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## Some problems in harmonic analysis on quantum groups

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## DOCTORAL THESIS

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# Some problems in harmonic analysis on quantum groups

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submitted by

**Simeng WANG**

in partial fulfillment of the requirements for the doctoral degrees in the  
**Université de Franche-Comté - École Doctorale Carnot-Pasteur**  
and  
**Institute of Mathematics, Polish Academy of Sciences**

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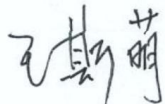
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I declare that this thesis is the result of my own work.

April 8, 2016

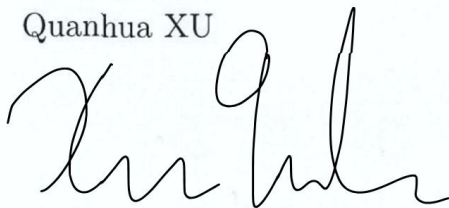
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We declare that the thesis is ready to be presented to the referees.

April 8, 2016

Quanhua XU



April 8, 2016

Adam SKALSKI





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# Résumé

Cette thèse étudie quelques problèmes d'analyse harmonique sur les groupes quantiques compacts. Elle consiste en trois parties.

La première partie présente la théorie  $L^p$  élémentaire des transformées de Fourier, les convolutions et les multiplicateurs sur les groupes quantiques compacts, y compris la théorie de Hausdorff-Young et les inégalités de Young.

Dans la seconde partie, nous caractérisons les opérateurs de convolution positifs sur un groupe quantique fini qui envoient  $L^p$  dans  $L^2$ , et donnons aussi quelques constructions sur les groupes quantiques compacts infinis. La méthode pour étudier les états non-dégénérés fournit une formule générale pour calculer les états idempotents associés aux images de Hopf, qui généralise un travail de Banica, Franz et Skalski.

La troisième partie est consacrée à l'étude des ensembles de Sidon, des ensembles  $\Lambda(p)$  et des notions associées pour les groupes quantiques compacts. Nous établissons différentes caractérisations des ensembles de Sidon, et en particulier nous démontrons que tout ensemble de Sidon est un ensemble de Sidon fort au sens de Picardello. Nous donnons quelques liens entre les ensembles de Sidon, les ensembles  $\Lambda(p)$  et les lacunarités pour les multiplicateurs de Fourier sur  $L^p$ , généralisant un travail de Blendek et Michaliček. Nous démontrons aussi l'existence des ensembles de type  $\Lambda(p)$  pour les systèmes orthogonaux dans les espaces  $L^p$  non commutatifs, et déduisons les propriétés correspondantes pour les groupes quantiques compacts. Nous considérons aussi les ensembles de Sidon centraux, et nous prouvons que les groupes quantiques compacts ayant les mêmes règles de fusion et les mêmes fonctions de dimension ont des ensembles de Sidon centraux identiques. Quelques exemples sont aussi étudiés dans cette thèse.

Les travaux présentés dans cette thèse se basent sur deux articles de l'auteur. Le premier s'intitule " $L_p$ -improving convolution operators on finite quantum groups" et a été accepté pour publication dans *Indiana University Mathematics Journal*, et le deuxième est un travail intitulé "Lacunary Fourier series for compact quantum groups" et a été publié en ligne dans *Communications in Mathematical Physics*.

## Mots-clefs

Groupes quantiques compacts, espaces  $L^p$  non commutatifs, opérateurs de convolution positifs, séries de Fourier, multiplicateurs de Fourier, ensembles de Sidon, ensembles  $\Lambda(p)$ .

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# Streszczenie

Niniejsza rozprawa dotyczy pewnych zagadnień teorii analizy harmonicznej na zwartych grupach kwantowych. Składa się z trzech części. Pierwsza przedstawia w kontekście grup kwantowych podstawy teorii transformat Fouriera w przestrzeniach  $L^p$ , splotów i mnożników, wraz z teorią Hausdorffa-Younga i nierównościami Younga.

W drugiej części charakteryzujemy dodatnie operatory splotu na skończonej grupie kwantowej poprawiające całkowalność z  $p$ -tą potęgą oraz przedstawiamy konstrukcje pewnych operatorów splotu poprawiających całkowalność z  $p$ -tą potęgą w przypadku nieskończeniowym. Metody opisujące stany niezdegenerowane pozwalają podać ogólny wzór na obliczanie stanów idempotentnych związanych z obrazami Hopfa, uogólniając tym samym wcześniejsze wyniki Banicy, Franza i Skalskiego.

Trzecia część jest poświęcona badaniem zbiorów Sidona, zbiorów  $\Lambda(p)$  i pewnych innych rodzajów zbiorów lakunarnych dla zwartych grup kwantowych. Dowodzimy w niej kilku równoważnych charakteryzacji zbiorów Sidona, pokazując w szczególności, że każdy zbiór Sidona w grupie dyskretnej jest silnym zbiorem Sidona w sensie Picardello. Podajemy liczne związki między zbiorami Sidona, zbiorami  $\Lambda(p)$  i lakunarnością rozumianą poprzez  $L^p$ -mnożniki Fourierowskie, uogólniając wcześniejsze rezultaty Blendeka i Michalička. Udowadniamy również istnienie zbiorów  $\Lambda(p)$  dla ogólnych układów ortogonalnych w nieprzemiennej przestrzeniach  $L^p$  i wnioskujemy z niego analogiczny wynik dla zwartych grup kwantowych. Omawiamy centralne zbiory Sidona, które okazują się pokrywać dla zwartych grup kwantowych o identycznych regułach fuzji i funkcji wymiaru. Wreszcie podajemy liczne przykłady rozważanych klas zbiorów.

## Słowa kluczowe

Zwarte grupy kwantowe, nieprzemienne przestrzenie  $L^p$ , operatory poprawiające całkowalność z  $p$ -tą potęgą, dodatnie operatory splotu, szeregi Fouriera, mnożniki Fouriera, zbiory Sidona, zbiory  $\Lambda(p)$ .

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# Abstract

This thesis studies some problems in the theory of harmonic analysis on compact quantum groups. It consists of three parts. The first part presents some elementary  $L^p$  theory of Fourier transforms, convolutions and multipliers on compact quantum groups, including the Hausdorff-Young theory and Young's inequalities. In the second part, we characterize positive convolution operators on a finite quantum group  $\mathbb{G}$  which are  $L^p$ -improving, and also give some constructions on infinite compact quantum groups. The methods for non-degenerate states yield a general formula for computing idempotent states associated to Hopf images, which generalizes earlier work of Banica, Franz and Skalski. The third part is devoted to the study of Sidon sets,  $\Lambda(p)$ -sets and some related notions for compact quantum groups. We establish several different characterizations of Sidon sets, and in particular prove that any Sidon set in a discrete group is a strong Sidon set in the sense of Picardello. We give several relations between Sidon sets,  $\Lambda(p)$ -sets and lacunarities for  $L^p$ -Fourier multipliers, generalizing a previous work by Blendek and Michaliček. We also prove the existence of  $\Lambda(p)$ -sets for orthogonal systems in noncommutative  $L^p$ -spaces, and deduce the corresponding properties for compact quantum groups. Central Sidon sets are also discussed, and it turns out that the compact quantum groups with the same fusion rules and the same dimension functions have identical central Sidon sets. Several examples are also included.

The thesis is principally based on two works by the author, entitled " $L^p$ -improving convolution operators on finite quantum groups" and "Lacunary Fourier series for compact quantum groups", which have been accepted for publication in *Indiana University Mathematics Journal* and *Communications in Mathematical Physics* respectively.

## Keywords

Compact quantum groups, noncommutative  $L^p$ -spaces,  $L^p$ -improving operators, positive convolution operators, Fourier series, Fourier multipliers, Sidon sets,  $\Lambda(p)$ -sets.



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# Introduction

Cette thèse étudie la théorie de l’analyse harmonique sur les groupes quantiques compacts. Ce travail est une extension naturelle de l’étude de l’analyse harmonique abstraite initiée au cours du siècle dernier. Historiquement, l’idée principale de l’analyse harmonique abstraite est d’étudier l’analyse sur les espaces de fonction ou les algèbres de groupes des groupes localement compacts, comme un analogue de la théorie de l’analyse de Fourier sur la droite des nombres réelle ou sur le cercle (voir par exemple [Eym64, HR70, Rud90]). Depuis une vingtaine d’années, la théorie des groupes quantiques topologiques s’est développée rapidement à travers la définition d’un groupe quantique compact de Woronowicz [Wor98] et le cadre des groupes quantiques localement compacts de Kustermans et Vaes [KV00], qui donne un cadre plus général et unifié à l’analyse harmonique abstraite. En particulier, la théorie de Hausdorff-Young et l’étude des multiplicateurs complètement borné pour les groupes quantiques localement compacts, ont connu des progrès importants au cours des dernières années (voir [Cas13, Coo10, Daw12, JNR09]).

Cette thèse vise à étudier l’analyse de Fourier sur les espace  $L^p$  associés aux groupes quantiques compacts. En comparaison avec l’étude abstraite des groupes quantiques localement compacts, nous présenterons une approche plus concise de la théorie élémentaire de l’analyse de Fourier spécifiquement dans le cadre des groupes quantiques compacts. En particulier, nous allons établir la théorie de Hausdorff-Young, les inégalités de Young et quelques faits fondamentaux sur les multiplicateurs de Fourier sur  $L^p$ . L’intérêt principal de la thèse sera quelques sujets typiques d’analyse harmonique, comme les opérateurs de convolution de  $L^p$  dans  $L^2$  et la lacunarité pour les multiplicateurs de Fourier.

Cette thèse consiste en quatre chapitres. Elle se base principalement sur deux travaux [Wan14, Wan16] de l’auteur, intitulés “ $L_p$ -improving convolution operators on finite quantum groups” et “Lacunary Fourier series for compact quantum groups”, qui ont été acceptées pour publication dans *Indiana Univ. Math. J.* et *Comm. Math. Phys.* respectivement. Dans cette introduction, nous allons d’abord donner un aperçu historique de l’analyse harmonique abstraite et la théorie des groupes quantiques, et puis nous allons décrire le contenu et les principaux résultats de la thèse.

## 0.1 Aperçu historique de l’analyse harmonique abstraite

L’analyse de Fourier classique a débuté par l’étude des fonctions périodiques sur la droite réelle  $\mathbb{R}$ . Elle est motivée par les études de la corde vibrante et du flux thermique en physique mathématique, où les équations différentielles correspondantes ont été résolus *via* les *séries de Fourier*. Si nous identifions les fonctions  $2\pi$ -périodiques comme les fonctions sur le groupe du cercle unité,  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ , les séries de Fourier sont données par la transformation suivante,

$$\mathcal{F} : L^1(\mathbb{T}) \rightarrow c_0(\mathbb{Z}), \quad \mathcal{F}(f)(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(z) z^{-n} dz, \quad n \in \mathbb{Z},$$



et nous allons considérer l'analyse classique des fonctions périodiques comme les résultats sur le groupe compact abélien  $\mathbb{T}$ .

Depuis les années 1930, le cadre de l'analyse de Fourier est les groupes abéliens localement compacts. L'observation se base sur deux résultats remarquables. Tout d'abord, Haar a prouvé, en 1932, que tout groupe localement compact possède une mesure invariante par translation (à gauche), appelée par la suite la mesure de Haar, qui est un analogue de la mesure de Lebesgue sur  $\mathbb{T}$  et  $\mathbb{R}$ . Deuxièmement, si l'on considère l'ensemble des *caractères* d'un groupe abélien localement compact  $G$ , à savoir

$$\hat{G} = \{\chi : G \rightarrow \mathbb{T} \text{ morphismes continues de groupes}\}, \quad (0.1)$$

muni de la multiplication scalaire et de la topologie compacte-ouverte,  $\hat{G}$  devient un groupe abélien localement compact, et de plus, il existe un isomorphisme entre  $G$  et  $\hat{\hat{G}}$ . Cette construction s'appelle la dualité de *Pontryagin* et a été créé par Pontryagin en 1936. Remarquons que nous avons l'identification

$$\hat{\mathbb{T}} = \mathbb{Z}, \quad \hat{\mathbb{R}} = \mathbb{R}.$$

La dualité de Pontryagin donne une réalisation de la transformation de Fourier pour les groupes abéliens localement compacts, définie par

$$\mathcal{F} : L^1(G) \rightarrow C_0(\hat{G}), \quad \mathcal{F}(f)(\chi) = \frac{1}{2\pi} \int_{\mathbb{T}} f(g) \overline{\chi(g)} dm(g), \quad \chi \in \hat{G},$$

où  $dm$  désigne la mesure de Haar sur  $G$ . Une théorie générale de l'analyse de Fourier peut être établie dans ce cadre, en parallèle à celle classique. Nous renvoyons à [Rud90] pour plus de détails.

Dans l'intérêt de cette thèse, soulignons le cas particulier où  $G$  est un groupe abélien compact. Dans ce cas-là, le dual de Pontryagin  $\Gamma = \hat{G}$  est un groupe abélien discret, et la mesure de Haar sur  $\Gamma$  est juste la mesure de comptage. Comme dans l'analyse classique sur  $\mathbb{T}$ , nous avons un développement en série de Fourier de chaque  $f \in L^1(G)$ ,

$$f \sim \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \gamma.$$

Un certain nombre de problèmes typiques en analyse harmonique peuvent être étudiés dans ce cadre, comme la décroissance de série de Fourier, les ensembles lacunaires, les convolutions et les multiplicateurs de Fourier. Nous donnerons un aperçu de ces considérations dans les sections suivantes.

L'étude de ces sujets dans un cadre plus général peut suivre deux directions, en remarquant dans le paragraphe ci-dessus que l'on peut considérer ou bien le groupe compact  $G$  ou bien le groupe discret  $\Gamma$ .

La première direction de généralisations débute par la suppression de la condition abélienne du groupe compact  $G$  dans le schéma ci-dessus. En conséquence, nous ne pouvons pas obtenir un groupe discret dual  $\hat{G}$  au sens de Pontryagin. Cependant, nous avons une substitution qui généralise la classe de caractères (0.1), à savoir

$$\text{Irr}(G) = \{\pi : G \rightarrow B(H) \text{ représentations unitaires continues irréductibles}\},$$

où les éléments sont pris modulo l'équivalence entre les représentations. Alors  $\text{Irr}(G)$  coïncide avec  $\hat{G}$  lorsque  $G$  est abélien. Selon la théorie de Peter-Weyl, créée dans les années 1930, l'ensemble  $\text{Irr}(G)$  donne une famille orthogonale de fonctions continues sur  $G$ , qui

engendre une sous-algèbre dense de  $C(G)$  et généralise la notion de séries trigonométriques sur le cercle. À partir de là nous pouvons établir une théorie complète de l'analyse de Fourier sur le groupe compact  $G$ , pour laquelle nous renvoyons à [Fol95, Chap.5]. Beaucoup de problèmes de l'analyse de Fourier classique, comme la sommabilité des séries de Fourier et la théorie de Littlewood-Paley, peuvent être discutés dans ce cadre général; en outre, l'analyse de Fourier sur les groupes compacts fournit un outil puissant pour l'étude de la théorie des représentations et de la physique mathématique, surtout pour les sujets liés aux groupes de Lie compacts. Nous renvoyons à [CW71, HR70, Ste70] pour plus de détails.

L'autre direction de développement concerne la suppression de la condition abélienne du groupe discret  $\Gamma$  dans le schéma abélien. De manière similaire, un groupe non abélien discret  $\Gamma$  n'admet pas de groupe compact dual  $\hat{\Gamma}$  au sens de Pontryagin. L'idée de l'analyse de Fourier dans ce cadre profite de l'émergence de la théorie de l'algèbre d'opérateurs fondée dans les années 1940. À chaque groupe discret  $\Gamma$  on peut associer une  $C^*$ -algèbre engendrée par les représentations régulières gauches sur  $\ell^2(\Gamma)$ , noté  $C_r^*(\Gamma)$ . Ce type d'algèbres fait partie des objets essentiels étudiés par la théorie de l'algèbre d'opérateurs jusqu'à aujourd'hui, et il est lié à beaucoup de domaines nouveaux des mathématiques. Le point de vue de l'analyse de Fourier s'appuie sur l'isomorphisme

$$C_r^*(\Gamma) \cong C(\hat{\Gamma}) \quad (0.2)$$

si  $\Gamma$  est abélien. D'autre part, depuis les années 1950, la théorie non commutative de l'intégration a été développée par Segal et Dixmier, qui permet de discuter l'analogue non commutatif des espaces  $L^p$  associés à l'algèbre  $C_r^*(\Gamma)$ . En se basant sur ces observations, nous pouvons voir  $C_r^*(\Gamma)$  comme un analogue des fonctions sur le "dual compact" de  $\Gamma$ , et établir un cadre convenable de l'analyse de Fourier non commutative sur les groupes discrets. Les premiers résultats novateurs dans ce domaine sont dues à Haagerup [Haa79], qui a été motivée par l'étude des propriétés d'approximation des algèbres de von Neumann de groupe. Depuis lors, l'analyse harmonique associée aux groupes non abéliens discrets a été largement discuté, menant à une forte interaction avec l'étude des propriétés d'approximation, l'algèbre d'opérateurs et les espaces d'opérateurs, et les espaces de Banach (voir par exemple [Boz85, BF84, CH89, FTP83, JR03, LDIS11, Pis95]). Dans les dernières décennies, la théorie  $L^p$  des multiplicateurs de Fourier sur ces algèbres a également été développée fructueusement dans [CPPR15, JMP14a, JMP14b, Har99].

Comme nous pouvons le voir ci-dessus, l'analyse de Fourier non commutative est basée sur une certaine généralisation de la dualité de Pontryagin; et les deux directions de recherche ci-dessus découlent essentiellement de différentes approches de cette généralisation, qui ont cependant beaucoup de points communs. On a alors une motivation naturelle pour trouver un cadre plus général unifiant ces différents points de vue. Ce travail a été initiée par Tannaka et Krein dans les années 1930, qui ont étudié la structure duale des groupes compacts *via* les classes de représentations  $\text{Irr}(G)$ . Une théorie plus complète a été ensuite développée par Kac, Vainerman, Enock et Schwartz autour des années 1970; ils ont introduit la notion d'*algèbre de Kac*, qui unifie les différents cadres mentionnés ci-dessus et donne un analogue parfait de la dualité de Pontryagin. Enfin dans les années 1980, une théorie plus générale et plus concise a été créée dans une série d'ouvrages par Woronowicz [Wor87a, Wor88, Wor89, Wor98]. La philosophie de la théorie de Woronowicz se trouve dans une idée typique de la *géométrie non commutative* (notamment dans [Con94]). Classiquement, nous pouvons apporter les observations essentielles suivantes.

(a) Pour un groupe compact  $G$ , la multiplication de groupe sur  $G$  induit une *comultiplication*  $\Delta$  sur l'algèbre  $C(G)$ , donnée par

$$\Delta : C(G) \rightarrow C(G) \otimes C(G), \quad (\Delta f)(g, h) = f(gh), \quad g, h \in G.$$

La paire  $(C(G), \Delta)$  permet de récupérer la structure de  $G$ .

(b) Pour un groupe discret  $\Gamma$ , la structure de  $\Gamma$  induit une certaine *comultiplication*  $\Delta$  sur l'algèbre  $C_r^*(\Gamma)$ . La paire  $(C_r^*(\Gamma), \Delta)$  permet de récupérer la structure de  $\Gamma$ . Rappelons que si  $\Gamma$  est abélien, la structure de  $\Gamma$  permet de plus de récupérer celle du groupe compact  $\hat{\Gamma}$  et  $C_r^*(\Gamma)$  est un analogue de l'algèbre  $C(\hat{\Gamma})$ .

Un *groupe quantique compact* de Woronowicz correspond à une structure similaire  $(A, \Delta)$ , où  $A$  est une  $C^*$ -algèbre unitale, et  $\Delta : A \rightarrow A \otimes A$  est un  $*$ -homomorphisme qui satisfait des propriétés similaires à (a) (voir Définition 1.2.1). En particulier, si  $A$  est commutative, alors  $A = C(G)$  pour un certain groupe compact  $G$ , et  $\Delta$  est donné comme dans (a); si  $A$  est cocommutative (c'est-à-dire  $\Delta$  est invariant par la volte  $x \otimes y \mapsto y \otimes x$ ), alors (à quelques technicités près)  $A = C_r^*(\Gamma)$  pour un certain groupe discret  $\Gamma$ , et  $\Delta$  est donné comme dans (b).

De plus, Woronowicz a établi une théorie analytique parfaite pour les groupes quantiques compacts. Il a prouvé l'existence d'un état de Haar "invariant par translation" sur le groupe quantique, et a développé une théorie du type de Peter-Weyl par rapport à cet état de Haar, qui généralise les objets correspondants pour les groupes compacts. En outre, étant donné un groupe quantique compact, nous pouvons construire un *groupe quantique discret* dual au sens des algèbres de Hopf à multiplicateur de Van Daele [VD94, VD96], ou au sens des groupes quantiques localement compacts de Kustermann-Vaes [KV00], et nous avons un théorème de dualité du type de Pontryagin dans ce cadre. En particulier, un groupe discret  $\Gamma$  peut être considéré comme l'ensemble des représentations irréductibles de son groupe quantique dual compact, et nous pouvons obtenir un langage unifié exprimant les transformées de Fourier, les convolutions et les multiplicateurs (voir Section 2.1).

En conséquence, les deux directions de recherche d'analyse harmonique abstraite mentionnées ci-dessus, séparées pendant plusieurs décennies, se réunissent de nouveau dans le nouveau cadre des groupes quantiques compacts. En revanche, ce nouveau cadre nous motive à chercher des aspects nouveaux de la théorie de l'analyse harmonique. En effet, l'apparition de groupes quantiques compacts peut apporter des exemples nouveaux à la théorie de l'algèbre des opérateurs. Ceci est réalisé *via* l'apparition de plusieurs objets nouveaux qui ressemblent aux groupes compacts, tels que la déformation ou la libération des groupes de Lie compacts dans [BS09, LS91, Wor87b], les symétries quantiques des espaces métriques (quantiques) compacts dans [BBN12, BGS11, VDW96, Wan98], les constructions de produit libre, tensoriel ou croisé dans [Wan95a, Wan95b]. En comparaison avec les exemples découlants des groupes, ces objets apportent leurs intérêts propres. Par exemple, bon nombre de ces exemples donnent lieu à des algèbres de von Neumann de type III ayant un état de Haar non tracial, ce qui ne peut pas arriver dans le cas classique. L'étude de ces nouveaux groupes quantiques lie entre eux plusieurs domaines modernes en mathématiques, comme l'algèbre d'opérateurs, les probabilités quantiques ou libres, la théorie ergodique et la physique mathématique.

Cela nous donne une belle motivation pour explorer l'analyse harmonique sur les groupes quantiques compacts. Au cours de ces dernières années, des sujets dans cette direction ont été largement discutés sous différents aspects. Mentionnons la théorie générale des transformées de Fourier étudiée dans [Kah10, PW90, VD07, Wor98], les multiplicateurs complètement bornés sur les groupes quantiques dans [Daw12, JNR09], et les opérateurs ou semigroupes de convolutions dans [CFK14, FKS15, LS08, LS11]. Toutefois, la théorie  $L^p$  correspondante de l'analyse de Fourier reste inexplorée à l'exception des études très abstraites telles que [Coo10, Cas13]. Dans cette thèse, nous présentons quelques nouveaux progrès dans cette direction et fournissons des approches plus concrètes et maniables con-

cernant la théorie  $L^p$  de l'analyse de Fourier, spécifiquement liées à la structure des groupes quantiques compacts.

## 0.2 Théorie générale

Soit  $\mathbb{G}$  un groupe quantique compact muni de son état de Haar  $h$ . Notons  $\text{Irr}(\mathbb{G})$  les classes d'équivalence de représentations irréductibles de  $\mathbb{G}$ , et  $Q = (Q_\pi)_{\pi \in \text{Irr}(\mathbb{G})}$  l'élément modulaire de  $\hat{\mathbb{G}}$ . En utilisant la notation standard, nous notons  $\text{Pol}(\mathbb{G})$  la  $*$ -algèbre de Hopf des polynômes sur  $\mathbb{G}$ ,  $C_r(\mathbb{G})$  la  $C^*$ -algèbre réduite des fonctions continues sur  $\mathbb{G}$ , et  $c_0(\hat{\mathbb{G}})$  la  $C^*$ -algèbre des fonctions sur  $\hat{\mathbb{G}}$  qui s'annulent à l'infini (voir Section 1.2.1 et 1.2.2). En outre, notons  $L^p(\mathbb{G})$  les espaces  $L^p$  non commutatifs de Kosaki sur  $\mathbb{G}$  par rapport à l'état de Haar  $h$ , et  $\ell^p(\hat{\mathbb{G}})$  les espaces correspondants sur le groupe quantique discret dual  $\hat{\mathbb{G}}$  par rapport au poids de Haar à gauche. On peut aussi établir la théorie similaire relativement au poids de Haar à droite par les modifications standards.

### Transformées de Fourier

La transformation de Fourier pour les groupes quantiques localement compacts a été étudié dans [Coo10], [Cas13] et [Kah10]. Dans le cadre des groupes quantiques compacts, nous pouvons donner une description plus explicite. Pour toute fonctionnelle linéaire  $\varphi$  sur  $\text{Pol}(\mathbb{G})$ , on peut définir une *série de Fourier*

$$\hat{\varphi}(\pi) = (\varphi \otimes \iota)((u^{(\pi)})^*) \in B(H_\pi), \quad \pi \in \text{Irr}(\mathbb{G}).$$

Ceci induit la définition de la série de Fourier pour tout  $x \in \text{Pol}(\mathbb{G})$ , donnée par

$$\hat{x}(\pi) = (h(\cdot x) \otimes \iota)((u^{(\pi)})^*) \in B(H_\pi), \quad \pi \in \text{Irr}(\mathbb{G}).$$

La transformée de Fourier  $\mathcal{F} : x \rightarrow \hat{x}$  peut être étendue aux espaces  $L^p$ . Nous pouvons établir la théorie de Hausdorff-Young suivante sur  $\mathbb{G}$ . (Voir aussi [Cas13, Coo10, PW90].)

**Proposition 0.2.1** (Section 2.1). (a)  $\mathcal{F}$  se prolonge en des contractions sur les espaces  $L^p$ , et

$$\mathcal{F}(L^1(\mathbb{G})) \subset c_0(\hat{\mathbb{G}}), \quad \mathcal{F}(L^p(\mathbb{G})) \subset \ell^q(\hat{\mathbb{G}}), \quad 1 \leq p \leq 2, \quad 1/p + 1/q = 1.$$

De plus,  $\mathcal{F}$  est un opérateur unitaire de  $L^2(\mathbb{G})$  sur  $\ell^2(\hat{\mathbb{G}})$ .

(b) Pour tout  $x \in L^2(\mathbb{G})$ , on a

$$x = \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi (\iota \otimes \text{Tr})[(1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}],$$

où la convergence de la série est au sens de  $L^2$ . Pour tout  $\pi \in \text{Irr}(\mathbb{G})$ , si on note  $\mathcal{E}_\pi$  la projection orthogonale de  $L^2(\mathbb{G})$  sur le sous-espace engendré par les coefficients de la matrice  $(u_{ij}^{(\pi)})_{i,j=1}^{n_\pi}$ , et on écrit  $\mathcal{E}_\pi x = \sum_{i,j} x_{ij}^{(\pi)} u_{ij}^{(\pi)}$  avec  $x_{ij}^{(\pi)} \in \mathbb{C}$ ,  $X_\pi = [x_{ji}^{(\pi)}]_{i,j}$ , alors

$$\hat{x}(\pi) = d_\pi^{-1} X_\pi Q_\pi^{-1}.$$

Nous introduisons également la notion d'*algèbre de Fourier* sur  $\mathbb{G}$ , définie par

$$A(\mathbb{G}) = \{x \in L^1(\mathbb{G}) : \hat{x} \in \ell^1(\hat{\mathbb{G}})\}.$$

**Proposition 0.2.2** (Proposition 2.1.5). *On a  $A(\mathbb{G}) \subset C_r(\mathbb{G})$  et*

$$\|x\|_\infty \leq \|\hat{x}\|_1, \quad x \in A(\mathbb{G}).$$

*De plus, si on note  $\|x\|_A = \|\hat{x}\|_1$  pour  $x \in A(\mathbb{G})$ , alors  $(A(\mathbb{G}), \|\cdot\|_A)$  est un espace de Banach isométriquement isomorphe à  $\ell^1(\hat{\mathbb{G}})$ .*

Ce résultat et sa preuve ont été récemment appliqués et généralisés dans [DPR16] pour discuter la relation avec les conditions de croissance polynomiale de  $\mathbb{G}$ .

### Inégalités de Young

Soient  $\varphi_1, \varphi_2$  deux formes linéaires sur  $\text{Pol}(\mathbb{G})$ . Nous définissons leur convolution par

$$\varphi \star \varphi' = (\varphi \otimes \varphi') \circ \Delta.$$

Cela induit aussi une définition de convolution pour  $x, x' \in \text{Pol}(\mathbb{G})$ , telle que

$$h(\cdot(x \star x')) = h(\cdot x) \star h(\cdot x') (\in \text{Pol}(\mathbb{G})).$$

Nous pouvons établir les inégalités de Young suivantes.

**Proposition 0.2.3** (Section 2.2). (a) *Soient  $1 \leq p, q, r \leq 2$  avec  $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $x, y \in \text{Pol}(\mathbb{G})$ . On a*

$$\|\tau_{i/p'}(y) \star x\|_r \leq \|x\|_p \|y\|_q.$$

(b) *Supposons de plus que  $\mathbb{G}$  est de type Kac. Soient  $1 \leq p, q, r \leq \infty$  avec  $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $x, y \in \text{Pol}(\mathbb{G})$ . On a*

$$\|y \star x\|_r \leq \|x\|_p \|y\|_q.$$

Les inégalités de Young et les sujets connexes pour les groupes quantiques localement compacts, plus généraux, seront discutées dans l'article à venir [LWW16].

### Multiplicateurs de Fourier sur $L^p$

Beaucoup de résultats de cette thèse s'appuient sur certaines propriétés très élémentaires des multiplicateurs de Fourier bornés sur  $L^p$  sur les groupes quantiques compact, qui n'ont pas été jusqu'à présent discutées dans la littérature. Bien que les multiplicateurs complètement bornés sur  $L^\infty$ , à la place des multiplicateurs bornés sur  $L^\infty$ , sont des objets plus naturels pour l'étude de l'analyse harmonique sur les groupes quantiques, les discussions sur les multiplicateurs bornés sur  $L^p$  sont encore naturels pour l'étude de l'analyse de Fourier de  $L^p$  sur les groupes quantiques, surtout pour  $p < \infty$ . Ainsi, nous incluons dans cette thèse un argument détaillé sur quelques faits élémentaires dans la Section 2.3. Ils seront fréquemment utilisés dans d'autres démonstrations de cette thèse. Ici, nous énumérons quelques propriétés importantes.

Pour  $a = (a_\pi)_\pi \in \prod_\pi B(H_\pi)$ , on définit les *multiplicateurs* à gauche et à droite  $m_a^L : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G})$ ,  $m_a^R : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G})$  associés à  $a$  (cf. [JNR09, Daw12]) par

$$(m_a^L \otimes \iota)u^{(\pi)} = (1 \otimes a_\pi)u^{(\pi)}, \quad (m_a^R \otimes \iota)u^{(\pi)} = u^{(\pi)}(1 \otimes a_\pi).$$

On dit que  $a$  est un *multiplicateur à gauche (resp., droit) borné* sur  $L^p(\mathbb{G})$  si  $m_a^L$  (resp.,  $m_a^R$ ) se prolonge en une application bornée sur  $L^p(\mathbb{G})$ , et on note  $M_L(L^p(\mathbb{G}))$  (resp.,  $M_R(L^p(\mathbb{G}))$ ) l'ensemble de tous ces multiplicateurs. On définit

$$M(L^p(\mathbb{G})) = \left\{ a \in \prod_\pi B(H_\pi) : Q^{-1/p} a Q^{1/p} \in M_L(L^p(\mathbb{G})), a \in M_R(L^p(\mathbb{G})) \right\}$$

muni de la norme

$$\|a\|_{M(L^p(\mathbb{G}))} = \max\{\|m_{Q^{-1/p}aQ^{1/p}}^L\|_{B(L^p(\mathbb{G}))}, \|m_a^R\|_{B(L^p(\mathbb{G}))}\}.$$

Notre première propriété concerne la question de comment construire un multiplicateur à gauche borné sur  $L^p(\mathbb{G})$  à partir d'un multiplicateur à droite borné.

**Proposition 0.2.4** (Lemma 2.3.3). *Soit  $1 \leq p \leq \infty$ . On a pour tout  $a \in \prod_{\pi} B(H_{\pi})$ ,*

$$\|m_a^R\|_{B(L^p(\mathbb{G}))} = \|m_{Q^{-1/2}a^*Q^{1/2}}^L\|_{B(L^p(\mathbb{G}))}.$$

Le second résultat concerne l'inégalité comparant les normes  $\|a\|_{\ell^{\infty}(\hat{\mathbb{G}})}$  et  $\|a\|_{M(L^p(\mathbb{G}))}$ . Il est bien connu que si  $\mathbb{G}$  est un groupe compact  $G$  ou le dual compact  $\hat{\Gamma}$  d'un groupe discret  $\Gamma$ , alors pour tout  $1 \leq p \leq \infty$  on a

$$\|a\|_{\infty} \leq \|a\|_{M(L^p(\mathbb{G}))}, \quad a \in M(L^p(\mathbb{G})).$$

Les inégalités sont aussi faciles à traiter pour les multiplicateurs complètement bornés sur  $L^{\infty}$ , mais il semble qu'elles ne sont pas évidentes pour les autres cas. La proposition suivante améliore un résultat de [Daw10, Proposition 8.8] pour les groupes quantiques compacts. Le résultat se base partiellement sur une communication personnelle de Marius Junge.

**Proposition 0.2.5** (Proposition 2.3.5). *Soit  $1 \leq p \leq \infty$ . Pour tout  $a \in \prod_{\pi} B(H_{\pi})$  et  $0 \leq \theta \leq 1$ , on a*

$$\|Q^{\frac{1}{4}-\frac{\theta}{2}}aQ^{-\frac{1}{4}+\frac{\theta}{2}}\|_{\infty} \leq \|m_a^R\|_{B(L^p(\mathbb{G}))}^{1/2} \|m_{Q^{-\theta}aQ^{\theta}}^L\|_{B(L^p(\mathbb{G}))}^{1/2}.$$

*En particulier, si  $\mathbb{G}$  est de type Kac, on a*

$$\|a\|_{\infty} \leq \|a\|_{M(L^p(\mathbb{G}))}.$$

### 0.3 Thèmes spécifiques

Les opérateurs de convolution et les multiplicateurs constituent des éléments essentiels de l'analyse de Fourier. Par exemple, pour une fonction périodique  $f : \mathbb{R} \rightarrow \mathbb{C}$ , la décroissance des coefficients de Fourier est étroitement liée aux propriétés d'intégrabilité de  $f$ , qui sont révélées par le lemme de Riemann-Lebesgue pour les fonctions de  $L^1$  et le théorème de Plancherel pour  $L^2$ . Pour  $1 < p < 2$ , un problème classique en analyse harmonique est de déterminer les conditions sur la fonction  $\zeta : \mathbb{Z} \rightarrow \mathbb{R}^+$  telle que

$$f \in L^p(\mathbb{T}) \Rightarrow \sum_{n \in \mathbb{Z}} \zeta(n) |\hat{f}(n)|^2 < +\infty.$$

Si on prend  $\zeta(n) = |\hat{\mu}(n)|^2$  pour une mesure  $\mu$ , ceci mène aux estimations de normes des opérateurs de convolution par une mesure, en particulier l'existence et le comportement des mesures boréliens qui convolent  $L^p(\mathbb{T})$  dans  $L^2(\mathbb{T})$  contractivement. Cela peut nous amener également au sujet de l'hypercontractivité et des inégalités de Sobolev logarithmiques qui sont étroitement liées à divers domaines des mathématiques [Gro06]. Des questions similaires peuvent être certainement discutées si nous remplaçons le groupe  $\mathbb{Z}$  ci-dessus par d'autres groupes discrets (quantiques). Dans cette thèse, nous caractériserons les opérateurs de convolution positifs sur un groupe quantique fini qui envoient  $L^p$  dans  $L^2$ ,

et donner quelques constructions et applications pour les groupes quantiques compacts infinis.

D'autre part, notons que si  $\zeta$  est une fonction caractéristique, le problème correspondant est lié aux notions de lacunarité et aux ensembles  $\Lambda(p)$ , qui ont été étudiées par exemple dans [Rud60]. Ces concepts ont été largement étendus et étudiés dans les dernières décennies, avec différents liens avec l'analyse harmonique, la théorie des espaces de Banach, les algèbres d'opérateurs et les propriétés géométriques ou analytiques de groupe. Dans le deuxième sous-section, nous allons présenter quelques résultats correspondants dans le contexte de groupes quantiques compacts.

### 0.3.1 Opérateurs de convolution de $L^p$ dans $L^2$

Comme évoqué précédemment, parmi les phénomènes étudiés sur le groupe du cercle  $\mathbb{T}$  se trouve l'existence et le comportement des mesures boréliennes positives qui convolent  $L^p(\mathbb{T})$  dans  $L^2(\mathbb{T})$  avec  $q > p$  finis pour un  $1 < p < \infty$  fixé. Un exemple par Oberlin [Obe82] est la mesure de Cantor-Lebesgue à support dans l'ensemble triadique de Cantor. Par une analyse approfondie de la structure de cette mesure, Oberlin a observé que ce résultat se réduit à prouver qu'il existe  $p < 2$  tel que

$$\|\mu \star f\|_2 \leq \|f\|_p, \quad f \in L^p(\mathbb{Z}/3\mathbb{Z})$$

où les normes  $L^p$  sont celles relatives à la mesure de comptage normalisée sur le groupe cyclique  $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$  à trois éléments et  $\mu$  est la mesure de probabilité de masse  $1/2$  en 0 et en 2. Motivé par ces résultats, Ritter a démontré en 1984 le résultat suivant.

**Théorème 0.3.1** ([Rit84]). (a) *Si  $G$  est un groupe fini et  $T_\mu : f \mapsto \mu \star f$  est l'opérateur de convolution relativement à une mesure de probabilité  $\mu$  sur  $G$ , alors*

$$\exists p < 2, \|T_\mu : L^p(G) \rightarrow L^2(G)\| = 1$$

*si et seulement si  $G$  est engendré par  $\{ij^{-1} : i, j \in \text{supp } \mu\}$ .*

(b) *Si de plus  $G$  est abélien, les conditions ci-dessus sont vraies si et seulement si  $|\hat{\mu}(\gamma)| < 1$  pour tout  $\gamma \in \hat{G} \setminus \{e\}$ .*

Dans cet article, nous donnons une approche alternative de ces sujets dans le cadre des groupes quantiques et des espaces  $L^p$  non commutatifs. En particulier, nous avons le résultat suivant.

**Théorème 0.3.2** (Theorem 3.3.4). *Soit  $\mathbb{G}$  un groupe quantique fini et  $\varphi$  un état sur  $C(\mathbb{G})$ . Notons  $\psi = (\varphi \circ S) \star \varphi$ . Les assertions suivantes sont équivalentes:*

(1) *il existe  $1 \leq p < 2$  tel que,*

$$\forall x \in C(\mathbb{G}), \|\varphi \star x\|_2 \leq \|x\|_p ;$$

(2)  *$\|\hat{\varphi}(\pi)\| < 1$  pour tout  $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}$  ;*

(3) *Pour tout  $x \in C(\mathbb{G})_+$  non nul, il existe  $n \geq 1$  tel que  $\psi^{\star n}(x) > 0$ .*

La dernière assertion peut s'interpréter en disant que le "support" de  $\varphi$  "engendre" le groupe quantique  $\mathbb{G}$ . Nous illustrerons *via* un exemple dans la Remarque 3.3.7 que l'hypothèse de la finitude du groupe quantique ci-dessus est cruciale et ne peut pas être enlevée.

En particulier, le résultat caractérise les multiplicateurs de Fourier-Schur sur les groupes finis qui envoient  $L^p$  dans  $L^2$ . Nous obtenons la généralisation suivante du Théorème 0.3.1(b).



**Corollaire 0.3.3** (Corollary 3.3.6). *Soit  $\Gamma$  un groupe fini et  $\varphi$  une fonction de type positif sur  $\Gamma$  avec  $\varphi(e) = 1$ . On considère le multiplicateur de Fourier-Schur associé déterminé par  $M_\varphi(\lambda(\gamma)) = \varphi(\gamma)\lambda(\gamma)$  pour  $\gamma \in \Gamma$ . Alors il existe  $1 \leq p < 2$  tel que*

$$\|M_\varphi x\|_2 \leq \|x\|_p, \quad x \in C^*(\Gamma)$$

*si et seulement si  $|\varphi(\gamma)| < 1$  pour tout  $\gamma \in \Gamma \setminus \{e\}$ .*

Nous devons souligner que notre argument s'appuie essentiellement sur les propriétés nouvelles et intéressantes sur des opérateurs unitaux préservant la trace sur les espaces  $L^p$  non commutatifs, découverts dans les travaux récents de Ricard et Xu [RX16]. En fait, le résultat suivant joue un rôle essentiel dans notre argument.

**Théorème 0.3.4** (Theorem 3.1.6). *Soient  $A$  une algèbre de von Neumann de dimension finie munie d'un état fidèle tracial  $\tau$ , et  $T : A \rightarrow A$  une application unitale positive sur  $A$  préservant la trace. Alors*

$$\exists 1 < p < 2, \forall x \in A, \|Tx\|_2 \leq \|x\|_p$$

*si et seulement si*

$$\sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\|Tx\|_2}{\|x\|_2} < 1.$$

Nous donnons deux démonstrations de ce résultat, dont une se base sur des arguments très élémentaires avec une hypothèse supplémentaire de 2-positivité et une autre, relativement courte, sur [RX16]. Nous montrons également que cette contractivité de  $L^p$  dans  $L^2$  reste stable par le produit libre. Cette méthode permet de construire le même type d'opérateurs de convolution pour les groupes quantiques infinis.

**Théorème 0.3.5.** (a)(Theorem 3.1.9) *Soit  $(A_i, \tau_i)$ ,  $1 \leq i \leq n$  une famille finie de  $C^*$ -algèbres de dimension finie munies d'états fidèles traciaux  $\tau_i$ , et soit  $(\mathcal{A}, \tau) = \bar{*}_{1 \leq i \leq n} (A_i, \tau_i)$  leur produit libre d'algèbres de von Neumann. Supposons que pour tout  $1 \leq i \leq n$ ,  $T_i : A_i \rightarrow A_i$  est une application unitale positive préservant la trace telle que*

$$\|T_i : L^p(A_i) \rightarrow L^2(A_i)\| = 1$$

*pour un certain  $1 < p < 2$ . Alors le produit libre (algébrique) d'applications  $T = \bar{*}_{1 \leq i \leq n} T_i$  sur  $\bar{*}_{1 \leq i \leq n} A_i$  se prolonge en une application telle que*

$$\|T : L^{p'}(\mathcal{A}) \rightarrow L^2(\mathcal{A})\| = 1$$

*pour un certain  $1 < p' < 2$ .*

(b)(Corollary 3.3.8) *Soient  $\mathbb{G}_1, \dots, \mathbb{G}_n$  des groupes quantiques finis et soit  $\varphi_i$  un état sur  $C(\mathbb{G}_i)$  pour  $i \in \{1, \dots, n\}$ . Notons  $\mathbb{G} = \mathbb{G}_1 \hat{*} \dots \hat{*} \mathbb{G}_n$ . Soit  $\varphi$  le produit libre conditionnel de  $\varphi_i$  relativement aux états de Haar. Si tout  $\varphi_i$  satisfait à l'un des conditions (1)-(3) du Théorème 0.3.2, alors l'opérateur de convolution associé donné par  $T : x \mapsto x \star \varphi$ ,  $x \in C(\mathbb{G})$  est un multiplicateur à gauche sur  $\mathbb{G}$  satisfaisant*

$$\|T : L^p(\mathbb{G}) \rightarrow L^2(\mathbb{G})\| = 1$$

*pour un certain  $1 < p < 2$ .*



Dans cette thèse, nous incluons également quelques propriétés simples des états non dégénérés sur les groupes quantiques compacts, ainsi que des applications. Nous démontrons dans le Lemme 3.2.3 que la limite de Cesàro de convolutions d'un état non dégénéré est l'état de Haar, ce qui contribue non seulement à la preuve de notre résultat principal, mais aussi généralise le résultat principal de [BFS12] concernant le calcul des états idempotent associés aux images de Hopf. Nous renvoyons à la Section 3.2 pour plus d'informations sur les images de Hopf.

**Théorème 0.3.6** (Theorem 3.2.4). *Soit  $\mathbb{G}$  un groupe quantique compact et soit  $A$  une  $C^*$ -algèbre unital avec un  $*$ -homomorphisme unital  $\pi : C(\mathbb{G}) \rightarrow A$ . Soit  $\mathbb{G}_\pi$  l'image de Hopf de  $\pi$  et  $q : C(\mathbb{G}) \rightarrow C(\mathbb{G}_\pi)$  l'application de quotient associée. Alors pour tout état fidèle  $\varphi$  sur  $A$ , on a*

$$h_{\mathbb{G}_\pi} \circ q = w^* - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\varphi \circ \pi)^{*k}.$$

Ce résultat a été récemment utilisé dans [BN16, Ban16b, Ban16a] pour étudier le modèle matriciel des groupes quantiques de permutations.

### 0.3.2 Lacunarité

L'étude de la lacunarité et en particulier les ensembles de Sidon et les ensembles  $\Lambda(p)$  est un sujet majeur et fascinant de l'analyse harmonique. Historiquement, la notion d'ensemble de Sidon provient des études des séries lacunaires spéciales sur le cercle  $\mathbb{T}$ . Cela a été généralisé par la suite en remplaçant  $\mathbb{T}$  avec un groupe compact abélien arbitraire  $G$ . Rappelons que pour un groupe compact abélien  $G$ , un ensemble de Sidon  $\mathbf{E}$  pour  $G$  est un sous-ensemble de son groupe dual  $\Gamma = \hat{G}$  tel que toute fonction continue sur  $G$  avec une transformée de Fourier de support  $\mathbf{E}$  possède une série de Fourier absolument convergente. La théorie de la lacunarité pour les groupes compacts abéliens s'est remarquablement développée dans les dernières décennies, dans une série de travaux de Rudin, Drury, Rider, Marcus, Pisier et autres (voir [Dru70, Rid75, Rud60, MP81, Pis78a, Pis78b, Pis83a, Pis83b] et les références qui s'y trouvent); elle s'est avérée profondément liée à la théorie des séries de Fourier aléatoires, aux conditions d'entropie métrique, aux espaces de multiplicateurs ainsi qu'à quelques autres domaines d'analyse harmonique et espaces de Banach.

