \tag{1.4 .7}$$

Theorem 1.4 .4. *The operator $(A, D(A))$ is the generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ and its resolvent is compact.*

Proof. The proof is similar to that of ([44], Theorem 2.8).

□

Proposition 1.4 .5. *The adjoint of $(A, D(A))$ in $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ is $(A^*, D(A^*))$ defined by*

$$D(A^*) = \{v \in \mathbf{V}^1_{\Gamma_d}(\Omega) \mid \exists p \in H^{\frac{1}{2}+\varepsilon_0}(\Omega) \textit{ s.t } \operatorname{div}\sigma(v,p) \in \mathbf{L}^2(\Omega),\ \sigma(v,p)n + u_s \cdot n v = 0 \textit{ on } \Gamma_n\},$$

and

$$A^*v = A_0 v + P(u_s \cdot \nabla)v - P(\nabla u_s)^T v.$$

Proof. See ([44], Theorem 2.11).

□

1.4 .3 The lifting operators

We introduce the lifting operators $L \in \mathcal{L}(\mathbf{H}_0^{\frac{3}{2}}(\Gamma_d) \times H^1(\Omega), \mathbf{H}^2_{\delta_0}(\Omega))$ and $L_p \in \mathcal{L}(\mathbf{H}_0^{\frac{3}{2}}(\Gamma_d) \times H^1(\Omega), H^1_{\delta_0}(\Omega))$ defined by

$$L(g,h) = w \quad \text{and} \quad L_p(g,h) = \pi, \tag{1.4 .8}$$

where (w, π) is the solution of the system

$$\begin{aligned} &\lambda_0 w - \operatorname{div}\sigma(w,\pi) + (u_s \cdot \nabla)w + (w \cdot \nabla)u_s = 0 \text{ in } \Omega,\\ &\operatorname{div} w = h \text{ in } \Omega, \quad w = g \text{ on } \Gamma_d, \quad \sigma(w,\pi)n = 0 \text{ on } \Gamma_n. \end{aligned}$$

We also introduce the operator $D_c \in \mathcal{L}(\mathbb{R}^{n_c}, \mathbf{H}^2_{\delta_0}(\Omega))$ defined by

$$D_c\mathbf{g} = \sum_{i=1}^{n_c} g_i L(w_i, 0), \quad \forall\, \mathbf{g} = (g_i)_{1 \le i \le n_c} \in \mathbb{R}^{n_c}. \tag{1.4 .9}$$

1.4 .4 Expression of the pressure

The goal of this section is to express the pressure π in equation (1.4 .1) in terms of w, F, g and h. The method used in [51] consists in calculating the divergence of the first equation of (1.4 .1) in order to get an elliptic equation for π. This method does not work here. Indeed, formally π is the solution of the equation

$$\begin{aligned} &\Delta\pi = -\lambda_0 h - \operatorname{div}((u_s \cdot \nabla)w + (w \cdot \nabla)u_s) + \operatorname{div} F \text{ in } \Omega, \quad \pi = 2\nu\varepsilon(w)n \cdot n \text{ on } \Gamma_n,\\ &\frac{\partial\pi}{\partial n} = 2\nu\operatorname{div}\varepsilon(w) \cdot n - \lambda_0 w \cdot n - ((u_s \cdot \nabla)w + (w \cdot \nabla)u_s) \cdot n + F \cdot n \text{ on } \Gamma_d. \end{aligned} \tag{1.4 .10}$$

Since $\mathrm{div}\,\varepsilon(w) \notin \mathbf{L}^2(\Omega)$, the boundary condition

$$\frac{\partial\pi}{\partial n} = 2\nu\mathrm{div}\,\varepsilon(w)\cdot n - \lambda_0 w\cdot n - ((u_s\cdot\nabla)w + (w\cdot\nabla)u_s)\cdot n + F\cdot n \text{ on } \Gamma_d,$$

is not well-posed. We are going to use another approach, the transposition method, to define an equation for the pressure π. First, we consider the equation

$$\Delta\psi = \zeta \text{ in } \Omega, \quad \frac{\partial\psi}{\partial n} = 0 \text{ on } \Gamma_d, \quad \psi = 0 \text{ on } \Gamma_n. \tag{1.4 .11}$$

Lemma 1.4 .2. *Let ζ belong to $L^2(\Omega)$. Then, equation (1.4 .11) admits a unique $\psi \in H^2(\Omega)$ with the estimate*

$$\|\psi\|_{H^2(\Omega)} \le C\|\zeta\|_{L^2(\Omega)}.$$

Proof. It follows from ([43], Theorem 6.5.4), ([43], Lemma 6.2.1) and ([36], Chapter 2).

□

For all $\zeta \in L^2(\Omega)$, we set

$$\begin{aligned}\ell(\zeta) \;=\; & \lambda_0\int_{\Gamma_d} g\cdot n\psi - \lambda_0\int_\Omega h\,\psi - \int_\Omega F\cdot\nabla\psi \\ & +2\nu\int_\Omega \varepsilon(w):\nabla^2\psi - 2\nu\int_{\Gamma_d}\varepsilon(w)n\cdot\nabla\psi + \int_\Omega\Big[(u_s\cdot\nabla)w + (w\cdot\nabla)u_s\Big]\cdot\nabla\psi,\end{aligned} \tag{1.4 .12}$$

where $\psi \in H^2(\Omega)$ is the solution of equation (1.4 .11).

Lemma 1.4 .3. *Let (w, F, g, h) belong to $\mathbf{H}^2_{\delta_0}(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}^{\frac{3}{2}}_0(\Gamma_d)\times H^1(\Omega)$. Then, the operator ℓ belongs to $\mathcal{L}(L^2(\Omega),\mathbb{R})$ and we have*

$$|\ell(\zeta)| \le C\left(\|g\|_{\mathbf{L}^2(\Gamma_d)} + \|h\|_{L^2(\Omega)} + \|F\|_{\mathbf{L}^2(\Omega)} + \|w\|_{\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega)}\right)\|\zeta\|_{L^2(\Omega)},$$

for all $\zeta \in L^2(\Omega)$.

Proof. The lemma follows from the following estimates.

$$\begin{aligned}
&\Big|\int_{\Gamma_d} g\cdot n\psi\Big| \le C\|g\|_{\mathbf{L}^2(\Gamma_d)}\|\psi\|_{L^2(\Gamma_d)} \le C\|g\|_{\mathbf{L}^2(\Gamma_d)}\|\zeta\|_{L^2(\Omega)},\\
&\Big|\int_\Omega h\psi\Big| \le C\|h\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} \le C\|h\|_{L^2(\Omega)}\|\zeta\|_{L^2(\Omega)},\\
&\Big|\int_\Omega F\cdot\nabla\psi\Big| \le C\|F\|_{\mathbf{L}^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} \le C\|F\|_{\mathbf{L}^2(\Omega)}\|\zeta\|_{L^2(\Omega)},\\
&\Big|\int_\Omega \varepsilon(w):\nabla^2\psi\Big| \le C\|\varepsilon(w)\|\|_{\mathbf{L}^2(\Omega)}\|\nabla^2\psi\|_{\mathbf{L}^2(\Omega)} \le C\|w\|_{\mathbf{H}^1(\Omega)}\|\zeta\|_{L^2(\Omega)},\\
&\Big|\int_{\Gamma_d}\varepsilon(w)n\cdot\nabla\psi\Big| \le C\|\varepsilon(w)\|_{\mathbf{L}^2(\Gamma_d)}\|\nabla\psi\|_{\mathbf{L}^2(\Gamma_0)} \le C\|w\|_{\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega)}\|\zeta\|_{L^2(\Omega)},\\
&\Big|\int_\Omega\Big[(u_s\cdot\nabla)w + (w\cdot\nabla)u_s\Big]\cdot\nabla\psi\Big| \le C\|w\|_{\mathbf{H}^1(\Omega)}\|\nabla\psi\|_{\mathbf{L}^2(\Omega)} \le C\|w\|_{\mathbf{H}^1(\Omega)}\|\zeta\|_{L^2(\Omega)}.
\end{aligned}$$

□

Theorem 1.4 .6. *If $(w,\pi) \in \mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ is a solution of system (1.4 .1) then π is the unique solution of the problem*

$$\textit{Find } q \in L^2(\Omega) \textit{ such that } \int_\Omega q\cdot\zeta = \ell(\zeta), \quad \forall\,\zeta \in L^2(\Omega). \tag{1.4 .13}$$

Proof. Let $(w,\pi)\in \mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ be a solution of system (1.4 .1) and let ζ belong to $L^2(\Omega)$. Multiplying the first equation of the system by $\nabla\psi$, where ψ is the solution of equation (1.4 .11), we obtain

$$\lambda_0\int_\Omega w\cdot\nabla\psi-\int_\Omega \operatorname{div}\sigma(w,\pi)\cdot\nabla\psi+\int_\Omega[(u_s\cdot\nabla)w+(w\cdot\nabla)u_s]\cdot\nabla\psi-\int_\Omega F\cdot\nabla\psi=0. \tag{1.4 .14}$$

By using integration by parts, we get

$$\lambda_0\int_\Omega w\cdot\nabla\psi=-\lambda_0\int_\Omega \operatorname{div} w\cdot\psi+\int_{\partial\Omega} w\cdot n\psi=-\lambda_0\int_\Omega h\,\psi+\lambda_0\int_{\Gamma_d} g\cdot n\psi, \tag{1.4 .15}$$

and

$$\begin{aligned}\int_\Omega \operatorname{div}\sigma(w,\pi)\cdot\nabla\psi &= -\int_\Omega \sigma(w,\pi):\nabla^2\psi+\int_{\partial\Omega}\sigma(w,\pi)n\cdot\nabla^2\psi\\ &= -2\nu\int_\Omega \varepsilon(w):\nabla^2\psi+\int_\Omega \pi\cdot\Delta\psi+\int_{\Gamma_d}\sigma(w,\pi)n\cdot\nabla\psi\\ &= -2\nu\int_\Omega \varepsilon(w):\nabla^2\psi+\int_{\Gamma_d}\sigma(w,\pi)n\cdot\nabla^2+\int_\Omega \pi\cdot\zeta.\end{aligned} \tag{1.4 .16}$$

From equations (1.4 .14), (1.4 .15) and (1.4 .15), it follows that π is a solution of equation (1.4 .13). Thanks to Lemma 1.4 .3 and the Lax-Milgram Theorem in $L^2(\Omega)$, we prove that (1.4 .13) admits the unique solution.

□

We introduce the following operators:

- $N_d\in\mathcal{L}(\mathbf{L}^2(\Gamma_d),H^1(\Omega))$ defined by $N_dg=q$ where q is the solution of the equation

$$\Delta q=0 \text{ in } \Omega,\quad \frac{\partial q}{\partial n}=g\cdot n \text{ on } \Gamma_d,\quad q=0 \text{ on } \Gamma_n. \tag{1.4 .17}$$

- $N_{\text{div}}\in\mathcal{L}(L^2(\Omega),H^1(\Omega))$ defined by $N_{\text{div}}h=q$ where q is the solution of the equation

$$\Delta q=h \text{ in } \Omega,\quad \frac{\partial q}{\partial n}=0 \text{ on } \Gamma_d,\quad q=0 \text{ on } \Gamma_n. \tag{1.4 .18}$$

- $N_p\in\mathcal{L}(\mathbf{L}^2(\Omega),H^1(\Omega))$ defined by $N_pF=q_1+q_2$ where q_1 and q_2 are the solution of the equations

$$\Delta q_1=\operatorname{div} F \text{ in } \Omega,\quad q_1=0 \text{ on } \partial\Omega, \tag{1.4 .19}$$

  and

$$\Delta q_2=0 \text{ in } \Omega,\quad \frac{\partial q_2}{\partial n}=(F-\nabla q_1)\cdot n \text{ on } \Gamma_d,\quad q_2=0 \text{ on } \Gamma_n. \tag{1.4 .20}$$

- $N_v\in\mathcal{L}(\mathbf{H}^2_{\delta_0}(\Omega),L^2(\Omega))$ defined by $N_vw=q$ where q is the solution of the variational problem

  Find $q\in L^2(\Omega)$ such that

$$\int_\Omega q\zeta=2\nu\int_\Omega\varepsilon(w):\nabla^2\psi-2\nu\int_{\Gamma_d}\varepsilon(w)n\cdot\nabla\psi+\int_\Omega\Big[(u_s\cdot\nabla)w+(w\cdot\nabla)u_s\Big]\cdot\nabla\psi\quad\forall\zeta\in L^2(\Omega),$$

  with ψ the solution of equation (1.4 .11).

Thanks to the Lax-Milgram Theorem, we can easily prove that the operators N_d, N_{div}, N_p and N_v are well-defined.

Remark 1.4 .1. *We have*

$$\nabla N_p = I - P.$$

Theorem 1.4 .7. *If* $(w,\pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ *is a the solution of system (1.4 .1) then we have*

$$\pi = -\lambda_0 N_d g - \lambda_0 N_{\mathrm{div}} h + N_p F + N_v w.$$

Proof. Thanks to Theorem 1.4 .6, for all ζ in $L^2(\Omega)$, we have

$$\begin{aligned}\int_\Omega \pi\zeta = \lambda_0 \int_{\Gamma_s} g\cdot n\psi - \lambda_0 \int_\Omega h\psi + \int_\Omega \psi \mathrm{div}\, F - \int_{\Gamma_d} F\cdot n\psi \\ +2\nu \int_\Omega \varepsilon(w) : \nabla^2\psi - 2\nu \int_{\Gamma_d} \varepsilon(w)n\cdot\nabla\psi + \int_\Omega \Big[(u_s\cdot\nabla)w + (w\cdot\nabla)u_s\Big]\cdot\nabla\psi,\end{aligned} \tag{1.4 .21}$$

where ψ is the solution of (1.4 .11). By integration by parts, we obtain

$$\int_{\Gamma_d} g\cdot n\psi = \int_{\Gamma_d} \frac{\partial N_d g}{\partial n}\psi = -\int_\Omega \zeta N_d g, \tag{1.4 .22}$$

and

$$-\int_\Omega h\psi = -\int_\Omega \Delta N_{\mathrm{div}} h\psi = -\int_\Omega \zeta N_{\mathrm{div}} h. \tag{1.4 .23}$$

Thanks to Remark 1.4 .1, we get

$$\int_\Omega F\cdot\nabla\psi = \int_\Omega \zeta N_p F. \tag{1.4 .24}$$

By combining equations (1.4 .21), (1.4 .22), (1.4 .23) and (1.4 .24), we obtain

$$\int_\Omega \pi\zeta = -\lambda_0 \int_\Omega \zeta N_d g - \lambda_0 \int_\Omega \zeta N_{\mathrm{div}} h + \int_\Omega \zeta N_p F + \int_\Omega \zeta N_v w, \quad \forall\, \zeta \in L^2(\Omega),$$

which implies that

$$\pi = -\lambda_0 N_d g - \lambda_0 N_{\mathrm{div}} h + N_p F + N_v w.$$

□

Lemma 1.4 .4. *Let* $(w,\pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ *be the solution of system (1.4 .1). Then* $N_v w$ *belongs to* $H^{\frac12+\varepsilon_0}(\Omega)$.

Proof. We start from the identity

$$\pi = -\lambda_0 N_d g - \lambda_0 N_{\mathrm{div}} h + N_p F + N_v w.$$

We know that $\pi \in H^1_{\delta_0}(\Omega) \subset H^{\frac12+\varepsilon_0}(\Omega)$ (see Lemma 1.4 .1), $N_p F$ belongs to $H^1(\Omega)$, $N_{\mathrm{div}} h$ belongs to $H^1(\Omega)$, and $N_d g$ belongs to $H^1(\Omega)$. Hence $N_v w$ belongs to $H^{\frac12+\varepsilon_0}(\Omega)$.

□

1.4 .5 Rewriting of the Oseen system with semigroup

Theorem 1.4 .8. *A pair* $(w,\pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ *is solution of system (1.4 .1) if and only if*

$$\begin{aligned}&(\lambda_0 I - A)Pw + (A - \lambda_0 I)PL(g,h) = PF,\\ &(I-P)w = \nabla N_d g + \nabla N_{\mathrm{div}} h,\\ &\pi = -\lambda_0 N_d g - \lambda_0 N_{\mathrm{div}} h + N_p F + N_v w.\end{aligned} \tag{1.4 .25}$$

Proof. Let $(w,\pi)\in \mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ be a solution of system (1.4 .1). We set

$$\hat{w}=w-L(g,h)\quad \text{ and }\quad \hat{\pi}=\pi-L_p(g,h).$$

The couple $(\hat{w},\hat{\pi})$ satisfies the system

$$\begin{aligned}&\lambda_0\hat{w}-\operatorname{div}\sigma(\hat{w},\hat{\pi})+(u_s\cdot\nabla)\hat{w}+(\hat{w}\cdot\nabla)u_s=F \text{ in } \Omega,\\ &\operatorname{div}\hat{w}=0 \text{ in } \Omega,\quad \hat{w}=0 \text{ on } \Gamma_d,\quad \sigma(\hat{w},\hat{\pi})n=0 \text{ on } \Gamma_n.\end{aligned}\tag{1.4 .26}$$

It follows that $\hat{w}$ belongs to $D(A)$ and

$$\lambda_0P\hat{w}-AP\hat{w}=PF.$$

Hence

$$(\lambda_0I-A)Pw+(A-\lambda_0I)PL(g,h)=PF.$$

The last two equations of (1.4 .25) come from Proposition 1.4 .3 and Theorem 1.4 .7. Now, we suppose that $(w,p)\in\mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ is solution of system (1.4 .25). Since

$$(I-P)w=\nabla N_dg+\nabla N_{\text{div}}h=(I-P)L(g,h),$$

we have

$$\hat{w}=w-L(g,h)\in D(A).\tag{1.4 .27}$$

Thus, there exists $\pi_1\in H^{\frac{1}{2}+\varepsilon_0}(\Omega)$ such that

$$A\hat{w}=P\operatorname{div}\sigma(\hat{w},\pi_1)-P(u_s\cdot\nabla)\hat{w}-P(\hat{w}\cdot\nabla)u_s,$$

and

$$\sigma(\hat{w},\pi_1)n=0 \text{ on } \Gamma_n.\tag{1.4 .28}$$

From the first equation of system (1.4 .25), it follows that

$$P\left[\lambda_0\hat{w}-\operatorname{div}\sigma(\hat{w},\pi_1)+(u_s\cdot\nabla)\hat{w}+(\hat{w}\cdot\nabla)u_s-F\right]=0.$$

Thus, there exists $\pi_2\in H^1_{\Gamma_n}(\Omega)$ such that

$$\lambda_0\hat{w}-\operatorname{div}\sigma(\hat{w},\pi_1+\pi_2)+(u_s\cdot\nabla)\hat{w}+(\hat{w}\cdot\nabla)u_s=F.\tag{1.4 .29}$$

From equations (1.4 .27), (1.4 .28) and (1.4 .29), we obtain the system

$$\begin{aligned}&\lambda_0\hat{w}-\operatorname{div}\sigma(\hat{w},\pi_1+\pi_2)+(u_s\cdot\nabla)\hat{w}+(\hat{w}\cdot\nabla)u_s=F \text{ in } \Omega,\\ &\operatorname{div}\hat{w}=0 \text{ in } \Omega,\quad \hat{w}=0 \text{ on } \Gamma_d,\quad \sigma(\hat{w},\pi_1+\pi_2)n=0 \text{ on } \Gamma_n.\end{aligned}\tag{1.4 .30}$$

We deduce that (w,π), with $\tilde{\pi}=\pi_1+\pi_2-L_p(g,h)$, is the solution of the system

$$\begin{aligned}&\lambda_0w-\operatorname{div}\sigma(w,\tilde{\pi})+(u_s\cdot\nabla)w+(w\cdot\nabla)u_s=F \text{ in } \Omega,\\ &\operatorname{div}w=h \text{ in } \Omega,\quad w=g \text{ on } \Gamma_d,\quad \sigma(w,\tilde{\pi})n=0 \text{ on } \Gamma_n.\end{aligned}\tag{1.4 .31}$$

Theorem 1.4 .7 gives

$$\tilde{\pi}=-\lambda_0N_dg-\lambda_0N_{\text{div}}h+N_pF+N_vw=\pi.$$

□

1.5 Reformulation of the linearized system

System (1.3 .3) linearized around $(0,0,0,0)$ reads as follows:

$$
\begin{aligned}
& v_t - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 - \omega v = 0 \text{ in } Q^\infty,\\
& \operatorname{div} v = A_3\eta_1 \text{ in } Q^\infty, \quad v = \eta_2 e_2 \text{ on } \Sigma_s^\infty, \quad v = \textstyle\sum_{i=1}^{n_c} g_i w_i \text{ on } \Sigma_b^\infty,\\
& v = 0 \text{ on } \Sigma_i^\infty, \quad \sigma(v,p)n = 0 \text{ on } \Sigma_n^\infty,\\
& \eta_{1,t} - \eta_2 - \omega\eta_1 = 0 \text{ on } \Sigma_s^\infty,\\
& \eta_{2,t} - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 - A_4\eta_1 - \omega\eta_2 = \gamma_s p \text{ on } \Sigma_s^\infty,\\
& \eta_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
& v(0) = v^0 \text{ in } \Omega, \quad \eta_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s.
\end{aligned}
\tag{1.5 .1}
$$

The goal of this section is to rewrite that system as a control system.

1.5 .1 Properties of the operators A_1, A_2, A_3 and A_4

Proposition 1.5 .1. *We have*

$$
\begin{aligned}
& A_1 \in \mathcal{L}(H_0^2(\Gamma_s), \mathbf{L}^2(\Omega)),\\
& A_2 \in \mathcal{L}(L^2(\Gamma_s), \mathbf{L}^2(\Omega)),\\
& A_3 \in \mathcal{L}(H_0^2(\Gamma_s), H^1(\Omega)),\\
& \textit{and } A_4 \in \mathcal{L}(H_0^2(\Gamma_s), H^{\varepsilon_0}(\Gamma_s)).
\end{aligned}
$$

Proof. For A_1, A_2 and A_3, the proposition follows from Lemmas 1.5 .1 and 1.5 .1 below, and for A_4, it follows from [[28], Proposition B.1].

□

Lemma 1.5 .1. *If (η, w) belongs to $L^2(\Gamma_s)\times\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)$ then ηw belongs to $\mathbf{L}^2(\Omega)$ and there exists $C>0$ such that*

$$\|\eta w\|_{\mathbf{L}^2(\Omega)} \le C\|\eta\|_{L^2(\Gamma_s)}\|w\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)}.$$

Proof. Thanks to the continuous embedding $\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)\hookrightarrow L^2(0,1;L^\infty(0,\ell))$, it follows that

$$\int_\Omega |\eta w|^2 = \int_0^1\left(\int_0^\ell |\eta(x)w(x,z)|^2\,\mathrm{dx}\right)\mathrm{dz} \le C\|\eta\|_{L^2(\Gamma_s)}\|w\|_{L^2(0,1;L^\infty(0,\ell))} \le C\|\eta\|_{L^2(\Gamma_s)}\|w\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)},$$

for some constant $C>0$.

□

Lemma 1.5 .2. *If (η, w) belongs to $H_0^1(\Gamma_s)\times\mathbf{L}^2_{\delta_0}(\Omega)$ then ηw belongs to $\mathbf{L}^2(\Omega)$ and there exists $C>0$ such that*

$$\|\eta w\|_{\mathbf{L}^2(\Omega)} \le C\|\eta\|_{H_0^1(\Gamma_s)}\|w\|_{\mathbf{L}^2_{\delta_0}(\Omega)}.$$

Proof. Since η belongs to $H_0^1(\Gamma_s)$, we have

$$|\eta(x)| = \left|\int_{x_j}^x \eta_x(s)\mathrm{ds}\right| \le \|\eta_x\|_{L^2(\Gamma_s)}\,|x-x_j|^{\frac{1}{2}},$$

for all $1 \leq j \leq 4$. Hence

$$\begin{aligned}\int_\Omega |\eta w|^2 &= \int_\Omega |\eta(x)|^2\,|w(x,z)|^2\,\mathrm{dxdz}\\ &\leq \|\eta_x\|^2_{L^2(\Gamma_s)}\int_\Omega \frac{|x-x_j|}{(|x-x_j|^2+|z-z_j|^2)^{2\delta_0}}\left|(|x-x_j|^2+|z-z_j|^2)^{\delta_0}w(x,z)\right|^2\mathrm{dxdz}\\ &\leq \|\eta_x\|^2_{L^2(\Gamma_s)}\int_\Omega (|x-x_j|^2+|z-z_j|^2)^{1-2\delta_0}\left|((x-x_j)^2+(z-z_j)^2)^{\delta_0}w(x,z)\right|^2\mathrm{dxdz}.\end{aligned}$$

From that inequality, we prove that there exists $C>0$ such that

$$\|\eta w\|_{\mathbf{L}^2(\Omega)} \leq C\|\eta\|_{H^1_0(\Gamma_s)}\|w\|_{\mathbf{L}^2_{\delta_0}(\Omega)}.$$

□

We denote by $A_1^* \in \mathcal{L}(\mathbf{L}^2(\Omega), H^{-2}(\Gamma_s))$, $A_2^* \in \mathcal{L}(\mathbf{L}^2(\Omega), L^2(\Gamma_s))$, $A_3^* \in \mathcal{L}(\mathbf{L}^2(\Omega), H^{-2}(\Gamma_s))$ and $A_4^* \in \mathcal{L}(L^2(\Gamma_s), H^{-2}(\Gamma_s))$ the adjoints of A_1, A_2, A_3 and A_4 respectively.

1.5 .2 Definition of the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$

For the structure, we introduce the unbounded operator $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ in $H^2_0(\Gamma_s)$ defined by

$$D(A_{\alpha,\beta}) = H^4(\Gamma_s)\cap H^2_0(\Gamma_s) \quad \text{and} \quad A_{\alpha,\beta} = \beta\Delta - \alpha\Delta^2. \tag{1.5 .2}$$

We equip the space $H^2_0(\Gamma_s)$ with the norm

$$(\eta,\xi)_{H^2_0(\Gamma_s)} = ((-A_{\alpha,\beta})^{\frac12}\eta, (-A_{\alpha,\beta})^{\frac12}\xi)_{L^2(\Gamma_s)} = \int_{\Gamma_s}(\beta\nabla\eta\cdot\nabla\xi + \alpha\Delta\eta\Delta\xi).$$

We introduce the operators $N_s \in \mathcal{L}(L^2(\Gamma_s), H^1(\Omega))$ and $N_c \in \mathcal{L}(\mathbb{R}^{n_c}, H^1(\Omega))$ defined by

$$N_s\eta = N_d(\eta e_2\chi_{\Gamma_s}) \quad \text{and} \quad N_c\mathbf{g} = \sum_{i=1}^{n_c} g_i N_d(w_i), \tag{1.5 .3}$$

for all $\eta \in L^2(\Gamma_s)$ and for all $\mathbf{g} = (g_i)_{1\leq i\leq n_c} \in \mathbb{R}^{n_c}$.

Lemma 1.5 .3. *The operator $I + \gamma_s N_s$ is an automorphism in $L^2(\Gamma_s)$.*

Proof. The proof is similar to that of ([51], Lemma 3.2).

□

We are going to define the infinitesimal generator $(\mathcal{A}, D(\mathcal{A}))$ in $Z = \mathbf{V}^0_{n,\Gamma_d}(\Omega)\times H^2_0(\Gamma_s)\times L^2(\Gamma_s)$ of the algebraic partial differential equation (1.5 .1). For that we first introduce

$$\begin{aligned}D(\mathcal{A}) = \{&(Pv,\eta_1,\eta_2) \in \mathbf{V}^1_{n,\Gamma_d}(\Omega)\times(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))\times H^2_0(\Gamma_s) \mid\\ &Pv - PL(\eta_2 e_2\chi_{\Gamma_s}, A_3\eta_1) \in D(A)\}.\end{aligned}$$

Lemma 1.5 .4. *If (Pv,η_1,η_2) belongs to $D(\mathcal{A})$ then $N_v(Pv + \nabla N_s\eta_2 + \nabla N_{\mathrm{div}}A_3\eta_1)$ belongs to $H^{\frac12+\varepsilon_0}(\Omega)$.*

Proof. Let (Pv,η_1,η_2) belong to $D(\mathcal{A})$. Then, there exists $F \in \mathbf{V}^0_{n,\Gamma_d}(\Omega)$ such that

$$(\lambda_0 I - A)Pv + (A-\lambda_0 I)PL(\eta_2 e_2\chi_{\Gamma_s}, A_3\eta_1) = F.$$

We set

$$v = Pv + \nabla N_s\eta_2 + \nabla N_{\mathrm{div}}A_3\eta_1,$$

and

$$\rho = -\lambda_0 N_s\eta_2 - \lambda_0 N_{\mathrm{div}} A_3\eta_1 + N_v v.$$

Then, according to Theorem 1.4 .8, we know that (v, ρ) is the unique solution of the equation

$$\begin{aligned} &\lambda_0 v - \operatorname{div}\sigma(v,\rho) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s = F \text{ in } \Omega,\\ &\operatorname{div} v = A_3\eta_1 \text{ in } \Omega, \quad v = \eta_2 e_2\chi_{\Gamma_s} \text{ on } \Gamma_d, \quad \sigma(w,\pi)n = 0 \text{ on } \Gamma_n. \end{aligned} \tag{1.5 .4}$$

According to Lemma 1.4 .4, we know that $N_v(Pv + \nabla N_s\eta_2 + \nabla N_{\mathrm{div}} A_3\eta_1)$ belongs to $H^{\frac{1}{2}+\varepsilon_0}(\Omega)$. $\square$

We are now in position to introduce the operator $\mathcal{A}$ defined in $D(\mathcal{A})$ by

$$\mathcal{A} = M_s^{-1}\begin{pmatrix} A & PA_1 + (\lambda_0 I - A)PL(0, A_3) & PA_2 + (\lambda_0 I - A)PL(e_2\chi_{\Gamma_s}, 0) \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} + \gamma_s N_p A_1 + A_4 & \gamma\Delta + \gamma_s N_p A_2 \end{pmatrix} + M_s^{-1}\mathcal{A}_p, \tag{1.5 .5}$$

with

$$M_s = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \gamma_s N_{\mathrm{div}} A_3 & I + \gamma_s N_s \end{pmatrix},$$

and $\mathcal{A}_p$ is defined in $D(\mathcal{A})$ by

$$\mathcal{A}_p\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma_s N_v(Pv + \nabla N_s\eta_2 + \nabla N_{\mathrm{div}} A_3\eta_1) \end{pmatrix}.$$

Lemma 1.5 .5. *Let (F, g, h) belong to $L^2(Q^\infty)\times H^{2,1}(\Sigma_d^\infty)\times(H^1(0,\infty; L^2(\Omega))\cap L^2(0,\infty; H^1(\Omega)))$. The pair $(w,\pi)\in H^{2,1}_{\delta_0}(Q^\infty)\times L^2(0,\infty; H^1_{\delta_0}(\Omega))$ is a solution of system*

$$\begin{aligned} &w_t - \operatorname{div}\sigma(w,\pi) + (u_s\cdot\nabla)w + (w\cdot\nabla)u_s = F \textit{ in } Q^\infty,\\ &\operatorname{div} w = h \textit{ in } Q^\infty, \quad v = g \textit{ on } \Sigma_d^\infty, \quad \sigma(w,\pi)n = 0 \textit{ on } \Sigma_n^\infty, \end{aligned} \tag{1.5 .6}$$

if and only if

$$\begin{aligned} &Pw_t = APw + (\lambda_0 I - A)PL(g,h) + PF,\\ &(I-P)w = \nabla N_d g + \nabla N_{\mathrm{div}} h,\\ &\pi = -N_d g_t - N_{\mathrm{div}} h_t + N_v(Pw + \nabla N_d g + \nabla N_{\mathrm{div}} h) + N_p F. \end{aligned} \tag{1.5 .7}$$

Proof. We rewrite the system (1.5 .6) as follows

$$\begin{aligned} &\lambda_0 w - \operatorname{div}\sigma(w,\pi) + (u_s\cdot\nabla)w + (w\cdot\nabla)u_s = -w_t + \lambda_0 w + F \text{ in } Q^\infty,\\ &\operatorname{div} w = h \text{ in } Q^\infty, \quad v = g \text{ on } \Sigma_d^\infty, \quad \sigma(w,\pi)n = 0 \text{ on } \Sigma_n^\infty. \end{aligned} \tag{1.5 .8}$$

The term $-w_t + \lambda_0 w + F$ belongs to $L^2(0,\infty; \mathbf{L}^2(\Omega))$, h belongs to $L^2(0,\infty; H^1(\Omega))$ and g belongs to $L^2(0,\infty; H_0^{\frac{3}{2}}(\Gamma_d))$. Thus, from Theorem 1.4 .8, it follows that $(w,\pi)\in H^{2,1}_{\delta_0}(Q^\infty)\times L^2(0,\infty; H^1_{\delta_0}(\Omega))$ is solution of system (1.5 .6) if and only if

$$\begin{aligned} &Pw_t = APw + (\lambda_0 I - A)PL(g,h) + PF,\\ &(I-P)w = \nabla N_d g + \nabla N_{\mathrm{div}} h,\\ &\pi = -\lambda_0 N_d g - \lambda_0 N_{\mathrm{div}} h + N_v w + N_p(-w_t + \lambda_0 w + F). \end{aligned} \tag{1.5 .9}$$

Thanks to the second equation of the above system and the identities

$$N_p v = N_d g + N_{\rm div} h \quad \text{and} \quad N_p v_t = N_d g_t + N_{\rm div} h_t,$$

we prove that

$$\pi = -N_d g_t - N_{\rm div} h_t + N_v(Pw + \nabla N_d g + \nabla N_{\rm div} h) + N_p F. \tag{1.5 .10}$$

□

Proposition 1.5 .2. *The quadruplet $(v, p, \eta_1, \eta_2) \in X_{\delta_0}$ is a solution of system (1.5 .1) if and only if*

$$\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = (\mathcal{A} + \omega I)\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}_1\mathbf{g} + \mathcal{B}_2(\mathbf{g}_t - \omega\mathbf{g}), \quad \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix}(0) = \begin{pmatrix} Pv^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix},\\
&(I-P)v = \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\rm div}A_3\eta_1,\\
&p = -N_s(\eta_{2,t} - \omega\eta_2) - N_{\rm div}(A_3\eta_{1,t} - \omega A_3\eta_1) + N_pA_1\eta_1 + N_pA_2\eta_2\\
&\quad +N_v(Pv + \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\rm div}A_3\eta_1) - N_c(\mathbf{g}_t - \omega\mathbf{g}),
\end{aligned} \tag{1.5 .11}$$

with

$$\mathcal{B}_1\mathbf{g} = \begin{pmatrix} (\lambda_0 I - A)PD_c\mathbf{g} \\ 0 \\ (I + \gamma_s N_s)^{-1}\gamma_s N_v \nabla N_c\mathbf{g} \end{pmatrix} \quad \textit{and} \quad \mathcal{B}_2\mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ -(I + \gamma_s N_s)^{-1}\gamma_s N_c\mathbf{g} \end{pmatrix}.$$

Proof. We suppose that $(v, p, \eta_1, \eta_2) \in X_{\delta_0}$ is a solution of system (1.5 .1). Then, $(v, p) \in H^2_{\delta_0}(Q^\infty) \times L^2(0, \infty; H^1_{\delta_0}(\Omega))$ is a solution of the system

$$\begin{aligned}
&v_t - \operatorname{div}\sigma(v, p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s = A_1\eta_1 + A_2\eta_2 \text{ in } Q^\infty,\\
&\operatorname{div} v(t) = A_3\eta_1 \text{ in } Q^\infty, \quad v = \eta_2 e_2 \text{ on } \Sigma_s^\infty, \quad v = \textstyle\sum_{i=1}^{n_c} g_i w_i \text{ on } \Sigma_b^\infty,\\
&v = 0 \text{ on } \Sigma_i^\infty, \quad \sigma(v, p)n = 0 \text{ on } \Sigma_n^\infty.
\end{aligned}$$

The term $A_1\eta_1 + A_2\eta_2$ belongs to $L^2(0, \infty; \mathbf{L}^2(\Omega))$, the term $\eta_2 e_2\chi_{\Gamma_s} + \sum_{i=1}^{n_c} g_i w_i$ belongs $L^2(0, \infty; \mathbf{H}_0^{\frac{3}{2}}(\Gamma_d))$ and the term $A_3\eta_1$ belongs to $L^2(0, \infty; H^1(\Omega))$. Moreover, we have

$$N_d(\eta_2 e_2\chi_{\Gamma_s} + \sum_{i=1}^{n_c} g_i w_i) = N_s\eta_2 + N_c\mathbf{g},$$

and

$$L(\eta_2 e_2\chi_{\Gamma_s} + \sum_{i=1}^{n_c} g_i w_i, A_3\eta_1) = L(\eta_2 e_2\chi_{\Gamma_s}, A_3\eta_1) + D_c\mathbf{g}.$$

Thus, from Lemma 1.5 .5, it follows that

$$\begin{aligned}
&Pv_t = APv + \omega Pv + (\lambda_0 I - A)PL(\eta_2 e_2\chi_{\Gamma_s}, A_3\eta_1) + (\lambda_0 I - A)D_c\mathbf{g} + PA_1\eta_1 + PA_2\eta_2,\\
&(I-P)v = \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\rm div}A_3\eta_1,\\
&p = -N_s(\eta_{2,t} - \omega\eta_2) - N_{\rm div}(A_3\eta_{1,t} - \omega A_3\eta_1) + N_pA_1\eta_1 + N_pA_2\eta_2\\
&\quad +N_v(Pv + \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\rm div}A_3\eta_1) - N_c(\mathbf{g}_t - \omega\mathbf{g}).
\end{aligned} \tag{1.5 .12}$$

Replacing the pressure p by the above expression in the equation satisfied by η_2, we obtain

$$\begin{aligned}(I+\gamma_s N_s)\eta_{2,t} + \gamma_s N_{\text{div}} A_3 \eta_{1,t} &= A_{\alpha,\beta}\eta_1 + \gamma_s N_p A_1 \eta_1 + A_4 \eta_1 + \gamma \Delta \eta_2 + \gamma_s N_p A_2 \eta_2 \\ &+ \gamma_s N_v (Pv + \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1) + \omega (I+\gamma_s N_s)\eta_2 + \gamma_s N_v \nabla N_c \mathbf{g} - \gamma_s N_c (\mathbf{g}_t - \omega \mathbf{g}).\end{aligned} \tag{1.5 .13}$$

Thanks to equations (1.5 .12) and (1.5 .13), we prove that (v, p, η_1, η_2) is a solution of system (1.5 .11). We prove the reverse statement by using Lemma 1.5 .5 again.

□

1.6 The extended system

We can see that the time derivative of $\boldsymbol{g}$ appears in system (1.5 .11). In order to get an evolution equation in the form of a classical control problem, we define an extended system by considering $\mathbf{g}$ as a new state variable and by introducing a new control variable

$$\mathbf{f} := \mathbf{g}_t - \Lambda \mathbf{g} - \omega \mathbf{g},$$

where $\Lambda = \text{diag}(\Lambda_1, \cdots, \Lambda_{n_c})$ is a diagonal matrix. We introduce the operator $M_{s,e} \in \mathcal{L}(Z)$ defined by

$$M_{s,e} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & \gamma_s N_{\text{div}} A_3 & I+\gamma_s N_s & \gamma_s N_c \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Lemma 1.6 .1. *The operator $M_{s,e}$ is an automorphism in Z_e and we have*

$$M_{s,e}^{-1} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -(I+\gamma_s N_s)^{-1}\gamma_s N_{\text{div}} A_3 & (I+\gamma_s N_s)^{-1} & -(I+\gamma_s N_s)^{-1}\gamma_s N_c \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Proof. It follows from Lemma 1.5 .3.

□

Lemma 1.6 .2. *The adjoint of the operator $M_{s,e}$ in Z_e is*

$$M_{s,e}^* = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & -(-A_{\alpha,\beta})^{-1} A_3^* N_s & 0 \\ 0 & 0 & I+\gamma_s N_s & 0 \\ 0 & 0 & -\left(\int_{\Gamma_b} w_i \cdot e_2 N_s\right)_i & I \end{pmatrix}.$$

Moreover, we have

$$M_{s,e}^{-*} := (M_{s,e}^*)^{-1} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & (-A_{\alpha,\beta})^{-1} A_3^* N_s (I+\gamma_s N_s)^{-1} & 0 \\ 0 & 0 & (I+\gamma_s N_s)^{-1} & 0 \\ 0 & 0 & \left(\int_{\Gamma_b} w_i \cdot e_2 N_s (I+\gamma_s N_s)^{-1}\right)_i & I \end{pmatrix}.$$

Proof. Let $(v, \eta_1, \eta_2, \mathbf{g})$ and $(\phi, \xi_1, \xi_2, \mathbf{h})$ in Z_e. We have

$$\begin{aligned}(M_{s,e}(v,\eta_1,\eta_2,\mathbf{g}),(\phi,\xi_1,\xi_2,\mathbf{h})) \quad &= \quad (v,\phi)_{\mathbf{V}^0_{n,\Gamma_d}(\Omega)} + (\eta_1,\xi_1)_{H^2_0(\Gamma_s)} + ((I+\gamma_s N_s)\eta_2,\xi_2)_{L^2(\Gamma_s)} \\ &\quad + (\gamma_s N_{\mathrm{div}} A_3\eta_1,\xi_2)_{L^2(\Gamma_s)} + (\gamma_s N_c \mathbf{g},\xi_2)_{L^2(\Gamma_s)} + \mathbf{g}\cdot\mathbf{h}.\end{aligned}$$

Since $I + \gamma_s N_s \in \mathcal{L}(L^2(\Gamma_s))$ is symmetric, we have

$$((I+\gamma_s N_s)\eta_2,\xi_2)_{L^2(\Gamma_s)} = (\eta_2,(I+\gamma_s N_s)\xi_2)_{L^2(\Gamma_s)}.$$

We also have

$$(\gamma_s N_{\mathrm{div}} A_3\eta_1,\xi_2)_{L^2(\Gamma_s)} = \int_{\Gamma_s} N_{\mathrm{div}} A_3\eta_1 \frac{\partial N_s\xi_2}{\partial n} = -(A_3\eta_1, N_s\xi_2)_{L^2(\Gamma_s)} = (\eta_1, -(-A_{\alpha,\beta})^{-1}A_3^* N_s\xi_2)_{H^2_0(\Gamma_s)},$$

and

$$(\gamma_s N_c\mathbf{g},\xi_2)_{L^2(\Gamma_s)} = \int_{\Gamma_s} N_c\mathbf{g}\frac{\partial N_s\xi_2}{\partial n} = -\sum_{i=1}^{n_c} g_i \int_{\Gamma_b} w_i\cdot e_2 N_s\xi_2.$$

We deduce that

$$(M_{s,e}(v,\eta_1,\eta_2,\mathbf{g}),(\phi,\xi_1,\xi_2,\mathbf{h})) = \Big((v,\eta_1,\eta_2,\mathbf{g}), M^*_{s,e}(\phi,\xi_1,\xi_2,\mathbf{h})\Big).$$

Thus, the first part of the lemma is proved. The second part follows from Lemma 1.5 .3. □

Now, we introduce the unbounded operator $(\mathcal{A}_e, D(\mathcal{A}_e))$ in $Z_e = Z\times\mathbb{R}^{n_c}$ defined by

$$\begin{aligned}D(\mathcal{A}_e) = \Big\{&(Pv,\eta_1,\eta_2,\mathbf{g})\in \mathbf{V}^1_{n,\Gamma_d}(\Omega)\times(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))\times H^2_0(\Gamma_s)\times\mathbb{R}^{n_c} \mid \\ &Pv - PL(\eta_2 e_2\chi_{\Gamma_s}, A_3\eta_1) - PD_c\mathbf{g}\in D(A)\Big\}\end{aligned}$$

and

$$\mathcal{A}_e = M_{s,e}^{-1}\begin{pmatrix} A & PA_1 + (\lambda_0 I - A)PL(0,A_3) & PA_2 + (\lambda_0 I - A)PL(e_2\chi_{\Gamma_s},0) & (\lambda_0 I - A)PD_c \\ 0 & 0 & I & 0 \\ 0 & A_{\alpha,\beta} + \gamma_s N_p A_1 + A_4 & \gamma\Delta + \gamma_s N_p A_2 & 0 \\ 0 & 0 & 0 & \Lambda \end{pmatrix} + \mathcal{A}_{p,e}, \tag{1.6 .1}$$

with

$$\mathcal{A}_{p,e}\begin{pmatrix} Pv\\ \eta_1\\ \eta_2\\ \mathbf{g}\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ (I+\gamma_s N_s)^{-1}\gamma_s N_v(Pv + \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\mathrm{div}} A_3\eta_1)\\ 0\end{pmatrix}.$$

As in Lemma 1.5 .4, we can prove that if $(Pv, \eta_1, \eta_2, \mathbf{g})$ belongs to $D(\mathcal{A}_e)$ then $N_v(Pv + \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\mathrm{div}}A_3\eta_1)$ belongs to $H^{\frac{1}{2}+\varepsilon_0}(\Omega)$. Thus, the operator $\mathcal{A}_{p,e}$ is well-defined in $D(\mathcal{A}_e)$. The first equation of system (1.5 .11) may be rewritten as

$$\frac{d}{dt}\begin{pmatrix} Pv\\ \eta_1\\ \eta_2\\ \mathbf{g}\end{pmatrix} = (\mathcal{A}_e + \omega I)\begin{pmatrix} Pv\\ \eta_1\\ \eta_2\\ \mathbf{g}\end{pmatrix} + \mathcal{B}_e\mathbf{f}, \quad \begin{pmatrix} Pv\\ \eta_1\\ \eta_2\\ \mathbf{g}\end{pmatrix}(0) = \begin{pmatrix} Pv^0\\ \eta_1^0\\ \eta_2^0\\ \mathbf{0}\end{pmatrix}, \tag{1.6 .2}$$

where the operator $\mathcal{B}_e \in \mathcal{L}(\mathbb{R}^{n_c}, Z_e)$ is defined by

$$\mathcal{B}_e = (0,0,0,I_{\mathbb{R}^{n_c}})^T. \tag{1.6 .3}$$

1.6 .1 Properties of the unbounded operator $(\mathcal{A}_e, D(\mathcal{A}_e))$

In this section, we are going to prove that $(\mathcal{A}_e, D(\mathcal{A}_e))$ is the generator of an analytic semigroup on Z_e and its resolvent is compact.

1.6 .2 Analyticity

Theorem 1.6 .1. *The unbounded operator $(\mathcal{A}_e, D(\mathcal{A}_e))$ is the generator of an analytic semigroup of class $\mathcal{C}^0$ on Z_e.*

First, we introduce the unbounded operator $(\widehat{\mathcal{A}}_e, D(\widehat{\mathcal{A}}_e))$ in Z_e defined by $D(\widehat{\mathcal{A}}_e) = D(\mathcal{A}_e)$ and

$$\widehat{\mathcal{A}}_e = \begin{pmatrix} A & (\lambda_0 I - A)PL(0, A_3) & (\lambda_0 I - A)PL(e_2\chi_{\Gamma_s}, 0) & (\lambda_0 I - A)PD_c \\ 0 & 0 & I & 0 \\ 0 & A_{\alpha,\beta} & \gamma\Delta & 0 \\ 0 & 0 & 0 & \Lambda \end{pmatrix}.$$

We set

$$K_s := (I + \gamma_s N_s)^{-1}.$$

We decompose $\mathcal{A}_e$ in the form $\mathcal{A}_e = \widehat{\mathcal{A}}_e + \widehat{\mathcal{A}}_{e,1} + \widehat{\mathcal{A}}_{e,2} + \widehat{\mathcal{A}}_{e,3} + \mathcal{A}_{p,e}$, where the operators $\widehat{\mathcal{A}}_{e,1}$, $\widehat{\mathcal{A}}_{e,2}$ and $\widehat{\mathcal{A}}_{e,3}$ are defined in $D(\widehat{\mathcal{A}}_e)$ by

$$\widehat{\mathcal{A}}_{e,1} \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (K_s - I)A_{\alpha,\beta}\eta_1 \\ 0 \end{pmatrix}, \quad \widehat{\mathcal{A}}_{e,2} \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma(K_s - I)\Delta\eta_2 \\ 0 \end{pmatrix},$$

$$\widehat{\mathcal{A}}_{e,3} \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} PA_1\eta_1 + PA_2\eta_2 \\ 0 \\ K_s\gamma_s N_p A_1\eta_1 + K_s A_4\eta_1 + K_s\gamma_s N_p A_2\eta_2 - K_s\gamma_s N_{\mathrm{div}} A_3\eta_2 - K_s\gamma_s N_c\Lambda\mathbf{g} \\ 0 \end{pmatrix},$$

for all $(Pv, \eta_1, \eta_2, \mathbf{g}) \in D(\widehat{\mathcal{A}}_e)$. We are going to prove that $(\widehat{\mathcal{A}}_e, D(\widehat{\mathcal{A}}_e))$ is the generator of an analytic semigroup of class $\mathcal{C}^0$ on Z_e, and that $\widehat{\mathcal{A}}_{e,1}$, $\widehat{\mathcal{A}}_{e,2}$ and $\widehat{\mathcal{A}}_{e,3}$ are $\widehat{\mathcal{A}}_e$-bounded with relative bound equal to zero.

Theorem 1.6 .2. *The unbounded operator $(\widehat{\mathcal{A}}_e, D(\widehat{\mathcal{A}}_e))$ is the generator of an analytic semigroup of class $\mathcal{C}^0$ on Z_e.*

Proof. The proof is similar to that of ([51], Theorem 3.6).

□

Proposition 1.6 .3. *There exist $0 < \theta_1, \theta_2 < 1$ such that $\widehat{\mathcal{A}}_{e,1}$ and $\widehat{\mathcal{A}}_{e,2}$ belong to $\mathcal{L}(D((-\widehat{\mathcal{A}}_e)^{\theta_1}), Z_e)$ and $\mathcal{L}(D((-\widehat{\mathcal{A}}_e)^{\theta_2}), Z_e)$ respectively.*

Proof. See ([51], Lemma 3.9).

□

Lemma 1.6 .3. *There exists $C > 0$ such that, for all $(Pv, \eta_1, \eta_2, \mathbf{g}) \in D(\widehat{\mathcal{A}}_e)$, we have the estimate*

$$\|N_v(Pv + \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\mathrm{div}} A_3\eta_1)\|_{H^{\frac{1}{2}+\varepsilon_0}(\Omega)} \leq C\|(Pv, \eta_1, \eta_2, \mathbf{g})\|_{D(\widehat{\mathcal{A}}_e)}.$$

Proof. Let λ be positive and $(Pv, \eta_1, \eta_2, \mathbf{g})$ be in $D(\widehat{\mathcal{A}}_e)$. We set

$$\begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \\ \mathbf{F}_c \end{pmatrix} := (\lambda M_{s,e} - M_{s,e}\widehat{\mathcal{A}}_e) \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix}.$$

As in Proposition 1.6 .6, we can prove that $(v, p, \eta_1, \eta_2, \mathbf{g})$ with

$$\begin{aligned} v &= Pv + \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\mathrm{div}}A_3\eta_1, \\ p &= -\lambda N_s\eta_2 - \lambda N_{\mathrm{div}}A_3\eta_1 + N_v(Pv + \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\mathrm{div}}A_3\eta_1), \end{aligned}$$

is a solution of the system

$$\begin{aligned} &\lambda v - \mathrm{div}\,\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s = F_f \text{ in } \Omega, \\ &\mathrm{div}\, v = A_3\eta_1 \text{ in } \Omega, \quad v = \eta_2 e_2 \text{ on } \Gamma_s, \quad v = \textstyle\sum_{i=1}^{n_c} g_i w_i \text{ on } \Gamma_b, \\ &v = 0 \text{ on } \Gamma_i, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n, \\ &\lambda\eta_1 - \eta_2 = F_s^1 \text{ on } \Gamma_s, \\ &\lambda\eta_2 - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 = F_s^2 \text{ on } \Gamma_s, \\ &\eta_1 = 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial\Gamma_s, \\ &\lambda\mathbf{g} - \Lambda\mathbf{g} = \mathbf{F}_c. \end{aligned} \tag{1.6 .4}$$

As in the proof of Proposition 1.6 .7, we can show that, for λ large enough, system (1.6 .4) admits a unique solution $(v, p, \eta_1, \eta_2, \mathbf{g}) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega) \times H^4(\Gamma_s) \cap H^2_0(\Gamma_s) \times H^2_0(\Gamma_s) \times \mathbb{R}^{n_c}$ satisfying the estimate

$$\|v\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|p\|_{H^1_{\delta_0}(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} + \|\mathbf{g}\|_{\mathbb{R}^{n_c}} \le C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}.$$

From that inequality and Lemma 1.4 .1, we obtain

$$\|p\|_{H^{\frac{1}{2}+\varepsilon_0}(\Omega)} \le C\|p\|_{H^1_{\delta_0}(\Omega)} \le C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e} \le C\|(Pv, \eta_1, \eta_2, \mathbf{g})\|_{D(\widehat{\mathcal{A}}_e)}.$$

Using Proposition 1.5 .1, we deduce that

$$\|N_v(Pv + \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\mathrm{div}}A_3\eta_1)\|_{H^{\frac{1}{2}+\varepsilon_0}(\Omega)} \le C\|(Pv, \eta_1, \eta_2, \mathbf{g})\|_{D(\widehat{\mathcal{A}}_e)}.$$

□

Lemma 1.6 .4. *The operator K_s belongs to $\mathcal{L}(H^{\varepsilon_0}(\Gamma_s))$.*

Proof. Let g belong to $H^{\varepsilon_0}(\Gamma_s)$. Thanks to Lemma 1.5 .3, there exists a unique $h \in L^2(\Gamma_s)$ such that

$$h + \gamma_s N_s h = g.$$

Since $\gamma_s N_s h$ belongs to $H^{\varepsilon_0}(\Gamma_s)$, we deduce that h belongs to $H^{\varepsilon_0}(\Gamma_s)$. Thus, the operator $I + \gamma_s N_s$ is an automorphism in $H^{\varepsilon_0}(\Gamma_s)$.

□

Proposition 1.6 .4. *The operator $\mathcal{A}_{p,e} \in \mathcal{L}(D(\widehat{\mathcal{A}}_e), L^2(\Gamma_s))$ is compact.*

Proof. Thanks to Proposition 1.5 .1, Lemmas 1.6 .3 and 1.6 .4, we prove that $\mathcal{A}_{p,e}$ belongs to $\mathcal{L}(D(\widehat{\mathcal{A}}_e), H^{\varepsilon_0}(\Gamma_s))$. From the compact embedding $H^{\varepsilon_0}(\Gamma_s) \hookrightarrow L^2(\Gamma_s)$, it follows that $\mathcal{A}_{p,e} \in \mathcal{L}(D(\widehat{\mathcal{A}}_e), L^2(\Gamma_s))$ is a compact operator.

□

Proposition 1.6 .5. *The operators $\widehat{\mathcal{A}}_{e,3}$ is $\widehat{\mathcal{A}}_e$-bounded with relative bound zero.*

Proof. We have to show that, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left\|\widehat{\mathcal{A}}_{e,3}(Pv,\eta_1,\eta_2,\mathbf{g})\right\|_{Z_e} \leq \varepsilon \left\|\widehat{\mathcal{A}}_e(Pv,\eta_1,\eta_2,\mathbf{g})\right\|_{Z_e} + C_\varepsilon \left\|(Pv,\eta_1,\eta_2,\mathbf{g})\right\|_{Z_e}, \quad \forall (Pv,\eta_1,\eta_2,\mathbf{g}) \in D(\widehat{\mathcal{A}}_e).$$

We argue by contradiction. We assume that there exist $\varepsilon > 0$ and a sequence $(Pv_k, \eta_{1,k}, \eta_{2,k}, \mathbf{g}_k) \in D(\widehat{\mathcal{A}}_e)$ such that

$$\left\|\widehat{\mathcal{A}}_{e,3}(Pv_k,\eta_{1,k},\eta_{2,k},\mathbf{g}_k)\right\|_{Z_e} \geq \varepsilon \left\|\widehat{\mathcal{A}}_e(Pv_k,\eta_{1,k},\eta_{2,k},\mathbf{g}_k)\right\|_{Z_e} + k \left\|(Pv_k,\eta_{1,k},\eta_{2,k},\mathbf{g}_k)\right\|_{Z_e}.$$

Therefore, without loss of generality, we can assume that

$$\left\|\widehat{\mathcal{A}}_{e,3}(Pv_k,\eta_{1,k},\eta_{2,k},\mathbf{g}_k)\right\|_{Z_e} = 1 \quad \text{and} \quad (Pv_k,\eta_{1,k},\eta_{2,k},\mathbf{g}_k) \longrightarrow 0 \text{ in } Z_e.$$

With Proposition 1.5 .1 and Lemma 1.6 .4, we have

$$\begin{aligned}\left\|\widehat{\mathcal{A}}_{e,3}(Pv_k,\eta_{1,k},\eta_{2,k},\mathbf{g}_k)\right\|_{Z_e} &= \|PA_1\eta_{1,k} + PA_2\eta_{2,k}\|_{\mathbf{V}^0_{n,\Gamma_0}(\Omega)} \\ &\quad + \|K_s\gamma_s N_p A_1\eta_{1,k} + K_s A_4\eta_{1,k} + K_s\gamma_s N_p A_2\eta_{2,k} - K_s\gamma_s N_{\mathrm{div}} A_3\eta_{2,k} - K_s\gamma_s N_c \Lambda \mathbf{g}_k\|_{L^2(\Gamma_s)} \\ &\leq C\|(\eta_{1,k},\eta_{2,k},\mathbf{g}_k)\|_{H^2(\Gamma_s)\times L^2(\Gamma_s)\times \mathbb{R}^{n_c}}.\end{aligned}$$

It follows that

$$\left\|\widehat{\mathcal{A}}_{e,3}(Pv_k,\eta_{1,k},\eta_{2,k},\mathbf{g}_k)\right\|_{Z_e} \longrightarrow 0,$$

which is in contradiction with

$$\left\|\widehat{\mathcal{A}}_{e,3}(Pv_k,\eta_{1,k},\eta_{2,k},\mathbf{g}_k)\right\|_{Z_e} = 1.$$

□

1.6 .3 Resolvent of $\mathcal{A}_e$ in Z_e.

The goal is to prove that the unbounded operator $(\mathcal{A}_e, D(\mathcal{A}_e)$ has a compact resolvent. First, we consider the system

$$\begin{aligned}&\lambda v - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 = F_f \text{ in } \Omega,\\ &\operatorname{div} v = 0 \text{ in } \Omega, \quad v = \eta_2 e_2 \text{ on } \Gamma_s, \quad v = \textstyle\sum_{i=1}^{n_c} g_i w_i \text{ on } \Gamma_b,\\ &v = A_3\eta_1 \text{ on } \Gamma_i, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n,\\ &\lambda\eta_1 - \eta_2 = F_s^1 \text{ on } \Gamma_s,\\ &\lambda\eta_2 - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 - A_4\eta_1 = \gamma_s p + F_s^2 \text{ on } \Gamma_s,\\ &\eta_1 = 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial\Gamma_s,\\ &\lambda\mathbf{g} - \Lambda\mathbf{g} = \mathbf{F}_c.\end{aligned} \tag{1.6 .5}$$

Proposition 1.6 .6. *Let $(F_f, F_s^1, F_s^2, \mathbf{F}_c)$ belong to Z_e and λ belong to the resolvent set of $\mathcal{A}_e$. The quadruplet $(v, p, \eta_1, \eta_2, \mathbf{g}) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega) \times (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s)$ is solution of system (1.6 .5) if and only if*

$$\begin{aligned}
&(\lambda M_{s,e} - M_{s,e}\mathcal{A}_e) \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \\ \mathbf{F}_c \end{pmatrix}, \\
&(I-P)v = \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1, \\
&p = -\lambda N_s \eta_2 - \lambda N_{\mathrm{div}} A_3 \eta_1 + N_p A_1 \eta_1 + N_p A_2 \eta_2 + N_v(Pv + \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1).
\end{aligned} \tag{1.6 .6}$$

Proof. It follows from Theorem 1.4 .8.

□

Proposition 1.6 .7. *Let $(F_f, F_s^1, F_s^2, \mathbf{F}_c)$ belong to Z_e and λ belong to the resolvent set of $\mathcal{A}_e$. If λ is large enough then system (1.6 .5) admits a unique solution $(v, p, \eta_1, \eta_2, \mathbf{g}) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega) \times (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s)$ with the estimate*

$$\|v\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|p\|_{H^1_{\delta_0}(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} + \|\mathbf{g}\|_{\mathbb{R}^{n_c}} \le C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}.$$

Proof. We proceed in several steps.
Step 1: System with divergence free.
We are going to rewrite system (1.6 .5) in the form of a system with divergence free. For that, we make the change of unknowns

$$\hat{v} = v - L\left(\sum_{i=1}^{n_c} g_i w_i, A_3\eta_1\right) \quad \text{and} \quad \hat{p} = p - L_p\left(\sum_{i=1}^{n_c} g_i w_i, A_3\eta_1\right). \tag{1.6 .7}$$

We have

$$\hat{v} = \eta_2 e_2 \text{ on } \Gamma_s,$$

which gives

$$\eta_2 = \hat{v}_2 \text{ on } \Gamma_s \quad \text{and} \quad \eta_1 = \frac{1}{\lambda}\hat{v}_2 + \frac{1}{\lambda}F_s^1 \text{ on } \Gamma_s.$$

We introduce the operator L_λ in $H^2_0(\Gamma_s)$ defined by

$$L_\lambda = \lambda^2 - (\beta + \gamma\lambda)\Delta + \alpha\Delta^2 - A_4 - \gamma_s L_p(0, A_3).$$

Thanks to the Lax-Milgram Theorem and Proposition 1.5 .1, we can prove that, for λ large enough, L_λ is an isomorphism from $H^2_0(\Gamma_s)$ into $H^{-2}(\Gamma_s)$. The operator L_λ is also an isomorphism from $H^4(\Gamma_s) \cap H^2_0(\Gamma_s)$ into $L^2(\Gamma_s)$. The system satisfied by $(\hat{v}, \hat{p}, \eta_1, \eta_2)$ can be rewritten in the form

$$\begin{aligned}
&\lambda\hat{v} - \operatorname{div}\sigma(\hat{v}, \hat{p}) + (u_s \cdot \nabla)\hat{v} + (\hat{v} \cdot \nabla)u_s - A_5\hat{v} = G_f \text{ in } \Omega, \\
&\operatorname{div}\hat{v} = 0 \text{ in } \Omega, \quad \hat{v} = \lambda L_\lambda^{-1}\gamma_s\hat{p}e_2 + \zeta e_2 \text{ on } \Gamma_s, \\
&\hat{v} = 0 \text{ on } \Gamma_b, \quad \hat{v} = 0 \text{ on } \Gamma_i, \quad \sigma(\hat{v}, \hat{p})n = 0 \text{ on } \Gamma_n, \\
&\lambda\eta_1 - \eta_2 = \xi_1 \text{ on } \Gamma_s, \\
&L_\lambda\eta_1 = \gamma_s\hat{p} + G_s \text{ on } \Gamma_s, \\
&\eta_1 = 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial\Gamma_s, \\
&\lambda\mathbf{g} - \Lambda\mathbf{g} = \mathbf{F}_c,
\end{aligned} \tag{1.6 .8}$$

where

$$A_5\hat{v} = \frac{1}{\lambda}A_1\gamma_s\hat{v}_2 + A_2\gamma_s\hat{v}_2 + L(0, A_3\gamma_s\hat{v}_2),$$
$$G_f = F + \frac{1}{\lambda}A_1F_s^1 + L(0, A_3F_s^1) - \lambda L\left(\sum_{i=1}^{n_c} g_iw_i, 0\right),$$
$$G_s = \gamma_sL_p\left(\sum_{i=1}^{n_c} g_iw_i, 0\right) + F_s^2 + \lambda F_s^1 - \gamma\Delta F_s^1,$$
$$\zeta = \lambda L_\lambda^{-1}G_s - F_s^1.$$

From Theorem 1.4 .2 and Proposition 1.5 .1, it follows that there exists $C > 0$ such that

$$\|A_5\hat{v}\|_{\mathbf{L}^2(\Omega)} \le C\|\gamma_sv_2\|_{H_0^2(\Gamma_s)},$$

and

$$\|G_f\|_{\mathbf{L}^2(\Omega)} + \|G_s\|_{L^2(\Gamma_s)} + \|\zeta\|_{H_0^2(\Gamma_s)} \le C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}.$$

Step 2: Existence of weak solutions for system (1.6 .8).
First, we consider the first three equations of system (1.6 .8)

$$\begin{aligned}
&\lambda\hat{v} - \operatorname{div}\sigma(\hat{v},\hat{p}) + (u_s\cdot\nabla)\hat{v} + (\hat{v}\cdot\nabla)u_s - A_5\hat{v} = G_f \text{ in } \Omega,\\
&\operatorname{div}\hat{v} = 0 \text{ in } \Omega, \quad \hat{v} = \lambda L_\lambda^{-1}(\gamma_s\hat{p})e_2 + \zeta e_2 \text{ on } \Gamma_s,\\
&\hat{v} = 0 \text{ on } \Gamma_b, \quad \hat{v} = 0 \text{ on } \Gamma_i, \quad \sigma(\hat{v},\hat{p})n = 0 \text{ on } \Gamma_n.
\end{aligned} \tag{1.6 .9}$$

We introduce the space

$$\mathbf{E} = \left\{\hat{v} \in \mathbf{V}^1(\Omega) \mid \hat{v} = 0 \text{ on } \Gamma_b\cup\Gamma_i,\ \hat{v}_1 = 0 \text{ on } \Gamma_s,\ \hat{v}_2|_{\Gamma_s} \in H_0^2(\Gamma_s)\right\},$$

equipped with the norm

$$\|\hat{v}\|_{\mathbf{E}} = \left(\|\hat{v}\|_{\mathbf{V}^1(\Omega)} + \|L_\lambda^{\frac{1}{2}}\hat{v}_2|_{\Gamma_s}\|_{L^2(\Gamma_s)}\right)^{\frac{1}{2}}.$$

Multiplying the first equation of (1.6 .9) by $\phi \in \mathbf{E}$, and after some integration by parts, we obtain

$$\lambda\int_\Omega \hat{v}\cdot\phi + 2\nu\int_\Omega \varepsilon(\hat{v}):\varepsilon(\phi) + \int_\Omega((u_s\cdot\nabla)\hat{v} + (\hat{v}\cdot\nabla)u_s)\cdot\phi - \int_\Omega A_5\hat{v}\cdot\phi + \int_{\Gamma_s}\hat{p}\phi_2 = \int_\Omega G_f\cdot\phi.$$

Using the equation

$$\lambda\gamma_s\hat{p} = L_\lambda\hat{v}_2 - L_\lambda\zeta \text{ in } H^{-2}(\Gamma_s),$$

we obtain

$$\begin{aligned}
&\lambda\int_\Omega \hat{v}\cdot\phi + 2\nu\int_\Omega \varepsilon(\hat{v}):\varepsilon(\phi) + \int_\Omega((u_s\cdot\nabla)\hat{v} + (\hat{v}\cdot\nabla)u_s)\cdot\phi - \int_\Omega A_5\hat{v}\cdot\phi + \frac{1}{\lambda}\int_{\Gamma_s}L_\lambda^{\frac{1}{2}}\hat{v}_2L_\lambda^{\frac{1}{2}}\phi_2\\
&\quad = \int_\Omega G_f\cdot\phi + \frac{1}{\lambda}\int_{\Gamma_s}L_\lambda^{\frac{1}{2}}\zeta L_\lambda^{\frac{1}{2}}\phi_2.
\end{aligned}$$

We set

$$a(\hat{v},\phi) = \lambda\int_\Omega \hat{v}\cdot\phi + 2\nu\int_\Omega \varepsilon(\hat{v}):\varepsilon(\phi) + \int_\Omega((u_s\cdot\nabla)\hat{v} + (\hat{v}\cdot\nabla)u_s)\cdot\phi - \int_\Omega A_5\hat{v}\cdot\phi + \frac{1}{\lambda}\int_{\Gamma_s}L_\lambda^{\frac{1}{2}}\hat{v}_2L_\lambda^{\frac{1}{2}}\phi_2,$$

and

$$l(\phi) = \int_\Omega G_f\cdot\phi + \frac{1}{\lambda}\int_{\Gamma_s}L_\lambda^{\frac{1}{2}}\zeta L_\lambda^{\frac{1}{2}}\phi_2.$$

Thanks to the Lax-Milgram Theorem, we prove that $\hat{v} \in \mathbf{E}$ is the unique solution of the variational problem

$$\text{Find } \hat{v} \in \mathbf{E} \text{ such that } a(\hat{v}, \phi) = l(\phi), \quad \forall \phi \in \mathbf{E},$$

and we have the estimate

$$\|\hat{v}\|_{\mathbf{E}} \leq C\left(\|G_f\|_{\mathbf{L}^2(\Omega)} + \|L_\lambda^{\frac{1}{2}}\zeta\|_{L^2(\Gamma_s)}\right) \leq C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}.$$

Step 3: Existence of strong solutions for system (1.6 .8).
The pair $(\hat{v}, \hat{p})$ is the solution of the system

$$\begin{aligned}
&\lambda_0\hat{v} - \operatorname{div}\sigma(\hat{v}, \hat{p}) + (u_s \cdot \nabla)\hat{v} + (\hat{v} \cdot \nabla)u_s = (\lambda_0 - \lambda)\hat{v} + G_f - A_5\hat{v} \text{ in } \Omega,\\
&\operatorname{div}\hat{v} = 0 \text{ in } \Omega, \quad \hat{v} = \hat{v}_2 e_2 \text{ on } \Gamma_s, \quad \hat{v} = 0 \text{ on } \Gamma_b,\\
&\hat{v} = 0 \text{ on } \Gamma_i, \quad \sigma(\hat{v}, \hat{p})n = 0 \text{ on } \Gamma_n,
\end{aligned} \tag{1.6 .10}$$

Since $\hat{v}$ belongs to $\mathbf{V}^1(\Omega)$, $(\lambda_0 - \lambda)\hat{v} + G_f - A_5\hat{v}$ belongs to $\mathbf{L}^2(\Omega)$ and $\hat{v}_2 e_2$ belongs to $\mathbf{H}_0^{\frac{3}{2}}(\Gamma_s)$. Thus, from Theorem 1.4 .2, it follows that $(\hat{v}, \hat{p})$ belongs to $\mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ and

$$\|\hat{v}\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|\hat{p}\|_{H^1_{\delta_0}(\Omega)} \leq C\|(\lambda_0 - \lambda)\hat{v} + G_f - A_5\hat{v}\|_{\mathbf{L}^2(\Omega)} + \|\hat{v}_2\|_{H_0^{\frac{3}{2}}(\Gamma_s)} \leq C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}. \tag{1.6 .11}$$

Moreover, from the last four equations of system (1.6 .8), we get

$$\|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} + \|\mathbf{g}\|_{\mathbb{R}^{n_c}} \leq C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}. \tag{1.6 .12}$$

We deduce that system (1.6 .8) admits a unique solution $(\hat{v}, \hat{p}, \eta_1, \eta_2, \mathbf{g}) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega) \times (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s)$ with the estimate

$$\|\hat{v}\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|\hat{p}\|_{H^1_{\delta_0}(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} + \|\mathbf{g}\|_{\mathbb{R}^{n_c}} \leq C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}. \tag{1.6 .13}$$

Step 4: Existence of strong solutions for system (1.6 .5).
According to Theorem 1.4 .2, $\left(L\left(\sum_{i=1}^{n_c} g_i w_i, A_3\eta_1\right), L_p\left(\sum_{i=1}^{n_c} g_i w_i, A_3\eta_1\right)\right)$ belongs to $\mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ and we have

$$\left\|L\left(\sum_{i=1}^{n_c} g_i w_i, A_3\eta_1\right)\right\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \left\|L_p\left(\sum_{i=1}^{n_c} g_i w_i, A_3\eta_1\right)\right\|_{H^1_{\delta_0}(\Omega)} \leq C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}. \tag{1.6 .14}$$

Thus, thanks to (1.6 .7) and (1.6 .13), system (1.6 .15) admits a unique solution (v, p, η_1, η_2) in

$$\mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega) \times \left(H^4(\Gamma_s) \cap H^2_0(\Gamma_s)\right) \times H^2_0(\Gamma_s),$$

and

$$\|v\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|p\|_{H^1_{\delta_0}(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} + \|\mathbf{g}\|_{\mathbb{R}^{n_c}} \leq C\|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}.$$

□

Theorem 1.6 .8. *The unbounded operator $(\mathcal{A}_e, D(\mathcal{A}_e))$ has a compact resolvent.*

Proof. Let $\lambda > 0$ belong to the resolvent of $\mathcal{A}_e$ and let $(F_f, F_s^1, F_s^2, \mathbf{F}_c)$ belong to Z_e. There exists $(Pv, \eta_1, \eta_2, \mathbf{g}) \in Z_e$ such that

$$(\lambda I - \mathcal{A}_e)\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = M_{s,e}^{-1}\begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \\ \mathbf{F}_c \end{pmatrix}.$$

Thanks to Proposition 1.6 .6, $(v, p, \eta_1, \eta_2, \mathbf{g})$ with

$$\begin{aligned} & v = Pv + \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1, \\ & p = -\lambda N_s \eta_2 - \lambda N_{\mathrm{div}} A_3 \eta_1 + N_p A_1 \eta_1 + N_p A_2 \eta_2 + N_v (Pv + \nabla N_s \eta_2 + \nabla N_c \mathbf{g} + \nabla N_{\mathrm{div}} A_3 \eta_1), \end{aligned}$$

is solution of the system

$$\begin{aligned} & \lambda v - \operatorname{div}\sigma(v,p) + (u_s \cdot \nabla) v + (v \cdot \nabla) u_s - A_1 \eta_1 - A_2 \eta_2 = F_f \text{ in } \Omega, \\ & \operatorname{div} v = A_3 \eta_1 \text{ in } \Omega, \quad v = \eta_2 e_2 \text{ on } \Gamma_s, \quad v = \textstyle\sum_{i=1}^{n_c} g_i w_i \text{ on } \Gamma_b, \\ & v = 0 \text{ on } \Gamma_i, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n, \\ & \lambda \eta_1 - \eta_2 = F_s^1 \text{ on } \Gamma_s, \\ & \lambda \eta_2 - \beta \Delta \eta_1 - \gamma \Delta \eta_2 + \alpha \Delta^2 \eta_1 - A_4 \eta_1 = \gamma_s p + F_s^2 \text{ on } \Gamma_s, \\ & \eta_1 = 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial\Gamma_s, \\ & \lambda \mathbf{g} - \Lambda \mathbf{g} = \mathbf{F}_c. \end{aligned} \tag{1.6 .15}$$

From Proposition (1.6 .7), $(v, p, \eta_1, \eta_2, \mathbf{g})$ belongs to $\mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega) \times (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s)$ and

$$\|v\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|p\|_{H^1_{\delta_0}(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} + \|\mathbf{g}\|_{\mathbb{R}^{n_c}} \le C \|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}.$$

From Lemma 1.4 .1 and Proposition 1.4 .3, it follows that $(Pv, \eta_1, \eta_2, \mathbf{g})$ belongs to

$$\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega) \times \Big(H^4(\Gamma_s) \cap H^2_0(\Gamma_s)\Big) \times H^2_0(\Gamma_s) \times \mathbb{R}^{n_c},$$

and

$$\|Pv\|_{\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} + \|\mathbf{g}\|_{\mathbb{R}^{n_c}} \le C \|(F_f, F_s^1, F_s^2, \mathbf{F}_c)\|_{Z_e}.$$

Thus, the resolvent of $\mathcal{A}_e$ is compact in Z_e because the embedding

$$\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega) \times \Big(H^4(\Gamma_s) \cap H^2_0(\Gamma_s)\Big) \times H^2_0(\Gamma_s) \times \mathbb{R}^{n_c} \hookrightarrow Z_e,$$

is compact.

$\square$

1.6 .4 Stabilization of the pair $(\mathcal{A}_e + \omega I, \mathcal{B}_e)$

Let $(\lambda_j)_{j \in \mathbb{N}^*}$ be the eigenvalues $\mathcal{A}$ and let J_u be the finite subset of N^* such that

$$\operatorname{Re} \lambda_j \ge -\omega, \quad \forall\, j \in J_u.$$

We consider the eigenvalue problem

$$
\begin{aligned}
&\lambda \text{ in } \mathbb{C},\ (\phi,\psi,\xi_1,\xi_2)\in \mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)\times \big(H^4(\Gamma_s)\cap H^2_0(\Gamma_s)\big)\times H^2_0(\Gamma_s),\\
&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0 \text{ in } \Omega,\\
&\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = \xi_2 e_2 \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_b,\\
&\phi = 0 \text{ on } \Gamma_i, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \text{ on } \Gamma_n,\\
&\lambda\xi_1 + \xi_2 - (-A_{\alpha,\beta})^{-1}A_4^*\xi_2 - (-A_{\alpha,\beta})^{-1}A_1^*\phi + (-A_{\alpha,\beta})^{-1}A_3^*\psi = 0 \text{ on } \Gamma_s,\\
&\lambda\xi_2 + \beta\Delta\xi_1 - \gamma\Delta\xi_2 - \alpha\Delta^2\xi_1 - A_2^*\phi - \gamma_s\psi = 0 \text{ on } \Gamma_s,\\
&\xi_1 = 0 \text{ on } \partial\Gamma_s, \quad \xi_{1,x} = 0 \text{ on } \partial\Gamma_s.
\end{aligned}
\tag{1.6 .16}
$$

We introduce the spaces $(E(\lambda_j))_{j\in J_u}$ defined by

$$
\begin{aligned}
E(\lambda_j) = \{&(\phi,\psi,\xi_1,\xi_2)\in \mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)\times \big(H^4(\Gamma_s)\cap H^2_0(\Gamma_s)\big)\times H^2_0(\Gamma_s)\ |\\
&(\lambda_j,(\phi,\psi,\xi_1,\xi_2)) \text{ is solution of the eigenvalue problem (1.6 .16)}\}.
\end{aligned}
$$

Let m belong to $C^2(\Gamma_b;[0,1])$ such that

$$
m = 1 \text{ on } \Gamma_c^+ \quad \text{and} \quad m = 0 \text{ on } \Gamma_b\setminus\Gamma_c,
$$

where Γ_c is a non empty open set of Γ_b and Γ_c^+ is a non empty open set of Γ_c. The function m is used to localize the action of the control in Γ_c. We set

$$
\begin{aligned}
W = \operatorname{Vect}\Big\{&m\mathrm{Re}\left(\sigma(\phi_j^k,\psi_j^k)e_2\right),\ m\mathrm{Im}\left(\sigma(\phi_j^k,\psi_j^k)e_2\right),\\
&m\mathrm{Re}(N_s\xi_{2,j}^k e_2),\ m\mathrm{Im}(N_s\xi_{2,j}^k e_2)\ |\ j\in J_u,\ 1\le k\le d_j\Big\},
\end{aligned}
\tag{1.6 .17}
$$

where $(\phi_j^k,\psi_j^k,\xi_{1,j}^k,\xi_{2,j}^k)_{1\le k\le d_j}$ is a basis of $E(\lambda_j)$. The number d_j is the dimension of $E(\lambda_j)$. It also corresponds to the geometrical multiplicity of the eigenvalue λ_j of $\mathcal{A}$. We choose the family $(w_i)_{1\le i\le n_c}$ such that

$$
\operatorname{Vect}\{(w_i)_{1\le i\le n_1}\} = W \quad \text{and} \quad \operatorname{Vect}\{(w_i)_{n_1+1\le i\le n_c}\} = W. \tag{1.6 .18}
$$

Assumption 1.6 .1. *We assume that*

$$
\Lambda = \operatorname{diag}(\alpha_1 I_{\mathbb{R}^{n_1}},\alpha_2 I_{\mathbb{R}^{n_2}}) \quad \textit{and} \quad \alpha_1,\ \alpha_2\notin\mathcal{A}, \tag{1.6 .19}
$$

with $\alpha_1\neq\alpha_2$ and $n_1+n_2=n_c$. We also assume that $-\omega\notin\mathcal{A}_e$.

Theorem 1.6 .9. *We assume that 1.4 .1 and 1.6 .1 are satisfied, and that $(w_i)_{1\le i\le n_c}$ is given by (1.6 .18). Then, the pair $(\mathcal{A}_e+\omega I,\mathcal{B}_e)$ is stabilizable.*

Proof. Thanks to ([10], Part III, Chapter 1, Proposition 3.3), the pair $(\mathcal{A}_e+\omega I,\mathcal{B}_e)$ is stabilizable if and only if

$$
\operatorname{Ker}(\lambda I-\mathcal{A}_e^*)\cap\operatorname{Ker}(\mathcal{B}_e^*) = \{0\} \text{ for all } \lambda\in\mathbb{C} \text{ such that } \mathrm{Re}\,\lambda\ge-\omega.
$$

Let $(P\phi,\xi_1,\xi_2,\mathbf{h})$ belong to $\operatorname{Ker}(\lambda I-\mathcal{A}_e^*)\cap\operatorname{Ker}(\mathcal{B}_e^*)$. We set

$$
\begin{pmatrix} P\tilde\phi\\ \tilde\xi_1\\ \tilde\xi_2\\ \tilde{\mathbf{h}}\end{pmatrix} := M_{s,e}^{-*}\begin{pmatrix} P\phi\\ \xi_1\\ \xi_2\\ \mathbf{h}\end{pmatrix} = \begin{pmatrix} P\phi\\ \xi_1+(-A_{\alpha,\beta})^{-1}A_3^*N_s(I+\gamma_sN_s)^{-1}\xi_2\\ (I+\gamma_sN_s)^{-1}\xi_2\\ \left(\int_{\Gamma_b} w_i\cdot N_s(I+\gamma_sN_s)^{-1}\xi_2\right)+\mathbf{h}\end{pmatrix}. \tag{1.6 .20}
$$

Since $(P\phi, \xi_1, \xi_2, \mathbf{h})$ belongs to $\mathrm{Ker}(\lambda I - \mathcal{A}_e^*)$ then, from Propositions 1.A.2 and 1.A.1, it follows that $(\tilde{\phi}, \tilde{\psi}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\mathbf{h}})$, with

$$\begin{aligned} \tilde{\phi} &= P\tilde{\phi} + \nabla N_s \tilde{\xi}_2, \\ \tilde{\psi} &= -\lambda N_s \tilde{\xi}_2 + N_\phi \tilde{\phi}, \end{aligned} \tag{1.6 .21}$$

is solution of the system

$$\begin{aligned}
&\lambda\tilde{\phi} - \operatorname{div}\sigma(\tilde{\phi}, \tilde{\psi}) - (u_s \cdot \nabla)\tilde{\phi} + (\nabla u_s)^T \tilde{\phi} = 0 \text{ in } \Omega, \\
&\operatorname{div}\tilde{\phi} = 0 \text{ in } \Omega, \quad \tilde{\phi} = \tilde{\xi}_2 e_2 \text{ on } \Gamma_s, \quad \tilde{\phi} = 0 \text{ on } \Gamma_b, \\
&\tilde{\phi} = 0 \text{ on } \Gamma_i, \quad \sigma(\tilde{\phi}, \tilde{\psi})n + u_s \cdot n\tilde{\phi} = 0 \text{ on } \Gamma_n, \\
&\lambda\tilde{\xi}_1 + \tilde{\xi}_2 - (-A_{\alpha,\beta})^{-1} A_4^* \tilde{\xi}_2 - (-A_{\alpha,\beta})^{-1} A_1^* \tilde{\phi} + (-A_{\alpha,\beta})^{-1} A_3^* \tilde{\psi} = 0 \text{ on } \Gamma_s, \\
&\lambda\tilde{\xi}_2 + \beta\Delta\tilde{\xi}_1 - \gamma\Delta\tilde{\xi}_2 - \alpha\Delta^2\tilde{\xi}_1 - A_2^*\tilde{\phi} - \gamma_s\tilde{\psi} = 0 \text{ on } \Gamma_s, \\
&\tilde{\xi}_1 = 0 \text{ on } \partial\Gamma_s, \quad \tilde{\xi}_{1,x} = 0 \text{ on } \partial\Gamma_s, \\
&\lambda\tilde{\mathbf{h}} - \Lambda\tilde{\mathbf{h}} - \left(\int_{\Gamma_b} \sigma(\tilde{\phi}, \tilde{\psi}) e_2 \cdot w_i \right)_i = 0.
\end{aligned} \tag{1.6 .22}$$

Moreover, as in Proposition 1.A.1, we prove that

$$\lambda M_s^* \begin{pmatrix} P\tilde{\phi} \\ \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix} - \mathcal{A}^* M_s^* \begin{pmatrix} P\tilde{\phi} \\ \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix} = 0, \tag{1.6 .23}$$

where $\mathcal{A}^*$ (resp. M_s^*) is the adjoint of $\mathcal{A}$ (resp. M_s) in Z. From the equation

$$\mathcal{B}_e^*(P\phi, \xi_1, \xi_2, \mathbf{h}) = \mathbf{h} = 0,$$

we deduce that

$$\tilde{\mathbf{h}} - \left(\int_{\Gamma_b} N_s \tilde{\xi}_2 e_2 \cdot w_i \right)_{1 \le i \le n_c} = 0. \tag{1.6 .24}$$

Now, we are going to show that $(\tilde{\phi}, \tilde{p}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\mathbf{h}}) = (0,0,0,0,0)$. Since the spectrum of $\mathcal{A}$ and Λ are disjoint, we can distinguish two cases.

Case 1 : $\lambda \notin \sigma(\mathcal{A})$. From equation (1.6 .23), it follows that $(P\tilde{\phi}, \tilde{\xi}_1, \tilde{\xi}_2) = (0,0,0)$. Thanks to equation (1.6 .21), we also have $(\tilde{\phi}, \tilde{\psi}) = (0,0)$. Finally, from equation $(1.6\ .22)_7$, we get $\tilde{\mathbf{h}} = 0$. Thus, we have $(\tilde{\phi}, \tilde{p}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\mathbf{h}}) = (0,0,0,0,0)$.

Case 2 : $\lambda \in \sigma(\mathcal{A})$. Then, we can solve the last equation of system (1.6 .22)

$$\begin{aligned}
\tilde{\mathbf{h}} &= (\lambda I - \Lambda)^{-1} \left(\int_{\Gamma_b} \sigma(\tilde{\phi}, \tilde{\psi}) e_2 \cdot w_i \right)_{1 \le i \le n_c} \\
&= \operatorname{diag}\left(\frac{1}{\lambda - \alpha_1} I_{\mathbb{R}^{n_1}}, \frac{1}{\lambda - \alpha_2} I_{\mathbb{R}^{n_2}} \right) \left(\int_{\Gamma_b} \sigma(\tilde{\phi}, \tilde{\psi}) e_2 \cdot w_i \right)_{1 \le i \le n_c}.
\end{aligned}$$

Thus, from equation (1.6 .24), we deduce that

$$\int_{\Gamma_b} \Big((\lambda - \alpha_1) N_s \tilde{\xi}_2 - \sigma(\tilde{\phi}, \tilde{\psi}) \Big) e_2 \cdot w_i = 0 \quad \forall\, 1 \le i \le n_1,$$

and

$$\int_{\Gamma_b} \Big((\lambda - \alpha_2) N_s \tilde{\xi}_2 - \sigma(\tilde{\phi}, \tilde{\psi}) \Big) e_2 \cdot w_i = 0 \quad \forall\, n_1 + 1 \le i \le n_c.$$

Thanks to the construction of $(w_i)_{1\leq i\leq n_c}$, we have

$$\Big((\lambda-\alpha_1)N_s\tilde{\xi}_2-\sigma(\tilde{\phi},\tilde{\psi})\Big)\,e_2=0 \text{ on } \Gamma_c^+ \quad \text{and} \quad \Big((\lambda-\alpha_2)N_s\tilde{\xi}_2-\sigma(\tilde{\phi},\tilde{\psi})\Big)\,e_2=0 \text{ on } \Gamma_c^+.$$

It follows that

$$(\alpha_1-\alpha_2)N_s\tilde{\xi}_2=0 \text{ on } \Gamma_c^+.$$

Since $\alpha_1\neq\alpha_2$, the above equality implies that

$$N_s\tilde{\xi}_2=0 \text{ on } \Gamma_c^+.$$

We deduce that

$$\tilde{\xi}_2=0 \text{ on } \Gamma_s \quad \text{and} \quad \sigma(\tilde{\phi},\tilde{\psi})e_2=0 \text{ on } \Gamma_c^+$$

Thus, thanks to the unique continuation property of Oseen system ([61], Theorem 1.3), it follows that $(\phi,\psi)=(0,0)$. From equation $(1.6\ .22)_7$, we obtain $\tilde{\mathbf{h}}=0$. Finally, $\tilde{\xi}_1$ is a solution of the equation

$$\begin{aligned}
&\beta\Delta\tilde{\xi}_1-\alpha\Delta^2\tilde{\xi}_1=0 \text{ on } \Gamma_s,\\
&\tilde{\xi}_1=0 \text{ on } \partial\Gamma_s, \quad \tilde{\xi}_{1,x}=0 \text{ on } \partial\Gamma_s,
\end{aligned}$$

implying that $\tilde{\xi}_1=0$. Thus, we have $(\tilde{\phi},\tilde{p},\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\mathbf{h}})=(0,0,0,0,0)$.

