



Entanglement and Decoherence in Loop Quantum Gravity

Alexandre Feller

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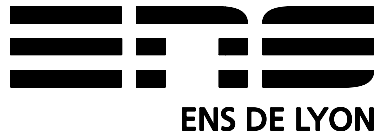
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Entanglement and Decoherence in Loop Quantum Gravity

Intrication et décohérence en gravitation quantique à boucles

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Remember...

Pour moi, les meilleurs remerciements que l'on peut adresser ne prennent pas la forme de quelques mots, formant à cet instant le début de la fin de trois belles années de travail doctoral, mais la forme de souvenirs, dans ma mémoire. Quoi qu'il en soit, je vais essayer ici de modestement les partager bien que cela paraîtra au lecteur parfois flou et obscur.

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Un peu plus tard, j'ai découvert les joies des soirées, du foyer et de la vie étudiante de l'École. Sans conteste, le centre névralgique de toute cette (ma) vie se trouve au babyfoot. Impossible de compter les heures passées là, ce serait indécent d'ailleurs. Lilou, Chris, Jerem, Percy, Dédé, Lulu, Sami, Anthony, tellement de rire, tellement de conneries, tellement de rage d'avoir perdu mes deux BO 69 que le mur de (l'ancien, RIP) foyer s'en est souvenu. Pour toutes ses soirées en Festive, en ville, ou tranquille chez les uns ou les autres, pour ces vacances à Nice au bord de la piscine, à manger, boire et chanter

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*And here be
dragons.*

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Introduction

Modern physics is built on two major physical theories, general relativity, describing the large scales of the Universe to our human scales, and quantum physics, describing the smallest scales of matter. However, we still lack a complete and coherent framework of physics marrying both theories. This is the purpose of a quantum theory of gravity which would offer a comprehensive framework to do physics at all distances and energies.

The natural question anybody could ask is why do we really need a quantum theory of gravity at all. We could simply try at first to make sense of a semi-classical theory describing matter with quantum field theories and spacetime with classical general relativity. Apart from the fact that such a theory mixes two very different kind of descriptions, a classical one for gravity and a quantum one for matter, issues arise when it comes to the sources of the gravitational field. Indeed, from the principle of equivalence, all sources of energy contribute to the gravitational field and are here described with quantum field theory by the energy-momentum operator. Is its average value the source of the gravitational field ? What about quantum fluctuations of matter ? What about quantum superpositions of spacetime then ? Those questions are poorly answered in such a semi-classical framework. In fact, many attempts to build semi-classical theories have been proven to be internally inconsistent: for instance, it can be shown that a semi-classical theory of electromagnetism (so without photons) leads to a violation of standard commutation relations of matter quantum operators. This of course does not mean that useful information cannot be extracted from semi-classical approaches (Hawking radiation or black hole entropy) but they are not the end of the story. Pushing this idea even further, the major reason appeals to the unity of physics. Basically the conceptual core of general relativity and quantum physics are too different. General relativity is a classical theory with a dynamical background and no preferred notion of time. Quantum physics on the other hand uses Hilbert spaces, operators defined with respect to a fixed background, a preferred notion of time and finally a notion of external observer that perform measurements. Another reason, among many others, concerns cosmology. Many evidences converge toward the idea that quantum effects should be important to understand the large structure of the Universe and its origin. It is necessary to have a quantum theory of gravity to understand the Universe. However, in this context, the observer cannot be thought as an exterior system but become a part of the whole quantum description. Questions regarding the meaning

of the quantum states, measurements, probabilities or what is an observer have to be addressed. Already posed on studies of the interpretation of quantum theory, they become a necessity in quantum gravity. This naturally asks the question if quantum gravity has something to say about the interpretation of quantum physics, if one is favored among the others.

Is there any experimental evidence that needs a full quantum theory of gravity? To this day no experiments require or probe quantum gravity physics. This makes research in quantum gravity quite different from standard theoretical physics which is most of the time guided by experiments and well established principles. Instead, we have here to rely on pure theoretical and philosophical thinking, building the theory from what we think it should be. We have to argue which principles should be deemed fundamental and others emergent. This requires a proper analysis of the foundations of the well established existing theories like quantum physics and general relativity. And the first conclusion we arrive at is that we do not yet fully and clearly understand their foundations, their meaning and the picture of the world they give us. This is an additional difficulty for quantum gravity. Yet such reflexions are essential for research, physics and science in general. The variety of point of views on those matters explains the huge amounts of ideas, toy models and incomplete theories in this field of theoretical physics. The mathematical tools used in the different proposal are heavily influenced by those prior choices ¹.

How do we go then into the quantum gravity realm? [Carlip, 2001] ² As already mentioned, a first step is to look at the physics of quantum fields in curved spacetime, or even use perturbative methods of quantum field theories [Donoghue, 1994]. Major predictions have been made from this approach like the Hawking radiation or black hole entropy, but this is not sufficient. Approaches to quantum gravity can be roughly speaking classified into two categories: one that look at general relativity as a low energy limit of a more fundamental theory and one that takes general relativity seriously and tries to properly quantize it. The two major candidates representing this two crude alternatives are string theory and loop quantum gravity. In this thesis we will work with the latter.

Loop quantum gravity is a non perturbative proposal of a theory of quantum gravity, historically based on a canonical quantization of general relativity, with background independence built in its core. Its major successes are the prediction of the discrete nature of space at the Planck scale and the development of quantum geometry. A major aspect to grasp in this approach is that the fundamental degrees of freedom of the theory do not live in spacetime and are not geometric in nature. They however define spacetime and geometry in a suitable sense. This leads to a key problem which without a proper answer forbids us to claim that the proposed theory is really a quantum theory of gravity: how do we recover the description of spacetime as a differentiable manifold? To be more specific,

¹An interesting point of view (which I do not completely grasp yet) about the role of the mathematical tools in the construction of physical theories and how they encode our prior philosophical ideas can be found in [Doring and Isham, 2010]

²See [Carlip et al., 2015] for a huge bibliography.

we can state two open problems in the loop quantum gravity proposal

- *The problem of localization.* This is the problem related to define a reference frame and localized systems in a diffeomorphism invariant theory. In such a theory, a notion of distance and time must be reconstructed from subsystems. Some of them are assigned the role of a clock and rulers, defining a reference frame. Evolution of observables of other subsystems relative to those coordinates have finally to be studied.
- *The semi-classical regime.* Recovering a classical and continuum theory of gravity in a proper limit must be understood. The main road toward formulating the continuum limit is through coarse-graining procedures of the discrete quantum states. Classicality is studied through the construction of coherent states interpolating between quantum and classical regimes. Understanding the notion of observer in the quantum regime is again crucial since only then we will be able to understand what a macroscopic and classical observer is.

In this no spacetime approach where the degrees of freedom are merely algebraic data, the question of the emergence is of paramount importance [Oriti, 2014]. Emergence is an essential idea in modern physics born from effective field theories, renormalization group and condensed matter physics. Basically the idea of emergence states that new unforeseen phenomena from the point of view of “microscopic” physics appear at larger length scales from the complex collective behavior of the fundamental degrees of freedom. A recent and quite astonishing example is the fractional quantum Hall effect coming from the complex Coulomb interaction of two dimensional electrons in a magnetic field at low temperature. In loop quantum gravity, this question of the emergence of classical spacetime from the fundamental none geometric degrees of freedom is essential to firmly ground the theory.

To clarify those questions, quantum information ideas and methods become increasingly important in quantum gravity studies. This is an understandable evolution. Indeed, general relativity is already at the classical level a theory of correlations. Physical predictions are about values of fields relative to other fields. This is beautifully illustrated by the physics of black holes where deep relations between gravity and information have been uncovered. Indeed, after the proof by Hawking that black holes were thermodynamical objects carrying an entropy, many thought experiments like the information loss paradox, black hole complementarity or more recently the firewall paradox were formulated and illustrate the paramount importance of quantum information ideas to understand gravitational physics. Moreover, this physics shows the holographic nature of gravity and challenged our standard notion of locality implemented in quantum field theories.

The work presented in this thesis follows this quantum information point of view to explore the problems of the emergence of geometry, locality and the quantum to classical transition in the context of loop quantum gravity. This thesis is structured as follows. It contains two parts, the first giving the necessary background to understand the core of the project which is presented in the second part. Chapter 1 introduces the basics of quantum

information with a strong focus on quantum entanglement and its use to understand the quantum to classical transition with decoherence theory. Both the conceptual part and the formal tools are discussed and used in the core of the thesis. The role of locality in quantum physics and quantum field theory is also emphasized. Chapter 2 deals with the gravitational aspects with a discussion of classical general relativity, its conceptual aspects and relations with quantum information in black holes physics. Once again, a particular attention is given to the notion of locality. Finally, the first part ends with Chapter 3 on loop quantum gravity. Its aim is to give the basics of its construction as a canonical quantization of a good formulation of general relativity and then review its kinematical states called spin network states which are the building block of the quantum description of space. We end up by a more precise discussion of the open issues left to understand in this theory and how this thesis project fits there. We then jump to the second part of the thesis which presents the core project at the interface between quantum information and loop quantum gravity. After presenting the whole idea of the project, Chapter 4 discusses the proposal to reconstruct geometry, distance and locality from correlations and entanglement stored in spin network states. The following Chapter 5 explores holographic ideas and the potential structure holographic physical states should have. This is an important question to make contact with the standard prediction of general relativity and the development of the holographic principle. From there, we focus our attention on the physics of the boundary of a region of space (more generally surfaces). Indeed, gravity can in a sense be fully understood from the dynamics and physics of two dimensional hypersurfaces. Chapter 6 proposes a definition of a quantum surface motivated by loop quantum gravity and analyzes a set of semi-classical dynamics. Both the isolated and the open regime are discussed, the latter being naturally more relevant from a physical point of view. All those models explore the issue dubbed the localization problem, from the problem of reconstruction of geometry to the problem of dynamics. The last Chapter 7 explores the second issue about the quantum to classical transition. It analyzes some decoherence mechanism of the boundary state induced by the bulk-boundary coupling.

A simple thought before diving into the subject. Having a well established quantum theory of gravity could bring a lot to physics, from a conceptual perspective but also maybe to clarify major open questions in theoretical physics like the problem of the expansion of the Universe (dark matter, dark energy), its origins or the unification of all the fundamental forces of Nature. But we do not have it yet, despite years of efforts. Yet they have led to a lot of questioning about the conceptual foundations of physics and have brought many new ideas. This is the essential point that maybe the true theory will not be based on general relativity, will go beyond quantum physics and that we need a different thought process, new mathematical framework and some new physical principles. Or not.

Part I

Background materials

Chapter 1

Quantum physics and information

The superposition principle and entanglement represent the new and counter-intuitive features of quantum theory over classical physics. Experiments are now able to prepare, manipulate and measure single quantum objects to test the quantum weirdness. One major goal is to use them as fundamental properties for quantum information processing and build a computer based on those quantum rules to improve computational tasks. Entanglement is the key ingredient here, at the same time source of computational errors and the solution to correct them with quantum error correction codes. On a more fundamental ground, entanglement has become the central aspect to understand the emergence of a classical objective reality from a quantum world through the process of decoherence.

Still, as discussed in the introduction, our current understanding of quantum physics is incomplete because of the elusive quantum theory of gravitation. In many different proposal, we are beginning to understand that entanglement between fundamental degrees of freedom may be crucial to the spacetime fabric and the emergence of classical geometry. Historically, the relationship between quantum information and gravity has been very fruitful to expose internal paradoxes in the standard modern physics framework based on quantum field theory and general relativity. The well known information loss paradox [Hawking, 1975] or the more recent firewall paradox rely [Almheiri et al., 2013] on the properties of quantum information and entanglement. An actual major research program in the quantum gravity community, from the AdS-CFT to the loop quantum gravity approaches, is to use entanglement to understand the quantum nature of spacetime [Van Raamsdonk, 2010]. For instance, the $ER = EPR$ [Maldacena and Susskind, 2013] proposal tries to establish a link between wormhole geometry (Einstein-Rosen bridges) and entanglement properties between two regions (Einstein-Podolski-Rosen). The goal is then to recover our intuitive notion of locality, distance and even, in the end, the full spacetime structure as a manifestation of entanglement and correlations of fundamental degrees of freedom. It is naturally possible that this microscopic notion of locality would not perfectly match the macroscopic notion of locality based on effective geometry.

Since we want to use entanglement and correlations to understand quantum gravity,

the first step is to have a clear overview of those quantum information notions, especially entanglement and decoherence. The presentation will follow the development of a pragmatic physical problem where entanglement and decoherence have an essential role: the problem of constructing a quantum theory of measurement. This is a major question on its own for the foundations of quantum physics. The idea here is to give some physical context while developing the tools we will need.

In section 1.1 we discuss the basic framework of quantum theory and entanglement. We then ask the question of modeling a quantum measurement process and raise three major issues rooted in entanglement. Section 1.2 discuss the theory of decoherence and the quantum to classical transition which answers most of those issues. Section 1.3 develops the formal tools to analyze open quantum systems, both from an exact and approximate point of view. The chapter ends with Sections 1.4 and 1.5 with a discussion of entanglement in quantum field theory and locality in quantum physics to pave the way toward the information loss and firewall paradoxes in black hole physics.

1.1 Entanglement - Quantum measurement

1.1.1 Quantum theory

Foundations of quantum theory raise many vivid debates among physicists. For instance what is a quantum state ? Is there any “good” interpretation of the theory ? How our classical world emerges from a quantum description ? Those questions are still looking for a definite answer. It is important to stress that there is no consensus among physicist on the matter and a given answer reflects mostly the opinion of the scientist. Yet, many experimental and theoretical advances have been made since the twenties [Haroche and Raimond, 2013]. For instance, quantum information has opened a new perspective to tackle those questions and entanglement is the lead character.

It is interesting to lay down the basic postulates of quantum theory we will use. This discussion follows mainly the work of Zurek [Zurek, 2008]. The standard approach to quantum theory was laid down by Bohr. The major issue was, and in fact still is, on the role of measurement and the meaning of the quantum state. As we will see, treating the measurement process in the quantum framework leads to non trivial issues and the famous Bohr-Einstein debates [Wheeler and Zurek, 1983]. Bohr’s view converges toward the idea that the quantum states can acquire meaning only through a classical language. This mostly require macroscopic systems obeying the laws of classical physics where information can be communicated, copied between different observers. Doing so solves the problem of the meaning of the quantum state which makes sense only relative to a classical setting and reflects the knowledge of the observer.

However, it seems odd to impose a clear frontier where quantum theory breaks down and classical physics should become valid. This asks the question where is the frontier and why it is here. Instead, we want to accept the universality of quantum physics at

all scales. And indeed, more experiments tend to push farther and farther the boundary introduced in the Bohr-Copenhagen interpretation, engineering quantum superpositions with larger systems away from the simple atomic scale [Hornberger et al., 2003]. What's more, evidences are that quantum physics in the early universe has macroscopic effects due to inflation. The aforementioned issues have to be thought again. We need to go beyond Bohr.

Thus I will discuss approaches that have the minimal set of assumptions and then see how we derive the standard results. We state that quantum principles are valid at all scales. Postulates regarding measurements are avoided as well as all ideas of quantum break down. Then what we know about the measuring process in quantum theory (resumed in the traditional measurement postulate) must be derived from first principles¹. That's where entanglement will enter the story both as the source and the solution of our problems.

The idea is to start from the following assumptions

- The Universe is composed of subsystems;
- the state of a system is described by an element $|\psi\rangle$ of an Hilbert space \mathcal{H}_S ;
- the evolution is given by a Schrodinger equation $i\hbar\partial_t|\psi\rangle = H_S|\psi\rangle$ with a Hamiltonian H_S .

Some remarks are in order. Hidden behind the linearity of the Schrodinger equation is the superposition principle, a major, and sometimes counter-intuitive, aspect of quantum physics. Simply stated, if a system can exist in different configurations, then it can also exist in a linear combination of those configurations. This leads to quantum interferences and all the strange properties of the quantum world.

Another important point here is the first one on subsystems which is often omitted or considered obvious. The whole idea to understand the measurement process from the quantum will rely on entanglement and open quantum system. Those are meaningless without the notion of subsystems. Defining subsystems is not such a trivial thing to do, especially in models with local symmetries like gauge field theories and general relativity [Donnelly and Freidel, 2016] since the total Hilbert space cannot be written as a tensor product. We will discuss this more on chapter 7 and admit for the time being that a proper definition of subsystems has been established.

The last remark concerns time. Here the evolution is given with respect to a physical time t and a true Hamiltonian. However, we know that in general relativity there is no global notion of time and, as we will see later, because of diffeomorphism invariance, no true Hamiltonian but only constraints. Thus the above formulation must be refined for constrained systems. Still, the statements we will make in the following remain valid, at least when we know how to solve the constraints and define a physically meaningful notion of time.

¹ This premiss leads naturally to relative states formulation of quantum theory [Everett, 1957; Zurek, 2008].

1.1.2 What is entanglement ?

Definition We wish now to describe a collection of systems \mathcal{S}_i that can be prepared at will and evolve according to a given dynamics. The total Hilbert space \mathcal{H} describing the states of those systems is, from the superposition principle, the tensor product

$$\mathcal{H} = \bigotimes_i \mathcal{H}_{\mathcal{S}_i} \quad (1.1)$$

Let's for now restrict ourselves with only two subsystems \mathcal{S}_A and \mathcal{S}_B to simplify the discussion, describing two observers Alice and Bob. Two kind of states can be distinguished. The first kind of states has a factorized structure like $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. In such a situation, we can assign a definite state to each subsystem separately, the first one being in the state $|\psi_A\rangle$ and the second in the state $|\psi_B\rangle$. Those states are called separable or factorized. Then there is the second kind of states which are simply those that cannot be written in a factorized manner. Those states are precisely entangled states. They have the generic form, in a basis $(|\phi_i^{A,B}\rangle)_i$

$$|\psi\rangle = \sum_{ij} a_{ij} |\phi_i^A\rangle |\phi_j^B\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \quad (1.2)$$

What makes them so special compared to factorized states ? First of all, we see that it is impossible to assign a definite state to each subsystems separately. Only the global state can have meaning. We cannot here reduce the global system to the sum of its parts like in classical physics. But what's more remarkable with entanglement is that it has stronger than classical correlations. There are many hints toward the fact that quantum information is completely different from classical information like for instance the no-cloning theorem or the no-deletion theorem, both consequence of linearity. But Bell's theorem is the major result showing that quantum correlations are strictly stronger than classical ones: Bell constructed an estimator of correlations and showed that its maximum value for a classical local theory is two whereas quantum theory allows to violate this bound. This violation was confirmed experimentally many times (even if some loopholes are still discussed). In fact, a result by Tsirelson states that quantum physics allows Bell's correlations to have a maximum value of $2\sqrt{2}$ but no one really understand why this particular number appears.

Reduced density matrix Like mentioned above, it is not possible to assign a definite state to a subsystem with entanglement. Still, it is possible to assign to each subsystem an object that encodes all the relevant information about the local measurements Alice or Bob can make on their system. It is called the reduced density matrix ρ_A . For Alice it is defined by $\rho_A = \text{tr}_B \rho_{AB}$ where ρ_{AB} is the density matrix describing the total system. To remind the reader, a density matrix is an Hermitian, positive operator with unit trace ρ generalizing the pure state description of a system to incorporate statistical aspects. For any local observable of Alice O_A , we can obtain its statistical properties from ρ_A only, like for instance its average $\langle O_A \rangle = \text{tr}(O_A \rho_A)$.

Entanglement measures It is interesting to have a more quantitative measure of the degree of entanglement and correlations between two systems. That's precisely what the Von Neumann entropy S does. Other entanglement witness can be used, mostly to have simpler calculations. For a system described by a density matrix ρ , the Von Neuman entropy is defined as

$$S(\rho) = -\text{tr}(\rho \ln \rho) \quad (1.3)$$

It is easy to see that the entropy is zero if the system is in a pure state. Thus, if the total system ($\mathcal{S}_A + \mathcal{S}_B$) is in a factorized pure state, the entropy of each subsystem calculated from the reduced density matrix is $S(\rho_{A(B)}) = 0$. However, if the total pure state is entangled, the reduced density matrix becomes a statistical mixture and gives a non zero entropy. It can be shown that $S(\rho_A) = S(\rho_B) > 0$. This non zero entropy between Alice and Bob is a witness of their entanglement.

More generally, if Alice and Bob are described by a generic density matrix (not necessarily a pure state), it can be also shown that if $\rho_{AB} = \rho_A \otimes \rho_B$, then $S(\rho_{AB}) = S(\rho_A) + S(\rho_B)$. When no correlations are present, the total entropy is the sum of the entropy of each independent subsystems. However, if the total density matrix is not factorized then we have only the subadditivity inequality

$$S(\rho_{AB}) < S(\rho_A) + S(\rho_B) \quad (1.4)$$

By looking at a subsystem only, information has been lost, precisely those correlations encoded in the global density matrix. Beware the subtlety here. For a generic ρ_{AB} , the Von Neumann entropy measures correlations between the subsystems but all correlations are not entanglement. Indeed ρ_{AB} could be a classical mixture of factorized states like $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$. In this case, we have no entanglement, only classical correlations and S is sensible to those also. Thus, the Von Neumann entropy really measures entanglement only when the total state is pure and correlations in the generic case.

To be even more precise and have a quantity that measures correlations, it is natural to consider the mutual information $I(A, B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ between Alice and Bob. It is zero when the observers are independent, positive otherwise ².

In the end, what we have to remember is that in quantum physics, the total state is not the sum of its parts. It contains more information and looking only at subsystems misses part of it.

1.1.3 Issues with quantum measurement

The standard approach to quantum physics treats measuring apparatus as classical devices not subject to the superposition principle. Quantum theory predicts correlations between different measuring events. We prepare a system in a state $|\psi_S\rangle$, let it evolve during some

²When the total state is pure, the maximum value of the mutual information is twice the entropy of $S(\rho_{A(B)})$.

time t , and then perform a measurement on it in a given basis ($|m_i\rangle$). Our predictions are on the probabilities of different outcomes, given by Born's rule,

$$p_i = |\langle m_i | \psi_s(t) \rangle|^2 \quad (1.5)$$

We thus have predictions on the statistics of measurement outcomes, conditioned to the preparation of the system. Quantum theory is a theory of probability on correlations. This rule is sufficient, we don't need the collapse of the wavefunction idea. Yet, in this formulation, we have supposed the existence of a classical measuring apparatus, measuring definite things and not subject to the superposition principle. It is out of the quantum regime.

Still, it is natural to seek a complete quantum description of the apparatus since, after all, they are made of atoms and molecules governed by quantum rules. However, it was early noticed in the development of quantum physics that treating the measuring process quantum mechanically leads to three important issues, coming from entanglement,

- The measurement basis problem. Why is there a preferred measurement basis at all?
- The interference problem. Why can't we observe quantum superposition at the macroscopic scale (Schrodinger's cat)?
- The single outcome problem. Why measurements only have one particular outcome?

Understanding those problems is a prerequisite to grasp the achievements of decoherence theory over the last decades but also to see what must still be done. Let's see how they come about. The simplest approach to quantum measurement was devised by Von Neumann and is called the pre-measurement evolution. It relies on the idea that the system and the apparatus will interact and that interaction creates correlations between them.

Consider a system \mathcal{S} interacting with a measuring device \mathcal{M} both described with quantum physics. We suppose that we have a conditional evolution. This means that if we prepare the system and the apparatus in the state $|s_i\rangle|0\rangle$ where ($|s_i\rangle$) is an orthogonal basis of the system and $|0\rangle$ is the ready state of the apparatus, then it will evolve in $|s_i\rangle|m_i\rangle$ with ($|m_i\rangle$) a basis of the apparatus. The measuring device "points" in the direction i , informing us that the system is in the state $|s_i\rangle$. Prepare now the initial global state in $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|s_i\rangle + |s_j\rangle)|0\rangle$ with $|s_{i,j}\rangle$ some states of the system. This state evolves into the entangled state

$$|\psi\rangle = \frac{|s_i\rangle|m_i\rangle + |s_j\rangle|m_j\rangle}{\sqrt{2}} \quad (1.6)$$

We have perfect correlations between the state of the system and the state of the measuring device. If our apparatus is in the state $|m_i\rangle$ then we know that the system is in $|s_i\rangle$. Expressed differently, the relative state $|\psi(S|m_i)\rangle$ of the system knowing the result m_i is $|\psi(S|m_i)\rangle = |s_i\rangle$. This is what we should expect from an ideal measurement process.

Even if it seems to start well, this entangled structure is ambiguous. Indeed, a perfectly correlated state like (1.6) can be written in any basis we want and still conserve the same correlation structure. To see this, the simplest example is with two qubits in the state $|\psi\rangle = \frac{|0_s\rangle|0_m\rangle + |1_s\rangle|1_m\rangle}{\sqrt{2}}$ written in the σ^z basis ($|0\rangle, |1\rangle$). However, it is legitimate in quantum theory to write a state in any basis. We can for instance choose a rotated basis $|\mathbf{n}\rangle$ where \mathbf{n} is any vector on the Bloch sphere. It can then be shown that $|\psi\rangle = \frac{|\mathbf{n}_s\rangle|\mathbf{n}_m\rangle + |-\mathbf{n}_s\rangle|-\mathbf{n}_m\rangle}{\sqrt{2}}$. The measurement basis ambiguity is now clear: from the correlation structure only, we cannot know at all what the measurement apparatus is supposed to measure. Is it in the z direction or the \mathbf{n} direction ?

The second issue is related to the fact that even if we have selected a preferred basis, the global state of the system and the apparatus is in a quantum superposition of different outcomes. However, we know that after a measurement, we end up in a statistical mixture of the different possible outcomes when the results are not read. We should end up with a reduced density matrix of the form $\rho = \frac{1}{2} (|s_i\rangle\langle s_i| \otimes |m_i\rangle\langle m_i| + |s_j\rangle\langle s_j| \otimes |m_j\rangle\langle m_j|)$. The fact that we have here a superposition means that interference effects are present (the density matrix is not diagonal) and that we could end up with Schrodinger's cat kind of situation which is not acceptable at the macroscopic level where measurement readings are supposed to take place. This interference or coherence problem is not well addressed in the Von Neumann scheme.

Finally, the last issue concerns the fact that in a read measurement, we end up with only one definite answer and not simply the statistical mixture of outcomes. We will see in the following that decoherence theory gives a clear idea on how to answer the measurement basis and interference problems but stays mute on the third. In fact, the outcome issue is believed to be a question on the interpretation of quantum physics.

1.2 Quantum to classical transition

1.2.1 Decoherence

The questions of the preferred basis and the inobservability of interference at the macroscopic scale find answers by introducing to the quantum description of the system and its apparatus the fact that they are never completely isolated from an environment composed of a tremendous number of unmonitored degrees of freedom. The continuous interaction with the environment leads to an effective destruction of quantum coherence in a very specific basis, depending on the specific microscopic interaction. A superposition will evolve toward a statistical mixture. In informational terms, information on the state of the system and the apparatus leaks into the environment which becomes effectively lost for the observer. This process is called decoherence.

We can understand this with a simple idealized example. Suppose we have a system \mathcal{S} in contact with an environment \mathcal{E} whose total dynamics is given by the interaction $H = S \otimes E$ with S and E observables of the system and the environment respectively.

We prepare the state in $|\psi(0)\rangle = |s_i\rangle|E_0\rangle$ with $|s_i\rangle$ eigenstates of S and $|E_0\rangle$ some environmental state like the vacuum. According to the Schrodinger equation, this will evolve into $|\psi(t)\rangle = |s_i\rangle|E_i(t)\rangle$ with $|E_i(t)\rangle = e^{-\frac{i}{\hbar}S_iEt}|E_0\rangle$. Now prepare the state in a quantum superposition $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|s_i\rangle + |s_j\rangle)|E_0\rangle$. This will evolve from the linearity of Schrodinger's equation into

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|s_i\rangle|E_i(t)\rangle + |s_j\rangle|E_j(t)\rangle) \quad (1.7)$$

This step is exactly the premeasurement evolution of Von Neumann. A first remark we can make is that information on the state has leaked into the environment as it is clear from the fact that $|E_i(t)\rangle$ depends on the eigenvalue s_i . For now we have not solved anything with the state (1.7) which is plagued with the same issue discussed in 1.1.3.

The new step here is to stipulate that the environment is impossible to monitor by the observer since it contains too many degrees of freedom. Then, the observer must assign to the system a reduced density matrix ρ_S obtained by tracing out all the environment degrees of freedom. That matrix ρ_S encodes all the information to predict what an observer can obtain from local measurements on the system. Doing so, we have

$$\rho_S = \frac{1}{2}(|s_i\rangle\langle s_i| + |s_j\rangle\langle s_j| + |s_i\rangle\langle s_j|\langle E_j(t)|E_i(t)\rangle + |s_j\rangle\langle s_i|\langle E_i(t)|E_j(t)\rangle) \quad (1.8)$$

We see now that off diagonal elements (also called coherence), which reflect the fact that we have a superposition and potential interference effects, are modulated by an overlap between states of the environment $\langle E_j(t)|E_i(t)\rangle$. Provided that the environment has encoded sufficient which-path information over time, meaning that the environmental states are almost perfectly distinguishable (more formally that they are orthogonal), we have the asymptotic behavior

$$\langle E_j(t)|E_i(t)\rangle \xrightarrow{t \rightarrow \infty} \delta_{ij} \quad (1.9)$$

We then see that the reduced density matrix evolves toward a statistical mixture, with no coherence left,

$$\rho_S(t) \xrightarrow{t \rightarrow \infty} \frac{1}{2}(|s_i\rangle\langle s_i| + |s_j\rangle\langle s_j|) \quad (1.10)$$

Thus by taking into account the environment, we have given a purely quantum answer to the second issue about the non observability of interference like in Schrodinger's cat state. Of course this relies on the behavior (1.9). In fact, most models of open quantum systems show that the relative states of the environment become exponentially distinguishable over time with a very short timescale like $\langle E_j(t)|E_i(t)\rangle \propto e^{-\frac{t}{\tau_d}}$. The timescale τ_d depends naturally on the form of the interaction but also on the initial state. For instance, if we prepare two wavepackets spaced by a distance $|x_1 - x_2|$ then τ_d depends very often on the inverse squared of the distance with a scattering model for the environment. The more spaced the packets are, the easier it is to distinguish them accurately.

1.2.2 Einselection

We have not yet addressed the first problem of the measurement basis. In fact, this issue is solved simultaneously with decoherence. Let's consider again the same simplified model. We saw that from the particular form of the interaction, the system prepared in the state $|s_i\rangle$ will stay in this state even after the interaction with the environment. In a sense, it is a predictable state, immune to the constant monitoring of the environment. However, this was not so when we prepared the system in a quantum superposition $\frac{1}{\sqrt{2}}(|s_i\rangle + |s_j\rangle)$. This one will not stay unaffected by the interaction with the environment since decoherence occurs. Such a superposition gets strongly entangled with the environment and the description of the system becomes effectively indistinguishable from a statistical mixture in the basis $(|s_i\rangle)$.

The point here is that there exists a class of states that are immune to decoherence, which don't get entangled with the environment. We thus have a preferred basis, chosen dynamically from the interaction between the system and the environment. This process is called by Zurek einselection [Zurek, 2003] for environment induced superselection rule and the preferred states are called pointer states. In our simple example, the preferred basis chosen by the environment is associated to the observable S in the interaction. It is information on this observable that is monitored by the environment.

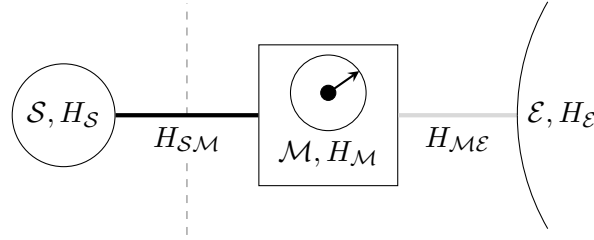


Figure 1.1: Typical quantum measurement scheme. The system and the apparatus strongly interact like the pre-measurement Von Neumann process. Then the environment which is weakly interacting with the apparatus only induces decoherence.

We can now revisit, still in a simplified way, the problem of measurement by re-introducing the measuring device \mathcal{M} explicitly which measures the system \mathcal{S} . Both are immersed in a huge environment \mathcal{E} . We suppose that only \mathcal{M} interacts with the environment and only after the rapid pre-measurement evolution with the system. Those hypothesis are here to simplify the mathematics and are not well justified for a more realistic model. Starting from $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|s_i\rangle + |s_j\rangle)|m_0\rangle|E_0\rangle$, the first step of the evolution is only the pre-measurement phase

$$|\psi\rangle = \frac{(|s_i\rangle|m_i\rangle + |s_j\rangle|m_j\rangle)}{\sqrt{2}}|E_0\rangle \quad (1.11)$$

We remember here that we had no way to know in which basis the measuring apparatus was operating because of the perfect correlation structure. We then suppose that the system does not evolve anymore and only \mathcal{M} and \mathcal{E} evolve through again the same kind of conditional evolution. We end up then with the global state,

$$|\psi\rangle = \frac{(|s_i\rangle|m_i\rangle|E_i\rangle + |s_j\rangle|m_j\rangle|E_j\rangle)}{\sqrt{2}} \quad (1.12)$$

Now we know that since the environment can't be monitored, we have to trace on its degrees of freedom to obtain the effective system-apparatus description and that this leads to decoherence in the basis ($|m_i\rangle$). We now have no ambiguities left on what the apparatus is measuring and this is selected by the microscopic interaction.

The discussion here is of course overly simplified but the ideas underlying the quantum to classical transition are given. First of all, the interaction that gives rise to the classical behavior is not that simple. The interaction with the many environmental degrees of freedom is quite complex and the proper dynamics of the system and the environment must be taken into account. Thus, most of the time, we do not have exact pointer states which are eigenstates of an observable due to the complicated full Hamiltonian. To be able to say anything, we have to quantify better the idea that pointer states are those states which are robust to decoherence. For more realistic situation, finding those pointer states requires to use a quantity d_{ent} that quantifies the degree of entanglement between the system and the environment. For instance the Von Neumann entropy is a good estimator. The idea is then to prepare the system in a given state $|\psi_0\rangle$ and calculate the evolution over time of the degree of entanglement $d_{\text{ent}}[|\psi_0\rangle]$. The minimum of this functional of the initial state gives us our approximate pointer states, those whose entanglement with the environment evolves the least. We can then map some classical zone in the full Hilbert space where systems will evolve inevitably because of decoherence. Naturally, we could ask if the determination of those protected pointer states is the same for different entanglement estimator. It is in fact believed there should not be any major difference, but this is not a proven fact. Luckily, for many realistic models such as quantum Brownian motion or the spin boson model, analytic results can be obtained, exact pointer states can be determined and come to confirm the whole process.

1.2.3 Objective reality

Introducing an unmonitored environment answers the question of the preferred basis in a quantum measurement and also the in-observability of quantum interference at the macroscopic scale. Yet, the situation we have considered here is still ideal. A more realistic approach would be to consider an ensemble of observers, making measurement in some fraction of the environment degrees of freedom in order to have information on the state of the system. Indeed, when we read something, we do not make a measurement on the paper itself. We have access to information through photons reflected on the paper. All those photons are part of the environment of the paper, leading to decoherence and some

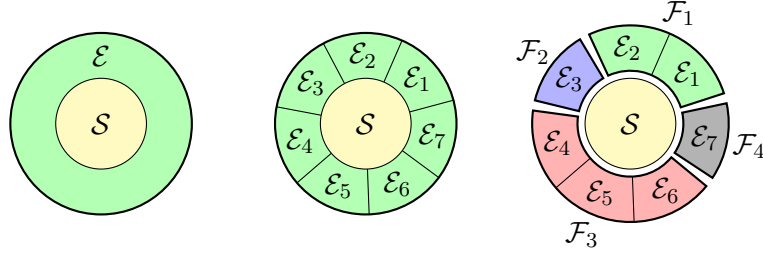


Figure 1.2: Decoherence theory studies the influence of a monolithic environment on the system, leading to its classical behavior. However, in order to be closer to our actual experience, we need to understand how different observers reach a consensus on the state of the system by having access to partial information on some small part of the whole environment. Quantum Darwinism starts with the more refined structure of the environment represented on the right.

small fragment of them are captured by our eyes. What's more, different readers reading the same paper will have access to a different subset of photons and yet agree on what's written on it.

A refinement of our previous decoherence picture is needed in order to understand the consensus between different observers and which states are better suited to diffuse their information on many different environmental channels. We thus need to consider not a monolithic environment but a collection of environment regrouped in some fragments from which an observer can extract information, like represented in Figure 1.2. This is the goal behind the quantum Darwinism approach which is an extension of the standard decoherence theory made by Zurek [Zurek, 2009].

To understand this approach, the idea is to look at the correlations between the system S and some fragment \mathcal{F} of the environment and see how those correlations evolve as a function of the size of \mathcal{F} . The natural information theoretic measure of those correlations (quantum and classical) is the mutual information $I[S, \mathcal{F}] = S[S] + S[\mathcal{F}] - S[S, \mathcal{F}]$. When the mutual information is of the order of the entropy of the system S , which is its classical upper bound in classical information theory, we know that an observer accessing such a fragment can reconstruct the state of the system.

The mutual information as a function of the size of the fragment can have two different behaviors. The first one is a purely quantum one where the mutual stays almost equal to zero until the fragment are half the size of the total environment where it then takes its maximum value of $2S[S]$. This is basically the behavior observed for a global quantum state picked at random in the Hilbert space. The second behavior is totally the opposite and happens for pointer states. Here, the mutual information saturates to the entropy of the system $S[S]$ for a very small size of the fragment and keeps this value until almost all the environment is being measured. The fact that from a very small part of the

environment an observer has access to all the information she needs to know the state of the system shows that this information is redundantly encoded in “many copies”, in many different channels, in the environment. This is this typical behavior we are looking for to obtain the consensus between different observers. The fact that it is obtained for pointer states is quite understandable. Indeed, only a state robust enough to the monitoring of the environment can live long enough to diffuse its information redundantly in the environment. On the contrary, a pure quantum state like a superposition will be decohered too quickly to hope for those records to happen at all.

What we learned here is that the problems of constructing a quantum theory of the measuring process and the emergence of a classical objective world can be clarified by taking into our description an unavoidable environment continuously monitoring the system. The problem of the preferred measuring basis and the observability of quantum interference are fairly understood. Still, there remains one issue that decoherence theory doesn't say anything about which is the problem of outcome. We still do not understand why there is unique outcome at all after a measurement. In fact, this issue is at the moment mostly discussed as a problem of interpretation of quantum theory where two camps have opposite views: one camp stays with the idea that there is only really one outcome in the end coming from some fundamental collapse mechanism and the other camp which stays with the basic postulates of quantum theory exposed in 1.1.1 which leads naturally to relative states formulation of quantum theory.

1.3 Open quantum system

1.3.1 General framework

We have just learned that taking into account the fact that any system is never completely isolated but in contact with an environment allows to clarify how the quantum to classical transition occurs. The purpose now is to lay down some formalism to deal with open quantum systems and set the general framework.

We are interested in a quantum system \mathcal{S} in interaction with an environment whose degrees of freedom are not observed. The state of the system is then generally described by a reduced density matrix

$$\rho_{\mathcal{S}}(t) = \text{tr}_{\mathcal{E}} (U(t)\rho_{\mathcal{SE}}U^{-1}(t)) \quad (1.13)$$

with $\rho_{\mathcal{SE}}$ the initial global state and $U(t)$ the full evolution operator depending on the Hamiltonian $H = H_{\mathcal{S}} + H_{\mathcal{E}} + H_{\mathcal{SE}}$. This operator contains all the information on the statistics of local measurements that can be done on \mathcal{S} . As we already saw, $\rho_{\mathcal{S}}$ is not in general a pure state due to the entanglement with the environment.

A nice representation of the influence of the environment, more precisely the imprint the system leaves into the environment degrees of freedom, is through the path integral representation and the Feynman-Vernon influence functional. Suppose that the system

follows a trajectory in classical configuration space³ γ which is then seen by the environment like a classical driving force. To each trajectory is then associated an evolution operator $U_{\mathcal{E}}[\gamma]$ conditioned on the trajectory. Starting at time t_i from a factorized state $|\gamma_i\rangle|E_0\rangle$ and summing over all path until time t_f , we obtain the global state

$$|\psi(t_f)\rangle = \sum_{\gamma(t), \gamma(t_i)=\gamma_i} \mathcal{A}[\gamma] |\gamma(t_f)\rangle \otimes U_{\mathcal{E}}[\gamma] |E_0\rangle \quad (1.14)$$

where $\mathcal{A}[\gamma]$ is the amplitude for the system to follow the path γ . We see here in a general way what we told in section 1.2.1 that information about the state of the system is leaking into the environment. It is reflected here in the final environment state $|E[\gamma]\rangle \equiv U_{\mathcal{E}}[\gamma]|E_0\rangle$ by its dependence on the path γ .

The reduced density matrix obtained by tracing out the environmental degrees of freedom has the following form

$$\rho_S(t_f) = \sum_{\gamma, \bar{\gamma}} \mathcal{A}[\gamma] \mathcal{A}^*[\bar{\gamma}] |\gamma(t_f)\rangle \langle \bar{\gamma}(t_f)| \times \langle E[\bar{\gamma}] | E[\gamma] \rangle \quad (1.15)$$

We see that coherence terms between different paths are modulated by an overlap $\mathcal{F}[\bar{\gamma}, \gamma] = \langle E[\bar{\gamma}] | E[\gamma] \rangle$ called the Feynman-Vernon influence functional. Its form naturally depends on the free dynamics of the environment $H_{\mathcal{E}}$ and also on the interaction $H_{\mathcal{SE}}$. In a decoherent scenario, the imprint left by the system in the environment is such that the overlap between two different relative environment states for two different trajectories will tend to zero. Stated differently, the relative states will become perfectly distinguishable over time. In this scenario, the system will be described by a statistical mixture of classical alternatives.

Computing the Feynman-Vernon influence functional is very tricky, like most path integral, apart for very simple models. Different approximation schemes must be constructed if we want to say something analytically. The most used equations, in atomic physics, condensed matter and quantum information, are the Lindblad and Born-Markov equations.

1.3.2 Born-Markov equation

A first approximation scheme is the Born-Markov approximation. This equation governs the dynamics of the reduced density matrix of the system under well-established approximations, even if not really physical in all situations. The idea is to start from the exact dynamics and perform an expansion of the equation of motion (the expansion parameter is to be discussed). Two approximations are introduced that rely on the fact that the environment is very large and contains many degrees of freedom. The following discussion is inspired by [Schlosshauer, 2008; Dalibard, 2006; Cohen-Tannoudji et al., 1996] .

³Here is the catch. In the path integral formulation, we need a classical setting to define the path integral while our goal is precisely to make the classical states emerge from the quantum dynamics. Some intuition is required to define things in the simplest way.

Let's look at $\rho_S(t) = \text{tr}_E(U(t)\rho_{SE}U^{-1}(t))$ in the interaction picture. The full quantum dynamics is given by $\frac{d}{dt}\rho_{SE}^{(I)}(t) = -i[H_{\text{int}}(t), \rho_{SE}^{(I)}(t)]$. Now the goal is to inject the integral form of this equation into the equation itself, giving $\frac{d}{dt}\rho^{(I)}(t) = -i[H_{\text{int}}(t), \rho^{(I)}(0)] + (-i)^2 \int_0^t [H_{\text{int}}(t), [H_{\text{int}}(t'), \rho_{SE}^{(I)}(t')]] dt'$. After tracing on the environment degrees of freedom, we end up with

$$\frac{d}{dt}\rho_S^{(I)}(t) = (-i)^2 \int_0^t \text{tr}_E [H_{\text{int}}(t), [H_{\text{int}}(t'), \rho_{SE}^{(I)}(t')]] dt' \quad (1.16)$$

Up to now, we have made no approximations at all, this equation is exact. But it is clear that such an integro-differential equation is not trivial to analyze. We have to resort to an approximation that cancel the dependance on the history of the full density matrix and have a local in time equation in $\rho_S^{(I)}(t)$.

Decorrelation approximation (Born-Markov)

The essential hypothesis behind the Markovian approximation that is used to study open quantum system is that the environment has so many degrees of freedom that we can neglect the building of correlations between it and the system over time. This is implemented in two steps,

- *Born approximation* The total density matrix $\rho_{SE}(t')$ can be replaced by an approximate factorized form

$$\rho_{SE}(t') = \rho_S(t') \otimes \bar{\rho}_E \quad (1.17)$$

where $\bar{\rho}_E$ is the equilibrium state of the environment. We neglect here the modification induced on the environment from the coupling with the system.

- *Markov approximation* The environment has a short term memory, meaning that the system-environment correlations decay on timescale much shorter than any observational or dynamical times. Formally, we can replace the non local in time $\rho_S(t')$ by its value at time t in the integral.

What we have done here is an expansion in the interaction, yet we have not explicitly expressed the expansion parameter. Naively, we could expect that vt/\hbar with v the order of magnitude of the interaction, to be the expansion parameter. For very long time, such an expansion would not make sense anymore because of the linear time dependance. In fact, the true expansion parameter is $v\tau_c/\hbar$ where τ_c is a correlation timescale. The hypothesis made above are justified in the case where

$$\frac{v\tau_c}{\hbar} \ll 1 \quad (1.18)$$

We now implement the two steps of the approximation scheme. The first one allows to write the integro-differential equation in ρ_S

$$\frac{d}{dt}\rho_S^{(I)}(t) = (-i)^2 \int_0^t \text{tr}_{\mathcal{E}} [H_{\text{int}}(t), [H_{\text{int}}(t'), \rho_S(t') \otimes \overline{\rho_{\mathcal{E}}}]] dt' \quad (1.19)$$

The second step will give us a purely differential equation. We work with an Hamiltonian of the form $H_{S\mathcal{E}} = \sum_{\alpha} S_{\alpha} \otimes E_{\alpha}$. Then, to have a better view of what is going on, we can introduce environment correlation functions at equilibrium $C_{\alpha\beta}(t, t') = \langle E_{\alpha}(t) E_{\beta}(t') \rangle_{\overline{\rho_{\mathcal{E}}}}$. Those correlation functions quantify how the measurement of the observable E_{β} at time t' will affect the measurement of E_{α} at time t in the equilibrium state. Informally, it quantifies how the environment is affected by its interaction with the system. So we can rewrite equation (1.19) like

$$\begin{aligned} \frac{d}{dt}\rho_S^{(I)}(t) &= (-i)^2 \int_0^t \sum_{\alpha\beta} \text{tr}_{\mathcal{E}} [S_{\alpha}(t) \otimes E_{\alpha}(t), [S_{\beta}(t') \otimes E_{\beta}(t'), \rho_S(t') \otimes \overline{\rho_{\mathcal{E}}}]] dt' \\ &= (-i)^2 \int_0^t \sum_{\alpha\beta} C_{\alpha\beta}(t - t') \left(S_{\alpha}(t) S_{\beta}(t') \rho_S^{(I)}(t') - S_{\beta}(t') \rho_S^{(I)}(t') S_{\alpha}(t) \right) \\ &\quad + C_{\beta\alpha}(t' - t) \left(\rho_S^{(I)}(t') S_{\beta}(t') S_{\alpha}(t) - S_{\alpha}(t) \rho_S^{(I)}(t') S_{\beta}(t) \right) \end{aligned}$$

The hypothesis that the environment has a short memory is seen in the behavior of the correlation functions as they decay to zero over a short correlation timescale τ_c

$$C_{\beta\alpha}(\tau) \underset{\tau_c \ll \tau}{\simeq} 0 \quad (1.20)$$

This justifies the approximation $\rho_S^{(I)}(t') \simeq \rho_S^{(I)}(t)$. The equation obtained is now a pure local in time differential equation in the reduced density matrix of the system. This equation is called the Redfield equation and can be used to analyze open quantum system. The difficulty now are purely mathematical because of the fact that the equation has non-constant coefficients. In fact, we can push further the Markovian approximation by taking the value of those coefficients to their limit as time goes to infinity $\int_0^t \simeq \int_0^{+\infty}$. In the Schrodinger picture, we end up with the Born-Markov equation (restoring \hbar),

$$\frac{d}{dt}\rho_S(t) = -\frac{i}{\hbar} [H_S, \rho_S(t)] - \frac{1}{\hbar^2} \sum_{\alpha} [S_{\alpha}, B_{\alpha} \rho_S(t)] + [\rho_S(t) C_{\alpha}, S_{\alpha}] \quad (1.21)$$

where we have defined the operators

$$B_{\alpha} = \int_0^{\infty} \sum_{\beta} C_{\alpha\beta}(\tau) S_{\beta}^{(I)}(-\tau) d\tau \quad C_{\alpha} = \int_0^{\infty} \sum_{\beta} C_{\beta\alpha}(\tau) S_{\beta}^{(I)}(-\tau) d\tau \quad (1.22)$$

This last step ends the derivation of the Born-Markov equation governing the dynamics of an open quantum system at the lowest order in the coupling. We see that the natural

frequency coming from the interaction with the environment is of the order $v^2\tau_c/\hbar^2$. In fact we have the hierarchy of frequency, valid when $\frac{v\tau_c}{\hbar} \ll 1$,

$$\frac{v^2\tau_c}{\hbar^2} \ll \frac{v}{\hbar} \ll \frac{1}{\tau_c} \quad (1.23)$$

The frequency v/\hbar is the natural frequency when the system is coupled to only one mode of the environment. The reduction to the frequency $v^2\tau_c/\hbar^2$ comes with the presence of many independent modes in the environment.

This equation is used quite often in quantum optics and condensed matter physics where the context is such that the environment fulfills all the criteria to give meaningful results. One of the drawback of this general form concerns the positivity of $\rho_S(t)$ [Whitney, 2008]. Indeed, since this reduced density matrix describes the system \mathcal{S} , it should be a positive operator in order to give positive probabilities. However it can happen, depending on the form of the dynamics, that equation (1.21) does not conserve the positive character of ρ_S over time. As worked out by Lindblad, only a specific type of Markovian equations called Lindblad equations determine physical reduced density matrix. The quantum Brownian motion is an example [Schlosshauer, 2008] of the failure of the brute Markovian approximation but remains meaningful on short timescales.