Le développement des sujets similaires dans le cadre plus général va naturellement en deux directions. La première est la théorie des ensembles de Sidon et des ensembles  $\Lambda(p)$  dans le cadre d'un groupe compact non abélien  $G$ ; ces ensembles sont des sous-ensembles particuliers des représentations irréductibles de  $G$ . Cette notion généralisée a été d'abord introduite dans [Hel58, FTR66, HR70] et puis a vu de nombreux liens vers l'analyse harmonique et la théorie de probabilités comme dans le cas abélien (voir [Bož74, MP81]). L'autre direction concerne les sous-ensembles d'un groupe discret non abélien arbitraire  $\Gamma$  avec des "fonctions" associées dans l'algèbre de von Neumann  $VN(\Gamma)$  du groupe  $\Gamma$ , comme dans [Pic73, FT77, Bož81]. Cette dernière est plus compliquée et le comportement des lacunarités est étroitement lié à la moyennabilité du groupe discret, qui exige des outils de la théorie des espaces d'opérateurs et mène à des sujets intéressants dans l'étude des multiplicateurs de Fourier complètement bornés sur  $L^p$  en analyse harmonique abstraite ([Har99, Pis95]). Aussi, pour les groupes non moyennables, l'estimation de type  $\Lambda(\infty)$  pour des sous-ensembles spécifiques joue souvent un rôle important dans l'étude de certaines conditions de croissance ([Jol90, JV91]), les propriétés d'approximation ([Haa79]), et les idéaux complétés des algèbres de Fourier ([BFZ16]).

Dans les deux directions mentionnées ci-dessus, les ensembles de Sidon admettent plusieurs caractérisations apparemment non liés et jouissent certaines relations spéciales

avec les ensembles  $\Lambda(p)$ , multiplicateurs, etc.. De plus, plusieurs propriétés fondamentales des ensembles de Sidon dans les deux contextes ont des formulations très similaires. Cela donne une motivation naturelle pour trouver une trame plus générale unifiant ces développements et pour chercher des nouveaux aspects théorique pour ces objets. En fait, à l'aide de la théorie de Woronowicz-Peter-Weyl pour les groupes quantiques compacts, nous pouvons discuter des questions similaires pour la lacunarité dans le cadre des groupes quantiques. Rappelons qu'*a priori* dans ce cadre général, un groupe discret  $\Gamma$  est considéré comme l'ensemble des représentations irréductibles du groupe quantique compact dual  $\mathbb{G} = \hat{\Gamma}$ , et la moyennabilité de  $\Gamma$  s'interprète souvent comme le comoyennabilité de  $\mathbb{G}$ .

Dans cette thèse, nous introduisons et développons certaines notions importantes et les propriétés associées à la lacunarités dans le cadre des groupes quantiques compacts. Les sujets auxquels nous nous intéressons sont comme les suivants.

### Généralisations et caractérisations des ensembles de Sidon

Soit  $\mathbb{G}$  un groupe quantique compact, et soit  $\mathbf{E}$  un sous-ensemble de  $\text{Irr}(\mathbb{G})$ . Notons

$$\ell^\infty(\mathbf{E}) = \{(a_\pi)_{\pi \in \text{Irr}(\mathbb{G})} \in \ell^\infty(\hat{\mathbb{G}}) : a_\pi = 0 \text{ si } \pi \in \text{Irr}(\mathbb{G}) \setminus \mathbf{E}\}, \quad c_0(\mathbf{E}) = c_0(\hat{\mathbb{G}}) \cap \ell^\infty(\mathbf{E}).$$

**Définition 0.3.7.** On dit qu'un sous-ensemble  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  est un *ensemble de Sidon* s'il existe  $K > 0$  tel que pour tout  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ , on a

$$\|\hat{x}\|_1 \leq K \|x\|_\infty.$$

Les caractérisations fondamentales suivantes étendent le résultat classique de [HR70, (37.2)] pour les groupes compacts.

**Théorème 0.3.8** (Theorem 4.1.3). *Soit  $\mathbb{G}$  un groupe quantique compact, et soit  $\mathbf{E}$  un sous-ensemble de  $\text{Irr}(\mathbb{G})$ . Les assertions suivantes sont équivalentes :*

- (1)  $\mathbf{E}$  est un ensemble de Sidon ;
- (2) pour tout  $a \in \ell^\infty(\mathbf{E})$ , il existe  $\varphi \in C_r(\mathbb{G})^*$  tel que  $\hat{\varphi}(\pi) = a_\pi$  pour tout  $\pi \in \mathbf{E}$  ;
- (3) pour tout  $a \in c_0(\mathbf{E})$ , il existe  $x \in L^1(\mathbb{G})$  tel que  $\hat{x}(\pi) = a_\pi$  pour tout  $\pi \in \mathbf{E}$  ;
- (4) tout  $x \in L^\infty(\mathbb{G})$  avec  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$  appartient à  $A(\mathbb{G})$  ;
- (5) tout  $x \in C_r(\mathbb{G})$  avec  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$  appartient à  $A(\mathbb{G})$ .

Ici, notre approche est différente de celle classique dans [HR70]. Cela en particulier répond à une question fondamentale dans l'étude de la lacunarité des groupes discrets, posé dans [FT77, Pic73], à savoir l'équivalence entre les ensembles de Sidon forts et les ensembles de Sidon dans les groupes discrets au sens de [Pic73]. En fait, dans ce dernier article, Picardello a défini deux types différents d'ensembles lacunaires qu'il a appelé ensembles de Sidon forts et ensembles de Sidon dans les groupes discrets. Il a montré l'équivalence entre ces deux notions pour les groupes discrets moyennables, mais le cas non moyennable restait ouvert jusqu'à notre travail.

Rappelons que l'algèbre de Fourier d'un groupe discret  $\Gamma$  est donnée par

$$A(\Gamma) = \{f(\cdot) = \langle \lambda(\cdot)\xi, \eta \rangle \in c_0(\Gamma) : \xi, \eta \in \ell^2(\Gamma)\}$$

où  $\lambda$  désigne la représentation régulière gauche. Notons qu'elle est un objet différent de l'algèbre de Fourier  $A(\hat{\Gamma})$  du groupe quantique compact  $\hat{\Gamma}$  donnée précédemment.

**Corollaire 0.3.9.** *Soit  $\Gamma$  un groupe discret. Les assertions suivantes sont équivalentes :*

- (a)  $E \subset \Gamma$  est un ensemble de Sidon ;
- (b)  $E \subset \Gamma$  est un ensemble de Sidon fort (au sens de [Pic73]), c'est-à-dire, pour tout  $a \in c_0(E)$ , il existe  $f$  dans l'algèbre de Fourier  $A(\Gamma)$  de  $\Gamma$  tel que  $f|_E = a$ .
- (c) Pour tout  $x \in VN(\Gamma)$  avec  $\hat{x}|_{\Gamma \setminus E} = 0$ , on a  $\sum_{\gamma \in \Gamma} |\hat{x}(\gamma)| < \infty$ .

Nous démontrons aussi que la propriété de sidonicité est stable par le produit cartésien ou le produit libre des groupes quantiques compacts. En dehors de l'approche décrite ci-dessus, il existe en effet différents points de vue sur les généralisations des ensembles de Sidon pour les groupes quantiques compacts non comoyennables, qui donnent différents types de lacunarités tels que les ensembles de Sidon faibles et les ensembles de Sidon inconditionnelles. Dans le Théorème 4.1.15 nous discuterons les relations entre ces différentes notions et montrerons leur équivalence pour les groupes quantiques compacts comoyennables.

### Ensembles $\Lambda(p)$

Nous étudions les ensembles  $\Lambda(p)$  et les ensembles d'interpolation pour les multiplicateurs de Fourier sur  $L^p$  pour un groupe quantique compact, définis comme suit.

**Définition 0.3.10.** Soit  $\mathbb{G}$  un groupe quantique compact, et soit  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  un sous-ensemble. Soit  $1 < p < \infty$ .

(a) On dit que  $\mathbf{E}$  est un *ensemble*  $\Lambda(p)$  s'il existe  $K > 0$  tel que pour tout  $x \in \text{Pol}(\mathbb{G})$  avec  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$ ,

$$\|x\|_p \leq K \|x\|_1;$$

(b) On dit que  $\mathbf{E}$  est un *ensemble d'interpolation* pour  $M(L^p(\mathbb{G}))$  si pour tout  $a \in \ell^\infty(\mathbf{E})$ , il existe un multiplicateur borné  $\tilde{a} \in M(L^p(\mathbb{G}))$  tel que  $\tilde{a}_\pi = a_\pi$  pour  $\pi \in \mathbf{E}$ .

Nous démontrons l'équivalence entre les ensembles  $\Lambda(p)$  et les ensembles d'interpolation pour  $M(L^p(\mathbb{G}))$ , qui généralise les travaux [Har99].

**Théorème 0.3.11** (Theorem 4.2.7). *Soit  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  un sous-ensemble. Supposons que  $2 < p < \infty$ . Alors  $\mathbf{E}$  est un ensemble  $\Lambda(p)$  si et seulement s'il est un ensemble d'interpolation pour  $M(L^p(\mathbb{G}))$ .*

L'argument est beaucoup plus délicat que dans le cas classique, et le résultat en effet résout un problème récent concernant les ensembles de Sidon. Pour un groupe compact ou (le dual d') un groupe discret, un argument simple (mais non trivial) montre que tout ensemble de Sidon est un ensemble  $\Lambda(p)$  pour  $1 < p < \infty$ . Le cas d'un groupe quantique compact général s'avère plus difficile. Une première tentative a été faite par Blendek et Michaliček [BM13] en 2013.

**Théorème 0.3.12** ([BM13]). *Soit  $\mathbb{G}$  un groupe quantique compact de type Kac. Pour tout  $\pi \in \text{Irr}(\mathbb{G})$ , on note  $\chi_\pi$  le caractère de  $\pi$ . Si  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  est un ensemble de Sidon satisfaisant la condition de Helgason-Sidon, alors il existe une constante  $K > 0$ , tel que pour tout  $(c_\pi)_{\pi \in \mathbf{E}} \subset \mathbb{C}$ , on a*

$$\left\| \sum_{\pi \in \mathbf{E}} c_\pi \chi_\pi \right\|_2 \leq K \left\| \sum_{\pi \in \mathbf{E}} c_\pi \chi_\pi \right\|_1.$$

Comparé aux résultats classiques, ce résultat est plus restrictif, mais sa preuve est technique et non triviale. Maintenant avec notre résultat présenté précédemment, il est facile de déduire que tout ensemble de Sidon pour un groupe quantique compact arbitraire est un ensemble  $\Lambda(p)$  pour  $1 < p < \infty$ , qui enlève complètement les restrictions indésirables ci-dessus.

**Corollaire 0.3.13** (Corollary 4.2.9). *Si  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  est un ensemble de Sidon, alors  $\mathbf{E}$  est un ensemble  $\Lambda(p)$  pour tout  $1 < p < \infty$ .*

Notre investigation a également donné certains faits qui se cachent dans les cas classiques. Notons qu'un groupe discret est forcément unimodulaire, mais il existe de nombreux groupes quantiques discrets qui n'est pas unimodulaire. Nous montrons que la condition  $\Lambda(p)$  entraîne un contrôle strict sur l'élément modulaire  $Q$  du groupe quantique dual.

**Proposition 0.3.1** (Proposition 4.2.5). *Si  $1 < p < \infty$  et  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  est un ensemble  $\Lambda(p)$ , alors*

$$\sup_{\pi \in \mathbf{E}} \|Q_\pi\| < \infty, \quad \sup_{\pi \in \mathbf{E}} \|Q_\pi^{-1}\| < \infty.$$

En général, les espaces  $L^p$  non commutatifs définis par Kosaki dépendent de paramètres d'interpolation différents. Pour un paramètre  $0 \leq \theta \leq 1$ , on note  $L^p_{(\theta)}(\mathbb{G})$  les espaces  $L^p$  associés. Selon le résultat ci-dessus, nous pouvons établir la propriété suivante.

**Proposition 0.3.2** (Proposition 4.2.14). *Soit  $\mathbb{G}$  un groupe quantique compact, et soient  $2 < p < \infty$ ,  $0 \leq \theta, \theta' \leq 1$ . Alors  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  est un ensemble  $\Lambda(p)$  pour  $L^p_{(\theta)}(\mathbb{G})$  si et seulement s'il est un ensemble  $\Lambda(p)$  pour  $L^p_{(\theta')}(\mathbb{G})$ .*

Nous discutons également l'existence des ensembles  $\Lambda(p)$ . Il est basé sur le résultat général suivant.

**Théorème 0.3.14** (Theorem 4.4.1). *Soit  $\mathcal{M}$  une algèbre de von Neumann muni d'un état normal fidèle  $\varphi$ , et soit  $L^p(\mathcal{M}, \varphi)$  les espaces  $L^p$  associés. Supposons que  $B = \{x_i \in \mathcal{M} : i \geq 1\}$  est un système orthogonal relativement à  $\varphi$  tel que  $\sup_i \|x_i\|_\infty < \infty$ . Alors pour tout  $2 < p < \infty$ , il existe un sous-ensemble infini  $\{x_{i_k} : k \geq 1\} \subset B$  et une constante  $C > 0$  tels que pour toute suite à support fini  $(c_k) \subset \mathbb{C}$  on a*

$$\left\| \sum_{k \geq 1} c_k x_{i_k} \right\|_p \leq C \left( \sum_{k \geq 1} |c_k|^2 \right)^{\frac{1}{2}}.$$

Ensuite, nous pouvons établir l'existence d'ensembles  $\Lambda(p)$  pour les groupes quantiques, sous une hypothèse de dimensions uniformément bornées.

**Théorème 0.3.15** (Theorem 4.2.15). *Soit  $\mathbb{G}$  un groupe quantique compact, et soit  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  un sous-ensemble infini avec  $\sup_{\pi \in \mathbf{E}} d_\pi < \infty$ . Alors pour tout  $1 < p < \infty$ , il existe un sous-ensemble infini  $\mathbf{F} \subset \mathbf{E}$  qui est un ensemble de  $\Lambda(p)$  pour  $\mathbb{G}$ .*

### Ensembles de Sidon centraux

Nous étudions également quelques notions et propriétés élémentaires concernant les ensembles de Sidon centraux pour les groupes quantiques compacts.

**Définition 0.3.16.** On dit qu'un sous-ensemble  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  est un *ensemble de Sidon central* s'il existe  $K > 0$  tel que pour toute suite à support fini  $(c_\pi) \subset \mathbb{C}$  et  $x = \sum_\pi c_\pi \chi_\pi$ , on a  $\|\hat{x}\|_1 \leq K \|x\|_\infty$ .

Notre argument contient certaines nouvelles caractérisations des groupes quantiques compacts de type Kac comme suit. Notons  $\text{Pol}^z(\mathbb{G}) = \{x = \sum_{\pi \in \text{Irr}(\mathbb{G})} c_\pi \chi_\pi \in \text{Pol}(\mathbb{G}) : c_\pi \in \mathbb{C}\}$  le sous-espace des polynômes centraux, et  $C_r^z(\mathbb{G})$  la fermeture en norme de  $\text{Pol}^z(\mathbb{G})$  dans  $C_r(\mathbb{G})$ .

**Proposition 0.3.3** (Lemma 4.3.3). *Soit  $\mathbb{G}$  un groupe quantique compact. Les assertions suivantes sont équivalentes :*

- (1)  $\mathbb{G}$  est de type Kac ;
- (2) toute fonctionnelle centrale  $\omega$  est bornée sur  $\text{Pol}(\mathbb{G})$  (par rapport à  $\|\cdot\|_\infty$ ) si et seulement si elle est bornée sur  $\text{Pol}^z(\mathbb{G})$  pour la même norme ;
- (3) il existe une espérance conditionnelle  $\mathcal{E}$  de  $C_r(\mathbb{G})$  sur  $C_r^z(\mathbb{G})$  telle que  $h \circ \mathcal{E} = h$ .

**Proposition 0.3.4** (Proposition 4.3.4). *Soit  $\mathbb{G}$  un groupe quantique compact de type Kac. Alors  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  est un ensemble de Sidon central si et seulement si pour toute suite bornée  $(a_\pi)_{\pi \in \mathbf{E}} \subset \mathbb{C}$ , il existe une fonctionnelle centrale bornée  $\varphi \in C_r(\mathbb{G})^*$  telle que  $\hat{\varphi}(\pi) = a_\pi \text{Id}_\pi$  pour  $\pi \in \mathbf{E}$ .*

L'argument répond également à une question dans le cas classique. En effet, pour un groupe quantique compact  $\mathbb{G}$  et un sous-ensemble  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$ , nous voyons que les deux propriétés suivantes *ne sont pas* équivalentes en général (Remark 4.3.5) :

- (a) il existe  $K > 0$  tel que pour tout unitaire  $a \in \ell^\infty(\mathbf{E})$  et pour tout  $x \in \text{Pol}(G)$  avec  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$ ,

$$\|m_a^L x\|_\infty \leq K \|x\|_\infty;$$

- (b) il existe une constante  $K > 0$  tel que pour tout  $(\varepsilon_\pi)_{\pi \in \mathbf{E}} \subset \{-1, 1\}$  et pour tout  $x \in \text{Pol}(G)$  avec  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$ ,

$$\left\| \sum_{\pi \in \mathbf{E}} d_\pi \varepsilon_\pi (\iota \otimes \text{Tr})((1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}) \right\|_\infty \leq K \|x\|_\infty.$$

En effet, la propriété (a) est équivalente au fait que  $\mathbf{E}$  est un ensemble de Sidon, mais (b) est équivalent au fait que  $\mathbf{E}$  est un ensemble de Sidon central.

Nous obtenons également que deux groupes quantiques compacts ayant les mêmes règles de fusion et les mêmes fonctions de dimension possèdent les ensembles de Sidon centraux identiques. Cela donne de nombreux exemples d'ensembles de Sidon centraux pour les groupes quantiques compacts *via* l'équivalence monoïdale, les déformations de Drinfeld-Jimbo, etc..

**Proposition 0.3.5** (Proposition 4.3.7). *Soient  $\mathbb{G}_1, \mathbb{G}_2$  deux groupes quantiques compacts. Supposons que  $\Phi : \text{Rep}(\mathbb{G}_1) \rightarrow \text{Rep}(\mathbb{G}_2)$  est une application injective préservant les règles de fusion, c'est-à-dire, pour tout  $\pi, \pi' \in \text{Rep}(\mathbb{G}_1)$  on a*

$$\Phi(\pi \otimes \pi') = \Phi(\pi) \otimes \Phi(\pi'), \quad \Phi(\oplus_{i=1}^n \pi_i) = \oplus_{i=1}^n \Phi(\pi_i), \quad \pi, \pi', \pi_i \in \text{Rep}(G), n \geq 1.$$

*Pour tout ensemble de Sidon central  $\mathbf{E} \subset \text{Irr}(\mathbb{G}_1)$ , si de plus il existe  $C > 0$  avec  $\dim(\Phi(\pi)) \leq C \dim(\pi)$  pour tout  $\pi \in \mathbf{E}$ , alors  $\Phi(\mathbf{E}) \subset \text{Irr}(\mathbb{G}_2)$  est un ensemble de Sidon central pour  $\mathbb{G}_2$ .*

### Quelques exemples

Maintenant nous recueillons quelques exemples typiques de la lacunarité dans la thèse.

- (1) (Exemple 4.1.17) Considérons le groupe quantique compact  $\mathbb{G} = \prod_{k \geq 1} U_{N_k}^+$ , où pour  $k \geq 1$  et  $N_k \geq 1$ ,  $U_{N_k}^+$  désigne le groupe quantique libre unitaire. Soient  $u^{(k)}$  la représentation fondamentale de  $U_{N_k}^+$  et  $\mathbf{E} = \{u^{(k)} : k \geq 1\} \subset \text{Irr}(\mathbb{G})$ . Alors  $\mathbf{E}$  est un ensemble de Sidon faible, et il est aussi un ensemble d'interpolation pour  $M(L^\infty(\mathbb{G}))$ . Il est donc un ensemble  $\Lambda(p)$ . Mais il *n'est pas* un ensemble de Sidon.

(2) (Exemple 4.1.18) Considérons la suite  $(q_n)_{n \geq 1} \subset [0, 1]$  avec  $q := \inf_n q_n > 0$  et le groupe quantique associé  $\mathbb{G} = \prod_{n \geq 1} \mathrm{SU}_{q_n}(2)$ , où  $\mathrm{SU}_q(2)$  est donné par la déformation de Drinfeld-Jimbo. On note  $u_n$  la représentation fondamentale de  $\mathrm{SU}_{q_n}(2)$ . Alors  $\mathbf{E} = \{u_n : n \geq 1\} \subset \mathrm{Irr}(\mathbb{G})$  est un ensemble de Sidon pour  $\mathbb{G}$ , et donc un ensemble  $\Lambda(p)$ . L'ordre optimal de la constante de sidonicité est  $o(q^{-1})$  quand  $q \rightarrow 0$ . Si  $q = 0$ , le sous-ensemble  $\mathbf{E}$  donné ci-dessus n'est ni un ensemble de Sidon ni un ensemble  $\Lambda(p)$  avec  $p > 2$ .

(3) (Exemple 4.3.8) Soit  $q \in [-1, 1] \setminus \{0\}$ . Considérons la  $q$ -déformation de Drinfeld-Jimbo  $\mathrm{SU}_q(n)$ ,  $n \geq 2$ . On note  $\pi_n$  la représentation fondamentale de  $\mathrm{SU}_q(n)$ . Pour tout  $0 < q < 1$  le sous-ensemble  $\{\pi_n : n \geq 2\}$  est un ensemble de Sidon central pour le groupe quantique compact  $\prod_{n \geq 2} \mathrm{SU}_q(n)$ .

(4) Pour tout  $0 < q < 1$  et tout groupe de Lie compact semi-simple et simplement connexe  $G$ , le groupe quantique compact  $G_q$  donné par la déformation de Drinfeld-Jimbo n'admet ni ensemble de Sidon infini ni ensemble de Sidon central (Corollary 4.3.9). Aussi, il n'existe pas d'ensemble  $\Lambda(p)$  infini avec  $p > 2$  pour  $\mathrm{SU}_q(2)$  (Proposition 4.2.17). Toutefois, le sous-ensemble  $\mathbf{E} \subset \mathrm{Irr}(\mathrm{SU}_q(2))$  indexé par  $\{n_k \in \mathbb{N} \cup \{0\} : k \geq 0\}$  tel que  $n_k = n_{k-1} + k$ , est un ensemble  $\Lambda(4)$  central (Proposition 4.2.16). Plus précisément, il existe  $K_q > 0$  tels que pour toute suite à support fini  $(c_n)_{n \in \mathbf{E}} \in \mathbb{C}$ ,

$$\left\| \sum_{n \in \mathbf{E}} c_n \chi_n \right\|_4 \leq K_q \left\| \sum_{n \in \mathbf{E}} c_n \chi_n \right\|_2.$$

### Quelques remarques et problèmes ouverts

Pour terminer cette introduction, remarquons qu'il serait également intéressant d'étudier la version complètement bornée des ensembles de Sidon ou des ensembles  $\Lambda(p)$ . Nous renvoyons aux travaux de Pisier [Pis95] pour la lacunarité complètement bornée dans les groupes discrets. Nous n'avons pas étudié ici ce sujet ; mais nous allons le poursuivre par ailleurs. En fait, notre argument sur les multiplicateurs de Fourier sur  $L^p$  dans la Section 2.3 et l'estimation des éléments modulaires dans la Section 4.2 sera utile pour l'étude dans cette direction.

D'autre part, il serait intéressant d'obtenir un analogue quantique du théorème de Drury-Rider [Dru70, Rid75], affirmant que l'union de deux ensembles de Sidon est encore un ensemble de Sidon. Aussi, le théorème de Pisier sur la caractérisation  $\Lambda(\psi_2)$  des ensembles de Sidon [MP81] reste inexploré dans le cadre des groupes quantiques. Il faut faire attention au fait que ces problèmes sont ouverts même pour les duals compacts des groupes discrets non abéliens, qui font partie des exemples les plus simples de groupes quantiques compacts.



# Introduction

This PhD thesis investigates the theory of harmonic analysis on compact quantum groups. This work is a natural extension of the study on abstract harmonic analysis initiated in the last century. Historically, the main idea of abstract harmonic analysis is to study the analysis on function spaces or group algebras of locally compact groups, as an analogue of Fourier analysis theory on the real line or on the circle (see for example [Eym64, HR70, Rud90]). In the last twenty years, the theory of topological quantum groups has developed rapidly through Woronowicz’s definition of a compact quantum group [Wor98] and the locally compact quantum group framework of Kustermans and Vaes [KV00], which provide a more general and unified framework of abstract harmonic analysis. In particular, the Hausdorff-Young theory and the study of completely bounded multipliers for locally compact quantum groups, have witnessed fruitful progress in recent years (see [Cas13, Co010, Daw12, JNR09]).

This thesis aims to study the Fourier analysis on  $L^p$ -spaces associated to compact quantum groups. Compared to the abstract study for locally compact quantum groups, we will present a more concise approach to the elementary Fourier analysis theory specifically in the setting of compact quantum groups. In particular, we will establish the Hausdorff-Young theory, the Young inequalities and some basic facts on  $L^p$ -Fourier multipliers. The main interest of the thesis will be in some typical topics in harmonic analysis, such as  $L^p$ -improving convolution operators and lacunarity for Fourier multipliers.

This thesis consists of four chapters. It is principally based on two works [Wan14, Wan16] by the author, entitled “ $L^p$ -improving convolution operators on finite quantum groups” and “Lacunary Fourier series for compact quantum groups”, which have been accepted for publication in *Indiana Univ. Math. J.* and *Comm. Math. Phys.* respectively. In this introduction, we will first recall some historical background on abstract harmonic analysis and quantum group theory, then we will describe the contents and main results of the thesis.

## 0.1 A historical review on abstract harmonic analysis

The classical Fourier analysis began with the study of periodic functions on the real line  $\mathbb{R}$ . It is motivated by the investigation of the vibrating string and heat flow in mathematical physics, where the relevant differential equations were solved in terms of *Fourier series*. If we identify the functions with a fixed periodicity as functions on the circle group  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ , the Fourier series are defined by the following transform,

$$\mathcal{F} : L^1(\mathbb{T}) \rightarrow c_0(\mathbb{Z}), \quad \mathcal{F}(f)(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(z)z^{-n}dz, \quad n \in \mathbb{Z},$$

and we will view classical analysis of periodic functions as results on the compact abelian group  $\mathbb{T}$ .



Since 1930s, an increasing number of mathematicians adopted the point of view that the theory of Fourier analysis should be considered in the class of all locally compact abelian groups. The observation is based on two remarkable results. Firstly, Haar proved in 1932 that every locally compact group has a (left) translation invariant measure, afterwards called the Haar measure, which is an analogue of Lebesgue measure on  $\mathbb{T}$  and  $\mathbb{R}$ . Secondly, if we consider the set of *characters* of a locally compact abelian group  $G$ , that is,

$$\hat{G} = \{\chi : G \rightarrow \mathbb{T} \text{ continuous group morphisms}\}, \quad (0.1)$$

then equipped with the pointwise multiplication and compact-open topology,  $\hat{G}$  becomes a locally compact abelian group, and moreover there is an isomorphism between  $G$  and  $\hat{\hat{G}}$ . This construction is called the *Pontryagin duality*, and was established by Pontryagin in 1936. We remark that we have the identifications

$$\hat{\mathbb{T}} = \mathbb{Z}, \quad \hat{\mathbb{R}} = \mathbb{R}.$$

And the Pontryagin duality indeed gives rise to a realization of Fourier transform for locally compact abelian groups, that is,

$$\mathcal{F} : L^1(G) \rightarrow C_0(\hat{G}), \quad \mathcal{F}(f)(\chi) = \frac{1}{2\pi} \int_{\mathbb{T}} f(g) \overline{\chi(g)} dm(g), \quad \chi \in \hat{G},$$

where  $dm$  denotes the Haar measure on  $G$ . A general theory of Fourier analysis can be established in this framework, in parallel to the classical one. We refer to [Rud90] for more specific topics towards this direction.

For the interest of this thesis, we would like to emphasize a special case, where  $G$  is a compact abelian group. In this case, the Pontryagin dual  $\Gamma = \hat{G}$  is a discrete abelian group, and the Haar measure on  $\Gamma$  is just the counting measure. As is in the classical analysis on  $\mathbb{T}$ , we have a Fourier series expansion for each  $f \in L^1(G)$ ,

$$f \sim \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \gamma.$$

A number of typical problems in harmonic analysis can be discussed in this setting, such as decay of Fourier series, lacunary sets, convolutions and Fourier multipliers. We will review some typical related topics in later sections.

The development of similar subjects in more general setting goes naturally into two lines. Let us keep in mind the scheme of Fourier analysis going through the compact group  $G$  and discrete group  $\Gamma$  in the above paragraph.

The first line of generalizations begins with removing the abelian condition of the compact group  $G$  in the above scheme. As a result, we cannot obtain a dual discrete group  $\hat{G}$  in the Pontryagin sense. However, we have a substitution which generalizes the class of characters (0.1), i.e.,

$$\text{Irr}(G) = \{\pi : G \rightarrow B(H) \text{ irreducible unitary continuous group representations}\},$$

where the elements are taken modulo equivalence between representations. Then  $\text{Irr}(G)$  coincides with  $\hat{G}$  when  $G$  is abelian. According to the Peter-Weyl theory established in 1930s, the set  $\text{Irr}(G)$  leads to an orthogonal family of continuous functions on  $G$ , which spans a dense subalgebra of  $C(G)$  and generalizes the notion of trigonometric series on the circle. From this we can establish a complete theory of Fourier analysis on the compact group  $G$ , for which we refer to [Fol95, Chap.5]. Many problems from the classical Fourier

analysis, such as the summability of Fourier series and Littlewood-Paley theory, can be discussed in this general framework; also, the Fourier analysis on compact groups provides a powerful tool in the study of representation theory and mathematical physics, especially for topics related to compact Lie groups. We refer to [CW71, HR70, Ste70] for more discussions.

The other line of development concerns removing the abelianness of the discrete group  $\Gamma$  in the abelian scheme. Similarly, a discrete non-abelian group  $\Gamma$  does not admit a compact dual group  $\hat{\Gamma}$  in the Pontryagin sense. The idea of Fourier analysis in this setting benefits from the emergence of operator algebra theory founded in 1940s. To each discrete group  $\Gamma$  one associates a  $C^*$ -algebra generated by left regular representations on  $\ell^2(\Gamma)$ , denoted by  $C_r^*(\Gamma)$ . This kind of algebras is among the key objects studied in operator algebra theory till today, and is related to many new fields in mathematics. The viewpoint of Fourier analysis relies on the isomorphism

$$C_r^*(\Gamma) \cong C(\hat{\Gamma}) \quad (0.2)$$

if  $\Gamma$  is abelian. On the other hand, since 1950s, the noncommutative integration theory was developed by Segal and Dixmier, which allows us to discuss the noncommutative analogue of  $L^p$ -spaces associated to the algebra  $C_r^*(\Gamma)$ . Based on these observations, we may view  $C_r^*(\Gamma)$  as an analogue of functions on the “compact dual” of  $\Gamma$ , and establish a suitable framework of noncommutative Fourier analysis on discrete groups. First ground-breaking results in this domain are due to Haagerup [Haa79], who was motivated by the study of approximation properties of group von Neumann algebras. Since then, the harmonic analysis associated to discrete non-abelian groups has been largely discussed, which leads to a strong interaction with the study of approximation properties, operator algebras and operator/Banach space theory (see for example [Boz85, BF84, CH89, FTP83, JR03, LDIS11, Pis95]). In recent decades, the  $L^p$ -theory for Fourier multipliers on these algebras has also been fruitfully developed in [CPPR15, JMP14a, JMP14b, Har99].

As we may see from above, the noncommutative Fourier analysis is based on a certain generalization of Pontryagin duality; and the above two research lines follows essentially from different approaches to this generalization, which however share many similarities. This gives rise to a natural motivation to find a more general framework to unify these different viewpoints. The work was initiated by Tannaka and Krein in 1930s, who discussed the dual structure of compact groups via the representation classes  $\text{Irr}(G)$ . A more complete theory was then developed by Kac, Vainerman, Enock and Schwartz around 1970s; they introduced the notion of *Kac algebras*, which firstly unifies the various frameworks mentioned above, and admits a perfect analogue of Pontryagin duality. Finally in 1980s, a more general and concise theory was established in a series of works by Woronowicz [Wor87a, Wor88, Wor89, Wor98]. The philosophy of Woronowicz’s theory lies in a typical viewpoint of *noncommutative geometry* (notably in [Con94]). Classically, we may make the following key observations.

(a) For a compact group  $G$ , the group multiplication on  $G$  induces a *comultiplication* map  $\Delta$  on the algebra  $C(G)$ , given by

$$\Delta : C(G) \rightarrow C(G) \otimes C(G), \quad (\Delta f)(g, h) = f(gh), \quad g, h \in G.$$

The pair  $(C(G), \Delta)$  recovers the structure of  $G$ .

(b) For a discrete group  $\Gamma$ , the structure of  $\Gamma$  induces a certain *comultiplication*  $\Delta$  on the algebra  $C_r^*(\Gamma)$ . The pair  $(C_r^*(\Gamma), \Delta)$  recovers the structure of  $\Gamma$ . Recall that if  $\Gamma$  is abelian, the structure of  $\Gamma$  further recovers that of the compact group  $\hat{\Gamma}$ , and  $C_r^*(\Gamma)$  is an analogue of the algebra  $C(\hat{\Gamma})$ .

Woronowicz's *compact quantum group* indeed corresponds to a similar structure  $(A, \Delta)$ , where  $A$  is a unital  $C^*$ -algebra, and  $\Delta : A \rightarrow A \otimes A$  is a  $*$ -homomorphism which satisfies similar properties as in (a) (see Definition 1.2.1). In particular, if  $A$  is commutative, then  $A = C(G)$  for some compact group  $G$ , and  $\Delta$  is given as in (a); if  $A$  is cocommutative (i.e.  $\Delta$  is invariant under the flip  $x \otimes y \mapsto y \otimes x$ ), then (modulo some technicalities)  $A = C_r^*(\Gamma)$  for some discrete group  $\Gamma$ , and  $\Delta$  is given as in (b).

Moreover, Woronowicz established a perfect analytic theory for compact quantum groups. He proved the existence of a “translate invariant” Haar state on the quantum group, and developed a Peter-Weyl type theory with respect to this Haar state, which generalizes the corresponding objects for compact groups. Furthermore, given a compact quantum group, we may construct a dual *discrete quantum group* in the sense of Van Daele's multiplier Hopf algebras [VD94, VD96], or in the sense of Kustermann-Vaes' locally compact quantum groups [KV00], and we have a Pontryagin type duality theorem in this setting. In particular, a discrete group  $\Gamma$  can be viewed as the set of irreducible representations of the dual compact quantum group, and we may obtain a unified language expressing the Fourier transforms and convolutions/multipliers (see Section 2.1).

As a result, the two research lines of abstract harmonic analysis mentioned above, being separated for several decades, join together again under the new framework of compact quantum groups. On the other hand, this new setting motivates us to seek more aspects of the harmonic analysis theory. Indeed, the appearance of compact quantum groups may bring new and fruitful examples in operator algebra theory. This is realized via the appearance of a number of new compact group-like objects, such as the deformation and liberation of compact Lie groups in [BS09, LS91, Wor87b], quantum symmetries of compact (quantum) metric spaces in [BBN12, BGS11, VDW96, Wan98], free/tensor and crossed product constructions in [Wan95a, Wan95b]. Compared to the examples arising from groups, these objects bring us their own interests. For instance, many of these examples give rise to type III von Neumann algebras, with a non-tracial Haar state, which cannot happen in the classical cases. The study of these new quantum groups links to a number of modern fields in mathematics, such as operator algebras, quantum/free probability, ergodic theory and mathematical physics.

This provides us with a nice motivation to explore the harmonic analysis on compact quantum groups. In recent years, topics towards this direction have been largely discussed from different aspects. Let us mention the general theory of Fourier transforms studied in [Kah10, PW90, VD07, Wor98], the completely bounded multipliers on quantum groups in [Daw12, JNR09], and the convolution operators/semigroups in [CFK14, FKS15, LS08, LS11]. However, the corresponding  $L^p$ -theory for Fourier analysis is very much unexplored except very abstract studies such as [Coo10, Cas13]. In this thesis, we will present new progress in this direction, and provide more concrete and manageable approaches regarding the  $L^p$ -theory of Fourier analysis, specifically related to the structure of compact quantum groups.

## 0.2 General theory

Let  $\mathbb{G}$  be a compact quantum group equipped with the Haar state  $h$ . Denote by  $\text{Irr}(\mathbb{G})$  the equivalence classes of irreducible representations of  $\mathbb{G}$ , and by  $Q = (Q_\pi)_{\pi \in \text{Irr}(\mathbb{G})}$  the modular element of  $\hat{\mathbb{G}}$ . Following the standard notation, we use  $\text{Pol}(\mathbb{G})$  to denote the Hopf  $*$ -algebra of polynomials on  $\mathbb{G}$ ,  $C_r(\mathbb{G})$  the reduced  $C^*$ -algebra of continuous functions on  $\mathbb{G}$ , and  $c_0(\hat{\mathbb{G}})$  the  $C^*$ -algebra of functions on  $\hat{\mathbb{G}}$  vanishing at infinity (see Section 1.2.1 and 1.2.2). Also, denote by  $L^p(\mathbb{G})$  the Kosaki noncommutative  $L^p$ -spaces on  $\mathbb{G}$  with respect to

the Haar state  $h$ , and  $\ell^p(\widehat{\mathbb{G}})$  the corresponding spaces on the dual discrete quantum group  $\widehat{\mathbb{G}}$  with respect to the left Haar weight. One may also establish the similar theory with respect to the right Haar weight up to standard modifications.

### Fourier transforms

The Fourier transform for locally compact quantum groups has been discussed in [Coo10], [Cas13] and [Kah10]. In the setting of compact quantum groups, we may give a more explicit description. For any linear functional  $\varphi$  on  $\text{Pol}(\mathbb{G})$ , we may define a *Fourier series*

$$\hat{\varphi}(\pi) = (\varphi \otimes \iota)((u^{(\pi)})^*) \in B(H_\pi), \quad \pi \in \text{Irr}(\mathbb{G}).$$

This induces the definition of Fourier series of each  $x \in \text{Pol}(\mathbb{G})$ , given by

$$\hat{x}(\pi) = (h(\cdot x) \otimes \iota)((u^{(\pi)})^*) \in B(H_\pi), \quad \pi \in \text{Irr}(\mathbb{G}).$$

The Fourier transform  $\mathcal{F} : x \rightarrow \hat{x}$  can be extended to  $L^p$ -spaces. We may establish the following Hausdorff-Young theory on  $\mathbb{G}$ . (See also [Cas13, Coo10, PW90] for related results.)

**Proposition 0.2.1** (Section 2.1). (a)  $\mathcal{F}$  extends to contractions on  $L^p$ -spaces, and

$$\mathcal{F}(L^1(\mathbb{G})) \subset c_0(\widehat{\mathbb{G}}), \quad \mathcal{F}(L^p(\mathbb{G})) \subset \ell^q(\widehat{\mathbb{G}}), \quad 1 \leq p \leq 2, \quad 1/p + 1/q = 1.$$

Moreover,  $\mathcal{F}$  is a unitary operator from  $L^2(\mathbb{G})$  onto  $\ell^2(\widehat{\mathbb{G}})$ .

(b) For all  $x \in L^2(\mathbb{G})$ , we have

$$x = \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi (\iota \otimes \text{Tr})[(1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}],$$

where the convergence of the series is in the  $L^2$ -sense. For any  $\pi \in \text{Irr}(\mathbb{G})$ , if we denote by  $\mathcal{E}_\pi$  the orthogonal projection of  $L^2(\mathbb{G})$  onto the subspace spanned by the matrix coefficients  $(u_{ij}^{(\pi)})_{i,j=1}^{n_\pi}$ , and write  $\mathcal{E}_\pi x = \sum_{i,j} x_{ij}^{(\pi)} u_{ij}^{(\pi)}$  with  $x_{ij}^{(\pi)} \in \mathbb{C}$ ,  $X_\pi = [x_{ji}^{(\pi)}]_{i,j}$ , then

$$\hat{x}(\pi) = d_\pi^{-1} X_\pi Q_\pi^{-1}.$$

We may also introduce the notion of *Fourier algebra* on  $\mathbb{G}$ , defined by

$$A(\mathbb{G}) = \{x \in L^1(\mathbb{G}) : \hat{x} \in \ell^1(\widehat{\mathbb{G}})\}.$$

**Proposition 0.2.2** (Proposition 2.1.5). We have  $A(\mathbb{G}) \subset C_r(\mathbb{G})$  and

$$\|x\|_\infty \leq \|\hat{x}\|_1, \quad x \in A(\mathbb{G}).$$

Moreover, if we let  $\|x\|_A = \|\hat{x}\|_1$  for  $x \in A(\mathbb{G})$ , then  $(A(\mathbb{G}), \|\cdot\|_A)$  is a Banach space isometrically isomorphic to  $\ell^1(\widehat{\mathbb{G}})$ .

This result and its proof have been recently applied and generalized in [DPR16] to discuss the relation with polynomial growth conditions of  $\mathbb{G}$ .

### Young's inequality

Let  $\varphi_1, \varphi_2$  be two linear functionals on  $\text{Pol}(\mathbb{G})$ . We define their convolution by

$$\varphi \star \varphi' = (\varphi \otimes \varphi') \circ \Delta.$$

This also induces a definition of convolution for  $x, x' \in \text{Pol}(\mathbb{G})$ , so that

$$h(\cdot(x \star x')) = h(\cdot x) \star h(\cdot x') \in \text{Pol}(\mathbb{G}).$$

We may establish the following Young's inequalities.

**Proposition 0.2.3** (Section 2.2). (a) Let  $1 \leq p, q, r \leq 2$  with  $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $x, y \in \text{Pol}(\mathbb{G})$ . We have

$$\|\tau_{i/p'}(y) \star x\|_r \leq \|x\|_p \|y\|_q.$$

(b) Assume additionally that  $\mathbb{G}$  is of Kac type. Let  $1 \leq p, q, r \leq \infty$  with  $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $x, y \in \text{Pol}(\mathbb{G})$ . We have

$$\|y \star x\|_r \leq \|x\|_p \|y\|_q.$$

The Young inequality and related topics for more general locally compact quantum groups are discussed in the forthcoming work [LWW16].

### $L^p$ -Fourier multipliers

Many of the results in the thesis rely on some very elementary properties of bounded  $L^p$ -Fourier multipliers on compact quantum groups, which have not been discussed so far in literature. Although the completely bounded  $L^\infty$ -multipliers, instead of bounded  $L^\infty$ -multipliers, are more natural objects for studying harmonic analysis on quantum groups, the discussions on bounded  $L^p$ -multipliers are still natural for studying  $L^p$ -Fourier analysis on quantum groups, especially for  $p < \infty$ . As a result we include in the thesis a detailed argument on some basic facts in Section 2.3. These are also frequently used in other proofs of this thesis. Here we list some main properties.

For  $a = (a_\pi)_\pi \in \prod_\pi B(H_\pi)$ , we define the left and right multipliers  $m_a^L : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G})$ ,  $m_a^R : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G})$  associated to  $a$  (cf. [JNR09, Daw12]) by

$$(m_a^L \otimes \iota)u^{(\pi)} = (1 \otimes a_\pi)u^{(\pi)}, \quad (m_a^R \otimes \iota)u^{(\pi)} = u^{(\pi)}(1 \otimes a_\pi).$$

We say that  $a$  is a *bounded left (resp., right) multiplier* on  $L^p(\mathbb{G})$  if  $m_a^L$  (resp.,  $m_a^R$ ) extends to a bounded map on  $L^p(\mathbb{G})$ , and denote the set of all such multipliers by  $M_L(L^p(\mathbb{G}))$  (resp.,  $M_R(L^p(\mathbb{G}))$ ). We define

$$M(L^p(\mathbb{G})) = \left\{ a \in \prod_\pi B(H_\pi) : Q^{-1/p} a Q^{1/p} \in M_L(L^p(\mathbb{G})), a \in M_R(L^p(\mathbb{G})) \right\}$$

equipped with the norm

$$\|a\|_{M(L^p(\mathbb{G}))} = \max\{\|m_{Q^{-1/p} a Q^{1/p}}^L\|_{B(L^p(\mathbb{G}))}, \|m_a^R\|_{B(L^p(\mathbb{G}))}\}.$$

Our first property concerns the question how to construct a left bounded  $L^p$ -multiplier from a right bounded  $L^p$ -multiplier.

**Proposition 0.2.4** (Lemma 2.3.3). Let  $1 \leq p \leq \infty$ . We have for all  $a \in \prod_\pi B(H_\pi)$ ,

$$\|m_a^R\|_{B(L^p(\mathbb{G}))} = \|m_{Q^{-1/2} a^* Q^{1/2}}^L\|_{B(L^p(\mathbb{G}))}.$$

The second result concerns the inequality comparing the norms  $\|a\|_{\ell^\infty(\hat{\mathbb{G}})}$  and  $\|a\|_{M(L^p(\mathbb{G}))}$ . It is well known that if  $\mathbb{G}$  is a compact group  $G$  or the dual compact quantum group  $\hat{\Gamma}$  of a discrete group  $\Gamma$ , then for all  $1 \leq p \leq \infty$  we have

$$\|a\|_\infty \leq \|a\|_{M(L^p(\mathbb{G}))}, \quad a \in M(L^p(\mathbb{G})).$$

The inequalities are also easy to deal with for completely bounded  $L^\infty$ -multipliers, but it seems that they are not obvious for other cases. The following proposition improves an earlier result in [Daw10, Proposition 8.8] for compact quantum groups. The result is partially based on a personal communication by Marius Junge.

**Proposition 0.2.5** (Proposition 2.3.5). *Let  $1 \leq p \leq \infty$ . For all  $a \in \prod_\pi B(H_\pi)$  and all  $0 \leq \theta \leq 1$ , we have*

$$\|Q^{\frac{1}{4}-\frac{\theta}{2}} a Q^{-\frac{1}{4}+\frac{\theta}{2}}\|_\infty \leq \|m_a^R\|_{B(L^p(\mathbb{G}))}^{1/2} \|m_{Q^{-\theta} a Q^\theta}^L\|_{B(L^p(\mathbb{G}))}^{1/2}.$$

*In particular, if additionally  $\mathbb{G}$  is of Kac type, we have*

$$\|a\|_\infty \leq \|a\|_{M(L^p(\mathbb{G}))}.$$

### 0.3 Specific topics

The convolution operators or multipliers constitute a central part of Fourier analysis. For instance, for a periodic function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , the decay of Fourier coefficients are closely related to the integrability properties of  $f$ , which are revealed by the Riemann-Lebesgue lemma for  $L^1$ -functions and the Plancherel theorem for  $L^2$ . Given  $1 < p < 2$ , a classical problem in harmonic analysis is to determine the condition on a multiplier function  $\zeta : \mathbb{Z} \rightarrow \mathbb{R}^+$  so that

$$f \in L^p(\mathbb{T}) \Rightarrow \sum_{n \in \mathbb{Z}} \zeta(n) |\hat{f}(n)|^2 < +\infty.$$

If we take  $\zeta(n) = |\hat{\mu}(n)|^2$  for a measure  $\mu$ , this leads to the norm estimates of convolution operators of measures, in particular the existence and behavior of Borel measures that convolve  $L^p(\mathbb{T})$  into  $L^2(\mathbb{T})$  contractively. This may further bring us to the topic of hypercontractivity and logarithmic Sobolov inequalities which are closely related to many other fields of mathematics [Gro06]. Similar questions can be certainly discussed if we replace the above  $\mathbb{Z}$  by other discrete (quantum) groups. In this thesis, we will characterize positive convolution operators on a finite quantum group  $\mathbb{G}$  which are  $L^p$ -improving, and give some constructions and applications for infinite compact quantum groups.