□

Now, we are going to construct a feedback law able to stabilize the pair $(\mathcal{A}_e+\omega I,\mathcal{B}_e)$. For that, we consider the following algebraic Riccati equation

$$\begin{aligned}
&\mathcal{P}_\omega\in\mathcal{L}(Z_e), \quad \mathcal{P}_\omega=\mathcal{P}_\omega^*>0,\\
&\mathcal{P}_\omega(\mathcal{A}_e+\omega I)+(\mathcal{A}_e+\omega I)^*\mathcal{P}_\omega-\mathcal{P}_\omega\mathcal{B}_e\mathcal{B}_e^*\mathcal{P}_\omega=0.
\end{aligned}$$

Theorem 1.6 .10. *We assume that 1.4 .1 and 1.6 .1 are satisfied. Then, the operator $\mathcal{K}_\omega=-\mathcal{B}_e^*\mathcal{P}_\omega\in\mathcal{L}(Z_e,\mathbb{R}^{n_c})$, provides a stabilizing feedback for $(\mathcal{A}_e+\omega I,\mathcal{B}_e)$. Moreover, the operator $\mathcal{A}_e+\omega I+\mathcal{B}_e\mathcal{K}_\omega$, with domain $D(\mathcal{A}_e+\omega I+\mathcal{B}_e\mathcal{K}_\omega)=D(\mathcal{A}_e)$, is the generator of an analytic semigroup exponentially stable on Z_e.*

Proof. See ([35], Theorem 3).

□

1.7 Stabilization of the extended linearized system

The goal of this section is to study the stability of the closed-loop nonhomogeneous linear system

$$\begin{aligned}
&v_t-\operatorname{div}\sigma(v,p)+(u_s\cdot\nabla)v+(v\cdot\nabla)u_s-A_1\eta_1-A_2\eta_2-\omega v=\mathcal{F}_f \text{ in } Q^\infty,\\
&\operatorname{div}v=\operatorname{div}F_{\text{div}}+A_3\eta_1 \text{ in } Q^\infty, \quad v=\eta_2e_2 \text{ on } \Sigma_s^\infty, \quad v=\textstyle\sum_{i=1}^{n_c}g_iw_i \text{ on } \Sigma_b^\infty,\\
&v=\hat{h} \text{ on } \Sigma_i^\infty, \quad \sigma(v,p)n=0 \text{ on } \Sigma_n^\infty,\\
&\eta_{1,t}-\eta_2-\omega\eta_1=0 \text{ on } \Sigma_s^\infty,\\
&\eta_{2,t}-\beta\Delta\eta_1-\gamma\Delta\eta_2+\alpha\Delta^2\eta_1-A_4\eta_1-\omega\eta_2=\gamma_sp+\mathcal{F}_s \text{ on } \Sigma_s^\infty,\\
&\eta_1=0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \eta_{1,x}=0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&\mathbf{g}_t-\Lambda\mathbf{g}-\omega\mathbf{g}=\mathcal{K}_\omega(Pv,\eta_1,\eta_2,\mathbf{g}),\\
&v(0)=v^0 \text{ in } \Omega, \quad \eta_1(0)=\eta_1^0 \text{ on } \Gamma_s, \quad \eta_2(0)=\eta_2^0 \text{ on } \Gamma_s, \quad \mathbf{g}(0)=0,
\end{aligned} \tag{1.7 .1}$$

where $\mathcal{K}_\omega$ is the feedback law stabilizing the pair $(\mathcal{A}+\omega I,\mathcal{B})$.

Theorem 1.7 .1. *We assume that 1.4 .1 and 1.6 .1 are satisfied. Let* $(F_f, F_{\text{div}}, F_s)$ *belong to* Y, $(v^0, \eta_1^0, \eta_2^0)$ *belong to* $H_{ic}(F_{\text{div}}(0))$ *and* $\hat{h}$ *belongs* $H_0^1(0,\infty; H_0^2(\Gamma_i))$. *We assume that there exists* $q \in L^2(0,\infty; H^1(\Omega))$ *such that*

$$F_{\text{div}} = 0 \ \text{on} \ \Sigma_s^\infty, \quad F_{\text{div}} = 0 \ \text{on} \ \Sigma_b^\infty, \quad F_{\text{div}} = 0 \ \text{on} \ \Sigma_i^\infty, \quad \sigma(w,q)n = 0 \ \text{on} \ \Sigma_n^\infty.$$

The solution $(v, p, \eta_1, \eta_2, \mathbf{g})$ *of system (1.7 .1) satisfies the estimate*

$$\|(v, p, \eta_1, \eta_2, \mathbf{g})\|_{X_{\delta_0,e}} \le CR(v^0, \eta_1^0, \eta_2^0, h, F_f, F_{\text{div}}, F_s),$$

where

$$R(v^0, \eta_1^0, \eta_2^0, h, F_f, F_{\text{div}}, F_s) := \|(v^0, \eta_1^0, \eta_2^0)\|_{ic} + \|\hat{h}\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} + \|(F_f, F_{\text{div}}, F_s)\|_Y + \|q\|_{L^2(0,\infty;H^1(\Omega))}.$$

Proof. We proceed in three steps.
Step 1: Reformation of the system
We suppose that $(v, p, \eta_1, \eta_2, \mathbf{g})$ belongs to $X_{\delta_0,e}$. We make the change of unknowns

$$\hat{v} = v - F_{\text{div}} - L(\hat{h}\chi_{\Gamma_i}, 0) \quad \text{and} \quad \hat{p} = p - q - L_p(\hat{h}\chi_{\Gamma_i}, 0), \tag{1.7 .2}$$

where $q \in L^2(0,\infty; H^1(\Omega))$ is such that

$$\sigma(F_{\text{div}}, q)n = 0 \text{ on } \Sigma_n^\infty.$$

The quadruplet $(\hat{v}, \hat{p}, \eta_1, \eta_2)$ is solution of the system

$$\begin{aligned}
&\hat{v}_t - \operatorname{div}\sigma(\hat{v},\hat{p}) + (u_s \cdot \nabla)\hat{v} + (\hat{v} \cdot \nabla)u_s - A_1\eta_1 - A_2\eta_2 - \omega\hat{v} = \widetilde{F}_f \text{ in } Q^\infty,\\
&\operatorname{div}\hat{v} = A_3\eta_1 \text{ in } Q^\infty, \quad \hat{v} = \eta_2 e_2 \text{ on } \Sigma_s^\infty, \quad \hat{v} = \textstyle\sum_{i=1}^{n_c} g_i w_i \text{ on } \Sigma_b^\infty,\\
&\hat{v} = 0 \text{ on } \Sigma_i^\infty, \quad \sigma(\hat{v},\hat{p})n = 0 \text{ on } \Sigma_n^\infty,\\
&\eta_{1,t} - \eta_2 - \omega\eta_1 = 0 \text{ on } \Sigma_s^\infty,\\
&\eta_{2,t} - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 - A_4\eta_1 - \omega\eta_2 = \gamma_s\hat{p} + F_s \text{ on } \Sigma_s^\infty,\\
&\eta_1 = 0 \text{ on } (0,\infty) \times \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0,\infty) \times \partial\Gamma_s,\\
&\mathbf{g}_t - \Lambda\mathbf{g} = \mathcal{K}_\omega(P\hat{v}, \eta_1, \eta_2, \mathbf{g}) + \mathbf{F}_c,\\
&\hat{v}(0) = \hat{v}^0 = v^0 - w(0) \text{ in } \Omega, \quad \eta_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s, \quad \mathbf{g}(0) = \mathbf{0},
\end{aligned} \tag{1.7 .3}$$

where

$$\widetilde{F}_f = F_f - F_{\text{div},t} + 2\nu\operatorname{div}\varepsilon(F_{\text{div}}) - (u_s\cdot\nabla)F_{\text{div}} - (F_{\text{div}}\cdot\nabla)u_s + \omega F_{\text{div}} - L(\hat{h}_t\chi_{\Gamma_i}, 0) - (\lambda_0 - \omega)L(\hat{h}\chi_{\Gamma_i}, 0),$$

and

$$\mathbf{F}_c = \mathcal{K}_\omega(PF_{\text{div}} + PL(\hat{h}\chi_{\Gamma_i}, 0), 0, 0, \mathbf{0}) + \gamma_s q + \gamma_s L_p(\hat{h}\chi_{\Gamma_i}, 0).$$

As in Proposition 1.5 .2, we prove that $(P\hat{v}, \eta_1, \eta_2)$ is solution of system

$$\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} P\hat{v} \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = (\mathcal{A}_e + \omega I + \mathcal{B}_e\mathcal{K}_\omega)\begin{pmatrix} P\hat{v} \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} + \begin{pmatrix} P\widetilde{F}_f \\ 0 \\ F_s + \gamma_s N_p \widetilde{F}_f \\ \mathbf{F}_c \end{pmatrix},\\
&\begin{pmatrix} P\hat{v} \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix}(0) = \begin{pmatrix} P\hat{v}(0) \\ \eta_1^0 \\ \eta_2^0 \\ \mathbf{0} \end{pmatrix},
\end{aligned} \tag{1.7 .4}$$

and

$$(I-P)\hat{v} = \nabla N_s\eta_2 + \nabla N_c\mathbf{g} + \nabla N_{\rm div}A_3\eta_1. \tag{1.7 .5}$$

Step 2: Existence of the solutions to system (1.7 .3).
We have the estimate

$$\|(P\hat{v}^0,\eta_1^0,\eta_2^0,\mathbf{0})\|_{[D(\mathcal{A}_e),Z_e]_{\frac{1}{2}}} + \left\|(P\widetilde{F}_f,0,F_s+\gamma_sN_p\widetilde{F}_f,\mathbf{F}_c)\right\|_{L^2(0,\infty;Z_e)} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s). \tag{1.7 .6}$$

Thus, since $(\mathcal{A}_e+\omega I+\mathcal{B}_e\mathcal{K}_\omega, D(\mathcal{A}_e+\omega I+\mathcal{B}_e\mathcal{K}_\omega))$ is the generator of an exponentially stable analytic semigroup on Z_e, it follows that

$$\|(P\hat{v},\eta_1,\eta_2,\mathbf{0})\|_{L^2(0,\infty;D(\mathcal{A}_e))\cap H^1(0,\infty;Z_e)} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s),$$

(see [9], Chapter1, Theorem 3.1). We deduce that

$$\|P\hat{v}\|_{H^1(0,\infty;\mathbf{L}^2(\Omega))} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s), \tag{1.7 .7}$$

and

$$\|\eta_1\|_{H^{4,2}(\Sigma_s^\infty)} + \|\eta_2\|_{H^{2,1}(\Sigma_s^\infty)} + \|\mathbf{g}\|_{H^1(0,\infty;\mathbb{R}^{n_c})} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s). \tag{1.7 .8}$$

From (1.7 .8) and (1.7 .5), it follows that $(I-P)\hat{v}$ belongs $H^1(0,\infty;\mathbf{L}^2(\Omega))$. It implies that $\hat{v} = P\hat{v}+(I-P)\hat{v} \in H^1(0,\infty;L^2(\Omega)$ and

$$\|\hat{v}\|_{H^1(0,\infty;\mathbf{L}^2(\Omega))} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s). \tag{1.7 .9}$$

For all $t>0$, $(\hat{v}(t),\hat{p}(t))$ is solution of system

$$\begin{aligned}
&\lambda_0\hat{v}(t) - \operatorname{div}\sigma(\hat{v}(t),\hat{p}(t)) + (u_s\cdot\nabla)\hat{v}(t) + (\hat{v}(t)\cdot\nabla)u_s = G(t) \text{ in } \Omega,\\
&\operatorname{div}\hat{v}(t) = A_3\eta_1(t) \text{ in } \Omega, \quad \hat{v}(t) = \eta_2(t)e_2 \text{ on } \Gamma_s, \quad \hat{v}(t) = \textstyle\sum_{i=1}^{n_c} g_i(t)w_i \text{ on } \Gamma_b,\\
&\hat{v}(t) = 0 \text{ on } \Gamma_i, \quad \sigma(\hat{v}(t),\hat{p}(t))n = 0 \text{ on } \Gamma_n,
\end{aligned} \tag{1.7 .10}$$

where

$$G = -\hat{v}_t + (\lambda_0+\omega)\hat{v} - (u_s\cdot\nabla)\hat{v} - (\hat{v}\cdot\nabla)u_s + A_1\eta_1 + A_2\eta_2 + \widetilde{F}_f.$$

From (1.7 .6), (1.7 .8) and (1.7 .9), it follows that G belongs to $L^2(0,\infty;\mathbf{L}^2(\Omega))$ and

$$\|G\|_{L^2(0,\infty;\mathbf{L}^2(\Omega))} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s).$$

Thus, thanks to Theorem 1.4 .2, $(\hat{v}(t),\hat{p}(t))$ belongs to $\mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ and

$$\|\hat{v}(t)\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|\hat{p}(t)\|_{H^1_{\delta_0}(\Omega)} \le C\left(\|G(t)\|_{\mathbf{L}^2(\Omega)} + \|\eta_2(t)\|_{H_0^{\frac{3}{2}}(\Gamma_s)} + \|A_3\eta_1(t)\|_{H^1(\Omega)}\right). \tag{1.7 .11}$$

We deduce that $(\hat{v},\hat{p})$ belongs to $L^2(0,\infty;\mathbf{H}^2_{\delta_0}(\Omega))\times L^2(0,\infty;H^1_{\delta_0}(\Omega))$ and

$$\|\hat{v}\|_{L^2(0,\infty;\mathbf{H}^2_{\delta_0}(\Omega))} + \|\hat{p}\|_{L^2(0,\infty;H^1_{\delta_0}(\Omega))} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s). \tag{1.7 .12}$$

By combining inequalities (1.7 .8) and (1.7 .10), we obtain

$$\|(\hat{v},\hat{p},\eta_1,\eta_2,\boldsymbol{g})\|_{X_{\delta_0,e}} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s). \tag{1.7 .13}$$

Step 3: Existence of the solutions to system (1.7 .1).
By using Theorem 1.4 .2, we prove that $(L(\hat{h}\chi_{\Gamma_i},0),L_p(\hat{h}\chi_{\Gamma_i},0))$ belongs to $H^{2,1}_{\delta_0}(Q^\infty)\times L^2(0,\infty;H^1_{\delta_0}(\Omega))$ and

$$\|L(\hat{h}\chi_{\Gamma_i},0)\|_{H^{2,1}_{\delta_0}(Q^\infty)} + \|L_p(\hat{h}\chi_{\Gamma_i},0)\|_{L^2(0,\infty;H^1_{\delta_0}(\Omega)} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s). \tag{1.7 .14}$$

Thus, thanks to equation (1.7 .2) and inequality (1.7 .13), we prove that

$$\|(v,p,\eta_1,\eta_2,\boldsymbol{g})\|_{X_{\delta_0,e}} \le CR(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s).$$

□

1.8 Stabilization of the extended nonlinear system

The goal of this section is to study the stability of the closed-loop extended nonlinear system

$$\begin{aligned}
&\hat{u}_t - \operatorname{div}\sigma(\hat{u},\hat{p}) + (u_s\cdot\nabla)\hat{u} + (\hat{u}\cdot\nabla)u_s - A_1\hat{\eta}_1 - A_2\hat{\eta}_2 - \omega\hat{u} = e^{-\omega t}\mathcal{F}_f[\hat{u},\hat{p},\hat{\eta}_1,\hat{\eta}_2] \text{ in } Q^\infty,\\
&\operatorname{div}\hat{u} = A_3\hat{\eta}_1 + e^{-\omega t}\operatorname{div}\mathcal{F}_{\mathrm{div}}[\hat{u},\hat{\eta}_1] \text{ in } Q^\infty, \quad \hat{u} = \hat{\eta}_2 e_2 \text{ on } \Sigma_s^\infty, \quad \hat{u} = \textstyle\sum_{i=1}^{n_c} g_i w_i \text{ on } \Sigma_b^\infty,\\
&\hat{u} = \hat{h} \text{ on } \Sigma_i^\infty, \quad \sigma(\hat{u},\hat{p})n = 0 \text{ on } \Sigma_n^\infty,\\
&\hat{\eta}_{1,t} - \hat{\eta}_2 - \omega\hat{\eta}_1 = 0 \text{ on } \Sigma_s^\infty,\\
&\hat{\eta}_{2,t} - \beta\Delta\hat{\eta}_1 - \gamma\Delta\hat{\eta}_2 + \alpha\Delta^2\hat{\eta}_1 - A_4\hat{\eta}_1 - \omega\hat{\eta}_2 = \gamma_s\hat{p} + e^{-\omega t}\mathcal{F}_s[\hat{u},\hat{\eta}_1] \text{ on } \Sigma_s^\infty,\\
&\hat{\eta}_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \hat{\eta}_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&\mathbf{g}_t - \Lambda\mathbf{g} = \mathcal{K}_\omega(P\hat{u},\eta_1,\eta_2,\mathbf{g}),\\
&\hat{u}(0) = \hat{u}^0 \text{ in } \Omega, \quad \hat{\eta}_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \hat{\eta}_2(0) = \eta_2^0 \text{ on } \Gamma_s, \quad \mathbf{g}(0) = 0,
\end{aligned} \tag{1.8 .1}$$

where $\mathcal{K}_\omega$ is the linear feedback law stabilizing system (1.5 .1). We are going to use the results on the nonhomogeneous linear system (1.7 .1) and a fixed point argument to prove that system (1.8 .1) is locally exponentially stable. For that, we introduce the mapping $\mathcal{G}$ defined by

$$\begin{array}{rccc}
\mathcal{G} \; : & X_{\delta_0,e} & \longrightarrow & X_{\delta_0,e}\\
& (u,q,\zeta_1,\zeta_2,\mathbf{h}) & \longmapsto & (v,p,\eta_1,\eta_2,\mathbf{g})
\end{array}$$

where $(v,p,\eta_1,\eta_2,\mathbf{g})$ is the solution of system (1.7 .1) with right-hand sides

$$(\mathcal{F}_f[u,q,\zeta_1,\zeta_2], \mathcal{F}_{\mathrm{div}}[u,\zeta_1], \mathcal{F}_s[u,\zeta_1]),$$

initial data $(\hat{u}^0,\eta_1^0,\eta_2^0)$ in $H_{ic}(\mathcal{F}_{\mathrm{div}}[\hat{u}^0,\eta_1^0])$ and boundary perturbation $\hat{h}\in H_0^1(0,\infty;H_0^2(\Gamma_i))$. We are going to prove that $\mathcal{G}$ is well-defined and that there exists $R>0$ such that $\mathcal{G}$ is a contraction in

$$B_{\delta_0,R} = \{(v,p,\eta_1,\eta_2,\mathbf{g})\in X_{\delta_0,e} \mid \|(v,p,\eta_1,\eta_2,\mathbf{g})\|_{X_{\delta_0,e}} \le R \quad \text{and} \quad \|e^{-\omega\cdot}\eta_1\|_{L^\infty(\Sigma_s^\infty)} \le \eta_{\max}\},$$

where $0<\eta_{\max}<1$. We set

$$B(v,p,\eta_1,\eta_2,\mathbf{g}) = \left(1 + \|(v,p,\eta_1,\eta_2,\mathbf{g})\|_{X_{\delta_0,e}}\right)\|(v,p,\eta_1,\eta_2,\mathbf{g})\|^2_{X_{\delta_0,e}}.$$

Now, let us prove some results useful to prove the contraction of the mapping $\mathcal{G}$.

Lemma 1.8 .1. *Let η be in $H^{4,2}(\Sigma_s^\infty)$. There exists $C>0$ such that*

$$\|\eta\|_{L^\infty(\Sigma_s^\infty)} + \|\eta\|_{\mathcal{C}(0,\infty;\mathcal{C}^1(\Gamma_s))} + \|\eta_x\|_{L^\infty(\Sigma_s^\infty)} + \|\eta_{xx}\|_{L^\infty(\Sigma_s^\infty)} \le C\|\eta\|_{H^{4,2}(\Sigma_s^\infty)},$$

and

$$\|\eta_t\|_{L^2(0,\infty;L^\infty(\Gamma_s))} + \|\eta_{tx}\|_{L^2(0,\infty;L^\infty(\Gamma_s))} + \|\eta_{xxx}\|_{L^2(0,\infty;L^\infty(\Gamma_s))} \le C\|\eta\|_{H^{4,2}(\Sigma_s^\infty)}.$$

Moreover, η belongs to $\mathcal{C}_b([0,\infty);\mathcal{C}^1(\overline{\Gamma}_s))$.

Proof. Since η belongs to $H^{4,2}(\Sigma_s^\infty)$ then by interpolation we have

$$\eta\in H^{2\theta}(0,\infty;H^{4(1-\theta)}(\Gamma_s)), \quad \forall\, 0<\theta<1,$$

which gives

$$\begin{aligned}
&\eta \in H^{\frac{7}{4}-\frac{\varepsilon}{2}}(0,\infty; H^{\frac{1}{2}+\varepsilon}(\Gamma_s)),\\
&\eta \in H^{\frac{5}{4}-\frac{\varepsilon}{2}}(0,\infty; H^{\frac{3}{2}+\varepsilon}(\Gamma_s)),\\
&\eta \in H^{\frac{3}{4}-\frac{\varepsilon}{2}}(0,\infty; H^{\frac{5}{2}+\varepsilon}(\Gamma_s)),\\
&\eta \in H^{\frac{1}{4}-\frac{\varepsilon}{2}}(0,\infty; H^{\frac{7}{2}+\varepsilon}(\Gamma_s)),
\end{aligned} \tag{1.8 .2}$$

where $0 < \varepsilon < \frac{1}{2}$. Thus, thanks to the continuous embeddings

$$H^{\frac{1}{2}+\varepsilon}(0,\infty) \hookrightarrow \mathcal{C}_b([0,\infty)) \quad \text{and} \quad H^{\frac{1}{2}+\varepsilon}(\Gamma_s) \hookrightarrow L^\infty(\Gamma_s),$$

we have on the one hand η, η_x and η_{xx} belong to $L^\infty(\Sigma_s^\infty)$ and the other terms η_t, η_{tx} and η_{xxx} belong to $L^2(0,\infty; L^\infty(\Gamma_s))$ with the expected estimates. From the continuous embeddings

$$H^{\frac{1}{2}+\varepsilon}(0,\infty) \hookrightarrow \mathcal{C}_b([0,\infty)) \quad \text{and} \quad H^{\frac{3}{2}+\varepsilon}(\Gamma_s) \hookrightarrow \mathcal{C}^1(\overline{\Gamma}_s),$$

it follows that η belongs to $\mathcal{C}_b([0,\infty); \mathcal{C}^1(\overline{\Gamma}_s))$.

□

We introduce the nonlinear term $q[v,\eta_1]$ defined by

$$q[v,\eta_1] = z\eta_{1,xx}v_1. \tag{1.8 .3}$$

Lemma 1.8 .2. *Let (v,η_1) belong to $\mathbf{H}^{2,1}_{\delta_0}(Q^\infty) \times H^{4,2}(\Sigma_s^\infty)$. Then $\mathcal{F}_{\mathrm{div}}[v,\eta_1]$ belongs to $H^{2,1}(Q^\infty)$, $q[v,\eta_1]$ belongs to $L^2(0,\infty; H^1(\Omega))$ and there exists $C > 0$ such that*

$$\|\mathcal{F}_{\mathrm{div}}[v,\eta_1]\|_{H^{2,1}(Q^\infty)} + \|q[v,\eta_1]\|_{L^2(0,\infty;H^1(\Omega))} \le C\|\eta_1\|_{H^{4,2}(\Sigma_s^\infty)}\|v\|_{H^{2,1}_{\delta_0}(Q^\infty)}.$$

Moreover, we have

$$\sigma(\mathcal{F}_{\mathrm{div}}[v,\eta_1], q[v,\eta_1])n = 0 \ \textit{on}\ \Sigma_n^\infty.$$

Proof. We have

$$\mathcal{F}_{\mathrm{div}}[v,\eta_1] = -\eta_1 v_1 e_1 + z\eta_{1,x}v_1 e_2,$$

and its derivatives are

$$\begin{aligned}
&\mathcal{F}_{\mathrm{div},x} = -\eta_{1,x}v_1e_1 - \eta_1 v_{1,x}e_1 + z\eta_{1,xx}v_1e_2 + z\eta_{1,xx}v_1e_2,\\
&\mathcal{F}_{\mathrm{div},z} = -\eta_1 v_{1,z}e_1 + \eta_{1,x}v_1e_2 + z\eta_{1,x}v_{1,z}e_2,\\
&\mathcal{F}_{\mathrm{div},xx} = -\eta_{1,xx}v_1e_1 - 2\eta_{1,x}v_{1,x}e_1 - \eta v_{1,xx}e_1 + z\eta_{1,xxx}v_1e_2 + 2z\eta_{1,xx}v_{1,x}e_2 + z\eta_{1,x}v_{1,xx}e_2,\\
&\mathcal{F}_{\mathrm{div},zz} = -\eta v_{1,zz}e_1 + 2\eta_{1,x}v_{1,z}e_2 + z\eta_{1,x}v_{1,zz}e_2,\\
&\mathcal{F}_{\mathrm{div},t} = -\eta_{1,t}v_1e_1 - \eta_1 v_{1,t}e_1 + z\eta_{1,tx}v_1e_2 + z\eta_{1,x}v_{1,t}e_2.
\end{aligned} \tag{1.8 .4}$$

The terms of $\mathcal{F}_{\mathrm{div}}[v,\eta_1]$ involving η_1, $\eta_{1,x}$ or $\eta_{1,xx}$ such that $\eta_{1,x}v_1$ can be estimated as follows:

$$\|\eta_{1,x}v_1\|_{L^2(Q^\infty)} \le \|\eta_{1,x}\|_{L^\infty(0,\infty;L^\infty(\Gamma_s))}\|v_1\|_{L^2(Q^\infty)},$$

and the terms of $\mathcal{F}_{\mathrm{div}}[v,\eta_1]$ involving $\eta_{1,t}$, $\eta_{1,tx}$ or $\eta_{1,xxx}$, such as $\eta_{1,xxx}v_1$, can be estimated as follows

$$\|\eta_{1,xxx}v_1\|_{L^2(Q^\infty)} \le \|\eta_{1,x}\|_{L^2(0,\infty;L^\infty(\Gamma_s))}\|v_1\|_{L^\infty(0,\infty;L^2(\Omega))}.$$

Thus, there exists $C > 0$ such that

$$\|\mathcal{F}_{\mathrm{div}}[v,\eta_1]\|_{H^{2,1}(Q^\infty)} \le C\|\eta_1\|_{H^{4,2}(\Sigma_s^\infty)}\|v\|_{H^{2,1}(Q^\infty)}.$$

Similarly to $\mathcal{F}_{\rm div}[v,\eta_1]$, we prove that

$$\|q[v,\eta_1]\|_{H^{2,1}(Q^\infty)} \leq C\|\eta_1\|_{H^{4,2}(\Sigma_s^\infty)}\|v\|_{H^{2,1}(Q^\infty)}.$$

By using equation (1.8 .4) and the boundary conditions for η_1, we prove that

$$\sigma(\mathcal{F}_{\rm div}[v,\eta_1], q[v,\eta_1]) = 0 \text{ on } \Sigma_n^\infty.$$

□

Lemma 1.8 .3. *Let $(v,p,\eta_1,\eta_2,\mathbf{g})$ belong to $B_{\delta_0,R}$. Then $\mathcal{F}_f[v,p,\eta_1,\eta_2]$ belongs to $L^2(Q^\infty)$ and there exists $C>0$ such that*

$$\|\mathcal{F}_f[v,p,\eta_1,\eta_2]\|_{L^2(Q^\infty)} \leq CB(v,p,\eta_1,\eta_2,\mathbf{g}).$$

Proof. To prove that $\mathcal{F}_f[v,p,\eta_1,\eta_2]$ belongs to $L^2(Q^\infty)$, we are going to show that all its terms belong to $L^2(Q^\infty)$. By using Lemma 1.8 .1, we can prove that all the geometrical nonlinear terms involving v or its first order derivatives belong to $L^2(Q^\infty)$. For example, we have

$$\|\eta_{1,xx}v_z\|_{\mathbf{L}^2(Q^\infty)} \leq C\,\|\eta_1\|_{L^\infty(\Sigma_s^\infty)}\,\|v_z\|_{\mathbf{L}^2(Q^\infty)} \leq CB(v,p,\eta_1,\eta_2,\mathbf{g}).$$

Let us prove that $\eta_{1,x}v_{xz}$, belongs to $L^2(Q^\infty)$. We have $\eta_{1,x}$ belongs to $H^1(0,\infty;H^1_0(\Gamma_s)) \hookrightarrow L^\infty(0,\infty;H^1_0(\Gamma_s))$ and v_{xz} belongs to $L^2(0,\infty;\mathbf{L}^2_{\delta_0}(\Omega))$. Therefore, thanks to Lemma 1.5 .2, we have

$$\|\eta_{1,x}v_{xz}\|_{L^2(Q^\infty)} \leq C\|\eta_{1,x}\|_{L^\infty(0,\infty;H^1_0(\Gamma_s))}\|v_{xz}\|_{L^2(0,\infty;\mathbf{L}^2_{\delta_0}(\Omega))} \leq CB(v,p,\eta_1,\eta_2,\mathbf{g}).$$

Similarly, we prove that all the geometrical nonlinear terms involving the derivatives of v of order 2 belong to $L^2(Q^\infty)$.
Now, we are going to estimate nonlinear terms coming from the Navier-Stokes equations as $(v\cdot\nabla)v$. Since $v\in L^2(0,\infty;\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega))\cap H^1(0,\infty;\mathbf{L}^2(\Omega))$, we have v belongs $L^\infty(0,\infty;\mathbf{H}^{\frac{3}{4}+\frac{\varepsilon_0}{2}}(\Omega))$ and ∇v belongs to $L^2(0,\infty;\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega))$. Thanks to [[28], Proposition B.1], we have

$$\|(v\cdot\nabla)v\|_{\mathbf{L}^2(\Omega)} \leq C\,\|v\|_{\mathbf{H}^{\frac{3}{4}+\frac{\varepsilon_0}{2}}(\Omega)}\,\|\nabla v\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)}.$$

Hence

$$\|(v\cdot\nabla)v\|_{\mathbf{L}^2(Q^\infty)} \leq C\,\|v\|_{L^\infty(0,\infty;\mathbf{H}^{\frac{3}{4}+\frac{\varepsilon_0}{2}}(\Omega))}\,\|\nabla v\|_{L^2(0,\infty;\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega))} \leq CB(v,p,\eta_1,\eta_2,\mathbf{g}).$$

Thus, we prove that $\mathcal{F}_f[v,p,\eta_1,\eta_2]$ belongs to $\mathbf{L}^2(Q^\infty)$ with

$$\|\mathcal{F}_f[v,p,\eta_1,\eta_2]\|_{\mathbf{L}^2(Q^\infty)} \leq CB(v,p,\eta_1,\eta_2,\mathbf{g}).$$

□

Lemma 1.8 .4. *Let $(v,p,\eta_1,\eta_2,\mathbf{g})$ belong to $B_{\delta_0,R}$. Then $\mathcal{F}_s[v,\eta_1]$ belongs to $L^2(Q^\infty)$ and there exists $C>0$ such that*

$$\|\mathcal{F}_s[v,\eta_1]\|_{L^2(Q^\infty)} \leq C\|\eta_1\|_{H^{4,2}(\Sigma_s^\infty)}\|v\|_{H^{2,1}_{\delta_0}(Q^\infty)}.$$

Proof. It is similar to that of the previous lemma.

□

Proposition 1.8 .1. *1. Let $(v, p, \eta_1, \eta_2, \mathbf{g})$ belong to $B_{\delta_0,R}$. Then, there exists $C > 0$ such that*

$$\|(\mathcal{F}_f[v, p, \eta_1, \eta_2], \mathcal{F}_{\mathrm{div}}[v, \eta_1], \mathcal{F}_s[v, \eta_1])\|_Y + \|q[v, \eta_1]\|_{L^2(0,\infty;L^2(\Omega)} \leq CB(v, p, \eta_1, \eta_2, \mathbf{g}). \tag{1.8 .5}$$

Moreover, if $(v, p, \eta_1, \eta_2, \mathbf{g})$ and $(u, q, \zeta_1, \zeta_2, \mathbf{h})$ belong to $B_{\delta_0,R}$ then we have

$$\begin{aligned}&\|(\mathcal{F}_f[v, p, \eta_1, \eta_2, \mathbf{g}], \mathcal{F}_{\mathrm{div}}[v, \eta_1], \mathcal{F}_s[v, \eta_1]) - (\mathcal{F}_f[u, q, \zeta_1, \zeta_2, \mathbf{h}], \mathcal{F}_{\mathrm{div}}[u, \zeta_1], \mathcal{F}_s[u, \zeta_1])\|_Y \\ &+\|q[v, \eta_1] - q[u, \zeta_1]\|_{L^2(0,\infty;L^2(\Omega)} \leq CR(1 + R) \|(v, p, \eta_1, \eta_2, \mathbf{g}) - (u, q, \zeta_1, \zeta_2, \mathbf{h})\|_{X_{\delta_0,e}} .\end{aligned} \tag{1.8 .6}$$

2. The mapping $\mathcal{G}$ is well-defined and for all $(v, p, \eta_1, \eta_2, \mathbf{g})$ and $(u, q, \zeta_1, \zeta_2, \mathbf{h})$ in $B_{\delta_0,R}$, we have

$$\|\mathcal{G}(v, p, \eta_1, \eta_2, \mathbf{g})\|_{X_{\delta_0,e}} \leq C \left(\|(\hat{u}^0, \eta_1^0, \eta_2^0)\|_{ic} + \|\hat{h}\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} + B(v, p, \eta_1, \eta_2, \mathbf{g})\right),$$

and

$$\|\mathcal{G}(v, p, \eta_1, \eta_2, \mathbf{g}) - \mathcal{G}(u, q, \zeta_1, \zeta_2, \mathbf{h})\|_{X_{\delta_0,e}} \leq CR(1 + R) \|(v, p, \eta_1, \eta_2, \mathbf{g}) - (u, q, \zeta_1, \zeta_2, \mathbf{h})\|_{X_{\delta_0,e}} . \tag{1.8 .7}$$

Proof. 1. Thanks to Lemmas 1.8 .2, 1.8 .3 and 1.8 .4, we prove that $(\mathcal{F}_f[v, p, \eta_1, \eta_2], \mathcal{F}_{\mathrm{div}}[v, \eta_1], \mathcal{F}_s[v, \eta_1])$ belongs to Y and there exists $C > 0$ such that

$$\|(\mathcal{F}_f[v, p, \eta_1, \eta_2], \mathcal{F}_{\mathrm{div}}[v, \eta_1], \mathcal{F}_s[v, \eta_1])\|_Y \leq CB(v, p, \eta_1, \eta_2, \mathbf{g}).$$

The inequality (1.8 .6) comes from the fact that we have at least quadratic terms.
2. From Theorem 1.7 .1, system (1.7 .1) with right-hand sides

$$(\mathcal{F}_f[v, p, \eta_1, \eta_2], \mathcal{F}_{\mathrm{div}}[v, \eta_1], \mathcal{F}_s[v, \eta_1]),$$

initial data $(\hat{u}^0, \eta_1^0, \eta_2^0) \in H_{ic,\mathcal{F}_{\mathrm{div}}[\hat{u}^0,\eta_1^0]}$ and boundary perturbation $\hat{h} \in H_0^1(0, \infty; H_0^2(\Gamma_i))$, have a unique controlled solution $\mathcal{G}(v, p, \eta_1, \eta_2, \mathbf{g})$ belonging to $X_{\delta_0,e}$ with the estimation

$$\begin{aligned}\|\mathcal{G}(v, p, \eta_1, \eta_2, \mathbf{g})\|_{X_{\delta_0,e}} \leq C \|(\hat{u}^0, \eta_1^0, \eta_2^0)\|_{ic} + C\|\hat{h}\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} \\ +C \|(\mathcal{F}_f[v, p, \eta_1, \eta_2], \mathcal{F}_{\mathrm{div}}[v, \eta_1], \mathcal{F}_s[v, \eta_1])\|_Y .\end{aligned}$$

Hence, using the first part, we obtain

$$\|\mathcal{G}(v, p, \eta_1, \eta_2, \mathbf{g})\|_{X_{\delta_0,e}} \leq C \left(\|(u^0, \eta_1^0, \eta_2^0)\|_{ic} + \|\hat{h}\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} + B(v, p, \eta_1, \eta_2, \mathbf{g})\right).$$

For the second inequality, we use the fact that

$$\mathcal{G}(v, p, \eta_1, \eta_2, \mathbf{g}) - \mathcal{G}(u, q, \zeta_1, \zeta_2, \mathbf{h}),$$

is the controlled solution of system (1.7 .1) with right-hand sides

$$(\mathcal{F}_f[v, p, \eta_1, \eta_2] - \mathcal{F}_f[u, q, \zeta_1, \zeta_2], \mathcal{F}_{\mathrm{div}}[v, \eta_1] - \mathcal{F}_{\mathrm{div}}[u, \eta_1], \mathcal{F}_s[v, \eta_1] - \mathcal{F}_s[u, \zeta_1]),$$

initial data $(0, 0, 0)$ and no boundary perturbation. Thus, we obtain from Theorem 1.7 .1 and the first part of this proposition

$$\|\mathcal{G}(v, p, \eta_1, \eta_2, \mathbf{g}) - \mathcal{G}(u, q, \zeta_1, \zeta_2, \mathbf{h})\|_{X_{\delta_0,e}} \leq CR(1 + R) \|(v, p, \eta_1, \eta_2, \mathbf{g}) - (u, q, \zeta_1, \zeta_2, \mathbf{h})\|_{X_{\delta_0,e}} .$$

□

Theorem 1.8 .2. *Let $\eta_{\max}$ be in $]0,1[$ and ω be positive. We assume that 1.4 .1 and 1.6 .1 are satisfied. Then, there exists $r>0$ such that for all $(\hat{u}^0,\eta_1^0,\eta_2^0)\in H_{ic}(\mathcal{F}_{\mathrm{div}}[\hat{u}^0,\eta_1^0])$ and $\hat{h}\in H_0^1(0,\infty;H_0^2(\Gamma_i))$ satisfying the estimate*

$$\left\|(\hat{u}^0,\eta_1^0,\eta_2^0)\right\|_{ic}+\|\hat{h}\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))}\leq r,$$

system (1.8 .1) admits a unique solution $(\hat{u},\hat{p},\eta_1,\eta_2,\mathbf{g})\in X_{e,\delta_0}$. Moreover, there exist $\varepsilon>0$ and $C>0$ such that

$$\|e^{-\omega t}\hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)}\leq\eta_{\max}\quad \text{and}\quad \|\hat{u}(t)\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon}(\Omega)}+\|\hat{\eta}_1(t)\|_{H^3(\Gamma_s)}+\|\hat{\eta}_2(t)\|_{H^1(\Gamma_s)}\leq C,$$

for all $t>0$.

Proof. Let $(v,p,\eta_1,\eta_2,\mathbf{g})$ and $(u,q,\zeta_1,\zeta_2,\mathbf{h})$ belong to $B_{\delta_0,R}$ with

$$R:=2C\left(\left\|(\hat{u}^0,\eta_1^0,\eta_2^0)\right\|_{ic}+\|\hat{h}\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))}\right),$$

where C is the constant of Proposition 1.8 .1. We set

$$(\tilde{v},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2,\tilde{\mathbf{g}}):=\mathcal{G}(v,p,\eta_1,\eta_2,\mathbf{g}).$$

From Proposition 1.8 .1, we have

$$\|\mathcal{G}(v,p,\eta_1,\eta_2,\mathbf{g})\|_{X_{\delta_0,e}}\leq\frac{R}{2}+C(1+R)R^2,$$

and

$$\|\mathcal{G}(v,p,\eta_1,\eta_2,\mathbf{g})-\mathcal{G}(u,q,\zeta_1,\zeta_2,\mathbf{h})\|_{X_{\delta_0,e}}\leq CR(1+R)\,\|(v,p,\eta_1,\eta_2,\mathbf{g})-(u,q,\zeta_1,\zeta_2,\mathbf{h})\|_{X_{\delta_0,e}}.$$

Thanks to Lemma 1.8 .1, we also have

$$\|e^{-\omega\cdot}\tilde{\eta}_1\|_{L^\infty(\Sigma_s^\infty)}\leq C\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_s^\infty)}.$$

We choose R_0 small enough such that

$$\frac{R_0}{2}+C(1+R_0)R_0^2<R_0,\quad CR_0(1+R_0)<1\quad\text{and}\quad CR_0<\eta_{\max}.$$

It follows that $\mathcal{G}$ is a contraction in B_{δ_0,R_0}. Thus, thanks to the Banach fixed point Theorem, there exists a unique quadruplet $(\hat{u},\hat{p},\hat{\eta}_1,\hat{\eta}_2)\in B_{\delta_0,R_0}$ such that

$$\mathcal{G}(\hat{u},\hat{p},\hat{\eta}_1,\hat{\eta}_2,\hat{\mathbf{g}})=(\hat{u},\hat{p},\hat{\eta}_1,\hat{\eta}_2,\hat{\mathbf{g}}).$$

We deduce that system (1.8 .1) admits a unique solution $(\hat{u},\hat{p},\hat{\eta}_1,\hat{\eta}_2)$ with the estimates

$$\|e^{-\omega\cdot}\hat{\eta}_1\|_{L^\infty(\Sigma_s^\infty)}\leq\eta_{\max}\quad\text{and}\quad\|(\hat{u},\hat{p},\hat{\eta}_1,\hat{\eta}_2,\hat{\mathbf{g}})\|_{X_{\delta_0,e}}\leq R_0.$$

From ([40], Theorem 2.1) and Lemma 1.4 .1, it follows that

$$\|\hat{u}(t)\|_{\mathbf{H}^{\frac{3}{4}+\frac{\varepsilon_0}{2}}(\Omega)}+\|\hat{\eta}_1(t)\|_{H^3(\Gamma_s)}+\|\hat{\eta}_2(t)\|_{H^1(\Gamma_s)}\leq CR_0.$$

□

1.9 Proof of the main theorem

Thanks to Theorem 1.8 .2, it follows that, in order to prove Theorem 1.3 .1, we have to check that the change of variables $\mathcal{T}_{\eta(t)}$ is well-defined as a $\mathcal{C}^1-$diffeomorphism from $\Omega_{\eta(t)}$ into Ω for all η belonging to $H^{4,2}(\Sigma_s^\infty)$ such that $\|\eta\|_{L^\infty(\Sigma_s^\infty)} \leq \eta_{\max}$ and for every $t \in (0,\infty)$. For that, we use the continuous embedding $H^{4,2}(\Sigma_s^\infty) \hookrightarrow \mathcal{C}(0,\infty;\mathcal{C}^1(\Gamma_s))$ (see Lemma 1.8 .1). Together with the estimate $\|\eta\|_{L^\infty(\Sigma_s^\infty)} \leq \eta_{\max}$, we show that $\mathcal{T}_{\eta(t)}$ is a $\mathcal{C}^1-$diffeomorphism from $\Omega_{\eta(t)}$ into Ω.

1.A Appendix A: Adjoint of $(\mathcal{A}_e, D(\mathcal{A}_e))$

We introduce the operator $N_\phi \in \mathcal{L}(\mathbf{H}^2_{\delta_0}(\Omega), L^2(\Omega))$ defined by $N_\phi \phi = q$ where q is the solution of the variational problem

Find $q \in L^2(\Omega)$ such that

$$\int_\Omega q\zeta = 2\nu\int_\Omega \varepsilon(\phi) : \nabla^2\psi - 2\nu\int_{\Gamma_d}\varepsilon(\phi)n\cdot\nabla\psi + \int_\Omega\Big[-(u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi\Big]\cdot\nabla\psi, \quad \forall\,\zeta \in L^2(\Omega),$$

where ψ is the solution of the equation

$$\Delta\psi = \zeta \text{ in } \Omega, \quad \frac{\partial\psi}{\partial n} = 0 \text{ on } \Gamma_d, \quad \psi = 0 \text{ on } \Gamma_n. \tag{1.A.1}$$

We also introduce the lifting operators $D \in \mathcal{L}(H_0^{\frac{3}{2}}(\Gamma_s), \mathbf{H}^2_{\delta_0}(\Omega))$ and $D_p \in \mathcal{L}(H_0^{\frac{3}{2}}(\Gamma_s), H^1_{\delta_0}(\Omega))$ defined by

$$Dg = \phi \quad \text{and} \quad D_p g = \psi, \tag{1.A.2}$$

where (ϕ,ψ) is the solution of the system

$$\begin{aligned}
&\lambda_0\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0 \text{ in } \Omega,\\
&\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = g e_2 \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_b,\\
&\phi = 0 \text{ on } \Gamma_i, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \text{ on } \Gamma_n.
\end{aligned} \tag{1.A.3}$$

Finally, we introduce the unbounded operator $(\mathcal{A}_e^\#, D(\mathcal{A}_e^\#))$ defined by

$$D(\mathcal{A}_e^\#) = \Big\{(P\phi,\xi_1,\xi_2,\mathbf{h}) \in \mathbf{V}^1_{n,\Gamma_d}(\Omega)) \times (H^4(\Gamma_s)\cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s) \times \mathbb{R}^{n_c} \mid P\phi - PD\xi_2 \in D(A^*)\Big\},$$

and

$$\mathcal{A}_e^\# = \begin{pmatrix} A^* & 0 & (\lambda_0 I - A^*)PD & 0\\ (-A_{\alpha,\beta})^{-1}A_1^* & 0 & -I + (-A_{\alpha,\beta})^{-1}(A_4^* + A_1^*\nabla N_s) & 0\\ \gamma_s N_\phi + A_2^* & -A_{\alpha,\beta} & \gamma\Delta + \gamma_s N_\phi\nabla N_s + A_2^*\nabla N_s & 0\\ \left(\displaystyle\int_{\Gamma_b} 2\nu\varepsilon(\cdot)e_2\cdot w_i\right)_i & 0 & -\left(\displaystyle\int_{\Gamma_b} 2\nu\varepsilon(\nabla N_s)e_2\cdot w_i\right)_i & \Lambda \end{pmatrix} + \mathcal{A}_{p,e}^\#.$$

The operator $\mathcal{A}_{p,e}^\#$ is defined in $D(\mathcal{A}_e^\#)$ by

$$\mathcal{A}_{p,e}^\#\begin{pmatrix} P\phi\\ \xi_1\\ \xi_2\\ \mathbf{h}\end{pmatrix} = \begin{pmatrix} 0\\ -(-A_{\alpha,\beta})^{-1}A_3^*\gamma_s N_\phi(P\phi + \nabla N_s\xi_1)\\ \gamma_s N_\phi(P\phi + \nabla N_s\xi_2)\\ -\left(\displaystyle\int_{\Gamma_b} N_\phi(P\phi + \nabla N_s\xi_2)e_2\cdot w_i\right)_i \end{pmatrix}, \quad \forall\,(P\phi,\xi_1,\xi_2,\mathbf{h}) \in D(\mathcal{A}_e^\#).$$

Proposition 1.A.1. *Let λ belong to $\mathbb{R}$ and let $(G_f, G_s^1, G_s^2, \mathbf{G}_c)$ belong to Z_e. The quadruplet $(\phi, \psi, \xi_1, \xi_2) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega) \times H^4(\Gamma_s) \cap H^2_0(\Gamma_s) \times H^2_0(\Gamma_s)$ is solution of the system*

$$\begin{aligned}
&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = G_f \ \textit{in } \Omega,\\
&\operatorname{div}\phi = 0 \ \textit{in } \Omega, \quad \phi = \xi_2 e_2 \ \textit{on } \Gamma_s, \quad \phi = 0 \ \textit{on } \Gamma_b,\\
&\phi = 0 \ \textit{on } \Gamma_i, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \ \textit{on } \Gamma_n,\\
&\lambda\xi_1 + \xi_2 - (-A_{\alpha,\beta})^{-1}A_4^*\xi_2 - (-A_{\alpha,\beta})^{-1}A_1^*\phi + (-A_{\alpha,\beta})^{-1}A_3^*\psi = G_s^1 \ \textit{on } \Gamma_s,\\
&\lambda\xi_2 + \beta\Delta\xi_1 - \gamma\Delta\xi_2 - \alpha\Delta^2\xi_1 - A_2^*\phi - \gamma_s\psi = G_s^2 \ \textit{on } \Gamma_s,\\
&\xi_1 = 0 \ \textit{on } \partial\Gamma_s, \quad \xi_{1,x} = 0 \ \textit{on } \partial\Gamma_s,\\
&\lambda\mathbf{h} - \Lambda\mathbf{h} - \left(\int_{\Gamma_b}\sigma(\phi,\psi)e_2\cdot w_i\right)_i = \mathbf{G}_c,
\end{aligned} \tag{1.A.4}$$

if and only if

$$\begin{aligned}
&\lambda M_{s,e}^*\begin{pmatrix} P\phi\\ \xi_1\\ \xi_2\\ \mathbf{h}\end{pmatrix} = \mathcal{A}_e^{\#}\begin{pmatrix} P\phi\\ \xi_1\\ \xi_2\\ \mathbf{h}\end{pmatrix} + \begin{pmatrix} G_f\\ G_s^1\\ G_s^2\\ \mathbf{G}_c\end{pmatrix},\\
&(I-P)\phi = \nabla N_s\xi_2,\\
&\psi = -\lambda N_s\xi_2 + N_\phi\nabla N_s\xi_2 + N_\phi P\phi.
\end{aligned} \tag{1.A.5}$$

Proof. Let $(\phi, \psi, \xi_1, \xi_2) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega) \times H^4(\Gamma_s) \cap H^2_0(\Gamma_s) \times H^2_0(\Gamma_s)$ be a solution of system (1.A.4). As in Theorem 1.4 .8, we prove that (ϕ,ψ) is a solution of the system

$$\begin{aligned}
&(\lambda I - A^*)P\phi + (A^* - \lambda_0 I)PD\xi_2 = G_f,\\
&(I-P)\phi = \nabla N_s\xi_2,\\
&\psi = -\lambda N_s\xi_2 + N_\phi(P\phi + \nabla N_s\xi_2).
\end{aligned}$$

Replacing the pressure ψ by the above expression in the equations satisfied by ξ_1 and ξ_2, we obtain

$$\begin{aligned}
\lambda\xi_1 - \lambda(-A_{\alpha,\beta})^{-1}A_3^*N_s\xi_2 = {}& -\xi_2 + (-A_{\alpha,\beta})^{-1}(A_4^* + A_1^*\nabla N_s)\xi_2 + (-A_{\alpha,\beta})^{-1}A_1^*P\phi\\
&-(-A_{\alpha,\beta})^{-1}A_3^*\gamma_s N_\phi(P\phi + \nabla N_s\xi_1) + G_s^1,
\end{aligned}$$

and

$$\lambda(I + \gamma_s N_s)\xi_2 = -A_{\alpha,\beta}\xi_1 + \gamma\Delta\xi_2 + \gamma_s N_\phi\nabla N_s\xi_2 + A_2^*P\phi + \gamma_s N_\phi(P\phi + \nabla N_s\xi_2) + G_s^2.$$

We also have

$$\sigma(\phi,\psi)e_2 = 2\nu\varepsilon(\phi)e_2 - \psi e_2 = 2\nu\varepsilon(P\phi + \nabla N_s\xi_2)e_2 + \lambda N_s\xi_2 e_2 - N_\phi(P\phi + \nabla N_s\xi_2)e_2.$$

Thus, the last equation of the system becomes

$$\begin{aligned}
\lambda\mathbf{h} - \Lambda\mathbf{h} - \lambda\left(\int_{\Gamma_b}(N_s\xi_2)e_2\cdot w_i\right)_i = {}& -\left(\int_{\Gamma_b}N_\phi(P\phi + \nabla N_s\xi_2)e_2\cdot w_i\right)_i\\
&+\left(\int_{\Gamma_b}2\nu\varepsilon(P\phi + \nabla N_s\xi_2)e_2\cdot w_i\right)_i + \mathbf{G}_c.
\end{aligned}$$

□

Proposition 1.A.2. *The adjoint of* $(\mathcal{A}_e, D(\mathcal{A}_e))$ *in* Z_e *is* $(\mathcal{A}_e^*, D(\mathcal{A}_e^*))$ *defined by*

$$D(\mathcal{A}_e^*) = \Big\{(P\phi, \xi_1, \xi_2, \mathbf{h}) \in \mathbf{V}^1_{n,\Gamma_d}(\Omega)) \times \big(H^4(\Gamma_s) \cap H^2_0(\Gamma_s)\big) \times H^2_0(\Gamma_s) \times \mathbb{R}^{n_c} \mid P\phi - PD\xi_2 \in D(A^*)\Big\},$$

and

$$\mathcal{A}_e^* = \mathcal{A}_e^{\#} M_{s,e}^{-*}.$$

Proof. Let $(F_f, F_s^1, F_s^2, \mathbf{F}_c)$ and $(G_f, G_s^1, G_s^2, \mathbf{G}_c)$ be in Z_e. We consider the two following systems

$$\begin{aligned}
&\lambda v - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 = F_f \text{ in } \Omega,\\
&\operatorname{div} v = A_3\eta_1 \text{ in } \Omega, \quad v = \eta_2 e_2 \text{ on } \Gamma_s, \quad v = \textstyle\sum_{i=1}^{n_c} g_i w_i \text{ on } \Gamma_b,\\
&v = 0 \text{ on } \Gamma_i, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n,\\
&\lambda\eta_1 - \eta_2 = F_s^1 \text{ on } \Gamma_s,\\
&\lambda\eta_2 - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 - A_4\eta_1 = \gamma_s p + F_s^2 \text{ on } \Gamma_s,\\
&\eta_1 = 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial\Gamma_s,\\
&\lambda\mathbf{g} - \Lambda\mathbf{g} = \mathbf{F}_c,
\end{aligned} \tag{1.A.6}$$

and

$$\begin{aligned}
&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = G_f \text{ in } \Omega,\\
&\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = \xi_2 e_2 \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_b,\\
&\phi = 0 \text{ on } \Gamma_i, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \text{ on } \Gamma_n,\\
&\lambda\xi_1 + \xi_2 - (-A_{\alpha,\beta})^{-1}A_4^*\xi_2 - (-A_{\alpha,\beta})^{-1}A_1^*\phi + (-A_{\alpha,\beta})^{-1}A_3^*\psi = G_s^1 \text{ on } \Gamma_s,\\
&\lambda\xi_2 + \beta\Delta\xi_1 - \gamma\Delta\xi_2 - \alpha\Delta^2\xi_1 - A_2^*\phi - \gamma_s\psi = G_s^2 \text{ on } \Gamma_s,\\
&\xi_1 = 0 \text{ on } \partial\Gamma_s, \quad \xi_{1,x} = 0 \text{ on } \partial\Gamma_s,\\
&\lambda\mathbf{h} - \Lambda\mathbf{h} - \left(\int_{\Gamma_b}\sigma(\phi,\psi)e_2\cdot w_i\right)_i = \mathbf{G}_c.
\end{aligned} \tag{1.A.7}$$

By integration by parts, we prove that

$$\begin{aligned}
\int_\Omega F_f\cdot\phi &= \int_\Omega G_f\cdot v - \int_{\Gamma_s}\psi\eta_2 - \sum_{i=1}^{n_c} g_i\int_{\Gamma_b}\sigma(\phi,\psi)e_2\cdot w_i + \int_{\Gamma_s} p\xi_2 - \int_\Omega A_1\eta_1\cdot\phi\\
&\quad - \int_{\Gamma_s} A_2\eta_2\cdot\phi + \int_\Omega A_3\eta_1\cdot\psi, \eta_1\\
&= \int_\Omega G_f\cdot v - \int_{\Gamma_s}\psi\eta_2 - \sum_{i=1}^{n_c} g_i\int_{\Gamma_b}\sigma(\phi,\psi)e_2\cdot w_i + \int_{\Gamma_s} p\xi_2 - \langle A_1^*\phi, \eta_1\rangle_{H^{-2}(\Gamma_s),H^2_0(\Gamma_s)}\\
&\quad - \int_{\Gamma_s}\eta_2 A_2^*\phi + \langle A_3^*\psi, \eta_1\rangle_{H^{-2}(\Gamma_s),H^2_0(\Gamma_s)}.
\end{aligned}$$

We also have

$$\begin{aligned}
\int_{\Gamma_s} G_s^1(-A_{\alpha,\beta})\eta_1 &= \int_{\Gamma_s}(-A_{\alpha,\beta})\xi_1(\eta_2 + F_s^1) + \int_{\Gamma_s}\xi_2(-A_{\alpha,\beta})\eta_1 - \int_{\Gamma_s}(-A_{\alpha,\beta})^{-1}A_4^*\xi_2(-A_{\alpha,\beta})\eta_1\\
&\quad - \int_{\Gamma_s}(-A_{\alpha,\beta})^{-1}A_1^*\phi(-A_{\alpha,\beta})\eta_1 + \int_{\Gamma_s}(-A_{\alpha,\beta})^{-1}A_3^*\psi(-A_{\alpha,\beta})\eta_1\\
&= \int_{\Gamma_s}(-A_{\alpha,\beta})\xi_1(\eta_2 + F_s^1) + \int_{\Gamma_s}\xi_2(-A_{\alpha,\beta})\eta_1 - \langle A_4^*\xi_2, \eta_1\rangle_{H^{-2}(\Gamma_s),H^2_0(\Gamma_s)}\\
&\quad - \langle A_1^*\phi, \eta_1\rangle_{H^{-2}(\Gamma_s),H^2_0(\Gamma_s)} + \langle A_3^*\psi, \eta_1\rangle_{H^{-2}(\Gamma_s),H^2_0(\Gamma_s)},
\end{aligned}$$

and

$$\int_{\Gamma_s}(G_s^2+\gamma_s\psi)\eta_2 = \int_{\Gamma_s}\xi_2 F_s^2 - \int_{\Gamma_s}(-A_{\alpha,\beta})\eta_1\xi_2 - \int_{\Gamma_s}(-A_{\alpha,\beta})\xi_1\eta_2 + \langle A_4^*\xi_2,\eta_1\rangle_{H^{-2}(\Gamma_s),H_0^2(\Gamma_s)} + \int_{\Gamma_s} p\xi_2 - \int_{\Gamma_s}\eta_2 A_2^*\phi.$$

By combining the three identities, we obtain the identity

$$\left(\begin{pmatrix} F_f\\ F_s^1\\ F_s^2\\ \mathbf{F}_c\end{pmatrix},\begin{pmatrix}\phi\\ \xi_1\\ \xi_2\\ \mathbf{h}\end{pmatrix}\right)_{Z_e} = \left(\begin{pmatrix} v\\ \eta_1\\ \eta_2\\ \mathbf{g}\end{pmatrix},\begin{pmatrix}G_f\\ G_s^1\\ G_s^2\\ \mathbf{G}_c\end{pmatrix}\right)_{Z_e}. \tag{1.A.8}$$

We have to interpret the identity. According to Proposition 1.6 .6, $(Pv,\eta_1,\eta_1,\mathbf{g})$ satisfies

$$(\lambda M_{s,e}-M_{s,e}\mathcal{A}_e)\begin{pmatrix}Pv\\ \eta_1\\ \eta_2\\ \mathbf{g}\end{pmatrix} = \begin{pmatrix} F_f\\ F_s^1\\ F_s^2\\ \mathbf{F}_c\end{pmatrix}.$$

Thanks to Proposition 1.A.1, $(P\phi,\xi_1,\xi_2,\mathbf{h})$ satisfies

$$(\lambda M_{s,e}^*-\mathcal{A}_e^{\#})\begin{pmatrix}P\phi\\ \xi_1\\ \xi_2\\ \mathbf{h}\end{pmatrix} = \begin{pmatrix} G_f\\ G_s^1\\ G_s^2\\ \mathbf{G}_c\end{pmatrix}.$$

Thus, identity (1.A.8) is equivalent to

$$\left((\lambda M_{s,e}-M_{s,e}\mathcal{A}_e)\begin{pmatrix}Pv\\ \eta_1\\ \eta_2\\ \mathbf{g}\end{pmatrix},\begin{pmatrix}P\phi\\ \xi_1\\ \xi_2\\ \mathbf{h}\end{pmatrix}\right)_{Z_e} = \left(\begin{pmatrix}Pv\\ \eta_1\\ \eta_2\\ \mathbf{g}\end{pmatrix},(\lambda M_{s,e}^*-\mathcal{A}_e^{\#})\begin{pmatrix}P\phi\\ \xi_1\\ \xi_2\\ \mathbf{h}\end{pmatrix}\right)_{Z_e}.$$

Therefore, the unbounded operator $(\mathcal{A}_e^*, D(\mathcal{A}_e^*))$ is the adjoint of $(\mathcal{A}_e, D(\mathcal{A}_e))$ in Z_e.

□

1.B Appendix B: Nonlinear terms

The nonlinear terms $\mathcal{F}_f$ and $\mathcal{F}_s$ in system (1.3 .3) are defined by

$$\begin{aligned}
\mathcal{F}_f[\hat u,\hat p,\hat\eta_1,\hat\eta_2] &= z\hat\eta_2\hat u_z - \nu z\hat\eta_{1,xx}\hat u_z - 2\nu z\hat\eta_{1,x}\hat u_{xz} - 2\nu\hat\eta_1\hat u_{zz} - \nu\nabla\mathrm{div}\,\mathcal{F}_{\mathrm{div}}[\hat u,\hat\eta_1] + z\hat\eta_{1,x}\hat p_z e_1\\
&+\hat\eta_1\hat p_z e_2 + ze^{-\omega t}\hat\eta_{1,x}\hat u_1\hat u_z + z\hat\eta_{1,x}\hat u_1 u_{s,z} + z\hat\eta_{1,x}u_{s,1}\hat u_z + e^{-\omega t}\hat\eta_1\hat u_2\hat u_z + \hat\eta_1\hat u_2 u_{s,z} + \hat\eta_1 u_{s,2}\hat u_z\\
&-\frac{z\hat\eta_1\hat\eta_2}{1+e^{-\omega t}\hat\eta_1}(e^{-\omega t}\hat u_z + u_{s,z}) + \frac{\nu z\hat\eta_1\hat\eta_{1,xx}}{1+e^{-\omega t}\hat\eta_1}(e^{-\omega t}\hat u_z+u_{s,z}) + \frac{2\nu z\hat\eta_{1,x}^2}{(1+e^{-\omega t}\hat\eta_1)^2}(e^{-\omega t}\hat u_z+u_{s,z})\\
&+\frac{2\nu z\hat\eta_1\hat\eta_{1,x}}{1+e^{-\omega t}\hat\eta_1}(e^{-\omega t}\hat u_{xz}+u_{s,xz}) + \frac{\nu(z^2\hat\eta_{1,x}^2-\hat\eta_1^2)}{(1+e^{-\omega t}\hat\eta_1)^2}(e^{-\omega t}\hat u_{zz}+u_{s,zz}) - \frac{z\hat\eta_1\hat\eta_{1,x}}{1+e^{-\omega t}\hat\eta_1}(e^{-\omega t}\hat p_z + p_{s,z})e_1\\
&-\frac{\hat\eta_1^2}{1+e^{-\omega t}\hat\eta_1}(e^{-\omega t}\hat p_z+p_{s,z})e_2 + \frac{2\nu(2+e^{-\omega t}\hat\eta_1)\hat\eta_1^2}{(1+e^{-\omega t}\hat\eta_1)^2}(e^{-\omega t}\hat u_{zz}+u_{s,zz}) + (\hat u\cdot\nabla)\hat u\\
&-\frac{z\hat\eta_1\hat\eta_{1,x}}{1+e^{-\omega t}\hat\eta_1}(e^{-\omega t}\hat u_1+u_{s,1})(e^{-\omega t}\hat u_z+u_{s,z}) - \frac{\hat\eta_1^2}{1+e^{-\omega t}\hat\eta_1}(e^{-\omega t}\hat u_2+u_{s,2})(e^{-\omega t}\hat u_z+u_{s,z}),
\end{aligned}$$

and

$$\mathcal{F}_s[\hat{u},\hat{\eta}_1] = \nu\left(\hat{\eta}_{1,x}\hat{u}_{1,z} + \hat{\eta}_{1,x}\hat{u}_{2,x} + 2\hat{\eta}_1\hat{u}_{2,z}\right) - \frac{\nu\hat{\eta}_1\hat{\eta}_{1,x}}{1+e^{-\omega t}\hat{\eta}_1}(e^{-\omega t}\hat{u}_{1,z} + u_{s,1,z})$$
$$-\frac{2\nu\hat{\eta}_1^2}{1+e^{-\omega t}\hat{\eta}_1}(e^{-\omega t}\hat{u}_{2,z} + u_{s,2,z}) - \frac{\nu z\hat{\eta}_{1,x}^2}{1+e^{-\omega t}\hat{\eta}_1}(e^{-\omega t}\hat{u}_{2,z} + u_{s,2,z}).$$

Chapter 2

Feedback stabilization of Navier-Stokes equations by boundary deformation

2.1 Introduction

We are interested in the stabilization of a fluid flow over and below a thick plate, locally around an unstable stationary flow, by boundary deformations of the plate. The reference configuration of the fluid domain is

$$\Omega = [0, L] \times [-\ell, \ell] \setminus [0, \ell_s] \times [-e, e],$$

where L, ℓ, ℓ_s and e are strictly positive. The boundary Γ of Ω is splitted into different parts

$$\Gamma = \Gamma_s \cup \Gamma_i \cup \Gamma_e \cup \Gamma_n,$$

where Γ_s is the part occupied by the structure, Γ_i is the part where inflow Dirichlet boundary condition is imposed, Γ_e and Γ_n are parts where Navier and Neumann boundary conditions are respectively prescribed (see *Figure* 2.1). The velocity u_s and the pressure p_s of the stationary flow are described by the incompressible Navier-Stokes equations

$$\begin{aligned}
&-\mathrm{div}\,\sigma(u_s, p_s) + (u_s \cdot \nabla)u_s = 0 \text{ in } \Omega,\\
&\mathrm{div}\, u_s = 0 \text{ in } \Omega, \quad u_s = 0 \text{ on } \Gamma_s, \quad u_s = g_s \text{ on } \Gamma_i, \qquad (2.1\ .1)\\
&u_s \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(u_s)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(u_s, p_s)n = 0 \text{ on } \Gamma_n,
\end{aligned}$$

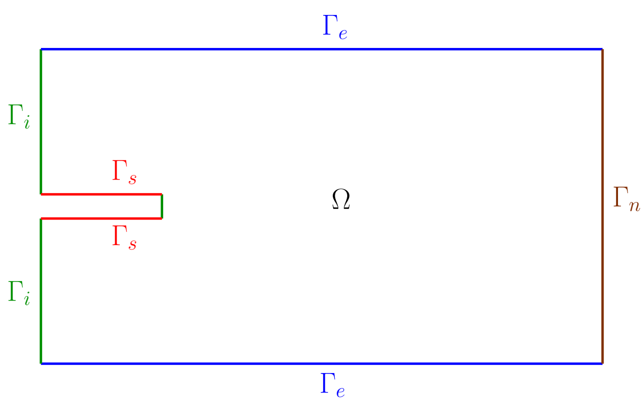


Figure 2.1 – Reference configuration of the fluid domain.

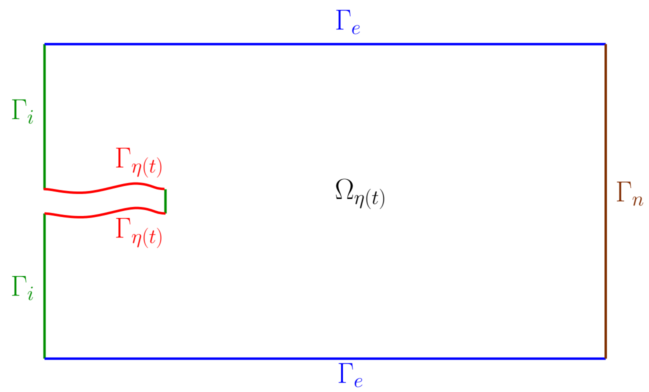


Figure 2.2 – Deformed configuration of the fluid domain.

where $g_s \in H_0^2(\Gamma_i;\mathbb{R}^2)$ is an inflow boundary condition. The stress tensor $\sigma(u_s,p_s)$ is defined by the expression

$$\sigma(u_s,p_s) = 2\,\nu\varepsilon(u_s) - p_sI, \quad \varepsilon(u_s) = \frac{1}{2}(\nabla u_s + (\nabla u_s)^T),$$

where ν is the fluid viscosity. We know that if the Reynolds number passes a certain critical value, the stationary flow (u_s,p_s) is unstable. We assume that we are in that case. We consider a perturbation h of the inflow boundary condition g_s. The goal is to stabilize the perturbed fluid flow around the stationary flow by deformation of the elastic boundary Γ_s of the fluid domain. We assume that the displacement η of the the structure is governed by an ordinary differential equation

$$\begin{aligned} &\eta(t,x) = \sum_{i=1}^{n_c} \eta^i(t)w_i(x), \\ &\boldsymbol{\eta}_{tt} = -\alpha\boldsymbol{\eta} - \gamma\boldsymbol{\eta}_t + \boldsymbol{f}, \\ &\boldsymbol{\eta}(0) = \boldsymbol{\eta}_1^0, \quad \boldsymbol{\eta}_t(0) = \boldsymbol{\eta}_2^0, \end{aligned} \tag{2.1 .2}$$

where $\boldsymbol{\eta} = [\eta^1,\cdots,\eta^{n_c}] \in \mathbb{R}^{n_c}$, $\boldsymbol{f} \in \mathbb{R}^{n_c}$, $\alpha > 0$ and $\gamma \geq 0$. The functions $(w_i)_{1\leq i\leq n_c} \subset H^4(\Gamma_s) \cap H_0^2(\Gamma_s)$ are constructed in Section 2.7 .3. We denote by $\Omega_{\eta(t)}$ the domain occupied by the fluid at time t and by $\Gamma_{\eta(t)}$ the boundary of $\Omega_{\eta(t)}$ occupied by the structure (see *Figure* 2.2). We use the notations

$$\begin{aligned} &Q_\eta^\infty = \bigcup_{t\in(0,\infty)} \Big(\{t\}\times\Omega_{\eta(t)}\Big), \quad \Sigma_\eta^\infty = \bigcup_{t\in(0,\infty)} \Big(\{t\}\times\Gamma_{\eta(t)}\Big), \\ &Q^\infty = (0,\infty)\times\Omega, \quad \Sigma_s^\infty = (0,\infty)\times\Gamma_s, \\ &\Sigma_i^\infty = (0,\infty)\times\Gamma_i, \quad \Sigma_e^\infty = (0,\infty)\times\Gamma_e, \\ &\Sigma_n^\infty = (0,\infty)\times\Gamma_n. \end{aligned}$$

We also use the notations

$$e_1 = (1,0) \quad \text{and} \quad e_2 = (0,1).$$

For any prescribed exponential decay rate $-\omega < 0$, and for h, u^0, $\boldsymbol{\eta}_1^0$ and $\boldsymbol{\eta}_2^0$, such that h, $u^0 - u_s$, $\boldsymbol{\eta}_1^0$ and $\boldsymbol{\eta}_2^0$ are small enough in appropriate spaces, we would like to find a control $\boldsymbol{f}$, in feedback form, such that the solution $(u, p, \boldsymbol{\eta})$ of system

$$
\begin{aligned}
& u_t - \operatorname{div}\sigma(u,p) + (u\cdot\nabla)u = 0 \text{ in } Q_\eta^\infty,\\
& \operatorname{div} u = 0 \text{ in } Q_\eta^\infty, \quad u = \textstyle\sum_{i=1}^{n_c} \eta_t^i w_i n \text{ on } \Sigma_\eta^\infty, \quad u = g_s + h \text{ on } \Sigma_i^\infty,\\
& u\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(u)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(u,p)n = 0 \text{ on } \Sigma_n^\infty,\\
& \boldsymbol{\eta}_{tt} = -\alpha\boldsymbol{\eta} - \gamma\boldsymbol{\eta}_t + \boldsymbol{f},\\
& u(0) = u_0 \text{ on } \Omega_{\eta_1^0}, \quad \boldsymbol{\eta}(0) = \boldsymbol{\eta}_1^0, \quad \boldsymbol{\eta}_t(0) = \boldsymbol{\eta}_2^0,
\end{aligned}
\tag{2.1 .3}
$$

converges to the stationary solution $(u_s, p_s, \mathbf{0})$ exponentially (see Theorem 2.3 .1). The equation $u = \sum_{i=1}^{n_c} \eta_t^i w_i n$ in system (2.1 .3) corresponds to the equality of the fluid velocity and the displacement velocity of the structure. In the second part of the chapter, we look for controls, in feedback form, able to stabilize semi-discrete systems associated to (2.1 .3). Thanks to numerical tests, we show that the linear feedback laws are able to locally stabilize the full discretization of the nonlinear system around the stationary solution. To determine the feedback law stabilizing the nonlinear system (2.1 .3), we proceed into the following steps.

Step 1: *System in a fixed domain and linearization.*

Thanks to the change of unknowns introduced in (2.3 .1), we rewrite system (2.1 .3) in a fixed domain

$$
\begin{aligned}
& \tilde u_t - \operatorname{div}\sigma(\tilde u,\tilde p) + (\tilde u\cdot\nabla)\tilde u = F_f[\tilde u,\tilde p,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2] \text{ in } Q^\infty,\\
& \operatorname{div}\tilde u = \operatorname{div} F_{\mathrm{div}}[\tilde u,\boldsymbol{\eta}_1] \text{ in } Q^\infty, \quad \tilde u = A_4\eta_2 n \text{ on } \Sigma_s^\infty, \quad \tilde u = h \text{ on } \Sigma_i^\infty,\\
& \tilde u = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\tilde u)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\tilde u,\tilde p)n = 0 \text{ on } \Sigma_n^\infty,\\
& \boldsymbol{\eta}_{1,t} - \boldsymbol{\eta}_2 = 0,\\
& \boldsymbol{\eta}_{2,t} + \alpha\boldsymbol{\eta}_1 + \gamma\boldsymbol{\eta}_2 = \boldsymbol{f},\\
& \tilde u(0) = \tilde u^0 \text{ in } \Omega, \quad \boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_1^0, \quad \boldsymbol{\eta}_2(0) = \boldsymbol{\eta}_2^0,
\end{aligned}
\tag{2.1 .4}
$$

with $\boldsymbol{\eta}_1 = \boldsymbol{\eta}$ and $\boldsymbol{\eta}_2 = \boldsymbol{\eta}_t$. The nonlinear terms $F_f[\tilde u,\tilde p,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2]$ and $F_{\mathrm{div}}[\tilde u,\boldsymbol{\eta}_1]$ are defined in (4.A.1). Since we want to stabilize the above system around the stationary solution $(u_s, p_s, \mathbf{0}, \mathbf{0})$ with an exponential decay rate $-\omega$, we make the change of unknowns

$$
\hat u = e^{\omega t}(\tilde u - u_s), \quad \hat p = e^{\omega t}(\tilde p - p_s), \quad \hat{\boldsymbol{\eta}}_1 = e^{\omega t}\boldsymbol{\eta}_1 \quad \text{and} \quad \hat{\boldsymbol{\eta}}_2 = e^{\omega t}\boldsymbol{\eta}_2.
$$

The system satisfied by $(\hat u, \hat p, \hat{\boldsymbol{\eta}}_1, \hat{\boldsymbol{\eta}}_2)$ is

$$
\begin{aligned}
& \hat u_t - \operatorname{div}\sigma(\hat u,\hat p) + (u_s\cdot\nabla)\hat u + (\hat u\cdot\nabla)u_s - A_1\hat{\boldsymbol{\eta}}_1 - A_2\hat{\boldsymbol{\eta}}_2 - \omega\hat u = e^{-\omega t}\mathcal{F}_f[\hat u,\hat p,\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_2] \text{ in } Q^\infty,\\
& \operatorname{div}\hat u = A_3\hat{\boldsymbol{\eta}}_1 + e^{-\omega t}\operatorname{div} F_{\mathrm{div}}[\hat u,\hat{\boldsymbol{\eta}}_1] \text{ in } Q^\infty, \quad \hat u = A_4\hat{\boldsymbol{\eta}}_2 n \text{ on } \Sigma_s^\infty, \quad \hat u = \hat h \text{ on } \Sigma_i^\infty,\\
& \hat u = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\hat u)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\hat u,\hat p)n = 0 \text{ on } \Sigma_n^\infty,\\
& \hat{\boldsymbol{\eta}}_{1,t} - \hat{\boldsymbol{\eta}}_2 - \omega\hat{\boldsymbol{\eta}}_1 = 0,\\
& \hat{\boldsymbol{\eta}}_{2,t} + \alpha\hat{\boldsymbol{\eta}}_1 + \gamma\hat{\boldsymbol{\eta}}_2 - \omega\hat{\boldsymbol{\eta}}_2 = \hat{\boldsymbol{f}},\\
& \hat u(0) = \hat u^0 \text{ in } \Omega, \quad \hat{\boldsymbol{\eta}}_1(0) = \boldsymbol{\eta}_1^0, \quad \hat{\boldsymbol{\eta}}_2(0) = \boldsymbol{\eta}_2^0.
\end{aligned}
\tag{2.1 .5}
$$

Step 2: *Reformulation of the linearized system and stabilizability.*

The goal is to rewrite the linearized system

$$\begin{aligned}
&v_t - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s + \omega v - A_1\boldsymbol{\eta}_1 - A_2\boldsymbol{\eta}_2 = 0 \text{ in } Q^\infty,\\
&\operatorname{div} v = A_3\boldsymbol{\eta}_1 \text{ in } Q^\infty, \quad v = A_4\boldsymbol{\eta}_2 n \text{ on } \Sigma_s^\infty, \quad v = 0 \text{ on } \Sigma_i^\infty,\\
&v\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(v)n\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(v,p)n = 0 \text{ on } \Sigma_n^\infty, \quad v(0) = v^0 \text{ in } \Omega,\\
&\boldsymbol{\eta}_{1,t} - \boldsymbol{\eta}_2 - \omega\boldsymbol{\eta}_1 = 0, \quad \boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_1^0,\\
&\boldsymbol{\eta}_{2,t} + \alpha\boldsymbol{\eta}_1 + \gamma\boldsymbol{\eta}_2 - \omega\boldsymbol{\eta}_2 = \boldsymbol{f}, \quad \boldsymbol{\eta}_2(0) = \boldsymbol{\eta}_2^0,
\end{aligned} \tag{2.1 .6}$$

as a control system where the operators A_1, A_2, A_3 and A_4 are defined in (2.4 .4). To do that, we have to eliminate the pressure of the fluid equations. In chapter 1, we have used the Leray projector which leads to split the velocity into two parts. Here, we are going to use a projector taking into account the fluid-structure couplings of the linearized system. This approach is followed in [5]. However, we consider here an oblique projector rather than an orthogonal. Thus, we also take into account the fluid-structure couplings of the adjoint of the linearized system. Thanks to that projector, denoted by Π, we prove that the linearized system is equivalent to the following one

$$\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix} = (\mathcal{A}+\omega I)\begin{pmatrix} v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix} + \mathcal{B}\boldsymbol{f}, \quad \begin{pmatrix} v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}(0) = \begin{pmatrix} v^0\\ \boldsymbol{\eta}_1^0\\ \boldsymbol{\eta}_2^0\end{pmatrix},\\
&p = -N_sA_4(\boldsymbol{\eta}_{2,t} - \omega\boldsymbol{\eta}_2) - N_{\mathrm{div}}A_3(\boldsymbol{\eta}_{1,t} - \omega\boldsymbol{\eta}_1) + N_pA_1\boldsymbol{\eta}_1 + N_pA_2\boldsymbol{\eta}_2 + N_v v.
\end{aligned} \tag{2.1 .7}$$

The operators $\mathcal{A}$ and $\mathcal{B}$ are defined in a space Z (see (2.6 .3) and (2.7 .4)) and the operators N_s, N_{div}, N_p and N_v are given in Section 2.5 . We prove that $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup with compact resolvent. This allows us to reduce the stabilizability of the pair $(\mathcal{A}+\omega I, \mathcal{B})$ to a unique continuation problem. Thanks to an appropriate choice of the family $(w_i)_{1\le i\le n_c}$, we prove that the pair $(\mathcal{A}+\omega I, \mathcal{B})$ is stabilizable.

Step 3: *Projection onto the unstable subspace associated to* $(\mathcal{A}+\omega I, \mathcal{B})$.

To construct a feedback control law which can be easily determined numerically, we are going to project system (2.1 .6) onto the unstable subspace associated to $\mathcal{A}+\omega I$. Let $(\lambda_j)_j$ be the eigenvalues of $\mathcal{A}$. We denote by $G_{\mathbb{R}}(\lambda_j)$ the real generalized eigenspace of $\mathcal{A}$ associated to λ_j. We choose a finite family $(\lambda_j)_{j\in J_u}$ containing at least all the unstable eigenvalues (with nonnegative real part) of $\mathcal{A}+\omega I$ and set

$$Z_u = \oplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j).$$

There exists a subspace Z_s of Z, invariant under $(e^{t\mathcal{A}})_{t\ge 0}$, such that

$$Z = Z_u \oplus Z_s.$$

We denote by π_u the projection from Z onto Z_u parallel to Z_s. We prove that there exist two families $(v_i, \boldsymbol{\eta}_{1,i}, \boldsymbol{\eta}_{2,i})_{1\le i\le d_u}$ and $(\phi_i, \boldsymbol{\xi}_{1,i}, \boldsymbol{\xi}_{2,i})_{1\le i\le d_u}$ satisfying the bi-orthogonality condition

$$\left(\begin{pmatrix} v_i\\ \boldsymbol{\eta}_{1,i}\\ \boldsymbol{\eta}_{2,i}\end{pmatrix}, \begin{pmatrix} \phi_i\\ \boldsymbol{\xi}_{1,i}\\ \boldsymbol{\xi}_{2,i}\end{pmatrix}\right) = \delta_{i,j}, \quad \forall\, 1\le i,j\le d_u,$$

such that

$$\pi_u \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \sum_{i=1}^{d_u} \left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi_i \\ \boldsymbol{\xi}_{1,i} \\ \boldsymbol{\xi}_{2,i} \end{pmatrix} \right) \begin{pmatrix} v_i \\ \boldsymbol{\eta}_{1,i} \\ \boldsymbol{\eta}_{2,i} \end{pmatrix}, \quad \forall\, (v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T \in Z,$$

and that the matrix Λ_u defined by

$$\Lambda_u = [\Lambda_{u,i,j}]_{1\leq i,j\leq d_u} \quad \text{and} \quad \Lambda_{u,i,j} = \left(\mathcal{A} \begin{pmatrix} v^i \\ \boldsymbol{\eta}_{1,i} \\ \boldsymbol{\eta}_{2,i} \end{pmatrix}, \begin{pmatrix} \phi_j \\ \boldsymbol{\xi}_{1,j} \\ \boldsymbol{\xi}_{2,j} \end{pmatrix} \right),$$

is constituted of real Jordan blocks. Moreover, we can prove that there exists p_i (resp. ψ_i) such that $(v_i, p_i, \boldsymbol{\eta}_{1,i}, \boldsymbol{\eta}_{2,i})$ (resp. $(\phi_i, \psi_i, \boldsymbol{\xi}_{1,i}, \boldsymbol{\xi}_{2,i})$) corresponds to the real or imaginary part of a generalized eigenfunctions of the eigenvalue problem (2.7 .7) associatied to the linearized system (2.1 .6) (resp. (2.7 .8) associated to the adjoint of (2.1 .6)). It follows from the relations between the generalized eigenfunctions of (2.7 .7) and (2.7 .5) on the one hand and the relations between the generalized eigenfunctions of (2.7 .8) and (2.7 .6) on the other hand. Thus, we can construct π_u without having to construct the projector Π which is not easy to approximate.

Step 4: *Stabilization of the linearized system.*

Since the pair $(\mathcal{A} + \omega I, \mathcal{B})$ is stabilizable then $(\Lambda_u + \omega I, B_u)$, where B_u is defined in (2.7 .22), is also stabilizable. Thus, the operator $K_u = -B_u^T P_u$, where P_u is the solution of the algebraic Riccati equation

$$\begin{aligned} &P_u \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_u}, \quad P_u = P_u^T > 0, \\ &P_u(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}})P_u - P_u B_u B_u^T P_u = 0, \end{aligned}$$

provides a stabilizing feedback law for $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, B_u)$. We deduce a feedback stabilizing the pair $(\mathcal{A} + \omega I, \mathcal{B})$ and then the linearized system.

Step 5: *Stabilization of the nonlinear system.*

We use a fixed point method to prove that the linear feedback law is able to locally stabilize nonlinear system (2.1 .4) or equivalently the original system (2.1 .3).

We follow a similar approach to find a feedback law able to stabilize semi-discrete systems.

2.2 Functional setting

We define the classical Sobolev spaces in two dimensions

$$\mathbf{L}^2(\Omega) = L^2(\Omega; \mathbb{R}^2) \quad \text{and} \quad \mathbf{H}^s(\Omega) = H^s(\Omega; \mathbb{R}^2), \quad \forall\, s > 0.$$

We set

$$\Gamma_0 = \Gamma_s \cup \Gamma_i \cup \Gamma_e.$$

We define the functional spaces

$$
\begin{aligned}
&\mathbf{V}^0_{n,\Gamma_0}(\Omega) = \{v \in \mathbf{L}^2(\Omega) \mid \operatorname{div} v = 0,\ v\cdot n = 0 \text{ on } \Gamma_0\},\\
&\mathbf{V}(\Omega) = \{v \in \mathbf{L}^2(\Omega) \mid \operatorname{div} v \in \mathbf{L}^2(\Omega)\},\\
&\mathbf{H}^1_{\Gamma_0}(\Omega) = \{v \in \mathbf{H}^1(\Omega) \mid v = 0 \text{ on } \Gamma_s \cup \Gamma_i,\ v\cdot n = 0 \text{ on } \Gamma_e\},\\
&\mathbf{V}^1_{n,\Gamma_0}(\Omega) = \mathbf{H}^1_{\Gamma_0}(\Omega) \times \mathbf{V}^0_{n,\Gamma_0}(\Omega),\\
&H = \mathbf{L}^2(\Omega) \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c},\\
&\widehat{Z} = \mathbf{V}^0_{n,\Gamma_0}(\Omega) \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c},\\
&Z = \Big\{(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in H \mid \operatorname{div} v = A_3\boldsymbol{\eta}_1 \text{ in } \Omega,\ v\cdot n = A_4\boldsymbol{\eta}_2 \text{ on } \Gamma_s,\ v\cdot n = 0 \text{ on } \Gamma_0 \setminus \Gamma_s\Big\}.
\end{aligned}
$$

For $w^0 \in \mathbf{H}^1(\Omega)$ given, we define the space

$$
\begin{aligned}
H_{ic}(w^0) = \{(v^0, \boldsymbol{\eta}^0_1, \boldsymbol{\eta}^0_2) \in H \mid\ & (v^0 - w^0, \boldsymbol{\eta}^0_1, \boldsymbol{\eta}^0_2) \in [D(\mathcal{A}), Z]_{\frac{1}{2}},\ \operatorname{div} v^0 = \operatorname{div} w^0 + A_3\boldsymbol{\eta}^0_1 \text{ in } \Omega,\\
& v = A_4\boldsymbol{\eta}^0_2 n \text{ in } \text{ on } \Gamma_s,\ v^0 = 0 \text{ on } \Gamma_i,\ v\cdot n = 0 \text{ on } \Gamma_e\},
\end{aligned}
$$

equipped with the norm

$$
\|(v^0, \boldsymbol{\eta}^0_1, \boldsymbol{\eta}^0_2)\|_{ic} = \|(v^0, \boldsymbol{\eta}^0_1, \boldsymbol{\eta}^0_2)\|_H + \|(v^0 - w^0, \boldsymbol{\eta}^0_1, \boldsymbol{\eta}^0_2)\|_{[D(\mathcal{A}),Z]_{\frac{1}{2}}}.
$$

The operators A_3 and A_4 are defined in (2.4 .4) and the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ is defined in (2.6 .3). Now, we are going to introduce the weighted Sobolev spaces. We denote by $\mathcal{C} = (C_j)_{1\le j\le 8}$ the set of the corners of Ω. For all $\delta, s \ge 0$ and for $n = 1$ or $n = 2$, we introduce the norms

$$
\|v\|^2_{H^2_\delta(\Omega;\mathbb{R}^n)} = \sum_{|\alpha|\le 2} \int_\Omega \prod_{j\in\mathcal{C}} r_j^{2\delta} |\partial_\alpha v|^2,
$$

where r_j stands for the distance to $C_j \in \mathcal{C}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ denotes a two-index, $|\alpha| = \alpha_1 + \alpha_2$ is its length, ∂_α denotes the corresponding partial differential operator. We denote by $H^2_\delta(\Omega;\mathbb{R}^n)$ the closure of $C^\infty(\overline{\Omega})$ in the norms $\|\cdot\|_{H^2_\delta(\Omega;\mathbb{R}^n)}$. We define weighted Sobolev spaces in two dimensions

$$
\mathbf{L}^2_\delta(\Omega) = H^0_\delta(\Omega;\mathbb{R}^2) \quad \text{and} \quad \mathbf{H}^s_\delta(\Omega) = H^s_\delta(\Omega;\mathbb{R}^2), \quad \forall\, s > 0,
$$

and the product space

$$
H_\delta = \mathbf{H}^2_\delta(\Omega) \times H^1_\delta(\Omega) \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c}.
$$

We introduce the spaces

$$
\begin{aligned}
&H^{2,1}(Q^\infty) = L^2(0,\infty;\mathbf{H}^2(\Omega)) \cap H^1(0,\infty;\mathbf{L}^2(\Omega)),\\
&H^{2,1}_\delta(Q^\infty) = L^2(0,\infty;\mathbf{H}^2_\delta(\Omega)) \cap H^1(0,\infty;\mathbf{L}^2(\Omega)),\\
&H^{4,2}(\Sigma^\infty_s) = L^2(0,\infty;H^4(\Gamma_s)) \cap H^2(0,\infty;L^2(\Gamma_s)),\\
&Y = L^2(0,\infty;\mathbf{L}^2(\Omega)) \times H^{2,1}(Q^\infty),\\
&X_\delta = H^{2,1}_\delta(Q^\infty) \times L^2(0,\infty;H^1_\delta(\Omega)) \times H^2(0,\infty;\mathbb{R}^{n_c}) \times H^1(0,\infty;\mathbb{R}^{n_c}).
\end{aligned}
$$

2.3 Main results

First, we are going to define the spaces of solutions of system (2.1 .3). For that, we introduce the change of variables

$$
\mathcal{T}_\eta : (t, x, y) \quad \longmapsto \quad (t, x, z) = \left(t, x, \frac{(\ell - e)y - \ell\eta(t, x, s(y)e)}{\ell - e - s(y)\eta(t, x, s(y)e)}\right), \tag{2.3 .1}
$$

and

$$\mathcal{T}_{\eta(t)} : (x,y) \quad \longmapsto \quad (x,z) = \left(x,\ \frac{(\ell-e)y - \ell\eta(t,x,s(y)e)}{\ell - e - s(y)\eta(t,x,s(y)e)}\right), \tag{2.3 .2}$$

where the function s is defined by

$$s(y) = -1 \quad \forall\, y < 0 \quad \text{and} \quad s(y) = 1 \quad \forall\, y \geq 0.$$

We define the mapping $\mathcal{T}_{\eta_1^0}$ in a similar way. The mapping $\mathcal{T}_{\eta(t)}$ transforms $\Omega_{\eta(t)}$ into the reference configuration Ω.