1.3.3 Lindblad expression

Formal form A master equation of the Lindblad form is an equation that describes the evolution of a Markovian open quantum system that conserves at all time the positivity and the complete positivity of the reduced density matrix. In a sense, they are the generic physical quantum equation of motions. They can be derived from a purely quantum information perspective [Haroche and Raimond, 2013]. The influence of the environment is described by jump operators L_μ , not necessary unitary. For instance, if we consider a two-level atom coupled to the electromagnetic field, we have emission and absorption processes that can take place. The jump operators would then be proportional to the creation and annihilation photon operators $L \propto a$. The generic form of the Lindblad equation is

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{\hbar} [H_S, \rho_S(t)] + \sum_{\mu=1}^{N^2-1} L_\mu \rho_S(t) L_\mu^\dagger - \frac{1}{2} \left(L_\mu^\dagger L_\mu \rho_S(t) + \rho_S(t) L_\mu^\dagger L_\mu \right) \quad (1.24)$$

The first term is the standard unitary evolution of a density matrix with H_S the free dynamics of the system, possibly Lamb shifted by the environment (this is the unitary contribution). The second encodes the effects of the environment in a non unitary fashion with $N - 1$ jump operators L_μ . The Born Markov can be reduced to this form in some specific cases.

How can such an equation lead to decoherence effects? To see this, imagine that we have one jump operator which is just the position operator $L = \sqrt{\kappa}x$ with κ a relaxation

time. Then the Lindblad equation reduces to

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{\hbar} [H_S, \rho_S(t)] - \frac{\kappa}{2} [x, [x, \rho_S(t)]] \quad (1.25)$$

Written in the position representation, the evolution of a non-diagonal matrix element of the reduced density matrix is affected by a term coming from the double commutator of the form $-\kappa/2(x - x')^2 \rho_S(x, x', t)$. Thus the coherences are attenuated by an exponential factor with a characteristic decoherence timescale $\tau_d^{-1} = \kappa(x - x')^2/2$. The pointer states are well localized states in position space. In the case of the two level system coupled to the electromagnetic field in the vacuum, the emission process is modeled by a jump operator proportional to the destruction operator of a photon, leading to coherent states as pointer states. In fact, when we have an open dynamics governed by only one jump operator L_μ , it can be argued [Haroche and Raimond, 2013] that the pointer states are mostly determined by the eigenstates of this jump operator.

Stochastic trajectories It is possible to give a nice representation of this equation in terms of a stochastic evolution of pure state [Dalibard et al., 1992]. Even if it is not necessary to understand what follows, this approach gives a different point of view on open quantum dynamics and is also the natural way of seeing things in experiments manipulating single quantum objects. The idea is to start from

$$\frac{d\rho_S(t)}{dt} = -i \left(\left(H - \frac{i}{2} \sum_k L_k^\dagger L_k \right) \rho_S(t) - \rho_S(t) \left(H - \frac{i}{2} \sum_k L_k^\dagger L_k \right) \right) + \sum_k L_k \rho_S(t) L_k^\dagger \quad (1.26)$$

Defining an effective Hamiltonian $H_{\text{eff}} = (H - \frac{i}{2} \sum_k L_k^\dagger L_k)$, which is not a true Hamiltonian because it is not an Hermitian operator, and decomposing the density matrix in some basis like $\rho_S(t) = \sum_i p_i |\psi_i(t)\rangle \langle \psi_i(t)|$, we end up with

$$\frac{d\rho_S(t)}{dt} = -i \sum_i p_i \left([H_{\text{eff}}, |\psi_i(t)\rangle \langle \psi_i(t)|] + \sum_k L_k |\psi_i(t)\rangle \langle \psi_i(t)| L_k^\dagger \right) \quad (1.27)$$

The first term can be interpreted as a Schrodinger evolution with the effective Hamiltonian while the second term as a measurement or jump by the operator L_k . A probabilistic or stochastic evolution of the pure state $|\psi_i(t)\rangle$ can be given. If a jump occurs in the time interval dt , the state evolve into the (properly normalized) state $|\psi_i^{(k)}(t + dt)\rangle = \sqrt{\frac{dt}{p_{ik}(dt)}} L_k |\psi_i\rangle$ with $p_{ik}(dt) = dt \langle \psi_i | L_k^\dagger L_k | \psi_i \rangle$ the probability the jump occurs. If no jump happened, the state evolves according to the Schrodinger equation with the effective Hamiltonian giving $|\psi_i(t + dt)\rangle = \frac{(1 - i dt H_{\text{eff}})}{\sqrt{1 - \sum_k p_{ik}(dt)}} |\psi_i\rangle$. Again the normalization is non-trivial due to the non hermicity of the effective Hamiltonian and we see that it depends

on the probability that no jump at all occurs. Finally for a small increment of time dt , the Lindblad equation can be written as

$$\rho_S(t+dt) = \sum_i p_i \left(\left(1 - \sum_k p_{ik}(dt) \right) |\psi_i(t+dt)\rangle\langle\psi_i(t+dt)| + \sum_k p_{ik}(dt) |\psi_i^{(k)}(t+dt)\rangle\langle\psi_i^{(k)}(t+dt)| \right) \quad (1.28)$$

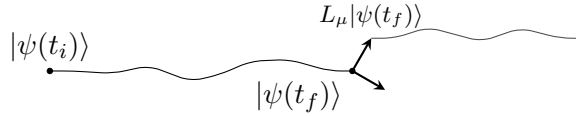


Figure 1.3: An example of a stochastic trajectory, with a unitary evolution between the times t_i and t_f and a jump at time t_f .

Thus the Lindblad equation can be interpreted as the statistical average on all possible trajectories the quantum state of the system can follow. Those trajectories are stochastic since at random times the environment acts like a measuring device and project the state. Figure 1.3 represents such trajectory. This is a very intuitive and neat picture to understand the open quantum dynamics of a system. What's more, this approach has a considerable advantage for numerical simulations since pure states and not density matrix are evaluated.

1.4 Entanglement in quantum field theories

1.4.1 Vacuum entanglement

Quantum field theory is the standard framework in which all the fundamental interactions except gravitation are described. It is the natural, and sometime argued the necessary, language to model a relativistic quantum system (at least the common low energy description of all more fundamental approaches to physics) and any system with no fixed number of particles or excitations. A quantum field theory is basically a quantum mechanical system with degrees freedom at each point of spacetime described by fields $\hat{\phi}(x)$. The picture that comes out of this description is the same as for many-body systems: we look at excitations of the fields over a ground state $|0\rangle$ called the vacuum.

The notion of entanglement in quantum field theory is usually meant as the correlation structure of the vacuum and not the bipartite entanglement we saw up until now. The reason is that it is not possible to separate the total Hilbert space into the tensor product of two subspaces $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_B$. The first issue in gauge theories is the presence of

constraints that physical states must satisfy [Donnelly and Freidel, 2016]. By looking at only subsystems, it is not possible to decide if the state of the subsystem satisfy the constraints. This issue is mostly ignored in many analysis and we will do the same for the time being. The second issue comes from UV divergences coming from the modes crossing the boundary but this can be dealt with by introducing a cut-off. We postpone the discussion of the separability of the Hilbert space in quantum field theory (and quantum gravity) to chapter 7.

In quantum field theory, the vacuum is a very correlated state. Indeed, for a free scalar field, the spatial two points functions at equal time behave as

$$\langle 0 | \hat{\phi}(x, 0) \hat{\phi}(y, 0) | 0 \rangle = \frac{m}{4\pi^2 |x - y|} K_1(m|x - y|) \underset{|x-y| \ll m^{-1}}{\sim} \frac{1}{|x - y|^2} \underset{|x-y| \gg m^{-1}}{\sim} \frac{e^{-m|x-y|}}{|x - y|} \quad (1.29)$$

with K_1 a Bessel function and the mass m playing the role of a correlation length. Thus we see that spatially separated regions of space are correlated in the vacuum which in the special gapless case $m = 0$ is long ranged. Such a behavior is expected to hold for any quantum field theory. This is in sharp contrast with the non relativistic free particle case where the vacuum is uncorrelated (it is the tensor product of all the single particle ground state).

The Reech-Schlieder theorem is a result also highlighting the non-trivial correlation structure of the vacuum. Consider a sub-region \mathcal{S} and its set of all local operators (not necessary unitary). Then any state of the full Hilbert space $\mathcal{H}_{\mathcal{SE}}$ can be arbitrarily approximated by a state constructed from the vacuum and local operators of \mathcal{S} only. This theorem is a statement about entanglement of the vacuum. Indeed, if the vacuum were factorized $|0\rangle = |0\rangle_{\mathcal{S}} \otimes |0\rangle_{\mathcal{E}}$, then it would be impossible by acting only on the sub-region \mathcal{S} to approximate entanglement between \mathcal{S} and \mathcal{E} since the final state would remain factorized⁴.

The last result we mention showing the entangled nature of the vacuum is related to the Minkowski vacuum seen from the point of view of uniformly accelerated observer. By considering some quantum field coupled to a uniformly accelerated observer or Rindler observer, it can be shown that the Minkowski vacuum, written in terms of Rindler modes, is an entangled states between the left and right wedges

$$|0\rangle = \prod_i \sum_{n=0}^{+\infty} e^{-\beta_U \hbar \omega_i} |n_i^L\rangle |n_i^R\rangle \quad (1.30)$$

where ω_i are the Rindler mode frequency and β_U is the Unruh inverse temperature $T_U = a\hbar/2\pi k_B$ where a is the acceleration of the detector. From this decomposition, it is

⁴ Another way to have some understanding of this result is to consider a state of the form $|\psi\rangle = \sum_{se} C_{se} |s\rangle |e\rangle$ with C_{es} an invertible matrix (statement about entanglement) in the finite dimensional case. Then the theorem can be proved easily [Harlow, 2016].

clear that the vacuum possesses some entanglement, a bipartite entanglement between the two causally disconnected region of Minkowski spacetime constructed from a uniformly accelerated observer.

1.4.2 Entropy in a field theory

So we have concluded that the ground state of a quantum field theory is a very entangled state. The natural following question is about entanglement entropy in a field theory which, as we already discussed, is a measure of correlations of a quantum state. The discussion will remain qualitative here and more quantitative results that corroborate the following statements can be found in the literature, especially in scale invariant field theories.

In finite dimensional quantum mechanics, the entanglement entropy in a random state scales as the volume of the region whereas the ground state of a local gapped Hamiltonian scales as the area of the region. This same behavior is recovered in quantum field theory. The problem is, as usual, that due to the presence of UV modes, the brute force calculation of the entanglement entropy diverges. In the vacuum and for a local theory in d dimensions, the scaling of the divergent part is expected to be of the form

$$S \simeq a_1 L^{d-2} + a_2 L^{d-4} + \dots \quad (1.31)$$

where L is the scale of the region considered and the a_i (UV divergent) coefficients are theory dependent ⁵. The first divergent part thus scales as the area of the region and comes from the UV modes crossing the boundary. In the ground state, the non divergent term is not expected to be extensive and consequently the entanglement entropy properly regularized should scale as the area. However, this is not true anymore in an excited state where the long distance (IR) modes contribution becomes more important. That again is the reason for the volume scaling in a random state since they correspond to highly excited states.

1.4.3 Tensor network ansatz

Up to now, we have looked at two different aspects of correlations, the first one was bipartite entanglement and the second was correlations in quantum field theories with correlations at all scales. The question that we can ask is: can we generalize bipartite entanglement into many-body entanglement and at the same time understand better and simulate the correlation structure of the vacuum of a quantum field theory? Asked in a different way, the question is how, given a collection of systems, do we define, characterize

⁵ An argument to obtain this scaling behavior is the following. In a local theory we expect $S_{\text{div}} \simeq \int_S f(K_{ab}, h_{ab}) \sqrt{h} \, d^{d-2} \sigma$ where K is the extrinsic curvature and h the induced metric on the boundary. Since $K \simeq 1/L$ we can expand in powers of L . In a pure state, the entanglement entropy of the boundary and the interior are the same. Since $K \simeq \nabla n$ where n is the normal, only even power of K can appear to satisfy the equality of entropies. Thus the result.

and use entanglement between them ? We won't discuss here many-body entanglement in details which is still an active (and unclear compared to the bipartite case) research area [Amico et al., 2008] but focus on one particular proposal, tensor network ansatz [Orus, 2014].

Consider a collection of N finite dimensional systems S_i described by the Hilbert space \mathcal{H}_{S_i} . Given a basis ($|i_k\rangle$) for the subsystem S_k , a state $|\psi\rangle$ in the total Hilbert space $\mathcal{H} = \bigotimes_k S_k$ can be written as

$$|\psi\rangle = \sum_{i_1, \dots, i_N} \alpha_{i_1, \dots, i_N} |i_1\rangle \otimes \dots \otimes |i_N\rangle \quad (1.32)$$

The entanglement structure of this state is encoded in the collection of amplitude α_{i_1, \dots, i_N} and can be arbitrarily general. The tensor network methods propose to deconstruct those amplitudes and build them from simple building blocks.

The simplest one are the Matrix Product States (MPS) ansatz which are useful to understand one dimensional condensed matter systems. The idea is to have local entanglement between the subsystems. To achieve this, consider a matrix $(M(k))_m^n$ for each subsystem k . We build our states with amplitudes of the form $\alpha_{i_1, \dots, i_N} = (M_{i_1})_{m_1}^{m_N} (M_{i_2})_{m_2}^{m_1} \dots (M_{i_N})_{m_N}^{m_{N-1}} = \text{tr}(M_{i_1} \dots M_{i_N})$. Focus on the contraction pattern of indices which is between adjacent subsystems only. This pattern induces nice properties like area law scaling for entanglement entropy, finite correlations, are good estimates for ground states of $1d$ system and are efficient for numerical studies. Constructed in a similar way, Projected Entangled Pairs States (PEPS) ansatz uses tensors with more indices (four mostly) and are good candidates to understand $2d$ systems and share the similar properties as MPS states. The philosophy of tensor network is by properly choosing the contraction pattern, we can obtain various correlations structure and simulate for instance correlations in quantum field theory.

Naturally, more complicated structure can be imagined and have been proposed to satisfy different purposes. As we will motivate in the next chapter the area law scaling for entanglement entropy (the fact that we have holographic states) is very important in a quantum gravitational context. Recent results [Vidal, 2008; Evenbly and Vidal, 2011] have proposed a class of tensor network states (called MERA for Multiscale Entanglement Renormalization Ansatz) which nicely simulate the ground state of critical systems and conformal field theories. Moreover, a geometrical interpretation can be given which open interesting links with a major quantum gravity approach, the AdS-CFT correspondence. This illustrates the potential relationship between entanglement and geometry at the quantum level.

1.5 Locality in quantum physics

We end this chapter by a brief discussion about locality in quantum physics. The notion of locality is a primitive concept in physics and more generally in science. Different intuitive

characterizations can be given and do not have in fact the same meaning:

- A modern characterization is given by the light cone structure of the special theory of relativity and the fact that no information can be transmitted faster than light. This is a spacetime notion.
- A more primitive characterization is given by the idea that results of an experiment done here do not depend on what happens out there. An experiment done in Paris do not depend on what happens at the center of our galaxy. (This notion of locality is what makes science a relevant field of knowledge). This is a spatial notion.

Entanglement questions those notions of locality. First, it seems to violate special relativity by allowing faster than light communication. It was one of the concerns at the basis of the EPR argument [Einstein et al., 1935] and tried to solve it by introducing local hidden variables ⁶. However, the solution proposed by Einstein has been excluded by the experimental violations of Bell's inequality. The quantum framework still holds. So what about locality and faster than light communication ? The solution is brought by the no signaling theorem. Suppose we have two systems Alice and Bob in some global arbitrary state and that Bob performs some local measurement in its region. Does Bob results influence the prediction of Alice local experiments ? By computing the probabilities of Alice measurements, it is straightforward to show that those probabilities do not depend in any way on what Bob is doing and the results he obtained; Alice's predictions depend only on her local operation and on her reduced density matrix. To reconstruct the correlations between the two, some classical communication (then ruled by the law of special relativity) must take place. Contrary to appearances, entanglement is not in contradiction with special relativity. It is in fact impossible to extract any information whatsoever from entanglement alone. This is the content of the no signaling theorem. To sum up, there is indeed some non-local features in the correlations encoded in a quantum state but they cannot be used to communicate instantaneously and they can only be uncovered through classical communications between observers.

It should be noted that entanglement relies on the possibility to factorize the total Hilbert space into subsystems. This tensorial product is sometimes associated with locality but it is better to distinguish it. This property is called separability and is related to entanglement alone. In standard quantum mechanics, locality and separability are distinct notions and their difference is perfectly illustrated by the EPR states. Still, we can ask the question if this distinction remains in a quantum gravity framework, especially for those approaches which try to define locality and geometry from correlations and entanglement. More will be discussed latter.

The compatibility between locality, relativity and quantum physics is beautifully summarized in the framework of quantum field theory. As we already said, the degrees of

⁶This is the spatial notion of locality that is used here. If an agent can determined by local measurement the value of a distant physical quantity without disturbing it, then an element of reality must be assigned to it : the local hidden variable.

freedom are described by fields on spacetime $\hat{\phi}(x)$ and interaction by local combinations of those fields. The fact that spacelike regions cannot influence each other is implemented by the criterion of commutativity $[\hat{\phi}(x), \hat{\phi}(y)] = 0$ for $(x - y)^2 > 0$. All this allows to construct a quantum model compatible with the principles of special relativity. It should be noted that we impose conditions on the observables of the theory (on the field operators) and not on the states. Entanglement can still be present but is no more a problem then it was in quantum mechanics. Second, the structure of all used quantum fields theories are such that the more primitive notion of locality, that the results of distant observation do not influence what we are doing here, is also implemented. Consider two observables A and $B(x)$ carried by Alice sitting at the origin and Bob at the position x respectively. Bob now travels away at infinity. Then, in the vacuum $|0\rangle$, the correlations tend to factorize at infinity. Formally, $\langle 0|AB(x)|0\rangle \xrightarrow{|x| \rightarrow +\infty} \langle 0|A|0\rangle\langle 0|B|0\rangle$. This result is called the clustering theorem. As Weinberg does in his course [Weinberg, 1995], it can be used to construct the formalism of field theories from scratch. All in all, we conclude that the different notions of locality and quantum theory are compatible even if at first sight entanglement seems to lead to their blatant violation.

However, when going into the realm of quantum gravity, the question of locality comes back in force. Indeed, locality is build in the spacetime structure but if we treat spacetime as a quantum object, what about locality then ? This is a fundamental question in quantum gravity research and no definite answers have been found. Extensions of the locality principle are analyzed, definitions from relative observables constructed from quantum geometry operators are envisioned , naturally leading to questions about its relative character. Going even further, some results tend to favor a fundamental non locality of physics at the quantum gravity scale. In the end, we do not yet know if locality is a fundamental principle of physics or an emergent notion from a full quantum theory of gravity based on other physical principles.

We gave in this chapter all the necessary tools about quantum information and open quantum systems to understand the models we will study in the context of quantum gravity. The following chapter continues on the background material focusing now on gravitational physics. The goal will be to better appreciate the relationships that exist between gravity, information theory and quantum physics.

Chapter 2

General Relativity

General relativity is our current theory of space, time and is the perfect synthesis of classical physics. Gravitation manifests itself in a geometric manner as the curvature of spacetime. As a classical theory, it is believed to be only an effective field theory of a more fundamental quantum theory of gravity. However, quantization of general relativity is a long standing problem of theoretical physics. Many proposal exists, like string theory, loop quantum gravity, the AdS-CFT correspondence or emergent models of gravity but no consensus yet exists. Before discussing one particular proposal, our present goal is to review the conceptual foundation of the theory. We want to highlight that general relativity, already at the classical level, is a theory of correlations. This will come as no surprise that, in the quantum regime, quantum information theory will play a major role.

Informational aspects in gravitational physics are especially transparent when black holes are involved. Black holes are very simple solutions of general relativity and one its most important prediction. Their physics is thought to be our best window on the quantum properties of spacetime and a potential source for its experimental exploration. Basically, once an observer cross the surface of the hole, called the event horizon, no external observer can ever receive a signal from him. Surprisingly, at equilibrium, they are completely characterized by few parameters, their total mass, their charge and their angular momentum whatever was the mechanism of its formation like the collapse of a star. This is valid at least in a purely classical theory. Putting quantum mechanics in the picture, a black hole becomes a thermodynamic system with a temperature, can radiates energy and evaporates in a finite amount of proper time for an external observer. We will see at the end of this discussion that this leads to interesting paradoxes linked to quantum information that are to this day still unsolved.

There are many different definitions of a black hole and its horizon. The standard one is in fact teleological and uses the global causal structure of spacetime. However, to better use our intuition and to discuss quantum aspects, quasi-local definitions are needed. An interesting local point of view for the external observer is the membrane paradigm which consider the boundary of the black hole like a physical surface with physical properties

like, for instance, viscosity.

The present discussion intends to review those different points to have the necessary background in mind when we will go toward the core of the thesis. Section 2.1 reviews the standard formulation of general relativity, how we can define local observables and the role of correlations. Section 2.2 and Section 2.3 discuss black holes, their standard properties at equilibrium, some quasi-local definition and finally how those objects, when used with quantum matter, lead to renown paradoxes illustrating perfectly the relationship between quantum physics, quantum information and gravitation. This will bring us to the holographic principle, a cornerstone idea in quantum gravity which has brought physicists to try to rethink the foundations of physics, especially our notion of locality. This chapter ends by opening the road toward models of quantum gravity in Section 2.5 before developing in details one non perturbative proposal, loop quantum gravity, in the next chapter.

2.1 Foundations of general relativity

2.1.1 Core concepts and original formulation

Basic concepts Understanding the foundations of a physical theory is not an easy task and is subject to many debates. Foundations of quantum physics is the best illustration of this. On the other hand, the foundations of general relativity is more rarely discussed among physicists, probably because it is a classical theory based on some intuitive principles. In fact, spending some time to think about what the theory is based on and what it is telling us about space and time reveals more subtleties than expected and in return offers some natural perspective for quantum physics.

General relativity describes space-time as a four dimensional differentiable manifold \mathcal{M} with one dimension associated with time and the three others with space. This mathematical structure is the natural object to describe our intuition about locality, the fact that we can speak about two systems to be as close as we want or far away from each other. This is a very general structure and we could ask about the necessity of being so general. That's what Einstein came to understand while searching for a relativistic theory of gravity through the principle of equivalence. It claims there exists a local observer with reference frame $e_\mu^I(x)$ at any point $x \in \mathcal{M}$ where the special relativistic laws of physics are valid. Stated geometrically, spacetime is locally flat, but not globally. In fact, two principles can be formulated where “laws of physics” include gravity itself or not, respectively called the strong and the weak equivalence principle. While the weak equivalence principle is expected to hold for any classical theory of gravity and has been tested experimentally to a relative precision of 10^{-12} , the strong version is not expected to hold for every theory. General relativity verifies the strong version also and this has been tested to a relative precision of 10^{-3} from observations of the Moon orbit.

Geometrical formulation From this principle, the local geometry at $x \in \mathcal{M}$ in the coordinates $(\xi^I)_I$ is given by the Minkowski metric η_{IJ} and the interval $ds^2 = \eta_{IJ}d\xi^I d\xi^J$. Now for an arbitrary observer with coordinate system $(x^\mu)_\mu$, the interval has the form $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ with

$$g_{\mu\nu}(x) = \eta_{IJ}e_\mu^I(x)e_\nu^J(x) \text{ and } e_\mu^I(x) = \frac{\partial \xi^I}{\partial x^\mu} \quad (2.1)$$

The tensor field $g_{\mu\nu}(x)$ is called the metric and encodes all the spacetime information between events. It has the same signature $(-, +, +, +)$ as the Minkowski metric encoding the fact that one direction corresponds to time and the three others to space.

The local reference frame fields $e_\mu^I(x)$ relates the physics of the local observer to the arbitrary one. Geometrically, to each $e_\mu^I(x)$ at point x can be associated a vector $e^I(x)$ describing the reference frame of the local observer. One component of this vector is naturally timelike and the three others are spacelike. In fact, the local observer has some liberties in choosing its reference frame. Every frame at a point x related by a Lorentz transformation Λ is a valid choice. We have thus a local Lorentz symmetry in the theory

$$e_\mu'^I = \Lambda_J^I e_\mu^J \quad (2.2)$$

We will come back later to the fields $e_\mu^I(x)$ as the more fundamental objects to understand gravity (to express the theory itself or even to do physics with fermions) and use only the metric $g_{\mu\nu}$ for now ¹.

The metric is the first geometrical object we have which gives an answer to the question on how far away two events are. Another question we can ask is about variation of quantities when we move them around on spacetime. The answer is found in an object called a connection ∇ or covariant derivative. In a given local basis, ∇ is completely characterized by its coefficient $\Gamma_{\alpha\beta}^\mu$ defined from $\nabla_{e^\alpha} e^\mu = \Gamma_{\alpha\beta}^\mu e^\beta$. Given a metric, there exists a unique (torsion free affine) connection compatible with it satisfying

$$\nabla_\alpha g_{\mu\nu} = 0 \quad (2.3)$$

The connection coefficients can be expressed as a function of the metric coefficients

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2}g^{\alpha\lambda}(\partial_\beta g_{\lambda\alpha} + \partial_\alpha g_{\beta\lambda} - \partial_\lambda g_{\alpha\beta}) \quad (2.4)$$

From the connection, we can construct different objects that characterize the geometry. The Riemann tensor is the central quantity which informs us on the curvature of spacetime. It is defined from the connection by looking at the transport of a vector along two different paths forming parallelogram. At the end point, the transported vectors are path dependent

¹From now on we will drop the x dependance of the field

and curvature corresponds to their difference. Formally, the Riemann curvature tensor $R_{\beta\gamma\delta}^\alpha$ is defined as

$$\forall v^\mu \in T(\mathcal{M}), [\nabla_\alpha, \nabla_\beta]v^\gamma = R_{\mu\alpha\beta}^\gamma v^\mu \quad (2.5)$$

where $T(\mathcal{M})$ is the tangent bundle. The necessary and sufficient condition for spacetime to be flat, in thus recover special relativity physics without gravitation, is for the Riemann tensor to be zero. In terms of the coefficient of the affine connection, the Riemann tensor has the components

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda \quad (2.6)$$

From a physical point of view, this tensor contains all the information on tidal effects caused by the presence of a gravitational field. In the Newtonian regime, the displacement vector ξ^i between two free fall observers satisfy the equation² $\frac{d^2 \xi^i}{dt^2} = -\xi^j \nabla_j \nabla^i \phi$ where ϕ is the gravitational potential satisfying a Poisson equation. The Riemann tensor is the covariant generalization of this second derivative of the gravitational potential. What characterizes gravitation compared to simple accelerations is precisely those tidal effects. That's why a zero Riemann tensor, thus no tidal effects, implies no gravitation. From the point of view of a local observer in free fall, the principle of equivalence says that physics is the same as without gravity. This is valid locally at least as long as tidal effects are negligible. The typical size is given by the curvature (whose dimension is the inverse of a squared length). Formally this comes from the following expansion of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + R_{\mu\nu\alpha\beta}(x - x_0)^\alpha (x - x_0)^\beta + o((x - x_0)^2) \quad (2.7)$$

Other geometrical quantities can be constructed from the curvature, encapsulating part of the information on the non trivial geometry of spacetime, by taking traces of the Riemann tensor. From its symmetries, two such objects can be constructed. The first is the Ricci tensor defined as

$$R_{\mu\nu} = R_{\mu\alpha\nu}^\alpha \quad (2.8)$$

Geometrically, the Ricci tensor evaluates the relative variation of a volume element relative to the flat case (this can be seen by expanding with (2.7) the determinant of the metric). Finally, by contracting further the Ricci curvature, we obtain the scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} \quad (2.9)$$

Now that we have a set of quantities describing the geometry of spacetime, we have to relate them somehow to the matter content of the Universe to form the fundamental

²In a given inertial reference frame, write the equations for the free fall observers described by coordinates x^i and $x^i + \xi^i$.

field equations. From the principle of equivalence, we can argue that all forms of energy contribute as the source of the gravitational field. They are all encoded in an energy-momentum symmetric tensor $T_{\mu\nu}$ which must satisfy the equation of motion

$$\nabla_\alpha T_{\mu\nu} = 0 \quad (2.10)$$

The simplest second order equation giving back Newton's theory of gravitation was found by Einstein in 1916. They are the central field equations of general relativity, called the Einstein field equations,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (2.11)$$

The tensor $G_{\mu\nu}$ is explicitly constructed to satisfy $\nabla_\alpha G_{\mu\nu} = 0$. Thus, those equations imply from their structure the equations of motion of matter fields as well as the form of the gravitational field. Einstein's equations contain all of classical physics by choosing the proper energy-impulsion tensor.

Action formulation The Einstein equation can be recovered from a variational principle. Like mentioned above, there are many different formulations of general relativity depending on what fundamental fields we use and if we are in Lagrangian or Hamiltonian formalism. We will deal with an Hamiltonian formulation in the next chapter.

The Lagrangian formulation is build from an action, integral on a region of spacetime V of a Lagrangian density. In fact, to have a well posed problem in general relativity, a boundary term, called the Hawking-Gibbons boundary term [Gibbons and Hawking, 1977] on ∂V , must be introduced in the action to recover the Einstein equation. This is an important term to define notions like mass or angular momentum of an isolated system [Poisson, 2004]. The field equation (2.11) can be derived from the Einstein-Hilbert action

$$S[g_{\mu\nu}] = \frac{c^4}{16\pi G} \int_V R \sqrt{-g} \, d^4x + \frac{c^4}{8\pi G} \int_{\partial V} (K - K_0) \sqrt{|h|} \, d^3y \quad (2.12)$$

Here g is the determinant of the metric, h the determinant of the induced metric on the boundary. K and K_0 are extrinsic curvatures of the boundary embedded in spacetime and in flat space respectively. The extrinsic curvature tensor K_{ij} is defined as the variation of the normal \mathbf{n} over an hypersurface ∂V , $K_{ij} = \nabla_j n_i$. It gives information on the embedding of an hypersurface into spacetime. K is its trace and is called the mean curvature of ∂V . The extremum of this action gives back the Einstein field equation. In the following chapter, we will see different equivalent actions given back the classical theory, some more manageable for the quantum treatment.

2.1.2 Symmetries, observables and the relative picture

The goal now is to clarify, in a formal way, the type of predictions general relativity can make and show that physical predictions are predictions on the correlations between physical fields.

General relativity is a spacetime theory which models spacetime with a four dimensional manifold \mathcal{M} described by geometrical fields like the metric. The physical meaning of this manifold is however more subtle than what is usually assumed or presented due to the particular symmetries of the theory. Points in \mathcal{M} have no physical meaning at all and serve only to parametrize the model. They have no ontological status. In fact, physical points are constructed not a priori but from the relative values of physical fields. We only have access to the value of a physical quantity only relative to another value of a physical quantity. In the language used by Einstein, we speak about spacetime coincidences. This set of coincidences represents better what we identify as spacetime. The goal of this section is to explain this in more details by discussing the meaning of coordinates in general relativity, what are the observables and the relative picture of spacetime. This discussion follows the work done mostly in [Rovelli, 1991; Westman and Sonego, 2009; Rovelli, 2007].

Einstein field equations have the properties to be generally invariant³. under the group of diffeomorphism of spacetime $\text{Diff}(\mathcal{M})$. The meaning of this statement is as follows: an equation is said to be invariant under some set of transformation if, when written explicitly (for instance in terms of $g_{\mu\nu}(x)$), it keeps the same functional form. This is an invariance property of the field equations and not the solutions. Such an invariance can be used to generate new solutions. Given a solution g and a diffeomorphism ϕ , then $\bar{g} = \phi_*g$ is again a solution of the problem, provided the initial conditions are preserved.

This general invariance questions the meaning of coordinates x^μ in general relativity. The standard view in classical physics is that the coordinates x^μ are defined through readings of clocks and rulers or more generally by some physical procedure. Nonetheless, we could also view those coordinates as mere mathematical parameters, values in some local chart in an abstract manifold and not some physically meaningful quantities. Both views are legitimate in non general relativistic physics. However, general invariance forces us to adopt the second point of view in general relativity. This was first realized by Einstein after his established his hole argument. The idea of the argument goes as follows. Suppose x^μ are physically meaningful. Consider some physical scalar property $p(x^\alpha)$ and two solutions of the same physical problem $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ related by a diffeomorphism f . Those solutions give some prediction on our property at the physical point x^μ respectively denoted $p(x^\alpha)$ and $\bar{p}(x^\alpha) = p(f(x^\alpha))$. However in general, this gives two different predictions of the property p at the given location x^α since $p(x^\alpha) \neq p(f(x^\alpha))$. Thus general invariance (and consequently Einstein's equations) cannot predict the value of p at a given position x^α . This conclusion comes from the hypothesis that the coordinates have some operationally defined meaning. We are forced to abandon this view and consider x^μ as just parameters on our manifold \mathcal{M} which is also devoid of physical significance.

The question now is what is observable in this generally invariant context. First of all, we are in a situation where a physical system is described by a whole class of solutions

³This is a different notion than general covariance. This one is a statement about the mathematical formalism used by writing our model in a language that treats all observers on the same footing. General covariance more like a principle of indifference.

related by diffeomorphism. An observable O should not depend on which element we choose $O[g] = O[\phi_*g]$. This is the strict analogue of the electromagnetic case when the field is described by A_μ : physical observable must be gauge-invariant. However, if we assume that those observables are defined on \mathcal{M} and that they are local we end up with only trivial solutions⁴. One way out is to define non local observables and it is perfectly legitimate to consider them. Still, from the point of view of a local observer, we need to have local observables. Thus the problem remains. One solution is to have a relative point of view: local observables are not defined at some point x^μ but relative to the value of other physical fields, to get rid of the parameters x^μ . Einstein accepted general covariance for a physical theory after he understood those aspects. Here is the method to construct properly physical local observables. Suppose that spacetime is described by a parameter manifold \mathcal{M} and a collection of field ψ . From those fields, construct in some local region $U \subset \mathcal{M}$ four scalar fields $q^\mu : U \rightarrow Q \subset \mathcal{S} \subset \mathbb{R}^4$ which are invertible $\det\left(\frac{\partial q^\mu}{\partial x^\nu}\right) \neq 0$. Those functions have their image in some subspace \mathcal{S} of \mathbb{R}^4 with no more specification for now. Consider then some physical property for instance $p : \mathcal{M} \rightarrow \mathbb{R}$. There, we construct a relational observable $p_r = p \circ q^{-1} : Q \rightarrow \mathbb{R}$,

$$\begin{array}{ccc} & \mathcal{M} & \\ q^\mu \swarrow & & \searrow p \\ Q \subset \mathcal{S} & \xrightarrow{p_r = p \circ q^{-1}} & \mathbb{R} \end{array} \quad (2.13)$$

This quantity is now gauge invariant, independent on the choice of the element of the equivalence class of solution. Indeed, consider another solution related by a diffeomorphism f . Construct the observable p' which is related to p by the relation $p' = p \circ f^{-1}$ and the coordinates $q' = q \circ f^{-1}$. Consider again the relative observable p'_r . Then we have directly $p'_r = p' \circ q'^{-1} = (p \circ f^{-1}) \circ (q \circ f^{-1})^{-1} = p \circ q^{-1} = p_r$. Thus, the function p_r is indeed a relative observable in general relativity⁵.

The set of all point coincidences, constructed from relative measurements, form a space \mathcal{S} that has a priori the structure of a four dimensional differentiable manifold not to be confused with the mathematical apparatus used to build the theory. This space is the one to be identified with our intuition of spacetime. An event corresponds to a relative reading of one value of a field compared to another. It is a gauge invariant data, an observable of the theory. Thus, the theory of general relativity is fundamentally a theory that's making relative predictions where the notion of spacetime is reconstructed from the theory itself as a properties of fields. To put it another way, we showed our statement that general relativity is a theory of correlations already at the classical level. In all practical cases however, we do not need to bother with such a construction and we can assume that all

⁴For instance, for scalar observables, only constant functions satisfy the condition.

⁵In the language of constrained systems, it was shown that those relative observables are Dirac observables.

well known calculations made are made with respect to coordinates properly defined in a relative way. Still, this discussion is important to understand the foundations of the theory.

2.1.3 Behind Einstein equations

Einstein equations (2.11) are the basic equations of general relativity and they identify geometry and matter. But can we have a more basic understanding of what they describe? Surprisingly we can from very simple equations that in a sense sum up all the physical content of those complicated tensorial equations. I give here two arguments based on spacelike considerations, one formulated by [Baez and Bunn, 2005] and the other by [Feynman et al., 2006; Jacobson, 2016].

The first situation considers a small ball of test particles initially at rest (relative velocity zero initially). Let $r^j(t)$ be the radius of the ball in the direction j . Then from the definition of the Riemann tensor $\lim_{t \rightarrow 0} \frac{\ddot{r}^j(t)}{r^j(t)} = -R_{\beta j \delta}^j v^\beta v^\delta = -R_{tjt}^j$ where v is the relative velocity of two nearby particles. We can then obtain the relative second order variation of the volume of the ball as $\lim_{V \rightarrow 0} \frac{\ddot{V}}{V} = -R_{tt}$. From the Einstein equations $R_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$ we conclude that

$$\lim_{V \rightarrow 0} \frac{\ddot{V}}{V} = -\frac{4\pi G}{c^4} (T_{tt} + \dots) = -\frac{4\pi G}{c^4} (\epsilon c^2 + p_x + p_y + p_z) \quad (2.14)$$

where ϵ is the energy density and p_i the pressure in the i direction. The difference from the Newtonian case is precisely that pressure (in fact all sorts of energies) is also a source of the gravitational field. In most cases, the energy density comes from the rest mass and we can write $\epsilon = \rho c^2$ with ρ the mass density, neglect the pressure contribution and have the Newtonian form $\lim_{V \rightarrow 0} \frac{\ddot{V}}{V} = -4\pi G\rho$ and recover Newton's law. Equation (2.14) encapsulates all the physical content of general relativity: indeed, we can reverse the logic, start from this equation and impose general covariance (that it must be valid in all reference frame) and recover Einstein equations.

The Feynman argument focus on area and radius deficit instead of volume. The original argument considered a small (compared to the local curvature length scale) region of fixed area A , a timelike direction t , and compared the measured radius r_{Meas} with the Euclidean one $r_{\text{Euc}} = \sqrt{\frac{A}{4\pi}}$. The difference of those two quantities, called the excess radius, is related to the amount of energy $M = T_{tt}V$ in the region of volume V ,

$$r_{\text{Meas}} - r_{\text{Euc}} = \frac{GM}{3c^2} \quad (2.15)$$

Once again, the full Einstein equations can be recovered by supposing the validity of this relation at all spacetime points and for all timelike directions. Different versions of this

relation can be derived by instead imposing the radius to be fixed and compare the areas or by imposing the volume and compare the areas [Jacobson, 2016],

$$\delta A|_V = \frac{32\pi^2 G r^4}{15} T_{00} \quad (2.16)$$

The interesting point of those discussions is, apart from the fact that we can recover general relativity from simple arguments, that those relations give some new insights and perspective on Einstein equations. In particular, we see that there is a link between the area of a region and the energy that it encloses. This is an important idea to keep in mind for the following.

Having now giving some fundamental background on the foundations of general relativity, we go on into a more precise discussion on one of its beautiful predictions: black holes. As we will see, their physics is to this day our best window into the realm of quantum gravity.

2.2 Black holes

2.2.1 Black holes at equilibrium

Black holes are simple and fascinating objects in general relativity. In an informal way, a black hole is a local region of spacetime where nothing, not even light, can escape. The boundary between the interior and the rest of the Universe is called the event horizon and is the principal characteristic of a black hole. This section reviews the principal characteristics of black holes which will be necessary in the following one in order to show that black hole physics also illustrates the important relationships between gravitation and information.

There exists many definitions of a black hole in general relativity. The standard one is a global definition and uses the complete causal structure of spacetime \mathcal{M} . Calling the future null infinity \mathcal{J}^+ and J^- the causal past of a region, the black hole region \mathcal{B} is defined as $\mathcal{B} = \mathcal{M} \setminus J^-(\mathcal{J}^+)$. This really encompasses the idea that the black hole region is the region from where light rays cannot reach infinity. The event horizon \mathcal{H} is then defined as the boundary $\mathcal{H} = \partial J^-(\mathcal{J}^+)$ and it can be shown that it is a null surface. Defined this way, a black hole and its horizon are global properties of spacetime and teleological in nature since the complete history of the Universe is needed to define them.

While not really intuitive, many results can be derived rigorously from this global definition. A major one in black hole physics is called the no-hair theorem. It states that the geometry of a stationary black hole in an empty Universe is characterized by only three parameters, the mass M , the total electric charge Q and the angular momentum J . The metric is then necessarily the Kerr-Newman metric. This is a very non intuitive results. Indeed, no matter the complexity of the formation process of the black hole, its equilibrium state is determined by very few parameters.

Imagine now the situation in which an observer throws a small system into a initially stationary black hole and waits long enough for the black hole to settle to its new stationary state. From the Einstein equations, the mechanics of this situation can be summarized into four laws [Bardeen et al., 1973; Carlip, 2014]

- 0. The surface gravity κ is constant on the horizon.
- 1. For two near stationary black holes, the variation of energy follows the equation

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega \delta J + \phi \delta Q. \quad (2.17)$$

with ϕ some electric potential and Ω the angular velocity.

- 2. The area A of the event horizon never decreases

$$\delta A \geq 0 \quad (2.18)$$

- 3. κ cannot be reduced to zero in a finite number of steps.

There is a clear analogy with the four laws of thermodynamics. For instance from the zeroth and first laws, it seems plausible to associate a temperature proportional to the surface gravity. Having this analogy, the first and mostly the second laws hint toward a proportionality relation between the area and a notion of black hole entropy. However at this stage, this is just a formal analogy. Indeed from a purely thermodynamics standpoint, a black hole is a system at absolute zero. The proportionality relation between the surface gravity and temperature already collapses from the beginning.

Before discussing further those tensions with thermodynamics in section 2.3 and the resolution found by Hawking with quantum physics, it is interesting to discuss some local definitions of a black holes. It is a necessary step to gain more intuition about black holes dynamics but also to move forward to the quantum regime.

2.2.2 Quasi-local approaches - Membrane paradigm

Most results in exact general relativity use the global definition. However, it suffers from major issues when we want to understand black holes in dynamical situations, for numerical studies or for a quantum description. Indeed, the major drawback of this definition is that it is a global notion relying on the whole causal structure of spacetime. It is non local in space and in time. For instance, it responds in advance to future perturbations (the rate of expansion grows while no matter is falling in) and no local physical experiment can probe the physics of the event horizon by definition. Concerning the laws of black hole dynamics, their formulation relies on different type of definition of physical quantities. The mass M and the angular momentum J are defined relative to spatial infinity whereas the surface gravity κ and the angular momentum Ω are defined on the horizon. It would be preferable to have laws expressed in terms of local physical quantities defined from

the geometry of the horizon only. It is necessary then to have some refinement on the definition of a black hole region and have it in some quasi-local way. In fact, there exists different quasi-local approaches. A hierarchy of local horizons has been constructed based on the idea of a trapped surface introduced by Penrose.

Before going into the different local definitions of a horizon, let us recall what is a trapped surface. Consider a closed spacelike surface S in spacetime (\mathcal{M}, g) . On each point of the surface, we have a light-cone with two null directions \mathbf{l} and \mathbf{k} respectively called outgoing and ingoing null normal. Consider the expansion, which we recall is the rate of change of the area element, defined as $\theta^{(\mathbf{n})} = q^{\mu\nu} \nabla_\mu n_\nu$ with $q_{\mu\nu}$ the induced metric on the surface and \mathbf{n} the normal. A closed spacelike surface S is said to be trapped if $\theta^{(\mathbf{k})} < 0$ and $\theta^{(\mathbf{l})} < 0$ (marginally trapped if $\theta^{(\mathbf{l})} = 0$). This is a local characteristic of the surface and captures some aspects of very strong gravitational fields. Indeed, the relevance of this concept for black hole physics is rooted in theorems of Penrose and Hawking. Under the weak energy condition (that the energy density is always positive for every observers), if there exists a trapped surface in spacetime then there exist a singularity (in the sense of a future inextensible null geodesic). Provided the cosmic censorship conjecture which stipulates that every singularity must be hidden behind an horizon (still an unproven fact in general relativity), there then exists a black hole in which the trapped surface is contained.

The two local spacetime structures that capture the essential physics of black holes at quasi-equilibrium are the notions of non-expanding and isolated horizons [Gourgoulhon and Jaramillo, 2006].

An hypersurface \mathcal{H} is a non-expanding horizon if

1. it is a null hypersurface with topology $S^2 \times \mathbb{R}$, and
2. the expansion of any null normal \mathbf{l} vanishes $\theta^{(\mathbf{l})} = 0$.

The first point is natural in regards of what is known about an event horizon which is a null surface and is contained in a local region of spacetime⁶ The second point states that the horizon is foliated by marginally trapped surfaces. It can also be shown that this hypothesis is equivalent to impose the time independence of the induced metric \mathbf{q} on the horizon and that the horizon (cross section) area is constant in time. Equilibrium is understood in this sense.

We can even go further and define a more stringent notion of isolation by imposing the time independence of the full geometry of the horizon encoded in the induced connection ∇ .

An hypersurface \mathcal{H} is an isolated horizon if

1. \mathcal{H} is a non-expanding horizon, and

⁶If no restriction on the topology is made then we cannot capture the idea that a black hole is localized and lightcones in Minkowski spacetime would not be excluded a priori.

2. the full geometry does not evolve along null normals $[\mathcal{L}_1, \nabla] = 0$.

The notion of isolated horizon is the last step in the hierarchy of local isolated surface. Local notions of mass, angular momentum can be defined. It encapsulates all the relevant information needed to recover the zero and first law of black hole mechanics. Apart from its purely local character, this notion doesn't restrict the whole spacetime to have some symmetries or stationary properties : contrary to the notion of Killing horizon for instance, an isolated horizon is a stationary surface living possibly in a dynamical spacetime. This is a more natural framework to think about black hole in an astrophysical context where things can happen outside the horizon. We will restrict here to this stationary case. However, in order to recover the second law of black hole mechanics but also to have a better understanding of the first law valid in fully dynamical situations, we need to go beyond the equilibrium case described by isolated horizon. This can be done by introducing dynamical horizons [Ashtekar and Krishnan, 2004].

2.2.3 Dynamical aspects

The physics of black hole presented so far is relevant for equilibrium and is described basically by the Kerr-Newman metric which depend only on the mass, the angular momentum and the charge of the hole (no-hair theorem) and the physics of near equilibrium holes is given by the four laws of black holes mechanics. The situation here is quite analogous to equilibrium thermodynamics (or statistical physics) which can be understood from a small set of principles. But what about the really out of equilibrium case like the collapse of a star? Like thermodynamics where no general principles can be given apart from the full study true dynamics in out of equilibrium situations, we have to go back to the Einstein equations to understand dynamical black holes.

Given the history of formation of a black hole from the collapse of a massive star, a natural question occurs. How come the final equilibrium state of the black hole is described by a small amount of parameters while it comes from the collapse of a star composed of myriads of degrees of freedom? Is there not an incompatibility? Part of the answer is provided by the quasi-normal ringing modes of a black hole [Kokkotas and Schmidt, 1999]. To introduce those modes, the simplest way is to imagine that we throw a small thing into a black hole. For instance, consider the black hole to be a Schwarzschild black hole. The total metric $g_{\mu\nu}$ is in this case the Schwarzschild metric $g_{\mu\nu}^{\text{Sch}}$ plus a small perturbation $h_{\mu\nu}$. Inserting this metric into the Einstein equations we can obtain an equation for the perturbation. Decomposing it in spherical harmonics, we can show that the metric has oscillation modes with some normal frequencies for the hole. The important part is that those frequencies have a complex part (thus the “quasi-”). This is the signature of a dissipative process: the amplitude of the normal mode relaxes in time, dissipating energy into gravitational waves into the environment of the black hole. Thus all the complexity of a dynamical process is washed out in time by the emission of quasi-normal modes. This relaxation mechanism explains how a very excited state of a black hole (no particular

symmetries, complex oscillation or rotation pattern) created by the collapse of a star ends up in a very simple equilibrium state. The no hair theorem is thus safe, relevant only for equilibrium (late time evolution) state of a black hole.

Quasi-normal modes of black holes have not been observed to this day. However, the signal of binary black holes merger, a part of which has been recently observed [Abbott and *et al.*, 2016] confirming directly the existence of gravitational waves and black holes, has an oscillatory part after the complete merger coming directly from quasi-normal ringing modes. For now the signals are too weak to observe this post merging part but future experiments or strong events could in the near future.

2.3 Black Holes, thermodynamics and information

2.3.1 Equilibrium thermodynamics of Black Holes

To sum up what we just saw, black holes in classical general relativity are very simple objects characterized by a limited number of parameters. This was turned into the famous citation of Wheeler that “black holes” have no hair. However, this result gives rise to tensions with thermodynamics and seems to provide a way to violate the second law of thermodynamics. Indeed, by throwing some amount of entropy into the black hole which then become effectively forever out of reach for an external observer, the total entropy measured by this external observer diminishes.

Bekenstein, a student of Wheeler, proposed to solve this issue by postulating that black holes are in fact carrying some entropy on the horizon scaling as its area and to generalize the second law by taking it into account. Such a statement can also be motivated by the first law of black hole mechanics from the term $\frac{\kappa}{8\pi G}\delta A$. The issue here is that in classical general relativity, a black hole is a system at absolute zero temperature. Thus the Bekenstein entropy cannot be viewed as a traditional thermodynamical entropy. The situation is here shaky.