On the other hand, note that for  $\zeta$  being a characteristic function, the corresponding problem is related to the notions of lacunary and  $\Lambda(p)$ -sets, which were studied for example in [Rud60]. These concepts were largely extended and studied in the past several decades, with various links to harmonic analysis, Banach space theory, operator algebras, and geometric/analytic properties of groups. In the second subsection, we will present some corresponding results in the framework of compact quantum groups.

#### 0.3.1 $L^p$ -improving convolution operators

As is remarked above, one among phenomena studied on the circle group  $\mathbb{T}$  is the existence and behavior of positive Borel measures that convolve  $L^p(\mathbb{T})$  into  $L^q(\mathbb{T})$  with finite  $q > p$

for a given  $1 < p < \infty$ , which are considered to be  $L^p$ -improving measures. An example due to Oberlin [Obe82] is the Cantor-Lebesgue measure supported by the usual middle-third Cantor set. Oberlin revealed that, after a careful analysis on the structure of this measure, this result can be reduced to proving that there exists  $p < 2$  such that

$$\|\mu \star f\|_2 \leq \|f\|_p, \quad f \in L^p(\mathbb{Z}/3\mathbb{Z})$$

where the  $L^p$ -norms are those taken with respect to the normalized counting measure on the cyclic group  $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$  with three elements and  $\mu$  is the probability measure with mass  $1/2$  at 0 and at 2. Motivated by these results, Ritter showed in 1984 the following result.

**Theorem 0.3.1** ([Rit84]). (a) *If  $G$  is a finite group and  $T_\mu : f \mapsto \mu \star f$  is the convolution operator associated to a probability measure  $\mu$  on  $G$ , then*

$$\exists p < 2, \quad \|T_\mu : L^p(G) \rightarrow L^2(G)\| = 1$$

*if and only if  $G$  is generated by  $\{ij^{-1} : i, j \in \text{supp } \mu\}$ .*

(b) *If additionally  $G$  is abelian, the above conditions holds if and only if  $|\hat{\mu}(\gamma)| < 1$  for any  $\gamma \in \hat{G} \setminus \{e\}$ .*

In this thesis we give an alternative approach to these topics in the context of quantum groups and noncommutative  $L^p$ -spaces. In particular, we have following result.

**Theorem 0.3.2** (Theorem 3.3.4). *Let  $\mathbb{G}$  be a finite quantum group and  $\varphi$  be a state on  $C(\mathbb{G})$ . Denote  $\psi = (\varphi \circ S) \star \varphi$ . The following assertions are equivalent:*

(1) *there exists  $1 \leq p < 2$  such that,*

$$\forall x \in C(\mathbb{G}), \quad \|\varphi \star x\|_2 \leq \|x\|_p ;$$

(2)  *$\|\hat{\varphi}(\pi)\| < 1$  for all  $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}$  ;*

(3) *For any nonzero  $x \in C(\mathbb{G})_+$ , there exists  $n \geq 1$  such that  $\psi^{\star n}(x) > 0$ .*

The last assertion should be interpreted as claiming that the “support” of  $\varphi$  “generates” the quantum group  $\mathbb{G}$ . We will illustrate by example in Remark 3.3.7 that the finiteness condition of the quantum group in the above conclusion is rather crucial and cannot be removed.

In particular, the result characterizes the Fourier-Schur multipliers on finite groups which have an  $L^p$ -improving property. We obtain the following generalization of Theorem 0.3.1(b).

**Corollary 0.3.3** (Corollary 3.3.6). *Let  $\Gamma$  be a finite group and  $\varphi$  be a positive definite function on  $\Gamma$  with  $\varphi(e) = 1$ . Let  $M_\varphi$  be the associated Fourier-Schur multiplier operator determined by  $M_\varphi(\lambda(\gamma)) = \varphi(\gamma)\lambda(\gamma)$  for all  $\gamma \in \Gamma$ . Then there exists  $1 \leq p < 2$  such that*

$$\|M_\varphi x\|_2 \leq \|x\|_p, \quad x \in C^*(\Gamma)$$

*if and only if  $|\varphi(\gamma)| < 1$  for any  $\gamma \in \Gamma \setminus \{e\}$ .*

We should emphasize that our argument relies essentially on new and interesting properties on the unital trace preserving operators on noncommutative  $L^p$ -spaces, discovered in the recent work of Ricard and Xu [RX16]. In fact, the following fact plays a key role in our argument.



**Theorem 0.3.4** (Theorem 3.1.6). *Let  $A$  be a finite dimensional von Neumann algebra equipped with a faithful tracial state  $\tau$ , and  $T : A \rightarrow A$  be a unital positive trace preserving map on  $A$ . Then*

$$\exists 1 < p < 2, \forall x \in A, \|Tx\|_2 \leq \|x\|_p$$

*if and only if*

$$\sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\|Tx\|_2}{\|x\|_2} < 1.$$

We provide two proofs of this result, where one is based on very elementary arguments with an additional assumption of 2-positivity and another, which is rather short, on [RX16]. We also show that the  $L^p$ -improving property remains stable under the free products. This method permits us to construct  $L^p$ -improving convolution operators for infinite quantum groups.

**Theorem 0.3.5.** (a)(Theorem 3.1.9) *Let  $(A_i, \tau_i)$ ,  $1 \leq i \leq n$  be a finite family of finite dimensional  $C^*$ -algebras equipped with faithful tracial states  $\tau_i$ , and set  $(\mathcal{A}, \tau) = \star_{1 \leq i \leq n} (A_i, \tau_i)$  to be the von Neumann algebraic free product. For each  $1 \leq i \leq n$ ,  $T_i : A_i \rightarrow A_i$  is a unital positive trace preserving map such that*

$$\|T_i : L^p(A_i) \rightarrow L^2(A_i)\| = 1$$

*for some  $1 < p < 2$ . Then the (algebraic) free product map  $T = \star_{1 \leq i \leq n} T_i$  on  $\star_{1 \leq i \leq n} A_i$  extends to a map such that*

$$\|T : L^{p'}(\mathcal{A}) \rightarrow L^2(\mathcal{A})\| = 1$$

*for some  $1 < p' < 2$ .*

(b)(Corollary 3.3.8) *Let  $\mathbb{G}_1, \dots, \mathbb{G}_n$  be finite quantum groups and let each  $\varphi_i$  be a state on  $C(\mathbb{G}_i)$ ,  $i \in \{1, \dots, n\}$ . Denote  $\mathbb{G} = \mathbb{G}_1 \hat{\star} \dots \hat{\star} \mathbb{G}_n$  and let  $\varphi$  be the conditional free product of  $\varphi_i$  with respect to Haar states. If each  $\varphi_i$  satisfies any one of the conditions (1)-(3) in Theorem 0.3.2, then the free product convolution operator given by  $T : x \mapsto x \star \varphi$ ,  $x \in C(\mathbb{G})$  is a unital left multiplier on  $\mathbb{G}$  satisfying*

$$\|T : L^p(\mathbb{G}) \rightarrow L^2(\mathbb{G})\| = 1$$

*for a certain  $1 < p < 2$ .*

In this thesis we also include some simple properties of non-degenerate states on compact quantum groups with applications. We prove in Lemma 3.2.3 that the convolution Cesàro limit of a non-degenerate state is the Haar state, which not only contributes to the proof of our main result, but also generalizes the main result of [BFS12] concerning the computation of idempotent states associated to Hopf images. We refer to Section 3.2 for more information on Hopf images.

**Theorem 0.3.6** (Theorem 3.2.4). *Let  $\mathbb{G}$  be a compact quantum group and  $A$  be a unital  $C^*$ -algebra with a unital  $*$ -homomorphism  $\pi : C(\mathbb{G}) \rightarrow A$ . Let  $\mathbb{G}_\pi$  be the Hopf image of  $\pi$  and  $q : C(\mathbb{G}) \rightarrow C(\mathbb{G}_\pi)$  be the corresponding quotient map. Then given any faithful state  $\varphi$  on  $A$ ,*

$$h_{\mathbb{G}_\pi} \circ q = w^* - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\varphi \circ \pi)^{\star k}.$$

This result was recently used in [BN16, Ban16b, Ban16a] to study the matrix model of quantum permutation groups.



### 0.3.2 Lacunarity

The study of lacunarity and particularly of Sidon sets and  $\Lambda(p)$ -sets is a major and fascinating subject of harmonic analysis. Historically, the notion of Sidon sets originated from discussions of special lacunary series on the circle  $\mathbb{T}$ . This has been generalized later by replacing  $\mathbb{T}$  with an arbitrary compact abelian group  $G$ . Recall that for a compact abelian group  $G$ , a Sidon set  $\mathbf{E}$  for  $G$  is a subset of the dual discrete group  $\Gamma = \hat{G}$  such that any continuous function on  $G$  with Fourier transform supported on  $\mathbf{E}$  has absolutely convergent Fourier series. The theory of Sidon sets and lacunarity for compact abelian groups has been remarkably developed in the past several decades, in a series of works of Rudin, Drury, Rider, Marcus, Pisier and others (see [Dru70, Rid75, Rud60, MP81, Pis78a, Pis78b, Pis83a, Pis83b] and references therein); it is shown to be deeply related to the theory of random Fourier series, metric entropy condition, multiplier spaces as well as some other topics in harmonic analysis and Banach space theory.

The development of similar subjects in more general setting goes naturally into two lines. The first one is the theory of Sidon and  $\Lambda(p)$ -sets in the setting of a non-abelian compact group  $G$ ; these sets are special subsets of irreducible representations of  $G$ . This generalized notion of lacunarity and Sidon sets was firstly introduced in [Hel58, FTR66, HR70] and then has seen many links to the harmonic analysis and probability theory as in the abelian case (see [Boz74, MP81]). The other line of development concerns subsets of an arbitrary non-abelian discrete group  $\Gamma$  with related "functions" in its group von Neumann algebra  $VN(\Gamma)$ , such as in [Pic73, FT77, Boz81]. The latter is more complicated and the behavior of lacunarity is closely related to the amenability of the discrete group, which also involves some tools from the operator space theory and leads to interesting topics in the study of completely bounded  $L^p$ -Fourier multipliers in abstract harmonic analysis ([Har99, Pis95]). Also, for non-amenable discrete groups, the  $\Lambda(\infty)$ -type estimation for particular subsets often plays an important role in the study of some growth conditions ([Jol90, JV91]), approximation properties ([Haa79]), and complemented ideals of Fourier algebras ([BFZ16]).

In the both directions mentioned above, the Sidon sets admit many seemingly non-related characterizations, and enjoy some special relations with  $\Lambda(p)$ -sets, multipliers, etc.. Moreover, many basic properties of Sidon sets in the two different settings have quite similar formulations. This gives rise to a natural motivation to find a more general framework to unify these developments from different viewpoints, and to seek new aspects of the theory for these objects. In fact, based on the Woronowicz-Peter-Weyl theory for compact quantum groups, we may discuss similar questions for lacunarity in the quantum group setting. Recall that a priori in this general view, a discrete group  $\Gamma$  is regarded as the set of irreducible representations of the dual compact quantum group  $\mathbb{G} = \hat{\Gamma}$ , and the amenability of  $\Gamma$  is often interpreted as the coamenability of  $\mathbb{G}$ .

In this thesis we introduce and develop some important notions and properties related to lacunarity in the framework of compact quantum groups. The subjects that we address are as follows.

#### Generalizations and characterizations of Sidon sets

Let  $\mathbb{G}$  be a compact quantum group and  $\mathbf{E}$  be a subset of  $\text{Irr}(\mathbb{G})$ . We denote

$$\ell^\infty(\mathbf{E}) = \{(a_\pi)_{\pi \in \text{Irr}(\mathbb{G})} \in \ell^\infty(\hat{\mathbb{G}}) : a_\pi = 0 \text{ if } \pi \in \text{Irr}(\mathbb{G}) \setminus \mathbf{E}\}, \quad c_0(\mathbf{E}) = c_0(\hat{\mathbb{G}}) \cap \ell^\infty(\mathbf{E}).$$

**Definition 0.3.7.** We say that a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a *Sidon set* if there exists  $K > 0$  such that for any  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ , we have

$$\|\hat{x}\|_1 \leq K\|x\|_{\infty}.$$

The following fundamental characterizations extends the classical result of [HR70, (37.2)] for compact groups.

**Theorem 0.3.8** (Theorem 4.1.3). *Let  $\mathbb{G}$  be a compact quantum group and  $\mathbf{E}$  be a subset of  $\text{Irr}(\mathbb{G})$ . The following assertions are equivalent:*

- (1)  $\mathbf{E}$  is a Sidon set;
- (2) for any  $a \in \ell^{\infty}(\mathbf{E})$ , there exists  $\varphi \in C_r(\mathbb{G})^*$  such that  $\hat{\varphi}(\pi) = a_{\pi}$  for all  $\pi \in \mathbf{E}$ ;
- (3) for any  $a \in c_0(\mathbf{E})$ , there exists  $x \in L^1(\mathbb{G})$  such that  $\hat{x}(\pi) = a_{\pi}$  for all  $\pi \in \mathbf{E}$ ;
- (4) any  $x \in L^{\infty}(\mathbb{G})$  with  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$  belongs to  $A(\mathbb{G})$ ;
- (5) any  $x \in C_r(\mathbb{G})$  with  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$  belongs to  $A(\mathbb{G})$ .

Here our approach is different from the classical one in [HR70]. This answers in particular a basic question in the study on lacunarity of discrete groups, raised in [FT77, Pic73], that is, the equivalence between strong Sidon sets and Sidon sets in discrete groups in the sense of [Pic73]. In fact, in the latter article Picardello defined two different kinds of lacunary sets that he called strong Sidon sets and Sidon sets in discrete groups. He proved the equivalence between these two notions for amenable discrete groups, but the non-amenable case had been left open before our work.

Recall that the Fourier algebra of a discrete group  $\Gamma$  is given by

$$A(\Gamma) = \{f(\cdot) = \langle \lambda(\cdot)\xi, \eta \rangle \in c_0(\Gamma) : \xi, \eta \in \ell^2(\Gamma)\}$$

where  $\lambda$  denotes the left regular representation. Note that it is a different object from the Fourier algebra  $A(\hat{\Gamma})$  of the compact quantum group  $\hat{\Gamma}$  introduced before.

**Corollary 0.3.9.** *Let  $\Gamma$  be a discrete group. The following assertions are equivalent:*

- (a)  $E \subset \Gamma$  is a Sidon set;
- (b)  $E \subset \Gamma$  is a strong Sidon set (in the sense of [Pic73]), i.e., for any  $a \in c_0(E)$ , there is  $f$  in the Fourier algebra  $A(\Gamma)$  of  $\Gamma$  such that  $f|_E = a$ .
- (c) For all  $x \in VN(\Gamma)$  with  $\hat{x}|_{\Gamma \setminus E} = 0$ , we have  $\sum_{\gamma \in \Gamma} |\hat{x}(\gamma)| < \infty$ .

We show as well that the Sidon property is stable under Cartesian/free products of compact quantum groups. Apart from the above approach, there are indeed various viewpoints on generalizations of Sidon sets for non-coamenable compact quantum groups, which lead to different types of lacunarities such as weak Sidon sets and unconditional Sidon sets. In Theorem 4.1.15 we will discuss the relations between these various notions and prove the equivalence among them for coamenable compact quantum groups.

### $\Lambda(p)$ -sets

We study the  $\Lambda(p)$ -sets and interpolation sets of bounded  $L^p$ -Fourier multipliers for a compact quantum group, defined as follows.

**Definition 0.3.10.** Let  $\mathbb{G}$  be a compact quantum group and  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  be a subset. Take  $1 < p < \infty$ .

(a) We say that  $\mathbf{E}$  is a  $\Lambda(p)$ -set if there exists  $K > 0$  such that for all  $x \in \text{Pol}(\mathbb{G})$  with  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$ ,

$$\|x\|_p \leq K\|x\|_1;$$

(b) We say that  $\mathbf{E}$  is an *interpolation set* of  $M(L^p(\mathbb{G}))$  if for any  $a \in \ell^{\infty}(\mathbf{E})$ , there exists a bounded multiplier  $\tilde{a} \in M(L^p(\mathbb{G}))$  such that  $\tilde{a}_{\pi} = a_{\pi}$  for all  $\pi \in \mathbf{E}$ .

We prove the equivalence between  $\Lambda(p)$ -sets and interpolation sets of bounded  $L^p$ -Fourier multipliers, generalizing the previous work [Har99].

**Theorem 0.3.11** (Theorem 4.2.7). *Let  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  be a subset. Assume  $2 < p < \infty$ . Then  $\mathbf{E}$  is a  $\Lambda(p)$ -set if and only if it is an interpolation set of  $M(L^p(\mathbb{G}))$ .*

The argument is much more tricky than in the classical case, and the result indeed solves a recent problem regarding the Sidon sets. For a compact group or (the dual of) a discrete group, a simple (but non-trivial) argument shows that any Sidon set is a  $\Lambda(p)$ -set for  $1 < p < \infty$ . The case for a general compact quantum group turns out to be more difficult. A first attempt was made by Blendek and Michaliček [BM13] in 2013.

**Theorem 0.3.12** ([BM13]). *Let  $\mathbb{G}$  be a compact quantum group of Kac type. For each  $\pi \in \text{Irr}(\mathbb{G})$ , denote by  $\chi_\pi$  the character of  $\pi$ . If  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a Sidon set satisfying the Helgason-Sidon condition, then there is a constant  $K > 0$ , such that for all  $(c_\pi)_{\pi \in \mathbf{E}} \subset \mathbb{C}$ , we have*

$$\left\| \sum_{\pi \in \mathbf{E}} c_\pi \chi_\pi \right\|_2 \leq K \left\| \sum_{\pi \in \mathbf{E}} c_\pi \chi_\pi \right\|_1.$$

Compared to the classical results, this is more restrictive, but the proof is quite technical and non-trivial. Now based on our result presented previously, it is easy to deduce that any Sidon set for an arbitrary compact quantum group is a  $\Lambda(p)$ -set for  $1 < p < \infty$ , which completely removes the undesirable restrictions above.

**Corollary 0.3.13** (Corollary 4.2.9). *If  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a Sidon set, then  $\mathbf{E}$  is a  $\Lambda(p)$ -set for all  $1 < p < \infty$ .*

Our investigation also leads to some facts which are hidden in the classical cases. Note that any discrete group must be unimodular, but there are many non-unimodular discrete quantum groups. We show that the  $\Lambda(p)$  condition yields a strict control on the modular element  $Q$  of the dual quantum group.

**Proposition 0.3.14** (Proposition 4.2.5). *If  $1 < p < \infty$  and  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a  $\Lambda(p)$ -set, then*

$$\sup_{\pi \in \mathbf{E}} \|Q_\pi\| < \infty, \quad \sup_{\pi \in \mathbf{E}} \|Q_\pi^{-1}\| < \infty.$$

In general, the noncommutative  $L^p$ -spaces introduced by Kosaki depend on different interpolation parameters. For each parameter  $0 \leq \theta \leq 1$ , we denote by  $L^p_{(\theta)}(\mathbb{G})$  the corresponding  $L^p$ -spaces. Based on the above result, we may establish the following fact.

**Proposition 0.3.15** (Proposition 4.2.14). *Let  $\mathbb{G}$  be a compact quantum group and let  $2 < p < \infty$ ,  $0 \leq \theta, \theta' \leq 1$ . Then  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a  $\Lambda(p)$ -set for  $L^p_{(\theta)}(\mathbb{G})$  if and only if it is a  $\Lambda(p)$ -set for  $L^p_{(\theta')}(\mathbb{G})$ .*

We also discuss the existence of  $\Lambda(p)$ -sets. It is based on the following general result.

**Theorem 0.3.16** (Theorem 4.4.1). *Let  $\mathcal{M}$  be a von Neumann algebra equipped with a normal faithful state  $\varphi$  and consider the associated  $L^p$ -spaces  $L^p(\mathcal{M}, \varphi)$ . Let  $B = \{x_i \in \mathcal{M} : i \geq 1\}$  be an orthogonal system with respect to  $\varphi$  such that  $\sup_i \|x_i\|_\infty < \infty$ . Then for each  $2 < p < \infty$ , there exists an infinite subset  $\{x_{i_k} : k \geq 1\} \subset B$  and a constant  $C > 0$  such that for all finitely supported sequences  $(c_k) \subset \mathbb{C}$  we have*

$$\left\| \sum_{k \geq 1} c_k x_{i_k} \right\|_p \leq C \left( \sum_{k \geq 1} |c_k|^2 \right)^{\frac{1}{2}}.$$

Then we may establish the existence of  $\Lambda(p)$ -sets for quantum groups, under an assumption of uniformly bounded dimensions.

**Theorem 0.3.17** (Theorem 4.2.15). *Let  $\mathbb{G}$  be a compact quantum group. Let  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  be an infinite subset with  $\sup_{\pi \in \mathbf{E}} d_\pi < \infty$ . Then for each  $1 < p < \infty$ , there exists an infinite subset  $\mathbf{F} \subset \mathbf{E}$  which is a  $\Lambda(p)$ -set for  $\mathbb{G}$ .*

### Central Sidon sets

We also investigate some basic notions and facts concerning central Sidon sets for compact quantum groups.

**Definition 0.3.18.** We say that a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a *central Sidon set* if there exists  $K > 0$  such that for any finite sequence  $(c_\pi) \subset \mathbb{C}$  and  $x = \sum_\pi c_\pi \chi_\pi$ , we have  $\|\hat{x}\|_1 \leq K \|x\|_\infty$ .

Our argument includes some new characterizations of compact quantum groups of Kac type as follows. Denote by  $\text{Pol}^z(\mathbb{G}) = \{x = \sum_{\pi \in \text{Irr}(\mathbb{G})} c_\pi \chi_\pi \in \text{Pol}(\mathbb{G}) : c_\pi \in \mathbb{C}\}$  the subspace of central polynomials and let  $C_r^z(\mathbb{G})$  be the norm closure of  $\text{Pol}^z(\mathbb{G})$  in  $C_r(\mathbb{G})$ .

**Proposition 0.3.19** (Lemma 4.3.3). *Let  $\mathbb{G}$  be a compact quantum group. The following assertions are equivalent:*

- (1)  $\mathbb{G}$  is of Kac type;
- (2) any central functional  $\omega$  is bounded on  $\text{Pol}(\mathbb{G})$  (with respect to  $\|\cdot\|_\infty$ ) if and only if it is bounded on  $\text{Pol}^z(\mathbb{G})$  with the same norm;
- (3) there exists a conditional expectation  $\mathcal{E}$  from  $C_r(\mathbb{G})$  onto  $C_r^z(\mathbb{G})$  such that  $h \circ \mathcal{E} = h$ .

**Proposition 0.3.20** (Proposition 4.3.4). *Let  $\mathbb{G}$  be a compact quantum group of Kac type. Then  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a central Sidon set if and only if for all bounded sequences  $(a_\pi)_{\pi \in \mathbf{E}} \subset \mathbb{C}$ , there exists a bounded central functional  $\varphi \in C_r(\mathbb{G})^*$  such that  $\hat{\varphi}(\pi) = a_\pi \text{Id}_\pi$  for  $\pi \in \mathbf{E}$ .*

The argument also answers a question in the classical case. Indeed, for a compact quantum group  $\mathbb{G}$  and a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$ , we see that the following two assertions are generally *not* equivalent (Remark 4.3.5):

- (a) there exists  $K > 0$  such that for any unitary  $a \in \ell^\infty(\mathbf{E})$  and for any  $x \in \text{Pol}(G)$  with  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$ ,

$$\|m_a^L x\|_\infty \leq K \|x\|_\infty;$$

- (b) there exists a constant  $K > 0$  such that for all  $(\varepsilon_\pi)_{\pi \in \mathbf{E}} \subset \{-1, 1\}$  and for any  $x \in \text{Pol}(G)$  with  $\hat{x}|_{\text{Irr}(\mathbb{G}) \setminus \mathbf{E}} = 0$ ,

$$\left\| \sum_{\pi \in \mathbf{E}} d_\pi \varepsilon_\pi (\iota \otimes \text{Tr})((1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}) \right\|_\infty \leq K \|x\|_\infty.$$

Indeed the assertion (a) is equivalent to the fact that  $\mathbf{E}$  is a Sidon set, but (b) is equivalent to the fact that  $\mathbf{E}$  is a central Sidon set.

We also obtain that any two compact quantum groups with the same fusion rules and the same dimension functions have identical central Sidon sets. This gives many examples of central Sidon sets for compact quantum groups via monoidal equivalence, Drinfeld-Jimbo deformations, etc..

**Proposition 0.3.21** (Proposition 4.3.7). *Let  $\mathbb{G}_1, \mathbb{G}_2$  be two compact quantum groups. Assume that  $\Phi : \text{Rep}(\mathbb{G}_1) \rightarrow \text{Rep}(\mathbb{G}_2)$  is an injective map preserving the fusion rules, that is, for all  $\pi, \pi' \in \text{Rep}(\mathbb{G}_1)$  we have*

$$\Phi(\pi \otimes \pi') = \Phi(\pi) \otimes \Phi(\pi'), \quad \Phi(\oplus_{i=1}^n \pi_i) = \oplus_{i=1}^n \Phi(\pi_i), \quad \pi, \pi', \pi_i \in \text{Rep}(G), n \geq 1.$$

*For any central Sidon set  $\mathbf{E} \subset \text{Irr}(\mathbb{G}_1)$ , if additionally there exists  $C > 0$  satisfying  $\dim(\Phi(\pi)) \leq C \dim(\pi)$  for all  $\pi \in \mathbf{E}$ , then  $\Phi(\mathbf{E}) \subset \text{Irr}(\mathbb{G}_2)$  is a central Sidon set for  $\mathbb{G}_2$ .*

### Some examples

Now we collect some typical examples of the lacunarity discussed in the thesis.

(1) (Example 4.1.17) Consider the compact quantum group  $\mathbb{G} = \prod_{k \geq 1} U_{N_k}^+$ , where for each  $k \geq 1$  and  $N_k \geq 1$ ,  $U_{N_k}^+$  denotes the free unitary group. Let  $u^{(k)}$  be the fundamental representation of  $U_{N_k}^+$  and put  $\mathbf{E} = \{u^{(k)} : k \geq 1\} \subset \text{Irr}(\mathbb{G})$ . Then  $\mathbf{E}$  is a weak Sidon set, and is an interpolation set of  $M(L^\infty(\mathbb{G}))$ . It is thus a  $\Lambda(p)$ -set. But it is *not* a Sidon set.

(2) (Example 4.1.18) Consider the sequence  $(q_n)_{n \geq 1} \subset [0, 1]$  with  $q := \inf_n q_n > 0$  and the associated quantum group  $\mathbb{G} = \prod_{n \geq 1} \text{SU}_{q_n}(2)$ , where  $\text{SU}_q(2)$  is given by the Drinfeld-Jimbo deformation. We denote by  $\pi_n$  the fundamental representation of  $\text{SU}_{q_n}(2)$ . Then  $\mathbf{E} = \{\pi_n : n \geq 1\} \subset \text{Irr}(\mathbb{G})$  is a Sidon set for  $\mathbb{G}$ , and hence a  $\Lambda(p)$ -set. The optimal order of Sidon constant is  $o(q^{-1})$  when  $q \rightarrow 0$ . If  $q = 0$ , the subset  $\mathbf{E}$  given above is not a Sidon set nor a  $\Lambda(p)$ -set with  $p > 2$ .

(3) (Example 4.3.8) Let  $q \in [-1, 1] \setminus \{0\}$  and consider the Drinfeld-Jimbo  $q$ -deformation  $\text{SU}_q(n)$ ,  $n \geq 2$ . We denote by  $\pi_n$  the fundamental representation of  $\text{SU}_q(n)$ . For any  $0 < q < 1$  the subset  $\{\pi_n : n \geq 2\}$  is a central Sidon set for the compact quantum group  $\prod_{n \geq 2} \text{SU}_q(n)$ .

(4) For all  $0 < q < 1$  and all simply connected compact semi-simple Lie group  $G$ , the compact quantum group  $G_q$  given by the Drinfeld-Jimbo deformation does not admit any infinite Sidon set nor central Sidon set (Corollary 4.3.9). Also, there does not exist any infinite  $\Lambda(p)$ -set with  $p > 2$  for  $\text{SU}_q(2)$  (Proposition 4.2.17). However, the subset  $\mathbf{E} \subset \text{Irr}(\text{SU}_q(2))$  indexed by  $\{n_k \in \mathbb{N} \cup \{0\} : k \geq 0\}$  such that  $n_k = n_{k-1} + k$ , is a central  $\Lambda(4)$ -set (Proposition 4.2.16). More precisely, there exists  $K_q > 0$  such that for any finitely supported sequence  $(c_n)_{n \in \mathbf{E}} \in \mathbb{C}$ ,

$$\left\| \sum_{n \in \mathbf{E}} c_n \chi_n \right\|_4 \leq K_q \left\| \sum_{n \in \mathbf{E}} c_n \chi_n \right\|_2.$$

### A few remarks and open problems

Before ending the introduction, we remark that, it would also be interesting to study the completely bounded version of Sidon sets or  $\Lambda(p)$ -sets. We refer to Pisier's work [Pis95] for completely bounded lacunarity in discrete groups. We have not studied here this topic; but we will pursue it elsewhere. In fact, our argument on  $L^p$ -Fourier multipliers in Section 2.3, and the estimation of modular elements in Section 4.2, will be helpful for the study towards this direction.

On the other hand, it is interesting to look for an quantum analogue of Drury-Rider's theorem [Dru70, Rid75] asserting that the union of two Sidon sets for a compact group is again a Sidon set. Also, Pisier's  $\Lambda(\psi_2)$  characterization of Sidon sets [MP81] is unexplored in the quantum group setting. We would like to remark that these problems are even open for the compact duals of discrete non-abelian groups, which are among the easiest examples of quantum groups.

# Chapter 1

## Preliminaries

This chapter collects the necessary preliminaries for the whole thesis. The first section is a brief introduction to basic properties of the noncommutative  $L^p$ -space associated to a von Neumann algebra with a normal faithful state. In the second section we present some preliminaries on compact quantum groups, including some basic notions and properties, as well as some typical constructions such as the free products and the Drinfeld-Jimbo deformation.

### 1.1 Noncommutative $L^p$ -spaces

In this section we recall some basic definitions and facts on noncommutative  $L^p$ -spaces. We refer to [Tak02, Tak03] for the theory of von Neumann algebras and to [PX03] for more details on noncommutative  $L^p$ -spaces. In this paper we will mainly use the construction via interpolation [Kos84] and we refer to [BL76] for all notions and notation from interpolation theory used below.

Let  $\mathcal{M}$  be a von Neumann algebra equipped with a distinguished normal faithful state  $\varphi$ . Denote by  $\mathcal{M}_*$  the predual space of  $\mathcal{M}$ . Define  $L^1(\mathcal{M}, \varphi) = \mathcal{M}_*$  and  $L^\infty(\mathcal{M}, \varphi) = \mathcal{M}$ . We identify  $\mathcal{M}$  as a subspace of  $\mathcal{M}_*$  by the following injection

$$j : \mathcal{M} \rightarrow \mathcal{M}_*, \quad j(x) = x\varphi := \varphi(\cdot x), \quad x \in \mathcal{M}.$$

It is known that  $j$  is a contractive injection with dense image. In this way we may view  $(\mathcal{M}, \mathcal{M}_*)$  as a compatible pair of Banach spaces and for  $1 < p < \infty$ , we introduce the corresponding noncommutative  $L^p$ -space as

$$L^p(\mathcal{M}, \varphi) = (\mathcal{M}, \mathcal{M}_*)_{1/p},$$

where  $(\cdot, \cdot)_{1/p}$  denotes the complex interpolation space. Denote by  $\|\cdot\|_p$  the norm on  $L^p(\mathcal{M}, \varphi)$ . Let  $H_\varphi$  be the Hilbert space in the GNS construction induced by  $\varphi$ . Then

$$L^2(\mathcal{M}, \varphi) = H_\varphi$$

with equal norms.

We will also need the following property (see [Jun02, Lemma 2.2]).

**Lemma 1.1.1.** *Let  $A$  be an ultraweakly dense  $*$ -subalgebra of  $\mathcal{M}$ . Then for  $1 \leq p < \infty$ ,  $A$  is dense in  $L^p(\mathcal{M}, \varphi)$  with respect to  $\|\cdot\|_p$ .*

Another useful and equivalent construction of noncommutative  $L^p$ -spaces is given by Haagerup. We refer to [Ter81, PX03, HJX10] for details. For the convenience of the reader, we will not formulate the precise definition but rather cite here several basic properties which will be sufficient for later use.

Assume that the von Neumann algebra  $\mathcal{M}$  acts on a Hilbert space  $H$  and denote by  $\sigma = \sigma^\varphi$  the modular automorphism group of  $\varphi$ . For  $1 \leq p \leq \infty$ , denote by  $L^{p,\mathbb{H}}(\mathcal{M}, \varphi)$  the corresponding Haagerup  $L^p$ -space with norm  $\|\cdot\|_{p,\mathbb{H}}$ . Recall that each element in  $L^{p,\mathbb{H}}(\mathcal{M}, \varphi)$  is realized as a densely defined operator on  $L^2(\mathbb{R}, H)$ . Also note that  $\mathcal{M}$  can be identified with  $L^{\infty,\mathbb{H}}(\mathcal{M}, \varphi)$ .

Let  $D$  be the density operator associated to  $\varphi$  and  $\text{tr}$  be the trace on  $L^{1,\mathbb{H}}(\mathcal{M}, \varphi)$ . We recall that  $D$  is a distinguished invertible positive selfadjoint operator on  $L^2(\mathbb{R}, H)$  and  $\text{tr}$  is a distinguished positive functional on  $L^{1,\mathbb{H}}(\mathcal{M}, \varphi)$ , which enjoy the following properties.

**Proposition 1.1.2.** (1) For all  $x \in \mathcal{M}, t \in \mathbb{R}, \sigma_t(x) = D^{it}x D^{-it}$ ;

(2) Let  $1 \leq p, q \leq \infty$  be such that  $1/p + 1/q = 1$ . Then for  $x \in L^{p,\mathbb{H}}(\mathcal{M}, \varphi), y \in L^{q,\mathbb{H}}(\mathcal{M}, \varphi)$ , we have  $xy, yx \in L^{1,\mathbb{H}}(\mathcal{M}, \varphi)$  and  $\text{tr}(xy) = \text{tr}(yx)$ ;

(3)  $D \in L^{1,\mathbb{H}}(\mathcal{M}, \varphi)$  and  $\varphi(x) = \text{tr}(xD)$  for  $x \in \mathcal{M}$ ;

(4) For  $x \in L^{1,\mathbb{H}}(\mathcal{M}, \varphi)$ ,  $\text{tr}(|x|) = \|x\|_{1,\mathbb{H}}$ . For  $x \in L^{p,\mathbb{H}}(\mathcal{M}, \varphi)$  with  $1 \leq p < \infty$ , we have  $|x|^p \in L^{1,\mathbb{H}}(\mathcal{M}, \varphi)$  and  $\|x\|_{p,\mathbb{H}} = \|x^*\|_{p,\mathbb{H}} = \| |x| \|_{p,\mathbb{H}} = \| |x|^p \|_{1,\mathbb{H}}^{1/p}$ ;

(5) ([Jun02, (1.3)]) For  $1 \leq p < \infty$  and for  $0 \leq x \leq y \in L^{p,\mathbb{H}}(\mathcal{M})$ , we have  $\|x\|_{p,\mathbb{H}} \leq \|y\|_{p,\mathbb{H}}$ .

Now we may formulate the usual Hölder inequality for these noncommutative  $L^p$ -spaces.

**Proposition 1.1.3.** Let  $1 \leq p, q, r \leq \infty$  be such that  $1/p + 1/q = 1/r$ . If  $x \in L_{p,\mathbb{H}}(\mathcal{M}, \varphi)$  and  $y \in L_{q,\mathbb{H}}(\mathcal{M}, \varphi)$ , then  $xy \in L_{r,\mathbb{H}}(\mathcal{M}, \varphi)$  and the following Hölder inequality holds:

$$\|xy\|_{r,\mathbb{H}} \leq \|x\|_{p,\mathbb{H}} \|y\|_{q,\mathbb{H}}.$$

In particular, if  $r = 1$ ,  $|\text{tr}(xy)| \leq \|x\|_{p,\mathbb{H}} \|y\|_{q,\mathbb{H}}$  for arbitrary  $x \in L_{p,\mathbb{H}}(\mathcal{M}, \varphi)$  and  $y \in L_{q,\mathbb{H}}(\mathcal{M}, \varphi)$ . This defines a natural duality bracket  $\langle x, y \rangle = \text{tr}(xy)$ , and for any  $1 \leq p < \infty$  we have

$$L_{p,\mathbb{H}}(\mathcal{M}, \varphi)^* = L_{q,\mathbb{H}}(\mathcal{M}, \varphi)$$

isometrically.

In this paper we will frequently identify the two  $L^p$ -spaces via the following isomorphism.

**Proposition 1.1.4** ([Kos84, Sect.9]). For all  $1 \leq p \leq \infty$ , the map

$$j^p : x \mapsto xD^{1/p}, x \in \mathcal{M}$$

extends to an isometry from  $L^p(\mathcal{M}, \varphi)$  onto  $L^{p,\mathbb{H}}(\mathcal{M}, \varphi)$ .

We remark that if  $\mathcal{M}$  is a finite von Neumann algebra equipped with a normal faithful tracial state  $\tau$ , then the space  $L_p(\mathcal{M}, \tau)$  for  $1 \leq p < \infty$  can be simply realized as the completion of  $(\mathcal{M}, \|\cdot\|_p)$ , where the norm  $\|\cdot\|_p$  is explicitly given by

$$\|x\|_p = [\tau(|x|^p)]^{1/p}, \quad x \in \mathcal{M}.$$

In this case, the density operator  $D$  associated to  $\tau$  is trivial and the functional  $\text{tr}$  coincides with  $\tau$ . The elements of  $L_p(\mathcal{M}, \tau)$  can be simply described by densely defined closed operators measurable with respect to  $(\mathcal{M}, \tau)$ , as in the commutative case.



We will need the noncommutative Khintchine inequality for Rademacher sequences. For a finite sequence  $(x_n)_{n \geq 1} \subset L^{p,\mathbb{H}}(\mathcal{M}, \varphi)$ , we introduce the notation

$$\|(x_n)\|_{L^p(\mathcal{M}; \ell_2^2)} = \left\| \left( \sum |x_n|^2 \right)^{1/2} \right\|_{p,\mathbb{H}}, \quad \|(x_n)\|_{L^p(\mathcal{M}; \ell_2^2)} = \left\| \left( \sum |x_n^*|^2 \right)^{1/2} \right\|_{p,\mathbb{H}},$$

and write

$$\|(x_n)\|_{C R_p[L^p(\mathcal{M})]} = \max\{\|(x_n)\|_{L^p(\mathcal{M}; \ell_2^2)}, \|(x_n)\|_{L^p(\mathcal{M}; \ell_2^2)}\}.$$

Denote by  $(\varepsilon_n)_{n \geq 1}$  a Rademacher sequence on a probability space  $(\Omega, P)$ , i.e., an independent sequence of random variables with  $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$  for all  $n$ . The following noncommutative Khintchine inequality for Haagerup's  $L^p$ -spaces is given in [JX03, Theorem 3.4].

**Theorem 1.1.5.** *There exists an absolute constant  $C > 0$  such that for all  $2 \leq p < \infty$  and all finite sequences  $(x_n)$  in  $L^{p,\mathbb{H}}(\mathcal{M}, \varphi)$ , we have*

$$\|(x_n)\|_{C R_p[L^p(\mathcal{M})]} \leq \left( \int_{\Omega} \left\| \sum_n \varepsilon_n(\omega) x_n \right\|_{L^{p,\mathbb{H}}(\mathcal{M}, \varphi)}^p dP(\omega) \right)^{1/p} \leq C \sqrt{p} \|(x_n)\|_{C R_p[L^p(\mathcal{M})]}.$$

Consequently  $L^{p,\mathbb{H}}(\mathcal{M}, \varphi)$  is of type 2, i.e., for the above sequences we have

$$\left( \int_{\Omega} \left\| \sum_n \varepsilon_n(\omega) x_n \right\|_{L^{p,\mathbb{H}}(\mathcal{M}, \varphi)}^p dP(\omega) \right)^{1/p} \leq C \sqrt{p} \left( \sum_n \|x_n\|_{L^{p,\mathbb{H}}(\mathcal{M}, \varphi)}^2 \right)^{1/2}.$$

## 1.2 Compact quantum groups

### 1.2.1 Basic notions

Let us first recall some well-known definitions and properties concerning compact quantum groups. We refer to [Wor98] and [MVD98] for more details.

**Definition 1.2.1.** Let  $A$  be a unital  $C^*$ -algebra. If there exists a unital  $*$ -homomorphism  $\Delta : A \rightarrow A \otimes A$  such that  $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$  and

$$\{\Delta(a)(1 \otimes b) : a, b \in A\} \quad \text{and} \quad \{\Delta(a)(b \otimes 1) : a, b \in A\}$$

are linearly dense in  $A \otimes A$ , then  $(A, \Delta)$  is called a *compact quantum group* and  $\Delta$  is called the *comultiplication* on  $A$ . We denote  $\mathbb{G} = (A, \Delta)$  and  $A = C(\mathbb{G})$ .

Any compact quantum group  $\mathbb{G}$  admits a unique *Haar state*  $h$  on  $C(\mathbb{G})$  such that for all  $x \in C(\mathbb{G})$ ,

$$(h \otimes \iota) \circ \Delta(x) = h(x)1 = (\iota \otimes h) \circ \Delta(x).$$

Consider an element  $u \in C(\mathbb{G}) \otimes B(H)$ , where  $H$  is a Hilbert space with  $\dim H = n$ . We identify  $C(\mathbb{G}) \otimes B(H) = \mathbb{M}_n(C(\mathbb{G}))$  and write  $u = [u_{ij}]_{i,j=1}^n$ . The matrix  $u$  is called an  *$n$ -dimensional representation* of  $\mathbb{G}$  if for all  $j, k = 1, \dots, n$  we have

$$\Delta(u_{jk}) = \sum_{p=1}^n u_{jp} \otimes u_{pk}.$$

Denote by  $\text{Irr}(\mathbb{G})$  the set of unitary equivalence classes of irreducible finite-dimensional unitary representations of  $\mathbb{G}$ . Also denote by  $\text{Rep}(\mathbb{G})$  the set of unitary equivalence classes of (not necessarily irreducible) finite-dimensional unitary representations of  $\mathbb{G}$ . For each



$\pi \in \text{Irr}(\mathbb{G})$ , we fix a representative  $u^{(\pi)} \in C(\mathbb{G}) \otimes B(H_\pi)$  of the class  $\pi$  where  $H_\pi$  is the finite dimensional Hilbert space on which  $u^{(\pi)}$  acts. In the sequel we write  $n_\pi = \dim \pi = \dim H_\pi$  for  $\pi \in \text{Irr}(\mathbb{G})$ .

For  $\pi, \pi' \in \text{Irr}(\mathbb{G})$ , define the tensor product representation  $\pi \otimes \pi'$  on  $H_\pi \otimes H_{\pi'}$  by

$$u^{(\pi \otimes \pi')} = \sum_{i,j,k,l} u_{ij}^{(\pi)} u_{kl}^{(\pi')} \otimes e_{ij}^{(\pi)} \otimes e_{kl}^{(\pi')},$$

where  $e_{ij}^{(\pi)}, e_{kl}^{(\pi')}$  denotes the matrix units of  $B(H_\pi)$  and  $B(H_{\pi'})$  respectively. For each  $\pi \in \text{Irr}(\mathbb{G})$ , there exists a unique  $\bar{\pi} \in \text{Irr}(\mathbb{G})$  such that the trivial representation  $1 \in \text{Irr}(\mathbb{G})$  is a subrepresentation of  $\pi \otimes \bar{\pi}$ . We call  $\bar{\pi}$  the adjoint of  $\pi$ . The notions of tensor product and adjoint then can be extended to all elements in  $\text{Rep}(\mathbb{G})$  by decomposing representations into irreducible ones.

For  $\pi \in \text{Rep}(\mathbb{G})$ , the character of  $\pi$  is the element  $\chi_\pi = \sum_{i=1}^{n_\pi} u_{ii}^{(\pi)} \in C(\mathbb{G})$ . One can show that the definition does not depend on the representative matrix  $u^{(\pi)}$ . For  $\pi, \pi' \in \text{Rep}(\mathbb{G})$ , we have

$$\chi_{\pi \oplus \pi'} = \chi_\pi + \chi_{\pi'}, \quad \chi_{\pi \otimes \pi'} = \chi_\pi \chi_{\pi'}, \quad \chi_{\bar{\pi}} = \chi_\pi^*. \quad (1.1)$$

Denote  $\text{Pol}(\mathbb{G}) = \text{span}\{u_{ij}^{(\pi)} : u^{(\pi)} = [u_{ij}^{(\pi)}]_{i,j=1}^{n_\pi}, \pi \in \text{Irr}(\mathbb{G})\}$ . This is a dense subalgebra of  $C(\mathbb{G})$ . A functional  $\epsilon$ , called the *counit*, can be defined on  $\text{Pol}(\mathbb{G})$ , that is, the linear functional defined by

$$\epsilon(u_{ij}^{(\pi)}) = \delta_{ij}, \quad u^{(\pi)} = [u_{ij}^{(\pi)}]_{i,j=1}^{n_\pi}, \quad \pi \in \text{Irr}(\mathbb{G}).$$

Consider the GNS representation  $(\pi_h, H_h)$  of the Haar state  $h$ , then  $\text{Pol}(\mathbb{G})$  can be viewed as a subalgebra of  $B(H_h)$ . Define  $C_r(\mathbb{G})$  (resp.,  $L^\infty(\mathbb{G})$ ) to be the C\*-algebra (resp., the von Neumann algebra) generated by  $\text{Pol}(\mathbb{G})$  in  $B(H_h)$ . Then  $h$  extends to a normal faithful state on  $L^\infty(\mathbb{G})$ . On the other hand, we may equip the following C\*-norm on  $\text{Pol}(\mathbb{G})$ ,

$$\|x\|_u = \sup\{p(x) : p \text{ is a C}^*\text{-seminorm on } \text{Pol}(\mathbb{G})\}, \quad x \in \text{Pol}(\mathbb{G}).$$

Then the corresponding completion of  $\text{Pol}(\mathbb{G})$  is a unital C\*-algebra, denoted by  $C_u(\mathbb{G})$ .