Definition 2.3 .1. *Let η belong to $H^{4,2}(\Sigma_s^\infty)$. We say that u belongs to $H_\delta^{2,1}(Q_\eta^\infty)$ if*

- *for all $t > 0$, the mapping $\mathcal{T}_{\eta(t)}$ is a $\mathcal{C}^1$-diffeomorphism from $\Omega_{\eta(t)}$ into Ω,*
- *$u \circ \mathcal{T}_\eta^{-1}$ belongs to $H_\delta^{2,1}(Q^\infty)$.*

Let δ_0 defined in Theorem 2.5 .2. The main theorem of the chapter is the following result of stabilisation for the nonlinear system (2.1 .3).

Theorem 2.3 .1. *Let ω be positive. We assume that 2.5 .1, 2.7 .2 and 2.7 .1 are satisfied. There exists $r > 0$ such that, for all $(u^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0, h) \in \mathbf{H}^1(\Omega_{\eta_1^0}) \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c} \times H^1(0,\infty; H_0^2(\Gamma_i))$ satisfying $(u^0 \circ \mathcal{T}_{\eta_1^0}^{-1} - u_s, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0) \in H_{ic}(F_{\mathrm{div}}[u^0 \circ \mathcal{T}_{\eta_1^0}^{-1} - u_s, \boldsymbol{\eta}_1^0])$ and*

$$\|(u^0 \circ \mathcal{T}_{\eta_1^0}^{-1} - u_s, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{ic} + \|e^{\omega t} h\|_{H^1(0,\infty;H_0^2(\Gamma_i))} \leq r,$$

we can find $\mathbf{f} \in H^1(0,\infty;\mathbb{R}^{n_c})$, in the feedback form, for which there exists a unique solution $(u, p, \boldsymbol{\eta})$ in

$$H_{\delta_0}^{2,1}(Q_\eta^\infty) \times L^2(0,\infty; H^1(\Omega_{\eta(t)})) \times H^2(0,\infty;\mathbb{R}^{n_c}),$$

to the nonlinear system (2.1 .3) satisfying

$$\left\|(u(t) \circ \mathcal{T}_{\eta(t)}^{-1} - u_s, \boldsymbol{\eta}(t), \boldsymbol{\eta}_t(t))\right\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon}(\Omega)\times\mathbb{R}^{n_c}\times\mathbb{R}^{n_c}} \leq C e^{-\omega t}, \quad \forall t > 0,$$

where $\varepsilon > 0$ and $C > 0$.

2.4 System in the reference configuration

In order to rewrite system (2.1 .3) in the reference configuration, we make the change of unknowns

$$\tilde{u}(t,x,z) := u(t, \mathcal{T}_{\eta(t)}^{-1}(x,z)) - u_s(x,z), \quad \tilde{p}(t,x,z) := p(t, \mathcal{T}_{\eta(t)}^{-1}(x,z)) - p_s(x,z),$$
$$\boldsymbol{\eta}_1(t) = [\boldsymbol{\eta}_{1,1}(t), \cdots, \boldsymbol{\eta}_{1,n_c}(t)] := \boldsymbol{\eta}(t), \quad \boldsymbol{\eta}_2(t) = [\boldsymbol{\eta}_{2,1}(t), \cdots, \boldsymbol{\eta}_{2,n_c}(t)] := \boldsymbol{\eta}_t(t).$$

Thus, system (2.1 .3) is equivalent to the following one

$$\begin{aligned}
&\tilde{u}_t - \operatorname{div}\sigma(\tilde{u},\tilde{p}) + (\tilde{u}\cdot\nabla)\tilde{u} = F_f[\tilde{u},\tilde{p},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2] \text{ in } Q^\infty,\\
&\operatorname{div}\tilde{u} = \operatorname{div}F_{\mathrm{div}}[\tilde{u},\boldsymbol{\eta}_1] \text{ in } Q^\infty, \quad \tilde{u} = \textstyle\sum_{i=1}^{n_c} \boldsymbol{\eta}_{2,i} w_i n \text{ on } \Sigma_s^\infty, \quad \tilde{u} = h \text{ on } \Sigma_i^\infty,\\
&\tilde{u} = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\tilde{u})n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\tilde{u},\tilde{p})n = 0 \text{ on } \Sigma_n^\infty,\\
&\boldsymbol{\eta}_{1,t} - \boldsymbol{\eta}_2 = 0,\\
&\boldsymbol{\eta}_{2,t} + \alpha\boldsymbol{\eta}_1 + \gamma\boldsymbol{\eta}_2 = \boldsymbol{f},\\
&\tilde{u}(0) = \tilde{u}^0 \text{ in } \Omega, \quad \boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_1^0, \quad \boldsymbol{\eta}_2(0) = \boldsymbol{\eta}_2^0,
\end{aligned} \tag{2.4 .1}$$

where $\tilde{u}^0 = u^0 \circ \mathcal{T}_{\eta_1^0}^{-1}$. The nonlinear terms are

$$\begin{aligned}F_f[\tilde{u},\tilde{p},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2]=&\frac{(\ell-s(z)z)\eta_2}{\ell-e-s(z)\eta_1}\tilde{u}_z-\frac{2\nu(\ell-s(z)z)\eta_{1,x}}{\ell-e-s(z)\eta_1}\tilde{u}_{xz}+\frac{\nu((z-\ell)^2\eta_{1,x}^2-\eta_1^2)}{(\ell-e+\eta_1)^2}\tilde{u}_{zz}+\frac{2\nu(\ell-s(z)z)\eta_{1,x}^2}{(\ell-e-s(z)\eta_1)^2}\tilde{u}_z\\ &-\frac{\nu(\ell-s(z)z)\eta_{1,xx}}{\ell-e-s(z)\eta_1}\tilde{u}_z-\frac{\nu(\ell-e)\eta_1}{(\ell-e-s(z)\eta_1)^2}\tilde{u}_{zz}-\nu\nabla\mathrm{div}\,F_{\mathrm{div}}[\tilde{u},\eta_1]+\frac{(\ell-s(z)z)\eta_{1,x}}{\ell-e-s(z)\eta_1}\tilde{p}_z e_1\\ &-\frac{s(z)\eta_1}{\ell-e-s(z)\eta_1}\tilde{p}_z e_2+\frac{(\ell+z)\eta_{1,x}}{\ell-e-s(z)\eta_1}\tilde{u}_1\tilde{u}_z-\frac{s(z)\eta_1}{\ell-e-s(z)\eta_1}\tilde{u}_2\tilde{u}_z,\end{aligned}$$

and

$$F_{\mathrm{div}}[\tilde{u},\boldsymbol{\eta}_1]=\frac{1}{\ell-e}\left(s(z)\eta_1\tilde{u}_1e_1+(\ell-s(z)z)\eta_{1,x}\tilde{u}_1e_2\right), \tag{2.4 .2}$$

with

$$\eta_1=\sum_{i=1}^{n_c}\boldsymbol{\eta}_{1,i}w_i \quad \text{and} \quad \eta_2=\sum_{i=1}^{n_c}\boldsymbol{\eta}_{2,i}w_i.$$

Now, we are reduced to the stabilisation of system (2.1 .4) around the stationary solution $(u_s,p_s,\mathbf{0},\mathbf{0})$ with any prescribed decay rate $-\omega<0$. In order to do that, we make new changes of unknowns

$$\begin{aligned}&\hat{u}(t,x,z):=e^{\omega t}[\tilde{u}(t,x,z)-u_s(x,z)], \quad \hat{p}(t,x,z):=e^{\omega t}[\tilde{p}(t,x,z)-p_s(x,z)],\\ &\hat{\boldsymbol{\eta}}_1(t)=[\hat{\boldsymbol{\eta}}_{1,1}(t),\cdots,\hat{\boldsymbol{\eta}}_{1,n_c}(t)]:=e^{\omega t}\boldsymbol{\eta}_1(t), \quad \hat{\boldsymbol{\eta}}_2(t)=[\hat{\boldsymbol{\eta}}_{2,1}(t),\cdots,\hat{\boldsymbol{\eta}}_{2,n_c}(t)]:=e^{\omega t}\boldsymbol{\eta}_2(t).\end{aligned}$$

Thus, we are led to prove the following result.

Theorem 2.4 .1. *Let $\eta_{\max}$ be in $(0,e)$ and ω be positive. We assume that 2.5 .1, 2.7 .2 and 2.7 .1 are satisfied. There exists $r>0$ such that, for all $(\hat{u}^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0,)\in H_{ic}(F_{\mathrm{div}}[\hat{u}^0,\boldsymbol{\eta}_1^0])$ and for all $\hat{h}\in H_0^1(0,\infty;H_0^2(\Gamma_i))$ satisfying the estimate*

$$\|(\hat{u}^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\|_{ic}+\|\hat{h}\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))}\le r,$$

we can find a control $\hat{\mathbf{f}}\in H^1(0,\infty;\mathbb{R}^{n_c})$, in feedback form, for which nonlinear system (2.4 .3) below admits a unique solution $(\hat{u},\hat{p},\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_2)\in X_{\delta_0}$ satisfying the estimate

$$\|e^{-\omega t}\hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)}\le\eta_{\max} \quad \text{and} \quad \|(\hat{u}(t),\hat{\boldsymbol{\eta}}_1(t),\hat{\boldsymbol{\eta}}_2(t))\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon}(\Omega)\times\mathbb{R}^{n_c}\times\mathbb{R}^{n_c}}\le C, \quad \forall t>0,$$

where $\hat{\eta}_1=\sum_{i=1}^{n_c}\hat{\boldsymbol{\eta}}_{1,i}w_i$.

Remark 2.4 .1. *As in Chapter 1 (see Section 1.9), the estimate*

$$\|e^{-\omega t}\hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)}\le\eta_{\max},$$

guarantees that the change of variables $\mathcal{T}_{e^{-\omega t}\hat{\eta}(t)}$ is a $\mathcal{C}^1-$diffeomorphism from $\Omega_{e^{-\omega t}\hat{\eta}(t)}$ into Ω for all $t>0$.

The system satisfied by $(\hat{u},\hat{p},\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_2)$ is

$$\begin{aligned}&\hat{u}_t-\mathrm{div}\,\sigma(\hat{u},\hat{p})+(u_s\cdot\nabla)\hat{u}+(\hat{u}\cdot\nabla)u_s-A_1\hat{\boldsymbol{\eta}}_1-A_2\hat{\boldsymbol{\eta}}_2-\omega\hat{u}=e^{-\omega t}\mathcal{F}_f[\hat{u},\hat{p},\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_2] \text{ in } Q^\infty,\\ &\mathrm{div}\,\hat{u}=A_3\hat{\boldsymbol{\eta}}_1+e^{-\omega t}\mathrm{div}\,F_{\mathrm{div}}[\hat{u},\hat{\boldsymbol{\eta}}_1] \text{ in } Q^\infty, \quad \hat{u}=A_4\hat{\boldsymbol{\eta}}_2 n \text{ on } \Sigma_s^\infty, \quad \hat{u}=\hat{h} \text{ on } \Sigma_i^\infty,\\ &\hat{u}=0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\hat{u})n\cdot\tau=0 \text{ on } \Sigma_e^\infty, \quad \sigma(\hat{u},\hat{p})n=0 \text{ on } \Sigma_n^\infty,\\ &\hat{\boldsymbol{\eta}}_{1,t}-\hat{\boldsymbol{\eta}}_2-\omega\hat{\boldsymbol{\eta}}_1=0,\\ &\hat{\boldsymbol{\eta}}_{2,t}+\alpha\hat{\boldsymbol{\eta}}_1+\gamma\hat{\boldsymbol{\eta}}_2-\omega\hat{\boldsymbol{\eta}}_2=\hat{\boldsymbol{f}},\\ &\hat{u}(0)=\hat{u}^0 \text{ in } \Omega, \quad \hat{\boldsymbol{\eta}}_1(0)=\boldsymbol{\eta}_1^0, \quad \hat{\boldsymbol{\eta}}_2(0)=\boldsymbol{\eta}_2^0,\end{aligned} \tag{2.4 .3}$$

where

$$\hat{u}^0 = \tilde{u}^0 - u_s, \quad \hat{\boldsymbol{f}}(t) = [\hat{\boldsymbol{f}}_i(t), \cdots, \hat{\boldsymbol{f}}_{n_c}] := e^{\omega t}\mathbf{f}(t) \quad \text{and} \quad \hat{h}(t,x,z) := e^{\omega t}h(t,x,z).$$

The nonlinear term $\mathcal{F}_f$ is given in Appendix 2.14 . We introduce the linear differential operators $\widetilde{A}_1$, $\widetilde{A}_2$ and $\widetilde{A}_3$ defined by

$$\begin{aligned}
\widetilde{A}_1\eta &= \frac{s(z)\eta}{\ell - e}(p_{s,z}e_2 + u_{s,2}u_{s,z} - 2\nu u_{s,zz} + \nu u_{s,1,xx}e_1 + \nu u_{s,1,xz}e_2) + \frac{\nu\eta_x}{\ell - e}(u_{s,1,z}e_2 - u_{s,1,x}e_1) \\
&\quad + \frac{(\ell - s(z)z)\eta_x}{\ell - e}(p_{s,z}e_1 - 2\nu u_{s,xz} + u_{s,1}u_{s,z} - \nu u_{s,1,zz}e_2 - \nu u_{s,1,xz}e_1) \\
&\quad - \frac{\nu(\ell - s(z)z)\eta_{xx}}{\ell - e}(u_{s,z} + u_{s,1,z}e_1), \\
\widetilde{A}_2\eta &= \frac{(\ell - s(z)z)\eta}{\ell - e}u_{s,z}, \\
\widetilde{A}_3\eta &= \frac{1}{\ell - e}(s(z)\eta u_{s,1,x} + (\ell - s(z)z)\eta_x u_{s,1,z}),
\end{aligned}$$

for all $\eta \in L^2(\Gamma_s)$. The operators A_1, A_2, A_3 and A_4 are defined by

$$A_1\widehat{\boldsymbol{\eta}}_1 = \sum_{i=1}^{n_c}\widehat{\boldsymbol{\eta}}_{1,i}\widetilde{A}_1 w_i, \quad A_2\widehat{\boldsymbol{\eta}}_2 = \sum_{i=1}^{n_c}\widehat{\boldsymbol{\eta}}_{2,i}\widetilde{A}_2 w_i, \quad A_3\widehat{\boldsymbol{\eta}}_1 = \sum_{i=1}^{n_c}\widehat{\boldsymbol{\eta}}_{1,i}\widetilde{A}_3 w_i \quad \text{and} \quad A_4\widehat{\boldsymbol{\eta}}_2 = \sum_{i=1}^{n_c}\widehat{\boldsymbol{\eta}}_{2,i} w_i. \tag{2.4 .4}$$

Proposition 2.4 .2. *We have*

$$\begin{aligned}
&\widetilde{A}_1 \in \mathcal{L}(H_0^2(\Gamma_s), \mathbf{L}^2(\Omega)), \\
&\widetilde{A}_2 \in \mathcal{L}(L^2(\Gamma_s), \mathbf{L}^2(\Omega)), \\
&\textit{and } \widetilde{A}_3 \in \mathcal{L}(H_0^2(\Gamma_s), H^1(\Omega)).
\end{aligned}$$

Proof. It is similar to that of Proposition 1.5 .1.

□

Corollary 2.4 .3. *We have*

$$\begin{aligned}
&A_1 \in \mathcal{L}(\mathbb{R}^{n_c}, \mathbf{L}^2(\Omega)), \\
&A_2 \in \mathcal{L}(\mathbb{R}^{n_c}, \mathbf{L}^2(\Omega)), \\
&A_3 \in \mathcal{L}(\mathbb{R}^{n_c}, H^1(\Omega)), \\
&\textit{and } A_4 \in \mathcal{L}(\mathbb{R}^{n_c}, H_0^2(\Gamma_s)).
\end{aligned}$$

We denote by $A_1^* \in \mathcal{L}(\mathbf{L}^2(\Omega), \mathbb{R}^{n_c})$, $A_2^* \in \mathcal{L}(\mathbf{L}^2(\Omega), \mathbb{R}^{n_c})$, $A_3^* \in \mathcal{L}(\mathbf{L}^2(\Omega), \mathbb{R}^{n_c})$ and $A_4^* \in \mathcal{L}(\mathbf{L}^2(\Gamma_s), \mathbb{R}^{n_c})$ the adjoints of A_1, A_2, A_3 and A_4 respectively.

2.5 The Oseen system

We consider the nonhomogeneous Oseen system

$$\begin{aligned}
&\lambda_0 w - \operatorname{div}\sigma(w,\pi) + (u_s\cdot\nabla)w + (w\cdot\nabla)u_s = F \text{ in } \Omega, \\
&\operatorname{div} w = h \text{ in } \Omega, \quad w = gn \text{ on } \Gamma_s, \quad w = 0 \text{ on } \Gamma_i, \\
&w\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(w)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(w,\pi)n = 0 \text{ on } \Gamma_n,
\end{aligned} \tag{2.5 .1}$$

where $F \in \mathbf{L}^2(\Omega)$, $g \in H_0^{\frac{3}{2}}(\Gamma_s)$, $h \in H^1(\Omega)$ and $\lambda_0 > 0$ obeys

$$\int_\Omega (\lambda_0 v \cdot v + 2\nu\varepsilon(v) : \varepsilon(v) + (u_s \cdot \nabla)v \cdot v + (v \cdot \nabla)u_s \cdot v) \geq \frac{\nu}{2}\|v\|^2_{\mathbf{V}^1_{\Gamma_0}}. \tag{2.5 .2}$$

The goal is to study the regularity of a solution (w, π) of that system and to characterize the pressure π in terms of w, F, g and h. First, we consider the Stokes system

$$\begin{aligned}
&-\mathrm{div}\, \sigma(v, p) = F, \quad \mathrm{div}\, v = 0 \text{ in } \Omega,\\
&v = 0 \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i,\\
&v \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v)n \cdot \tau = 0 \text{ on } \Gamma_e,\\
&\sigma(v, p)n = 0 \text{ on } \Gamma_n.
\end{aligned} \tag{2.5 .3}$$

We recall the spaces

$$\mathbf{H}^1_{\Gamma_0}(\Omega) = \{v \in \mathbf{H}^1(\Omega) \mid v = 0 \text{ on } \Gamma_s \cup \Gamma_i \text{ and } v \cdot n = 0 \text{ on } \Gamma_e\},$$

and

$$\mathbf{V}^1_{\Gamma_0}(\Omega) = \mathbf{H}^1_{\Gamma_0}(\Omega) \times \mathbf{V}^0_{n,\Gamma_0}(\Omega).$$

We shall say that $(v, p) \in \mathbf{V}^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ is a weak solution of (2.5 .3) if and only if it satisfies the following mixed variational formulation

$$\begin{aligned}
&\text{Find } (v, p) \in \mathbf{H}^1_{\Gamma_0}(\Omega) \times L^2(\Omega), \text{ such that },\\
&a(v, \phi) - b(\phi, p) = \int_\Omega F \cdot \phi, \quad \forall \phi \in \mathbf{H}^1_{\Gamma_0}(\Omega),\\
&b(v, \psi) = 0, \quad \forall \psi \in L^2(\Omega),
\end{aligned}$$

where

$$a(v, \phi) = 2\nu \int_\Omega \varepsilon(v) : \varepsilon(\phi) \quad \text{and} \quad b(v, \psi) = \int_\Omega \mathrm{div}(v)\psi.$$

Theorem 2.5 .1. *Let F be in $\mathbf{L}^2(\Omega)$. Then equation (2.5 .3) admits a unique weak solution $(v, p) \in \mathbf{V}^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ satisfying the estimate*

$$\|v\|_{\mathbf{V}^1_{\Gamma_0}(\Omega)} + \|p\|_{L^2(\Omega)} \leq C\|F\|_{\mathbf{L}^2(\Omega)}.$$

Proof. It follows from ([43], Theorem 9.1.5). □

We also have regularity results in weighted Sobolev spaces.

Theorem 2.5 .2. *Let F be in $\mathbf{L}^2(\Omega)$. There exists $0 < \delta_0 < \frac{1}{2}$ such that the solution $(v, p) \in \mathbf{V}^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ to equation (2.5 .3) belongs $(v, p) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ with the estimate*

$$\|v\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|p\|_{H^1_{\delta_0}(\Omega)} \leq C\|F\|_{\mathbf{L}^2(\Omega)}.$$

Proof. According to ([43], page 397), in order to determine the exponent δ_0, we have to consider the equations below depending on the type of the boundary conditions imposed on the two edges of a vertice C_j and their angle θ.

Case 1: *Junction between a Dirichlet and a Navier boundary condition with $\theta = \frac{\pi}{2}$.*

$$cos(\lambda\frac{\pi}{2})(\lambda + sin(\lambda\pi)) = 0. \tag{2.5 .4}$$

Case 2: *Junction between two Dirichlet boundary conditions with* $\theta = \frac{\pi}{2}$.

$$sin(\lambda\frac{\pi}{2})(\lambda^2 - sin^2(\lambda\frac{\pi}{2})) = 0. \tag{2.5 .5}$$

Case 3: *Junction between two Dirichlet boundary conditions with* $\theta = \frac{3\pi}{2}$.

$$sin(\lambda\frac{3\pi}{2})(\lambda^2 - sin^2(\lambda\frac{3\pi}{2})) = 0. \tag{2.5 .6}$$

Case 4: *Junction between a Neumann and a Navier boundary condition with* $\theta = \frac{\pi}{2}$.

$$sin(\lambda\frac{\pi}{2})(-\lambda + sin(\lambda\pi)) = 0. \tag{2.5 .7}$$

From ([43], Theorem 9.4.5), the solution (v, p) to (2.5 .3) belongs to $\mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ if there is no solution of the above equations in the strip $0 < \mathrm{Re}(\lambda) < 1 - \delta_0$. For the equations (2.5 .4), (2.5 .5) and (2.5 .7), there is no solution in the strip $0 < \mathrm{Re}(\lambda) < 1$. The solution λ_c to equation (2.5 .6) with the smallest positive real part is such that $\mathrm{Re}(\lambda_c) \approx 0.54$. Thus, there exists $0 < \delta_0 < \frac{1}{2}$ such that the solution (v, p) belongs $(v, p) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$. Moreover, we have the estimate

$$\|v\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|p\|_{H^1_{\delta_0}(\Omega)} \leq C\|F\|_{\mathbf{L}^2(\Omega)}.$$

□

We can assume the following result for the stationary Navier-Stokes equations (see [44]).

Assumption 2.5 .1. *System (2.1 .1) admits a solution* $(u_s, p_s) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$.

Lemma 2.5 .1. *There exists* $\varepsilon_0 \in (0, 1/2)$ *such that* $\mathbf{H}^2_{\delta_0}(\Omega) \subset \mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega)$ *and* $H^1_{\delta_0}(\Omega) \subset H^{\frac{1}{2}+\varepsilon_0}(\Omega)$.

Proof. It follows from ([43], Lemma 6.2.1).

□

Now, we are able to give regularity results for the nonhomogeneous Oseen system.

Theorem 2.5 .3. *For all* $(F, h, g) \in \mathbf{L}^2(\Omega) \times H^1(\Omega) \times H^{\frac{3}{2}}_0(\Gamma_d)$, *equation (2.5 .1) admits a unique solution* $(w, \pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ *with the estimate*

$$\|w\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|\pi\|_{H^1_{\delta_0}(\Omega)} \leq C\left(\|F\|_{\mathbf{L}^2(\Omega)} + \|h\|_{H^1(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Gamma_s)}\right).$$

Proof. The proof is similar to that of Theorem 1.4 .2.

□

In order to characterize the pressure π of system (2.5 .1), we introduce the following operators.

- $N_s \in \mathcal{L}(L^2(\Gamma_s), H^1(\Omega))$ is defined by $N_s g = q$ where q is the solution of the equation

$$\begin{aligned} &\Delta q = 0 \text{ in } \Omega, \quad \frac{\partial q}{\partial n} = g \text{ on } \Gamma_s, \\ &\frac{\partial q}{\partial n} = 0 \text{ on } \Gamma_0 \setminus \Gamma_s, \quad q = 0 \text{ on } \Gamma_n. \end{aligned} \tag{2.5 .8}$$

- $N_{\mathrm{div}} \in \mathcal{L}(L^2(\Omega), H^1(\Omega))$ is defined by $N_{\mathrm{div}} h = q$ where q is the solution of the equation

$$\begin{aligned} &\Delta q = h \text{ in } \Omega, \\ &\frac{\partial q}{\partial n} = 0 \text{ on } \Gamma_0, \quad q = 0 \text{ on } \Gamma_n. \end{aligned} \tag{2.5 .9}$$

- $N_p \in \mathcal{L}(\mathbf{L}^2(\Omega), H^1(\Omega))$ is defined by $N_p F = q_1 + q_2$ where q_1 and q_2 are the solution of the equations

$$\Delta q_1 = \operatorname{div} F \text{ in } \Omega, \quad q_1 = 0 \text{ on } \partial\Omega, \tag{2.5 .10}$$

and

$$\Delta q_2 = 0 \text{ in } \Omega, \quad \frac{\partial q_2}{\partial n} = (F - \nabla q_1) \cdot n \text{ on } \Gamma_0, \quad q_2 = 0 \text{ on } \Gamma_n. \tag{2.5 .11}$$

- $N_v \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}}(\Omega), L^2(\Omega))$ is defined by $N_v w = q$ where q is the solution of the variational problem

Find $q \in L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega q\zeta &= 2\nu \langle \varepsilon(w), \nabla^2 \psi \rangle_{\mathbf{H}^{\frac{1}{2}-\varepsilon_0}(\Omega), \mathbf{H}^{-\frac{1}{2}+\varepsilon_0}(\Omega)} - 2\nu \int_{\Gamma_0} \varepsilon(w) n \cdot \nabla \psi \\ &+ \int_\Omega \Big[(u_s \cdot \nabla) w + (w \cdot \nabla) u_s \Big] \cdot \nabla \psi, \quad \forall \zeta \in L^2(\Omega), \end{aligned}$$

with $\psi \in H^{\frac{3}{2}+\varepsilon_0}(\Omega)$ the solution of the equation

$$\Delta \psi = \zeta \text{ in } \Omega, \quad \frac{\partial \psi}{\partial n} = 0 \text{ on } \Gamma_0, \quad \psi = 0 \text{ on } \Gamma_n, \tag{2.5 .12}$$

(see Lemma below).

Lemma 2.5 .2. *If ζ belongs to $L^2(\Omega)$ then equation (2.5 .12) admits a unique solution $\psi \in H^{\frac{3}{2}+\varepsilon_0}(\Omega)$ and there exists $C > 0$ such that*

$$\|\psi\|_{H^{\frac{3}{2}+\varepsilon_0}(\Omega)} \leq C \|\zeta\|_{L^2(\Omega)}.$$

Proof. It follows from ([43], Theorem 6.5.4), ([43], Lemma 6.2.1) and ([36], Chapter 2). □

Theorem 2.5 .4. *If $(w, \pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ is a solution of system (2.5 .1) then*

$$\pi = -\lambda_0 N_s g - \lambda_0 N_{\mathrm{div}} h + N_p F + N_v w.$$

Proof. The proof is similar to that of Theorem 1.4 .4. □

2.6 Definition and properties of the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$

2.6 .1 The projector Π

We introduce the space

$$H = \mathbf{L}^2(\Omega) \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c},$$

equipped with the scalar product

$$\left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} \right) = \int_\Omega v \cdot \phi + \boldsymbol{\eta}_1 \cdot \boldsymbol{\xi}_1 + \boldsymbol{\eta}_2 \cdot \boldsymbol{\xi}_2.$$

We also introduce the subspaces

$$Z = \Big\{(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in H \mid \operatorname{div} v = A_3\boldsymbol{\eta}_1 \text{ in } \Omega,\ v\cdot n = A_4\boldsymbol{\eta}_2 \text{ on } \Gamma_s,\ v\cdot n = 0 \text{ on } \Gamma_0\setminus\Gamma_s\Big\},$$

and

$$Z_{sup} = \Big\{ (\nabla q,\, 0,\, 0)\ ;\ q \in H^1_{\Gamma_n}(\Omega)\Big\}.$$

We have

$$H = Z \oplus Z_{sup}.$$

We denote by Π the projection from H onto Z parallel to Z_{sup} and by Π^* the adjoint of Π in H.

Proposition 2.6 .1. *For all $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in H$, we have*

$$\Pi \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} v - \nabla N_p v + \nabla N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 + \nabla N_s A_4 \boldsymbol{\eta}_2 \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \tag{2.6 .1}$$

and

$$\Pi^* \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} v - \nabla N_p v \\ \boldsymbol{\eta}_1 - A_3^* N_p v \\ \boldsymbol{\eta}_2 + A_4^* N_p v \end{pmatrix}. \tag{2.6 .2}$$

Proof. The first part is immediate. Now let $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ belong to H. From equation (2.6 .1), it follows that

$$\left(\Pi \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right) = \int_\Omega v\cdot\phi - \int_\Omega (\nabla N_p v - \nabla N_{\mathrm{div}} A_3\boldsymbol{\eta}_1 - \nabla N_s A_4 \boldsymbol{\eta}_2)\cdot\phi + \boldsymbol{\eta}_1\cdot\boldsymbol{\xi}_1 + \boldsymbol{\eta}_2\cdot\boldsymbol{\xi}_2.$$

If v and ϕ are regular enough, we have

$$\begin{aligned} \int_\Omega \nabla N_p v\cdot\phi &= -\int_\Omega N_p v\cdot \operatorname{div}\phi + \int_{\Gamma_0} \phi\cdot n N_p v \\ &= -\int_\Omega \Delta N_p v\cdot N_p\phi + \int_{\Gamma_0} \frac{\partial N_p v}{\partial n}\cdot N_p\phi \\ &= -\int_\Omega \operatorname{div} v\cdot N_p\phi + \int_{\Gamma_0} v\cdot n N_p\phi \\ &= \int_\Omega v\cdot\nabla N_p\phi. \end{aligned}$$

Similarly, we prove that

$$\int_\Omega \nabla N_{\mathrm{div}} A_3\boldsymbol{\eta}_1\cdot\phi = -\boldsymbol{\eta}_1\cdot A_3^* N_p\phi \quad \text{and} \quad \int_\Omega \nabla N_s A_4\boldsymbol{\eta}_2\cdot\phi = \boldsymbol{\eta}_2\cdot A_4^* N_p\phi.$$

Thus, by density, we obtain

$$\left(\Pi \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}\right) = \left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \Pi^* \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}\right),$$

for all $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ belonging to H.

□

We introduce the spaces

$$\widehat{Z} = \mathbf{V}^0_{n,\Gamma_0}(\Omega) \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c},$$

and

$$\widehat{Z}_{sup} = \Big\{ (\nabla q,\, -A_3^* q,\, A_4^* q) \;;\; q \in H^1_{\Gamma_n}(\Omega) \Big\}.$$

We have

$$H = \widehat{Z} \oplus \widehat{Z}_{sup}.$$

Proposition 2.6 .2. *The operator Π^* is the projection from H onto $\widehat{Z}$ parallel to $\widehat{Z}_{sup}$.*

2.6 .2 Definition of the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$

We introduce the space

$$\mathcal{V} = \Big\{ (v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in H \mid v \in \mathbf{H}^1(\Omega),\, \operatorname{div} v = A_3 \boldsymbol{\eta}_1 \text{ in } \Omega,\, v = A_4 \boldsymbol{\eta}_2 \text{ on } \Gamma_s,\, v = 0 \text{ on } \Gamma_i,\, v \cdot n = 0 \text{ on } \Gamma_e \Big\}.$$

The unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ in Z is defined by

$$D(\mathcal{A}) = \Big\{ (v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in \mathcal{V} \mid \exists p \in H^{\frac{1}{2}+\varepsilon_0}(\Omega) \text{ such that } \operatorname{div} \sigma(v,p) \in \mathbf{L}^2(\Omega),\, \sigma(v,p)n = 0 \text{ on } \Gamma_n \Big\},$$

and

$$\mathcal{A} \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \Pi \begin{pmatrix} \operatorname{div} \sigma(v,p) - (u_s \cdot \nabla) v - (v \cdot \nabla) u_s + A_1 \boldsymbol{\eta}_1 + A_2 \boldsymbol{\eta}_2 \\ \boldsymbol{\eta}_2 \\ -\alpha \boldsymbol{\eta}_1 - \gamma \boldsymbol{\eta}_2 \end{pmatrix}. \tag{2.6 .3}$$

2.6 .3 Analyticity of $(\mathcal{A}, D(\mathcal{A}))$

In order to prove the analyticity of $\mathcal{A}$, we will decompose into the form $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_p$, where $\mathcal{A}_0$ and $\mathcal{A}_p$ are defined below. Next, we prove that $\mathcal{A}_0$ is the infinitesimal generator of an analytic semigroup and that $\mathcal{A}_p$ is $\mathcal{A}_0$-bounded with relative bound zero. First, we introduce the operator A_s in $\mathbb{R}^{n_c} \times \mathbb{R}^{n_c}$ defined by

$$A_s = \begin{pmatrix} I - A_3^* N_{\text{div}} A_3 & -A_3^* N_s A_4 \\ A_4^* N_{\text{div}} A_3 & I + A_4^* N_s A_4 \end{pmatrix}.$$

Lemma 2.6 .1. *The operator A_s is an automorphism in $\mathbb{R}^{n_c} \times \mathbb{R}^{n_c}$.*

Proof. We are going to prove that A_s is symmetric and positive. Let $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ belong to $\mathbb{R}^{n_c} \times \mathbb{R}^{n_c}$. We have

$$\begin{aligned} \left(A_s \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} \right)_{\mathbb{R}^{n_c} \times \mathbb{R}^{n_c}} &= (\boldsymbol{\eta}_1 - A_3^* N_{\text{div}} A_3 \boldsymbol{\eta}_1 - A_3^* N_s A_4 \boldsymbol{\eta}_2) \cdot \boldsymbol{\xi}_1 \\ &\quad + (\boldsymbol{\eta}_2 + A_4^* N_s A_4 \boldsymbol{\eta}_2 + A_4^* N_{\text{div}} A_3 \boldsymbol{\eta}_1) \cdot \boldsymbol{\xi}_2. \end{aligned}$$

We have

$$\begin{aligned} A_3^* N_{\text{div}} A_3 \boldsymbol{\eta}_1 \cdot \boldsymbol{\xi}_1 &= \int_\Omega N_{\text{div}} A_3 \boldsymbol{\eta}_1 \cdot A_3 \boldsymbol{\xi}_1 \\ &= \int_\Omega N_{\text{div}} A_3 \boldsymbol{\eta}_1 \cdot \Delta N_{\text{div}} A_3 \boldsymbol{\xi}_1 \\ &= \int_\Omega A_3 \boldsymbol{\eta}_1 \cdot N_{\text{div}} A_3 \boldsymbol{\xi}_1 \\ &= \boldsymbol{\eta}_1 \cdot A_3^* N_{\text{div}} A_3 \boldsymbol{\xi}_1, \end{aligned}$$

and

$$\begin{aligned} A_3^* N_s A_4 \boldsymbol{\eta}_2 \cdot \boldsymbol{\xi}_1 &= \int_\Omega N_s A_4 \boldsymbol{\eta}_2 \cdot A_3 \boldsymbol{\xi}_1 \\ &= \int_\Omega N_s A_4 \boldsymbol{\eta}_2 \cdot \Delta N_{\mathrm{div}} A_3 \boldsymbol{\xi}_1 \\ &= -\int_{\Gamma_s} A_4 \boldsymbol{\eta}_2 \cdot N_{\mathrm{div}} A_3 \boldsymbol{\xi}_1 \\ &= -\boldsymbol{\eta}_2 \cdot A_4^* N_{\mathrm{div}} A_3 \boldsymbol{\xi}_1. \end{aligned}$$

Similarly, we prove that

$$A_4^* N_s A_4 \boldsymbol{\eta}_2 \cdot \boldsymbol{\xi}_2 = \boldsymbol{\eta}_2 \cdot A_4^* N_s A_4 \boldsymbol{\xi}_2,$$

and

$$A_4^* N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \cdot \boldsymbol{\xi}_2 = -\boldsymbol{\eta}_1 \cdot A_3^* N_s A_4 \boldsymbol{\xi}_2.$$

Thus, we have

$$\left(A_s \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} \right)_{\mathbb{R}^{nc} \times \mathbb{R}^{nc}} = \left(\begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, A_s \begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} \right)_{\mathbb{R}^{nc} \times \mathbb{R}^{nc}}.$$

implying that A_s is symmetric. Now, let us prove that A_s is positive. We have

$$\begin{aligned} A_3^* N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_1 &= \int_\Omega N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \cdot A_3 \boldsymbol{\eta}_1 \\ &= \int_\Omega N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \cdot \Delta N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \\ &= -\int_\Omega \nabla N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \cdot \nabla N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1, \end{aligned}$$

and

$$\begin{aligned} A_3^* N_s A_4 \boldsymbol{\eta}_2 \cdot \boldsymbol{\eta}_1 &= \int_\Omega N_s A_4 \boldsymbol{\eta}_2 \cdot A_3 \boldsymbol{\eta}_1 \\ &= \int_\Omega N_s A_4 \boldsymbol{\eta}_2 \cdot \Delta N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \\ &= -\int_{\Gamma_s} \nabla N_s A_4 \boldsymbol{\eta}_2 \cdot \nabla N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1. \end{aligned}$$

Similarly, we prove that

$$A_4^* N_s A_4 \boldsymbol{\eta}_2 \cdot \boldsymbol{\eta}_2 = \int_\Omega \nabla N_s A_4 \boldsymbol{\eta}_2 \cdot \nabla N_s A_4 \boldsymbol{\eta}_2,$$

and

$$A_4^* N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_2 = \int_\Omega \nabla N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \cdot \nabla N_s A_4 \boldsymbol{\eta}_2.$$

We deduce that

$$\begin{aligned} \left(A_s \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} \right)_{\mathbb{R}^{nc} \times \mathbb{R}^{nc}} &= \boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 \cdot \boldsymbol{\eta}_2 + \int_\Omega \nabla N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \cdot \nabla N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \\ &\quad + \int_\Omega \nabla N_s A_4 \boldsymbol{\eta}_2 \cdot \nabla N_s A_4 \boldsymbol{\eta}_2 + 2 \int_\Omega \nabla N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 \cdot \nabla N_s A_4 \boldsymbol{\eta}_2. \\ &= \boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 \cdot \boldsymbol{\eta}_2 + \int_\Omega (\nabla N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 + \nabla N_s A_4 \boldsymbol{\eta}_2)^2. \end{aligned}$$

Hence, A_s is positive.

$\square$

We introduce the operators $(\mathcal{A}_0, D(\mathcal{A}_0))$ and $(\mathcal{A}_p, D(\mathcal{A}_p))$ in Z defined by

$$D(\mathcal{A}_0) = D(\mathcal{A}_p) = D(\mathcal{A}),$$

$$\mathcal{A}_0 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \Pi \begin{pmatrix} \operatorname{div} \sigma(v,p) - (u_s \cdot \nabla)v - (v \cdot \nabla)u_s + A_1\boldsymbol{\eta}_1 + A_2\boldsymbol{\eta}_2 \\ \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{pmatrix},$$

and

$$\mathcal{A}_p \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \Pi \begin{pmatrix} 0 \\ \boldsymbol{\eta}_2 - \boldsymbol{\zeta}_1 \\ -\alpha\boldsymbol{\eta}_1 - \gamma\boldsymbol{\eta}_2 - \boldsymbol{\zeta}_2 \end{pmatrix},$$

with

$$\begin{pmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{pmatrix} = A_s^{-1} \begin{pmatrix} -A_3^* N_v v \\ A_4^* N_v v \end{pmatrix}.$$

Lemma 2.6 .2. *Let $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ belong to $D(\mathcal{A}_0)$ and $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ belong to $\mathcal{V}$. Then, we have*

$$\begin{aligned} \left(\mathcal{A}_0 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right) &= -2\nu \int_\Omega \varepsilon(v) : \varepsilon(\phi) - \int_\Omega (u_s \cdot \nabla)v \cdot \phi - \int_\Omega (v \cdot \nabla)u_s \cdot \phi \\ &\quad + \int_\Omega A_1\boldsymbol{\eta}_1 \cdot \phi + \int_\Omega A_2\boldsymbol{\eta}_2 \cdot \phi + 2\nu \int_{\Gamma_s} A_3\boldsymbol{\eta}_1 \cdot A_4\boldsymbol{\xi}_2. \end{aligned}$$

Proof. There exists $p \in H^{\frac{1}{2}+\varepsilon_0}(\Omega)$ such that

$$\mathcal{A}_0 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \Pi \begin{pmatrix} \operatorname{div} \sigma(v,p) - (u_s \cdot \nabla)v - (v \cdot \nabla)u_s + A_1\boldsymbol{\eta}_1 + A_2\boldsymbol{\eta}_2 \\ \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{pmatrix}, \tag{2.6 .4}$$

with

$$\begin{pmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{pmatrix} = A_s^{-1} \begin{pmatrix} -A_3^* N_v v \\ A_4^* N_v v \end{pmatrix}. \tag{2.6 .5}$$

Without loss of generality, we can assume that p is such that

$$\operatorname{div} \sigma(v,p) - (u_s \cdot \nabla)v - (v \cdot \nabla)u_s + A_1\boldsymbol{\eta}_1 + A_2\boldsymbol{\eta}_2 \in \mathbf{V}^0_{n,\Gamma_0}(\Omega). \tag{2.6 .6}$$

Indeed, let F belong to $\mathbf{L}^2(\Omega)$ such that

$$\operatorname{div} \sigma(v,p) - (u_s \cdot \nabla)v - (v \cdot \nabla)u_s + A_1\boldsymbol{\eta}_1 + A_2\boldsymbol{\eta}_2 = F \text{ in } \Omega,$$

and

$$\sigma(v,p)n = 0 \text{ on } \Gamma_n.$$

Then, we have

$$\operatorname{div} \sigma(v,\tilde{p}) - (u_s \cdot \nabla)v - (v \cdot \nabla)u_s + A_1\boldsymbol{\eta}_1 + A_2\boldsymbol{\eta}_2 = F - N_pF,$$

and

$$\sigma(v,\tilde{p})n = 0 \text{ on } \Gamma_n,$$

with $\tilde{p} = p - \nabla N_pF$ and $F - N_pF$ belongs to $\mathbf{V}^0_{n,\Gamma_0}(\Omega)$. Using Theorem 2.5 .4, we prove that

$$p = N_v v. \tag{2.6 .7}$$

Since $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ belong to $\mathcal{V}$, we have

$$N_p\phi = N_{\rm div} A_3 \boldsymbol{\xi}_1 + N_s A_4 \boldsymbol{\xi}_2.$$

Thus, from Proposition 2.6 .1, it follows that

$$\Pi^* \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} \phi - \nabla N_p\phi \\ \boldsymbol{\xi}_1 - A_3^* N_p\phi \\ \boldsymbol{\xi}_2 + A_4^* N_p\phi \end{pmatrix} = \begin{pmatrix} \phi - \nabla N_p\phi \\ \boldsymbol{\xi}_1 - A_3^* N_{\rm div} A_3 \boldsymbol{\xi}_1 - A_3^* N_s A_4 \boldsymbol{\xi}_2 \\ \boldsymbol{\xi}_2 + A_4^* N_{\rm div} A_3 \boldsymbol{\xi}_1 + A_4^* N_s A_4 \boldsymbol{\xi}_2 \end{pmatrix}. \tag{2.6 .8}$$

Thanks to equations (2.6 .4), (2.6 .5), (2.6 .6), (2.6 .7) and (2.6 .8), we obtain

$$\begin{aligned} \left(\mathcal{A}_0 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right) &= \int_\Omega (\operatorname{div}\sigma(v,p) - (u_s\cdot\nabla)v - (v\cdot\nabla)u_s + A_1\boldsymbol{\eta}_1 + A_2\boldsymbol{\eta}_2)\cdot\phi \\ &\quad - A_3^* N_v v\cdot\boldsymbol{\eta}_1 + A_4^* N_v v\cdot\boldsymbol{\eta}_2. \end{aligned} \tag{2.6 .9}$$

By integration by parts, we get

$$\begin{aligned} \int_\Omega \operatorname{div}\sigma(v,p)\cdot\phi &= -2\nu\int_\Omega \varepsilon(v):\varepsilon(\phi) + \int_\Omega p\operatorname{div}\phi + \int_{\partial\Omega}\sigma(v,p)n\cdot\phi \\ &= -2\nu\int_\Omega \varepsilon(v):\varepsilon(\phi) + \int_\Omega pA_3\boldsymbol{\xi}_1 + \int_{\Gamma_s}(2\nu A_3\boldsymbol{\eta}_1 - p)\cdot A_4\boldsymbol{\xi}_2 \\ &= -2\nu\int_\Omega \varepsilon(v):\varepsilon(\phi) + A_3^* N_v v\cdot\boldsymbol{\xi}_1 + 2\nu\int_{\Gamma_s} A_3\boldsymbol{\eta}_1\cdot A_4\boldsymbol{\xi}_2 - A_4^* N_v v\cdot\boldsymbol{\xi}_2. \end{aligned}$$

From equation (2.6 .9), it follows that

$$\begin{aligned} \left(\mathcal{A}_0 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right) &= -2\nu\int_\Omega \varepsilon(v):\varepsilon(\phi) - \int_\Omega (u_s\cdot\nabla)v\cdot\phi - \int_\Omega (v\cdot\nabla)u_s\cdot\phi \\ &\quad + \int_\Omega A_1\boldsymbol{\eta}_1\cdot\phi + \int_\Omega A_2\boldsymbol{\eta}_2\cdot\phi + 2\nu\int_{\Gamma_s} A_3\boldsymbol{\eta}_1\cdot A_4\boldsymbol{\xi}_2. \end{aligned} \tag{2.6 .10}$$

□

Theorem 2.6 .3. *The unbounded operator $(\mathcal{A}_0, D(\mathcal{A}_0))$ is the generator of an analytic semigroup of class $\mathcal{C}^0$ on Z.*

Proof. We introduce the bilinear form $\mathbf{a}$ defined by

$$\begin{aligned} \mathbf{a}\left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right) &= -2\nu\int_\Omega \varepsilon(v):\varepsilon(\phi) - \int_\Omega (u_s\cdot\nabla)v\cdot\phi - \int_\Omega (v\cdot\nabla)u_s\cdot\phi \\ &\quad + \int_\Omega A_1\boldsymbol{\eta}_1\cdot\phi + \int_\Omega A_2\boldsymbol{\eta}_2\cdot\phi + 2\nu\int_{\Gamma_s} A_3\boldsymbol{\eta}_1\cdot A_4\boldsymbol{\xi}_2, \end{aligned} \tag{2.6 .11}$$

for all $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ in $\mathcal{V}$. Using classical inequalities in Sobolev spaces, we prove that there exists $\lambda_1 > 0$ and $C_1 > 0$ such that

$$\lambda_1 \left\| \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} \right\|_Z^2 - \mathbf{a}\left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}\right) \geq C_1 \left\| \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} \right\|_{\mathcal{V}}^2, \quad \forall\, (v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in \mathcal{V}. \tag{2.6 .12}$$

We introduce the unbounded operator $(\widehat{\mathcal{A}}_0, D(\widehat{\mathcal{A}}_0))$ in Z by

$$D(\widehat{\mathcal{A}}_0) = \{(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in \mathcal{V} \mid a((v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2), \cdot) \in \mathcal{L}(Z, \mathbb{R})\},$$

and

$$\left(\widehat{\mathcal{A}}_0 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right) = \mathbf{a}\left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right),$$

for all $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in D(\mathcal{A}_0)$ and for all $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathcal{V}$. According to [[10], Theorem 2.12], $(\widehat{\mathcal{A}}_0, D(\widehat{\mathcal{A}}_0))$ is the generator of an analytic semigroup on Z. Thus, to complete the proof, we are going to prove that $D(\widehat{\mathcal{A}}_0) = D(\mathcal{A}_0)$ and $\widehat{\mathcal{A}}_0 = \mathcal{A}_0$. From Lemma 2.6 .2, it follows that

$$\left(\mathcal{A}_0 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right) = \left(\widehat{\mathcal{A}}_0 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right),$$

for all $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in D(\mathcal{A}_0)$ and for all $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathcal{V}$. To finish, we have to prove that $D(\widehat{\mathcal{A}}_0) = D(\mathcal{A}_0)$. It is obvious that $D(\widehat{\mathcal{A}}_0) \subset D(\mathcal{A}_0)$. Now, let $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ belong to $D(\mathcal{A}_0)$. Then, there exists $(w, \boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) \in Z$ such that

$$\mathbf{a}\left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right) = \left(\begin{pmatrix} w \\ \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right), \quad \forall\, (\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathcal{V}.$$

In particular, we have

$$\int_\Omega \Big(-2\nu\varepsilon(v) : \varepsilon(\phi) - (u_s \cdot \nabla)v \cdot \phi - (v \cdot \nabla)u_s \cdot \phi + A_1\boldsymbol{\eta}_1 \cdot \phi + A_2\boldsymbol{\eta}_2 \cdot \phi \Big) = \int_\Omega w \cdot \phi, \quad (2.6\ .13)$$

for all $\phi \in C_c^\infty(\Omega) \cap \mathbf{V}^1_{\Gamma_0}(\Omega)$. Due to [[24], Chapter 1, Theorem 2.3], there exists $p \in L^2(\Omega)$ such that

$$\operatorname{div} \sigma(v,p) - (u_s \cdot \nabla)v - (v \cdot \nabla)u_s \cdot \phi + A_1\boldsymbol{\eta}_1 + A_2\boldsymbol{\eta}_2 = w. \quad (2.6\ .14)$$

Using equations (2.6 .13) and (2.6 .14), we obtain

$$\int_{\Gamma_n} \sigma(v,p)n \cdot \phi = 0, \quad \forall\, \phi \in \mathbf{V}^1_{\Gamma_0}(\Omega),$$

implying that

$$\sigma(v,p)n = 0 \text{ on } \Gamma_n.$$

Thus, $(v,p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ is a solution of the system

$$\begin{aligned} &-\operatorname{div} \sigma(v,p) + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s = w \text{ in } \Omega, \\ &\operatorname{div} v = A_3\boldsymbol{\eta}_1 \text{ in } \Omega, \quad v = A_4\boldsymbol{\eta}_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i, \\ &v \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n. \end{aligned} \quad (2.6\ .15)$$

Thanks to Theorem 2.5 .3, (v,p) belongs to $\mathbf{H}^2_\delta(\Omega) \times H^1_\delta(\Omega)$. Moreover, from equation (2.6 .14), it follows that $\operatorname{div} \sigma(v,p)$ belongs to $\mathbf{L}^2(\Omega)$. Thus, $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ belongs to $D(\mathcal{A}_0)$. The proof is complete.

$\square$

Theorem 2.6 .4. *The unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ is the generator of an analytic semigroup of class $\mathcal{C}^0$ on Z.*

Proof. Since $\mathcal{A}_0$ is analytic, it is sufficient to prove that the operator $\mathcal{A}_p \in \mathcal{L}(D(\mathcal{A}_0), Z)$ is compact. Let $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ belong to $D(\mathcal{A}_0)$ and let (F_f, F_s^1, F_s^2) belong to Z such that

$$\lambda_1 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} - \mathcal{A}_0 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \end{pmatrix}. \tag{2.6 .16}$$

We set

$$p = -\lambda_1 N_s A_4 \boldsymbol{\eta}_2 - \lambda_1 N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 + N_p A_1 \boldsymbol{\eta}_1 + N_p A_2 \boldsymbol{\eta}_2 + N_v v + N_p F_f. \tag{2.6 .17}$$

Then, we can prove that $(v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is a solution of the system

$$\begin{aligned}
&\lambda_1 v - \operatorname{div} \sigma(v,p) + (u_s \cdot \nabla) v + (v \cdot \nabla) u_s - A_1 \boldsymbol{\eta}_1 - A_2 \boldsymbol{\eta}_2 = F_f \text{ in } \Omega,\\
&\operatorname{div} v = A_3 \boldsymbol{\eta}_1 \text{ in } \Omega, \quad v = A_4 \boldsymbol{\eta}_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i,\\
&v \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v) n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v,p) n = 0 \text{ on } \Gamma_n,\\
&\lambda_1 \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} - A_s^{-1} \begin{pmatrix} -A_3^* N_v v \\ A_4^* N_v v \end{pmatrix} = \begin{pmatrix} F_s^1 \\ F_s^2 \end{pmatrix}.
\end{aligned} \tag{2.6 .18}$$

Thanks to integrations by parts, we prove that $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is the solution of the variational problem:

Find $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ such that

$$\lambda_1 \left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} \right) - \mathbf{a} \left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} \right) = \left(\begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} \right), \quad \forall (\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathcal{V}, \tag{2.6 .19}$$

(see (2.6 .11) for the definition of $\mathbf{a}$). The bilinear form $\lambda_1(\cdot,\cdot) - \mathbf{a}(\cdot,\cdot)$ is coercive (see (2.6 .12)). Thus, thanks to the Lax-Milgram Theorem, it follows there exists $C > 0$ such that

$$\left\| \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} \right\|_Z \leq C \left\| \begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \end{pmatrix} \right\|_Z. \tag{2.6 .20}$$

Due to Theorem 2.5 .3 and inequality (2.6 .20), the pressure p belongs to $H^1_{\delta_0}(\Omega)$ and

$$\|p\|_{H^1_{\delta_0}(\Omega)} \leq C \left\| \begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \end{pmatrix} \right\|_Z. \tag{2.6 .21}$$

Thanks to equation (2.6 .17) and estimate (2.6 .21), it follows that

$$\|N_v v\|_{H^1_{\delta_0}(\Omega)} \leq C \left\| \begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \end{pmatrix} \right\|_Z. \tag{2.6 .22}$$

From estimates (2.6 .20) and (2.6 .22), we obtain

$$\left\| \mathcal{A}_p \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} \right\|_{\mathcal{V}} \leq C \left\| \begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \end{pmatrix} \right\|_Z \leq C \left\| \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} \right\|_{D(\mathcal{A})},$$

which means that $\mathcal{A}_p$ belongs to $\mathcal{L}(D(\mathcal{A}_0), \mathcal{V})$. Since the embedding $\mathcal{V} \subset Z$ is compact, the operator $\mathcal{A}_p \in \mathcal{L}(D(\mathcal{A}_0), Z)$ is compact.

□

2.6 .4 Resolvent of $\mathcal{A}$ in Z

The goal is to prove that the operator $(\mathcal{A}, D(\mathcal{A}))$ has a compact resolvent. First, we establish a preliminary result.

Proposition 2.6 .5. *Let (F_f, F_s^1, F_s^2) belong to H. The quadruplet $(v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in H_{\delta_0}$ is a solution of the system*

$$\begin{aligned}
&\lambda_1 v - \operatorname{div} \sigma(v,p) + (u_s \cdot \nabla) v + (v \cdot \nabla) u_s - A_1 \boldsymbol{\eta}_1 - A_2 \boldsymbol{\eta}_2 = F_f \ \textit{in } \Omega,\\
&\operatorname{div} v = A_3 \boldsymbol{\eta}_1 \ \textit{in } \Omega, \quad v = A_4 \boldsymbol{\eta}_2 n \ \textit{on } \Gamma_s, \quad v = 0 \ \textit{on } \Gamma_i,\\
&v \cdot n = 0 \ \textit{on } \Gamma_e, \quad \varepsilon(v) n \cdot \tau = 0 \ \textit{on } \Gamma_e, \quad \sigma(v,p) n = 0 \ \textit{on } \Gamma_n,\\
&\lambda_1 \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 = F_s^1,\\
&\lambda_1 \boldsymbol{\eta}_2 + \alpha \boldsymbol{\eta}_1 + \gamma \boldsymbol{\eta}_2 = F_s^2,
\end{aligned} \tag{2.6 .23}$$

if and only if

$$\begin{aligned}
&(\lambda_1 I - \mathcal{A}) \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \Pi \begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \end{pmatrix},\\
&p = -\lambda_1 N_s A_4 \boldsymbol{\eta}_2 - \lambda_1 N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 + N_p A_1 \boldsymbol{\eta}_1 + N_p A_2 \boldsymbol{\eta}_2 + N_v v + N_p F_f.
\end{aligned} \tag{2.6 .24}$$

Proof. According to Theorem 2.5 .4, it follows that if $(v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in H_{\delta_0}$ is a solution of system (2.6 .23) then

$$p = -\lambda_1 N_s A_4 \boldsymbol{\eta}_2 - \lambda_1 N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 + N_p A_1 \boldsymbol{\eta}_1 + N_p A_2 \boldsymbol{\eta}_2 + N_v v + N_p F_f.$$

Therefore, it is easy to prove that if $(v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in H_{\delta_0}$ is a solution of system (2.6 .23) then $(v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is a solution of system (2.6 .24). Now, we suppose that $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is a solution of (2.6 .24). Using Theorem 2.5 .4, we prove that (v, p) is a solution of the system

$$\begin{aligned}
&\lambda_1 v - \operatorname{div} \sigma(v,p) + (u_s \cdot \nabla) v + (v \cdot \nabla) u_s - A_1 \boldsymbol{\eta}_1 - A_2 \boldsymbol{\eta}_2 = F_f \ \text{in } \Omega,\\
&\operatorname{div} v = A_3 \boldsymbol{\eta}_1 \ \text{in } \Omega, \quad v = A_4 \boldsymbol{\eta}_2 n \ \text{on } \Gamma_s, \quad v = 0 \ \text{on } \Gamma_i,\\
&v \cdot n = 0 \ \text{on } \Gamma_e, \quad \varepsilon(v) n \cdot \tau = 0 \ \text{on } \Gamma_e, \quad \sigma(v,p) n = 0 \ \text{on } \Gamma_n.
\end{aligned} \tag{2.6 .25}$$

It follows that

$$\mathcal{A} \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \Pi \begin{pmatrix} \operatorname{div} \sigma(v,p) - (u_s \cdot \nabla) v - (v \cdot \nabla) u_s + A_1 \boldsymbol{\eta}_1 + A_2 \boldsymbol{\eta}_2 \\ \boldsymbol{\eta}_2 \\ -\alpha \boldsymbol{\eta}_1 - \gamma \boldsymbol{\eta}_2 \end{pmatrix}.$$

Thus, there exists $q \in H^1_{\Gamma_n}(\Omega)$ such that

$$\lambda_1 \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} - \begin{pmatrix} \operatorname{div} \sigma(v,q) - (u_s \cdot \nabla) v - (v \cdot \nabla) u_s + A_1 \boldsymbol{\eta}_1 + A_2 \boldsymbol{\eta}_2 \\ \boldsymbol{\eta}_2 \\ -\alpha \boldsymbol{\eta}_1 - \gamma \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \end{pmatrix} + \begin{pmatrix} \nabla q \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $(v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is a solution of the system

$$\begin{aligned}
&\lambda_1 v - \operatorname{div} \sigma(v,p) + (u_s \cdot \nabla) v + (v \cdot \nabla) u_s - A_1 \boldsymbol{\eta}_1 - A_2 \boldsymbol{\eta}_2 = F_f + \nabla q \ \text{in } \Omega,\\
&\operatorname{div} v = A_3 \boldsymbol{\eta}_1 \ \text{in } \Omega, \quad v = A_4 \boldsymbol{\eta}_2 n \ \text{on } \Gamma_s, \quad v = 0 \ \text{on } \Gamma_i,\\
&v \cdot n = 0 \ \text{on } \Gamma_e, \quad \varepsilon(v) n \cdot \tau = 0 \ \text{on } \Gamma_e, \quad \sigma(v,q) n = 0 \ \text{on } \Gamma_n,\\
&\lambda_1 \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 = F_s^1,\\
&\lambda_1 \boldsymbol{\eta}_2 + \alpha \boldsymbol{\eta}_1 + \gamma \boldsymbol{\eta}_2 = F_s^2.
\end{aligned}$$

System (2.6 .25) gives

$$\lambda_1 v - \operatorname{div}\sigma(v,q) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\boldsymbol{\eta}_1 - A_2\boldsymbol{\eta}_2 = F_f \text{ in } \Omega,$$

implying that $\nabla q = 0$. We deduce that $q = 0$ on Ω because q belongs to $H^1_{\Gamma_n}(\Omega)$. Thus, $(v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is a solution of system (2.6 .23).

□

Theorem 2.6 .6. *The unbounded operator* $(\mathcal{A}, D(\mathcal{A}))$ *has a compact resolvent in* Z*.*

Proof. Let (F_f, F^1_s, F^2_s) belong to Z. There exists $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in Z$ such that

$$(\lambda_1 I - \mathcal{A})\begin{pmatrix} Pv \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} F_f \\ F^1_s \\ F^2_s \end{pmatrix}.$$

From Proposition 2.6 .5, it follows that $(v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ with

$$p = -\lambda_1 N_s A_4 \boldsymbol{\eta}_2 - \lambda_1 N_{\rm div} A_3 \boldsymbol{\eta}_1 + N_p A_1 \boldsymbol{\eta}_1 + N_p A_2 \boldsymbol{\eta}_2 + N_v v + N_p F_f,$$

is the solution of the system

$$\begin{aligned}
&\lambda_1 v - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\boldsymbol{\eta}_1 - A_2\boldsymbol{\eta}_2 = F_f \text{ in } \Omega,\\
&\operatorname{div} v = A_3\boldsymbol{\eta}_1 \text{ in } \Omega, \quad v = A_4\boldsymbol{\eta}_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i,\\
&v\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n,\\
&\lambda_1\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 = F^1_s,\\
&\lambda_1\boldsymbol{\eta}_2 + \alpha\boldsymbol{\eta}_1 + \gamma\boldsymbol{\eta}_2 = F^2_s.
\end{aligned} \tag{2.6 .26}$$

By solving the last two equations, we get

$$\|\boldsymbol{\eta}_1\|_{\mathbb{R}^{n_c}} + \|\boldsymbol{\eta}_2\|_{\mathbb{R}^{n_c}} \le C\left(\left\|F^1_s\right\|_{\mathbb{R}^{n_c}} + \left\|F^2_s\right\|_{\mathbb{R}^{n_c}}\right). \tag{2.6 .27}$$

Next, we consider the system

$$\begin{aligned}
&\lambda_1 v - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s = A_1\boldsymbol{\eta}_1 + A_2\boldsymbol{\eta}_2 + F_f \text{ in } \Omega,\\
&\operatorname{div} v = A_3\boldsymbol{\eta}_1 n \text{ in } \Omega, \quad v = A_4\boldsymbol{\eta}_2 \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i\\
&v\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n.
\end{aligned} \tag{2.6 .28}$$

Thanks to Theorem 2.5 .3, the above system admits a solution (v, p) satisfying the estimate

$$\|v\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|p\|_{H^1_{\delta_0}(\Omega)} \le C\left(\|A_1\boldsymbol{\eta}_1 + A_2\boldsymbol{\eta}_2 + F_f\|_{\mathbf{L}^2(\Omega)} + \|A_4\boldsymbol{\eta}_1\|_{H^{\frac{3}{2}}_0(\Omega)} + \|A_3\boldsymbol{\eta}_1\|_{H^1(\Omega)}\right).$$

From Corollary 2.4 .3 and Lemma 2.5 .1, it follows that

$$\|v\|_{\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega)} + \|\boldsymbol{\eta}_1\|_{\mathbb{R}^{n_c}} + \|\boldsymbol{\eta}_2\|_{\mathbb{R}^{n_c}} \le C\left\|(F_f, F^1_s, F^2_s)\right\|_Z.$$

Thus, the resolvent of $(\mathcal{A}, D(\mathcal{A}))$ is compact because the embedding $\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega)\times\mathbb{R}^{n_c}\times\mathbb{R}^{n_c} \hookrightarrow Z$ is compact.

□

2.6 .5 Adjoint of $(\mathcal{A}, D(\mathcal{A}))$

The goal of this section is to determine the adjoint $(\mathcal{A}^*, D(\mathcal{A}^*))$ of $(\mathcal{A}, D(\mathcal{A}))$ in Z. For that, we first introduce the unbounded operator $(\mathcal{A}^\#, D(\mathcal{A}^\#))$ defined in $\widehat{Z}$ by

$$D(\mathcal{A}^\#) = \Big\{(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \widehat{Z} \mid \exists \psi \in H^{\frac{1}{2}+\varepsilon_0}(\Omega) \text{ such that } \operatorname{div} \sigma(\phi, \psi) \in \mathbf{L}^2(\Omega),$$
$$\text{and } \sigma(\phi, \psi)n + u_s \cdot n\phi = 0 \text{ on } \Gamma_n\Big\},$$

and

$$\mathcal{A}^\# \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \Pi^* \begin{pmatrix} \operatorname{div} \sigma(\phi, \psi) + (u_s \cdot \nabla)\phi - (\nabla u_s)^T \phi \\ -\alpha \boldsymbol{\xi}_2 + A_1^* \phi - A_3^* \psi \\ \boldsymbol{\xi}_1 - \gamma \boldsymbol{\xi}_2 + A_2^* \phi + A_4^* \psi \end{pmatrix}.$$

Next, we introduce the operator $N_\phi \in \mathcal{L}(\mathbf{H}^2_{\delta_0}(\Omega), L^2(\Omega))$, defined by $N_\phi \phi = \psi$, where ψ is the solution of the variational problem

$$\text{Find } \psi \in L^2(\Omega) \text{ such that}$$
$$\int_\Omega \psi\zeta = 2\nu \langle \varepsilon(\phi), \nabla^2 \tilde{\psi} \rangle_{\mathbf{H}^{\frac{1}{2}-\varepsilon_0}(\Omega), \mathbf{H}^{-\frac{1}{2}+\varepsilon_0}(\Omega)} - 2\nu \int_{\Gamma_0} \varepsilon(\phi)n \cdot \nabla \tilde{\psi}$$
$$+ \int_\Omega \Big[-(u_s \cdot \nabla)\phi + (\nabla u_s)^T \phi \Big] \cdot \nabla \tilde{\psi}, \quad \forall \zeta \in L^2(\Omega),$$

with $\tilde{\psi}$ the solution of the equation

$$\Delta \tilde{\psi} = \zeta \text{ in } \Omega, \quad \frac{\partial \tilde{\psi}}{\partial n} = 0 \text{ on } \Gamma_0, \quad \tilde{\psi} = 0 \text{ on } \Gamma_n.$$

Lemma 2.6 .3. *Let G belong to $\mathbf{L}^2(\Omega)$ and λ belong to $\mathbb{R}$. If $(\phi, \psi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ is a solution of the system*

$$\begin{aligned} &\lambda\phi - \operatorname{div} \sigma(\phi, \psi) - (u_s \cdot \nabla)\phi + (\nabla u_s)^T \phi = G \text{ in } \Omega, \\ &\operatorname{div} \phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_i, \\ &\phi \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(\phi)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi, \psi)n + u_s \cdot n\phi = 0 \text{ on } \Gamma_n, \end{aligned} \tag{2.6 .29}$$

then

$$\psi = N_\phi \phi + N_p G.$$

Proof. The proof is similar to that of Theorem 1.4 .4.

□

Proposition 2.6 .7. *Let (G_f, G_s^1, G_s^2) belong to H. The quadruplet $(\phi, \psi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in H_{\delta_0}$ is a solution of system*

$$\begin{aligned} &\lambda\phi - \operatorname{div} \sigma(\phi, \psi) - (u_s \cdot \nabla)\phi + (\nabla u_s)^T \phi = G_f \text{ in } \Omega, \\ &\operatorname{div} \phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_i, \\ &\phi \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(\phi)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi, \psi)n + u_s \cdot n\phi = 0 \text{ on } \Gamma_n, \\ &\lambda \boldsymbol{\xi}_1 + \alpha \boldsymbol{\xi}_2 - A_1^* \phi + A_3^* \psi = G_s^1, \\ &\lambda \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1 + \gamma \boldsymbol{\xi}_2 - A_2^* \phi - A_4^* \psi = G_s^2, \end{aligned} \tag{2.6 .30}$$

if and only if

$$
\begin{aligned}
&(\lambda I - \mathcal{A}^{\#}) \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \Pi^* \begin{pmatrix} G_f \\ G_s^1 \\ G_s^2 \end{pmatrix}, \\
&\phi = N_v \phi + N_p G_f .
\end{aligned}
\tag{2.6 .31}
$$

Proof. The proof is similar to that of Proposition 2.6 .5.

□

Now, we introduce the operator $M \in \mathcal{L}(Z)$ defined by

$$
M = \begin{pmatrix} M_f & 0 & 0 \\ -A_3^* N_p & I & 0 \\ A_4^* N_p & 0 & I \end{pmatrix}. \tag{2.6 .32}
$$

where

$$
M_f = I - \nabla N_{\mathrm{div}} A_3 A_3^* N_p + \nabla N_s A_4 A_4^* N_p.
$$

Lemma 2.6 .4. *The operator M_f is an automorphism in $\mathbf{V}(\Omega)$.*

Proof. To prove the lemma, we are going to show that M_f is symmetric and positive. Let v and ϕ belong to $\mathbf{V}(\Omega)$. We have

$$
\begin{aligned}
\int_\Omega \nabla N_{\mathrm{div}} A_3 A_3^* N_p v \cdot \phi &= -\int_\Omega N_{\mathrm{div}} A_3 A_3^* N_p v \cdot \operatorname{div} \phi + \int_{\Gamma_s} \phi \cdot n N_{\mathrm{div}} A_3 A_3^* N_p v \\
&= -\int_\Omega N_{\mathrm{div}} A_3 A_3^* N_p v \cdot \Delta N_p \phi + \int_{\Gamma_s} \frac{\partial N_p \phi}{\partial n} N_{\mathrm{div}} A_3 A_3^* N_p v \\
&= -\int_\Omega A_3 A_3^* N_p v \cdot N_p \phi \\
&= -\int_\Omega N_p v \cdot A_3 A_3^* N_p \phi \\
&= -\int_\Omega N_p v \cdot \Delta N_{\mathrm{div}} A_3 A_3^* N_p \phi \\
&= -\int_\Omega \operatorname{div} v \cdot N_{\mathrm{div}} A_3 A_3^* N_p \phi + \int_{\Gamma_s} v \cdot n N_{\mathrm{div}} A_3 A_3^* N_p \phi \\
&= \int_\Omega v \cdot \nabla N_{\mathrm{div}} A_3 A_3^* N_p \phi .
\end{aligned}
$$

Similarly, we prove that

$$
\int_\Omega \nabla N_s A_4 A_4^* N_p v \cdot \phi = \int_\Omega v \cdot \nabla N_s A_4 A_4^* N_p \phi .
$$

Thus, M_f is symmetric. For the positivity, we notice that

$$
\int_\Omega \nabla N_{\mathrm{div}} A_3 A_3^* N_p v \cdot v = -A_3^* N_p v \cdot A_3^* N_p v \quad \text{and} \quad \int_\Omega \nabla N_s A_4 A_4^* N_p v \cdot v = A_4^* N_p v \cdot A_4^* N_p v .
$$

Thus, due to its definition, M_f is positive.

□

Lemma 2.6 .5. *The operator M is an automorphism in $\mathbf{V}(\Omega) \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c}$ and we have*

$$M^{-1} = \begin{pmatrix} M_f^{-1} & 0 & 0 \\ A_3^* N_p M_f^{-1} & I & 0 \\ -A_4^* N_p M_f^{-1} & 0 & I \end{pmatrix}.$$

Proof. It follows from Lemma 2.6 .4. □

Lemma 2.6 .6. *We have*

$$I - \nabla N_p = (I - \nabla N_p) M_f^{-1}.$$

Proof. We have

$$I - \nabla N_p = (I - \nabla N_p) M_f M_f^{-1} = (I - \nabla N_p)(M_f^{-1} - \nabla N_{\mathrm{div}} A_3 A_3^* N_p M_f^{-1} + \nabla N_s A_4 A_4^* N_p M_f^{-1}).$$

It is easy to show that

$$\nabla N_p(\nabla N_{\mathrm{div}} A_3 A_3^* N_p M_f^{-1}) = \nabla N_{\mathrm{div}} A_3 A_3^* N_p M_f^{-1},$$

and

$$\nabla N_p(\nabla N_s A_4 A_4^* N_p M_f^{-1}) = \nabla N_s A_4 A_4^* N_p M_f^{-1}.$$

We deduce that

$$I - \nabla N_p = (I - \nabla N_p) M_f^{-1}.$$

Lemma 2.6 .7. *We have*

$$\Pi^* M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}, \tag{2.6 .33}$$

for all $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \widehat{Z}$.

Proof. From Proposition 2.6 .1, it follows that

$$\Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} \phi + \nabla N_{\mathrm{div}} A_3 \boldsymbol{\xi}_1 + \nabla N_s A_4 \boldsymbol{\xi}_2 \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix},$$

where

$$\tilde{\phi} = \phi + \nabla N_{\mathrm{div}} A_3 \boldsymbol{\xi}_1 + \nabla N_s A_4 \boldsymbol{\xi}_2.$$

Thus, we obtain

$$M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} M_f^{-1} \tilde{\phi} \\ \boldsymbol{\xi}_1 + A_3^* N_p M_f^{-1} \tilde{\phi} \\ \boldsymbol{\xi}_2 - A_4^* N_p M_f^{-1} \tilde{\phi} \end{pmatrix}.$$

Thanks to Proposition 2.6 .1, we deduce that

$$\Pi^* M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} (I - \nabla N_p) M_f^{-1} \tilde{\phi} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}.$$

From Lemma 2.6 .6, we have

$$(I - \nabla N_p) M_f^{-1} \tilde{\phi} = (I - \nabla N_p) \tilde{\phi} = (I - \nabla N_p)(\phi + \nabla N_{\mathrm{div}} A_3 \boldsymbol{\xi}_1 + \nabla N_s A_4 \boldsymbol{\xi}_2) = \phi. \tag{2.6 .34}$$

Thus, we obtain

$$\Pi^* M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}.$$

□

Lemma 2.6 .8. *We have*

$$M^{-1} \Pi \Pi^* \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \tag{2.6 .35}$$

for all $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in Z$.

Proof. From Proposition 2.6 .1, it follows that

$$\Pi^* \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} v - \nabla N_p v \\ \boldsymbol{\eta}_1 - A_3^* N_p v \\ \boldsymbol{\eta}_2 + A_4^* N_p v \end{pmatrix},$$

and

$$\Pi \Pi^* \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} v - \nabla N_p v - \nabla N_p (v - \nabla N_p v) + \nabla N_{\mathrm{div}} A_3 (\boldsymbol{\eta}_1 - A_3^* N_p v) + \nabla N_s A_4 (\boldsymbol{\eta}_2 + A_4^* N_p v) \\ \boldsymbol{\eta}_1 - A_3^* N_p v \\ \boldsymbol{\eta}_2 + A_4^* N_p v \end{pmatrix}.$$

We have

$$N_p v + N_p (v - \nabla N_p v) = N_p v,$$

and since $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ belongs to Z, we obtain

$$N_p v = N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 + N_s A_4 \boldsymbol{\eta}_1.$$

Hence

$$\Pi \Pi^* \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} v - \nabla N_{\mathrm{div}} A_3 A_3^* N_p v + \nabla N_s A_4 A_4^* N_p v \\ \boldsymbol{\eta}_1 - A_3^* N_p v \\ \boldsymbol{\eta}_2 + A_4^* N_p v \end{pmatrix} = M \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix},$$

and then

$$M^{-1} \Pi \Pi^* \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}.$$

□

Lemma 2.6 .9. *If* $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ *belongs to* $\widehat{Z}$ *then* $M^{-1}\Pi(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ *belongs to* Z.

Proof. From Proposition 2.6 .1, it follows that

$$\Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} \tilde{\phi} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix},$$

where

$$\tilde{\phi} = \phi + \nabla N_{\mathrm{div}} A_3 \boldsymbol{\xi}_1 + \nabla N_s A_4 \boldsymbol{\xi}_2.$$

Thus, we obtain

$$M^{-1}\Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} M_f^{-1}\tilde{\phi} \\ \boldsymbol{\xi}_1 + A_3^* N_p M_f^{-1}\tilde{\phi} \\ \boldsymbol{\xi}_2 - A_4^* N_p M_f^{-1}\tilde{\phi} \end{pmatrix},$$

Thanks to Proposition 2.6 .1, we deduce that

$$\Pi M^{-1}\Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} M_f^{-1}\tilde{\phi} - \nabla N_p M_f^{-1}\tilde{\phi} + \nabla N_{\mathrm{div}} A_3(\boldsymbol{\xi}_1 + A_3^* N_p M_f^{-1}\tilde{\phi}) + \nabla N_s A_4(\boldsymbol{\xi}_2 - A_4^* N_p M_f^{-1}\tilde{\phi}) \\ \boldsymbol{\xi}_1 + A_3^* N_p M_f^{-1}\tilde{\phi} \\ \boldsymbol{\xi}_2 - A_4^* N_p M_f^{-1}\tilde{\phi} \end{pmatrix}.$$

Since

$$\tilde{\phi} = M_f M_f^{-1}\tilde{\phi} = M_f^{-1}\tilde{\phi} - \nabla N_{\mathrm{div}} A_3 A_3^* N_p M_f^{-1}\tilde{\phi} + \nabla N_s A_4 A_4^* N_p M_f^{-1}\tilde{\phi},$$

we have

$$\begin{aligned} \Pi M^{-1}\Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} &= \begin{pmatrix} 2M_f^{-1}\tilde{\phi} - \nabla N_p M_f^{-1}\tilde{\phi} + \nabla N_{\mathrm{div}} A_3 \boldsymbol{\xi}_1 + \nabla N_s A_4 \boldsymbol{\xi}_2 - \tilde{\phi} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} \\ &= \begin{pmatrix} M_f^{-1}\tilde{\phi} + (I - \nabla N_p) M_f^{-1}\tilde{\phi} - \phi \\ \boldsymbol{\xi}_1 + A_3^* N_p M_f^{-1}\tilde{\phi} \\ \boldsymbol{\xi}_2 - A_4^* N_p M_f^{-1}\tilde{\phi} \end{pmatrix}. \end{aligned}$$

From Lemma 2.6 .6, we have

$$(I - \nabla N_p) M_f^{-1}\tilde{\phi} = (I - \nabla N_p)\tilde{\phi} = (I - \nabla N_p)(\phi + \nabla N_{\mathrm{div}} A_3 \boldsymbol{\xi}_1 + \nabla N_s A_4 \boldsymbol{\xi}_2) = \phi.$$

We deduce that

$$\Pi M^{-1}\Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} M_f^{-1}\tilde{\phi} \\ \boldsymbol{\xi}_1 + A_3^* N_p M_f^{-1}\tilde{\phi} \\ \boldsymbol{\xi}_2 - A_4^* N_p M_f^{-1}\tilde{\phi} \end{pmatrix} = M^{-1}\Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}.$$

$\square$

Proposition 2.6 .8. *The adjoint of $(\mathcal{A}, D(\mathcal{A}))$ in Z is $(\mathcal{A}^*, D(\mathcal{A}^*))$ defined by*

$$D(\mathcal{A}^*) = \{(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in Z \mid \Pi^*(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in D(\mathcal{A}^{\#})\}, \tag{2.6 .36}$$

and

$$\mathcal{A}^* = M^{-1}\Pi\mathcal{A}^{\#}\Pi^*. \tag{2.6 .37}$$

Proof. Let (F_f, F_s^1, F_s^2) and (G_f, G_s^1, G_s^2) be in Z. We consider the two following systems

$$\begin{aligned} &\lambda v - \operatorname{div}\sigma(v,p) + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s - A_1\boldsymbol{\eta}_1 - A_2\boldsymbol{\eta}_2 = F_f \text{ in } \Omega, \\ &\operatorname{div} v = A_3\boldsymbol{\eta}_1 \text{ in } \Omega,\ v = \eta_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i, \\ &v \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n, \\ &\lambda\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 = F_s^1, \\ &\lambda\boldsymbol{\eta}_2 + \alpha\boldsymbol{\eta}_1 + \gamma\boldsymbol{\eta}_2 = F_s^2, \end{aligned} \tag{2.6 .38}$$

and

$$\begin{aligned}
&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = G_f \text{ in } \Omega,\\
&\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_i,\\
&\phi\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(\phi)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \text{ on } \Gamma_n,\\
&\lambda\boldsymbol{\xi}_1 + \alpha\boldsymbol{\xi}_2 - A_1^*\phi + A_3^*\psi = G_s^1,\\
&\lambda\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1 + \gamma\boldsymbol{\xi}_2 - A_2^*\phi - A_4^*\psi = G_s^2.
\end{aligned} \tag{2.6 .39}$$

By integration by parts, we have

$$\begin{aligned}
\int_\Omega F_f\cdot\phi &= \int_\Omega G_f\cdot v - \int_{\Gamma_s} A_4\boldsymbol{\eta}_2\psi - \int_\Omega A_1\boldsymbol{\eta}_1\cdot\phi - \int_\Omega A_2\boldsymbol{\eta}_2\cdot\phi + \int_\Omega A_3\boldsymbol{\eta}_1\psi\\
&= \int_\Omega G_f\cdot v - \boldsymbol{\eta}_2\cdot A_4^*\psi - \boldsymbol{\eta}_1\cdot A_1^*\phi - \boldsymbol{\eta}_2\cdot A_2^*\phi + \boldsymbol{\eta}_1\cdot A_3^*\psi.
\end{aligned}$$

We also have

$$F_s^1\cdot\xi_1 + F_s^2\cdot\xi_2 = G_s^1\cdot\boldsymbol{\eta}_1 + G_s^2\cdot\boldsymbol{\eta}_2 + \boldsymbol{\eta}_2\cdot A_4^*\psi + \boldsymbol{\eta}_1\cdot A_1^*\phi + \boldsymbol{\eta}_2\cdot A_2^*\phi - \boldsymbol{\eta}_1\cdot A_3^*\psi.$$

By combining the two identities, we obtain the identity

$$\left(\begin{pmatrix} F_f\\ F_s^1\\ F_s^2\end{pmatrix}, \begin{pmatrix}\phi\\ \boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\end{pmatrix}\right) = \left(\begin{pmatrix} G_f\\ G_s^1\\ G_s^2\end{pmatrix}, \begin{pmatrix} v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}\right). \tag{2.6 .40}$$

Moreover, we notice that $(v,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ belongs to Z and $(\phi,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ belongs to $\widehat{Z}$. We have to interpret identity (2.6 .40). First, from Lemmas 2.6 .8 and 2.6 .9, it follows that

$$D(\mathcal{A}^*) = \{M^{-1}\Pi(\phi,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \mid (\phi,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\in D(\mathcal{A}^\#)\}. \tag{2.6 .41}$$

From Proposition 2.6 .5, we have

$$(\lambda I - \mathcal{A})\begin{pmatrix} v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix} = \begin{pmatrix} F_f\\ F_s^1\\ F_s^2\end{pmatrix}. \tag{2.6 .42}$$

Thanks to equation (2.6 .42), Lemma 2.6 .7 and Proposition 2.6 .1, it follows that

$$\begin{aligned}
\left(\begin{pmatrix} F_f\\ F_s^1\\ F_s^2\end{pmatrix}, \begin{pmatrix}\phi\\ \boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\end{pmatrix}\right) &= \left((\lambda I - \mathcal{A})\begin{pmatrix} v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}, \Pi^* M^{-1}\Pi\begin{pmatrix}\phi\\ \boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\end{pmatrix}\right)\\
&= \left(\Pi\,(\lambda I - \mathcal{A})\begin{pmatrix} v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}, M^{-1}\Pi\begin{pmatrix}\phi\\ \boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\end{pmatrix}\right)\\
&= \left((\lambda I - \mathcal{A})\begin{pmatrix} v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}, M^{-1}\Pi\begin{pmatrix}\phi\\ \boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\end{pmatrix}\right).
\end{aligned} \tag{2.6 .43}$$

From Proposition 2.6 .7, we have

$$(\lambda I - \mathcal{A}^\#)\begin{pmatrix}\phi\\ \boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\end{pmatrix} = \Pi^*\begin{pmatrix} G_f\\ G_s^1\\ G_s^2\end{pmatrix}. \tag{2.6 .44}$$

Thanks to equation (2.6 .44), Proposition 2.6 .1 and Lemma 2.6 .7, it follows that

$$\begin{aligned}
\left(\begin{pmatrix} G_f \\ G_s^1 \\ G_s^2 \end{pmatrix}, \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}\right) &= \left(\Pi^* \begin{pmatrix} G_f \\ G_s^1 \\ G_s^2 \end{pmatrix}, \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}\right) \\
&= \left((\lambda I - \mathcal{A}^\#) \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}, \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}\right) \\
&= \left((\lambda I - \Pi^* M^{-1} \Pi \mathcal{A}^\#) \Pi^* M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}, \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}\right) \qquad (2.6\ .45) \\
&= \left((\lambda I - M^{-1} \Pi \mathcal{A}^\# \Pi^*) M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}, \Pi \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}\right) \\
&= \left((\lambda I - M^{-1} \Pi \mathcal{A}^\# \Pi^*) M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}, \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}\right).
\end{aligned}$$

Finally, from equation (2.6 .40), (2.6 .43) and (2.6 .45), we obtain

$$\left((\lambda I - \mathcal{A}) \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right) = \left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, (\lambda I - M^{-1} \Pi \mathcal{A}^\# \Pi^*) M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}\right),$$

for all $(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in D(\mathcal{A})$ and for all $(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in D(\mathcal{A}^\#)$. Using (2.6 .41), from the above identity, and with classical arguments, we can conclude that $(\mathcal{A}^*, D(\mathcal{A}^*))$ is the adjoint of $(\mathcal{A}, D(\mathcal{A}))$ in Z. □

2.7 Stabilization of the linearized system

2.7 .1 Reformulation of the linearized system

The goal is to rewrite the linearized system

$$\begin{aligned}
& v_t - \operatorname{div} \sigma(v,p) + (u_s \cdot \nabla) v + (v \cdot \nabla) u_s + \omega v - A_1 \boldsymbol{\eta}_1 - A_2 \boldsymbol{\eta}_2 = 0 \text{ in } Q^\infty, \\
& \operatorname{div} v = A_3 \boldsymbol{\eta}_1 \text{ in } Q^\infty, \quad v = A_4 \boldsymbol{\eta}_2 n \text{ on } \Sigma_s^\infty, \quad v = 0 \text{ on } \Sigma_i^\infty, \\
& v \cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(v) n \cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(v,p) n = 0 \text{ on } \Sigma_n^\infty, \quad v(0) = v^0 \text{ in } \Omega, \qquad (2.7\ .1) \\
& \boldsymbol{\eta}_{1,t} - \boldsymbol{\eta}_2 - \omega \boldsymbol{\eta}_1 = 0, \quad \boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_1^0, \\
& \boldsymbol{\eta}_{2,t} + \alpha \boldsymbol{\eta}_1 + \gamma \boldsymbol{\eta}_2 - \omega \boldsymbol{\eta}_2 = \boldsymbol{f}, \quad \boldsymbol{\eta}_2(0) = \boldsymbol{\eta}_2^0,
\end{aligned}$$

as a control system.

Theorem 2.7 .1. *The quadruplet $(v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in X_{\delta_0}$ is solution of system (2.7 .1) if and only if*

$$\frac{d}{dt} \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = (\mathcal{A} + \omega I) \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} + \mathcal{B} \boldsymbol{f}, \quad \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}(0) = \begin{pmatrix} v^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{pmatrix}, \qquad (2.7\ .2)$$

and

$$p = -N_s A_4(\boldsymbol{\eta}_{2,t} - \omega\boldsymbol{\eta}_2) - N_{\text{div}} A_3(\boldsymbol{\eta}_{1,t} - \omega\boldsymbol{\eta}_1) + N_p A_1\boldsymbol{\eta}_1 + N_p A_2\boldsymbol{\eta}_2 + N_v v, \tag{2.7 .3}$$

with

$$\mathcal{B} = \Pi \begin{pmatrix} 0 \\ 0 \\ I_{\mathbb{R}^{n_c}} \end{pmatrix}. \tag{2.7 .4}$$

Proof. It follows from Proposition 2.6 .5.