A path in the right direction was made by Hawking by incorporating quantum theory and analyzed the behavior of quantum fields in a classical curved background [Hawking, 1975]. What Hawking discovered was a major insights of the physics at the interface of gravity and the quantum. In this semi-classical description, a black hole is now a true thermodynamical system with a non zero temperature given by the Hawking temperature T_H . For a black hole of mass M

$$T_H = \frac{\hbar c}{2\pi k_B} \kappa = \frac{\hbar c^3}{8\pi G M k_B} = 6.169 \times 10^{-8} \text{ K} \times \frac{M_\odot}{M} \quad (2.19)$$

This formula for the temperature encompasses all of physics, from quantum theory, statistical physics, relativity and gravitation. Yet, it is clear that for astrophysical system like a solar mass black hole, the temperature is so low that it cannot be simply observed. At least at a theoretical level we have now a true thermodynamical temperature. Consequently,

the second law of black hole mechanics gives us the form of the formula of black hole thermodynamical entropy

$$S_{\text{BH}}(A) = \frac{k_B c^3}{G \hbar} \frac{A}{4} = \frac{k_B}{4} \frac{A}{l_P^2}, \text{ with } l_P = 1,6 \cdot 10^{-35} \text{ m} \quad (2.20)$$

The entropy scales as the area of the horizon in Planck units. It is not an extensive quantity scaling as the volume of the region. Yet it is not that surprising from the point of view of an external observer where he sees all matter falling into the hole accumulate on the horizon. Those results have been derived in a tremendous amount of ways [Carlip, 2014] giving them more weight as a true physical effect waiting for experimental confirmation. More precisely, what Hawking showed is that black holes radiate a thermal black body spectrum⁷. Since the black hole loses energy at infinity, it is no longer an eternal ever growing system.

2.3.2 New dynamical aspects

The thermodynamical nature of black hole renewed interests for the dynamics. The major new physical process is black hole evaporation. Indeed, a black hole loses energy over time by emitting Hawking radiation. After consuming all its energy, a black hole should evaporate. The evaporation rate scales as $A^{3/2}$ or as $1/M^2$. The lifetime of a solar mass black hole is of the order 10^{67} yr. Like Hawking temperature, this effect is so small that no experiments have been able to observe it.

A lot of questions remain about the true final state of a black hole after the evaporation process. Is there nothing left at all? We will see in the following that it gives rise to some important paradoxes. Other proposal, motivated in part by quantum gravity, suggest that there is some remnant object at the end of the evaporation process like the Planck star proposal [Rovelli and Vidotto, 2014b] or black stars [Barceló et al., 2008]

Finally, the fact that black hole have a true thermodynamical entropy brings the idea, following the Boltzmann's view on entropy, that the horizon should be described by a collection of microscopic degrees of freedom. This is in apparent conflict with the no hair theorem of black holes at equilibrium. Recent ideas were put forward looking for black hole hairs, like for instance soft graviton [Strominger, 2014; Hawking et al., 2016] coming from symmetries at infinity..

2.3.3 Information and firewall paradoxes

This evaporation process leads to tensions with quantum physics. The first one that was observed was with unitarity. Suppose that the whole process of black hole evolution and evaporation can be described by a unitary quantum theory. Suppose that Alice, an

⁷The full computation shows that the spectrum is modulated by a energy dependent factor. We then speak of a gray-body emission spectrum.

external experimentalist throw a quantum system in a pure state into the black hole. In classical general relativity, that system would be lost forever behind the horizon. But as Hawking showed this is not the case with quantum system. Information leaks out of the black hole in the form of thermal radiation. After the death of the black hole (its complete evaporation), no horizon is left and all the information that was hidden came out in the form of thermal radiation. However this is problematic and in complete contradiction with a unitary evolution: no correlations are encoded in a thermal radiation and it is not possible to have a unitary evolution sending a pure state (possibly correlated) to an uncorrelated one. Such an evolution is a non unitary process. This is the first paradox that was encountered and debated over many years. The standard approach to this issue now is to keep unitarity as more fundamental (quantum physics prevails) and suppose that in a full quantum theory of gravity the radiation emitted by the black hole would not be thermal but would encode in a very subtle (scrambled) way correlations and look like thermal. The thermal character found by Hawking is considered as an artifact of the semi-classical calculations (classical gravity, quantum matter).

Whatever the form of the true Hawking radiation, we still have an issue with the no cloning theorem. Indeed, imagine again our experimentalist Alice dropping a system into the black hole to give it to Bob. From the point of view of the system, once the horizon is crossed, it will stay with Bob. However for the external experimentalist, the information on the system must come out in the form of Hawking radiation. Suppose then that Alice waits enough time to be able to reconstruct the information about the system she sent in the past. It is possible to perform this without waiting all the information to come out and correlations can be reconstructed at the moment when half of it is out. After this decoding tasks is performed she decides to jump into the black hole region. In there, she will be able to communicate with the system lost long ago and in the end have two copies of it, the original one and the reconstructed one. This is in contradiction with the no cloning theorem of quantum physics. To circumvent this problem, the point of view of black hole complementarity was advocated by Susskind [Susskind et al., 1993] and t’Hooft [’t Hooft, 1990]. The idea is that whatever Alice is doing, it will be impossible for her to have access to both copies simultaneously. Even if she waits up until the evaporation time to capture almost all Hawking radiation to reconstruct the quantum state and then jumps into the black hole to access the system inside, this “other copy” would have already reached the singularity and thus be inaccessible. The final point of the “resolution” is that the situation outside for Alice and the one inside for Bob are just the same physical situation seen by two different observers.

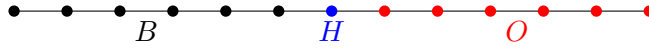


Figure 2.1: A black hole spacetime seen as a collection of qubits with the bulk region B , H the qubit horizon ready to come out as Hawking radiation O .

Recently though, this point of view was challenged by a new paradox coined the

firewall paradox [Almheiri et al., 2013] build around entanglement in quantum theory and quantum information. Two ingredients are used in the reasoning. The first one is that the vacuum in quantum field theory has short range entanglement. More precisely, pairs of particles inside and outside of the horizon are in a Bell state. The second one models a black hole as a scrambling machine of n qubits. Consider a (Haar quasi-) random state of n qubits where k of them form the outside region of the black hole. The state of the outside observer ρ_k is obtained by tracing out the $(n - k)$ qubits in the black hole region. Two cases have to be distinguished. Either $k < n/2$, meaning the outside observer Alice has access to less than half the total information. Then basically $\rho_k = \mathbb{1}_k$. We can think of this as the fact that the observer does not have enough room to store the information inside the black hole. Or we have the case where $k > n/2$ in which Alice can begin to extract information from the bulk. When more than half the qubits come out as Hawking radiation, Alice can reconstruct correlations. Here is the problem. Divide now space as the outside qubits O , the boundary qubit H and the inside qubits B like in Figure 2.1. From what we just discuss we expect entanglement between O and H and by a suitable unitary transformation, Alice can measure perfect Bell correlations. After that decoding task is performed, she jumps into the black hole in free fall. From the principle of equivalence, she is seeing the quantum field theory vacuum and we know from the first point that perfect correlations exist between the qubit boundary H and on inside qubit B . The correlation is observable as Alice crosses the horizon. However, this situation where three systems are perfectly entangled by pairs violates a result of quantum theory called the monogamy of entanglement. Some hypothesis made above must be dropped to resolve the situation. Is it the principle of equivalence ? Some principle of quantum physics ? The situation is still under discussions but seems to be related to the notion of locality and to local degrees of freedom in general relativity. Some argued that no such paradox arises in a quantum theory of gravity [Perez, 2015], for instance because of the non local nature of the fundamental degrees of freedom (like in the string theory or loop gravity), or that the decoding correlation tasks needs too much resources [Harlow and Hayden, 2013]. In the end, all those paradoxes show that there is a deep connection between gravity and information in quantum physics.

2.4 Locality and the holographic principle in general relativity

The relationship between thermodynamics and black holes is very surprising, especially the proportionality of the entropy of a black hole and its area. Indeed, in standard statistical physics, the entropy of a system is extensive, scaling as the volume not the area. If we consider for instance qubits on a lattice of spacing a in a volume V , we have V/a^3 qubits and a total number of states $N = 2^{V/a^3}$. Since the maximum entropy is the logarithm of N , we recover the volume scaling. A similar result holds in quantum field theories (properly regulated) and cannot then be used to explain this anomalous scaling for black

holes.

As a result, general relativity is said to be holographic. Another way to grasp this is to remember that it is not possible to write observables in general relativity as integral of a density over a volume. For instance, there is no local covariant way to write the energy of a gravitational field. The root lies in the equivalence principle from which we know we can erase locally a gravitational field. In an Hamiltonian formalism, this translates into the fact that the Hamiltonian for physical configuration is zero for a closed gravitational system without boundaries. However, in the presence of a boundary, a new contribution appears to the energy while the bulk contribution remains zero. Thus a quasi-local notion of energy can be defined with and at the boundary.

We could state that this holography is limited to gravitational physics and does not question the foundations of traditional field theories. This is not the case. Consider for instance a ball of matter of area A , of entropy $S_m(A)$ and of mass M smaller than the mass of a black hole $M_{BH}(A)$ of area A . Collapse then a shell of light of entropy S_l in such a way that the final state is now a black hole of area A . At equilibrium, the final state entropy is given by Hawking formula (2.20). Assuming that the second law of thermodynamics holds, we conclude that

$$S_m(A) + S_l \leq S_{BH}(A) \quad (2.21)$$

Thus the entropy of any region of area A is bound by an entropy scaling as the area. This is called the holographic bound. This argument can be criticized on the ground that the ball of matters fills a spacelike region and thus that the bound is not necessary covariant. A covariant entropy bound can in fact be derived for causal diamonds [Bousso, 2002] and this general results can be projected to entropy bound on spacelike region in specific cases like the one above.

Those results really question the fundamental notion of locality embodied in standard quantum field theories. Indeed, they have too many degrees of freedom, one in every point in space, leading to a volume rather than area scaling law. t'Hooft and Susskind proposed to raise those results to a fundamental physical principle of quantum gravity

The maximum number of degrees of freedom describing any region of spacetime scales as the area of this region.

In general, this principle remains a conjecture but the most advanced implementation of it is realized in the AdS/CFT correspondence [Maldacena, 1999] which describes the bulk physics in an Anti-De Sitter spacetime in terms of a conformal field theory in lower dimension.

2.5 Toward a quantum theory of gravity

2.5.1 Spacetime thermodynamics

What we saw so far has given insights to understand in what way we should explore the physics of a quantum theory of gravitation. We emphasized the particular role of correlations and information and also that of locality. Those aspects are, in some sense, consequences of general relativity. Another attack angle is not to focus on the consequences of the theory but on how we can derive it from first principles. This is the idea of the work of Jacobson [Jacobson, 1995; Chirco et al., 2014] who gave a thermodynamics derivation of the Einstein equations.

The derivation is based on the following assumptions: in the (local) Minkowski vacuum, an accelerated a observer sees a thermal state at the Unruh temperature $T = \hbar a / 2\pi$; the entanglement entropy density per unit area is universal with the value $s = 1/4\hbar G$ for any causal horizon; the Clausius relation $\delta S = \delta E / T$ holds; all those assumptions are valid in any reference frame. Starting there, the Einstein can be derived [Jacobson, 1995]⁸. The most important relation consequence of those postulates is [Frodden et al., 2013]

$$\delta E = \frac{a\hbar}{2\pi} s \delta A = \frac{a}{8\pi G} \delta A \quad (2.22)$$

This is a necessary and sufficient condition relation to recover the Einstein equations (we need only to write it in terms of generally covariant quantities). Remark that it is a relation between energy and geometry. In some sense, it sum up all of the dynamics of general relativity as we will see in 3.4. Let's note again the appearance of the area of a boundary which hints again at the specific role of boundaries in gravitation and quantum gravity.

The very fact that Einstein equations can be derived from entanglement entropy is in itself an astonishing result. Interpreting black hole entropy as entanglement entropy of matter and gravitational degrees of freedom is not new but was plagued with the important problem of universality (also called species problem): the entanglement entropy should depend on the number of species in the model which can be enormous. This universality hypothesis is central for the argument here: imposing this constraint implies necessarily for the geometry to be dynamical and not fixed. Indeed, the variation of $S = sA$ gives $\delta S = \delta s A + s \delta A$ and with universality $\delta s = 0$ can be simplified into $\delta S = s \delta A$. The variation of entropy can only come from a variation of the area and thus geometry. The universality issue was clarified in [Bianchi and Satz, 2013] in the perturbative quantum gravity regime: this universality comes from the principle of equivalence and the universality of gravitation. It is a dynamical property coming from the Einstein equations. One can then suspect that some aspects of the Einstein equations have been assumed in this attempts at deriving them.

⁸This is a null version of the timelike argument of [Baez and Bunn, 2005] we analyzed.

Like the black body radiation of black holes, this thermodynamical derivation can be seen as an evidence (but not a proof) of a more fundamental description of gravity with underlying microscopic degrees of freedom.

2.5.2 Effective field theory

The natural paths to construct a quantum theory are either to use a covariant formulation using path integral techniques or canonical formulation using Hilbert spaces. However, because of background independence, all the standard techniques cannot be used as such for the quantization of general relativity.

In a covariant approach, we would need to give some meaning to the partition function

$$Z = \int e^{\frac{i}{\hbar} S[g_{\mu\nu}]} \mathcal{D}g_{\mu\nu} \quad (2.23)$$

There is no known way to make sense of the path integral over metric with the constraint of diffeomorphism invariance. What's more, the Einstein Hilbert action is non polynomial in the field. To apply known methods of quantum field theory, an idea [Donoghue, 1995] is to break general covariance by writing the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where η is now a background Minkowski metric and h the gravitational perturbation. In terms of the h field, it is possible to formally define an interacting quantum field theory and analyze it with perturbative methods. Additional terms satisfying the symmetries can even be added as corrections to the Einstein Hilbert action. Predictions can then be made for scattering processes or calculate corrections to the Newtonian potential [Donoghue, 1994]. The central result of this effective field theory approach is that the fundamental excitation of the gravitational field is a massless spin 2 particles, called the graviton. However, this field theory approach is limited by the fact that it is a non renormalizable theory. Thus, the physics of gravitons is only meaningful at low energy scale but cannot naturally be extrapolated at arbitrary high energy. Still, it is important to stress that this effective method captures part of the relevant physics of quantum gravity.

Apart from this covariant or path integral approach to quantization from which only perturbative information can be extracted, the older canonical approach offers a non perturbative way to construct a quantum theory. The idea would be to have states living in a Hilbert space, operators describing physical observables and a Schrodinger equation for the time evolution. Initiated mostly by Dirac, Arnowitt, Deser and Misner (ADM) for general relativity, this path was long unappreciated because once again of the specific difficulties inherent to general relativity. Loop quantum gravity can be seen as the continuation of this approach.

The following chapter reviews the construction of the loop approach, the well established results about quantum geometry and the remaining open issues researchers are trying to understand and that are the basic motivations behind this work.

Chapter 3

Loop Quantum Gravity

Loop quantum gravity is a proposal of a non perturbative background independent theory of quantum gravity. The canonical approach is based on a canonical quantization of general relativity formulated in an Hamiltonian framework and gives a neat kinematical picture of the quantum geometry of $3d$ space. The covariant approaches are on the contrary better suited to understand dynamical aspects. Background independence means that there is no assumed background defining the theory, like for instance in standard quantum field theories or string theories. Consequently, the whole geometry of spacetime must be reconstructed from the quantum state.

General relativity is traditionally presented in its Lagrangian metric formulation. However, for many purposes, like initial value analysis, numerical relativity or canonical quantization, an Hamiltonian formulation is required (the original one is called the ADM Hamiltonian formulation [[Arnowitt et al., 1959](#)]). An important feature that comes out of the Hamiltonian analysis is that general relativity is a complete constrained theory: it is defined from constraints only and in particular the Hamiltonian is zero on physical states. This is a consequence of diffeomorphism invariance and the fact that the time parameter has no physical meaning and can be re-parametrized at will. All the dynamics is hidden in the constraints. Our standard non general relativistic view of a physical Hamiltonian and true time evolution is hidden in the correlations between observables [[Rovelli, 2002](#); [Tambornino, 2012](#)].

Starting from this Hamiltonian constrained formulation, we can try to apply the program of canonical quantization. There are two possible routes when we have constraints. The first one is to solve the constraints at the classical level, exhibits the physical degrees of freedom and then quantize the theory in the standard manner. This is for instance what is done in the canonical quantization of electromagnetism in the Coulomb gauge. The alternative route to quantization is the following:

- Define a quantum theory without imposing the constraints at the classical level. The Hilbert space of the theory is not the physical one and is called the kinematical Hilbert space.

- Promote the classical constraints to quantum operators.
- Find physical states by solving the constraints (looking at their kernel).

The canonical loop quantum gravity quantization program follows this second strategy (for textbooks, see [Rovelli, 2007; Rovelli and Vidotto, 2014a; Thiemann, 2007]). A nice picture of the quantum structure of space has been given in this theory. The fundamental states, called spin network states, of this quantum geometry give a discrete picture of space with quantized values of area and volume in terms of the Planck length.

At the moment, the complete canonical theory is still under construction. Indeed, the Hamiltonian constraint remains to be properly defined and the physical states of quantum geometry and their time evolution to be understood. Covariant approaches of loop gravity (called in the literature spin foams models) have shed some interesting insights on the quantum evolution [Perez, 2013]. The second major issue is about the recovery of the classical theory of general relativity and the continuum limit through coarse-graining procedures.

The goal of this chapter is to give elements about the construction of the theory, the predictions it can make and the remaining issues that motivate the work of this thesis. This chapter is structured as follows. Section 3.1 exposes the classical Hamiltonian framework of general relativity. Section 3.2 constructs the quantum theory and Section 3.3 discusses the geometrical meaning of the spin network states. The chapter ends with Section 3.4 on the open issues of the quantum dynamics and the quantum to classical transition.

3.1 Hamiltonian formulation

Before beginning any quantization procedure, we need to have a well defined Hamiltonian formulation of our theory. General relativity can be expressed in many different ways depending on the fundamental fields we use. The metric $g_{\mu\nu}$ or the local basis e_μ^I can be taken as the fundamental fields, the connection ∇ can be chosen as another independent field and we can choose to work in a Lagrangian or an Hamiltonian framework. The goal of the first section is to express general relativity as a gauge theory similar to Yang Mills gauge theories. This will allow us to use all the methods of standard quantum field theories. We will need to work through all this as it forms the basis of the loop quantization approach. Detailed explanations and calculations are given in the following reviews [Gourgoulhon, 2012; Poisson, 2004; Dona and Speziale, 2013; Thiemann, 2007].

3.1.1 Hamiltonian ADM formulation

3+1 decomposition The Hamiltonian ADM form [Arnowitt et al., 1959] is constructed from a 3 + 1 split of spacetime to parametrize the configuration space. The idea is to recover the picture of space evolving in time. We consider a foliation of spacetime by a family of spacelike three dimensional hypersurface Σ_t parametrized by a scalar field t

called coordinate time. Spacetime has the topology of $\mathcal{M} = \Sigma \times \mathbb{R}$ and can be written formally like $\mathcal{M} = \bigcup_{t \in \mathbb{R}} \Sigma_t$. Σ_t corresponds to a snapshot of space at time t and it the set $\Sigma_t = \{p \in \mathcal{M}, t(p) = t\}$. This split formalizes our intuitive idea of evolution in time. A natural coordinate system adapted to the foliation is given by $(t, x^a)_{a=1,2,3}$ where the coordinates x^a parametrize a spatial splice Σ_t . Then the spacetime metric $g_{\mu\nu}$ will be decomposed into intrinsic geometrical objects associated to the spatial slice and to extrinsic one associated to its embedding and their time evolution.

Some remarks are in order. First, such a splitting seems to violate the diffeomorphism invariance of the theory. This is not the case as long as we look at observable that are independent of the choice of coordinates. We simply use here our freedom to parametrize spacetime as wish. Second, not all spacetime admits such a decomposition and some solution of the Einstein equation are excluded. However, those kind of solutions are considered to be non physical (they violate causality for instance). Spacetimes with reasonable properties are called globally hyperbolic and we restrict ourselves to those for the time being. Third, the coordinate time t has a priori no physical meaning at all and serves only as a parameter.

We can construct now some kinematical objects. The normal \mathbf{n} to a space slice Σ_t embedded \mathcal{M} is defined from the time coordinate as $\mathbf{n} = -N\nabla t$ with N called the lapse function. This vector is a timelike unitary vector and can thus be seen as the four velocity of an observer (called eulerian or fiducial). As such, the space Σ_t corresponds to its local rest frame at time t , the set of simultaneous events. What's more the physical intuition behind the lapse function N corresponds to the proper time $\delta\tau$ between two events with the same spatial coordinates on different times $\delta\tau = N\delta t$. Associated to the adapted coordinate system $(t, x^a)_{a=1,2,3}$ we have vectors forming a natural basis with \mathbf{n} for instance associated to the time direction. We can then decompose the vector

$$\mathbf{t} = \mathbf{N} + N\mathbf{n} \quad (3.1)$$

The tangential vector \mathbf{N} on Σ_t is called the shift vector. It corresponds to the displacement between two instants of the observer. Finally, the intrinsic geometry of the spatial slice is described by the induced metric q_{ab} defined from $ds^2|_{\Sigma_t} = g_{\mu\nu}dx^\mu dx^\nu = q_{ab}dx^a dx^b$. The set of all those quantities, the shift N , the lapse \mathbf{N} and the induced metric q are the variables used in the ADM formulation of general relativity.

The full spacetime metric g can be decomposed in the adapted basis of the foliation and get

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \mathbf{N}^2 & N_b \\ N_a & q_{ab} \end{pmatrix} \quad (3.2)$$

This decomposition is analogous to the Pythagorean theorem in its infinitesimal and Minkowskian version. Indeed, the interval takes the form $ds^2 = (\text{proper distance on } \Sigma_t)^2 - (\text{proper time between } \Sigma_t/\Sigma_{t+dt})^2 = -N^2(dt)^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt)$. From this

ADM parametrization, the Einstein-Hilbert action can be written as

$$S[q_{ab}, N, N^a] = \frac{c^4}{16\pi G} \int_{t_i}^{t_f} \left[\int_{\Sigma_t} \left({}^3R + K^{ab} K_{ab} - K^2 \right) N \sqrt{q} \, d^3x + 2 \int_{S_t} (k - k_0) N \sqrt{\sigma} \, d^2x \right] \quad (3.3)$$

Here 3R is the curvature of the spatial metric q_{ab} , K^{ab} is the extrinsic curvature of the spatial slice and K its trace. The second term of the action correspond to the $3 + 1$ decomposition of the boundary term with S_t the spatial slice of the boundary, k and k_0 its extrinsic curvatures when embedded in the Σ_t and the flat space respectively and σ_{ij} the induced metric on S_t .

Hamiltonian analysis We can proceed now to the Hamiltonian analysis from this ADM formulation. The canonical momenta associated to the induced metric q_{ab} is obtained from (3.3)

$$p^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}} = \frac{c^4}{16\pi G} \left(K^{ab} - K q^{ab} \right) \sqrt{q} \quad (3.4)$$

Being a canonical conjugate of q_{ab} we see that the extrinsic geometry measures the speed of evolution of the $3d$ metric. The fundamental idea is that the extrinsic geometry is conjugated to the intrinsic geometry. Concerning the lapse and the shift, we see that they do not have a canonical momenta since their time derivative do not appear in the action. This fact will lead to a constrained Hamiltonian theory. The Hamiltonian can be evaluated from a Legendre transform and reads [Hawking and Horowitz, 1996; Poisson, 2004]

$$H = \frac{c^4}{16\pi G} \int_{\Sigma_t} (H_a N^a - H N) \sqrt{q} \, d^3x - 2 \int_{S_t} N(k - k_0) - N_a r_b \left(K^{ab} - k q^{ab} \right) \sqrt{\sigma} \, d^2x \quad (3.5)$$

with H called the Hamiltonian constraint and H_a the vector constraint having the following expressions

$$H_a = -2\nabla_b \left(\frac{p_a^b}{\sqrt{q}} \right) \sqrt{q} \quad H = \left(G_{abcd} \frac{p^{ab}}{\sqrt{q}} \frac{p^{cd}}{\sqrt{q}} - {}^3R \right) \sqrt{q} \quad (3.6)$$

where G_{abcd} is some function of the metric coefficients. Like in electromagnetism where the Gauss constraints generates the $U(1)$ gauge transformation, here the vector constraints generates the spatial diffeomorphism and the Hamiltonian constraint encodes the dynamics of the theory since it relates different spatial slices at different times.

The kinematical phase space is parametrized by the variables (q_{ab}, p^{ab}) equipped with the Poisson bracket structure

$$\{p^{ab}(t, x), q_{cd}(t, x')\} = \delta_{(c}^a \delta_{d)}^b \delta(x - x') \quad (3.7)$$

The physical phase space is then defined as the subspace spanned by the constraints. In fact they are an infinite set of constraints, one for each point on the spatial slice. They form an algebra, called the Dirac algebra,

$$\{H_a(x), H_b(y)\} = H_a(y)\partial_b\delta(x-y) - H_b\delta'_a(x-y) \quad (3.8)$$

$$\{H_a(x), H(y)\} = H(x)\partial_a\delta(x-y) \quad (3.9)$$

$$\{H(x), H(y)\} = H^a(y)\partial_a\delta(x-y) - H^a(y)\partial_a\delta'_a(x-y) \quad (3.10)$$

The right hand side of the equations vanishes on physical solutions, no other constraints are needed. H^a and H are for this reason called first class constraints. They generate the diffeomorphism group on physical solutions (but are more general for generic configurations and defines the algebra of hypersurface deformations). Indeed, this can be seen properly by averaging the constraints on test fields. Given a vector \mathbf{N} and a function N , we define the smeared fields $H[\mathbf{N}] = \int_{\Sigma_t} H^a(x)N_a(x) \, d^3x$ and $H(N) = \int_{\Sigma_t} H(x)N(x) \, d^3x$. We have then the Poisson brackets

$$\{H[\mathbf{N}], q_{ab}\} = \mathcal{L}_{\mathbf{N}}q_{ab} \quad \{H[\mathbf{N}], p_{ab}\} = \mathcal{L}_{\mathbf{N}}p_{ab} \quad (3.11)$$

$$\{H(N), q_{ab}\} = \mathcal{L}_{N\mathbf{n}}q_{ab} \quad \{H(N), p_{ab}\} = \mathcal{L}_{N\mathbf{n}}p_{ab} + \frac{1}{2}q^{ab}NH - 2N\sqrt{q}q^{c[a}q^{b]q}R_{cd} \quad (3.12)$$

with \mathcal{L} the Lie derivative (not to be confused with the Lagrangian density). The first set of brackets (3.11) means that the vector constraint is indeed the generator of the spatial diffeomorphism. The second set (3.12) is almost the same apart from additional terms that vanish for physical solutions and thus conclude that the scalar constraint generates time diffeomorphism on physical solutions.

Without any boundary term, we see that physical solutions are determined by the set of constraints $H_a = 0$, $H = 0$ and from (3.5) a zero Hamiltonian. General relativity is a completely constrained theory. As expected, we do not have a dynamical evolution in the standard sense since the time parameter t is non physical and can be reparametrized by an arbitrary choice of the lapse function. In the presence of the boundary term, we have a true Hamiltonian which has the form $H = -\frac{c^4}{8\pi G} \int_{S_t} N(k - k_0) - N_a r_b (K^{ab} - kq^{ab}) \sqrt{\sigma} \, d^2x$ and notion like the energy of an isolated system (ADM energy) and the angular momentum can be defined from it [Hawking and Horowitz, 1996; Poisson, 2004]. This is a first hint [Freidel, 2015] toward the idea of the holographic nature of gravity highlighted in 2.4. To remind the reader, this means that all the physics of a region of spacetime is encoded on a boundary theory. That's what happens on the Hamiltonian which is zero in the bulk and non zero on a boundary.

3.1.2 Local frame (tetrad) and Connection formulations

So far, we have looked at general relativity from its metric formulation both in the Lagrangian and Hamiltonian formulations. Other formulations of the theory can be given which are more manageable to treat for instance coupling to fermions or to perform canonical quantization.

Local frame formulation We already saw that from the principle of equivalence, that at each point $x \in \mathcal{M}$ of spacetime, it is possible to consider a free fall observer $(\xi^I)_I$ for which the geometry is locally Minkowskian. It is related to the metric of an arbitrary observer $(x^\mu)_\mu$ as

$$g_{\mu\nu}(x) = \eta_{IJ} e_\mu^I(x) e_\nu^J(x) \text{ and } e_\mu^I(x) = \frac{\partial \xi^I}{\partial x^\mu} \quad (3.13)$$

A hidden local $\text{SO}(3, 1)$ symmetry is unraveled in this local basis point of view. Indeed, we have the liberty to define our local inertial reference frame up to a Lorentz transformation Λ_J^I ; it relates different free fall observers. We have then

$$e_\mu^{I'} = \Lambda_J^I e_\mu^J \quad (3.14)$$

Instead of seeing general relativity as a metric theory, it is possible to describe it in terms of those local fields $e_\mu^I(x)$ called tetrad.

To do so, we need to define the proper geometrical object associated with the local fields to describe the geometry of spacetime. Since we now work in the tangent space¹, we can define a new connection, called the spin connection $\omega_{\mu J}^I$, which can be used to define a covariant differentiation D_μ

$$D_\mu v^I(x) = \partial_\mu v^I(x) + \omega_{\mu J}^I v^J(x) \quad (3.15)$$

It can be used to generalize the covariant differentiation \mathcal{D}_μ to objects having both spacetime indices and tangent indices naturally as

$$\mathcal{D}_\mu e_\nu^I = \partial_\mu e_\nu^I + \omega_{\mu J}^I e_\nu^J - \Gamma_{\nu\mu}^\rho e_\rho^I \quad (3.16)$$

Geometrical objects like the curvature, the Ricci tensor and scalar can be written from the tetrad and the spin connection. In a gauge theory, a field strength is associated to the gauge group, like for instance the Maxwell tensor for the $U(1)$ gauge group of electromagnetism. For the spin connection, the field strength is directly related to the Riemann tensor constructed for the metric associated to the tetrad [Dona and Speziale, 2013], showing that general relativity is a Lorentz gauge theory,

$$F_{\mu\nu}^{IJ}(\omega(e)) = e^{I\rho} e^{J\sigma} R_{\mu\nu\rho\sigma}(e) \quad (3.17)$$

The bulk part of the Einstein-Hilbert action (2.12) can then be written in the differential form language as

$$S[e_\mu^I] = \frac{c^4}{16\pi G} \int_V \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}(\omega(e)) \quad (3.18)$$

Like mentioned previously, this is not simply a technical complication but a necessary step to couple gravity with fermions. Indeed, the spin of a particle is defined with respect to the Lorentz group in the local frame and the tetrad is used to go into an arbitrary one.

¹The tetrad defines an isomorphism between the tangent bundle and a Lorentz principal bundle. The connection is defined on this principal bundle.

Connection formulation In its metric formulation, general relativity is a second order theory meaning that second partial derivatives appear in the action. It is in fact possible to express the theory in a first order formalism. The Einstein Hilbert action $S[g_{\mu\nu}] = \frac{c^4}{16\pi G} \int g^{\mu\nu} R_{\mu\nu}[\Gamma(g_{\mu\nu})] \sqrt{-g} d^4x$ depends on the metric through the Ricci tensor which can be expressed with the connection coefficients Γ and its first derivatives. The connection coefficients can themselves be expressed from the compatibility condition as a function of the metric and its first derivatives (hence the appearance of second derivative of the metric in the action). The idea of Palatini was to lift the connection as an independent variable of the theory not initially related to the metric at all. This gives the connection or first order formulation of general relativity. The first order bulk Palatini action is then

$$S[g_{\mu\nu}, \Gamma_{\alpha\beta}^\mu] = \frac{c^4}{16\pi G} \int g^{\mu\nu} R_{\mu\nu}[\Gamma] \sqrt{-g} d^4x \quad (3.19)$$

A priori this theory is different from general relativity. However, the Einstein equation is recovered when all the equations of motions of the Palatini action are written. Variation with respect to the metric gives an equation analogous to the Einstein equation $G_{\mu\nu}(\Gamma) = \frac{8\pi G}{c^4} T_{\mu\nu}$ when matter is properly added to the total action. This is not yet an equation for the metric here since we do not know how it relates to the connection. What's very surprising is that the variation with respect to the connection gives an equation of motion that is exactly the compatibility requirement (??) for $\Gamma[g_{\mu\nu}]$. From the action and the equations of motion, we directly have the fact that the connection must be the unique torsion free connection compatible with the metric. By inserting it back into the first equation of motion, the Einstein equation for the metric of general relativity is recovered. Notice also that in this formulation, only first order derivatives are used, at the expense of more variables defining the theory.

The interest about this formulation is that we bring general relativity closer to the standard field theories used in high energy physics which are the prototype of connection theory on spacetime. This is the first step to express gravitation as a Yang Mills theory.

Combining the two One of the final step in the reformulation of the general theory of relativity is to combine both formulations into one model by using the local frame fields and a connection formulation. The Palatini formulation of the tetrad action (3.18) is (we drop the dimensional factor to conform to the standard writing in the literature)

$$S[e_\mu^I, \omega_\mu^{IJ}] = \int_V \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}(\omega) \quad (3.20)$$

For a discussion of the boundary term, see [Freidel et al., 2017]. The first remark that can be made is that another term compatible with the symmetries can be added to this action without changing the classical equations of motion. It has the form $\delta_{I[K} \delta_{L]J} e^I \wedge e^J \wedge F^{KL}(\omega)$. Adding a coupling constant $1/\gamma$ called the Immirzi parameter, we end with the (Holst)

action [Holst, 1996]

$$S[e_\mu^I, \omega_\mu^{IJ}] = \frac{1}{2} \int_V \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}(\omega) + \frac{1}{\gamma} \int_V \delta_{I[K} \delta_{L]J} e^I \wedge e^J \wedge F^{KL}(\omega) \quad (3.21)$$

Second, the action (3.21) shows the relationship between gravity and topological field theories called BF theories. Those are defined by an action of the form $S_{BF} = \int B_{IJ} \wedge F^{IJ}(A)$ with the B field acting as a Lagrange multiplier and F as the curvature of the connection A . The equations of motion impose flat curvature. Such a theory has no local degrees of freedom. General relativity seems quite different from this. However, in three dimension, general relativity is really a topological theory with no local degrees of freedom (no gravitational waves) and is exactly a BF theory [Witten, 1988; Carlip, 2005]². Equation (3.21) shows that in four dimension, general relativity is a constrained BF theory with the constraint being $B_{IJ} = \epsilon_{IJKL} e^K \wedge e^L$. The advantage of looking at gravity from this angle is the possibility to use well-known techniques of topological quantum field theories to study gravitation at the quantum level. Covariant approaches like spinfoam models start from this perspective.

3.2 Loop quantization

3.2.1 Ashtekar-Barbero formulation

We now have all the necessary elements to go into the loop quantization program. The idea now is again to change our set of variables and write the theory (3.21) with them. The first step is to proceed to a canonical quantization by again assuming a foliation of spacetime with spacelike hypersurface Σ . In the ADM decomposition, the tetrad is projected into

$$e_0^I = N n^I + N^a e_a^I \quad (3.22a)$$

$$q_{ab} = \delta_{ij} e_a^i e_b^j \quad (3.22b)$$

The variables of the theory are now the lapse N , the shift N^a and the triad e_a^i (basis of a local spatial reference frame). The difficulty here is that the set of constraints in this formulation is second class. However, a particular class of choice of gauge and variables can circumvent this issue by recovering a set of first class constraints that can for the most part be solved in the quantum regime.

We start by partially fixing the gauge. We require the timelike tetrad vector e_0^I to be in the same direction as the time direction given by our chosen foliation. We impose

$$e_0^I = (1, 0, 0, 0) \quad (3.23)$$

²The step from second to first order requires a choice of whether both orientations as well as degenerate configuration are allowed or one as to restrict to one orientation only. The first case is not equivalent to general relativity even in three dimensions

The local Lorentz symmetry uncovered with the tetrad is now a local spatial rotation symmetry $SO(3)$ which encode the liberty to choose our spatial reference frame up to any rotation. The Ashtekar-Barbero change of variables uses a densitized triad E_i^a and a new connection A_a^i

$$E_i^a = e e_i^a \quad (3.24a)$$

$$A_a^i = \gamma \omega_a^{0i} + \frac{1}{2} \epsilon_{jk}^i \omega_a^{jk} \quad (3.24b)$$

The connection A_a^i is not a spacetime connection. It depends explicitly on the embedding of the spatial hypersurface into spacetime [Charles and Livine, 2015]. The major result is that those variables are conjugate variables $\{A_a^i(x), E_j^b(y)\} = \gamma \delta_b^a \delta_j^i \delta^3(x, y)$. The notations are chosen to be similar to electromagnetism where the vector potential and the electric field are conjugate variables. The action and the constraints can then be written as

$$S(A_a^i, E_i^a, N, N^a) = \frac{1}{\gamma} \int \left[\dot{A}_a^i E_i^a - A_0^i D_a E_i^a - N H - N^a H_a \right] d^3x dt \quad (3.25a)$$

$$G_i = D_a E_i^a \quad (3.25b)$$

$$H_a = \frac{1}{\gamma} F_{ab}^j E_j^b - \frac{1 + \gamma^2}{\gamma} K_a^i G_i \quad (3.25c)$$

$$H = \left[F_{ab}^j - (\gamma^2 + 1) \epsilon_{mn}^j K_a^m K_b^n \right] \frac{\epsilon_j^{kl} E_k^a E_l^b}{\det(E)} + \frac{1 + \gamma^2}{\gamma} G^i \partial_a \frac{E_i^a}{\det(E)} \quad (3.25d)$$

We have finally arrived at the last formulation of general relativity which is used for the loop quantization. General relativity is expressed through a set of (seven) first class constraints. The first one (3.25b) is an $SU(2)$ Gauss constraint similar to the one found in electromagnetism or Yang-Mills theories. This extra constraint compared to the ADM case is present due to the choice of triad variables as our fundamental fields. The other two are the same as the ADM case.

An interesting choice of the Immirzi parameter would be $\gamma = i$. This is the historical choice. In this case, the connection becomes a true spacetime connection. What's more, we see that the spatial and Hamiltonian constraints simplify drastically. The issue of such a formulation is that we go to a complex theory and that new constraints called reality constraints must be dealt with in order to recover real gravity. For a real Immirzi parameter, we stay with a real theory with no additional reality constraints to implement but at the expense of non polynomial terms in the Hamiltonian constraint.

3.2.2 Quantization

General relativity is now expressed in a suitable way to perform a canonical quantization. The subtlety here is that it is a totally constrained theory, meaning that all its physical predictions are encoded in constraints. The final quantum theory will be obtained once

all the constraints are solved. Historically, the ADM program was launched for this very purpose, leading Dirac to his analysis of constrained Hamiltonian theory. The implementation of the Hamiltonian constraint in the metric formulation lead to the famous Wheeler-De Witt equation of quantum gravity. However, this approach did not lead to fruitful results due to the complex mathematics of the theory or even to make sense of the equations (problem of time for instance).

Like mentioned in the introductory part of this chapter, the loop approach is a direct descendent of this canonical approach using a new formulation of the general relativity. It then follows the strategy of Dirac to first define a kinematical Hilbert space \mathcal{H}^{Kin} by a suitable choice of representation of the configuration variables, then to promote constraints to quantum operators and then solve them. For general relativity, we have basically three steps. After defining the loop algebra for the kinematical space, we implement the Gauss constraint giving the gauge invariant Hilbert space $\mathcal{H}^{\text{Gauge}}$. We then look for (spatial) diffeomorphism invariant states defining $\mathcal{H}^{\text{Diff}}$. This Hilbert space is sometimes also refereed as the kinematical Hilbert space since it describes the set of quantum states of space (the hypersurface Σ in the semi-classical regime). The last and most difficult step is to solve the dynamics of the theory by implementing the Hamiltonian constraint (temporal diffeomorphism) to end up with the physical quantum gravity Hilbert space $\mathcal{H}^{\text{Phys}}$. This part is still not clearly understood while many proposal exists. The difficulty is rooted into our poor understanding of what is time in quantum gravity. For the moment, we will review the construction of the kinematical setting of the theory by discussing each step one at the time, summed up in the following sequence

$$\mathcal{H}^{\text{Kin}} \xrightarrow[\text{Gauss}]{} \mathcal{H}^{\text{Gauge}} \xrightarrow[\text{Diffeo}]{} \mathcal{H}^{\text{Diff}} \xrightarrow[\text{Hamilt}]{} \mathcal{H}^{\text{Phys}} \quad (3.26)$$

Schrodinger representation In the Ashtekar formulation, the basics variables are thus a spatial $\text{SU}(2)$ connection A_a^i and its momentum conjugate the densitized triad E_i^a . We now jump to the quantum theory by a textbook canonical quantization. In the Schrodinger representation, we work with a wavefunctional of the connection $\psi[A_a^i]$. We choose to represent the connection operator multiplicatively and the momentum by a derivative action to properly translate the Poisson bracket into the quantum regime

$$\hat{A}_a^i(x)\psi[A] = A_a^i(x)\psi[A] \quad (3.27a)$$

$$\hat{E}_i^a(x)\psi[A] = -i\gamma \frac{\delta\psi}{\delta A_a^i}[A] \quad (3.27b)$$

What we have to do next is to implement the constraints. To have an intuitive understanding of the method, by begin first with the Gauss constraint encoding the $\text{SU}(2)$ gauge invariance. In the connection representation, we know how transform the connection under the action of an $\text{SU}(2)$ element g . Quantum mechanically this translates as $U(g)\psi[A] = \psi[g.A]$ where $U(g)$ is a unitary representation of the group element and

the dot \cdot corresponds to the group action on the space of connections. States solving the constraints have then to satisfy the relation

$$\forall g \in \text{SU}(2), \psi[g.A] = \psi[A] \quad (3.28)$$

Loop excitations What gauge invariant function can we construct? The classical theory gives us a simple gauge invariant function called the holonomy. The idea is to consider the parallel transport along a close loop C . Here is the mathematical definition. Let's parametrize the path by coordinates $x^a(s)$. We define a function of the group $h_C(s)$ as

$$h_C(0) = \mathbb{1} \quad (3.29a)$$

$$\frac{dh_C}{ds}(s) = i \frac{dx^a}{ds}(s) A_a^i(x(s)) \frac{\sigma_i}{2} h_C(s) \quad (3.29b)$$

Thus $h_C(s)$ represents the evolution in $\text{SU}(2)$ (in the fundamental representation with Pauli matrices σ_i) between the origin and the point s . The non-trivial aspect comes from the fact that for a closed loop, $h_C(1)$ has no reason to be equal to $h_C(0)$. This is in fact a signature of the curvature of spacetime. That it is a signature of curvature is perfectly illustrated in the common example of the parallel transport of a vector along a loop on a sphere to introduce the notion of curvature. For now, this function of the connection is not a gauge invariant object. Indeed, under some rotation $r(t)$, it transforms as $h_C(s) \rightarrow r^{-1}(x(s))h_C(s)r(x(0))$. It depends only on the rotations at the end points of the curve. For a loop, $h_C(1)$ transforms as $h_C(1) \rightarrow r^{-1}(x(0))h_C(1)r(x(0))$ and is just conjugated by the rotation at the origin. By considering the trace of it, we end up with a function $\chi_C[A]$ of the connection that is now a gauge invariant function

$$\chi_C[A] = \text{tr } h_C(1) \quad (3.30)$$

In the context of lattice gauge theories, those functions are called Wilson loops. Their fundamental importance is that, at least for a compact group like $\text{SU}(2)$, the full gauge invariant information on the connection is encoded in those functions [Thiemann, 2007]. Holonomies (their traces) are classical gauge invariant functions that can be translated into the quantum regime. In loop quantum gravity, holonomies are the fundamental excitations of the gravitational field (giving the theory its name). However, they are not the simplest way to represent the wavefunctions of the theory and a simple generalization into spin networks is more manageable.

Holonomy-Flux algebra It is no surprise if what we just described resembles electromagnetism which is a $U(1)$ gauge theory. In its Hamiltonian form, the canonical variables are the potential vector A_a and its canonical conjugate the electric field E^a . The potential vector alone is a gauge dependent field. The physically meaningful function to look at is its circulation around a given path (again a gauge dependent function) or more precisely

its circulation around closed loops (a gauge invariant physical quantity, already without taking the trace here because of the simplicity of the $U(1)$ group). We recover here the very definition of holonomies. Interpreted in terms of the magnetic field, holonomies corresponds to the magnetic flux crossing any region of space with support on the loop. All this appears naturally in the Aharonov-Bohm effect. Concerning the electric field, the quantity to evaluate is again its flux through surfaces. The most interesting case is for closed surface where the value of the flux must be equal to the charge inside (Gauss law). Thus in electromagnetism, the two natural functions of the conjugate variables are the holonomies of A_a (or the magnetic flux) and the electric E_a fluxes.

In the case of gravitation, everything is similar. Holonomies of the connection A_a^i form a particular subset of phase space functions used to define the quantum theory. They are the regulated functions of the connection A_a^i . To have a good regulated algebra we need again to consider also the regulated form of the conjugated variables E_i^a which is simply its flux through a given surface S with normal n^i . In integrated form we have the regulated functions

$$h_C(A) = \mathcal{P} \exp \left(\int_C A \right) = \mathcal{P} \exp \left(\int_C \frac{dx^a}{ds}(s) A_a^i(x(s)) \frac{i\sigma_i}{2} ds \right) \quad (3.31a)$$

$$E_n(S) = \int_S E_i^a n^i dS_a = \int_S E_i^a n^i \epsilon_{abc} dx^b \wedge dx^c \quad (3.31b)$$

where \mathcal{P} is the path ordered exponential (like the time ordered exponential in field theories but in a spatial sense here). We can consider then the Poisson bracket of an holonomy with a flux. For the simple case where the holonomy crosses the surface only once, the bracket is

$$\{h_C(A), E_n(S)\} = h_C^{\text{in}}(A) (\sigma_i n^i) h_C^{\text{out}}(A) \quad (3.32)$$

where $h_C^{\text{in}}(A)$ is the holonomy from the beginning up to the crossing point and $h_C^{\text{out}}(A)$ is the holonomy from the crossing point into the end of the path. The holonomy-flux algebra is the algebra of phase space functions we choose as our fundamental set of observables for the quantum theory. We will see in the next section how we define those operators at the quantum level.

3.3 Kinematical setting: spin network

3.3.1 Kinematical Hilbert space

Construction Now that we have formal understanding of the quantum set up, we seek a more efficient way to look at the Hilbert space of the quantum theory. We start first by disregarding the constraints of gauge and diffeomorphism invariance and construct the kinematical space only. This can be done step by step by remembering how things work for instance in quantum electrodynamics. In QED, we begin by considering the truncation

of the full Hilbert space with the Hilbert space \mathcal{H}_n with a fixed number of photons n with states properly symmetrized to take the Bose statistics into account. We can then construct the total Fock space as an infinite tensor product over the number of photons $\mathcal{H} = \otimes_{n=0}^{+\infty} \mathcal{H}_n \simeq \lim_{N \rightarrow +\infty} \otimes_{n=0}^N \mathcal{H}_n$. The loop quantum gravity Hilbert space is constructed by following the same steps: we consider a specific truncation properly “symmetrized” and then go to some limit to obtain the kinematical Hilbert space.

The key ingredient to obtain a good truncation is with the notion of cylindrical functions that corresponds to a natural generalization of the loop construction. Those are functionals that depend on the connection only through its (open) holonomies $h_e[A]$ along some links e of an oriented graph Γ . Denoting by L the number of links, a cylindrical function is a function $\psi : \text{SU}(2)^L \rightarrow \mathbb{C}$ that defines a quantum state $|\psi_\Gamma\rangle$ with wavefunction $\langle A | \psi_\Gamma \rangle = \psi(h_{e_1}(A), \dots, h_{e_L}(A))$.

To have an Hilbert space, we need a scalar product. A natural scalar product can be defined with the Haar measure $d\mu$ of a compact group G , here $\text{SU}(2)$. It is invariant under left and right composition, $\forall h \in G$, $\int f(g) d\mu(g) = \int f(hg) d\mu(g) = \int f(gh) d\mu(g)$. A scalar product between two functions of the group can be defined as

$$\langle f_1 | f_2 \rangle = \int f_1^*(g) f_2(g) d\mu(g) \quad (3.33)$$

The space of cylindrical functions equipped with this scalar product forms an Hilbert space $\tilde{\mathcal{H}}_\Gamma$ isomorphic to the Hilbert space of square integrable functions $L^2(\text{SU}(2)^L)$.

For the moment, this Hilbert space contains equivalence classes of states describing the same physics. Indeed, consider a subgraph Γ' of Γ . Then $\tilde{\mathcal{H}}_{\Gamma'}$ can be identified with a subspace of $\tilde{\mathcal{H}}_\Gamma$. Such a mapping defines an equivalence relation \sim . The true truncated Hilbert space \mathcal{H}_Γ that removes the redundancy is then the quotient $\mathcal{H}_\Gamma = \tilde{\mathcal{H}}_\Gamma / \sim$.

Finally the full kinematical Hilbert space is obtained from a limiting procedure called projective limit or, to be simpler, as the tensor product of all graphs having their support in Σ is $\mathcal{H} = \oplus_{\Gamma \subset \Sigma} \mathcal{H}_\Gamma$. Its scalar product is induced from the Haar measure, with the subtlety that if two functions do not share the same graph, we have to extend their definition trivially in the finer graph obtained from the union of the original two and work on this new one. In the end, it was shown rigorously that this construction defines an Hilbert space on distributional valued connections A on Σ with a measure $d\mu$ called the Ashtekar-Lewandowski measure $L^2(A, d\mu)$.