In this thesis, we will mainly be interested in the  $L^p$ -spaces associated to a compact quantum group  $\mathbb{G}$ . Let  $h$  be the Haar state on  $\mathbb{G}$ . Throughout the paper, for any  $1 \leq p \leq \infty$  we will use the notation  $L^p(\mathbb{G}) := L^p(L^\infty(\mathbb{G}), h)$  for Kosaki's noncommutative  $L^p$ -spaces and  $L^{p,\text{H}}(\mathbb{G}) := L^{p,\text{H}}(L^\infty(\mathbb{G}), h)$  for Haagerup's noncommutative  $L^p$ -spaces introduced in the previous section. It is easy to see that the algebra of polynomials  $\text{Pol}(\mathbb{G})$  is a common dense subspace of all  $L^p$ -spaces associated to  $\mathbb{G}$  with  $1 \leq p < \infty$  by Lemma 1.1.1.

## 1.2.2 Modular properties and Duality

It is known that there exists a linear antihomomorphism  $S$  on  $\text{Pol}(\mathbb{G})$ , called the *antipode* of  $\mathbb{G}$ , determined by

$$S(u_{ij}^{(\pi)}) = (u_{ji}^{(\pi)})^*, \quad u^{(\pi)} = [u_{ij}^{(\pi)}]_{i,j=1}^{n_\pi}, \quad \pi \in \text{Irr}(\mathbb{G}).$$

It is well-known that for each  $\pi \in \text{Irr}(\mathbb{G})$  there exists a unique positive invertible operator  $Q_\pi \in B(H_\pi)$  with  $\text{Tr}(Q_\pi) = \text{Tr}(Q_\pi^{-1}) := d_\pi$  that intertwines  $u^{(\pi)}$  and  $(S^2 \otimes \iota)(u^{(\pi)})$ . Then the Haar state can be calculated as follows,

$$h(u_{ij}^{(\pi)} (u_{lm}^{(\pi')})^*) = \delta_{\pi\pi'} \delta_{il} \frac{(Q_\pi)_{mj}}{d_\pi}, \quad h((u_{ij}^{(\pi)})^* u_{lm}^{(\pi')}) = \delta_{\pi\pi'} \delta_{jm} \frac{(Q_{\pi'}^{-1})_{li}}{d_{\pi'}}, \quad (1.2)$$

where  $\pi' \in \text{Irr}(\mathbb{G})$ ,  $1 \leq i, j \leq n_\pi$ ,  $1 \leq l, m \leq n_{\pi'}$ . The number  $d_\pi$  is called the *quantum dimension* of  $\pi$ . The Woronowicz characters on  $\text{Pol}(\mathbb{G})$  are defined as

$$f_z(u_{ij}^{(\pi)}) = (Q_\pi^z)_{ij}, \quad z \in \mathbb{C}, \pi \in \text{Irr}(\mathbb{G}), 1 \leq i, j \leq n_\pi.$$

We denote

$$\Delta^{(2)} := (\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta.$$

The modular automorphism group of the Haar state  $h$  on  $L^\infty(\mathbb{G})$  is determined by the following formula:

$$\sigma_z(x) = (f_{iz} \otimes \iota \otimes f_{iz})\Delta^{(2)}(x), \quad x \in \text{Pol}(\mathbb{G}), z \in \mathbb{C},$$

in other words,

$$(\sigma_z \otimes \iota)(u^{(\pi)}) = (1 \otimes Q_\pi^{iz})u^{(\pi)}(1 \otimes Q_\pi^{iz}), \quad \pi \in \text{Irr}(\mathbb{G}). \quad (1.3)$$

The antipode  $S$  has the following polar decomposition

$$S = R \circ \tau_{-\frac{i}{2}} = \tau_{-\frac{i}{2}} \circ R, \quad (1.4)$$

where  $R$  is a  $*$ -antiautomorphism of  $C_r(\mathbb{G})$  and  $(\tau_z)_{z \in \mathbb{C}}$  is the analytic extension of the one-parameter group  $(\tau_t)_{t \in \mathbb{R}}$  of  $*$ -automorphisms defined as

$$\tau_z(x) = (f_{iz} \otimes \iota \otimes f_{-iz})\Delta^{(2)}(x) \quad (1.5)$$

for  $x \in \text{Pol}(\mathbb{G})$ , or in other words,

$$(\tau_z \otimes \iota)(u^{(\pi)}) = (1 \otimes Q_\pi^{iz})u^{(\pi)}(1 \otimes Q_\pi^{-iz}), \quad \pi \in \text{Irr}(\mathbb{G}), \quad (1.6)$$

and moreover

$$S^2 = \tau_{-i}, \quad \Delta \circ R = \Sigma \circ (R \otimes R) \circ \Delta. \quad (1.7)$$

Also we have

$$\begin{aligned} S((\iota \otimes h)(\Delta(b)(1 \otimes a))) &= (\iota \otimes h)((1 \otimes b)\Delta(a)), \\ S((h \otimes \iota)((b \otimes 1)\Delta(a))) &= (h \otimes \iota)(\Delta(b)(a \otimes 1)). \end{aligned} \quad (1.8)$$

We say that  $\mathbb{G}$  is of *Kac type* if its Haar state is tracial, and say that  $\mathbb{G}$  is *finite* if  $C(\mathbb{G})$  is finite-dimensional. The following facts are well-known.

**Proposition 1.2.2** ([Wor98, Theorem 1.5]). *Let  $\mathbb{G}$  be a compact quantum group. The Haar state  $h$  on  $C(\mathbb{G})$  is tracial if and only if  $Q_\alpha = \text{Id}_{H_\alpha}$  for all  $\alpha \in \text{Irr}(\mathbb{G})$  in the formula (1.2) and if and only if the antipode  $S$  satisfies  $S^2(x) = x$  for all  $x \in \text{Pol}(\mathbb{G})$ . In particular, if the above conditions are satisfied and  $h$  is faithful on  $C(\mathbb{G})$ , then  $S$  extends to a  $*$ -antihomomorphism on  $C(\mathbb{G})$  which is positive and bounded of norm one.*

**Proposition 1.2.3** ([VD97]). *If  $\mathbb{G}$  is a finite quantum group, then the Haar state is tracial on  $C(\mathbb{G})$ .*

The dual quantum group  $\hat{\mathbb{G}}$  of  $\mathbb{G}$  is defined via its ‘‘algebras of functions’’,

$$c_0(\hat{\mathbb{G}}) = \bigoplus_{\pi \in \text{Irr}(\mathbb{G})}^{c_0} B(H_\pi), \quad \ell^\infty(\hat{\mathbb{G}}) = \bigoplus_{\pi \in \text{Irr}(\mathbb{G})} B(H_\pi),$$

where  $\bigoplus_{\pi} B(H_\pi)$  refers to the direct sum of  $B(H_\pi)$ , i.e. the bounded families  $(x_\pi)_\pi$  with each  $x_\pi$  in  $B(H_\pi)$ , and  $\bigoplus_{\pi \in \text{Irr}(\mathbb{G})}^{c_0} B(H_\pi)$  corresponds to the subalgebra of bounded families

converging to 0 at infinity. Also set  $c_c(\hat{\mathbb{G}})$  to be the corresponding algebraic direct sum and denote by  $\prod_\pi B(H_\pi)$  the usual Cartesian product. We will not recall the full quantum group structure on  $\hat{\mathbb{G}}$  as we do not need it in the following. We only remark that the (left) Haar weight  $\hat{h}$  on  $\hat{\mathbb{G}}$  can be explicitly given by (see e.g. [VD96, Section 5])

$$\hat{h} : \ell^\infty(\hat{\mathbb{G}}) \ni x \mapsto \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \text{Tr}(Q_\pi p_\pi x),$$

where  $p_\pi$  is the projection onto  $H_\pi$  and  $\text{Tr}$  denotes the usual trace on  $B(H_\pi)$  for each  $\pi$ . In fact,  $Q = (Q_\pi)_\pi$  plays a role of the *modular element* of the discrete quantum group  $\hat{\mathbb{G}}$ .

We will not concentrate on the noncommutative  $L^p$ -spaces associated to discrete quantum groups for a general  $p$  in this paper, but we may add several concrete words without using the scheme in Section 1.1 (note that in Section 1.1, we assume the von Neumann algebra to be equipped with a normal faithful *state* for simplicity). We define the  $L^1$ -space  $\ell^1(\hat{\mathbb{G}})$  on  $\hat{\mathbb{G}}$  associated to  $\hat{h}$  as

$$\ell^1(\hat{\mathbb{G}}) = \{x \in c_0(\hat{\mathbb{G}}) : \|x\|_1 := \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \text{Tr}(|p_\pi x Q_\pi|) < \infty\}.$$

By the property of  $\text{Tr}$  on  $B(\oplus_\pi H_\pi)$ , it is easy to see that  $\ell^1(\hat{\mathbb{G}})$  is a Banach space and the injection

$$j' : \ell^1(\hat{\mathbb{G}}) \rightarrow \ell^\infty(\hat{\mathbb{G}})_*, \quad j'(x) = x\hat{h} := \hat{h}(\cdot x), \quad x \in \ell^1(\hat{\mathbb{G}})$$

is an isometric isomorphism. With respect to this compatible couple, the complex interpolation space

$$\ell^p(\hat{\mathbb{G}}) = (\ell^\infty(\hat{\mathbb{G}}), \ell^1(\hat{\mathbb{G}}))_{1/p}, \quad 1 \leq p < \infty,$$

can be indeed realized as

$$\ell^p(\hat{\mathbb{G}}) = \{x \in c_0(\hat{\mathbb{G}}) : \|x\|_p := \left( \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \text{Tr}(|p_\pi x Q_\pi^{1/p}|^p) \right)^{1/p} < \infty\}.$$

One can easily see that  $c_c(\hat{\mathbb{G}})$  is a dense subset in  $\ell^p(\hat{\mathbb{G}})$  for  $1 \leq p < \infty$ . Also, the space  $\ell^2(\hat{\mathbb{G}})$  coincides with the Hilbert space in the GNS construction induced by the Haar weight  $\hat{h}$  on  $\ell^\infty(\hat{\mathbb{G}})$ .

### 1.2.3 Free products and tensor products

We firstly recall some constructions of free product of  $C^*$ -algebras, for which we refer to [VDN92] and [NS06] for details. Consider a family of unital  $C^*$ -algebras  $(A_i, \phi_i)_{i \in I}$  with distinguished faithful states  $\phi_i$  and associated GNS constructions  $(\pi_i, H_i)$ . Set  $\dot{A}_i = \ker \phi_i$  and  $\dot{a}_i = a_i - \phi_i(a_i)1$  for each  $i$  and  $a_i \in A_i$ . Construct a vector space

$$A = \mathbb{C}\mathbf{1} \oplus \bigoplus_{n \geq 1} \left( \bigoplus_{i_1 \neq i_2 \neq \dots \neq i_n} \dot{A}_{i_1} \otimes \dot{A}_{i_2} \otimes \dots \otimes \dot{A}_{i_n} \right). \quad (1.9)$$

We equip  $A$  with an algebra structure such that  $\mathbf{1}$  is the identity and the multiplication of a letter  $a \in \dot{A}_i$  with an elementary tensor  $a_1 \otimes a_2 \otimes \dots \otimes a_n$  in  $\dot{A}_{i_1} \otimes \dot{A}_{i_2} \otimes \dots \otimes \dot{A}_{i_n}$  is defined as

$$a \cdot (a_1 \otimes a_2 \otimes \dots \otimes a_n) = \begin{cases} a \otimes a_1 \otimes a_2 \otimes \dots \otimes a_n, & \text{if } i_1 \neq i, \\ (aa_1) \otimes a_1 \otimes a_2 \otimes \dots \otimes a_n \\ \quad + \phi_{i_1}(aa_1)a_2 \otimes \dots \otimes a_n, & \text{if } i_1 = i. \end{cases}$$

Moreover, we give an involution on  $A$  by

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n)^* = a_n^* \otimes a_2^* \otimes \cdots \otimes a_1^*.$$

In this sense  $A$  becomes a  $*$ -algebra, and each  $A_i$  can be viewed as a  $*$ -subalgebra in  $A$  by identifying  $A_i$  with  $\mathbb{C}\mathbf{1} \oplus \mathring{A}_i$  in the big direct sum. We call  $A$  the *algebraic free product* of  $(A_i)_{i \in I}$ .

It then can be shown that the algebra  $A$  admits a faithful  $*$ -representation  $(\pi, H, \xi)$  such that  $\pi|_{A_i} = \pi_i$  for each  $i \in I$  and  $\phi(\cdot) := \langle \pi(\cdot)\xi, \xi \rangle$  restricted on  $A_i$  coincides with  $\phi_i$ . Moreover the state  $\phi$  is faithful on  $A$ . Then the *reduced  $C^*$ -algebraic free product* of  $(A_i)_{i \in I}$  is the  $C^*$ -algebra generated by  $\pi(A)$  in  $B(H)$ , i.e., the norm closure of  $\pi(A)$  in  $B(H)$ , denoted by  $*_{i \in I}^{c_0} A_i$ ; and the state extends to  $*_{i \in I}^{c_0} A_i$ , called the *free product state* of  $(\phi_i)_{i \in I}$  and denoted by  $*_{i \in I} \phi_i$ . If moreover each  $A_i = \mathcal{M}_i$  is a von Neumann algebra and each  $\phi_i$  is normal, then the weak closure of  $\pi(A)$  in  $B(H)$ , is defined to be the *von Neumann algebraic free product* of  $(\mathcal{M}_i)_{i \in I}$ , denoted by  $\bar{*}_{i \in I} \mathcal{M}_i$ , and the free product state  $\phi = *_{i \in I} \phi_i$  is also normal. Also, we remark that if each  $\phi_i$  is a tracial state, then  $\phi = *_{i \in I} \phi_i$  is also tracial.

Let  $A_i$  and  $B_i$  be unital  $C^*$ -algebras with distinguished faithful states  $\phi_i$  and  $\psi_i$  ( $i \in I$ ) respectively, and let  $T_i : A_i \rightarrow B_i$  be a unital state preserving map for each  $i \in I$ . Set  $(A, \phi) = *_{i \in I} (A_i, \phi_i)$  and  $(B, \psi) = *_{i \in I} (B_i, \psi_i)$ . Then it is obvious that

$$T(a_1 a_2 \cdots a_n) = T_{i_1}(a_1) \cdots T_{i_n}(a_n) \quad (a_k \in \mathring{A}_{i_k}, \forall k, i_1 \neq i_2 \neq \cdots \neq i_n)$$

defines a unital state preserving map from the *algebraic free products*  $(A, \phi)$  to  $(B, \psi)$ . We denote by  $T = *_{i \in I} T_i$ , and call it the *free product map* of the  $T_i$ 's.

Similarly, we may define the  *$c$ -free (conditionally free) product state* in the sense of Bożejko, Leinert and Speicher [BLS96]. Let  $(A_i, \phi_i)$  be as above and let  $\rho_i$  be further states respectively on  $A_i$  for each  $i$ . The conditional free product of  $(\rho_i)_i$  is the functional  $\omega := *_{(\psi_i)} \rho_i$  on  $(A, \phi) = *_{i \in I} (A_i, \phi_i)$  defined by the prescription  $\omega(1) = 1$  and

$$\omega(a_1 \cdots a_n) = \rho_{i(1)}(a_1) \cdots \rho_{i(n)}(a_n)$$

for all  $n \geq 1$ ,  $i(1) \neq \cdots \neq i(n)$  elements in  $I$  and  $a_j \in \ker \phi_{i(j)}$  for  $j = 1, \dots, n$ . It is shown in [BLS96, Theorem 2.2] that the conditional free product of states is again a state.

Now we may introduce the dual free product of compact quantum groups. The following construction is given by [Wan95a].

**Proposition 1.2.4.** *Let  $(\mathbb{G}_i : i \in I)$  be a family of compact quantum groups with comultiplications  $\Delta_i$  and Haar states  $h_i$  respectively. There exists a unique comultiplication  $\Delta$  on  $*_{i \in I}^{c_0} C_r(\mathbb{G}_i)$  such that the pair  $(*_{i \in I}^{c_0} C_r(\mathbb{G}_i), \Delta)$  forms a compact quantum group, denoted by  $\hat{*}_{i \in I} \mathbb{G}_i$  and we have*

$$\Delta|_{C_r(\mathbb{G}_i)} = (\iota_i \otimes \iota_i) \circ \Delta_i, \quad i \in I$$

where  $\iota_i$  are the natural embedding of  $C_r(\mathbb{G}_i)$  into  $*_{i \in I}^{c_0} C_r(\mathbb{G}_i)$ . Moreover the Haar state on  $\mathbb{G}$  is the free product state  $*_{i \in I}^{c_0} h_i$ . And

$$\text{Irr}(\hat{*}_{i \in I} \mathbb{G}_i) = 1 \cup \{u^{(\pi_1)} \otimes \cdots \otimes u^{(\pi_k)} : k \in \mathbb{N}, i(j) \in I, i(j) \neq i(j+1), \pi_j \in \text{Irr}(\mathbb{G}_{i(j)}), u^{(\pi_j)} \neq 1\},$$

where  $u^{(\pi_1)} \otimes \cdots \otimes u^{(\pi_k)} \in M_{n_{\pi_1}} \otimes \cdots \otimes M_{n_{\pi_k}} \otimes (\text{Pol}(\hat{*}_{i \in I} \mathbb{G}_i))$ ,

$$(u^{(\pi_1)} \otimes \cdots \otimes u^{(\pi_k)})_{(l_1, \dots, l_k), (m_1, \dots, m_k)} = u_{l_1, m_1}^{(\pi_1)} \cdots u_{l_k, m_k}^{(\pi_k)}.$$

Similar construction also applies to the tensor product according to [Wan95b]. We summarize it in the following proposition.

**Proposition 1.2.5.** *Let  $(\mathbb{G}_i : i \in I)$  be a family of compact quantum groups with comultiplications  $\Delta_i$  and Haar states  $h_i$  respectively. There exists a unique comultiplication  $\Delta$  on the minimal  $C^*$ -tensor product  $\otimes_{i \in I} C_r(\mathbb{G}_i)$  such that the pair  $(\otimes_{i \in I} C_r(\mathbb{G}_i), \Delta)$  forms a compact quantum group, denoted by  $\prod_{i \in I} \mathbb{G}_i$  and we have*

$$\Delta|_{C_r(\mathbb{G}_i)} = (\iota_i \otimes \iota_i) \circ \Delta_i, \quad i \in I$$

where  $\iota_i$  are the natural embedding of  $C_r(\mathbb{G}_i)$  into  $\otimes_{i \in I} C_r(\mathbb{G}_i)$ . Moreover the Haar state on  $\mathbb{G}$  is the tensor product state  $\otimes_{i \in I} h_i$ . And

$$\text{Irr}\left(\prod_{i \in I} \mathbb{G}_i\right) = 1 \cup \{u^{(\pi_1)} \otimes \cdots \otimes u^{(\pi_k)} : k \in \mathbb{N}, i(j) \in I, i(j) < i(j+1), \pi_j \in \text{Irr}(\mathbb{G}_{i(j)}), u^{(\pi_j)} \neq 1\},$$

where  $u^{(\pi_1)} \otimes \cdots \otimes u^{(\pi_k)} \in M_{n_{\pi_1}} \otimes \cdots \otimes M_{n_{\pi_k}} \otimes (\text{Pol}(\prod_{i \in I} \mathbb{G}_i))$ ,

$$(u^{(\pi_1)} \otimes \cdots \otimes u^{(\pi_k)})_{(l_1, \dots, l_k), (m_1, \dots, m_k)} = u_{l_1, m_1}^{(\pi_1)} \otimes \cdots \otimes u_{l_k, m_k}^{(\pi_k)}.$$

### 1.2.4 Coamenability

Consider a compact quantum group  $\mathbb{G}$ . We say that  $\mathbb{G}$  is *coamenable* if the counit  $\epsilon : \text{Pol}(\mathbb{G}) \rightarrow \mathbb{C}$  extends to a state on  $C_r(\mathbb{G})$ . On the other hand, for linear functionals  $\varphi, \varphi'$  on  $\text{Pol}(\mathbb{G})$ , we define the convolution product

$$\varphi \star \varphi' = (\varphi \otimes \varphi') \circ \Delta. \quad (1.10)$$

Then it is easy to see that

$$\|\varphi_1 \star \varphi_2\| \leq \|\varphi_1\| \|\varphi_2\|,$$

where the norm is induced from  $C_r(\mathbb{G})^*$  or  $L^\infty(\mathbb{G})^*$ , when the respective functionals admit bounded extensions. Equipped with this convolution product, the predual space  $L^\infty(\mathbb{G})_*$  forms a Banach algebra. The following characterization of coamenability is given in [BT03, Theorem 3.1].

**Proposition 1.2.6.** *A compact quantum group  $\mathbb{G}$  is coamenable if and only if the Banach algebra  $L^\infty(\mathbb{G})_*$  has a bounded right approximate unit with norm not more than 1, and if and only if the identity map on  $\text{Pol}(\mathbb{G})$  extends to a  $*$ -isomorphism from  $C_u(\mathbb{G})$  to  $C_r(\mathbb{G})$ .*

We remark that any compact group  $G$  or compact quantum group  $\mathbb{G}$  with an amenable discrete group  $\Gamma$  as the dual quantum group, is coamenable.

### 1.2.5 Drinfeld-Jimbo deformation, quantum $\text{SU}(N)$ groups

Let  $G$  be a simply connected semi-simple compact Lie group. It follows from the work of Levendorskii and Soibelman [LS91, Soi90] that given any  $q > 0$  one can define a compact quantum group  $G_q$ , called the *Drinfeld-Jimbo  $q$ -deformation* of  $G$  such that the fusion rules, the classical dimension function and the coamenability do not depend on  $q$ . More precisely, we may state the following property (see for example [NT13, Theorem 2.4.7], [Ban99] and references therein).

**Proposition 1.2.7.** *Let  $0 < q < 1$ .*

(1) *There exists a bijection  $\Phi : \text{Rep}(G) \rightarrow \text{Rep}(G_q)$  such that*

$$\Phi(\pi \otimes \pi') = \Phi(\pi) \otimes \Phi(\pi'), \quad \Phi(\oplus_{i \in I} \pi_i) = \oplus_{i \in I} \Phi(\pi_i), \quad \pi, \pi', \pi_i \in \text{Rep}(G)$$

*and  $\dim \Phi(\pi) = \dim \pi$  for all  $\pi \in \text{Irr}(G)$ .*

(2)  *$G_q$  is coamenable.*

Take  $G$  to be the special unitary group  $\text{SU}(N)$  of degree  $N$ . We denote by  $\text{SU}_q(N)$  the compact quantum group  $G_q$  for  $0 < q < 1$ . Let us recall some facts of the representation theory of  $\text{SU}_q(2)$ . The elements of  $\text{Irr}(\text{SU}_q(2))$  can be indexed by  $n \in \mathbb{N} \cup \{0\}$  and each representation  $u^{(n)}$  is of dimension  $n + 1$ . The associated matrix  $Q_n$  in (1.2) can be represented as a diagonal under some appropriate basis (see for example Theorem 17 in [KS97, Sect.4.3.2]):

$$Q_n = \begin{bmatrix} q^{-n} & & & & \\ & q^{-n+2} & & & \\ & & \ddots & & \\ & & & q^{n-2} & \\ & & & & q^n \end{bmatrix}. \quad (1.11)$$

Write  $\chi_n = \sum_i u_{ii}^{(n)}$  to be the character of  $u^{(n)}$ . We recall the property below. See for example Proposition 6.2.10 in [Tim08] for the proof.

**Proposition 1.2.8.** (1) *For  $m, m' \in \mathbb{N} \cup \{0\}$ ,  $\chi_m \chi_{m'} = \chi_{|m-m'|} + \chi_{|m-m'|+1} + \cdots + \chi_{m+m'}$ ;*  
 (2) *For  $n \in \mathbb{N} \cup \{0\}$ ,  $\chi_n = \chi_n^*$ .*



## Chapter 2

# Introduction to Fourier analysis on compact quantum groups

This chapter studies some very elementary facts of Fourier analysis on compact quantum groups. The Fourier transform for locally compact quantum groups has been discussed in [Coo10], [Cas13] and [Kah10]. In the setting of compact quantum groups, we give an explicit calculation of Fourier series and convolutions for compact quantum groups in Section 2.1 and 2.2, parallel to the case of classical compact groups, which we cannot find clearly in other literature. In Section 2.3, we study some very elementary properties of bounded  $L^p$ -Fourier multipliers, which have not been discussed so far in literature as well. In particular, we will discuss how to construct a left bounded  $L^p$ -multiplier from a right bounded  $L^p$ -multiplier, and also the inequality comparing the norms  $\|a\|_{\ell^\infty(\hat{\mathbb{G}})}$  and  $\|a\|_{M(L^p(\mathbb{G}))}$ . These results are of essential use in other chapters.

### 2.1 Fourier series and Fourier algebras

Let a compact quantum group  $\mathbb{G}$  be fixed. For a linear functional  $\varphi$  on  $\text{Pol}(\mathbb{G})$ , we define the *Fourier transform*  $\hat{\varphi} = (\hat{\varphi}(\pi))_{\pi \in \text{Irr}(\mathbb{G})} \in \oplus_{\pi} B(H_{\pi})$  by

$$\hat{\varphi}(\pi) = (\varphi \otimes \iota)((u^{(\pi)})^*) \in B(H_{\pi}), \quad \pi \in \text{Irr}(\mathbb{G}). \quad (2.1)$$

In particular, any  $x \in L^\infty(\mathbb{G})$  (or  $L^2(\mathbb{G})$ ) induces a functional  $xh := h(\cdot x)$  on  $\text{Pol}(\mathbb{G})$  defined by  $y \mapsto h(yx)$ , and the Fourier transform  $\hat{x} = (\hat{x}(\pi))_{\pi \in \text{Irr}(\mathbb{G})}$  of  $x$  is given by

$$\hat{x}(\pi) = (h(\cdot x) \otimes \iota)((u^{(\pi)})^*) \in B(H_{\pi}), \quad \pi \in \text{Irr}(\mathbb{G}).$$

The above definition is slightly different from that of [Cas13] or [Kah10]. Indeed, we replace the unitary  $u^{(\pi)}$  by  $(u^{(\pi)})^*$  in the above formulas. This is just to be compatible with standard definitions in classical analysis on compact groups such as in [Fol95, Section 5.3], and this will not cause any essential difference. On the other hand, the notation  $\hat{\varphi}$  has a slight conflict with the dual Haar weight  $\hat{h}$  on  $\hat{\mathbb{G}}$ . One can however distinguish them by the elements on which it acts, so we hope that this will not cause any ambiguity for the reader.

In the sequel let  $\mathcal{F} : f \mapsto \hat{f}$  denote the Fourier transform.

**Proposition 2.1.1.**  *$\mathcal{F}$  is a contraction from  $L^\infty(\mathbb{G})^*$  to  $\ell^\infty(\hat{\mathbb{G}})$ , and moreover  $\mathcal{F}$  sends  $L^1(\mathbb{G})$  injectively into  $c_0(\hat{\mathbb{G}})$ .*



*Proof.* For  $\varphi \in L^\infty(\mathbb{G})^*$ , recall that  $\|\varphi \otimes \iota\| = \|\varphi\|$ , so we have

$$\|\hat{\varphi}\| = \sup_{\pi \in \text{Irr}(\mathbb{G})} \|\hat{\varphi}(\pi)\| = \sup_{\pi \in \text{Irr}(\mathbb{G})} \|(\varphi \otimes \iota)((u^{(\pi)})^*)\| \leq \sup_{\pi \in \text{Irr}(\mathbb{G})} \|\varphi \otimes \iota\| \|u^{(\pi)}\| = \|\varphi\|.$$

So  $\mathcal{F}$  is a contraction. Recall that  $\text{Pol}(\mathbb{G})$  is dense in  $L^1(\mathbb{G}) = L^\infty(\mathbb{G})_*$  and note that  $\mathcal{F}(\text{Pol}(\mathbb{G})) \subset c_c(\hat{\mathbb{G}})$ , so  $\mathcal{F}(L^1(\mathbb{G})) \subset \overline{\mathcal{F}(\text{Pol}(\mathbb{G}))} \subset \overline{c_c(\hat{\mathbb{G}})} = c_0(\hat{\mathbb{G}})$ . The injectivity of  $\mathcal{F}$  follows from the ultraweak density of  $\text{Pol}(\mathbb{G})$  in  $L^\infty(\mathbb{G})$ .  $\square$

It is easy to establish the Fourier inversion formula and the Plancherel theorem for  $L^2(\mathbb{G})$ .

**Proposition 2.1.2.** (a) For all  $x \in L^2(\mathbb{G})$ , we have

$$x = \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi (\iota \otimes \text{Tr})[(1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}], \quad (2.2)$$

where the convergence of the series is in the  $L^2$ -sense. For any  $\pi \in \text{Irr}(\mathbb{G})$ , if we denote by  $\mathcal{E}_\pi$  the orthogonal projection of  $L^2(\mathbb{G})$  onto the subspace spanned by the matrix coefficients  $(u_{ij}^{(\pi)})_{i,j=1}^{n_\pi}$ , and write  $\mathcal{E}_\pi x = \sum_{i,j} x_{ij}^{(\pi)} u_{ij}^{(\pi)}$  with  $x_{ij}^{(\pi)} \in \mathbb{C}$ ,  $X_\pi = [x_{ji}^{(\pi)}]_{i,j}$ , then

$$\hat{x}(\pi) = d_\pi^{-1} X_\pi Q_\pi^{-1}.$$

(b)  $\mathcal{F}$  is a unitary operator from  $L^2(\mathbb{G})$  onto  $\ell^2(\hat{\mathbb{G}})$ .

*Proof.* (a) Denote by  $E_\pi$  the subspace spanned by the matrix units  $(u_{ij}^{(\pi)})_{i,j=1}^{n_\pi}$  for  $\pi \in \text{Irr}(\mathbb{G})$ . Then  $\text{Pol}(\mathbb{G})$  is spanned by all the  $E_\pi$ ,  $\pi \in \text{Irr}(\mathbb{G})$ . It is easy to see from Hölder's inequality that  $\text{Pol}(\mathbb{G})$  is  $\|\cdot\|_2$ -dense in  $L_\infty(\mathbb{G})$ , and also recall that  $L_2(\mathbb{G})$  is the  $\|\cdot\|_2$ -completion of  $L_\infty(\mathbb{G})$ , so  $\text{Pol}(\mathbb{G})$  is  $\|\cdot\|_2$ -dense in  $L_2(\mathbb{G})$  and  $L_2(\mathbb{G})$  is a Hilbert direct sum of the orthogonal subspaces  $(E_\pi)_{\pi \in \text{Irr}(\mathbb{G})}$ . So each  $x \in L_2(\mathbb{G})$  can be written as

$$x = \sum_{\pi \in \text{Irr}(\mathbb{G})} \mathcal{E}_\pi x = \sum_{\pi \in \text{Irr}(\mathbb{G})} \sum_{i,j} x_{ij}^{(\pi)} u_{ij}^{(\pi)}, \quad (x_{ij}^{(\pi)} \in \mathbb{C}) \quad (2.3)$$

where  $\mathcal{E}_\pi x := \sum_{i,j} x_{ij}^{(\pi)} u_{ij}^{(\pi)}$  is the orthogonal projection of  $x$  onto  $E_\pi$ .

Now for  $\pi \in \text{Irr}(\mathbb{G})$ , write  $u^{(\pi)} = \sum_{l,m} u_{lm}^{(\pi)} \otimes e_{lm}^{(\pi)}$  and  $X_\pi = [x_{ji}^{(\pi)}]_{i,j}$  where  $e_{lm}^{(\pi)}$  denote the canonical matrix units of  $B(H_\pi)$ . Then

$$\begin{aligned} \hat{x}(\pi) &= (h(\cdot x) \otimes \iota)((u^{(\pi)})^*) = (h(\cdot \mathcal{E}_\pi x) \otimes \iota)((u^{(\pi)})^*) + (h(\cdot \mathcal{E}_\pi^\perp x) \otimes \iota)((u^{(\pi)})^*) \quad (2.4) \\ &= \sum_{i,j,l,m} x_{ij}^{(\pi)} \left( (h(\cdot u_{ij}^{(\pi)}) \otimes \iota) \left( (u_{lm}^{(\pi)})^* \otimes e_{ml}^{(\pi)} \right) + 0 \right) \\ &= \sum_{i,j,l,m} x_{ij}^{(\pi)} h \left( (u_{lm}^{(\pi)})^* u_{ij}^{(\pi)} \right) e_{ml}^{(\pi)} \\ &= d_\pi^{-1} \sum_{i,j,l} x_{ij}^{(\pi)} (Q_\pi^{-1})_{il} e_{jl}^{(\pi)} = d_\pi^{-1} X_\pi Q_\pi^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} d_\pi (\iota \otimes \text{Tr})[(1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}] &= \sum_{i,j,l,m} (\iota \otimes \text{Tr})[(1 \otimes x_{ji}^{(\pi)} e_{ij}^{(\pi)}) (u_{lm}^{(\pi)} \otimes e_{lm}^{(\pi)})] \\ &= \sum_{i,j,m} x_{ji}^{(\pi)} (\iota \otimes \text{Tr})(u_{jm}^{(\pi)} \otimes e_{im}^{(\pi)}) = \sum_{i,j} x_{ji}^{(\pi)} u_{ji}^{(\pi)} = \mathcal{E}_\pi x. \end{aligned}$$

Combining the last equality with (2.3) proves the desired (2.2).

(b) Let

$$x = \sum_{\pi \in \text{Irr}(\mathbb{G})} \mathcal{E}_\pi x = \sum_{\pi \in \text{Irr}(\mathbb{G})} \sum_{i,j} x_{ij}^{(\pi)} u_{ij}^{(\pi)} \in L_2(\mathbb{G}).$$

For each  $\pi \in \text{Irr}(\mathbb{G})$ ,

$$\begin{aligned} \|\mathcal{E}_\pi x\|_2^2 &= h((\mathcal{E}_\pi x)^*(\mathcal{E}_\pi x)) = \sum_{i,j,l,m} \overline{x_{ij}^{(\pi)}} x_{lm}^{(\pi)} h((u_{ij}^{(\pi)})^* u_{lm}^{(\pi)}) \\ &= d_\pi^{-1} \sum_{i,j,l} \overline{x_{ij}^{(\pi)}} x_{lj}^{(\pi)} (Q_\pi^{-1})_{li} \end{aligned}$$

and also by (2.4)

$$d_\pi^2 \text{Tr}(Q_\pi \hat{x}(\pi)^* \hat{x}(\pi)) = \text{Tr}(X_\pi^* X_\pi Q_\pi^{-1}) = \sum_{i,j,l} \overline{x_{ij}^{(\pi)}} x_{lj}^{(\pi)} (Q_\pi^{-1})_{li}.$$

Hence by Parseval's identity,

$$\begin{aligned} \|x\|_2^2 &= \sum_{\pi} \|\mathcal{E}_\pi x\|_2^2 = \sum_{\pi} d_\pi^{-1} \sum_{i,j,l} \overline{x_{ij}^{(\pi)}} x_{lj}^{(\pi)} (Q_\pi^{-1})_{li} \\ &= \sum_{\pi} d_\pi^{-1} d_\pi^2 \text{Tr}(Q_\pi \hat{x}(\pi)^* \hat{x}(\pi)) = \|\hat{x}\|_2^2. \end{aligned}$$

Thus  $\mathcal{F}$  maps isometrically  $L_2(\mathbb{G})$  into  $\ell_2(\hat{\mathbb{G}})$ . From (2.2) and the isometric relation we see that the range of  $\mathcal{F}$  contains the subset of all finitely supported families  $(a_\pi) \in \oplus_{\pi} B(H_\pi)$ , which is dense in  $\ell_2(\hat{\mathbb{G}})$ . Therefore  $\mathcal{F}$  is surjective and hence unitary.  $\square$

Note that by the complex interpolation we immediately get the Hausdorff-Young inequality, which is established in [Coo10, Cas13] for more general frameworks.

**Proposition 2.1.3.** *Let  $1 \leq p < \infty$  and  $1/p + 1/p' = 1$ . Then  $\mathcal{F}$  extends to a contraction from  $L^p(\mathbb{G})$  to  $\ell^{p'}(\hat{\mathbb{G}})$ .*

**Example 2.1.4.** (1) Let  $G$  be a compact group and define

$$\Delta(f)(s, t) = f(st), \quad f \in C(G), \quad s, t \in G.$$

Then  $\mathbb{G} = (C(G), \Delta)$  is a compact quantum group. The elements in  $\text{Irr}(\mathbb{G}) := \text{Irr}(G)$  coincide with the usual strongly continuous irreducible unitary representations of  $G$ . Any continuous functional  $\varphi$  on  $C(G)$  corresponds to a complex Radon measure  $\mu$  on  $G$  by the Riesz representation theorem. By definition (2.1), the Fourier series of  $\mu$  is given by

$$\hat{\mu}(\pi) = (\varphi \otimes \iota)(\pi(\cdot)^*) = \int_G \pi(g)^* d\mu(g), \quad \pi \in \text{Irr}(G).$$

In particular for  $f \in L_2(G)$ , we have

$$\hat{f}(\pi) = \int_G \pi(g)^* f(g) dg, \quad \pi \in \text{Irr}(G)$$

and we have the Fourier expansion and the Plancherel formula

$$f = \sum_{\pi \in \text{Irr}(G)} d_\pi \text{Tr}(\hat{f}(\pi) \pi), \quad \|f\|_2^2 = \sum_{\pi \in \text{Irr}(G)} d_\pi \|\hat{f}(\pi)\|_{\text{HS}}^2$$

where  $d_\pi$  is the dimension of the Hilbert space on which the representation  $\pi$  acts and  $\|\cdot\|_{\text{HS}}$  denotes the usual Hilbert-Schmidt norm. We refer to [Fol95, Section 5.3] and [HR70, pp.77-87] for more information.

(2) Let  $\Gamma$  be a discrete group with its neutral element  $e$  and  $C_r^*(\Gamma)$  be the associated reduced group C\*-algebra generated by  $\lambda(\Gamma) \subset B(\ell_2(\Gamma))$ , where  $\lambda$  denotes the left regular representation. The “dual”  $\mathbb{G} = \hat{\Gamma}$  of  $\Gamma$  is a compact quantum group such that  $C(\mathbb{G})$  is the group C\*-algebra  $C_r^*(\Gamma)$  equipped with the comultiplication  $\Delta : C_r^*(\Gamma) \rightarrow C_r^*(\Gamma) \otimes C_r^*(\Gamma)$  defined by

$$\Delta(\lambda(\gamma)) = \lambda(\gamma) \otimes \lambda(\gamma), \quad \gamma \in \Gamma.$$

The Haar state of  $\mathbb{G}$  is the unique trace  $\tau$  on  $C_r^*(\Gamma)$  such that  $\tau(1) = 1$  and  $\tau(\lambda(\gamma)) = 0$  for  $\gamma \in \Gamma \setminus \{e\}$ . The elements of  $\lambda(\Gamma)$  give all irreducible unitary representations of  $\mathbb{G}$ , which are all of dimension 1. It is easy to check from definition that for any  $f \in C_r^*(\Gamma)$ ,

$$\hat{f}(\gamma) = \tau(f\lambda(\gamma)^*), \quad \gamma \in \Gamma.$$

And any  $f \in L_2(\mathbb{G})$  has an expansion such that

$$f = \sum_{\gamma \in \Gamma} \hat{f}(\gamma)\lambda(\gamma), \quad \|f\|_2^2 = \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^2.$$

In the following paragraph we present a special subspace of  $C_r(\mathbb{G})$ , which, in the classical case, is known as the *Fourier algebra* introduced by Eymard [Eym64], as explained in [HR70, p.367]. Recall that we always identify  $C_r(\mathbb{G}) \subset L^\infty(\mathbb{G})$  as a subspace of  $L^1(\mathbb{G})$ .

**Proposition 2.1.5.** *Let  $A(\mathbb{G}) = \{x \in L^1(\mathbb{G}) : \hat{x} \in \ell^1(\hat{\mathbb{G}})\}$ . Then  $A(\mathbb{G}) \subset C_r(\mathbb{G})$  and*

$$\|x\|_\infty \leq \|\hat{x}\|_1, \quad x \in A(\mathbb{G}). \quad (2.5)$$

Moreover, if we let  $\|x\|_A = \|\hat{x}\|_1$  for  $x \in A(\mathbb{G})$ , then  $(A(\mathbb{G}), \|\cdot\|_A)$  is a Banach space isometrically isomorphic to  $\ell^1(\hat{\mathbb{G}})$ .

*Proof.* Firstly we show that

$$\|x\|_\infty \leq \|\hat{x}\|_1, \quad x \in \text{Pol}(\mathbb{G}). \quad (2.6)$$

Choose an  $x \in \text{Pol}(\mathbb{G})$ . For each  $\pi \in \text{Irr}(\mathbb{G})$ , by the identification  $L^\infty(\mathbb{G}) = (L^\infty(\mathbb{G})_*)^*$  and the Hölder inequality on  $\text{Tr}$  we have

$$\begin{aligned} \|(\iota \otimes \text{Tr}) \left[ (1 \otimes \hat{x}(\pi)Q_\pi)u^{(\pi)} \right]\|_\infty &= \sup_{\omega \in L^\infty(\mathbb{G})_*, \|\omega\|=1} \left| \omega \left( (\iota \otimes \text{Tr}) \left[ (1 \otimes \hat{x}(\pi)Q_\pi)u^{(\pi)} \right] \right) \right| \\ &= \sup_{\omega \in L^\infty(\mathbb{G})_*, \|\omega\|=1} \left| \text{Tr} \left( \hat{x}(\pi)Q_\pi \left[ (\omega \otimes \iota)u^{(\pi)} \right] \right) \right| \\ &\leq \sup_{\omega \in L^\infty(\mathbb{G})_*, \|\omega\|=1} \text{Tr}(|\hat{x}(\pi)Q_\pi|) \|(\omega \otimes \iota)u^{(\pi)}\| \\ &\leq \text{Tr}(|\hat{x}(\pi)Q_\pi|). \end{aligned}$$

Therefore by (2.2),

$$\|x\|_\infty \leq \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \|(\iota \otimes \text{Tr}) \left[ (1 \otimes \hat{x}(\pi)Q_\pi)u^{(\pi)} \right]\|_\infty \leq \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \text{Tr}(|\hat{x}(\pi)Q_\pi|) = \|\hat{x}\|_1.$$

Now given  $y \in A(\mathbb{G})$ , we note that  $c_c(\hat{\mathbb{G}})$  is dense in  $\ell^1(\hat{\mathbb{G}})$  and  $\mathcal{F}(\text{Pol}(\mathbb{G})) = c_c(\hat{\mathbb{G}})$ , so we may take  $x_n \in \text{Pol}(\mathbb{G})$  such that  $\|\hat{x}_n - \hat{y}\|_1 \rightarrow 0$ . So  $(\hat{x}_n)_n$  is  $\|\cdot\|_1$ -Cauchy, and hence

by (2.6) the sequence  $(x_n)_n$  is  $\|\cdot\|_\infty$ -Cauchy in  $\text{Pol}(\mathbb{G})$ . Since  $\text{Pol}(\mathbb{G})$  is dense in  $C_r(\mathbb{G})$ , the sequence  $(x_n)_n$  in  $\text{Pol}(\mathbb{G})$  converges to some  $x \in C_r(\mathbb{G})$ . Note that  $\|\cdot\|_1 \leq \|\cdot\|_\infty$  on  $L^\infty(\mathbb{G})$ , we have also  $\|x_n - x\|_1 \rightarrow 0$ . Then according to the contractive property of  $\mathcal{F}$ , it holds that  $\hat{x}(\pi) = \lim_n \hat{x}_n(\pi) = \hat{y}(\pi)$  for all  $\pi \in \text{Irr}(\mathbb{G})$ . So  $y = x \in C_r(\mathbb{G})$  according to the injectivity of  $\mathcal{F}$ , and  $\|y\|_\infty = \lim_n \|x_n\|_\infty \leq \lim_n \|\hat{x}_n\|_1 = \|\hat{y}\|_1$ . As a result  $A(\mathbb{G}) \subset C_r(\mathbb{G})$  and (2.5) is proved.

Now we set  $\|x\|_A = \|\hat{x}\|_1$  for all  $x \in A(\mathbb{G})$ , then  $(A(\mathbb{G}), \|\cdot\|_A)$  is clearly a normed space and the map  $x \mapsto \hat{x}$  sends  $(A(\mathbb{G}), \|\cdot\|_A)$  isometrically into  $\ell^1(\hat{\mathbb{G}})$ . To see that it is surjective, we shall show that for any  $a = (a_\pi)_{\pi \in \text{Irr}(\mathbb{G})} \in \ell^1(\hat{\mathbb{G}})$  there exists  $x \in L^1(\mathbb{G})$  with  $\hat{x} = a$ . In fact, if we consider a sequence  $x_n \in \text{Pol}(\mathbb{G})$  such that  $\|\hat{x}_n - a\|_1 \rightarrow 0$ , then the existence of  $x \in L^1(\mathbb{G})$  with  $\hat{x} = a$  follows simply from the same argument as in the preceding paragraph.  $\square$

In some literature the Fourier algebra  $A(\mathbb{G})$  is also defined simply to be the space  $\ell^1(\hat{\mathbb{G}})$ . Note that the identification that we use yields also the product on  $A(\mathbb{G})$  (convolution product of  $\ell^1(\hat{\mathbb{G}}) \cong \ell^\infty(\hat{\mathbb{G}})_*$ , well-known in the theory of locally compact quantum groups), which makes  $A(\mathbb{G})$  a Banach algebra.

## 2.2 Convolutions

We recall the standard definition of convolution products given by Woronowicz. Let  $x \in \text{Pol}(\mathbb{G})$  and  $\varphi_1, \varphi_2$  be linear functionals on  $\text{Pol}(\mathbb{G})$ . We define

$$\begin{aligned}\varphi \star \varphi' &= (\varphi \otimes \varphi') \circ \Delta, \\ x \star \varphi &= (\varphi \otimes \iota)\Delta(x), \\ \varphi \star x &= (\iota \otimes \varphi)\Delta(x).\end{aligned}$$

Observing the embedding  $x \mapsto h(\cdot x)$  from  $\text{Pol}(\mathbb{G})$  into  $\text{Pol}(\mathbb{G})^*$ , the convolution products defined above are related as follows according to (1.8) (see also [VD07, Proposition 2.2]): on  $\text{Pol}(\mathbb{G})$  we have

$$h(\cdot x) \star \varphi = h(\cdot [(\varphi \circ S) \star x]), \quad \varphi \star h(\cdot x) = h(\cdot [x \star (\varphi \circ S^{-1})]). \quad (2.7)$$

We note that for a linear functional  $\varphi$  on  $\text{Pol}(\mathbb{G})$  and  $\pi \in \text{Irr}(\mathbb{G})$ ,

$$(\varphi \circ S^{-1})^\wedge(\pi) = ((\varphi \circ S^{-1}) \otimes \iota)((u^{(\pi)})^*) = \left[ \varphi(u_{ij}^{(\pi)}) \right]_{i,j}. \quad (2.8)$$

In particular,

$$(\varphi^* \circ S^{-1})^\wedge(\pi) = \hat{\varphi}(\pi)^*, \quad (2.9)$$

where  $\varphi^*$  denotes the usual adjoint of  $\varphi$ , i.e.,  $\varphi^*(x) = \overline{\varphi(x^*)}$ . On the other hand, a straightforward calculation shows that

$$(\varphi_1 \star \varphi_2)^\wedge(\pi) = \hat{\varphi}_2(\pi) \hat{\varphi}_1(\pi). \quad (2.10)$$

Then by (2.7) we see that for a linear functional  $\varphi$  on  $\text{Pol}(\mathbb{G})$  and  $x \in \text{Pol}(\mathbb{G})$ ,  $\pi \in \text{Irr}(\mathbb{G})$ ,

$$((\iota \otimes \varphi)\Delta(x))^\wedge(\pi) = (\varphi \circ S^{-1})^\wedge(\pi) \hat{x}(\pi), \quad ((\varphi \otimes \iota)\Delta(x))^\wedge(\pi) = \hat{x}(\pi) (\varphi \circ S)^\wedge(\pi). \quad (2.11)$$

Using (1.6) and (1.7) we may rewrite the second equality above as

$$((\varphi \otimes \iota)\Delta(x))^\wedge(\pi) Q_\pi = \hat{x}(\pi) Q_\pi (\varphi \circ S^{-1})^\wedge(\pi). \quad (2.12)$$

The convolutions can be extended to  $L^p$ -spaces under some assumptions. As is seen in the previous construction, we will in the sequel identify  $\text{Pol}(\mathbb{G})$  with a subspace of  $\text{Pol}(\mathbb{G})^*$  via the embedding

$$j : x \mapsto xh := h(\cdot x).$$

In particular we define the convolution for  $x, x' \in \text{Pol}(\mathbb{G})$ ,

$$x \star x' := (xh) \star (x'h) (\in \text{Pol}(\mathbb{G})).$$

We may establish the following Young's inequality.