□

2.7 .2 Projection onto the unstable subspace associated to $\mathcal{A} + \omega I$

Let $(\lambda_j)_{j\in\mathbb{N}^*}$ the eigenvalues of $\mathcal{A}$. We denote by $G_{\mathbb{R}}(\lambda)$ the real generalized eigenspace of $\mathcal{A}$ and by $G^*_{\mathbb{R}}(\lambda)$ the real generalized eigenspace of $\mathcal{A}^*$. We define the unstable subspaces

$$Z_u = \oplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j) \quad \text{and} \quad Z^*_u = \oplus_{j\in J_u} G^*_{\mathbb{R}}(\lambda_j),$$

where J_u is a finite subset of $\mathbb{N}^*$ such that

$$\operatorname{Re}\lambda_j \geq -\omega, \quad \forall\, j \in J_u.$$

We have the decomposition

$$Z = Z_u \oplus Z_s,$$

where Z_s is a subspace of Z invariant under $(e^{t\mathcal{A}})_{t\geq 0}$. We denote by π_u the projection onto Z_u along Z_s and set $\pi_s = I - \pi_u$. The goal is to express the projector π_u in terms of the solutions to the following eigenvalue problems

$$\begin{aligned}
&\lambda \in \mathbb{C}, \quad (v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in H_{\delta_0},\\
&\lambda v - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\boldsymbol{\eta}_1 - A_2\boldsymbol{\eta}_2 = 0 \text{ in } \Omega,\\
&\operatorname{div} v = A_3\boldsymbol{\eta}_1 \text{ in } \Omega, \quad v = A_4\boldsymbol{\eta}_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i,\\
&v\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n,\\
&\lambda\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 = 0,\\
&\lambda\boldsymbol{\eta}_2 + \alpha\boldsymbol{\eta}_1 + \gamma\boldsymbol{\eta}_2 = 0,
\end{aligned} \tag{2.7 .5}$$

and

$$\begin{aligned}
&\lambda \in \mathbb{C}, \quad (\phi, \psi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in H_{\delta_0},\\
&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0 \text{ in } \Omega,\\
&\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_i,\\
&\phi\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(\phi)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \text{ on } \Gamma_n,\\
&\lambda\boldsymbol{\xi}_1 + \alpha\boldsymbol{\xi}_2 - A^*_1\phi + A^*_3\psi = 0,\\
&\lambda\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1 + \gamma\boldsymbol{\xi}_2 - A^*_2\phi - A^*_4\psi = 0.
\end{aligned} \tag{2.7 .6}$$

We also consider the eigenvalue problems

$$\lambda \in \mathbb{C}, \quad (v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in D(\mathcal{A}), \quad \lambda\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \tag{2.7 .7}$$

and

$$\lambda \in \mathbb{C}, \quad (\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in D(\mathcal{A}^*), \quad \lambda \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \mathcal{A}^* \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix}. \tag{2.7 .8}$$

Definition 2.7 .1. *A triplet $(v_k, \boldsymbol{\eta}_{1,k}, \boldsymbol{\eta}_{2,k}) \in D(\mathcal{A})$ is a generalized eigenfunction for problem (2.7 .7) of order k associated to a solution $(\lambda, (v_0, \boldsymbol{\eta}_{1,0}, \boldsymbol{\eta}_{2,0}))$ of (2.7 .7) if $(v_k, \boldsymbol{\eta}_{1,k}, \boldsymbol{\eta}_{2,k})$ is obtained by solving the chain of equations*

$$(\lambda I - \mathcal{A}) \begin{pmatrix} v_j \\ \boldsymbol{\eta}_{1,j} \\ \boldsymbol{\eta}_{2,j} \end{pmatrix} = - \begin{pmatrix} v_{j-1} \\ \boldsymbol{\eta}_{1,j-1} \\ \boldsymbol{\eta}_{2,j-1} \end{pmatrix}, \quad \text{for } 1 \leq j \leq k.$$

A quadruplet $(v_k, p_k, \boldsymbol{\eta}_{1,k}, \boldsymbol{\eta}_{2,k}) \in H_{\delta_0}$ is a generalized eigenfunction for problem (2.7 .5) of order k associated to a solution $(\lambda, (v_0, p_0, \boldsymbol{\eta}_{1,0}, \boldsymbol{\eta}_{2,0}))$ of (2.7 .5) if $(v_k, \boldsymbol{\eta}_{1,k}, \boldsymbol{\eta}_{2,k})$ is obtained by solving the chain of equations

$$\begin{aligned}
&\lambda v_j - \operatorname{div} \sigma(v_j, p_j) + (u_s \cdot \nabla) v_j + (v_j \cdot \nabla) u_s - A_1 \boldsymbol{\eta}_{1,j} - A_2 \boldsymbol{\eta}_{2,j} = -v_j \text{ in } \Omega, \\
&\operatorname{div} v_j = A_3 \boldsymbol{\eta}_{1,j} \text{ in } \Omega, \quad v_j = A_4 \boldsymbol{\eta}_{2,j} n \text{ on } \Gamma_s, \quad v_j = 0 \text{ on } \Gamma_i, \\
&v_j \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v_j) n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v_j, p_j) n = 0 \text{ on } \Gamma_n, \\
&\lambda \boldsymbol{\eta}_{1,j} - \boldsymbol{\eta}_{2,j} = -\boldsymbol{\eta}_{1,j}, \\
&\lambda \boldsymbol{\eta}_{2,j} + \alpha \boldsymbol{\eta}_{1,j} + \gamma \boldsymbol{\eta}_{2,j} = -\boldsymbol{\eta}_{2,j},
\end{aligned}$$

for $1 \leq j \leq k$. We have similar definitions for the adjoint eigenvalue problems (2.7 .8) and (2.7 .6).

Proposition 2.7 .2. *A pair $(\lambda, (v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)) \in \mathbb{C} \times H_{\delta_0}$ is a solution of eigenvalue problem (2.7 .5) if and only if $(\lambda, (v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)) \in \mathbb{C} \times D(\mathcal{A})$ is a solution of eigenvalue problem (2.7 .7) and*

$$p = -\lambda N_s A_4 \boldsymbol{\eta}_2 - \lambda N_{\mathrm{div}} A_3 \boldsymbol{\eta}_1 + N_v v.$$

A quadruplet $(v_k, p_k, \boldsymbol{\eta}_{1,k}, \boldsymbol{\eta}_{2,k}) \in H$ is a generalized eigenfunction of order k associated to a solution $(\lambda, (v_0, p_0, \boldsymbol{\eta}_{1,0}, \boldsymbol{\eta}_{2,0}))$ of (2.7 .5) if and only if $(\lambda, (v_k, \boldsymbol{\eta}_{1,k}, \boldsymbol{\eta}_{2,k})) \in D(\mathcal{A})$ is a generalized eigenfunction of order k associated to a solution $(\lambda, (v_0, \boldsymbol{\eta}_{1,0}, \boldsymbol{\eta}_{2,0}))$ of (2.7 .7) and

$$p_k = -\lambda N_s A_4 \boldsymbol{\eta}_{2,k} - \lambda N_{\mathrm{div}} A_3 \boldsymbol{\eta}_{1,k} + N_v v_k.$$

Proof. It follows from Proposition 2.6 .5.

□

Proposition 2.7 .3. *A pair $(\lambda, (\phi, \psi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)) \in \mathbb{C} \times H_{\delta_0}$ is a solution of eigenvalue problem (2.7 .6) if and only if $(\lambda, M^{-1}\Pi(\phi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)) \in \mathbb{C} \times D(\mathcal{A}^*)$ is a solution of eigenvalue problem (2.7 .8) and*

$$\begin{aligned}
&(I - \Pi) \begin{pmatrix} \phi_k \\ \boldsymbol{\xi}_{1,k} \\ \boldsymbol{\xi}_{2,k} \end{pmatrix} = - \begin{pmatrix} \nabla N_{\mathrm{div}} A_3 \boldsymbol{\xi}_{1,k} + \nabla N_s A_4 \boldsymbol{\xi}_{2,k} \\ 0 \\ 0 \end{pmatrix}, \\
&\psi = N_\phi \phi.
\end{aligned}$$

A quadruplet $(\phi_k, \psi_k, \boldsymbol{\xi}_{1,k}, \boldsymbol{\xi}_{2,k}) \in H_{\delta_0}$ is a generalized eigenfunction of order k associated to a solution $(\lambda, (\phi_0, \psi_0, \boldsymbol{\xi}_{1,0}, \boldsymbol{\xi}_{2,0}))$ of (2.7 .6) if and only if $(\lambda, (\phi_k, \boldsymbol{\xi}_{1,k}, \boldsymbol{\xi}_{2,k})) \in D(\mathcal{A}^*)$ is a generalized *eigenfunction of order k associated to a solution $(\lambda,(\phi_0,\boldsymbol{\xi}_{1,0},\boldsymbol{\xi}_{2,0}))$ of (2.7 .8) and*

$$(I-\Pi)\begin{pmatrix}\phi_k\\ \boldsymbol{\xi}_{1,k}\\ \boldsymbol{\xi}_{2,k}\end{pmatrix}=-\begin{pmatrix}\nabla N_{\mathrm{div}}A_3\boldsymbol{\xi}_{1,k}+\nabla N_s A_4\boldsymbol{\xi}_{2,k}\\ 0\\ 0\end{pmatrix},$$
$$\psi_k=N_\phi\phi_k.$$

Proof. It follows from Propositions 2.6 .7 and 2.6 .8.

□

Theorem 2.7 .4. *There exist two families $(v_i,\boldsymbol{\eta}_{1,i},\boldsymbol{\eta}_{2,i})_{1\le i\le d_u}$ and $(\phi_i,\boldsymbol{\xi}_{1,i},\boldsymbol{\xi}_{2,i})_{1\le i\le d_u}$ of H such that:*

- *The family $(v_i,\boldsymbol{\eta}_{1,i},\boldsymbol{\eta}_{2,i})_{1\le i\le d_u}$ is a basis of Z_u.*
- *The family $(M^{-1}\Pi(\phi_i,\boldsymbol{\xi}_{1,i},\boldsymbol{\xi}_{2,i}))_{1\le i\le d_u}$ is a basis of Z_u^*.*
- *We have the bi-orthogonality condition*

$$\left(\begin{pmatrix}v_i\\ \boldsymbol{\eta}_{1,i}\\ \boldsymbol{\eta}_{2,i}\end{pmatrix},\begin{pmatrix}\phi_i\\ \boldsymbol{\xi}_{1,i}\\ \boldsymbol{\xi}_{2,i}\end{pmatrix}\right)=\delta_{i,j},\quad \forall\, 1\le i,j\le d_u. \tag{2.7 .9}$$

- *The projector π_u is characterized by*

$$\pi_u\begin{pmatrix}v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}=\sum_{i=1}^{d_u}\left(\begin{pmatrix}v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix},\begin{pmatrix}\phi_i\\ \boldsymbol{\xi}_{1,i}\\ \boldsymbol{\xi}_{2,i}\end{pmatrix}\right)\begin{pmatrix}v_i\\ \boldsymbol{\eta}_{1,i}\\ \boldsymbol{\eta}_{2,i}\end{pmatrix},\quad \forall\,(v,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)^T\in Z.$$

- *The matrix Λ_u defined by*

$$\Lambda_u=[\Lambda_{u,i,j}]_{1\le i,j\le d_u}\quad \text{and}\quad \Lambda_{u,i,j}=\left(\mathcal{A}\begin{pmatrix}v_i\\ \boldsymbol{\eta}_{1,i}\\ \boldsymbol{\eta}_{2,i}\end{pmatrix},\begin{pmatrix}\phi_j\\ \boldsymbol{\xi}_{1,j}\\ \boldsymbol{\xi}_{2,j}\end{pmatrix}\right),$$

  is constituted of real Jordan blocks.

Proof. Since $\mathcal{A}$ is the generator of an analytic semigroup with compact resolvent, there exist a basis $(v_i,\boldsymbol{\eta}_{1,i},\boldsymbol{\eta}_{2,i})_{1\le i\le d_u}$ of Z_u constituted of generalized eigenfunctions of $\mathcal{A}$ and a basis $(\tilde{\phi}_i,\tilde{\boldsymbol{\xi}}_{1,i},\tilde{\boldsymbol{\xi}}_{2,i})_{1\le i\le d_u}$ of Z_u^* constituted of generalized eigenfunctions of $\mathcal{A}^*$, such that

$$\left(\begin{pmatrix}v_i\\ \boldsymbol{\eta}_{1,i}\\ \boldsymbol{\eta}_{2,i}\end{pmatrix},\begin{pmatrix}\tilde{\phi}_i\\ \tilde{\boldsymbol{\xi}}_{1,i}\\ \tilde{\boldsymbol{\xi}}_{2,i}\end{pmatrix}\right)=\delta_{i,j},\quad \forall\, 1\le i,j\le d_u, \tag{2.7 .10}$$

and

$$\pi_u\begin{pmatrix}v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}=\sum_{i=1}^{d_u}\left(\begin{pmatrix}v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix},\begin{pmatrix}\tilde{\phi}_i\\ \tilde{\boldsymbol{\xi}}_{1,i}\\ \tilde{\boldsymbol{\xi}}_{2,i}\end{pmatrix}\right)\begin{pmatrix}v_i\\ \boldsymbol{\eta}_{1,i}\\ \boldsymbol{\eta}_{2,i}\end{pmatrix},\quad \forall\,(v,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)^T\in Z, \tag{2.7 .11}$$

(see ([22], Lemma 6.2)). We define $(\phi_i,\boldsymbol{\xi}_{1,i},\boldsymbol{\xi}_{2,i})$ by

$$\Pi\begin{pmatrix}\phi_i\\ \boldsymbol{\xi}_{1,i}\\ \boldsymbol{\xi}_{2,i}\end{pmatrix}=M\begin{pmatrix}\tilde{\phi}_i\\ \tilde{\boldsymbol{\xi}}_{1,i}\\ \tilde{\boldsymbol{\xi}}_{2,i}\end{pmatrix},$$

and

$$(I-\Pi)\begin{pmatrix}\phi_i\\ \boldsymbol{\xi}_{1,i}\\ \boldsymbol{\xi}_{2,i}\end{pmatrix} = -\begin{pmatrix}\nabla N_{\mathrm{div}} A_3\tilde{\boldsymbol{\xi}}_{1,i} + \nabla N_s A_4 \tilde{\boldsymbol{\xi}}_{2,i}\\ 0\\ 0\end{pmatrix}.$$

Using Lemma 2.6 .7, we prove that

$$\left(\begin{pmatrix}v_i\\ \boldsymbol{\eta}_{1,i}\\ \boldsymbol{\eta}_{2,i}\end{pmatrix}, \begin{pmatrix}\phi_i\\ \boldsymbol{\xi}_{1,i}\\ \boldsymbol{\xi}_{2,i}\end{pmatrix}\right) = \delta_{i,j}, \quad \forall\, 1\le i,j\le d_u,$$

and

$$\pi_u\begin{pmatrix}v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix} = \sum_{i=1}^{d_u}\left(\begin{pmatrix}v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}, \begin{pmatrix}\phi_i\\ \boldsymbol{\xi}_{1,i}\\ \boldsymbol{\xi}_{2,i}\end{pmatrix}\right)\begin{pmatrix}v_i\\ \boldsymbol{\eta}_{1,i}\\ \boldsymbol{\eta}_{2,i}\end{pmatrix}, \quad \forall\,(v,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)^T\in Z.$$

The property of the matrix Λ_u comes from the construction of $(v_i,\boldsymbol{\eta}_{1,i},\boldsymbol{\eta}_{2,i})_{1\le i\le d_u}$ and $(\phi_i,\boldsymbol{\xi}_{1,i},\boldsymbol{\xi}_{2,i})_{1\le i\le d_u}$. □

Remark 2.7 .1. *From Proposition 2.7 .2, it follows that there exists $p_i\in H^1_{\delta_0}(\Omega)$ such that $(v_i,p_i,\boldsymbol{\eta}_{1,i},\boldsymbol{\eta}_{2,i})\in H_{\delta_0}$ is a real or imaginary part of a generalized eigenfunction of (2.7 .5) for all $1\le i\le d_u$. Similarly, from Proposition 2.7 .3, it follows that, there exists $\psi_i\in H^1_{\delta_0}(\Omega)$ such that $(\phi_i,\psi_i,\boldsymbol{\xi}_{1,i},\boldsymbol{\xi}_{2,i})\in H_{\delta_0}$ is a real or imaginary part of a generalized eigenfunction of (2.7 .6).*

2.7 .3 Stabilizability of the pair $(\mathcal{A}+\omega I,\mathcal{B})$

We are going to construct the family $(w_i)_{1\le i\le n_c}$ in such way that the pair $(\mathcal{A}+\omega I,\mathcal{B})$ is stabilizable. We denote by $\widetilde{A}_1^*\in\mathcal{L}(\mathbf{L}^2(\Omega),(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))')$, $\widetilde{A}_2^*\in\mathcal{L}(\mathbf{L}^2(\Omega),(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))')$ and $\widetilde{A}_3^*\in\mathcal{L}(L^2(\Omega),(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))')$ the adjoints of $\widetilde{A}_1\in\mathcal{L}(H^4(\Gamma_s)\cap H^2_0(\Gamma_s),\mathbf{L}^2(\Omega))$, $\widetilde{A}_2\in\mathcal{L}(H^4(\Gamma_s)\cap H^2_0(\Gamma_s),\mathbf{L}^2(\Omega))$ and $\widetilde{A}_3\in\mathcal{L}(H^4(\Gamma_s)\cap H^2_0(\Gamma_s),L^2(\Omega))$.

Proposition 2.7 .5. *We have*

$$\begin{aligned}
&A_1^*\phi = \Big(\langle w_i,\widetilde{A}_1^*\phi\rangle_{H^4(\Gamma_s)\cap H^2_0(\Gamma_s),(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))'}\Big)_i,\\
&A_2^*\phi = \Big(\langle w_i,\widetilde{A}_2^*\phi\rangle_{H^4(\Gamma_s)\cap H^2_0(\Gamma_s),(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))'}\Big)_i,\\
&A_3^*\psi = \Big(\langle w_i,\widetilde{A}_3^*\psi\rangle_{H^4(\Gamma_s)\cap H^2_0(\Gamma_s),(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))'}\Big)_i,\\
&\text{and } A_4^*\eta = \Big(\langle w_i,\eta\rangle_{H^4(\Gamma_s)\cap H^2_0(\Gamma_s),(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))'}\Big)_i,
\end{aligned} \tag{2.7 .12}$$

for all $\phi\in\mathbf{L}^2(\Omega)$, for all $\psi\in L^2(\Omega)$ and for all $\eta\in L^2(\Gamma_s)$.

Proof. We have

$$(A_1\boldsymbol{\eta}_1,\phi)_{\mathbf{L}^2(\Omega)} = \sum_{i=1}^{n_c}\boldsymbol{\eta}_{1,i}(\widetilde{A}_1 w_i,\phi)_{\mathbf{L}^2(\Omega)} = \boldsymbol{\eta}_1\cdot\Big(\langle w_i,\widetilde{A}_1^*\phi\rangle_{H^4(\Gamma_s)\cap H^2_0(\Gamma_s),(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))'}\Big)_i.$$

The other identites are proved in a similar way.

□

We consider the eigenvalue problem

$$\begin{aligned}
&\lambda\in\mathbb{C},\quad (\phi,\psi)\in\mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega),\\
&\lambda\phi-\operatorname{div}\sigma(\phi,\psi)-(u_s\cdot\nabla)\phi+(\nabla u_s)^T\phi=0 \text{ in } \Omega,\\
&\operatorname{div}\phi=0 \text{ in } \Omega,\quad \phi=0 \text{ on } \Gamma_s,\quad \phi=0 \text{ on } \Gamma_i,\\
&\phi\cdot n=0 \text{ on } \Gamma_e,\quad \varepsilon(\phi)n\cdot\tau=0 \text{ on } \Gamma_e,\quad \sigma(\phi,\psi)n+u_s\cdot n\phi=0 \text{ on } \Gamma_n.
\end{aligned} \tag{2.7 .13}$$

We introduce the spaces $(E(\lambda_j))_{j\in J_u}$ defined by

$$E(\lambda_j) = \{(\phi,\psi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega) \mid (\lambda_j,(\phi,\psi)) \text{ is solution of eigenvalue problem (2.7 .13)}\}. \tag{2.7 .14}$$

Finally, we introduce the Riesz operator $R_s \in \mathcal{L}((H^4(\Gamma_s)\cap H^2_0(\Gamma_s))', H^4(\Gamma_s)\cap H^2_0(\Gamma_s))$ defined by $R_s\eta = \xi$, where ξ is such that

$$\langle \zeta,\eta\rangle_{H^4(\Gamma_s)\cap H^2_0(\Gamma_s),(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))'} = (\zeta,\xi)_{H^4(\Gamma_s)\cap H^2_0(\Gamma_s)}.$$

We choose the family $(w_i)_{1\le i\le n_c}$ in $H^4(\Gamma_s)\cap H^2_0(\Gamma_s)$ such that

$$\begin{aligned}\operatorname{Vect}\{w_i \mid 1\le i\le n_c\} = \operatorname{Vect}\Big\{&\operatorname{Re}[R_s(\widetilde{A}_1^*\phi_j^k - \widetilde{A}_3^*\psi_j^k + \lambda_j\widetilde{A}_2^*\phi_j^k + \lambda_j\phi_j^k)],\\ &\operatorname{Im}[R_s(\widetilde{A}_1^*\phi_j^k - \widetilde{A}_3^*\psi_j^k + \lambda_j\widetilde{A}_2^*\phi_j^k + \lambda_j\phi_j^k)] \mid j\in J_u, 1\le k\le d_j\Big\},\end{aligned} \tag{2.7 .15}$$

where $(\phi_j^k,\psi_j^k)_{1\le k\le d_j}$ is a basis of $E(\lambda_j)$. The number d_j is the dimension of $E(\lambda_j)$. It also corresponds to the geometrical multiplicity of the eigenvalue λ_j of A.

Assumption 2.7 .1. *We assume that $-\omega \notin \sigma(\mathcal{A})$, $0\notin\sigma(A)$ and $\sigma(A)\cap\sigma(A_s)=\emptyset$.*

Assumption 2.7 .2. *Let λ be in $\mathbb{C}^*$ such that $\operatorname{Re}\lambda \ge -\omega$. Assume that (ϕ,ψ) is solution of the equation*

$$\begin{aligned}&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0 \text{ in } \Omega,\\ &\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma_s, \quad \psi = 0 \text{ on } \Gamma_s.\end{aligned}$$

If

$$\lambda(\widetilde{A}_2^*\phi + \psi) = \widetilde{A}_3^*\psi - \widetilde{A}_1^*\phi, \tag{2.7 .16}$$

then $(\phi,\psi)=(0,0)$.

Remark 2.7 .2. *When $u_s = 0$ (case of the Stokes system), the unique continuation property is proved in [46] and [47]. Unfortunately, the proofs given in these two papers cannot be adapted in the case $u_s\neq 0$. Indeed, a necessary and may be not sufficient condition to adapt those proofs is to assume that u_s is analytic. However, we can numerically see that this property is verified.*

Theorem 2.7 .6. *We assume that 2.5 .1, 2.7 .2 and 2.7 .1 are satisfied, and that $(w_i)_{1\le i\le n_c}$ is given by (2.7 .15). Then, the pair $(\mathcal{A}+\omega I,\mathcal{B})$ is stabilizable.*

Proof. Thanks to [[10], Part III, Chapter 1, Proposition 3.3], the pair $(\mathcal{A}+\omega I,\mathcal{B})$ is stabilizable if and only if

$$\operatorname{Ker}(\lambda I - \mathcal{A}^*)\cap\operatorname{Ker}(\mathcal{B}^*) = \{0\} \text{ for all } \lambda\in\mathbb{C} \text{ such that } \operatorname{Re}\lambda \ge -\omega.$$

Let $M^{-1}\Pi(\phi,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ belong to $\operatorname{Ker}(\lambda I-\mathcal{A}^*)\cap\operatorname{Ker}(\mathcal{B}^*)$. From Proposition 2.6 .7, it follows that there exists $\psi\in L^2(\Omega)$ such that $(\phi,\psi,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ is solution of system

$$\begin{aligned}&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0 \text{ in } \Omega,\\ &\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_i,\\ &\phi\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(\phi)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \text{ on } \Gamma_n,\\ &\lambda\boldsymbol{\xi}_1 + \alpha\boldsymbol{\xi}_2 - A_1^*\phi + A_3^*\psi = 0,\\ &\lambda\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1 + \gamma\boldsymbol{\xi}_2 - A_2^*\phi - A_4^*\psi = 0.\end{aligned} \tag{2.7 .17}$$

Moreover, since

$$\mathcal{B}^* M^{-1} \Pi \begin{pmatrix} \phi \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = 0, \tag{2.7 .18}$$

we can prove that

$$\boldsymbol{\xi}_2 = 0. \tag{2.7 .19}$$

We are going to show that $(\phi, \psi, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = (0, 0, 0, 0)$. For that, we distinguish two cases:

Case 1: $\lambda \notin \sigma(A^*)$. In that case, we have $(\phi, \psi) = (0, 0)$. Thus, from the last equation of system (2.7 .17) and equation (2.7 .19), we get $\boldsymbol{\xi}_1 = 0$.

Case 2: $\lambda \in \sigma(A^*)$. It follows that

$$\begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = (\lambda I - A_s^*)^{-1} \begin{pmatrix} A_1^* \phi - A_3^* \psi \\ A_2^* \phi + A_4^* \psi \end{pmatrix}.$$

We deduce that

$$\boldsymbol{\xi}_1 = \frac{\lambda + \gamma}{\lambda^2 + \gamma\lambda + \alpha}(A_1^* \phi - A_3^* \psi) - \frac{\alpha}{\lambda^2 + \gamma\lambda + \alpha}(A_2^* \phi + A_4^* \psi), \tag{2.7 .20}$$

and

$$\boldsymbol{\xi}_2 = \frac{1}{\lambda^2 + \gamma\lambda + \alpha}(A_1^* \phi - A_3^* \psi) + \frac{\lambda}{\lambda^2 + \gamma\lambda + \alpha}(A_2^* \phi + A_4^* \psi). \tag{2.7 .21}$$

Since $\boldsymbol{\xi}_2 = 0$, we have

$$A_1^* \phi - A_3^* \psi + \lambda A_2^* \phi + \lambda A_4^* \psi = 0.$$

Thanks to Proposition 2.7 .5 and equation (2.7 .12), we obtain

$$\langle w_i, \widetilde{A}_1^* \phi - \widetilde{A}_3^* \psi + \lambda \widetilde{A}_2^* \phi + \lambda \psi \rangle_{H_0^2(\Gamma_s), H^{-2}(\Gamma_s)}, \quad \forall\, 1 \le i \le n_c,$$

and due the construction of the family $(w_i)_{1 \le i \le n_c}$, we get

$$\lambda(\widetilde{A}_2^* \phi + \psi) = \widetilde{A}_3^* \psi - \widetilde{A}_1^* \phi.$$

Thanks to Assumption 2.7 .2, if follows that $(\phi, \psi) = (0, 0)$. From the last equation of system (2.7 .17), we obtain $\boldsymbol{\xi}_1 = 0$.

□

2.7 .4 The feedback law

We are going to determine a feedback law of finite dimension able to stabilize system (2.7 .1). For that, we introduce the matrix B_u defined by

$$B_u := [B_{u,i,j}]_{1 \le i \le d_u, 1 \le j \le n_c} \quad \text{and} \quad B_{u,i,j} := \left(\mathcal{B} \begin{pmatrix} v_i \\ \boldsymbol{\eta}_{1,i} \\ \boldsymbol{\eta}_{2,i} \end{pmatrix}, \begin{pmatrix} \phi_j \\ \boldsymbol{\xi}_{1,j} \\ \boldsymbol{\xi}_{2,j} \end{pmatrix} \right) = \boldsymbol{\eta}_{2,i} \cdot \boldsymbol{\xi}_{2,j}. \tag{2.7 .22}$$

Thanks to Theorems 2.7 .4 and 2.7 .6, it follows that the pair $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, B_u)$ is stabilizable. Thus, the operator $K_u = [K_u^{i,j}]_{1 \le i \le n_c, 1 \le j \le d_u}$ defined by $K_u = -B_u^T P_u$, where P_u is the solution of the algebraic Riccati equation

$$P_u \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_u}, \quad P_u = P_u^T > 0,$$
$$P_u(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}}) P_u - P_u B_u B_u^T P_u = 0,$$

provides a stabilizing feedback law for $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, B_u)$. We introduce the operator $\mathcal{K} \in \mathcal{L}(H, \mathbb{R}^{n_c})$ defined by

$$\mathcal{K}\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \left(\sum_{j=1}^{d_u} K_u^{i,j} \left(\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix}, \begin{pmatrix} \phi_j \\ \boldsymbol{\xi}_{1,j} \\ \boldsymbol{\xi}_{2,j} \end{pmatrix} \right) \right)_{1 \le i \le n_c}.$$

Theorem 2.7 .7. *We assume that 2.5 .1, 2.7 .2 and 2.7 .1 are satisfied. The operator $\mathcal{K}$ provides a stabilizing feedback for $(\mathcal{A}+\omega I, \mathcal{B})$. Moreover, the operator $\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}$, with domain $D(\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}) = D(\mathcal{A})$, is the generator of an exponentially stable analytic semigroup on Z.*

Proof. A pair $(\lambda, (v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2))$ is a solution of the eigenvalue problem

$$(\mathcal{A}+\omega I+\mathcal{B}\mathcal{K})\begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \lambda \begin{pmatrix} v \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix},$$

if and only if

$$\begin{pmatrix} \mathcal{A}_{s,\omega} & \mathcal{B}_s\mathcal{K} \\ 0 & \mathcal{A}_{u,\omega} + \mathcal{B}_u\mathcal{K} \end{pmatrix} \begin{pmatrix} \pi_s(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \\ \pi_u(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \end{pmatrix} = \lambda \begin{pmatrix} \pi_s(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \\ \pi_u(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \end{pmatrix},$$

where

$$\mathcal{A}_{u,\omega} = \pi_u(\mathcal{A}+\omega I), \quad \mathcal{A}_{s,\omega} = \pi_s(\mathcal{A}+\omega I), \quad \mathcal{B}_u = \pi_u\mathcal{B} \quad \text{and} \quad \mathcal{B}_s = \pi_s\mathcal{B}.$$

Therefore, if λ is an eigenvalue of $\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}$ then either λ is an eigenvalue of $\mathcal{A}_{s,\omega}$ or λ is an eigenvalue of $\mathcal{A}_{u,\omega}+\mathcal{B}_u\mathcal{K}$. It is clear that if λ is an eigenvalue of $\mathcal{A}_{s,\omega}$ then $\mathrm{Re}(\lambda) < 0$. We can prove that $\lambda \in \sigma(\mathcal{A}_{u,\omega}+\mathcal{B}_u\mathcal{K})$ if and only if $\lambda \in \sigma(\Lambda_u + \omega I + B_u K_u)$. Thus, if λ is an eigenvalue of $\mathcal{A}_{u,\omega}+\mathcal{B}_u\mathcal{K}$ then $\mathrm{Re}(\lambda) < 0$. We deduce that $\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}$ is a generator of an infinitesimal semigroup which is stable on Z.
The operators $\mathcal{A}_{s,\omega}$ and $\mathcal{A}_{u,\omega}+\mathcal{B}_u\mathcal{K}$ are the generators of analytic semigroups on Z_s and Z_u respectively. Moreover, the operator

$$\begin{pmatrix} 0 & \mathcal{B}_s\mathcal{K} \\ 0 & 0 \end{pmatrix},$$

is a bounded perturbation with relative bound equal to zero of the the analytic operator

$$\begin{pmatrix} \mathcal{A}_{s,\omega} & 0 \\ 0 & \mathcal{A}_{u,\omega}+\mathcal{B}_u\mathcal{K} \end{pmatrix}.$$

Thus, the operator $\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}$ is a generator of an exponentially stable analytic semigroup. □

2.8 The closed-loop nonhomogeneous linearized system

The goal is to prove regularity results for the solutions to the closed-loop nonhomogeneous linear system

$$\begin{aligned}
&v_t - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - \omega v - A_1\boldsymbol{\eta}_1 - A_2\boldsymbol{\eta}_2 = F_f \text{ in } Q^\infty,\\
&\operatorname{div} v = A_3\boldsymbol{\eta}_1 + \operatorname{div} F_{\mathrm{div}} \text{ in } Q^\infty, \quad v = A_4\boldsymbol{\eta}_2 n \text{ on } \Sigma_s^\infty, \quad v = \hat{h} \text{ on } \Sigma_i^\infty,\\
&v\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(v)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(v,p)n = 0 \text{ on } \Sigma_n^\infty,\\
&\boldsymbol{\eta}_{1,t} - \boldsymbol{\eta}_2 - \omega\boldsymbol{\eta}_1 = 0,\\
&\boldsymbol{\eta}_{2,t} + \alpha\boldsymbol{\eta}_1 + \gamma\boldsymbol{\eta}_2 - \omega\boldsymbol{\eta}_2 = \mathcal{K}(v, \boldsymbol{\eta}_1, \boldsymbol{\eta}_1)^T,\\
&v(0) = v^0 \text{ in } \Omega, \quad \boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_1^0, \quad \boldsymbol{\eta}_2(0) = \boldsymbol{\eta}_2^0,
\end{aligned} \tag{2.8 .1}$$

where $\mathcal{K}$ is the feedback law stabilizing the pair $(\mathcal{A}+\omega I, \mathcal{B})$.

Theorem 2.8 .1. *We assume that 2.5 .1, 2.7 .2 and 2.7 .1 are satisfied. Let* $(F_f, F_{\rm div})$ *belong to* Y *and let* $(v^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)$ *belong to* $H_{ic}(F_{\rm div}(0))$, h *belong to* $H_0^1(0,\infty; H_0^2(\Gamma_i))$. *We assume that* $F_{\rm div}$ *satisfies the following boundary conditions*

$$\begin{aligned}&F_{\rm div}=0 \ \text{on}\ \Sigma_s^\infty, \quad F_{\rm div}=0 \ \text{on}\ \Sigma_i^\infty, \quad F_{\rm div}\cdot n=0 \ \text{on}\ \Sigma_e^\infty, \quad \varepsilon(F_{\rm div})n\cdot\tau=0 \ \text{on}\ \Sigma_e^\infty,\\ &\varepsilon(F_{\rm div})n=0 \ \text{on}\ \Sigma_n^\infty.\end{aligned} \tag{2.8 .2}$$

The solution $(v,p,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ *of system (2.8 .1) satisfies the estimate*

$$\|(v,p,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\|_{X_{\delta_0}} \le C\left(\left\|(v^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\right\|_{ic} + \|h\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} + \|(\mathcal{F}_f, F_{\rm div})\|_Y\right).$$

Proof. First, we make the change of unknown $\hat v = v - F_{\rm div}$. The quadruplet $(\hat v, p, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is solution of the system

$$\begin{aligned}&\hat v_t - \operatorname{div}\sigma(\hat v,p) + (u_s\cdot\nabla)\hat v + (\hat v\cdot\nabla)u_s - \omega\hat v - A_1\boldsymbol{\eta}_1 - A_2\boldsymbol{\eta}_2 = \widetilde F_f \text{ in } Q^\infty,\\ &\operatorname{div}\hat v = A_3\boldsymbol{\eta}_1 \text{ in } Q^\infty, \quad \hat v = A_4\boldsymbol{\eta}_2 n \text{ on } \Sigma_s^\infty, \quad \hat v = 0 \text{ on } \Sigma_i^\infty, \quad \hat v\cdot n = 0 \text{ on } \Sigma_f^\infty,\\ &\varepsilon(\hat v)n\cdot\tau = 0 \text{ on } \Sigma_f^\infty, \quad \sigma(\hat v,p)n = 0 \text{ on } \Sigma_n^\infty, \quad \hat v(0) = \hat v^0 = v^0 - w(0) \text{ in } \Omega,\\ &\boldsymbol{\eta}_{1,t} - \boldsymbol{\eta}_2 - \omega\boldsymbol{\eta}_1 = 0, \quad \boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_1^0,\\ &\boldsymbol{\eta}_{2,t} + \alpha\boldsymbol{\eta}_1 + \gamma\boldsymbol{\eta}_2 - \omega\boldsymbol{\eta}_2 = \mathcal{K}(\hat v,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)^T + \widetilde F_s, \quad \boldsymbol{\eta}_2(0) = \boldsymbol{\eta}_2^0,\end{aligned} \tag{2.8 .3}$$

where

$$\widetilde F_f = F_f - F_{{\rm div},t} + 2\nu\operatorname{div}\varepsilon(F_{\rm div}) - (u_s\cdot\nabla)F_{\rm div} - (F_{\rm div}\cdot\nabla)u_s + \omega F_{\rm div},$$

and

$$\widetilde F_s = \mathcal{K}(F_{\rm div},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)^T.$$

We have $(\hat v^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0) \in [D(\mathcal{A}), Z]_{\frac12}$, $\widetilde F_f \in L^2(0,\infty;\mathbf{L}^2(\Omega))$ and $\widetilde F_s \in L^2(0,\infty;\mathbb{R}^{n_c})$ with the estimate

$$\begin{aligned}&\|(\hat v^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\|_{[D(\mathcal{A}),Z]_{\frac12}} + \|\widetilde F_f\|_{L^2(0,\infty;\mathbf{L}^2(\Omega))} + \|\widetilde F_s\|_{L^2(0,\infty;\mathbb{R}^{n_c})}\\ &\le C\left(\|(v^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\|_{ic} + \|h\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} + \|(F_f,F_{\rm div})\|_Y\right).\end{aligned}$$

As in Proposition 2.7 .1, we prove that $(\hat v,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ is solution of the equation

$$\frac{d}{dt}\begin{pmatrix}\hat v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix} = (\mathcal{A}+\omega I+\mathcal{B}\mathcal{K})\begin{pmatrix}\hat v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix} + \Pi\begin{pmatrix}\widetilde F_f\\ 0\\ \widetilde F_s\end{pmatrix}, \quad \begin{pmatrix}\hat v\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}(0) = \begin{pmatrix}\hat v(0)\\ \boldsymbol{\eta}_1^0\\ \boldsymbol{\eta}_2^0\end{pmatrix}. \tag{2.8 .4}$$

Since $(\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}, D(\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}))$ is the generator of an analytic semigroup exponentially stable on Z, there exists $C>0$ such that

$$\begin{aligned}\|(\hat v,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\|_{L^2(0,\infty;D(\mathcal{A}))} &\le C\left(\left\|(\hat v^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\right\|_{[D(\mathcal{A}),Z]_{\frac12}} + \left\|\Pi(\widetilde F_f,0,\widetilde F_s)\right\|_{L^2(0,\infty;Z)}\right)\\ &\le C\left(\|(v^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\|_{ic} + \|h\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} + \|(F_f,w)\|_Y\right).\end{aligned}$$

From ([43], Theorem 9.4.5), it follows that the pressure p belongs to $L^2(0,\infty;H_\delta^{\frac12}(\Omega))$ with the estimate

$$\|p\|_{L^2(0,\infty;H_\delta^{\frac12}(\Omega))} \le C\left(\left\|(v^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\right\|_{ic} + \|h\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} + \|(F_f,F_{\rm div})\|_Y\right).$$

By combining the three inequalities, we get

$$\|(v,p,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\|_{X_{\delta_0}} \le C\left(\left\|(v^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\right\|_{ic} + \|h\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} + \|(F_f,F_{\mathrm{div}})\|_Y\right).$$

□

2.9 Stabilization of the nonlinear system

The goal is to prove theorem 2.4 .1. For that, we consider the following closed-loop nonlinear system

$$\begin{aligned}
&\hat{u}_t - \operatorname{div}\sigma(\hat{u},\hat{p}) + (u_s\cdot\nabla)\hat{u} + (\hat{u}\cdot\nabla)u_s - A_1\hat{\boldsymbol{\eta}}_1 - A_2\hat{\boldsymbol{\eta}}_2 - \omega\hat{u} = \mathcal{F}_f[\hat{u},\hat{p},\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_2] \text{ in } Q^\infty,\\
&\operatorname{div}\hat{u} = A_3\hat{\boldsymbol{\eta}}_1 + \operatorname{div}F_{\mathrm{div}}[\hat{u},\hat{\boldsymbol{\eta}}_1] \text{ in } Q^\infty, \quad \hat{u} = \hat{\boldsymbol{\eta}}_2 n \text{ on } \Sigma_s^\infty, \quad \hat{u} = \hat{h} \text{ on } \Sigma_i^\infty,\\
&\hat{u}\cdot n = 0 \text{ on } \Sigma_f^\infty, \quad \varepsilon(\hat{u})n\cdot\tau = 0 \text{ on } \Sigma_f^\infty, \quad \sigma(\hat{u},\hat{p})n = 0 \text{ on } \Sigma_n^\infty, \quad \hat{u}(0) = \hat{u}^0 - u_s \text{ in } \Omega,\\
&\hat{\boldsymbol{\eta}}_{1,t} - \hat{\boldsymbol{\eta}}_2 - \omega\hat{\boldsymbol{\eta}}_1 = 0, \quad \hat{\boldsymbol{\eta}}_1(0) = \hat{\boldsymbol{\eta}}_1^0,\\
&\hat{\boldsymbol{\eta}}_{2,t} + \alpha\hat{\boldsymbol{\eta}}_1 + \gamma\hat{\boldsymbol{\eta}}_2 - \omega\hat{\boldsymbol{\eta}}_2 = \mathcal{K}(\hat{u},\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_1)^T, \quad \hat{\boldsymbol{\eta}}_2(0) = \hat{\boldsymbol{\eta}}_2^0,
\end{aligned} \tag{2.9 .1}$$

where $\mathcal{K}$ is the linear feedback law stabilizing system (2.7 .1). We are going to prove the stability of that system.

Theorem 2.9 .1. *Let $\eta_{\max}$ be in $(0,e)$ and ω be positive. We assume that 2.5 .1, 2.7 .2 and 2.7 .1 are satisfied. There exists $r>0$ such that for all $(\hat{u}^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\in H_{ic}(F_{\mathrm{div}}[\hat{u}^0,\boldsymbol{\eta}_1^0])$ and for all $\hat{h}\in H_0^1(0,\infty;H_0^2(\Gamma_i))$ satisfying the estimate*

$$\left\|(\hat{u}^0,\boldsymbol{\eta}_1^0,\boldsymbol{\eta}_2^0)\right\|_{ic} + \|\hat{h}\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} \le r,$$

there exists a unique solution $(\hat{u},\hat{p},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\in X_{\delta_0}$ to system (2.9 .1) satisfying the estimate

$$\|e^{-\omega t}\hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)} \quad \text{and} \quad \|(\hat{u}(t),\hat{\boldsymbol{\eta}}_1(t),\hat{\boldsymbol{\eta}}_2(t))\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon}(\Omega)\times\mathbb{R}^{n_c}\times\mathbb{R}^{n_c}} \le C, \quad \forall t>0,$$

where $\hat{\eta}_1 = \sum_{i=1}^{n_c}\hat{\boldsymbol{\eta}}_{1,i}w_i$.

Proof. The proof is similar to that of Theorem 1.8 .2. It is based on Theorem 2.8 .1 and the Banach fixed point Theorem.

□

2.10 Semi-discrete approximations

To approximate systems (2.4 .1) and (2.7 .1) by a finite elements method, we introduce finite dimensional subspaces $X_h\subset\mathbf{H}^1(\Omega)$ for the velocity, $P_h\subset L^2(\Omega)$ for the pressure, $S_h\subset H^2(\Gamma_s)$ for the control functions (w_i), $D_h\subset\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)$ for the multiplier associated to the Dirichlet boundary conditions on Γ_s. We denote by $(\phi_i)_{1\le i\le n_v}$ a basis of X_h, $(q_i)_{1\le i\le n_p}$ a basis of P_h, $(\zeta_i)_{1\le i\le n_s}$ a basis of S_h, $(\mu_i)_{1\le i\le n_\tau}$ a basis of D_h. We set

$$v=\sum_{i=1}^{n_v}v_i\phi_i, \quad v^0=\sum_{i=1}^{n_v}v_i^0\phi_i, \quad p=\sum_{i=1}^{n_p}p_iq_i, \quad \tau=\sum_{i=1}^{n_\tau}\tau_i\mu_i, \quad w_j=\sum_{i=1}^{n_s}w_j^i\zeta_i.$$

If we denote boldface letters the vectors of the corresponding coordinates, we have

$$\boldsymbol{v} = (v_1, \cdots, v_{n_v})^T, \quad \boldsymbol{v}^0 = (v_1^0, \cdots, v_{n_v}^0)^T, \quad \boldsymbol{p} = (p_1, \cdots, p_{n_p})^T, \quad \boldsymbol{\tau} = (\tau_1, \cdots, \tau_{n_\tau})^T,$$
$$\boldsymbol{\theta} = (p_1, \cdots, p_{n_p}, \tau_1, \cdots, \tau_{n_\tau})^T, \quad \mathbf{W} = \begin{bmatrix} w_1^1 & \cdots & w_1^{n_c} \\ \vdots & \vdots & \vdots \\ w_{n_s}^1 & \cdots & w_{n_s}^{n_c} \end{bmatrix}.$$

We introduce the matrix

$$\begin{aligned}
&(A_{vv})_{ij} = -2\nu \int_\Omega \varepsilon(\phi_i) : \varepsilon(\phi_j) - \int_\Omega (u_s \cdot \nabla)\phi_j \cdot \phi_i - \int_\Omega (\phi_j \cdot \nabla) u_s \cdot \phi_i, && \forall\, 1 \le i \le n_v,\ 1 \le i \le n_v,\\
&(A_{v\eta_1})_{ij} = \sum_{k=1}^{n_s} w_j^k \int_\Omega \widetilde{A}_1 \zeta_k \cdot \phi_i, \quad (A_{v\eta_2})_{ij} = \sum_{k=1}^{n_s} w_j^k \int_\Omega \widetilde{A}_2 \zeta_k \cdot \phi_i, && \forall\, 1 \le i \le n_v,\ 1 \le j \le n_c,\\
&(A_{vp})_{ij} = \int_\Omega q_j \operatorname{div} \phi_i,, && \forall\, 1 \le i \le n_v,\ 1 \le j \le n_p,\\
&(A_{v\tau})_{ij} = \int_{\Gamma_s} \mu_j \cdot \phi_i, && \forall\, 1 \le i \le n_v,\ 1 \le j \le n_\tau,\\
&(A_{\eta_1 p})_{ij} = -\sum_{k=1}^{n_s} w_i^k \int_\Omega \widetilde{A}_3 \zeta_k \cdot q_j, && \forall\, 1 \le i \le n_c,\ 1 \le j \le n_p,\\
&(A_{\eta_2 \tau})_{ij} = -\sum_{k=1}^{n_s} w_i^k \int_{\Gamma_s} \zeta_k n \cdot \mu_j, && \forall\, 1 \le i \le n_c,\ 1 \le j \le n_\tau,\\
&(M_{vv})_{ij} = \int_\Omega \phi_j \cdot \phi_i, && \forall\, 1 \le i \le n_v,\ 1 \le j \le n_v,\\
&A_{v\theta} = \begin{bmatrix} A_{vp} & A_{v\tau} \end{bmatrix}, \quad A_{\eta_1\theta} = \begin{bmatrix} A_{\eta_1 p} & 0 \end{bmatrix}, \quad A_{\eta_2\theta} = \begin{bmatrix} 0 & A_{\eta_2\tau} \end{bmatrix}.
\end{aligned} \tag{2.10 .1}$$

We set

$$n_1 = n_v + 2n_c, \quad n_\theta = n_p + n_\tau \quad \text{and} \quad n = n_1 + n_\theta.$$

We introduce the mass matrix M_L and the stiffness matrix A_L are defined by

$$M_L = \begin{bmatrix} M_{vv} & 0 & 0 & 0 \\ 0 & I_{\mathbb{R}^{n_c}} & 0 & 0 \\ 0 & 0 & I_{\mathbb{R}^{n_c}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_L = \begin{bmatrix} A_{vv} & A_{v\eta_1} & A_{v\eta_2} & A_{v\theta} \\ 0 & 0 & I_{\mathbb{R}^{n_c}} & 0 \\ 0 & -\alpha I_{\mathbb{R}^{n_c}} & -\gamma I_{\mathbb{R}^{n_c}} & 0 \\ A_{v\theta}^T & A_{\eta_1\theta}^T & A_{\eta_2\theta}^T & 0 \end{bmatrix}. \tag{2.10 .2}$$

The finite dimensional approximation of system (2.7 .1) (with $\omega = 0$) is

$$\begin{aligned}
&M_L \frac{d}{dt} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\theta} \end{bmatrix} = A_L \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{f} \\ 0 \end{bmatrix},\\
&\boldsymbol{v}(0) = \boldsymbol{v}^0, \quad \boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_1^0, \quad \boldsymbol{\eta}_2(0) = \boldsymbol{\eta}_2^0.
\end{aligned} \tag{2.10 .3}$$

We can rewrite system (2.10 .3) as follows

$$
\begin{aligned}
&M\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = A\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} + \widehat{A}_{v\eta\theta}\boldsymbol{\theta} + B\boldsymbol{f}, \quad \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}(0) = \begin{bmatrix}\boldsymbol{v}^0\\ \boldsymbol{\eta}_1^0\\ \boldsymbol{\eta}_2^0\end{bmatrix},\\
&A_{v\eta\theta}^T\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = 0,
\end{aligned}
\tag{2.10 .4}
$$

where

$$
\begin{aligned}
&M = \begin{bmatrix} M_{vv} & 0 & 0\\ 0 & I_{\mathbb{R}^{n_c}} & 0\\ 0 & 0 & I_{\mathbb{R}^{n_c}}\end{bmatrix}, \quad A = \begin{bmatrix} A_{vv} & A_{v\eta_1} & A_{v\eta_2}\\ 0 & 0 & I_{\mathbb{R}^{n_c}}\\ 0 & -\alpha I_{\mathbb{R}^{n_c}} & -\gamma I_{\mathbb{R}^{n_c}}\end{bmatrix},\\
&B = \begin{bmatrix}0\\ 0\\ I_{\mathbb{R}^{n_c}}\end{bmatrix}, \quad \widehat{A}_{v\eta\theta} = \begin{bmatrix}A_{v\theta}\\ 0\\ 0\end{bmatrix}, \quad A_{v\eta\theta} = \begin{bmatrix}A_{v\theta}\\ A_{\eta_1\theta}\\ A_{\eta_2\theta}\end{bmatrix}.
\end{aligned}
\tag{2.10 .5}
$$

The semi-discrete approximation of the nonlinear system (2.4 .3) is

$$
\begin{aligned}
&M_L\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{\eta}}_1\\ \tilde{\boldsymbol{\eta}}_2\\ \tilde{\boldsymbol{\theta}}\end{bmatrix} = \widetilde{A}_L\begin{bmatrix}\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{\eta}}_1\\ \tilde{\boldsymbol{\eta}}_2\\ \tilde{\boldsymbol{\theta}}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \boldsymbol{f}\\ \boldsymbol{h}\end{bmatrix} + \mathcal{F}_{ns}[\tilde{\boldsymbol{u}}] + \mathcal{F}_g[\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\eta}}_2],\\
&\tilde{\boldsymbol{u}}(0) = \tilde{\boldsymbol{u}}^0, \quad \tilde{\boldsymbol{\eta}}_1(0) = \tilde{\boldsymbol{\eta}}_1^0, \quad \tilde{\boldsymbol{\eta}}_2(0) = \tilde{\boldsymbol{\eta}}_2^0,
\end{aligned}
\tag{2.10 .6}
$$

where

$$
\widetilde{A}_L = \begin{bmatrix} \widetilde{A}_{vv} & 0 & 0 & A_{v\theta}\\ 0 & 0 & I_{\mathbb{R}^{n_c}} & 0\\ 0 & -\alpha I_{\mathbb{R}^{n_c}} & -\gamma I_{\mathbb{R}^{n_c}} & \\ A_{v\theta}^T & 0 & A_{\eta_2\theta}^T & 0\end{bmatrix},
$$

and

$$
\widetilde{A}_{vv} = \left(-2\nu\int_\Omega \varepsilon(\phi_i) : \varepsilon(\phi_j)\right)_{1\le i,j\le n_v}.
$$

The nonlinear terms $\mathcal{F}_{ns}[\tilde{\boldsymbol{u}}]$ and $\mathcal{F}_g[\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\eta}}_2]$ are defined by

$$
\mathcal{F}_{ns}[\tilde{\boldsymbol{u}}] = \begin{bmatrix}\mathcal{F}_{ns}^u\\ 0\\ 0\\ 0\end{bmatrix} \quad \text{and} \quad \mathcal{F}_g[\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\eta}}_2] = \begin{bmatrix}\mathcal{F}_g^u\\ 0\\ 0\\ \mathcal{F}_g^\theta\end{bmatrix},
$$

where

$$\mathcal{F}^u_{ns} = -\left(\sum_{i=1}^{n_v}\sum_{j=1}^{n_v} \tilde{u}_i\tilde{u}_j(\phi_i\cdot\nabla)\phi_j\cdot\phi_k\right)_{1\le k\le n_v},$$

$$\mathcal{F}^u_g = \left(F_f\left[\sum_{i=1}^{n_v}\tilde{u}_i\phi_i, \sum_{j=1}^{n_p}\tilde{p}_j q_j, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\eta}}_2\right]\cdot\phi_k\right)_{1\le k\le n_v},$$

$$\mathcal{F}^\theta_g = \left(F_{\mathrm{div}}\left[\sum_{i=1}^{n_v}\tilde{u}_i\phi_i, \sum_{j=1}^{n_p}\tilde{p}_j q_j, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\eta}}_2\right]\cdot\phi_k\right)_{1\le k\le n_v}.$$

2.11 Reformulation of the finite dimensional linear system

The goal is to rewrite system (2.10 .4) as a controlled system. For that, we have to eliminate the multiplier θ from the first equation. To do that, we are going to introduce the projection into $\mathrm{Ker}(A^T_{\upsilon\eta\theta}M^{-1})$ parallel to $\mathrm{Im}(\widehat{A}_{\upsilon\eta\theta})$.

Proposition 2.11 .1. *1. The projector $\mathbb{P}$ the projection into $\mathrm{Ker}(A^T_{\upsilon\eta\theta})$ parallel to $\mathrm{Im}(M^{-1}\widehat{A}_{\upsilon\eta\theta})$ is*

$$\mathbb{P} = I - M^{-1}\widehat{A}_{\upsilon\eta\theta}(A^T_{\upsilon\eta\theta}M^{-1}\widehat{A}_{\upsilon\eta\theta})^{-1}A^T_{\upsilon\eta\theta}.$$

2. The projector $\mathbb{P}^T$ the projection into $\mathrm{Ker}(\widehat{A}^T_{\upsilon\eta\theta}M^{-1})$ parallel to $\mathrm{Im}(A_{\upsilon\eta\theta})$ is

$$\mathbb{P}^T = I - A_{\upsilon\eta\theta}(\widehat{A}^T_{\upsilon\eta\theta}M^{-1}A_{\upsilon\eta\theta})^{-1}\widehat{A}^T_{\upsilon\eta\theta}M^{-1}.$$

3. The projector $\widehat{\mathbb{P}}$ the projection into $\mathrm{Ker}(\widehat{A}^T_{\upsilon\eta\theta})$ parallel to $\mathrm{Im}(M^{-1}A_{\upsilon\eta\theta})$ is

$$\widehat{\mathbb{P}} = I - M^{-1}A_{\upsilon\eta\theta}(\widehat{A}^T_{\upsilon\eta\theta}M^{-1}A_{\upsilon\eta\theta})^{-1}\widehat{A}^T_{\upsilon\eta\theta}.$$

4. The projector $\widehat{\mathbb{P}}^T$ the projection into $\mathrm{Ker}(A^T_{\upsilon\eta\theta}M^{-1})$ parallel to $\mathrm{Im}(\widehat{A}_{\upsilon\eta\theta})$ is

$$\widehat{\mathbb{P}}^T = I - \widehat{A}_{\upsilon\eta\theta}(A^T_{\upsilon\eta\theta}M^{-1}\widehat{A}_{\upsilon\eta\theta})^{-1}A^T_{\upsilon\eta\theta}M^{-1}.$$

Moreover, we have

$$\widehat{\mathbb{P}}^T\widehat{A}_{\upsilon\eta\theta} = 0,\quad (\widehat{\mathbb{P}}^T)^2 = \widehat{\mathbb{P}}^T,\quad \widehat{\mathbb{P}}^T M = M\mathbb{P},\quad \widehat{\mathbb{P}}^T M = (\widehat{\mathbb{P}}^T)^2 M = \widehat{\mathbb{P}}^T M\mathbb{P},\quad M^{-1}\widehat{\mathbb{P}}^T = \mathbb{P}M^{-1},$$
$$\mathbb{P}^T A_{\upsilon\eta\theta} = 0,\quad (\mathbb{P}^T)^2 = \mathbb{P}^T,\quad \mathbb{P}^T M = M\widehat{\mathbb{P}},\quad \mathbb{P}^T M = (\mathbb{P}^T)^2 M = \mathbb{P}^T M\widehat{\mathbb{P}},\quad M^{-1}\mathbb{P}^T = \widehat{\mathbb{P}}M^{-1}.$$

Proof. We can easily prove that the operator

$$\widehat{\mathbb{P}}^T = I - \widehat{A}_{\upsilon\eta\theta}(A^T_{\upsilon\eta\theta}M^{-1}\widehat{A}_{\upsilon\eta\theta})^{-1}A^T_{\upsilon\eta\theta}M^{-1},$$

is such that $(\widehat{\mathbb{P}}^T)^2 = \widehat{\mathbb{P}}^T$. Therefore, $\widehat{\mathbb{P}}^T$ is the projection of $\mathbb{R}^{n_1}$ into $\mathrm{Im}(\widehat{\mathbb{P}}^T)$ parallel to $\mathrm{Ker}(\widehat{\mathbb{P}}^T)$. We have to show that $\mathrm{Im}(\widehat{\mathbb{P}}^T) = \mathrm{Ker}(A^T_{\upsilon\eta\theta}M^{-1})$ and $\mathrm{Ker}(\widehat{\mathbb{P}}^T) = \mathrm{Im}(\widehat{A}_{\upsilon\eta\theta})$. It is clear that $\mathrm{Ker}(A^T_{\upsilon\eta\theta}M^{-1}) \subset \mathrm{Im}(\widehat{\mathbb{P}}^T)$ and $\mathrm{Ker}(\widehat{\mathbb{P}}^T) = \mathrm{Im}(\widehat{A}_{\upsilon\eta\theta})$. To finish, we have to prove that $\mathrm{Im}(\widehat{\mathbb{P}}^T) \subset \mathrm{Ker}(A^T_{\upsilon\eta\theta}M^{-1})$. If X belongs to $\mathrm{Im}(\widehat{\mathbb{P}}^T)$ then we have $X = \widehat{\mathbb{P}}^T X$ which implies that

$$\widehat{A}_{\upsilon\eta\theta}(A^T_{\upsilon\eta\theta}M^{-1}\widehat{A}_{\upsilon\eta\theta})^{-1}A^T_{\upsilon\eta\theta}M^{-1}X = 0.$$

Since $\widehat{A}_{v\eta\theta}$ is of rank n_θ, we have $A^T_{v\eta\theta}M^{-1}X = 0$ which implies that $X \in \mathrm{Ker}(A^T_{v\eta\theta}M^{-1})$. We deduce that $\mathrm{Im}(\widehat{\mathbb{P}}^T) \subset \mathrm{Ker}(A^T_{v\eta\theta}M^{-1})$ and therefore $\widehat{\mathbb{P}}^T$ is the projection into $\mathrm{Ker}(A^T_{v\eta\theta}M^{-1})$ parallel to $\mathrm{Im}(\widehat{A}_{v\eta\theta})$. It is immediate that $\widehat{\mathbb{P}}$ is the projection into $\mathrm{Ker}(\widehat{A}^T_{v\eta\theta})$ parallel to $\mathrm{Im}(M^{-1}A_{v\eta\theta})$. We have similar proofs for $\mathbb{P}^T$, $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{P}}^T$. The rest of the proof is easy.

□

Remark 2.11 .1. *The projection $\mathbb{P}$ (resp. $\widehat{\mathbb{P}}$) is an approximation of the projector Π (resp. Π^*) introduced in Section 2.6 .1. Moreover, the adjoint of $\mathbb{P}$ is $\widehat{\mathbb{P}}$ if $\mathbb{R}^{n_1}$ is equipped with the scalar product $(M\cdot,\cdot)_{\mathbb{R}^{n_1}}$.*

We introduce the operator $\mathbf{A}$ defined by

$$\mathbf{A} = \mathbb{P}M^{-1}A.$$

Proposition 2.11 .2. *System (2.10 .4) is equivalent to the system*

$$\begin{aligned}
&\frac{d}{dt}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbf{A}\mathbb{P}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \mathbf{B}\mathbf{f}, \quad \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix}(0) = \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}, \\
&\boldsymbol{\theta} = -(A^T_{v\eta\theta}M^{-1}\widehat{A}_{v\eta\theta})^{-1}A^T_{v\eta\theta}M^{-1}A\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} - (A^T_{v\eta\theta}M^{-1}\widehat{A}_{v\eta\theta})^{-1}A^T_{v\eta\theta}M^{-1}B\mathbf{f},
\end{aligned} \tag{2.11 .1}$$

where

$$\mathbf{B} = \mathbb{P}M^{-1}B.$$

Proof. Let $(\boldsymbol{v},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\theta})$ be a solution of system (2.10 .4). First, we note that $(\boldsymbol{v},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) = \mathbb{P}(\boldsymbol{v},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$. Thus, multiplying the first equation of system (2.10 .4) by $\mathbb{P}^TM^{-1}$ and $A^T_{v\eta\theta}M^{-1}$, we get

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbb{P}M^{-1}A\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \mathbb{P}M^{-1}B\boldsymbol{f} = \mathbf{A}\mathbb{P}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \mathbf{B}\mathbf{f},$$

and

$$\boldsymbol{\theta} = -(A^T_{v\eta\theta}M^{-1}\widehat{A}_{v\eta\theta})^{-1}A^T_{v\eta\theta}M^{-1}A\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} - (A^T_{v\eta\theta}M^{-1}\widehat{A}_{v\eta\theta})^{-1}A^T_{v\eta\theta}M^{-1}B\mathbf{f}.$$

□

2.12 Stabilization of the finite dimensional linear system

2.12 .1 Spectral decomposition of the operator $\mathbf{A}\mathbb{P}$

We are looking for a decomposition of $\mathbb{R}^{n_1}$ into a sum of the generalized eigenspaces of the operators $\mathbf{A}\mathbb{P}$ and $\mathbf{A}^*\widehat{\mathbb{P}}$ where $\mathbf{A}^* = \widehat{\mathbb{P}}M^{-1}A^T$. We can decompose $\mathbb{R}^{n_1}$ in the form

$$\mathbb{R}^{n_1} = \mathrm{Ker}(A^T_{v\eta\theta}) \oplus \mathrm{Im}(M^{-1}\widehat{A}_{v\eta\theta}) \quad \text{and} \quad \mathbb{R}^{n_1} = \mathrm{Ker}(\widehat{A}^T_{v\eta\theta}) \oplus \mathrm{Im}(M^{-1}A_{v\eta\theta}),$$

We assume that $0 \notin \sigma(A)$. Then, it follows that

$$\mathrm{Ker}(\mathbf{A}\mathbb{P}) = \mathrm{Ker}(\mathbb{P}) = \mathrm{Im}(M^{-1}\widehat{A}_{v\eta\theta}) \quad \text{and} \quad \mathrm{Ker}(\mathbf{A}^*\widehat{\mathbb{P}}) = \mathrm{Ker}(\widehat{\mathbb{P}}) = \mathrm{Im}(M^{-1}A_{v\eta\theta}).$$

Thus, 0 is an eigenvalue of the operator $\mathbf{A}\mathbb{P}$ (resp. $\mathbf{A}^*\widehat{\mathbb{P}}$) and $\mathrm{Im}(M^{-1}\widehat{A}_{\upsilon\eta\theta})$ (resp. $\mathrm{Im}(M^{-1}A_{\upsilon\eta\theta})$) is the corresponding eigenspace. In order to decompose $\mathrm{Ker}(A^T_{\upsilon\eta\theta})$ (resp. $\mathrm{Ker}(\widehat{A}^T_{\upsilon\eta\theta})$) into the other eigenspaces of the operator $\mathbf{A}\mathbb{P}$ (resp. $\mathbf{A}^*\widehat{\mathbb{P}}$), we consider the following eigenvalue problems

$$\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathrm{Ker}(A^T_{\upsilon\eta\theta}), \quad \mathbf{A}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} = \boldsymbol{\lambda}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix}, \tag{2.12 .1}$$

and

$$\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \in \mathrm{Ker}(\widehat{A}^T_{\upsilon\eta\theta}), \quad \mathbf{A}^*\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} = \boldsymbol{\lambda}\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix}, \tag{2.12 .2}$$

Theorem 2.12 .1. *A pair $(\boldsymbol{\lambda}, (\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)) \in \mathbb{C}^* \times \mathrm{Ker}(A^T_{\upsilon\eta\theta})$ is solution to the eigenvalue problem (2.12 .1) if and only if $(\boldsymbol{\lambda}, (\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}))$ with*

$$\boldsymbol{\theta} = -(A^T_{\upsilon\eta\theta}M^{-1}\widehat{A}_{\upsilon\eta\theta})^{-1}A^T_{\upsilon\eta\theta}M^{-1}A\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix},$$

is a solution to the eigenvalue problem

$$\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}) \in \mathbb{C}^n, \quad A_L\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ \boldsymbol{\theta} \end{bmatrix} = \boldsymbol{\lambda}M_L\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ \boldsymbol{\theta} \end{bmatrix}, \tag{2.12 .3}$$

Similarly, a pair $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)) \in \mathbb{C}^* \times \mathrm{Ker}(\widehat{A}^T_{\upsilon\eta\theta})$ is solution to the eigenvalue problem (2.12 .2) if and only if $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}))$ with

$$\boldsymbol{\rho} = -(\widehat{A}^T_{\upsilon\eta\theta}M^{-1}A_{\upsilon\eta\theta})^{-1}\widehat{A}^T_{\upsilon\eta\theta}M^{-1}A^T\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix},$$

is a solution to the eigenvalue problem

$$\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}) \in \mathbb{C}^n, \quad A_L^T\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \\ \boldsymbol{\rho} \end{bmatrix} = \boldsymbol{\lambda}M_L\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \\ \boldsymbol{\rho} \end{bmatrix}. \tag{2.12 .4}$$

Proof. The proof of the equivalence between the direct eigenvalue problems (2.12 .1) and (2.12 .3) is similar to that of systems (2.10 .4) and (2.11 .1) (see Proposition 2.11 .2). Now, we are going to prove the equivalence between the adjoint eigenvalue problems (2.12 .2) and (2.12 .4). It is easy to prove that if $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}))$ is a solution of (2.12 .4) then $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2))$ is a solution (2.12 .2). Now, we suppose that $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)) \in \mathbb{C}^* \times \mathrm{Ker}(\widehat{A}^T_{\upsilon\eta\theta})$ is a solution (2.12 .2). Then, we have

$$\widehat{\mathbb{P}}\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} \quad \text{and} \quad \widehat{\mathbb{P}}\left(\boldsymbol{\lambda}\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} - M^{-1}A^T\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix}\right) = 0.$$

It follows that

$$\boldsymbol{\lambda}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\kappa}_1\\ \boldsymbol{\kappa}_2\end{bmatrix} - M^{-1}A^T\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\kappa}_1\\ \boldsymbol{\kappa}_2\end{bmatrix} = -(I-\widehat{\mathbb{P}})M^{-1}A^T\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\kappa}_1\\ \boldsymbol{\kappa}_2\end{bmatrix}.$$

Thanks to Proposition 2.11 .1, we obtain

$$(I-\widehat{\mathbb{P}})M^{-1}A^T\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\kappa}_1\\ \boldsymbol{\kappa}_2\end{bmatrix} = M^{-1}A_{\upsilon\eta\theta}(\widehat{A}^T_{\upsilon\eta\theta}M^{-1}A_{\upsilon\eta\theta})^{-1}\widehat{A}^T_{\upsilon\eta\theta}M^{-1}A^T\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\kappa}_1\\ \boldsymbol{\kappa}_2\end{bmatrix} = -M^{-1}A_{\upsilon\eta\theta}\boldsymbol{\rho}.$$

Hence

$$\boldsymbol{\lambda}M\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\kappa}_1\\ \boldsymbol{\kappa}_2\end{bmatrix} = A^T\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\kappa}_1\\ \boldsymbol{\kappa}_2\end{bmatrix} + A_{\upsilon\eta\theta}\boldsymbol{\rho}. \tag{2.12 .5}$$

We also have

$$\widehat{A}^T_{\upsilon\eta\theta}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\kappa}_1\\ \boldsymbol{\kappa}_2\end{bmatrix} = 0. \tag{2.12 .6}$$

Thus, $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}))$ is a solution of (2.12 .4).

□

Definition 2.12 .1. *A vector* $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i) \in \mathrm{Ker}(A^T_{\upsilon\eta\theta})$ *is a generalized eigenvector of order* i *for problem (2.12 .1) associated with a solution* $(\boldsymbol{\lambda}, (\boldsymbol{z}^0, \boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0))$ *of (2.12 .1) if* $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)$ *is obtained by solving the chain of equations*

$$(\boldsymbol{\lambda}I - \mathbf{A})\begin{bmatrix}\boldsymbol{z}^j\\ \boldsymbol{\xi}_1^j\\ \boldsymbol{\xi}_2^j\end{bmatrix} = -\begin{bmatrix}\boldsymbol{z}^{j-1}\\ \boldsymbol{\xi}_1^{j-1}\\ \boldsymbol{\xi}_2^{j-1}\end{bmatrix}, \quad \textit{for } 1\le j\le i.$$

A vector $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i) \in \mathbb{C}^n$ *is a generalized eigenvector of order* i *associated with a solution* $(\boldsymbol{\lambda}, (\boldsymbol{z}^0, \boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0, \boldsymbol{\theta}^0))$ *of (2.12 .3) if it is obtained by solving the chain of equations*

$$(\boldsymbol{\lambda}M_L - A_L)\begin{bmatrix}\boldsymbol{z}^j\\ \boldsymbol{\xi}_1^j\\ \boldsymbol{\xi}_2^j\\ \boldsymbol{\theta}^j\end{bmatrix} = -M_L\begin{bmatrix}\boldsymbol{z}^{j-1}\\ \boldsymbol{\xi}_1^{j-1}\\ \boldsymbol{\xi}_2^{j-1}\\ \boldsymbol{\theta}^{j-1}\end{bmatrix}, \quad \textit{for } 1\le j\le i.$$

We have similar definitions for the problems (2.12 .2) and (2.12 .4).

Theorem 2.12 .2. *A vector* $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i) \in \mathrm{Ker}(A^T_{\upsilon\eta\theta})$ *is a generalized eigenvector associated with the solution* $(\boldsymbol{\lambda}, (\boldsymbol{z}^0, \boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0)) \in \mathbb{C}^* \times \mathrm{Ker}(A^T_{\upsilon\eta\theta})$ *of (2.12 .1) if and only if* $(\boldsymbol{\lambda}, (\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i))$ *with*

$$\boldsymbol{\theta}^i = -(A^T_{\upsilon\eta\theta}M^{-1}\widehat{A}_{\upsilon\eta\theta})^{-1}A^T_{\upsilon\eta\theta}M^{-1}A\begin{bmatrix}\boldsymbol{z}^i\\ \boldsymbol{\xi}_1^i\\ \boldsymbol{\xi}_2^i\end{bmatrix},$$

is a generalized eigenvector associated with the solution $(\boldsymbol{\lambda}, (\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}))$ *of problem (2.12 .3).*

A vector $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i) \in \mathrm{Ker}(\widehat{A}_{\upsilon\eta\theta}^T)$ *is a generalized eigenvector associated with the solution* $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)) \in \mathbb{C}^* \times \mathrm{Ker}(\widehat{A}_{\upsilon\eta\theta}^T)$ *of (2.12 .2) if and only if* $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}))$ *with*

$$\boldsymbol{\rho}^i = -(\widehat{A}_{\upsilon\eta\theta}^T M^{-1} A_{\upsilon\eta\theta})^{-1} \widehat{A}_{\upsilon\eta\theta}^T M^{-1} A^T \begin{bmatrix} \boldsymbol{\psi}^i \\ \boldsymbol{\kappa}_1^i \\ \boldsymbol{\kappa}_2^i \end{bmatrix},$$

is a generalized eigenvector associated with the solution $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}))$ *of problem (2.12 .2).*

Theorem 2.12 .3. *There exists a basis* $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)_{1\le i\le n_1}$ *of* $\mathbb{C}^{n_1}$ *constituted of eigenvectors and generalized eigenvectors of (2.12 .1) and a basis* $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{1\le i\le n_1}$ *of* $\mathbb{C}^{n_1}$ *constituted of eigenvectors and generalized eigenvectors of (2.12 .2) such that*

$$\Lambda_{\mathbb{C}} = \boldsymbol{F}^{-1}\mathbf{A}\mathbb{P}\boldsymbol{F} \quad \text{and} \quad \Lambda_{\mathbb{C}}^T = \boldsymbol{\Phi}^{-1}\mathbf{A}^*\widehat{\mathbb{P}}\boldsymbol{\Phi},$$

where $\Lambda_{\mathbb{C}}$ *is a decomposition of* $\mathbf{A}\mathbb{P}$ *into complex Jordan blocks,* $\boldsymbol{E}$ *is the matrix whose columns are* $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)_{1\le i\le n_1}$ *and* $\boldsymbol{\Phi}$ *is the matrix whose columns are* $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{1\le i\le n_1}$.

Proof. Using the complex Jordan decomposition of real matrices, we know that there exists a matrix $\boldsymbol{F} \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_1}$ constituted of eigenvectors and generalized eigenvectors of problem (2.12 .1) such that

$$\Lambda_{\mathbb{C}} = \boldsymbol{F}^{-1}\mathbf{A}\mathbb{P}\boldsymbol{F}.$$

We set

$$\boldsymbol{\Phi} = M^{-1}\boldsymbol{F}^{-T}.$$

Thus, the matrices $\boldsymbol{F}$ and $\boldsymbol{\Phi}$ satisfy the bi-orthogonality condition

$$\boldsymbol{F}^T M \boldsymbol{\Phi} = I.$$

Now, we are going to prove that

$$\Lambda_{\mathbb{C}}^T = \boldsymbol{\Phi}^{-1}\mathbf{A}^*\widehat{\mathbb{P}}\boldsymbol{\Phi}.$$

From the identities

$$M^{-1}\boldsymbol{\Phi}^{-T} = \boldsymbol{F} \quad \text{and} \quad \boldsymbol{F}\Lambda_{\mathbb{C}} = \mathbf{A}\mathbb{P}\boldsymbol{F} = \mathbb{P}M^{-1}A\mathbb{P}\boldsymbol{F},$$

we deduce that

$$M^{-1}\boldsymbol{\Phi}^{-T}\Lambda_{\mathbb{C}} = \mathbb{P}M^{-1}A\mathbb{P}M^{-1}\boldsymbol{\Phi}^{-T},$$

and

$$\Lambda_{\mathbb{C}}\boldsymbol{\Phi}^T = \boldsymbol{\Phi}^T M\mathbb{P}M^{-1}A\mathbb{P}M^{-1}.$$

Since $M\mathbb{P} = \widehat{\mathbb{P}}^T M$ and $\mathbb{P}M^{-1} = M^{-1}\widehat{\mathbb{P}}^T$, we have

$$\Lambda_{\mathbb{C}}\boldsymbol{\Phi}^T = \boldsymbol{\Phi}^T\widehat{\mathbb{P}}^T A M^{-1}\widehat{\mathbb{P}}^T$$

Taking the conjugated transpose, we obtain

$$\boldsymbol{\Phi}\Lambda_{\mathbb{C}}^T = \widehat{\mathbb{P}}M^{-1}A^T\widehat{\mathbb{P}}\boldsymbol{\Phi}.$$

Hence

$$\Lambda_{\mathbb{C}}^T = \boldsymbol{\Phi}^{-1}\mathbf{A}^*\widehat{\mathbb{P}}\boldsymbol{\Phi}.$$

□

Theorem 2.12 .4. *There exists two basis* $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1\le i\le n_1}$ *and* $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{1\le i\le n_1}$ *of* $\mathbb{R}^{n_1}$ *satisfying the bi-orthogonality condition*

$$(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)^T M(\boldsymbol{\phi}^j, \boldsymbol{\zeta}_1^j, \boldsymbol{\zeta}_2^j) = \delta_{ij} \quad \textit{for } 1 \le i, j \le n_1,$$

and such that

$$\Lambda = \boldsymbol{E}^{-1}\mathbf{A}\mathbb{P}\boldsymbol{E} \quad \textit{and} \quad \Lambda^T = \boldsymbol{\Xi}^{-1}\mathbf{A}^*\widehat{\mathbb{P}}\boldsymbol{\Xi},$$

where Λ *is a decomposition of* $\mathbf{A}\mathbb{P}$ *into real Jordan blocks,* $\boldsymbol{E}$ *is the matrix whose columns are* $((\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i))_{1\le i\le n_1}$ *and* $\boldsymbol{\Xi}$ *is the matrix whose columns are* $((\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i))_{1\le i\le n_1}$.

Proof. We denote by $(\boldsymbol{\lambda}_j)_j$ the eigenvalues of the operator $\mathbf{A}\mathbb{P}$. To construct the basis $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1\le i\le n_1}$ and $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{1\le i\le n_1}$, we proceed in the following way.
Case 1: $\boldsymbol{\lambda}_j$ *is real.* If $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)$ and $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)$ are generalized eigenvectors associated to $\boldsymbol{\lambda}_j$, we can assume that they are real vectors. Thus, we set

$$(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i) = (\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i) \quad \text{and} \quad (\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i) = (\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i).$$

Case 2: $\boldsymbol{\lambda}_j$ *is complex.* There exists k such that $\boldsymbol{\lambda}_k = \bar{\boldsymbol{\lambda}}_j$ is also an eigenvalue of $\mathbf{A}\mathbb{P}^T$. If $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)$ and $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)$ are generalized eigenvectors associated to $\boldsymbol{\lambda}_j$, $(\boldsymbol{z}^k, \boldsymbol{\xi}_1^k, \boldsymbol{\xi}_2^k)$ and $(\boldsymbol{\psi}^k, \boldsymbol{\kappa}_1^k, \boldsymbol{\kappa}_2^k)$ are eigenvectors or generalized eigenvectors associated to $\boldsymbol{\lambda}_k$, then we may assume that $(\bar{\boldsymbol{z}}^i, \bar{\boldsymbol{\xi}}_1^i, \bar{\boldsymbol{\xi}}_2^i) = (\boldsymbol{z}^k, \boldsymbol{\xi}_1^k, \boldsymbol{\xi}_2^k)$ and $(\bar{\boldsymbol{\psi}}^i, \bar{\boldsymbol{\kappa}}_1^i, \bar{\boldsymbol{\kappa}}_2^i) = (\boldsymbol{\psi}^k, \boldsymbol{\kappa}_1^k, \boldsymbol{\kappa}_2^k)$. Thus, we can set

$$(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i) = \sqrt{2}\mathrm{Re}(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i) \quad \text{and} \quad (\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i) = \sqrt{2}\mathrm{Re}(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i),$$

and

$$(\boldsymbol{v}^m, \boldsymbol{\eta}_1^m, \boldsymbol{\eta}_2^m) = \sqrt{2}\mathrm{Im}(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i) \quad \text{and} \quad (\boldsymbol{\phi}^m, \boldsymbol{\zeta}_1^m, \boldsymbol{\zeta}_2^m) = \sqrt{2}\mathrm{Im}(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i).$$

Thus, we have constructed two basis $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1\le i\le n_1}$ and $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{1\le i\le n_1}$ of $\mathbb{R}^{n_1}$ satisfying the bi-orthogonality condition

$$(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)^T M(\boldsymbol{\phi}^j, \boldsymbol{\zeta}_1^j, \boldsymbol{\zeta}_2^j) = \delta_{ij} \quad \text{for } 1 \le i, j \le n_1,$$

and such that

$$\Lambda = \boldsymbol{E}^{-1}\mathbf{A}\mathbb{P}\boldsymbol{E} \quad \text{and} \quad \Lambda^T = \boldsymbol{\Xi}^{-1}\mathbf{A}^*\widehat{\mathbb{P}}\boldsymbol{\Xi}.$$

□

Corollary 2.12 .5. *We have*

$$\Lambda = \boldsymbol{\Xi}^T\widehat{\mathbb{P}}^T A\mathbb{P}\boldsymbol{E} \quad \textit{and} \quad \Lambda^T = \boldsymbol{E}^T\mathbb{P}^T A\widehat{\mathbb{P}}\boldsymbol{\Xi}.$$

Proof. From the bi-orthogonality of $\boldsymbol{E}$ and $\boldsymbol{\Xi}$, we have

$$\Lambda = \boldsymbol{E}^{-1}\mathbf{A}\mathbb{P}\boldsymbol{E} = \boldsymbol{\Xi}^T M\mathbb{P}M^{-1}A\mathbb{P}\boldsymbol{E}.$$

Since $\mathbb{P}M^{-1} = M^{-1}\widehat{\mathbb{P}}^T$, we get

$$\Lambda = \boldsymbol{\Xi}^T\widehat{\mathbb{P}}^T A\mathbb{P}\boldsymbol{E}.$$

Similarly, we prove that

$$\Lambda^T = \boldsymbol{E}^T\mathbb{P}^T A\widehat{\mathbb{P}}\boldsymbol{\Xi}.$$

□

2.12 .2 Projected systems

Let $(\boldsymbol{\lambda}_j)_{j\in\mathbb{N}}$ the eigenvalues of the operator $\mathbf{A}\mathbb{P}$. We denote by $\mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_j)$ the real generalized eigenspace of $\mathbf{A}\widehat{\mathbb{P}}$ and $\mathbf{G}_{\mathbb{R}}^*(\boldsymbol{\lambda}_j)$ the real generalized eigenspace of $\mathbf{A}^*\widehat{\mathbb{P}}$ associated to the eigenvalue $\boldsymbol{\lambda}_j$. Let α_u be positive. We define the unstable subspaces

$$\mathbb{Z}_u = \oplus_{j\in\mathbf{J}_u}\mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_j) \quad \text{and} \quad \mathbb{Z}_u^* = \oplus_{j\in\mathbf{J}_u}G_{\mathbb{R}}^*(\boldsymbol{\lambda}_j)$$

where $\mathbf{J}_u$ is a finite subset of $\mathbb{N}$ such that

$$-\alpha_u < \operatorname{Re}\boldsymbol{\lambda}_j \quad \forall\, j\in\mathbf{J}_u.$$

We denote by d_0 the dimension of $\mathbb{K} = \operatorname{Ker}(\mathbf{A}\mathbb{P})$. Without loss of generality, we assume that

$$\mathbb{Z}_u = \operatorname{Vect}\{(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1\le i\le d_u}\} \quad \text{and} \quad \mathbb{K} = \operatorname{Vect}\{(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{d_u+d_s+1\le i\le n_1}\},$$

where $d_s = n_1 - d_s - d_0$. Therefore, we have

$$\mathbb{R}^{n_1} = \mathbb{Z}_u \oplus \mathbb{Z}_s \oplus \mathbb{K},$$

where

$$\mathbb{Z}_s = \operatorname{Vect}\{(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i))_{d_u+1\le i\le d_u+d_s}\},$$

We denote by

- $\boldsymbol{E}_u \in \mathbb{R}^{n_1}\times\mathbb{R}^{d_u}$ the matrix whose columns are $((\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i))_{1\le i\le d_u}$,
- $\boldsymbol{E}_s \in \mathbb{R}^{n_1}\times\mathbb{R}^{d_s}$ the matrix whose columns are $((\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i))_{d_u+1\le i\le d_u+d_s}$,
- $\boldsymbol{\Xi}_u \in \mathbb{R}^{n_1}\times\mathbb{R}^{d_u}$ the matrix whose columns are $((\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i))_{1\le i\le d_u}$,
- $\boldsymbol{\Xi}_s \in \mathbb{R}^{n_1}\times\mathbb{R}^{d_s}$ the matrix whose columns are $((\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i))_{d_u+1\le i\le d_u+d_s}$.

Thanks to the bi-orthogonality condition

$$\boldsymbol{E}^T M \boldsymbol{\Xi} = I,$$

we can prove that

$$\boldsymbol{E}_u^T M\boldsymbol{\Xi}_u = I_{\mathbb{R}^{d_u}}, \quad \boldsymbol{E}_s^T M\boldsymbol{\Xi}_s = I_{\mathbb{R}^{d_s}}, \quad \boldsymbol{E}_s^T M\boldsymbol{\Xi}_u = 0_{\mathbb{R}^{d_s}\times\mathbb{R}^{d_u}}, \quad \boldsymbol{E}_u^T M\boldsymbol{\Xi}_s = 0_{\mathbb{R}^{d_u}\times\mathbb{R}^{d_s}}. \tag{2.12 .7}$$

We set

$$\Lambda_u = \boldsymbol{\Xi}_u^T A\boldsymbol{E}_u, \quad \Lambda_u^T = \boldsymbol{E}_u^T A^T\boldsymbol{\Xi}_u, \quad \Lambda_s = \boldsymbol{\Xi}_s^T A\boldsymbol{E}_s, \quad \Lambda_s^T = \boldsymbol{E}_s^T A^T\boldsymbol{\Xi}_s. \tag{2.12 .8}$$

From Corollary 2.12 .5, it follows that

$$\Lambda = \begin{bmatrix} \Lambda_u & 0_{\mathbb{R}^{d_s}\times\mathbb{R}^{d_u}} & 0_{\mathbb{R}^{d_0}\times\mathbb{R}^{d_u}} \\ 0_{\mathbb{R}^{d_u}\times\mathbb{R}^{d_s}} & \Lambda_s & 0_{\mathbb{R}^{d_0}\times\mathbb{R}^{d_s}} \\ 0_{\mathbb{R}^{d_u}\times\mathbb{R}^{d_0}} & 0_{\mathbb{R}^{d_s}\times\mathbb{R}^{d_0}} & 0_{\mathbb{R}^{d_0}\times\mathbb{R}^{d_0}} \end{bmatrix} \quad \text{and} \quad \Lambda^T = \begin{bmatrix} \Lambda_u^T & 0_{\mathbb{R}^{d_s}\times\mathbb{R}^{d_u}} & 0_{\mathbb{R}^{d_0}\times\mathbb{R}^{d_u}} \\ 0_{\mathbb{R}^{d_u}\times\mathbb{R}^{d_s}} & \Lambda_s^T & 0_{\mathbb{R}^{d_0}\times\mathbb{R}^{d_s}} \\ 0_{\mathbb{R}^{d_u}\times\mathbb{R}^{d_0}} & 0_{\mathbb{R}^{d_s}\times\mathbb{R}^{d_0}} & 0_{\mathbb{R}^{d_0}\times\mathbb{R}^{d_0}} \end{bmatrix}.$$

We introduce the operators $\Pi_u \in \mathcal{L}(\mathbb{R}^{n_1}, \mathbb{Z}_u)$ and $\Pi_s \in \mathcal{L}(\mathbb{R}^{n_1}, \mathbb{Z}_s)$ defined by

$$\Pi_u = \boldsymbol{E}_u\boldsymbol{\Xi}_u^T M \quad \text{and} \quad \Pi_s = \boldsymbol{E}_s\boldsymbol{\Xi}_s^T M.$$

Proposition 2.12 .6. *The operator Π_u is the projection of $\mathbb{R}^{n_1}$ onto $\mathbb{Z}_u$ parallel to $\mathbb{Z}_s \oplus \mathbb{K}$ and the operator Π_s is the projection of $\mathbb{R}^{n_1}$ onto $\mathbb{Z}_s$ parallel to $\mathbb{Z}_u \oplus \mathbb{K}$.*
Moreover, we have the identities

$$\Pi_u \mathbb{P} = \Pi_u, \quad \Pi_s \mathbb{P} = \Pi_s, \quad \Pi_u + \Pi_s = \mathbb{P}.$$

Proof. We have

$$\Pi_u^2 = \boldsymbol{E}_u(\boldsymbol{\Xi}_u^T M \boldsymbol{E}_u)\boldsymbol{\Xi}_u^T M = \boldsymbol{E}_u \boldsymbol{\Xi}_u^T M = \Pi_u,$$

because $\boldsymbol{\Xi}_u^T M \boldsymbol{E}_u = I_{\mathbb{R}^{d_u}}$ (see equation (2.12 .7)). Therefore, Π_u is the projection onto $\mathrm{Im}(\Pi_u)$ parallel to $\mathrm{Ker}(\Pi_u)$. Thus, to prove that Π_u is the projection onto $\mathbb{Z}_u$ parallel to $\mathbb{Z}_s \oplus \mathbb{K}$, we have to show that $\mathrm{Im}(\Pi_u) = \mathbb{Z}_u$ and $\mathrm{Ker}(\Pi_u) = \mathbb{Z}_s \oplus \mathbb{K}$. Using the bi-orthogonality condition $\boldsymbol{\Xi}_u^T M \boldsymbol{E}_u = I_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_u}}$, we prove that $\mathrm{Im}(\Pi_u) = \mathbb{Z}_u$. Using the identities

$$\boldsymbol{\Xi}_u^T M \boldsymbol{E}_s = 0_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_s}}, \quad \widehat{\mathbb{P}} \boldsymbol{\Xi}_u = \boldsymbol{\Xi}_u \quad \text{and} \quad \widehat{\mathbb{P}}^T M = M\mathbb{P},$$

we prove $Z_s \oplus \mathbb{K} \subset \mathrm{Ker}(\Pi_u)$ and since $\dim(\mathbb{Z}_s \oplus \mathbb{K}) = \dim(\mathrm{Ker}(\Pi_u))$, we have $\mathrm{Ker}(\Pi_u) = \mathbb{Z}_s \oplus \mathbb{K}$. We prove similarly that Π_s is the projection onto $\mathbb{Z}_s$ parallel to $\mathbb{Z}_u \oplus \mathbb{K}$.
Since

$$M\mathbb{P} = \widehat{\mathbb{P}}^T, \quad \widehat{\mathbb{P}} \boldsymbol{\Xi}_u = \boldsymbol{\Xi}_u \quad \text{and} \quad \widehat{\mathbb{P}} \boldsymbol{\Xi}_u = \boldsymbol{\Xi}_u,$$

we have

$$\Pi_u \mathbb{P} = \boldsymbol{E}_u \boldsymbol{\Xi}_u^T M \mathbb{P} = \boldsymbol{E}_u \boldsymbol{\Xi}_u^T \widehat{\mathbb{P}}^T M = \boldsymbol{E}_u \boldsymbol{\Xi}_u^T M = \Pi_u,$$

and

$$\Pi_s \mathbb{P} = \boldsymbol{E}_s \boldsymbol{\Xi}_s^T M \mathbb{P} = \boldsymbol{E}_s \boldsymbol{\Xi}_s^T \widehat{\mathbb{P}}^T M = \boldsymbol{E}_s \boldsymbol{\Xi}_s^T M = \Pi_s.$$

The identity $\Pi_u + \Pi_s = \mathbb{P}$ comes the fact that both the operators $\mathbb{P}$ and $\Pi_u + \Pi_s$ are projections onto $\mathbb{Z}_u \oplus \mathbb{Z}_s$ parallel to $\mathbb{K}$.