Fourier basis The way we have constructed the kinematical Hilbert space, from functions of the $\text{SU}(2)$ group, is not very useful for practical calculations. If it were the $U(1)$ group, we known that such function can be decomposed on the orthogonal basis $(e^{in\theta})_n$ and corresponds to the Fourier decomposition of a periodic function. Such a result can be generalized to any compact group like $\text{SU}(2)$ thanks to the Peter-Weyl theorem. It states simply that an orthogonal basis of functions on a group are given by the set of matrix elements of its irreducible representations. For $\text{SU}(2)$ those representations live in a space

V^j labeled by spin $j \in \mathbb{N}/2$. Then a function $f(g)$ of $SU(2)$ is decomposed on the Wigner matrix $D^j(g)$ as

$$f(g) = \sum_{j,m,n} f_j^{m,n} D_{m,n}^j(g) \quad (3.34)$$

Such a decomposition applies to the Hilbert space \mathcal{H}_Γ

$$\psi(g_1, \dots, g_L) = \sum_{[j_e, m_e, n_e]} \psi_{j_1, \dots, j_L}^{m_1, n_1, \dots, m_L, n_L} D_{m_1 n_1}^{j_1}(g_1) \cdots D_{m_L n_L}^{j_L}(g_L) \quad (3.35)$$

The basis element, labeled by the spin j and the quantum numbers m and n can be represented by a colored graph where the edges carry the information on the holonomy. In terms of Hilbert space, this amounts to the decomposition $\mathcal{H}_\Gamma \simeq L^2(SU(2)^L) = \bigoplus_{j_l} \bigotimes_l (V_{j_l}^* \otimes V_{j_l})$. Here the star corresponds to the adjoint representation. For a given link, the two Hilbert spaces are associated to the two nodes, the source s_n and the target t_n , of the link. The tensor product gives the states of the spin network given a collection of spins on the graph while the sum takes into account all possible spin configurations. In fact, by defining $V_n = \bigotimes_{s_l=n} V_{j_l}^* \otimes_{t_l=n} V_{j_l}$, we write it as

$$\mathcal{H}_\Gamma = \bigoplus_{j_l} \bigotimes_n V_n \quad (3.36)$$

Holonomy and flux operators A natural question that we can ask now is about the physical interpretation of this Fourier basis. Like in standard quantum mechanics, we expect it to diagonalize the momentum operator. Does it diagonalizes the flux operator then?

In this Schrodinger representation, we can give a natural representation of the holonomy-flux algebra. Operators corresponding to the holonomies act in a multiplicative manner while the densitized triad as a derivative,

$$\hat{h}_e^j[A] |\psi_\Gamma\rangle = h_e^j[A] |\psi_\Gamma\rangle \quad (3.37a)$$

$$\hat{E}_n(S) |\psi_\Gamma\rangle = -i\hbar\gamma \int n_i \frac{\delta}{\delta A_a^i} d\sigma |\psi_\Gamma\rangle \quad (3.37b)$$

On a simple holonomy, the momentum action is simply $\hat{E}_n(S) h_e[A] = \pm i\hbar\gamma h_e^{\text{in}}[A] (\tau_i n^i) h_e^{\text{out}}[A]$. The action of a flux operator amounts to an insertion of Pauli matrices (or $SU(2)$ generators) at the crossing point of the path and the surface. The sign depends on the relative orientation. If now we look at the action of two contracted flux $\hat{E}_i \hat{E}^i$ where the i directions are defined relative to a given reference frame at the crossing, then $\hat{E}_i \hat{E}^i h_e[A] = -\hbar^2 \gamma^2 h_e^{\text{in}}[A] (\tau_i \tau^i) h_e^{\text{out}}[A]$. The Casimir operator appears $C^2 = \tau_i \tau^i$ and commutes with all group elements so that using the composition rule of two holonomies

$$\hat{E}_i \hat{E}^i h_e[A] = -\gamma^2 \hbar^2 C^2 h_e[A] \quad (3.38)$$

This simple diagonal action will find a direct application to the area operator as we will see. Furthermore, it is straightforward to see that the action of two fluxes of different directions do not commute. The action of the algebra we have defined extends to any Fourier basis element with again a diagonal action with eigenvalue $\hbar^2 \gamma^2 j(j+1)$ and to the full kinematical Hilbert space. A remarkable result has shown that this representation on the kinematical Hilbert space is unique.

3.3.2 Gauge invariant Hilbert spaces

Gauss constraint The Hilbert space we have constructed so far is called kinematical since no constraints have been taken into account yet. We begin by the Gauss constraint and look for the gauge invariant sector. We already found a gauge invariant solution for the loop case. We need to see how this translates with the states $|\psi_\Gamma\rangle$. We look for states in the kernel of the Gauss constraint. Remember the transformation law for an holonomy along an edge e , $g_e \rightarrow r^{-1}(t_e)g_e r(s_e)$ with s_e and t_e the source and target nodes of the edge e respectively. In terms of the Fourier basis, this amounts to the product of the three irreducible representation matrices. Thus, a gauge transformation acts on the source and the target of a link, or stated differently on the nodes of the graph. Remembering (3.28), we are looking for wavefunctions satisfying $\psi(g_1, \dots, g_L) = \psi(h_{t_1}^{-1}g_1 h_{s_1}, \dots, h_{t_L}^{-1}g_L h_{s_L})$. The mathematical procedure to obtain such wavefunctions is through a group averaging procedure. We do not need to go into a detailed analysis. The end result is quite simple.

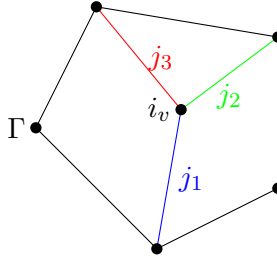


Figure 3.1: Example of a colored graph Γ with spins $j_{1,2,3}$ on the edges and the interwiner i_v living at the node.

In the Fourier basis, it is not too complicated to convince ourselves that the gauge equality is satisfied by inserting at each node n of the graph an $SU(2)$ invariant tensor $i_n : \otimes_{e_{in}} V^{j_e} \rightarrow \otimes_{e_{out}} V^{j_e}$, called intertwiners. A basis of the gauge invariant Hilbert space based on a graph Γ is represented by a colored graph with spins j_e on the edges and intertwiners i_n at the nodes. Those states $|\Gamma, j_e, i_n\rangle$ are called spin network states and live in the Hilbert space $\mathcal{H}_\Gamma^{\text{Gauge}} \simeq L^2(SU(2)^L / SU(2)^N)$ with L the number of edges and N the number of nodes of the graph. In the Fourier basis, we select the gauge invariant

part of the space V_n denoted by $\text{Inv}_{\text{SU}(2)}(V_n)$. We end up with the decomposition

$$\mathcal{H}_\Gamma^{\text{Gauge}} = \bigoplus_{j_l} \bigotimes_n \text{Inv}_{\text{SU}(2)}(V_n) \quad (3.39)$$

Another basis of this Hilbert space is the one we introduced in our motivations on the quantization procedure with the trace of holonomies on the graph. In the Fourier basis, they correspond to what is called in representation group theory the character of the representation.

The last step is once again to use projective techniques to define the continuum gauge invariant Hilbert space $\mathcal{H}^{\text{Gauge}}$. The gauge invariant scalar product is the one inherited from the Ashtekar-Lewandowski measure.

Spatial diffeomorphism Now that we have an $\text{SU}(2)$ gauge invariant Hilbert space, we would like to implement the spatial diffeomorphism constraint. We have to find diffeomorphism invariant states or in other terms we have to quotient $\mathcal{H}^{\text{Gauge}}$ by the action of diffeomorphisms.

The action of spatial diffeomorphism on spin network is simple: it amounts to move the graph that support the state. For a diffeomorphism ϕ of the canonical surface, its action on a cylindrical function is $\phi.\psi_\Gamma[A] \rightarrow \psi_\Gamma[\phi^{-1}A] = \psi_{\phi.\Gamma}[A]$ where $\phi.\Gamma$ is the transported graph under the action of the diffeomorphism. The idea is then simply to consider the equivalence classes of cylindrical functions under this action. The measure and the scalar product go through the quotient without difficulties. The major difference is that two states with different graphs were previously orthogonal whereas now they are the same state if the graphs are diffeomorphic. In the end, we have an Hilbert space $\mathcal{H}^{\text{Diff}}$ with a basis formed by spin networks labeled by spins, intertwiners and an equivalence class of graphs.

When working in this Hilbert space, we have to be aware that we do not live anymore in the canonical hypersurface Σ like up until now. The states in $\mathcal{H}^{\text{Diff}}$ define space itself, they define Σ . This is the natural consequence of diffeomorphism invariance. Consequently, we must be cautious when talking about distance, locality or regions of space here.

3.3.3 Geometry of spin networks

Now that we have defined the kinematical, gauge and diffeomorphism Hilbert spaces of the theory, we need some physical measurable quantities of the quantum gravitational field. In loop quantum gravity, two geometric observables which form a complete set are well understood, the area and volume operators. To have a proper definition of a surface and a region, we will work first of all in $\mathcal{H}^{\text{Gauge}}$ describing the gauge invariant states of the canonical hypersurface Σ . We will see that the area and volume operators give a beautiful geometric interpretation of a spin network state and of the quantum geometry of space.

To define the area operator, we start from its classical expression. For a surface S parametrized by the coordinates σ^i embedded in canonical space Σ with coordinates x^a , the area $A(S)$ is written as

$$A(S) = \int_S \sqrt{\det \left(q_{ab} \frac{\partial x^a}{\partial \sigma^i} \frac{\partial x^b}{\partial \sigma^j} \right)} d^2\sigma = \int_S \sqrt{h} dS \quad (3.40)$$

This is nothing else than the standard volume element for the induced metric h_{ij} on S . Now we need to express this in term of the Ashtekar variables. Consider a coordinate system where the spatial metric q_{ab} is diagonal in space and time (where the time direction is given by the normal vector n_a). Then $q_{ab} = \text{diag}(q_{00}, h_{ij})$. Straightforwardly, $q = q_{00}h$ or for our purpose $h = qq_{00}^{-1}$. Using $qq^{ab} = E_i^a E_j^b \delta^{ij}$, we have $qq^{ab}n_a n_b = E_i^a E_j^b \delta^{ij} n_a n_b = h$. We end up with the desired result, expressing the area in terms of the densitized triad,

$$A(S) = \int_S \sqrt{E_k^a E^{bk} n_a n_b} d^2\sigma \quad (3.41)$$

The goal is now to promote this expression to a quantum operator. At the quantum level, the triad acts as a derivative operator or, when acting on an holonomy as a Pauli matrix insertion operator. The action of the scalar product $E_k^a E_k^b$ is easily obtained on an holonomy crossing only once the surface S : indeed it is diagonal with the eigenvalue $\hbar^2 j(j+1)$ for an holonomy in the spin j representation. The similarity with the angular momentum results in quantum mechanics is clear. The idea then is simply to consider a discretization of the surface S into small enough patches so that each patch is punctured by only one edge of the spin network. The classical area can be written as the limit

$$A(S) = \lim_{A_i \rightarrow 0} \gamma \sum_i \sqrt{E^k(A_i) E_k(A_i)} \quad (3.42)$$

Going quantum, no issues arise from the presence of the square-root which can be properly defined here. For a generic surface intersecting the network only at its links³, we define the area quantum operator as

$$\hat{A}(S) = \lim_{A_i \rightarrow 0} \gamma \sum_i \sqrt{\hat{E}^k(A_i) \hat{E}_k(A_i)} \quad (3.43)$$

Its eigenvalues are, putting back the gravitational units,

$$A(S) = 8\pi G \hbar \gamma \sum_{e_i \perp S} \sqrt{j_i(j_i + 1)} \quad (3.44)$$

Here $e_i \perp S$ denotes the links crossing the surface and j_i the spin characterizing the $\text{SU}(2)$ representation carried by the link e_i .

³The definition and the eigenvalue are a bit more technical when we intersect nodes or have tangential links, but this known.

Key remarks are in order. First, spin network states are eigenstates of the area operator since its action is diagonal on them. This is the first step toward the geometrical interpretation of spin network states: links of the graph carry information about the area of surfaces. The second major remark is that the area spectrum is quantized and has a gap. This means that there is a minimal value of the area above the vacuum. This is a non trivial prediction of the theory and goes in the opposite direction of our standard mindset where geometry and thus area can become arbitrarily small.

Concerning the definition of the area operator, there still exists some ambiguity, like in any quantum theory, because of operator ordering. We above chose the standard definition motivated by the classical result and the close analogy with the theory of angular momentum in quantum mechanics (which is just $SU(2)$ representation theory). In fact, other formalisms of loop quantum gravity favor other definitions. For instance, in the framework we will present and use later in this thesis, it seems more natural to define the area operator such that its eigenvalues are $A(S) = 8\pi G\hbar\gamma \sum_{e_i \perp S} j_i$. The difference is that this time the eigenvalue are evenly spaced.

Another natural geometrical operator can be defined by quantization: the volume of a region R of space $V(R) = \int_R \sqrt{h} \, d^3x = \int_R \sqrt{\left| \frac{1}{3!} E_i^a E_j^b E_k^c \epsilon_{abc} \epsilon^{ijk} \right|}$. Its proper mathematical definition is a bit more technical than the area operator and we refer to the literature for a complete discussion of it. The final result is that the volume operator acts only on the nodes of the graph and is diagonal in the spin network basis. Its spectrum is again quantized (the exact eigenvalues are again more complicated than the area) with eigenvalues proportional to the Planck volume l_P^3 .

Both the area and volume operators form a complete set of observables in loop quantum gravity and provide the geometrical interpretation of spin network states. We saw that at the Planck scale the space geometry is discrete and that each spin network state describes a quantum geometry. The graph encodes relational information between regions of space, an edge dressed with a spin j_e encodes information on area and an intertwiner i_n on a node n encodes information on volume.

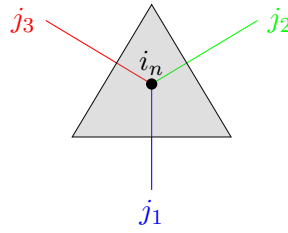


Figure 3.2: In two dimension, a tri-valent part of a spin network is dual to a triangle in the geometrical interpretation of spin networks. The intertwiner encode volume information on the dual polyhedron, the edges carry area and normal information of the faces of the polyhedron.

Before ending this discussion, some comments are in order. In the above definitions, the notions of area and volume of a region of space was crucial. If we work only on $\mathcal{H}_\Gamma^{\text{Gauge}}$ meaning when only the Gauss constraint is satisfied, we still work with the spatial slice Σ used at the classical level and we can define regions with respect to this geometry. However, from the moment we satisfy the spatial diffeomorphism constraint, we lose this possibility and only the quantum state itself must be used to define a notion of region. Remember that we already encountered similar difficulties at the classical level to define local regions. How can we make sense of the results above in the full diffeomorphism invariant space $\mathcal{H}^{\text{Diff}}$? Do our results about discrete geometry hold? The answer is positive as long as we can define spatially diffeomorphic geometric operators. This can be done in a relational way similar to the classical resolution. Indeed, what we can be done to define a region is to construct it from the values of other fields like matter fields. For instance, we can consider the region in space with non zero electric field and then measure its volume. Such a relational operator would have a similar spectrum as the one defined above. Thus, the discrete geometry picture makes sense when the notions of regions, its volume and areas are properly defined in a relational way relative to a reference frame constructed from physical fields of the theory.

3.4 Dynamics and the quantum to classical transition

3.4.1 The dynamics

Up to now, we have only considered kinematical aspects of the quantum theory and have forgotten all dynamical aspects. In fact, dynamics of loop quantum gravity is the black spot of the theory. We do not yet have a comprehensive understanding of it even after years of research. We have two possibilities to study the dynamics: from the Hamiltonian point of view by trying to implement at the quantum level the Hamiltonian constraint or from the Lagrangian (path integral) perspective.

Canonical approach In the canonical approach, we continue the process started in the last sections to define and implement the constraints of general relativity. The dynamics is encoded in the Hamiltonian constraint (3.25d) which generates the time diffeomorphism. The first well-defined Hamiltonian constraint at the quantum level was proposed by [Thiemann, 1996] (see e.g. [Bonzom and Laddha, 2012] for a review of the various approaches through toy models). Its regularized form is built on a choice of triangulation of space and a re-expression of the triad E_i^a in terms of Poisson bracket and a re-expression of the connection A_a^i and its curvature F_{ab}^i in terms of well chosen holonomies of the triangulation (see [Dona and Speziale, 2013] for a simple discussion of this construction). The important point here is that this construction gives a well defined quantum Hamiltonian constraint, can be defined in a continuum limit and realized on shell in a non anomalous way the Dirac algebra at the quantum level.

Nonetheless, this proposal is plagued with a number of issue. A first issue is that the construction is ambiguous in different regards: different ordering in the operators can be chosen, different regularization choice can be made which change the action of the operator. Another one is that the action of the Thiemann constraint is ultralocal. It acts only at one node and creates new nodes and edges with zero total volume. It has a graph changing action (which is fine) but do not act on the new nodes it creates. This is quite odd since it appears that such an action do not allow any propagation of information. Finally, very few exact solutions of the Hamiltonian constraint are known at the moment.

Covariant approach To have a better view on the dynamics, it is preferable to use a covariant language which launched the development of spinfoam models of quantum gravity. This difficulty to apprehend the dynamics is rooted in the problem of time in general relativity and quantum gravity. The canonical approach which is based on a choice of foliation and the time gauge do not give a simple answer to this question. The idea of spinfoam models is to treat again thing in a four dimensional and covariant way, incorporating time and dynamics from the beginning.

The starting point is the kinematical framework of canonical loop quantum gravity and dynamics is defined through a path integral approach. The transition amplitudes can be interpreted as histories of spin network states. To define the amplitude, the models start from the action (3.18) which is the action of general relativity expressed as a constrained topological field theory (quantizing topological theory is a well known process). We refer to the literature for a proper introduction to this large topic [Baez, 2000; Perez, 2013].

Instead, we give here some details about an important relation called the simplicity constraints which in a sense impose that the quantum variables used are geometric quantities rather than just mere functions of the canonical coordinates. From the action (3.18), on a spacelike boundary Σ , the canonical conjugate of ω is the two form $\Pi = \star e \wedge e + \frac{1}{\gamma} e \wedge e$. In the same way that the electromagnetic tensor can be decomposed into the electric field and the magnetic field when an inertial observer is chosen. Here the choice of reference frame is given by the choice of the normal n_I . In the gauge where $n_I = (1, 0, 0, 0)$, which reduces the Lorentz symmetry to the rotation symmetry, we define $K^i = \Pi^{i0}$ (electric field analogue) and $L^i = \frac{1}{2} \epsilon_{jk}^i \Pi^{jk}$ (magnetic field analogue). Those are in fact vectors on Σ . From the definition of the canonical momentum and that $n_I e^I|_{\Sigma} = 0$, we can derive the relation $n_I (\star \Pi)^{IJ} = \frac{1}{\gamma} n_I \Pi^{IJ}$ or with our notations

$$K^i = \gamma L^i \tag{3.45}$$

From the fact that Π is the momentum of the connection, it generates the Lorentz transformations. The boost are generated by (correctly normalized) $\frac{1}{8\pi\gamma G} K$. Moreover, from the definition of areas, the norm of L^i is an area surface element for a surface normal to n^i . Using the simplicity constraint, we end up with a relationship between boost and

area

$$K = \frac{A}{8\pi G} \quad (3.46)$$

If we consider a uniformly accelerated observer with acceleration a , like for instance an observer hovering near a black hole horizon, we know that its energy is proportional to the boost generator with proportionality factor a . Thus, we end up with an important relationship between the energy of a uniformly accelerated observer and area element, which will be used to understand horizon or more generally surface dynamics in loop quantum gravity

$$E = \frac{aA}{8\pi G} \quad (3.47)$$

An important remark to end this discussion is that what we derived here makes sense only if we have a notion of spacetime, surface and similar geometrical objects. In a full quantum theory like loop quantum gravity such notions have to be reconstructed from the quantum state itself and require a notion of continuum limit. Thus the previous discussion is just a step to clarify the content of the classical continuum theory.

To end this very brief discussion on the dynamics in loop quantum gravity, we stress the importance to solve this issue, first to have a complete well defined theory from which predictions can be extracted and second to prove that we indeed have a quantum theory of gravity giving back in a proper limit classical general relativity.

3.4.2 The continuum and classical limits

With a lack of a well defined dynamics in loop quantum gravity, it is difficult to properly investigate the question of recovering general relativity. There are in fact two limits that we have to take in order to understand how general relativity emerges from the quantum theory.

The first one is the continuum limit. Indeed, while the theory is defined in a continuum sense, the geometric operators have a discrete spectrum and therefore the quantum geometry is discrete. In which approximate sense do we recover the smooth manifold of general relativity? The direction taken to solve this issue is by properly defining a coarse-graining procedure [Livine, 2014; Livine and Terno, 2005; Dittrich et al., 2013; Dittrich, 2014; Dittrich and Geiller, 2015]. Even if we do not have a dynamics at hand, we can propose some procedures on how to coarse-grain spin network states. This in itself is a challenge since, working in a background independent way, we do not have a geometrical background to define our renormalization algorithm. In return, we could expect that a better understanding of the coarse-graining of spin network states will give some important insights into the dynamics of the theory.

The second limit is the classical limit. One facet of this is to obtain a set of quantum states interpolating properly between the quantum geometry and a classical geometry.

In quantum electrodynamics those are called coherent states and can be defined in different equivalent way as eigenstates of the destruction operator for every field modes for instance or as the states that saturate Heisenberg uncertainties relations. Similar states have been proposed in the context of loop quantum gravity [Freidel and Livine, 2011a; Stottmeister and Thiemann, 2015a,b,c]. A second facet, seldom explored in the loop quantum gravity community, is really the quantum to classical transition itself or how quantum superpositions do not appear at the macro-scale. This question is (almost) perfectly handled by decoherence theory and specific questions relative to gravity can be asked like for instance the one we will discuss later in this thesis relative to an intrinsic decoherence induce by gravity on itself.

Part II

Entanglement and Decoherence in Quantum Gravity

The project

Now that we have reviewed all the necessary background physics, we can go deep into the core work of this thesis. Here is a summary of the story. In the loop quantum gravity framework and more generally in all proposals of quantum gravity theories, we extracted two main issues: the problem of localization or defining notions of reference frame, distance, locality and dynamics and the the problem of the quantum to classical transition. The idea of the present work is to give some toy models and ideas that clarify those fundamental issues from a quantum information and condensed matter perspective. The general flow is to go from a generic analysis of the geometry of space from spin network states toward the physical properties of subsystems and their boundary. Crudely presented, the first two chapters are oriented on the problem of localization while the last two are oriented on the quantum to classical transition.

We begin by investigating in chapter 4 how we can reconstruct geometry (notion of distance, locality) from correlations of spin network states. By constructing an ansatz class of physical states inspired by condensed matter and statistical physics, we are able to test this proposal in a well controlled environment, discuss the role of phase transition and the consequences for the dynamics in loop quantum gravity.

From general relativity, we know that the entropy of a black hole scales as the area of its boundary. This result has been raised to a fundamental principle in quantum gravity. The following chapter 5 analyzes a class of holographic states. Holography is here understood as a consequence of the entanglement entropy between a partition of the spin network. The source of this non trivial entanglement comes from holonomies crossing the boundary which can be interpreted as the presence of boundary degrees of freedom when the environment degrees of freedom have been traced over.

We move then to the study of boundaries and their interaction with the bulk in loop quantum gravity. Chapter 6 proposes an analysis of the semi-classical dynamics of a quantum surface. A surface is modeled as a collection of degrees of freedom (called punctures) with a relational structure induced by the complete spin network it is embedded in. The first dynamics we investigate are open dynamics where a relaxation process of a suitable geometrical quantity (the closure constraint) can occur or not. The second kind of dynamics is in the isolated regime where the boundary degrees of freedom interact only with each other. Once again, phase transitions are present and the response to local

perturbations is analyzed.

Finally, we end this work by an analysis of decoherence theory in quantum gravity. We investigate decoherence of boundaries induced by the bulk-boundary coupling. We propose a natural model where the environment (composed of all the degrees of freedom of the rest of the Universe) can deform the surface while, for now, conserving the total area. We are able to confront the different approaches to open quantum systems, analyze their domain of validity, and understand in this particular model some physical properties of pointer states.

Chapter 4

Toward geometry from correlations

Loop quantum gravity is a proposal for a non perturbative theory of quantum gravity. The kinematical states given by spin network states describe quantum geometry. They are not embedded in space but define space itself. But we must not jump to the conclusion that “distant” parts of the graph describe far away regions because of diffeomorphism invariance. How then is the notion of distance, systems close to each other, recovered from the algebraic data of the spin network state? More generally how can we reconstruct geometry from the quantum state in a background independent theory? Developing such a notion of distance is important to define properly coarse-graining procedure not relying on an a priori background but only from the spin network state itself. The emergent semi-classical geometry would then allow to better understand the quantum to classical transition.

In the previous chapters, we began to understand that there exists a deep relation between gravitation and quantum information properties, at least to formulate paradoxes. But we can go further than that. For instance, in a quantum gravity context and in loop quantum gravity, it has been suggested that entanglement entropy between the interior and exterior of a spatial region should explain black hole entropy. In fact, this result should be valid for any region, not specifically for black hole geometry. Similar conclusions are drawn for instance in the AdS-CFT correspondence approach to quantum gravity [[Ryu and Takayanagi, 2006](#); [Nishioka et al., 2009](#)].

In fact, as we saw from the thermodynamics reconstruction of Einstein equations and the fact that general relativity is fundamentally a theory of correlations, it has been suggested [[Van Raamsdonk, 2010](#); [Bianchi and Myers, 2014](#)] that entanglement, correlations and entropy could be used to reconstruct the whole spacetime. This is the basic proposal we want to explore in the context of loop quantum gravity by building toy models implementing this idea. The fact that we are limited to toy models is that the full quantum theory is still out of reach. The idea is then to explore the consequences and draw some lessons for the full quantum gravity theory concerning the form of the Hamiltonian constraint, the physical states or the role of phase transitions.

The chapter is organized as follows. Section 4.1 discusses in more details the proposal at a purely conceptual level, trying to clarify some natural questions. Section 4.2, based on our published work [Feller and Livine, 2016], develops a toy model in loop quantum gravity from a condensed matter perspective. Finally Section 4.3 sums up the situation and proposes some directions to develop this idea further in loop gravity to make contact with Einstein equations.

4.1 Reconstructing geometry from correlations

4.1.1 How entanglement and geometry can be related?

The vacuum of a quantum field theory (on a fixed background) is a very entangled state, with correlations at arbitrary small distances. This is a non-trivial aspect of those theories if we have in mind the structure of the ground state of a non relativistic quantum mechanical system where the ground state is just the product of the lowest energy state of its constituents. Now, let's imagine an experiment that disentangles a given region of space. Consequently, we end up in an excited state of the theory. This will cost some amount of energy. In a gravitational context, all form of energies will contribute to the curvature of spacetime and change its geometry. The disentangling procedure creates curvature and modify the distances between regions.

Having that in mind, it is tempting to push further the logic and investigate, in a full quantum gravity context like the loop approach, the idea of reconstructing geometry from correlations and entanglement. The goal would then be to understand the quantum metric induced by a spin network state, without any reference to a background manifold. We stress again that spin networks define space and do not live in it. Two regions of the spin network would be considered close when correlations are strong and far away otherwise. Factorized states would correspond to independent regions of space, like two universes evolving independently.

Remembering again field theories 1.4, we know that the two points correlation functions $\langle 0 | \hat{\phi}(x, 0) \hat{\phi}(y, 0) | 0 \rangle$ depend on the distance between x and y and have at long distance an algebraic decay. Condensed matter systems, like spin systems which are quite similar to spin network states, provide another example of algebraic correlations depending on the phase of the system. The idea would then simply be to invert such a relation, meaning that we would define a notion of distance $d(x, y)$ between points from the correlations itself.

One critique of this proposal that could be formulated concerns Bell pairs with two systems being in a maximally entangled state while being far away from each other that can be produced in laboratories on demand. First, if we have in mind matter degrees of freedom in a Bell pair, such pairs evolve in an already defined geometry while we are talking about correlations for quantum gravity degrees of freedom. Then, thinking about Bell correlations for those degrees of freedom will be the analogous of wormholes in a geometric view. This is the direction taken for the $ER = EPR$ idea. Another critique

that we could hastily formulate is that we could imagine states where every regions are perfectly entangled with all the others leading to a trivial geometry (everything at the same place). Such a situation is however forbidden by quantum physics and is known as the monogamy of entanglement: it is not possible for a system to be perfectly entangled with two others.

In our presentation of entanglement, we made a clear distinction between locality and the separability of the Hilbert space (the fact that we can define subsystems). In the context of quantum gravity and the proposal of reconstruction of geometry from correlations, we see that those two notions become in fact indistinguishable: it has no meaning to speak about locality without entanglement/subsystems and entangled states (non-trivial states) of subsystems leads to a notion of locality.

4.1.2 Toward a relative locality?

A recent idea that has been develop into the relative locality framework is that the notion of locality, distance or more generally spacetime is a relative notion that depend on the observer [Amelino-Camelia et al., 2011]. Our standard framework assumes that once an observer reconstructs its surrounding geometry from physical measurements, every observer performing their own measurement will reconstruct the same geometry. The full structure is what we call spacetime and is the same for every observer. The relative locality proposal challenges this last aspect and states that the reconstructed spacetime depends on the observer and that different observer can reconstruct different spacetimes. This of course would be observable at only some high energies and we would recover an effective absolute spacetime at low energy scales.

In some sense, the geometry from correlations proposal implements this idea in quantum gravity. Indeed correlations depend on the observer. It is quite simple to illustrate this. Consider the situation where we have a system $S = \bigcup_i S_i$ composed a many subsystems S_i and an observer O embedded in a environment (the rest of the degrees of freedom in the Universe). Some state $|\psi\rangle$ in the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_O \otimes \mathcal{H}_E$ is then decomposed in some basis of the system, the observer and the environment. To simplify the discussion (not write sums) let's look at a decohered state $|\psi\rangle = \sum_s a_s |s\rangle |O_s\rangle |E_O\rangle$. We have perfect correlations between the state of the system and the one of the observer. The relative state between the system and the observer is $|\psi(S|O_s)\rangle = |s\rangle$. We could suppose that the state of the system is factorized $|s\rangle = \otimes_i |s_i\rangle$ giving a trivial geometrical picture as a collection of independent spaces. Now consider another observer O' looking at the system S . The total Hilbert space is now decomposed into $\mathcal{H}_S \otimes \mathcal{H}_{O'} \otimes \mathcal{H}_{E'}$. The observer O' has its preferred basis $|O_{s'}\rangle$. We can then expand $|O_s\rangle = \sum_{s'} t_{ss'} |O_{s'}\rangle$. Then the state of the system relative to O' is $|\psi(S|O_{s'})\rangle = \sum_s a_s t_{ss'} |s\rangle$. This observer now sees a quantum superposition and entanglement between the different subsystems S_i . She will reconstruct a spacetime with a non-trivial geometry coming from the non-trivial correlations.

We mentioned that such a sharp difference for different observers should not be observable at our energy scales. While we cannot clarify much this distinction yet, we

already discuss this issue of consensus between different observers in the context of the quantum to classical transition with quantum Darwinism 1.2.3. We stated that consensus between observers is reached for pointer states. Translated into the gravity framework, it amounts to say that when a semi-classical geometry emerges, it would not be possible to measure the relative locality aspects and we would recover an effective absolute spacetime. To measure the relative character, we need to have access to enough information to reconstruct the quantum correlations. This means for instance to be able to probe high energy physics since, if disregarded, those high energy degrees of freedom would act as an environment leading to decoherence.

While this discussion is still at an informal level, we can understand that this information theoretic reconstruction point of view can enable us to build some bridges with other proposed ideas and fields of quantum physics.

4.1.3 Trivial correlations of spin network states

Going now into loop quantum gravity, let's remind that the kinematical states are spin network states defined on a graph dressed with spins on the edges and intertwiners at the vertices. Those data carry geometrical information about area and volume. More explicitly, the spin network state $|\Gamma, j_e, i_v\rangle$, based on the graph Γ and dressed with spins j_e and intertwiners i_v , defines a wave function $\psi(g_e)$ on the space of discrete connections of $SU(2)^E/SU(2)^V$

$$\psi^{\{j_e, i_v\}}(g_e) = \text{tr} \bigotimes_e D^{j_e}(g_e) \otimes \bigotimes_v i_v \quad (4.1)$$

where the trace contracts the intertwiners with the Wigner matrix $D^{j_e}(g_e)$ representing the group element g_e along the links between vertices. The scalar product between those states are obtained using the orthogonality relation of the Wigner matrices

$$\langle \Gamma, \bar{j}_e, \bar{i}_v | \Gamma, j_e, i_v \rangle = \prod_e \frac{\delta_{j_e, \bar{j}_e}}{2j_e + 1} \prod_v \langle j_v | i_v \rangle \quad (4.2)$$

Those spin network states, being tensor product of intertwiners, have trivial correlations. We already saw that the vacuum of a quantum field theory is a highly entangled state. In the Rindler form (1.30), the vacuum is seen as a quantum superposition of Fock states living on both Rindler wedge with a Boltzmann like amplitude, $|0\rangle = \prod_i \sum_{n=0}^{+\infty} e^{-\beta_U \hbar \omega_i} |n_i^L\rangle |n_i^R\rangle$. This global pure state gives a thermal behavior for a subsystem giving back the Hawking radiation effect. Having this structure in mind, what we intend to do is to construct states as superposition of those basis elements 4.1, with a Boltzman like amplitude that matches to some known condensed matter models with non-trivial correlations and then understand its implication for quantum gravity.

4.2 Ising spin network state

The purpose of the following model is to explore this geometry from correlations idea in a well controlled toy model in loop quantum gravity. The idea is to develop a condensed matter point of view, where many models are well understood, to extract geometrical data out of spin network states. Questions regarding physical states, their structure, the Hamiltonian constraints they satisfy or phase transitions can be addressed. Since the full theory is not yet known, as any theory of quantum gravity, toy models investigations like this are intended to illustrate generic ideas about how a quantum theory of gravity should work. The lessons extracted should serve as guiding principle for future investigations.

Like just mentioned, physical solutions satisfying the Hamiltonian constraints are not known and difficulties remain even to implement this constraint properly in the quantum regime. To circumvent this issue, a different strategy is proposed: build test spin networks states on regular lattices with a natural *a priori* notion of distance and compare it with the reconstructed distance emerging from correlations.

4.2.1 Definition

Spin network states have a structure very similar to condensed matter systems. A lot is known in this field regarding correlations, phase diagram and phase transitions for a rich class of models. The simplest one is the Ising model. The idea then is to construct Ising spin network states, giving a mapping between the condensed matter model and loop quantum gravity states. We build our spin network states based on three clear simplifications:

1. We work on a fixed graph, discarding graph superposition and graph changing dynamics for now, and we will focus on working with a fixed regular lattice.
2. We freeze all the spins on all the graph edges. We fix them to their smallest possible value, $\frac{1}{2}$, which corresponds to the most basic excitation of geometry in loop quantum gravity, thus representing a quantum geometry directly at the Planck scale.
3. We restrict ourselves to 4-valent vertices, which represent the basic quanta of volume in loop quantum gravity, dual to quantum tetrahedra.

Since the spins are frozen, the only degrees of freedom left are the intertwiners living at the 4-valent vertices. They form a square lattice in 2d or a diamond lattice in 3d. For the remaining we will give details only for the two dimensional case since the results are straightforwardly generalized. Those regular lattices carry a natural *a priori* notion of distance given by the graph distance.

In this setting, the space of 4-valent intertwiners between four spins $1/2$ is two dimensional: it can be decomposed into spin 0 and spin 1 states by combining the spins by pairs, as on Figure 4.1. Different such decompositions exist and Figure 4.2 gives a graphical representation of them. There are three such decompositions, depending on which spins

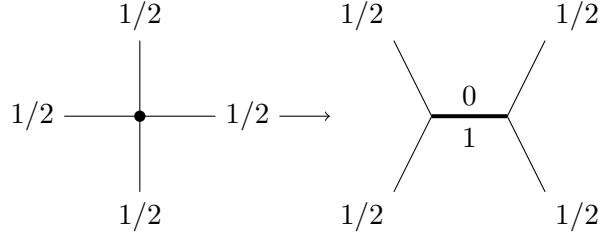


Figure 4.1: Graphic representation of the 4-valent spin $1/2$ and one of its possible decomposition into a spin 0 and 1 states.

are paired together, which we dub s , t and u channels as in particle physics and quantum field theory. Writing \mathcal{V} for the 2-dimensional Hilbert space of a spin $1/2$, we denote its basis states $(|\uparrow\rangle, |\downarrow\rangle)$. We call $|0_s\rangle$ and $|1_s\rangle$ the spin 0 and 1 states in the s channel. They can be explicitly written in terms of the up and down states of the four spins:

$$\begin{aligned}
 |0_s\rangle &= \frac{1}{2} (|\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle) \\
 |1_s\rangle &= \frac{1}{\sqrt{3}} (|\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle \\
 &\quad - \frac{|\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle}{2})
 \end{aligned} \tag{4.3}$$

Those two states obviously form a basis of the intertwiner space. We give can the transformation matrices between this basis and the two other channels:

$$\begin{aligned}
 \begin{pmatrix} |0_t\rangle \\ |1_t\rangle \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} |0_s\rangle \\ |1_s\rangle \end{pmatrix} \\
 \begin{pmatrix} |0_u\rangle \\ |1_u\rangle \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} |0_s\rangle \\ |1_s\rangle \end{pmatrix}
 \end{aligned} \tag{4.4}$$

The intertwiner basis that actually interests us is not attached to one of these channels but is defined in terms of the square volume operator \hat{U}_v of loop quantum gravity. Since the spins, and thus area quanta, are fixed, the only freedom left in the spin network states are the volume quanta defined by the intertwiners. This will provide the geometrical interpretation of our spin network states as excitations of volumes located at each lattice node. For a 4-valent vertex, this operator is defined as

$$\hat{U}_v = \left(\frac{\sqrt{2}}{3} \right)^2 (8\pi G \hbar \gamma)^3 \left(-i [\vec{J}_1 \cdot \vec{J}_2, \vec{J}_1 \cdot \vec{J}_3] \right) \tag{4.5}$$

where \vec{J}_i are the spin operators acting on the i link. For clarity, we will for the remaining of this chapter use the natural quantum gravity units for which $G = c = \hbar = 1$. This

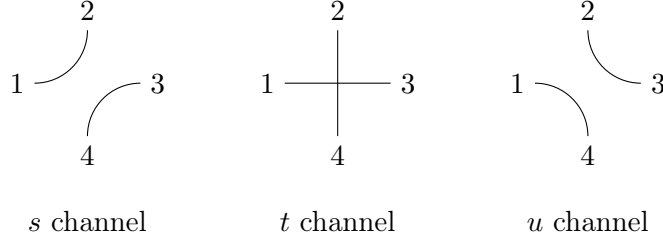


Figure 4.2: Graphic representation of the different decompositions of the 4-valent spin $1/2$ intertwiner. A spin 0 or 1 state can be associated to each of these decompositions.

operator is Hermitian but not positive; it also registers the space orientation (or more precisely the orientation of the bivector space over \mathbb{R}^3 , which is itself isomorphic to \mathbb{R}^3). The volume itself can then be obtained by taking the squareroot of the absolute value of \hat{U} . Geometrically, 4-valent intertwiners are interpreted as representing quantum tetrahedron, which become the building block of the quantum geometry in loop quantum gravity and spinfoam models [Baez and Barrett, 1999; Barbieri, 1998]. Using the change of basis relations (4.4), U_v takes the following form in the s channel basis

$$\hat{U}_v = i \frac{\sqrt{3}}{4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (4.6)$$

The eigenstates $|u_{\uparrow,\downarrow}\rangle$ can be obtained directly has $|u_{\uparrow,\downarrow}\rangle = \frac{1}{\sqrt{2}} (|0_s\rangle \mp i|1_s\rangle)$ with associated eigenvalues in Planck units $\pm \frac{\sqrt{3}}{4}$. This (square) volume is the smallest non-trivial possible value of a chunk of space. The only freedom is the orientation of the chunk of volume corresponding to the intertwiner state.

We consider these two oriented volume states $|u_{\uparrow,\downarrow}\rangle$ as the two levels of an effective qubit. Let us point out that these two level states are a priori only defined up to phases, and we could actually choose arbitrary phases $|u_{\uparrow,\downarrow}\rangle \rightarrow e^{i\theta_{\uparrow,\downarrow}} |u_{\uparrow,\downarrow}\rangle$. Choosing $\sqrt{2}|u_{\uparrow,\downarrow}\rangle = |0_s\rangle \mp i|1_s\rangle$ fixes those phases and actually selects the s -channel over the t or u channels.

We can now define a pure spin network state which maps its quantum fluctuations on the thermal fluctuations of a given classical statistical model -for instance the Ising model- by

$$|\psi\rangle = \sum_{\{\sigma_v = \pm\}} A[\sigma_v] e^{i\Theta[\sigma_v]} \psi^{\sigma_v 1/2} \quad (4.7)$$

with $A[\sigma_v] = e^{\frac{J}{2} \sum_{\langle v, v' \rangle} \sigma_v \sigma_{v'} + \frac{1}{2} \sum_v B_v \sigma_v}$

where we sum over all the possible spin σ_v configurations modulated by an arbitrary phase $\Theta[\sigma_v]$ and an amplitude $A[\sigma_v]$ which has been chosen to be an Ising nearest-neighbors

model with a coupling constant J and magnetic fields B_v . We have mapped the Ising spin $\sigma_v = \pm$ onto the space orientation of the square volume eigenvalues $|u_{\uparrow,\downarrow}\rangle$. The state $\psi^{\sigma_v 1/2}$ represents a particular configuration of the spin network and the full state is a quantum superposition of them all. Defined as such, the state is unnormalized but its norm is easily computed to be the Ising partition function Z_{Ising}

$$Z_{\text{Ising}} = \sum_{\{\sigma_v\}} e^{J \sum_e \sigma_{s(e)} \sigma_{t(e)}} = \sum_{\{\sigma_v\}} e^{J \sum_{\langle v, v' \rangle} \sigma_v \sigma_{v'}} \quad (4.8)$$

using the usual condensed matter notation $\langle v, v' \rangle$ for nearest neighbor vertices or the usual loop gravity notation $s(e), t(e)$ respectively for the source and target vertices of every (oriented) edge e of the graph. In principle we could define such states with any amplitude involving Ising spins and generalize our Ising spin network states to any other condensed matter models built from 2-level systems.

The intertwiner states living at each vertex are now entangled and carry non-trivial correlations. More precisely this state exhibits Ising correlations between two vertex i, j

$$\langle \sigma_i \sigma_j \rangle = \frac{1}{Z_{\text{Ising}}} \sum_{[\sigma_v]} \sigma_i \sigma_j |A[\sigma_v]|^2 \quad (4.9)$$

Those correlations are between two volume operators at different vertex which are in fact components of the 2-point function of the gravitational field. So understanding how those correlations can behave in a non-trivial way is a first step toward understanding the behavior of the full 2-point gravity correlations and for instance recover the inverse square law of the propagator.

We see that in order to emulate correlations of a classical statistical system, the amplitudes appearing in the pure state must be the square root of the classical Boltzmann factor. More generally, a spin dependent phase $\Theta[\sigma_v]$ could be introduced in the definition of the state itself as done in (4.7). This phase could for example consist in complex valued local magnetic fields, which occur in the Lee-Yang zeroes theorem and are relevant to some decoherence models [Wei and Liu, 2012a]. For instance, if we change the phases in the definition of the two states $|u_{\uparrow,\downarrow}\rangle$, this would clearly change the Ising spin network state defined above, but the phases would entirely be re-absorbed into the Ising Hamiltonian as purely imaginary magnetic fields. Such a phase term doesn't actually change the spin correlations, $\langle \cdot \rangle_{\Theta} = \langle \cdot \rangle_{\Theta=0}$. They will nevertheless affect the expectation values of dual operators that shift the spins and will affect the Hamiltonian constraints satisfied by the state as we will see below. For now, we consider this phase to be equal to unity for the sake of simplicity.

Similar states have been studied in a quantum information approach to condensed matter physics and appear to have nice properties with respect to entanglement entropy or even for quantum computation purposes [Verstraete et al., 2006; Wolf et al., 2008]. The spirit of this approach consists in constructing quantum states with well controlled physical properties and then understand them as a ground state of a particular dynamic. This is the same perspective we are advocating for loop quantum gravity here.

4.2.2 Low and high temperature expansion definition

In order to better understand the structure of the Ising spin networks and how to build them from the basic loop quantum gravity operators, we look at them for the perspective of the low and high temperature loop expansions of the Ising model. This is also interesting from the point of view that the duality between the loop expansions at low and high temperatures allows to characterize exactly the critical point of the Ising model, especially for the 2d square lattice, which is self-dual (see e.g. [Baxter, 1982]).

Low temperature expansion The partition function of the Ising model admits different representations which are more relevant in different temperature regimes. We restrict ourselves here to the case with no local magnetic field. In the low temperature regime, typically below the critical temperature, the ground state of the system is an ordered phase. It is then natural to extract the ground state contribution from the partition function and focus explicitly on the excitation contributions. Such a representation is called the low temperature expansion, or cluster expansion. For a planar graph, as the square lattice, it reads

$$\begin{aligned} Z_{\text{Ising}} &= \sum_{\{\sigma_v = \pm\}} e^{J \sum_e \sigma_{s(e)} \sigma_{t(e)}} \\ &= 2e^{\frac{zJN}{2}} \sum_{C \in \mathcal{C}} e^{-2JP_C} = 2e^{\frac{zJN}{2}} \sum_{\gamma^* \in \mathcal{G}^*} e^{-2JP_{\gamma^*}} \end{aligned} \quad (4.10)$$

where z is the valence or order of the graph's nodes, fixed here at $z = 4$ for the square lattice. We are summing over all clusters C , that is all subsets of vertices of the graph Γ . The expansion is obtained by distinguishing the up spins from the down spins. A cluster C is equivalent to an even subgraph $\gamma^* \subset \Gamma^*$ of the dual lattice, that is such as the valence of each node of the subgraph is even, i.e. 0 (the node does not belong to the subgraph), 2 (the subgraph goes through that node) or 4 (which is the maximal value). This even subgraph on the dual graph is simply the boundary of the cluster, as illustrated on Figure 4.3. $P_C = P_{\gamma^*}$ is the number of edges of the dual subgraph or equivalently the total perimeter of the cluster. One can decompose an even subgraph in terms possibly intersecting loops, with loop intersection corresponding to the 4-valent nodes of the subgraph.

This low temperature expansion leads to an alternative definition of the Ising spin network, which we note $|\psi_{\text{LT}}\rangle$. We start from the totally ordered state $|u_\uparrow u_\uparrow \dots\rangle$ and act with an operator that flips all spins inside a given cluster C . This reads:

$$\begin{aligned} |\psi_{\text{LT}}\rangle &= \sqrt{2}e^{JN} \sum_C e^{-JP_C} |C\rangle \\ |C\rangle &\equiv \hat{\tau}_x^C |u_\uparrow u_\uparrow \dots\rangle \equiv \prod_{v \in C} \hat{\tau}_x^v |u_\uparrow u_\uparrow \dots\rangle \end{aligned} \quad (4.11)$$

where $\hat{\tau}$ is the notation used for the Pauli matrices. $\hat{\tau}_x$ is the flip operator that sends the up spin $|u_\uparrow\rangle$ on a down spin $|u_\downarrow\rangle$ and vice versa. It can be constructed in terms of

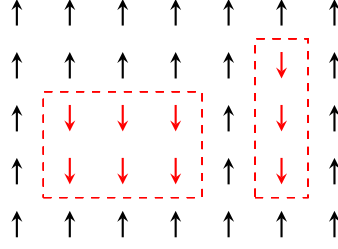


Figure 4.3: Example of cluster configuration that appears in the low temperature expansion with two spin down regions.

geometric operators, from area and scalar product operators acting on the intertwiner space.

High temperature expansion In the high temperature regime, there exist various expansions and we focus on the loop expansion of the partition function. It is constructed using the following well-known linearization identity for \mathbb{Z}_2 variables

$$e^{J\sigma} = (\cosh J + \sigma \sinh J), \quad \forall \sigma = \pm 1. \quad (4.12)$$

Expanding the nearest neighbor exponentials leads to a loop expansion of the Ising partition function

$$\begin{aligned} Z_{\text{Ising}} &= (\cosh J)^E \sum_{\{\sigma_v\}} \prod_e (1 + \sigma_{s(e)} \sigma_{t(e)} \tanh J) \\ &= 2^V (\cosh J)^E \sum_{\gamma \in \mathcal{G}} (\tanh J)^{P_\gamma}. \end{aligned} \quad (4.13)$$

We are summing over all even subgraphs γ of the initial lattice Γ . These subgraphs, as said earlier, can be seen as sets of possibly intersecting loops on the square lattice. Then P_γ is the number of edges, or equivalently the perimeter, of γ . We proceed similarly from (4.7) and we define the high temperature form of the Ising spin network state $|\psi_{\text{HT}}\rangle$ as

$$\begin{aligned} |\psi_{\text{HT}}\rangle &= \left(\cosh \frac{J}{2} \right)^{\frac{E}{2}} \sum_{\gamma \subset \Gamma} \left(\tanh \frac{J}{2} \right)^{\frac{P_\gamma}{2}} |\gamma\rangle \\ |\gamma\rangle &\equiv \prod_{v \in \gamma} (\hat{\tau}_v^z)^{\epsilon_v^\gamma} |++\dots\rangle, \end{aligned} \quad (4.14)$$

where we have switched the intertwiner basis to $|\pm\rangle = \frac{1}{\sqrt{2}}(|u_\uparrow\rangle \pm |u_\downarrow\rangle)$. We are now summing over all possible subgraphs γ (defined as arbitrary subsets of edges of the initial graph Γ) and not restricting ourselves to even subgraphs. The index ϵ_v^γ actually registers the parity of the valency of a vertex v with respect to the subgraph: $\epsilon_v^\gamma = +1$ if v is

attached to an even number of edges of the subgraph γ , while $\epsilon_v^\gamma = -1$ if it is attached to an odd number of edges. It is a non-trivial consistency check to show directly that the norm $\langle \psi_{\text{HT}} | \psi_{\text{HT}} \rangle$ is actually simply the Ising partition function. This high temperature state is constructed not from the full spin-up state but from the plus state $|+\rangle = \frac{1}{\sqrt{2}}(|u_\uparrow\rangle + |u_\downarrow\rangle)$ which actually sums over all possible combinations of spins up and down. This seems natural since the high temperature regime of the Ising model is totally disordered.