**Proposition 2.2.1.** *Let  $1 \leq p, q, r \leq 2$  with  $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $x, y \in \text{Pol}(\mathbb{G})$ , we have*

$$\|\tau_{i/p'}(y) \star x\|_r \leq \|x\|_p \|y\|_q.$$

*Proof.* Note that by definition it is easy to deduce

$$\|y \star x\|_1 \leq \|x\|_1 \|y\|_1, \quad x, y \in \text{Pol}(\mathbb{G}).$$

By Proposition 2.1.2 and above discussions we have

$$\|y \star x\|_2 = \|\hat{x}\hat{y}\|_2 \leq \|\hat{x}\|_\infty \|\hat{y}\|_2 \leq \|x\|_1 \|y\|_2.$$

Hence, by the complex interpolation we get

$$\|y \star x\|_p \leq \|x\|_1 \|y\|_p, \quad 1 \leq p \leq 2, x, y \in \text{Pol}(\mathbb{G}).$$

So in order to establish the proposition, by a standard complex interpolation argument, it suffices to consider the case  $r = 2$ . We take  $\frac{3}{2} = \frac{1}{p} + \frac{1}{q}$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ . By the construction of  $(f_z)$  and (2.8) we have  $(f_{iz} \circ S^{-1})^\wedge = Q^{iz}$ . So together with (2.11), (2.12) and (1.5) we get

$$\mathcal{F}(\tau_z(y)) = Q^{-iz} \hat{y} Q^{iz}.$$

Together with Proposition 2.1.3, we have

$$\begin{aligned} \|\tau_{i/p'}(y) \star x\|_2 &= \|\hat{x} Q^{\frac{1}{p'}} \hat{y} Q^{-\frac{1}{p'}}\|_2 = \left( \sum_{\pi} d_{\pi} \text{Tr}(|\hat{x} Q^{\frac{1}{p'}} \hat{y} Q^{-\frac{1}{p'}}|^2) \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{\pi} d_{\pi} \text{Tr}(|\hat{x} Q^{\frac{1}{p'}}|^{p'}) \right)^{\frac{1}{p'}} \left( \sum_{\pi} d_{\pi} \text{Tr}(|\hat{y} Q^{-\frac{1}{p'}}|^{q'}) \right)^{\frac{1}{q'}} \\ &= \|\hat{x}\|_{p'} \|\hat{y}\|_{q'} \leq \|x\|_p \|y\|_q. \end{aligned}$$

So we establish the proposition.  $\square$

We do not know if the above type of Young's inequality can be extended for all  $1 \leq r \leq \infty$ . In the following we give an affirmative observation on the Kac case.

**Proposition 2.2.2.** *Assume that  $\mathbb{G}$  is of Kac type. Let  $1 \leq p, q, r \leq \infty$  with  $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $x, y \in \text{Pol}(\mathbb{G})$ , we have*

$$\|y \star x\|_r \leq \|x\|_p \|y\|_q.$$

*Proof.* Since  $\mathbb{G}$  is of Kac type, the antipode  $S$  is of norm 1 on  $L^p(\mathbb{G})$  for all  $1 \leq p \leq \infty$ . So by (2.7), we have

$$\|y \star x\|_\infty \leq \|x\|_1 \|y\|_\infty, \quad x, y \in \text{Pol}(\mathbb{G}).$$

And by complex interpolation we get

$$\|y \star x\|_p \leq \|x\|_1 \|y\|_p, \quad 1 \leq p \leq \infty, x, y \in \text{Pol}(\mathbb{G}). \quad (2.13)$$

Following the same scheme as the previous proposition, it suffices to establish the desired inequality for  $r = \infty$ . Take  $1/p + 1/q = 1$ . By the duality between  $L^1(\mathbb{G})$  and  $L^\infty(\mathbb{G})$ , we show that for all  $z \in \text{Pol}(\mathbb{G})$  with  $\|z\|_1 = 1$ ,

$$|h(z(y \star x))| \leq \|x\|_p \|y\|_q.$$

Indeed, by (2.7) and (2.13),

$$|h(z(y \star x))| = |h((Sz \star y)Sx)| \leq \|Sz \star y\|_q \|Sx\|_p \leq \|x\|_p \|y\|_q,$$

as desired.  $\square$

The Young inequality and related topics for more general locally compact quantum groups are discussed in [LWW16].

### 2.3 $L^p$ -Fourier multipliers

For  $a = (a_\pi)_\pi \in \prod_\pi B(H_\pi)$ , we define the left and right *multipliers*  $m_a^L : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G})$ ,  $m_a^R : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G})$  associated to  $a$  (cf. [JNR09, Daw12]) by

$$(m_a^L \otimes \iota)u^{(\pi)} = (1 \otimes a_\pi)u^{(\pi)}, \quad (m_a^R \otimes \iota)u^{(\pi)} = u^{(\pi)}(1 \otimes a_\pi), \quad (2.14)$$

which, according to the Proposition 2.1.2, yields that

$$(m_a^L x)^\wedge(\pi)Q_\pi = \hat{x}(\pi)Q_\pi a_\pi, \quad (m_a^R x)^\wedge(\pi)Q_\pi = a_\pi \hat{x}(\pi)Q_\pi \quad \pi \in \text{Irr}(\mathbb{G}). \quad (2.15)$$

Denote  $Q = (Q_\pi)_\pi \in \prod_\pi B(H_\pi)$  and let  $1 \leq p \leq \infty$ . We say that  $a$  is a *bounded left (resp., right) multiplier* on  $L^p(\mathbb{G})$  if  $m_a^L$  (resp.,  $m_a^R$ ) extends to a bounded map on  $L^p(\mathbb{G})$ , and denote the set of all such multipliers by  $M_L(L^p(\mathbb{G}))$  (resp.,  $M_R(L^p(\mathbb{G}))$ ). We define

$$M(L^p(\mathbb{G})) = \left\{ a \in \prod_\pi B(H_\pi) : Q^{-1/p} a Q^{1/p} \in M_L(L^p(\mathbb{G})), a \in M_R(L^p(\mathbb{G})) \right\}$$

equipped with the norm

$$\|a\|_{M(L^p(\mathbb{G}))} = \max\{\|m_{Q^{-1/p} a Q^{1/p}}^L\|_{B(L^p(\mathbb{G}))}, \|m_a^R\|_{B(L^p(\mathbb{G}))}\}.$$

**Lemma 2.3.1.** *For any  $a \in \prod_\pi B(H_\pi)$ , we have*

$$\|a\|_\infty = \|m_{Q^{-1/2} a Q^{1/2}}^L\|_{B(L^2(\mathbb{G}))} = \|m_a^R\|_{B(L^2(\mathbb{G}))}.$$

*In particular, we have the following isometric isomorphism*

$$M(L^2(\mathbb{G})) = \ell^\infty(\hat{\mathbb{G}}).$$

*Proof.* Let  $a \in \ell^\infty(\hat{\mathbb{G}})$ . By equality (2.15) and Proposition 2.1.2, we have for all  $x \in \text{Pol}(\mathbb{G})$ ,

$$\begin{aligned} \|m_{Q^{-1/2}aQ^{1/2}}^L x\|_2^2 &= \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \text{Tr} \left( |\hat{x}(\pi)Q_\pi^{1/2}a_\pi Q_\pi^{-1/2}|^2 Q_\pi \right) \\ &= \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \text{Tr} \left( \hat{x}(\pi)Q_\pi^{1/2}a_\pi a_\pi^* Q_\pi^{1/2} \hat{x}(\pi)^* \right) \\ &\leq \|a\|_\infty^2 \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \text{Tr} \left( \hat{x}(\pi)Q_\pi \hat{x}(\pi)^* \right) = \|a\|_\infty^2 \|x\|_2^2. \end{aligned}$$

Therefore we get

$$Q^{-1/2}aQ^{1/2} \in M_L(L^2(\mathbb{G})), \quad \|m_{Q^{-1/2}aQ^{1/2}}^L\|_{B(L^2(\mathbb{G}))} \leq \|a\|_\infty. \quad (2.16)$$

Conversely, take  $a \in \prod_\pi B(H_\pi)$  with  $Q^{-1/2}aQ^{1/2} \in M_L(L^2(\mathbb{G}))$ . For  $\pi \in \text{Irr}(\mathbb{G})$  and for each eigenvalue  $\lambda$  of the operator  $a_\pi a_\pi^*$ , we choose an orthogonal projection  $E \in B(H_\pi)$  such that  $a_\pi a_\pi^* E = \lambda E$ . According to Proposition 2.1.2, we choose an  $x \in \text{Pol}(\mathbb{G})$  with

$$\hat{x}(\pi) = EQ_\pi^{-1/2}, \quad \hat{x}(\pi') = 0, \quad \pi' \neq \pi.$$

Then we have

$$\begin{aligned} \|m_{Q^{-1/2}aQ^{1/2}}^L x\|_2^2 &= d_\pi \text{Tr} \left( |\hat{x}(\pi)Q_\pi^{1/2}a_\pi Q_\pi^{-1/2}|^2 Q_\pi \right) \\ &= d_\pi \text{Tr} \left( a_\pi^* Q_\pi^{1/2} \hat{x}(\pi)^* \hat{x}(\pi) Q_\pi^{1/2} a_\pi \right) = d_\pi \text{Tr} \left( a_\pi^* E a_\pi \right) = \lambda d_\pi \text{Tr} (E) \\ &= \lambda d_\pi \text{Tr} (E^* E) = \lambda d_\pi \text{Tr} \left( Q_\pi^{1/2} \hat{x}(\pi)^* \hat{x}(\pi) Q_\pi^{1/2} \right) = \lambda \|x\|_2^2. \end{aligned}$$

So we see that  $\lambda \leq \|m_{Q^{-1/2}aQ^{1/2}}^L\|_{B(L^2(\mathbb{G}))}^2$ . Taking the supremum over all such eigenvalues  $\lambda$ , we get

$$a \in \ell^\infty(\hat{\mathbb{G}}), \quad \|a\|_\infty \leq \|m_{Q^{-1/2}aQ^{1/2}}^L\|_{B(L^2(\mathbb{G}))}. \quad (2.17)$$

Combining the two inequalities (2.16) and (2.17) we prove that for all  $a \in \prod_\pi B(H_\pi)$  we have

$$\|a\|_\infty = \|m_{Q^{-1/2}aQ^{1/2}}^L\|_{B(L^2(\mathbb{G}))}.$$

A similar argument gives

$$\|a\|_\infty = \|m_a^R\|_{B(L^2(\mathbb{G}))}.$$

So we establish the lemma.  $\square$

It is well known that if  $\mathbb{G}$  is a compact group  $G$  or the dual compact quantum group  $\hat{\Gamma}$  of a discrete group  $\Gamma$ , then for all  $1 \leq p \leq \infty$  we have

$$\|a\|_\infty \leq \|a\|_{M(L^p(\mathbb{G}))}, \quad a \in M(L^p(\mathbb{G})). \quad (2.18)$$

We refer to [Har99, HR70] for related discussions. This is however not clear for an arbitrary compact quantum group. Note that if the operators  $m_a^L$  and  $m_a^R$  are completely bounded on  $L^\infty(\mathbb{G})$ , then from (2.14) we see that

$$\|a\|_\infty = \|((1 \otimes a_\pi)u^{(\pi)})\|_{\oplus(C(\mathbb{G}) \otimes B(H_\pi))} \leq \|m_a^L\|_{CB(L^\infty(\mathbb{G}))}, \quad \|a\|_\infty \leq \|m_a^R\|_{CB(L^\infty(\mathbb{G}))},$$

where  $\|\cdot\|_{CB(L^\infty(\mathbb{G}))}$  denote the completely bounded norm in the sense of [ER00]. If the multiplier  $a \in M(L^\infty(\mathbb{G}))$  is not necessarily completely bounded, it is even not clear to see whether the sequence  $a \in \prod_\pi B(H_\pi)$  is bounded or not. We refer to [Daw10, Section 8.2] for some related discussions. In the following Proposition 2.3.5 we will give an affirmative answer in the case that  $\mathbb{G}$  is of Kac type, and indeed give a similar estimate for the general case, which also improves an early result in [Daw10, Proposition 8.8] for compact quantum groups. Let us first establish some easy but useful lemmas.

**Lemma 2.3.2.** *Let  $1 \leq p \leq \infty$ . We have the following assertions:*

- (a) *For all  $t \in \mathbb{R}$ , the operator  $\sigma_t$  extends to an isometry on  $L^p(\mathbb{G})$ .*
- (b) *We have  $\sigma_{-i/p} \circ R = R \circ \sigma_{i/p}$ , and they extend to an isometry on  $L^p(\mathbb{G})$ .*
- (c) *The map  $x \mapsto \sigma_{-i/p}(x^*)$  extends to an isometry on  $L^p(\mathbb{G})$ .*

*Proof.* (a) Since  $h \circ \sigma = h$ , it is easy to see that the operators  $\sigma_t$  and  $\sigma_{-t}$  extend to isometries on  $L^1(\mathbb{G})$  and  $L^\infty(\mathbb{G})$  for all  $t \in \mathbb{R}$ . So by interpolation  $\sigma_t$  also extends to an isometry on  $L^p(\mathbb{G})$  for all  $t \in \mathbb{R}$ .

(b) It is well known and easy to see from (1.3), (1.4) and (1.6) that

$$S \circ \sigma_z = \sigma_{-z} \circ S, \quad R \circ \sigma_z = \sigma_{-z} \circ R, \quad z \in \mathbb{C}.$$

Also recall that  $R$  is a  $*$ -antiautomorphism on  $L^\infty(\mathbb{G})$  with  $h \circ R = h$ , and  $h(xy) = h(y\sigma_{-i}(x))$  for  $x, y \in \text{Pol}(\mathbb{G})$ . So for  $x \in \text{Pol}(\mathbb{G})$ , we have

$$\begin{aligned} \|x\|_1 &= \sup_{y \in \text{Pol}(\mathbb{G}), \|y\|_\infty=1} |h(yx)| = \sup_{y \in \text{Pol}(\mathbb{G}), \|y\|_\infty=1} |h(R(x)R(y))| \\ &= \sup_{y \in \text{Pol}(\mathbb{G}), \|y\|_\infty=1} |h(R(y)\sigma_{-i}(R(x)))| = \|\sigma_{-i}(R(x))\|_1. \end{aligned}$$

Thus we see that  $\sigma_{-i} \circ R$  extends to an isometry on  $L^1(\mathbb{G})$ . Therefore by (a),  $\sigma_{-i-t} \circ R$  and  $R \circ \sigma_{i+t}$  also extend to isometries on  $L^1(\mathbb{G})$  for  $t \in \mathbb{R}$ . Then the assertion follows directly from the Stein interpolation theorem (see e.g. [Lun09, Theorem 2.7]).

(c) This follows directly from Proposition 1.1.2 (1)(4).  $\square$

**Lemma 2.3.3.** *Let  $1 \leq p \leq \infty$ . Consider  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  and*

$$X = \text{span} \left\{ u_{ij}^{(\pi)}, (u_{ij}^{(\pi)})^* : \pi \in \mathbf{E}, 1 \leq i, j \leq n_\pi \right\}.$$

*We equip the space  $X$  with the norm  $\|\cdot\|_p$ . Then we have for all  $a \in \prod_\pi B(H_\pi)$ ,*

$$\|m_a^R\|_{B(X)} = \|m_{Q^{-1/2}a^*Q^{1/2}}^L\|_{B(X)}.$$

*Proof.* Note that the subspace  $X$  is invariant under  $R$ ,  $\sigma$  and  $\tau$ , and that the maps  $m_a^R$  and  $m_{Q^{-1/2}a^*Q^{1/2}}^L$  send  $X$  into  $X$  itself. We define a linear functional  $\varphi$  on  $\text{Pol}(\mathbb{G})$  by  $\varphi(u_{ij}^{(\pi)}) = \overline{(a_\pi)_{ji}}$  for  $\pi \in \text{Irr}(\mathbb{G})$  and  $1 \leq i, j \leq n_\pi$ . Then we see that  $\widehat{\varphi^*} = a$  where  $\varphi^* = \overline{\varphi(\cdot^*)}$ . Take  $x \in X$ . By (2.11), (2.12), we have

$$m_a^R(\sigma_{-i/p}(x^*)) = (\iota \otimes (\varphi^* \circ S))\Delta(\sigma_{-i/p}(x^*)).$$

Recall from (1.4) and (1.7) that

$$S = \tau_{-i/2} \circ R, \quad R^2 = \text{id}, \quad \Sigma \circ \Delta \circ R = (R \otimes R) \circ \Delta.$$

So we get

$$\begin{aligned} (\iota \otimes (\varphi^* \circ S))\Delta(\sigma_{-i/p}(x^*)) &= (R \otimes (\varphi^* \circ \tau_{-i/2}))(R \otimes R)\Delta(\sigma_{-i/p}(x^*)) \\ &= ((\varphi^* \circ \tau_{-i/2}) \otimes R)\Delta(R \circ \sigma_{-i/p}(x^*)). \end{aligned}$$

Writing  $R = (R \circ \sigma_{i/p}) \circ \sigma_{-i/p}$  and using the previous lemma, we then have

$$\|m_a^R(\sigma_{-i/p}(x^*))\|_p = \|\sigma_{-i/p}((\varphi^* \circ \tau_{-i/2}) \otimes \iota)\Delta(R \circ \sigma_{-i/p}(x^*))\|_p$$



$$\begin{aligned}
&= \left\| \sigma_{-i/p} \left( \left[ ((\varphi \circ \tau_{i/2}) \otimes \iota) \Delta(R \circ \sigma_{i/p}(x)) \right]^* \right) \right\|_p \\
&= \left\| ((\varphi \circ \tau_{i/2}) \otimes \iota) \Delta(R \circ \sigma_{i/p}(x)) \right\|_p.
\end{aligned}$$

Note that by (2.8) and (1.6), we have

$$(\varphi \circ \tau_{i/2} \circ S^{-1})^\wedge = Q^{-1/2} a^* Q^{1/2}.$$

So together with (2.12) and (2.15), we have

$$((\varphi \circ \tau_{i/2}) \otimes \iota) \Delta(R \circ \sigma_{i/p}(x)) = m_{Q^{-1/2} a^* Q^{1/2}}^L(R \circ \sigma_{i/p}(x))$$

and hence the above computations yield

$$\|m_a^R(\sigma_{-i/p}(x^*))\|_p = \|m_{Q^{-1/2} a^* Q^{1/2}}^L(R \circ \sigma_{i/p}(x))\|_p.$$

Since the maps  $x \mapsto \sigma_{-i/p}(x^*)$  and  $x \mapsto R \circ \sigma_{i/p}(x)$  are isometries on  $X$  according to the previous lemma, we obtain the desired conclusion.  $\square$

**Lemma 2.3.4.** *For  $z \in \mathbb{C}$  and  $x \in \text{Pol}(\mathbb{G})$ , we have*

$$\mathcal{F}(\sigma_z(x)) = Q^{iz} \hat{x} Q^{iz}.$$

*In particular for  $a \in \prod_{\pi \in \text{Irr}(\mathbb{G})} B(H_\pi)$  and  $t \in \mathbb{R}$ ,*

$$\sigma_t \circ m_a^L \circ \sigma_{-t} = m_{Q^{-it} a Q^{it}}^L, \quad \sigma_t \circ m_a^R \circ \sigma_{-t} = m_{Q^{it} a Q^{-it}}^R.$$

*Proof.* Recall that

$$\sigma_z(x) = (f_{iz} \otimes \iota \otimes f_{iz}) \Delta^{(2)}(x), \quad x \in \text{Pol}(\mathbb{G}), z \in \mathbb{C}.$$

And by the construction of  $(f_z)$  and (2.8) we have  $(f_{iz} \circ S^{-1})^\wedge = Q^{iz}$ . So together with (2.11) and (2.12) we get

$$\mathcal{F}(\sigma_z(x)) = Q^{iz} \hat{x} Q^{iz}.$$

Consequently, for  $x \in \text{Pol}(\mathbb{G})$ ,

$$\mathcal{F}(\sigma_t \circ m_a^L \circ \sigma_{-t}(x)) Q = Q^{it} \mathcal{F}(m_a^L \circ \sigma_{-t}(x)) Q^{1+it} = Q^{it} \mathcal{F}(\sigma_{-t}(x)) Q a Q^{it} = \hat{x} Q Q^{it} a Q^{-it}.$$

That is,

$$\sigma_t \circ m_a^L \circ \sigma_{-t} = m_{Q^{-it} a Q^{it}}^L,$$

as desired. The equality for  $m_a^R$  follows similarly.  $\square$

Now we are able to give an analogue of inequality (2.18) for general compact quantum groups. The proof below is based on a personal communication by Marius Junge.

**Proposition 2.3.5.** *Let  $1 \leq p \leq \infty$ . For all  $a \in \prod_{\pi} B(H_\pi)$  and all  $0 \leq \theta \leq 1$ , we have*

$$\|Q^{\frac{1}{4} - \frac{\theta}{2}} a Q^{-\frac{1}{4} + \frac{\theta}{2}}\|_\infty \leq \|m_a^R\|_{B(L^p(\mathbb{G}))}^{1/2} \|m_{Q^{-\theta} a Q^\theta}^L\|_{B(L^p(\mathbb{G}))}^{1/2}.$$

*Proof.* Assume that  $m_a^R$  is bounded on  $L^p(\mathbb{G})$ . Consider the map

$$T : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G}), \quad y \mapsto \sigma_{-i/p}([m_a^R(\sigma_{-i/p}(x^*))])^*.$$

By Lemma 2.3.2,  $T$  extends to a bounded operator on  $L^p(\mathbb{G})$  with  $\|T\|_{B(L^p(\mathbb{G}))} = \|m_a^R\|_{B(L^p(\mathbb{G}))}$ . We let  $1 \leq p' \leq \infty$  with  $1/p + 1/p' = 1$  and denote  $\langle \cdot, \cdot \rangle_{p',p}$  the duality bracket between  $L^{p'}(\mathbb{G})$  and  $L^p(\mathbb{G})$  defined by

$$\langle x, y \rangle_{p',p} = \text{tr}(xD^{1/p'}yD^{1/p}), \quad x, y \in \text{Pol}(\mathbb{G}).$$

First let  $1 \leq p < \infty$ . Then consider the adjoint map  $T^* : L^p(\mathbb{G})^* \rightarrow L^p(\mathbb{G})^*$  of  $T$  and recall the formula  $\sigma_t(x) = D^{it}xD^{-it}$  in Proposition 1.1.2. We have for  $x, y \in \text{Pol}(\mathbb{G})$ ,

$$\begin{aligned} \langle T^*x, y \rangle_{p',p} &= \langle x, Ty \rangle_{p',p} = \text{tr}(xD^{1/p'}TyD^{1/p}) = \text{tr}(xD[m_a^R(\sigma_{-i/p}(y^*))])^* \\ &= h([m_a^R(\sigma_{-i/p}(y^*))]^*x). \end{aligned}$$

According to Proposition 2.1.2 and (2.15), we note that

$$\begin{aligned} h([m_a^R(\sigma_{-i/p}(y^*))]^*x) &= \hat{h}([m_a^R(\sigma_{-i/p}(y^*))]^\wedge)^* \hat{x} = \hat{h}([a(\sigma_{-i/p}(y^*))]^\wedge)^* \hat{x} \\ &= \hat{h}([\sigma_{-i/p}(y^*)]^\wedge)^* a^* \hat{x} = h((\sigma_{-i/p}(y^*))^* m_{a^*}^R x). \end{aligned}$$

Observe that  $(\sigma_{-i/p}(y^*))^* = D^{-1/p}yD^{1/p}$ . Therefore the above equalities give

$$\langle T^*x, y \rangle_{p',p} = h((\sigma_{-i/p}(y^*))^* m_{a^*}^R x) = \text{tr}(m_{a^*}^R x D^{1/p'} y D^{1/p}) = \langle m_{a^*}^R x, y \rangle_{p',p}.$$

So  $m_{a^*}^R$  extends to an isometry on  $L^{p'}(\mathbb{G})$  with

$$\|m_{a^*}^R\|_{B(L^{p'}(\mathbb{G}))} = \|T\|_{B(L^p(\mathbb{G}))} = \|m_a^R\|_{B(L^p(\mathbb{G}))}. \quad (2.19)$$

If  $p = \infty$ , we consider the restriction of operator  $T^*|_{L^1(\mathbb{G})} : L^1(\mathbb{G}) \rightarrow L^1(\mathbb{G})$  instead of  $T^* : L^\infty(\mathbb{G})^* \rightarrow L^\infty(\mathbb{G})^*$ , and repeat the above argument. Then (2.19) also holds for  $p = \infty$ .

On the other hand, we assume that  $m_{Q^{-\theta}aQ^\theta}^L$  is bounded on  $L^p(\mathbb{G})$ . By Lemma 2.3.3, we have

$$\|m_{Q^{-\frac{1}{2}+\theta}a^*Q^{\frac{1}{2}-\theta}}^R\|_{B(L^p(\mathbb{G}))} = \|m_{Q^{-\theta}aQ^\theta}^L\|_{B(L^p(\mathbb{G}))}. \quad (2.20)$$

Note also that by Lemma 2.3.2 and Lemma 2.3.4, for all  $t \in \mathbb{R}$ ,

$$\begin{aligned} \|m_{Q^{(-\frac{1}{2}+\theta)(1+it)}a^*Q^{(\frac{1}{2}-\theta)(1+it)}}^R\|_{B(L^p(\mathbb{G}))} &= \|\sigma_{(-\frac{1}{2}+\theta)t} \circ m_{Q^{-\frac{1}{2}+\theta}a^*Q^{\frac{1}{2}-\theta}}^R \circ \sigma_{(\frac{1}{2}-\theta)t}\|_{B(L^p(\mathbb{G}))} \\ &= \|m_{Q^{-\frac{1}{2}+\theta}a^*Q^{\frac{1}{2}-\theta}}^R\|_{B(L^p(\mathbb{G}))}. \end{aligned}$$

Hence, applying the Stein interpolation theorem, (2.19) and (2.20) yields

$$\begin{aligned} \|m_{Q^{-\frac{1}{4}+\frac{\theta}{2}}a^*Q^{\frac{1}{4}-\frac{\theta}{2}}}^R\|_{B(L^2(\mathbb{G}))} &\leq \|m_{a^*}^R\|_{B(L^{p'}(\mathbb{G}))}^{\frac{1}{2}} \|m_{Q^{-\frac{1}{2}+\theta}a^*Q^{\frac{1}{2}-\theta}}^R\|_{B(L^p(\mathbb{G}))}^{\frac{1}{2}} \\ &= \|m_a^R\|_{B(L^p(\mathbb{G}))}^{\frac{1}{2}} \|m_{Q^{-\theta}aQ^\theta}^L\|_{B(L^p(\mathbb{G}))}^{\frac{1}{2}}. \end{aligned}$$

Now applying Lemma 2.3.1, we get

$$\|Q^{\frac{1}{4}-\frac{\theta}{2}}aQ^{-\frac{1}{4}+\frac{\theta}{2}}\|_\infty \leq \|m_a^R\|_{B(L^p(\mathbb{G}))}^{\frac{1}{2}} \|m_{Q^{-\theta}aQ^\theta}^L\|_{B(L^p(\mathbb{G}))}^{\frac{1}{2}},$$

as desired.  $\square$

In particular, we may state the following corollary.

**Corollary 2.3.6.** *Assume that  $\mathbb{G}$  is of Kac type. Let  $1 \leq p \leq \infty$ . For all  $a \in \prod_\pi B(H_\pi)$ , we have*

$$\|a\|_\infty \leq \|a\|_{M(L^p(\mathbb{G}))}.$$



# Chapter 3

## Convolutions of states and $L^p$ -improving operators

In this chapter we aim to characterize positive convolution operators on a finite quantum group  $\mathbb{G}$  which are  $L^p$ -improving. More precisely, we prove that the convolution operator  $T_\varphi : x \mapsto \varphi \star x$  given by a state  $\varphi$  on  $C(\mathbb{G})$  satisfies

$$\exists 1 < p < 2, \quad \|T_\varphi : L^p(\mathbb{G}) \rightarrow L^2(\mathbb{G})\| = 1$$

if and only if the Fourier series  $\hat{\varphi}$  satisfy  $\|\hat{\varphi}(\pi)\| < 1$  for all nontrivial irreducible unitary representations  $\pi$ , if and only if the state  $(\varphi \circ S) \star \varphi$  is non-degenerate (where  $S$  is the antipode). We also prove that these  $L^p$ -improving properties are stable under taking free products, which gives a method to construct  $L^p$ -improving multipliers on infinite compact quantum groups. Our methods for describing non-degenerate states yield a general formula for computing idempotent states associated to Hopf images, which generalizes earlier work [BFS12] of Banica, Franz and Skalski.

### 3.1 $L^p$ -improving operators and spectral gaps

Let  $A$  be a finite dimensional  $C^*$ -algebra equipped with a faithful tracial state  $\tau$ . The associated noncommutative  $L^p$ -spaces will be denoted by  $L^p(A)$ . For a subset  $E \subset A$ , we denote by  $E_+$  the positive part of  $E$ .

Recall that  $A$  can be identified with a direct sum of matrix algebras, that is, there exist some finite dimensional Hilbert spaces  $H_1, \dots, H_m$  such that the following  $*$ -isomorphism holds

$$A \simeq B(H_1) \oplus \dots \oplus B(H_m).$$

We will not distinguish the above two  $C^*$ -algebras in the sequel. For each  $i \in \{1, \dots, m\}$ , let  $\xi_1^i, \dots, \xi_{n_i}^i$  be an orthonormal basis for  $H_i$ , and define the operator  $e_{pq}^i \in B(H_i)$  by  $e_{pq}^i(v) = \langle v, \xi_q^i \rangle_{H_i} \xi_p^i$  for all  $v \in H_i$  and  $p, q \in \{1, \dots, n_i\}$ . Take any  $x = x_1 \oplus \dots \oplus x_m \in A$  with  $x_i \in B(H_i)$  for each  $i \in \{1, \dots, m\}$ , and let  $\lambda_1^i, \dots, \lambda_{n_i}^i$  be the eigenvalues of  $|x_i| \in B(H_i)$  ( $1 \leq i \leq m$ ) ranged in non-increasing order and counted according to multiplicity. We can find a direct sum of unitaries  $u = u_1 \oplus u_2 \oplus \dots \oplus u_m$  with  $u_i \in B(H_i)$  for each  $i$  such that  $|x_i|(u_i \xi_k^i) = \lambda_k^i(u_i \xi_k^i)$  for all  $k \in \{1, \dots, n_i\}$  and  $i \in \{1, \dots, m\}$ , that is,  $u^*|x|u = \sum_i \sum_{k=1}^{n_i} \lambda_k^i e_{kk}^i$ . If we write  $\beta_k^i = \tau(e_{kk}^i) \in [0, 1]$  for  $k \in \{1, \dots, n_i\}$  and

$i \in \{1, \dots, m\}$ , then the  $L^p$ -norm of  $x$  for  $1 \leq p < \infty$  is

$$\|x\|_p^p = \tau(u^*|x|^p u) = \tau\left(\sum_{i=1}^m \sum_{k=1}^n (\lambda_k^i)^p e_{kk}^i\right) = \sum_{i=1}^m \sum_{k=1}^n (\lambda_k^i)^p \beta_k^i. \quad (3.1)$$

We will prove in this section the result below.

**Theorem 3.1.1.** *Let  $A$  be a finite dimensional  $C^*$ -algebra equipped with a faithful tracial state  $\tau$ , and  $T : A \rightarrow A$  be a unital 2-positive trace preserving map on  $A$ . Then*

$$\exists 1 \leq p < 2, \quad \|Tx\|_2 \leq \|x\|_p, \quad x \in A$$

if and only if

$$\sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\|Tx\|_2}{\|x\|_2} < 1. \quad (3.2)$$

*Remark 3.1.2.* Equivalently we can rewrite the above condition (3.2) as

$$\sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\langle |T|x, x \rangle}{\|x\|_2^2} < 1,$$

which means exactly that the whole eigenspace of  $|T|$  for the eigenvalue 1 is just  $\mathbb{C}1$ . In this sense we refer to the above inequality as a spectral gap phenomenon of  $T$ .

Recall that the  $L^2$ -norms assert some differential properties. The following lemma is elementary.

**Lemma 3.1.3.** *Let  $A$  be a  $C^*$ -algebra with a state  $\varphi$  and  $T : A \rightarrow A$  be a positive map on  $A$ . Let  $O \subset A_h$  be an open set in the space  $A_h$  of all selfadjoint elements in  $A$ . The function  $f : O \ni x \mapsto \varphi((Tx)^2)$  is infinitely (Fréchet) differentiable in  $O$  and for  $x \in O$ ,  $f'(x) = \varphi(TxT \cdot) + \varphi(T \cdot Tx)$ ,  $f'' \equiv 2\varphi(T \cdot T \cdot)$ ,  $f^{(n)} \equiv 0$ ,  $n \geq 3$ .*

In general a norm estimate can be reduced to the argument on positive cones.

**Lemma 3.1.4** ([RX16, Remark 9]). *Let  $\mathcal{M}$  be a von Neumann algebra and  $T : L^p(\mathcal{M}) \rightarrow L^q(\mathcal{M})$  be a bounded linear map for  $1 \leq p, q \leq \infty$ . Assume that  $T$  is 2-positive in the sense that  $\text{Id}_{\mathbb{M}_2} \otimes T$  maps the positive cone of  $L^p(\mathbb{M}_2 \otimes \mathcal{M})$  to that of  $L^q(\mathbb{M}_2 \otimes \mathcal{M})$ . Then*

$$\|Tx\|_q \leq \|T(|x|)\|_q^{1/2} \|T(|x^*|)\|_q^{1/2}, \quad x \in L^p(\mathcal{M}).$$

Consequently,

$$\|T\| = \sup\{\|Tx\|_q : x \in L^p(\mathcal{M})_+, \|x\|_p \leq 1\}.$$

Now we give the proof of the theorem.

*Proof of the theorem.* Assume firstly  $1 \leq p < 2$  and  $\|Tx\|_2 \leq \|x\|_p$  for all  $x \in A$ . Note that  $\|Tx\|_2 = \||T|x\|_2$ . Observe that  $T^*$  is also a positive trace preserving map on  $A$ , and hence so is  $|T|$ . We choose an element  $x \in A$  such that  $\tau(x) = 0$  and  $|T|x = \lambda x$ , with

$$\lambda = \sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\|Tx\|_2}{\|x\|_2} = \sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\||T|x\|_2}{\|x\|_2}.$$

Since  $|T|$  is a positive map on  $A$  and  $\lambda \in \mathbb{R}$ , we may assume that  $x = x^*$ . For any self-adjoint element  $y \in A$  and  $1 \leq q < \infty$ , it is easy to compute that

$$\left. \frac{d^2}{d\varepsilon^2} \|1 + \varepsilon y\|_q \right|_{\varepsilon=0} = (q-1)\tau(y^2) > 0.$$

Note also that by assumption

$$\|1 + \lambda \varepsilon x\|_2 \leq \|1 + \varepsilon x\|_p, \quad \varepsilon > 0.$$

Then taking the second derivative at  $\varepsilon = 0$  we get  $\lambda^2 \leq (p-1) < 1$ , as desired.

Now we suppose (3.2) holds. Set  $\mathring{A} = \{x \in A \mid \tau(x) = 0\}$  and take  $\sigma = \{x \in A_+ \mid \tau(x) = 1\} = (1 + \mathring{A})_+$  which is exactly the set of positive elements in the unit sphere of  $L^1(A)$ . We first show that there exists  $1 \leq p < 2$  and a neighborhood  $U$  of 1 such that

$$\forall x \in U \cap \sigma, \quad \|Tx\|_2 \leq \|x\|_p. \quad (3.3)$$

To begin with, we consider

$$F(x) = \|Tx\|_2 - \|x\|_2, \quad x \in A_+.$$

Using the previous lemma we see that  $F$  is infinitely differentiable at any  $x \in A_+ \setminus \{0\}$  and

$$\begin{aligned} F'(x)(y) &= \|Tx\|_2^{-1} \tau((Tx)(Ty)) - \|x\|_2^{-1} \tau(xy), \quad y \in A \\ F''(x)(y_1, y_2) &= -\|Tx\|_2^3 \tau((Tx)(Ty_1)) \tau((Tx)(Ty_2)) + \|Tx\|_2^{-1} \tau((Ty_1)(Ty_2)) \\ &\quad + \|x\|_2^{-3} \tau(xy_1) \tau(xy_2) - \|x\|_2^{-1} \tau(y_1 y_2), \quad y_1, y_2 \in A. \end{aligned}$$

Since  $T$  is unital and preserves the trace, it follows that for  $y \in \mathring{A}$ ,

$$F'(1)(y) = 0, \quad F''(1)(y, y) = \|Ty\|_2^2 - \|y\|_2^2.$$

Then consider the second order Taylor expansion of  $F$  at 1. We can find a  $\delta_1 > 0$  such that for all  $\|y\|_2 \leq \delta_1$ ,  $y \in \mathring{A}$ , we have  $1 + y \in A_+$  and

$$\begin{aligned} F(1 + y) &= F(1) + F'(1)(y) + \frac{1}{2} F''(1)(y, y) + R_1(y) \\ &= \frac{1}{2} (\|Ty\|_2^2 - \|y\|_2^2) + R_1(y), \quad R_1(y) = o(\|y\|_2^2). \end{aligned}$$

Recall that by (3.2),  $\|Ty\|_2^2 - \|y\|_2^2 < 0$  for  $y \in \mathring{A}$ . Thus by continuity,

$$c := \sup\{\|Ty\|_2^2 - \|y\|_2^2 : y \in \mathring{A}, \|y\|_2 = 1\} < 0.$$

Since the function  $y \mapsto \|Ty\|_2^2 - \|y\|_2^2$  is 2-homogeneous, we get

$$\forall y \in \mathring{A}, \quad \|Ty\|_2^2 - \|y\|_2^2 \leq c \|y\|_2^2.$$

Take  $\delta_0 \in (0, \delta_1)$  such that

$$\forall y \in \mathring{A}, \|y\|_2 \leq \delta_0, \quad \frac{|R_1(y)|}{\|y\|_2^2} < \frac{|c|}{4}.$$

Then for  $y \in \mathring{A}$ ,  $\|y\|_2 \leq \delta_0$ ,

$$(*) \quad F(1 + y) = \frac{1}{2} (\|Ty\|_2^2 - \|y\|_2^2) + R_1(y) \leq \frac{c}{4} \|y\|_2^2.$$

On the other hand, consider

$$G(x) = \|x\|_2 - \|x\|_p, \quad x = 1 + y, y = y^* \in A, \|y\|_2 < \delta_0.$$

Let  $y = y^* \in A$  with  $\|y\|_2 < \delta_0$ , then by (3.1) we may take some  $K \in \mathbb{N}$  and  $\beta_1, \dots, \beta_K \in [0, 1]$  such that the  $L^p$ -norm of  $x = 1 + y$  for  $1 \leq p < \infty$  is exactly

$$(**) \quad \|1 + y\|_p = \left( \sum_{i=1}^K \beta_i (1 + \lambda_i)^p \right)^{\frac{1}{p}}$$

where  $(\lambda_i)_i \subset \mathbb{R}$  is the list of eigenvalues of  $y$ . So in order to estimate  $G$ , we consider the function  $g$  on  $\mathbb{R}^K$  defined as

$$g(\xi) = \left( \sum_{i=1}^K \beta_i (1 + \xi_i)^2 \right)^{\frac{1}{2}} - \left( \sum_{i=1}^K \beta_i (1 + \xi_i)^p \right)^{\frac{1}{p}}, \quad \xi = (\xi_1, \dots, \xi_K) \in \mathbb{R}^K.$$

A straightforward calculation gives

$$\frac{\partial g}{\partial \xi_i}(0) = 0, \quad \frac{\partial^2 g}{\partial \xi_i \partial \xi_j}(0) = (p-2)\beta_i \beta_j, \quad \frac{\partial^2 g}{\partial \xi_i^2}(0) = (2-p)(\beta_i - \beta_i^2), \quad 1 \leq i \neq j \leq K.$$

So by the Taylor formula

$$g(\xi) = \frac{1}{2} \sum_i (2-p)(\beta_i - \beta_i^2) \xi_i^2 + \frac{1}{2} \sum_{j \neq k} (p-2)\beta_j \beta_k \xi_j \xi_k + R_2(\xi), \quad R_2(\xi) = o(\|\xi\|^2).$$

If  $2 - \frac{|c|}{8} \leq p \leq 2$  and  $0 < \delta < \delta_0$  is such that  $|R_2(\xi)| \leq \frac{|c|}{8} \sum_{i=1}^K \beta_i \xi_i^2$  whenever  $\sum_{i=1}^K \beta_i \xi_i^2 \leq \delta^2$ , then for any  $\xi \in \mathbb{R}^K$  with  $\sum_{i=1}^K \beta_i \xi_i^2 \leq \delta^2$ ,

$$|g(\xi)| \leq \frac{1}{2} (2-p) \sum_{i=1}^K (\beta_i - \beta_i^2) \xi_i^2 + \frac{1}{2} (2-p) \sum_{i=1}^K \beta_i^2 \xi_i^2 + \frac{|c|}{8} \sum_{i=1}^K \beta_i \xi_i^2 < \frac{|c|}{4} \sum_{i=1}^K \beta_i \xi_i^2.$$

This, together with (\*\*), implies that, putting  $\lambda = (\lambda_1, \dots, \lambda_K)$ ,

$$G(1 + y) = g(\lambda) \leq \frac{|c|}{4} \sum_{i=1}^K \beta_i \lambda_i^2 = \frac{|c|}{4} \|y\|_2^2, \quad \|y\|_2 \leq \delta.$$

Combined with (\*) we deduce

$$\|Tx\|_2 - \|x\|_p = F(1 + y) + G(1 + y) \leq 0, \quad x = 1 + y, \quad y \in \mathring{A}, \quad \|y\|_2 \leq \delta,$$

for all  $p \geq 2 - \frac{|c|}{8} := p_1$ . So  $U = \{1 + y \mid y = y^* \in A, \|y\|_2 < \delta\}$  is the desired neighborhood in (3.3).

Now we can derive the inequality for all  $x \in \sigma$ . For  $x \in \sigma \setminus U \subset (1 + \mathring{A})_+ \setminus \{1\}$ , we write  $x = 1 + y$  with  $y \in \mathring{A} \setminus \{0\}$  and then by (3.2) and the trace preserving property we have  $\|Tx\|_2^2 = 1 + \|Ty\|_2^2 < 1 + \|y\|_2^2 = \|x\|_2^2$ . Note also that  $\sigma$  is compact, so we can find  $M < 1$  such that  $\|Tx\|_2 / \|x\|_2 < M$  for all  $x \in \sigma \setminus U$ . Given  $p < 2$ , let  $C_p$  be the optimal constant for the inequality  $\|x\|_2 \leq C_p \|x\|_p$  for  $x \in A$ , then  $C_p \rightarrow 1$  when  $p \rightarrow 2$ . Take  $p_0 \geq p_1$  such that  $C_{p_0} \leq M^{-1}$ . We get then

$$\forall x \in \sigma \setminus U, \quad \frac{\|Tx\|_2}{\|x\|_p} \leq M C_p \leq 1, \quad p_0 \leq p \leq 2.$$

As a result, for all  $p \in [p_0, 2]$ , it holds that

$$\|Tx\|_2 \leq \|x\|_p, \quad x \in \sigma.$$

Since the norm is homogeneous and  $T$  is 2-positive, the above inequality holds for all  $x \in A$  as well.  $\square$

Apart from the above elementary proof, we would like to give an alternative simpler approach which yields a little bit stronger conclusion. The argument, however, depends heavily on the following recent and deep result on the convexity of  $L^p$ -spaces:

**Theorem 3.1.5** ([RX16, Theorem 1]). *Let  $\mathcal{M}$  be a von Neumann algebra equipped with a faithful semifinite normal trace  $\phi$ . Let  $\mathcal{N}$  be a von Neumann subalgebra such that the restriction of  $\phi$  to  $\mathcal{N}$  is semifinite. Denote by  $\mathcal{E}$  the unique  $\phi$ -preserving conditional expectation from  $\mathcal{M}$  onto  $\mathcal{N}$ . For  $1 < p \leq 2$ , we have*

$$\|x\|_p^2 \geq \|\mathcal{E}x\|_p^2 + (p-1)\|x - \mathcal{E}x\|_p^2, \quad x \in L^p(\mathcal{M}).$$

For  $2 < p < \infty$ , the inequality is reversed.

Immediately we may deduce Theorem 3.1.1 as follows. Note that the result below is slightly stronger than the statement of Theorem 3.1.1.