□

Proposition 2.12 .7. *If $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$ is solution of system (2.10 .4), then*

$$(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})^T = \boldsymbol{\Xi}_u^T M (\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T \quad \textit{and} \quad (\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})^T = \boldsymbol{\Xi}_s^T M (\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T$$

obey to the system

$$\begin{aligned}
&\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} = \Lambda_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \mathbb{B}_u \boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix}(0) = \boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}, \\
&\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} = \Lambda_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} + \mathbb{B}_s \boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix}(0) = \boldsymbol{\Xi}_s^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix},
\end{aligned} \tag{2.12 .9}$$

where

$$\mathbb{B}_u = \boldsymbol{\Xi}_u^T B \quad \textit{and} \quad \mathbb{B}_s = \boldsymbol{\Xi}_s^T B.$$

Conversely, if the pair $((\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u}), (\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s}))$ is solution of system (2.12 .9), then $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$, with

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \boldsymbol{E}_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \boldsymbol{E}_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix},$$

$$\boldsymbol{\theta} = -(A_{v\eta\theta}^T M^{-1} \widehat{A}_{v\eta\theta})^{-1} A_{v\eta\theta}^T M^{-1} A \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix},$$

is solution of system (2.10 .4).

Proof. Since

$$\widehat{\mathbb{P}}^T \widehat{A}_{v\eta\theta} = 0, \quad \widehat{\mathbb{P}}\boldsymbol{\Xi}_u = \boldsymbol{\Xi}_u \quad \text{ and } \widehat{\mathbb{P}}\boldsymbol{\Xi}_s = \boldsymbol{\Xi}_s,$$

we have

$$\boldsymbol{\Xi}_u^T \widehat{A}_{v\eta\theta} = 0 \quad \text{ and } \quad \boldsymbol{\Xi}_s^T \widehat{A}_{v\eta\theta} = 0.$$

Thus, multiplying the first equation of system (2.10 .4) by $\boldsymbol{\Xi}_u^T$, we get

$$\frac{d}{dt}\boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \boldsymbol{\Xi}_u^T A \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \boldsymbol{\Xi}_u^T \widehat{A}_{v\eta\theta}\boldsymbol{\theta} + \boldsymbol{\Xi}_u^T B\mathbf{f},$$

Since

$$\boldsymbol{\Xi}_u^T A = \Lambda_u \boldsymbol{\Xi}_u^T M.$$

it follows that $(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})$ is the solution of the equation

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} = \Lambda_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \mathbb{B}_u \boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix}(0) = \boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}.$$

We prove similarly that $(\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})$ is the solution of the equation

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} = \Lambda_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} + \mathbb{B}_s \boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix}(0) = \boldsymbol{\Xi}_s^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}.$$

Now, we suppose that the pair $((\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u}), (\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s}))$ is solution of system (2.12 .9). We want to prove that $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$, with

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \boldsymbol{E}_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \boldsymbol{E}_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix},$$

$$\boldsymbol{\theta} = -(A_{v\eta\theta}^T M^{-1}\widehat{A}_{v\eta\theta})^{-1} A_{v\eta\theta}^T M^{-1} A \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix},$$

is solution of system (2.10 .4). Thanks to **Proposition 2.11 .2**, we are reduced to prove that $(\boldsymbol{v},\, \boldsymbol{\eta}_1,\, \boldsymbol{\eta}_2)$ is the solution of the equation

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbf{A}\mathbb{P}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \mathbf{B}\mathbf{f}.$$

We have

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} &= \frac{d}{dt}\left(\boldsymbol{E}_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \boldsymbol{E}_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix}\right) \\
&= \boldsymbol{E}_u \frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \boldsymbol{E}_s \frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} \\
&= \boldsymbol{E}_u \Lambda_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \boldsymbol{E}_s \Lambda_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} + \boldsymbol{E}_u \mathbb{B}_u \boldsymbol{f} + \boldsymbol{E}_s \mathbb{B}_s \boldsymbol{f} \\
&= \mathbf{A}\mathbb{P}\boldsymbol{E}_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \mathbf{A}\mathbb{P}\boldsymbol{E}_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} + (\boldsymbol{E}_u \mathbb{B}_u + \boldsymbol{E}_s \mathbb{B}_s)\boldsymbol{f}
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\boldsymbol{E}_u \mathbb{B}_u + \boldsymbol{E}_s \mathbb{B}_s &= \boldsymbol{E}_u \boldsymbol{\Xi}_u^T B + \boldsymbol{E}_s \boldsymbol{\Xi}_s^T B \\
&= (\boldsymbol{E}_u \boldsymbol{\Xi}_u^T M + \boldsymbol{E}_s \boldsymbol{\Xi}_s^T M) M^{-1} B \\
&= (\Pi_u + \Pi_s) M^{-1} B \\
&= \mathbf{B}.
\end{aligned}$$

Thus, we obtain

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbf{A}\mathbb{P}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \mathbf{B}\boldsymbol{f}.$$

□

2.12 .3 Linear feedback law

The goal is to find a feedback law able to stabilize system (2.10 .4). Due to Proposition 2.12 .7, it is necessary and sufficient to find a feedback law able to stabilize the equation satisfies by $(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})$. That is why we make the following assumption.

$$\text{For all } \omega > -\mathrm{Re}\sigma(\Lambda_u) \text{ the pair } (\Lambda_u + \omega_{\mathbb{R}^{d_u}}, \mathbb{B}_u) \text{ is stabilizable.} \tag{2.12 .10}$$

Hence, for all $Q \in \mathcal{L}(\mathbb{R}^{d_u})$ satisfying $Q = Q^T \geq 0$ and $-\omega < \mathrm{Re}\sigma(\Lambda_u)$, the following algebraic Riccati equation

$$\begin{aligned}
&\mathbb{P}_{u,\omega} \in \mathcal{L}(\mathbb{R}^{d_u}), \quad \mathbb{P}_{u,\omega} = \mathbb{P}_{u,\omega}^* \geq 0, \\
&\mathbb{P}_{u,\omega}(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u + \omega I_{\mathbb{R}^{d_u}})^* \mathbb{P}_{u,\omega} - \mathbb{P}_{u,\omega}\mathbb{B}_u\mathbb{B}_u^*\mathbb{P}_{u,\omega} + Q = 0, \\
&\mathbb{P}_{u,\omega} \text{ is invertible and } \Lambda_u + \omega I_{\mathbb{R}^{d_u}} - \mathbb{B}_u\mathbb{B}_u^*\mathbb{P}_{u,\omega} \text{ is exponentially stable,}
\end{aligned}$$

admits a unique solution $\mathbb{P}_{u,\omega}$. The operator $\mathbb{K}_{u,\omega} = -\mathbb{B}_u^*\mathbb{P}_{u,\omega}$ provides a feedback law stabilizing the pair $(\Lambda_u + \omega_{\mathbb{R}^{d_u}}, \mathbb{B}_u)$.

2.12 .4 Stability of the closed-loop linear system

We are going to prove that the linear closed-loop system

$$
\begin{aligned}
&M\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = A\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} + \widehat{A}_{v\eta\theta}\boldsymbol{\theta} + B\mathbb{K}_\omega\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}, \quad \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}(0) = \begin{bmatrix}\boldsymbol{v}^0\\ \boldsymbol{\eta}_1^0\\ \boldsymbol{\eta}_2^0\end{bmatrix},\\
&A_{v\theta}^T\boldsymbol{v} + A_{\eta_1\theta}^T\boldsymbol{\eta}_1 + A_{\eta_2\theta}^T\boldsymbol{\eta}_2 = 0,
\end{aligned}
\tag{2.12 .11}
$$

where $\mathcal{K}_\omega = \mathbb{K}_{u,\omega}\boldsymbol{\Xi}_u^T M$, is exponentially stable.

Theorem 2.12 .8. *Let $(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)$ be in $\mathrm{Ker}(A_{v\eta\theta})$. The solution $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$ of system (2.12 .11) satisfies the estimates*

$$
\begin{aligned}
\|\Pi_u(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)\|_{\mathbb{R}^n} &\le Ce^{-\omega t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}},\\
\|\Pi_s(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)\|_{\mathbb{R}^n} &\le Ce^{-\alpha_u t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}}.
\end{aligned}
$$

Proof. From Proposition 2.12 .7, $e^{\omega t}(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u}) = e^{\omega t}\boldsymbol{\Xi}_u^T M\mathbb{P}^T(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T$ is the solution of the system

$$
\frac{d}{dt}e^{\omega t}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix} = (\Lambda_u + \omega I_{\mathbb{R}^{d_u}} - \mathbb{B}_u\mathbb{B}_u^*\mathbb{P}_{u,\omega})e^{\omega t}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix}, \quad e^{\omega t}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix}(0) = \boldsymbol{\Xi}_u^T M\begin{bmatrix}\boldsymbol{v}^0\\ \boldsymbol{\eta}_1^0\\ \boldsymbol{\eta}_2^0\end{bmatrix}.
$$

Since $\Lambda_u + \omega I_{\mathbb{R}^{d_u}} - \mathbb{B}_u\mathbb{B}_u^*\mathbb{P}_{u,\omega}$ is exponentially stable, there exists $C > 0$ such that

$$
\|(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})\|_{\mathbb{R}^{d_u}} \le Ce^{-\omega t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}}.
$$

The triplet $(\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})^T = \boldsymbol{\Xi}_s^T M(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T$ obeys to the system

$$
\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix} = \Lambda_s\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix} + \mathbb{B}_s\mathbb{K}_{u,\omega}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix}, \quad \begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix}(0) = \boldsymbol{\Xi}_s^T M\begin{bmatrix}\boldsymbol{v}^0\\ \boldsymbol{\eta}_2^0\\ \boldsymbol{\eta}_2^0\end{bmatrix}.
$$

Since $\mathrm{Re}(\sigma(\Lambda_u)) < -\alpha_u$, there exists $C > 0$ such that

$$
\|(\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})\|_{\mathbb{R}^{d_s}} \le Ce^{-\alpha_u t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}}.
$$

□

2.13 Numerical results for the nonlinear linear system

We are going to show, by numerical experiments, that the linear feedback law is able the stabilize nonlinear system. First, let us introduce the parameters of the geometrical configuration used in the numerical experiments. The fluid domain is contained in the rectangle $[0, 50e] \times [-21e, 21e]$, where $2e$ is the thickness of the plate. The length of the plate is $10e$. The inflow condition g_s imposed on the boundary $\Gamma_i = \{0\}\times[-21e, -e]\cup\{10e\}\times[-e, e]\cup\{0\}\times[e, 21e]$ is an approximation

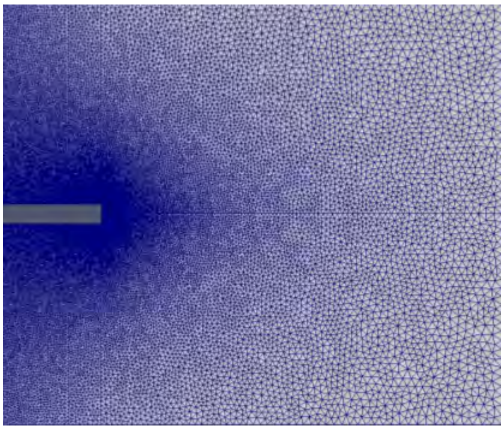


Figure 2.3 – Triangular mesh of the reference configuration of the fluid domain.

of the Blasius boundary layer profile defined by

$$g_s(x) = \begin{cases} U_m & \forall\, e+b \le x \le 21e, \\ U_m \left(2\dfrac{x-e}{b} - 2\left(\dfrac{x-e}{b}\right)^3 + \left(\dfrac{x-e}{b}\right)^4\right) & \forall\, e \le x \le e+b, \\ 0 & \forall\, -e \le x \le e, \\ U_m \left(2\dfrac{x+e}{b} - 2\left(\dfrac{x+e}{b}\right)^3 + \left(\dfrac{x+e}{b}\right)^4\right) & \forall\, -e-b \le x \le -e, \\ U_m & \forall\, -21e \le x \le -e-b, \end{cases} \tag{2.13 .1}$$

where U_m is the maximum velocity at the inflow. The Reynolds number Rey of the fluid and the thickness b of the boundary layer are defined by

$$\mathrm{Rey} = \frac{2eU_m}{\nu} \quad \text{and} \quad b = \frac{14e}{\sqrt{\mathrm{Rey}}}.$$

We choose

$$e = 0.05, \quad U_m = 1.0 \quad \text{and} \quad \mathrm{Rey} = 200.$$

We use a triangular mesh of 54960 cells, symmetric with respect to the horizontal axis $z = 0$ (see *Figure* 2.3). For the space discretization, we use generalized Taylor-Hood finite elements P_2-P_1-P_1 for the velocity, the pressure and the Lagrange multipliers leading to 249968 degrees of freedom. The time discretization is treated by a backward difference formula of order 2. The geometrical nonlinearities are treated explicitly while a Newton algorithm is used to deal with the nonlinearities coming from the fluid. The simulations are done with Getfem++ Library.

Spectrum of the discrete Oseen operator

We use an Arnoldi method combined with a 'shift and inverse' transformation implemented in Arpack Library to compute the spectrum of the linearized Navier-Stokes equation. We fix the shift parameter at 10 and the size of the small Hessenberg matrix at 400. In *Table* 2.1, we report the first eigenvalues of the linearized Navier-Stokes system. The spectrum the linearized Navier-Stokes equation is characterized by two complex conjugate unstable eigenvalues $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2 = \bar{\boldsymbol{\lambda}}_1$ (see *Figure* 2.4).

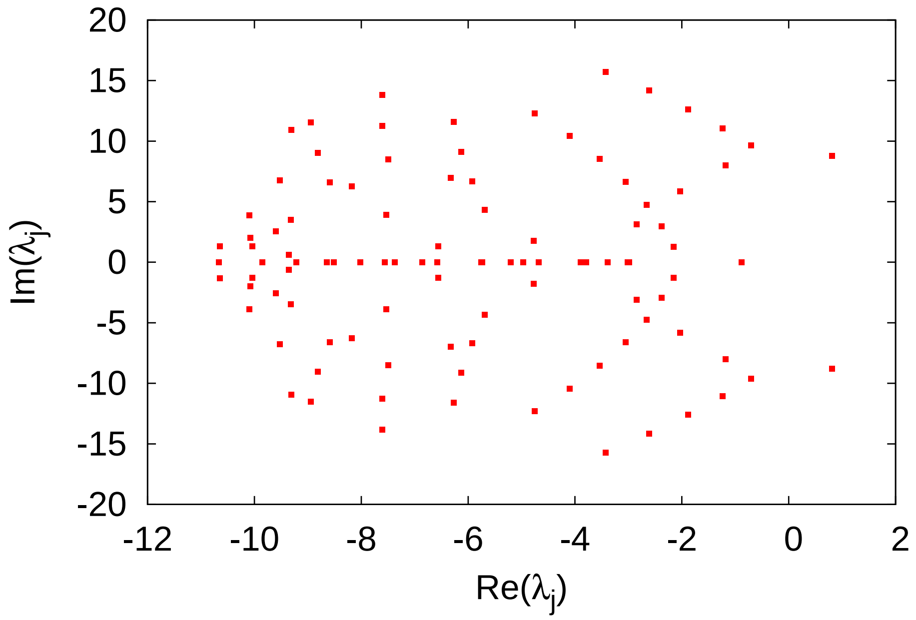


Figure 2.4 – Spectrum of the linearized Navier-Stokes operator for Rey = 200.

j	$1-2$	$3-4$	5	$6-7$	$8-9$	$10-11$	$12-13$
$\boldsymbol{\lambda}_j$	$0.81\pm 8.79i$	$-0.70\pm 9.63i$	-0.88	$-1.18\pm 7.99i$	$-1.23\pm 11.04i$	$-1.88\pm 12.60i$	$-2.03\pm 5.83i$

Table 2.1 – First eigenvalues of the linearized Navier-Stokes operator for Rey = 200.

Efficiency of the feedback law

We are going to show that the linear feedback law constructed in Section 2.12 .3 is able to stabilize the nonlinear system (2.10 .6). First, we introduce the deformations functions w_1 and w_2 on Γ_s by

$$w_1(x) = m_s(x)\mathrm{Re}\,\sigma(\phi^1, \psi^1)n \cdot n|_{\Gamma_s} \quad \text{and} \quad w_2(x) = m_s(x)\mathrm{Im}\,\sigma(\phi^1, \psi^1)n \cdot n|_{\Gamma_s},$$

(see *Figure* 2.5), where m_s is a truncation function on Γ_s, used in order to satisfy boundary conditions for w_1 and w_2. The function m_s is defined by

$$m_s(x) = g\left(\frac{3x}{10e}\right),$$

where g is defined by

$$g(x) = \begin{cases} G(x) & \forall\, x \le \dfrac{3}{2}, \\ G(3-x) & \forall\, x > \dfrac{3}{2}, \end{cases}$$

with

$$G(x) = \begin{cases} 0 & \forall\, x < 0, \\ x^3(6x^2 - 15x + 10) & \forall\, 0 \le x \le 1, \\ 1 & \forall\, x > 1, \end{cases}$$

(see *Figure* 2.6).

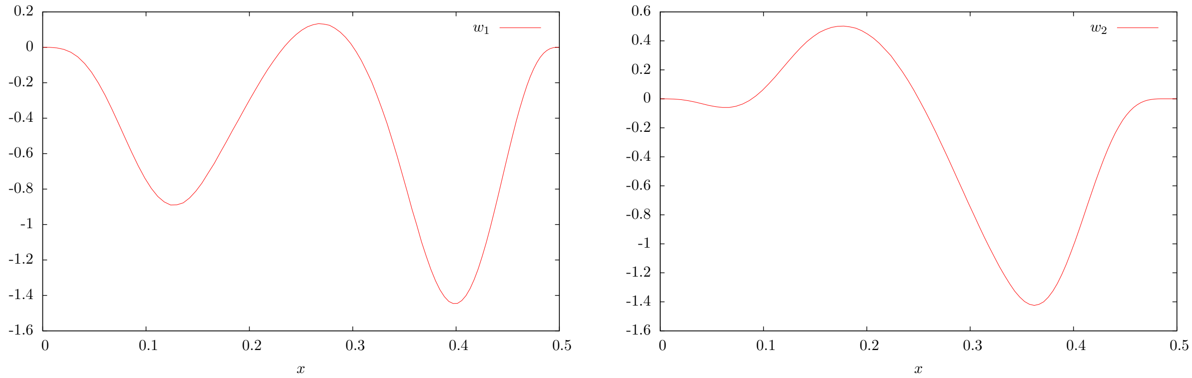


Figure 2.5 – The two deformation functions w_1 and w_2.

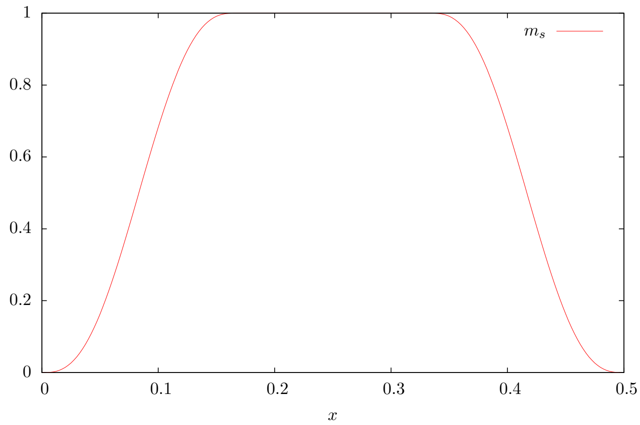


Figure 2.6 – Graph of the truncation m_s.

We introduce a boundary perturbation

$$h(t,x) = \mu(t)g(x), \tag{2.13 .2}$$

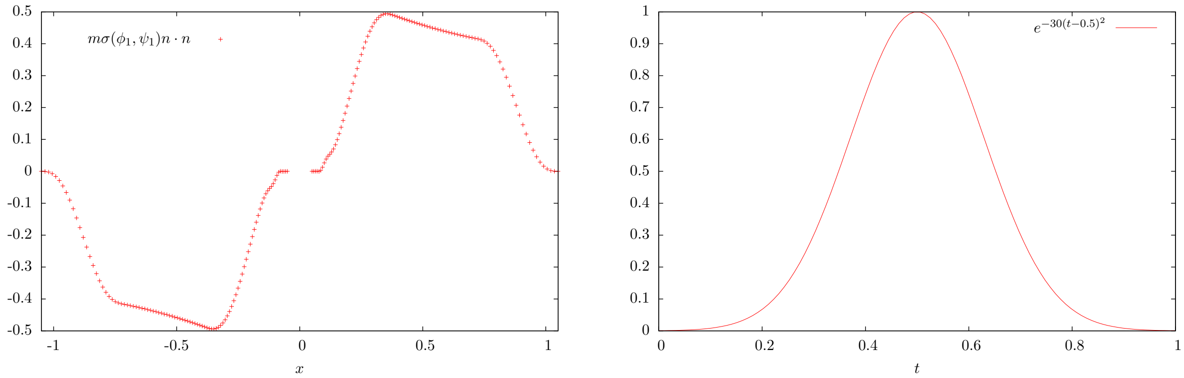
in the inflow boundary $(0,\infty) \times \Gamma_i$, localized in time, and defined by

$$\mu(t) = \beta_{pert} e^{-30(t-0.5)^2} \quad \text{and} \quad g(x) = \Big(m\sigma(\phi^1,\psi^1)n \cdot n, 0\Big)^T,$$

with $\beta_{pert} > 0$ is a parameter to vary the amplitude of the perturbation and m is a truncation function on Γ_i that can be defined in a same manner as m_s to impose the compatibility conditions at the junction between Γ_i and Γ_s (see *Figure* 2.8).

The perturbation h based on $\sigma(\phi^1,\psi^1)n.n$ depends on the eigenfunction corresponding to the most unstable eigenvalue. It is one of the most destabilizing normal boundary perturbations. In *Figure* 2.7, we plot the graph of $m\sigma(\phi^1,\psi^1)n \cdot n$ on Γ_i and the graph of μ.

Numerical experiments are realized for a given perturbation corresponding to $\beta_{pert} = 0.1$ (that corresponds to 10% of the maximum value of the inflow profile). Different values of the parameters α, γ and ω are considered. The influence of the parameters α and γ (dumping) are observed varying $\alpha = \gamma \in \{-3,-5,-10\}$ for a fixed value of $\omega = 0$. Finally, the rule of ω is studied by considering the best choice already observed ($\alpha = \gamma = -10$) and by taking ω not equal to 0 but equal to 6.



(a) Graph of the function $m\sigma(\boldsymbol{\phi}, \boldsymbol{\psi})$.

(b) Graph of the function μ.

Figure 2.7 – Graphs of the functions $m\sigma(\boldsymbol{\phi}, \boldsymbol{\psi})$ and μ.

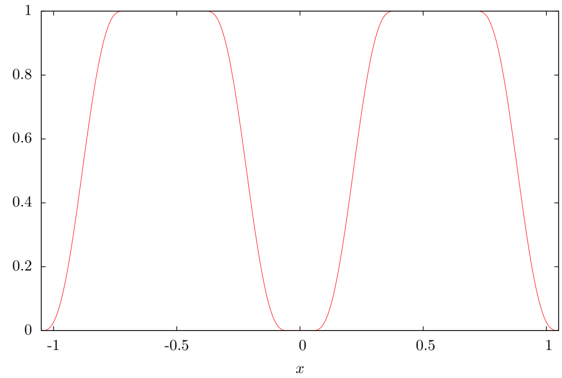
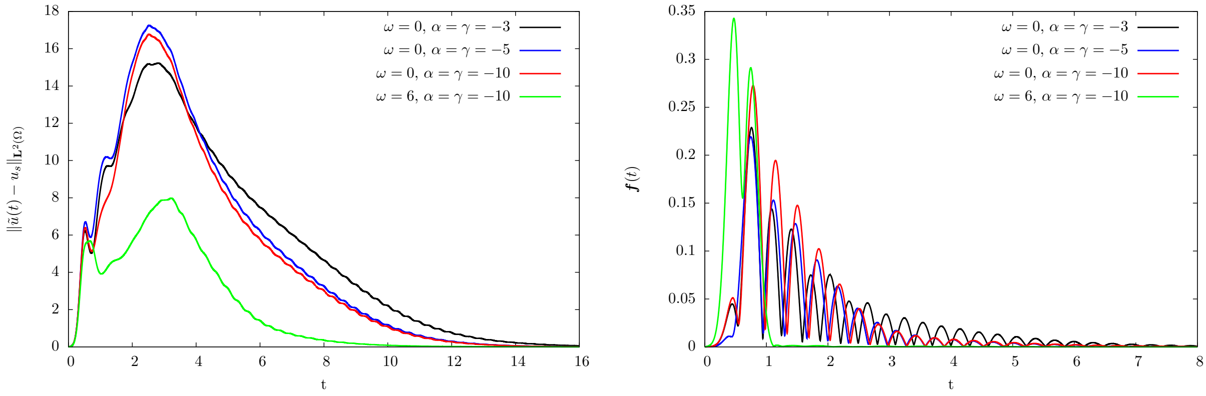


Figure 2.8 – Graph of the truncation function m.

In *Figure* 2.9, we plot the evolution of the L^2-norm of the difference between the controlled solution $\tilde{u}$ of the nonlinear system (2.4 .1) and the stationary solution u_s of the Navier-Stokes equations for $\alpha = \gamma \in \{-3, -5, -10\}$ and $\beta_{pert} = 0.1$, and the corresponding L^2-norm of the control. We observe that all choices are able to stabilize the flow. The decay rate is improved with smaller values of α and γ while the L^2-norm of the control decreases. However this good behavior is balanced by the fact that the maximum value of the control is obtained when α and γ are small.

In *Figure* 2.10, we report the L^∞-norm of the displacement of the structure for the lower and the upper part. As expected, the displacements decrease when α and γ decrease (that correspond to increase the dumping).

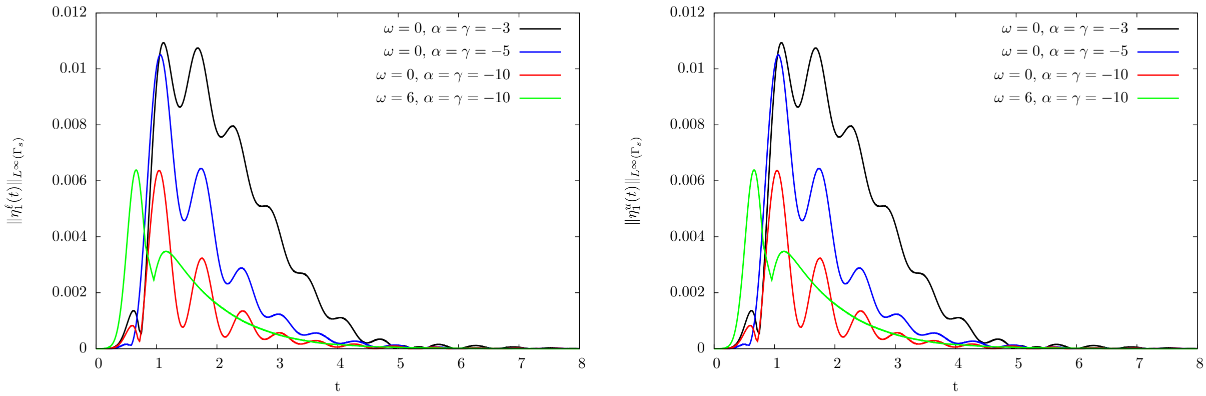
Finally, in *Figure* 2.9 and *Figure* 2.10, we report the results observed when $\omega = 6$ (green curves). We observe that the decay rate is improved with higher value of the shift parameter ω. As expected, the L^∞-norm of the control increases with ω however it is effectively applied during a smaller period. When a shift is used, the displacement of the structure is reduced and is the smallest one. This phenomena is very interesting in practice.



(a) Evolution of the L^2-norm of $\tilde{u} - u_s$.

(b) Evolution of the L^2-norm of the control.

Figure 2.9 – Evolution of the L^2-norm of $\tilde{u} - u_s$ and $\boldsymbol{f}$.



(a) Evolution of the L^∞-norm of the displacement of the structure (lower part).

(b) Evolution of the L^∞-norm of the displacement of the structure (upper part).

Figure 2.10 – Evolution of the L^∞-norm of the displacement η_1 of different controls.

2.14 Appendix A: Nonlinear term

The nonlinear term $\mathcal{F}_f$ in system (2.4 .3) is defined by

$$
\begin{aligned}
\mathcal{F}_f[\hat{u},\hat{p},\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_2] = {} & \frac{(\ell-s(z)z)\hat{\eta}_2\hat{u}_z}{\ell-e} - \frac{\nu(\ell-s(z)z)\hat{\eta}_{1,xx}\hat{u}_z}{\ell-e} - \frac{2\nu(\ell-s(z)z)\hat{\eta}_{1,x}\hat{u}_{xz}}{\ell-e} \\
& + \frac{2\nu s(z)\hat{\eta}_1\hat{u}_{zz}}{\ell-e} - \nu\nabla\mathrm{div}\,\mathcal{F}_{\mathrm{div}}[\hat{u},\hat{\eta}_1] + \frac{(\ell-s(z)z)\hat{\eta}_{1,x}\hat{p}_z e_1}{\ell-e} - \frac{s(z)\hat{\eta}_1\hat{p}_z e_2}{\ell-e} \\
& + \frac{(\ell-s(z)z)e^{-\omega t}\hat{\eta}_{1,x}\hat{u}_1\hat{u}_z}{\ell-e} + \frac{(\ell-s(z)z)\hat{\eta}_{1,x}\hat{u}_1 u_{s,z}}{\ell-e} + \frac{(\ell-s(z)z)\hat{\eta}_{1,x}u_{s,1}\hat{u}_z}{\ell-e} \\
& - \frac{e^{-\omega t}s(z)\hat{\eta}_1\hat{u}_2\hat{u}_z}{\ell-e} - \frac{s(z)\hat{\eta}_1\hat{u}_2 u_{s,z}}{\ell-e} - \frac{s(z)\hat{\eta}_1 u_{s,2}\hat{u}_z}{\ell-e} + \frac{(\ell-s(z)z)s(z)\hat{\eta}_1\hat{\eta}_2(e^{-\omega t}\hat{u}_z+u_{s,z})}{(\ell-e)(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)} \\
& + \frac{\nu(\ell-s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,xx}(e^{-\omega t}\hat{u}_z+u_{s,z})}{(\ell-e)(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)} + \frac{2\nu(\ell-s(z)z)\hat{\eta}_{1,x}^2(e^{-\omega t}\hat{u}_z+u_{s,z})}{(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)^2} \\
& - \frac{2\nu(\ell-s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{u}_{xz}+u_{s,xz})}{(\ell-e)(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)} + \frac{\nu((\ell-s(z)z)^2\hat{\eta}_{1,x}^2-\hat{\eta}_1^2)(e^{-\omega t}\hat{u}_{zz}+u_{s,zz})}{(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)^2} \\
& + \frac{2\nu(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)\hat{\eta}_1^2(e^{-\omega t}\hat{u}_{zz}+u_{s,zz})}{(\ell-e)(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)^2} + \frac{(\ell-s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{p}_z+p_{s,z})e_1}{(\ell-e)(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)} \\
& - \frac{\hat{\eta}_1^2(e^{-\omega t}\hat{p}_z+p_{s,z})e_2}{(\ell-e)(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)} + \frac{(\ell-s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{u}_1+u_{s,1})(e^{-\omega t}\hat{u}_z+u_{s,z})}{(\ell-e)(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)} \\
& - \frac{\hat{\eta}_1^2(e^{-\omega t}\hat{u}_2+u_{s,2})(e^{-\omega t}\hat{u}_z+u_{s,z})}{(\ell-e)(\ell-e-s(z)e^{-\omega t}\hat{\eta}_1)},
\end{aligned}
$$

with

$$\hat{\eta}_1 = \sum_{i=1}^{n_c}\hat{\boldsymbol{\eta}}_{1,i}w_i \quad \text{and} \quad \hat{\eta}_2 = \sum_{i=1}^{n_c}\hat{\boldsymbol{\eta}}_{2,i}w_i.$$

Chapter 3

Feedback stabilization of a 2D fluid-structure model by beam deformation: theoretical analysis

3.1 Introduction

We consider a stabilization problem for a fluid-structure model where the structure is located at the boundary of the fluid domain. The model couples the incompressible Navier-Stokes equations with a damped beam equation.
Since the structure is deformed under the action of the fluid, the domain occupied by the fluid at time t depends on the displacement η of the structure and we denote it by $\Omega_{\eta(t)}$ (see *Figure* 3.1). The elastic beam equation is written in the reference configuration, denoted by Γ_s. The deformed configuration of Γ_s at time t will be denoted by $\Gamma_{\eta(t)}$.
The reference configuration for the fluid is denoted by Ω. The boundary Γ of Ω is split into different parts

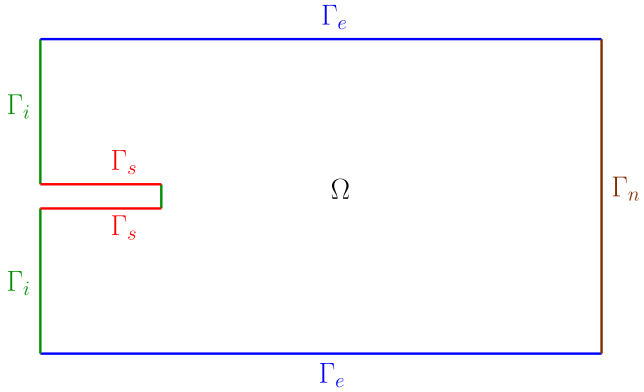
$$\Gamma = \Gamma_s \cup \Gamma_i \cup \Gamma_e \cup \Gamma_n,$$

where Γ_s is the part occupied by the structure, Γ_i is the part where inflow Dirichlet boundary condition is imposed, Γ_n is the part where a Neumann boundary condition is prescribed. For simplicity, a Navier boundary condition is prescribed in Γ_e rather then a Neumann boundary condition. We use the notations

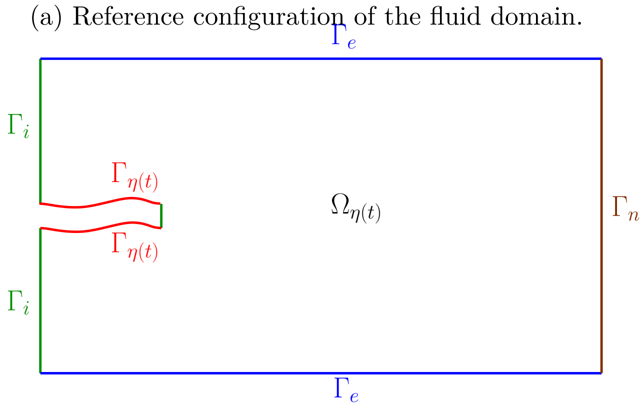
$$\begin{aligned}
&Q_\eta^\infty = \bigcup_{t\in(0,\infty)} \Big(\{t\}\times\Omega_{\eta(t)}\Big), \quad \Sigma_\eta^\infty = \bigcup_{t\in(0,\infty)} \Big(\{t\}\times\Gamma_{\eta(t)}\Big),\\
&Q^\infty = (0,\infty)\times\Omega, \quad \Sigma_s^\infty = (0,\infty)\times\Gamma_s,\\
&\Sigma_i^\infty = (0,\infty)\times\Gamma_i, \quad \Sigma_e^\infty = (0,\infty)\times\Gamma_e,\\
&\Sigma_n^\infty = (0,\infty)\times\Gamma_n.
\end{aligned}$$

We also use the notations $e_1 = (1,0)$ and $e_2 = (0,1)$. The incompressible Navier-Stokes equations are written as follows:

$$\begin{aligned}
&u_t - \operatorname{div}\sigma(u,p) + (u\cdot\nabla)u = 0 \text{ in } Q_\eta^\infty,\\
&\operatorname{div} u = 0 \text{ in } Q_\eta^\infty, \quad u = \eta_t n \text{ on } \Sigma_\eta^\infty, \quad u = u_i \text{ on } \Sigma_i^\infty,\\
&u\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(u)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(u,p)n = 0 \text{ on } \Sigma_n^\infty,\\
&u(0) = u^0 \text{ on } \Omega_{\eta_1^0},
\end{aligned} \tag{3.1 .1}$$



(a) Reference configuration of the fluid domain.



(b) Deformed configuration of the fluid domain.

Figure 3.1 – Configurations of the fluid domain

where u and p stand for the velocity and the pressure of the fluid. The stress tensor $\sigma(u,p)$ is defined by

$$\sigma(u,p) = 2\,\nu\varepsilon(u) - pI, \quad \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

where ν is the viscosity of the fluid. The damped beam equation reads as follows:

$$\begin{aligned}
&\eta_{tt} - \beta\Delta\eta - \gamma\Delta\eta_t + \alpha\Delta^2\eta = -\sigma(u,p)|_{\Gamma_{\eta(t)}}\, n_{\eta(t)} \cdot n + f_b \text{ on } \Sigma_s^\infty,\\
&\eta = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \eta_x = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&\eta(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \eta_t(0) = \eta_2^0 \text{ on } \Gamma_s,
\end{aligned} \tag{3.1 .2}$$

where $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$ are parameters of the structure already indicated in Chapter 1, and $n_{\eta(t)}$ is the unit normal of $\Gamma_{\eta(t)}$ exterior to $\Omega_{\eta(t)}$. Clamped boundary conditions are imposed for the beam equation. In this setting, u_i is the inflow boundary condition and f_b is a force exerted on the structure. We assume that u_i and f_b are in the form

$$u_i = g_s + h \quad \text{and} \quad f_b = f_s + f,$$

where $g_s \in H_0^2(\Gamma_i;\mathbb{R}^2)$ and $f_s \in L^2(\Gamma_s)$ are stationary data (independent of time), h is a perturbation of the inflow boundary condition g_s, and f is a control function of finite dimension n_c:

$$f(t,x,y) = \sum_{i=1}^{n_c} f_i(t)w_i(x,y),$$

where the functions $w_i \in L^2(\Gamma_s)$ are defined in (3.7 .4). Let (u_s, p_s) be a solution of the stationary Navier-Stokes equations

$$\begin{aligned} &-\mathrm{div}\,\sigma(u_s,p_s) + (u_s\cdot\nabla)u_s = 0 \text{ in } \Omega_{\eta_s},\\ &\mathrm{div}\,u_s = 0 \text{ in } \Omega, \quad u_s = 0 \text{ on } \Gamma_s, \quad u_s = g_s \text{ on } \Gamma_i,\\ &u_s\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(u_s)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(u_s,p_s)n = 0 \text{ on } \Gamma_n, \end{aligned} \tag{3.1 .3}$$

(see 3.4 .1) We choose $f_s = p_s|\Gamma_s$. Thus, $(u_s, p_s, 0)$ is a stationary solution of system (3.1 .1)-(3.1 .2). We assume that $(u_s, p_s, 0)$ is unstable stationary solution of that system. For any prescribed exponential decay rate $-\omega < 0$ and for given h, u^0, η_1^0 and η_2^0, such that h, $u^0 - u_s$, η_1^0 and η_2^0 are small enough in appropriate functional spaces, we would like to find f in feedback form so that the solution to the control system converges exponentially to the stationary solution in a sense that we make precise later on. We recall that this problem comes from a stabilization problem of a fluid flow in a wind tunnel over a thick plate. To achieve this goal we are going to follow a classical approach.

Step 1: *System in a fixed domain.*

Thanks to the change of unknowns introduced in (3.3 .3), we rewrite system (3.1 .1)-(3.1 .2) in a fixed domain

$$\begin{aligned} &\hat u_t - \mathrm{div}\,\sigma(\hat u,\hat p) + (u_s\cdot\nabla)\hat u + (\hat u\cdot\nabla)u_s - A_1\hat\eta_1 - A_2\hat\eta_2 - \omega\hat u = e^{-\omega t}\mathcal{F}_f[\hat u,\hat p,\hat\eta_1,\hat\eta_2] \text{ in } Q^\infty,\\ &\mathrm{div}\,\hat u = A_3\hat\eta_1 + e^{-\omega t}\mathrm{div}\,\mathcal{F}_{\mathrm{div}}[\hat u,\hat\eta_1] \text{ in } Q^\infty, \quad \hat u = \hat\eta_2 n \text{ on } \Sigma_s^\infty, \quad \hat u = \hat h \text{ on } \Sigma_i^\infty,\\ &\hat u = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\hat u)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\hat u,\hat p)n = 0 \text{ on } \Sigma_n^\infty,\\ &\hat\eta_{1,t} - \hat\eta_2 - \omega\hat\eta_1 = 0 \text{ on } \Sigma_s^\infty,\\ &\hat\eta_{2,t} - \beta\Delta\hat\eta_1 - \gamma\Delta\hat\eta_2 + \alpha\Delta^2\hat\eta_1 - A_4\hat\eta_1 - \omega\hat\eta_2 = \gamma_s\hat p + e^{-\omega t}\mathcal{F}_s[\hat u,\hat\eta_1] + \hat f \text{ on } \Sigma_s^\infty,\\ &\hat\eta_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \hat\eta_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\ &\hat u(0) = \hat u^0 \text{ in } \Omega, \quad \hat\eta_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \hat\eta_2(0) = \eta_2^0 \text{ on } \Gamma_s, \end{aligned} \tag{3.1 .4}$$

where the nonlinear terms $\mathcal{F}_f[\hat u,\hat p,\hat\eta_1,\hat\eta_2]$ and $\mathcal{F}_s[\hat u,\hat\eta_1]$ are given in Appendix 3.A and the nonlinear term $\mathcal{F}_{\mathrm{div}}[\hat u,\hat\eta_1]$ is defined in (3.3 .5). The linear differential operators A_1, A_2, A_3 and A_4 are defined in (3.3 .6).

Step 2: *Reformulation of the linearized system.*

Thanks to the Leray projector P, we prove that the linearized system

$$\begin{aligned} &v_t - \mathrm{div}\,\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 - \omega v = 0 \text{ in } Q^\infty,\\ &\mathrm{div}\,v = A_3\eta_1 \text{ in } Q^\infty, \quad v = \eta_2 n \text{ on } \Sigma_s^\infty, \quad v = 0 \text{ on } \Sigma_i^\infty,\\ &v\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(v)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(v,p)n = 0 \text{ on } \Sigma_n^\infty,\\ &\eta_{1,t} - \eta_2 - \omega\eta_1 = 0 \text{ on } \Sigma_s^\infty,\\ &\eta_{2,t} - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 - A_4\eta_1 - \omega\eta_2 = p + f \text{ on } \Sigma_s^\infty,\\ &\eta_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\ &v(0) = v^0 \text{ in } \Omega, \quad \eta_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s, \end{aligned} \tag{3.1 .5}$$

is equivalent to the following one

$$
\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = (\mathcal{A}+\omega I)\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}\mathbf{f}, \quad \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix}(0) = \begin{pmatrix} Pv^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \\
&(I-P)v = \nabla N_s\eta_2 + \nabla N_{\text{div}}A_3\eta_1, \\
&p = -N_s(\eta_{2,t}-\omega\eta_2) - N_{\text{div}}A_3(\eta_{1,t}-\omega\eta_1) + N_pA_1\eta_1 + N_pA_2\eta_2 \\
&\quad + N_v(Pv + \nabla N_s\eta_2 + \nabla N_{\text{div}}A_3\eta_1).
\end{aligned}
\tag{3.1 .6}
$$

The operators $\mathcal{A}$ and $\mathcal{B}$ are defined in (3.4 .11) and (3.4 .14) respectively, and the operators N_s, N_{div}, N_p and N_v are defined in Section 3.4 .2. We prove that $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup with compact resolvent. This allows us to reduce the stabilizability of the pair $(\mathcal{A}+\omega I, \mathcal{B})$ to a unique continuation problem.

Step 3: *Equivalence between eigenvalue problems*

For numerical viewpoint (see Chapter 4), the eigenvalues of $\mathcal{A}$ are not easy to determine because $\mathcal{A}$ involves several operators, as P, which cannot not be easily approximated. This is why we look for the eigenvalue problem associated with the linearized system (3.1 .5). We study the relations between the eigenvalue problems associated $\mathcal{A}$ and the eigenvalue problems associated to (3.1 .5). Using these relations, we will able to construct the feedback law for the linearized system which can be determined without the Leray projector P.

Step 4: *Projection of the linearized system.*

To construct a feedback control law which can be easily determined numerically, we are going to project system (3.1 .5) onto the unstable subspace for $\mathcal{A}+\omega I$. Let $(\lambda_j)_j$ be the eigenvalues of $\mathcal{A}$. We denote by $G_{\mathbb{R}}(\lambda_j)$ the real generalized eigenspace of $\mathcal{A}$ and $G_{\mathbb{R}}^*(\lambda_j)$ the real generalized eigenspace of $\mathcal{A}^*$ associated to λ_j. We choose a finite family $(\lambda_j)_{j\in J_u}$ containing at least all the unstable eigenvalues (with nonnegative real part) and set

$$Z_u = \oplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j) \quad \text{and} \quad Z_u^* = \oplus_{j\in J_u} G_{\mathbb{R}}^*(\lambda_j).$$

We know that there exist subspaces Z_s and Z_s^*, invariant subspaces under $(e^{t\mathcal{A}})_{t\geq 0}$ and $(e^{t\mathcal{A}^*})_{t\geq 0}$ respectively, such that

$$Z = Z_u \oplus Z_s \quad \text{and} \quad Z = Z_u^* \oplus Z_s^*.$$

We denote by π_u the projection in Z onto Z_u along Z_s and set $\pi_s = I - \pi_u$. In Proposition 3.7 .1, we prove that there exist two families $(v_i, \eta_{1,i}, \eta_{2,i})_{1\leq i\leq d_u}$ and $(\phi_i, \xi_{1,i}, \xi_{2,i})_{1\leq i\leq d_u}$ satisfying the bi-orthogonality conditions

$$\left(\begin{pmatrix} v_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, \begin{pmatrix} \phi_i \\ \xi_{1,i} \\ \xi_{2,i} \end{pmatrix}\right) = \left(\begin{pmatrix} Pv_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, M_s^*\begin{pmatrix} P\phi_i \\ \xi_{1,i} \\ \xi_{2,i} \end{pmatrix}\right) = \delta_{i,j}, \quad \forall\, 1\leq i,j\leq d_u,$$

such that

$$\pi_u\begin{pmatrix} v \\ \eta_1 \\ \eta_2 \end{pmatrix} = \sum_{i=1}^{d_u}\left(\begin{pmatrix} v \\ \eta_1 \\ \eta_2 \end{pmatrix}, M_s^*\begin{pmatrix} P\phi_i \\ \xi_{1,i} \\ \xi_{2,i} \end{pmatrix}\right)\begin{pmatrix} Pv_i \\ \eta_{1,i} \\ \eta_{2,i} \end{pmatrix}, \quad \forall\,(v,\eta_1,\eta_2)^T \in Z,$$

and that the matrix Λ_u defined by

$$\Lambda_u = [\Lambda_{u,i,j}]_{1\le i,j\le d_u} \quad \text{and} \quad \Lambda_{u,i,j} = \left(\mathcal{A}\begin{pmatrix} Pv^i \\ \eta_{1,i} \\ \eta_{2,i}\end{pmatrix}, M_s^*\begin{pmatrix} P\phi_j \\ \xi_{1,j} \\ \xi_{2,j}\end{pmatrix}\right),$$

is constituted of real Jordan blocks. The operator $M_s \in \mathcal{L}(Z)$ is defined in (3.4 .12). Next, we prove that (Pv, η_1, η_2) is a solution of the first equation of system (3.1 .6) if and only if

$$\zeta_u := [((Pv, \eta_1, \eta_2), M_s^*(P\phi_i, \xi_{1,i}, \xi_{2,i}))]_{1\le i\le d_u} \quad \text{and} \quad (v_s, \eta_{1,s}, \eta_{2,s}) := \pi_s(Pv, \eta_1, \eta_2),$$

satisfy

$$\begin{aligned} &\frac{d\zeta_u}{dt} = (\Lambda_u + \omega I)\zeta_u + B_u \boldsymbol{f}, \quad \zeta_u(0) = \zeta_u^0, \\ &\frac{d}{dt}\begin{pmatrix} v_s \\ \eta_{1,s} \\ \eta_{2,s}\end{pmatrix} = \mathcal{A}_{s,\omega}\begin{pmatrix} v_s \\ \eta_{1,s} \\ \eta_{2,s}\end{pmatrix} + \mathcal{B}_s \boldsymbol{f}, \quad \begin{pmatrix} v_s \\ \eta_{1,s} \\ \eta_{2,s}\end{pmatrix}(0) = \pi_s\begin{pmatrix} v^0 \\ \eta_1^0 \\ \eta_2^0\end{pmatrix}, \end{aligned} \tag{3.1 .7}$$

where

$$\mathcal{A}_{s,\omega} := \pi_s(\mathcal{A} + \omega I), \quad \mathcal{B}_s := \pi_s\mathcal{B} \quad \text{and} \quad \zeta_u^0 := [((Pv^0, \eta_1^0, \eta_2^0), M_s^*(P\phi_i, \xi_{1,i}, \xi_{2,i}))]_{1\le i\le d_u}.$$

The matrix B_u is defined in (3.7 .1).

Step 5: *Stabilization of the linearized system.*

To stabilize the linearized system, it is necessary and sufficient to stabilize the equation satisfied by ζ_u. Thanks to an appropriate choice of the family $(w_i)_{1\le i\le n_c}$, we prove that the pair $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, B_u)$ is stabilizable. Thus, the operator $K_u = -B_u^T P_u$, where P_u is the solution of the algebraic Riccati equation

$$\begin{aligned} &P_u \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_u}, \quad P_u = P_u^T > 0, \\ &P_u(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}})P_u - P_u B_u B_u^T P_u = 0, \end{aligned}$$

provides a stabilizing feedback law for $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, B_u)$.

Step 6: *Stabilization of the nonlinear system.*

We use a fixed point method to prove that the linear feedback law is also able to locally stabilize the original nonlinear system.

3.2 Functional setting

We introduce the classical Sobolev spaces in two dimensions

$$\mathbf{L}^2(\Omega) = L^2(\Omega;\mathbb{R}^2) \quad \text{and} \quad \mathbf{H}^s(\Omega) = H^s(\Omega;\mathbb{R}^2), \quad \forall\, s > 0.$$

We also define the functional spaces

$$\begin{aligned} &\mathbf{V}^0_{n,\Gamma_0}(\Omega) = \{v \in \mathbf{L}^2(\Omega) \mid \operatorname{div} v = 0,\ v\cdot n = 0 \text{ on } \Gamma_0\}, \\ &\mathbf{V}(\Omega) = \{v \in \mathbf{L}^2(\Omega) \mid \operatorname{div} v \in \mathbf{L}^2(\Omega)\}, \\ &\mathbf{H}^s_{\Gamma_0}(\Omega) = \{v \in \mathbf{H}^s(\Omega) \mid v = 0 \text{ on } \Gamma_s \cup \Gamma_i,\ v\cdot n = 0 \text{ on } \Gamma_e\}, \\ &\mathbf{V}^s_{n,\Gamma_0}(\Omega) = \mathbf{H}^s(\Omega) \cap \mathbf{V}^0_{n,\Gamma_0}(\Omega), \\ &\mathbf{V}^s_{\Gamma_0}(\Omega) = \mathbf{H}^s_{\Gamma_0}(\Omega) \cap \mathbf{V}(\Omega), \\ &H = \mathbf{L}^2(\Omega) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s), \\ &Z = \mathbf{V}^0_{n,\Gamma_0}(\Omega) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s), \end{aligned}$$

with $\Gamma_0 = \Gamma_s \cup \Gamma_i \cup \Gamma_e$ and $s > \frac{1}{2}$. We denote by $\mathcal{C} = (C_j)_{1\leq j\leq 8}$ the set of the corners of Ω. For all $\delta, s \geq 0$ and for $n = 1$ or $n = 2$, we introduce the norms

$$\|v\|^2_{H^2_\delta(\Omega;\mathbb{R}^n)} = \sum_{|\alpha|\leq 2} \int_\Omega \prod_{j\in\mathcal{C}} r_j^{2\delta} |\partial_\alpha v|^2,$$

where r_j stands for the distance to $C_j \in \mathcal{C}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ denotes a two-index, $|\alpha| = \alpha_1 + \alpha_2$ is its length, ∂_α denotes the corresponding partial differential operator. We denote by $H^2_\delta(\Omega;\mathbb{R}^n)$ the closure of $C^\infty(\overline{\Omega})$ in the norms $\|\cdot\|_{H^2_\delta(\Omega;\mathbb{R}^n)}$. We define weighted Sobolev spaces in two dimensions

$$\mathbf{L}^2_\delta(\Omega) = H^0_\delta(\Omega;\mathbb{R}^2) \quad \text{and} \quad \mathbf{H}^s_\delta(\Omega) = H^s_\delta(\Omega;\mathbb{R}^2), \quad \forall\, s > 0,$$

and the product space

$$H_\delta = \mathbf{H}^2_\delta(\Omega) \times H^1_\delta(\Omega) \times \Big(H^4(\Gamma_s) \cap H^2_0(\Gamma_s)\Big) \times H^2_0(\Gamma_s).$$

We introduce the spaces

$$\begin{aligned}
&H^{2,1}(Q^\infty) = L^2(0,\infty;\mathbf{H}^2(\Omega)) \cap H^1(0,\infty;\mathbf{L}^2(\Omega)),\\
&H^{2,1}_\delta(Q^\infty) = L^2(0,\infty;\mathbf{H}^2_\delta(\Omega)) \cap H^1(0,\infty;\mathbf{L}^2(\Omega)),\\
&H^{4,2}(\Sigma_s^\infty) = L^2(0,\infty;H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \cap H^2(0,\infty;L^2(\Gamma_s)),\\
&H^{2,1}(\Sigma_s^\infty) = L^2(0,\infty;H^2_0(\Gamma_s)) \cap H^1(0,\infty;L^2(\Gamma_s)),\\
&Y = L^2(0,\infty;\mathbf{L}^2(\Omega)) \times H^{2,1}(Q^\infty) \times L^2(0,\infty;L^2(\Gamma_s)),\\
&X_\delta = H^{2,1}_\delta(Q^\infty) \times L^2(0,\infty;H^1_\delta(\Omega)) \times H^{4,2}(\Sigma_s^\infty) \times H^{2,1}(\Sigma_s^\infty),
\end{aligned}$$

and for $w^0 \in \mathbf{L}^2(\Omega)$ given

$$\begin{aligned}
H_{ic}(w^0) = \{&(v^0,\eta_1^0,\eta_2^0) \in \mathbf{H}^1(\Omega) \times (H^3(\Gamma_s)\cap H^2_0(\Gamma_s)) \times H^1_0(\Gamma_s) \mid (Pv^0 - Pw^0, \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), Z]_{\frac{1}{2}},\\
&\operatorname{div} v^0 = \operatorname{div} w^0 + A_3\eta_1^0 \text{ in } \Omega, \quad v^0 = \eta_2^0 n \text{ on } \Gamma_s, \quad v^0 = 0 \text{ on } \Gamma_i, \quad v^0\cdot n = 0 \text{ on } \Gamma_e\},
\end{aligned} \tag{3.2 .1}$$

equipped with the norm

$$\|(v^0,\eta_1^0,\eta_2^0)\|_{ic} = \|(v^0,\eta_1^0,\eta_2^0)\|_{\mathbf{H}^1(\Omega)\times\left(H^3(\Gamma_s)\cap H^2_0(\Gamma_s)\right)\times H^1_0(\Gamma_s)} + \|(Pv^0 - Pw^0, \eta_1^0, \eta_2^0)\|_{[D(\mathcal{A}),Z]_{\frac{1}{2}}}.$$

The Leray projector P is defined in Section 3.4 .1. The operators $\mathcal{A}$ and A_3 are defined in (3.4 .11) and (3.3 .6) respectively.

3.3 System in the reference configuration and main results

Let δ_0 defined in Theorem 3.4 .1. The main result of this chapter is the following.

Theorem 3.3 .1. *Let ω be positive. We assume that 3.4 .1, 3.7 .2 and 3.7 .1 are satisfied. There exists $r > 0$ such that, for all $(u^0,\eta_1^0,\eta_2^0) \in \mathbf{H}^1(\Omega_{\eta_1^0}) \times H^3(\Gamma_s)\cap H^2_0(\Gamma_s) \times H^1_0(\Gamma_s)$ and $h \in H^1_0(0,\infty;H^2_0(\Gamma_i))$ satisfying $(u^0\circ\mathcal{T}^{-1}_{\eta_1^0} - u_s, \eta_1^0, \eta_2^0) \in H_{ic}(\mathcal{F}_{\mathrm{div}}[\hat{u}^0,\eta_1^0])$ and*

$$\|u^0\circ\mathcal{T}^{-1}_{\eta_1^0} - u_s, \eta_1^0, \eta_2^0\|_{ic} + \|\hat{h}\|_{H^1_0(0,\infty;H^2_0(\Gamma_i))} \leq r,$$

we can find $\mathbf{f} \in H^1(0,\infty;\mathbb{R}^{n_c})$, in feedback form, for which there exists a unique solution (u,p,η) in

$$H^{2,1}_{\delta_0}(Q^\infty_\eta) \times L^2(0,\infty;H^1(\Omega_{\eta(t)})) \times H^{4,2}(\Sigma_s^\infty),$$

to the nonlinear system (3.1 .1)-(3.1 .2), satisfying

$$\left\|(u(t)\circ \mathcal{T}_{\eta(t)}^{-1} - u_s, \eta(t), \eta_t(t))\right\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon}(\Omega)\times H^3(\Gamma_s)\times H^1(\Gamma_s)} \leq Ce^{-\omega t}, \quad \forall t > 0,$$

where $\varepsilon > 0$ *and* $C > 0$.

The mappings $\mathcal{T}_\eta$ and $\mathcal{T}_{\eta(t)}$ are defined by

$$\mathcal{T}_\eta : (t,x,y) \quad \longmapsto \quad (t,x,z) = \left(t, x, \frac{(\ell-e)y - \ell\eta(t,x,s(y)e)}{\ell - e - s(y)\eta(t,x,s(y)e)}\right), \tag{3.3 .1}$$

and

$$\mathcal{T}_{\eta(t)} : (x,y) \quad \longmapsto \quad (x,z) = \left(t, x, \frac{(\ell-e)y - \ell\eta(t,x,s(y)e)}{\ell - e - s(y)\eta(t,x,s(y)e)}\right), \tag{3.3 .2}$$

where the sign function s is defined by

$$s(y) = -1 \text{ if } y < 0 \quad \text{ and } \quad s(y) = 1 \text{ if } y \geq 0.$$

The mapping $\mathcal{T}_{\eta_1^0}$ is defined in a similar way. We notice that $\mathcal{T}_{\eta(t)}$ transforms $\Omega_{\eta(t)}$ into the reference configuration Ω. The definition of the space $H^{2,1}_{\delta_0}(Q^\infty_\eta)$ is given in 2.3 .1.
In order the prove that theorem, we make the change of unknowns

$$\begin{aligned}
&\hat{u}(t,x,z) := e^{\omega t}[u(t, \mathcal{T}_{\eta(t)}^{-1}(x,z)) - u_s(x,z)], \quad \hat{p}(t,x,z) := e^{\omega t}[p(t, \mathcal{T}_{\eta(t)}^{-1}(x,z)) - p_s(x,z)],\\
&\hat{\eta}_1(t,x) := e^{\omega t}\eta(t,x), \quad \hat{\eta}_2(t,x) := e^{\omega t}\eta_t(t,x), \quad \hat{\boldsymbol{f}}(t) = (\hat{f}_i(t))_{1\leq i\leq n_c} := e^{\omega t}\mathbf{f}(t).
\end{aligned} \tag{3.3 .3}$$

Thus, we are lead to prove the following result.

Theorem 3.3 .2. *Let* $\eta_{\max}$ *be in* $(0,e)$ *and* ω *be positive. We assume that 3.4 .1, 3.7 .2 and 3.7 .1 are satisfied. There exists* $r > 0$ *such that, for all* $(\hat{u}^0, \eta_1^0, \eta_2^0) \in H_{ic}(\mathcal{F}_{\text{div}}[\hat{u}^0, \eta_1^0])$ *and* $\hat{h}) \in H^1_0(0,\infty; H^2_0(\Gamma_i))$ *satisfying the estimate*

$$\|\hat{u}^0, \eta_1^0, \eta_2^0\|_{ic} + \|\hat{h}\|_{H^1_0(0,\infty;H^2_0(\Gamma_i))} \leq r,$$

we can find a control $\hat{\mathbf{f}} \in H^1(0,\infty;\mathbb{R}^{n_c})$*, in feedback form, for which nonlinear system (3.3 .4) below admits a unique bounded solution* $(\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2) \in X_{\delta_0}$ *with the estimate*

$$\|e^{-\omega t}\hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)} \leq \eta_{\max} \quad \text{ and } \|(\hat{u}(t), \hat{\eta}(t), \hat{\eta}_t(t))\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon}(\Omega)\times H^3(\Gamma_s)\times H^1(\Gamma_s)} \leq C, \quad \forall t > 0,$$

where $\varepsilon > 0$ *and* $C > 0$.

Remark 3.3 .1. *As in Chapter* 1 *(see Section 1.9), the estimate*

$$\|e^{-\omega t}\hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)} \leq \eta_{\max},$$

guarantees that the change of variables $\mathcal{T}_{e^{-\omega t}\hat{\eta}(t)}$ *is a* $\mathcal{C}^1-$*diffeomorphism from* $\Omega_{e^{-\omega t}\hat{\eta}(t)}$ *into* Ω *for all* $t > 0$.

The system satisfied by $(\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2)$ is

$$
\begin{aligned}
&\hat{u}_t - \operatorname{div} \sigma(\hat{u}, \hat{p}) + (u_s \cdot \nabla)\hat{u} + (\hat{u} \cdot \nabla) u_s - A_1\hat{\eta}_1 - A_2\hat{\eta}_2 - \omega \hat{u} = e^{-\omega t} \mathcal{F}_f[\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2] \text{ in } Q^\infty,\\
&\operatorname{div} \hat{u} = A_3\hat{\eta}_1 + e^{-\omega t} \operatorname{div} \mathcal{F}_{\mathrm{div}}[\hat{u}, \hat{\eta}_1] \text{ in } Q^\infty, \quad \hat{u} = \hat{\eta}_2 n \text{ on } \Sigma_s^\infty, \quad \hat{u} = \hat{h} \text{ on } \Sigma_i^\infty,\\
&\hat{u} = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\hat{u})n \cdot \tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\hat{u}, \hat{p})n = 0 \text{ on } \Sigma_n^\infty,\\
&\hat{\eta}_{1,t} - \hat{\eta}_2 - \omega \hat{\eta}_1 = 0 \text{ on } \Sigma_s^\infty,\\
&\hat{\eta}_{2,t} - \beta \Delta \hat{\eta}_1 - \gamma \Delta \hat{\eta}_2 + \alpha \Delta^2 \hat{\eta}_1 - A_4 \hat{\eta}_1 - \omega \hat{\eta}_2 = \gamma_s \hat{p} + e^{-\omega t} \mathcal{F}_s[\hat{u}, \hat{\eta}_1] + \hat{f} \text{ on } \Sigma_s^\infty,\\
&\hat{\eta}_1 = 0 \text{ on } (0, \infty) \times \partial\Gamma_s, \quad \hat{\eta}_{1,x} = 0 \text{ on } (0, \infty) \times \partial\Gamma_s,\\
&\hat{u}(0) = \hat{u}^0 \text{ in } \Omega, \quad \hat{\eta}_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \hat{\eta}_2(0) = \eta_2^0 \text{ on } \Gamma_s,
\end{aligned}
\tag{3.3 .4}
$$

where

$$\hat{h}(t, x, z) := e^{\omega t} h(t, x, z) \quad \text{and} \quad \hat{u}^0 = u^0 \circ \mathcal{T}_{\eta_1^0}^{-1} - u_s.$$

The operator γ_s is the trace operator on Γ_s. The nonlinear terms $\mathcal{F}_f$ and $\mathcal{F}_s$ are given in Appendix 3.A and $\mathcal{F}_{\mathrm{div}}[\hat{u}, \hat{\eta}_1]$ is defined by

$$\mathcal{F}_{\mathrm{div}}[\hat{u}, \hat{\eta}_1] = \frac{1}{\ell - e} \left(s(z)\hat{\eta}_1 \hat{u}_1 e_1 + (\ell - s(z)z)\hat{\eta}_{1,x} \hat{u}_1 e_2 \right). \tag{3.3 .5}$$

The linear differential operators A_1, A_2, A_3 and A_4 are defined by

$$
\begin{aligned}
A_1\hat{\eta}_1 &= \frac{s(z)\hat{\eta}_1}{\ell - e} \left(p_{s,z} e_2 + u_{s,2} u_{s,z} - 2\nu u_{s,zz} + \nu u_{s,1,xx} e_1 + \nu u_{s,1,xz} e_2\right) + \frac{\nu \hat{\eta}_{1,x}}{\ell - e} \left(u_{s,1,z} e_2 - u_{s,1,x} e_1\right)\\
&\quad + \frac{(\ell - s(z)z)\hat{\eta}_{1,x}}{\ell - e} \left(p_{s,z} e_1 - 2\nu u_{s,xz} + u_{s,1} u_{s,z} - \nu u_{s,1,zz} e_2 - \nu u_{s,1,xz} e_1\right)\\
&\quad - \frac{\nu(\ell - s(z)z)\hat{\eta}_{1,xx}}{\ell - e} \left(u_{s,z} + u_{s,1,z} e_1\right),\\
A_2\hat{\eta}_2 &= \frac{(\ell - s(z)z)\hat{\eta}_2}{\ell - e} u_{s,z},\\
A_3\hat{\eta}_1 &= \frac{1}{\ell - e} (s(z)\hat{\eta}_1 u_{s,1,x} + (\ell - s(z)z)\hat{\eta}_{1,x} u_{s,1,z}),\\
A_4\hat{\eta}_1 &= \nu \left(\frac{s(z)\hat{\eta}_1}{\ell - e} u_{s,2,z} - \hat{\eta}_{1,x} u_{s,1,z} \right).
\end{aligned}
\tag{3.3 .6}
$$

The linearized system around $(u_s, p_s, 0, 0)$ of (3.3 .4) is

$$
\begin{aligned}
&v_t - \operatorname{div} \sigma(v, p) + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s - A_1\eta_1 - A_2\eta_2 - \omega v = 0 \text{ in } Q^\infty,\\
&\operatorname{div} v = A_3\eta_1 \text{ in } Q^\infty, \quad v = \eta_2 n \text{ on } \Sigma_s^\infty, \quad v = 0 \text{ on } \Sigma_i^\infty,\\
&v \cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(v)n \cdot \tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(v, p)n = 0 \text{ on } \Sigma_n^\infty,\\
&\eta_{1,t} - \eta_2 - \omega \eta_1 = 0 \text{ on } \Sigma_s^\infty,\\
&\eta_{2,t} - \beta \Delta \eta_1 - \gamma \Delta \eta_2 + \alpha \Delta^2 \eta_1 - A_4\eta_1 - \omega \eta_2 = \gamma_s p + f \text{ on } \Sigma_s^\infty,\\
&\eta_1 = 0 \text{ on } (0, \infty) \times \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0, \infty) \times \partial\Gamma_s,\\
&v(0) = v^0 \text{ in } \Omega, \quad \eta_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s.
\end{aligned}
\tag{3.3 .7}
$$

3.4 Reformulation of the linearized system

3.4 .1 The Oseen and lifting operators

We consider the nonhomogenous Oseen system

$$\begin{aligned}
&\lambda_0 w - \operatorname{div}\sigma(w,\pi) + (u_s\cdot\nabla)w + (w\cdot\nabla)u_s = F \text{ in } \Omega,\\
&\operatorname{div} w = h \text{ in } \Omega, \quad w = gn \text{ on } \Gamma_s, \quad w = 0 \text{ on } \Gamma_i,\\
&w\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(w)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(w,\pi)n = 0 \text{ on } \Gamma_n.
\end{aligned} \tag{3.4 .1}$$

where $F\in\mathbf{L}^2(\Omega)$, $g\in H_0^{\frac{3}{2}}(\Gamma_s)$, $h\in H^1(\Omega)$ and $\lambda_0>0$ is such that the coercivity inequality

$$\lambda_0\int_\Omega |v|^2 + 2\nu\int_\Omega |\varepsilon(v)|^2 + \int_\Omega [(u_s\cdot\nabla)v + (v\cdot\nabla)u_s]\cdot v \geq \frac{\nu}{2}\|v\|_{\mathbf{V}^1_{\Gamma_0}(\Omega)}, \quad \forall\, v\in\mathbf{V}^1_{\Gamma_0}(\Omega), \tag{3.4 .2}$$

is satisfied.

Theorem 3.4 .1. *There exists $0<\delta_0<\frac{1}{2}$ such that, for all $(F,h,g)\in\mathbf{L}^2(\Omega)\times H^1(\Omega)\times H_0^{\frac{3}{2}}(\Gamma_d)$, equation (3.4 .1) admits a unique solution $(w,\pi)\in\mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ with the estimate*

$$\|w\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|\pi\|_{H^1_{\delta_0}(\Omega)} \leq C\Big(\|F\|_{\mathbf{L}^2(\Omega)} + \|h\|_{H^1(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Gamma_s)}\Big).$$

Proof. See Theorem 2.5 .3.

□

In that theorem, we have supposed that the assumption below is satisfied (see 2.5 .1).

Assumption 3.4 .1. *System (3.1 .3) admits a solution $(u_s,p_s)\in\mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$.*

Lemma 3.4 .1. *There exists $\varepsilon_0\in(0,1/2)$ such that $\mathbf{H}^2_{\delta_0}(\Omega)\subset\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega)$ and $H^1_{\delta_0}(\Omega)\subset H^{\frac{1}{2}+\varepsilon_0}(\Omega)$.*

Proof. It follows from ([43], Lemma 6.2.1).

□

Let us denote by P the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}^0_{n,\Gamma_0}(\Omega)$. The operator P is the Leray projector associated to our mixed boundary conditions. Moreover, for all v in $\mathbf{L}^2(\Omega)$, we have

$$Pv = v - \nabla q_1 - \nabla q_2,$$

where q_1 and q_2 are the solutions of the equations

$$\Delta q_1 = \operatorname{div} v \text{ in } \Omega, \quad q_1 = 0 \text{ on } \partial\Omega,$$

and

$$\Delta q_2 = 0, \quad \frac{\partial q_2}{\partial n} = (v-\nabla q_1)n \text{ on } \Gamma_0, \quad q_2 = 0 \text{ on } \Gamma_n.$$

Proposition 3.4 .2. *If v belongs to $\mathbf{H}^2_{\delta_0}(\Omega)$ then Pv and $(I-P)v$ belong to $\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)$. Moreover, we have*

$$(I-P)v = \nabla q,$$

where q is the solution of the equation

$$\Delta q = \operatorname{div} v \text{ in } \Omega, \quad \frac{\partial q}{\partial n} = v\cdot n \text{ on } \Gamma_0, \quad q = 0 \text{ on } \Gamma_n. \tag{3.4 .3}$$

Proof. Thanks to ([43], Theorem 6.5.4), ([43], Lemma 6.2.1) and ([36], Chapter 2), equation (3.4 .3) admits a unique solution $q \in H^{\frac{3}{2}+\varepsilon_0}(\Omega)$. Thus, we can easily show that $Pv = v - \nabla q \in H^{\frac{1}{2}+\varepsilon_0}(\Omega) \cap \mathbf{V}^0_{n,\Gamma_0}(\Omega)$ and $(I-P)v = \nabla q \in H^{\frac{1}{2}+\varepsilon_0}(\Omega)$.

□

We introduce the Oseen operator $(A, D(A))$ in $\mathbf{V}^0_{n,\Gamma_0}(\Omega)$ defined by

$$\begin{aligned} D(A) = \{ v \in \mathbf{V}^{\frac{3}{2}+\varepsilon_0}_{n,\Gamma_0}(\Omega) \mid \exists p \in H^{\frac{1}{2}+\varepsilon_0}(\Omega) \text{ s.t } \operatorname{div} \sigma(v,p) \in \mathbf{L}^2(\Omega),\ \varepsilon(v)n\cdot\tau = 0 \text{ on } \Gamma_e, \\ \sigma(v,p)n = 0 \text{ on } \Gamma_n \}, \\ \text{and} \\ Av = P\operatorname{div}\sigma(v,p) - P[(u_s\cdot\nabla)v + (v\cdot\nabla)u_s]. \end{aligned} \tag{3.4 .4}$$

Theorem 3.4 .3. *The operator $(A, D(A))$ is the generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma_0}(\Omega)$.*

Proof. The proof is similar to that of ([44], Theorem 2.8).

□

Proposition 3.4 .4. *The adjoint of $(A, D(A))$ in $\mathbf{V}^0_{n,\Gamma_0}(\Omega)$ is the operator $(A^*, D(A^*))$ defined by*

$$\begin{aligned} D(A^*) = \{ v \in \mathbf{V}^{\frac{3}{2}+\varepsilon_0}_{n,\Gamma_0}(\Omega) \mid \exists p \in H^{\frac{1}{2}+\varepsilon_0}(\Omega) \ s.t \operatorname{div} \sigma(v,p) \in \mathbf{L}^2(\Omega),\ \varepsilon(v)n\cdot\tau = 0 \ on\ \Gamma_e, \\ \sigma(v,p)n + u_s\cdot n v = 0 \ on\ \Gamma_n \}, \end{aligned}$$

and

$$A^*v = P\operatorname{div}\sigma(v,p) + P(u_s\cdot\nabla)v - P(\nabla u_s)^T v.$$

Proof. See ([44], Theorem 2.11).

□

Finally, we introduce the lifting operators $L \in \mathcal{L}(H_0^{\frac{3}{2}}(\Gamma_s)\times H^1(\Omega), \mathbf{H}^2_{\delta_0}(\Omega))$ and $L_p \in \mathcal{L}(H_0^{\frac{3}{2}}(\Gamma_s)\times H^1(\Omega), H^1_{\delta_0}(\Omega))$ defined by

$$L(g,h) = w \quad \text{and} \quad L_p(g,h) = \pi, \tag{3.4 .5}$$

where (w,π) is the solution of

$$\begin{aligned} &\lambda_0 w - \operatorname{div}\sigma(w,\pi) + (u_s\cdot\nabla)w + (w\cdot\nabla)u_s = 0 \text{ in } \Omega, \\ &\operatorname{div} w = h \text{ in } \Omega, \quad w = gn \text{ on } \Gamma_s, \quad w = 0 \text{ on } \Gamma_i, \\ &w\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(w)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(w,\pi)n = 0 \text{ on } \Gamma_n. \end{aligned}$$

3.4 .2 Expression of the pressure

The goal is to express the pressure π of system (3.4 .1) in terms of w, F, g and h. For that, we introduce the following operators.

- $N_s \in \mathcal{L}(L^2(\Gamma_s), H^1(\Omega))$ is defined by $N_s g = q$ where q is the solution of the equation

$$\begin{aligned} \Delta q = 0 \text{ in } \Omega, \quad \frac{\partial q}{\partial n} = g \text{ on } \Gamma_s, \\ \frac{\partial q}{\partial n} = 0 \text{ on } \Gamma_0\setminus\Gamma_s, \quad q = 0 \text{ on } \Gamma_n. \end{aligned} \tag{3.4 .6}$$

- $N_{\rm div} \in \mathcal{L}(L^2(\Omega), H^1(\Omega))$ is defined by $N_{\rm div}h = q$ where q is the solution of the equation

$$\begin{aligned} &\Delta q = h \text{ in } \Omega, \\ &\frac{\partial q}{\partial n} = 0 \text{ on } \Gamma_0, \quad q = 0 \text{ on } \Gamma_n. \end{aligned} \tag{3.4 .7}$$

- $N_p \in \mathcal{L}(\mathbf{L}^2(\Omega), H^1(\Omega))$ is defined by $N_pF = q_1 + q_2$ where q_1 and q_2 are the solution of the equations

$$\Delta q_1 = \operatorname{div} F \text{ in } \Omega, \quad q_1 = 0 \text{ on } \partial\Omega, \tag{3.4 .8}$$

and

$$\Delta q_2 = 0 \text{ in } \Omega, \quad \frac{\partial q_2}{\partial n} = (F - \nabla q_1)\cdot n \text{ on } \Gamma_0, \quad q_2 = 0 \text{ on } \Gamma_n. \tag{3.4 .9}$$

- $N_v \in \mathcal{L}(\mathbf{H}^{\frac{3}{2}}(\Omega), L^2(\Omega))$ is defined by $N_vw = q$ where q is the solution of the variational problem

Find $q \in L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega q\zeta = 2\nu\langle \varepsilon(w), \nabla^2\psi\rangle_{\mathbf{H}^{\frac{1}{2}-\varepsilon_0}(\Omega),\mathbf{H}^{-\frac{1}{2}+\varepsilon_0}(\Omega)} - 2\nu\int_{\Gamma_0}\varepsilon(w)n\cdot\nabla\psi \\ + \int_\Omega \Big[(u_s\cdot\nabla)w + (w\cdot\nabla)u_s\Big]\cdot\nabla\psi, \quad \forall\,\zeta\in L^2(\Omega), \end{aligned}$$

where $\psi \in H^{\frac{3}{2}+\varepsilon_0}(\Omega)$ is the solution of the equation

$$\Delta\psi = \zeta \text{ in } \Omega, \quad \frac{\partial\psi}{\partial n} = 0 \text{ on } \Gamma_0, \quad \psi = 0 \text{ on } \Gamma_n, \tag{3.4 .10}$$

(see Lemma 2.5 .2).

Remark 3.4 .1. *We have*

$$\nabla N_p = I - P.$$

Theorem 3.4 .5. *If $(w,\pi) \in \mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ is the solution of system (3.4 .1) then we have*

$$\pi = -\lambda_0 N_s g - \lambda_0 N_{\rm div}h + N_pF + N_vw.$$

Proof. The proof is similar to that of Theorem 1.4 .7.

□

Lemma 3.4 .2. *If $(w,\pi) \in \mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ is the solution of system (3.4 .1) then we have N_vw belongs to $H^{\frac{1}{2}+\varepsilon_0}(\Omega)$.*

Proof. It is similar to that of Lemma 1.4 .4.

□

3.4 .3 Definition of the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$

For the structure, we introduce the unbounded operator $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ in $H^2_0(\Gamma_s)$ defined by

$$D(A_{\alpha,\beta}) = H^4(\Gamma_s)\cap H^2_0(\Gamma_s) \quad \text{ and } A_{\alpha,\beta} = \beta\Delta - \alpha\Delta^2.$$

We equip the space $H^2_0(\Gamma_s)$ with the norm

$$(\eta,\xi)_{H^2_0(\Gamma_s)} = ((-A_{\alpha,\beta})^{\frac{1}{2}}\eta, (-A_{\alpha,\beta})^{\frac{1}{2}}\xi)_{L^2(\Gamma_s)} = \int_{\Gamma_s}(\beta\nabla\eta\cdot\nabla\xi + \alpha\Delta\eta\Delta\xi).$$

Lemma 3.4 .3. *The operator $I + \gamma_s N_s$ is an isomorphism in $L^2(\Gamma_s)$.*

Proof. The proof is similar to that of ([51], Lemma 3.2).

□

Proposition 3.4 .6. *We have*

$$\begin{aligned}
&A_1 \in \mathcal{L}(H_0^2(\Gamma_s), \mathbf{L}^2(\Omega)),\\
&A_2 \in \mathcal{L}(L^2(\Gamma_s), \mathbf{L}^2(\Omega)),\\
&A_3 \in \mathcal{L}(H_0^2(\Gamma_s), H^1(\Omega)),\\
&\textit{and } A_4 \in \mathcal{L}(H_0^2(\Gamma_s), L^2(\Gamma_s)).
\end{aligned}$$

Proof. It is similar to that of Proposition 1.5 .1.

□

We define the space

$$H = \mathbf{L}^2(\Omega) \times H_0^2(\Gamma_s) \times L^2(\Gamma_s),$$

equipped with the norm

$$\left(\left(\begin{pmatrix} v \\ \eta_1 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} \phi \\ \xi_1 \\ \xi_2 \end{pmatrix}\right)\right) = (v, \phi)_{\mathbf{L}^2(\Omega)} + (\eta_1, \xi_1)_{H_0^2(\Gamma_s)} + (\eta_2, \xi_2)_{L^2(\Gamma_s)}.$$

We also define the subspace

$$Z = \mathbf{V}^0_{n,\Gamma_0}(\Omega) \times H_0^2(\Gamma_s) \times L^2(\Gamma_s).$$

We introduce the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ in Z defined by

$$D(\mathcal{A}) := \{(Pv, \eta_1, \eta_2) \in \mathbf{V}^{\frac{1}{2}+\varepsilon_0}_{n,\Gamma_0}(\Omega)) \times \left(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\right) \times H_0^2(\Gamma_s) \mid Pv - PL(\eta_2, A_3\eta_1) \in D(A)\},$$

$$\mathcal{A} = M_s^{-1} \begin{pmatrix} A & PA_1 + (\lambda_0 I - A)PL(0, A_3) & PA_2 + (\lambda_0 I - A)PL(n\chi_{\Gamma_s}, 0) \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} + \gamma_s N_p A_1 + A_4 & \gamma\Delta + \gamma_s N_p A_2 \end{pmatrix} + M_s^{-1}\mathcal{A}_p, \tag{3.4 .11}$$

with

$$M_s = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \gamma_s N_{\mathrm{div}} A_3 & I + \gamma_s N_s \end{pmatrix}. \tag{3.4 .12}$$

The operator $\mathcal{A}_p$ is defined on $D(\mathcal{A})$ by

$$\mathcal{A}_p \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma_s N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\mathrm{div}} A_3 \eta_1) \end{pmatrix}, \quad \forall\, (Pv, \eta_1, \eta_2) \in D(\mathcal{A}).$$

and thanks to the lemma below, it is well-defined.

Remark 3.4 .2. *We notice that, in M_s, $\gamma_s N_s$ corresponds to the so-called added mass effect. Here, the added mass operator M_s is not symmetric.*

Lemma 3.4 .4. *If* (Pv, η_1, η_2) *belongs to* $D(\mathcal{A})$ *then* $N_v(Pv + \nabla N_s\eta_2 + \nabla N_{\mathrm{div}} A_3\eta_1)$ *belongs to* $H^{\frac{1}{2}+\varepsilon_0}(\Omega)$.

Proof. The proof is similar to that of Lemma 1.5 .4.

□

Proposition 3.4 .7. *The quadruplet* $(v, p, \eta_1, \eta_2) \in X_{\delta_0}$ *is a solution of system (3.3 .7) if and only if*

$$\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = (\mathcal{A} + \omega I)\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}\mathbf{f}, \quad \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix}(0) = \begin{pmatrix} Pv^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \\
&(I - P)v = \nabla N_s\eta_2 + \nabla N_{\mathrm{div}} A_3\eta_1, \\
&p = -N_s(\eta_{2,t} - \omega\eta_2) - N_{\mathrm{div}} A_3(\eta_{1,t} - \omega\eta_1) + N_p A_1\eta_1 + N_p A_2\eta_2 \\
&\quad + N_v(Pv + \nabla N_s\eta_2 + \nabla N_{\mathrm{div}} A_1\eta_1),
\end{aligned} \tag{3.4 .13}$$

with

$$\mathcal{B}\mathbf{f} = \sum_{i=1}^{n_c} f_i \begin{pmatrix} 0 \\ 0 \\ (I + \gamma_s N_s)^{-1} w_i \end{pmatrix}. \tag{3.4 .14}$$

Proof. It is similar to that of Proposition 1.5 .2.

□

3.5 Properties of the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$

Theorem 3.5 .1. *The unbounded operator* $(\mathcal{A}, D(\mathcal{A}))$ *is the generator of an analytic semigroup of class* $\mathcal{C}^0$ *on* Z.

Proof. The proof is similar to that of Theorem 1.6 .1.

□

Theorem 3.5 .2. *The unbounded operator* $(\mathcal{A}, D(\mathcal{A}))$ *has a compact resolvent.*

Proof. Let $\lambda > 0$ belong to the resolvent set of $\mathcal{A}$ and let (F_f, F_s^1, F_s^2) belong to Z. There exists $(Pv, \eta_1, \eta_2) \in Z$ such that

$$(\lambda M_s - M_s\mathcal{A})\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} F_f \\ F_s^1 \\ F_s^2 \end{pmatrix}.$$

Thanks to Proposition 3.6 .1, (v, p, η_1, η_2) with

$$\begin{aligned}
&v = Pv + \nabla N_s\eta_2 + \nabla N_{\mathrm{div}} A_3\eta_1, \\
&p = -\lambda N_s\eta_2 - \lambda N_{\mathrm{div}} A_3\eta_1 + N_p A_1\eta_1 + N_p A_2\eta_2 + N_v(Pv + \nabla N_s\eta_2 + \nabla N_{\mathrm{div}} A_3\eta_1),
\end{aligned}$$

is solution to system

$$\begin{aligned}
&\lambda v - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 = F_f \text{ in } \Omega,\\
&\operatorname{div} v = A_3\eta_1 \text{ in } \Omega, \quad v = \eta_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i,\\
&v\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n,\\
&\lambda\eta_1 - \eta_2 = F_s^1 \text{ on } \Gamma_s,\\
&\lambda\eta_2 - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 - A_4\eta_1 = \gamma_s p + F_c \text{ on } \Gamma_s,\\
&\eta_1 = 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial\Gamma_s.
\end{aligned} \tag{3.5 .1}$$

Using the same method as in the proof of Proposition 1.6 .7, we can show that, for λ large, system (3.5 .1) admits a unique solution $(v,p,\eta_1,\eta_2) \in H_{\delta_0}$ with the estimate

$$\|v\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|p\|_{H^1_{\delta_0}(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} \le C\|(F_f, F_s^1, F_s^2)\|_Z.$$

Form Proposition 3.4 .2, we deduce that

$$\|Pv\|_{\mathbf{V}^{\frac12+\varepsilon_0}_{n,\Gamma_0}(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} \le C\|(F_f, F_s^1, F_s^2)\|_Z.$$

Thus, the resolvent of $\mathcal{A}$ is compact in Z because the embedding $\mathbf{V}^{\frac12+\varepsilon_0}_{n,\Gamma_0}(\Omega)\times(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))\times H^2_0(\Gamma_s) \subset Z$ is compact.

□

We denote by $A_1^* \in \mathcal{L}(\mathbf{L}^2(\Omega), H^{-2}(\Gamma_s))$, $A_2^* \in \mathcal{L}(\mathbf{L}^2(\Omega), L^2(\Gamma_s))$, $A_3^* \in \mathcal{L}(\mathbf{L}^2(\Omega), H^{-2}(\Gamma_s))$ and $A_4^* \in \mathcal{L}(L^2(\Gamma_s), H^{-2}(\Gamma_s))$ the adjoints of A_1, A_2, A_3 and A_4 respectively.

Lemma 3.5 .1. *The adjoint of the operator M_s in Z is*

$$M_s^* = \begin{pmatrix} I & 0 & 0\\ 0 & I & -(-A_{\alpha,\beta})^{-1}A_3^*N_s\\ 0 & 0 & I+\gamma_s N_s \end{pmatrix}.$$

Proof. It is similar to that of Lemma 1.6 .2.