From the perspective of loop quantum gravity, the $|+\rangle$ state is actually $|0_s\rangle$ and the $\hat{\tau}^z$ operator acting at a vertex v is simply the normalized squared volume operator $\hat{\tau}^z = \frac{4}{\sqrt{3}}\hat{U}_v$. This allows to define this high temperature expansion of the Ising spin network entirely in terms of geometric operators.

4.2.3 Loop representation

Loop quantum gravity is based on the reformulation of general relativity as a $SU(2)$ gauge field theory. Its basic observables, and then operators at the quantum level, are the holonomies. From the viewpoint of the spin network wave-functions, the gauge invariance is ensured by the intertwiners. It is always enlightening to understand how to reconstruct some spin network states from the basic holonomy operators and we will investigate below how to derive such a loop decomposition for our Ising spin network states.

So the Ising spin network wave-function for a vanishing phase $\Theta[\sigma_v] = 0$ reads

$$\psi(g_e) = \sum_{[\sigma_v]} A[\sigma_v] \text{tr} \left(\bigotimes_e D^{1/2}(g_e) \otimes \bigotimes_v \sigma_v \right) \quad (4.15)$$

where $D^{1/2}(g_e)$ are the spin-1/2 Wigner matrix representation of the $SU(2)$ group element g_e associated to every oriented edge e . Changing the orientation of an edge simply switches g_e to its inverse g_e^{-1} . Using the relation between the eigenstates of the squared volume operator and the spin zero intertwiner states in the three pairing channels, $|u_{\uparrow,\downarrow}\rangle = \frac{2}{3} \left(|0_s\rangle + e^{\mp \frac{i\pi}{3}} |0_t\rangle e^{\mp \frac{i2\pi}{3}} |0_u\rangle \right)$, we obtain a decomposition of the Ising states over tessellations \mathcal{T} on the lattice:

$$\psi(g_e) = \sum_{\mathcal{T}} \sum_{[\sigma_v]} A[\sigma_v] e^{i\frac{\pi}{3}\theta(\mathcal{T}, [\sigma_v])} \prod_{\mathcal{L} \in \mathcal{T}} \chi_{1/2} \left(\prod_{e \in \mathcal{L}} g_e \right) \quad (4.16)$$

where the phase $\theta(\mathcal{T}, [\sigma_v])$ depend on both the tessellation and the Ising spins, and $\chi_{1/2}$ is the character (i.e the trace) of the fundamental representation. We call here tessellation a set of loops covering every link of the lattice once and only once (or equivalently a partition of the edge set of the lattice in terms of closed loops). Defining an auxiliary variable $t_v = 0, 1, 2$, as a value for the three channels s, t, u respectively, see figure 4.2, the phase takes a simple form, $\theta(\mathcal{T}, [\sigma_v]) \equiv \theta([t_v], [\sigma_v]) = -\sum_v t_v \sigma_v$. Here, we have omitted a factor $(2/3)^V$ in the normalization of the wave-function (4.16) coming from the decomposition of the up and down states $|u_{\uparrow,\downarrow}\rangle$ in terms of the spin-0 states in the s, t and u channels.

It is interesting to note that the amplitude associated to a given tessellation is a partition function of a 2D Ising model with local imaginary magnetic fields. For a given tessellation, or equivalently channel values t_v for every vertex, the effective Hamiltonian is

$$H_{\text{loop}} = -\frac{J}{2} \sum_{\langle v,w \rangle} \sigma_v \sigma_w - \sum_v \left(B_v + i\frac{\pi}{3} t_v \right) \sigma_v \quad (4.17)$$

An analytical solution of the 2d Ising model with magnetic fields is actually still unknown. In one dimension, for $B_v = ih$, there exist an infinite number of couples (J, h) for which the correlation length diverges [Mussardo, 2009]. In fact, studying models with imaginary magnetic fields gives information on the onset of phase transitions, relaxation or decoherence timescales [Yang and Lee, 1952; Lee and Yang, 1952; Wei and Liu, 2012b]. Indeed, the Lee-Yang theorem explores the zeros of the partition function which in turn are related to the divergence of thermodynamical quantities like the free energy.

Figure 4.4 gives an example of a particular tessellation which as defined above is a set of loops passing once and only once through each link of the lattice. As a potential candidate for the generic solutions of the Hamiltonian constraint, this loop decomposition gives us information on the loop structure of physical states. Generically, the emergence of non-trivial correlations is a consequence of the quantum superposition of all possible sets of extended loops covering the spin network. Having the loop representation, it is

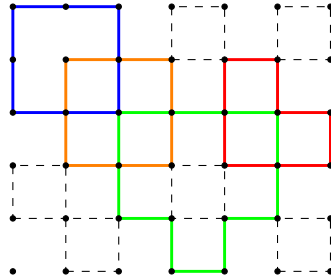


Figure 4.4: One example of a tessellation that appears in the loop decomposition of the Ising spin network state (for clarity some loops have been omitted at the boundary). The thick loops are examples of extended loops contrary to the dashed ones. It is those that are at the origin of the non-trivial correlations.

interesting to check the results mentioned in the previous section and see that, apart from a normalization factor, we have the same expressions of the norm and correlations. The method here is to write all the loop contributions as a product at each vertex v of a matrix element $\langle \sigma'_v | M | \sigma_v \rangle$. This matrix is quite reminiscent of the transfer matrix method used in statistical physics. In our case, it is proportional to the identity as is expected from the very construction of the state.

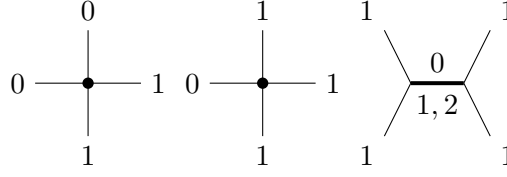


Figure 4.5: Graphical representation of the spin 1 intertwiners which are used in to extract parallel transport information from the density.

4.2.4 Hamiltonian constraints

Definition and algebra

Up to now, we have focused on defining Ising spin network states such that their n -point functions map onto the correlations of the standard Ising model and on how to generate them using geometric and holonomy operators, but they remain kinematical states of loop quantum gravity. It would be more interesting and physically relevant if we could build physical states solving the Hamiltonian constraint operators or if we could interpret our Ising spin networks as at least approximate physical states in some regime.

Here we will build some local Hamiltonian constraint operators using the basic geometric operators of loop quantum gravity, such that the Ising spin network states are their only unique solutions. Of course, these do not have any a priori link with actual gravity or any proposal of dynamics for loop quantum gravity. Moreover, we are working on a fixed graph and one might expect the quantum gravity dynamics to act on the spin network graph. Nevertheless, despite all these shortcomings, there might be some lessons to draw for such a toy construction.

First it illustrates the type of Hamiltonian constraints that would lead to Ising-like spin network states and thus to admitting non-trivial long range correlations in a continuum limit. Second we will build our constraint operators from the volume operators acting on nearest neighbor vertices, or equivalently around loop on the dual lattice. This suggest a change of perspective from the usual methods to construct regularized Hamiltonian constraint operators in loop quantum gravity, which focus on the holonomy operators around loops on the actual lattice.

Let us thus construct some Hamiltonian constraint operators characterizing the Ising spin networks. From the condensed matter perspective, a first approach could be to look at the parent Hamiltonian technique and build a Hamiltonian operator whose ground state is our considered Ising state [Verstraete et al., 2006]. We will follow a simpler path, looking at the action of the basic Pauli matrix operators on the Ising spins and using them to define simple local Hermitian Hamiltonian constraint operators characterizing uniquely our Ising states.

Let us consider the Pauli operator $\hat{\tau}_v^{x,y,z}$ acting on the Ising spin at the vertex v and

first focus on the switching operator $\hat{\tau}_v^x |u_\uparrow\rangle = |u_\downarrow\rangle$, $\hat{\tau}_v^x |u_\downarrow\rangle = |u_\uparrow\rangle$. Its action on the Ising state can be re-absorbed in the amplitude:

$$\begin{aligned}\hat{\tau}_v^x |\psi\rangle &= \sum_{\{\sigma_i\}} A[\sigma_i] |-\sigma_v\rangle \otimes \bigotimes_{i \neq v} |\sigma_i\rangle \\ &= \sum_{\{\sigma_i\}} A[\sigma_i] e^{-J \sum_{\langle v,w \rangle} \sigma_v \sigma_w} e^{-B_v \sigma_v} \bigotimes_i |\sigma_i\rangle \\ \text{with } A[\sigma_i] &= e^{\frac{J}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \frac{1}{2} \sum_i B_i \sigma_i}.\end{aligned}\tag{4.18}$$

Up to the magnetic field term, the change of sign of the Ising spin at the vertex v translates into the insertion of an additional factor involving its nearest neighbor spins, $e^{-J \sum_{\langle v,w \rangle} \sigma_v \sigma_w}$. We turn this factor in its linearized form, $e^{-J \sigma_v \sigma_w} = \cosh(J) - \sigma_v \sigma_w \sinh(J)$, which allows us to define the following Hamiltonian constraint operator at the vertex v :

$$\hat{H}_v^x = \hat{\tau}_v^x - \hat{h}_v \prod_{\langle v,w \rangle} \hat{h}_{vw}\tag{4.19}$$

where we use for simplicity the auxiliary operators ¹:

$$\hat{h}_v = \cosh B_v - \hat{\tau}_v^z \sinh B_v\tag{4.20}$$

$$\hat{h}_{vw} = \cosh J - \hat{\tau}_v^z \hat{\tau}_w^z \sinh J\tag{4.21}$$

We have just showed that these constraints annihilate our Ising spin network state:

$$\hat{H}_v^x |\psi\rangle = 0.\tag{4.22}$$

Let us remember that the operator $\hat{\tau}_v^z$ is simply the normalized square volume operator acting at the vertex v .

Similarly, the action of the other Pauli matrix operator $\hat{\tau}_v^y$ leads us to define another set of Hamiltonian constraint operators \hat{H}_v^y :

$$\hat{H}_v^y = \hat{\tau}_v^y + i \hat{\tau}_v^z \hat{h}_v \prod_{\langle v,w \rangle} \hat{h}_{vw} = -i \hat{\tau}_v^z \hat{H}_v^x\tag{4.23}$$

Since the operators $\hat{\tau}_v^{x,y,z}$ can all be constructed from the area and volume operators, those Hamiltonian constraints are entirely expressed in terms of geometric operators. This differs slightly from the usual construction of the Hamiltonian constraint operators in loop

¹ These auxiliary operators are easily invertible:

$$\begin{aligned}\hat{h}_v^{-1} &= \cosh B_v + \hat{\tau}_v^z \sinh B_v \\ \hat{h}_{vw}^{-1} &= \cosh J + \hat{\tau}_v^z \hat{\tau}_w^z \sinh J\end{aligned}$$

quantum gravity, based on Thiemann's trick [Thiemann, 1996] and which involves the holonomy operator (as regularization for the curvature) around loops of the spin network graph. Here, the product of operators $\prod_{\langle v,w \rangle} \hat{h}_{vw}$ over the nearest neighbors of the vertex v can be also considered as living on a loop but on the dual lattice, as illustrated on fig.4.6. We will discuss more this shift of perspective below in section 4.2.4.

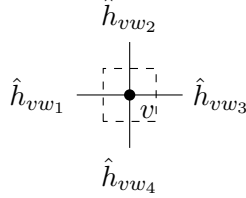


Figure 4.6: A nearest neighbor vertex operator can be seen as a loop operator on the dual lattice around the vertex v .

We note here that the possible local spin dependent phases that could be added in the definition of the state (Eq.4.7) which do not change the correlation functions would change the Hamiltonian constraints previously defined.

By construction, the Ising spin network state is solution of all these constraints, $\hat{H}_v^{x,y}|\psi\rangle = 0$. To check that there are not any other constraints, we check the algebra they generate. The constraints for two vertices which are not nearest neighbor vanish. The commutation relations on a single site v or for nearest neighbors $\langle v, w \rangle$ read:

$$\begin{aligned}
 [\hat{H}_v^x, \hat{H}_v^y] &= \hat{\tau}_v^x \hat{H}_v^y \\
 [\hat{H}_v^x, \hat{H}_w^x] &= 2 \sinh J \hat{h}_{vw} \hat{\tau}_{[v}^y \hat{H}_w^y] \\
 [\hat{H}_v^y, \hat{H}_w^y] &= 2 \sinh J \hat{h}_{vw} \hat{\tau}_{[v}^x \hat{H}_w^x] \\
 [\hat{H}_v^x, \hat{H}_w^y] &= 2 \sinh J \hat{h}_{vw} \hat{\tau}_{[v}^y \hat{H}_w^x]
 \end{aligned} \tag{4.24}$$

Every other commutation relations are zero. So the constraint algebra does not generate any further constraints satisfied by our Ising spin network states.

Our Hamiltonian constraints \hat{H}_v exist for all values of the Ising coupling J . They do not depend on the specific phase of the Ising model, it doesn't see a priori the structure of the correlations or the phase transition. We think that this setting would be the perfect testing ground for any coarse-graining scheme for spin network states in loop quantum gravity (e.g. [Livine, 2014; Livine and Terno, 2005; Dittrich et al., 2013; Dittrich, 2014; Dittrich and Geiller, 2015]). Indeed the phase diagram, coarse-graining and phase transition of the 2d Ising model is entirely under control and we know what to expect from the coarse-graining flow. The coarse-graining procedure for loop quantum gravity should reproduce the same flow on the correlations of the Ising spin network states. It

will then be enlightening to understand what happens at the level of the Hamiltonian constraint operators, how the emergence of large scale correlations is taken into account and what triggers or signals the phase transition at the critical Ising coupling.

Another approach would be to have Hamiltonian constraints specifically tuned to the coarse-graining properties of the Ising partition function, which would not determine the Ising spin network states for arbitrary values of the coupling J but that would select specifically the critical point. Such Hamiltonian operator would reflect the exact coarse-graining flow of the Ising model. This would be much more complicated to realize than our present proposal. It would either involve a graph changing dynamics or implement a self-duality property of the Ising partition function at the critical temperature, probably through the square-star or triangle-star relations [Baxter, 1982].

Unicity

So far, we have introduced a set of constraints $\hat{H}_v^{x,y}|\psi\rangle = 0$ of which the Ising spin network state is a solution. We now show that it is the unique solution of those constraints (up to a global phase factor).

The state of the intertwiners has the following general form

$$|\psi\rangle = \sum_{[\sigma_v]} \alpha_{[\sigma_v]} \bigotimes_v |\sigma_v\rangle \quad (4.25)$$

where $\alpha_{[\sigma_v]}$ are the coefficients of the state on the intertwiner basis. Imposing that the state solves the constraints $\hat{H}_v^{x,y}|\psi\rangle = 0$ leads us to a detailed balance type condition between a configuration and another with only one spin flip difference at a vertex v ,

$$\alpha_{\sigma_1, \dots, -\sigma_v, \dots, \sigma_N} = e^{-J \sum_{\langle v, w \rangle} \sigma_v \sigma_w} \alpha_{\sigma_1, \dots, \sigma_v, \dots, \sigma_N} \quad (4.26)$$

The solution of this relation is found as follows. Beginning at a particular configuration, namely the one with all spins up u_\uparrow , we get the amplitude of an arbitrary configuration by flipping the relevant spins in that balance equation. The exponential factor that appears only depends on the number of pairs of anti-parallel spins. Using the low temperature expansion notation of section 4.2.2 with down spin clusters C , we obtain that

$$\alpha_{[\sigma_v]} = \alpha e^{-JP_C}, \quad |\alpha|^2 \left(\sum_C e^{-2JP_C} \right) = 1 \quad (4.27)$$

where the amplitude α associated to the totally ordered configuration with only u_\uparrow spins is fixed by the normalization condition. We recognize the low temperature expansion of the Ising partition function. So we conclude that the amplitude is then proportional to the Boltzmann factor of the nearest neighbors Ising model with coupling $J/2$. Thus the constraints have a unique solution up to a global phase given by the Ising spin network state. We note that because of the relation between the constraints \hat{H}_v^x and \hat{H}_v^y , we only

require either one to reach this conclusion. For more complex qubit models, this might not be the case.

This discussion also enlightens the action of the Hamiltonian constraints and the dynamics they could create. Their role is to impose a detailed balance condition between two different configurations of intertwiners. Such behavior is often encountered when studying stochastic systems relaxing toward equilibrium such as for instance Glauber dynamics for Ising models which consists exactly at looking at local spin flips.

Insight for Loop Quantum Gravity

The primary purpose of our Ising spin network states is to provide a toy model framework to study coarse-graining in loop quantum gravity. Working in the controlled environment of the Ising model with its explicitly known correlations seems to be a perfect testing ground to investigate coarse-graining procedures, continuum limit definitions and phases transitions. But beyond this aspect, it turns out that they could also bring some insight into the structure of the dynamics of loop quantum gravity.

Indeed, as we have pointed out earlier in section 4.2.4, the Hamiltonian constraint operators that we introduce for the Ising states are different from the typical construction in loop quantum gravity, based on holonomy operators around loops of the spin network graph or creating such loops [Thiemann, 1996; Brunnemann and Thiemann, 2006] (also see [Livine and Tambornino, 2013] for a more recent reformulation of the holonomy and grasping operators in spinorial terms). These holonomies usually enter the constraints as a regularization of components of the curvature tensor. Here the natural structure of our Hamiltonian operators involve volume operators on dual loops (living on the dual graph) acting on all nearest neighbors of a given vertex. Although our present construction has clearly no a priori link with gravity, it seems closer to the intuition of gravitational waves deforming volumes and shapes from vertices to vertices. This suggests to look for a reformulation of the loop quantum gravity dynamics in such terms, or more generally to investigate the relation (e.g. under the form of a dual expansion) between the two types of Hamiltonian. This might help seeing gravitation waves (or at least geometry deformation waves) emerge in loop quantum gravity directly at the level of the Hamiltonian constraint algebra and not only in a large scale semi-classical limit.

From our viewpoint, this would require, first un-freezing the spins on the spin network edges to allow from holonomy operators to shift those spins, second to work out how our Hamiltonian constraints for the Ising spin network states can be re-written as some fixed-spin projection of more general operators built of holonomies. To this purpose, the decomposition of the Ising states in terms of basic holonomies might be a good starting point. Then we hope to generalize this discussion to the full loop quantum gravity framework outside the very restricted toy model of the present Ising spin network states.

4.2.5 Phase diagram, distance and coarse-graining

Orientation alignment

The effective spins we used were defined using the two eigenstates of the square volume operator which physically correspond to two different possible orientations of the fundamental volume. It is interesting to identify a regime in which all orientations would be aligned, for instance either all in the positive sector of the \hat{U}_v operators or in their negative sector. This is exactly what happens for the Ising model, in both 2d and 3d, at low temperature or equivalently at high enough coupling J in our setting. Indeed beyond the critical coupling, $J_c = \frac{\ln(1+\sqrt{2})}{2}$ for the regular square lattice [Baxter, 1982], the Ising model predicts an ordered state, with all the Ising spins align with each other. More precisely, a phase transition occurs for the Ising model from a disordered to an ordered state. Figure 4.7 shows the generic behavior of the magnetization $\langle \sigma_v \rangle$ as a function of the coupling J (see eqn.(4.29) for exact formulas in the 2d case). Above the critical coupling, the system acquires an average orientation direction, randomly picked when passing the phase transition. Of course the system is perfectly ordered only in the infinite coupling limit (zero temperature in statistical physics) and gets quantum fluctuations away from it.

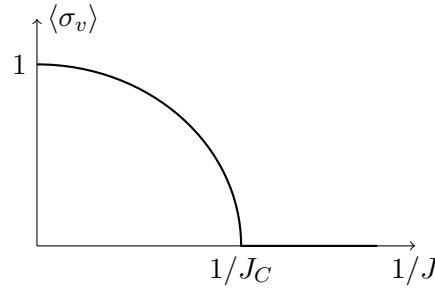


Figure 4.7: Generic representation of the behavior of the magnetization (in normalized units) as a function of the coupling passing through a second order phase transition.

As for the Ising spin network states, in the infinite coupling limit, the state reduces to the sum over the two limit states, the one where the intertwiners are all in the positive volume orientation state plus the one where the intertwiners are all in the negative volume orientation state. As the coupling decreases, opposite spin clusters appear and their typical size increases with the correlation lengths. Towards the phase transition, we have clusters of all sizes with no apparent ordering or preferred orientation. Beyond the phase transition, at lower coupling, we are in the disordered phase, with no orientation alignment. An ordered universe, with aligned volume orientation, would therefore live in the ordered phase at high coupling J (i.e low temperature).

Distance from correlations

In a non-perturbative background-independent approach to quantum gravity, the very notions of distance, locality or metric need to be reconstructed from scratch in the absence of a background geometry. Understanding the emergent geometry from the quantum state is still an ongoing issue and we hope to be able to recover all this geometric information from the correlations and entanglement in the quantum state [Livine and Terno, 2005].

So the idea we pursue here is that a notion of distance emerges from the correlations in the spin network state. Using the known results on the Ising model, we get the Ising spin network correlations in terms of the natural lattice distance and then work backwards attempting to see to what extent the distance between two spins can get be extracted from the sole knowledge of the correlations between those spins. By construction, the Ising spin networks have exactly the same known correlation behavior than the classical Ising model, namely

$$\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \propto \begin{cases} C \exp \frac{-|i-j|}{\xi}, & J \neq J_c \\ \frac{1}{|i-j|^{1/4}}, & J = J_c \end{cases} \quad (4.28)$$

where ξ is the correlation length (in units of the lattice spacing) and C a positive constant. Here the distance between the vertices $|i-j|$ is the graph distance as naturally defined on the 2d regular square lattice. The exact form of the magnetization and the behavior of the correlation length near the transition are known but won't be used in the following ².

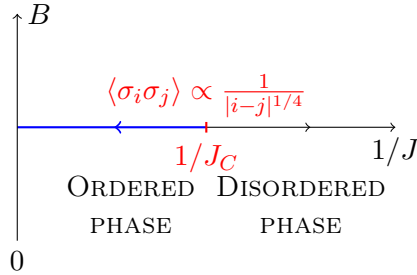


Figure 4.8: Phase diagram with renormalization flow of the Ising spin network state with the algebraic decaying correlation function at the critical coupling.

It is possible to invert the correlation formula so as to express the distance as a function of the correlation itself. The distance will be generically a monotonous decreasing function

²For reference, below the critical coupling

$$\begin{aligned} \langle \sigma_v \rangle^2 &= \left[1 - \left(\frac{1 - (\tanh J)^2}{2(\tanh J)^2} \right) \right]^{1/4} \\ \xi &\propto \left| \frac{1}{J} - \frac{1}{J_c} \right|^{-1} \end{aligned} \quad (4.29)$$

of the proper correlation between the two spins, $\langle \sigma_i \sigma_j \rangle_p \equiv \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$, but the precise function will depend on the considered phase and regime. Away from the critical point, we have an exponential decay of the correlation and we can define the distance as $d(i, j) \equiv -\xi \log \langle \sigma_i \sigma_j \rangle_p$ up to an additive constant. The correlation scale ξ becomes the new length unit. At the critical point, things become especially interesting. The correlation length blows to infinity and we have an algebraic decay of correlation. We can now define the distance as $d(i, j) \equiv 1/\langle \sigma_i \sigma_j \rangle_p^4$.

From the point of view of gravity, the most interesting case is clearly an algebraic decay of the correlations. Ideally, we'd like to derive a quadratic decay of correlation, just as in standard quantum field theory, in order to retrieve Newton's gravity law. In order to get this, we expect of course to have to change our Ising model state to another better suited statistical model, but also move to a 3d lattice or graph structures, unfreeze the spins of the spin network and so on. Nevertheless, the present Ising spin network state allows to illustrate a couple of important points

- Reconstructing the distance from the (2-point) correlation depends not only of the considered statistical model on the specific phase of that model.
- We naturally have a algebraic decay of the correlation in terms of the distance, and vice-versa, at the phase transition. At that point, the state also admits a non-trivial continuum limit.

Finally, in order to truly validate this notion of reconstructed distance, we need in general to check the triangle inequality satisfied by the distances between three points, which would involve the 3-point fluctuations. Here we know that the reconstructed distance is the initial graph distance, which satisfies this requirement. But if we would consider another spin network state, not necessarily related to a known local statistical model, this would have to be checked.

Coarse-graining

One of the main challenge of the loop quantum gravity program is to define and understand its renormalization flow, from the Planck scale to large scales and derive the low energy behavior as semi-classical general relativity with a systematic method to compute the perturbative quantum corrections. To this purpose, coarse graining procedures for the spin network states and the dynamics of the theory are essential tools [Livine and Terno, 2005; Dittrich et al., 2013; Livine, 2014; Dittrich, 2014].

Understanding coarse-graining in loop quantum gravity is not an easy task because of the absence of a background geometry or structure and the complicated nature of the Hamiltonian constraints. In the present framework of the Ising spin network state, we have fixed a background graph, to a regular 2d square lattice up to now (we will deal with its three-dimensional generalization below in the next section 4.2.5), and we can use this structure to define the coarse-graining flow as in standard statistical physics and

condensed matter models. Thus this provides a neat toy model to test some coarse-graining procedures of loop quantum gravity.

In statistical and condensed matter physics, renormalization group and coarse graining methods [Goldenfeld, 1992] are widely used to understand critical phenomena. We wish to import these methods and results to quantum gravity to understand better the emergence of critical regimes. The usual formulation of loop quantum gravity focuses on the Hamiltonian constraints, their algebra and their flow generating the time evolution. Here, for our Ising spin network states, the Hamiltonian constraints that we introduced do not see a priori the phase transition: nothing obvious happens at the level of the algebra of the constraints when the coupling reaches its critical value. The information about the critical regime and phase transition is truly in the coarse-graining flow. We could, for example, apply the tensor network renormalization tools to our Hamiltonian constraints, such as it was done for the so-called spin-net toy models in [Dittrich et al., 2013], in order to derive their coarse-graining flow and check if it fits as expected the known flow of the 2d Ising model.

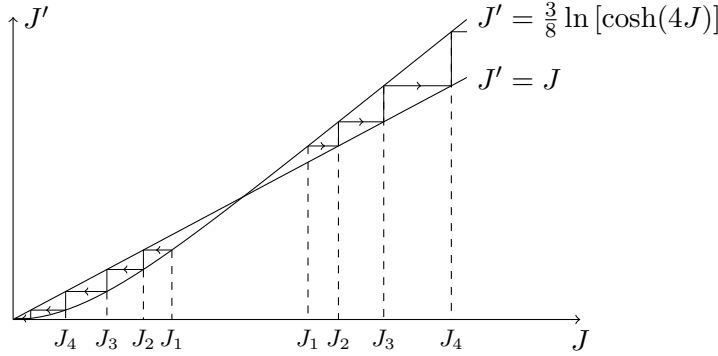


Figure 4.9: Renormalization group flow for the 2d Ising model using the method of decimation. The critical point is totally repulsive.

Since the spin correlations of the Ising spin network states are the same as the Ising model, we know their coarse-graining flow from the standard techniques in statistical physics. For instance, Figure 4.9 show its renormalization group flow using the basic isotropic decimation method: we plot the effective coupling J' at larger scale (including the nearest and next nearest neighbor interactions) as a function of the smaller scale coupling J . We can follow the renormalization flow of the effective coupling by iterating this map and the critical point is identified as its non-trivial fixed point. In the critical regime, the state presents scale invariance which renders irrelevant the micro structure of the theory and allows for a well-defined continuum limit. However, this fixed point is repulsive for the Ising model: as we coarse-grain, the effective coupling runs away from the critical regime towards either the low or high temperature fixed points. We hope to later identify a better statistical physics model such that the coarse-graining flow would run instead under the coarse-graining flow towards the critical regime with algebraic decaying correlation and

continuum limit. The corresponding spin network state, built along the same lines as our Ising spin networks, would then be better suited to the definition of the continuum limit and semi-classical regime of loop quantum gravity.

Going tridimensional

So far, we restrain the discussion of the gravitational state to a two dimensional lattice for exposition simplicity but it happens that the generalization to 3d is straightforward. Keeping the requirement that the lattice be 4-valent (so as to keep a two-dimensional intertwiner space and the map to effective qubits), the natural regular lattice is the diamond lattice as illustrated on Fig.4.10. Under the usual geometrical interpretation of loop quantum gravity, this lattice can be seen as dual to a triangulation of the 3d space in terms of tetrahedra dual to each vertex. This can be seen as an extension of the more used cubic lattice better suited to loop quantum gravity. The Ising spin network state and

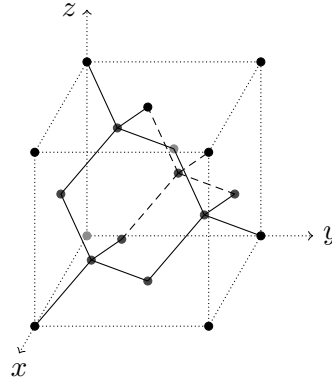


Figure 4.10: Elementary cell of the diamond lattice which is the natural one in 3d in loop quantum gravity for a 4-valent spin network.

the whole set of results which followed are then identical : the wave function eqn.(4.16), the Hamiltonian constraints and their algebra eqn.(4.24) are the same.

In 3d, the Ising model also exhibits a phase transition. Even if the exact solution is still unknown and is the subject of active research, much can be learnt about this behavior around the critical coupling. Like its 2d counterpart, the Ising spin network state has an ordered phase which corresponds to an orientation alignment of the elementary chunk of space. Concerning the 2-points correlation functions, information about its long distance behavior at the phase transition or near it can be obtained using methods of quantum field theory. In d dimensions we have [Baxter, 1982]

$$\langle \sigma_i \sigma_j \rangle \propto \frac{1}{(2\pi)^{d-2}} \frac{1}{(|i-j|)^{d-2}} K_{\frac{d-2}{2}} \left(\frac{|i-j|}{\xi} \right) \quad (4.30)$$

where $K(r)$ are modified Bessel functions and ξ is the correlation length. We mention that the correlation does not depend in the long distance regime on the "magnetic field" B (even local ones B_i if we choose to define the state in this way). For the three dimensional case, we have the simple and exact expression

$$\langle \sigma_i \sigma_j \rangle \propto \frac{1}{4\pi|i-j|} e^{-\frac{|i-j|}{\xi}} \quad (4.31)$$

Figure 4.11 represents the phase diagram of the 3d Ising spin network (in the presence of a "magnetic field"). At the critical coupling, the correlations have a one over the spacing power law decay at long distance, still not the inverse square law but less exotic than the $2d$ behavior. Understanding distance from correlations in 3d is much more vital than in the $2d$ setting since in 3d there exist loops that can wind around a vertex without visiting other distant vertex (in a relational sense).

The expression for the correlation is obtained in a mean field setting where only a correlation length appears. However statistical systems possess another length scale, the lattice spacing, which leads to correction of this expression through the anomalous dimension. In quantum gravity the Planck length (or whatever fundamental length scale a complete quantum theory of gravity will predict) that plays this role and it would be interesting to understand its influence on the behavior of the correlations.

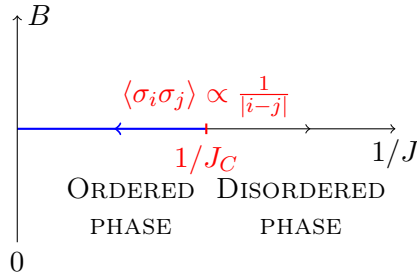


Figure 4.11: The 3d phase diagram of the spin network state on the honeycomb lattice has a phase transition whose correlations behave with a one over the separation law.

4.3 Conclusion

The model that we constructed in the context of loop quantum gravity using the Ising model as reference can be used as a toy model to test coarse-graining methods in loop gravity and to discuss the role of phase diagram and phase transition. What's more, we used it to illustrate the distance from correlations point of view, a necessity in a background independent context where geometry must emerge from the quantum state. We were able to define a nice notion of distance depending on the phase with our ansatz Ising spin network state. We were also able to discuss a set of Hamiltonian constraints

that must be satisfied by our state, the algebra they satisfy and their evolution under coarse-graining.

To deepen our understanding of the distance from correlations, a first step would be to go beyond the 2-points correlation functions and look for instance at many-body correlation functions. By checking for instance triangular inequalities, we could better understand in what extent we recover a metric structure. A strategy similar to the one developed in [Cao et al., 2017] can be followed. This time instead of focusing on a particular set of observables and their correlations, we could use the mutual information (for a holographic state) between two regions $I(A, B)$ of the spin network. It in fact provide an upper bound on the correlations between any operators of regions A and B

$$\frac{(\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle)^2}{2\|O_A\|^2\|O_B\|^2} \leq I(A, B) \quad (4.32)$$

It can be used to construct an effective “metric” graph between the subsystem. Then considering a small perturbation of the state, the variation of entropy can be link to a variation of energy $\delta S = \delta \langle K \rangle$ (relation known as the entanglement first law) where K is known as the modular Hamiltonian defining in all generality as $\rho = e^{-K}$ for any density matrix. This variation has then to be linked to a curvature variation in the geometric picture [Jacobson, 2016]. Such a procedure could be tested in loop quantum gravity.

Chapter 5

Area law from entanglement entropy

Up to now, we have explored the physics of the whole space and have discussed the idea of reconstructing the notion of distance from correlations of the spin network state. This was the first chapter of the story. From now on, we will focus our attention to regions of space. This is the natural way to go. As we saw in different sections, the notion of subsystems, being fundamental to even do science at all, is crucial to speak about entanglement and understand the emergence of classical physics. Admitting that we have a proper notion of subsystem or region of space, the first thing we have to understand is how the area law entanglement entropy comes out of the quantum theory.

The area law of black hole entropy and its universality¹ is a central prediction of general relativity. Its implications are profound. We saw that from it we can rederive the Einstein equations from thermodynamical arguments or that it questions the notion of locality we are accustomed to in quantum field theories and is embodied by the holographic principle. Yet its meaning is still unclear and it is expected from any quantum theory of gravity to clarify it.

Area laws also appear in different areas of physics. In condensed matter for instance, the ground state of a local gapped Hamiltonian fulfills the area law entropy. Holographic states represent in fact a very specific non-trivial class of states in the full Hilbert space of a quantum mechanical system. Indeed, for a random state in the Hilbert space, the entanglement entropy of a subregion scales as the volume of that region, not the area of the boundary. This by the way illustrates the restrictive power of the holographic principle and the fact that holography may not be only a kinematical property of a model but also a dynamical one.

Once again, since the full theory of loop quantum gravity and especially its dynamical part are not yet properly known, the philosophy we will use to clarify this issue is through well motivated ansatz of physical holographic states. We will follow the idea that this entropy has its origin in correlations of the quantum states, coming from the entanglement

¹This law is valid for causal horizon. The universality mentioned here is that the proportionality factor between the area and the entropy is independent of the type of horizon we consider.

at the boundary.

The chapter is organized as follows. Section 5.1 will discuss area laws in different physical context and especially the standard determination in loop gravity. Section 5.2.1 will review a condensed matter model, the toric code model, which has the striking particularity of being holographic and a topological model in the same class as general relativity. Then Section 5.2 will discuss our approach per se, based on the work [Feller and Livine, shed].

5.1 Area law entropy

An area law for entanglement entropy appears now in many different areas of physics and especially in condensed matter systems. A system is said to obey an area law entanglement entropy when

$$S = \lambda A + \dots \quad (5.1)$$

where λ is a model-dependent constant independent of the area. Generically, the vacuum state of a local gapped Hamiltonian (short range interaction) fulfills this law. It is proved for one dimensional systems and strongly believed to be true for higher dimensions. We should stress that such a scaling law is non-trivial. Indeed, consider a random state (for the Haar measure) $|\psi\rangle$ in the Hilbert space $\mathbb{C}^m \otimes \mathbb{C}^n$. Then the average entanglement entropy of the first subsystem in \mathbb{C}^m scales $\langle S \rangle = \ln m - \frac{m}{2n}$ [Page, 1993]. For a typical state, the entanglement entropy scales as the dimension of the Hilbert space or if we think about a set of finite dimensional systems in a volume V , the entropy scales linearly as the volume (is extensive). Considering the Von Neumann entropy is interesting for a ground state analysis but a better measure of correlations for thermal states is with the mutual information. Again for a one dimensional system at temperature β , it was shown that the mutual information obeys, for a local Hamiltonian $H = H_A + H_B + H_{AB}$, the inequality $I(A, B) \leq \beta \operatorname{tr}(H_{AB}(\rho_A \rho_B - \rho_{AB}))$.

The area law was historically recovered in loop quantum gravity from a semi-classical input. The idea is to perform a canonical analysis of general relativity with a spacetime containing an isolated horizon. This supplementary structure introduces boundary contributions which to be more precise is a $SU(2)$ (or $U(1)$ depending on the gauge fixing [Ashtekar et al., 1998]) Chern-Simons term [Engle et al., 2010]. This translates in the quantum theory into boundary degrees of freedom, called punctures, living in a boundary Hilbert space. We here only quantize the degrees of freedom sitting at the horizon in the classical theory. The resulting theory is not in the end a sector of the full loop quantum gravity theory but is more like a toy model or a symmetry reduced model like loop quantum cosmology. The entropy is then simply the logarithm of the dimension of this Hilbert space which scales as the area or the number of degrees of freedom on the boundary. However, the proportionality factor of the semiclassical result (the density α we used in the thermodynamics argument) is not recovered from first principles and has

to be adjusted by a clever choice of the Immirzi parameter. More recent results, to be discussed in the next chapter, seems to provide a better derivation independent of the Immirzi parameter and giving back the Hawking result

While those results provide some evidence that loop quantum gravity can indeed give back the black hole area entropy, all those derivations require semi-classical input (isolated horizon boundary conditions or energy-area relation). Our purpose here is to get rid of those semi-classical inputs and look at quantum states and their entanglement entropy between a bi-partition of the graph. For BF theories, an area law for the entanglement entropy was established [Livine and Terno, 2009]. Before discussion our proposal for holographic states in loop quantum gravity, it is best to motivate our choice by giving some background on a particular condensed matter model, the toric code model.

5.2 Holographic state in LQG

5.2.1 The toric code model

The toric code model is a topological model of spin $1/2$ living on the links of a general 2D lattice. The anyonic structure of the excitations makes it useful for fault-tolerant quantum computation [Kitaev, 2003; Hamma et al., 2005]. This model can be shown to be equivalent to a BF theory on the discrete group \mathbb{Z}_2 , a highly interesting fact since gravity can be formulated as a (constrained) BF theory.

Considering a bipartite partition of the lattice, it was found that the entanglement entropy between those regions is proportional to the boundary area (plus a topological term) which is reminiscent of the holographic principle. We will review here the basic results useful for our following discussion on spin network states.

Let's define n_s and n_p the number of vertex and plaquettes respectively. The dynamic of the model is constructed with the vertex operators $A_s = \bigotimes_{j \in s} \sigma_j^x$, tensor product of Pauli matrices with the vertex s as a source, and plaquette operators $B_p = \bigotimes_{j \in \partial p} \sigma_j^z$, tensor product of Pauli matrices around the plaquette p . The Hamiltonian is then

$$H = - \sum_{s=1}^{n_s} A_s - \sum_{p=1}^{n_p} B_p \quad (5.2)$$

We stress here the fact that the plaquette and vertex operators are subject to a particular constraint $\prod_p B_p = \prod_s A_s = \mathbb{1}$. Every term commute with all the others which makes it easier to find the ground state(s) $|\psi_0\rangle$ of the system by looking at the fundamental of each operators (states diagonalizing all the operators with highest eigenvalues)

$$A_s |\psi_0\rangle = B_p |\psi_0\rangle = |\psi_0\rangle \quad (5.3)$$

The lowest energy state of the vertex operators can be seen as a gas of loops. It implements the Gauss law enforcing gauge invariance at each vertex. Taking into account the plaquette

operators, which deform smoothly a loop into another, the ground state $|\psi_0\rangle$ of the system is a superposition of all those loops with equal weights and imposes the flatness of the \mathbb{Z}_2 holonomy. Denoting by \mathcal{A} the group generated by the vertex operators, we have

$$|\psi_0\rangle = \frac{1}{\sqrt{2^{n_s-1}}} \sum_{g \in \mathcal{A}} g|0\rangle \quad (5.4)$$

In fact, because a plaquette operator only smoothly deforms a loop, the fundamental (minimum energy) subspace is degenerate, with dimension 2^{2g} for an orientable genus g surface. This degeneracy is at the heart of the topological character of this model which cannot be lifted by local perturbations.

To cast things in the proper form useful for generalizing to gravity (for the $SU(2)$ group), we write explicitly the expanded form of the ground states in terms of loops. Written in terms of projectors $(\mathbb{1} + B_p)/\sqrt{2}$, $|\psi_0\rangle = \frac{1}{\sqrt{2^{n_p+1}}} \prod_p (\mathbb{1} + B_p) |0\rangle$, the loop expansion is straightforward. It simply suffices to expand the product of operators. Thanks to the fact that $\sigma_i^2 = \mathbb{1}$, the product of two plaquette operators B_{p_1} and B_{p_2} sharing one link is equivalent to an operator on the disjoint union of the plaquettes $B_{\partial(p_1 \cup p_2)}$. We have finally the loop superposition form

$$|\psi_0\rangle = \frac{1}{\sqrt{2^{n_p-1}}} \sum_{\mathcal{C}} \bigotimes_{\mathcal{L} \in \mathcal{C}} |1_{e \in \mathcal{L}}, 0_{e \notin \mathcal{L}}\rangle \quad (5.5)$$

Here the set \mathcal{C} is the set of all configuration of non intersecting loops having no links in common.

As mentioned above, one of the interesting result of the toric code model is the area scaling law of the entanglement entropy for the ground state between a bipartite partition of the lattice [Hamma et al., 2005]. Consider a region S and its exterior E , the global system being in a ground state, see Fig.5.1. The reduced density matrix of the region S is needed to obtain the entropy. The loop structure coming from the state (5.5) is composed of three kinds of loops, those contained completely in S or E and those belonging to both. We then have the following result: the entanglement entropy associated to a given region S whose (contractible) frontier possesses n_{SE} degrees of freedom in the ground state is

$$S = n_{SE} - 1 \quad (5.6)$$

This entropy is proportional to the number of degrees of freedom on the boundary and scales as the area. The minus one is a topological contribution and is model dependent [Kitaev and Preskill, 2006].

What we intend to do now is to study in the context of loop quantum gravity the class of wavefunctions having the same loop structure as the one for the toric code model and study the entanglement entropy and correlations they contain.

5.2.2 Definition

The loop decomposition

The holographic principle is one of the few accepted feature every quantum theory of gravity should have. Simply stated, the physics of region can be totally reconstructed from a boundary theory (with one dimension less) [Bousso, 2002]. We saw that the entanglement entropy of the toric code model has this same behavior. The purpose here is then quite simple: we want to adapt the ground state structure of this model for the spin network state on a given random 2d lattice with edge degrees of freedom fixed to the fundamental spin 1/2 excitation (this condition can be relaxed by choosing any spin j).

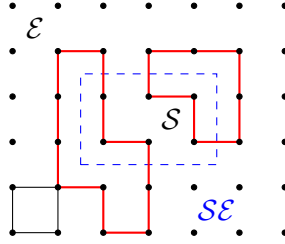


Figure 5.1: Illustration of one possible loop structure appearing in the superposition defining the Kitaev state motivated by the ground state structure of the toric code model. Three kind of loops are distinguished when a subsystem is chosen and only loops crossing the boundary give a non zero entanglement.

The natural $SU(2)$ gauge invariant object in loop quantum gravity is the holonomy, here $\chi_{1/2}(\prod_{e \in \mathcal{L}} g_e)$ for a given loop \mathcal{L} and group element $g_e \in SU(2)$ for each edge. We thus define by analogy the state which has the same loop structure than (5.5). In fact, canonical models of statistical physics such as the Ising model or $O(N)$ models, hint toward adding new amplitude contributions like a perimeter $P(\mathcal{C})$ contribution $\gamma^{P(\mathcal{C})}$ or/and a number of loops $N(\mathcal{C})$ contribution $\beta^{N(\mathcal{C})}$. So the natural general states we are interested in are given by the wave function

$$\psi_{\alpha, \beta, \gamma}(g_e) = \sum_{\mathcal{C}} \alpha^{A(\mathcal{C})} \beta^{N(\mathcal{C})} \gamma^{P(\mathcal{C})} \prod_{\mathcal{L} \in \mathcal{C}} \chi_{1/2} \left(\prod_{e \in \mathcal{L}}^{\rightarrow} g_e \right) \quad (5.7)$$

A given configuration \mathcal{C} is composed of non-intersecting loops \mathcal{L} having no links in common while $A(\mathcal{C})$, $N(\mathcal{C})$ and $P(\mathcal{C})$ are respectively the total area, the number of loops and the perimeter of the configuration and $\alpha, \beta, \gamma \in \mathbb{C}$ are complex amplitudes.

Our goal is to study this class of states, the scaling law of the entanglement entropy in a partition of the spin network and then the correlation two-points functions between spins associated to different edges. We start with the simple case $\beta = \gamma = 1$ and see that the entropy scales as expected with the number of degrees of freedom of the boundary. However, the correlations will appear to be topological, motivating the introduction of

a more general class of states with amplitudes function of the perimeter of the loops or their number.

In fact, we could have first thought of a simpler state constructed as a product of all holonomies of each plaquette (with a potential contribution from a boundary for a finite size graph) as

$$\psi(g_e) = \prod_p \chi_{1/2} \left(\prod_{e \in p}^{\rightarrow} g_e \right) \chi_{1/2} \left(\prod_{e \in \partial}^{\rightarrow} g_e \right)$$

At first sight, it would appear that such a state would display some non-trivial correlations. However this is not the case both for the holonomy and spin two-points functions in the infinite size limit.

Behavior under coarse-graining

The state $\psi_{\alpha,\beta,\gamma}$ have very nice properties under some coarse-graining procedures due the very particular loop structure we chose. For a graph Γ , one procedure is to simply eliminate a link e_0 constructing the new graph $\Gamma \setminus e_0$ and another is to pinch the link to a node defining the pinched graph $\Gamma - e_0$.

From the wave function $\psi_{\alpha,\beta,\gamma}^\Gamma$, the pure elimination of a link is done by a simple average. Here, since the loops composing the state are always non overlapping, the integration over e_0 amounts to remove all loops containing it.

$$\int_{\text{SU}(2)} \psi_{\alpha,\beta,\gamma}^\Gamma(g_{e_0}, g_e) dg_{e_0} = \psi_{\alpha,\beta,\gamma}^{\Gamma \setminus e_0}(g_e) \quad (5.8)$$

Thus the coarse-grained state corresponds exactly to the state on the coarse-grained graph $\Gamma \setminus e_0$. We have a stability under this coarse-graining procedure.

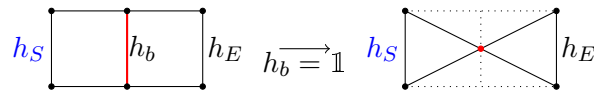


Figure 5.2: The pinch coarse-graining method is an invariant procedure only for the case $\gamma = 1$, meaning the state doesn't contain perimeter information.

The second method is to pinch the link. This is done by imposing the holonomy on e_0 to be equal to the identity. The coarse-grained state is $\psi_{\alpha,\beta,\gamma}^\Gamma(g_e) \Big|_{g_{e_0}=\mathbb{1}}$. For a given loop containing e_0 , pinching the link doesn't change the area or the number of loops, but only its perimeter. Separating configuration containing the link e_0 or not, forming respectively

the sets \mathcal{C}_0 and $\mathcal{C} \setminus e_0$, we have

$$\begin{aligned} \psi_{\alpha,\beta,\gamma}^\Gamma(g_e)|_{g_{e_0}=1} &= \sum_{\mathcal{C} \setminus e_0} \alpha^{A(\mathcal{C})} \gamma^{P(\mathcal{C})} \prod_{\mathcal{L} \in \mathcal{C}} \beta \chi_{1/2} \left(\prod_{e \in \mathcal{L}}^{\rightarrow} g_e \right) \\ &+ \gamma \sum_{\mathcal{C}_0} \alpha^{A(\mathcal{C})} \gamma^{P(\mathcal{C})} \prod_{\mathcal{L} \in \mathcal{C}} \beta \chi_{1/2} \left(\prod_{e \in \mathcal{L}}^{\rightarrow} g_e \right) \end{aligned}$$

The invariance under coarse-graining is recovered at the condition that $\gamma = 1$, meaning that the perimeter of the loops doesn't matter: $\psi_{\alpha,\beta,\gamma}^\Gamma(g_e)|_{g_{e_0}=1} = \psi_{\alpha,\beta,\gamma}^{\Gamma-e_0}(g_e)$.

5.2.3 Entanglement entropy

The next step is to compute the entanglement (Von Neuman) entropy $S = -\text{tr}(\rho_S \ln \rho_S)$ between a bipartite partition of the graph. The system \mathcal{S} of interest will be a bounded connected region and the rest of the graph forms the environment \mathcal{E} whose degrees of freedom are traced out. The boundary group elements will be by convention incorporated into the system and won't be traced over. For simplicity, we will restrict the evaluation of the entropy for $\beta = \gamma = 1$.