**Theorem 3.1.6.** *Let  $A$  be a finite dimensional  $C^*$ -algebra equipped with a faithful tracial state  $\tau$ , and  $T : A \rightarrow A$  be a unital positive trace preserving map on  $A$ . Then*

$$\exists 1 < p < 2, \forall x \in A, \|Tx\|_2 \leq \|x\|_p \tag{3.4}$$

if and only if

$$\sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\|Tx\|_2}{\|x\|_2} < 1.$$

*Proof.* The necessity has been already proved in the proof of Theorem 3.1.1. Now assume

$$\lambda := \sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\|Tx\|_2}{\|x\|_2} < 1.$$

Let  $x \in A$  and  $y = x - \tau(x)1$ . Write  $a = \tau(x)$ . Since  $T$  is trace preserving,  $\tau(Ty) = \tau(y) = 0$ . For  $p \leq 2$  we denote by  $c_p$  the best constant with  $\|\cdot\|_2 \leq c_p \|\cdot\|_p$ . Then  $(p-1)/c_p^2 \rightarrow 1$  when  $p \rightarrow 2$ . Take  $p < 2$  such that  $(p-1)/c_p^2 > \lambda^2$ , then we have

$$\begin{aligned} \|Tx\|_2^2 &= \|a1 + Ty\|_2^2 = |a|^2 + \|Ty\|_2^2 \leq |a|^2 + \lambda^2 \|y\|_2^2 \\ &\leq |a|^2 + \lambda^2 c_p^2 \|y\|_p^2 \leq |a|^2 + (p-1) \|y\|_p^2 \leq \|x\|_p^2, \end{aligned}$$

whence (3.4). □

*Remark 3.1.7.* Let  $A$  be a finite dimensional  $C^*$ -algebra equipped with a faithful tracial state  $\tau$ , and  $T : A \rightarrow A$  be a unital trace preserving map on  $A$ . Consider the restriction of  $T$  to the subspace  $\{x \in A : \tau(x) = 0\}$  of  $A$  and its adjoint, then we see that

$$\sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\|Tx\|_2}{\|x\|_2} = \sup_{x \in A \setminus \{0\}, \tau(x)=0} \frac{\|T^*x\|_2}{\|x\|_2}.$$

Then the above theorem also implies that if there exists  $1 < p < 2$  such that

$$\forall x \in A, \|Tx\|_2 \leq \|x\|_p,$$

then

$$\forall x \in A, \|T^*x\|_2 \leq \|x\|_p,$$

and equivalently for  $2 < q < \infty$  with  $1/p + 1/q = 1$ ,

$$\forall x \in A, \|Tx\|_q \leq \|x\|_2.$$



It is easy to see that the free product of unital trace preserving completely positive maps can be extended to the  $L^p$ -spaces using the interpolation between  $L^1$  and  $L^\infty$ . But in general it is a delicate problem for the extension of algebraic free product of unital trace preserving maps onto the associated  $L^p$ -spaces. Here we provide a method to construct unital trace preserving  $L^p$ -improving operators on the free product of finite-dimensional  $C^*$ -algebras. To see this we need the following trivial claim.

*Claim 3.1.8.* Let  $\mathcal{M}$  be a finite von Neumann algebra equipped with a faithful tracial state  $\tau$ . If the vectors  $e_1, \dots, e_m \in \mathcal{M}$  are orthonormal in  $L^2(\mathcal{M}, \tau)$  and we denote  $c = \max_{1 \leq k \leq m} \|e_k\|_\infty^2$ , then for  $\alpha_1, \dots, \alpha_m \in \mathbb{C}$  and  $2 \leq q \leq \infty$ ,

$$\left\| \sum_{k=1}^m \alpha_k e_k \right\|_q \leq (cm)^{\frac{1}{2} - \frac{1}{q}} \left\| \sum_{k=1}^m \alpha_k e_k \right\|_2.$$

*Proof.* Note that

$$\left\| \sum_{k=1}^m \alpha_k e_k \right\|_\infty \leq c^{1/2} \sum_{k=1}^m |\alpha_k| \leq c^{1/2} m^{1/2} \left( \sum_{k=1}^m |\alpha_k|^2 \right)^{1/2},$$

which gives the claim for  $q = \infty$ . The inequality for  $2 \leq q \leq \infty$  then follows from the Hölder inequality.  $\square$

**Theorem 3.1.9.** Let  $(A_i, \tau_i)$ ,  $1 \leq i \leq n$  be a finite family of finite dimensional  $C^*$ -algebras equipped with faithful tracial states  $\tau_i$  and set  $(\mathcal{A}, \tau) = \bar{*}_{1 \leq i \leq n} (A_i, \tau_i)$  to be the von Neumann algebraic free product. For each  $1 \leq i \leq n$ ,  $T_i : A_i \rightarrow A_i$  is a unital positive trace preserving map such that

$$\|T_i : L^p(A_i) \rightarrow L^2(A_i)\| = 1$$

for some  $1 < p < 2$ . Then the (algebraic) free product map  $T = \bar{*}_{1 \leq i \leq n} T_i$  on  $\bar{*}_{1 \leq i \leq n} A_i$  extends to a contractive map from  $L^{p'}(\mathcal{A})$  to  $L^2(\mathcal{A})$  for some  $1 < p' < 2$ .

*Proof.* By the previous theorem and remark,

$$\lambda = \max_{1 \leq i \leq n} \sup_{x \in \dot{A}_i} \frac{\|T_i x\|_2}{\|x\|_2} = \max_{1 \leq i \leq n} \sup_{x \in \dot{A}_i} \frac{\|T_i^* x\|_2}{\|x\|_2} < 1. \quad (3.5)$$

Consider  $R = T^*$  and  $R_i = T_i^*$  for all  $1 \leq i \leq n$ , then  $R = R_1 * \dots * R_n$ . By density, consider  $x \in \bar{*}_{1 \leq i \leq n} (A_i, \tau_i)$  in the algebraic free product and we will show that

$$\|Rx\|_q \leq \|x\|_2$$

for some  $q > 2$  independent of the choice of  $x$ . Now fix some  $r \geq 1$ . For each  $i$ , choose a family  $(e_k^{(i)})_{k=1}^{n_i}$  of eigenvectors of  $|R_i|$  which forms an orthonormal basis of  $\dot{A}_i$  under  $\tau_i$ , then  $E_r = \{e_{k_1}^{i_1} \dots e_{k_r}^{i_r} : 1 \leq k_j \leq n_j, 1 \leq j \leq r, i_1 \neq \dots \neq i_r\}$  forms an orthonormal basis of  $\oplus_{i_1 \neq \dots \neq i_r} \dot{A}_{i_1} \otimes \dots \otimes \dot{A}_{i_r}$  consisting of eigenvectors of  $|R|$ . Note that  $|E_r| \leq n^r m^r$  for  $m = \max_j n_j$ . Write additionally  $c = \max_{k,i} \|e_k^{(i)}\|_\infty^2$ . Then for any  $y_r \in \oplus_{i_1 \neq \dots \neq i_r} \dot{A}_{i_1} \otimes \dots \otimes \dot{A}_{i_r}$  the above claim yields

$$\|y_r\|_q \leq (cnm)^{r(\frac{1}{2} - \frac{1}{q})} \|y_r\|_2. \quad (3.6)$$

Write  $x = \tau(x)1 + \sum_{r \geq 1} x_r$  where  $x_r \in \oplus_{i_1 \neq \dots \neq i_r} \dot{A}_{i_1} \otimes \dots \otimes \dot{A}_{i_r}$ . Note that  $\|Rx_r\|_2 \leq \lambda^r \|x_r\|_2$  according to (3.5) and the choice of  $E_r$ . Together with Theorem 3.1.5 and (3.6),

$$\begin{aligned} \|Rx\|_q^2 &\leq |\tau(x)|^2 + (q-1) \left\| \sum_{r \geq 1} Rx_r \right\|_q^2 \leq |\tau(x)|^2 + (q-1) \left( \sum_{r \geq 1} \|Rx_r\|_q \right)^2 \\ &\leq |\tau(x)|^2 + (q-1) \left( \sum_{r \geq 1} (cnm)^{r(\frac{1}{2}-\frac{1}{q})} \|Rx_r\|_2 \right)^2 \\ &\leq |\tau(x)|^2 + (q-1) \left( \sum_{r \geq 1} (cnm)^{r(\frac{1}{2}-\frac{1}{q})} \lambda^r \|x_r\|_2 \right)^2. \end{aligned}$$

Observe that  $(q-1)(cnm)^{\frac{1}{2}-\frac{1}{q}}$  tends to 1 whenever  $q \rightarrow 2$  and that  $\lambda < 1$ , so we may choose  $2 < q < \infty$  such that  $\lambda(cn)^{\frac{1}{2}-\frac{1}{q}} \leq (q-1)^{-1}$ . For such a  $q$  we then have

$$\begin{aligned} \|Rx\|_q^2 &\leq |\tau(x)|^2 + (q-1) \left( \sum_{r \geq 1} (q-1)^{-r} \|x_r\|_2 \right)^2 \\ &\leq |\tau(x)|^2 + (q-1) \sum_{k \geq 1} (q-1)^{-2k} \sum_{r \geq 1} \|x_r\|_2^2 \\ &< |\tau(x)|^2 + \sum_{r \geq 1} \|x_r\|_2^2 = \|x\|_2^2. \end{aligned}$$

Take  $1 < p' < 2$  such that  $1/p' + 1/q = 1$ . Then we get  $\|T : L^{p'}(\mathcal{A}) \rightarrow L^2(\mathcal{A})\| = 1$ .  $\square$

## 3.2 Non-degenerate states and applications to Hopf images

In this short section we give the key lemma on non-degenerate states, which will be of use for our main results. We need the following observation adapted from [Wor98, Lemma 2.1]. The result is mentioned in [Sol05].

**Lemma 3.2.1.** *Let  $\mathbb{G}$  be a compact quantum group and  $A = C(\mathbb{G})$ . Suppose that  $(\rho_i)_{i \in I}$  is a family of states on  $A$  separating the points of  $A_+$ , i.e.,  $\forall x \in A_+ \setminus \{0\}, \exists i \in I, \rho_i(x) > 0$ . If  $\rho$  is a state on  $A$  such that*

$$\forall i \in I, \quad \rho \star \rho_i = \rho_i \star \rho = \rho,$$

*then  $\rho$  is the Haar state  $h$  of  $\mathbb{G}$ .*

*Proof.* Set

$$\mathcal{I} = \{q \in A \otimes A : \forall i \in I, (\rho_i \otimes \rho)(q^*q) = 0\}.$$

Then  $\mathcal{I}$  is a closed left ideal of  $A \otimes A$ . Define

$$\Psi_L(x) = (\rho \otimes \iota)\Delta(x) - \rho(x)1, \quad x \in A.$$

Since  $\Psi_L$  is a difference of two unital completely positive maps, we see that  $\Psi_L$  is a completely bounded map with norm at most 2. We will prove that

$$(\Psi_L \otimes \iota)\Delta(A) \subset \mathcal{I}.$$

In fact, given  $x \in A$ , by the coassociativity of  $\Delta$  we have

$$\begin{aligned} q &:= (\Psi_L \otimes \iota)\Delta(x) = (\rho \otimes \iota \otimes \iota)(\iota \otimes \Delta)\Delta(x) - 1 \otimes [(\rho \otimes \iota)\Delta(x)] \\ &= \Delta((\rho \otimes \iota)\Delta(x)) - 1 \otimes [(\rho \otimes \iota)\Delta(x)]. \end{aligned}$$

Thus

$$\begin{aligned} q^*q &= \Delta([( \rho \otimes \iota)\Delta(x)]^*(\rho \otimes \iota)\Delta(x)) - \Delta((\rho \otimes \iota)\Delta(x))^*[1 \otimes (\rho \otimes \iota)\Delta(x)] \\ &\quad - [1 \otimes ((\rho \otimes \iota)\Delta(x))^*]\Delta((\rho \otimes \iota)\Delta(x)) + 1 \otimes ([(\rho \otimes \iota)\Delta(x)]^*[(\rho \otimes \iota)\Delta(x)]) \end{aligned}$$

and hence for any  $i \in I$  we may write

$$(\rho_i \otimes \rho)(q^*q) = q_1 - q_2 - q_3 + q_4$$

where by the convolution invariance assumption and the coassociativity of  $\Delta$  we have

$$\begin{aligned} q_1 &= (\rho_i \otimes \rho)\Delta([( \rho \otimes \iota)\Delta(x)]^*(\rho \otimes \iota)\Delta(x)) = \rho([( \rho \otimes \iota)\Delta(x)]^*(\rho \otimes \iota)\Delta(x)), \\ q_2 &= q_3^*, \\ q_3 &= (\rho_i \otimes \rho)([1 \otimes ((\rho \otimes \iota)\Delta(x))^*]\Delta((\rho \otimes \iota)\Delta(x))) \\ &= \rho([( \rho \otimes \iota)\Delta(x)]^*(\rho_i \otimes \iota)((\rho \otimes \iota \otimes \iota)(\iota \otimes \Delta)\Delta(x))) \\ &= \rho([( \rho \otimes \iota)\Delta(x)]^*(\rho \otimes \rho_i \otimes \iota)((\Delta \otimes \iota)\Delta(x))) = \rho([( \rho \otimes \iota)\Delta(x)]^*(\rho \otimes \iota)\Delta(x)), \\ q_4 &= (\rho_i \otimes \rho)(1 \otimes [(\rho \otimes \iota)\Delta(x)]^*(\rho \otimes \iota)\Delta(x)) = \rho([( \rho \otimes \iota)\Delta(x)]^*(\rho \otimes \iota)\Delta(x)). \end{aligned}$$

Note that  $q_1 = q_2 = q_3 = q_4$ . So  $(\rho_i \otimes \rho)(q^*q) = 0$  and  $(\Psi_L \otimes \iota)\Delta(A) \subset \mathcal{I}$  is proved.

Now by the density of  $(1 \otimes A)\Delta(A)$  in  $A \otimes A$  and the complete boundedness of  $\Psi_L$ , it follows that  $\Psi_L(A) \otimes 1 \subset \overline{(1 \otimes A)(\Psi_L \otimes \iota)\Delta(A)}$  is also contained in the closed left ideal  $\mathcal{I}$ , which means that for any  $i \in I$  and  $x \in A$ ,

$$\rho_i(\Psi_L(x)^*\Psi_L(x)) = \rho_i \otimes \rho(\Psi_L(x)^*\Psi_L(x) \otimes 1) = 0.$$

Recall that  $(\rho_i)_{i \in I}$  separates the points of  $A_+$ , so we have  $\Psi_L(x) = 0$  and  $(\rho \otimes \iota)\Delta(x) = \rho(x)1$  for all  $x \in A$ .

A similar argument applies as well to the map  $\Psi_R(x) = (\iota \otimes \rho)\Delta(x) - \rho(x)1$ ,  $x \in A$ . So  $\rho = h$  is the Haar state.  $\square$

*Remark 3.2.2.* We remark that in case  $\mathbb{G}$  is a finite quantum group, we can provide a simpler proof of the above lemma. Indeed, since  $C(\mathbb{G})$  is finite-dimensional, its dual space  $C(\mathbb{G})^*$  is also finite-dimensional, and we can take a maximal linear independent family  $\{\rho_{i_1}, \dots, \rho_{i_s}\} \subset (\rho_i)_{i \in I}$  which forms a basis of the subspace spanned by  $(\rho_i)_{i \in I}$  in  $C(\mathbb{G})^*$ . Given a nonzero  $x \in A_+$ , there is an  $i \in I$  such that  $\rho_i(x) > 0$ . Write  $\rho_i = \sum_{k=1}^s a_k \rho_{i_k}$  ( $a_k \in \mathbb{C}$ ), then we see clearly that there exists at least one  $k \in \{1, \dots, s\}$  such that  $\rho_{i_k}(x) \neq 0$  in order that  $\rho_i(x) > 0$ . Then  $\rho' = \frac{1}{s} \sum_{k=1}^s \rho_{i_k}$  is faithful on  $C(\mathbb{G})$  and  $\rho \star \rho' = \rho' \star \rho = \rho$ , thus  $\rho$  is the Haar state by [Wor98, Lemma 2.1].

We immediately obtain the following fact.

**Lemma 3.2.3.** *Let  $\mathbb{G}$  be a compact quantum group and  $\varphi$  be a state on  $C(\mathbb{G})$ . If  $\varphi$  is non-degenerate on  $C(\mathbb{G})$  in the sense that for all nonzero  $x \in C(\mathbb{G})_+$  there exists  $k \geq 0$  such that  $\varphi^{\star k}(x) > 0$ , then*

$$w^* \text{-} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{\star k} = h.$$

*If additionally  $h$  is faithful on  $C(\mathbb{G})$ , then the converse also holds.*

*Proof.* If the above limit holds and  $h$  is faithful on  $C(\mathbb{G})$ , then clearly  $\varphi$  is non-degenerate since if there existed a nonzero  $x \geq 0$  such that  $\varphi^{*n}(x) = 0$  for all  $n$ , then we would have  $\lim_n \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(x) = 0$ , which contradicts the faithfulness of  $h$ . On the other hand, if  $\varphi$  is non-degenerate, the family of states  $\{\frac{1}{n} \sum_{k=1}^n \varphi^{*k} : n \geq 1\}$  separates the points of  $C(\mathbb{G})_+$ , so any accumulation point of this family becomes the unique Haar state by our previous lemma.  $\square$

We would like to digress momentarily to see an application of above lemmas to some problems concerning Hopf images. Let  $A$  be a unital  $C^*$ -algebra and  $\pi : C(\mathbb{G}) \rightarrow A$  be a unital  $*$ -homomorphism. The *Hopf image* of  $\pi$ , firstly introduced by Banica and Bichon in [BB10], is the largest algebra  $C(\mathbb{G}_\pi)$  for some compact quantum subgroup  $\mathbb{G}_\pi \subset \mathbb{G}$  such that  $\pi$  factorizes through  $C(\mathbb{G}_\pi)$ . In this paper we will use the following equivalent characterization recently given in [SS16]: for each  $k$  consider

$$\pi_k = \underbrace{(\pi \otimes \cdots \otimes \pi)}_k \circ \Delta^{(k-1)} : C(\mathbb{G}) \rightarrow A^{\otimes k}$$

and let  $\mathcal{I} = \bigcap_{k=1}^{\infty} \ker \pi_k$ , then there exists a compact quantum group  $\mathbb{G}_\pi = (B, \Delta_\pi)$  with an  $*$ -homomorphism  $\pi_q : B \rightarrow A$  such that

$$B = C(\mathbb{G})/\mathcal{I}, \quad \Delta_\pi \circ q = (q \otimes q) \circ \Delta, \quad \pi = \pi_q \circ q \quad (3.7)$$

where  $q : C(\mathbb{G}) \rightarrow C(\mathbb{G})/\mathcal{I}$  denotes the quotient map. The algebra  $B = C(\mathbb{G}_\pi)$  is exactly the Hopf image of  $\pi$ . Now let  $h_\mathbb{G}, h_{\mathbb{G}_\pi}$  be the Haar states on  $\mathbb{G}, \mathbb{G}_\pi$  respectively. A related question raised in [BFS12] is the computation of the associated idempotent state  $h_{\mathbb{G}_\pi} \circ q$  on  $\mathbb{G}$ . Simply based on Lemma 3.2.3, the following property generalizes the main result of [BFS12].

**Theorem 3.2.4.** *Let  $\mathbb{G}$  be a compact quantum group and  $A$  be a unital  $C^*$ -algebra with a unital  $*$ -homomorphism  $\pi : C(\mathbb{G}) \rightarrow A$ . Let  $\mathbb{G}_\pi$  be the compact quantum group constructed above. Then given any faithful state  $\varphi$  on  $A$ ,*

$$h_{\mathbb{G}_\pi} \circ q = w^* \text{-} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\varphi \circ \pi)^{*k}.$$

*Proof.* Let  $\varphi$  be a faithful state on  $A$  and let  $\mathcal{I} = \bigcap_{k=1}^{\infty} \ker \pi_k$  with  $\pi_k, k \geq 1$  constructed as above. Let us show that  $\varphi \circ \pi_q$  is non-degenerate on  $B = C(\mathbb{G}_\pi)$ . Consider any  $x = q(y)$  with  $y \in C(\mathbb{G})_+$  satisfying

$$\forall k \geq 1, \quad (\varphi \circ \pi_q)^{*k}(x) = 0.$$

Since, by (3.7),

$$\begin{aligned} (\varphi \circ \pi_q)^{*k} &= [(\varphi \otimes \cdots \otimes \varphi) \circ (\pi_q \otimes \cdots \otimes \pi_q)] \Delta^{(k-1)}(x) \\ &= [(\varphi \otimes \cdots \otimes \varphi) \circ ((\pi_q \circ q) \otimes \cdots \otimes (\pi_q \circ q))] \Delta^{(k-1)}(y) = (\varphi \otimes \cdots \otimes \varphi)(\pi_k(y)) \end{aligned}$$

and since  $\varphi$  is faithful, we get  $y \in \bigcap_{k=1}^{\infty} \ker \pi_k$ , which means that  $x = q(y) = 0$ . As a result  $\varphi \circ \pi_q$  is non-degenerate. Therefore we have  $h_{\mathbb{G}_\pi} = w^* \text{-} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\varphi \circ \pi_q)^{*k}$ , and hence using (3.7) again,

$$h_{\mathbb{G}_\pi} \circ q = w^* \text{-} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\varphi \circ \pi_q)^{*k} \circ q = w^* \text{-} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\varphi \circ \pi)^{*k},$$

as desired.  $\square$

### 3.3 $L^p$ -improving convolutions operators on finite quantum groups

In this section we aim to give several characterizations of  $L^p$ -improving convolutions given by states on finite quantum groups, and also give the constructions for the free product of finite quantum groups. We will start with some discussions on multipliers on compact quantum groups. In this section we keep the notation of multipliers  $m_a^L$ ,  $m_a^R$  and convolutions  $\varphi_1 \star \varphi_2$  given in Chapter 2.

**Lemma 3.3.1.** *Let  $\mathbb{G}$  be a compact quantum group of Kac type. Suppose  $a \in \ell_\infty(\widehat{\mathbb{G}})$  such that  $m_a^L$  (resp.  $m_a^R$ ) extends to a unital left (resp., right) multiplier on  $L^2(\mathbb{G})$  and  $b = aa^*$ . Then  $\lim_n \frac{1}{n} \sum_{k=1}^n (m_b^L)^k x = h(x)1$  for all  $x \in L^2(\mathbb{G})$  if and only if  $\lim_n \frac{1}{n} \sum_{k=1}^n (m_b^R)^k x = h(x)1$  for all  $x \in L^2(\mathbb{G})$  if and only if  $\|a_\pi\| < 1$  for all  $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}$ .*

*Proof.* Without loss of generality we only discuss the left multiplier  $m_a^L$ . Assume that  $\|a_\pi\| < 1$  for all  $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}$ . By Lemma 2.3.1 we note that  $m_b^L$  extends to a bounded map of norm one on  $L^2(\mathbb{G})$ . We first consider the case  $x \in \text{Pol}(\mathbb{G})$ , so that  $\hat{x}$  is finitely supported. Let  $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}$  and  $\|a_\pi\| < 1$ . Then

$$\left\| \left( \frac{1}{n} \sum_{k=1}^n (m_b^L)^k x \right)^\wedge(\pi) \right\|_2 = \left\| \frac{1}{n} \sum_{k=1}^n \hat{x}(\pi) (a_\pi a_\pi^*)^k \right\|_2 \leq \frac{1}{n} \sum_{k=1}^n \|a_\pi\|^{2k} \|\hat{x}(\pi)\|_2 \rightarrow 0$$

whenever  $n \rightarrow \infty$ . And for  $\pi = 1$ ,

$$\left( \frac{1}{n} \sum_{k=1}^n (m_b^L)^k x \right)^\wedge(1) = \hat{x}(1) = h(x).$$

Thus by the Plancherel theorem

$$\left\| \frac{1}{n} \sum_{k=1}^n (m_b^L)^k x - h(x)1 \right\|_2^2 = \sum_{\pi \neq 1} d_\pi \left\| \left( \frac{1}{n} \sum_{k=1}^n (m_b^L)^k x \right)^\wedge(\pi) \right\|_2^2 \rightarrow 0$$

when  $n \rightarrow \infty$ . Since  $\frac{1}{n} \sum_{k=1}^n (m_b^L)^k$  is a contraction on  $L^2(\mathbb{G})$  and  $\text{Pol}(\mathbb{G})$  is dense in  $L^2(\mathbb{G})$ , we get the convergence  $\lim_n \frac{1}{n} \sum_{k=1}^n (m_b^L)^k x = h(x)1$  for all  $x \in L^2(\mathbb{G})$ .

Conversely, if  $\exists \pi_0 \in \text{Irr}(\mathbb{G}) \setminus \{1\}$ ,  $\|a_{\pi_0}\| = 1$ , then viewing  $b_{\pi_0}$  as a matrix in  $\mathbb{M}_{n_{\pi_0}}$ , we observe that  $1 \in \sigma(b_{\pi_0})$  and there exists a nonzero  $x_{\pi_0} \in \mathbb{M}_{n_{\pi_0}}$  such that  $x_{\pi_0} b_{\pi_0} = x_{\pi_0}$ . Take  $x \in L^2(\mathbb{G})$  such that  $\hat{x}(1) = 1$ ,  $\hat{x}(\pi_0) = x_{\pi_0}$ ,  $\hat{x}(\pi) = 0$  for  $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1, \pi_0\}$ . Then  $m_b^L x = x$  and hence  $\frac{1}{n} \sum_{k=1}^n (m_b^L)^k x = x$  does not converge to  $h(x)1$ .  $\square$

*Remark 3.3.2.* In case the compact quantum group  $\mathbb{G}$  is not of Kac type, the above argument still remains valid for right multipliers.

The following first main result is now in reach. We will consider the case where  $\mathbb{G}$  is a finite quantum group.

**Theorem 3.3.3.** *Let  $\mathbb{G}$  be a finite quantum group. Suppose  $a \in \ell_\infty(\widehat{\mathbb{G}})$  is such that  $m_a^L$  (resp.,  $m_a^R$ ) is a unital left (resp., right) multiplier on  $C(\mathbb{G})$ . Then the following assertions are equivalent:*

- (1) *there exists  $1 \leq p < 2$  such that,*

$$\forall x \in C(\mathbb{G}), \|m_a^L x\|_2 \leq \|x\|_p ;$$

(2) there exists  $1 \leq p < 2$  such that,

$$\forall x \in C(\mathbb{G}), \|m_a^R x\|_2 \leq \|x\|_p;$$

(3)  $\|a_\pi\| < 1$  for all  $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}$ ;

(4)  $\lim_n \frac{1}{n} \sum_{k=1}^n (m_b^L)^k x = h(x)1$  for all  $x \in C(\mathbb{G})$  when  $b = aa^*$ ;

(5)  $\lim_n \frac{1}{n} \sum_{k=1}^n (m_b^R)^k x = h(x)1$  for all  $x \in C(\mathbb{G})$  when  $b = aa^*$ .

*Proof.* Without loss of generality we only discuss the left multiplier  $m_a^L$  and prove the equivalence (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

It is easy to see from Plancherel's theorem that (3) is just (3.2) for  $T = m_a^L$ . In fact note that for  $x \in C(\mathbb{G})$ ,  $h(x) = 0$  if and only if  $\hat{x}(1) = 0$ , so (3) implies (3.2) via Plancherel's theorem. On the other hand, suppose by contradiction that there exists  $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}$  such that  $\|a_\pi\| = 1$ . By Proposition 2.1.2 we may take a nonzero  $x \in C(\mathbb{G})$  such that  $\hat{x}(1) = 0$ ,  $\hat{x}(\pi) = 0$  when  $\|a_\pi\| < 1$ , and  $\|\hat{x}(\pi)a_\pi\|_2 = \|\hat{x}(\pi)\|_2$  when  $\|a_\pi\| = 1$ . Then  $\|m_a^L x\|_2 = \|x\|_2$  with  $h(x) = 0$ . As a consequence the equivalence (1) $\Leftrightarrow$ (3) follows from Theorem 3.1.6.

The equivalence between (3) and (4) was proved in the previous lemma. Therefore the theorem is established.  $\square$

Now we turn to the corresponding convolution problems. Let  $\varphi \in C(\mathbb{G})^*$  for a compact quantum group  $\mathbb{G}$ . Recall the formula (2.10), and then we have

$$\left[ \varphi^{*n}((u_{ji}^{(\pi)})^*) \right] = (\varphi^{*n})^\wedge(\pi) = \hat{\varphi}(\pi)^n. \quad (3.8)$$

Note that the convergence  $\frac{1}{n} \sum_{k=1}^n (m_{\hat{\varphi}^*}^L)^k(x) \rightarrow h(x)1$  for all  $x \in \text{Pol}(\mathbb{G})$ , by (2.15) can be reformulated in terms of Fourier coefficients as

$$\begin{aligned} \left( \frac{1}{n} \sum_{k=1}^n (m_{\hat{\varphi}^*}^L)^k(x) \right)^\wedge(1) &= h(x)\hat{\varphi}(1)^* = h(x)1, \\ \lim_n \left( \frac{1}{n} \sum_{k=1}^n (m_{\hat{\varphi}^*}^L)^k(x) \right)^\wedge(\pi) &= \lim_n \frac{1}{n} \sum_{k=1}^n (\hat{\varphi}(\pi)^*)^k \hat{x}(\pi) = 0, \quad \pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}. \end{aligned}$$

This is to say,

$$\hat{\varphi}(1) = 1, \quad \lim_n \frac{1}{n} \sum_{k=1}^n \hat{\varphi}(\pi)^k = 0, \quad \pi \in \text{Irr}(\mathbb{G}) \setminus \{1\},$$

which, according to (3.8), is equivalent to  $\frac{1}{n} \sum_{k=1}^n \varphi^{*k}(u_{ij}^{(\pi)}) \rightarrow h(u_{ij}^{(\pi)})$  for all  $\pi \in \text{Irr}(\mathbb{G})$  and  $1 \leq i, j \leq n_\pi$ , or in other words,

$$\frac{1}{n} \sum_{k=1}^n \varphi^{*k}(x) \rightarrow h(x), \quad n \rightarrow \infty, \quad x \in \text{Pol}(\mathbb{G}).$$

Any state  $\varphi$  on  $C(\mathbb{G})$  induces two convolution operators on  $C(\mathbb{G})$

$$T_\varphi : C(\mathbb{G}) \ni x \mapsto x \star \varphi = (\varphi \otimes \iota)\Delta(x), \quad T'_\varphi : C(\mathbb{G}) \ni x \mapsto \varphi \star x = (\iota \otimes \varphi)\Delta(x).$$

If additionally  $\mathbb{G}$  is of Kac type and the Haar state is faithful on  $C(\mathbb{G})$ , then by Proposition 1.2.2 the antipode  $S$  extends to a positive linear operator on  $C(\mathbb{G})$  and  $S = S^{-1}$ , and hence  $\varphi \circ S = \varphi \circ S^{-1}$  is also a state. In this case by (2.11) we have

$$(x \star \varphi)^\wedge(\pi) = \hat{x}(\pi)\hat{\varphi}(\pi)^*, \quad (\varphi \star x)^\wedge(\pi) = \hat{\varphi}(\pi)^* \hat{x}(\pi)$$

for  $\pi \in \text{Irr}(\mathbb{G})$ ,  $x \in C(\mathbb{G})$ . So

$$T_\varphi = m_{\hat{\varphi}^*}^L, \quad T'_\varphi = m_{\hat{\varphi}^*}^R$$

are unital completely positive left and right multipliers, respectively.

Now with these remarks and Lemma 3.2.3 in hand, we may reformulate Theorem 3.3.3 in terms of convolution operators using the above arguments.

**Theorem 3.3.4.** *Let  $\mathbb{G}$  be a finite quantum group and  $\varphi$  be a state on  $C(\mathbb{G})$ . Denote  $\psi = (\varphi \circ S) \star \varphi$ . The following assertions are equivalent:*

(1) *there exists  $1 \leq p < 2$  such that,*

$$\forall x \in C(\mathbb{G}), \|\varphi \star x\|_2 \leq \|x\|_p;$$

(2) *there exists  $1 \leq p < 2$  such that,*

$$\forall x \in C(\mathbb{G}), \|x \star \varphi\|_2 \leq \|x\|_p;$$

(3)  $\|\hat{\varphi}(\pi)\| < 1$  for all  $\pi \in \text{Irr}(\mathbb{G}) \setminus \{1\}$ ;

(4)  $\lim_n \frac{1}{n} \sum_{k=1}^n \psi^{\star k} = h$ ;

(5) *For any nonzero  $x \in C(\mathbb{G})_+$ , there exists  $n \geq 1$  such that  $\psi^{\star n}(x) > 0$ .*

Note that if  $C(\mathbb{G})$  is commutative, i.e.  $C(\mathbb{G}) = C(G)$  where  $G$  is a finite group, then  $\varphi \in C(G)^*$  corresponds to a Radon measure  $\mu$  via the Riesz representation theorem. The above condition (5) in the theorem just asserts that  $G$  is the union of  $D_n := \text{supp}(\nu^{\star n})$ ,  $n \geq 1$ , where  $\nu$  denotes the Radon measure corresponding to  $\psi$ . It is easy to see that  $D_1 = \{i^{-1}j : i, j \in \text{supp}(\mu)\}$  and  $D_n = D_1^n$ . So the above corollary covers Ritter's result [Rit84].

**Corollary 3.3.5.** *Let  $G$  be a finite group and  $\mu$  be a probability measure on  $G$ . Then there is a  $1 \leq p < 2$  such that*

$$\|x \star \mu\|_2 \leq \|x\|_p, \quad x \in L^p(G)$$

*if and only if  $G$  is equal to the subgroup generated by  $\{i^{-1}j : i, j \in \text{supp}(\mu)\}$ .*

On the other hand, let  $\Gamma$  be a finite group with neutral element  $e$  and  $C^*(\Gamma)$  be the associated group C\*-algebra generated by  $\lambda(\Gamma) \subset B(\ell_2(\Gamma))$ , where  $\lambda$  denotes the left regular representation. Recall that if  $\mathbb{G} = \hat{\Gamma}$ , then  $C(\mathbb{G})$  is the group C\*-algebra  $C^*(\Gamma)$  equipped with the comultiplication  $\Delta : C^*(\Gamma) \rightarrow C^*(\Gamma) \otimes C^*(\Gamma)$  defined by

$$\Delta(\lambda(\gamma)) = \lambda(\gamma) \otimes \lambda(\gamma), \quad \gamma \in \Gamma.$$

Note that any state  $\Phi$  on  $C(\mathbb{G})$  corresponds to a positive definite function  $\varphi$  on  $\Gamma$  with  $\varphi(e) = 1$  via the relation  $\Phi(\lambda(\gamma)) = \varphi(\gamma)$  for all  $\gamma \in \Gamma$ . Therefore we have

$$\Phi \star \lambda(\gamma) = \lambda(\gamma) \star \Phi = (\Phi \otimes \iota)\Delta(\lambda(\gamma)) = (\Phi \otimes \iota)(\lambda(\gamma) \otimes \lambda(\gamma)) = \varphi(\gamma)\lambda(\gamma),$$

so the convolution operators associated to  $\Phi$  are just the Fourier-Schur multipliers on  $\Gamma$  associated to  $\varphi$ . Our precedent argument in particular yields the following result extending [Rit84, Theorem 2(a)].

**Corollary 3.3.6.** *Let  $\Gamma$  be a finite group and  $\varphi$  be a positive definite function on  $\Gamma$  with  $\varphi(e) = 1$ . Let  $M_\varphi$  be the associated Fourier-Schur multiplier operator determined by  $M_\varphi(\lambda(\gamma)) = \varphi(\gamma)\lambda(\gamma)$  for all  $\gamma \in \Gamma$ . Then there exists  $1 \leq p < 2$  such that*

$$\|M_\varphi x\|_2 \leq \|x\|_p, \quad x \in C^*(\Gamma)$$

*if and only if  $|\varphi(\gamma)| < 1$  for any  $\gamma \in \Gamma \setminus \{e\}$ .*

*Remark 3.3.7.* We remark that the finite-dimensional condition cannot be removed in any of our results, including Theorem 3.1.1, Theorem 3.3.3, Corollary 3.3.4-3.3.6. Here we give a counterexample illustrating this. Let  $\mathbb{T}$  be the unit circle in the complex plane, then  $\mathbb{T}$  gives an infinite compact quantum group. Define an operator  $T : C(\mathbb{T}) \rightarrow C(\mathbb{T})$  by

$$T(f) = (1 - \lambda)\tau(f) + \lambda f, \quad f \in C(\mathbb{T}),$$

where  $0 < \lambda < 1$  and  $\tau$  denotes the usual integral against the normalized Lebesgue measure on  $\mathbb{T}$ . Then  $T$  is obviously unital completely positive since so are  $\tau$  and the identity map. It is a left multiplier satisfying  $T(f)\hat{\ }^{\wedge}(0) = \hat{f}(0)$  and  $T(f)\hat{\ }^{\wedge}(n) = \lambda\hat{f}(n)$  for  $n \neq 0$ . Also we may view  $T$  as a convolution operator associated to the state  $f \mapsto (1 - \lambda)\tau(f) + \lambda f(1)$  on  $C(\mathbb{T})$ , which is faithful since  $\tau$  is faithful. Note that  $T$  admits the spectral gap inequality (3.2) as well, and in fact,  $\|Tf\|_2 = \lambda\|f\|_2 < \|f\|_2$  for any  $f \in C(\mathbb{T})$  with  $\tau(f) = 0$ . However, there doesn't exist any  $p < 2$  such that  $\|Tf\|_2 \leq \|f\|_p$  for all  $f \in C(\mathbb{T})$ . Indeed if such a  $p$  existed, then for any  $f \in C(\mathbb{T})$ , we would have

$$\begin{aligned} \|f\|_2^2 &\geq \|f\|_p^2 \geq \|Tf\|_2^2 = \tau(f)^2 + \lambda^2\|f - \tau(f)\|_2^2 \\ &\geq \lambda^2(\tau(f)^2 + \|f - \tau(f)\|_2^2) = \lambda^2\|f\|_2^2, \end{aligned}$$

which yields an impossible equivalence between the norms  $\|\cdot\|_2$  and  $\|\cdot\|_p$ .

In spite of the above general remark, Theorem 3.1.9 still gives constructions of  $L^p$ -improving positive convolution operators on infinite compact quantum groups. Let  $\mathbb{G}_1, \dots, \mathbb{G}_n$  be finite quantum groups and let each  $\varphi_i$  be a state on  $C(\mathbb{G}_i)$ ,  $i \in \{1, \dots, n\}$ . Denote  $\mathbb{G} = \mathbb{G}_1 \hat{*} \dots \hat{*} \mathbb{G}_n$  with the Haar state  $h$  and consider the convolution operators  $T_i : x \mapsto x \star \varphi_i$ ,  $x \in C(\mathbb{G}_i)$ . Note that the free product map  $T = *_{1 \leq i \leq n} T_i$  on  $C(\mathbb{G})$  is just the convolution operator given by the c-free product state  $\varphi = *(_{h_1, \dots, h_n}) \varphi_i$ , i.e.,

$$T(x) = (\varphi \otimes \iota)\Delta(x), \quad x \in C(\mathbb{G}). \quad (3.9)$$

In fact, we note that if  $h(a) = 0$ , then  $h(a_{(1)}) = h(a_{(2)}) = 0$  by (1.2.1), where  $\Delta(a) := \sum a_{(1)} \otimes a_{(2)}$  is the Sweedler notation. Now for a reduced word  $x = x^1 \dots x^m$  with  $x^k \in C(\mathbb{G}_{i_k})$  such that  $h(x^k) = 0$ ,  $i_1 \neq \dots \neq i_n$ ,  $i_k \in \{1, \dots, n\}$  for each  $k = 1, \dots, m$ , we have

$$\begin{aligned} T(x) &= T_{i_1}(x^1) \cdots T_{i_m}(x^m) = \sum \varphi_{i_1}(x_{(1)}^1)x_{(2)}^1 \cdots \sum \varphi_{i_m}(x_{(1)}^m)x_{(2)}^m \\ &= \sum \varphi(x_{(1)})x_{(2)} = (\varphi \otimes \iota)\Delta(x) \end{aligned}$$

where we have used the fact that the comultiplication  $\Delta$  is a homomorphism. Then the equality (3.9) follows from a standard density argument. Now taking in Theorem 3.1.9 each  $T_i$  to be a convolution operator on a finite quantum group, we get the following corollary.

**Corollary 3.3.8.** *Let  $\mathbb{G}_1, \dots, \mathbb{G}_n$  be finite quantum groups and let each  $\varphi_i$  be a state on  $C(\mathbb{G}_i)$ ,  $i \in \{1, \dots, n\}$ . Denote  $\mathbb{G} = \mathbb{G}_1 \hat{*} \dots \hat{*} \mathbb{G}_n$  and  $\varphi = *(_{h_1, \dots, h_n}) \varphi_i$ . If each  $\varphi_i$  satisfies*



any one of the conditions (1)-(5) in Corollary 3.3.4, then the free product convolution operator given by  $T : x \mapsto x \star \varphi$ ,  $x \in C(\mathbb{G})$  is a unital left multiplier on  $\mathbb{G}$  satisfying

$$\|T : L^p(\mathbb{G}) \rightarrow L^2(\mathbb{G})\| = 1$$

for a certain  $1 < p < 2$ .

**Example 3.3.9.** Now we give a simple method to create non-trivial  $L^p$ -improving positive convolutions (i.e. the associated state is different from the Haar state) on finite and infinite compact quantum groups. Let  $\mathbb{G}$  be a finite quantum group and  $h$  the Haar state on it. To any state  $\varphi$  on  $C(\mathbb{G})$  and any  $0 < \lambda < 1$ , we can associate a state  $\rho$  on  $C(\mathbb{G})$  given by

$$\rho = \lambda\varphi + (1 - \lambda)h.$$

This is a faithful state which is in particular non-degenerate, and hence by Theorem 3.3.4 the convolution operator  $T_\rho : x \mapsto x \star \rho$ ,  $x \in C(\mathbb{G})$  satisfies

$$\|T_\rho : L^p(\mathbb{G}) \rightarrow L^2(\mathbb{G})\| = 1$$

for a certain  $1 < p < 2$  according to Theorem 3.3.4. Moreover by Corollary 3.3.8, the convolution operator  $T_{\rho'} : x \mapsto x \star \rho'$ ,  $x \in C(\mathbb{G} \hat{\ast} \mathbb{G})$  given by the free product state  $\rho' = \rho \ast \rho$  satisfies

$$\|T_{\rho'} : L^{p'}(\mathbb{G} \hat{\ast} \mathbb{G}) \rightarrow L^2(\mathbb{G} \hat{\ast} \mathbb{G})\| = 1$$

for some  $1 < p' < 2$ .

# Chapter 4

## Lacunarity

This chapter is devoted to the study of Sidon sets,  $\Lambda(p)$ -sets and some related notions for compact quantum groups. We establish several different characterizations of Sidon sets, and in particular prove that any Sidon set in a discrete group is a strong Sidon set in the sense of Picardello. We give several relations between Sidon sets,  $\Lambda(p)$ -sets and lacunarities for  $L^p$ -Fourier multipliers, generalizing a previous work by Blendek and Michaliček. We also prove the existence of  $\Lambda(p)$ -sets for orthogonal systems in noncommutative  $L^p$ -spaces, and deduce the corresponding properties for compact quantum groups. Central Sidon sets are also discussed, and it turns out that the compact quantum groups with the same fusion rules and the same dimension functions have identical central Sidon sets. Several examples are also included.

### 4.1 Sidon sets

Let  $\mathbb{G}$  be a compact quantum group and  $\mathbf{E}$  be a subset of  $\text{Irr}(\mathbb{G})$ . Define

$$L_{\mathbf{E}}^{\infty}(\mathbb{G}) = \{x \in L^{\infty}(\mathbb{G}) : \hat{x}(\pi) = 0 \text{ if } \pi \in \text{Irr}(\mathbb{G}) \setminus \mathbf{E}\},$$

$$C_{\mathbf{E}}(\mathbb{G}) = C_r(\mathbb{G}) \cap L_{\mathbf{E}}^{\infty}(\mathbb{G}), \quad \text{Pol}_{\mathbf{E}}(\mathbb{G}) = \text{Pol}(\mathbb{G}) \cap L_{\mathbf{E}}^{\infty}(\mathbb{G}),$$

and

$$\ell^{\infty}(\mathbf{E}) = \{(a_{\pi})_{\pi \in \text{Irr}(\mathbb{G})} \in \ell^{\infty}(\hat{\mathbb{G}}) : a_{\pi} = 0 \text{ if } \pi \in \text{Irr}(\mathbb{G}) \setminus \mathbf{E}\},$$

$$\ell^1(\mathbf{E}) = \ell^1(\hat{\mathbb{G}}) \cap \ell^{\infty}(\mathbf{E}), \quad c_0(\mathbf{E}) = c_0(\hat{\mathbb{G}}) \cap \ell^{\infty}(\mathbf{E}), \quad c_c(\mathbf{E}) = c_c(\hat{\mathbb{G}}) \cap \ell^{\infty}(\mathbf{E}).$$

Then the subspaces  $L_{\mathbf{E}}^{\infty}(\mathbb{G}), C_{\mathbf{E}}(\mathbb{G}), \ell^{\infty}(\mathbf{E}), \ell^1(\mathbf{E})$  are all closed subspaces. Note that we may identify the duality between spaces

$$c_0(\mathbf{E})^* = \ell^1(\mathbf{E}), \quad \ell^1(\mathbf{E})^* = \ell^{\infty}(\mathbf{E})$$

via the bracket  $\langle a, b \rangle = \hat{h}(ba) = \sum_{\pi} d_{\pi} \text{Tr}(a_{\pi} Q_{\pi} b_{\pi})$  for  $a \in \ell^1(\mathbf{E}), b \in \ell^{\infty}(\mathbf{E})$ .

**Definition 4.1.1.** We say that a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a *Sidon set* (with constant  $K$ ) if there exists  $K > 0$  such that for any  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ , we have

$$\|\hat{x}\|_1 \leq K \|x\|_{\infty}.$$

*Remark 4.1.2.* Any finite subset of  $\text{Irr}(\mathbb{G})$  is clearly a Sidon set. Let  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  be a Sidon set and  $\mathbf{F} \subset \text{Irr}(\mathbb{G})$  be finite. Then  $\mathbf{E} \cup \mathbf{F}$  is also a Sidon set. In fact, take  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$  and  $y \in \text{Pol}_{\mathbf{F}}(\mathbb{G})$ . Since  $\text{Pol}_{\mathbf{F}}(\mathbb{G})$  is finite-dimensional, it is complemented in  $L^\infty(\mathbb{G})$ . Therefore, there exist two constants  $K_1, K_2 > 0$  such that

$$\begin{aligned} \|\hat{x} + \hat{y}\|_1 &\leq \|\hat{x}\|_1 + \|\hat{y}\|_1 \leq K_1\|x\|_\infty + K_1\|y\|_\infty \leq K_1\|x + y\|_\infty + 2K_1\|y\|_\infty \\ &\leq K_1\|x + y\|_\infty + 2K_1K_2\|x + y\|_\infty = (K_1 + 2K_1K_2)\|x + y\|_\infty. \end{aligned}$$

We first give the following fundamental characterizations of Sidon set, extending the classical result of [HR70, (37.2)] for compact groups. On the other hand, this result establishes the equivalence between the so-called strong Sidon sets (i.e. sets with condition (3) in the theorem below) and Sidon sets in non-amenable discrete groups, which had been open since the work of Picardello [Pic73] in 1970s. Our approach is different from the idea of [HR70, Pic73]. In hindsight, the proof in [HR70] depends essentially on the coamenability of the compact group, which does not apply to the more general cases in the quantum setting. Instead, we use a simpler argument via duality.

**Theorem 4.1.3.** *Let  $\mathbb{G}$  be a compact quantum group and  $\mathbf{E}$  be a subset of  $\text{Irr}(\mathbb{G})$ . The following assertions are equivalent:*

- (1)  $\mathbf{E}$  is a Sidon set;
- (2) for any  $a \in \ell^\infty(\mathbf{E})$ , there exists  $\varphi \in C_r(\mathbb{G})^*$  such that  $\hat{\varphi}(\pi) = a_\pi$  for all  $\pi \in \mathbf{E}$ ;
- (3) for any  $a \in c_0(\mathbf{E})$ , there exists  $x \in L^1(\mathbb{G})$  such that  $\hat{x}(\pi) = a_\pi$  for all  $\pi \in \mathbf{E}$ ;
- (4)  $L_{\mathbf{E}}^\infty(\mathbb{G}) \subset A(\mathbb{G})$ ;
- (5)  $C_{\mathbf{E}}(\mathbb{G}) \subset A(\mathbb{G})$ ;
- (6) there exists  $K > 0$  such that for any  $x \in C_{\mathbf{E}}(\mathbb{G})$ ,  $\|\hat{x}\|_1 \leq K\|x\|_\infty$ ;
- (7) there exists  $K > 0$  such that for any  $x \in L_{\mathbf{E}}^\infty(\mathbb{G})$ ,  $\|\hat{x}\|_1 \leq K\|x\|_\infty$ .