□

We introduce the lifting operator $D \in \mathcal{L}(H_0^{\frac32}(\Gamma_s), \mathbf{H}^2_{\delta_0}(\Omega))$ defined by

$$Dg = \phi, \tag{3.5 .2}$$

where (ϕ,ψ) is the solution of the system

$$\begin{aligned}
&\lambda_0\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0 \text{ in } \Omega,\\
&\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = gn \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_i,\\
&\phi\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(\phi)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \text{ on } \Gamma_n.
\end{aligned} \tag{3.5 .3}$$

We also introduce the operator $N_\phi \in \mathcal{L}(\mathbf{H}^2_{\delta_0}(\Omega), L^2(\Omega))$ defined by

$$N_\phi\phi = \psi, \tag{3.5 .4}$$

where ψ is the solution of the variational problem

$$\begin{aligned}
&\text{Find } \psi \in L^2(\Omega) \text{ such that}\\
&\int_\Omega \psi\zeta = 2\nu\langle\varepsilon(\phi), \nabla^2\tilde\psi\rangle_{\mathbf{H}^{\frac12-\varepsilon_0}(\Omega),\mathbf{H}^{-\frac12+\varepsilon_0}(\Omega)} - 2\nu\int_{\Gamma_0}\varepsilon(\phi)n\cdot\nabla\tilde\psi\\
&\qquad + \int_\Omega \Big[-(u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi\Big]\cdot\nabla\tilde\psi, \quad \forall\,\zeta \in L^2(\Omega),
\end{aligned}$$

with $\tilde{\psi} \in H^{\frac{3}{2}+\varepsilon_0}(\Omega)$ the solution of equation (3.4 .10). We set

$$\mathcal{A}^{\#} := \begin{pmatrix} A^* & 0 & (\lambda_0 I - A^*)PD \\ (-A_{\alpha,\beta})^{-1}(A_1^* - A_3^* N_\phi) & 0 & -I + (-A_{\alpha,\beta})^{-1}(A_4^* - A_3^* N_\phi \nabla N_s + A_1^* \nabla N_s) \\ \gamma_s N_\phi + A_2^* & -A_{\alpha,\beta} & \gamma\Delta + \gamma_s N_\phi \nabla N_s + A_2^* \nabla N_s \end{pmatrix}.$$

Proposition 3.5 .3. *The adjoint $(\mathcal{A}^*, D(\mathcal{A}^*))$ of $(\mathcal{A}, D(\mathcal{A}))$ in Z is defined by*

$$D(\mathcal{A}^*) = \Big\{(P\phi, \xi_1, \xi_2) \in \mathbf{V}^{\frac{1}{2}+\varepsilon_0}_{n,\Gamma_0}(\Omega)) \times (H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s) \mid P\phi - PD\xi_2 \in D(A^*)\Big\},$$

and

$$\mathcal{A}^* = \mathcal{A}^{\#}\, M_s^{-*}.$$

Proof. The proof is similar to that of Proposition 1.A.2. □

3.6 Eigenvalue problems

The goal of this section is to find relations between the eigenvalue problem associated to $\mathcal{A}$ and the eigenvalue problem associated to system (3.3 .7), i.e.

$$\lambda \in \mathbb{C}, \quad (Pv, \eta_1, \eta_2) \in D(\mathcal{A}), \quad \lambda \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A} \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix}, \tag{3.6 .1}$$

and

$$\begin{aligned} &\lambda \in \mathbb{C}, \quad (v, p, \eta_1, \eta_2) \in H_{\delta_0}, \\ &\lambda v - \operatorname{div}\sigma(v,p) + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s - A_1\eta_1 - A_2\eta_2 = 0 \text{ in } \Omega, \\ &\operatorname{div} v = A_3\eta_1 \text{ in } \Omega, \quad v = \eta_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i, \\ &v \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v,p)n = 0 \text{ on } \Gamma_n, \\ &\lambda\eta_1 - \eta_2 = 0 \text{ on } \Gamma_s, \\ &\lambda\eta_2 - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 - A_4\eta_1 - \gamma_s p = 0 \text{ on } \Gamma_s, \\ &\eta_1 = 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial\Gamma_s. \end{aligned} \tag{3.6 .2}$$

We also consider both the adjoint eigenvalue problems, i.e.

$$\lambda \in \mathbb{C}, \quad (P\phi, \xi_1, \xi_2) \in D(\mathcal{A}^*), \quad \lambda \begin{pmatrix} P\phi \\ \xi_1 \\ \xi_2 \end{pmatrix} = \mathcal{A}^* \begin{pmatrix} P\phi \\ \xi_1 \\ \xi_2 \end{pmatrix}, \tag{3.6 .3}$$

and

$$\begin{aligned} &\lambda \in \mathbb{C}, \quad (\phi, \psi, \xi_1, \xi_2) \in H_{\delta_0}, \\ &\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s \cdot \nabla)\phi + (\nabla u_s)^T\phi = 0 \text{ in } \Omega, \\ &\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = \xi_2 n \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_i, \\ &\phi \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(\phi)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi,\psi)n + u_s \cdot n\phi = 0 \text{ on } \Gamma_n, \\ &\lambda\xi_1 + \xi_2 - (-A_{\alpha,\beta})^{-1}A_4^*\xi_2 - (-A_{\alpha,\beta})^{-1}A_1^*\phi + (-A_{\alpha,\beta})^{-1}A_3^*\psi = 0 \text{ on } \Gamma_s, \\ &\lambda\xi_2 + \beta\Delta\xi_1 - \gamma\Delta\xi_2 - \alpha\Delta^2\xi_1 - A_2^*\phi - \gamma_s\psi = 0 \text{ on } \Gamma_s, \\ &\xi_1 = 0 \text{ on } \partial\Gamma_s, \quad \xi_{1,x} = 0 \text{ on } \partial\Gamma_s. \end{aligned} \tag{3.6 .4}$$

Definition 3.6 .1. *A triplet $(Pv_k, \eta_{1,k}, \eta_{2,k}) \in D(\mathcal{A})$ is a generalized eigenfunction for problem (3.6 .1) of order k associated to a solution $(\lambda, (Pv_0, \eta_{1,0}, \eta_{2,0}))$ of (3.6 .1) if $(Pv_k, \eta_{1,k}, \eta_{2,k})$ is obtained by solving the chain of equations*

$$(\lambda I - \mathcal{A}) \begin{pmatrix} Pv_j \\ \eta_{1,j} \\ \eta_{2,j} \end{pmatrix} = - \begin{pmatrix} Pv_{j-1} \\ \eta_{1,j-1} \\ \eta_{2,j-1} \end{pmatrix}, \qquad \text{for } 1 \leq j \leq k.$$

A quadruplet $(v_k, p_k, \eta_{1,k}, \eta_{2,k}) \in H_{\delta_0}$ is a generalized eigenfunction for problem (3.6 .2) of order k associated to a solution $(\lambda, (v_0, p_0, \eta_{1,0}, \eta_{2,0}))$ of (3.6 .2) if $(v_k, p_k \eta_{1,k}, \eta_{2,k})$ is obtained by solving the chain of systems

$$\begin{aligned}
&\lambda v_j - \operatorname{div} \sigma(v_j, p_j) + (u_s \cdot \nabla) v_j + (v_j \cdot \nabla) u_s - A_1 \eta_{1,j} - A_2 \eta_{2,j} = -v_{j-1} \text{ in } \Omega, \\
&\operatorname{div} v_j = A_3 \eta_{1,j} \text{ in } \Omega, \quad v_j = \eta_{2,j} n \text{ on } \Gamma_s, \quad v_j = 0 \text{ on } \Gamma_i, \\
&v_j \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v_j) n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v_j, p_j) n = 0 \text{ on } \Gamma_n, \\
&\lambda \eta_{1,j} - \eta_{2,j} = -\eta_{1,j-1} \text{ on } \Gamma_s, \\
&\lambda \eta_{2,j} - \beta \Delta \eta_{1,j} - \gamma \Delta \eta_{2,j} + \alpha \Delta^2 \eta_{1,j} - A_4 \eta_{1,j} - \gamma_s p_j = -\eta_{2,j-1} \text{ on } \Gamma_s, \\
&\eta_{1,j} = 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,j,x} = 0 \text{ on } \partial\Gamma_s,
\end{aligned}$$

for $1 \leq j \leq k$. We have similar statements for adjoint eigenvalue problems (3.6 .3) and (3.6 .4).

3.6 .1 Equivalence between direct eigenvalue problems

Lemma 3.6 .1. *Let (F, h, g) belong to $\mathbf{L}^2(\Omega) \times H^1(\Omega) \times H_0^{\frac{3}{2}}(\Gamma_s)$ and λ belong to $\mathbb{C}$. The pair $(v, p) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ is solution of system*

$$\begin{aligned}
&\lambda v - \operatorname{div} \sigma(v, p) + (u_s \cdot \nabla) v + (v \cdot \nabla) u_s = F \text{ in } \Omega, \\
&\operatorname{div} v = h \text{ in } \Omega, \quad v = gn \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i, \\
&v \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v) n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v, \pi) n = 0 \text{ on } \Gamma_n,
\end{aligned} \tag{3.6 .5}$$

if and only if

$$\begin{aligned}
&(\lambda I - A) P v + (A - \lambda_0 I) P L(g, h) = PF, \\
&(I - P) v = \nabla N_s g + \nabla N_{\mathrm{div}} h, \\
&\pi = -\lambda N_s g - \lambda N_{\mathrm{div}} h + N_p F + N_v v.
\end{aligned}$$

Proof. It is similar to that of Theorem 1.4 .8.

□

Proposition 3.6 .1. *Let (F_f, F_s^1, F_s^2) belong to $\mathbf{L}^2(\Omega) \times H_0^2(\Gamma_s) \times L^2(\Gamma_s)$ and λ belong to $\mathbb{C}$,. The quadruplet $(v, p, \eta_1, \eta_2) \in H_{\delta_0}$ is solution of the system*

$$\begin{aligned}
&\lambda v - \operatorname{div} \sigma(v, p) + (u_s \cdot \nabla) v + (v \cdot \nabla) u_s - A_1 \eta_1 - A_2 \eta_2 = F_f \text{ in } \Omega, \\
&\operatorname{div} v = A_3 \eta_1 \text{ in } \Omega, \quad v = \eta_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i, \\
&v \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v) n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v, p) n = 0 \text{ on } \Gamma_n, \\
&\lambda \eta_1 - \eta_2 = F_s^1 \text{ on } \Gamma_s, \\
&\lambda \eta_2 - \beta \Delta \eta_1 - \gamma \Delta \eta_2 + \alpha \Delta^2 \eta_1 - A_4 \eta_1 - \gamma_s p = F_s^2 \text{ on } \Gamma_s, \\
&\eta_1 = 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial\Gamma_s,
\end{aligned} \tag{3.6 .6}$$

if and only if

$$\lambda \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A} \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} + M_s^{-1} \begin{pmatrix} PF_f \\ F_s^1 \\ F_s^2 + \gamma_s N_p F_f \end{pmatrix},$$
$$(I-P)v = \nabla N_s \eta_2 + \nabla N_{\mathrm{div}} A_3 \eta_1,$$
$$p = -\lambda N_s \eta_2 - \lambda N_{\mathrm{div}} A_3 \eta_1 + N_p A_1 \eta_1 + N_p A_2 \eta_2 + N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\mathrm{div}} A_3 \eta_1) + N_p F_f. \tag{3.6 .7}$$

Proof. It follows from Lemma 3.6 .1.

□

Theorem 3.6 .2. *A couple* $(\lambda, (v, p, \eta_1, \eta_2)) \in \mathbb{C} \times H_{\delta_0}$ *is a solution of (3.6 .2) if and only if* $(\lambda, (Pv, \eta_1, \eta_2)) \in \mathbb{C} \times D(\mathcal{A})$ *is a solution of (3.6 .1) and*

$$(I-P)v = \nabla N_s \eta_2 + \nabla N_{\mathrm{div}} A_3 \eta_1,$$
$$p = -\lambda N_s \eta_2 - \lambda N_{\mathrm{div}} A_3 \eta_1 + N_p A_1 \eta_1 + N_p A_2 \eta_2 + N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\mathrm{div}} A_3 \eta_1).$$

Proof. It follows from Proposition 3.6 .1

□

Theorem 3.6 .3. *A quadruplet* $(v_k, p_k, \eta_{1,k}, \eta_{2,k}) \in H_{\delta_0}$ *is a generalized eigenfunction associated with a solution* $(\lambda, (v_0, p_0, \eta_{1,0}, \eta_{2,0}))$ *of (3.6 .2) if and only if* $(Pv_k, \eta_{1,k}, \eta_{2,k}) \in D(\mathcal{A})$ *is a generalized eigenfunction for (3.6 .1) associated with a solution* $(\lambda, (Pv_0, \eta_{1,0}, \eta_{2,0}))$ *and*

$$(I-P)v_k = \nabla N_s \eta_{2,k} + \nabla N_{\mathrm{div}} A_3 \eta_{1,k},$$
$$p_k = -\lambda N_s \eta_{2,k} - \lambda N_{\mathrm{div}} A_3 \eta_{1,k} + N_p A_1 \eta_{1,k} + N_p A_2 \eta_{2,k} + N_v(Pv_k + \nabla N_s \eta_{2,k} + \nabla N_{\mathrm{div}} A_3 \eta_{1,k})$$
$$-N_s \eta_{2,k-1} - N_{\mathrm{div}} A_3 \eta_{1,k-1}.$$

Proof. Thanks to Proposition 3.6 .1, $(v_k, p_k, \eta_{1,k}, \eta_{2,k})$ is a generalized eigenfunction of order k associated with a solution $(\lambda, (v_0, p_0, \eta_{1,0}, \eta_{2,0}))$ of (3.6 .2) if and only if

$$\lambda \begin{pmatrix} Pv_k \\ \eta_{1,k} \\ \eta_{2,k} \end{pmatrix} = \mathcal{A} \begin{pmatrix} Pv_k \\ \eta_{1,k} \\ \eta_{2,k} \end{pmatrix} - M_s^{-1} \begin{pmatrix} Pv_{k-1} \\ \eta_{1,k-1} \\ \eta_{2,k-1} + \gamma_s N_p v_{k-1} \end{pmatrix},$$
$$(I-P)v_k = \nabla N_s \eta_{2,k} + \nabla N_{\mathrm{div}} A_3 \eta_{1,k},$$
$$p_k = -\lambda N_s \eta_{2,k} - \lambda N_{\mathrm{div}} A_3 \eta_{1,k} + N_p A_1 \eta_{1,k} + N_p A_2 \eta_{2,k} + N_v(Pv_k + \nabla N_s \eta_{2,k} + \nabla N_{\mathrm{div}} A_3 \eta_{1,k}) - N_p v_{k-1},$$

where $(v_{k-1}, p_{k-1}, \eta_{1,k-1}, \eta_{2,k-1})$ is a generalized eigenfunction of order $k-1$. Since

$$N_p v_{k-1} = N_s \eta_{2,k-1} + N_{\mathrm{div}} A_3 \eta_{1,k-1},$$

it follows that

$$\lambda \begin{pmatrix} Pv_k \\ \eta_{1,k} \\ \eta_{2,k} \end{pmatrix} = \mathcal{A} \begin{pmatrix} Pv_k \\ \eta_{1,k} \\ \eta_{2,k} \end{pmatrix} - \begin{pmatrix} Pv_{k-1} \\ \eta_{1,k-1} \\ \eta_{2,k-1} \end{pmatrix},$$
$$(I-P)v_k = \nabla N_s \eta_{2,k} + \nabla N_{\mathrm{div}} A_3 \eta_{1,k},$$
$$p_k = -\lambda N_s \eta_{2,k} - \lambda N_{\mathrm{div}} A_3 \eta_{1,k} + N_p A_1 \eta_{1,k} + N_p A_2 \eta_{2,k} + N_v(Pv_k + \nabla N_s \eta_{2,k} + \nabla N_{\mathrm{div}} A_3 \eta_{1,k})$$
$$-N_s \eta_{2,k-1} - N_{\mathrm{div}} A_3 \eta_{1,k-1}.$$

We deduce, by induction, that $(v_k, p_k, \eta_{1,k}, \eta_{2,k})$ is a generalized eigenfunction of order k associated with a solution $(\lambda, (v_0, p_0, \eta_{1,0}, \eta_{2,0}))$ of (3.6 .2) if and only if $(Pv_k, \eta_{1,k}, \eta_{2,k})$ is a generalized eigenfunction of order k associated with a solution $(\lambda, (Pv_0, \eta_{1,0}, \eta_{2,0}))$ of (3.6 .1).

□

3.6 .2 Equivalence between adjoint eigenvalue problems

Lemma 3.6 .2. *Let (G, g) belong to $\mathbf{L}^2(\Omega) \times H_0^{\frac{3}{2}}(\Gamma_s)$ and λ belong to $\mathbb{C}$. A pair $(\phi, \psi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ is solution of the system*

$$\begin{aligned}
&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = G \ \textit{in } \Omega,\\
&\operatorname{div}\phi = 0 \ \textit{in } \Omega, \quad \phi = gn \ \textit{on } \Gamma_s, \quad \phi = 0 \ \textit{on } \Gamma_i,\\
&\phi\cdot n = 0 \ \textit{on } \Gamma_e, \quad \varepsilon(\phi)n\cdot\tau = 0 \ \textit{on } \Gamma_e, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \ \textit{on } \Gamma_n,
\end{aligned} \tag{3.6 .8}$$

if and only if

$$\begin{aligned}
&(\lambda I - A^*)P\phi + (A^* - \lambda_0 I)PDg + PG_f,\\
&(I-P)\phi = \nabla N_s g,\\
&\psi = -\lambda N_s g + N_\phi\phi + N_p G.
\end{aligned}$$

Proof. It is similar to that of Theorem 1.4 .8.

□

Proposition 3.6 .4. *Let (G_f, G_s^1, G_s^2) belongs to $\mathbf{L}^2(\Omega) \times H_0^2(\Gamma_s) \times L^2(\Gamma_s)$ and λ belong to $\mathbb{C}$. The quadruplet $(\phi, \psi, \xi_1, \xi_2) \in H_{\delta_0}$ is solution of the system*

$$\begin{aligned}
&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = G_f \ \textit{in } \Omega,\\
&\operatorname{div}\phi = 0 \ \textit{in } \Omega, \quad \phi = \xi_2 n \ \textit{on } \Gamma_s, \quad \phi = 0 \ \textit{on } \Gamma_i,\\
&\phi\cdot n = 0 \ \textit{on } \Gamma_e, \quad \varepsilon(\phi)n\cdot\tau = 0 \ \textit{on } \Gamma_e, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \ \textit{on } \Gamma_n,\\
&\lambda\xi_1 + \xi_2 - (-A_{\alpha,\beta})^{-1}A_4^*\xi_2 - (-A_{\alpha,\beta})^{-1}A_1^*\phi + (-A_{\alpha,\beta})^{-1}A_3^*\psi = G_s^1 \ \textit{on } \Gamma_s,\\
&\lambda\xi_2 + \beta\Delta\xi_1 - \gamma\Delta\xi_2 - \alpha\Delta^2\xi_1 - A_2^*\phi - \gamma_s\psi = G_s^2 \ \textit{on } \Gamma_s,\\
&\xi_1 = 0 \ \textit{on } \partial\Gamma_s, \quad \xi_{1,x} = 0 \ \textit{on } \partial\Gamma_s,
\end{aligned} \tag{3.6 .9}$$

if and only if

$$\begin{aligned}
&\lambda M_s^* \begin{pmatrix} P\phi \\ \xi_1 \\ \xi_2 \end{pmatrix} = \mathcal{A}^* M_s^* \begin{pmatrix} P\phi \\ \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} PG_f \\ G_s^1 - (-A_{\alpha,\beta})^{-1}A_3^*\gamma_s N_p G_f \\ G_s^2 + \gamma_s N_p G_f \end{pmatrix},\\
&(I-P)\phi = \nabla N_s\xi_2,\\
&\psi = -\lambda N_s\xi_2 + N_\phi(P\phi + \nabla N_s\xi_2) + N_p G_f.
\end{aligned} \tag{3.6 .10}$$

Proof. It follows from Lemma 3.6 .2.

□

Theorem 3.6 .5. *A couple $(\lambda, (\phi, \psi, \xi_1, \xi_2)) \in \mathbb{C} \times H_{\delta_0}$ is a solution of eigenvalue problem (3.6 .4) if and only if $(\lambda, M_s^*(P\phi, \xi_1, \xi_2)) \in \mathbb{C} \times D(\mathcal{A}^*)$ is a solution of eigenvalue problem (3.6 .3) and*

$$\begin{aligned}
&(I-P)\phi = \nabla N_s\xi_2,\\
&\psi = -\lambda N_s\xi_2 + N_\phi(P\phi + \nabla N_s\xi_2).
\end{aligned}$$

Proof. It follows from (3.6 .4).

□

Theorem 3.6 .6. *A quadruplet $(\phi_k, \psi_k, \xi_{1,k}, \xi_{2,k}) \in H_{\delta_0}$ is a generalized eigenfunction of order k associated with a solution $(\lambda, (\phi_0, \psi_0, \xi_{1,0}, \xi_{2,0}))$ of (3.6 .4) if and only if $(P\phi_k, \xi_{1,k}, \xi_{2,k}) \in D(\mathcal{A}^*)$ is a generalized eigenfunction of order k associated with a solution $(\lambda, (P\phi_0, \xi_{1,0}, \xi_{2,0}))$ of (3.6 .3) and*

$$(I-P)\phi_k = \nabla N_s \xi_{2,k},$$
$$\psi_k = -\lambda N_s \xi_{2,k} + N_\phi(P\phi_k + \nabla N_s \xi_{2,k}) + N_s \xi_{2,k-1}.$$

Proof. Thanks to Proposition 3.6 .4, $(\phi_k, \psi_k, \xi_{1,k}, \xi_{2,k})$ is a generalized eigenfunction of order k associated with a solution $(\lambda, (\phi_0, \psi_0, \xi_{1,0}, \xi_{2,0}))$ of (3.6 .4) if and only if

$$\lambda M_s^* \begin{pmatrix} P\phi_k \\ \xi_{1,k} \\ \xi_{2,k} \end{pmatrix} = \mathcal{A}^* M_s^* \begin{pmatrix} P\phi_k \\ \xi_{1,k} \\ \xi_{2,k} \end{pmatrix} - \begin{pmatrix} P\phi_{k-1} \\ \xi_{1,k-1} - (-A_{\alpha,\beta})^{-1} A_3^* \gamma_s N_p \phi_{k-1} \\ \xi_{2,k-1} + \gamma_s N_p \phi_{k-1} \end{pmatrix},$$
$$(I-P)\phi_k = \nabla N_s \xi_{2,k},$$
$$\psi_k = -\lambda N_s \xi_{2,k} + N_\phi(P\phi_k + \nabla N_s \xi_{2,k}) - N_p \phi_{2,k-1},$$

where $(\phi_{k-1}, \psi_{k-1}, \xi_{1,k-1}, \xi_{2,k-1})$ is a generalized eigenfunction of order $k-1$. Since

$$N_p \phi_{2,k-1} = N_s \xi_{2,k-1},$$

we have

$$\lambda M_s^* \begin{pmatrix} P\phi_k \\ \xi_{1,k} \\ \xi_{2,k} \end{pmatrix} = \mathcal{A}^* M_s^* \begin{pmatrix} P\phi_k \\ \xi_{1,k} \\ \xi_{2,k} \end{pmatrix} - M_s^* \begin{pmatrix} P\phi_{k-1} \\ \xi_{1,k-1} \\ \xi_{2,k-1} \end{pmatrix},$$
$$(I-P)\phi_k = \nabla N_s \xi_{2,k},$$
$$\psi_k = -\lambda N_s \xi_{2,k} + N_\phi(P\phi_k + \nabla N_s \xi_{2,k}) - N_s \xi_{2,k-1}.$$

We deduce, by induction, that $(\phi_k, \psi_k, \xi_{1,k}, \xi_{2,k})$ is a generalized eigenfunction of order k associated with a solution $(\lambda, (\phi_0, \psi_0, \xi_{1,0}, \xi_{2,0}))$ of (3.6 .4) if and only if $M_s^*(P\phi_k, \xi_{1,k}, \xi_{2,k})$ is a generalized eigenfunction of order k associated with a solution $(\lambda, M_s^*(P\phi_0, \xi_{1,0}, \xi_{2,0}))$ of (3.6 .3).

□

3.7 Stabilization of the linearized system

3.7 .1 Projected systems

Let $(\lambda_j)_{j\in\mathbb{N}^*}$ the eigenvalues of $\mathcal{A}$. We denote by $G_{\mathbb{R}}(\lambda_j)$ the real generalized eigenspace of $\mathcal{A}$ and $G_{\mathbb{R}}^*(\lambda_j)$ the real generalized eigenspace of $\mathcal{A}^*$ associated to the eigenvalue λ_j. We define the unstable subspaces

$$Z_u = \oplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j) \quad \text{and} \quad Z_u^* = \oplus_{j\in J_u} G_{\mathbb{R}}^*(\lambda_j),$$

where J_u is a finite subset of $\mathbb{N}^*$ such that

$$\mathrm{Re}\,\lambda_j \geq -\omega, \quad \forall\, j \in J_u.$$

Moreover, there exists two subspace Z_s and Z_s^*, invariant under $(e^{t\mathcal{A}})_{t\geq 0}$ and $(e^{t\mathcal{A}^*})_{t\geq 0}$, such that

$$Z = Z_u \oplus Z_s \quad \text{and} \quad Z = Z_u^* \oplus Z_s^*,$$

We have identified Z^* with Z. We denote by π_u be the projection of Z onto Z_u along Z_s and by π_s be the projection of Z onto Z_s along Z_u. We denote by d_u the dimension of the subspace Z_u. In the following proposition, we characterize π_u.

Proposition 3.7 .1. *There exist two families $(v_i, \eta_{1,i}, \eta_{2,i})_{1\le i\le d_u}$ and $(\phi_i, \xi_{1,i}, \xi_{2,i})_{1\le i\le d_u}$ such that:*

- *The family $(Pv_i, \eta_{1,i}, \eta_{2,i})_{1\le i\le d_u}$ is a basis of Z_u.*
- *The family $(M_s^*(P\phi_i, \xi_{1,i}, \xi_{2,i}))_{1\le i\le d_u}$ is a basis of Z_u^*.*
- *We have the bi-orthogonality conditions*

$$\left(\left(\begin{array}{c} v_i \\ \eta_{1,i} \\ \eta_{2,i} \end{array}\right), \left(\begin{array}{c} \phi_i \\ \xi_{1,i} \\ \xi_{2,i} \end{array}\right)\right) = \left(\left(\begin{array}{c} Pv_i \\ \eta_{1,i} \\ \eta_{2,i} \end{array}\right), M_s^*\left(\begin{array}{c} P\phi_i \\ \xi_{1,i} \\ \xi_{2,i} \end{array}\right)\right) = \delta_{i,j}, \quad \forall\, 1\le i,j\le d_u.$$

- *The projector π_u is characterized by*

$$\pi_u\left(\begin{array}{c} v \\ \eta_1 \\ \eta_2 \end{array}\right) = \sum_{i=1}^{d_u}\left(\left(\begin{array}{c} v \\ \eta_1 \\ \eta_2 \end{array}\right), M_s^*\left(\begin{array}{c} P\phi_i \\ \xi_{1,i} \\ \xi_{2,i} \end{array}\right)\right)\left(\begin{array}{c} Pv_i \\ \eta_{1,i} \\ \eta_{2,i} \end{array}\right), \quad \forall\,(v,\eta_1,\eta_2)^T\in Z.$$

- *The matrix Λ_u defined by*

$$\Lambda_u = [\Lambda_{u,i,j}]_{1\le i,j\le d_u} \quad \textit{and} \quad \Lambda_{u,i,j} = \left(\mathcal{A}\left(\begin{array}{c} Pv^i \\ \eta_{1,i} \\ \eta_{2,i} \end{array}\right), M_s^*\left(\begin{array}{c} P\phi_j \\ \xi_{1,j} \\ \xi_{2,j} \end{array}\right)\right),$$

  is constituted of real Jordan blocks.

Proof. Since $\mathcal{A}$ is the generator of an analytic semigroup with compact resolvent, there exist a basis $(\tilde v_i, \eta_{1,i}, \eta_{2,i})_{1\le i\le d_u}$ of Z_u constituted of real or imaginary parts of the generalized eigenfunctions of $\mathcal{A}$ and a basis $(M_s^*(\tilde\phi_i, \xi_{1,i}, \xi_{2,i}))_{1\le i\le d_u}$ of Z_u^* constituted of the real or imaginary parts of the generalized eigenfunctions of $\mathcal{A}$, such that

$$\left(\left(\begin{array}{c} \tilde v_i \\ \eta_{1,i} \\ \eta_{2,i} \end{array}\right), M_s^*\left(\begin{array}{c} \tilde\phi_i \\ \xi_{1,i} \\ \xi_{2,i} \end{array}\right)\right) = \delta_{i,j}, \quad \forall\, 1\le i,j\le d_u,$$

and

$$\pi_u\left(\begin{array}{c} v \\ \eta_1 \\ \eta_2 \end{array}\right) = \sum_{i=1}^{d_u}\left(\left(\begin{array}{c} v \\ \eta_1 \\ \eta_2 \end{array}\right), M_s^*\left(\begin{array}{c} \tilde\phi_i \\ \xi_{1,i} \\ \xi_{2,i} \end{array}\right)\right)\left(\begin{array}{c} \tilde v_i \\ \eta_{1,i} \\ \eta_{2,i} \end{array}\right), \quad \forall\,(v,\eta_1,\eta_2)^T\in Z,$$

(see ([22], Lemma 6.2)). We set

$$v_i = \tilde v_i + \nabla N_s\eta_{2,i} + \nabla N_{\mathrm{div}}A_3\eta_{1,i} \quad \text{and} \quad \phi_i = \tilde\phi_i + \nabla N_s\xi_{2,i}.$$

We notice that

$$Pv_i = \tilde v_i \quad \text{and} \quad P\phi_i = \tilde\phi_i.$$

Then, we have

$$\pi_u\left(\begin{array}{c} v \\ \eta_1 \\ \eta_2 \end{array}\right) = \sum_{i=1}^{d_u}\left(\left(\begin{array}{c} Pv \\ \eta_1 \\ \eta_2 \end{array}\right), M_s^*\left(\begin{array}{c} P\phi_i \\ \xi_{1,i} \\ \xi_{2,i} \end{array}\right)\right)\left(\begin{array}{c} Pv_i \\ \eta_{1,i} \\ \eta_{2,i} \end{array}\right), \quad \forall\,(v,\eta_1,\eta_2)^T\in Z.$$

We also have

$$\begin{aligned}
(v_i,\phi_i)_{\mathbf{L}^2(\Omega)} &= (\tilde v_i,\tilde\phi_i)_{\mathbf{V}^0_{n,\Gamma_0}(\Omega)} + (\nabla N_s\eta_{2,i} + \nabla N_{\rm div}A_3\eta_{1,i}, \nabla N_s\xi_{2,i})_{\mathbf{L}^2(\Omega)}\\
&= (\tilde v_i,\tilde\phi_i)_{\mathbf{V}^0_{n,\Gamma_0}(\Omega)} + (\eta_{2,i},\gamma_s N_s\xi_{2,i})_{L^2(\Gamma_s)} - (A_3\eta_{1,i}, N_s\xi_{2,i})_{L^2(\Omega)}\\
&= (\tilde v_i,\tilde\phi_i)_{\mathbf{V}^0_{n,\Gamma_0}(\Omega)} + (\eta_{2,i},\gamma_s N_s\xi_{2,i})_{L^2(\Gamma_s)} + (\eta_{1,i}, -(-A_{\alpha,\beta})^{-1}A_3^*N_s\xi_{2,i})_{H^2_0(\Gamma_s)}\\
&= (Pv_i,P\phi_i)_{\mathbf{V}^0_{n,\Gamma_0}(\Omega)} + (\eta_{2,i},\gamma_s N_s\xi_{2,i})_{L^2(\Gamma_s)} + (\eta_{1,i}, -(-A_{\alpha,\beta})^{-1}A_3^*N_s\xi_{2,i})_{H^2_0(\Gamma_s)}.
\end{aligned}$$

Hence

$$\left(\begin{pmatrix} v_i\\ \eta_{1,i}\\ \eta_{2,i}\end{pmatrix}, \begin{pmatrix} \phi_i\\ \xi_{1,i}\\ \xi_{2,i}\end{pmatrix}\right) = \left(\begin{pmatrix} Pv_i\\ \eta_{1,i}\\ \eta_{2,i}\end{pmatrix}, M_s^*\begin{pmatrix} P\phi_i\\ \xi_{1,i}\\ \xi_{2,i}\end{pmatrix}\right) = \delta_{i,j}, \quad \forall\, 1\le i,j\le d_u.$$

The last property comes from the construction of $(Pv_i,\eta_{1,i},\eta_{2,i})_{1\le i\le d_u}$ and $(M_s^*(P\phi_i,\xi_{1,i},\xi_{2,i}))_{1\le i\le d_u}$. □

Remark 3.7 .1. *For all $1\le i\le d_u$, from Theorem 3.6 .3, it follows that there exists $p_i\in H^1_{\delta_0}(\Omega)$ such that $(v_i,p_i,\eta_{1,i},\eta_{2,i})\in H_{\delta_0}$ is a real or imaginary part of a generalized eigenfunction of (3.6 .2). Similarly, from Theorem 3.6 .6, it follows that, there exists $\psi_i\in H^1_{\delta_0}(\Omega)$ such that $(\phi_i,\psi_i,\xi_{1,i},\xi_{2,i})\in H_{\delta_0}$ is a real or imaginary part of a generalized eigenfunction of (3.6 .4).*

The operator π_u can be extended to H, and we have

$$\pi_u(v,\eta_1,\eta_2) = \sum_{i=1}^{d_u}\left(\begin{pmatrix} Pv\\ \eta_1\\ \eta_2\end{pmatrix}, M_s^*\begin{pmatrix} P\phi_i\\ \xi_{1,i}\\ \xi_{2,i}\end{pmatrix}\right)\begin{pmatrix} Pv_i\\ \eta_{1,i}\\ \eta_{2,i}\end{pmatrix}, \quad \forall (v,\eta_1,\eta_2)\in H.$$

We introduce the matrix B_u defined by

$$B_u = [B_{u,i,j}]_{1\le i\le d_u, 1\le j\le n_c} \quad \text{and} \quad B_{u,i,j} = (w_i,\xi_{2,j})_{L^2(\Gamma_s)}. \tag{3.7 .1}$$

We set

$$\mathcal{A}_{s,\omega} = \pi_s(\mathcal{A}+\omega I) \quad \text{and} \quad \mathcal{B}_s = \pi_s\mathcal{B}.$$

We notice that

$$\|e^{t\mathcal{A}_{s,\omega}}\|_{\mathcal{L}(Z)} \le Ce^{-\varepsilon_s t} \quad \forall t>0,$$

where $0<\varepsilon_s<\mathrm{dist}(\mathrm{Re}\,\sigma(\mathcal{A}_{s,\omega}),0)$.

Proposition 3.7 .2. *The triplet (Pv,η_1,η_2) is the solution of the first equation of system (3.4 .13) if and only if*

$$\zeta_u := [((Pv,\eta_1,\eta_2), M_s^*(P\phi_i,\xi_{1,i},\xi_{2,i}))]_{1\le i\le d_u} \quad \textit{and} \quad (v_s,\eta_{1,s},\eta_{2,s}) := \pi_s(Pv,\eta_1,\eta_2),$$

satisfy

$$\begin{aligned}
&\frac{d\zeta_u}{dt} = (\Lambda_u+\omega I)\zeta_u + B_u\boldsymbol{f}, \quad \zeta_u(0)=\zeta_u^0,\\
&\frac{d}{dt}\begin{pmatrix} v_s\\ \eta_{1,s}\\ \eta_{2,s}\end{pmatrix} = \mathcal{A}_{s,\omega}\begin{pmatrix} v_s\\ \eta_{1,s}\\ \eta_{2,s}\end{pmatrix} + \mathcal{B}_s\boldsymbol{f}, \quad \begin{pmatrix} v_s\\ \eta_{1,s}\\ \eta_{2,s}\end{pmatrix}(0) = \pi_s\begin{pmatrix} v^0\\ \eta_1^0\\ \eta_2^0\end{pmatrix},
\end{aligned} \tag{3.7 .2}$$

where

$$\zeta_u^0 := [((Pv^0,\eta_1^0,\eta_2^0), M_s^*(P\phi_i,\xi_{1,i},\xi_{2,i}))]_{1\le i\le d_u}.$$

Thanks to that Proposition, the stabilization of the linearized system is reduced to the stabilization of the pair $(\Lambda_u+\omega I, B_u)$.

3.7 .2 Stabilizability of the pair $(\Lambda_u + \omega I, B_u)$

First, we introduce the unbounded operator $(A_s, D(A_s))$ in $H_0^2(\Gamma_s) \times L^2(\Gamma_s)$ defined by

$$D(A_s) = \Big(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\Big) \times H_0^2(\Gamma_s) \quad \text{and} \quad A_s = \begin{pmatrix} 0 & I \\ A_{\alpha,\beta} + A_4 & \gamma\Delta \end{pmatrix}.$$

Let λ be in $\mathbb{C}$. We introduce the unbounded operator $(L_\lambda, D(L_\lambda))$ defined on $L^2(\Gamma_s)$ by $D(L_\lambda) = H_0^2(\Gamma_s)$ and

$$L_\lambda = I + (-A_{\alpha,\beta})^{-1}(\lambda^2 - \gamma\lambda\Delta - A_4^*).$$

Lemma 3.7 .1. *If λ does not belong to $\sigma(A_s)$, then L_λ is an isomorphism from $H^4(\Gamma_s) \cap H_0^2(\Gamma_s)$ into $L^2(\Gamma_s)$ and we have*

$$(\lambda I - A_s^*)^{-1} = \begin{pmatrix} (-A_{\alpha,\beta})^{-1}(\lambda - \gamma\Delta)L_\lambda^{-1} & (-A_{\alpha,\beta})^{-1}(\lambda^2 - \gamma\lambda\Delta)L_\lambda^{-1}(-A_{\alpha,\beta})^{-1} - (-A_{\alpha,\beta})^{-1} \\ L_\lambda^{-1} & \lambda L_\lambda^{-1}(-A_{\alpha,\beta})^{-1} \end{pmatrix}.$$

We introduce the spaces $(E(\lambda_j))_{j \in J_u}$ defined by

$$E(\lambda_j) = \{(\phi, \psi, \xi_1, \xi_2) \in H_{\delta_0} \mid (\lambda_j, (\phi, \psi, \xi_1, \xi_2)) \text{ is solution of eigenvalue problem (3.6 .4)}\}. \tag{3.7 .3}$$

We choose the family $(w_i)_{1 \le i \le n_c}$ such that

$$\text{Vect}\{w_i | 1 \le i \le n_c\} = \text{Vect}\{\operatorname{Re}\xi_{2,j}^k, \operatorname{Im}\xi_{2,j}^k \mid j \in J_u, 1 \le k \le d_j\}, \tag{3.7 .4}$$

where $(\phi_j^k, \psi_j^k, \xi_{1,j}^k, \xi_{2,j}^k)_{1 \le i \le d_j}$ is a basis of $E(\lambda_j)$. The number d_j is the dimension of $E(\lambda_j)$. It also corresponds to the geometrical multiplicity of the eigenvalue λ_j of A.

Assumption 3.7 .1. *We assume that $-\omega \notin \sigma(\mathcal{A})$, $0 \notin \sigma(A)$ and $\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda \ge -\omega\} \cap \{\lambda \in \sigma(A_s) \mid \operatorname{Re}\lambda \ge -\omega\} = \emptyset$.*

We also assume the following property for the Oseen system (see 2.7 .2).

Assumption 3.7 .2. *Let λ be in $\mathbb{C}^*$ such that $\operatorname{Re}\lambda \ge -\omega$. Assume that (ϕ, ψ) is solution of the equation*

$$\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s \cdot \nabla)\phi + (\nabla u_s)^T\phi = 0 \ \textit{in}\ \Omega,$$
$$\operatorname{div}\phi = 0 \ \textit{in}\ \Omega, \quad \phi = 0 \ \textit{on}\ \Gamma_s, \quad \psi = 0 \ \textit{on}\ \Gamma_s.$$

If

$$\lambda(A_2^*\phi + \gamma_s\psi) = A_3^*\psi - A_1^*\phi, \tag{3.7 .5}$$

then $(\phi, \psi) = (0, 0)$.

Theorem 3.7 .3. *We assume that 3.4 .1, 3.7 .2 and 3.7 .1 are satisfied, and that $(w_i)_{1 \le i \le n_c}$ is given by (3.7 .4). Then, the pair $(\Lambda_u + \omega I_{\mathbb{R}^{n_c}}, B_u)$ is stabilizable.*

Proof. Thanks to ([10], Part III, Chapter 1, Proposition 3.3), the pair $(\Lambda_u + \omega I_{\mathbb{R}^{n_c}}, B_u)$ is stabilizable if and only if

$$\text{Ker}(\lambda I - \mathcal{A}^*) \cap \text{Ker}(\mathcal{B}^*) = \{0\} \text{ for all } \lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \ge -\omega.$$

Let $M_s^*(P\phi, \xi_1, \xi_2)$ belong to $\mathrm{Ker}(\lambda I - \mathcal{A}^*) \cap \mathrm{Ker}(\mathcal{B}^*)$ From Proposition 3.6 .4, it follows that there exists $\psi \in H^1_{\delta_0}(\Omega)$ such that

$$\begin{aligned}
&\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0 \text{ in } \Omega,\\
&\operatorname{div}\phi = 0 \text{ in } \Omega, \quad \phi = \xi_2 n \text{ on } \Gamma_s, \quad \phi = 0 \text{ on } \Gamma_i, \quad \phi\cdot n = 0 \text{ on } \Gamma_e,\\
&\varepsilon(\phi)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi,\psi)n + u_s\cdot n\phi = 0 \text{ on } \Gamma_n,\\
&\lambda\xi_1 + \xi_2 - (-A_{\alpha,\beta})^{-1}A_4^*\xi_2 - (-A_{\alpha,\beta})^{-1}A_1^*\phi + (-A_{\alpha,\beta})^{-1}A_3^*\psi = 0 \text{ on } \Gamma_s,\\
&\lambda\xi_2 + \beta\Delta\xi_1 - \gamma\Delta\xi_2 - \alpha\Delta^2\xi_1 - A_2^*\phi = \gamma_s\psi \text{ on } \Gamma_s,\\
&\xi_1 = 0 \text{ on } \partial\Gamma_s, \quad \xi_{1,x} = 0 \text{ on } \partial\Gamma_s.
\end{aligned} \tag{3.7 .6}$$

Since

$$\mathcal{B}^* M_s^* \begin{pmatrix} P\phi \\ \xi_1 \\ \xi_2 \end{pmatrix} = 0,$$

we have

$$\left(\int_{\Gamma_s} w_i\cdot\xi_2\right)_{1\le i\le n_c} = 0. \tag{3.7 .7}$$

We are going to prove that $(\phi,\psi,\xi_1,\xi_2) = (0,0,0,0)$. From equation (3.7 .7) and due to the construction of $(w_i)_{1\le i\le n_c}$, it follows that $\xi_2 = 0$. Next, we can distinguish two cases.

Case 1: $\lambda \notin \sigma(A)$. Then, we have $(\phi,\psi) = (0,0)$ and by using equation $(3.7\ .6)_4$ we also have $\xi_1 = 0$.

Case 2: $\lambda \in \sigma(A)$. In that case, λ does not belong to $\sigma(A_s)$. Therefore, we have

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = (\lambda I - A_s^*)^{-1} \begin{pmatrix} (-A_{\alpha,\beta})^{-1}(A_1^*\phi - A_3^*\psi) \\ A_2^*\phi + \gamma_s\psi \end{pmatrix},$$

which implies

$$\xi_2 = L_\lambda^{-1}(-A_{\alpha,\beta})^{-1}(A_1^*\phi - A_3^*\psi + \lambda A_2^*\phi + \lambda\gamma_s\psi).$$

Since $\xi_2 = 0$, we have

$$\lambda(A_2^*\phi + \gamma_s\psi) = A_3^*\psi - A_1^*\phi.$$

Thus, thanks to assumption (3.7 .2), we have $(\phi,\psi) = (0,0)$. Equation $(3.7\ .6)_4$ gives $\xi_1 = 0$.

□

3.7 .3 The feedback law

We are going to determine a feedback law of finite dimension able to stabilize system (3.3 .7). For that, it is necessary and sufficient to find a feedback law stabilizing system $(3.7\ .2)_1$. Since $(\Lambda_u + \omega I, B_u)$ is stabilizable, the following algebraic Riccati equation

$$\begin{aligned}
&P_u \in \mathbb{R}^{d_u}\times\mathbb{R}^{d_u}, \quad P_u = P_u^T > 0,\\
&P_u(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}})P_u - P_u B_u B_u^T P_u = 0,
\end{aligned}$$

admits a unique solution P_u. Moreover, the operator $K_u = [K_u^{i,j}]_{1\le i\le n_c, 1\le j\le d_u}$ defined by $K_u = -B_u^T P_u$ provides a stabilizing feedback law for $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, B_u)$. We introduce the operator $\mathcal{K}_p \in \mathcal{L}(Z, \mathbb{R}^{n_c})$ defined by

$$\mathcal{K}_p \begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = \left(\sum_{j=1}^{d_u} K_u^{i,j} \left(\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix}, M_s^* \begin{pmatrix} P\phi_j \\ \xi_{1,j} \\ \xi_{2,j} \end{pmatrix} \right) \right)_{1\le i\le n_c} .$$

Theorem 3.7 .4. *We assume that 3.4 .1, 3.7 .2 and 3.7 .1 are satisfied. Then, the operator $\mathcal{K}_p$ provides a stabilizing feedback for $(\mathcal{A}+\omega I, \mathcal{B})$. Moreover, the operator $\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}_p$, with domain $D(\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}_p) = D(\mathcal{A})$, is the generator of an exponentially stable analytic semigroup on Z.*

Proof. The proof is similar to that of Theorem 2.7 .7.

□

3.8 Stabilisation of the nonhomogeneous linearized system

The goal of this section is to stabilize the nonhomogeneous linearized system associated to (3.3 .4)and to give some estimates of the controlled solutions. First, we introduce the feedback law $\mathcal{K} \in \mathcal{L}(H, \mathbb{R}^{n_c})$ defined by

$$\mathcal{K} \begin{pmatrix} v \\ \eta_1 \\ \eta_2 \end{pmatrix} = \left(\sum_{j=1}^{d_u} K_u^{i,j} \left(\begin{pmatrix} v \\ \eta_1 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} \phi_j \\ \xi_{1,j} \\ \xi_{2,j} \end{pmatrix} \right) \right)_{1\le i\le n_c} . \tag{3.8 .1}$$

Remark 3.8 .1. *The feedback law $\mathcal{K}$ does not depend of the Leray projector P.*

We are going to prove that $\mathcal{K}$ is able to stabilize the nonhomogeneous linearized system. For that, we consider the corresponding closed-loop system

$$\begin{aligned}
& v_t - \operatorname{div}\sigma(v,p) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 - \omega v = F_f \text{ in } Q^\infty, \\
& \operatorname{div} v = A_3\eta_1 + \operatorname{div} F_{\mathrm{div}} \text{ in } Q^\infty, \quad v = \eta_2 n \text{ on } \Sigma_s^\infty, \quad v = \hat{h} \text{ on } \Sigma_i^\infty, \\
& v\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(v)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(v,p)n = 0 \text{ on } \Sigma_n^\infty, \\
& \eta_{1,t} - \eta_2 - \omega\eta_1 = 0 \text{ on } \Sigma_s^\infty, \\
& \eta_{2,t} - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 - \omega\eta_2 = \gamma_s p + F_s + \textstyle\sum_{i=1}^{n_c} [\mathcal{K}(v,\eta_1,\eta_1)^T]_i w_i \text{ on } \Sigma_s^\infty, \\
& \eta_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \\
& v(0) = v^0 \text{ in } \Omega, \quad \eta_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s,
\end{aligned} \tag{3.8 .2}$$

where $[\mathcal{K}(v,\eta_1,\eta_1)^T]_i$ is the i-*th* component of the vector $\mathcal{K}(v,\eta_1,\eta_1)^T \in \mathbb{R}^{n_c}$.

Theorem 3.8 .1. *We assume that 3.4 .1, 3.7 .2 and 3.7 .1 are satisfied. Let $(F_f, F_{\mathrm{div}}, F_s)$ belong to Y, $\hat{h}$ belong to $H_0^1(0,\infty; H_0^2(\Gamma_i))$ and $(v^0, \eta_1^0, \eta_2^0)$ belong to $H_{ic}(F_{\mathrm{div}}(0))$. We assume that F_{div} satisfies the following boundary conditions*

$$\begin{aligned}
& F_{\mathrm{div}} = 0 \ \text{on } \Sigma_s^\infty, \quad F_{\mathrm{div}} = 0 \ \text{on } \Sigma_i^\infty, \quad F_{\mathrm{div}}\cdot n = 0 \ \text{on } \Sigma_e^\infty, \\
& \varepsilon(F_{\mathrm{div}})n\cdot\tau = 0 \ \text{on } \Sigma_e^\infty, \quad \varepsilon(F_{\mathrm{div}})n = 0 \ \text{on } \Sigma_n^\infty.
\end{aligned} \tag{3.8 .3}$$

The solution (v,p,η_1,η_2) of system (3.8 .2) satisfies the estimate

$$\|(v,p,\eta_1,\eta_2)\|_{X_{\delta_0}} \le CR(v^0, \eta_1^0, \eta_2^0, h, F_f, F_{\mathrm{div}}, F_s),$$

where

$$R(v^0,\eta_1^0,\eta_2^0,h,F_f,F_{\rm div},F_s) := \|(v^0,\eta_1^0,\eta_2^0)\|_{ic} + \|\hat h\|_{H^1_0(0,\infty;H^2_0(\Gamma_i))} + \|(F_f,F_{\rm div},F_s)\|_Y\,.$$

Proof. First, we make the change of unknowns

$$\hat v = v - F_{\rm div} - L(\hat h\chi_{\Gamma_i},0) \quad \text{and} \quad \hat p = p - L_p(\hat h\chi_{\Gamma_i},0). \tag{3.8 .4}$$

Thus, the quadruplet $(\hat v,\hat p,\eta_1,\eta_2)$ is solution of the system

$$\begin{aligned}
&\hat v_t - \operatorname{div}\sigma(\hat v,\hat p) + (u_s\cdot\nabla)\hat v + (\hat v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 - \omega\hat v = \widetilde F_f \text{ in } Q^\infty,\\
&\operatorname{div}\hat v = A_3\eta_1 \text{ in } Q^\infty, \quad \hat v = \eta_2 n \text{ on } \Sigma_s^\infty, \quad \hat v = 0 \text{ on } \Sigma_i^\infty,\\
&\hat v\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\hat v)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\hat v,\hat p)n = 0 \text{ on } \Sigma_n^\infty,\\
&\eta_{1,t} - \eta_2 - \omega\eta_1 = 0 \text{ on } \Sigma_s^\infty,\\
&\eta_{2,t} - \beta\Delta\eta_1 - \gamma\Delta\eta_2 + \alpha\Delta^2\eta_1 - \omega\eta_2 = \gamma_s p + \widetilde F_s + \textstyle\sum_{i=1}^{n_c}[\mathcal K(\hat v,\eta_1,\eta_1)^T]_i w_i \text{ on } \Sigma_s^\infty,\\
&\eta_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&\hat v(0) = \hat v^0 = v^0 - w(0) \text{ in } \Omega, \quad \eta_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s,
\end{aligned} \tag{3.8 .5}$$

where

$$\widetilde F_f = F_f - w_t + 2\nu\operatorname{div}\varepsilon(w) - (u_s\cdot\nabla)w - (w\cdot\nabla)u_s + \omega w + \lambda_0 L(\hat h\chi_{\Gamma_i},0) + L(\hat h_t\chi_{\Gamma_i},0),$$

and

$$\widetilde F_s = F_s + \sum_{i=1}^{n_c}[\mathcal K(w + L(\hat h\chi_{\Gamma_i},0),0,0)^T]_i w_i + \gamma_s L_p(\hat h\chi_{\Gamma_i},0).$$

As in Proposition 3.4 .7, we prove that $(P\hat v,\eta_1,\eta_2)$ is solution to the system

$$\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} P\hat v\\ \eta_1\\ \eta_2\end{pmatrix} = (\mathcal A + \omega I)\begin{pmatrix} P\hat v\\ \eta_1\\ \eta_2\end{pmatrix} + \mathcal B\mathcal K\begin{pmatrix} \hat v\\ \eta_1\\ \eta_2\end{pmatrix} + \begin{pmatrix} P\widetilde F_f\\ 0\\ \widetilde F_s + \gamma_s N\widetilde F_f\end{pmatrix},\\
&\begin{pmatrix} P\hat v\\ \eta_1\\ \eta_2\end{pmatrix}(0) = \begin{pmatrix} P\hat v(0)\\ \eta_1^0\\ \eta_2^0\end{pmatrix},
\end{aligned} \tag{3.8 .6}$$

and

$$(I-P)\hat v = \nabla N_s\eta_2 + \nabla N_{\rm div}A_3\eta_1. \tag{3.8 .7}$$

By integration by parts, we get

$$\begin{aligned}
(\hat v,\phi_i)_{\mathbf L^2(\Omega)} &= (P\hat v,P\phi_i)_{\mathbf V^0_{n,\Gamma_0}(\Omega)} + (\nabla N_s\eta_2 + \nabla N_{\rm div}A_3\eta_1,\nabla N_s\xi_{2,i})_{\mathbf L^2(\Omega)}\\
&= (P\hat v,P\phi_i)_{\mathbf V^0_{n,\Gamma_0}(\Omega)} + (\eta_2,\gamma_s N_s\xi_{2,i})_{L^2(\Gamma_s)} - (A_3\eta_1,N_s\xi_{2,i})_{L^2(\Omega)}\\
&= (P\hat v,P\phi_i)_{\mathbf V^0_{n,\Gamma_0}(\Omega)} + (\eta_2,\gamma_s N_s\xi_{2,i})_{L^2(\Gamma_s)} + (\eta_1,-(-A_{\alpha,\beta})^{-1}A_3^*N_s\xi_{2,i})_{H^2_0(\Gamma_s)}.
\end{aligned}$$

Thus, we have

$$\left(\begin{pmatrix} \hat v\\ \eta_1\\ \eta_2\end{pmatrix},\begin{pmatrix} \phi_j\\ \xi_{1,j}\\ \xi_{2,j}\end{pmatrix}\right) = \left(\begin{pmatrix} P\hat v\\ \eta_1\\ \eta_2\end{pmatrix}, M_s^*\begin{pmatrix} P\phi_j\\ \xi_{1,j}\\ \xi_{2,j}\end{pmatrix}\right).$$

Hence

$$\mathcal{K}\begin{pmatrix}\hat{v}\\ \eta_1\\ \eta_2\end{pmatrix} = \mathcal{K}_p\begin{pmatrix}P\hat{v}\\ \eta_1\\ \eta_2\end{pmatrix}.$$

From that identity, it follows that $(P\hat{v}, \eta_1, \eta_2)$ is a solution of the equation

$$\frac{d}{dt}\begin{pmatrix}P\hat{v}\\ \eta_1\\ \eta_2\end{pmatrix} = (\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}_p)\begin{pmatrix}P\hat{v}\\ \eta_1\\ \eta_2\end{pmatrix} + \begin{pmatrix}P\widetilde{F}_f\\ 0\\ \widetilde{F}_s+\gamma_s N\widetilde{F}_f\end{pmatrix},$$
$$\begin{pmatrix}P\hat{v}\\ \eta_1\\ \eta_2\end{pmatrix}(0) = \begin{pmatrix}P\hat{v}(0)\\ \eta_1^0\\ \eta_2^0\end{pmatrix},$$

where the unbounded operator $(\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}_p, D(\mathcal{A}+\omega I+\mathcal{B}\mathcal{K}_p))$ is the generator of an exponentially stable analytic semigroup on Z (see Theorem 3.7 .4). The sequel of the proof is similar to that of Theorem 1.6 .10.

□

3.9 Stabilization of the nonlinear system

The goal is to prove Theorem 3.3 .2. For that, we are going to show that the feedback law $\mathcal{K}$, stabilizing system (3.3 .7), is also able to locally stabilize nonlinear system (3.3 .4). We consider the nonlinear closed-loop system

$$\begin{aligned}
&\hat{u}_t - \operatorname{div}\sigma(\hat{u},\hat{p}) + (u_s\cdot\nabla)\hat{u} + (\hat{u}\cdot\nabla)u_s - A_1\hat{\eta}_1 - A_2\hat{\eta}_2 - \omega\hat{u} = \mathcal{F}_f[\hat{u},\hat{p},\hat{\eta}_1,\hat{\eta}_2] \text{ in } Q^\infty,\\
&\operatorname{div}\hat{u} = A_3\hat{\eta}_1 + \operatorname{div}\mathcal{F}_{\mathrm{div}}[\hat{u},\hat{\eta}_1] \text{ in } Q^\infty, \quad \hat{u} = \hat{\eta}_2 n \text{ on } \Sigma_s^\infty, \quad \hat{u} = \hat{h} \text{ on } \Sigma_i^\infty,\\
&\hat{u}\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\hat{u})n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\hat{u},\hat{p})n = 0 \text{ on } \Sigma_n^\infty,\\
&\hat{\eta}_{1,t} - \hat{\eta}_2 - \omega\hat{\eta}_1 = 0 \text{ on } \Sigma_s^\infty,\\
&\hat{\eta}_{2,t} - \beta\Delta\hat{\eta}_1 - \gamma\Delta\hat{\eta}_2 + \alpha\Delta^2\hat{\eta}_1 - A_4\hat{\eta}_1 - \omega\hat{\eta}_2 = \gamma_s\hat{p} + \mathcal{F}_s[\hat{u},\hat{\eta}_1] + \textstyle\sum_{i=1}^{n_c}[\mathcal{K}(\hat{u},\hat{\eta}_1,\hat{\eta}_1)^T]_i w_i \text{ on } \Sigma_s^\infty,\\
&\hat{\eta}_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \hat{\eta}_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&\hat{u}(0) = \hat{u}^0 \text{ in } \Omega, \quad \hat{\eta}_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \hat{\eta}_2(0) = \eta_2^0 \text{ on } \Gamma_s.
\end{aligned} \tag{3.9 .1}$$

Thus, to prove Theorem 3.3 .2, it is sufficient to prove that the system above is locally stable.

Theorem 3.9 .1. *Let $\eta_{\max}$ be in $(0,e)$ and ω be positive. There exists $r>0$ such that for all $(\hat{u}^0,\hat{\eta}_1^0,\hat{\eta}_2^0)\in H_{ic}(\mathcal{F}_{\mathrm{div}}[\hat{u}^0,\hat{\eta}_1^0])$ and $\hat{h}\in H_0^1(0,\infty;H_0^2(\Gamma_i))$ satisfying the estimate*

$$\left\|(\hat{u}^0,\hat{\eta}_1^0,\hat{\eta}_2^0)\right\|_{ic} + \|\hat{h}\|_{H_0^1(0,\infty;H_0^2(\Gamma_i))} \le r,$$

there exists a solution $(\hat{u},\hat{p},\eta_1,\eta_2)\in X_{\delta_0}$ to system (3.9 .1) with the estimate

$$\|e^{-\omega t}\hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)} \le \eta_{\max} \quad \text{and} \quad \|(\hat{u}(t),\hat{\eta}(t),\hat{\eta}_t(t))\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon}(\Omega)\times H^3(\Gamma_s)\times H^1(\Gamma_s)} \le C, \quad \forall t>0,$$

where $\varepsilon>0$ and $C>0$.

Proof. The proof is similar to that of Theorem 1.8 .2.

□

3.A Appendix A: Nonlinear terms

The nonlinear terms $\mathcal{F}_f$ and $\mathcal{F}_s$ in system (3.3 .4) are defined by

$$\begin{aligned}
\mathcal{F}_f[\hat{u},\hat{p},\hat{\eta}_1,\hat{\eta}_2] = {}& \frac{(\ell - s(z)z)\hat{\eta}_2\hat{u}_z}{\ell - e} - \frac{\nu(\ell - s(z)z)\hat{\eta}_{1,xx}\hat{u}_z}{\ell - e} - \frac{2\nu(\ell - s(z)z)\hat{\eta}_{1,x}\hat{u}_{xz}}{\ell - e} \\
&+ \frac{2\nu s(z)\hat{\eta}_1\hat{u}_{zz}}{\ell - e} - \nu\nabla \mathrm{div}\,\mathcal{F}_{\mathrm{div}}[\hat{u},\hat{\eta}_1] + \frac{(\ell - s(z)z)\hat{\eta}_{1,x}\hat{p}_z e_1}{\ell - e} - \frac{s(z)\hat{\eta}_1\hat{p}_z e_2}{\ell - e} \\
&+ \frac{(\ell - s(z)z)e^{-\omega t}\hat{\eta}_{1,x}\hat{u}_1\hat{u}_z}{\ell - e} + \frac{(\ell - s(z)z)\hat{\eta}_{1,x}\hat{u}_1 u_{s,z}}{\ell - e} + \frac{(\ell - s(z)z)\hat{\eta}_{1,x}u_{s,1}\hat{u}_z}{\ell - e} \\
&- \frac{e^{-\omega t}s(z)\hat{\eta}_1\hat{u}_2\hat{u}_z}{\ell - e} - \frac{s(z)\hat{\eta}_1\hat{u}_2 u_{s,z}}{\ell - e} - \frac{s(z)\hat{\eta}_1 u_{s,2}\hat{u}_z}{\ell - e} + \frac{(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_2(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} \\
&+ \frac{\nu(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,xx}(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} + \frac{2\nu(\ell - s(z)z)\hat{\eta}_{1,x}^2(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)^2} \\
&- \frac{2\nu(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{u}_{xz} + u_{s,xz})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} + \frac{\nu((\ell - s(z)z)^2\hat{\eta}_{1,x}^2 - \hat{\eta}_1^2)(e^{-\omega t}\hat{u}_{zz} + u_{s,zz})}{(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)^2} \\
&+ \frac{2\nu(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)\hat{\eta}_1^2(e^{-\omega t}\hat{u}_{zz} + u_{s,zz})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)^2} + \frac{(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{p}_z + p_{s,z})e_1}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} \\
&- \frac{\hat{\eta}_1^2(e^{-\omega t}\hat{p}_z + p_{s,z})e_2}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} + \frac{(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{u}_1 + u_{s,1})(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} \\
&- \frac{\hat{\eta}_1^2(e^{-\omega t}\hat{u}_2 + u_{s,2})(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_s[\hat{u},\hat{\eta}_1] = {}& \nu\left(\hat{\eta}_{1,x}\hat{u}_{1,z} + \hat{\eta}_{1,x}\hat{u}_{2,x} + \frac{2s(z)\hat{\eta}_1\hat{u}_{2,z}}{\ell - e}\right) - \frac{\nu\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{u}_{1,z} + u_{s,1,z})}{\ell - e - s(z)e^{-\omega t}\hat{\eta}_1} \\
&- \frac{2\nu\hat{\eta}_1^2(e^{-\omega t}\hat{u}_{2,z} + u_{s,2,z})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} - \frac{\nu(\ell - e)\eta_{1,x}^2(e^{-\omega t}\hat{u}_{2,z} + u_{s,2,z})}{\ell - e - s(z)e^{-\omega t}\hat{\eta}_1}.
\end{aligned}$$

Chapter 4

Feedback stabilization of a 2D fluid-structure model by beam deformation: semi-discrete approximation and numerical results

4.1 Introduction

We study the stabilization of a semi-discrete approximation of the fluid-structure interaction model introduced in Chapter 3. The model describes the evolution of an incompressible fluid flow around an elastic structure. The equations of the model are

$$\begin{aligned}
&u_t - \operatorname{div}\sigma(u,p) + (u\cdot\nabla)u = 0 \text{ in } Q_\eta^\infty,\\
&\operatorname{div} u = 0 \text{ in } Q_\eta^\infty, \quad u = \eta_t n \text{ on } \Sigma_\eta^\infty, \quad u = g_s + h \text{ on } \Sigma_i^\infty,\\
&u\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(u)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(u,p)n = 0 \text{ on } \Sigma_n^\infty,\\
&\eta_{tt} - \beta\Delta\eta - \gamma\Delta\eta_t + \alpha\Delta^2\eta = -\sigma(u,p)|_{\Gamma_{\eta(t)}}\, n_{\eta(t)}\cdot n + f_s + f \text{ on } \Sigma_s^\infty,\\
&\eta = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \eta_x = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&u(0) = u^0 \text{ on } \Omega_{\eta_1^0}, \quad \eta(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \eta_t(0) = \eta_2^0 \text{ on } \Gamma_s,
\end{aligned} \tag{4.1 .1}$$

where u and p are, respectively, the velocity and the pressure of the fluid, η is the displacement of the structure (see chapter 3 for more details). We have already studied the stabilization of the continuous model by controls of finite dimension in chapter 3. Here, we look for controls, in feedback form, able to stabilize discrete approximations of that coupled system.
Thanks to the change of variable $\mathcal{T}_\eta$ defined in (3.3 .1), we rewrite the coupled system in a fixed domain

$$\begin{aligned}
&\tilde u_t - \operatorname{div}\sigma(\tilde u,\tilde p) + (\tilde u\cdot\nabla)\tilde u = F_f[\tilde u,\tilde p,\tilde\eta_1,\tilde\eta_2] \text{ in } Q^\infty,\\
&\operatorname{div}\tilde u = \operatorname{div} w[\tilde u,\tilde\eta_1] \text{ in } Q^\infty, \quad \tilde u = \tilde\eta_2 n \text{ on } \Sigma_s^\infty, \quad \tilde u = u_i \text{ on } \Sigma_i^\infty,\\
&\tilde u\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\tilde u)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\tilde u,\tilde p)n = 0 \text{ on } \Sigma_n^\infty,\\
&\tilde\eta_{1,t} - \tilde\eta_2 = 0 \text{ on } \Sigma_s^\infty,\\
&\tilde\eta_{2,t} - \beta\Delta\tilde\eta_1 - \gamma\Delta\tilde\eta_2 + \alpha\Delta^2\tilde\eta_1 = \gamma_s\tilde p + f_s + F_s[\tilde u,\tilde\eta_1] + f \text{ on } \Sigma_s^\infty,\\
&\tilde\eta_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \tilde\eta_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&\tilde u(0) = \tilde u^0 \text{ in } \Omega, \quad \tilde\eta_1(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \tilde\eta_2(0) = \eta_2^0 \text{ on } \Gamma_s,
\end{aligned} \tag{4.1 .2}$$

where $(\tilde u,\tilde p)$ is the image of (u,p) by the change of variables, $\tilde u^0$ is the image of u^0 and $(\tilde\eta_1,\tilde\eta_2)=(\eta,\eta_t)$. The geometrical nonlinear terms F_f, $F_{\rm div}$ and F_s are defined in Appendix 4.A. The control function f is of finite dimension n_c

$$f(t,x)=\sum_{i=1}^{n_c} f_i(t)w_i(x),$$

where the family $(w_i)_{1\le i\le n_c}$ is defined in (3.7 .4). The linearization of system (4.1 .2) around the unstable stationary solution $(u_s,p_s,0,0)$ is

$$\begin{aligned}
&v_t-\operatorname{div}\sigma(v,p)+(u_s\cdot\nabla)v+(v\cdot\nabla)u_s-A_1\eta_1-A_2\eta_2=0 \text{ in } Q^\infty,\\
&\operatorname{div} v=A_3\eta_1 \text{ in } Q^\infty,\quad v=\eta_2 n \text{ on } \Sigma_s^\infty,\quad v=0 \text{ on } \Sigma_i^\infty,\\
&v\cdot n=0 \text{ on } \Sigma_e^\infty,\quad \varepsilon(v)n\cdot\tau=0 \text{ on } \Sigma_e^\infty,\quad \sigma(v,p)n=0 \text{ on } \Sigma_n^\infty,\\
&\eta_{1,t}-\eta_2=0 \text{ on } \Sigma_s^\infty,\\
&\eta_{2,t}-\beta\Delta\eta_1-\gamma\Delta\eta_2+\alpha\Delta^2\eta_1-A_4\eta_1=\gamma_s p+f \text{ on } \Sigma_s^\infty,\\
&\eta_1=0 \text{ on } (0,\infty)\times\partial\Gamma_s,\quad \eta_{1,x}=0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&v(0)=v^0 \text{ in } \Omega,\quad \eta_1(0)=\eta_1^0 \text{ on } \Gamma_s,\quad \eta_2(0)=\eta_2^0 \text{ on } \Gamma_s,
\end{aligned}\tag{4.1 .3}$$

where γ_s is the trace operator on Γ_s. The linear differential operators A_1, A_2, A_3 and A_4 are defined in (3.3 .6).
The goal is to find feedback controls able to locally stabilize approximations of system (4.1 .2) around the unstable stationary solution $(u_s,p_s,0,0)$ with any exponentially decay rate $-\omega<0$. We are going to construct a linear feedback law, which is easy to compute, able to locally stabilize the nonlinear system. To achieve this goal we are going to follow a strategy similar to that used for the continuous model. It can be summarized in the following steps.

Step 1: *Semi-discrete approximation of the linearized system (4.1 .3) and the nonlinear system (4.1 .2)*

We use a finite element method to discretize systems (4.1 .3) and (4.1 .2). The semi-discrete approximation of the linearized system (4.1 .3) is written under the form

$$\begin{aligned}
&M_L\frac{d}{dt}\begin{bmatrix}\boldsymbol v\\ \boldsymbol\eta_1\\ \boldsymbol\eta_2\\ \boldsymbol\theta\end{bmatrix}=A_L\begin{bmatrix}\boldsymbol v\\ \boldsymbol\eta_1\\ \boldsymbol\eta_2\\ \boldsymbol\theta\end{bmatrix}+\begin{bmatrix}0\\0\\M_{\eta\eta}\mathbf W\boldsymbol f\\0\end{bmatrix},\\
&\begin{bmatrix}\boldsymbol v\\ \boldsymbol\eta_1\\ \boldsymbol\eta_2\end{bmatrix}(0)=\begin{bmatrix}\boldsymbol v^0\\ \boldsymbol\eta_1^0\\ \boldsymbol\eta_2^0\end{bmatrix},
\end{aligned}\tag{4.1 .4}$$

or, equivalently

$$\begin{aligned}
&M\frac{d}{dt}\begin{bmatrix}\boldsymbol v\\ \boldsymbol\eta_1\\ \boldsymbol\eta_2\end{bmatrix}=A\begin{bmatrix}\boldsymbol v\\ \boldsymbol\eta_1\\ \boldsymbol\eta_2\end{bmatrix}+\widetilde A_{v\eta\theta}\boldsymbol\theta+B\boldsymbol f,\\
&\widehat A^T_{v\eta\theta}\begin{bmatrix}\boldsymbol v\\ \boldsymbol\eta_1\\ \boldsymbol\eta_2\end{bmatrix}=0,\quad \begin{bmatrix}\boldsymbol v\\ \boldsymbol\eta_1\\ \boldsymbol\eta_2\end{bmatrix}(0)=\begin{bmatrix}\boldsymbol v^0\\ \boldsymbol\eta_1^0\\ \boldsymbol\eta_2^0\end{bmatrix},
\end{aligned}\tag{4.1 .5}$$

where $\boldsymbol v$, $\boldsymbol\eta_1$ and $\boldsymbol\eta_1$ are approximations of v, η_1 and η_2 respectively. The equations $v=\eta_2 n$ on Γ_s and $v=0$ on Γ_i are treated in a weak form by introducing two Lagrange multipliers

$\boldsymbol{\tau}_s$ and $\boldsymbol{\tau}_{in}$. To simplify the writing of the semi-discrete approximation, we concatenate the multipliers with the other multiplier $\boldsymbol{p}$ (approximation of the pressure p) by introducing $\boldsymbol{\theta} = (\boldsymbol{p}, \boldsymbol{\tau}_s, \boldsymbol{\tau}_{in})$. The matrices A_L, M_L, A, M, $\widetilde{A}_{v\eta\theta}$, $\widehat{A}_{v\eta\theta}$, $M_{\eta\eta}$, B and $\mathbf{W}$ are defined in Section 4.2 .

Step 2: *Reformulation of the finite-dimensional linear system*

To rewrite system (4.1 .5) as a control system, we have to eliminate the multiplier $\boldsymbol{\theta}$ from the equations. To eliminate the multiplier from the equation of $\boldsymbol{v}$, we use an operator $\mathbb{P}_v$ which plays a role similar to that of the Leray projector for the infinite-dimensional system. Indeed, thanks to $\mathbb{P}_v$, we can split the equation satisfied by $\boldsymbol{v}$ into two equations, an ordinary differential equation for $\mathbb{P}_v^T\boldsymbol{v}$ and an algebraic equation for $(I-\mathbb{P}_v^T)\boldsymbol{v}$. Next, we eliminate the multiplier from the equation satisfied by $\boldsymbol{\eta}_2$ by expressing it in terms of $\mathbb{P}_v^T\boldsymbol{v}$, $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$. Thus, we prove that system (4.1 .5) is equivalent to the following Differential Algebraic Equation (DAE):

$$\begin{aligned}
&\frac{d}{dt}\mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbf{A}\mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \mathbf{B}\mathbf{f}, \quad \mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix}(0) = \mathbb{P}^T \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}, \\
&(I-\mathbb{P}_v^T)\boldsymbol{v} = -M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\eta}_1 - M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\eta}_2, \\
&\boldsymbol{\theta} = -N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\eta}_{1,t} - N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\eta}_{2,t} - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_1} - A_{\theta\theta}A_{\eta_1\theta}^T)\boldsymbol{\eta}_1 \\
&\qquad - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_2} - A_{\theta\theta}A_{\eta_2\theta}^T)\boldsymbol{\eta}_2 - N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\boldsymbol{v},
\end{aligned} \tag{4.1 .6}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbb{P}$ are defined in Section 4.3 .2. Thus, to stabilize system (4.1 .4), it is necessary and sufficient to stabilize the first equation of system (4.1 .6).

Step 3: *Equivalence between the eigenvalue problems*

The construction of the linear feedback law is based on the spectral decomposition of the operator $\mathbf{A}\mathbb{P}^T$. But, it is numerically difficult to compute that matrix. To overcome this difficulty, we study the relations between the eigenvalue problem associated with $\mathbf{A}\mathbb{P}^T$ and the eigenvalue problems associated with the pair (M_L, A_L). Using the relations between these eigenvalue problems, we will be able to construct the feedback law without having to compute the matrices $\mathbf{A}$ and $\mathbb{P}^T$.

Step 4: *Projected systems*

Thanks to spectral decomposition of the operator $\mathbf{A}\mathbb{P}^T$ and the relations between the eigenvalue problems associated to $\mathbf{A}\mathbb{P}^T$ and (M_L, A_L), we prove that there exist matrices $\boldsymbol{E}_u$, $\boldsymbol{E}_s$, $\boldsymbol{\Xi}_u$ and $\boldsymbol{\Xi}_s$, obtained from the eigenvectors of (M_L, A_L), such that $\mathbb{P}^T(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is solution of the first equation of system (4.1 .6) if and only if

$$(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})^T = \boldsymbol{\Xi}_u^T M(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T \quad \text{and} \quad (\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})^T = \boldsymbol{\Xi}_s^T M(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T,$$

obey to the system

$$\begin{aligned}
&\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} = \Lambda_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \mathbb{B}_u\boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix}(0) = \boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_1^0 \end{bmatrix}, \\
&\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} = \Lambda_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} + \mathbb{B}_s\boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix}(0) = \boldsymbol{\Xi}_s^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix},
\end{aligned} \tag{4.1 .7}$$

where

$$\Lambda_u = \boldsymbol{\Xi}_u^T A \boldsymbol{E}_u, \quad \Lambda_s = \boldsymbol{\Xi}_s^T A \boldsymbol{E}_s, \quad \mathbb{B}_u = \boldsymbol{\Xi}_u^T B \quad \text{and} \quad \mathbb{B}_s = \boldsymbol{\Xi}_s^T B.$$

Moreover, the matrices Λ_u and Λ_s are constituted of real Jordan blocks and

$$\mathrm{Re}\, \sigma(\Lambda_s) < -\alpha_u,$$

for some $\alpha_u > 0$.

Step 5: *Linear feedback law*

To stabilize the first equation of system (4.1 .6), it is necessary and sufficient to stabilize the equation satisfied by $(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})$. Thus, assuming that, for all $\omega > -\mathrm{Re}(\sigma(\Lambda_u))$, the pair $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, \mathbb{B}_u)$ is stabilizable, we can find a feedback law by solving the following algebraic Riccati equation

$$\begin{aligned} &\mathbb{P}_u \in \mathcal{L}(\mathbb{R}^{d_u}), \quad \mathbb{P}_u = \mathbb{P}_u^T > 0, \\ &\mathbb{P}_u(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u + \omega I_{\mathbb{R}^{d_u}})^T \mathbb{P}_u - \mathbb{P}_u \mathbb{B}_u \mathbb{B}_u^T \mathbb{P}_u = 0. \end{aligned}$$

4.2 Semi-discrete approximations

To approximate systems (4.1 .2) and (4.1 .3) by a finite element method, we introduce finite-dimensional subspaces $X_h \subset \{v \in \mathbf{H}^1(\Omega) \mid v \cdot n = 0 \text{ on } \Gamma_e\}$ for the velocity, $P_h \subset L^2(\Omega)$ for the pressure, S_h of $H^2(\Gamma_s)$ for the displacement and its velocity, $D_{s,h} \subset \mathbf{H}^{-\frac{1}{2}}(\Gamma_s)$ (resp. $D_{in,h} \subset \mathbf{H}^{-\frac{1}{2}}(\Gamma_i)$) for the multiplier associated to the Dirichlet boundary conditions on Γ_s (resp. Γ_i).

We denote by $(\phi_i)_{1\le i\le n_v}$ a basis of X_h, $(q_i)_{1\le i\le n_p}$ a basis of P_h, $(\zeta_i)_{1\le i\le n_s}$ a basis of S_h, $(\mu_{s,i})_{1\le i\le n_{d_s}}$ a basis of $D_{s,h}$ and $(\mu_{in,i})_{1\le i\le n_{d_{in}}}$ a basis of $D_{in,h}$.

We set

$$\begin{aligned} &v = \sum_{i=1}^{n_v} v_i \phi_i, \quad v^0 = \sum_{i=1}^{n_v} v_i^0 \phi_i, \quad \eta_1 = \sum_{i=1}^{n_s} \eta_1^i \zeta_i, \quad \eta_2 = \sum_{i=1}^{n_s} \eta_2^i \zeta_i, \\ &p = \sum_{i=1}^{n_p} p_i q_i, \quad \tau_s = \sum_{i=1}^{n_{d_s}} \tau_{s,i} \mu_{s,i}, \quad \tau_{in} = \sum_{i=1}^{n_{d_{in}}} \tau_{in,i} \mu_{in,i}, \quad w_j = \sum_{i=1}^{n_s} w_j^i \zeta_i. \end{aligned}$$

If we denote with boldface letters the vectors of the corresponding coordinates, we have

$$\begin{aligned} &\boldsymbol{v} = (v_1, \cdots, v_{n_v})^T, \quad \boldsymbol{v}^0 = (v_1^0, \cdots, v_{n_v}^0)^T, \quad \boldsymbol{\eta}_1 = (\eta_1^1, \cdots, \eta_1^{n_s})^T, \quad \boldsymbol{\eta}_2 = (\eta_2^1, \cdots, \eta_2^{n_s})^T, \\ &\boldsymbol{f} = (f_1, \cdots, f_{n_c})^T, \quad \boldsymbol{p} = (p_1, \cdots, p_{n_p})^T, \quad \boldsymbol{\tau}_s = (\tau_{s,1}, \cdots, \tau_{n_{d_s}})^T, \quad \boldsymbol{\tau}_{in} = (\tau_{in,1}, \cdots, \tau_{in,n_{d_{in}}})^T, \\ &\boldsymbol{\theta} = (p_1, \cdots, p_{n_p}, \tau_{s,1}, \cdots, \tau_{n_s}, \tau_{in,1}, \cdots, \tau_{in,n_{d_{in}}})^T, \quad \mathbf{W} = \begin{bmatrix} w_1^1 & \cdots & w_1^{n_c} \\ \vdots & \vdots & \vdots \\ w_{n_s}^1 & \cdots & w_{n_s}^{n_c} \end{bmatrix}. \end{aligned}$$

For the fluid, we introduce the stiffness matrices A_{vv}, A_{vp}, $A_{v\tau_s}$, $A_{v\tau_{in}}$, $A_{v\theta}$ and the mass matrix

M_{vv} defined by

$$
\begin{aligned}
&(A_{vv})_{ij} = -2\nu \int_\Omega \varepsilon(\phi_i) : \varepsilon(\phi_j) - \int_\Omega (u_s \cdot \nabla)\phi_j \cdot \phi_i - \int_\Omega (\phi_j \cdot \nabla) u_s \cdot \phi_i, && \forall\, 1 \le i \le n_v,\, 1 \le j \le n_v,\\
&(A_{vp})_{ij} = \int_\Omega q_j \mathrm{div}\, \phi_i, && \forall\, 1 \le i \le n_v,\, 1 \le j \le n_p,\\
&(A_{v\tau_s})_{ij} = \int_{\Gamma_s} \mu_{s,j} \cdot \phi_i, && \forall\, 1 \le i \le n_v,\, 1 \le j \le n_{d_s},\\
&(A_{v\tau_{in}})_{ij} = \int_{\Gamma_i} \mu_{in,j} \cdot \phi_i, && \forall\, 1 \le i \le n_v,\, 1 \le j \le n_{d_{in}},\\
&(M_{vv})_{ij} = \int_\Omega \phi_j \cdot \phi_i, && \forall\, 1 \le i \le n_v,\, 1 \le j \le n_v,\\
&A_{v\theta} = \begin{bmatrix} A_{vp} & A_{v\tau_s} & A_{v\tau_{in}} \end{bmatrix}.
\end{aligned}
$$

For the structure, we introduce the stiffness matrices $A_{\eta_1\eta_2}$, $A_{\eta_2\eta_1}$, $A_{\eta_2\eta_2}$ and the mass matrix $M_{\eta\eta}$ defined by

$$
\begin{aligned}
&(A_{\eta_1\eta_2})_{ij} = \int_{\Gamma_s} \zeta_j \cdot \zeta_i, && \forall\, 1 \le i \le n_s,\, 1 \le j \le n_s,\\
&(A_{\eta_2\eta_1})_{ij} = -\int_{\Gamma_s} (\beta \nabla\zeta_j \cdot \nabla\zeta_i + \alpha \Delta\zeta_j \cdot \Delta\zeta_i) + \int_{\Gamma_s} A_4 \zeta_j \cdot \zeta_i, && \forall\, 1 \le i \le n_s,\, 1 \le j \le n_s,\\
&(A_{\eta_2\eta_2})_{ij} = -\gamma \int_{\Gamma_s} \nabla\zeta_j \cdot \nabla\zeta_i, && \forall\, 1 \le i \le n_s,\, 1 \le j \le n_s,\\
&(M_{\eta\eta})_{ij} = \int_{\Gamma_s} \zeta_j \cdot \zeta_i, && \forall\, 1 \le i \le n_s,\, 1 \le j \le n_s.
\end{aligned}
$$

We also introduce the coupling matrices $A_{v\eta_1}$, $A_{v\eta_2}$, $A_{\eta_1 p}$, $A_{\eta_2\tau}$, $A_{\eta_1\theta}$ and $A_{\eta_2\theta}$ defined by

$$
\begin{aligned}
&(A_{v\eta_1})_{ij} = \int_\Omega A_1 \zeta_j \cdot \phi_i, && \forall\, 1 \le i \le n_v,\, 1 \le j \le n_s,\\
&(A_{v\eta_2})_{ij} = \int_\Omega A_2 \zeta_j \cdot \phi_i, && \forall\, 1 \le i \le n_v,\, 1 \le j \le n_s,\\
&(A_{\eta_1 p})_{ij} = -\int_\Omega A_3 \zeta_i \cdot q_j, && \forall\, 1 \le i \le n_s,\, 1 \le j \le n_p,\\
&(A_{\eta_2\tau})_{ij} = -\int_{\Gamma_s} \zeta_i n \cdot \mu_{s,j}, && \forall\, 1 \le i \le n_s,\, 1 \le j \le n_{d_s},\\
&A_{\eta_1\theta} = \begin{bmatrix} A_{\eta_1 p} & 0 & 0 \end{bmatrix}, \quad A_{\eta_2\theta} = \begin{bmatrix} 0 & A_{\eta_2\tau_s} & A_{\eta_2\tau_{in}} \end{bmatrix}.
\end{aligned}
$$

We set

$$
n_1 = n_v + 2n_s, \quad n_\theta = n_p + n_{d_s} + n_{d_{in}} \quad \text{and} \quad n = n_1 + n_\theta.
$$

We introduce the mass matrix M_L and the stiffness matrix A_L defined by

$$
M_L = \begin{bmatrix} M_{vv} & 0 & 0 & 0 \\ 0 & M_{\eta\eta} & 0 & 0 \\ 0 & 0 & M_{\eta\eta} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_L = \begin{bmatrix} A_{vv} & A_{v\eta_1} & A_{v\eta_2} & A_{v\theta} \\ 0 & 0 & A_{\eta_1\eta_2} & 0 \\ 0 & A_{\eta_2\eta_1} & A_{\eta_2\eta_2} & A_{\eta_2\theta} \\ A_{v\theta}^T & A_{\eta_1\theta}^T & A_{\eta_2\theta}^T & 0 \end{bmatrix}. \tag{4.2 .1}
$$

The matrices M_L and A_L can be rewritten in the form

$$M_L = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_L = \begin{bmatrix} A & \widetilde{A}_{v\eta\theta} \\ \widehat{A}^T_{v\eta\theta} & 0 \end{bmatrix}, \tag{4.2 .2}$$

where

$$\widetilde{A}_{v\eta\theta} = \begin{bmatrix} A_{v\theta} \\ 0 \\ A_{\eta_2\theta} \end{bmatrix}, \quad \widehat{A}^T_{v\eta\theta} = \begin{bmatrix} A_{v\theta} \\ A_{\eta_1\theta} \\ A_{\eta_2\theta} \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} M_{vv} & 0 & 0 \\ 0 & M_{\eta\eta} & 0 \\ 0 & 0 & M_{\eta\eta} \end{bmatrix}. \tag{4.2 .3}$$

The finite-dimensional approximation of system (4.1 .3) is

$$\begin{aligned} &M_L \frac{d}{dt}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\theta} \end{bmatrix} = A_L \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_{\eta\eta}\mathbf{W}\boldsymbol{f} \\ 0 \end{bmatrix}, \\ &\boldsymbol{v}(0) = \boldsymbol{v}^0, \quad \boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_1^0, \quad \boldsymbol{\eta}_2(0) = \boldsymbol{\eta}_2^0. \end{aligned} \tag{4.2 .4}$$

We can also rewrite system (4.2 .4) as follows

$$\begin{aligned} &M\frac{d}{dt}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = A \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \widetilde{A}_{v\eta\theta}\boldsymbol{\theta} + B\boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix}(0) = \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}, \\ &\widehat{A}^T_{v\eta\theta}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = 0, \end{aligned} \tag{4.2 .5}$$

where

$$A = \begin{bmatrix} A_{vv} & A_{v\eta_1} & A_{v\eta_2} \\ 0 & 0 & A_{\eta_1\eta_2} \\ 0 & A_{\eta_2\eta_1} & A_{\eta_2\eta_2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ M_{\eta\eta}\mathbf{W} \end{bmatrix}. \tag{4.2 .6}$$

The finite-dimensional approximation of the nonlinear system (4.1 .2) is

$$\begin{aligned} &M_L \frac{d}{dt}\begin{bmatrix} \tilde{\boldsymbol{u}} \\ \tilde{\boldsymbol{\eta}}_1 \\ \tilde{\boldsymbol{\eta}}_2 \\ \tilde{\boldsymbol{\theta}} \end{bmatrix} = \widehat{A}_L \begin{bmatrix} \tilde{\boldsymbol{u}} \\ \tilde{\boldsymbol{\eta}}_1 \\ \tilde{\boldsymbol{\eta}}_2 \\ \tilde{\boldsymbol{\theta}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_{\eta\eta}\mathbf{W}\boldsymbol{f} \\ \boldsymbol{h} \end{bmatrix} + F_{ns}[\tilde{\boldsymbol{u}}] + F_g[\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\eta}}_2], \\ &\tilde{\boldsymbol{u}}(0) = \tilde{\boldsymbol{u}}^0, \quad \tilde{\boldsymbol{\eta}}_1(0) = \tilde{\boldsymbol{\eta}}_1^0, \quad \tilde{\boldsymbol{\eta}}_2(0) = \tilde{\boldsymbol{\eta}}_2^0, \end{aligned} \tag{4.2 .7}$$

where

$$\widehat{A}_L = \begin{bmatrix} \widehat{A}_{vv} & 0 & 0 & A_{v\theta} \\ 0 & 0 & A_{\eta_1\eta_2} & 0 \\ 0 & \widehat{A}_{\eta_2\eta_1} & A_{\eta_2\eta_2} & A_{\eta_2\theta} \\ A^T_{v\theta} & 0 & A^T_{\eta_2\theta} & 0 \end{bmatrix},$$

and

$$\widehat{A}_{vv} = \left(-2\nu \int_\Omega \varepsilon(\phi_i) : \varepsilon(\phi_j)\right)_{1\le i,j\le n_v} \quad \text{and} \quad (\widehat{A}_{\eta_2\eta_1})_{ij} = -\int_{\Gamma_s} (\beta\nabla\zeta_j \cdot \nabla\zeta_i + \alpha\Delta\zeta_j \cdot \Delta\zeta_i).$$

The nonlinear terms $F_{ns}[\tilde{\boldsymbol{u}}]$ and $F_g[\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\eta}}_2]$ are defined by

$$F_{ns}[\tilde{\boldsymbol{u}}] = \begin{bmatrix} F_{ns}^u \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad F_g[\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\eta}}_1, \tilde{\boldsymbol{\eta}}_2] = \begin{bmatrix} F_g^u \\ 0 \\ F_g^{\eta_2} \\ F_g^{\theta} \end{bmatrix},$$

where

$$F_{ns}^u = -\left(\int_\Omega \sum_{i=1}^{n_v} \sum_{j=1}^{n_v} \tilde{u}_i \tilde{u}_j (\phi_i \cdot \nabla)\phi_j \cdot \phi_k\right)_{1 \le k \le n_v},$$

$$F_g^u = \left(\int_\Omega F_f \left[\sum_{i=1}^{n_v} \tilde{u}_i \phi_i, \sum_{j=1}^{n_p} \tilde{p}_j q_j, \sum_{l=1}^{n_s} \tilde{\eta}_{1,l} \zeta_l, \sum_{m=1}^{n_s} \tilde{\eta}_{2,m} \zeta_m\right] \cdot \phi_k\right)_{1 \le k \le n_v},$$

$$\mathcal{F}_g^{\eta_2} = \left(\int_{\Gamma_s} F_s \left[\sum_{i=1}^{n_v} \tilde{u}_i \phi_i, \sum_{j=1}^{n_s} \tilde{\eta}_{1,j} \zeta_j\right] \cdot \zeta_k\right)_{1 \le k \le n_s},$$

$$\mathcal{F}_g^{\theta} = \left[\left(\int_\Omega \operatorname{div} F_{\text{div}} \left[\sum_{i=1}^{n_v} \tilde{u}_i \phi_i, \sum_{j=1}^{n_s} \tilde{\eta}_{1,j} \zeta_j\right] \cdot q_k\right)_{1 \le k \le n_p}, 0, \cdots, 0\right].$$

The inflow boundary condition u_i is taking into account in the vector $\boldsymbol{h} \in \mathbb{R}^{n_\theta}$.