To compute the entropy of \mathcal{S} , we need its reduced density matrix defined as

$$\rho_S(\tilde{g}_e, g_e) = \int \bar{\psi}_\alpha(\tilde{g}_{e \in \mathcal{S}}, h_{e \notin \mathcal{S}}) \psi_\alpha(h_{e \notin \mathcal{S}}, g_{e \in \mathcal{S}}) dh_{e \notin \mathcal{S}} \quad (5.9)$$

The method to evaluate the entropy $S = -\text{tr}(\rho_S \ln \rho_S)$ is based on the replica trick [Callan and Wilczek, 1994]. Computing the successive power of the reduced density matrix ρ_S^n , $n \in \mathbb{N}$, we then obtain the entropy by $S = -\left. \frac{\partial \text{tr} \rho_S^n}{\partial n} \right|_{n=1}$.

The first step is to compute the reduced density matrix. Denoting respectively \mathcal{C}_S , \mathcal{C}_E and \mathcal{C}_{SE} the loops belonging to S , E or both, we have

$$\begin{aligned} \rho_S(g, g') &= \frac{\mathcal{N}_E(\alpha)}{\mathcal{N}(\alpha)} \sum_{\substack{\mathcal{C}_S \cup \mathcal{C}_{SE} \\ \mathcal{C}'_S \cup \mathcal{C}_{SE}}} \bar{\alpha}^{A(\mathcal{C}'_S \cup \mathcal{C}_{SE})} \alpha^{A(\mathcal{C}_S \cup \mathcal{C}_{SE})} \times \prod_{\substack{\mathcal{L}_S \in \mathcal{C}_S \\ \mathcal{L}'_S \in \mathcal{C}'_S}} \chi_{1/2}(\mathcal{L}_S(g)) \chi_{1/2}(\mathcal{L}'_S(g')) \\ &\prod_{\mathcal{L}_{SE} \in \mathcal{C}_{SE}} \left(\frac{1}{2} \chi_{1/2}(\mathcal{L}_{SE}(g, g')) \right) \end{aligned} \quad (5.10)$$

with $\mathcal{N}(\alpha)$ the norm and $\mathcal{N}_E(\alpha) = \sum_{\mathcal{C}_E} |\alpha|^{2A(\mathcal{C}_E)}$ the factor coming out of the partial trace on the environment. We see that two contributions appear, one with only loops in S and another coming from loops crossing the boundary. This last term is responsible for the entanglement between S and E .

Figure 5.3 shows an example of a configuration appearing in the reduced density matrix. To understand simply the form of ρ_S , let's imagine we have only two-loops configuration,

one copy for the bra and ket of the density matrix. Each configuration is composed of non overlapping and non intersecting loops. Nonetheless, each copy can overlap since their are independent. Now, tracing out the E degrees of freedom imposes that the parts in E from each copies to be exactly the same, otherwise the average gives zero. Complications come from loop crossing the boundary. The average of the \mathcal{E} part of crossing loops gives a contribution of the form $\int_{\text{SU}(2)} \chi_{1/2}(gh) \chi_{1/2}(g'h) dh = \frac{1}{2} \chi_{1/2}(gg'^{-1})$. This is at the origin of the boundary holonomies in (5.10). Now considering again all the allowed configurations, we see that for a given bulk/boundary plaquette choice like in Fig.5.3, there is a huge redundancy coming from the E plaquettes. After the partial trace, this leads to the overall $\mathcal{N}_E(\alpha)$ prefactor.

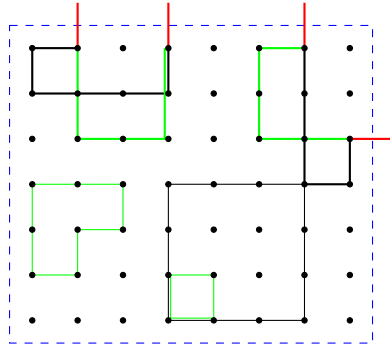


Figure 5.3: Illustration of one possible loop structure after the partial trace over the environment has been performed. The reduced density matrix is not factorized anymore and a non-trivial boundary contribution leads to entanglement.

The next step is to compute the successive power and take the trace. At the end a simple formula remains,

$$\text{tr } \rho_S^n(g, g') = \left(\frac{\mathcal{N}_E(\alpha)}{\mathcal{N}(\alpha)} \right)^n \mathcal{N}_S^{n-1}(\alpha) \sum_{\mathcal{C}_S \cup \mathcal{C}_{SE}} \frac{|\alpha^{A(\mathcal{C}_S \cup \mathcal{C}_{SE})}|^{2n}}{(4^{n-1})^{\#\mathcal{C}_{SE}}}$$

The entropy of the region \mathcal{S} is then directly obtained by differentiation and we have that the leading order term scales as the area of the boundary of the region. The entanglement entropy is given finally by

$$S = n_{SE} f(|\alpha|^2) + \frac{2 \ln 2}{(1 + |\alpha|^2)^{n_{SE}}} \sum_{\mathcal{C}_{SE}} \#\mathcal{L}_{SE} |\alpha|^{2A(\mathcal{L}_{SE})} \quad (5.11)$$

with n_{SE} the number of degrees of freedom at the boundary and $f(|\alpha|^2) = \ln(1 + |\alpha|^2) - \frac{|\alpha|^2}{1 + |\alpha|^2} \ln(|\alpha|^2)$. This formula is quite general and is valid for an arbitrary graph as long as the loop structure of the spin network state is the same.

Boundary degrees of freedom - Purification

We came to understand that the entanglement in the subsystem \mathcal{S} prepared in the state (5.10) can be traced back to loops crossing the boundary. In fact, we can understand the state (5.10) as resulting from tracing out additional boundary degrees of freedom. This idea goes in the same spirit as recent studies on local subsystems in gauge theories and gravity [Donnelly, 2008; Donnelly and Freidel, 2016; Freidel et al., 2017].

To purify the state, consider at each puncture a new degree of freedom, for instance a new fictitious edge. We work in the extended Hilbert space $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_e^{\otimes N}$ with \mathcal{H}_e the Hilbert space associated to the new edge, N the total number of puncture and $\mathcal{H}_{\mathcal{S}}$ the Hilbert space of the system. We then construct a pure state as superposition of loops in the bulk and paths joining pairs of punctures, see for instance Figure 5.3. For a path \mathcal{P} , we use the holonomy (properly oriented)

$$\chi_{1/2}(\mathcal{P}) = \chi_{1/2}(h_{s_b} g_{\mathcal{S}} h_{t_b}) \quad (5.12)$$

with h_{s_b, t_b} associated to the pair of boundary degrees of freedom, the source and target of the path respectively. The reduced density matrix (5.10) can then be purified by considering the state $|\psi_{SB}\rangle$ (B for boundary) with wave-function

$$\psi_{SB}(g) = \sum_{\mathcal{C}} \alpha^{A(\mathcal{C})} \beta^{N(\mathcal{C})} \gamma^{P(\mathcal{C})} \prod_{\mathcal{L} \in \mathcal{C}} \chi_{1/2}(\mathcal{L}) \prod_{\mathcal{P} \in \mathcal{C}} \chi_{1/2}(\mathcal{P}) \quad (5.13)$$

Then $\rho_{\mathcal{S}} = \text{tr}(|\psi_{SB}\rangle\langle\psi_{SB}|)$. We have purified the reduced density matrix of the system. One could argue that there are many ways to purify a quantum state and could question its relevance here. After all the original state (5.7) is a perfectly valid purification. What is really interesting here is the method. We can think of the local subsystem on its own by doubling the boundary degrees of freedom and construct pure state in an extended Hilbert space. The physical state is recovered by tracing out the additional boundary degrees of freedom. This matches exactly the results of [Donnelly and Freidel, 2016] by a direct analysis of the reduced density matrix of a sub-region of the spin network state. Naturally we here have no particular information on those additional degrees of freedom and they should be determined by a proper analysis of boundary terms in the classical and quantum theory.

This elementary discussion illustrates simply the fact that the extended Hilbert space method can be seen from a quantum information perspective as a clever way to purify a state of a local region and consequently why it has something to say about entanglement, correlations and entropy.

On correlations

This holographic behavior is a good sign for this class of states to be good candidates for physical states solutions of the Hamiltonian constraint of loop quantum gravity. What's more, for physical solutions, we expect the correlations between geometrical observables

to be non-trivial. This is where the limit $\beta = \gamma = 1$ fails. Indeed, the spin (or holonomies) two-points correlation functions are topological in the sense that they do not depend on the graph distance between the edges.

Let's look for instance at the spin two-points functions $\langle \hat{j}_e \hat{j}_{e'} \rangle - \langle \hat{j}_e \rangle \langle \hat{j}_{e'} \rangle$. This spin operator is defined by its action on a spin network state with the help of the Peter-Weyl theorem $(\hat{j}_e \psi)(g, g_e) = \sum_{j_e} (2j_e + 1) j_e \int \chi_{j_e}(g_e h^{-1}) \psi(g, h) dh$. The method to evaluate the averages goes as follows. First we have only the spin 1/2 component of the average that gives a non zero contribution, so that we have

$$\begin{aligned} \langle \hat{j}_e \rangle &= \frac{\int_{SU(2)} \chi_{1/2}(g_e h_e^{-1}) \psi_\alpha(g, h_e) \psi_\alpha(g_e, g) dg dh_e dg_e}{\mathcal{N}(\alpha)} \\ &= \sum_{\mathcal{C}, \mathcal{C}'} \alpha^{A(\mathcal{C})} \bar{\alpha}^{A(\mathcal{C}')} \int_{SU(2)} \chi_{1/2}(g_e h_e^{-1}) \\ &\quad \times \prod_{\mathcal{L} \in \mathcal{C}} \chi_{1/2}(h_e, g) \prod_{\mathcal{L}' \in \mathcal{C}'} \chi_{1/2}(g_e, g) dg dh_e dg_e \end{aligned}$$

We integrate over g_e . If $g_e \notin \mathcal{C}'$ the integral gives zero. Otherwise we have simply $\int_{SU(2)} \chi_{1/2}(g_e h_e^{-1}) \chi_{1/2}(g_e h) dg_e = \frac{1}{2} \chi_{1/2}(h h_e)$; substitute h_e for g_e with a factor one half. Finally, denoting by \mathcal{C}'_e a configuration of loops containing the link e

$$\langle \hat{j}_e \rangle = \frac{1}{2\mathcal{N}(\alpha)} \sum_{\mathcal{C}, \mathcal{C}'_e} \alpha^{A(\mathcal{C})} \bar{\alpha}^{A(\mathcal{C}'_e)} \quad (5.14)$$

$$\begin{aligned} &\prod_{\mathcal{L} \in \mathcal{C}, \mathcal{L}'_e \in \mathcal{C}'_e} \underbrace{\int_{SU(2)} \chi_{1/2} \left(\prod_{e \in \mathcal{L}}^{\rightarrow} g_e \right) \chi_{1/2} \left(\prod_{e \in \mathcal{L}'_e}^{\rightarrow} g_e \right) dg_e}_{=0 \text{ unless } \mathcal{L} = \mathcal{L}'_e} \\ &= \frac{1}{2\mathcal{N}(\alpha)} \sum_{\mathcal{C}_e} |\alpha|^{A(2\mathcal{C}_e)} = \frac{|\alpha|^2}{(1 + |\alpha|^2)^2} \end{aligned} \quad (5.15)$$

The explicit evaluation of $\langle \hat{j}_e \hat{j}_{e'} \rangle$ follows the same steps. Distinguishing the two cases when the spins belong to the same loop or not, see Figure 5.4, we have respectively $\langle \hat{j}_e \hat{j}_{e'} \rangle = \frac{1}{4} \frac{|\alpha|^2}{(1 + |\alpha|^2)^2}$ and $\langle \hat{j}_e \hat{j}_{e'} \rangle = \frac{|\alpha|^4}{(1 + |\alpha|^2)^4}$. In both cases, the correlation $\langle \hat{j}_e \hat{j}_{e'} \rangle = \langle \hat{j}_e \rangle \langle \hat{j}_{e'} \rangle$ is not in any way a function of the distance between the edges which is particularly clear when the edges don't belong to the same loop where the correlation is strictly zero.

From the structure of state, we should have naively expected the correlations to scale in some way as the graph distance between the edges e and e' . This is in fact not the case since the averages counts every loops meeting the edges in a democratic way (both configuration in Figure 5.4 give the same correlations). Introducing a contribution to the amplitude proportional for instance to the number of loops can be a solution to this issue. The limit $\beta = \gamma = 1$ has thus to be reconsidered to account for non-trivial correlations.

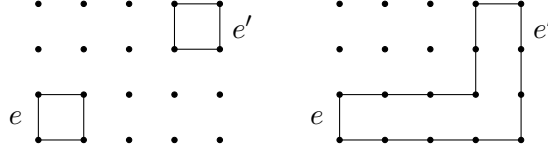


Figure 5.4: Trivial correlations arise because there is no distinction between configurations presented in the figure.

Example

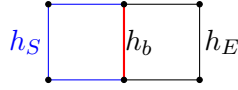


Figure 5.5: Illustrative example for the evaluation of the entanglement entropy for a two-loops state.

As an illustrative example, consider a two-loops state whose wave function is $\psi(h_S, h_b, h_E) = 1 + \alpha\chi(h_S h_b) + \alpha\chi(h_E h_b^{-1}) + \alpha^2\chi(h_S h_E)$. The reduced density matrix, obtained by taking two copies of the state and tracing out over the environment has the form

$$\begin{aligned} \rho_S(h_b, h_S, h'_b, h'_S) &= \int \psi^*(h'_S, h'_b, h_E) \psi(h_S, h_b, h_E) dh_E \\ &= 1 + \alpha\chi(h_S h_b) + \bar{\alpha}\chi(h'_S h'_b) + |\alpha|^2\chi(h_S h_b)\chi(h'_S h'_b) \\ &\quad + \frac{|\alpha|^2}{2} (\chi(h_b h_b'^{-1}) + \alpha\chi(h_b h_S'^{-1}) + \bar{\alpha}\chi(h_S h_b'^{-1}) + |\alpha|^2\chi(h_S h_S'^{-1})) \end{aligned} \quad (5.16)$$

The computation of the successive power of the reduced density matrix and the trace is then straightforward. We have $\text{tr} \rho_S^n(g, g') = \frac{(1+|\alpha|^2)^n}{\mathcal{N}(\alpha)} \left(1 + \frac{|\alpha|^{2n}}{4^{n-1}}\right)$ and the entanglement entropy follows formula (5.11).

5.2.4 Finding non-trivial correlations

Distinguishing loops

We saw in the last section why correlations were trivial for the restricted state studied for entanglement entropy. This was coming from the fact that there was no distinction between loops passing through both links or not, see Figure 5.4. To understand how the solution comes about, let's look first at a simpler state constructed as the superposition of single loop holonomy

$$\psi(g_e) = \sum_{\mathcal{L}} \alpha^{A(\mathcal{L})} \chi_{1/2} \left(\prod_{e \in \mathcal{L}}^{\rightarrow} g_e \right) \quad (5.17)$$

This term is the first non-trivial term of 5.7 in an expansion of β (for $\gamma = 1$). The spin two-points correlation function is straightforwardly evaluated as

$$\begin{aligned} \langle \hat{j}_e \hat{j}_{e'} \rangle &= \int \chi_{1/2}(g_e h_e^{-1}) \chi_{1/2}(g_{e'} h_{e'}^{-1}) \\ &\quad \psi(h_e, h_{e'}, g) \psi(g_e, g_{e'}, g) dg dh_{e,e'} dg_{e,e'} \\ &= \frac{1}{4} \frac{\mathcal{N}(e, e')}{\mathcal{N}} \end{aligned} \quad (5.18)$$

with $\mathcal{N}(e, e') = \sum_{\mathcal{L}_{ee'}} |\alpha|^{A(\mathcal{L}_{ee'})}$ is a sum over all loops passing through both edge e and e' . Now in this case, the correlations will scale non-trivially on the minimal area between the edges since we must consider loops passing through both links at the same time. Here is the main difference between this state and the previous one.

We can go even further and analyze the entanglement entropy of a local region for this state. However, the entanglement entropy doesn't follow an area law, doesn't even scale as a function of the boundary degrees of freedom. Indeed, the area scaling came from the term $\ln\left(\frac{\mathcal{N}}{\mathcal{N}_S \mathcal{N}_E}\right)$ and the multiplicative nature of $\mathcal{N} = \mathcal{N}_S \mathcal{N}_E \mathcal{N}_{SE}$ whereas for 5.17, \mathcal{N} is additive. Thus the same contribution $\ln\left(\frac{\mathcal{N}}{\mathcal{N}_S \mathcal{N}_E}\right)$ is not only a function of the boundary degrees of freedom.

The proposal

The previous discussions show that two ingredients are necessary to obtain states with non-trivial correlations and an entanglement entropy for a localized region to scale as the area of the boundary (at least to be a function of boundary degrees of freedom). Area law entanglement entropy came from the loop structure of the toric code model, more precisely all configurations of non-intersecting and overlapping loops enter the superposition. Non-trivial correlations came on the other hand from the fact that a clear distinction between loops passing by both edges e and e' and those that do not was made. The solution presents itself when we come back to our original state (5.7) with amplitude scaling as the area and the number of loops (we omit the perimeter contribution since it can obstruct coarse-graining invariance),

$$\psi_{\alpha, \beta}(g_e) = \sum_{\mathcal{C}} \alpha^{A(\mathcal{C})} \prod_{\mathcal{L} \in \mathcal{C}} \left(\beta \chi_{1/2} \left(\prod_{e \in \mathcal{L}}^{\rightarrow} g_e \right) \right) \quad (5.19)$$

The two requirements are here met. The β factor corresponds exactly to a number of loops contribution. It is straightforward to generalize the following discussion to an arbitrary superposition of holonomies $f(g) = \sum_{j=1}^{\infty} p_j \chi_j(g)$; the formal expressions remains the same as before.

Let's review its features. Concerning correlations, we can now distinguish a dominant term in the two-points function. Indeed, what rendered the correlations topological initially

was that there was no distinction between the cases when the edges e and e' belong to the same loop or different ones. With the additional β contribution, we can now pinpoint a dominant term which is the one with minimal area and only one loop connecting the edges. Denoting by L_x and L_y the horizontal and vertical graph distance respectively

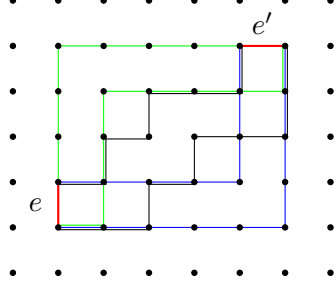


Figure 5.6: Set of possible minimal paths (not all drawn) joining the edges e and e' in red.

connecting two given links e and e' , the number of such minimal loops is $\binom{A_{\min}}{L_y}$. Thus, the dominant contribution to the spin correlations is

$$\langle \hat{j}_e \hat{j}_{e'} \rangle = \frac{1}{4} |\beta|^2 |\alpha|^{2A_{\min}} \binom{A_{\min}}{L_y} + o(|\beta|^2, |\alpha|^{2A_{\min}}) \quad (5.20)$$

$$\stackrel{N \rightarrow +\infty}{=} \frac{1}{4} |\beta|^2 (2|\alpha|^2)^{A_{\min}} \frac{e^{-\frac{(L_x - L_y)^2}{2A_{\min}}}}{\sqrt{A_{\min} \pi / 2}} \quad (5.21)$$

The correlations are now non-topological. The correlations are highest when the number of minimal paths joining the edges is maximal. This can be seen as entropic competition between the number of paths linking the edges e and e' and an energetic term $|\alpha|^{2A_{\min}}$. The more connected the edges are the more correlated they are. In the light of the distance from correlation point of view [Livine and Terno, 2005; Feller and Livine, 2016], the edges get closer when more independent minimal paths exist.

The entropy can be obtained following the same steps as presented previously by computing the successive power of the reduced density matrix and by employing the replica formula for the Von Neumann entropy. We have in the end an entanglement entropy function only of the boundary degrees of freedom,

$$\begin{aligned} S = \ln(\mathcal{N}_{SE}(\alpha, \beta)) - \frac{1}{\mathcal{N}(\alpha, \beta)} \sum_{\mathcal{C}_{SE}} [A(\mathcal{C}_{SE}) \ln(|\alpha|^2)] \\ + N(\mathcal{C}_{SE}) \ln \left(\frac{|\beta|^2}{4} \right) \Big] |\alpha|^{2A(\mathcal{C}_{SE})} |\beta|^{2N(\mathcal{C}_{SE})} \end{aligned} \quad (5.22)$$

In the special case where $\alpha = 1$ and $|\beta| = 2$, we have a very simple expression for the entanglement entropy,

$$S = \ln(\mathcal{N}_{SE}(\alpha, \beta)) \quad (5.23)$$

So in the end, we see that the state (5.19) is a good candidate to be a physical state, at least mirrors some features that true physical solution of the Hamiltonian constraint may have, since it has correlations that are function of some measure of distance in the graph through the minimal area and as an area law scaling entanglement entropy.

5.3 Conclusion

This goal of this chapter was to study a class of states in loop quantum gravity whose entanglement entropy for a bounded region scales as the area of the boundary (number of degrees of freedom) and whose correlation functions between distant spins are non-trivial. Focusing on entanglement in spin network states to recover an area law is not the standard approach in loop quantum gravity which amounts to simply evaluate the dimension of a boundary Hilbert space constructed from a semi-classical analysis. Here we started from the kinematical theory and recovered an area law for a specific class of states.

Their structure is motivated by a condensed matter model, the toric code model, where the ground state can be seen as a condensate of loops on the lattice. Our ansatz mimics this structure, being defined as a superposition of non intersecting loops of arbitrary size. To each configuration, an amplitude function of the area, the perimeter or the number of loops is considered. We showed that indeed the entanglement entropy of a region scales as the area of its boundary using the replica trick method. The source of entanglement is seen to be exclusively due to loops crossing the boundary and the fact that entanglement depends only on boundary degrees. This analysis serves also to illustrate extended Hilbert space ideas coming from research on local subsystems in gauge and gravity theories by seeing it as a clever way to purify a state. On the side of the correlations, their non triviality come from the fact that some loops are distinct from the other. What's more, we showed that correlations grow as the number of minimal paths joining the two spins is larger.

The holography we recovered here is purely kinematical, coming from the specific structure of the quantum state. The question of having a dynamics that gives rise to holographic states is still open. In fact, the condensed matter result that the ground state of a local gapped Hamiltonian is holographic while we know that not all states are strongly suggests that holography is intrinsically a property of the dynamics. A nice and new direction tries to link coarse-graining of the bulk physics into the boundary and the dynamics of loop quantum gravity [Livine, 2017]. This “dynamics through coarse-graining” would give rise to a one to one correspondence between physical states and boundary states in a dynamical manner.

Chapter 6

Surface dynamics

The last two chapters focused on the structure of the whole quantum state of space geometry in order to understand the emergence of geometrical notions from quantum information and recover holographic behavior. In fact, as we already discussed in our presentation of general relativity, the holographic principle is a guiding idea in many approaches of quantum gravity. Remember that the idea is that the whole theory can be described through degrees of freedom living on a boundary theory.

This principle deeply intertwines with the coarse-graining of geometry and renormalization of its dynamics in quantum gravity. For instance, considering a small bounded region of the 3d space which is to be coarse-grained to a single point (in a renormalisation group scheme à la Wilson), the dynamics of the bulk geometry (within that region) can be projected onto its boundary and described through degrees of freedom living on that boundary surface. These surface degrees of freedom contain all the relevant data for the interaction of that 3d region with the exterior. Then the dynamics of that surface will be understood as the renormalized dynamics of local effective degrees of freedom living at the coarse-grained point.

This underlines the importance of studying the surface dynamics in quantum gravity. At the classical level, this leads to a reformulation of general relativity in hydrodynamics and thermodynamics terms. Indeed, when considering a $2d$ surface (null, space-like or time-like), the Einstein equations reduce to the $2d$ Navier-Stokes equation and physical quantities such as viscosity or surface charge density can be defined and encode various components of the curvature tensor. This point of view has been particularly developed for black hole leading to the membrane paradigm for black hole horizons [[Damour, 1982](#); [Price and Thorne, 1986](#); [Freidel and Yokokura, 2015](#)]. Reciprocally, as we mentioned, it is possible to reverse this logic and reconstruct all of general relativity from the (thermo)dynamics of surfaces and boundaries. Following this logic, understanding surface dynamics in the quantum regime appears to be a crucial task. The purpose of this chapter is to begin to understand the dynamics of surfaces, properly defined, in loop quantum gravity. This is a huge enterprise on its own. That's why we focus first in our analysis on semi-classical

dynamics. Once again, our goal is to gain more intuition by building well motivated toy models on what a good quantum dynamics should be.

The chapter is structured as follows. Section 6.1 reviews quickly the importance of boundaries and surfaces in gravity with the appearance of boundary degrees of freedom already at the classical level and discusses the standard approaches to understand black holes (and their entropy) in loop quantum gravity. Section 6.2, based on our published work [Feller and Livine, 2017a], presents a collection of classical dynamics for surfaces with natural physical properties.

6.1 Horizon and quantum surface

6.1.1 Boundary degrees of freedom

We are now familiar with the importance of boundaries in gravitational physics from the results mentioned in the introduction above to the holographic nature of gravitation. Let's recall for instance that when a boundary is present, we have a true non zero Hamiltonian 3.1 from which notions like the energy or the angular momentum of a region can be defined. The purpose is here to give arguments for the belief that boundaries in classical and quantum gravity (or any gauge theory) carry new degrees of freedom.

The simplest way to see this is to think about an holonomy. Indeed we know that holonomies are gauge-invariant observables in a gauge theories. Said differently, they are physical degrees of freedom. But what happens when we consider a $2d$ boundary crossing a loop? The loop is now cut in half and the holonomy will be defined on a open line ending on the boundary. This is a gauge variant quantity and the end points on the boundary carry new degrees of freedom called punctures. The gauge invariance symmetry is promoted into a physical symmetry at the boundary. This is the simplest argument.

More formally, boundary degrees of freedom can become apparent at the level of the classical action combined with gauge symmetry at the boundary. By looking at the action of a gauge theory in the presence of $2d$ boundaries, gauge invariance is broken unless new terms in the action are added in order to restore the symmetry. Those terms, dictated by gauge invariance, correspond to additional degrees of freedom on the boundary. Moreover it was shown that those degrees of freedom must transform under a new symmetry called surface symmetry as a result again of gauge invariance [Donnelly and Freidel, 2016]. One physical aspect of the presence of those boundary degrees of freedom and their symmetries can be seen, in a quantum analogue, in the entanglement entropy between the bulk and the boundary where an area contribution appears.

6.1.2 Black holes in LQG

One of the first test of any quantum theory of gravity is to recover from a microscopic description the black hole entropy law. As we already discussed, the original approach in loop quantum gravity took first some classical input [Ashtekar et al., 1998] from the

isolated horizon framework 2.2.2. The presence of the boundary gives rise to new terms in the action. By performing an Hamiltonian analysis, we end up with new degrees of freedom on the boundary described by a $(U(1)$ or $SU(2)$ depending on the gauge fixation [Engle et al., 2010]) Chern-Simons term in the action. Going into the quantum regime the states of those boundary degrees of freedom live in a boundary Hilbert space. The entropy of a black hole is then simply evaluated as the logarithm of the dimension of this Hilbert space and the area law is recovered for large black holes from this simple counting.

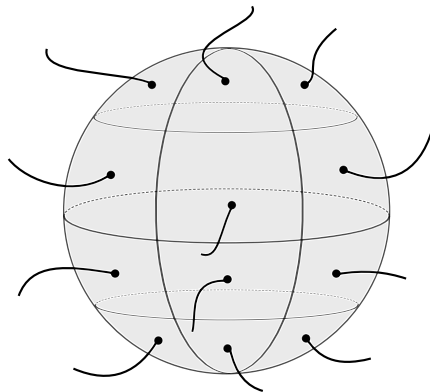


Figure 6.1: Section of an horizon in loop quantum gravity seen as a punctured surface.

In a quantum gravity context, a natural question we have to ask is what is a quantum black hole. A provocative answer would be: we do not know. In light of the evaporation process and Hawking radiation, the idea of a black hole as a region hiding information forever seems untenable and can at most be considered as a classical approximation of some specific situations but not a fundamental one. Modern quantum approaches on black holes revolve around the idea of seeing black holes as (efficient) scrambling machines. We already used this to discuss the firewall paradox. Information in a black hole is completely mixed up in a complex way, appearing for an external observer and for all practical purposes lost forever. However this is a coarse-grained view and fundamentally information can be recovered since they are in the end just quantum systems. Yet to this day there exists no proper definition a quantum system to be a quantum black hole which gives, in a classical approximation, a black hole spacetime.

Going back to entropy in quantum gravity, it in fact appears that the area law scaling is more general in loop quantum gravity than just the isolated horizon analysis. By considering the thermodynamical properties of surfaces defined as a set of N punctures and performing statistical calculations, any surface have an entropy scaling as its area. Two less stringent classical inputs are required here: the value of the temperature which is taken to be the Hawking temperature near an horizon and the form of the energy which is just proportional to the area (3.47). Since the area operator is fully known in quantum

gravity, the statistical properties can be calculated exactly in the microcanonical, canonical [Ghosh and Perez, 2011] and grand canonical ensemble [Ghosh et al., 2014; Asin et al., 2015]. Entropy scaling as the area is recovered in all cases, and questions about phase transitions or the statistics of the punctures can be studied. The following models will use the same kind of ideas except that we will have a more dynamical approach where the punctures can interact with each other or an environment.

6.1.3 Generic quantum surface

Let's define now a generic quantum surface in the context of loop quantum gravity. Consider a spin network based on a graph Γ . We consider a bounded region of space, that is a finite set of vertices together with all the edges linking them to each other. We can define such a region by considering an arbitrary embedding to the graph into the Euclidean 3d space and choosing the vertices within a region of the 3d space with the topology of a 3-ball. Then the boundary surface of the region is defined as the set of edges linking one of the region's vertex with an exterior vertex.

For a surface punctured by N edges, the quantum surface Hilbert space is the tensor product of N spins

$$\mathcal{H}_S = \bigoplus_{\{j_e\}_{e=1..N}} \bigotimes_{e=1}^N V_{j_e} \quad (6.1)$$

where V_j is the Hilbert space of dimension $\dim V_j = d_j = (2j + 1)$ associated to the spin- j representation. The quantum surface is made of N elementary surface patches, each corresponding to a single edge and carrying a spin j_e . This spin gives the elementary surface area ¹ as $a_e = \gamma j_e l_P^2$ in Planck units with Immirzi parameter γ , and the quantum state of that elementary surface patch lives in V_{j_e} . Above, we have summed all possible spins attached the N edges making the boundary surface.

We have a natural action of $SU(2)$ on that Hilbert space \mathcal{H}_S with group elements acting simultaneously on all the surface patches V_{j_e} . From the point of view of the spin network, this is a little bit more subtle to come by. Each vertex of the spin network carries an action of $SU(2)$, which corresponds to local change of frame (thought as gauge transformations). The various surface patches making the whole boundary are a priori not linked to the same vertex and thus are “desynchronized”. To synchronize their frames and have a single $SU(2)$ action on the whole boundary surface, one needs to gauge-fix the bulk and effectively reduce it to a single vertex. Following the procedure described in [Charles and Livine, 2016], one chooses a root vertex within the region and a maximal tree of the subgraph within the region: that tree will define unique paths from the root vertex to every boundary edge. Gauge-fixing the $SU(2)$ holonomies to the identity on

¹Another area spectrum for loop quantum gravity is given by the square-root of the $SU(2)$ Casimir operator, $a_e = \gamma \sqrt{j_e(j_e + 1)} l_P^2$ and differs from the simpler prescription we took by an operator ordering.

the tree edges will collapse the bounded region to a single vertex and the $SU(2)$ action around the root vertex will be directly transported to the boundary surface: this defines a common reference frame for all the surface patches on the whole surface. In some sense, the tree defines the rigid bulk structure supporting the boundary surface and identifies a growth process from the root vertex in the bulk and the dual surface surrounding it to the boundary surface of the whole region, thereby ensuring that the final surface is closed.

Decomposing the surface Hilbert space \mathcal{H}_S into irreducible representations of that global $SU(2)$ action, we write

$$\mathcal{H}_S = \bigoplus_{J \in \mathbb{N}} \mathcal{R}_J^{(N)} = \bigoplus_{J \in \mathbb{N}} \bigoplus_{C \in \mathbb{N}/2} \mathcal{R}_{J,C}^{(N)}, \quad (6.2)$$

$$\text{with } \mathcal{R}_J^{(N)} = \bigoplus_{\sum_e j_e = J} \bigotimes_{e=1}^N V_{j_e}, \quad \mathcal{R}_{J,C}^{(N)} = \mathcal{P}_C[\mathcal{R}_J^{(N)}]$$

J is the sum of all the spins j_e and gives the total surface area. C is called the closure defect. It is the spin to which all the spins j_e recouple. It is the spin of the global $SU(2)$ action on the surface, meaning that if we call $\vec{J}^{(e)}$ the $\mathfrak{su}(2)$ generators acting on each elementary surface patch, then their sum \vec{J} is of norm C

$$\vec{J} = \sum_e \vec{J}^{(e)}, \quad \vec{J}^{(e)2} = j_e(j_e + 1), \quad \vec{J}^2 = C(C + 1) \quad (6.3)$$

The projector \mathcal{P}_C enforces that recoupling condition.

If the bounded region is a single vertex v of the spin network, and thus the surface is simply the set of edges attached to that vertex, then the closure defect vanishes, $C_v = 0$. This is the intertwiner condition that ensures the gauge invariance of the spin network states. However as soon as the bounded region is made of several vertices, the closure defect C can be non-trivial and accounts for the possible excitations of the curvature within the region (defined as non-trivial holonomies around loops inside the region). And it was shown in [Freidel and Livine, 2010; Livine, 2013] that each Hilbert space $\mathcal{R}_{J,C}^{(N)}$, at fixed total area J and closure defect C , carries an irreducible representation of the unitary group $U(N)$. Of course, for a generic open surface, that is not necessarily a closed surface bounding a region of space, there are absolutely no constraints on the closure defect.

6.1.4 Spinors for Discrete Surfaces

Considering a single surface patch, the Hilbert space $\mathcal{H} = \bigoplus_j V_j$ can be seen as the quantization of a pair of harmonic oscillators, following Schwinger representation of the $\mathfrak{su}(2)$ algebra. This has led to the spinor representation of loop quantum gravity [Freidel and Speziale, 2010a; Borja et al., 2011; Livine and Tambornino, 2012; Dupuis et al., 2011].

Here we review how this provides a classical phase space description for quantum surfaces as defined above.

Considering a discrete surface made of N surface patches, we introduce for each surface patch a spinor $|z_i\rangle \in \mathbb{C}^2$ with the label i running from 1 to N . The two components of the spinors correspond to the pair of harmonic oscillators and are provided with the canonical symplectic structure

$$|z_i\rangle = \begin{pmatrix} z_i^0 \\ z_i^1 \end{pmatrix}, \quad \{z_i^A, \bar{z}_j^B\} = -i\delta^{AB}\delta_{ij} \quad (6.4)$$

Each spinor z_i defines a 3-vector \vec{X}_i by projecting it onto the Pauli matrices

$$\vec{X}_i \equiv \langle z_i | \vec{\sigma} | z_i \rangle, \quad X_i \equiv |\vec{X}_i| = \langle z_i | z_i \rangle, \quad (6.5)$$

$$|z_i\rangle\langle z_i| = \frac{1}{2} \left(X_i \mathbb{1} + \vec{X}_i \cdot \vec{\sigma} \right)$$

We also introduce the dual spinors $|z_i]$ obtained by acting on the original spinor with the $SU(2)$ structure map

$$|z_i] = \epsilon | \bar{z}_i \rangle = \begin{pmatrix} -\bar{z}_i^1 \\ \bar{z}_i^0 \end{pmatrix} \quad (6.6)$$

These dual spinors give the opposite 3-vectors

$$[z_i | \vec{\sigma} | z_i] = -\vec{X}_i, \quad [z_i | z_i] = \langle z_i | z_i \rangle \quad (6.7)$$

The 3-vectors defined by the spinors are interpreted as the flux vectors carried by the spin network and puncturing the surface. Geometrically, they give the normal vectors to the surface. Moreover a spinor carries one extra degree of freedom compared to the 3-vector. It is the phase of the spinor and encodes the extrinsic curvature angle in the twisted geometry interpretation of spin networks [Freidel and Speziale, 2010c,a].

So, as drawn on Figure 6.2, a discrete surface is described classically as set of N (flat) surface patches, each defined by its corresponding normal vector \vec{X}_i , whose norm X_i gives the area of the surface patch and whose direction $\hat{X}_i \in \mathcal{S}_2$ is orthogonal to the surface (plane). The surface total area is given by the sum of all those norms

$$A_S = \sum_i X_i = \sum_i \langle z_i | z_i \rangle \quad (6.8)$$

The components of each normal vector form a $\mathfrak{su}(2)$ Lie algebra

$$\{X_i^a, X_i^b\} = 2\epsilon^{abc} X_i^c \quad (6.9)$$

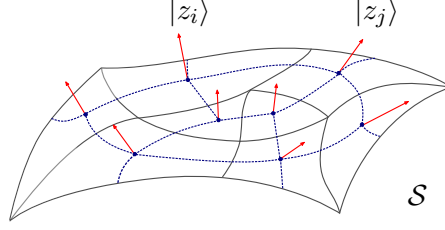


Figure 6.2: A surface \mathcal{S} constructed from N patches is defined as a collection of N spinors $|z_i\rangle$. The spinor defines the normal vector to each patch, with the norm $\langle z_i|z_i\rangle$ giving the area of the patch i . We further define a notion of locality on the surface as a graph (in dotted blue lines) linking nearest neighbor surface patches, thought of as induced by the spin network state encoding the quantum state of the bulk 3d geometry.

They generate $SU(2)$ transformation on the corresponding spinor. Now summing all those 3-vectors defines the closure defect associated to the surface

$$\vec{C} = \sum_i \vec{X}_i = \sum_i \langle z_i | \vec{\sigma} | z_i \rangle \quad (6.10)$$

It generates global $SU(2)$ transformations acting simultaneously on all the spinors

$$\{C^a, C^b\} = 2\epsilon^{abc} C^c \quad (6.11a)$$

$$\{\vec{C}, |z_i\rangle\} = i\vec{\sigma}|z_i\rangle, \quad e^{i\vec{u}\cdot\vec{C}}|z_i\rangle = g|z_i\rangle \quad (6.11b)$$

with $g = e^{i\vec{u}\cdot\vec{\sigma}} \in SU(2)$. Geometrically these are 3d rotations of the surface, rotating the normal vectors \vec{X}_i by a global $SO(3)$ transformation.

Both the total area and closure defect are encoded in a single Hermitian matrix \mathcal{X}

$$\mathcal{X} = \sum_i |z_i\rangle\langle z_i| = \frac{1}{2} \left(A \mathbb{1} + \vec{C} \cdot \vec{\sigma} \right) \quad (6.12)$$

where the vector \vec{C} is seen as a Bloch vector.

In the simplest case, when the bulk region is made of a single vertex (or more generally, when the spin network within the region is a tree, without loops) and the surface is simply the dual surface surrounding that vertex, the spinors satisfy the closure constraints

$$\vec{C} = \sum_i \vec{X}_i = 0. \quad (6.13)$$

These constraints are first class and generate the $SU(2)$ gauge transformations acting at the vertex. In that case, the symplectic quotient of the spinor phase space by the closure constraints $\mathbb{C}^{2N}/\!/SU(2)$ defines the phase space of framed polyhedra with N faces, that convex polyhedra in the Euclidean 3d space up to 3d rotations and translations and with an extra phase in $U(1)$ attached to each face (defining a local 2d frame) [Freidel and Livine, 2010; Freidel and Speziale, 2010b; Freidel and Livine, 2011b; Livine, 2013]. Those phases are the canonical variables conjugate to the faces' areas.

As soon as the region is composite, containing several vertices and at least one closed loop, the vector \vec{C} does not vanish anymore and we have a closure defect. Then the two basic global observables describing the discrete surface are the area A and the closure vector \vec{C} , which can be understood respectively as its monopole and dipole moments. When investigating global dynamics of the surface, it is natural to check how it affects these two geometric observables.

Finally, in order to go back to the quantum theory, each spinor is to be quantized as a spin and we recover spin networks and quantum surfaces defined as a collection of spins: we proceed to the canonical quantization of the spinor components, the 3-vector components X_i^a become the $\mathfrak{su}(2)$ generators J_i^a acting on the elementary surface i and the norm $X_i = |\vec{X}_i|$ becomes the spin carried by that surface patch and giving its quantized area.

6.1.5 Bulk-induced Locality as Surface Graph

We have introduced the observables E_{ij} and F_{ij} and described them as local deformation modes of the surface. This only makes sense if there is a notion of locality on the surface, for which the pair of surface patches i and j are close. In the continuum, the bulk metric for the nearby 3d geometry induced a 2d metric on the boundary surface. In the discrete setting of loop quantum gravity, we expect the spin network state, which encodes the quantum state of the 3d geometry, to induce a notion of locality on the discrete boundary surface.

Following this logic, we postulate a graph structure on the surface: we represent the elementary surface patches as nodes and draw links between nearest neighbors. Then one can use this notion of nearest neighbors to define local interactions on the discrete surface (see Figure 6.2 for an illustration), through for instance E_{ij} or F_{ij} operators acting on nearest neighbor surface patches.

One possible definition of this surface graph is to project the spin network state onto the surface: one draws a link between surface patches if the corresponding spin network edges meet at a spin network vertex in the bulk (on either side of the surface), as shown on Figure 6.3. Technically, this means that a holonomy operator acting on a loop starting at that bulk vertex and going along those two edges will act and deform that pair of surface patches. This minimalistic definition is probably naive, but it provides a first concrete proposal for the notion of bulk-induced locality on a quantum surface in loop quantum gravity.

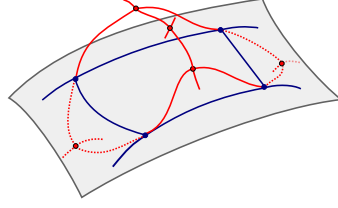


Figure 6.3: The spin network links (in red) puncture the surface and are projected onto it. They define a surface graph with links between nearest neighbor surface patches: two patches are neighbors if the corresponding spin network links meet at a vertex in the bulk.

As an example, in the simplest case that the bulk region contains a single spin network vertex, all the surrounding surface patches are considered as nearest neighbors: the surface graph is completely connected and all the observables E_{ij} or F_{ij} are legitimate local deformation modes. As soon as the bulk region will contain more spin network vertices, the notion of locality induced on the boundary surface will have to be refined and the connectivity of the surface patches will decrease.

6.2 Classical dynamics

Ideally, the dynamics of a quantum surface would be given by solving the Hamiltonian constraint of the theory. Since the implementation of the constraint algebra as quantum operators is still under active research, we adopt a more indirect point of view. We focus on a classical analysis, describing the surface classically with a spinor phase space representation [Livine, 2013] and define generic models of surface dynamics, in terms of Hamiltonians polynomial in the spinor variables, that would serve as templates for quantum dynamics in LQG. Since the spinor variables directly labels coherent spin network states, these models of classical dynamics can be further understood as defining the dynamics of coherent states of the quantum geometry. Notice that we are still working at a discrete level and that in order to recover general relativity a continuum limit is still needed.

We will consider different types of surface dynamics. We place ourselves in the logic of the renormalization of quantum gravity, where a fundamental dynamics leads to effective dynamics with various new terms, emerging at different scales and exploring various modes of deformation of the (boundary) surface geometry. We will also suppose that a notion of time t can be used. Being still at a classical setting, we can understand this time as a relational clock variable for instance.

We investigate two classes of dynamics. On the one hand, we consider global surface dynamics. In this case, we scrub the notion of locality on the surface and consider all the elementary surface patches on the same footing. This is in line with the point of view of coarse-graining spin networks: a bounded 3d region is coarse-grained to a single spin

network vertex, dressed with a closure defect [Charles and Livine, 2016] which accounts for the curvature excitations within the region. Our model of global surface dynamics will focus on the dynamics of the closure defect associated to the surface. We will introduce two different templates: a dissipative model where the closure defect relaxes towards the closure constraint (vanishing closure defect) and a Hamiltonian model of precession of the closure defect.

The second class of dynamics deals with local surface dynamics. The notion of locality materializes as the organisation of the surface patches into a network with nearest-neighbor interactions. We will introduce a generic model of isolated dynamics, where both the surface area and closure defect are invariant, defined by a Bose-Hubbard Hamiltonian with a local potential and a hopping term leading to area fluctuations on the surface. This model has a rich phase structure in the quantum regime, which will be interesting to apply to black hole horizons in loop quantum gravity.

6.2.1 Global dynamics

A surface in quantum gravity is not an isolated system. It is an object that lives in the 3d space and that is in constant interaction with the 3d geometry and its degrees of freedom. So we need to envision the surface dynamics as a system in interaction with the bulk geometry thought of as its environment. At the classical level, this leads to the possibility of a dissipative dynamics, not necessarily encoded as a Hamiltonian dynamics. And at the quantum level, this would lead to decoherence phenomena.

In this section, we will focus on models of global dynamics of the quantum surface. This is meant to be especially relevant in the context of coarse-graining loop quantum gravity: the surface bounds a region of the 3d space which is coarse-grained to a single vertex, so that the surface degrees of freedom are considered as described an effective dressed vertex of a coarse-grained spin network state. So we focus on the main two global geometric observables: the area and the closure defect. This leads us to two basic models. First, we present a dissipation model where the closure defect will relax to the closure constraint while the area decreases to a minimal value, somehow defining a notion of “rest area” for the surface [Livine, 2014]. Second, we present a forced rotation model, with a precession of the closure defect. Both models are to be thought of as effective dynamics induced by the bulk dynamics interacting with the surface.

Dissipation toward the closure constraint

Let us place ourselves in the coarse-graining scenario for loop quantum gravity: a region bounded by a closed surface is to be coarse-grained to a single vertex. In general, the composite nature of the region leads to a non-trivial closure defect for the boundary surface. This closure defect accounts for possible curvature within the region’s bulk. This is in contrast with the description of a single vertex of a spin network, which enforces

a closure constraint at the vertex. That closure constraint actually ensures that we can embed the local 3d geometry around each vertex into the flat 3d Euclidean space.

Here, we introduce a model of effective dynamics ², which relaxes a non-trivial closure defect back to the closure constraint. It can be understood as erasing and flattening the potential curvature excitations which have built in the bulk. We will implement this through a continuous Lorentz transformation, boosting along the direction of the closure defect, asymptotically leading back to the position at rest with a vanishing closure defect.

Considering a discrete surface made of N patches, we recall the Hermitian matrix \mathcal{X} encoding both area and closure defect $\mathcal{X} = \sum_i |z_i\rangle\langle z_i| = \frac{1}{2} (A + \vec{C} \cdot \vec{\sigma})$, and introduce its traceless component $c\tilde{X} = \mathcal{X} - (\text{tr } \mathcal{X}) \frac{1}{2} = \frac{1}{2} \vec{C} \cdot \vec{\sigma}$. With these notations, we define a first order equation of motion:

$$\begin{aligned} \partial_t |z_i\rangle &= -\gamma c\tilde{X} |z_i\rangle \\ &= -\gamma \sum_j \langle z_j | z_i \rangle |z_j\rangle + \frac{\gamma}{2} \sum_j \langle z_j | z_j \rangle |z_i\rangle \end{aligned} \quad (6.14)$$

with an arbitrary real parameter $\gamma \in \mathbb{R}$ setting the relaxation speed. These are non-linear evolution equations. And as we see from the definition of $c\tilde{X} \propto \vec{C} \cdot \vec{\sigma}$, the only fixed point of the evolution is when the closure defect vanishes, in which case the time derivatives vanish too.

Let us integrate the equations of motion (6.14) and obtain the explicit evolution of the discrete surface. This is not obvious, because the matrix $c\tilde{X}$ depends on the spinors themselves and the equations of motion is non-linear. The simplest route toward the solution is to identify constants of motion. It turns out that the SU(2)-invariant observables $F_{ij} = [z_i | z_j]$ all remain constant during the evolution. These are very strong constraints on the evolution. Indeed if two collections of spinors, z_i and w_i , have equal scalar products F_{ij} , then they are equal to each other up to a global Lorentz transform $\Lambda \in \text{SL}(2, \mathbb{C})$

$$\begin{aligned} \forall i, j, [z_i | z_j] &= [w_i | w_j] \\ \implies \exists \Lambda \in \text{SL}(2, \mathbb{C}) \text{ s.t. } \forall i, |w_i\rangle &= \Lambda |z_i\rangle \end{aligned} \quad (6.15)$$

This implies that the evolution of the spinors is entirely given by Lorentz transformations $\Lambda(t) \in \text{SL}(2, \mathbb{C})$ acting on their initial values

$$\forall i, |z_i(t)\rangle = \Lambda(t) |z_i(t=0)\rangle \quad (6.16)$$

This Lorentz transformation does not depend on the label i but acts globally on all the spinors. This leads to the evolution of the matrix \mathcal{X} , which contains both the area

² We do not attempt to describe the microscopic evolution of the bulk given by some exact quantum gravity dynamics, and we focus on the effective dynamics induced on the boundary.

and the closure defect

$$\mathcal{X} = \frac{1}{2} \left(A \mathbb{1} + \vec{C} \cdot \vec{\sigma} \right) = \Lambda \mathcal{X}_0 \Lambda^\dagger \quad (6.17)$$

where we write $\mathcal{X}_0 = \mathcal{X}(t=0)$ for the initial condition. In particular, its determinant $\det \mathcal{X} = \det \Lambda \mathcal{X}_0 \Lambda^\dagger = \det \mathcal{X}_0$ is constant since $\text{SL}(2, \mathbb{C})$ matrices have unit determinant. This gives an important constant of motion

$$\partial_t [\det \mathcal{X}] = \frac{1}{4} \partial_t [A^2 - \vec{C}^2] = 0 \quad \Rightarrow \quad A^2 - \vec{C}^2 = A_\infty^2 \quad (6.18)$$

where the notation A_∞ will be justified below. Furthermore, the 4-vector (A, \vec{C}) actually transforms as a relativistic vector under the $\text{SO}(3, 1)$ transformation defined by Λ .