*Proof.* From Proposition 2.1.5, it is easy to see that the condition (4) implies the surjectivity of the inverse Fourier transform  $\mathcal{F}^{-1} : \ell^1(\mathbf{E}) \rightarrow L_{\mathbf{E}}^\infty(\mathbb{G})$ . Then the equivalence (4)  $\Leftrightarrow$  (7) follows from the open mapping theorem. Similarly, we may obtain (5)  $\Leftrightarrow$  (6). Also, the implications (7)  $\Rightarrow$  (6)  $\Rightarrow$  (1) are trivial. Let us show (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (7). In the following set  $\text{Pol}_{\mathbf{E}}(\mathbb{G})^c = \{x \in \text{Pol}(\mathbb{G}) : x^* \in \text{Pol}_{\mathbf{E}}(\mathbb{G})\}$ .

(1)  $\Rightarrow$  (2). Take  $a \in \ell^\infty(\mathbf{E})$ . We consider the functional

$$\varphi : \text{Pol}_{\mathbf{E}}(\mathbb{G})^c \rightarrow \mathbb{C}, \quad x \mapsto \widehat{h(a^*x^*)}.$$

According to (1), we have

$$|\varphi(x)| = |\widehat{h(a^*x^*)}| \leq \|a^*\|_\infty \|\widehat{x^*}\|_1 \leq K\|a^*\|_\infty \|x^*\|_\infty = K\|a\|_\infty \|x\|_\infty,$$

so  $\varphi$  is continuous on  $\text{Pol}_{\mathbf{E}}(\mathbb{G})^c$  and it has a Hahn-Banach extension to  $C_r(\mathbb{G})$ . We still denote the extension by  $\varphi$ . Recall that by Proposition 2.1.2,

$$(u_{ji}^{(\pi)})^\wedge(\pi) = d_\pi^{-1} e_{ij}^{(\pi)} Q_\pi^{-1}.$$

We have for  $\pi \in \mathbf{E}$ ,

$$\begin{aligned} \hat{\varphi}(\pi) &= (\varphi \otimes \iota)((u^{(\pi)})^*) = \sum_{1 \leq i, j \leq n(\pi)} \varphi((u_{ji}^{(\pi)})^*) e_{ij}^{(\pi)} \\ &= \sum_{1 \leq i, j \leq n(\pi)} \widehat{h(a^*(u_{ji}^{(\pi)})^\wedge(\pi))} e_{ij}^{(\pi)} = d_\pi^{-1} \sum_{1 \leq i, j \leq n(\pi)} \widehat{h(a^* e_{ij}^{(\pi)} Q_\pi^{-1})} e_{ij}^{(\pi)} \end{aligned}$$

$$= \sum_{1 \leq i, j \leq n(\pi)} \overline{\text{Tr}(a_\pi^* e_{ij}^{(\pi)})} e_{ij}^{(\pi)} = \sum_{1 \leq i, j \leq n(\pi)} \overline{(a_\pi^*)_{ji}} e_{ij}^{(\pi)} = a_\pi,$$

as desired.

(2)  $\Rightarrow$  (3). We consider the map

$$\sigma : L^1(\mathbb{G}) \rightarrow c_0(\mathbf{E}), \quad \psi \mapsto \hat{\psi}|_{\mathbf{E}}.$$

Then the second adjoint map  $(\sigma^*)^*$  is given by

$$\sigma^{**} : L^\infty(\mathbb{G})^* \rightarrow \ell^\infty(\mathbf{E}), \quad \psi \mapsto \hat{\psi}|_{\mathbf{E}}.$$

Note that the condition (2) means nothing but the surjectivity of  $\sigma^{**}$ . Recall the general fact that for a bounded map  $T$  between two Banach spaces,  $T$  is surjective if and only if  $T^{**}$  is surjective (cf. for example [Meg98, 3.1.22]). So  $\sigma$  is also surjective, whence the condition (3).

(3)  $\Rightarrow$  (1). The assertion (3) implies that the bounded map  $\sigma : \psi \mapsto \hat{\psi}|_{\mathbf{E}}$  from  $L^1(\mathbb{G})$  to  $c_0(\mathbf{E})$  is surjective. By the open mapping theorem, we may find a constant  $K > 0$  such that for all  $a \in c_0(\mathbf{E})$ , there exists  $x \in L^1(\mathbb{G})$  satisfying  $\hat{x}(\pi) = a_\pi$  for all  $\pi \in \mathbf{E}$  and  $\|x\|_1 \leq K\|a\|_\infty$ .

Now consider  $y \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$  and let us show that  $\|\hat{y}\|_1 \leq K\|y\|_\infty$ . Equivalently, let us prove that for all  $a \in c_c(\mathbf{E})$  with  $\|a\|_\infty \leq 1$ , we have

$$|\hat{h}(a^* \hat{y})| \leq K\|y\|_\infty. \quad (4.1)$$

In the following let  $y$  and  $a$  be fixed as above. Choose a  $\psi \in L^1(\mathbb{G})$  satisfying  $\hat{\psi}(\pi) = a_\pi$  for all  $\pi \in \mathbf{E}$  and  $\|\psi\|_1 \leq K$ . Note also that  $\psi|_{\text{Pol}_{\mathbf{E}}(\mathbb{G})^c} = (\mathcal{F}^{-1}(a)h)|_{\text{Pol}_{\mathbf{E}}(\mathbb{G})^c}$ . Hence together with Proposition 2.1.2 and the choice of  $y$ , we get

$$|\hat{h}(a^* \hat{y})| = |\hat{h}(\hat{y}^* a)| = |h(y^* \mathcal{F}^{-1}(a)) + (\psi - \mathcal{F}^{-1}(a)h)(y^*)| = |\psi(y^*)| \leq K\|y\|_\infty.$$

Therefore the desired inequality (4.1) follows.

(3)  $\Rightarrow$  (7). We have proved that (3) implies (1) and on the other hand, for any set  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  satisfying (1), we have the following observation: if  $\psi \in L^1(\mathbb{G})$  satisfies  $\hat{\psi}|_{\mathbf{E}} = 0$ , then for all  $x \in L_{\mathbf{E}}^\infty(\mathbb{G})$ , we have  $\psi(x^*) = 0$ . In fact, let  $(y_n) \subset \text{Pol}(\mathbb{G})$  be a sequence such that  $y_n h$  converges to  $\psi$  in  $L^1(\mathbb{G})$  and let  $x \in L_{\mathbf{E}}^\infty(\mathbb{G})$ . We then note that  $\hat{\psi}^* \hat{x} = 0$ ,  $(\hat{y}_n - \hat{\psi})^* \hat{x} \in c_c(\mathbf{E})$  and by (2.9) and (2.11) we have

$$\begin{aligned} |\hat{h}(\hat{y}_n^* \hat{x})| &= |\hat{h}((\hat{y}_n - \hat{\psi})^* \hat{x})| \leq \|(\hat{y}_n - \hat{\psi})^* \hat{x}\|_1 = \|\mathcal{F}((\iota \otimes (y_n h - \psi))^* \Delta(x))\|_1 \\ &\leq K\|(\iota \otimes (y_n h - \psi))^* \Delta(x)\|_\infty \leq K\|y_n h - \psi\|_1 \|x\|_\infty \rightarrow 0, \end{aligned}$$

where  $K$  is the Sidon constant for  $\mathbf{E}$ . Consequently by the choice of  $y_n$  and Proposition 2.1.2 we have

$$\psi(x^*) = \lim_n h(x^* y_n) = \lim_n \overline{\hat{h}(\hat{y}_n^* \hat{x})} = 0,$$

as claimed. Then the implication (3)  $\Rightarrow$  (7) follows from the same argument as in (3)  $\Rightarrow$  (1) above.  $\square$

*Remark 4.1.4.* Note that for a Sidon set  $\mathbf{E}$ , by the assertion (4) and Proposition 2.1.5, we have

$$L_{\mathbf{E}}^\infty(\mathbb{G}) = C_{\mathbf{E}}(\mathbb{G}).$$

By the assertion (7),  $C_{\mathbf{E}}(\mathbb{G})$  is isomorphic to a Banach subspace of  $\ell^1(\mathbf{E})$  via the Fourier transform. Since  $c_c(\mathbf{E})$  is dense in  $\ell^1(\mathbf{E})$ ,  $C_{\mathbf{E}}(\mathbb{G})$  and  $\ell^1(\mathbf{E})$  are indeed isomorphic. In particular, the subspace  $\text{Pol}_{\mathbf{E}}(\mathbb{G})$  is dense in  $C_{\mathbf{E}}(\mathbb{G})$ .

*Remark 4.1.5.* Since by (2.9) we know that  $(\varphi^* \circ S^{-1})^\wedge(\pi) = \hat{\varphi}(\pi)^*$  for  $\pi \in \text{Irr}(\mathbb{G})$ , the assertion (2) in the above theorem can be replaced by the following one:

(2') for any  $a \in \ell^\infty(\mathbf{E})$ , there exists  $\varphi \in C_r(\mathbb{G})^*$  such that  $(\varphi \circ S^{-1})^\wedge(\pi) = a_\pi$  for all  $\pi \in \mathbf{E}$ .

*Remark 4.1.6.* Note that we have shown in the proof that, if  $\mathbf{E}$  is a Sidon set of constant  $K$ , then the obtained elements  $\varphi$  and  $x$  in (2) and (3) can be chosen to have the norms not more than  $K\|a\|_\infty$ , respectively; conversely arguing as in the beginning of the proof (3)  $\Rightarrow$  (1), if (2) or (3) holds, we may find a constant  $K > 0$  such that the norms of  $\varphi$  or  $x$  is not more than  $K\|a\|_\infty$  respectively, and the obtained Sidon constant in (1) is exactly  $K$ .

*Remark 4.1.7.* In view of some technical tricks concerning the non-traciality of Haar states, we would like to present a second proof of the implication (1)  $\Rightarrow$  (7) in the above theorem under an additional assumption that  $\mathbb{G}$  is coamenable, roughly following the idea in [HR70, (37.2)]. The subtle point, which is trivially hidden in the commutative and cocommutative cases, is the fact that to directly deduce the convergence of Fourier series of  $x$  from (4.2) below as in [HR70, (37.2)], one needs to know that the convolution (1.10) defines a bounded map from  $L^1(\mathbb{G}) \times L^\infty(\mathbb{G})$  into  $L^\infty(\mathbb{G})$  by restriction. This is generally not clear for a compact quantum group which is not of Kac type, caused by the unboundedness of the antipode  $S$ . Now let us assume the coamenability of  $\mathbb{G}$ , and show the implication (1)  $\Rightarrow$  (7):

Assume (1). Take  $x \in L^\infty_{\mathbf{E}}(\mathbb{G})$ . Since  $\mathbb{G}$  is coamenable and the subspace  $\text{Pol}(\mathbb{G})$  is dense in  $L^1(\mathbb{G})$ , we may choose a net  $(y_i)_{i \in I} \subset \text{Pol}(\mathbb{G})$  with  $\|y_i\|_1 \leq 1$  such that  $\lim_i \|x \star y_i - x\|_1 = 0$ . Hence for each  $\pi \in \mathbf{E}$ ,

$$\|\hat{y}_i(\pi)\hat{x}(\pi) - \hat{x}(\pi)\|_{B(H_\pi)} \leq \|\hat{y}_i\hat{x} - \hat{x}\|_\infty \leq \|x \star y_i - x\|_1 \rightarrow 0.$$

Since the norms on a finite dimensional space are equivalent, we have for all  $\pi \in \mathbf{E}$ ,

$$\lim_i \text{Tr}(|(\hat{y}_i(\pi)\hat{x}(\pi) - \hat{x}(\pi))Q_\pi|) = 0.$$

Therefore for any finite subset  $\mathbf{F} \subset \mathbf{E}$  and for any  $\varepsilon > 0$ , we may find some  $a \in c_c(\mathbf{E})$  with  $\|a\|_\infty \leq 1$  and

$$\sum_{\pi \in \mathbf{F}} d_\pi \text{Tr}(|(a_\pi \hat{x}(\pi) - \hat{x}(\pi))Q_\pi|) < \varepsilon. \quad (4.2)$$

Since  $\mathbf{E}$  is a Sidon set, we may find by previous arguments (proof of (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)) a functional  $\varphi \in L^\infty(\mathbb{G})_* = L^1(\mathbb{G})$  with  $\|\varphi\|_1 \leq K$  and  $\hat{\varphi}(\pi) = a_\pi^*$  for all  $\pi \in \mathbf{E}$ . Take  $y \in \text{Pol}(\mathbb{G})$  such that  $\|\varphi - y\|_1 < \varepsilon$ . Then  $\|a_\pi - ((yh)^* \circ S^{-1})^\wedge(\pi)\|_{B(H_\pi)} = \|\hat{\varphi}(\pi)^* - \hat{y}(\pi)^*\|_{B(H_\pi)} < \varepsilon$  for  $\pi \in \mathbf{E}$ . Further together with (2.11),

$$\begin{aligned} \sum_{\pi \in \mathbf{F}} d_\pi \text{Tr}(|\hat{x}(\pi)Q_\pi|) &\leq \sum_{\pi \in \mathbf{F}} d_\pi \text{Tr}(|a_\pi \hat{x}(\pi)Q_\pi|) + \sum_{\pi \in \mathbf{F}} d_\pi \text{Tr}(|(a_\pi \hat{x}(\pi) - \hat{x}(\pi))Q_\pi|) \\ &\leq \sum_{\pi \in \mathbf{F}} d_\pi \text{Tr}(|((yh)^* \circ S^{-1})^\wedge(\pi)\hat{x}(\pi)Q_\pi|) + \varepsilon \sum_{\pi \in \mathbf{F}} d_\pi \text{Tr}(|\hat{x}(\pi)Q_\pi|) + \varepsilon \\ &\leq \|\mathcal{F}((\iota \otimes (yh)^*)\Delta(x))\|_1 + \varepsilon \sum_{\pi \in \mathbf{F}} d_\pi \text{Tr}(|\hat{x}(\pi)Q_\pi|) + \varepsilon \\ &\leq K\|(\iota \otimes (yh)^*)\Delta(x)\|_\infty + \varepsilon \sum_{\pi \in \mathbf{F}} d_\pi \text{Tr}(|\hat{x}(\pi)Q_\pi|) + \varepsilon \\ &\leq K(K + \varepsilon)\|x\|_\infty + \varepsilon \sum_{\pi \in \mathbf{F}} d_\pi \text{Tr}(|\hat{x}(\pi)Q_\pi|) + \varepsilon \end{aligned}$$

where we have applied the property of the Sidon set to the element  $(\iota \otimes (yh)^*)\Delta(x) \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ . Since  $\mathbf{F}$  and  $\varepsilon$  are arbitrarily chosen, we get  $\|\hat{x}\|_1 \leq K^2\|x\|_\infty$ , as desired. Note that the constant  $K^2$  obtained here is worse than that in the previous proof.

As a corollary we may give a quick proof of the non-surjectivity of the Fourier transform  $\mathcal{F} : L^1(\mathbb{G}) \rightarrow c_0(\hat{\mathbb{G}})$  for infinite compact quantum group  $\mathbb{G}$ .

**Corollary 4.1.8.** *Let  $\mathbb{G}$  be a compact quantum group. The following conditions are equivalent:*

- (1)  $\mathbb{G}$  is finite, i.e.,  $L^\infty(\mathbb{G})$  is a finite-dimensional space;
- (2)  $\text{Irr}(\mathbb{G})$  is a Sidon set;
- (3)  $\mathcal{F} : L^1(\mathbb{G}) \rightarrow c_0(\hat{\mathbb{G}})$  is surjective;
- (4)  $\mathcal{F} : L^\infty(\mathbb{G})^* \rightarrow \ell^\infty(\hat{\mathbb{G}})$  is surjective.

*Proof.* The equivalence (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4) has been already given in Theorem 4.1.3. The implication (1)  $\Rightarrow$  (2) is trivial. Assume (2) holds, then the previous theorem yields that there exists a constant  $K > 0$  such that

$$\forall x \in L^\infty(\mathbb{G}), \quad \|x\|_\infty \leq \|\hat{x}\|_1 \leq K\|x\|_\infty. \quad (4.3)$$

Suppose by contradiction with (1) that  $\mathbb{G}$  is not finite. Then we may choose an infinite countable subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  and let  $A$  be the  $*$ -subalgebra generated by  $\{\chi_\pi : \pi \in \mathbf{E}\}$  in  $\text{Pol}(\mathbb{G})$ . Write

$$\mathbf{F} = \{\pi \in \text{Irr}(\mathbb{G}) : \exists \pi_1, \dots, \pi_n \in \mathbf{E}, \pi \text{ is a subrepresentation of } \sigma_1 \otimes \dots \otimes \sigma_n, \sigma_i = \pi_i \text{ or } \bar{\pi}_i\},$$

then  $\mathbf{F}$  is countable and  $A \subset \text{Pol}_{\mathbf{F}}(\mathbb{G})$ . Consider the von Neumann subalgebra  $\mathcal{M}$  generated by  $A$  in  $L^\infty(\mathbb{G})$ . Then by the weak density of  $A$  in  $\mathcal{M}$ , for each  $x \in \mathcal{M}$  and each  $\pi \in \text{Irr}(\mathbb{G}) \setminus \mathbf{F}$  we have  $\hat{x}(\pi) = 0$ . So by the above inequality (4.3) each  $x \in \mathcal{M}$  can be approximated in  $\|\cdot\|_\infty$  by elements in  $\text{Pol}_{\mathbf{F}}(\mathbb{G})$ , and in particular  $\mathcal{M}$  is separable, which gives a contradiction since the von Neumann algebra  $\mathcal{M}$  is infinite-dimensional as so is  $A$ .  $\square$

*Remark 4.1.9.* Together with the condition (5) in Theorem 4.1.3, the above argument also shows that for any infinite discrete quantum group  $\mathbb{H}$ , the Fourier transform  $\mathcal{F} : \ell^1(\mathbb{H}) \rightarrow C(\hat{\mathbb{H}})$  in the sense of [Kah10, Cas13] is not surjective. The above result is a particular case of the general fact that the predual of an infinite-dimensional von Neumann algebra cannot be equipped with an equivalent  $C^*$ -norm.

The following properties give some general methods of constructing infinite Sidon sets for compact quantum groups.

**Proposition 4.1.10.** *Let  $(\mathbb{G}_i)_{i \in I}$  be a family of compact quantum groups and assume that  $\mathbf{E}_i \subset \text{Irr}(\mathbb{G}_i)$  is a Sidon set with constant  $C_i$  for each  $i \in I$  and that  $C := \sup_{i \in I} C_i < \infty$ . Then  $\cup_{i \in I} \mathbf{E}_i \subset \text{Irr}(\prod_{i \in I} \mathbb{G}_i)$  is a Sidon set for  $\prod_{i \in I} \mathbb{G}_i$ .*

*Proof.* This follows directly from the assertion (2) in Theorem 4.1.3 and Remark 4.1.6. Let  $a \in \ell^\infty(\cup_{i \in I} \mathbf{E}_i)$ . Without loss of generality we assume  $\|a\|_\infty = 1/C$ . For each  $i \in I$ , we may find  $\varphi_i \in C_r(\mathbb{G}_i)^*$  such that  $\hat{\varphi}_i$  coincides with  $a$  on  $\mathbf{E}_i$  and  $\|\varphi_i\| \leq 1$ . Take  $\varphi = \otimes_{i \in I} \varphi_i$ , then  $\varphi$  extends to a bounded functional on  $C_r(\prod_{i \in I} \mathbb{G}_i)$ . Hence  $\hat{\varphi}(\pi) = a_\pi$  for all  $\pi \in \cup_{i \in I} \mathbf{E}_i$ . So  $\cup_{i \in I} \mathbf{E}_i$  is a Sidon set.  $\square$

**Proposition 4.1.11.** *Let  $\mathbb{G}_1, \mathbb{G}_2$  be compact quantum groups. Assume that  $\mathbf{E} \subset \text{Irr}(\mathbb{G}_1)$  and  $\mathbf{F} \subset \text{Irr}(\mathbb{G}_2)$  are Sidon sets. Then  $\mathbf{E} \cup \mathbf{F}$  is a Sidon set for  $\mathbb{G}_1 \hat{*} \mathbb{G}_2$ .*

*Proof.* Denote  $h_1$  and  $h_2$  the Haar states for  $\mathbb{G}_1$  and  $\mathbb{G}_2$  respectively. Let  $K_1$  be the Sidon constant for  $\mathbf{E}$  and  $K_2$  that for  $\mathbf{F}$ . By Remark 4.1.2, we may assume that  $1 \notin \mathbf{E}$  and  $1 \notin \mathbf{F}$ . Now for any  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}_1 \hat{*} \mathbb{G}_2)$  and  $y \in \text{Pol}_{\mathbf{F}}(\mathbb{G}_1 \hat{*} \mathbb{G}_2)$ , we have  $h_1(x) = 0$ ,  $h_2(y) = 0$ , and  $x$  and  $y$  are free. Then it is well known and easy to see from the construction of reduced free products that  $\max\{\|x\|_{\infty}, \|y\|_{\infty}\} \leq \|x + y\|_{\infty}$  (see [Voi98, Jun05, RX06] for more information on the norm estimates related to freeness). Hence

$$\|\hat{x} + \hat{y}\|_1 \leq \|\hat{x}\|_1 + \|\hat{y}\|_1 \leq K_1\|x\|_{\infty} + K_2\|y\|_{\infty} \leq (K_1 + K_2)\|x + y\|_{\infty}.$$

This proves that  $\mathbf{E} \cup \mathbf{F}$  is a Sidon set of constant  $K_1 + K_2$ .  $\square$

*Remark 4.1.12.* Note that one cannot expect to extend the above proposition to an infinite family of compact quantum groups as in Proposition 4.1.10. An easy example is the set of infinitely many free generators of the free group  $\mathbb{F}_{\infty}$ , which is not a Sidon set. More details of this example will be presented in Remark 4.1.16.

As is seen in [Pic73, Boz81], there are several alternative possible ways to generalize the notion of Sidon sets for non-amenable cases. Let us briefly discuss them in the quantum group setting. We follow the terminologies in [Boz81, Har99].

**Definition 4.1.13.** (1) We say that a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a *weak Sidon set* (with constant  $K$ ) if there exists  $K > 0$  such that for any  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ , we have

$$\|\hat{x}\|_1 \leq K\|x\|_{C_u(\mathbb{G})}.$$

(2) We say that  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is an *interpolation set of  $M(L^{\infty}(\mathbb{G}))$*  (resp., of  $M_L(L^{\infty}(\mathbb{G}))$ , of  $M_R(L^{\infty}(\mathbb{G}))$ ) with constant  $K$  if for any  $a \in \ell^{\infty}(\mathbf{E})$ , there exists a bounded multiplier  $\tilde{a} \in M(L^{\infty}(\mathbb{G}))$  (resp.,  $\tilde{a} \in M_L(L^{\infty}(\mathbb{G}))$ ,  $\tilde{a} \in M_R(L^{\infty}(\mathbb{G}))$ ) with  $\|\tilde{a}\|_{M(L^{\infty}(\mathbb{G}))} \leq K\|a\|_{\infty}$  (resp.,  $\|m_{\tilde{a}}^L\| \leq K\|a\|_{\infty}$ ,  $\|m_{\tilde{a}}^R\| \leq K\|a\|_{\infty}$ ) such that  $\tilde{a}_{\pi} = a_{\pi}$  for all  $\pi \in \mathbf{E}$ .

(3) We say that a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a *left* (resp., *right*) *unconditional Sidon set* (with constant  $K$ ) if there exists  $K > 0$  such that for any unitary  $a \in \ell^{\infty}(\mathbf{E})$  and for any  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ ,  $\|m_a^L x\|_{\infty} \leq K\|x\|_{\infty}$  (resp.,  $\|m_a^R x\|_{\infty} \leq K\|x\|_{\infty}$ ).

*Remark 4.1.14.* (1) Following the same idea as in the proof (1)  $\Rightarrow$  (2) and (3)  $\Rightarrow$  (1) in Theorem 4.1.3, one can see easily that a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a weak Sidon set of constant  $K$  if and only if for all  $a \in \ell^{\infty}(\mathbf{E})$ , there exists  $\varphi \in C_u(\mathbb{G})^*$  such that  $\|\varphi\| \leq K\|a\|_{\infty}$  and  $\hat{\varphi}(\pi) = a_{\pi}$  for all  $\pi \in \mathbf{E}$ . Evidently, a Sidon set for  $\mathbb{G}$  is necessarily a weak Sidon set.

(2) If  $\mathbb{G}$  is of Kac type, by Lemma 2.3.3 we see that the classes of interpolation sets of  $M_L(L^{\infty}(\mathbb{G}))$  and of  $M_R(L^{\infty}(\mathbb{G}))$  coincide. Also we note that if  $\mathbb{G}$  is coamenable, these two classes coincide as well, which can be seen from the following theorem. Some more properties of these classes of interpolation sets will be discussed in the next section.

**Theorem 4.1.15.** *Let  $\mathbb{G}$  be a compact quantum group and  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  be a subset. Let  $K > 0$ . If*

- (1)  $\mathbf{E}$  is a weak Sidon set of constant  $K$ ;

then

- (2)  $\mathbf{E}$  is an interpolation set of  $M(L^{\infty}(\mathbb{G}))$  of constant  $K$ ;

and in particular,

- (3)  $\mathbf{E}$  is a left unconditional Sidon set of constant  $K$ .

Moreover, if additionally  $\mathbb{G}$  is coamenable, then the conditions (1), (2) and (3) are all equivalent to:

(4)  $\mathbf{E}$  is a Sidon set of constant  $K$ .

*Proof.* The proof follows the same line as in [Pic73, Bož81] and we only present the sketch. Assume that  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a weak Sidon set. Then by the above remark and Remarks 4.1.5-4.1.6 any  $a \in \ell^\infty(\mathbf{E})$  is a restriction of  $\hat{\varphi}$  for some  $\varphi \in C_u(\mathbb{G})^*$  of norm no more than  $K\|a\|_\infty$ . Note that  $C_r(\mathbb{G})$  is a quotient space of  $C_u(\mathbb{G})$  and hence the dual space  $C_r(\mathbb{G})^*$  embeds isometrically into  $C_u(\mathbb{G})^*$ . Therefore by the density of  $\text{Pol}(\mathbb{G})$  in  $L^1(\mathbb{G}) = L^\infty(\mathbb{G})_*$ , one can easily see from (2.10) that  $\psi \mapsto \varphi \star \psi$  and  $\psi \mapsto \psi \star \varphi$  give two bounded multipliers on  $L^1(\mathbb{G})$ . Then by duality we obtain the desired multipliers for (2). Thus the implication (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) is established.

Now assume additionally  $\mathbb{G}$  is coamenable and show that (3)  $\Rightarrow$  (4). Take  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$  and for each  $\pi \in \mathbf{E}$  let  $a_\pi$  be a unitary matrix such that  $|\hat{x}(\pi)Q_\pi| = a_\pi \hat{x}(\pi)Q_\pi$ . Then

$$\begin{aligned} \|\hat{x}\|_1 &= \sum_{\pi \in \mathbf{E}} d_\pi \text{Tr}(|\hat{x}(\pi)Q_\pi|) = \sum_{\pi \in \mathbf{E}} d_\pi \text{Tr}(a_\pi \hat{x}(\pi)Q_\pi) \\ &= \epsilon \left( \sum_{\pi \in \mathbf{E}} d_\pi (\iota \otimes \text{Tr})[(1 \otimes a_\pi \hat{x}(\pi)Q_\pi)u^{(\pi)}] \right) = \epsilon(m_a x) \\ &\leq \|m_a x\|_\infty \leq K\|x\|_\infty, \end{aligned}$$

as desired.  $\square$

*Remark 4.1.16.* (1) The left unconditional Sidon set in the assertion (3) above can be obviously replaced by the right one.

(2) The coamenability is crucial in the above proposition. In fact, denote by  $\mathbb{F}_\infty$  the free group with infinitely many generators and let  $\mathbb{G}$  be the quantum group with dual  $\hat{\mathbb{G}} = \mathbb{F}_\infty$ . Take  $\mathbf{E}$  to be the generators of  $\mathbb{F}_\infty$ , and recall the Haagerup inequality [Lei74, Bož75a]: for finitely many elements  $\gamma_1, \dots, \gamma_n \in \mathbf{E}$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ ,

$$\left( \sum_{k=1}^n |\alpha_k|^2 \right)^{1/2} \leq \left\| \sum_{k=1}^n \alpha_k \lambda(\gamma_k) \right\|_{VN(\mathbb{F}_\infty)} \leq 2 \left( \sum_{k=1}^n |\alpha_k|^2 \right)^{1/2}.$$

So  $\mathbf{E}$  satisfies (2) and (3) in the proposition, but in this case obviously (1) fails to hold.

**Example 4.1.17.** Consider the compact quantum group  $\mathbb{G} = \prod_{k \geq 1} \mathbb{G}_k$ , where for each  $k \geq 1$  and  $N_k \geq 1$ ,  $\mathbb{G}_k = U_{N_k}^+$  denotes the free unitary group of Wang [VDW96]. Recall that  $C_u(U_{N_k}^+)$  is the universal  $C^*$ -algebra generated by  $N_k^2$  elements  $\{u_{ij}^{(k)} : 1 \leq i, j \leq N_k\}$  such that the matrix  $u^{(k)} = [u_{ij}^{(k)}]$  is unitary. Take  $\mathbf{E} = \{u^{(k)} : k \geq 1\} \subset \text{Irr}(\mathbb{G})$ . Then  $\mathbf{E}$  is a weak Sidon set, and hence is an interpolation set of  $M(L^\infty(\mathbb{G}))$  and an unconditional Sidon set. In fact, let  $U_{N_k}$  be the  $N_k \times N_k$  unitary matrix group and  $w^{(k)} : U_{N_k} \rightarrow \mathbb{M}_{N_k}(\mathbb{C})$ ,  $w \mapsto w$  be its fundamental representation. It is easy to see that  $\{w^{(k)} : k \geq 1\}$  is a Sidon set of constant 1 for  $G = \prod_{k \geq 1} U_{N_k}$  ([HR70, (37.5)]). Thus by the universal property of  $U_{N_k}^+$  we have for all finitely supported sequences  $(A_k) \in \prod_k \mathbb{M}_{N_k}$ ,

$$\sum_{k \geq 1} \text{Tr}(|A_k|) \leq \left\| \sum_{k \geq 1} \text{Tr}(A_k w^{(k)}) \right\|_{C(G)} \leq \left\| \sum_{k \geq 1} (\iota \otimes \text{Tr})[(1 \otimes A_k)u^{(k)}] \right\|_{C_u(\mathbb{G})}.$$

Therefore  $\mathbf{E}$  is a weak Sidon set. However,  $\mathbf{E}$  is *not* a Sidon set. Indeed, write  $x = \sum_k x_k \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$  with  $x_k \in \text{Pol}(U_{N_k}^+)$ . Vergnioux [Ver07] and Brannan [Bra12, Theorem 6.3] showed that there exists  $C > 0$  such that  $\|x_k\|_\infty \leq C\|\hat{x}_k\|_2$  for all  $k$ . Hence  $\|x\|_\infty \leq C \sum_k \|\hat{x}_k\|_2$  and the inequality in Definition 4.1.1 cannot hold.



**Example 4.1.18.** Consider the sequence  $(q_n)_{n \geq 1} \subset [q, 1]$  with  $q := \inf_n q_n > 0$  and the associated quantum group  $\mathbb{G} = \prod_{n \geq 1} \text{SU}_{q_n}(2)$ . Recall that  $\text{SU}_{q_n}(2)$  is coamenable and for each  $n$ ,  $C(\text{SU}_{q_n}(2))$  is the universal  $C^*$ -algebra generated by elements  $\alpha_n$  and  $\gamma_n$  such that the matrix

$$u_n = \begin{bmatrix} \alpha_n & -q_n \gamma_n^* \\ \gamma_n & \alpha_n^* \end{bmatrix}$$

is unitary. The matrix  $u_n \in \mathbb{M}_2(C(\text{SU}_{q_n}(2)))$  defines a unitary representation of  $\text{SU}_{q_n}(2)$ , and the matrix  $Q_{u_n}$  associated to  $u_n$ , simply written as  $Q_n$ , has the eigenvalues  $q_n, q_n^{-1}$ . Let  $d_n = \text{Tr}(Q_n) = q_n + q_n^{-1}$ . Then  $\mathbf{E} = \{u_n : n \geq 1\} \subset \text{Irr}(\mathbb{G})$  is a Sidon set for  $\mathbb{G}$ . To see this, by Proposition 4.1.10 and Theorem 4.1.15, it suffices to show that for each  $n \geq 1$ , the singleton  $\{u_n\}$  has a uniform right unconditional Sidon constant  $1 + q^{-1}$ , which means that for all  $A \in \mathbb{M}_2(\mathbb{C})$  and all unitaries  $V \in \mathbb{M}_2(\mathbb{C})$ , we have

$$\|d_n(\iota \otimes \text{Tr})[(1 \otimes V A Q_n)u_n]\|_\infty \leq (1 + q^{-1})\|d_n(\iota \otimes \text{Tr})[(1 \otimes A Q_n)u_n]\|_\infty. \quad (4.4)$$

Indeed, since the map  $x \mapsto d_n^{-1}(\iota \otimes \text{Tr})(x(1 \otimes Q_n))$  is unital completely positive and the functional  $\text{Tr}$  is tracial, we may use the Cauchy-Schwarz inequality and Proposition 2.1.2 to get

$$\begin{aligned} \|d_n(\iota \otimes \text{Tr})[(1 \otimes V A Q_n)u_n]\|_\infty^2 &= d_n^4 \|d_n^{-1}(\iota \otimes \text{Tr})[u_n(1 \otimes V A Q_n)]\|_\infty^2 \\ &\leq d_n^4 \|d_n^{-1}(\iota \otimes \text{Tr})[(u_n(1 \otimes V A))^*(u_n(1 \otimes V A))(1 \otimes Q_n)]\|_\infty \\ &= d_n^3 \text{Tr}(A^* A Q_n) = d_n^2 \|d_n(\iota \otimes \text{Tr})[(1 \otimes A Q_n)u_n]\|_2^2 \\ &\leq (q_n + q_n^{-1})^2 \|d_n(\iota \otimes \text{Tr})[(1 \otimes A Q_n)u_n]\|_\infty^2 \end{aligned}$$

which establishes (4.4) if we note that  $q \leq q_n \leq 1$ . The order  $o(q^{-1})$  of the constant obtained in the above inequality is optimal when  $q \rightarrow 0$ : we see that

$$\text{Tr}(|e_{21}|) = 1, \quad \|(\iota \otimes \text{Tr})[(1 \otimes e_{21})u_n]\|_\infty = \|-q_n \gamma_n^*\| \leq q_n,$$

which means the Sidon constant  $K$  with  $\|\hat{x}\|_\infty \leq K\|x\|_\infty$  for  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$  cannot be less than  $q_n^{-1}$ . Also as a result, if  $q_n \rightarrow 0$ , the subset  $\mathbf{E}$  given above is not a Sidon set.

## 4.2 $\Lambda(p)$ -sets

In this section we aim to investigate  $\Lambda(p)$ -sets, and in particular we will establish the relations between Sidon sets and  $\Lambda(p)$ -sets.

### 4.2.1 $\Lambda(p)$ -sets and $L^p$ -Fourier multipliers

In the following we define the  $\Lambda(p)$ -sets for compact quantum groups, which follows from a direct quantum adaptation of the classical notion of  $\Lambda(p)$ -sets for compact groups.

**Definition 4.2.1.** Let  $\mathbb{G}$  be a compact quantum group and  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  be a subset. Let  $\chi_\pi = \sum_i u_{ii}^{(\pi)}$  be the character of  $\pi \in \text{Irr}(\mathbb{G})$ . For  $1 < p < \infty$ , we say that  $\mathbf{E}$  is a  $\Lambda(p)$ -set with constant  $K$  if there exists  $K > 0$  such that for all  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ ,

$$\|x\|_p \leq K\|x\|_1,$$

and we say that  $\mathbf{E}$  is a *central*  $\Lambda(p)$ -set with constant  $K$  if there exists  $K > 0$  such that for all finitely supported sequences  $(c_\pi)_{\pi \in \mathbf{E}} \subset \mathbb{C}$  and  $x = \sum_{\pi \in \mathbf{E}} c_\pi \chi_\pi \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ ,

$$\|x\|_p \leq K\|x\|_1.$$

*Remark 4.2.2.* Let  $1 < p < \infty$  and  $1 < p_0 < p$ . Notice that in order to see a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a  $\Lambda(p)$ -set, it suffices to check the existence of a constant  $K > 0$  with

$$\|x\|_p \leq K\|x\|_{p_0}, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}).$$

This is due to the fact that  $(L^p(\mathbb{G}))_{1 \leq p \leq \infty}$  is a complex interpolation scale so that  $\|x\|_{p_0} \leq \|x\|_1^\theta \|x\|_p^{1-\theta}$  for some  $0 < \theta < 1$ . On the other hand, we see that any  $\Lambda(p)$ -set must be a  $\Lambda(p')$ -set for  $1 < p' < p < \infty$ . Similar observations are also valid for central  $\Lambda(p)$ -sets. And as in the classical case, we will be mainly interested in the  $\Lambda(p)$ -sets for  $2 < p < \infty$ .

In order to characterize the  $\Lambda(p)$ -sets, let us consider the following notions of interpolation sets of bounded multipliers on  $L^p(\mathbb{G})$ , which also generalize Definition 4.1.13.

**Definition 4.2.3.** Let  $\mathbb{G}$  be a compact quantum group and let  $1 \leq p \leq \infty$ . We say that  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is an *interpolation set* of  $M(L^p(\mathbb{G}))$  (resp., of  $M_L(L^p(\mathbb{G}))$ , of  $M_R(L^\infty(\mathbb{G}))$ ) with constant  $K$  if for any  $a \in \ell^\infty(\mathbf{E})$ , there exists a bounded multiplier  $\tilde{a} \in M(L^p(\mathbb{G}))$  (resp.,  $\tilde{a} \in M_L(L^p(\mathbb{G}))$ ,  $\tilde{a} \in M_R(L^p(\mathbb{G}))$ ) with  $\|\tilde{a}\|_{M(L^p(\mathbb{G}))} \leq K\|a\|_\infty$  (resp.,  $\|m_{\tilde{a}}^L\|_{B(L^p(\mathbb{G}))} \leq K\|a\|_\infty$ ,  $\|m_{\tilde{a}}^R\|_{B(L^p(\mathbb{G}))} \leq K\|a\|_\infty$ ) such that  $\tilde{a}_\pi = a_\pi$  for all  $\pi \in \mathbf{E}$ .

*Remark 4.2.4.* Let  $1 \leq p \leq \infty$  and  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$ . We remark that, if for any  $a \in \ell^\infty(\mathbf{E})$ , there exists a bounded multiplier  $\tilde{a} \in M(L^p(\mathbb{G}))$ , then automatically there exists a constant  $K > 0$  with  $\|\tilde{a}\|_{M(L^p(\mathbb{G}))} \leq K\|a\|_\infty$ , and  $\mathbf{E}$  is an interpolation set of  $M(L^p(\mathbb{G}))$ . In fact, if for any  $a \in \ell^\infty(\mathbf{E})$ , there exists a bounded multiplier  $\tilde{a} \in M(L^p(\mathbb{G}))$  with  $\tilde{a}_\pi = a_\pi$  for all  $\pi \in \mathbf{E}$ , then by Proposition 2.3.5, we have

$$\|Q^{\frac{1}{4} - \frac{1}{2p}} a Q^{-\frac{1}{4} + \frac{1}{2p}}\|_\infty < \infty.$$

In particular, we choose an appropriate basis of  $\oplus_\pi H_\pi$  so that  $Q_\pi$  is diagonal under this basis for  $\pi \in \text{Irr}(\mathbb{G})$ , and take  $a_\pi = e_{ij}$  with  $1 \leq i, j \leq n_\pi$ , then the above inequality yields that

$$\sup_{\pi \in \mathbf{E}} \|Q_\pi\| < \infty, \quad \sup_{\pi \in \mathbf{E}} \|Q_\pi^{-1}\| < \infty.$$

Also by Proposition 2.3.5, it is easy to see that  $M(L^p(\mathbb{G}))$  is a Banach subspace of  $B(L^p(\mathbb{G}))$ . So by the open mapping theorem and Proposition 2.3.5, we may always find a constant  $K > 0$  such that the inequality

$$\|\tilde{a}\|_{M(L^p(\mathbb{G}))} \leq K\|a\|_\infty$$

is automatically satisfied. Thus  $\mathbf{E}$  is automatically an interpolation set of  $M(L^p(\mathbb{G}))$ . But we do not know whether the similar observation can be made for interpolation sets of  $M_L(L^p(\mathbb{G}))$  and those of  $M_R(L^p(\mathbb{G}))$ .

These kinds of lacunarities have some special restrictive properties in the non-Kac case. The following result will be of use later.

**Proposition 4.2.5.** *Let  $\mathbb{G}$  be a compact quantum group and  $1 \leq p \leq \infty$ . Assume that  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  satisfies one of the following four conditions:*

- (1)  $p \neq \infty$  and  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is an interpolation set of  $M_L(L^p(\mathbb{G}))$  with constant  $K$ ;
- (2)  $p \neq \infty$  and  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is an interpolation set of  $M_R(L^p(\mathbb{G}))$  with constant  $K$ ;
- (3)  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is an interpolation set of  $M(L^p(\mathbb{G}))$  with constant  $K$ ;
- (4) there exist constants  $K > 0$  and  $\theta, \theta' \in \mathbb{R}$  with  $\theta + \theta' - \frac{1}{p} - \frac{1}{2} \neq 0$  such that for all  $a \in c_c(\mathbf{E})$ ,

$$\|m_a^L x\|_p \leq K\|Q^\theta a Q^{-\theta}\|_\infty \|x\|_p, \quad \|m_a^R x\|_p \leq K\|Q^{\theta'} a Q^{-\theta'}\|_\infty \|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}).$$

Then there exist constants  $K_1, K_2 > 0$  such that for all  $\pi \in \mathbf{E}$  and  $1 \leq i, j, k, l \leq n_\pi$ , we have

$$\|u_{ij}^{(\pi)}\|_p \leq K_1 \|u_{kl}^{(\pi)}\|_p, \quad (4.5)$$

and moreover

$$\max\{\|Q_\pi\|, \|Q_\pi^{-1}\|\} \leq K_2. \quad (4.6)$$

*Proof.* In the proof we always choose an appropriate basis of  $\oplus_\pi H_\pi$  so that  $Q_\pi$  is diagonal under this basis for  $\pi \in \text{Irr}(\mathbb{G})$ . We first prove (4.5) and (4.6) under the assumption (4).

Assume that  $K > 0$  and  $\theta, \theta' \in \mathbb{R}$  with  $\theta + \theta' - \frac{1}{p} - \frac{1}{2} \neq 0$  such that for all  $a \in c_c(\mathbf{E})$ ,

$$\|m_a^L x\|_p \leq K \|Q^\theta a Q^{-\theta}\|_\infty \|x\|_p, \quad \|m_a^R x\|_p \leq K \|Q^{\theta'} a Q^{-\theta'}\|_\infty \|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}).$$

Let  $\pi \in \mathbf{E}$  and  $1 \leq i, j, k, l \leq n_\pi$ . Then in particular we have

$$\|m_{e_{ki}}^L x\|_p \leq K (Q_\pi)_{kk}^\theta (Q_\pi)_{ii}^{-\theta} \|x\|_p, \quad \|m_{e_{jl}}^R x\|_p \leq K (Q_\pi)_{jj}^{\theta'} (Q_\pi)_{ll}^{-\theta'} \|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}).$$

So by Proposition 2.1.2 we have

$$\begin{aligned} \|u_{ij}^{(\pi)}\|_p &= \|(\iota \otimes \text{Tr})[(1 \otimes e_{ji})u^{(\pi)}]\|_p = \|m_{e_{jl}}^R m_{e_{ki}}^L (\iota \otimes \text{Tr})[(1 \otimes e_{lk})u^{(\pi)}]\|_p \\ &\leq K^2 (Q_\pi)_{kk}^\theta (Q_\pi)_{ii}^{-\theta} (Q_\pi)_{jj}^{\theta'} (Q_\pi)_{ll}^{-\theta'} \|(\iota \otimes \text{Tr})[(1 \otimes e_{lk})u^{(\pi)}]\|_p \\ &= K^2 (Q_\pi)_{kk}^\theta (Q_\pi)_{ii}^{-\theta} (Q_\pi)_{jj}^{\theta'} (Q_\pi)_{ll}^{-\theta'} \|u_{kl}^{(\pi)}\|_p. \end{aligned} \quad (4.7)$$

In particular we get for any  $1 \leq i, j \leq n_\pi$ ,

$$K^{-2} (Q_\pi)_{jj}^{\theta+\theta'} (Q_\pi)_{ii}^{-\theta-\theta'} \|u_{ji}^{(\pi)}\|_p \leq \|u_{ij}^{(\pi)}\|_p \leq K^2 (Q_\pi)_{jj}^{\theta+\theta'} (Q_\pi)_{ii}^{-\theta-\theta'} \|u_{ji}^{(\pi)}\|_p. \quad (4.8)$$

Recall that  $Q_\pi$  is chosen diagonal, and note that by Lemma 2.3.2 and the formula (1.3),

$$\|u_{ij}^{(\pi)}\|_p = \|\sigma_{-i/p}((u_{ij}^{(\pi)})^*)\|_p = (Q_\pi)_{ii}^{-\frac{1}{p}} (Q_\pi)_{jj}^{-\frac{1}{p}} \|(u_{ij}^{(\pi)})^*\|_p = (Q_\pi)_{ii}^{-\frac{1}{p}} (Q_\pi)_{jj}^{-\frac{1}{p}} \|S(u_{ji}^{(\pi)})\|_p.$$

But using the polar decomposition of  $S$  in (1.4) and (1.6) and recalling that  $Q_\pi$  is chosen diagonal, we have

$$\|S(u_{ji}^{(\pi)})\|_p = \|R(\tau_{-\frac{i}{2}}(u_{ji}^{(\pi)}))\|_p = \|\tau_{-\frac{i}{2}}(u_{ji}^{(\pi)})\|_p = (Q_\pi)_{jj}^{\frac{1}{2}} (Q_\pi)_{ii}^{-\frac{1}{2}} \|u_{ji}^{(\pi)}\|_p.$$

The above three inequalities yield that

$$(Q_\pi)_{ii}^{\theta+\theta'-\frac{1}{p}-\frac{1}{2}} (Q_\pi)_{jj}^{-\theta-\theta'+\frac{1}{p}+\frac{1}{2}} \leq K^2, \quad (Q_\pi)_{ii}^{-\theta-\theta'+\frac{1}{p}+\frac{1}{2}} (Q_\pi)_{jj}^{\theta+\theta'-\frac{1}{p}-\frac{1}{2}} \leq K^2$$

Note that  $i$  and  $j$  are arbitrarily chosen and that  $\|Q_\pi\| \geq 1, \|Q_\pi^{-1}\| \geq 1$ , so the above inequalities yield

$$\max\{\|Q_\pi\|, \|Q_\pi^{-1}\|\} \leq K^{2/|\theta+\theta'-\frac{1}{p}-\frac{1}{2}|}.$$

Combining this with (4.7), we also get

$$\|u_{ij}^{(\pi)}\|_p \leq K^{2+2/|\theta+\theta'-\frac{1}{p}-\frac{1}{2}|} \|u_{kl}^{(\pi)}\|_p,$$

as desired.