4.3 Reformulation of the finite-dimensional linear system

4.3 .1 Approximation of the Oseen operator and of the lifting operator

We consider the system

$$\begin{aligned} M_{vv} \boldsymbol{z}_t &= A_{vv} \boldsymbol{z} + A_{v\theta} \theta + F, \\ A_{v\theta}^T \boldsymbol{z} &= \boldsymbol{g}, \quad \boldsymbol{z}(0) = \boldsymbol{z}^0, \end{aligned} \tag{4.3 .1}$$

where $F \in \mathbb{R}^{n_v}$, $\boldsymbol{g} \in \mathbb{R}^{n_\theta}$ and $\boldsymbol{z}^0 \in \mathbb{R}^{n_v}$. System (4.3 .1) is an approximation of Oseen equations. The goal is to rewrite it into two equations, an ordinary differential equation and an algebraic equation. To eliminate the multiplier θ from the first equation of (4.3 .1), we are going to introduce the projection into $\operatorname{Ker}(A_{v\theta}^T)$ parallel to $\operatorname{Im}(M_{vv}^{-1} A_{v\theta})$.

Proposition 4.3 .1. *1. The projection in $\mathbb{R}^{n_v}$ onto $\operatorname{Ker}(A_{v\theta}^T)$ parallel to $\operatorname{Im}(M_{vv}^{-1} A_{v\theta})$ is*

$$\mathbb{P}_v^T = I - M_{vv}^{-1} A_{v\theta} (A_{v\theta}^T M_{vv}^{-1} A_{v\theta})^{-1} A_{v\theta}^T.$$

2. The projection in $\mathbb{R}^{n_v}$ onto $\operatorname{Ker}(A_{v\theta}^T M_{vv}^{-1})$ parallel to $\operatorname{Im}(A_{v\theta})$ is

$$\mathbb{P}_v = I - A_{v\theta} (A_{v\theta}^T M_{vv}^{-1} A_{v\theta})^{-1} A_{v\theta}^T M_{vv}^{-1}.$$

Moreover, we have

$$\mathbb{P}_v A_{v\theta} = 0, \quad \mathbb{P}_v^2 = \mathbb{P}_v, \quad \mathbb{P}_v M_{vv} = M_{vv} \mathbb{P}_v^T, \quad \mathbb{P}_v M_{vv} = \mathbb{P}_v^2 M_{vv} = \mathbb{P}_v M_{vv} \mathbb{P}_v^T, \quad M_{vv}^{-1} \mathbb{P}_v = \mathbb{P}_v^T M_{vv}^{-1}.$$

Proof. It is similar to that of Proposition 2.11 .1. □

The projector $\mathbb{P}_v^T$ is an approximation of the Leray projector. However, there are two differences between the operators. Unlike the Leray projection, $\mathbb{P}_v^T$ is not symmetric when $\mathbb{R}^{n_v}$ is equipped with the usual inner product, but when $\mathbb{R}^{n_v}$ is equipped with the weighted product $(M_{vv}^{-1}\cdot,\cdot)_{\mathbb{R}^{n_v}}$. This is why $\mathbb{P}_v \neq \mathbb{P}_v^T$. The other difference is that the projector $\mathbb{P}_v^T$ is used to eliminate the pressure and the additional multiplier in system (4.3 .1), while the Leray projector is only used to eliminate the pressure. Now, we introduce the operator $\mathbb{A}$ in $\mathbb{R}^{n_v}$ defined by

$$\mathbb{A} = \mathbb{P}_v^T M_{vv}^{-1} A_{vv}. \tag{4.3 .2}$$

Finally, we introduce the lifting operators $\mathbb{L}$ and $\mathbb{L}_\theta$ in $\mathbb{R}^{n_\theta}$ defined by

$$\mathbb{L}\boldsymbol{g} = \boldsymbol{w} \quad \text{and} \quad \mathbb{L}_\theta \boldsymbol{g} = \boldsymbol{\pi} \tag{4.3 .3}$$

where $(\boldsymbol{w}, \boldsymbol{\pi})$ is the solution of the system

$$A_{vv}\boldsymbol{w} + A_{v\theta}\boldsymbol{\pi} = 0, \quad A_{v\theta}^T \boldsymbol{w} = \boldsymbol{g}, \tag{4.3 .4}$$

(see Lemma 4.3 .1 below). The operator $\mathbb{A}$ is the approximation of the Oseen operator defined in (3.4 .4). The operators $\mathbb{L}$ and $\mathbb{L}_\theta$ are the approximations of the lifting operators defined in (3.4 .5). We assume that $0 \notin \sigma(A_{vv})$.

Lemma 4.3 .1. *Let $\boldsymbol{g}$ be in $\mathbb{R}^{n_\theta}$. Then, system (4.3 .4) admits a unique solution*

$$(\boldsymbol{w}, \boldsymbol{\pi}) = (A_{vv}^{-1}A_{v\theta}(A_{v\theta}^T A_{vv}^{-1}A_{v\theta})^{-1}\boldsymbol{g}, -(A_{v\theta}^T A_{vv}^{-1}A_{v\theta})^{-1}\boldsymbol{g}).$$

Proof. Multiplying the first equation of system (4.3 .4) by $A_{v\theta}^T A_{vv}^{-1}$, we obtain

$$\boldsymbol{\pi} = -(A_{v\theta}^T A_{vv}^{-1}A_{v\theta})^{-1}A_{v\theta}^T\boldsymbol{w} = -(A_{v\theta}^T A_{vv}^{-1}A_{v\theta})^{-1}\boldsymbol{g},$$

then

$$\boldsymbol{w} = A_{vv}^{-1}A_{v\theta}(A_{v\theta}^T A_{vv}^{-1}A_{v\theta})^{-1}\boldsymbol{g}.$$

□

We set

$$N_{\theta\theta} = (A_{v\theta}^T M_{vv}^{-1}A_{v\theta})^{-1}. \tag{4.3 .5}$$

Lemma 4.3 .2. *We have*

$$\mathbb{L} = A_{vv}^{-1}A_{v\theta}N_{\theta\theta} \quad \textit{and} \quad (I - \mathbb{P}_v^T)\mathbb{L} = M_{vv}^{-1}A_{v\theta}N_{\theta\theta}.$$

Proof. The first equality follows from Lemma 4.3 .1. The last equality comes from the fact that

$$\mathbb{P}_v^T = I - M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{v\theta}^T,$$

(see Proposition 4.3 .1).

□

Now, we are in position to reformulate system (4.3 .1).

Proposition 4.3 .2. *System (4.3 .1) is equivalent to*

$$\begin{aligned}
&\mathbb{P}_v^T\boldsymbol{z}_t = \mathbb{A}\mathbb{P}_v^T\boldsymbol{z} + \mathbb{P}_v^T M_{vv}^{-1}F - \mathbb{A}\mathbb{P}_v^T\mathbb{L}\boldsymbol{g}, \quad \mathbb{P}_v^T\boldsymbol{z}(0) = \mathbb{P}_v^T\boldsymbol{z}^0,\\
&(I - \mathbb{P}_v^T)\boldsymbol{z} = M_{vv}^{-1}A_{v\theta}N_{\theta\theta}\boldsymbol{g},\\
&\boldsymbol{\theta} = -N_{\theta\theta}\left(-\boldsymbol{g}_t + A_{\theta\theta}\boldsymbol{g} + A_{v\theta}^T M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\boldsymbol{z} + A_{v\theta}^T M_{vv}^{-1}F\right),
\end{aligned} \tag{4.3 .6}$$

where

$$A_{\theta\theta} = A_{v\theta}^T M_{vv}^{-1}A_{vv}M_{vv}^{-1}A_{v\theta}N_{\theta\theta}. \tag{4.3 .7}$$

Proof. Let $(\boldsymbol{z}, \boldsymbol{\theta})$ be a solution of system (4.3 .1). First, we have

$$(I - \mathbb{P}_v^T)\boldsymbol{z} = M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{v\theta}^T \boldsymbol{z} = M_{vv}^{-1} A_{v\theta} N_{\theta\theta} \boldsymbol{g}.$$

Next, we multiply the first equation of the system (4.3 .1) by $M_{vv}^{-1}\mathbb{P}_v$

$$M_{vv}^{-1}\mathbb{P}_v M_{vv} \boldsymbol{z}_t = M_{vv}^{-1}\mathbb{P}_v A_{vv}\boldsymbol{z} + M_{vv}^{-1}\mathbb{P}_v A_{v\theta}\boldsymbol{\theta} + M_{vv}^{-1}\mathbb{P}_v F.$$

Since

$$\mathbb{P}_v M_{vv} = M_{vv}\mathbb{P}_v^T, \quad M_{vv}^{-1}\mathbb{P}_v = \mathbb{P}_v^T M_{vv}^{-1} \quad \text{and} \quad \mathbb{P}_v A_{v\theta} = 0,$$

we obtain

$$\mathbb{P}_v^T \boldsymbol{z}_t = \mathbb{P}_v^T M_{vv}^{-1} A_{vv}\boldsymbol{z} + \mathbb{P}_v^T M_{vv}^{-1} F.$$

Moreover, we have

$$\boldsymbol{z} = \mathbb{P}_v^T\boldsymbol{z} + (I - \mathbb{P}_v^T)\boldsymbol{z} = \mathbb{P}_v^T\boldsymbol{z} + (I - \mathbb{P}_v^T)\mathbb{L}\boldsymbol{g} = \mathbb{P}_v^T\boldsymbol{z} - \mathbb{P}_v^T\mathbb{L}\boldsymbol{g} + \mathbb{L}\boldsymbol{g},$$

and

$$\mathbb{P}_v^T M_{vv}^{-1} A_{vv}\mathbb{L}\boldsymbol{g} = \mathbb{P}_v^T M_{vv}^{-1} A_{vv} A_{vv}^{-1} A_{v\theta} N_{\theta\theta}\boldsymbol{g} = M_{vv}^{-1}\mathbb{P}_v A_{v\theta} N_{\theta\theta}\boldsymbol{g} = 0.$$

Thus, $\mathbb{P}_v^T\boldsymbol{z}$ is a solution to the equation

$$\mathbb{P}_v^T\boldsymbol{z}_t = \mathbb{A}\mathbb{P}_v^T\boldsymbol{z} - \mathbb{A}\mathbb{P}_v^T\mathbb{L}\boldsymbol{g} + \mathbb{P}_v^T M_{vv}^{-1} F.$$

To finish, we have to determine the multiplier $\boldsymbol{\theta}$. Multiplying the first equation of the system by $A_{v\theta}^T M_{vv}^{-1}$, we obtain

$$\begin{aligned}
N_{\theta\theta}^{-1}\boldsymbol{\theta} &= A_{v\theta}^T\boldsymbol{z}_t - A_{v\theta}^T M_{vv}^{-1} A_{vv}\boldsymbol{z} - A_{v\theta}^T M_{vv}^{-1} F \\
&= \boldsymbol{g}_t - A_{v\theta}^T M_{vv}^{-1} A_{vv}(I - \mathbb{P}_v^T)\boldsymbol{z} - A_{v\theta}^T M_{vv}^{-1} A_{vv}\mathbb{P}_v^T\boldsymbol{z} - A_{v\theta}^T M_{vv}^{-1} F \\
&= \boldsymbol{g}_t - A_{v\theta}^T M_{vv}^{-1} A_{vv} M_{vv}^{-1} A_{v\theta} N_{\theta\theta}\boldsymbol{g} - A_{v\theta}^T M_{vv}^{-1} A_{vv}\mathbb{P}_v^T\boldsymbol{z} - A_{v\theta}^T M_{vv}^{-1} F \\
&= \boldsymbol{g}_t - A_{\theta\theta}\boldsymbol{g} - A_{v\theta}^T M_{vv}^{-1} A_{vv}\mathbb{P}_v^T\boldsymbol{z} - A_{v\theta}^T M_{vv}^{-1} F.
\end{aligned}$$

Hence

$$\boldsymbol{\theta} = -N_{\theta\theta}\left(-\boldsymbol{g}_t + A_{\theta\theta}\boldsymbol{g} + A_{v\theta}^T M_{vv}^{-1} A_{vv}\mathbb{P}_v^T\boldsymbol{z} + A_{v\theta}^T M_{vv}^{-1} F\right).$$

□

4.3 .2 Definition of the operator A

First, we introduce the operator $\mathbb{P}$ in $\mathbb{R}^{n_v} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_s}$ defined by

$$\mathbb{P} = \begin{bmatrix} \mathbb{P}_v & 0 & 0 \\ 0 & I_{\mathbb{R}^{n_s}} & 0 \\ 0 & 0 & I_{\mathbb{R}^{n_s}} \end{bmatrix}. \tag{4.3 .8}$$

We set

$$A_{v\eta\theta} = \begin{bmatrix} A_{v\theta} \\ 0 \\ 0 \end{bmatrix}, \tag{4.3 .9}$$

and

$$M_s = \begin{bmatrix} M_{vv} & 0 & 0 \\ 0 & M_{\eta\eta} & 0 \\ 0 & N_{\eta_2\eta_1} & M_{\eta\eta} + N_{\eta_2\eta_2} \end{bmatrix}, \tag{4.3 .10}$$

where

$$N_{\eta_2\eta_2} = A_{\eta_2\theta} N_{\theta\theta} A_{\eta_2\theta}^T \quad \text{and} \quad N_{\eta_2\eta_1} = A_{\eta_2\theta} N_{\theta\theta} A_{\eta_1\theta}^T. \tag{4.3 .11}$$

Proposition 4.3 .3. *The operator $\mathbb{P}^T$ is the projector in $\mathbb{R}^{n_v} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_s}$ onto* $\mathrm{Ker}(A^T_{v\eta\theta})$ *parallel to* $\mathrm{Im}(M_s^{-1}A_{v\eta\theta})$ *and the operator $\mathbb{P}$ is the projector in $\mathbb{R}^{n_v} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_s}$ onto* $\mathrm{Ker}(A^T_{v\eta\theta}M_s^{-1})$ *parallel to* $\mathrm{Im}(A_{v\eta\theta})$.
Moreover, we have

$$\mathbb{P}A_{v\eta\theta} = 0, \quad \mathbb{P}^2 = \mathbb{P}, \quad \mathbb{P}M_s = M_s\mathbb{P}^T, \quad \mathbb{P}M_s = \mathbb{P}^2M_s = \mathbb{P}M_s\mathbb{P}^T, \quad M_s^{-1}\mathbb{P} = \mathbb{P}^TM_s^{-1}.$$

Proof. It follows from Proposition 4.3 .1. ∎

We introduce the operator $\mathbf{A}$ in $\mathbb{R}^{n_v} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_s}$ defined by

$$\mathbf{A} = \mathbb{P}^TM_s^{-1}A_0, \tag{4.3 .12}$$

where

$$A_0 = \begin{bmatrix} A_{vv} & A_{v\eta_1} + A_{vv}\mathbb{P}_v^T\mathbb{L}A^T_{\eta_1\theta} & A_{v\eta_2} + A_{vv}\mathbb{P}_v^T\mathbb{L}A^T_{\eta_2\theta} \\ 0 & 0 & A_{\eta_1\eta_2} \\ A_{\eta_2 v} & \widetilde{A}_{\eta_2\eta_1} & \widetilde{A}_{\eta_2\eta_2} \end{bmatrix}, \tag{4.3 .13}$$

and

$$\begin{aligned} A_{\eta_2 v} &= -A_{\eta_2\theta}N_{\theta\theta}A^T_{v\theta}M_{vv}^{-1}A_{vv}, \\ \widetilde{A}_{\eta_2\eta_1} &= A_{\eta_1\eta_2} - A_{\eta_2\theta}N_{\theta\theta}(A^T_{v\theta}M_{vv}^{-1}A_{v\eta_1} - A_{\theta\theta}A^T_{\eta_1\theta}), \\ \widetilde{A}_{\eta_2\eta_2} &= A_{\eta_2\eta_2} - A_{\eta_2\theta}N_{\theta\theta}(A^T_{v\theta}M_{vv}^{-1}A_{v\eta_2} - A_{\theta\theta}A^T_{\eta_2\theta}). \end{aligned} \tag{4.3 .14}$$

Proposition 4.3 .4. *System (4.2 .4) is equivalent to the system*

$$\begin{aligned} &\frac{d}{dt}\mathbb{P}^T\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbf{A}\mathbb{P}^T\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \mathbf{Bf}, \quad \mathbb{P}^T\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix}(0) = \mathbb{P}^T\begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}, \\ &(I - \mathbb{P}_v^T)\boldsymbol{v} = -M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A^T_{\eta_1\theta}\boldsymbol{\eta}_1 - M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\eta}_2, \\ &\boldsymbol{\theta} = -N_{\theta\theta}A^T_{\eta_1\theta}\boldsymbol{\eta}_{1,t} - N_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\eta}_{2,t} - N_{\theta\theta}(A^T_{v\theta}M_{vv}^{-1}A_{v\eta_1} - A_{\theta\theta}A^T_{\eta_1\theta})\boldsymbol{\eta}_1 \\ &\qquad -N_{\theta\theta}(A^T_{v\theta}M_{vv}^{-1}A_{v\eta_2} - A_{\theta\theta}A^T_{\eta_2\theta})\boldsymbol{\eta}_2 - N_{\theta\theta}A^T_{v\theta}M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\boldsymbol{v}, \end{aligned} \tag{4.3 .15}$$

where

$$\mathbf{B} = M_s^{-1}B. \tag{4.3 .16}$$

Proof. Let $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$ be a solution of system (4.2 .4). The couple $(\boldsymbol{v}, \boldsymbol{\theta})$ is solution of the system

$$\begin{aligned} M_{vv}\boldsymbol{v}_t &= A_{vv}\boldsymbol{v} + A_{v\theta}\boldsymbol{\theta} + A_{v\eta_1}\boldsymbol{\eta}_1 + A_{v\eta_2}\boldsymbol{\eta}_2, \\ A^T_{v\theta}\boldsymbol{v} &= -(A^T_{\eta_1\theta}\boldsymbol{\eta}_1 + A^T_{\eta_2\theta}\boldsymbol{\eta}_2), \quad \boldsymbol{v}(0) = \boldsymbol{v}^0. \end{aligned} \tag{4.3 .17}$$

Thus, thanks to Proposition 4.3 .2, we have

$$\begin{aligned} &\mathbb{P}_v^T\boldsymbol{v}_t = \mathbb{A}\mathbb{P}_v^T\boldsymbol{v} + \mathbb{A}\mathbb{P}_v^T\mathbb{L}(A^T_{\eta_1\theta}\boldsymbol{\eta}_1 + A^T_{\eta_2\theta}\boldsymbol{\eta}_2) + \mathbb{P}_v^TM_{vv}^{-1}(A_{v\eta_1}\boldsymbol{\eta}_1 + A_{v\eta_2}\boldsymbol{\eta}_2), \\ &(I - \mathbb{P}_v^T)\boldsymbol{v} = -M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A^T_{\eta_1\theta}\boldsymbol{\eta}_1 - M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\eta}_2, \\ &\boldsymbol{\theta} = -N_{\theta\theta}A^T_{\eta_1\theta}\boldsymbol{\eta}_{1,t} - N_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\eta}_{2,t} - N_{\theta\theta}(A^T_{v\theta}M_{vv}^{-1}A_{v\eta_1} - A_{\theta\theta}A^T_{\eta_1\theta})\boldsymbol{\eta}_1 \\ &\qquad -N_{\theta\theta}(A^T_{v\theta}M_{vv}^{-1}A_{v\eta_2} - A_{\theta\theta}A^T_{\eta_2\theta})\boldsymbol{\eta}_2 - N_{\theta\theta}A^T_{v\theta}M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\boldsymbol{v}. \end{aligned}$$

We also have

$$\begin{aligned} A_{\eta_2\theta}\boldsymbol{\theta} &= -A_{\eta_2\theta}N_{\theta\theta}A^T_{\eta_1\theta}\boldsymbol{\eta}_{1,t} - A_{\eta_2\theta}N_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\eta}_{2,t} - A_{\eta_2\theta}N_{\theta\theta}\left(A^T_{v\theta}M^{-1}_{vv}A_{v\eta_1} - A_{\theta\theta}A^T_{\eta_1\theta}\right)\boldsymbol{\eta}_1 \\ &\quad -A_{\eta_2\theta}N_{\theta\theta}\left(A^T_{v\theta}M^{-1}_{vv}A_{v\eta_2} - A_{\theta\theta}A^T_{\eta_2\theta}\right)\boldsymbol{\eta}_2 - A_{\eta_2\theta}N_{\theta\theta}A^T_{v\theta}M^{-1}_{vv}A_{vv}\mathbb{P}^T_v\boldsymbol{v} \\ &= -N_{\eta_2\eta_1}\boldsymbol{\eta}_{1,t} - N_{\eta_2\eta_2}\boldsymbol{\eta}_{2,t} + (\widetilde{A}_{\eta_2\eta_1} - A_{\eta_2\eta_1})\boldsymbol{\eta}_1 + (\widetilde{A}_{\eta_2\eta_2} - A_{\eta_2\eta_2})\boldsymbol{\eta}_2 + A_{\eta_2 v}\mathbb{P}^T_v\boldsymbol{v}. \end{aligned} \tag{4.3 .18}$$

The system satisfied by $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ is

$$\begin{aligned} M_{\eta\eta}\boldsymbol{\eta}_{1,t} &= A_{\eta_1\eta_2}\boldsymbol{\eta}_2, \\ M_{\eta\eta}\boldsymbol{\eta}_{2,t} &= A_{\eta_2\eta_1}\boldsymbol{\eta}_1 + A_{\eta_2\eta_2}\boldsymbol{\eta}_2 + A_{\eta_2\theta}\boldsymbol{\theta}. \end{aligned} \tag{4.3 .19}$$

Replacing $A_{\eta_2\theta}\boldsymbol{\theta}$ by the expression in (4.3 .18), we obtain

$$\begin{aligned} &M_{\eta\eta}\boldsymbol{\eta}_{1,t} = A_{\eta_1\eta_2}\boldsymbol{\eta}_2, \\ &(M_{\eta\eta} + N_{\eta_2\eta_2})\boldsymbol{\eta}_{2,t} + N_{\eta_2\eta_1}\boldsymbol{\eta}_{1,t} = \widetilde{A}_{\eta_2\eta_1}\boldsymbol{\eta}_1 + \widetilde{A}_{\eta_2\eta_2}\boldsymbol{\eta}_2 + A_{\eta_2 v}\mathbb{P}^T_v\boldsymbol{v}. \end{aligned}$$

□

4.4 Equivalence between the eigenvalue problems

We are going to study the links between the eigenvalue problems associated to the operator $\mathbf{A}$ and the eigenvalue problem associated to the pair (A_L, M_L), i.e.

$$\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathrm{Ker}(A^T_{v\eta\theta}), \quad \mathbf{A}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} = \boldsymbol{\lambda}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix}, \tag{4.4 .1}$$

and

$$\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}) \in \mathbb{C}^n, \quad A_L\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ \boldsymbol{\theta} \end{bmatrix} = \boldsymbol{\lambda} M_L\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ \boldsymbol{\theta} \end{bmatrix}. \tag{4.4 .2}$$

We will also study the links between the adjoint eigenvalue problems

$$\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \in \mathrm{Ker}(A^T_{v\eta\theta}), \quad \mathbf{A}^{\#}\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} = \boldsymbol{\lambda}\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix}, \tag{4.4 .3}$$

and

$$\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}) \in \mathbb{C}^n, \quad A^T_L\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \\ \boldsymbol{\theta} \end{bmatrix} = \boldsymbol{\lambda} M_L\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \\ \boldsymbol{\rho} \end{bmatrix}, \tag{4.4 .4}$$

where the operator $\mathbf{A}^{\#}$ is defined by

$$\mathbf{A}^{\#} = \mathbb{P}^T M_s^{-T} A_0^T. \tag{4.4 .5}$$

We recall that $A_{v\eta\theta}$ is defined in (4.3 .9) and that $\mathrm{Ker}(A^T_{v\eta\theta}) = \mathrm{Im}(\mathbb{P}^T)$.

Definition 4.4 .1. *A vector* $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i) \in \mathrm{Ker}(A_{v\eta\theta}^T)$ *is a generalized eigenvector of order* i *for problem (4.4 .1) associated with a solution* $(\boldsymbol{\lambda}, (\boldsymbol{z}^0, \boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0))$ *of (4.4 .1) if* $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)$ *is obtained by solving the chain of equations*

$$(\boldsymbol{\lambda} I - \mathbf{A}) \begin{bmatrix} \boldsymbol{z}^j \\ \boldsymbol{\xi}_1^j \\ \boldsymbol{\xi}_2^j \end{bmatrix} = - \begin{bmatrix} \boldsymbol{z}^{j-1} \\ \boldsymbol{\xi}_1^{j-1} \\ \boldsymbol{\xi}_2^{j-1} \end{bmatrix}, \quad \textit{for } 1 \leq j \leq i.$$

A vector $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i) \in \mathbb{C}^n$ *is a generalized eigenvector of order* i *associated with a solution* $(\boldsymbol{\lambda}, (\boldsymbol{z}^0, \boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0, \boldsymbol{\theta}^0))$ *of (4.4 .2) if it is obtained by solving the chain of equations*

$$(\boldsymbol{\lambda} M_L - A_L) \begin{bmatrix} \boldsymbol{z}^j \\ \boldsymbol{\xi}_1^j \\ \boldsymbol{\xi}_2^j \\ \boldsymbol{\theta}^j \end{bmatrix} = -M_L \begin{bmatrix} \boldsymbol{z}^{j-1} \\ \boldsymbol{\xi}_1^{j-1} \\ \boldsymbol{\xi}_2^{j-1} \\ \boldsymbol{\theta}^{j-1} \end{bmatrix}, \quad \textit{for } 1 \leq j \leq i.$$

We have similar definitions for the problems (4.4 .3) and (4.4 .4).

4.4 .1 Equivalence between the direct eigenvalue problems

First, we will establish preliminary results that useful to study the equivalence between (4.4 .1) and (4.4 .2).

Lemma 4.4 .1. *Let* $F \in \mathbb{R}^{n_v}$ *and* $\mathbf{g} \in \mathbb{R}^{n_\theta}$. *The couple* $(\hat{\boldsymbol{z}}, \hat{\boldsymbol{\theta}})$ *is a solution of the system*

$$\begin{aligned} &\boldsymbol{\lambda} M_{vv} \hat{\boldsymbol{z}} = A_{vv} \hat{\boldsymbol{z}} + A_{v\theta} \hat{\boldsymbol{\theta}} + F, \\ &A_{v\theta}^T \hat{\boldsymbol{z}} = \boldsymbol{g}, \end{aligned} \tag{4.4 .6}$$

if and only if

$$\begin{aligned} &\boldsymbol{\lambda} \mathbb{P}_v^T \hat{\boldsymbol{z}} = \mathbb{A} \mathbb{P}_v^T \hat{\boldsymbol{z}} - \mathbb{A} \mathbb{P}_v^T \mathbb{L} \boldsymbol{g} + \mathbb{P}_v^T M_{vv}^{-1} F, \\ &(I - \mathbb{P}_v^T) \hat{\boldsymbol{z}} = M_{vv}^{-1} A_{v\theta} N_{\theta\theta} \boldsymbol{g}, \\ &\hat{\boldsymbol{\theta}} = -N_{\theta\theta} \left(-\boldsymbol{\lambda} \boldsymbol{g} + A_{\theta\theta} \boldsymbol{g} + A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \hat{\boldsymbol{z}} + A_{v\theta}^T M_{vv}^{-1} F \right). \end{aligned} \tag{4.4 .7}$$

Proof. The proof is similar to that of Proposition 4.3 .2.

□

Proposition 4.4 .1. *Let* $(\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta})$ *be in* $\mathbb{C}^n$ *such that*

$$A_{v\theta}^T \boldsymbol{z} + A_{\eta_1\theta}^T \boldsymbol{\xi}_1 + A_{\eta_2\theta}^T \boldsymbol{\xi}_2 = 0.$$

The quadruplet $(\hat{\boldsymbol{z}}, \hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2, \hat{\boldsymbol{\theta}}) \in \mathbb{C}^n$ *is solution of the equation*

$$(\boldsymbol{\lambda} M_L - A_L) \begin{bmatrix} \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \\ \hat{\boldsymbol{\theta}} \end{bmatrix} = -M_L \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ \boldsymbol{\theta} \end{bmatrix}, \tag{4.4 .8}$$

if and only if

$$
\begin{aligned}
&(\boldsymbol{\lambda} I-\mathbf{A})\mathbb{P}^T\begin{bmatrix}\hat{\boldsymbol{z}}\\ \hat{\boldsymbol{\xi}}_1\\ \hat{\boldsymbol{\xi}}_2\end{bmatrix}=-\mathbb{P}^T\begin{bmatrix}\boldsymbol{z}\\ \boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\end{bmatrix},\\
&(I-\mathbb{P}_v^T)\hat{\boldsymbol{z}}=-M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_1\theta}^T\hat{\boldsymbol{\xi}}_1-M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\xi}}_2,\\
&\hat{\boldsymbol{\theta}}=-\boldsymbol{\lambda}N_{\theta\theta}A_{\eta_1\theta}^T\hat{\boldsymbol{\xi}}_1-\boldsymbol{\lambda}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\xi}}_2-N_{\theta\theta}(A_{v\theta}^TM_{vv}^{-1}A_{v\eta_1}-A_{\theta\theta}A_{\eta_1\theta}^T)\hat{\boldsymbol{\xi}}_1\\
&\quad-N_{\theta\theta}(A_{v\theta}^TM_{vv}^{-1}A_{v\eta_2}-A_{\theta\theta}A_{\eta_2\theta}^T)\hat{\boldsymbol{\xi}}_2-N_{\theta\theta}A_{v\theta}^TM_{vv}^{-1}A_{vv}\mathbb{P}_v^T\hat{\boldsymbol{z}}+N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1+N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2,
\end{aligned}
\tag{4.4 .9}
$$

where the operator $A_{\theta\theta}$ is defined in (4.3 .7).

Proof. Let $(\hat{\boldsymbol{z}},\hat{\boldsymbol{\xi}}_1,\hat{\boldsymbol{\xi}}_2,\hat{\boldsymbol{\theta}})$ be a solution of system (4.4 .8). The couple $(\hat{\boldsymbol{z}},\hat{\boldsymbol{\theta}})$ is solution of the system

$$
\begin{aligned}
&\boldsymbol{\lambda}M_{vv}\hat{\boldsymbol{z}}=A_{vv}\hat{\boldsymbol{z}}+A_{v\theta}\hat{\boldsymbol{\theta}}+A_{v\eta_1}\hat{\boldsymbol{\xi}}_1+A_{v\eta_2}\hat{\boldsymbol{\xi}}_2+M_{vv}\boldsymbol{z},\\
&A_{v\theta}^T\hat{\boldsymbol{z}}=-(A_{\eta_1\theta}^T\hat{\boldsymbol{\xi}}_1+A_{\eta_2\theta}^T\hat{\boldsymbol{\xi}}_2).
\end{aligned}
\tag{4.4 .10}
$$

Thanks to Lemma 4.4 .1, we have

$$
\begin{aligned}
&\boldsymbol{\lambda}\mathbb{P}_v^T\hat{\boldsymbol{z}}=\mathbb{A}\mathbb{P}_v^T\hat{\boldsymbol{z}}+\mathbb{A}\mathbb{P}_v^T\mathbb{L}(A_{\eta_1\theta}^T\hat{\boldsymbol{\xi}}_1+A_{\eta_2\theta}^T\hat{\boldsymbol{\xi}}_2)+\mathbb{P}_v^TM_{vv}^{-1}(A_{v\eta_1}\hat{\boldsymbol{\xi}}_1+A_{v\eta_2}\hat{\boldsymbol{\xi}}_2)+\mathbb{P}_v^TM_{vv}^{-1}M_{vv}\boldsymbol{z},\\
&(I-\mathbb{P}_v^T)\hat{\boldsymbol{z}}=-M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_1\theta}^T\hat{\boldsymbol{\xi}}_1-M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\xi}}_2,\\
&\hat{\boldsymbol{\theta}}=-\boldsymbol{\lambda}N_{\theta\theta}A_{\eta_1\theta}^T\hat{\boldsymbol{\xi}}_1-\boldsymbol{\lambda}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\xi}}_2-N_{\theta\theta}(A_{v\theta}^TM_{vv}^{-1}A_{v\eta_1}-A_{\theta\theta}A_{\eta_1\theta}^T)\hat{\boldsymbol{\xi}}_1\\
&\quad-N_{\theta\theta}(A_{v\theta}^TM_{vv}^{-1}A_{v\eta_2}-A_{\theta\theta}A_{\eta_2\theta}^T)\hat{\boldsymbol{\xi}}_2-N_{\theta\theta}A_{v\theta}^TM_{vv}^{-1}A_{vv}\mathbb{P}_v^T\hat{\boldsymbol{z}}-N_{\theta\theta}A_{v\theta}^TM_{vv}^{-1}M_{vv}\boldsymbol{z}.
\end{aligned}
$$

Since

$$A_{v\theta}^T\boldsymbol{z}+A_{\eta_1\theta}^T\boldsymbol{\xi}_1+A_{\eta_2\theta}^T\boldsymbol{\xi}_2=0,$$

we have

$$N_{\theta\theta}A_{v\theta}^T\boldsymbol{z}=-N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1-N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2.$$

Hence

$$
\begin{aligned}
&\hat{\boldsymbol{\theta}}=-\boldsymbol{\lambda}N_{\theta\theta}A_{\eta_1\theta}^T\hat{\boldsymbol{\xi}}_1-\boldsymbol{\lambda}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\xi}}_2-N_{\theta\theta}(A_{v\theta}^TM_{vv}^{-1}A_{v\eta_1}-A_{\theta\theta}A_{\eta_1\theta}^T)\hat{\boldsymbol{\xi}}_1\\
&\quad-N_{\theta\theta}(A_{v\theta}^TM_{vv}^{-1}A_{v\eta_2}-A_{\theta\theta}A_{\eta_2\theta}^T)\hat{\boldsymbol{\xi}}_2-N_{\theta\theta}A_{v\theta}^TM_{vv}^{-1}A_{vv}\mathbb{P}_v^T\hat{\boldsymbol{z}}+N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1+N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2,
\end{aligned}
$$

then

$$
\begin{aligned}
A_{\eta_2\theta}\hat{\boldsymbol{\theta}}\;=\;&-\boldsymbol{\lambda}A_{\eta_2\theta}N_{\theta\theta}A_{\eta_1\theta}^T\hat{\boldsymbol{\xi}}_1-\boldsymbol{\lambda}A_{\eta_2\theta}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\xi}}_2-A_{\eta_2\theta}N_{\theta\theta}(A_{v\theta}^TM_{vv}^{-1}A_{v\eta_1}-A_{\theta\theta}A_{\eta_1\theta}^T)\hat{\boldsymbol{\xi}}_1\\
&-A_{\eta_2\theta}N_{\theta\theta}(A_{v\theta}^TM_{vv}^{-1}A_{v\eta_2}-A_{\theta\theta}A_{\eta_2\theta}^T)\hat{\boldsymbol{\xi}}_2-A_{\eta_2\theta}N_{\theta\theta}A_{v\theta}^TM_{vv}^{-1}A_{vv}\mathbb{P}_v^T\hat{\boldsymbol{z}}\\
&+A_{\eta_2\theta}N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1+A_{\eta_2\theta}N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2\\
=\;&-\boldsymbol{\lambda}N_{\eta_2\eta_1}\hat{\boldsymbol{\xi}}_1-\boldsymbol{\lambda}N_{\eta_2\eta_2}\hat{\boldsymbol{\xi}}_2+(\widetilde{A}_{\eta_2\eta_1}-A_{\eta_2\eta_1})\hat{\boldsymbol{\xi}}_1+(\widetilde{A}_{\eta_2\eta_2}-A_{\eta_2\eta_2})\hat{\boldsymbol{\xi}}_2+A_{\eta_2v}\mathbb{P}_v^T\hat{\boldsymbol{z}}\\
&+N_{\eta_2\eta_1}\boldsymbol{\xi}_1+N_{\eta_2\eta_2}\boldsymbol{\xi}_2.
\end{aligned}
\tag{4.4 .11}
$$

The system satisfied by $\hat{\boldsymbol{\xi}}_1$ and $\hat{\boldsymbol{\xi}}_2$ is

$$
\begin{aligned}
&\boldsymbol{\lambda}M_{\eta\eta}\hat{\boldsymbol{\xi}}_1=A_{\eta_1\eta_2}\hat{\boldsymbol{\xi}}_2+M_{\eta\eta}\boldsymbol{\xi}_1,\\
&\boldsymbol{\lambda}M_{\eta\eta}\hat{\boldsymbol{\xi}}_2=A_{\eta_2\theta}\hat{\boldsymbol{\theta}}+A_{\eta_2\eta_1}\hat{\boldsymbol{\xi}}_1+A_{\eta_2\eta_2}\hat{\boldsymbol{\xi}}_2+M_{\eta\eta}\boldsymbol{\xi}_2.
\end{aligned}
$$

Replacing $A_{\eta_2\theta}\hat{\boldsymbol{\theta}}$ by (4.4 .11), we obtain

$$\begin{aligned}
&\boldsymbol{\lambda} M_{\eta\eta}\hat{\boldsymbol{\xi}}_1 = A_{\eta_1\eta_2}\hat{\boldsymbol{\xi}}_2 + M_{\eta\eta}\boldsymbol{\xi}_1,\\
&\boldsymbol{\lambda}(M_{\eta\eta}+N_{\eta_2\eta_2})\hat{\boldsymbol{\xi}}_2 + \boldsymbol{\lambda} N_{\eta_2\eta_1}\hat{\boldsymbol{\xi}}_1 = \widetilde{A}_{\eta_2\eta_1}\hat{\boldsymbol{\xi}}_1 + \widetilde{A}_{\eta_2\eta_2}\hat{\boldsymbol{\xi}}_2 + A_{\eta_2 v}\mathbb{P}_v^T\hat{\boldsymbol{z}} + N_{\eta_2\eta_1}\boldsymbol{\xi}_1 + (M_{\eta\eta}+N_{\eta_2\eta_2})\boldsymbol{\xi}_2.
\end{aligned}$$

Thus, $(\mathbb{P}_v^T\hat{\boldsymbol{z}}, \hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2)$ is solution to the equation

$$(\boldsymbol{\lambda} I - \mathbf{A})\mathbb{P}^T \begin{bmatrix} \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_2 \end{bmatrix} = -\mathbb{P}^T \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix}.$$

□

Now, we are in position to prove the equivalence between the two direct eigenvalue problems.

Theorem 4.4 .2. *A pair $(\boldsymbol{\lambda}, (\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\theta}))$ is a solution of eigenvalue problem (4.4 .2) if and only if $(\boldsymbol{\lambda}, (\mathbb{P}_v^T\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)) \in \mathbb{C}^* \times \mathrm{Ker}(A_{v\eta\theta}^T)$ is a solution of eigenvalue problem (4.4 .1) and*

$$\begin{aligned}
(I-\mathbb{P}_v^T)\boldsymbol{z} &= -M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1 - M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2,\\
\boldsymbol{\theta} &= -\boldsymbol{\lambda} N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1 - \boldsymbol{\lambda} N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2 - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_1} - A_{\theta\theta}A_{\eta_1\theta}^T)\boldsymbol{\xi}_1\\
&\quad - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_2} - A_{\theta\theta}A_{\eta_2\theta}^T)\boldsymbol{\xi}_2 - N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\boldsymbol{z}.
\end{aligned}$$

Proof. It follows from Proposition 4.4 .1.

□

Theorem 4.4 .3. *A quadruplet $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i)$ is a generalized eigenvector of order i associated with the solution $(\boldsymbol{\lambda}, (\boldsymbol{z}^0, \boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0, \boldsymbol{\theta}^0))$ of problem (4.4 .2) if and only if $(\mathbb{P}_v^T\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i) \in \mathrm{Ker}(A_{v\eta\theta}^T)$ is a generalized eigenvector of order i associated with the solution $(\boldsymbol{\lambda}, (\mathbb{P}_v^T\boldsymbol{z}^0, \boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0)) \in \mathbb{C}^* \times \mathrm{Ker}(A_{v\eta\theta}^T)$ of (4.4 .1) and*

$$\begin{aligned}
(I-\mathbb{P}_v^T)\boldsymbol{z}^i &= -M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1^i - M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2^i,\\
\boldsymbol{\theta}^i &= -\boldsymbol{\lambda} N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1^i - \boldsymbol{\lambda} N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2^i - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_1} - A_{\theta\theta}A_{\eta_1\theta}^T)\boldsymbol{\xi}_1^i\\
&\quad - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_2} - A_{\theta\theta}A_{\eta_2\theta}^T)\boldsymbol{\xi}_2^i - N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\boldsymbol{z}^i\\
&\quad - N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1^{i-1} - N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2^{i-1}.
\end{aligned}$$

The vector $(\mathbb{P}_v^T\boldsymbol{z}^{i-1}, \boldsymbol{\xi}_1^{i-1}, \boldsymbol{\xi}_2^{i-1}) \in \mathrm{Ker}(A_{v\eta\theta}^T)$ is a generalized eigenvector of order $i-1$ associated with $(\boldsymbol{\lambda}, (\mathbb{P}_v^T\boldsymbol{z}^0, \boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0))$.

Proof. Thanks to Proposition 4.4 .1, it follows that $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i)$ is a generalized eigenvector of order i associated with the solution $(\boldsymbol{\lambda}, (\boldsymbol{z}^0, \boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0, \boldsymbol{\theta}^0))$ of problem (4.4 .2) if and only if

$$\begin{aligned}
&(\boldsymbol{\lambda} I - \mathbf{A})\mathbb{P}^T \begin{bmatrix} \boldsymbol{z}^i \\ \boldsymbol{\xi}_1^i \\ \boldsymbol{\xi}_2^i \end{bmatrix} = -\mathbb{P}^T \begin{bmatrix} \boldsymbol{z}^{i-1} \\ \boldsymbol{\xi}_1^{i-1} \\ \boldsymbol{\xi}_2^{i-1} \end{bmatrix},\\
&(I-\mathbb{P}_v^T)\boldsymbol{z}^i = -M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1^i - M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2^i,\\
&\boldsymbol{\theta}^i = -\boldsymbol{\lambda} N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1^i - \boldsymbol{\lambda} N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2^i - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_1} - A_{\theta\theta}A_{\eta_1\theta}^T)\boldsymbol{\xi}_1^i\\
&\quad - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_2} - A_{\theta\theta}A_{\eta_2\theta}^T)\boldsymbol{\xi}_2^i - N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\boldsymbol{z}^i\\
&\quad - N_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\xi}_1^{i-1} - N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\xi}_2^{i-1}.
\end{aligned}$$

We deduce the theorem by induction.

□

4.4 .2 Equivalence between the adjoint eigenvalue problems

We introduce the operator $\mathbb{L}^{\#}$ in $\mathbb{R}^{n_\theta}$ defined by

$$\mathbb{L}^{\#}\boldsymbol{g} = \boldsymbol{\phi}, \tag{4.4 .12}$$

where $(\boldsymbol{\psi}, \boldsymbol{\rho})$ is the solution of the equation

$$A_{vv}^T\boldsymbol{\psi} + A_{v\theta}\boldsymbol{\rho} = 0, \quad A_{v\theta}^T\boldsymbol{\psi} = \boldsymbol{g}.$$

As in Lemma 4.3 .2, we can prove that

$$\mathbb{L}^{\#} = A_{vv}^{-T}A_{v\theta}N_{\theta\theta} \quad \text{and} \quad (I - \mathbb{P}_v^T)\mathbb{L}^{\#} = M_{vv}^{-1}A_{v\theta}N_{\theta\theta}. \tag{4.4 .13}$$

We introduce the operator $\mathbb{A}^{\#}$ in $\mathbb{R}^{n_v}$ defined by

$$\mathbb{A}^{\#} = \mathbb{P}_v^T M_{vv}^{-1}A_{vv}^T. \tag{4.4 .14}$$

We also introduce the operator $A_{\theta\theta}^{\#}$ in $\mathbb{R}^{n_\theta}$ defined by

$$A_{\theta\theta}^{\#} = A_{v\theta}^T M_{vv}^{-1}A_{vv}^T M_{vv}^{-1}A_{v\theta}N_{\theta\theta}. \tag{4.4 .15}$$

Lemma 4.4 .2. *Let $G \in \mathbb{R}^{n_v}$ and $\boldsymbol{h} \in \mathbb{R}^{n_\theta}$. The couple $(\boldsymbol{\psi}, \boldsymbol{\rho})$ is a solution of the system*

$$\begin{aligned} \boldsymbol{\lambda} M_{vv}\boldsymbol{\psi} &= A_{vv}^T\boldsymbol{\psi} + A_{v\theta}\boldsymbol{\rho} + G, \\ A_{v\theta}^T\boldsymbol{\psi} &= \boldsymbol{h}. \end{aligned} \tag{4.4 .16}$$

if and only if

$$\begin{aligned} \boldsymbol{\lambda}\mathbb{P}_v^T\boldsymbol{\psi} &= \mathbb{A}^{\#}\mathbb{P}_v^T\boldsymbol{\psi} - \mathbb{A}^{\#}\mathbb{P}_v^T\mathbb{L}^{\#}\boldsymbol{h} + \mathbb{P}_v^T M_{vv}^{-1}G, \\ (I - \mathbb{P}_v^T)\boldsymbol{\psi} &= M_{vv}^{-1}A_{v\theta}N_{\theta\theta}\boldsymbol{h}, \\ \boldsymbol{\rho} &= -N_{\theta\theta}\left(-\boldsymbol{\lambda}\boldsymbol{h} + A_{\theta\theta}^{\#}\boldsymbol{h} + A_{v\theta}^T M_{vv}^{-1}A_{vv}^T\mathbb{P}_v^T\boldsymbol{\psi} + A_{v\theta}^T M_{vv}^{-1}G\right). \end{aligned} \tag{4.4 .17}$$

Proof. It is similar to that of Lemma 4.4 .1.

□

Proposition 4.4 .4. *Let $(\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)$ be in $\mathbb{C}^n$ such that*

$$A_{v\theta}^T\boldsymbol{\psi} + A_{\eta_2\theta}^T\boldsymbol{\kappa}_2 = 0.$$

The quadruplet $(\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\kappa}}_1, \hat{\boldsymbol{\kappa}}_2, \hat{\boldsymbol{\rho}}) \in \mathbb{C}^n$ is solution of the equation

$$(\boldsymbol{\lambda} M_L - A_L^T)\begin{bmatrix} \hat{\boldsymbol{\psi}} \\ \hat{\boldsymbol{\kappa}}_1 \\ \hat{\boldsymbol{\kappa}}_2 \\ \hat{\boldsymbol{\rho}} \end{bmatrix} = -M_L\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \\ \boldsymbol{\rho} \end{bmatrix}, \tag{4.4 .18}$$

if and only if

$$\begin{aligned} &(\boldsymbol{\lambda} I - \mathbf{A}^{\#})\mathbb{P}^T\begin{bmatrix} \hat{\boldsymbol{\psi}} \\ \hat{\boldsymbol{\kappa}}_1 \\ \hat{\boldsymbol{\kappa}}_2 \end{bmatrix} = -\mathbb{P}^T\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix}, \\ &(I - \mathbb{P}_v^T)\hat{\boldsymbol{\psi}} = -M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2, \\ &\hat{\boldsymbol{\rho}} = -\boldsymbol{\lambda} N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 + N_{\theta\theta}A_{\theta\theta}^{\#}A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 - N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}^T\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2. \end{aligned} \tag{4.4 .19}$$

Proof. Let $(\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\kappa}}_1, \hat{\boldsymbol{\kappa}}_2, \hat{\boldsymbol{\rho}}) \in \mathbb{C}^n$ be a solution of (4.4 .18). Then, $(\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\rho}})$ is solution of the system

$$\begin{aligned} \boldsymbol{\lambda} M_{vv}\hat{\boldsymbol{\psi}} &= A_{vv}^T\hat{\boldsymbol{\psi}} + A_{v\theta}\hat{\boldsymbol{\rho}} + M_{vv}\boldsymbol{\psi}, \\ A_{v\theta}^T\hat{\boldsymbol{\psi}} &= -A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2. \end{aligned} \tag{4.4 .20}$$

Thanks to Lemma 4.4 .2, we have

$$\begin{aligned} &\boldsymbol{\lambda}\mathbb{P}_v^T\hat{\boldsymbol{\psi}} = \mathbb{A}^\#\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + \mathbb{A}^\#\mathbb{P}_v^T\mathbb{L}^\# A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 + \mathbb{P}_v^T M_{vv}^{-1}M_{vv}\boldsymbol{\psi}, \\ &(I - \mathbb{P}_v^T)\hat{\boldsymbol{\psi}} = -M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2, \\ &\hat{\boldsymbol{\rho}} = -\boldsymbol{\lambda} N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 + N_{\theta\theta}A_{\theta\theta}^\# A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 - N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}^T\mathbb{P}_v^T\hat{\boldsymbol{\psi}} - N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}M_{vv}\boldsymbol{\psi}. \end{aligned}$$

The equation satisfied by $\mathbb{P}_v^T\hat{\boldsymbol{\psi}}$ is

$$\boldsymbol{\lambda}\mathbb{P}_v^T\hat{\boldsymbol{\psi}} = \mathbb{A}^\#\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + \mathbb{A}^\#\mathbb{P}_v^T\mathbb{L}^\# A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 + \mathbb{P}_v^T\boldsymbol{\psi}. \tag{4.4 .21}$$

Since

$$A_{v\theta}^T\boldsymbol{\psi} + A_{\eta_2\theta}^T\boldsymbol{\kappa}_2 = 0,$$

we have

$$N_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} = -N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2.$$

Hence

$$\hat{\boldsymbol{\rho}} = -\boldsymbol{\lambda} N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 + N_{\theta\theta}A_{\theta\theta}^\# A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 - N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}^T\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2,$$

$$\begin{aligned} A_{\eta_1\theta}\hat{\boldsymbol{\rho}} &= -\boldsymbol{\lambda} A_{\eta_1\theta}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 + A_{\eta_1\theta}N_{\theta\theta}A_{\theta\theta}^\# A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 - A_{\eta_1\theta}N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + A_{\eta_1\theta}N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2 \\ &= -\boldsymbol{\lambda} N_{\eta_2\eta_1}^T\hat{\boldsymbol{\kappa}}_2 + A_{\eta_1\theta}N_{\theta\theta}A_{\theta\theta}^\# A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 - A_{\eta_1\theta}N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}^T\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + N_{\eta_2\eta_1}^T\boldsymbol{\kappa}_2, \end{aligned} \tag{4.4 .22}$$

and

$$\begin{aligned} A_{\eta_2\theta}\hat{\boldsymbol{\rho}} &= -\boldsymbol{\lambda} A_{\eta_2\theta}N_{\theta\theta}A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 + A_{\eta_2\theta}N_{\theta\theta}A_{\theta\theta}^\# A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 - A_{\eta_2\theta}N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}^T\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + A_{\eta_2\theta}N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2 \\ &= -\boldsymbol{\lambda} N_{\eta_2\eta_2}\hat{\boldsymbol{\kappa}}_2 + A_{\eta_2\theta}N_{\theta\theta}A_{\theta\theta}^\# A_{\eta_2\theta}^T\hat{\boldsymbol{\kappa}}_2 - A_{\eta_2\theta}N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + N_{\eta_2\eta_2}\boldsymbol{\kappa}_2. \end{aligned} \tag{4.4 .23}$$

The couple $(\hat{\boldsymbol{\kappa}}_1, \hat{\boldsymbol{\kappa}}_2)$ is a solution to the system

$$\begin{aligned} \boldsymbol{\lambda} M_{\eta\eta}\hat{\boldsymbol{\kappa}}_1 &= A_{\eta_2\eta_1}^T\hat{\boldsymbol{\kappa}}_2 + A_{v\eta_1}^T\hat{\boldsymbol{\psi}} + A_{\eta_1\theta}\hat{\boldsymbol{\rho}} + M_{\eta\eta}\boldsymbol{\kappa}_1, \\ \boldsymbol{\lambda} M_{\eta\eta}\hat{\boldsymbol{\kappa}}_2 &= A_{\eta_1\eta_2}^T\hat{\boldsymbol{\kappa}}_1 + A_{\eta_2\eta_2}^T\hat{\boldsymbol{\kappa}}_2 + A_{v\eta_2}^T\hat{\boldsymbol{\psi}} + A_{\eta_2\theta}\hat{\boldsymbol{\rho}} + M_{\eta\eta}\boldsymbol{\kappa}_2. \end{aligned}$$

Replacing $A_{\eta_1\theta}\hat{\boldsymbol{\rho}}$ by (4.4 .22) and $A_{\eta_2\theta}\hat{\boldsymbol{\rho}}$ by (4.4 .23) in the equations of $\hat{\boldsymbol{\kappa}}_1$ and $\hat{\boldsymbol{\kappa}}_2$ respectively, we obtain

$$\begin{aligned} &\boldsymbol{\lambda} M_{\eta\eta}\hat{\boldsymbol{\kappa}}_1 + \boldsymbol{\lambda} N_{\eta_2\eta_1}^T\hat{\boldsymbol{\kappa}}_2 = A_{\eta_2\eta_1}^T\hat{\boldsymbol{\kappa}}_2 + C_{\eta_1\eta_2}\hat{\boldsymbol{\kappa}}_2 + A_{v\eta_1}^T\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + C_{\eta_1 v}\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + M_{\eta\eta}\boldsymbol{\kappa}_1 + N_{\eta_2\eta_1}^T\boldsymbol{\kappa}_2, \\ &\boldsymbol{\lambda}(M_{\eta\eta} + N_{\eta_2\eta_2})\hat{\boldsymbol{\kappa}}_2 = A_{\eta_1\eta_2}^T\hat{\boldsymbol{\kappa}}_1 + A_{\eta_2\eta_2}^T\hat{\boldsymbol{\kappa}}_2 + C_{\eta_2\eta_2}\hat{\boldsymbol{\kappa}}_2 + A_{v\eta_2}^T\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + C_{\eta_2 v}\mathbb{P}_v^T\hat{\boldsymbol{\psi}} + (M_{\eta\eta} + N_{\eta_2\eta_2})\boldsymbol{\kappa}_2, \end{aligned} \tag{4.4 .24}$$

where

$$\begin{aligned} C_{\eta_1\eta_2} &:= A_{\eta_1\theta}N_{\theta\theta}A_{\theta\theta}^\# A_{\eta_2\theta}^T - A_{v\eta_1}^T M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T, \\ C_{\eta_2\eta_2} &:= A_{\eta_2\theta}N_{\theta\theta}A_{\theta\theta}^\# A_{\eta_2\theta}^T - A_{v\eta_2}^T M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A_{\eta_2\theta}^T, \\ C_{\eta_1 v} &:= -A_{\eta_1\theta}N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}^T, \\ C_{\eta_2 v} &:= -A_{\eta_2\theta}N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}^T. \end{aligned}$$

Thanks to (4.4 .21) and (4.4 .24), we have

$$(\boldsymbol{\lambda} I - \mathbb{P}^T M_s^{-T} C)\mathbb{P}^T \begin{bmatrix} \hat{\boldsymbol{\psi}} \\ \hat{\boldsymbol{\kappa}}_1 \\ \hat{\boldsymbol{\kappa}}_2 \end{bmatrix} = -\mathbb{P}^T \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix},$$

where

$$C := \begin{bmatrix} A_{vv}^T & 0 & A_{vv}^T \mathbb{P}_v^T \mathbb{L}^\# A_{\eta_2\theta}^T \\ A_{v\eta_1}^T + C_{\eta_1 v} & 0 & A_{\eta_2\eta_1}^T + C_{\eta_1\eta_2} \\ A_{v\eta_2}^T + C_{\eta_2 v} & A_{\eta_1\eta_2}^T & A_{\eta_2\eta_2}^T + C_{\eta_2\eta_2} \end{bmatrix}.$$

We recall that $\mathbf{A}^\# = \mathbb{P}^T M_s^{-T} A_0^T$ with

$$A_0 = \begin{bmatrix} A_{vv} & A_{v\eta_1} + A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_1\theta}^T & A_{v\eta_2} + A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_2\theta}^T \\ 0 & 0 & A_{\eta_1\eta_2} \\ A_{\eta_2 v} & \widetilde{A}_{\eta_2\eta_1} & \widetilde{A}_{\eta_2\eta_2} \end{bmatrix},$$

and

$$\begin{aligned} A_{\eta_2 v} &= -A_{\eta_2\theta} N_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}, \\ \widetilde{A}_{\eta_2\eta_1} &= A_{\eta_1\eta_2} - A_{\eta_2\theta} N_{\theta\theta} (A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1} - A_{\theta\theta} A_{\eta_1\theta}^T), \\ \widetilde{A}_{\eta_2\eta_2} &= A_{\eta_2\eta_2} - A_{\eta_2\theta} N_{\theta\theta} (A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2} - A_{\theta\theta} A_{\eta_2\theta}^T). \end{aligned}$$

To complete the proof we are going to prove that $\mathbb{P}^T M_s^{-T} C \mathbb{P}^T = \mathbb{P}^T M_s^{-T} A_0^T \mathbb{P}^T$. We have

$$A_0^T = \begin{bmatrix} A_{vv}^T & 0 & A_{\eta_2 v}^T \\ A_{\eta_1 v}^T + (A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_1\theta}^T)^T & 0 & \widetilde{A}_{\eta_2\eta_1}^T \\ A_{v\eta_2}^T + (A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_2\theta}^T)^T & A_{\eta_1\eta_2}^T & \widetilde{A}_{\eta_2\eta_2}^T \end{bmatrix}.$$

Hence

$$(A_0^T - C)\mathbb{P}^T = \begin{bmatrix} 0 & 0 & A_{\eta_2 v}^T - A_{vv}^T \mathbb{P}_v^T \mathbb{L}^\# A_{\eta_2\theta}^T \\ ((A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_1\theta}^T)^T - C_{\eta_1 v})\mathbb{P}_v^T & 0 & \widetilde{A}_{\eta_2\eta_1}^T - A_{\eta_2\eta_1}^T - C_{\eta_1\eta_2} \\ ((A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_2\theta}^T)^T - C_{\eta_2 v})\mathbb{P}_v^T & 0 & \widetilde{A}_{\eta_2\eta_2}^T - A_{\eta_2\eta_2}^T - C_{\eta_2\eta_2} \end{bmatrix}. \tag{4.4 .25}$$

Thanks to equation (4.4 .13), we obtain

$$\begin{aligned} A_{vv}^T \mathbb{P}_v^T \mathbb{L}^\# A_{\eta_2\theta}^T &= A_{vv}^T \mathbb{L}^\# A_{\eta_2\theta}^T - A_{vv}^T (I - \mathbb{P}_v^T) \mathbb{L}^\# A_{\eta_2\theta}^T \\ &= A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T - A_{vv}^T M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T \\ &= A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T + A_{\eta_2\theta}^T, \end{aligned} \tag{4.4 .26}$$

$$\begin{aligned} A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_1\theta}^T &= A_{vv} \mathbb{L} A_{\eta_1\theta}^T - A_{vv}(I - \mathbb{P}_v^T) \mathbb{L} A_{\eta_1\theta}^T \\ &= A_{v\theta} N_{\theta\theta} A_{\eta_1\theta}^T - A_{vv} M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_1\theta}^T \\ &= A_{v\theta} N_{\theta\theta} A_{\eta_1\theta}^T + C_{\eta_1 v}^T, \end{aligned} \tag{4.4 .27}$$

and

$$\begin{aligned} A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_2\theta}^T &= A_{vv} \mathbb{L} A_{\eta_2\theta}^T - A_{vv}(I - \mathbb{P}_v^T) \mathbb{L} A_{\eta_2\theta}^T \\ &= A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T - A_{vv} M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T \\ &= A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T + C_{\eta_2 v}^T, \end{aligned} \tag{4.4 .28}$$

(the operator $A_{\eta_2\theta}$ is defined in (4.3 .14)). It follows that

$$\begin{aligned} &A^T_{\eta_2 v} - A^T_{vv}\mathbb{P}^T_v\mathbb{L}^{\#}A^T_{\eta_2\theta} = -A_{v\theta}N_{\theta\theta}A^T_{\eta_2\theta}, \\ &((A_{vv}\mathbb{P}^T_v\mathbb{L}A^T_{\eta_1\theta})^T - C_{\eta_1 v})\mathbb{P}^T_v = A_{\eta_1\theta}N_{\theta\theta}A^T_{v\theta}\mathbb{P}^T_v = 0, \\ &((A_{vv}\mathbb{P}^T_v\mathbb{L}A^T_{\eta_2\theta})^T - C_{\eta_2 v})\mathbb{P}^T_v = A_{\eta_2\theta}N_{\theta\theta}A^T_{v\theta}\mathbb{P}^T_v = 0, \end{aligned} \tag{4.4 .29}$$

because $\mathbb{P}_v A_{v\theta} = 0$ (see Proposition 4.3 .1). We recall that

$$A_{\theta\theta} = A^T_{v\theta}M^{-1}_{vv}A_{vv}M^{-1}_{vv}A_{v\theta}N_{\theta\theta} \quad \text{and} \quad A^{\#}_{\theta\theta} = A^T_{v\theta}M^{-1}_{vv}A^T_{vv}M^{-1}_{vv}A_{v\theta}N_{\theta\theta},$$

Then, we can easily prove that

$$A_{\eta_1\theta}N_{\theta\theta}A^{\#}_{\theta\theta}A^T_{\eta_2\theta} = A_{\eta_1\theta}A^T_{\theta\theta}N_{\theta\theta}A^T_{\eta_2\theta} \quad \text{and} \quad A_{\eta_2\theta}N_{\theta\theta}A^{\#}_{\theta\theta}A^T_{\eta_2\theta} = A_{\eta_2\theta}A^T_{\theta\theta}N_{\theta\theta}A^T_{\eta_2\theta}.$$

Hence

$$\widetilde{A}^T_{\eta_2\eta_1} - A^T_{\eta_2\eta_1} - C_{\eta_1\eta_2} = A_{\eta_1\theta}A^T_{\theta\theta}N_{\theta\theta}A^T_{\eta_2\theta} - A_{\eta_1\theta}N_{\theta\theta}A^{\#}_{\theta\theta}A^T_{\eta_2\theta} = 0, \tag{4.4 .30}$$

and

$$\widetilde{A}^T_{\eta_2\eta_2} - A^T_{\eta_2\eta_2} - C_{\eta_2\eta_2} = A_{\eta_2\theta}A^T_{\theta\theta}N_{\theta\theta}A^T_{\eta_2\theta} - A_{\eta_2\theta}N_{\theta\theta}A^{\#}_{\theta\theta}A^T_{\eta_2\theta} = 0. \tag{4.4 .31}$$

From equations (4.4 .25), (4.4 .29), (4.4 .30) and (4.4 .30), we obtain

$$(A^T_0 - C)\mathbb{P}^T = \begin{bmatrix} 0 & 0 & -A_{v\theta}N_{\theta\theta}A^T_{\eta_2\theta} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.4 .32}$$

We deduce that

$$\mathbb{P}^T M_s^{-T}(A^T_0 - C)\mathbb{P}^T = \begin{bmatrix} 0 & 0 & -\mathbb{P}^T_v M^{-1}_{vv}A_{v\theta}N_{\theta\theta}A^T_{\eta_2\theta} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{4.4 .33}$$

because $\mathbb{P}^T_v M^{-1}_{vv}A_{v\theta} = M_{vv}\mathbb{P}_v A_{v\theta} = 0$ (see Proposition 4.3 .1). Thus, we have

$$\mathbb{P}^T M_s^{-T} C\mathbb{P}^T = \mathbb{P}^T M_s^{-T} A^T_0 \mathbb{P}^T.$$

□

Now, we are going to prove the equivalence between the eigenvalue problems.

Theorem 4.4 .5. *A pair $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}))$ is a solution of eigenvalue problem (4.4 .4) if and only if $(\boldsymbol{\lambda}, (\mathbb{P}^T_v\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)) \in \mathbb{C}^* \times \mathrm{Ker}(A^T_{v\eta\theta})$ is a solution of eigenvalue problem (4.4 .3) and*

$$\begin{aligned} (I - \mathbb{P}^T_v)\boldsymbol{\psi} &= -M^{-1}_{vv}A_{v\theta}N_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\kappa}_2, \\ \boldsymbol{\rho} &= -\boldsymbol{\lambda}N_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\kappa}_2 + N_{\theta\theta}A^{\#}_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\kappa}_2 - N_{\theta\theta}A^T_{v\theta}M^{-1}_{vv}A_{vv}\mathbb{P}^T_v\boldsymbol{\psi}. \end{aligned}$$

Proof. It follows from Proposition 4.4 .4

□

Theorem 4.4 .6. *A quadruplet* $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i)$ *is a generalized eigenvector of order* i *associated with the solution* $(\boldsymbol{\lambda}, (\boldsymbol{\psi}^0, \boldsymbol{\kappa}_1^0, \boldsymbol{\kappa}_2^0, \boldsymbol{\rho}^0))$ *of problem (4.4 .4) if and only if* $(\mathbb{P}_v^T \boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i) \in \mathrm{Ker}(A_{v\eta\theta}^T)$ *is a generalized eigenvector of order* i *associated with the solution* $(\boldsymbol{\lambda}, (\mathbb{P}_v^T \boldsymbol{\psi}^0, \boldsymbol{\kappa}_1^0, \boldsymbol{\kappa}_2^0)) \in \mathbb{C}^* \times \mathrm{Ker}(A_{v\eta\theta}^T)$ *of (4.4 .3) and*

$$
\begin{aligned}
&(I - \mathbb{P}_v^T)\boldsymbol{\psi}^i = -M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^i,\\
&\boldsymbol{\rho}^i = -\boldsymbol{\lambda} N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^i + N_{\theta\theta} A_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^i - N_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{\psi}^i + N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^{i-1}.
\end{aligned}
$$

The vector $(\mathbb{P}_v^T \boldsymbol{\psi}^{i-1}, \boldsymbol{\kappa}_1^{i-1}, \boldsymbol{\kappa}_2^{i-1}) \in \mathrm{Ker}(A_{v\eta\theta}^T)$ *is a generalized eigenvector of order* $i-1$.

Proof. Thanks to Proposition 4.4 .4, it follows that $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i)$ is a generalized eigenvector of order i associated with the solution $(\boldsymbol{\lambda}, (\boldsymbol{\psi}^0, \boldsymbol{\kappa}_1^0, \boldsymbol{\kappa}_2^0, \boldsymbol{\rho}^0))$ of problem (4.4 .4) if and only if

$$
\begin{aligned}
&(\boldsymbol{\lambda} I - \mathbf{A}^{\#})\mathbb{P}^T \begin{bmatrix} \boldsymbol{\psi}^i \\ \boldsymbol{\kappa}_1^i \\ \boldsymbol{\kappa}_2^i \end{bmatrix} = -\mathbb{P}^T \begin{bmatrix} \boldsymbol{\psi}^{i-1} \\ \boldsymbol{\kappa}_1^{i-1} \\ \boldsymbol{\kappa}_2^{i-1} \end{bmatrix},\\
&(I - \mathbb{P}_v^T)\boldsymbol{\psi}^i = -M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^i,\\
&\boldsymbol{\rho}^i = -\boldsymbol{\lambda} N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^i + N_{\theta\theta} A_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^i - N_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{\psi}^i + N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^{i-1}.
\end{aligned}
$$

We complete the proof by induction.

□

4.5 Stabilization of the finite-dimensional linear system

4.5 .1 Spectral decomposition of the operator $\mathbf{A}\mathbb{P}^T$

We are looking for a decomposition of $\mathbb{R}^{n_1}$ into a sum of the generalized eigenspaces of the operator $\mathbf{A}\mathbb{P}^T$. We assume that $0 \notin \sigma(\mathbf{A})$. We can decompose $\mathbb{R}^{n_1}$ as follows

$$
\mathbb{R}^{n_1} = \mathrm{Ker}(A_{v\eta\theta}^T) \oplus \mathrm{Im}(M_s^{-1} A_{v\eta\theta}),
$$

with

$$
\mathrm{Ker}(A_{v\eta\theta}^T) = \mathrm{Im}(\mathbb{P}^T) \quad \text{and} \quad \mathrm{Im}(M_s^{-1} A_{v\eta\theta}) = \mathrm{Ker}(\mathbb{P}^T).
$$

Thus, since $0 \notin \sigma(\mathbf{A})$, we have $\mathrm{Ker}(\mathbf{A}\mathbb{P}^T) = \mathrm{Ker}(\mathbb{P}^T) = \mathrm{Im}(M_s^{-1} A_{v\eta\theta})$. Therefore, 0 is an eigenvalue of the operator $\mathbf{A}\mathbb{P}^T$ and $\mathrm{Im}(M_s^{-1} A_{v\eta\theta})$ is the corresponding eigenspace. In order the decompose $\mathrm{Ker}(A_{v\eta\theta}^T)$ into the other eigenspaces of the operator $\mathbf{A}\mathbb{P}^T$, we consider the eigenvalue problem (4.4 .1) and the adjoint problem (4.4 .3). We are going prove that the spectral decomposition of the operators $\mathbf{A}\mathbb{P}^T$ and $\mathbf{A}^{\#}\mathbb{P}^T$ can be deduced from the study of the eigenvalue problems (4.4 .2) and (4.4 .4). First, we show the following result.

Lemma 4.5 .1. *Let* $(\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathbb{C}^{n_1}$ *and* $(\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \in \mathbb{C}^{n_1}$ *such that*

$$
A_{v\theta}^T \boldsymbol{z} + A_{\eta_1\theta}^T \boldsymbol{\xi}_1 + A_{\eta_2\theta}^T \boldsymbol{\xi}_2 = 0 \quad \textit{and} \quad A_{v\theta}^T \boldsymbol{\psi} + A_{\eta_2\theta} \boldsymbol{\kappa}_2 = 0.
$$

Then, we have

$$
(\boldsymbol{z}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)^T (\mathbb{P} M_s^T - M)(\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) = 0.
$$

Proof. We have

$$\mathbb{P}M_s^T - M = \begin{bmatrix} (\mathbb{P}_v - I)M_{vv} & 0 & 0 \\ 0 & 0 & N_{\eta_2\eta_1}^T \\ 0 & 0 & N_{\eta_2\eta_2} \end{bmatrix} = \begin{bmatrix} -A_{v\theta}N_{\theta\theta}A_{v\theta}^T & 0 & 0 \\ 0 & 0 & N_{\eta_2\eta_1}^T \\ 0 & 0 & N_{\eta_2\eta_2} \end{bmatrix},$$

because

$$\mathbb{P}_v - I = -A_{v\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\theta})^{-1} A_{v\theta}^T M_{vv}^{-1} = -A_{v\theta}N_{\theta\theta}A_{v\theta}^T M_{vv}^{-1},$$

(see Proposition 4.3 .1) It follows that

$$\begin{aligned}(\boldsymbol{z},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)^T(\mathbb{P}M_s^T - M)(\boldsymbol{\psi},\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2) &= -\boldsymbol{z}^T A_{v\theta}N_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} + \boldsymbol{\xi}_1^T A_{\eta_1\theta}N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2 + \boldsymbol{\xi}_2^T A_{\eta_2\theta}N_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2 \\ &= -\boldsymbol{z}^T A_{v\theta}N_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} - \boldsymbol{\xi}_1^T A_{\eta_1\theta}N_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} - \boldsymbol{\xi}_2^T A_{\eta_2\theta}N_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} \\ &= -(\boldsymbol{z}^T A_{v\theta} + \boldsymbol{\xi}_1^T A_{\eta_1\theta} + \boldsymbol{\xi}_2^T A_{\eta_2\theta})N_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} \\ &= 0.\end{aligned}$$

□

We set

$$n_0 = \dim(\mathrm{Ker}(A_{v\eta\theta})). \tag{4.5 .1}$$

Theorem 4.5 .1. *There exist two families* $(\boldsymbol{z}^i,\boldsymbol{\xi}_1^i,\boldsymbol{\xi}_2^i,\boldsymbol{\theta}^i)_{1\le i\le n}$ *and* $(\boldsymbol{\psi}^i,\boldsymbol{\kappa}_1^i,\boldsymbol{\kappa}_2^i,\boldsymbol{\rho}^i)_{1\le i\le n}$ *of* $\mathbb{C}^n$ *constituted of eigenvectors and generalized eigenvectors of (4.4 .2) and (4.4 .4) respectively, such that*

- *The union of the families* $(\mathbb{P}_v^T\boldsymbol{z}^i,\boldsymbol{\xi}_1^i,\boldsymbol{\xi}_2^i)_{1\le i\le n_0}$ *and* $(\boldsymbol{z}^i,\boldsymbol{\xi}_1^i,\boldsymbol{\xi}_2^i)_{n_0+1\le i\le n_1}$ *is a basis of* $\mathbb{C}^{n_1}$ *constituted of eigenvectors and generalized eigenvectors of (4.4 .1).*
- *The union of the families* $(\mathbb{P}_v^T\boldsymbol{\psi}^i,\boldsymbol{\kappa}_1^i,\boldsymbol{\kappa}_2^i)_{1\le i\le n_0}$ *and* $(\boldsymbol{\psi}^i,\boldsymbol{\kappa}_1^i,\boldsymbol{\kappa}_2^i)_{n_0+1\le i\le n_1}$ *is a basis of* $\mathbb{C}^{n_1}$ *constituted of eigenvectors and generalized eigenvectors of (4.4 .3).*
- *We have the decompositions*

$$\Lambda_{\mathbb{C}} = \boldsymbol{F}^{-1}\mathbf{A}\mathbb{P}^T\boldsymbol{F} \quad \textit{and} \quad \Lambda_{\mathbb{C}}^T = \boldsymbol{\Phi}^{-1}\mathbf{A}^{\#}\mathbb{P}^T\boldsymbol{\Phi},$$

where $\Lambda_{\mathbb{C}}$ *is a decomposition of* $\mathbf{A}\mathbb{P}^T$ *into complex Jordan blocks. The matrices* $\boldsymbol{F}$ *and* $\boldsymbol{\Phi}$ *are in the form* $\boldsymbol{F} = [\mathbb{P}^T\boldsymbol{F}_1\ \boldsymbol{F}_0]$ *and* $\boldsymbol{\Phi} = [\mathbb{P}^T\boldsymbol{\Phi}_1\ \boldsymbol{\Phi}_0]$ *where* $\boldsymbol{F}_1$ *is the matrix whose columns are* $(\boldsymbol{z}^i,\boldsymbol{\xi}_1^i,\boldsymbol{\xi}_2^i)_{1\le i\le n_0}$, $\boldsymbol{F}_0$ *is the matrix whose columns are* $(\boldsymbol{z}^i,\boldsymbol{\xi}_1^i,\boldsymbol{\xi}_2^i)_{n_0+1\le i\le n_1}$, $\boldsymbol{\Phi}_1$ *is the matrix whose columns are* $(\boldsymbol{\psi}^i,\boldsymbol{\kappa}_1^i,\boldsymbol{\kappa}_2^i)_{1\le i\le n_0}$ *and* $\boldsymbol{\Phi}_0$ *is the matrix whose columns are* $(\boldsymbol{\psi}^i,\boldsymbol{\kappa}_1^i,\boldsymbol{\kappa}_2^i)_{n_0+1\le i\le n_1}$. *Moreover, we have the following bi-orthogonality conditions*

$$\boldsymbol{F}^T M_s^T \boldsymbol{\Phi} = I \quad \textit{and} \quad \boldsymbol{F}_1^T M \boldsymbol{\Phi}_1 = I.$$

Proof. Step 1: Decomposition of $\mathbf{A}\mathbb{P}^T$ *and* $\mathbf{A}^{\#}\mathbb{P}^T$. Using the complex Jordan decomposition of real matrices, we know that there exists a matrix $\widehat{\boldsymbol{F}} \in \mathbb{C}^{n_1}\times\mathbb{C}^{n_1}$ constituted of eigenvectors and generalized eigenvectors of problem (4.4 .1) such that

$$\Lambda_{\mathbb{C}} = \widehat{\boldsymbol{F}}^{-1}\mathbf{A}\mathbb{P}^T\widehat{\boldsymbol{F}}.$$

We set

$$\widehat{\boldsymbol{\Phi}} = M_s^{-T}\widehat{\boldsymbol{F}}^{-T}.$$

Thus, the matrices $\widehat{\boldsymbol{F}}$ and $\widehat{\boldsymbol{\Phi}}$ satisfy the bi-orthogonality condition

$$\widehat{\boldsymbol{F}}^T M_s^T \widehat{\boldsymbol{\Phi}} = I.$$

Now, we are going to prove that

$$\Lambda_{\mathbb{C}}^T = \widehat{\boldsymbol{\Phi}}^{-1} \mathbf{A}^{\#} \mathbb{P}^T \widehat{\boldsymbol{\Phi}}.$$

From the identities

$$M_s^{-1} \widehat{\boldsymbol{\Phi}}^{-T} = \widehat{\boldsymbol{F}} \quad \text{and} \quad \widehat{\boldsymbol{F}} \Lambda_{\mathbb{C}} = \mathbf{A} \mathbb{P}^T \widehat{\boldsymbol{F}} = \mathbb{P}^T M_s^{-1} A_0 \mathbb{P}^T \widehat{\boldsymbol{F}},$$

we deduce that

$$M_s^{-1} \widehat{\boldsymbol{\Phi}}^{-T} \Lambda_{\mathbb{C}} = \mathbb{P}^T M_s^{-1} A_0 \mathbb{P}^T M_s^{-1} \widehat{\boldsymbol{\Phi}}^{-T},$$

and

$$\Lambda_{\mathbb{C}} \widehat{\boldsymbol{\Phi}}^T = \widehat{\boldsymbol{\Phi}}^T M_s \mathbb{P}^T M_s^{-1} A_0 \mathbb{P}^T M_s^{-1}.$$

Since $M_s \mathbb{P}^T = \mathbb{P} M_s$ and $\mathbb{P}^T M_s^{-1} = M_s^{-1} \mathbb{P}$, we have

$$\Lambda_{\mathbb{C}} \widehat{\boldsymbol{\Phi}}^T = \widehat{\boldsymbol{\Phi}}^T \mathbb{P} A_0 M_s^{-1} \mathbb{P}.$$

Taking the conjugated transpose, we obtain

$$\widehat{\boldsymbol{\Phi}} \Lambda_{\mathbb{C}}^T = \mathbb{P}^T M_s^{-T} A_0^T \mathbb{P}^T \widehat{\boldsymbol{\Phi}} = \mathbf{A}^{\#} \mathbb{P}^T \widehat{\boldsymbol{\Phi}}.$$

Hence

$$\Lambda_{\mathbb{C}}^T = \widehat{\boldsymbol{\Phi}}^{-1} \mathbf{A}^{\#} \mathbb{P}^T \widehat{\boldsymbol{\Phi}}.$$

Step 2: Construction of $\boldsymbol{F}$ and $\boldsymbol{\Phi}$. We denote by $(\hat{\boldsymbol{z}}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)_i$ (resp. $(\hat{\boldsymbol{\psi}}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_i$) the columns of the matrix $\widehat{\boldsymbol{F}}$ (resp. $\widehat{\boldsymbol{\Phi}}$). Without loss of generality, we assume that $(\hat{\boldsymbol{z}}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)_{1\le i\le n_0}$ and $(\hat{\boldsymbol{\psi}}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{1\le i\le n_0}$ belong to $\mathrm{Im}(\mathbb{P}^T)$. We denote by $\boldsymbol{F}_0$ (resp. $\boldsymbol{\Phi}_0$) the matrix whose columns are $(\hat{\boldsymbol{z}}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)_{n_0+1\le i\le n_1}$ (resp. $(\hat{\boldsymbol{\psi}}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{n_0+1\le i\le n_1}$). Finally, we denote by $\boldsymbol{\lambda}_i$ the eigenvalue of $\mathbf{A}\mathbb{P}^T$ associated to $(\boldsymbol{z}_0^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)$. Thanks to Theorem 4.4 .3, $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i)_i$ with

$$\begin{aligned}
\boldsymbol{z}^i &= \hat{\boldsymbol{z}}^i - M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\xi}_1^i - M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\xi}_2^i,\\
\boldsymbol{\theta}^i &= -\boldsymbol{\lambda}_i N_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\xi}_1^i - \boldsymbol{\lambda}_i N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\xi}_2^i - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1} - A_{\theta\theta} A_{\eta_1\theta}^T)\boldsymbol{\xi}_1^i\\
&\quad - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2} - A_{\theta\theta} A_{\eta_2\theta}^T)\boldsymbol{\xi}_2^i - N_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \hat{\boldsymbol{z}}^i\\
&\quad - N_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\xi}_1^{i-1} - N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\xi}_2^{i-1},
\end{aligned}$$

is a generalized eigenvector of problem (4.4 .2). From Theorem 4.4 .6, $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i)_i$ with

$$\begin{aligned}
\boldsymbol{\psi}^i &= \hat{\boldsymbol{\psi}}^i - M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^i,\\
\boldsymbol{\rho}^i &= -\boldsymbol{\lambda}_i N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^i + N_{\theta\theta} A_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^i - N_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \hat{\boldsymbol{\psi}}^i + N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2^{i-1},
\end{aligned}$$

is a generalized eigenvector of problem (4.4 .4). We denote by $\boldsymbol{F}_1$ (resp. $\boldsymbol{\Phi}_1$) the matrix whose columns are $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i)_{1\le i\le n_0}$ (resp. $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{1\le i\le n_0}$). By construction, we have

$$\boldsymbol{F} := [\mathbb{P}^T \boldsymbol{F}_1 \ \boldsymbol{F}_0] = \boldsymbol{F}_0 \quad \text{and} \quad \boldsymbol{\Phi} := [\mathbb{P}^T \boldsymbol{\Phi}_1 \ \boldsymbol{\Phi}_0] = \boldsymbol{\Phi}_0,$$

Step 3: Bi-orthogonality conditions. We have to show that

$$\boldsymbol{F}^T M_s^T \boldsymbol{\Phi} = I \quad \text{and} \quad \boldsymbol{F}_1^T M \boldsymbol{\Phi}_1 = I.$$

Due to the bi-orthogonality condition satisfied by $\boldsymbol{F}_0$ and $\boldsymbol{\Phi}_0$, we have

$$\boldsymbol{F}^T M_s^T \boldsymbol{\Phi} = I \quad \text{and} \quad (\mathbb{P}^T \boldsymbol{F}_1)^T M_s^T \mathbb{P}^T \boldsymbol{\Phi}_1 = \boldsymbol{F}_1^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Phi}_1 = I. \tag{4.5 .2}$$

Since $\mathbb{P} M_s^T \mathbb{P}^T = \mathbb{P} M_s^T$, we have

$$\boldsymbol{F}_1^T M \boldsymbol{\Phi}_1 = \boldsymbol{F}_1^T \mathbb{P} M_s^T \boldsymbol{\Phi}_1 + \boldsymbol{F}_1^T (M - \mathbb{P} M_s^T) \boldsymbol{\Phi}_1 = \boldsymbol{F}_1^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Phi}_1 + \boldsymbol{F}_1^T (M - \mathbb{P} M_s^T) \boldsymbol{\Phi}_1.$$

Therefore, thanks to Lemma 4.5 .1 and equation (4.5 .2), we get

$$\boldsymbol{F}_1^T M \boldsymbol{\Phi}_1 = I.$$

□

Theorem 4.5 .2. *There exist two families $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}_r^i)_{1 \le i \le n_1}$ and $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}_r^i)_{1 \le i \le n_1}$ of $\mathbb{R}^n$ such that*

- *The union of the families $(\mathbb{P}_v^T \boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1 \le i \le n_0}$ and $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{n_0+1 \le i \le n_1}$ is a basis of $\mathbb{R}^{n_1}$.*
- *The union of the families $(\mathbb{P}_v^T \boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{1 \le i \le n_0}$ and $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{n_0+1 \le i \le n_1}$ is a basis of $\mathbb{R}^{n_1}$.*
- *We have the decompositions*

$$\Lambda = \boldsymbol{E}^{-1} \mathbf{A} \mathbb{P}^T \boldsymbol{E} \quad \text{and} \quad \Lambda^T = \boldsymbol{\Xi}^{-1} \mathbf{A}^\# \mathbb{P}^T \boldsymbol{\Xi},$$

where Λ is a decomposition of $\mathbf{A}\mathbb{P}^T$ into real Jordan blocks. The matrices $\boldsymbol{E}$ and $\boldsymbol{\Xi}$ are in the form $\boldsymbol{E} = [\mathbb{P}^T \boldsymbol{E}_1 \ \boldsymbol{E}_0]$ and $\boldsymbol{\Xi} = [\mathbb{P}^T \boldsymbol{\Xi}_1 \ \boldsymbol{\Xi}_0]$ where $\boldsymbol{E}_1$ is the matrix whose columns are $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1 \le i \le n_0}$, $\boldsymbol{E}_0$ is the matrix whose columns are $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{n_0+1 \le i \le n_1}$, $\boldsymbol{\Xi}_1$ is the matrix whose columns are $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{1 \le i \le n_0}$ and $\boldsymbol{\Xi}_0$ is the matrix whose columns are $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{n_0+1 \le i \le n_1}$. Moreover, we have the following bi-orthogonality conditions

$$\boldsymbol{E}^T M_s^T \boldsymbol{\Xi} = I \quad \text{and} \quad \boldsymbol{E}_1^T M \boldsymbol{\Xi}_1 = I.$$

Proof. We denote by $(\boldsymbol{\lambda}_j)_j$ the eigenvalues of the pair (M_L, A_L). To construct the basis $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}_r^i)_{1 \le i \le n_1}$ and $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}_r^i)_{1 \le i \le n_1}$, we proceed in the following way.
Case 1: $\boldsymbol{\lambda}_j$ is real. If $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i)$ and $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i)$ are generalized eigenvectors associated to $\boldsymbol{\lambda}_j$, we can assume that they are real vectors. Thus, we set

$$(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}_r^i) = (\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i) \quad \text{and} \quad (\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}_r^i) = (\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i).$$

Case 2: $\boldsymbol{\lambda}_j$ is complex. There exists k such that $\boldsymbol{\lambda}_k = \bar{\boldsymbol{\lambda}}_j$ is also an eigenvalue of (M_L, A_L). If $(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i)$ and $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i)$ are generalized eigenvectors associated to $\boldsymbol{\lambda}_j$, $(\boldsymbol{z}^k, \boldsymbol{\xi}_1^k, \boldsymbol{\xi}_2^k, \boldsymbol{\theta}^k)$ and $(\boldsymbol{\psi}^k, \boldsymbol{\kappa}_1^k, \boldsymbol{\kappa}_2^k, \boldsymbol{\rho}^k)$ are eigenvectors or generalized eigenvectors associated to $\boldsymbol{\lambda}_k$, then we may assume that $(\bar{\boldsymbol{z}}^i, \bar{\boldsymbol{\xi}}_1^i, \bar{\boldsymbol{\xi}}_2^i, \bar{\boldsymbol{\theta}}^i) = (\boldsymbol{z}^k, \boldsymbol{\xi}_1^k, \boldsymbol{\xi}_2^k, \boldsymbol{\theta}^k)$ and $(\bar{\boldsymbol{\psi}}^i, \bar{\boldsymbol{\kappa}}_1^i, \bar{\boldsymbol{\kappa}}_2^i, \bar{\boldsymbol{\rho}}^i) = (\boldsymbol{\psi}^k, \boldsymbol{\kappa}_1^k, \boldsymbol{\kappa}_2^k, \boldsymbol{\rho}^k)$. Thus, we can set

$$(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}_r^i) = \sqrt{2}\mathrm{Re}(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i) \quad \text{and} \quad (\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}_r^i) = \sqrt{2}\mathrm{Re}(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i),$$

and

$$(\boldsymbol{v}^m, \boldsymbol{\eta}_1^m, \boldsymbol{\eta}_2^m, \boldsymbol{\theta}_r^m) = \sqrt{2}\mathrm{Im}(\boldsymbol{z}^i, \boldsymbol{\xi}_1^i, \boldsymbol{\xi}_2^i, \boldsymbol{\theta}^i) \quad \text{and} \quad (\boldsymbol{\phi}^m, \boldsymbol{\zeta}_1^m, \boldsymbol{\zeta}_2^m, \boldsymbol{\rho}_r^i) = \sqrt{2}\mathrm{Im}(\boldsymbol{\psi}_0^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i).$$

Thus, we have constructed two families $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}_r^i)_{1 \le i \le n_1}$ and $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}_r^i)_{1 \le i \le n_1}$ of $\mathbb{R}^n$ satisfying the bi-orthogonality conditions

$$\boldsymbol{E}^T M_s^T \boldsymbol{\Xi} = I \quad \text{and} \quad \boldsymbol{E}_1^T M \boldsymbol{\Xi}_1 = I,$$

and such that

$$\Lambda = \boldsymbol{E}^{-1} \mathbf{A} \mathbb{P}^T \boldsymbol{E} \quad \text{and} \quad \Lambda^T = \boldsymbol{\Xi}^{-1} \mathbf{A}^\# \mathbb{P}^T \boldsymbol{\Xi}.$$

□

Corollary 4.5 .3. *We have the following decompositions of* $\mathbf{A}\mathbb{P}^T$ *and* $\mathbf{A}^{\#}\mathbb{P}^T$

$$\Lambda = \boldsymbol{\Xi}^T M_s \mathbf{A}\mathbb{P}^T \boldsymbol{E} \quad \text{and} \quad \Lambda^T = \boldsymbol{E}^T M_s^T \mathbf{A}^{\#}\mathbb{P}^T \boldsymbol{\Xi}.$$

Proof. It follows from the bi-orthogonality of $\boldsymbol{E}$ and $\boldsymbol{\Xi}$.