To get the explicit evolution, we can compute the equation of motion for the full matrix \mathcal{X} which is $\partial_t \mathcal{X} = \sum_i \partial_t |z_i\rangle \langle z_i| + |z_i\rangle \partial_t \langle z_i| = -\gamma c \tilde{X} \mathcal{X} - \gamma \mathcal{X} c \tilde{X}^\dagger = -2\gamma \mathcal{X} c \tilde{X}$. Decomposing this equation onto the identity and the Pauli matrices gives the equations of motion for the area and the closure defect

$$\frac{1}{2} \partial_t A = -\gamma \vec{C}^2 = \gamma (A_\infty^2 - A^2) \quad (6.19a)$$

$$\frac{1}{2} \partial_t \vec{C} = -\gamma A \vec{C} \quad (6.19b)$$

These coupled non-linear equations can be solved using the constancy of $(A^2 - \vec{C}^2)$. The solution is the area converging to its asymptotic value A_∞ as a hyperbolic tangent and the closure defect exponentially vanishing while remaining parallel to its initial value \vec{C}_0

$$A(t) = A_\infty \coth \left(\frac{t}{\tau_r} + \alpha_0 \right) \quad (6.20a)$$

$$\vec{C}(t) = \frac{A_\infty}{\sinh \left(\frac{t}{\tau_r} + \alpha_0 \right)} \frac{\vec{C}_0}{|\vec{C}_0|} \quad (6.20b)$$

for positive times $t \geq 0$, with $\alpha_0 > 0$ giving the initial area at $t=0$ and $\tau_r^{-1} = 2\gamma A_\infty$ defining the characteristic relaxation time. This relaxation time τ_r becomes shorter as the damping rate γ is taken high or the area large.

Starting from an initial configuration (A_0, \vec{C}_0) with a non-trivial closure defect, the evolution acts as a Lorentz transformation on the relativistic 4-vector (A, \vec{C}) boosting it back asymptotically to its rest configuration $(A_\infty, \vec{0})$ with a vanishing closure defect and a rest area $A_\infty = \sqrt{A_0^2 - \vec{C}_0^2} < A_0$, as shown on the plots on Figure 6.4. The Lorentz transformation can be made explicit, re-writing the evolution in terms of a boost rapidity η

$$\begin{cases} A &= A_\infty \cosh \eta, \\ |\vec{C}| &= A_\infty \sinh \eta, \end{cases} \quad \eta = \ln \left(\frac{1 + e^{-\alpha_0} e^{-\frac{t}{\tau_r}}}{1 - e^{-\alpha_0} e^{-\frac{t}{\tau_r}}} \right) \quad (6.21)$$

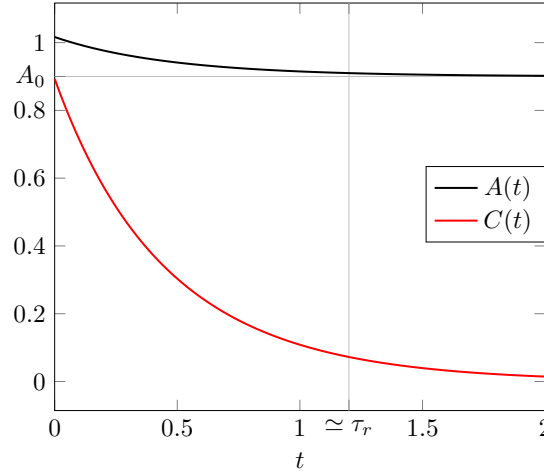


Figure 6.4: Evolution in time of the area and closure defect: the closure defect $C \equiv |\vec{C}|$ relaxes toward the closure constraint $C \rightarrow 0$ whereas the area relaxes toward the rest area A_∞ in a characteristic time τ_r .

$$\Lambda = e^{\frac{\eta}{2} \hat{C}_0 \cdot \vec{\sigma}} \in \text{SL}(2, \mathbb{C}) \quad \text{with} \quad \hat{C}_0 = \frac{\vec{C}_0}{|\vec{C}_0|} \quad (6.22)$$

The Lorentz transformation Λ is a pure boost along the direction of the closure defect, and its rapidity η asymptotically vanishes in the late time limit $t \rightarrow \infty$, allowing to recover the closure constraint $\vec{C} \rightarrow 0$. This relaxation to the rest frame is entirely a dissipative process.

To summarize, the spinors flow along a $\text{SL}(2, \mathbb{C})$ orbit from an arbitrary non-trivial closure defect back to a rest frame satisfying the closure constraint. The total boundary area also evolves towards its minimal rest area. In the context of coarse-graining spin networks, this dissipative flow erases the curvature excitations within the region's bulk.

Precessing the Closure Defect

Let us imagine a slight variation of the previous model, coupling the spinors to their dual

$$\partial_t |z_i\rangle = -\gamma c \tilde{X} |z_i\rangle \quad (6.23)$$

If the closure constraint is satisfied, $\vec{C} = 0$, then the matrix $c\tilde{X}$ vanishes too and we have a fixed point. However, this fixed point will not be attractive. Actually, this system does not relax to the closure constraint and we get a forced rotational motion of the closure defect. This precession explores a completely complementary regime to the dissipative model studied above.

In order to analyze the model's dynamics, we need to look deeper into the geometric interpretation of the spinors living on each surface patch. Each spinor $z_i \in \mathbb{C}^2$ defines the surface normal vector \vec{X}_i , but it further defines a whole orthonormal basis in \mathbb{R}^3

$$\vec{X}_i = \langle z_i | \vec{\sigma} | z_i \rangle \quad (6.24a)$$

$$\vec{Y}_i = \frac{1}{2} (\langle z_i | \vec{\sigma} | z_i \rangle + [z_i | \vec{\sigma} | z_i]) \quad (6.24b)$$

$$\vec{Z}_i = \frac{i}{2} (\langle z_i | \vec{\sigma} | z_i \rangle - [z_i | \vec{\sigma} | z_i]) \quad (6.24c)$$

where the resulting three vectors $\vec{X}_i, \vec{Y}_i, \vec{Z}_i$ have the same norm and are orthogonal to each other. While \vec{X}_i is the projection of the Hermitian matrix $|z_i\rangle\langle z_i|$ on the Pauli matrices, $(\vec{Y}_i - i\vec{Z}_i)$ is the projection of the traceless matrix $|z_i\rangle\langle z_i|$.

We introduce the sum over all surface patches of those three vectors. The sum $\vec{C} = \sum_i \vec{X}_i$ being the closure defect, while we get two new vectors, $\vec{Y} = \sum_i \vec{Y}_i$ and $\vec{Z} = \sum_i \vec{Z}_i$. Actually we can repackage all these, together with the total area, in terms of 2×2 matrices, by introducing the traceless matrix $\mathcal{Y} = \sum_i |z_i\rangle\langle z_i|$

$$\mathcal{X} = \frac{1}{2}(A\mathbb{1} + \vec{C} \cdot \vec{\sigma}), \quad \mathcal{Y} = \frac{1}{2}(\vec{Y} - i\vec{Z}) \cdot \vec{\sigma} \quad (6.25)$$

Although, for each spinor z_i , the triplet of vectors $(\vec{X}_i, \vec{Y}_i, \vec{Z}_i)$ is an orthonormal basis, the three vectors $(\vec{C}, \vec{Y}, \vec{Z})$ are a priori not orthogonal to each other. We nevertheless have the inequalities that their norms C, Y, Z are all less or equal to the total area A .

Let us assume that the parameter γ is real. The general case of a complex coupling, together with the whole details on the derivation and solution of the equations of motion, can be found in [Feller and Livine, 2017a]. For $\gamma \in \mathbb{R}$, the equations of motion read

$$\partial_t \vec{C} = \gamma \vec{Z} \times \vec{C}, \quad \partial_t A = -\gamma \vec{Y} \cdot \vec{C}, \quad (6.26)$$

$$\partial_t \vec{Y} = -\gamma A \vec{C}, \quad \partial_t \vec{Z} = 0 \quad (6.27)$$

The direction \vec{Z} remains constant during the evolution. The closure defect \vec{C} rotates around the \vec{Z} direction and no relaxation occurs. This precession dynamics is exactly the same as a spin in a constant magnetic field. \vec{Z} plays the role of the effective magnetic field and \vec{C} the role of a classical spin. In that context, such dynamics would be obtained from an Hamiltonian $H \propto \vec{C} \cdot \vec{Z}$, but the present model is more intricate and carries more degrees of freedom and can not be derived from this Hamiltonian ³. Let us turn to the evolution of the area. Its evolution is coupled to the vector \vec{Y} and we have a new invariant

³Let us consider the Hamiltonian flow generated by

$$H \equiv \gamma \vec{C} \cdot \vec{Z} = \frac{i\gamma}{2} \text{tr } \mathcal{X}(\mathcal{Y} - \mathcal{Y}^\dagger)$$

of motion, $(A^2 - \vec{Y}^2)$. This points towards a possible interpretation of this model in terms of Lorentz transformations, but we have not identified explicitly such a representation. Instead, we have looked for a closed differential equation satisfied by the area. The idea is to compute the successive derivatives of the area, and consequently of the scalar product $(\vec{Y} \cdot \vec{C})$, and finally obtain a fourth order differential equation satisfied by the area. Its solutions have four possible modes

$$A(t) = \alpha_{\pm} e^{\pm \eta t} + \beta_{\pm} e^{\pm i \omega t} + \alpha_0 \quad (6.28)$$

in terms of the roots of the corresponding quartic polynomial:

$$\eta = \frac{\gamma}{\sqrt{2}} \left[\sqrt{(C^2 - Z^2)^2 + 4(\vec{C} \cdot \vec{Z})^2} + (C^2 - Z^2) \right]^{\frac{1}{2}}$$

$$\omega = \frac{\gamma}{\sqrt{2}} \left[\sqrt{(C^2 - Z^2)^2 + 4(\vec{C} \cdot \vec{Z})^2} - (C^2 - Z^2) \right]^{\frac{1}{2}}$$

Let us keep in mind that C^2 , Z^2 and $(\vec{C} \cdot \vec{Z})$ are all constants of motion. Although the motion of the closure defect \vec{C} is purely oscillatory and periodic, the evolution of the area has exponential modes and oscillatory modes. The constants of integration $(\alpha_{\pm}, \beta_{\pm}, \alpha_0)$ depend entirely on the initial conditions ⁴ and determine which evolution modes the area actually follow. On the one hand, it is natural for the area to have an oscillatory mode, since the motion of the closure defect is also oscillatory. On the other hand, a forced rotational motion can also pulse a constant flow of energy inducing a hyperbolic trajectory. One can indeed check that both regimes are effectively realized by choosing suitable initial conditions.

To summarize, this new dynamics we have introduced may describe a complementary regime to the dissipative dynamics we defined earlier. It induces a straightforward rotation of the closure defect, without affecting its norm. If we interpret the closure defect as a measure of the local curvature or (gravitational) energy density within the region's bulk, this would model an object or region with a rotating energy-momentum. Moreover, the evolution of the surface area has two possible modes: an oscillatory mode forced by the rotation and an exponential mode leading to a hyperbolic trajectory for the area. Such

It almost leads to the equations of motion (6.23) that we postulated for the spinors

$$\{H, |z_i\rangle\} = -\gamma c \tilde{X} |z_i\rangle + \frac{\gamma}{2} (\mathcal{Y}^\dagger - \mathcal{Y}) |z_i\rangle$$

⁴ The initial conditions are the initial values of all the spinors z_i , but actually focusing on the evolution of the area and closure defect, we only need to focus on $(A, \vec{C}, \vec{Y}, \vec{Z})$ described by 10 real parameters. We have 5 constants of motion, the vector \vec{Z} , the norm $|\vec{C}|$ and the scalar product $\vec{C} \cdot \vec{Z}$, plus the 5 constants of integration parameterizing the trajectory of the area. Once the evolution of A is given, the trajectory of \vec{Y} is entirely determined.

global dynamics is very likely to be relevant to models of cosmological evolution or of astrophysical objects (with the increasing area potentially describing an exploding object or region of space), modulo of course the continuum limit.

6.2.2 Local dynamics

Up to now, we have looked into global dynamics of the surface, coupling all the surface patches together to produce a global motion for the area and the closure defect associated to the overall surface. We have explored dissipative effects and forced rotation of the surface modeling, in an effective manner, the presence of unmonitored bulk degrees of freedom thought of as the environment to the surface.

In this section, we propose to investigate local dynamics on the surface, with local fluctuations of the elementary surface patches through coupling between nearest neighbors. Moreover, since we have already introduced dynamical models for the area and closure defect, we will focus here on an isolated regime, with the surface at equilibrium with constant area and closure defect. In particular, this regime would model the dynamics of the horizon for (quantum) black holes.

The relation of nearest neighbors between surface patches is formalized as a surface graph or network. The interactions between nearest neighbors are thought as resulting from bulk operators (e.g. holonomy operators) ending on the surface or going through the surface. We will not attempt to explore the details of the bulk-boundary interactions and study the projection of bulk dynamics onto the boundary surface. We instead take the point of view of effective dynamics. Using the spinor variables to describe the state of the discrete surface, we proceed to a natural polynomial expansion of the Hamiltonian in spinor variables and we analyze the physics and dynamics induced by each possible term starting from the lowest order. The surface dynamics induced by any possible regular bulk dynamics could in principle be decomposed in such a way.

We will see that at the lowest order (the quadratic order in the spinors), the general ansatz for a Hamiltonian is a Bose-Hubbard model, with an interaction between area patches and a local potential, built from the basic area-preserving deformation operators E_{ij} . Quartic order terms will lead to Ising-like Hamiltonian and so on when going to higher orders.

Fixed Area Dynamics: the Bose-Hubbard model on the Horizon

We would like to investigate the surface dynamics in the fixed area regime. We have in mind the application to the dynamics of isolated horizons (and thus to black hole horizons). Indeed, as we already saw, the energy associated to a isolated horizon is directly proportional to its area. Taking such a simple Hamiltonian,

$$H \equiv \kappa A = \kappa \sum_i E_{ii} = \kappa \sum_i \langle z_i | z_i \rangle \quad (6.29)$$

leads to a almost completely stationary dynamics for the surface. The normal vectors \vec{X}_i do not evolve. The area of each surface patch are constant, as well as the total area. The only degrees of freedom that evolve are the phases of the spinors, which oscillate at a frequency set by the coupling factor κ $|z_i(t)\rangle = e^{i\kappa t}|z_i(t=0)\rangle$ Here we would like to go beyond this stationary regime and introduce a framework where we can study perturbations of the surface. For instance, we would like to be able to analyze how perturbations propagate on the surface of a black hole when a system is thrown through the black hole horizon or when a Hawking radiation particle evaporates from the horizon.

The natural next-to-leading order dynamics is to introduce a hopping term between surface patches. We identify nearest neighbor patches and define a Hamiltonian realizing local area quanta exchanges on the surface. This lead to an intrinsic local dynamics on the quantum surface

Hopping Dynamics: the $u(N)$ Hamiltonian The lowest order $SU(2)$ -invariant Hamiltonian, defining area-preserving local interactions between nearest neighbor on the surface graph, is quadratic in the spinors and given by a linear combination of the E_{ij} observables (defined as the scalar product between spinors)

$$H^{\{J_{ij}\}} = - \sum_{\langle i,j \rangle} J_{ij} \langle z_i | z_j \rangle$$

where the J_{ij} are the interaction couplings between nearest neighbor patches $\langle i, j \rangle$. Since there is no a priori reason to distinguish links on the surface graph, we work in the homogeneous case with a global coupling $J \in \mathbb{R}$

$$H = -J \sum_{\langle i,j \rangle} \langle z_i | z_j \rangle \quad (6.30)$$

where the coupling matrix J_{ij} is taken to be proportional to the surface graph adjacency matrix.

Considering nearest neighbor interactions is standard in tight binding models in condensed matter physics. In our context, the notion of locality on the surface and the dynamics of the surface are induced by the evolution of the bulk geometry, so a more general type of interaction could of course be envisioned. Nevertheless, working with “block-by-block” dynamics in the bulk, as usually in discrete geometry models for quantum gravity (in order to keep a causal evolution), leads naturally to nearest neighbor interactions on the surface. Thus this choice is not only a matter of simplicity in defining our template surface dynamics for loop quantum gravity, but it is also enough in order to identify universality classes of surface dynamics induced by causal bulk dynamics.

By writing the equations of motion, it is straightforward to obtain the solutions which are simply Bloch waves for a periodic one dimensional lattice. By labeling the site by the integer l running from 0 to $N-1$, we have a basis of solution is given by

$${}^{(l)}z_k^A = Z^A e^{i(q_l k - \omega_l t)} \quad (6.31)$$

for an arbitrary fixed spinor Z , with wavelength and frequency

$$q_l = \frac{2\pi l}{N} \quad \omega_l = -2J \cos \frac{2\pi l}{N} \quad (6.32)$$

This is for the moment very simple. What is already interesting with this model is the response of the surface to a local perturbation having for instance in mind a dynamical situation where a small object is thrown into a black hole and we are interested in the response of the horizon. Let us check the response of the system to a local perturbation and look at the evolution of a localized excitation at a site K . The initial condition at $t = 0$ is the set of spinors $z_k(t = 0) = Z\delta_{kK}$ for a given spinor Z . Taking the Fourier transform and computing the evolution gives

$$z_k = \frac{Z}{N} \sum_{l=0}^{N-1} e^{\frac{2i\pi l}{N}(k-K) - 2itJ \cos \frac{2\pi l}{N}} \quad (6.33)$$

For large N number of sites (i.e. of surface patches), this is approximated by a Bessel function (Riemann integral approximation of a sum),

$$z_k \underset{N \rightarrow \infty}{\sim} Z J_{2(k-K)}(2tJ) \underset{t \rightarrow \infty}{\sim} \frac{Z}{\sqrt{\pi t J}} \cos\left(2tJ - \frac{\pi}{4}\right) \quad (6.34)$$

with J_n are the Bessel functions of the first kind.

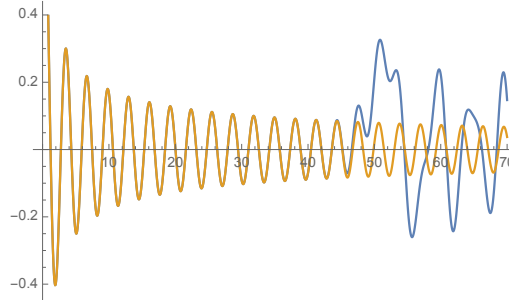


Figure 6.5: For a 1d lattice of size $N = 100$, we study the propagation and diffusion of a perturbation initially localized at the site $k = 0$. We look at the evolution of the amplitude at the initial site in terms of the time tJ . The amplitude follows very closely its Bessel approximation with oscillations decreasing in $t^{-\frac{1}{2}}$ until tJ reaches $N/2$ and the oscillations start growing again.

More precisely, as long as the time $2tJ \lesssim N$ is shorter than the compact size of the 1d lattice, then the Bessel approximation describes very well the evolution of the system, as shown on Figure 6.5. The initial peak at $k = K$ spreads out on all the Fourier modes. At the initial site, the amplitude is maximal at $t = 0$, then oscillates while decreasing in

$1/\sqrt{t}$. At another site k , the amplitude is evanescent until the perturbation reaches it around $tJ \sim |k - K|$, at which time it reaches its maximal value before oscillating and decreasing again in $1/\sqrt{t}$, as shown on Figure 6.6.

As soon as the time is of the same order as the system size, $2tJ \sim N$, we see the effect of working on a compact lattice and the actual amplitude departs from its Bessel approximation. At the initial site, at that critical time, the amplitude's oscillations increases again and periodically reaches an almost maximal value (see Figure 6.6 and Figure 6.7). We note the same behavior on all sites.

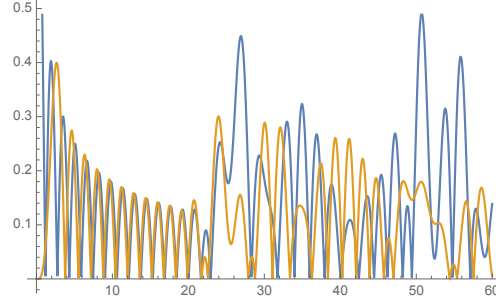


Figure 6.6: For a 1d lattice of size $N = 50$, we study the propagation and diffusion of a perturbation initially localized at the site $k = 0$. We look at the amplitudes at the sites $k = 0$, $k = 4$ and $k = 11$. At early times, the amplitudes follow their Bessel approximation. Initially the perturbation propagates until it reaches the site $k = 4$ around $tJ \sim 4$ and similarly for the site $k = 11$. Then the amplitudes oscillate and tend to synchronize, and decrease as $t^{-\frac{1}{2}}$ overall. Then around $tJ \sim N/2$, this nice simple behavior breaks due to the compactness of the system.

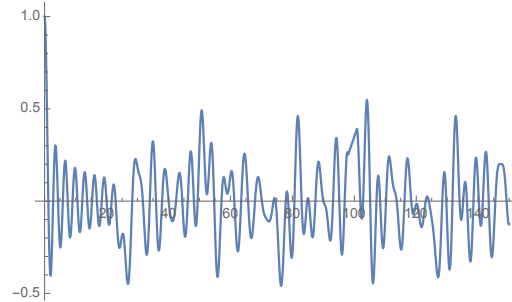


Figure 6.7: Evolution of the amplitude at the initial site of a local perturbation at the site $k = 0$ for a 1d lattice of size $N = 50$ in terms of the time tJ .

So the local perturbation excites all the Fourier modes. At early times, we do not see the finite size effects and we have a diffusive behavior. The perturbation propagates from its initial site K : once it reaches a given site k at a time $tJ \sim |k - K|$, the amplitude

keeps oscillating while decreasing in $1/\sqrt{t}$. This way, the perturbation spreads out on the whole lattice. Then when $tJ \sim N/2$, the finite size effects kick in and we observe some kind of interference between the oscillations, with the amplitude reaching its maximal value periodically.

Here, we use a one-dimensional lattice, but our analysis would extend without complication to a regular two-dimensional lattice.

The diffusion of the perturbation is an important feature when thinking about the application of such models of local surface dynamics to black hole horizons: it would allow the horizon to relax to equilibrium after a local perturbation. This comes by because the proper mode of evolution are collective wave modes. The initial local perturbation excites all those collective modes, thus leading to its diffusion throughout the lattice until finite size effects periodically cause some resonance and almost re-localize the perturbation. We will refine this analysis below when extending our very simple model to make it more realistic by including a local potential.

Introducing a Local Potential: Bose-Hubbard Dynamics Up to now, we have discussed all possible quadratic terms for an area-preserving Hamiltonian: the global area and a hopping term creating area exchange between nearest neighbor on the surface. The hopping Hamiltonian can be considered as a free propagation for area degrees of freedom. Following the logic of a polynomial expansion in the spinor variables, we can supplement it with a local potential term, quartic in the spinors (i.e. quadratic in the E_{ij} and F_{ij} observables). Thus we introduce a Bose-Hubbard Hamiltonian, with a local repulsion term for area quanta at each site

$$H_{BH} = -J \sum_{\langle i,j \rangle} \langle z_i | z_j \rangle + \frac{U}{2} \sum_i \langle z_i | z_i \rangle^2 \quad (6.35)$$

or explicitly expanding the spinors into their two components $A = 0, 1$

$$H_{BH} = -J \sum_{\langle i,j \rangle} (\bar{z}_i^0 z_j^0 + \bar{z}_i^1 z_j^1) + \frac{U}{2} \sum_i (|z_i^0|^2 + |z_i^1|^2)^2$$

This is our main proposal of a template for the loop quantum gravity dynamics of a discrete surface of N patches in the fixed area regime. This is a generalization of the standard Bose-Hubbard model used in atomic physics to two coupled atomic species, since here the spinors we use here to model the surface have two independent components.

The equations of motion resulting from (6.35) are the Gross-Pitaevskii equation on a lattice for each component of the spinors,

$$i\partial_t |z_i\rangle = -J \sum_{\langle k,i \rangle} |z_k\rangle + U \langle z_i | z_i \rangle |z_i\rangle \quad (6.36)$$

This equation is usually used to describe the ground state of Bose-Einstein condensate and superfluid dynamics. Indeed the ground state of a Bose-Einstein condensate can be approximated by a coherent state whose parameter obeys the same kind of equation. Here the states $|z_i\rangle$ can also be seen as a coherent state of the quantum discrete geometry.

Let us come back to the case of a 1d lattice in order to illustrate simply the main features of the model. The stationary waves -the Bloch waves- obtained in the previous section when the potential vanishes $U = 0$ are still stationary solutions of the Bose-Hubbard dynamics,

$${}^{(l)}z_k^A = Z^A e^{i(q_l k - \omega_l t)} \quad q_l = \frac{2\pi l}{N} \quad (6.37)$$

except that the dispersion relation is modified by the interaction potential,

$$\omega_l = \omega_l^{U=0} + U \langle Z|Z \rangle \quad \omega_l^{U=0} = -2J \cos q_l \quad (6.38)$$

with an explicit non-linear dependence on the amplitude of the wave. The superfluid nature of those waves is highlighted when looking at small wave perturbations. We look at small perturbations of the Bloch waves for the Gross-Pitaevskii equation: we choose a base mode l , add a perturbation with a slightly shifted momentum at $l \pm \delta l$ and study the stability of the mode k with respect to such variation.

We start by looking at perturbations around the $l = 0$ mode. This case, with $q_{l=0} = 0$, corresponds to a homogeneous wave $e^{-i\omega_0 t}$ constant in space and thus defines a homogeneous potential, equal for all surface patches. In this homogeneous case, small perturbations leads to phonon-like excitations with a Bogoliubov spectrum $\lambda_q \sim qv_s$ and a speed of sound $v_s = \sqrt{2JU \langle Z|Z \rangle}$. This shows the stability of the waves for velocities smaller than the sound velocity. Moreover, according to Landau's criteria, the Bogoliubov spectrum ensures superfluidity: as long as an object travels in the fluid at a speed smaller than the speed of sound $v < v_s$, the motion is favored energetically over the excitations of perturbations and will happen without any dissipation. On the other hand, for higher base mode $l > 0$, contrary to the homogeneous case, the condensate is unstable against perturbations. This hints new physics is involved and signals the onset of a phase transition.

We postpone a detailed analysis of this quantum model for surface dynamics for future investigation. It would very likely have interesting predictions for the behavior of quantum horizons in quantum gravity, their dynamics and their phase diagram. Without performing a full analysis, we can nevertheless have a glance of what to expect by looking at the properties of the one-component Bose-Hubbard model on a regular lattice. It is a model that has been greatly studied ⁵ in condensed matter and whose features are well-understood in atomic physics (for reviews and textbooks [Krutitsky, 2016; Bloch

⁵ Here we work in a canonical framework with a fixed area, i.e. a fixed total number of area quanta. The number of quanta at each site (on each surface patch) is an integer. To match this with the computations done in a grand-canonical framework, the chemical potential is fixed so that the average occupation number at each site $\langle n \rangle$ is an integer.

et al., 2008; Sachdev, 2011]). In particular, it exhibits a quantum phase transition at zero temperature $T = 0$ between a superfluid phase and a Mott phase controlled by the parameter U/J . We note $(U/J)_c$ the critical value of this ratio. For the Bose-Hubbard model in d dimensions, this transition belongs to the universality class of the XY model in $d+1$ dimensions. As temperature is turned on, the fluid changes into a simple Bose gas phase, which can be referred to as the normal phase of the system.

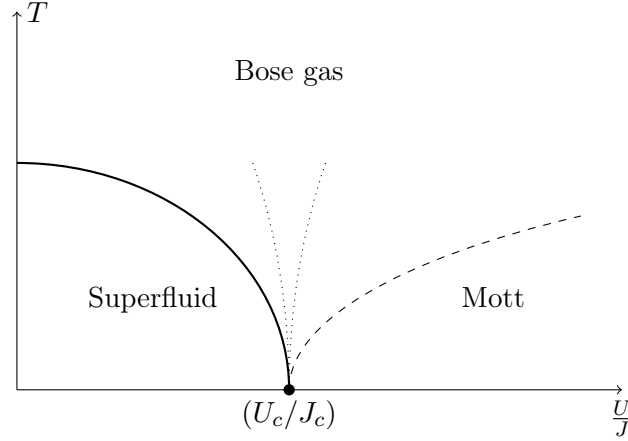


Figure 6.8: Phase diagram of the Bose-Hubbard model for a fixed integer average number of quanta per site. Two important phases are present, the superfluid phase at low temperature and the Bose gas - Mott phase. At $T = 0$ the system undergoes a quantum phase transition.

Working on a square lattice, the critical line in d dimensions is given by ⁶

$$\frac{k_B T_c^{(d)}}{J} = A^{(d)} \left[\left(\frac{U}{J} \right)_c - \frac{U}{J} \right]^{z\nu}, \quad A^{(d=2)} \sim \frac{1}{2} \quad (6.39)$$

where z is the node connectivity of the lattice ($z = 2d$ for a square lattice in d dimension) and ν is the critical exponent of the correlation length of the XY model, $\xi \sim \delta^{-\nu}$. The critical exponent is $\nu = \frac{1}{2}$ near the critical point for all dimensions $d > 2$, except in two dimensions $d = 2$ where we have $\nu \sim \frac{2}{3}$. The phase diagram is drawn below on Figure 6.8.

In fact, only the superfluid/Bose gas phase transition is associated to a symmetry breaking and truly define different phases. The Bose gas and the Mott phase cannot be

⁶ In two dimensions $d = 2$, the critical line is slightly different. When we take the limit of a vanishing potential, $U/J \rightarrow 0$, the critical line goes back to a vanishing temperature $T \rightarrow 0$. In that weakly interacting regime, we have a Berezinsky-Kosterlitz-Thouless phase transition with:

$$\frac{k_B T_c^{(d=2)}}{J} \underset{U \rightarrow 0}{\sim} \frac{4\pi \langle n \rangle}{-\ln 2\xi \frac{U}{J}}.$$

distinguished as such. The Mott phase is characterized by the presence of a gap Δ and the exponential damping of thermodynamical quantities.

Coming back to surface dynamics, we see that we should expect a phase transition in the dynamics at fixed area as the temperature of the surface grows, from a superfluid phase to a Bose gas. For instance, the response of the surface to local perturbations will be different in those phases. For the Mott phase or the Bose gas, the propagation of perturbations is ballistic whereas it is diffusive in the superfluid phase as we have already seen in the previous section in the simple model of hopping dynamics. Indeed in the Bose gas, we can have localized perturbations that propagate through the gas as particles, while the superfluid basic excitations are collective modes on the surface. This should have interesting applications to the physics of quantum black holes, as we discuss in the next section below.

We have given the properties of the Bose-Hubbard on a regular lattice, but there is no *a priori* restriction on the type of surface graph, defining the locality on the surface. As we have already argued, the surface graph is supposed to be induced by the structure and dynamics of the spin network state underlying the bulk geometry. If we focus on the surface and forget all knowledge of the bulk geometry, we can consider an alternative point of view: the exact phases and transitions of the Bose-Hubbard model for the surface dynamics crucially depend on the details of the surface graph. Then we infer the type of surface graph we need from the properties of the surface we expect.

For instance, for a symmetric and smooth surface, as we expect for a black hole horizon, it seems natural to consider that the surface graph is a regular lattice. But in fact, nothing *a priori* forbids to consider random graphs instead of regular ones. From the condensed matter perspective, this amounts to introduce disorder in the system. Disorder typically blocks the diffusion of waves and leads to a localization -Anderson localization- around the defects. We speculate that this would naturally lead to the possibility of localized excitations on the surface.

Application to Quantum Black Hole Horizons In loop quantum gravity, the black hole horizon, as a space(-time) boundary, is pictured as a quantum surface, punctured by the spin network states defining the bulk geometry, with each puncture representing a basic surface patch and carrying quanta of area. These surface patches represent the microscopic degrees of freedom of the black hole. This led to the paradigm of the horizon as a gas of punctures, which allows to recover the area-entropy law.

Here we propose to refine this basic picture, by introducing an intrinsic dynamics to the punctures on the horizon. Since isolated horizons are naturally in the isolated regime at fixed area, we propose to model the surface dynamics with the Bose-Hubbard model we introduced in the previous section: area quanta can now propagate along the horizon, hopping from puncture to puncture, with a repulsive local potential.

This would lead to two main predictions:

- A modification of the energy spectrum of the black hole, which would not be simply

proportional to the area but will have corrections due to the Bose-Hubbard Hamiltonian, and which would imply corrections to the Hawking evaporation spectrum.

- A non-trivial phase diagram for quantum black holes, with (at least) a superfluid phase and a Bose gas phase, which would depend on the notion of locality on the horizon and the surface graph induced by the near-horizon geometry.

First of all, the choice of surface graph crucially affects the properties, and in particular the phase diagram, of the Bose-Hubbard model living on it. Thus we should identify the phase(s) relevant to black hole physics and used this to constrain the surface graphs corresponding to a black hole horizon.

Nevertheless, considering the spherical symmetry and homogeneity of a black hole horizon, it seems natural to model it with a regular surface graph. So, assuming that we work on a (almost) regular graph, and assuming that the Bose-Hubbard couplings J and U are constant (determined by the quantum gravity exact dynamics and not depending on the black hole mass or horizon area), the Bose-Hubbard phase diagram gives us is a critical temperature T_c , which depends on the ratio U/J and on the (average) node valency of the surface graph. This critical temperature is to be compared to the Hawking temperature T_H of the black hole. When the mass is large, and thus the Hawking temperature is small, we will be in the superfluid horizon phase. While when the mass is smaller and smaller, and the Hawking temperature exceeds the critical temperature, we will have a phase transition and enter the Bose gas phase.

The main difference between these two phases is how they respond to local perturbations. The superfluid phase has a diffusive behavior, perfectly suited to black holes physics, with local perturbations exciting collective modes spread out on the whole surface. On the other hand, the Bose gas phase has a ballistic behavior with local perturbations able to travel along the surface, barely deformed. So a superfluid horizon would tend to relax ⁷ faster back to a homogeneous horizon after a perturbation such as an incoming mass or particle evaporation, while a Bose gas horizon would retain for a longer time the local perturbation.

We should study the details of this scenario, identify the correct physical meaning for the Bose-Hubbard couplings J and U , check if we obtain reasonable values for the critical temperature, and finally if and how this phase transition scenario fits with the black hole entropy and the apparent “information loss paradox” in quantum gravity.

⁷ Actually there can not be a true full relaxation without dissipation. If we look at a local perturbation of the horizon, due either to an incoming mass or the evaporation of a Hawking photon from the horizon, we should consider not only the intrinsic evolution of the degrees of freedom on the surface but also its coupling to the near-horizon geometry. For instance the emission of quasi-normal modes play an essential role in the relaxation of the black hole horizon to its equilibrium state with a homogeneous horizon (see e.g. [Hod, 2007]).

6.3 Conclusion

We have looked in this chapter at quantum surfaces in loop quantum gravity. Defined as a collection of elementary surface patch, a surface is further endowed with the extra structure of a graph which defines a network of nearest neighbor patches on the surface. This surface graph defines a notion of locality, thought as induced by the spin network states encoding the near-surface bulk geometry.

From this definition of a discrete surface, we have launched the program of defining and analyzing surface dynamics in loop quantum gravity. Describing each elementary surface patch by a spinor at the classical level or an irreducible representation of $SU(2)$ (spin) at the quantum level, we have introduced generic templates for both global and local surface dynamics.

The global dynamics focuses on the evolution of the area and closure defect of the surface due to its interaction with the bulk geometry, thought as an environment to the system. For a closed surface bounding a space region (with the trivial topology of a 3-ball), the closure defect can be interpreted as a measure of curvature in the region's bulk and vanishes by definition when the region consists in a single spin network vertex.

We explore two different models: a dissipative regime and a forced rotation regime. In the dissipative model, the surface spinors flow along a $SL(2, \mathbb{C})$ orbit and asymptotically converges back to a vanishing closure defect. In some sense, the curvature excitations within the region dissipates through the surface and the boundary area converges towards a minimal value at rest. This is the first explicit mechanism to dynamically recover the closure constraint for a composite region in loop quantum gravity. On the other hand, the forced rotation model describes a precession motion, for which the closure defect has a constant norm and rotates around a fixed axis (defined by the initial state of the surface). The area follows a hyperbolic trajectory, together with some oscillatory modes. This could most certainly be used to model rotating astrophysical objects in loop quantum gravity or be used in a cosmological context.

The local dynamics focuses on an isolated regime for which the total surface area remains fixed and describes intrinsic fluctuations of the surface geometry. We introduced a generic ansatz for a Hamiltonian, going to quartic order in the spinors (or quadratic order in the gauge-invariant observables), defined by a generalized Bose-Hubbard model (with two atomic species). It consists in a hopping term, encoding the free propagation of area quanta along the surface, plus a (repulsive) local potential (given by the squared number of area quanta). This can be supplemented with a Ising-like term favoring (or disfavoring) the alignment of the (normal) direction of neighboring surface patches.

The physics of such a Bose-Hubbard model is rich, with phase transitions between a superfluid and a Mott phase at zero temperature and between a superfluid and a Bose gas as the temperature increases. Moreover the dynamical properties of the model crucially depend on the surface graph defining the network of nearest neighbor surface patches. This promises interesting applications to quantum black holes, with a modified energy

spectrum and a possible transition from a superfluid horizon to a Bose gas horizon as the black hole becomes smaller (i.e. as its mass decreases), which should be relevant to study in more details.

The next step will be to implement both global and local dynamics in the full quantum regime of loop quantum gravity, combining the dissipative relaxation to the closure constraint to wave propagations on the surface. Not only we should analyze the phase diagrams of these models, but we should connect explicitly them to the quantum gravity dynamics, either by realizing them as effective surface dynamics in some regimes of the canonical loop quantum gravity or of spinfoam models, or at least by relating their coupling parameters (relaxation time, Bose-Hubbard couplings,...) to parameters from the full quantum gravity theory. Then we could also extend our surface models to a variable number of surface patches (or punctures) by introducing a chemical potential as in statistical mechanics. This would certainly lead to more realistic models for black hole horizons in a grand-canonical framework.

Finally, we believe that it is crucial to understand how the structure of the surface graph, encoding the notion of locality on the surface, affects the evolution and (thermo)dynamical properties of the quantum surface. This seems necessary, in the context of implementing the holographic principle, in order to clarify the conditions for a proper definition of holographic screen in loop quantum gravity, and could lead to surprises, such as (Anderson) localization or (Bose) glass phases due to disorder and randomness in the surface graph.

Chapter 7

Decoherence in gravity

We come now to the last part of the project. Just to remind the reader, our fundamental goal is to explore the idea that geometry is an emergent notion coming from quantum information properties of the fundamental quantum gravity degrees of freedom. We explored first the global structure of the quantum state with an emphasis on the distance from correlations and holographic principles. Then we moved forward to the specific study of surfaces which are natural geometrical entities to understand since they may encode in a sense the complete gravitational physics. The last chapter focuses on the semi-classical dynamics of surfaces from a loop quantum gravity point of view.

As we now know, a major issue in the loop quantum gravity proposal is to recover from the quantum theory general relativity as a semi-classical approximation and understand quantum deviations from it. Only after a complete resolution of this question we will be able to state that the loop gravity proposal is a good candidate for a quantum theory of gravity. One approach is to properly formulate coarse-graining procedure in a background independent context and define good coherent states interpolating a classical geometry. Still, this is in fact only one part of a proper analysis of the semi-classical regime of a quantum theory. Indeed, an important question is to properly understand the quantum to classical transition, meaning how a macroscopic observer cannot observe quantum superposition and how emerge from a dynamical process the classical states. This is for instance well understood in quantum electrodynamics.

What we propose to explore in this last chapter is a decoherence model in the context of loop quantum gravity by studying the open quantum dynamics of a quantum surface. Ideally this would require to have the full Hamiltonian constraint to properly model the effects of the environment but since we do not have it at our disposition, we will work in an effective way.

The chapter is structured as follows. Section 7.1 discusses some context about the relationships between decoherence, coarse-graining and gravitation. Section 7.2, based on our published work [Feller and Livine, 2017b], discusses decoherence effect on the boundary induced by the bulk-boundary coupling.

7.1 Decoherence with gravity

7.1.1 Local subsystems

The whole idea we are exploring in this work is that entanglement and correlations are the central notions to understand the quantum nature of space, time, distance and, more generally, of geometry. Still, for this approach to make any sense at all, we must be able to distinguish subsystems in the universe and have a separable Hilbert space structure. Such a tensor product structure is non-trivial in any theory with a local symmetry like Yang-Mills theories or general relativity because of the presence of constraints.

Let's illustrate the relationship between locality and constraints with a simple example in quantum electrodynamics. Suppose that we have an operator that can create an electron¹. In quantum electrodynamics, the electric field must satisfy the Gauss law which states that the flux of the electric field through a boundary is equal to the total charge inside the region. Can our electronic creation operator be local in space? In order to satisfy the Gauss law it must be non local. Indeed, it must also create the electric field generated by the charge everywhere in space in order for the flux of any region, however large it may be, to be equal to the created charge. In this situation, both the conservation of charge and the finite propagation of fields are violated. To save locality, what we can do is create a pair of particle and anti-particle. They must be created at the same point in space, again to satisfy Gauss law. Thanks to the screening effect, the electric field is zero away from the creation region initially and, as time goes on, the electromagnetic field can propagate following the rules of special relativity and locality. This example illustrates in a simple way the tension between constraints from local symmetries and locality.

To recover a separable Hilbert space in a constrained theory, different alternatives are possible. The simplest is to regularize the theory on a lattice where each site carry some degrees of freedom. A region is then defined as a collection of sites and the Hilbert space factorizes. In such an approach, gauge invariant states form a subspace, not necessary factorized, of the full Hilbert space. An alternative approach to the lattice one is called the extended Hilbert space. The idea is to consider additional degrees of freedom at the boundary (also called entangling surface) forming an extended Hilbert space which is itself separable (we already encountered this idea in our analysis of the area law). The gauge invariant Hilbert space \mathcal{H} is just a subspace. Taking a state in the extended Hilbert space and tracing out the additional degrees of freedom, which are subject to some symmetries depending on the theory we consider, allows to recover gauge invariance. In the end, the true physical Hilbert space is the extended one. This approach predicts a very general form of the entanglement entropy of a region with a boundary contribution coming from the additional degrees of freedom [Donnelly and Freidel, 2016].

A second issue for the separability property in field theories comes from UV divergences. We know that the two-points correlation function diverges as the distance goes to zero. The simplest solution is again to put the theory on a lattice where the fundamental length

¹Forget for the moment that such an operator do not conserve the charge.

scale regularizes automatically the divergences. However this is a limited solution a priori. Still, in a quantum gravity setting, the Planck length plays the role of a fundamental length scale and possibly a fundamental cut-off. In loop quantum gravity, geometry is discrete, not continuous, with a fundamental length scale set by the Planck length and the Immirzi parameter. It is argued that this discreteness cures the divergences of field theories. The conclusion here is that when properly incorporating gravitation, divergences do not threaten the separability of the Hilbert space.

7.1.2 Decoherence and Coarse-graining

Before introducing our model, we digress a bit about the relationships between coarse-graining procedures and decoherence. This is a relevant inquiry since coarse-graining is the central concept to understand the emergence of classical geometry and most probably also the interplay between holography and dynamics.

Open quantum dynamics and coarse-graining procedures have a lot in common. Both amount to tracing out inaccessible degrees of freedom. For open quantum systems those form the environment modes and we consider only the evolution of the system degrees of freedom through the reduced density matrix $\rho_S(t) = \text{tr}(\rho_{SE}(t))$. Ideally no approximations are made in this average dynamics and the evolution is most of the time non-unitary and non-local in time. Only with some approximation like the Markovian approximation can we recover an evolution that is local in time. The presence of the environment induces renormalization, dissipation and decoherence.

Concerning coarse-graining procedures, given some scale Λ , the idea is to look at the dynamics of the slow moving modes $\phi_L(k)$ by tracing out fast moving modes $\phi_H(k)$ in such a way as to obtain an effective theory that is unitary. The complete theory is studied with the partition function

$$Z = \int e^{i(S[\phi_H, \phi_L] + S[\phi_L] + S[\phi_H])} \mathcal{D}\phi_H \mathcal{D}\phi_L \quad (7.1)$$

The effective theory is then obtained by considering the effective action

$$\int e^{i(S[\phi_H, \phi_L] + S[\phi_H])} \mathcal{D}\phi_H = e^{iS_\Lambda[\phi_L]} \quad (7.2)$$

To have a more direct contact with the traditional decoherence formalism, let's look at this from the point of view of the wave functional for a field theory. Given an initial state $|\Psi[\phi^i]\rangle$ at time t_i , it will evolve at time t_f into

$$|\Psi[\phi(t_f)]\rangle = \int_{\phi(t_i)=\phi^i} |\phi(t_f)\rangle e^{iS[\phi]} \mathcal{D}\phi \quad (7.3)$$

Now we choose a scale Λ and suppose that the initial state is uncorrelated $|\Psi[\phi^i]\rangle =$

$|\Psi[\phi_L^i]\rangle \otimes |\Psi[\phi_H^i]\rangle$. Then the final state is re-written as

$$|\Psi[\phi(t_f)]\rangle = \int_{\phi_L(t_i)=\phi_L^i} |\phi_L(t_f)\rangle e^{iS[\phi_L]} \underbrace{\int_{\phi_H(t_i)=\phi_H^i} |\phi_H(t_f)\rangle e^{i(S[\phi_H, \phi_L] + S[\phi_H])} \mathcal{D}\phi_H \mathcal{D}\phi_L}_{U_\Lambda[\phi_L]|\Psi[\phi_H(t_i)]\rangle} \quad (7.4)$$

The operator U_Λ gives the evolution of the fast mode given a classical configuration of the slow modes seen as the imprint left on the environment. In a coarse-graining procedure, we trace out the fast modes and look only at the slow modes thought as the only relevant modes at the scale set by the observer. In the general case, we then have a reduced density matrix

$$\rho_\Lambda[\phi_L(t_f), \phi'_L(t_f)] = \int e^{iS[\phi_L] - iS[\phi'_L]} |\phi_L(t_f)\rangle \langle \phi'_L(t_f)| \langle \Psi[\phi'_H(t_i)] | U_\Lambda^\dagger[\phi'_L] U_\Lambda[\phi_L] | \Psi[\phi_H(t_i)] \rangle \mathcal{D}\phi_L \mathcal{D}\phi'_L \quad (7.5)$$

At this stage, we have the exact reduced density matrix of the slow modes where we see that the presence of the fast modes modulate the transition amplitudes by an overlap function of the interaction between fast and slow modes and the trajectories (ϕ_L, ϕ'_L) . This overlap is called the Feynman-Vernon functional and encodes effects like decoherence. At this stage, there is no difference between a coarse-graining procedure in field theories and the average of environmental degrees of freedom for open quantum systems. In fact, a coarse-graining procedure can be interpreted in the open system perspective by seeing the length scale Λ as the precision of an observer's probe and where details finer than this scale become inaccessible to her device.

The major difference between the dynamics of traditional open quantum systems and coarse-grained effective theories comes from the form the overlap takes. In effective field theories, the central hypothesis made is the separation of scales. This means that during the evolution, no (few) correlations appear between the physics at very small scales and the physics at larger scales. We have a weak coupling constant between the modes at different energies. The Feynman-Vernon influence functional is then just a phase difference and we can define a unitary effective action for the low energy modes. On the contrary, the context of open quantum system is such that we have an interaction in a strong coupling regime. We do not have a separation of scales and strong correlations between the low energy modes (the system) and the high energy modes (the environment) occur. The Feynman-Vernon influence functional is not anymore simply a phase but has a modulus whose form depends on the imprints the state of the system left in the environment. The take-home message of this qualitative discussion is that the philosophy behind effective field theories and open quantum systems is the same: we integrate over unobservable (irrelevant) degrees of freedom for the system. The branching between the unitary description of effective field theories and the non-unitary one in open quantum

system comes from the interaction. This discussion is at this stage qualitative; for a more quantitative analysis, we can look at the physics of the spin-boson system [Leggett et al., 1987] or for a field theory perspective at [Lombardo and Mazzitelli, 1996].

7.2 Surface state decoherence

The purpose of this section is to explore decoherence effect on the boundary of a region induced by the bulk-boundary coupling. The first section analyzes rapidly a simple toy model that shows indeed that bulk-boundary coupling can induce decoherence of the boundary before moving on, in the rest of this section, to the construction and analysis of a more involved model based on loop quantum gravity, from which we can extract some geometrical pointer states. We conclude by discussing the relevance of the many different approaches to open quantum systems.

7.2.1 Local bulk induced decoherence

Let's consider N qubits with an Ising like local interaction $H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z$ in three dimensional space in, for instance, a square array. The total Hilbert space is given by the tensor product $\mathcal{H} = \otimes_{i=1}^N \mathcal{H}_{1/2}$ and a basis is given by the set of states $(\otimes_{i=1}^N |\sigma_i\rangle)$ with $\sigma_i = \pm 1$ (the set of binary words of N bits). Let's choose a region composed of a set of qubits forming the system S while the rest forms the environment E . The total Hamiltonian can be written in a more manageable way as $H = H_S + H_E + H_{\text{int}}$ with

$$H_{\text{int}} = -J \sum_{l \in \partial S} \sigma_{s(l)}^z \sigma_{t(l)}^z \quad (7.6)$$

where l corresponds to a link that crosses the boundary ∂S of the region S . To simplify the writing, we suppose that the graph is oriented and that each link crossing the boundary points outward. Then $s(l)$ and $t(l)$ correspond respectively to the source and target qubit of the link.

From the form of this interaction only, we can determine the pointer states selected by the dynamics. First, each part of the Hamiltonian commutes with the others. Pointer states are then given by the interaction term (7.6) only. Again each terms of the sum commutes with the others. Then from (7.6), we conclude easily that the pointer states of the system are eigenvalues of the set $(\sigma_{s(l)}^z)_{l \in \partial S}$. Those are states with a given value of the boundary energy $\hat{\partial E} = \bigotimes_{l \in \partial S} \sigma_{s(l)}^z$. A natural decomposition of the system Hilbert space is then $\mathcal{H}_S = \bigotimes_{\partial E} \mathcal{H}_S^{\partial E}$ where ∂E is a eigenvalue of $\hat{\partial E}$ and $\mathcal{H}_S^{\partial E}$ is the set of states with a given boundary energy.