□

4.5 .2 Projected systems

Let $(\boldsymbol{\lambda}_j)_{j\in\mathbb{N}}$ be the eigenvalues of the operator $\mathbf{A}\mathbb{P}^T$. We denote by $\mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_j)$ the real generalized eigenspace of $\mathbf{A}\mathbb{P}^T$ and $\mathbf{G}_{\mathbb{R}}^*(\boldsymbol{\lambda}_j)$ the real generalized eigenspace of $\mathbf{A}^{\#}\mathbb{P}^T$ associated to the eigenvalue $\boldsymbol{\lambda}_j$. We define the unstable subspaces

$$\mathbb{Z}_u = \oplus_{j\in\mathbf{J}_u}\mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_j) \quad \text{and} \quad \mathbb{Z}_u^* = \oplus_{j\in\mathbf{J}_u} G_{\mathbb{R}}^*(\boldsymbol{\lambda}_j),$$

where $\mathbf{J}_u$ is a finite subset of $\mathbb{N}$ such that

$$-\alpha_u < \operatorname{Re}\boldsymbol{\lambda}_j \quad \forall\, j \in \mathbf{J}_u,$$

for some $\alpha_u > 0$. Without loss of generality, we assume that

$$\mathbb{Z}_u = \operatorname{Vect}\left\{(\mathbb{P}_v^T \boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1\le i\le d_u}\right\},$$

and we denote $d_s = n_0 - d_u$. Therefore, we have

$$\mathbb{R}^{n_1} = \mathbb{Z}_u \oplus \mathbb{Z}_s \oplus \mathbb{K},$$

where

$$\mathbb{Z}_s = \operatorname{Vect}\left\{(\mathbb{P}_v^T \boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{d_u+1\le i\le n_0}\right\} \quad \text{and} \quad \mathbb{K} = \operatorname{Vect}\left\{(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{n_0+1\le i\le n_1}\right\}.$$

We denote by

- $\boldsymbol{E}_u \in \mathbb{R}^{n_1}\times\mathbb{R}^{d_u}$ the matrix whose columns are $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1\le i\le d_u}$,
- $\boldsymbol{E}_s \in \mathbb{R}^{n_1}\times\mathbb{R}^{d_s}$ the matrix whose columns are $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{d_u+1\le i\le n_0}$,
- $\boldsymbol{\Xi}_u \in \mathbb{R}^{n_1}\times\mathbb{R}^{d_u}$ the matrix whose columns are $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{1\le i\le d_u}$,
- $\boldsymbol{\Xi}_s \in \mathbb{R}^{n_1}\times\mathbb{R}^{d_s}$ the matrix whose columns are $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{d_u+1\le i\le n_0}$,
- $\boldsymbol{\Theta}_u \in \mathbb{R}^{n_\theta}\times\mathbb{R}^{d_u}$ the matrix whose columns are $(\boldsymbol{\theta}_r^i)_{1\le i\le d_u}$,
- $\boldsymbol{\Theta}_s \in \mathbb{R}^{n_\theta}\times\mathbb{R}^{d_s}$ the matrix whose columns are $(\boldsymbol{\theta}_r^i)_{d_u+1\le i\le n_0}$,
- $\boldsymbol{\Phi}_u \in \mathbb{R}^{n_\theta}\times\mathbb{R}^{d_u}$ the matrix whose columns are $(\boldsymbol{\rho}_r^i)_{1\le i\le d_u}$,
- $\boldsymbol{\Phi}_s \in \mathbb{R}^{n_\theta}\times\mathbb{R}^{d_s}$ the matrix whose columns are $(\boldsymbol{\rho}_r^i)_{d_u+1\le i\le n_0}$,

Lemma 4.5 .2. *We have the relations*

$$\begin{aligned}
&\boldsymbol{E}_u^T\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Xi}_u = I, \quad \boldsymbol{E}_s^T\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Xi}_s = I, \quad \boldsymbol{E}_0^T M_s^T\boldsymbol{\Xi}_0 = I,\\
&\boldsymbol{E}_u^T\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Xi}_s = 0, \quad \boldsymbol{E}_s^T\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Xi}_u = 0, \quad \boldsymbol{E}_0^T\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Xi}_u = 0,\\
&\boldsymbol{E}_u^T\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Xi}_0 = 0, \quad \boldsymbol{E}_s^T\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Xi}_0 = 0, \quad \boldsymbol{E}_0^T\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Xi}_s = 0.
\end{aligned}$$

Proof. It follows from the fact the matrices $\boldsymbol{E} = [\mathbb{P}^T \boldsymbol{E}_u \, \mathbb{P}^T \boldsymbol{E}_s \, \boldsymbol{E}_0]$ and $\boldsymbol{\Xi} = [\mathbb{P}^T \boldsymbol{\Xi}_u \, \mathbb{P}^T \boldsymbol{\Xi}_s \, \boldsymbol{\Xi}_0]$ satisfy the bi-orthogonality condition

$$\boldsymbol{E}^T M_s^T \boldsymbol{\Xi} = I.$$

$\square$

Lemma 4.5 .3. *We have*

$$\Lambda = \begin{bmatrix} \Lambda_u & 0_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_s}} & 0_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_0}} \\ 0_{\mathbb{R}^{d_s} \times \mathbb{R}^{d_u}} & \Lambda_s & 0_{\mathbb{R}^{d_s} \times \mathbb{R}^{d_0}} \\ 0_{\mathbb{R}^{d_0} \times \mathbb{R}^{d_u}} & 0_{\mathbb{R}^{d_0} \times \mathbb{R}^{d_s}} & 0_{\mathbb{R}^{d_0} \times \mathbb{R}^{d_0}} \end{bmatrix} \quad \text{and} \quad \Lambda^T = \begin{bmatrix} \Lambda_u^T & 0_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_s}} & 0_{\mathbb{R}^{d_u} \times \mathbb{R}^{d_0}} \\ 0_{\mathbb{R}^{d_s} \times \mathbb{R}^{d_u}} & \Lambda_s & 0_{\mathbb{R}^{d_s} \times \mathbb{R}^{d_0}} \\ 0_{\mathbb{R}^{d_0} \times \mathbb{R}^{d_u}} & 0_{\mathbb{R}^{d_0} \times \mathbb{R}^{d_s}} & 0_{\mathbb{R}^{d_0} \times \mathbb{R}^{d_0}} \end{bmatrix},$$

where

$$\Lambda_u = \boldsymbol{\Xi}_u^T \mathbb{P} M_s \mathbf{A} \mathbb{P}^T \boldsymbol{E}_u, \quad \Lambda_u^T = \boldsymbol{E}_u^T \mathbb{P} M_s^T \mathbf{A}^\# \mathbb{P}^T \boldsymbol{\Xi}_u, \quad \Lambda_s = \boldsymbol{\Xi}_s^T \mathbb{P} M_s \mathbf{A} \mathbb{P}^T \boldsymbol{E}_s, \quad \Lambda_s^T = \boldsymbol{E}_s^T \mathbb{P} M_s^T \mathbf{A}^\# \mathbb{P}^T \boldsymbol{\Xi}_s. \tag{4.5 .3}$$

Proof. It follows from Corollary 4.5 .3.

$\square$

Lemma 4.5 .4. *We have the relations*

$$\boldsymbol{E}_u^T M \boldsymbol{\Xi}_u = I, \quad \boldsymbol{E}_s^T M \boldsymbol{\Xi}_s = I, \quad \boldsymbol{E}_u^T M \boldsymbol{\Xi}_s = 0, \quad \boldsymbol{E}_s^T M \boldsymbol{\Xi}_u = 0.$$

Proof. It follows from the fact the matrices $\boldsymbol{E}_1 = [\boldsymbol{E}_u \, \boldsymbol{E}_s]$ and $\boldsymbol{\Xi}_1 = [\boldsymbol{\Xi}_u \, \boldsymbol{\Xi}_s]$ satisfy the bi-orthogonality condition

$$\boldsymbol{E}_1^T M \boldsymbol{\Xi}_1 = I.$$

$\square$

Proposition 4.5 .4. *We have*

$$\Lambda_u = \boldsymbol{\Xi}_u^T A \boldsymbol{E}_u, \quad \Lambda_u^T = \boldsymbol{E}_u^T A^T \boldsymbol{\Xi}_u, \quad \Lambda_s = \boldsymbol{\Xi}_s^T A \boldsymbol{E}_s, \quad \Lambda_s^T = \boldsymbol{E}_s^T A^T \boldsymbol{\Xi}_s.$$

Proof. By construction of the matrices $\boldsymbol{E}_u$, $\boldsymbol{\Xi}_u$, $\boldsymbol{\Theta}_u$ and $\boldsymbol{\Phi}_u$, we have

$$\begin{bmatrix} A & \widetilde{A}_{\upsilon\eta\theta} \\ \widehat{A}_{\upsilon\eta\theta}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_u \\ \boldsymbol{\Theta}_u \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_u \\ \boldsymbol{\Theta}_u \end{bmatrix} \Lambda_u \quad \text{i.e.} \quad A\boldsymbol{E}_u + \widetilde{A}_{\upsilon\eta\theta} \boldsymbol{\Theta}_u = M \boldsymbol{E}_u \Lambda_u, \quad \widehat{A}_{\upsilon\eta\theta}^T \boldsymbol{E}_u = 0,$$

and

$$\begin{bmatrix} A^T & \widehat{A}_{\upsilon\eta\theta} \\ \widetilde{A}_{\upsilon\eta\theta}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Xi}_u \\ \boldsymbol{\Phi}_u \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Xi}_u \\ \boldsymbol{\Phi}_u \end{bmatrix} \Lambda_u^T \quad \text{i.e.} \quad A^T \boldsymbol{\Xi}_u + \widehat{A}_{\upsilon\eta\theta} \boldsymbol{\Phi}_u = M \boldsymbol{\Xi}_u \Lambda_u^T, \quad \widetilde{A}_{\upsilon\eta\theta}^T \boldsymbol{\Xi}_u = 0.$$

We deduce that

$$\begin{aligned} \boldsymbol{\Xi}_u^T M \boldsymbol{E}_u \Lambda_u &= \boldsymbol{\Xi}_u^T A \boldsymbol{E}_u + \boldsymbol{\Xi}_u^T \widetilde{A}_{\upsilon\eta\theta} \boldsymbol{\Theta}_u, \\ &= \boldsymbol{\Xi}_u^T A \boldsymbol{E}_u + (\widetilde{A}_{\upsilon\eta\theta}^T \boldsymbol{\Xi}_u)^T \boldsymbol{\Theta}_u \\ &= \boldsymbol{\Xi}_u^T A \boldsymbol{E}_u, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{E}_u^T M \boldsymbol{\Xi}_u \Lambda_u^T &= \boldsymbol{E}_u^T A^T \boldsymbol{\Xi}_u + \boldsymbol{E}_u^T \widehat{A}_{\upsilon\eta\theta} \boldsymbol{\Phi}_u \\ &= \boldsymbol{E}_u^T A^T \boldsymbol{\Xi}_u + (\widehat{A}_{\upsilon\eta\theta}^T \boldsymbol{E}_u)^T \boldsymbol{\Phi}_u \\ &= \boldsymbol{E}_u^T A^T \boldsymbol{\Xi}_u. \end{aligned}$$

From Lemma 4.5 .4, it follows that

$$\Lambda_u = \boldsymbol{\Xi}_u^T A \boldsymbol{E}_u \quad \text{and} \quad \Lambda_u^T = \boldsymbol{E}_u^T A^T \boldsymbol{\Xi}_u.$$

Similarly, we prove that

$$\Lambda_s = \boldsymbol{\Xi}_s^T A \boldsymbol{E}_s \quad \text{and} \quad \Lambda_s^T = \boldsymbol{E}_s^T A^T \boldsymbol{\Xi}_s.$$

□

We introduce the operators $\Pi_u \in \mathcal{L}(\mathbb{R}^{n_1}, \mathbb{Z}_u)$ and $\Pi_s \in \mathcal{L}(\mathbb{R}^{n_1}, \mathbb{Z}_s)$ defined by

$$\Pi_u = \mathbb{P}^T \boldsymbol{E}_u \boldsymbol{\Xi}_u^T \mathbb{P} M_s \quad \text{and} \quad \Pi_s = \mathbb{P}^T \boldsymbol{E}_s \boldsymbol{\Xi}_s^T \mathbb{P} M_s.$$

Proposition 4.5 .5. *The operator Π_u is the projection of $\mathbb{R}^{n_1}$ onto $\mathbb{Z}_u$ parallel to $\mathbb{Z}_s \oplus \mathbb{K}$ and the operator Π_s is the projection of $\mathbb{R}^{n_1}$ onto $\mathbb{Z}_s$ parallel to $\mathbb{Z}_u \oplus \mathbb{K}$.*

Proof. We have

$$\Pi_u^2 = \mathbb{P}^T \boldsymbol{E}_u (\boldsymbol{\Xi}_u^T \mathbb{P} M_s \mathbb{P}^T \boldsymbol{E}_u) \boldsymbol{\Xi}_u^T \mathbb{P} M_s = \mathbb{P}^T \boldsymbol{E}_u \boldsymbol{\Xi}_u^T \mathbb{P} M_s = \Pi_u,$$

because $\boldsymbol{\Xi}_u^T \mathbb{P} M_s \mathbb{P}^T \boldsymbol{E}_u = I_{\mathbb{R}^{d_u}}$ (see equation (**??**)). Therefore, Π_u is the projection onto $\mathrm{Im}(\Pi_u)$ parallel to $\mathrm{Ker}(\Pi_u)$. Thus, to prove that Π_u is the projection onto $\mathbb{Z}_u$ parallel to $\mathbb{Z}_s$, we have to show that $\mathrm{Im}(\Pi_u) = \mathbb{Z}_u$ and $\mathrm{Ker}(\Pi_u) = \mathbb{Z}_s \oplus \mathbb{K}$. It is immediate that $\mathrm{Im}(\Pi_u) = \mathbb{Z}_u$ and $\mathbb{Z}_s \oplus \mathbb{K} \subset \mathrm{Ker}(\Pi_u)$, and since $\dim(\mathbb{Z}_s \oplus \mathbb{K}) = \dim(\mathrm{Ker}(\Pi_u))$, we have $\mathrm{Ker}(\Pi_u) = \mathbb{Z}_s$. We prove similarly that Π_s is the projection of $\mathbb{R}^{n_1}$ onto $\mathbb{Z}_s$ parallel to $\mathbb{Z}_u \oplus \mathbb{K}$.

□

Proposition 4.5 .6. *If $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$ is solution of system (4.2 .4), then the triplets $(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})$ and $(\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})$ defined by*

$$(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})^T = \boldsymbol{\Xi}_u^T M (\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T \quad \textit{and} \quad (\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})^T = \boldsymbol{\Xi}_s^T M (\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T,$$

obey the system

$$\begin{aligned}
&\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} = \Lambda_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \mathbb{B}_u \boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix}(0) = \boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}, \\
&\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} = \Lambda_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} + \mathbb{B}_s \boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix}(0) = \boldsymbol{\Xi}_s^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix},
\end{aligned} \tag{4.5 .4}$$

where

$$\mathbb{B}_u = \boldsymbol{\Xi}_u^T B \quad \textit{and} \quad \mathbb{B}_s = \boldsymbol{\Xi}_s^T B.$$

Conversely, if the pair $((\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u}), (\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s}))$ is solution of system (4.5 .4), then $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$, with

$$\begin{aligned}
&\mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbb{P}^T \boldsymbol{E}_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \mathbb{P}^T \boldsymbol{E}_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix}, \\
&(I - \mathbb{P}_v^T)\boldsymbol{v} = -M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 - M_{vv}^{-1} A_{v\theta} N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\eta}_2, \\
&\boldsymbol{\theta} = -N_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\eta}_{1,t} - N_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\eta}_{2,t} - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1} - A_{\theta\theta} A_{\eta_1\theta}^T)\boldsymbol{\eta}_1 \\
&\qquad - N_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2} - A_{\theta\theta} A_{\eta_2\theta}^T)\boldsymbol{\eta}_2 - N_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{v},
\end{aligned}$$

is solution of system (4.2 .4).

Proof. Let $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$ be a solution of system (4.2 .4). We have

$$\begin{aligned} &M \frac{d}{dt}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = A \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \widehat{A}_{v\eta\theta}\boldsymbol{\theta} + B\boldsymbol{f}, \qquad \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix}(0) = \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}, \\ &\widehat{A}^T_{v\eta\theta}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = 0. \end{aligned} \tag{4.5 .5}$$

Multiplying the first equation of system (4.5 .5) by $\boldsymbol{\Xi}_u^T$, we obtain

$$\frac{d}{dt}\boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \boldsymbol{\Xi}_u^T A \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \boldsymbol{\Xi}_u^T \widehat{A}_{v\eta\theta}\boldsymbol{\theta} + \boldsymbol{\Xi}_u^T B\boldsymbol{f}.$$

We have

$$A^T\boldsymbol{\Xi}_u + \widehat{A}_{v\eta\theta}\boldsymbol{\Phi}_u = M\boldsymbol{\Xi}_u\Lambda_u^T \quad \text{and} \quad \widetilde{A}^T_{v\eta\theta}\boldsymbol{\Xi}_u = 0,$$

(see proof of Propostion 4.5 .4). It follows that

$$\begin{aligned} \frac{d}{dt}\boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} &= \Lambda_u \boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} - \boldsymbol{\Phi}_u^T \widehat{A}^T_{v\eta\theta} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + (\widehat{A}^T_{v\eta\theta}\boldsymbol{\Xi}_u)\boldsymbol{\theta} + \boldsymbol{\Xi}_u^T B\boldsymbol{f} \\ &= \Lambda_u \boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \boldsymbol{\Xi}_u^T B\boldsymbol{f}. \end{aligned}$$

Thus, $(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})$ is the solution of the equation

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} = \Lambda_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \mathbb{B}_u\boldsymbol{f}, \qquad \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix}(0) = \boldsymbol{\Xi}_u^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix},$$

We can prove similarly that $(\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})$ is the solution of the equation

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} = \Lambda_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} + \mathbb{B}_s\boldsymbol{f}, \qquad \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix}(0) = \boldsymbol{\Xi}_s^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}.$$

Now, we suppose that the pair $((\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u}), (\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s}))$ is solution of system (4.5 .4). We want to prove that $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$, with

$$\begin{aligned} &\mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbb{P}^T\boldsymbol{E}_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \mathbb{P}^T\boldsymbol{E}_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix}, \\ &(I - \mathbb{P}_v^T)\boldsymbol{v} = -M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A^T_{\eta_1\theta}\boldsymbol{\eta}_1 - M_{vv}^{-1}A_{v\theta}N_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\eta}_2, \\ &\boldsymbol{\theta} = -N_{\theta\theta}A^T_{\eta_1\theta}\boldsymbol{\eta}_{1,t} - N_{\theta\theta}A^T_{\eta_2\theta}\boldsymbol{\eta}_{2,t} - N_{\theta\theta}(A^T_{v\theta}M_{vv}^{-1}A_{v\eta_1} - A_{\theta\theta}A^T_{\eta_1\theta})\boldsymbol{\eta}_1 \\ &\qquad -N_{\theta\theta}(A^T_{v\theta}M_{vv}^{-1}A_{v\eta_2} - A_{\theta\theta}A^T_{\eta_2\theta})\boldsymbol{\eta}_2 - N_{\theta\theta}A^T_{v\theta}M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\boldsymbol{v}, \end{aligned}$$

is solution of system (4.2 .4). Thanks to Proposition 4.3 .4, it follows that it is sufficient to prove that $\mathbb{P}^T(\boldsymbol{v},\, \boldsymbol{\eta}_1,\, \boldsymbol{\eta}_2)$ is the solution of the equation

$$\frac{d}{dt}\mathbb{P}^T\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = \mathbf{A}\mathbb{P}^T\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} + \mathbf{B}\mathbf{f}.$$

We have

$$\begin{aligned}
\frac{d}{dt}\mathbb{P}^T\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} &= \frac{d}{dt}\left(\mathbb{P}^T\boldsymbol{E}_u\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix} + \mathbb{P}^T\boldsymbol{E}_s\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix}\right)\\
&= \mathbb{P}^T\boldsymbol{E}_u\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix} + \mathbb{P}^T\boldsymbol{E}_s\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix}\\
&= \mathbb{P}^T\boldsymbol{E}_u\Lambda_u\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix} + \mathbb{P}^T\boldsymbol{E}_s\Lambda_s\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix} + \mathbb{P}^T\boldsymbol{E}_u\mathbb{B}_u\boldsymbol{f} + \mathbb{P}^T\boldsymbol{E}_s\mathbb{B}_s\boldsymbol{f}\\
&= \mathbf{A}\mathbb{P}^T\boldsymbol{E}_u\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix} + \mathbf{A}\mathbb{P}^T\boldsymbol{E}_s\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix} + (\mathbb{P}^T\boldsymbol{E}_u\boldsymbol{\Xi}_u^T + \mathbb{P}^T\boldsymbol{E}_s\boldsymbol{\Xi}_s^T)B\boldsymbol{f},\\
&= \mathbf{A}\mathbb{P}^T\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} + \mathbf{B}\boldsymbol{f},
\end{aligned}$$

because

$$(\mathbb{P}^T\boldsymbol{E}_u\boldsymbol{\Xi}_u^T + \mathbb{P}^T\boldsymbol{E}_s\boldsymbol{\Xi}_s^T)B = (\mathbb{P}^T\boldsymbol{E}_u\boldsymbol{\Xi}_u^T\mathbb{P}M_s + \mathbb{P}^T\boldsymbol{E}_s\boldsymbol{\Xi}_s^T\mathbb{P}M_s)\mathbf{B} = (\Pi_u + \Pi_s)\mathbf{B} = \mathbf{B}.$$

□

4.5 .3 Degree of stabilizability

We are going to show how to choose the eigenmodes for stabilization stabilize. Let $\boldsymbol{\lambda}_j$ be an eigenvalue of $\mathbf{A}\mathbb{P}^T$. There exists a family of indices $\mathbf{J}_{\boldsymbol{\lambda}_j}$, of cardinal $m_j = \dim(\mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_j))$, such that

$$\mathbb{Z}_j = \mathrm{Vect}\left\{(\mathbb{P}_v^T\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{i\in\mathbf{J}_{\boldsymbol{\lambda}_j}}\right\} \quad \text{and} \quad \mathbb{Z}_j^* = \mathrm{Vect}\left\{(\mathbb{P}_v^T\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{i\in\mathbf{J}_{\boldsymbol{\lambda}_j}}\right\}.$$

We denote by $\boldsymbol{E}_j \in \mathbb{R}^{n_1}\times\mathbb{R}^{m_j}$ the matrix whose columns are $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{i\in\mathbf{J}_{\boldsymbol{\lambda}_j}}$ and $\boldsymbol{\Xi}_j \in \mathbb{R}^{n_1}\times\mathbb{R}^{m_j}$ the matrix whose columns are $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{i\in\mathbf{J}_{\boldsymbol{\lambda}_j}}$. In order to avoid dependence on multiplicative constants, we choose the vectors $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)$ such a way that

$$(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)^T M(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i) = 1, \quad \forall\, i \in \mathbf{J}_{\boldsymbol{\lambda}_j}.$$

We set

$$\Lambda_j = \boldsymbol{\Xi}_j^T A\boldsymbol{E}_j \quad \text{and} \quad \mathbb{B}_j = \boldsymbol{\Xi}_j^T B.$$

The vector $(\boldsymbol{v}_j, \boldsymbol{\eta}_{1,j}, \boldsymbol{\eta}_{2,j})^T = e^{\omega t}\boldsymbol{\Xi}_j^T M(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T$ is the solution of the equation

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}_j\\ \boldsymbol{\eta}_{1,j}\\ \boldsymbol{\eta}_{2,j}\end{bmatrix} = (\Lambda_j + \omega I_{\mathbb{R}^{m_j}})\begin{bmatrix}\boldsymbol{v}_j\\ \boldsymbol{\eta}_{1,j}\\ \boldsymbol{\eta}_{2,j}\end{bmatrix} + \mathbb{B}_j\hat{\boldsymbol{f}}, \quad \begin{bmatrix}\boldsymbol{v}_j\\ \boldsymbol{\eta}_{1,j}\\ \boldsymbol{\eta}_{2,j}\end{bmatrix}(0) = \begin{bmatrix}\boldsymbol{v}_j^0\\ \boldsymbol{\eta}_{1,j}^0\\ \boldsymbol{\eta}_{2,j}^0\end{bmatrix}, \tag{4.5 .6}$$

where

$$\hat{\boldsymbol{f}} = e^{\omega t}\boldsymbol{f} \quad \text{ and } \quad \begin{bmatrix} \boldsymbol{v}_j^0 \\ \boldsymbol{\eta}_{1,j}^0 \\ \boldsymbol{\eta}_{2,j}^0 \end{bmatrix} = \Xi_j^T M \begin{bmatrix} \boldsymbol{v}^0 \\ \boldsymbol{\eta}_1^0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}.$$

The control of minimal norm stabilizing system (4.5 .6) is

$$\hat{\boldsymbol{f}} = -\mathbb{B}_j^* \mathbb{P}_{j,\omega} \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{\eta}_{1,j} \\ \boldsymbol{\eta}_{2,j} \end{bmatrix},$$

where $\mathbb{P}_{j,\omega}$ is the solution of the Riccati equation

$$\begin{aligned}
&\mathbb{P}_{j,\omega} \in \mathcal{L}(\mathbb{R}^{m_j}), \quad \mathbb{P}_{j,\omega} = \mathbb{P}_{j,\omega}^T \geq 0, \\
&\mathbb{P}_{j,\omega}(\Lambda_u + \omega I_{\mathbb{R}^{m_j}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{m_j}})\mathbb{P}_{j,\omega} - \mathbb{P}_{j,\omega}\mathbb{B}_j\mathbb{B}_j^T\mathbb{P}_{j,\omega} = 0, \\
&\mathbb{P}_{j,\omega} \text{ is invertible and } \Lambda_u + \omega I_{\mathbb{R}^{m_j}} - \mathbb{B}_j\mathbb{B}_j^T\mathbb{P}_{j,\omega} \text{ is exponentially stable.}
\end{aligned}$$

The square norm of the control is

$$\|\hat{\boldsymbol{f}}\|^2_{L^2(0,\infty;R^{n_c})} = \left(\mathbb{P}_{j,\omega} \begin{bmatrix} \boldsymbol{v}_j^0 \\ \boldsymbol{\eta}_{1,j}^0 \\ \boldsymbol{\eta}_{2,j}^0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{v}_j^0 \\ \boldsymbol{\eta}_{1,j}^0 \\ \boldsymbol{\eta}_{2,j}^0 \end{bmatrix} \right),$$

and the norm corresponding to the worst normalized initial condition is

$$\max_{\|(\boldsymbol{v}_j^0, \boldsymbol{\eta}_{1,j}^0, \boldsymbol{\eta}_{2,j}^0)\|_{\mathbb{R}^{m_j}} = 1} \left(\mathbb{P}_{j,\omega} \begin{bmatrix} \boldsymbol{v}_j^0 \\ \boldsymbol{\eta}_{1,j}^0 \\ \boldsymbol{\eta}_{2,j}^0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{v}_j^0 \\ \boldsymbol{\eta}_{1,j}^0 \\ \boldsymbol{\eta}_{2,j}^0 \end{bmatrix} \right) = \max \sigma(\mathbb{P}_{j,\omega}).$$

Therefore, more the number $\max \sigma(\mathbb{P}_{j,\omega})$ is large more we need a strong control to stabilize system (4.5 .6). We define the degree of stabilizability d_j of the eigenvalue $\boldsymbol{\lambda}_j$ by

$$d_j = \frac{1}{\max \sigma(\mathbb{P}_{j,\omega})}.$$

4.5 .4 Linear feedback law

The goal is to find a feedback law able to stabilize system (4.2 .4) or equivalently (4.3 .15). Due to Proposition 4.5 .6, it is necessary and sufficient to find a feedback law able to stabilize the equation satisfied by $(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})$. That is why we make the following assumption.

$$\text{For all } \omega > -\mathrm{Re}\,\sigma(\Lambda_u) \text{ the pair } (\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, \mathbb{B}_u) \text{ is stabilizable.} \tag{4.5 .7}$$

Hence, for all $Q \in \mathcal{L}(\mathbb{R}^{d_u})$ satisfying $Q = Q^T \geq 0$ and $-\omega < \mathrm{Re}\,\sigma(\Lambda_u)$, the following algebraic Riccati equation

$$\begin{aligned}
&\mathbb{P}_{u,\omega} \in \mathcal{L}(\mathbb{R}^{d_u}), \quad \mathbb{P}_{u,\omega} = \mathbb{P}_{u,\omega}^T \geq 0, \\
&\mathbb{P}_{u,\omega}(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}})\mathbb{P}_{u,\omega} - \mathbb{P}_{u,\omega}\mathbb{B}_u\mathbb{B}_u^T\mathbb{P}_{u,\omega} + Q = 0, \\
&\mathbb{P}_{u,\omega} \text{ is invertible and } \Lambda_u + \omega I_{\mathbb{R}^{d_u}} - \mathbb{B}_u\mathbb{B}_u^T\mathbb{P}_{u,\omega} \text{ is exponentially stable,}
\end{aligned}$$

admits a unique solution $\mathbb{P}_{u,\omega}$. The operator $\mathbb{K}_{u,\omega} = -\mathbb{B}_u^T\mathbb{P}_{u,\omega}$ provides a feedback law stabilizing the pair $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, \mathbb{B}_u)$.

4.5 .5 Stability of the closed-loop linear system

We are going to prove that the linear closed-loop system

$$M\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = A\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} + \widetilde{A}_{v\eta\theta}\boldsymbol{\theta} + B\mathbb{K}_\omega\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}, \quad \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}(0) = \begin{bmatrix}\boldsymbol{v}^0\\ \boldsymbol{\eta}_1^0\\ \boldsymbol{\eta}_2^0\end{bmatrix},$$
$$\widehat{A}^T_{v\eta\theta}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = 0, \tag{4.5 .8}$$

where $\mathcal{K}_\omega = \mathbb{K}_{u,\omega}\boldsymbol{\Xi}_u^T M$, is exponentially stable.

Theorem 4.5 .7. *Let $(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0) \in \mathbb{R}^{n_1}$ be such that*

$$A^T_{v\theta}\boldsymbol{v}^0 + A^T_{\eta_1\theta}\boldsymbol{\eta}_1^0 + A^T_{\eta_2\theta}\boldsymbol{\eta}_2^0 = 0.$$

The solution $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$ of system (4.5 .8) satisfies the estimates

$$\begin{aligned}
&\|\Pi_u(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)\|_{\mathbb{R}^{n_1}} \le Ce^{-\omega t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}},\\
&\|\Pi_s(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)\|_{\mathbb{R}^{n_1}} \le Ce^{-\alpha_u t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}},\\
&\|(I - \mathbb{P}_v^T)\boldsymbol{v}\|_{\mathbb{R}^{n_v}} \le Ce^{-\alpha_u t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}}.
\end{aligned}$$

Proof. From Proposition 4.5 .6, $e^{\omega t}(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})^T = e^{\omega t}\boldsymbol{\Xi}_u^T M(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T$ is the solution of the system

$$\frac{d}{dt}e^{\omega t}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix} = (\Lambda_u + \omega I_{\mathbb{R}^{d_u}} - \mathbb{B}_u\mathbb{B}_u^T\mathbb{P}_{u,\omega})e^{\omega t}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix}, \quad \begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix}(0) = \boldsymbol{\Xi}_u^T M\begin{bmatrix}\boldsymbol{v}^0\\ \boldsymbol{\eta}_1^0\\ \boldsymbol{\eta}_2^0\end{bmatrix}.$$

Since $\Lambda_u + \omega I_{\mathbb{R}^{d_u}} - \mathbb{B}_u\mathbb{B}_u^T\mathbb{P}_{u,\omega}$ is exponentially stable, there exists $C > 0$ such that

$$\|(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})\|_{\mathbb{R}^{d_u}} \le Ce^{-\omega t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}}.$$

The triplet $(\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})^T = \boldsymbol{\Xi}_s^T M(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T$ obeys to the system

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix} = \Lambda_s\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix} + \mathbb{B}_s\mathbb{K}_{u,\omega}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix}, \quad \begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix}(0) = \boldsymbol{\Xi}_s^T M\begin{bmatrix}\boldsymbol{v}^0\\ \boldsymbol{\eta}_2^0\\ \boldsymbol{\eta}_2^0\end{bmatrix}.$$

Since $\mathrm{Re}\,\sigma(\Lambda_u) < -\alpha_u$, there exists $C > 0$ such that

$$\|(\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})\|_{\mathbb{R}^{d_s}} \le Ce^{-\alpha_u t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}}.$$

Finally, we have

$$\|(I - \mathbb{P}_v^T)\boldsymbol{v}\|_{\mathbb{R}^{n_v}} \le C\|(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)\|_{\mathbb{R}^{n_s}\times\mathbb{R}^{n_s}} \le Ce^{-\alpha_u t}\|(\boldsymbol{v}^0, \boldsymbol{\eta}_1^0, \boldsymbol{\eta}_2^0)\|_{\mathbb{R}^{n_1}}.$$

□

4.6 Full discretization of the nonlinear system

They are mainly two approaches to solve numerically fluid-structure interaction systems. The first, called *monolithic* approach, consists to solve the coupled system like a unique problem (see [56, 30]). The second, called *partitioned* approach, use two different codes to solve separately the equations of the fluid and the equations of the structure (see [21]). The *monolithic* approach is generally stable contrary to the *partitioned* approach. However, the implementation is easier in the latter case because we can use classical tools developed for the fluid and the structure problems.
In most of numerical codes, the equations of the fluid are rewritten in an Arbitrary Lagrangian Eulerian (ALE) formulation. For our problem, we use the Lagrangian formulation of the equations of the fluid because the feedback law is constructed from that formulation. The spatial semi-discretization of the nonlinear system is given in Section 4.2 (see equation (4.2 .7)). The time discretization is treated by the backward difference formula of order 2 (BDF2). The geometrical nonlinearities are treated explicitly and a Newton algorithm is used for the nonlinearities coming from the equations of the fluid. We denote by dt the time step length. The full discretization of the nonlinear system is

$$\left(\frac{3}{2dt}M_L - \widehat{A}_L\right)\begin{bmatrix}\tilde{\boldsymbol{u}}^{n+2}\\ \tilde{\boldsymbol{\eta}}_1^{n+2}\\ \tilde{\boldsymbol{\eta}}_2^{n+2}\\ \tilde{\boldsymbol{\theta}}^{n+2}\end{bmatrix} = \frac{2}{dt}M_L\begin{bmatrix}\tilde{\boldsymbol{u}}^{n+1}\\ \tilde{\boldsymbol{\eta}}_1^{n+1}\\ \tilde{\boldsymbol{\eta}}_2^{n+1}\\ \tilde{\boldsymbol{\theta}}^{n+1}\end{bmatrix} - \frac{1}{2dt}M_L\begin{bmatrix}\tilde{\boldsymbol{u}}^{n}\\ \tilde{\boldsymbol{\eta}}_1^{n}\\ \tilde{\boldsymbol{\eta}}_2^{n}\\ \tilde{\boldsymbol{\theta}}^{n}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ M_{\eta\eta}\mathbf{W}\boldsymbol{f}^{n+1}\\ \boldsymbol{h}^{n+2}\end{bmatrix}$$

$$+F_{ns}[\tilde{\boldsymbol{u}}^{n+2}] + F_g\begin{bmatrix}\tilde{\boldsymbol{u}}^{n+1}\\ \tilde{\boldsymbol{\eta}}_1^{n+1}\\ \tilde{\boldsymbol{\eta}}_2^{n+1}\\ \tilde{\boldsymbol{\theta}}^{n+1}\end{bmatrix},$$

for all $n \in \mathbb{N}$.

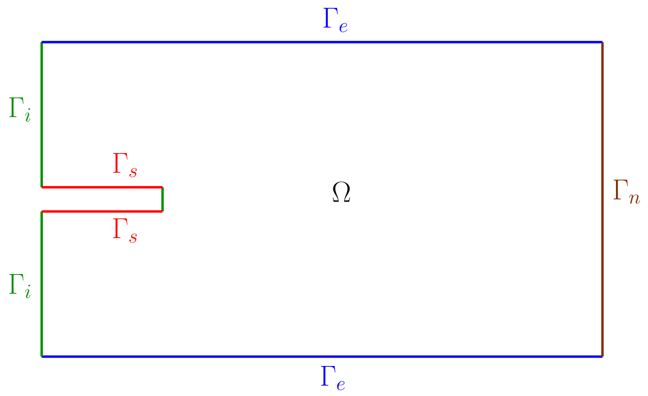
4.7 Numerical results for the nonlinear system

We are going to show, by numerical experiments, that the linear feedback law is able the stabilize nonlinear system (4.2 .4).

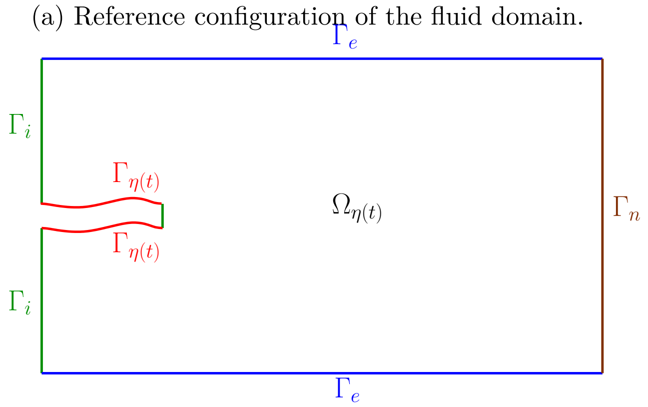
4.7 .1 Data of the numerical experiments

The fluid domain is contained in the rectangle $[0, 50e] \times [-21e, 21e]$, where $2e = 0.1$ is the thickness of the plate (see Figure 4.1). The length of the plate is $10e$. The stationary inflow condition g_s imposed on the boundary $\Gamma_i = \{0\} \times [-21e, -e] \cup \{10e\} \times [-e, e] \cup \{0\} \times [e, 21e]$ is an approximation of the Blasius boundary layer profile defined by

$$g_s(x) = \begin{cases} U_m & \forall\, e+b \leq x \leq 21e,\\ U_m\left(2\dfrac{x-e}{b} - 2\left(\dfrac{x-e}{b}\right)^3 + \left(\dfrac{x-e}{b}\right)^4\right) & \forall\, e \leq x \leq e+b,\\ 0 & \forall\, -e \leq x \leq e,\\ U_m\left(2\dfrac{x+e}{b} - 2\left(\dfrac{x+e}{b}\right)^3 + \left(\dfrac{x+e}{b}\right)^4\right) & \forall\, -e-b \leq x \leq -e,\\ U_m & \forall\, -21e \leq x \leq -e-b,\end{cases}$$



(a) Reference configuration of the fluid domain.



(b) Deformed configuration of the fluid domain.

Figure 4.1 – Configurations of the fluid domain

where U_m is the maximum velocity at the inflow. The Reynolds number Rey of the fluid and the thickness b of the boundary layer are defined by

$$\text{Rey} = \frac{2eU_m}{\nu} \quad \text{and} \quad b = \frac{14e}{\sqrt{\text{Rey}}}.$$

We choose

$$\text{Rey} = 200 \quad \text{and} \quad U_m = 1.0.$$

For the structure, we choose the parameters

$$\beta = \alpha = 10^{-3} \quad \text{and} \quad \gamma = 10^{-1}.$$

We use a triangular mesh of 54960 cells, symmetric with respect to the horizontal axis $z = 0$ (see **Figure** 4.2). We derive from that a 1D mesh where the solutions of the beam equation are computed. For the space discretization, we use generalized Taylor-Hood finite elements $\mathbb{P}_2$-$\mathbb{P}_1$-$\mathbb{P}_1$ for the velocity, the pressure and the Lagrange multipliers, and Hermite finite elements for the displacement of the structure leading to 250752 total degrees of freedom.

4.7 .2 Computation of the spectra

We use an Arnoldi method combined with a *shift and inverse* transformation implemented in Arpack Library to compute the spectrum of the different systems. We fix the shift parameter at 10 and the size of the small Hessenberg matrix at 400. We have computed the spectrum of the structure, the fluid and the coupled system (see *Figure* 4.3).

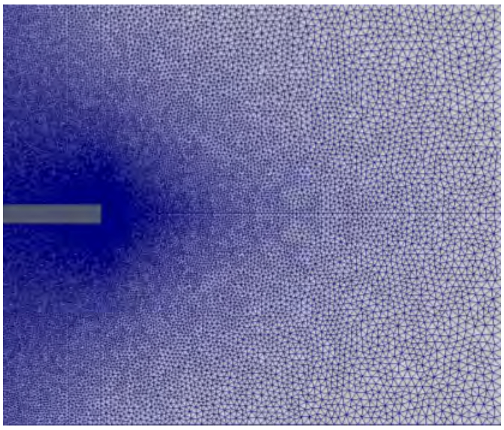


Figure 4.2 – Triangular mesh of the reference configuration of the fluid domain.

j	$1-2$	3	4	$5-6$	7	8	$9-10$	11	12
$\boldsymbol{\lambda}_j$	$-2.86\pm1.30i$	-4.09	-5.74	$-12.92\pm3.27i$	-16.34	-19.99	$-30.32\pm5.50i$	-36.77	-42.49

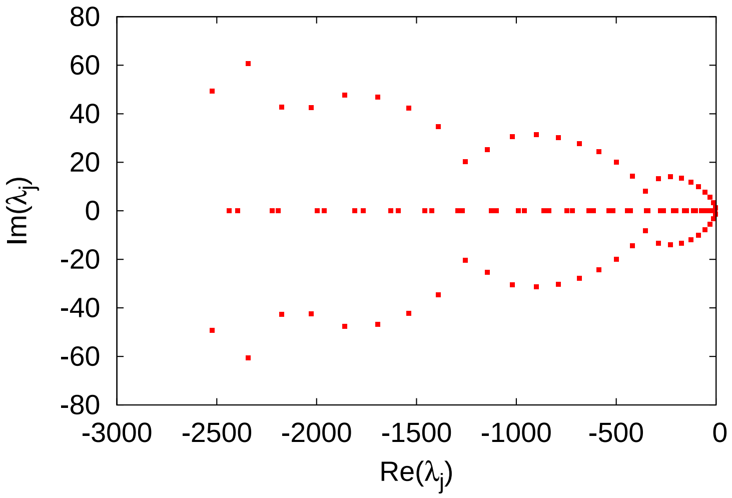
Table 4.1 – First eigenvalues of the structure system.

j	$1-2$	$3-4$	5	$6-7$	$8-9$	$10-11$	$12-13$
$\boldsymbol{\lambda}_j$	$0.81\pm8.79i$	$-0.70\pm9.63i$	-0.88	$-1.18\pm7.99i$	$-1.23\pm11.04i$	$-1.88\pm12.60i$	$-2.03\pm5.83i$

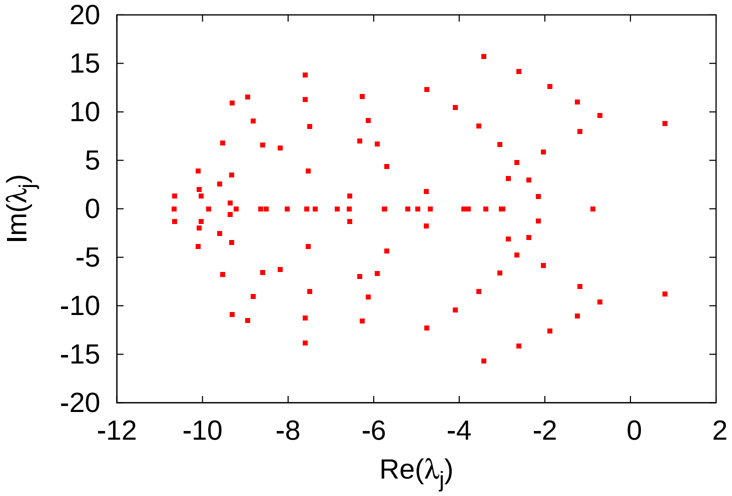
Table 4.2 – First eigenvalues of the fluid system.

j	$1-2$	3	$4-5$	6	$7-8$	$9-10$	$11-12$
$\boldsymbol{\lambda}_j$	$0.81\pm8.78i$	-0.59	$-0.72\pm9.59i$	-0.83	$-0.88\pm0.79i$	$-1.08\pm1.86i$	$-1.22\pm7.97i$
$d_{\boldsymbol{\lambda}_j}$	$0.21\,10^{-7}$	$5.55\,10^{-6}$	$0.26\,10^{-7}$	$9.09\,10^{-6}$	$0.13\,10^{-6}$	$0.27\,10^{-6}$	$0.61\,10^{-7}$

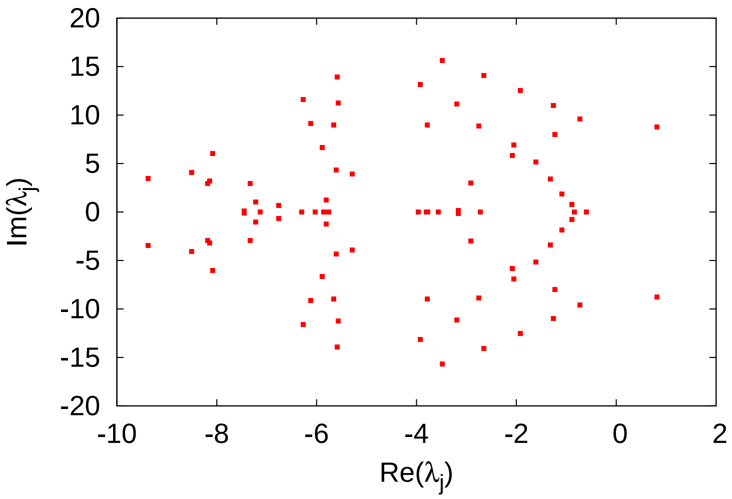
Table 4.3 – First eigenvalues of the coupled system and the degrees of stabilizability for $\omega = 6$.



(a) Spectrum of the structure.

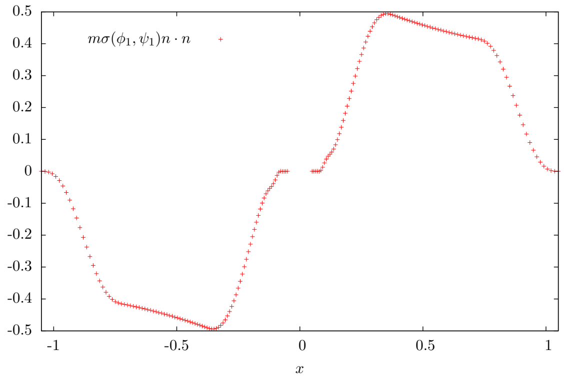


(b) Spectrum of the fluid.

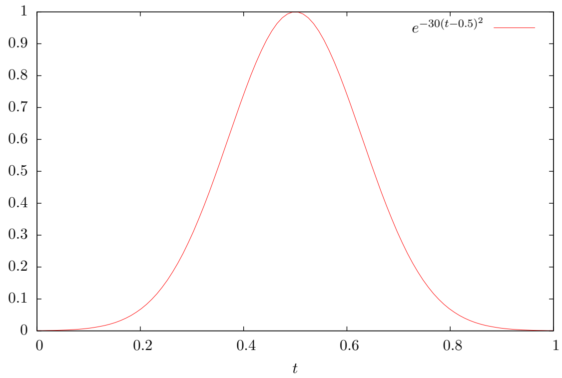


(c) Spectrum of the coupled system.

Figure 4.3 – Spectra.



(a) Graph of the function g.



(b) Graph of the function μ.

Figure 4.4 – Graphs of the functions g and μ.

We report, in the *Tables* 4.1, 4.2 and 4.3, the eigenvalues of the structure, the fluid and coupled systems close to the imaginary axis. The spectra of the fluid and the coupled systems are characterized by two complex conjugate unstable eigenvalues while all the eigenvalues of the structure are with negative real part.

4.7 .3 Efficiency of the controls law.

To test the efficiency of the controls on the nonlinear system, we use the boundary perturbation h defined in Chapter 2 to destabilize the nonlinear system (see (2.13 .2)). We recall that h is of the form

$$h(t,x) = \mu(t)g(x),$$

with

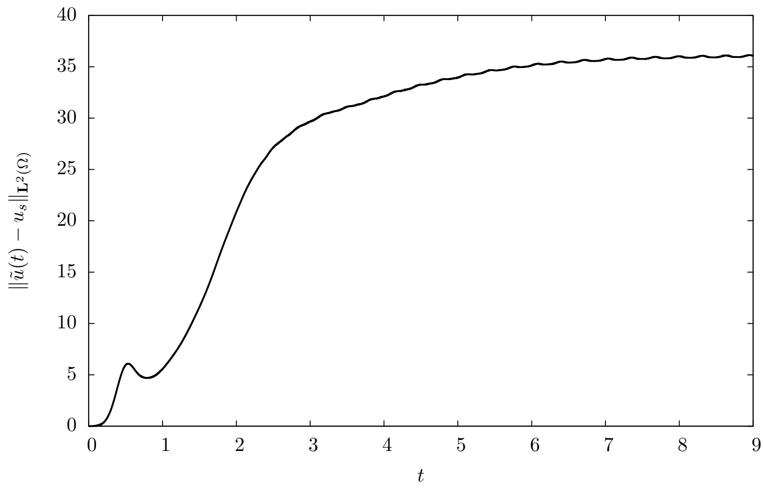
$$\mu(t) = \beta_{pert} e^{-30(t-0.5)^2}.$$

We plot in *Figure* 4.4 the curves of the functions g and μ. The parameter β_{pert} is used to vary the amplitude of the perturbation.

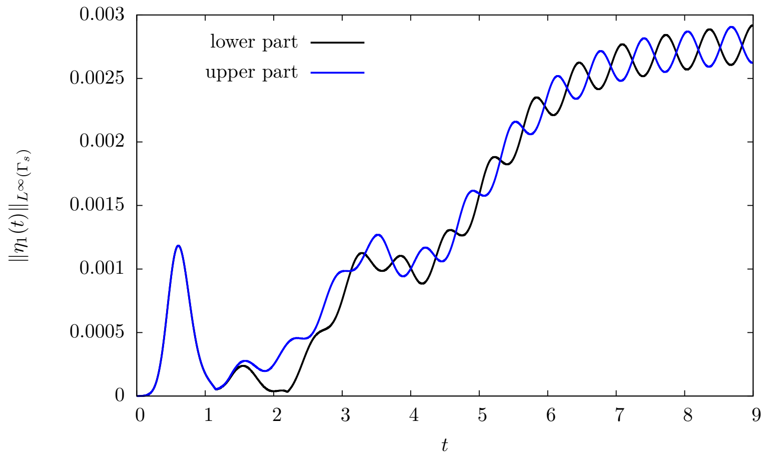
Small perturbation

We choose the parameter $\beta_{pert} = 0.1$. Thus, the maximal value of the perturbation h corresponds to 10% of that of g_s. In *Figure* 4.5, we plot the evolution of the L^2-norm of the evolution of the difference between the velocity $\tilde{u}$ of the uncontrolled nonlinear system and the velocity u_s of the stationary system and the evolution of the L^∞-norm of the displacement of the lower and upper part of the structure.

Now, we are going to test the efficiency of the controls on the nonlinear coupled system in three cases: $\mathbb{Z}_u = \mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_1)$ and $\omega = 0$, $\mathbb{Z}_u = \mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_1)$ and $\omega = 6$, and the last $\mathbb{Z}_u = \mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_1) \oplus \mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_3)$ and $\omega = 6$. The results are reported in *Figure* 4.6. We have plotted the evolution of the L^2-norm of the difference between the velocity $\tilde{u}$ of the controlled nonlinear system and the velocity u_s of the stationary system, the evolution of L^2-norm of the control and the evolution of the L^∞ norm of the displacement of the lower and upper part of the structure. We observe that when we increase the shift parameter ω, the decay rate is clearly improved, the corresponding control is stronger but it is effectively applied during a smaller period. One good effect is reported on the displacement of the structure which is reduced when ω growths. We notice that considering higher dimension of $\mathbb{Z}_u$ does not improve the control of the flow and the displacement of the structure.

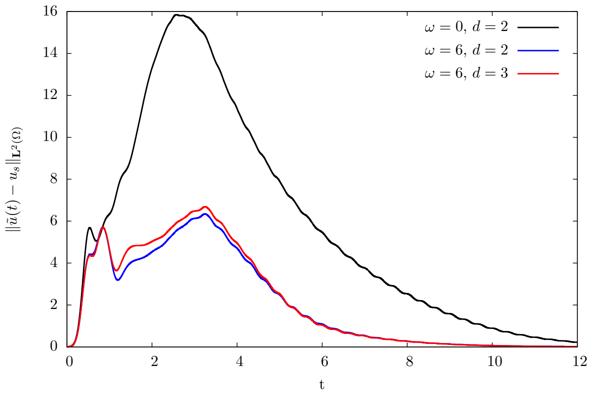


(a) Evolution of the L^2-norm of $\tilde{u} - u_s$.

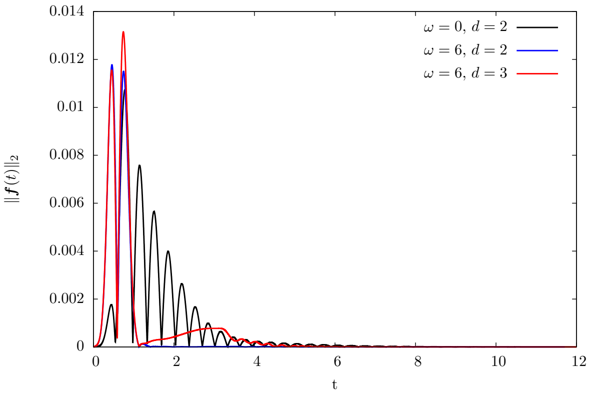


(b) Evolution of the L^∞-norm of η_1 without control.

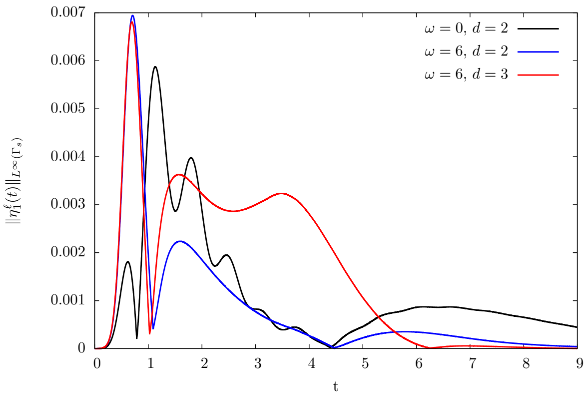
Figure 4.5 – Evolution of the L^2-norm of $\tilde{u} - u_s$ and the L^∞-norm of η_1 without control ($\beta_{pert} = 0.1$).



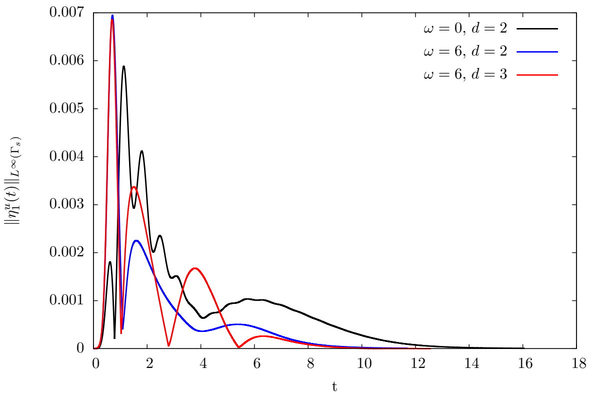
(a) Evolution of the L^2-norm of $\tilde{u} - u_s$.



(b) Evolution of the L^2-norm of the control.

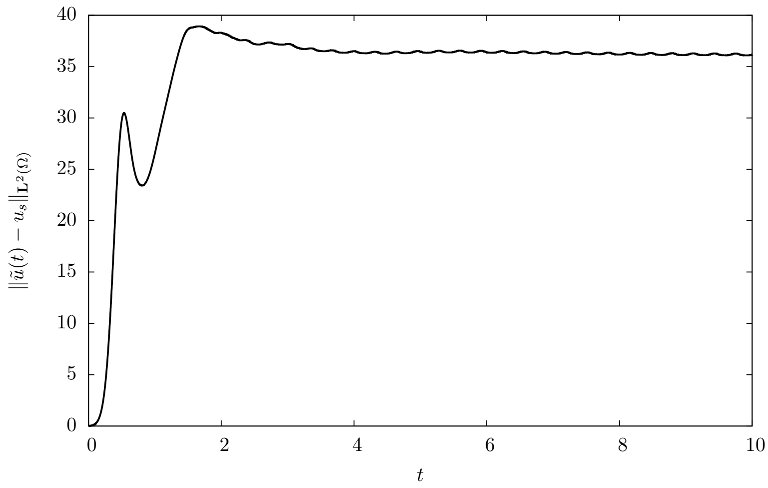


(c) Evolution of the L^∞-norm of the displacement of the structure (lower part).

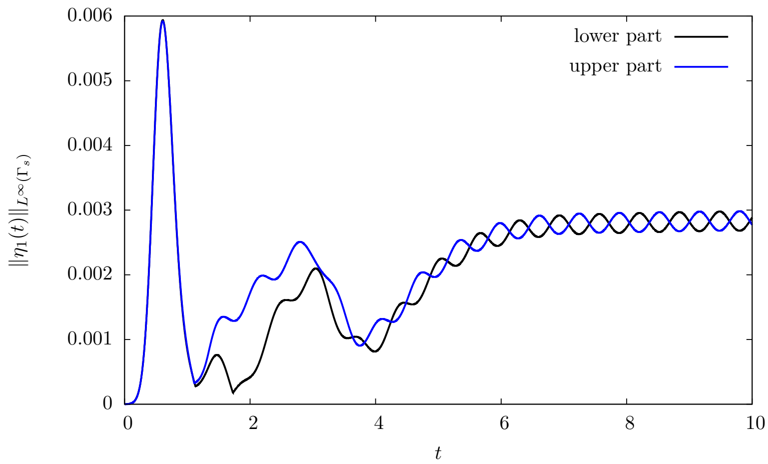


(d) Evolution of the L^∞-norm of the displacement of the structure (upper part).

Figure 4.6 – Evolution of the L^2-norm of $\tilde{u} - u_s$ and $\boldsymbol{f}$, and the L^∞-norm of η_1 of different controls ($\beta_{pert} = 0.1$).



(a) Evolution of the L^2-norm of $\tilde{u} - u_s$.

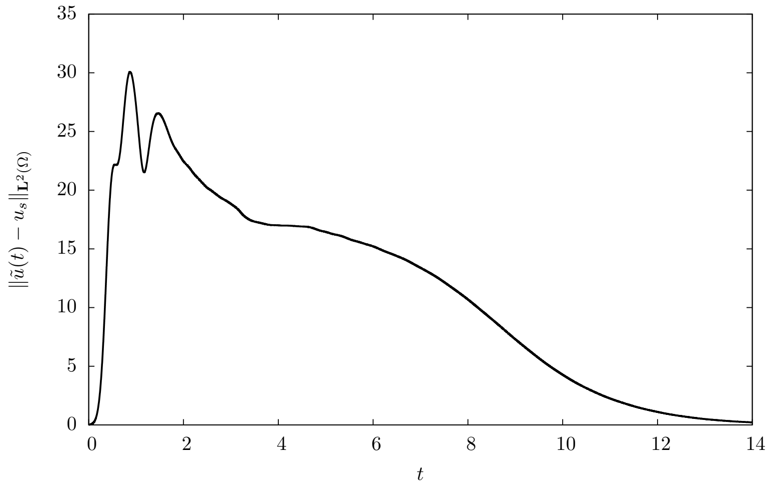


(b) Evolution of the L^∞-norm of η_1.

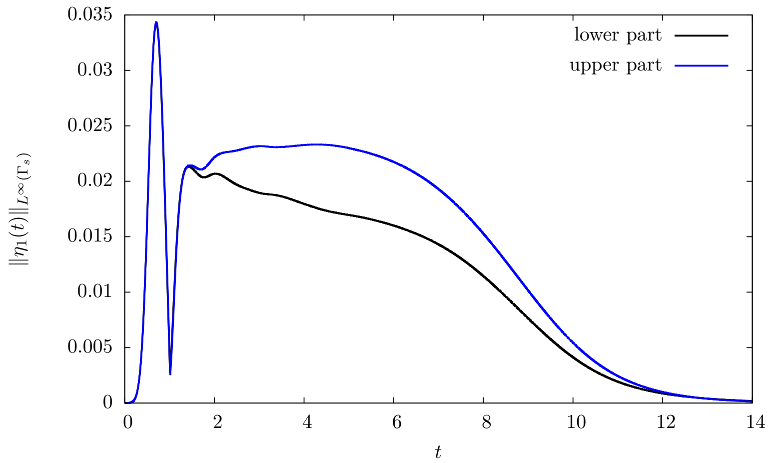
Figure 4.7 – Evolution of the L^2-norm of $\tilde{u} - u_s$ and the L^∞-norm of η_1 without control ($\beta_{pert} = 0.5$).

Large perturbation

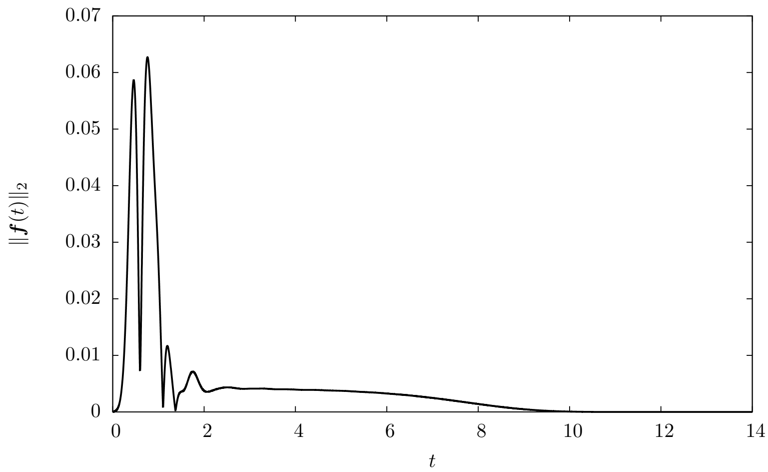
We choose the parameter $\beta_{pert} = 0.5$. Thus, the maximal value of the perturbation h corresponds to 50% of that of g_s. In *Figure* 4.7, we plot the evolution of the L^2-norm of the evolution of the difference between the velocity $\tilde{u}$ of the uncontrolled nonlinear system and the velocity u_s of the stationary system and the evolution of the L^∞-norm of the displacement of the lower and upper part of the structure. Here, for the control, we only consider the case $\mathbb{Z}_u = \mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_1)$ which seems to be the best choice and $\omega = 6$. In *Figure* 4.8, we plot the evolution of the L^2-norm of the difference between the velocity $\tilde{u}$ of the controlled nonlinear system and the velocity u_s of the stationary system, the evolution of L^2-norm of the control and the evolution of the L^∞ norm of the displacement of the lower and upper part of the structure. As expected, we observe that the control and the displacement of the structure increase with the perturbation β_{pert}. At the end of the section, we display the vorticity of u (the fluid flow in the moving domain) with and without control at different times.



(a) Evolution of the L^2-norm of $\tilde{u} - u_s$.

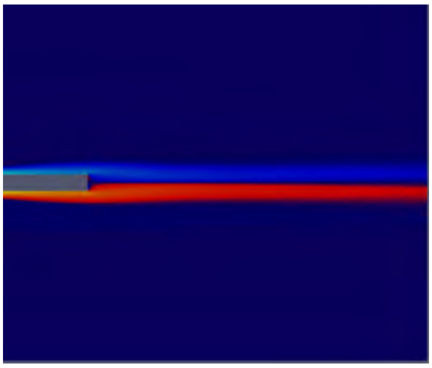


(b) Evolution of the L^∞-norm of η_1.

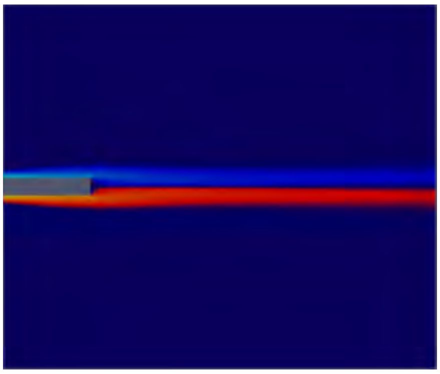


(c) Evolution of the norm of the control.

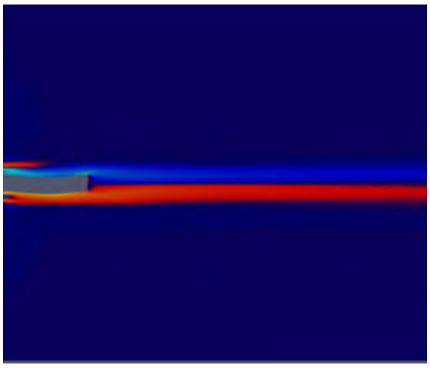
Figure 4.8 – Evolution of the L^2-norm of $\tilde{u} - u_s$ and the L^∞-norm of η_1 with control ($\beta_{pert} = 0.5$).



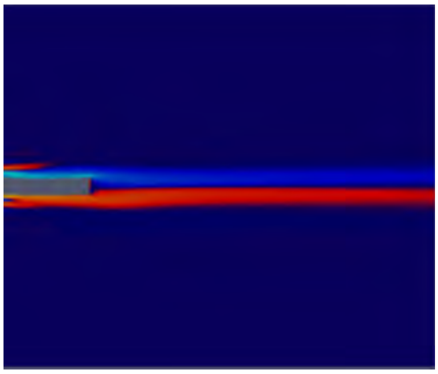
Vorticity of u at times $t = 0$.



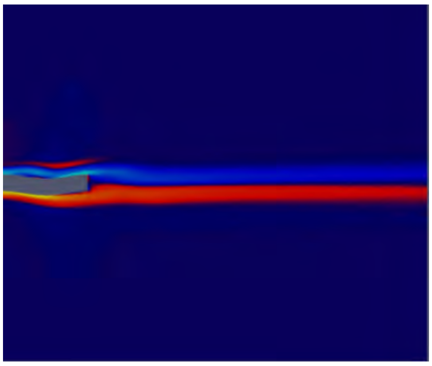
Vorticity of u at times $t = 0$ (without control).



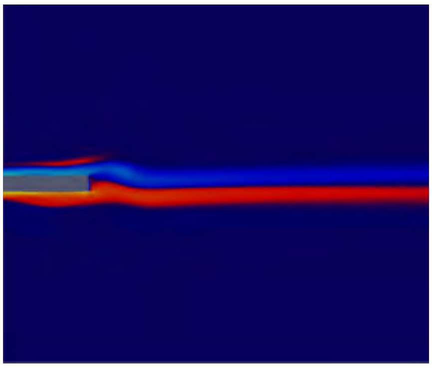
Vorticity of u at times $t = 0.5$.



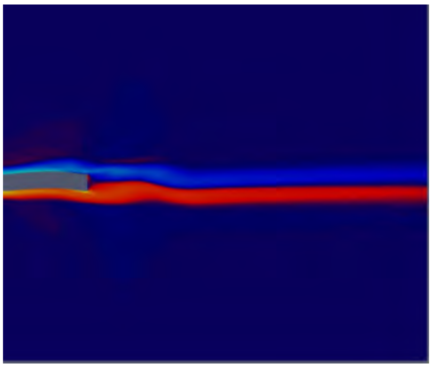
Vorticity of u at times $t = 0.5$ (without control).



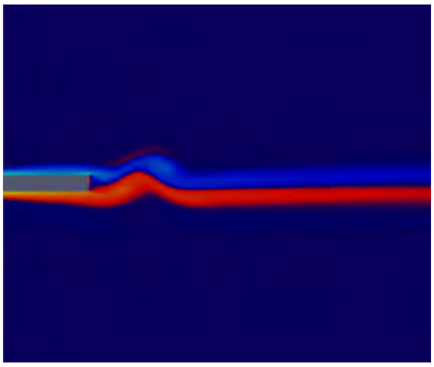
Vorticity of u at times $t = 0.85$.



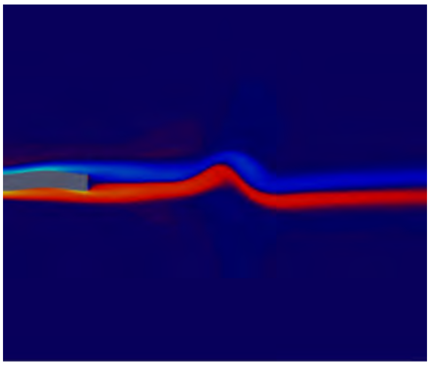
Vorticity of u at times $t = 0.85$ (without control).



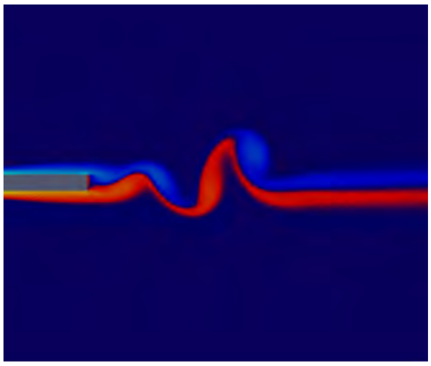
Vorticity of u at times $t = 1.2$.



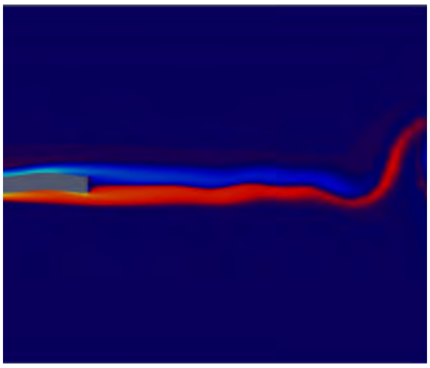
Vorticity of u at times $t = 1.2$ (without control).



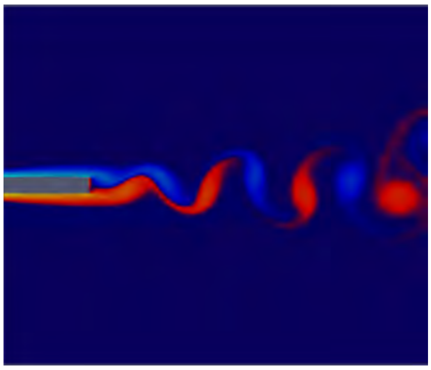
Vorticity of u at times $t = 1.9$.



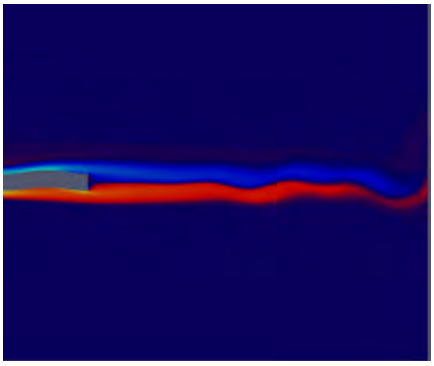
Vorticity of u at times $t = 1.9$ (without control).



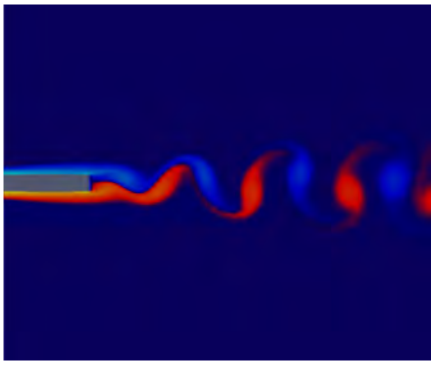
Vorticity of u at times $t = 3.2$.



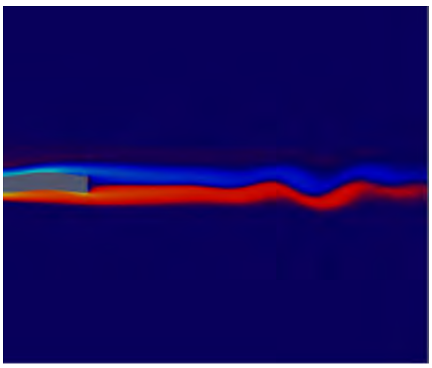
Vorticity of u at times $t = 3.2$ (without control).



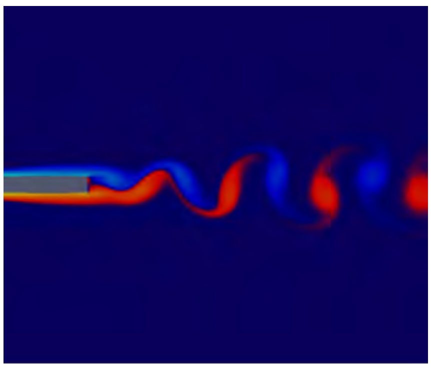
Vorticity of u at times $t = 3.5$.



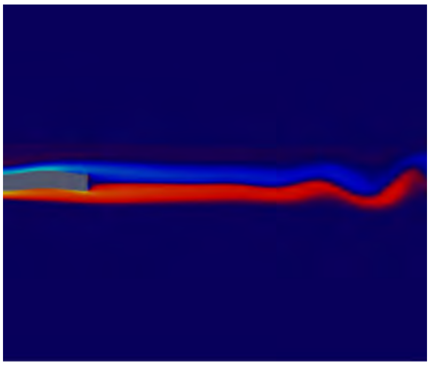
Vorticity of u at times $t = 3.5$ (without control).



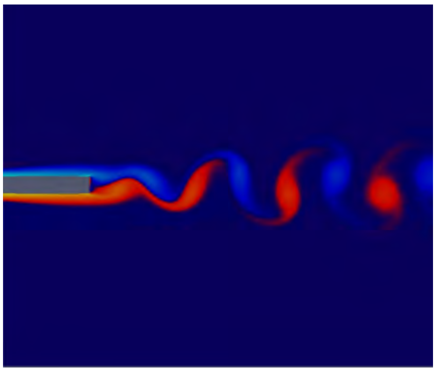
Vorticity of u at times $t = 4$.



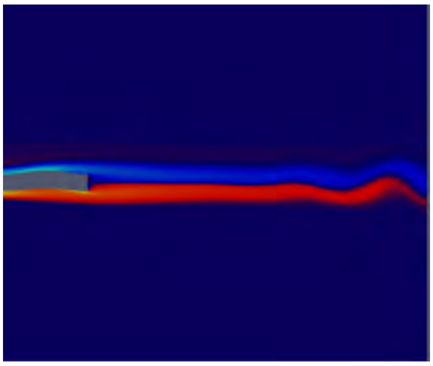
Vorticity of u at times $t = 4$ (without control).



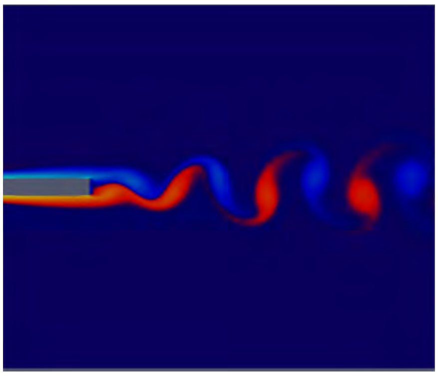
Vorticity of u at times $t = 5$.



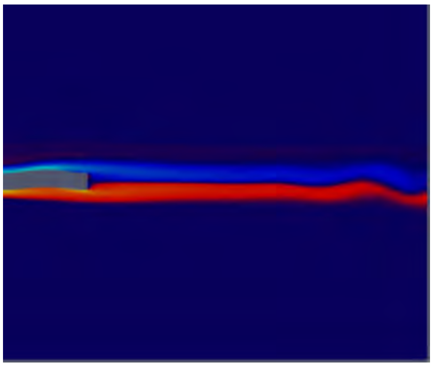
Vorticity of u at times $t = 5$ (without control).



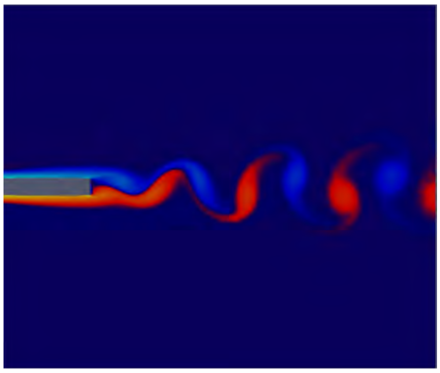
Vorticity of u at times $t = 5.5$.



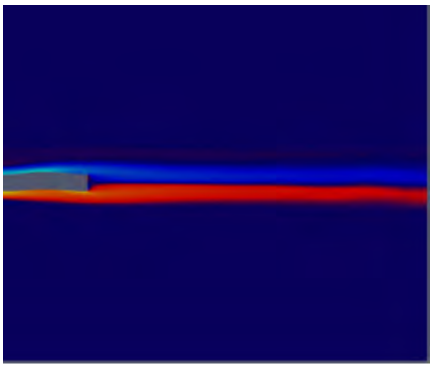
Vorticity of u at times $t = 5.5$ (without control).



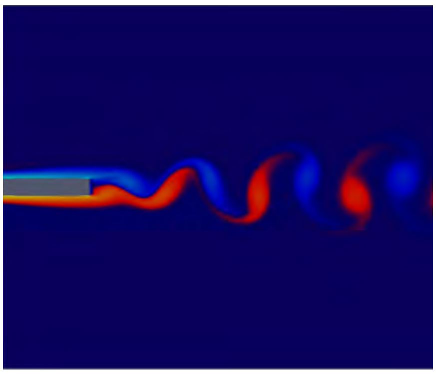
Vorticity of u at times $t = 6$.



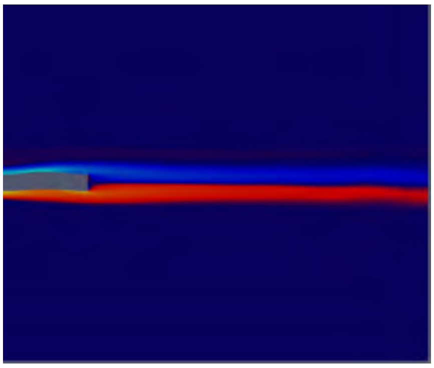
Vorticity of u at times $t = 6$ (without control).



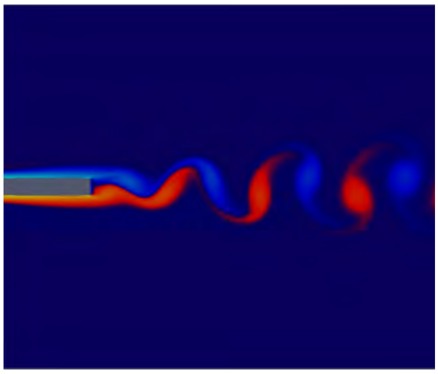
Vorticity of u at times $t = 7$.



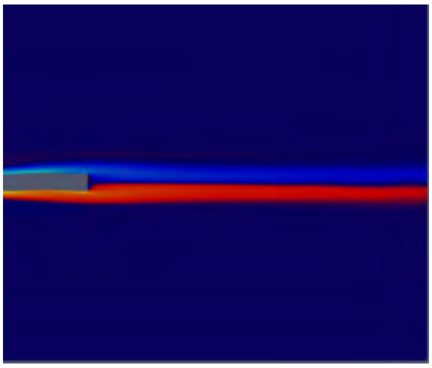
Vorticity of u at times $t = 7$ (without control).



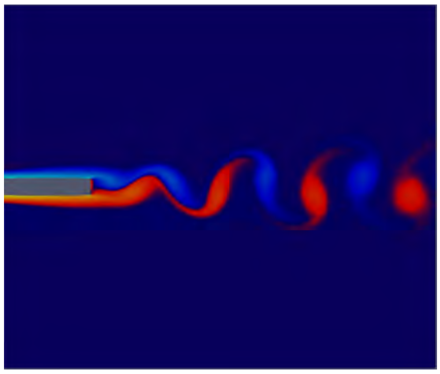
Vorticity of u at times $t = 8$.



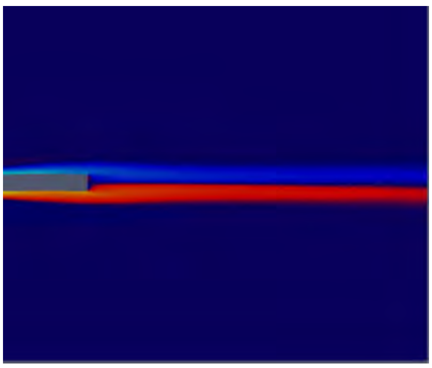
Vorticity of u at times $t = 8$ (without control).



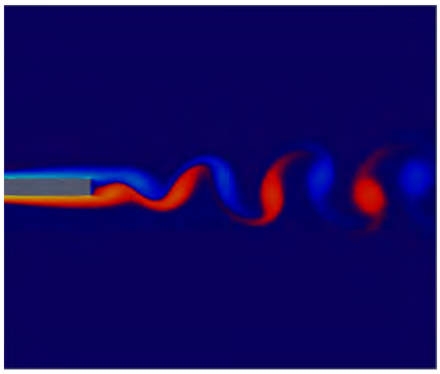
Vorticity of u at times $t = 9$.



Vorticity of u at times $t = 9$ (without control).



Vorticity of u at times $t = 10$.



Vorticity of u at times $t = 10$ (without control).

4.A Appendix A: The geometrical nonlinear terms

The nonlinear terms F_f, F_{div} and F_s in system (4.1 .2) are defined by

$$\begin{aligned}
F_f[\tilde{u},\tilde{p},\eta_1,\eta_2] =& \frac{(\ell - s(z)z)\eta_2}{\ell - e - s(z)\eta_1}\tilde{u}_z - \frac{2\nu(\ell - s(z)z)\eta_{1,x}}{\ell - e - s(z)\eta_1}\tilde{u}_{xz} + \frac{\nu((z-\ell)^2\eta_{1,x}^2 - \eta_1^2)}{(\ell - e + \eta_1)^2}\tilde{u}_{zz} + \frac{2\nu(\ell - s(z)z)\eta_{1,x}^2}{(\ell - e - s(z)\eta_1)^2}\tilde{u}_z \\
& - \frac{\nu(\ell - s(z)z)\eta_{1,xx}}{\ell - e - s(z)\eta_1}\tilde{u}_z - \frac{\nu(\ell - e)\eta_1}{(\ell - e - s(z)\eta_1)^2}\tilde{u}_{zz} - \nu\nabla \mathrm{div}\, F_{\mathrm{div}}[\tilde{u},\eta_1] + \frac{(\ell - s(z)z)\eta_{1,x}}{\ell - e - s(z)\eta_1}\tilde{p}_z e_1 \\
& - \frac{s(z)\eta_1}{\ell - e - s(z)\eta_1}\tilde{p}_z e_2 + \frac{(\ell + z)\eta_{1,x}}{\ell - e - s(z)\eta_1}\tilde{u}_1\tilde{u}_z - \frac{s(z)\eta_1}{\ell - e - s(z)\eta_1}\tilde{u}_2\tilde{u}_z, \\
F_{\mathrm{div}}[\tilde{u},\eta_1] =& \frac{1}{\ell - e}\left(s(z)\eta_1\tilde{u}_1 e_1 + (\ell - s(z)z)\eta_{1,x}\tilde{u}_1 e_2\right), \\
F_s[\tilde{u},\eta_1] =& \nu\left(\frac{(\ell - e)\eta_{1,x}}{\ell - e - s(z)\eta_1}\tilde{u}_{1,z} + \eta_{1,x}\tilde{u}_{2,x} - s(z)\frac{2\eta_1 - (\ell - s(z)z)\eta_{1,x}^2}{\ell - e - s(z)\eta_1}\tilde{u}_{2,z}\right),
\end{aligned} \tag{4.A.1}$$

where s is the sign function defined by

$$s(y) = -1 \text{ if } y < 0 \quad \text{and} \quad s(y) = 1 \text{ if } y \geq 0.$$

Chapter 5

Conclusion and perspectives

L'objectif de cette thèse était d'étudier la stabilisation, du point de vue théorique et numérique, de modèles d'interaction fluide-structure par des contrôles de dimension finie agissant sur la frontière du domaine fluide. Dans le chapitre 1, nous avons traité le cas où le contrôle agit comme une condition de Dirichlet sur les équations de fluide (contrôle par aspiration/soufflage). Dans les chapitres 2 et 3, nous avons étudié le cas où le contrôle est une force appliquée sur la structure (contrôle par déformation de paroi). Rappelons que le modèle du chapitre 2 est un modèle simplifié de celui du chapitre 3. En effet, l'équation d'Euler-Bernoulli pour la structure a été remplacée par une équation différentielle ordinaire de dimension finie qui, de plus, ne prend pas en compte l'action du fluide sur la structure. Le chapitre 4 et une partie du chapitre 2 ont été consacrés à la détermination de lois de contrôle permettant de stabiliser les approximations semi-discrètes des modèles des chapitres 3 et 2 respectivement. Nous avons vérifié l'efficacité de ces lois de contrôle grâce à des tests numériques. Contrairement à ce que nous pensions, les lois de contrôle du chapitre 2 ne sont pas plus efficaces que celles du chapitre 4. En effet, les résultats numériques des chapitres 2 et 4 sont assez similaires. Pour terminer ce manuscrit, nous allons présenté quelques perspectives.

1. *Propriété de continuation unique*

   Dans les chapitres 2 et 3, nous avons montré que la stabilisabilité du modèle couplé conduit à un problème de continuation unique sur les fonctions propres des équations de Navier-Stokes linéarisés. Nous n'avons pas pu résoudre théoriquement ce problème. Néanmoins, nous avons vérifié numériquement la stabilisabilité. De plus, le problème a été résolu dans [46, 47] pour le système de Stokes (i.e. l'équation de Navier-Stokes linéarisé autour de la solution nulle) mais la preuve ne peut pas être adaptée à notre cas, car elle nous conduit à supposer que la solution des équations de Navier-Stokes est analytique. Nous explorons donc d'autres approches.

2. *Estimations de modèles*

   Dans cette thèse, la loi de contrôle requiert la connaissance de tout l'état du système à chaque instant, ce qui est irréaliste dans les applications. Nous souhaiterions donc construire des estimateurs qui, à partir de la mesure de certaines quantités comme la pression du fluide sur la structure, permettrait de reconstruire l'état du système. La détermination d'estimateurs pour des problèmes fluide-structure est un sujet très actuel de recherche. Nous citons par exemple [18, 11].

3. *Modèles 3D*

   Nous souhaitons étendre les résultats obtenus à des modèles fluide-structure en 3D. L'écoulement sera toujours décrit par les équations de Navier-Stokes incompressibles.

L'équation d'Euler-Bernoulli pour la structure sera remplacée par une équation de plaque. Nous pensons que, si le domaine fluide est suffisamment régulier, les résultats obtenus dans le cas 2D pourraient être étendus assez facilement au cas 3D. Par contre, si le domaine fluide est peu régulier, la situation est plus délicate. En effet, nous avons vu, dans le cas 2D, que la géométrie du domaine ou/et la nature des conditions limites peut conduire à une perte de régularité pour les solutions des équations de Navier-Stokes.

4. *Formulation ALE des équations du fluide*

   La détermination des lois de contrôle permettant de stabiliser des modèles d'interaction fluide-structure est généralement basée sur une formulation eulérienne des équations du fluide. Par contre, la plupart des algorithmes de résolution numérique de ces modèles sont développés à partir d'une formulation Arbitraire Lagrangienne Eulérienne (ALE) des équations du fluide. Une perspective de travail pourrait être de déterminer des formulations ALE des systèmes en boucle fermée que nous avons obtenus aux chapitres 2 et 4 et les tester numériquement.

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Stabilisation et simulation de modèles d'interaction fluide-structure

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Résumé : L'objet de cette thèse est l'étude de la stabilisation de modèles d'interaction fluide-structure par des contrôles de dimension finie agissant sur la frontière du domaine fluide. L'écoulement du fluide est décrit par les équations de Navier-Stokes incompressibles tandis que l'évolution de la structure, située à la frontière du domaine fluide, satisfait une équation d'Euler-Bernoulli avec amortissement. Dans le chapitre 1, nous étudions le cas où le contrôle est une condition aux limites de Dirichlet sur les équations du fluide (contrôle par soufflage/aspiration). Nous obtenons des résultats de stabilisation locale du système non-linéaire autour d'une solution stationnaire instable de ce système. Dans les chapitres 2 et 3, nous nous intéressons au cas où le contrôle est une force appliquée sur la structure (contrôle par déformation de paroi). Dans le chapitre 2, nous considérons un modèle simplifié, où l'équation d'Euler-Bernoulli pour la structure est remplacée par un système de dimension finie. Nous construisons des lois de contrôle pour les systèmes de dimension infinie, ou pour leurs approximations semi-discrètes, capables de stabiliser les systèmes linéarisés avec un taux de décroissance exponentielle prescrit, et localement les systèmes non-linéaires. Nous présenterons des résultats numériques permettant de vérifier l'efficacité de ces lois de contrôles.

Mots-clés : interaction fluide-structure, équations de Navier-Stokes, équation d'Euler-Bernoulli, contrôle frontière, éléments finis, simulation numérique.

Abstract : The aim of this thesis is to study the stabilization of fluid-structure interaction models by finite dimensional controls acting at the boundary of the fluid domain. The fluid flow is described by the incompressible Navier-Stokes equations while the displacement of the structure, localized at the boundary of the fluid domain, satisfies a damped Euler-Bernoulli beam equation. First, we study the case where the control is a Dirichlet boundary condition in the fluid equations (control by suction/blowing). We obtain local feedback stabilization results around an unstable stationary solution of this system. Chapters 2 and 3 are devoted to the case where control is a force applied to the structure (control by boundary deformation). We consider, in chapter 2, a simplified model where the Euler-Bernoulli equation for the structure is replaced by a system of finite dimension. We construct feedback control laws for the infinite dimensional systems, or for their semi-discrete approximations, able to stabilize the linearized systems with a prescribed exponential decay rate, and locally the nonlinear systems. We present some numerical results showing the efficiency of the feedback laws.

Keywords : fluid-structure interaction, Navier-Stokes equations, Euler-Bernoulli beam equation, boundary control, finite element, numerical simulation.