The simple conclusion here is that a local interaction induces a decoherence on the boundary of a region. In the previous discussion, when we spoke about the system being a region, we had in mind the natural picture of some finite part of the Universe.

However, there is no counter-indication to have the reverse point of view. Indeed, in the spirit of a black hole spacetime, the system is in general the outside observer while the inaccessible degrees of freedom form the bulk local region behind the horizon. Then the simple argument presented here can be interpreted as a decoherence induced from the bulk by the (local) bulk-boundary interaction. This is this kind of phenomenon that we want to analyze in a context closer to the loop quantum gravity framework.

7.2.2 Surface geometry and quantization

To begin this analysis, we need a bit more mathematical tools on the description of a quantum surface in loop quantum gravity. Having that, we will see that it is very natural to set up a model of open dynamics for a quantum surface with fixed area in loop quantum gravity.

Spin as harmonic oscillators The geometry of a two dimensional surface \mathcal{S} can be described from two different point of views: the intrinsic one which relies on the Riemannian curvature and the extrinsic one. The latter presupposes an embedding of the surface in a higher dimensional space like \mathbb{R}^3 . The extrinsic curvature (also called second fundamental form) is defined as the variation of the surface normal vector $\mathbf{N} \in \mathbb{R}^3$ along the manifold \mathcal{S} . This normal vector also gives the integration measure on the manifold. This description is privileged by the canonical quantization of geometry in loop quantum gravity.

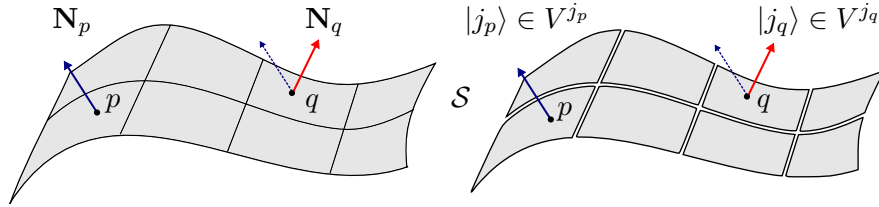


Figure 7.1: Geometry of a 2d surface \mathcal{S} in terms of the extrinsic curvature seen as the variation of the normal. The quantum theory describes \mathcal{S} as a discretized set of patches \mathcal{S}_i whose quantum states live in V^{j_i} a spin j_i representation of $SU(2)$.

The loop quantum gravity approach to the quantization of such a geometry is twofold. First we consider a discretization of \mathcal{S} in terms of elementary surfaces (a face or a patch) \mathcal{S}_i . Each patch is defined by its surface normal $\mathbf{N}_i \in \mathbb{R}^3$, whose norm is the area of the surface. It is further provided with a phase space defined by a $\mathfrak{su}(2)$ Poisson Bracket $\{\mathbf{N}_i^a, \mathbf{N}_i^b\} = \gamma \frac{8\pi G}{c^3} \epsilon^{abc} \mathbf{N}_i^c$. This phase space is then canonically quantized to the operator commutator $[J_i^a, J_i^b] = \gamma \frac{8\pi G \hbar}{c^3} \epsilon^{abc} J_i^c$. This is the basic postulate of loop quantum gravity. The proportionality factor has the dimension of an area and is related to the Planck area

$G\hbar/c^3 = l_P^2$, the only dimensional quantity that appears in quantum gravity and γ is the Immirzi parameter, a dimensionless number that fixes the scale of the theory.

The quantum state of each elementary surface patch \mathcal{S}_i is then a vector of an irreducible $SU(2)$ representation V^{j_i} . The spin j_i then gives the area of that surface in Planck units γl_P^2 . The Hilbert space of a N patches surface with fixed spins $j_1 \dots j_N$ is then

$$\mathcal{H}_{j_1 \dots j_N} = V^{j_1} \otimes \dots \otimes V^{j_N} \quad (7.7)$$

Intertwiners are defined as the $SU(2)$ -invariant subspace of this Hilbert space:

$${}^0\mathcal{H}_{j_1 \dots j_N} = \text{Inv}_{SU(2)} [V^{j_1} \otimes \dots \otimes V^{j_N}] \quad (7.8)$$

These singlet states are understood as the quantum counterpart of classical polyhedra [Livine, 2013]. We will focus on a surface with fixed area $A = \sum_{p=1}^N j_p$ with Hilbert space

$$\mathcal{H}_N^A = \bigoplus_{A=\sum_{p=1}^N j_p} \mathcal{H}_{j_1 \dots j_N}. \quad (7.9)$$

Its $SU(2)$ -invariant subspace ${}^0\mathcal{H}_N^A$ describes the Hilbert space of the set of all polyhedron of area A . As it was shown in [Freidel and Livine, 2010] that these intertwiner spaces ${}^0\mathcal{H}_N^A$ each carry an irreducible representation of the unitary group $U(N)$, which can be understood as the group of deformations of quantum polyhedra at fixed total boundary area A . We will recall the definition of the $\mathfrak{u}(N)$ generators below as the basic deformation operators for a quantum surface.

Finally, the total Hilbert space associated to a quantum surface \mathcal{S} with N patches is

$$\mathcal{H}_N = \bigoplus_{A \in \mathbb{N}} \mathcal{H}_N^A = \bigoplus_{\{j_p\}} \mathcal{H}_{j_1 \dots j_N} = \bigoplus_{\{j_p\}} V^{j_1} \otimes \dots \otimes V^{j_N}. \quad (7.10)$$

In this framework, studying the dynamics is naturally done through the study of the deformations of the surface. In particular, the area of each face can change which means in the quantum theory changing the spin j_e attached to the face. The common $SU(2)$ representation used in angular momentum theory is not adapted for this purpose. But the Schwinger representation of the $\mathfrak{su}(2)$ Lie algebra in terms of harmonic oscillators is and we review its construction here [Freidel and Livine, 2010].

Let's focus on one spin (i.e. one elementary surface patch) and introduce two harmonic oscillators a and b whose commutation relations are naturally $[a, a^\dagger] = [b, b^\dagger] = 1$. It is then straightforward to show that

$$J_z = \frac{1}{2}(a^\dagger a - b^\dagger b) \quad J_+ = J_-^\dagger = a^\dagger b \quad (7.11)$$

satisfy the $\mathfrak{su}(2)$ -algebra. The total energy of the oscillators $\mathcal{E} = \frac{1}{2}(a^\dagger a + b^\dagger b)$ allows to write $\mathbf{J}^2 = \mathcal{E}(\mathcal{E} + 1)$, so that the total energy gives exactly the spin j , i.e. the area of the

elementary surface patch. Similarly, the energy difference corresponds to the magnetic quantum number m . The Hilbert space we are working with is then $\mathcal{H}_{HO} \otimes \mathcal{H}_{HO} = \oplus_j V^j$. Using standard notations, we have the correspondence between the spin and the harmonic oscillators states

$$|j, m\rangle = |n_a, n_b\rangle \quad j = \frac{1}{2}(n_a + n_b) \quad m = \frac{1}{2}(n_a - n_b) \quad (7.12)$$

We can at once see that the action of a or b decreases the spin and thus the area by $1/2$. The Schwinger representation admits natural operators allowing to move between different spin representation of the $\mathfrak{su}(2)$ -algebra, a feature more complicated to achieve with the standard representation.

Now consider a surface with N faces described by spins $(j_i)_{i=1,\dots,N}$. We then indeed need N pairs of harmonic oscillators $(a_i, b_i)_{i=1,\dots,N}$ to describe the surface state living in the Hilbert space $\mathcal{H}_N = \mathcal{H}_{HO}^{\otimes 2N}$. This representation naturally allows us to define a new set of operators that deform the surface. We define the operator E_{ij} that destroys a quantum of area at the face j and creates one at i by² :

$$E_{ij} = a_i^\dagger a_j + b_i^\dagger b_j \quad (7.13)$$

Clearly those operators deform the surface \mathcal{S} , preserve the total area and are invariant under $SU(2)$ rotations. They then act on each space \mathcal{H}_N^A without affecting the area A . The total area $A = \sum_{p=1}^N j_p$ is related to the total energy of the oscillators as the eigenvalue of $\hat{A} = \frac{1}{2} \sum_{p=1}^N E_{pp}$. Throughout the text we use the simplified notation $E_{pp} = E_p$. The operators E_{ij} also satisfy the $\mathfrak{u}(N)$ algebra [Livine, 2013]

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} \quad (7.14)$$

The group $U(N)$ can thus be seen as the group of area preserving deformations of a discrete quantum surface with N faces. The (quadratic) Casimir operator \mathcal{C} of this $\mathfrak{u}(N)$ algebra is

$$\mathcal{C}^2 = \sum_{ij} E_{ij}^\dagger E_{ij} = 2\hat{A}(\hat{A} + N - 2) + 2\mathbf{J} \cdot \mathbf{J} \quad (7.15)$$

where $\mathbf{J} = \sum_{p=1}^N \mathbf{J}_p$ is the total spin operator and \mathbf{J}_p are the operators for each patch defined in (7.11). The operators \mathbf{J} generate global $SU(2)$ transformations on all spins simultaneously, corresponding to an overall 3d-rotation of the whole surface. When this global $SU(2)$ Casimir vanishes, $\mathbf{J}^2 = 0$. We are back on the $SU(2)$ -invariant subspace ${}^0\mathcal{H}_N^A$ of quantum polyhedra. But in general \mathbf{J}^2 is not zero and is dubbed the “closure

²We could define operator that only destroy or create quantum of area but we don't need them yet for the present study. They are defined as $F_{ij} = a_i b_j - a_j b_i$ and their Hermitian conjugate F_{ij}^\dagger [Freidel and Livine, 2011a] and are used to define coherent intertwiner states.

defect” [Livine, 2014; Charles and Livine, 2016]. This closure defect appears naturally when coarse-graining the spin network state as seen in the previous chapter. Nonetheless its physical significance is not yet perfectly understood but it is suspected to be related to curvature and torsion in the coarse-grained region induced by some quasi-local energy density. The important point for our concerns is that the eigenstates of the operator \mathbf{J}^2 will be at the heart of our discussion of pointer states of the quantum surface. Having now the kinematical scene for the system and natural operators to define its dynamics, we go on to discuss a very special class of states that play a central role for the semi-classical understanding of the theory.

Coherent states Coherent states play a very special role in the understanding of the quantum/classical transition in many areas of physics and also in quantum gravity. They allow to interpolate a classical geometry from its quantum description. $SU(2)$ coherent states are the natural ones to use in loop quantum gravity. Following [Perelomov, 1977], a coherent state $|j, g\rangle$ is defined by applying an $SU(2)$ rotation g to a state analogous to the vacuum in quantum optics that minimizes the uncertainty relations such as the highest weight state $|j, m = j\rangle$,

$$|j, g\rangle = g|j, j\rangle, \quad g \in SU(2) \quad (7.16)$$

A key property of coherent states is that they remain coherent under the action of a $SU(2)$ -rotation. This follows directly from their very definition,

$$h|j, g\rangle = |j, hg\rangle \quad (7.17)$$

Different ways exist to index coherent states. Instead of using the $SU(2)$ rotation g , a coherent state can equivalently be labeled using spinors $z \in \mathbb{C}^2$. The highest weight vector is the spinor $|\uparrow\rangle$ and can be mapped to any arbitrary unit spinor by a rotation $g \in SU(2)$, so that a $SU(2)$ matrix contains the same information as a unit spinor $\langle z|z\rangle = 1$. Explicitly the parametrization of $SU(2)$ coherent states by spinors goes as:

$$|\uparrow\rangle = \left|j = \frac{1}{2}, m = \frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |z\rangle = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \quad g = \frac{1}{\sqrt{\langle z|z\rangle}} \begin{pmatrix} z^0 & -\bar{z}^1 \\ z^1 & \bar{z}^0 \end{pmatrix}, \quad g|\uparrow\rangle = |z\rangle, \quad (7.18)$$

$$|j, \uparrow\rangle = |j, j\rangle, \quad |j, z\rangle = \left(\sqrt{\langle z|z\rangle}\right)^{2j} g|j, \uparrow\rangle \quad (7.19)$$

Spinors for loop quantum gravity have been extensively studied in [Freidel and Livine, 2010, 2011a; Livine, 2013]. The explicit decomposition of a coherent state on the standard basis $|j, m\rangle$ used in angular momentum theory is

$$|j, z\rangle = \sum_{m=-j}^j \sqrt{\binom{2j}{j+m}} (z^0)^{j+m} (z^1)^{j-m} |j, m\rangle. \quad (7.20)$$

Then the norm (and scalar product) between coherent states can be calculated in terms of the simpler scalar product between spinors by the formula:

$$\langle j', z' | j, z \rangle = \delta_{j'j} \langle z' | z \rangle^{2j} \quad (7.21)$$

Such coherent states are the basic tools for the construction of more interesting states such as coherent intertwiner states or $U(N)$ coherent states for the semi-classical analysis of loop quantum gravity at fixed graph.

7.2.3 The general model

The setting

We are now ready to define our framework for the open quantum dynamics of a surface. We are interested in the dynamics of a quantum surface $\mathcal{S} = (j_1 \dots j_N)$ whose total area $A = \sum_{p=1}^N j_p$ is supposed to be a constant of motion, having in mind a black hole at equilibrium³. We recall that in loop quantum gravity \mathcal{S} is just a part of a spin network state and the remaining degrees of freedom will here be considered as its environment. The Hilbert space $\mathcal{H}_{\mathcal{S}}$ we work with is

$$\mathcal{H}_{\mathcal{S}} = \bigoplus_{A=\sum_{p=1}^N j_p} \left(\bigotimes_{p=1}^N V^{j_p} \right) \quad (7.22)$$

Using the deformation formalism, we can use the operators E_{ij} to construct a natural interaction $H_{\mathcal{S}\mathcal{E}}$ between \mathcal{S} and \mathcal{E} where the environment excites each deformation E_{ij} through an operator V_{ij} so that

$$H_{\mathcal{S}\mathcal{E}} = \sum_{i,j=1}^N E_{ij} \otimes V_{ij} \quad (7.23)$$

For now, we don't specify the explicit form of those operators. Hermiticity requires that $V_{ij}^\dagger = V_{ji}$. Such an interaction can be seen to emerge from the the bulk/ boundary (and the exterior) correlations. In an effective manner, we encode in this interaction how spins on the boundary are excited by other gravitational degrees of freedom. Moreover, it could also encode the effect of any matter fields present inside or outside the region. The typical environments considered in decoherence studies are bath of harmonic oscillators. That is what we will suppose in the remaining of this chapter. The free Hamiltonian $H_{\mathcal{E}}$ of the environment is thus the energy of a set of harmonic oscillators. From the Hawking radiation, thermal states for the environment seem to be the most natural. Concerning the free dynamics of the system $\mathcal{H}_{\mathcal{S}}$ we suppose that it has a contribution proportional

³Technically, a black hole at equilibrium being a very large semi-classical object will not be an eigenstate of the area operator since its conjugate variable would be undetermined.

to the area of the surface so that $H_S = \sum_{i=1}^N E_{ii}$. Since the area is fixed in our problem, such a term has no contribution to the global dynamics. Of course, it would be natural to include higher order contribution of the E_{ii} operators (as for instance a Bose-Hubbard type term).

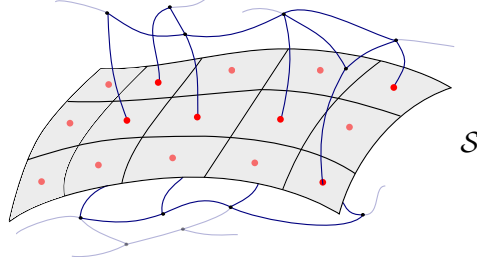


Figure 7.2: A surface \mathcal{S} is defined as a subset of a spin network while its environment is the remaining. The origin of the interaction between the patches comes from the structure of the graph.

A first hint

Having now set the dynamics of quantum surface, we would like to understand the influence the environment has on the evolution of states of the system, especially if decoherence occurs for certain privileged states or geometrical quantities.

The standard approach to analyze this open quantum dynamics is to formulate a master equation for the reduced density matrix involving only operators acting on the system encoding the effects of the environment. This is a non-trivial problem. We already saw in the second chapter 1.3 that the Lindblad equation describes the open quantum dynamics of a system by incorporating the effect of the environment into jump operators. Following this route, we would like to analyze here the relevance of this equation with the deformation operators E_{ij} as jump operators for the dynamics of the quantum surface. This would be a natural choice in light of the dynamics (7.23) where the deformation modes are excited by the environment,

$$\frac{d\rho_S}{dt} = \sum_{i,j=1}^N E_{ij}\rho_S E_{ij}^\dagger - \frac{1}{2} \left(E_{ij}^\dagger E_{ij}\rho_S + \rho_S E_{ij}^\dagger E_{ij} \right) = \sum_{i,j=1}^N E_{ij}\rho_S E_{ij}^\dagger - \frac{1}{2} (\mathcal{C}^2 \rho_S + \rho_S \mathcal{C}^2) \quad (7.24)$$

Decoherence can be analyzed from this equation. The straightest route for this is to choose an entanglement witness like the Von Neumann entropy or for simplicity the purity and

look at the evolution of this witness for some quantum superposition as initial condition. The states that entangle the least with the environment are the most classical ones and called approximate pointer states.

For our problem, we can look for instance at the evolution induced by (7.24) on an initial quantum superposition of highest weight states $|\psi_J^A\rangle$. They verify $E_1|\psi_J^A\rangle = (A+J)|\psi_J^A\rangle$, $E_2|\psi_J^A\rangle = (A-J)|\psi_J^A\rangle$ and $E_{ki}|\psi_J^A\rangle = 0$ for $k < i$ with A and J respectively representing the area of the surface and its closure defect. They are the $U(N)$ analogue of the $|j, j\rangle$ state of $SU(2)$. The short time evolution of the purity of the coherence of the reduced density matrix $\text{tr}(\rho_{JJ'}\rho_{J'J})$ with initial state $\frac{|\psi_J^A\rangle + |\psi_{J'}^A\rangle}{\sqrt{2}}$ can be directly obtained from (??)

$$\left. \frac{d}{dt} \text{tr}(\rho_{JJ'}\rho_{J'J}) \right|_{t=0} = -[A(N-2) + J + J' + (J-J')^2] \text{tr} \rho_{JJ'}\rho_{J'J} \quad (7.25)$$

The damping factor is always positive and composed of three terms. The last one $(J-J')^2$ is the one we were looking for which induces a decoherence effect on quantum superposition of geometry with different closure defect. The second shows that the states most immune to entanglement with the environment are the geometries without defect. Finally, the first shows that the greater the area is the more entangled the system will be.

This setting would be the ideal situation to study the dynamics of an open quantum surface. Nonetheless, many questions remain to be clarified in order to confirm the validity of the above. What we will see here from exact and approximate special cases is that

- The Markovian hypothesis must be discussed and its validity clarified. We will see that for a non dynamical environment, this hypothesis is restricted to work only on short timescales.
- Recoherence is not excluded for superposition of states with different values of the defect (because of the compactness of $SU(2)$). This cannot be seen with a short time analysis. This phenomenon disappears when a large and dynamical environment is considered.

7.2.4 Beginning with a toy model

In this section, we study the open dynamics of a toy model of a quantum surface with two faces and focus on decoherence effects. The end goal is to have a clear understanding of the long time behavior of the system, to exhibit the pointer states and their physical significance. Those steps will serve as the basis for the analysis of a more realistic model of the open surface dynamics in quantum gravity. We limit our exact study to the measurement limit by neglecting the free dynamics of the environment. Thus the domain of validity of the following results is in fact limited to timescales smaller than any dynamical times of the environment.

Motivation

From the interaction $H_{\mathcal{SE}} = \sum_{i,j=1}^N E_{ij} \otimes V_{ij}$ we can motivate the introduction of the toy model by looking explicitly at the $N = 2$ patches model. We work in the subspace of $\mathcal{H}_2 = \mathcal{H}_{HO}^{\otimes 4}$ with fixed area. By introducing the operators

$$L_z = \frac{E_{11} - E_{22}}{2}, \quad L_+ = E_{12}, \quad L_- = L_+^\dagger \quad (7.26)$$

we can rewrite the interaction in the form

$$\begin{aligned} H_{\mathcal{SE}} = & \left(L_z \otimes \frac{V_{11} - V_{22}}{2} \right) + \left(L_x \otimes \frac{V_{12} + V_{21}}{2} \right) + \left(L_y \otimes \frac{V_{21} - V_{12}}{2i} \right) \\ & + \left(\frac{E_{11} + E_{22}}{2} \otimes \frac{V_{11} + V_{22}}{2} \right) \end{aligned} \quad (7.27)$$

Using those definitions, we can check the form of the $U(N)$ Casimir operator (7.15) explicitly and obtain with the notation of this section that $\sum_{ij} E_{ij}^\dagger E_{ij} = \frac{E^2}{2} + 2\mathbf{L} \cdot \mathbf{L}$ with E the total energy of the oscillators describing the patches. Thus we have $\mathbf{L}^2 = \mathbf{J}^2$ and the eigenvalues of \mathbf{L} correspond exactly to the closure defect of the surface. Nonetheless, $\mathbf{L} \neq \mathbf{J}$ except in the special case where the spins are decoupled.

For concreteness, we choose in the remaining the environment to be a bath of harmonic oscillators. We will look at the case of only one oscillator at first and then generalize to an arbitrary number. So the model Hamiltonian we consider is

$$H_{\mathcal{SE}} = \mathbf{L} \otimes \mathbf{p} + \left(\frac{E_{11} + E_{22}}{2} \otimes \frac{V_{11} + V_{22}}{2} \right) \quad (7.28)$$

with \mathbf{p} the momentum operator of the harmonic oscillator. We then have a dynamics with the three directions of a spin coupled to the environment with an additional coupling to the energy of the oscillators and the spin. Since the area (i.e. the total energy of the oscillators) is fixed, the second term is non-dynamical⁴. The first term of this interaction is the non-trivial one and involves three non-commuting observables coupled to the environment.

The program of this section is first to look at the potential decoherence effect induced by the interaction $\mathbf{L} \otimes \mathbf{p}$ with a single oscillator as the environment and then explore the consequences of a large environment.

The reduced density matrix

We focus now on the study of the spin part of the interaction $H_{\mathcal{SE}} = \mathbf{L} \otimes \mathbf{p}$ with a single mode environment and want to understand the decoherence it induces. We look

⁴ If we had not fix the area, this second term in $(E_{11} + E_{22}) \otimes (V_{11} + V_{22})$ would very likely imply a decoherence of quantum superposition of the area, and thus lead to a classical notion of surface area at late time.

at a possible decoherence on the value of the total spin j which is the quantum number associated to the operator \mathbf{L}^2 , the same \mathbf{L} that appears in the interaction. Since we have in mind the dynamics of the horizon of a black hole, we naturally consider our system to be the spin. On the other hand, it would be also legitimate to reverse the problem and focus on the induced dynamics of the oscillator describing the exterior observer.

Going back to the surface, the idea is then to study the evolution of a superposition of coherent states of the system

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|j, g\rangle + |j', g'\rangle) \quad (7.29)$$

The initial state of the harmonic environment is supposed to be the vacuum and uncorrelated to the state of the system so that the global initial state is

$$|\psi_{\mathcal{SE}}\rangle = |\psi\rangle \otimes |0\rangle \quad (7.30)$$

The evolution of this state is obtained most simply by developing the vacuum state on the momentum basis of the environment $|0\rangle = \int_{\mathbb{R}^3} \phi(\mathbf{p}) |\mathbf{p}\rangle d\mathbf{p}$ where $\phi(\mathbf{p}) \propto e^{-\lambda p^2}$ is a Gaussian wave function (the parameter λ is not specified explicitly for it will be easier to obtain more general results keeping it) since we have $U(t)|j, g\rangle |\mathbf{p}\rangle = |j, e^{-\frac{it}{\hbar} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2}} g\rangle |\mathbf{p}\rangle$, with $\boldsymbol{\sigma} \equiv (\sigma_i)$ and σ_i the Pauli matrices normalized to $\sigma_i^2 = \mathbb{1}$. The system remains in a coherent state when the environment is in an eigenstate of the momentum operator.

The central object we want to calculate is the reduced density matrix of the system $\rho_S(t) = \text{tr}_{\mathcal{E}} (U(t) \rho_{\mathcal{SE}}(0) U^{-1}(t))$ which characterizes completely the dynamics of the system alone. Decoherence effects will be seen by analyzing the long time evolution of the $j \neq j'$ matrix elements and by showing that they tend to zero. By introducing projection operators P_j on the subspace of spin j , we can then focus on certain elements of the reduced density matrix $\rho_S^{jj'}(t) = P_j \rho_S(t) P_{j'}$. Those operators $\rho_S^{jj'}(t)$ contain all the information about the coherence between superposition of spins j and j' .

On the decoherence timescales

We will not discuss, here, the detailed analysis presented in [Feller and Livine, 2017b] and we present here the general results coming out of this study.

The form of the projected reduced density matrix can be obtained exactly at any time and Figure 7.3 represents its typical behavior in the two distinct cases of a boson-boson type superposition and a boson-fermion type superposition. Since the former has a non-zero limit as time goes to infinity, a coherence always remains between those states.

The short timescale behavior is dominated by a Gaussian decay with a damping time inversely proportional to the “squared distance” between the spins $(j - j')^2$ but also to

⁴Considering a thermal state for the environment would be more accurate from our knowledge of Hawking radiation. But since in this first investigation $H_{\mathcal{E}} = 0$, this case is not relevant.

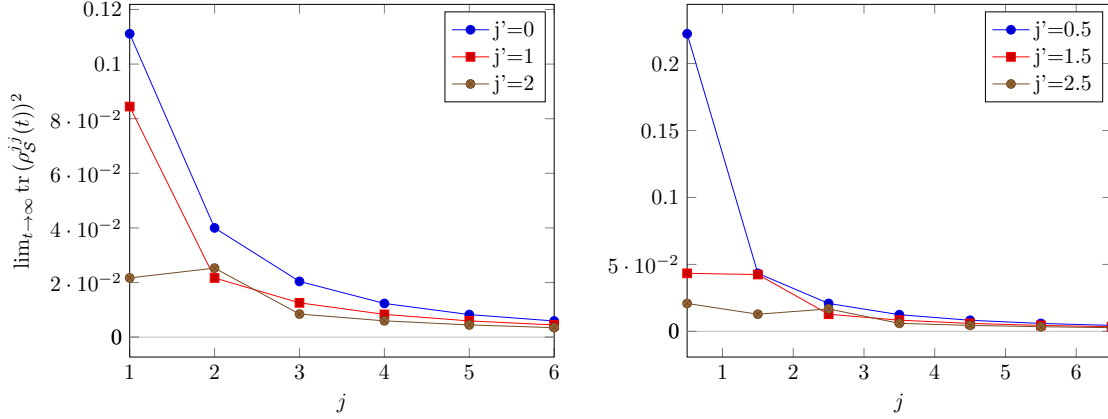


Figure 7.3: Some numerical values of the limit of the coherence at infinity. Each value is a rational number that can be obtained by evaluating formula (??). A coherence remains only for integer or half-integer superpositions but tends rapidly to zero as the spins get higher.

their sum $j + j'$. This evolution can be obtained by a straightforward expansion at leading order in the time t of the exact equation,

$$\text{tr} \rho_S^{jj'}(t) \rho_S^{j'j}(t) \underset{t \rightarrow 0}{\simeq} \text{tr} \rho_S^{00} \rho_S^{00} (1 - [(j + j') + (j - j')^2]) \frac{t^2}{4} \quad (7.31)$$

This suggests at first sight a decoherence between states with different spins. Moreover, the state most immune to the interaction with the environment is the rotation invariant $j = 0$ state as suggested by the $j + j'$ damping. This is natural in light of the interaction which couples the three rotation operators L_i to the environment.

However, in the long run, coherence re-appears in the superposition and different conclusions must be drawn. Re-coherence is a natural phenomenon when a finite size environment (with all free dynamics taken into account) is considered. The associated timescale depends on the number of modes of the environment. Only in the limit of an infinite size environment can we obtain a true decoherence but for all practical purposes the timescale can be extremely long.

As stated previously, for the problem at hand, a coherence remains between superposition of two integer or two half-integer spins. The limit value of the norm depends of course on the spins of the superposition. For instance we had the scaling law in $1/(2j + 1)^2$ for the case with $j' = 0$. For integer/half-integer spins superposition the coherence dies out as time goes to infinity but with a typical timescale completely independent of the spins.

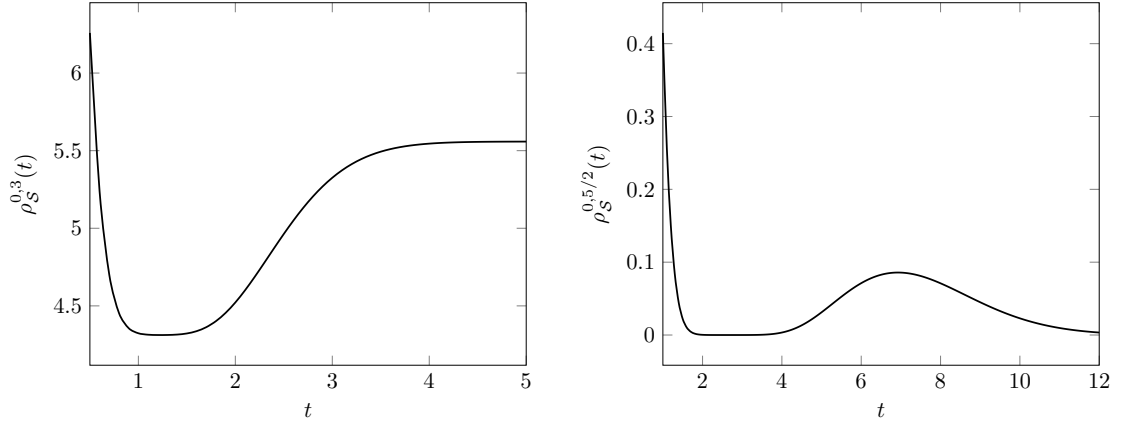


Figure 7.4: Numerical evaluations of the two typical behaviors of the norm of the spin coherence (the beginning is Gaussian and has been omitted in the plots to highlights the non-trivial structures). At first the coherence tends to diminish. However a re-coherence occurs and depending on the nature of the superposition the coherence saturates to a non-zero value (boson/boson like superposition) or tends to zero (fermion/boson like superposition).

7.2.5 Master equation approaches

Most models of open quantum systems and studies of decoherence are not exactly solvable and approximate methods have to be developed. Master equations based on Born-Markov approximations are the ones most commonly used for analyzing open quantum dynamics in quantum optics and condensed matter physics. They are equations for the reduced density matrix of the system taking into account the effects of the environment to first order. They are relevant for understanding the behavior of the system at a time t much longer than any correlation times τ_c but still shorter than dynamical timescales T : $\tau_c \ll t \ll T$. This is the essence of the Markov approximation. A large environment is needed to neglect the changes of the state of the environment due to the coupling to the system and correlations up to second order.

In the following, we apply the master equation methods to the problem of open quantum surface dynamics by first deriving the Born-Markov master equation. This step will motivate a more phenomenological approach by postulating jump operators for the Lindblad equation. The results of those different approaches are then compared to the exact results obtained previously.

Born-Markov equation

An approximate equation for the reduced density matrix of the system can be derived by an expansion of the exact equation of motion $\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)]$ [Haroche and Raimond, 2013]. For an interaction written as $H_{SE} = \sum_i S_i \otimes E_i$, it has the general form

$$\begin{aligned} \frac{d\rho_S(t)}{dt} = & -\frac{i}{\hbar}[H_S, \rho_S(t)] \\ & + \frac{1}{\hbar^2} \sum_i U_i(t) \rho_S(t) S_i + S_i \rho_S(t) U_i^\dagger(t) - S_i U_i(t) \rho_S(t) - \rho_S(t) U_i^\dagger(t) S_i \end{aligned} \quad (7.32)$$

with the operators $U_i(t) = \int_0^t \sum_j g_{ij}(\tau) S_j(-\tau) d\tau$ encoding the action of the environment on the system and depending on its correlation functions $g_{ij}(\tau) = \langle E_i(t) E_j(t - \tau) \rangle_{\rho_E}$.

To go further, the behavior in time of the correlation functions must be discussed. It depends naturally on the proper dynamics of the environment H_E and on the state ρ_E . For a dynamical environment, the correlation functions decay over a timescale τ_c called correlation time or memory time. Denoting by v an order of magnitude of an element of matrix of the interaction, equation (7.32) is an expansion in the parameter $v\tau_c/\hbar$. The order of magnitude of the coupling in the Born-Markov equation is $v^2\tau_c/\hbar$ which is then much smaller than the memory frequency τ_c^{-1} in the short memory time approximation. The complete Born-Markov equation is then obtained by approximating the integral in $U_i(t)$ by its value at infinite time giving in the end a pure local in time equation of motion. However, if the environment were small or non dynamical, the natural expansion parameter would be vt/\hbar and the results of the Born-Markov equation would be inaccurate on long timescales and the time dependence of the correlation functions must be kept. This is an issue we will discuss further in the section comparing the different approaches.

Now for the specific problem we are interested in, we use the interaction (7.23) and express the Born-Markov equation. The equation is here simplified by the fact that we neglect the proper dynamics of the surface. In particular, the operators U_{ij} have the simple form $U_{ij} = \sum_{kl} \left(\int_0^t g_{ij,kl}(\tau) d\tau \right) E_{kl}$. After some straightforward algebra using the $U(N)$ commutation relations, we have the equation

$$\begin{aligned} \frac{d}{dt} \rho_S(t) = & - \sum_{ijk} E_{ij} \rho_S(t) \int_0^t (g_{ik,kj}(\tau) - g_{ki,jk}(\tau)) d\tau \\ & + \rho_S(t) E_{ij} \int_0^t (g_{ik,kj}(-\tau) - g_{ki,jk}(-\tau)) d\tau \\ & + \sum_{ijkl} E_{kl} [E_{ij}, \rho_S(t)] \int_0^t g_{ij,kl}(\tau) d\tau + [\rho_S(t), E_{ij}] E_{kl} \int_0^t g_{kl,ij}(-\tau) d\tau \end{aligned} \quad (7.33)$$

To go further we have to specify the form of the correlation functions. First it is natural to expect the correlation functions to be symmetric in time. To be more specific, let's

imagine we have an harmonic environment and that the operator V_{ij} creates a photon at j with creation operator γ_j^\dagger (a quanta of area is destroyed) and absorbs one at i with destruction operator γ_i (a quanta of area is created) so $V_{ij} = \gamma_i^\dagger \gamma_j$. For the environment in the vacuum state or thermal state (any Gaussian states), the Wick theorem applies and allows to develop the correlation functions. Replacing those correlation functions into the master equation is then straightforward. For an isotropic, homogeneous non dynamical environment, we obtain the simplest form of the equation

$$\frac{d\rho_S}{dt} = \sum_{i,j=1}^N \kappa(t) \left[E_{ij} \rho_S E_{ij}^\dagger - \frac{1}{2} \left(E_{ij}^\dagger E_{ij} \rho_S + \rho_S E_{ij}^\dagger E_{ij} \right) \right] \quad (7.34)$$

where $\kappa(t) = t\kappa$ with κ a constant function of the correlation function. This master equation has the Lindblad form. In the full Born-Markov approximation, $\kappa(t)$ would be independent of time and a decoherence would be expected a priori with an exponential decay e^{-t/τ_d} with τ_d a decoherence timescale. Here however, the linear time dependence caused by the non dynamical character of the environment (non markovianity) leads to a decoherence with a Gaussian behavior e^{-t^2/τ_d^2} . This form is in full agreement with the short time exact calculations (7.31).

7.2.6 Lindblad approach

Once again, we focus on the simplest $N = 2$ patches model with the spin interaction part and take a phenomenological approach to it with the Lindblad master equation. The jump operators (Lindblad operators) are the spin L_i operators and no free dynamics is supposed to occur for the system. We should not forget that we really consider the Schwinger representation here and that we work not in the Hilbert space at a given spin j . Superposition of states with different values of the spin j are permitted. Let's emphasize their are some subtleties regarding the correlation functions and the definition of the jump operators in order to compare those master equation approaches to the exact dynamics proposed in the last section due to the hypothesis of a non-dynamical environment $H_E = 0$. We keep in mind this important point but discuss now in a phenomenological way a Lindblad equation with J_i jump operators as done traditionally in quantum optics models.

The master equation we propose to study is thus

$$\frac{d\rho_S}{dt} = \sum_{i=x,y,z} L_i \rho_S L_i - \frac{1}{2} (L_i L_i \rho_S + \rho_S L_i L_i) = \sum_{i=x,y,z} L_i \rho_S L_i - \frac{1}{2} (\mathbf{L}^2 \rho_S + \rho_S \mathbf{L}^2) \quad (7.35)$$

For the surface dynamics we are ultimately interested in, we want to understand if their is a decoherence phenomenon on a superposition with different values of the spin j . Since \mathbf{L}^2 commutes with the jump operators, the environment does not induce transitions between states with different spins and no dissipation occurs. To focus on coherence between

different spin states, we can look again at the projection of the reduced density matrix $\rho_{kl} = P_k \rho P_l$ with $P_{k,l}$ the projection operator on the subspace of spin k and l respectively.

$$\frac{d\rho_{kl}}{dt} = \sum_{i=x,y,z} L_i \rho_{kl} L_i - \frac{1}{2} (k(k+1) + l(l+1)) \rho_{kl}$$

Searching for pointer states (approximate pointer states generally) requires to evaluate an entanglement witness such as the Von Neumann entropy or the purity of the states $\text{tr} \rho^2(t)$ ⁵. For our purpose we will mostly focus on the purity of the projected reduced density matrix.

$$\frac{d \text{tr} \rho_{kl} \rho_{kl}^\dagger}{dt} = \sum_{i=x,y,z} 2 \text{tr} L_i \rho_{kl} L_i \rho_{kl}^\dagger - (k(k+1) + l(l+1)) \text{tr} \rho_{kl} \rho_{kl}^\dagger \quad (7.36)$$

Let's for instance look at the short time evolution of the superposition $|\psi\rangle = \frac{|k,k\rangle + |l,l\rangle}{\sqrt{2}}$,

$$\left. \frac{d \text{tr} \rho_{kl} \rho_{kl}^\dagger}{dt} \right|_{t=0} = [2kl - (k(k+1) + l(l+1))] \text{tr} \rho_{kl} \rho_{kl}^\dagger = -[(k-l)^2 + (k+l)] \text{tr} \rho_{kl} \rho_{kl}^\dagger \quad (7.37)$$

We thus qualitatively see that a superposition of different spin states leads to a more rapid entanglement with the environment than a state with definite spin. Moreover we see that only a rotation invariant state is immune to entanglement (at first order) with the environment whereas even a state with a definite spin gets entangled with its environment (the higher the spin the more entangled). This behavior can be generalized to an arbitrary initial pure state

$$\left. \frac{d \text{tr} \rho_{kl} \rho_{kl}^\dagger}{dt} \right|_{t=0} \leq -[(k-l)^2 + (k+l)] \text{tr} \rho_{kl} \rho_{kl}^\dagger \quad (7.38)$$

Let's discuss now the relations between the different approaches and in particular why the conclusions appear not to be the same. We have explored in two different ways a possible decoherence effect for quantum superposition of states with different spins associated to the closure defect. The first method was based on an exact calculation for the $N = 2$ patches model and the second used the traditional methods of Markovian master equations.

⁵ The choice of one or the other should in a proper limit gives the same approximate results. In fact, to the purity is associated a linear entropy $S_{\text{lin}} = 1 - \text{tr} \rho^2$. This is the first term in the expansion of the Von Neumann entropy. Computing the trace of the successive powers of the density matrix, Renyi entropies can be defined and finally using the replica trick the Von Neumann entropy can be obtained.

- The ingredient for master equations to work is to have a large enough dynamical environment for its correlation functions to vanish on a timescale smaller than any relaxation or observational times. Qualitatively said, the environment is without memory. In this context, we have shown that the surface (approximate) pointer states are those with a given value of the closure defect and that the decoherence factor has an exponential decay e^{-t/τ_d} .

The behavior predicted by the exact approach on a short-time scale eq.(7.31) is in fact Gaussian. This is easily understood when remembering that the exact dynamics was studied for a non-dynamical environment which then acts as a classical fluctuating potential. The integrals in (7.33) cannot be extended to infinite time. We thus have memory effects and a linear dependance in time in $\kappa(t)$ and so linear time dependent jump operators. The Born-Markov analysis eq.(7.33)(7.34) would only be meaningful on short time-scale and would naturally lead to Gaussian decay functions e^{-t^2/τ_d^2} . The difference between Gaussian and exponential decay is thus traced back the memory of the environment controlled by its dynamics.

- The predictions on the decoherence effect differ for the two methods and only match on a short timescale. In particular the exact analysis shows that a recoherence occurs with a non-zero limit (a limit still approaching zero as the spins get higher). If, as expected, the closure defect is associated to a quasi-local energy density and the curvature or torsion it generates, the spin is also expected to be high enough for a black hole. Thus for all practical purposes, we can conclude on an approximate decoherence on the closure defect.

7.3 Conclusion

Decoherence is now a cornerstone of quantum physics to clarify the quantum to classical transition. In a theory of quantum gravity, the geometry is a dynamical and fluctuating field and quantum superpositions of geometry are perfectly allowed states. Their non-observability in the classical regime remains to be clarified in the semi-classical analysis of loop quantum gravity. Our first investigation focus on the open dynamics of a quantum surface coupled to an environment comprising all the other gravitational and matter degrees of freedom of the Universe. This bulk-boundary coupling induces a decoherence and our aim was to understand the emergence of some geometrical super-selection sector.

Through the deformation formalism of quantum geometry, we proposed toy models for the open dynamics of a quantum surface in the context of loop quantum gravity and a natural coupling between a bath of harmonic oscillators and the deformations of the surface. We looked for a decoherence on the closure defect of a surface with fixed area using two different methods: one exact method analyzing the physical effect on a superposition of the interaction part of the Hamiltonian (quantum measurement limit) and the other using master equations approaches under Born-Markov approximations. The two approaches

agree on the short timescale and indicate a decoherence of quantum superposition of states with different spins associated to the closure defect. The decoherence factor is here a Gaussian decaying with a timescale inversely proportional to the spin difference. However due to the different treatment of the structure of the environment, the conclusions differ as time goes to infinity. The exact treatment neglects the proper dynamics of the environment which thus has an infinite correlation time (constant correlation functions) and leads to a re-coherence of integer/integer or half-integer/half-integer superpositions. Nonetheless this non-zero limit is for all practical purposes irrelevant when large spins are considered, which is potentially the case for black holes.

The take-home message from this analysis of decoherence of toy models in the context of quantum gravity and quantum geometry is that bulk and exterior degrees of freedom of the gravitational field (and matter fields) induce a decoherence on geometrical quantities on the boundary.

Conclusion - Perspectives

Here we come to the conclusion of this work. Quantum gravity is still after decades of investigation an open problem of theoretical physics. Loop quantum gravity is a non-perturbative proposal based historically on a canonical quantization of general relativity. The key insight coming from it is that geometry at the Planck scale is discrete. The quantum state of space are described by spin network states encoding geometry in its algebraic structure. At the moment, two issues have to be solved in this theory: the problem of localization, which amounts to properly define distance, time, reference frame and the dynamics of physical observables, and the problem of the semi-classical regime, which amounts to recovering classical continuum general relativity in a proper approximation scheme.

The approach we advocate to tackle the issue is to have a quantum information perspective combining it with an holographic perspective on gravitation. The working philosophy was to build toy models motivated by different areas of physics, going from loop quantum gravity to condensed matter physics. The reason why we cannot attack those questions from first principles is that we do not have at the moment a complete theory of quantum gravity and our principal motivation is indeed to shed some lights on what the full quantum theory should look like. The general flow starts with the general idea of defining the geometry of space from entanglement and goes progressively into the specific analysis of the boundary of regions and surfaces, looking at their entropy, their proper dynamics and potential decoherence mechanism induced by an environment. This work was divided into four chapters. The first one proposes a class of ansatz states useful to analyze the distance from correlations point of view, the role of phase transition in quantum gravity and the issue of the quantum dynamics and the Hamiltonian constraint. The second one follows the same strategy and looks at the potential form of holographic states in loop quantum gravity. The last two chapters focused on the physics of surfaces seen as a set of interacting punctures. We began by studying a set of semi-classical dynamics with nice physical properties like relaxation to equilibrium or phase transitions. The last part dealt with the open quantum dynamics of boundaries with a particular focus on decoherence effects on geometrical properties of the boundary induced by the bulk-boundary coupling.

This work opens a lot of new perspectives to investigate in the quantum gravity

context.

- On the distance from correlations proposal, loop quantum gravity appears to be the perfect test grounds for a more general treatment using mutual information. For instance, it seems reasonable to implement the procedure presented in [Cao et al., 2017] in the context of loop quantum gravity.
- On holography and surface dynamics, the natural way to go is to have a better understanding of those models with black hole physics and the fundamental parameters of loop gravity. For instance, we could have a better understanding of black hole phase transitions by having an interacting quantum model of punctures (Bose-Hubbard model for instance) and make predictions on potential corrections to the Hawking spectrum. Moreover, this would most probably allow to have a better view on the relationships between Hawking radiation (given by the spectrum of our model) and quasi-normal modes of black holes (given by the analysis of perturbations). This would give the first dynamical model of black holes in loop quantum gravity.
- On open quantum dynamics and decoherence, many directions can be taken. We are in need of a more general framework to understand decoherence phenomena in this context. We could pursue our approach by combining it with the recent results on the extended Hilbert space approach [Donnelly and Freidel, 2016] by analyzing general open quantum dynamics with constraints. The idea would be to consider constrained open quantum systems, lift the constraint with extended Hilbert space ideas and define then open quantum dynamics and analyze the consequences of the additional degrees of freedom. It could even be tested in simpler theories like electromagnetism. Questions regarding intrinsic decoherence effects could then be answered, if spontaneous localization occurs, what are the natural pointer states of geometry (can we recover the coherent states proposed in the literature in a dynamical setting?) and if ultimately decoherence in quantum gravity is different from standard decoherence theory.
- Finally, a question that will naturally arise is the question of causality. We advocate to look at spatial geometry from the point of view of correlations. But what about spacetime geometry where relativistic causality is of paramount importance? Here is the surprising result of quantum information theory [Lieb and Robinson, 1972]: even in a non-relativistic context, for a short range interaction, correlations in the system propagate at a finite velocity, defining an effective light cone in the system. This is a very encouraging results for gravity and the reconstruction of geometry from correlations. In some sense to be clarified, causality can fit into this framework and is seen as an emergent property of the system from the underlying dynamics. Even more interesting is the fact that this results was recently generalized for open quantum systems obeying a Markovian dynamics [Poulin, 2010]. We can then envisioned a framework combining all our different angles by having an open

quantum dynamics inducing holography from which spacetime geometry (locality, distance, causality) emerges from correlation properties.

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Entanglement and Decoherence in Loop Quantum Gravity

A quantum theory of gravitation aims at describing the gravitational interaction at every scales of energy and distance. However, understanding the emergence of our classical spacetime is still an open issue in many proposals. This thesis analyzes this problem in loop quantum gravity with tools borrowed from quantum information theory. This is done in several steps. Since loop quantum gravity is still under construction, a pragmatic point of view is advocated and an ansatz for physical states of the gravitational field is studied at first, motivated from condensed matter physics and simple intuitions. We analyze the proposal of reconstructing geometry from correlations. Lessons on the quantum dynamics and the Hamiltonian constraint are extracted. The second aspect of this work focuses on the physics of sub-systems and especially the physics of their boundary. We begin by calculating the entanglement entropy between the interior and the exterior of the region, recovering the holographic law known from classical black hole physics. Then different boundary dynamics are studied, both in the isolated and open cases, which shed lights again on the fundamental dynamics. Finally, the last aspect of this research studies the dynamics of the boundary interacting with an environment whose degrees of freedom (gravitational or matter) forming the rest of the Universe and especially the decoherence it induces. This allows to discuss the quantum-to-classical transition and understand, in a given model, the pointer states of geometry.

Intrication et Décohérence en Gravitation Quantique à Boucles

Une théorie de gravitation quantique propose de décrire l'interaction gravitationnelle à toutes les échelles de distance et d'énergie. Cependant, comprendre l'émergence de notre espace-temps classique reste un problème toujours ouvert. Cette thèse s'y attaque en gravité quantique à boucles à partir d'outils de l'information quantique. Ceci est fait en plusieurs étapes. La gravité quantique à boucles étant toujours une théorie en cours de développement, un point de vue pragmatique est adopté en étudiant une classe d'états physiques du champ gravitationnel, motivée à la fois par des intuitions simples et les résultats de la physique à N corps. Une analyse de la reconstruction de la géométrie à partir des corrélations peut être faite et des leçons peuvent être tirées sur la forme de la dynamique fondamentale. Dans un second temps, la physique des sous-systèmes est analysée en commençant d'abord par évaluer l'entropie d'intrication entre l'intérieur et l'extérieur de la région, permettant ainsi de retrouver la loi holographique de l'entropie des trous noirs et donnant une forme possible des états holographiques de la théorie. Plusieurs dynamiques de la frontière, vue comme un système isolé ou ouvert, sont ensuite analysées, éclairant de nouveau la forme de la dynamique fondamentale. Enfin, la dernière étape de ces recherches étudie la dynamique de la frontière en interaction avec un environnement formé des degrés de liberté (de matière ou gravitationnels) formant le reste de l'Univers et la décohérence sur la frontière qu'il induit. Ceci permet de discuter la transition quantique/classique et de mettre en lumière, dans un modèle donné, les états pointeurs de la géométrie.