Now we assume that (1) holds, that is,  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is an interpolation set of  $M_L(L^p(\mathbb{G}))$ . Let  $a \in c_c(\mathbf{E})$ . Then  $a$  extends to a bounded left multiplier on  $L^p(\mathbb{G})$ , and hence by Lemma 2.3.3 we have

$$\|m_a^L x\|_p \leq K \|a\|_\infty \|x\|_p, \quad \|m_a^R x\|_p \leq K \|Q^{\frac{1}{2}} a Q^{-\frac{1}{2}}\|_\infty \|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}).$$

Taking  $\theta = 0, \theta' = 1/2$  in (4), then we obtain the desired inequalities (4.5) and (4.6). And the proof under the assumption (2) follows from a similar argument.

The assumption (3) can be viewed as a particular case of (4) by taking  $\theta = 1/p, \theta' = 0$ . So we establish the proposition.  $\square$

Combining this proposition with Lemma 2.3.3, we deduce the following observation.

**Lemma 4.2.6.** *Let  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  be a subset. Let  $1 \leq p < \infty$ . Then  $\mathbf{E}$  is an interpolation set of  $M_L(L^p(\mathbb{G}))$  if and only if it is an interpolation set of  $M_R(L^p(\mathbb{G}))$ .*

Now we are able to characterize the  $\Lambda(p)$ -sets via the interpolation sets of bounded multipliers for  $2 < p < \infty$ .

**Theorem 4.2.7.** *Let  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  be a subset. Assume  $2 < p < \infty$ . The following assertions are equivalent:*

- (1)  $\mathbf{E}$  is a  $\Lambda(p)$ -set;
- (2) there exists a constant  $K > 0$  such that for all  $a \in \ell^\infty(\mathbf{E})$ ,

$$\|m_a^L x\|_p \leq K \|a\|_\infty \|x\|_p, \quad \|m_a^R x\|_p \leq K \|a\|_\infty \|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G});$$

- (3)  $\mathbf{E}$  is an interpolation set of  $M_L(L^p(\mathbb{G}))$ ;
- (4)  $\mathbf{E}$  is an interpolation set of  $M_R(L^p(\mathbb{G}))$ ;
- (5)  $\mathbf{E}$  is an interpolation set of  $M(L^p(\mathbb{G}))$ .

If additionally the subset  $\mathbf{E}$  is symmetric in the sense that  $\pi \in \mathbf{E}$  if and only if  $\bar{\pi} \in \mathbf{E}$ , then the above assertions are also equivalent to:

- (6) there exists a constant  $K > 0$  such that for all  $a \in \ell^\infty(\mathbf{E})$ ,

$$\|m_a^L x\|_p \leq K \|a\|_\infty \|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G});$$

- (7) there exists a constant  $K > 0$  such that for all  $a \in \ell^\infty(\mathbf{E})$ ,

$$\|m_a^R x\|_p \leq K \|a\|_\infty \|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}).$$

*Proof.* Note that the equivalence (3)  $\Leftrightarrow$  (4) has been already given in the previous lemma and the implication (5)  $\Rightarrow$  (4) is trivial. Since (3) and (4) are equivalent, the implication (3)  $\Rightarrow$  (2) is also obvious.

(1)  $\Rightarrow$  (5). Assume that  $\mathbf{E}$  is a  $\Lambda(p)$ -set with constant  $K$ . Then together with Lemma 2.3.1, we see that for all  $a \in \ell^\infty(\mathbf{E})$ ,

$$\|m_a^R x\|_p \leq K \|m_a^R x\|_2 \leq K \|a\|_\infty \|x\|_2 \leq K \|a\|_\infty \|x\|_p, \quad x \in \text{Pol}(\mathbb{G}), \quad (4.9)$$

which yields that  $m_a^R$  extends to a bounded operator on  $L^p(\mathbb{G})$ . So  $\mathbf{E}$  is an interpolation set of  $M_R(L^p(\mathbb{G}))$ , and in particular by Proposition 4.2.5, we may find a constant  $K'$  such that

$$\max\{\|Q_\pi\|, \|Q_\pi^{-1}\|\} \leq K', \quad \pi \in \mathbf{E}.$$

Hence by Lemma 2.3.3, (4.9) also yields that for all  $a \in \ell^\infty(\mathbf{E})$ ,

$$\|m_a^L x\|_p \leq K \|m_a^L x\|_2 \leq K K' \|a\|_\infty \|x\|_2 \leq K K' \|a\|_\infty \|x\|_p, \quad x \in \text{Pol}(\mathbb{G}).$$

As a result, both maps  $m_a^L$  and  $m_a^R$  for  $a \in \ell^\infty(\mathbf{E})$  extend to bounded operators on  $L^p(\mathbb{G})$ , which means that  $\mathbf{E}$  is an interpolation set of  $M(L^p(\mathbb{G}))$ .

(2)  $\Rightarrow$  (1). Assume that (2) holds and we take  $K > 0$  to be the constant satisfying

$$\|m_a^L x\|_p \leq K \|a\|_\infty \|x\|_p, \quad \|m_a^R x\|_p \leq K \|a\|_\infty \|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}), \quad a \in \ell^\infty(\mathbf{E}). \quad (4.10)$$

For each  $\pi \in \text{Irr}(\mathbb{G})$ , since the operator  $Q_\pi$  is positive, we may fix a basis in  $H_\pi$  such that the associated matrix of  $Q_\pi$  is diagonal, and denote by  $u^{(\pi)} \in \mathbb{M}_{n_\pi}(C(\mathbb{G}))$  the representation matrix under this basis.

Let us first show that for  $\pi \in \text{Irr}(\mathbb{G})$  and  $1 \leq i \leq n_\pi$ , we have some constant  $C_0 > 0$  such that

$$\|u_{ii}^{(\pi)}\|_p^2 \leq C_0 p \|u_{ii}^{(\pi)}\|_2^2 = C_0 p d_\pi^{-1} (Q_\pi^{-1})_{ii}. \quad (4.11)$$

Take the  $n_\pi \times n_\pi$  matrices

$$a = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ (Q_\pi^{-\frac{1}{2}})_{11} & (Q_\pi^{-\frac{1}{2}})_{22} & \cdots & (Q_\pi^{-\frac{1}{2}})_{n_\pi n_\pi} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & \cdots & 0 & (Q_\pi^{-\frac{1}{2}})_{11} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (Q_\pi^{-\frac{1}{2}})_{22} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & (Q_\pi^{-\frac{1}{2}})_{n_\pi n_\pi} & 0 & \cdots & 0 \end{bmatrix}$$

where the nonzero coefficients are in the  $i$ -th row of  $a$ , and in the  $i$ -th column of  $b$ . Note that  $\|a\|_\infty = d_\pi^{1/2}$ . Let  $y = \sum_{j=1}^{n_\pi} (Q_\pi^{-\frac{1}{2}})_{jj} u_{ij}^{(\pi)}$ . Then by Proposition 2.1.2,

$$y = (\iota \otimes \text{Tr})[(1 \otimes b)u^{(\pi)}], \quad u_{ii}^{(\pi)} = d_\pi^{-1} m_a^R y. \quad (4.12)$$

According to (4.10) we have

$$\|u_{ii}^{(\pi)}\|_p \leq K d_\pi^{-\frac{1}{2}} \|y\|_p. \quad (4.13)$$

Let  $\varepsilon_j^{(\pi)} (1 \leq j \leq n_\pi)$  be a sequence of independent Rademacher variables on a probability space  $(\Omega, P)$  and write the  $n_\pi \times n_\pi$  matrix

$$\mathbf{e} = \begin{bmatrix} \varepsilon_1^{(\pi)} & 0 & \cdots & 0 \\ 0 & \varepsilon_2^{(\pi)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_{n_\pi}^{(\pi)} \end{bmatrix}.$$

Again by (4.10) and by Theorem 1.1.5 we have with some universal constant  $C > 0$ ,

$$\begin{aligned} \|y\|_p &= \int_\Omega \|m_{\mathbf{e}}^R y\|_p dP \leq K \int_\Omega \|m_{\mathbf{e}}^R y\|_p dP = K \int_\Omega \left\| \sum_{j=1}^{n_\pi} (Q_\pi^{-\frac{1}{2}})_{jj} \varepsilon_j^{(\pi)} u_{ij}^{(\pi)} \right\|_p dP \\ &\leq CK \sqrt{p} \|((Q_\pi^{-\frac{1}{2}})_{jj} u_{ij}^{(\pi)} D^{\frac{1}{p}})\|_{CR_p[L^p(\mathbb{G})]}, \end{aligned} \quad (4.14)$$

where

$$\begin{aligned} &\|((Q_\pi^{-\frac{1}{2}})_{jj} u_{ij}^{(\pi)} D^{\frac{1}{p}})\|_{CR_p[L^p(\mathbb{G})]} \\ &= \max\left\{ \left\| (D^{\frac{1}{p}} \sum_{j=1}^{n_\pi} (Q_\pi^{-1})_{jj} (u_{ij}^{(\pi)})^* u_{ij}^{(\pi)} D^{\frac{1}{p}})^{\frac{1}{2}} \right\|_{p, \mathbf{H}}, \left\| \left( \sum_{j=1}^{n_\pi} (Q_\pi^{-1})_{jj} u_{ij}^{(\pi)} D^{\frac{2}{p}} (u_{ij}^{(\pi)})^* \right)^{\frac{1}{2}} \right\|_{p, \mathbf{H}} \right\}. \end{aligned}$$

Recall (1.7) and that the matrix  $Q_\pi$  under the chosen basis is diagonal. We have

$$\begin{aligned} \sum_{j=1}^{n_\pi} (Q_\pi^{-1})_{jj} (u_{ij}^{(\pi)})^* u_{ij}^{(\pi)} &= \sum_{j=1}^{n_\pi} (Q_\pi^{-1})_{jj} S(u_{ji}^{(\pi)}) u_{ij}^{(\pi)} = \sum_{j=1}^{n_\pi} (Q_\pi^{-1})_{jj} (S^{-1} \circ S^2)(u_{ji}^{(\pi)}) u_{ij}^{(\pi)} \quad (4.15) \\ &= \sum_{j=1}^{n_\pi} (Q_\pi^{-1})_{jj} S^{-1}((Q_\pi)_{jj} u_{ji}^{(\pi)} (Q_\pi^{-1})_{ii}) u_{ij}^{(\pi)} \\ &= S^{-1} \left( \sum_{j=1}^{n_\pi} (Q_\pi^{-1})_{ii} (u_{ji}^{(\pi)})^* u_{ji}^{(\pi)} \right) = (Q_\pi^{-1})_{ii} S^{-1}(1) = (Q_\pi^{-1})_{ii} \end{aligned}$$

where the last line above follows from the fact that  $u^{(\pi)}$  is unitary. Recall Proposition 1.1.2 (1) and (1.3). We then have

$$u_{ij}^{(\pi)} D^{\frac{1}{p}} = D^{\frac{1}{p}} \sigma_{\frac{1}{p}}(u_{ij}^{(\pi)}) = D^{\frac{1}{p}} (Q_\pi^{-\frac{1}{p}})_{ii} u_{ij}^{(\pi)} (Q_\pi^{-\frac{1}{p}})_{jj},$$

and hence

$$D^{\frac{1}{p}} (u_{ij}^{(\pi)})^* = (Q_\pi^{-\frac{1}{p}})_{ii} (Q_\pi^{-\frac{1}{p}})_{jj} (u_{ij}^{(\pi)})^* D^{\frac{1}{p}}.$$

Therefore

$$\sum_{j=1}^{n_\pi} (Q_\pi^{-1})_{jj} u_{ij}^{(\pi)} D^{\frac{2}{p}} (u_{ij}^{(\pi)})^* = D^{\frac{1}{p}} \sum_{j=1}^{n_\pi} (Q_\pi^{-\frac{2}{p}})_{ii} (Q_\pi^{-\frac{2}{p}-1})_{jj} u_{ij}^{(\pi)} (u_{ij}^{(\pi)})^* D^{\frac{1}{p}}.$$

Recall that by Proposition 4.2.5 we have  $K_1 = \sup_{\pi \in \mathbf{E}} \|Q_\pi\| < \infty$ ,  $K_2 = \sup_{\pi \in \mathbf{E}} \|Q_\pi^{-1}\| < \infty$ , so the above expression can be estimated as

$$\sum_{j=1}^{n_\pi} (Q_\pi^{-1})_{jj} u_{ij}^{(\pi)} D^{\frac{2}{p}} (u_{ij}^{(\pi)})^* \leq K_2^{\frac{4}{p}+1} K_1 (Q_\pi^{-1})_{ii} D^{\frac{1}{p}} \sum_{j=1}^{n_\pi} u_{ij}^{(\pi)} (u_{ij}^{(\pi)})^* D^{\frac{1}{p}} = K_2^{\frac{4}{p}+1} K_1 (Q_\pi^{-1})_{ii} D^{\frac{2}{p}}.$$

Together with (4.15) we deduce

$$\|((Q_\pi^{-\frac{1}{2}})_{jj} u_{ij}^{(\pi)})\|_{CR_p[L^p(\mathbb{G})]}^2 \leq K_2^{\frac{4}{p}+1} K_1 (Q_\pi^{-1})_{ii}.$$

Back to (4.14) we get  $\|y\|_p^2 \leq C^2 K^2 K_2^{\frac{4}{p}+1} K_1 p (Q_\pi^{-1})_{ii}$  and by (4.13) we obtain

$$\|u_{ii}^{(\pi)}\|_p^2 \leq C^2 K^4 K_2^{\frac{4}{p}+1} K_1 p d_\pi^{-1} (Q_\pi^{-1})_{ii},$$

whence (4.11), as desired.

Now take  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ . Choose by polar decomposition and diagonalization two sequences of unitary matrices  $v = (v_\pi)_{\pi \in \mathbf{E}}$ ,  $v' = (v'_\pi)_{\pi \in \mathbf{E}} \in \ell^\infty(\mathbf{E})$  such that  $c_\pi := v_\pi v'_\pi \hat{x}(\pi) Q_\pi v_\pi^*$  is a diagonal matrix for each  $\pi$ . Write

$$c_\pi = \begin{bmatrix} c_{11}^{(\pi)} & 0 & \cdots & 0 \\ 0 & c_{22}^{(\pi)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{n_\pi n_\pi}^{(\pi)} \end{bmatrix}, \quad c_{11}^{(\pi)}, \dots, c_{n_\pi n_\pi}^{(\pi)} \in \mathbb{C}$$

and denote  $\mathbf{c} = (c_\pi)_{\pi \in \mathbf{E}}$ . By (4.10)-(4.11) and Theorem 1.1.5 we have with some universal constant  $C > 0$ ,

$$\|x\|_p = \|m_{v'^* v^*}^R m_v^L \left( \sum_{\pi \in \mathbf{E}} d_\pi (\iota \otimes \text{Tr}) [(1 \otimes c_\pi) u^{(\pi)}] \right)\|_p \leq K^2 \left\| \sum_{\pi \in \mathbf{E}} d_\pi (\iota \otimes \text{Tr}) [(1 \otimes c_\pi) u^{(\pi)}] \right\|_p$$

$$\begin{aligned}
&\leq K^3 \int_{\Omega} \|m_{\mathfrak{c}}^R \left( \sum_{\pi \in \mathbf{E}} d_{\pi} (\iota \otimes \text{Tr}) [(1 \otimes c_{\pi}) u^{(\pi)}] \right)\|_p dP = K^3 \int_{\Omega} \left\| \sum_{\pi \in \mathbf{E}} d_{\pi} \sum_{j=1}^{n_{\pi}} \varepsilon_j^{(\pi)} c_{jj}^{(\pi)} u_{jj}^{(\pi)} \right\|_p dP \\
&\leq K^3 C \sqrt{p} \left( \sum_{\pi \in \mathbf{E}} \sum_{j=1}^{n_{\pi}} d_{\pi}^2 (c_{jj}^{(\pi)})^2 \|u_{jj}^{(\pi)}\|_p^2 \right)^{\frac{1}{2}} \leq C_0^{\frac{1}{2}} C p K^3 \left( \sum_{\pi \in \mathbf{E}} \sum_{j=1}^{n_{\pi}} d_{\pi}^2 (c_{jj}^{(\pi)})^2 \|u_{jj}^{(\pi)}\|_2^2 \right)^{\frac{1}{2}} \\
&= C_0^{\frac{1}{2}} C p K^3 \left\| \sum_{\pi \in \mathbf{E}} \sum_{j=1}^{n_{\pi}} d_{\pi} c_{jj}^{(\pi)} u_{jj}^{(\pi)} \right\|_2
\end{aligned}$$

where the last equality follows from the fact that  $\{u_{ii}^{(\pi)} : 1 \leq i \leq n_{\pi}\}$  are orthogonal with respect to  $h$  according to (1.2). Using Lemma 2.3.1 we have

$$\left\| \sum_{\pi \in \mathbf{E}} \sum_{j=1}^{n_{\pi}} d_{\pi} c_{jj}^{(\pi)} u_{jj}^{(\pi)} \right\|_2 = \|m_{vv}^R m_{v^*}^L x\|_2 \leq K_1 K_2 \|x\|_2.$$

Hence the above two inequalities together yield

$$\|x\|_p \leq C_0^{\frac{1}{2}} C p K^3 K_1 K_2 \|x\|_2.$$

So we prove that  $\mathbf{E}$  is a  $\Lambda(p)$ -set.

Finally let us discuss (6) and (7). In fact, if  $\mathbf{E}$  is symmetric, we note that  $X = \text{Pol}_{\mathbf{E}}(\mathbb{G})$  satisfies the assumption of Lemma 2.3.3. So if (6) holds, by Lemma 2.3.3 we have for all  $a \in \ell^{\infty}(\mathbf{E})$ ,

$$\|m_a^R x\|_p \leq K \|Q^{1/2} a Q^{-1/2}\|_{\infty} \|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}). \quad (4.16)$$

and hence using Proposition 4.2.5 and the assumption in (6), we obtain the inequality

$$\max\{\|Q_{\pi}\|, \|Q_{\pi}^{-1}\|\} \leq K', \quad \pi \in \mathbf{E}$$

for some constant  $K' > 0$ . Thus coming back to (4.16) again, we see that the condition (7) holds as well. Conversely, we may also show in the same way that (7) implies (6). So in other words the assertions (6) and (7) are equivalent, and in particular they are equivalent to the assertion (2).  $\square$

### 4.2.2 $\Lambda(p)$ -sets and Sidon sets

It is well known and not difficult to see that when  $\mathbb{G}$  is a compact group  $G$  or the dual quantum group of a discrete group  $\Gamma$ , any Sidon set  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  (or more generally, an interpolation set of  $M(L^{\infty}(\mathbb{G}))$ ) is a  $\Lambda(p)$ -set for  $1 < p < \infty$  ([HR70, Har99]). The same question for an arbitrary compact quantum group is however more delicate. To the best knowledge of the author, the only effort towards this direction before our work is the following property recently given by Blendek and Michaliček, as a main result in [BM13]: if  $\mathbb{G}$  is a compact quantum group of *Kac type* and if  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a Sidon set satisfying the *Helgason-Sidon* condition, then there exists  $K > 0$  such that for all finitely supported sequences  $(c_{\pi})_{\pi \in \mathbf{E}} \subset \mathbb{C}$  and  $x = \sum_{\pi \in \mathbf{E}} c_{\pi} \chi_{\pi} \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ ,

$$\|x\|_2 \leq K \|x\|_1.$$

Observe that this result requires many more restrictions on the subset  $\mathbf{E}$  than in the classical cases while the obtained inequality is much weaker. However, compared to the classical one, its proof utilizes quite nontrivial tools such as “modified Rademacher functions”.

Now based on our results in the previous subsection, we are able to give an alternative and more concise argument, which completely removes the non-expected restrictions in

[BM13] and moreover directly establishes the nice relation between Sidon sets and  $\Lambda(p)$ -sets for all  $1 < p < \infty$ . In fact as in the classical case, we have the stronger result below. Recall that in Theorem 4.1.15 we have some generalized notions of the Sidon set.

**Theorem 4.2.8.** *Assume that  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is an interpolation set of  $M(L^\infty(\mathbb{G}))$ . Then  $\mathbf{E}$  is a  $\Lambda(p)$ -set for all  $1 < p < \infty$ .*

*Proof.* Assume that  $\mathbf{E}$  is an interpolation set of  $M(L^\infty(\mathbb{G}))$  with constant  $K$ . Let  $a \in \ell^\infty(\mathbf{E})$ . Then  $a$  extends to a bounded multiplier  $\tilde{a} \in M(L^\infty(\mathbb{G}))$ , and by Proposition 2.3.5,

$$\|Q^{1/4}\tilde{a}Q^{-1/4}\|_\infty \leq \|\tilde{a}\|_{M(L^\infty(\mathbb{G}))} \leq K\|a\|_\infty.$$

Consider

$$T_z = m_{a^{(z)}}^L, \quad a_\pi^{(z)} = Q_\pi^{-\frac{z}{2} + \frac{1}{4}} \tilde{a}_\pi Q_\pi^{\frac{z}{2} - \frac{1}{4}}, \quad \pi \in \text{Irr}(\mathbb{G}), \quad z = t_1 + it_2, \quad 0 \leq t_1 \leq 1, t_2 \in \mathbb{R}.$$

Observe that by Lemma 2.3.1, the operators  $m_{a^{(z)}}^L$  for  $z = 1 + it$  are bounded on  $L^2(\mathbb{G})$  with norm

$$\|m_{a^{(z)}}^L\|_{B(L^2(\mathbb{G}))} = \|Q^{1/4}\tilde{a}Q^{-1/4}\|_\infty \leq K\|a\|_\infty.$$

So by the Stein interpolation theorem (see e.g. [Lun09, Theorem 2.7]) for  $2 \leq p < \infty$  we have  $\|T_{2/p}x\|_p \leq K_0\|a\|_\infty\|x\|_p$  for  $x \in L^p(\mathbb{G})$ . Let  $K_1 = \sup_{\pi \in \mathbf{E}} \|Q_\pi\|$ ,  $K_2 = \sup_{\pi \in \mathbf{E}} \|Q_\pi^{-1}\|$ . Both  $K_1$  and  $K_2$  are finite by Proposition 4.2.5. Rewrite  $b_\pi = Q_\pi^{-\frac{1}{p} + \frac{1}{4}} a_\pi Q_\pi^{\frac{1}{p} - \frac{1}{4}}$  and the above argument yields that for  $b \in \ell^\infty(\mathbf{E})$  and  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ ,

$$\|m_b^L x\|_p \leq K\|a\|_\infty\|x\|_p = K_0 \sup_{\pi \in \mathbf{E}} \|Q_\pi^{\frac{1}{p} - \frac{1}{4}} b_\pi Q_\pi^{-\frac{1}{p} + \frac{1}{4}}\|_\infty \|x\|_p \leq K_0 K_1^{\frac{1}{p} - \frac{1}{4}} K_2^{\frac{1}{p} - \frac{1}{4}} \|b\|_\infty \|x\|_p.$$

Similar argument also applies to  $m_a^R$  (where the above  $T_z$  should be simply taken as  $m_{Q^{1/4}\tilde{a}Q^{-1/4}}^R$  identically). Now we may take  $K > 0$  to be the constant satisfying

$$\|m_a^L x\|_p \leq K\|a\|_\infty\|x\|_p, \quad \|m_a^R x\|_p \leq K\|a\|_\infty\|x\|_p, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}), \quad a \in \ell^\infty(\mathbf{E}).$$

Thus  $\mathbf{E}$  is a  $\Lambda(p)$ -set for all  $1 < p < \infty$  according to Theorem 4.2.7.  $\square$

By Theorem 4.1.15, we have the following corollary, as desired.

**Corollary 4.2.9.** *If  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a Sidon set, then  $\mathbf{E}$  is a  $\Lambda(p)$ -set for all  $1 < p < \infty$ .*

Below we give two more generalizations of the main result in [BM13]. For convenience we introduce the following definitions.

**Definition 4.2.10.** We say that  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a *central interpolation set* of  $M(L^\infty(\mathbb{G}))$  if for each bounded scalar sequence  $(c_\pi)_{\pi \in \mathbf{E}} \subset \mathbb{C}$ , there exists a bounded sequence  $(\tilde{c}_\pi)_{\pi \in \text{Irr}(\mathbb{G})} \subset \mathbb{C}$  with  $\tilde{c}_\pi = c_\pi$  for  $\pi \in \mathbf{E}$  such that  $\tilde{c} = (\tilde{c}_\pi \text{Id}_\pi)_{\pi \in \text{Irr}(\mathbb{G})}$  is a bounded multiplier on  $L^\infty(\mathbb{G})$ .

We remark that if  $\mathbf{E}$  is a Sidon set or an interpolation set of  $M(L^\infty(\mathbb{G}))$ , or if  $\mathbb{G}$  is of Kac type and  $\mathbf{E}$  is a central Sidon set which will be introduced in Section 4.3, then  $\mathbf{E}$  is such a central interpolation set of  $M(L^\infty(\mathbb{G}))$ .

**Theorem 4.2.11.** *Let  $\mathbb{G}$  be a compact quantum group. Assume that  $\mathbf{E}$  is a central interpolation set of  $M(L^\infty(\mathbb{G}))$ .*

- (1) *If  $1 < p < \infty$  and  $\sup_{\pi \in \mathbf{E}} \|\chi_\pi\|_p < \infty$ , then  $\mathbf{E}$  is a central  $\Lambda(p)$ -set;*
- (2) *If  $\sup_{\pi \in \mathbf{E}} d_\pi < \infty$ , then  $\mathbf{E}$  is a  $\Lambda(p)$ -set for  $1 < p < \infty$ .*



*Proof.* As is in the previous proof, for each  $\pi \in \text{Irr}(\mathbb{G})$ , we fix a basis in  $H_\pi$  such that the associated matrix of  $Q_\pi$  is diagonal, and denote by  $u^{(\pi)} \in \mathbb{M}_{n_\pi}(C(\mathbb{G}))$  the representation matrix under this basis.

Denote by  $(\varepsilon_\pi)_{\pi \in \mathbf{E}}$  a Rademacher sequence on a probability space  $(\Omega, P)$  and write  $\mathbf{e} = (\varepsilon_\pi \text{Id}_\pi)_{\pi \in \mathbf{E}} \in \ell^\infty(\mathbf{E})$ . The same argument as in the proof of Theorem 4.2.8 yields that we may find  $K > 0$  such that for  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ ,  $\omega \in \Omega$  and  $2 < p < \infty$ ,

$$\|x\|_p = \|m_{\mathbf{e}(\omega)}^R x\|_p \leq K \|m_{\mathbf{e}(\omega)}^R x\|_p.$$

Integrating the inequality over  $\omega \in \Omega$ , we get

$$\|x\|_p \leq K \int_{\Omega} \left\| \sum_{\pi \in \mathbf{E}} d_\pi \varepsilon_\pi (\iota \otimes \text{Tr}) [(1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}] \right\|_p dP.$$

By Theorem 1.1.5, we have with some  $K_0 > 0$ ,

$$\begin{aligned} & \int_{\Omega} \left\| \sum_{\pi \in \mathbf{E}} d_\pi \varepsilon_\pi (\iota \otimes \text{Tr}) [(1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}] \right\|_p dP \\ & \leq K_0 \sqrt{p} \left( \sum_{\pi \in \mathbf{E}} \|d_\pi (\iota \otimes \text{Tr}) [(1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}]\|_p^2 \right)^{1/2}. \end{aligned}$$

Therefore

$$\|x\|_p \leq K K_0 \sqrt{p} \left( \sum_{\pi \in \mathbf{E}} \|d_\pi (\iota \otimes \text{Tr}) [(1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}]\|_p^2 \right)^{1/2}. \quad (4.17)$$

(1) If  $2 < p < \infty$  and  $x = \sum_{\pi \in \mathbf{E}} c_\pi \chi_\pi \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$  with  $(c_\pi)_{\pi \in \mathbf{E}} \subset \mathbb{C}$ , then the above inequality (4.17) reads

$$\|x\|_p \leq K K_0 \sqrt{p} \left( \sum_{\pi \in \mathbf{E}} |c_\pi|^2 \|\chi_\pi\|_p^2 \right)^{1/2}.$$

Note that by (1.2) and the choice of basis in  $H_\pi$ , we have  $\|\chi_\pi\|_2^2 = h(\chi_\pi^* \chi_\pi) = 1$ . Also recall that  $\chi_\pi$  are orthogonal with respect to  $h$ . Thus the condition  $K_1 = \sup_{\pi \in \mathbf{E}} \|\chi_\pi\|_p < \infty$  implies

$$\|x\|_p \leq K K_1 K_0 \sqrt{p} \left( \sum_{\pi \in \mathbf{E}} |c_\pi|^2 \|\chi_\pi\|_2^2 \right)^{1/2} = \|x\|_2.$$

Therefore  $\mathbf{E}$  is a central  $\Lambda(p)$ -set.

(2) Assume  $K_2 = \sup_{\pi \in \mathbf{E}} d_\pi < \infty$ . We need to show that the right term of the inequality (4.17) is not more than  $\|x\|_2$ , up to a constant independent of  $x$ . Note that by traciality of  $\text{Tr}$ , we have

$$(\iota \otimes \text{Tr}) [(1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}] = (\iota \otimes \text{Tr}) [u^{(\pi)} (1 \otimes \hat{x}(\pi) Q_\pi)]$$

Also, we note that the map  $d_\pi^{-1} (\iota \otimes \text{Tr}(\cdot (1 \otimes Q_\pi)))$  is unital completely positive on  $C(\mathbb{G}) \otimes B(H_\pi)$  for each  $\pi \in \text{Irr}(\mathbb{G})$ , so by the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left| d_\pi^{-1} (\iota \otimes \text{Tr}) [u^{(\pi)} (1 \otimes \hat{x}(\pi) Q_\pi)] D^{1/p} \right|^2 \\ & = D^{1/p} \left| d_\pi^{-1} (\iota \otimes \text{Tr}) [u^{(\pi)} (1 \otimes \hat{x}(\pi) Q_\pi)] \right|^2 D^{1/p} \\ & \leq D^{1/p} d_\pi^{-1} (\iota \otimes \text{Tr}(\cdot Q_\pi)) [(1 \otimes \hat{x}(\pi)^*) (u^{(\pi)})^* u^{(\pi)} (1 \otimes \hat{x}(\pi))] D^{1/p} \\ & = d_\pi^{-1} D^{1/p} \text{Tr}(|\hat{x}(\pi)|^2 Q_\pi) D^{1/p} = d_\pi^{-1} \text{Tr}(|\hat{x}(\pi)|^2 Q_\pi) D^{2/p}. \end{aligned}$$

Hence together with Proposition 1.1.2 (5),

$$\begin{aligned} & \sum_{\pi \in \mathbf{E}} \|d_\pi(\iota \otimes \text{Tr})[(1 \otimes \hat{x}(\pi)Q_\pi)u^{(\pi)}]\|_p^2 = \sum_{\pi \in \mathbf{E}} \| |d_\pi(\iota \otimes \text{Tr})[(1 \otimes \hat{x}(\pi)Q_\pi)u^{(\pi)}]D^{1/p}|^2 \|_{p/2, \mathbb{H}} \\ & \leq K_2^2 \sum_{\pi \in \mathbf{E}} d_\pi^2 \| |d_\pi^{-1}(\iota \otimes \text{Tr})[(1 \otimes \hat{x}(\pi)Q_\pi)u^{(\pi)}]D^{1/p}|^2 \|_{p/2, \mathbb{H}} \\ & \leq K_2^2 \sum_{\pi \in \mathbf{E}} d_\pi^2 \| |d_\pi^{-1} \text{Tr}(|\hat{x}(\pi)|^2 Q_\pi)D^{2/p} \|_{p/2, \mathbb{H}} = \sum_{\pi \in \mathbf{E}} d_\pi \text{Tr}(|\hat{x}(\pi)|^2 Q_\pi) = \|\hat{x}\|_2^2 = \|x\|_2^2. \end{aligned}$$

Now back to (4.17) we get  $\|x\|_p \leq KK_0K_2\sqrt{p}\|x\|_2$ , as desired.  $\square$

*Remark 4.2.12.* Let us make a few remarks on the constant of  $\Lambda(p)$ -sets. In the proof of Theorem 4.2.7 and Theorem 4.2.8, we have shown that, if  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is an interpolation set of  $M(L^p(\mathbb{G}))$  with constant  $K$  for some  $1 < p < \infty$ , then  $\mathbf{E}$  is a  $\Lambda(p)$ -set with constant  $c_1K^{c_2}p$  with two universal constants  $c_1, c_2 > 0$ ; and if  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is an interpolation set of  $M(L^\infty(\mathbb{G}))$ , then  $\mathbf{E}$  is a  $\Lambda(p)$ -set with constant  $c'_1K^{c'_2}p$  with two universal constants  $c'_1, c'_2 > 0$  for all  $1 < p < \infty$ . It seems that these constants should be improved. Indeed, it is well-known that if  $\mathbb{G}$  is a compact group or the dual quantum group of a discrete group, the constants above can be improved to  $c_1K^{c_2}\sqrt{p}$  and  $c'_1K^{c'_2}\sqrt{p}$  respectively (see [HR70, Har99]). On the other hand, we see that, for some subclass of Sidon sets as in Theorem 4.2.11, we may also obtain the constant in the form  $c_1K^{c_2}\sqrt{p}$ .

**Example 4.2.13.** (1) Following the notation in Example 4.1.17, we consider  $\mathbb{G} = \prod_{k \geq 1} \mathbb{G}_k$ ,  $\mathbb{G}_k = U_{N_k}^+$  and  $\mathbf{E} = \{u^{(k)} : k \geq 1\} \subset \text{Irr}(\mathbb{G})$ . We saw in Example 4.1.17 that  $\mathbf{E}$  is an interpolation set of  $M(L^\infty(\mathbb{G}))$ , and hence by Theorem 4.2.8  $\mathbf{E}$  is a  $\Lambda(p)$ -set for all  $1 < p < \infty$ . Alternatively, recall the Haagerup type inequality shown in Brannan [Bra12, Theorem 6.3]: for  $k \geq 1$ ,  $x_{ij} \in \mathbb{C}$ ,  $1 \leq i, j \leq N_k$

$$\left\| \sum_{i,j=1}^{N_k} x_{ij} u_{ij}^{(k)} \right\|_p \leq C \left\| \sum_{i,j=1}^{N_k} x_{ij} u_{ij}^{(k)} \right\|_2$$

for a universal constant  $C$ . So by Theorem 1.1.5 and the standard argument as in the beginning of the proof of Theorem 4.2.11, we have the following Khintchine type inequality: for a universal constant  $K$  and for all  $2 \leq p < \infty$  and all finitely supported sequences  $(A_k) \in \prod_k \mathbb{M}_{N_k}$ ,

$$\begin{aligned} \left\| \sum_{k \geq 1} N_k (\iota \otimes \text{Tr})[(1 \otimes A_k)u^{(k)}] \right\|_{L^p(\mathbb{G})} & \leq K\sqrt{p} \left( \sum_k \|N_k (\iota \otimes \text{Tr})[(1 \otimes A_k)u^{(k)}]\|_{L^p(\mathbb{G})}^2 \right)^{1/2} \\ & \leq CK\sqrt{p} \left( \sum_k N_k \text{Tr}(|A_k|^2) \right)^{1/2}. \end{aligned}$$

If  $n_k = 1$  for all  $k$ , then the above inequality reduces to the classical Khintchine inequalities.

(2) Follow the notation in Example 4.1.18. We consider the sequence  $(q_n)_{n \geq 1} \subset [q, 1]$  with  $q := \inf_n q_n > 0$  and the associated quantum group  $\mathbb{G} = \prod_{n \geq 1} \text{SU}_{q_n}(2)$ . In Example 4.1.18 we proved that  $\mathbf{E} = \{u_n : n \geq 1\} \subset \text{Irr}(\mathbb{G})$  is a Sidon set for  $\mathbb{G}$ . Note that the associated matrix  $Q_n := Q_{u_n}$  is diagonal with entries  $\{q_n^{-1}, q_n\}$  under the standard basis and hence  $\sup_n \dim_q(u_n) = \sup_n (q_n + q_n^{-1}) \leq 1 + q^{-1} < \infty$ . So by Theorem 4.2.11,  $\mathbf{E}$  is also a  $\Lambda(p)$ -set for  $2 \leq p < \infty$  and we obtain the following Khintchine type inequalities: there exists a constant  $K > 0$  (depending on  $c$ ) so that for all finitely supported sequences  $(A_n) \in \prod_{n \geq 1} \mathbb{M}_2$ ,

$$\left( \sum_{n \geq 1} d_n \text{Tr}(|A_n|^2 Q_n) \right)^{1/2} \leq \left\| \sum_{n \geq 1} d_n (\iota \otimes \text{Tr})[(1 \otimes A_n Q_n)u_n] \right\|_{L^p(\mathbb{G})} \leq K\sqrt{p} \left( \sum_{n \geq 1} d_n \text{Tr}(|A_n|^2 Q_n) \right)^{1/2}$$

where  $d_n = q_n + q_n^{-1}$ . Note that by Proposition 4.2.5 and Theorem 4.2.7, if  $q = 0$  and  $q_n \rightarrow 0$ , the subset  $\mathbf{E}$  defined as above cannot be a  $\Lambda(p)$ -set for any  $2 < p < \infty$ .

### 4.2.3 Independence of the interpolation parameters for $L_{(\theta)}^p(\mathbb{G})$

In this subsection we would like to show that our definition of  $\Lambda(p)$ -sets does not depend on different interpolation parameters of  $L^p$ -spaces associated to a compact quantum group  $\mathbb{G}$ . Recall that in Chapter 1, we introduced the space  $L^p(\mathbb{G})$  as the complex interpolation space  $(L^\infty(\mathbb{G}), L^\infty(\mathbb{G})_*)_{1/p}$  associated to the compatible couple  $(L^\infty(\mathbb{G}), L^\infty(\mathbb{G})_*)$  given by the embedding

$$L^\infty(\mathbb{G}) \hookrightarrow L^\infty(\mathbb{G})_*, \quad x \mapsto h(\cdot x), \quad x \in L^\infty(\mathbb{G}).$$

However in [Kos84] Kosaki provides some other possibilities to define the complex interpolation scale  $(L^p(\mathbb{G}))_{1 \leq p \leq \infty}$ . More precisely, fix a parameter  $0 \leq \theta \leq 1$ , we may consider the compatible couple  $(L^\infty(\mathbb{G}), L^\infty(\mathbb{G})_*)^{(\theta)}$  given by the embedding

$$L^\infty(\mathbb{G}) \hookrightarrow L^\infty(\mathbb{G})_*, \quad x \mapsto h(\cdot \sigma_{-\theta i}(x)), \quad x \in L^\infty(\mathbb{G}),$$

and define the complex interpolation space

$$L_{(\theta)}^p(\mathbb{G}) = (L^\infty(\mathbb{G}), L^\infty(\mathbb{G})_*)_{1/p}^{(\theta)}.$$

Note that  $L_{(0)}^p(\mathbb{G})$  coincides with the space  $L^p(\mathbb{G})$  defined before. In the language of Haagerup's  $L^p$ -spaces, we have considered the embedding

$$L^\infty(\mathbb{G}) \hookrightarrow L^{1,\mathbf{H}}(\mathbb{G}), \quad x \mapsto D^\theta x D^{1-\theta} = \sigma_{-\theta i}(x) D, \quad x \in L^\infty(\mathbb{G}),$$

and for each  $1 \leq p \leq \infty$  we have the isometric isomorphism

$$L_{(\theta)}^p(\mathbb{G}) \rightarrow L^{p,\mathbf{H}}(\mathbb{G}), \quad x \mapsto D^{\frac{\theta}{p}} x D^{\frac{1-\theta}{p}}, \quad x \in L^\infty(\mathbb{G}).$$

In particular,

$$\|x\|_{L_{(\theta)}^p(\mathbb{G})} = \|\sigma_{-\frac{\theta}{p} i}(x) D^{\frac{1}{p}}\|_{L^{p,\mathbf{H}}(\mathbb{G})} = \|\sigma_{-\frac{\theta}{p} i}(x)\|_{L_{(0)}^p(\mathbb{G})}, \quad x \in \text{Pol}(\mathbb{G}). \quad (4.18)$$

The spaces  $L_{(\theta)}^p(\mathbb{G})$  and  $L_{(\theta')}^p(\mathbb{G})$  for different parameters  $\theta, \theta'$  are isometric as Banach spaces, but one needs to be careful with the parameter  $\theta$  when doing the Fourier analysis on  $\mathbb{G}$ , for which we refer to [Cas13, Section 7] for some related discussions. So returning back to the topic on  $\Lambda(p)$ -sets, it is natural to ask if the notion of  $\Lambda(p)$ -sets is independent of the choice of the parameter  $\theta$ . In the following we give an affirmative answer.

**Proposition 4.2.14.** *Let  $\mathbb{G}$  be a compact quantum group and let  $2 < p < \infty$ ,  $0 \leq \theta \leq 1$ . Then  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a  $\Lambda(p)$ -set for  $L_{(\theta)}^p(\mathbb{G})$ , that is, there exists a constant  $K > 0$  with*

$$\|x\|_{L_{(\theta)}^p(\mathbb{G})} \leq K \|x\|_{L_{(\theta)}^2(\mathbb{G})}, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}),$$

*if and only if it is a  $\Lambda(p)$ -set for  $L_{(0)}^p(\mathbb{G})$  in the sense of Definition 4.2.1.*

*Proof.* Assume that  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a  $\Lambda(p)$ -set for  $L_{(\theta)}^p(\mathbb{G})$  with the constant  $K$  given above. Then for all  $a \in \ell^\infty(\mathbf{E})$ , together with the equality (4.18) we have for  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ ,

$$\|m_a^R x\|_{L_{(\theta)}^p(\mathbb{G})} \leq K \|m_a^R x\|_{L_{(\theta)}^2(\mathbb{G})} = K \|\sigma_{-i\frac{\theta}{2}}(m_a^R x)\|_{L_{(0)}^2(\mathbb{G})}. \quad (4.19)$$

By Lemma 2.3.4, we know that

$$\mathcal{F}(\sigma_{-i\frac{\theta}{2}}(m_a^R x)) = Q^{\frac{\theta}{2}} a \hat{x} Q^{\frac{\theta}{2}}.$$

Hence by Proposition 2.1.2,

$$\|\sigma_{-i\frac{\theta}{2}}(m_a^R x)\|_{L_{(0)}^2(\mathbb{G})}^2 = \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \text{Tr}(|Q_\pi^{\frac{\theta}{2}} a \hat{x}(\pi) Q_\pi^{\frac{\theta}{2}}|^2 Q_\pi). \quad (4.20)$$

Let  $\pi \in \mathbf{E}$  and without loss of generality we choose an appropriate basis of  $H_\pi$  so that  $Q_\pi$  is diagonal under this basis. Assume that  $a \in c_c(\mathbf{E})$  and take  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ , then the above inequalities gives

$$\begin{aligned} \|m_a^R x\|_{L_{(\theta)}^p(\mathbb{G})}^2 &\leq K^2 \|Q^{\frac{\theta}{2}} a Q^{-\frac{\theta}{2}}\|_\infty^2 \sum_{\pi \in \mathbf{E}} d_\pi \text{Tr}(|Q_\pi^{\frac{\theta}{2}} \hat{x}(\pi) Q_\pi^{\frac{\theta}{2}}|^2 Q_\pi) \\ &= K^2 \|Q^{\frac{\theta}{2}} a Q^{-\frac{\theta}{2}}\|_\infty^2 \|\sigma_{-i\frac{\theta}{2}}(x)\|_{L_{(0)}^2(\mathbb{G})}^2 = K^2 \|Q^{\frac{\theta}{2}} a Q^{-\frac{\theta}{2}}\|_\infty^2 \|x\|_{L_{(\theta)}^2(\mathbb{G})}^2 \\ &\leq K^2 \|Q^{\frac{\theta}{2}} a Q^{-\frac{\theta}{2}}\|_\infty^2 \|x\|_{L_{(\theta)}^p(\mathbb{G})}^2. \end{aligned}$$

And a similar inequality can be proved for the map  $m_a^L$ . Then following the idea in the proof of Proposition 4.2.5, we may find a constant  $K' > 0$  such that

$$\max\{\|Q_\pi\|, \|Q_\pi^{-1}\|\} \leq K', \quad \pi \in \mathbf{E}.$$

Therefore together with (4.18), Lemma 2.3.4 and Lemma 2.3.1, we get for all  $x \in \text{Pol}_{\mathbf{E}}(\mathbb{G})$ ,

$$\begin{aligned} \|x\|_{L_{(\theta)}^p(\mathbb{G})} &= \|\sigma_{i\frac{\theta}{p}}(x)\|_{L_{(\theta)}^p(\mathbb{G})} \leq K \|\sigma_{i\frac{\theta}{p}}(x)\|_{L_{(\theta)}^2(\mathbb{G})} = K \|\sigma_{i(\frac{1}{p}-\frac{1}{2})\theta}(x)\|_{L_{(0)}^2(\mathbb{G})} \\ &= K \|\mathcal{F}(\sigma_{i(\frac{1}{p}-\frac{1}{2})\theta}(x))\|_{\ell^2(\hat{\mathbb{G}})} = K \|Q^{\frac{1}{2}-\frac{1}{p}} \hat{x} Q^{\frac{1}{2}-\frac{1}{p}}\|_{\ell^2(\hat{\mathbb{G}})} \\ &= K \|m_{Q^{\frac{1}{2}-\frac{1}{p}}}^R m_{Q^{\frac{1}{2}-\frac{1}{p}}}^L x\|_{L_{(0)}^2(\mathbb{G})} \leq K (K')^{1-\frac{2}{p}} \|x\|_{L_{(\theta)}^2(\mathbb{G})}. \end{aligned}$$

So  $\mathbf{E}$  is a  $\Lambda(p)$ -sets for  $L_{(0)}^p(\mathbb{G})$ .

Note that if conversely  $\mathbf{E}$  is a  $\Lambda(p)$ -set for  $L_{(0)}^p(\mathbb{G})$ , then we have already shown in Proposition 4.2.5 that there exists a constant  $K' > 0$  such that

$$\max\{\|Q_\pi\|, \|Q_\pi^{-1}\|\} \leq K', \quad \pi \in \mathbf{E}.$$

So a similar estimation as above yields that  $\mathbf{E}$  must be a  $\Lambda(p)$ -sets for  $L_{(\theta)}^p(\mathbb{G})$ . Therefore the proof is complete.  $\square$

Finally we remark that all the discussions in the previous sections can be in fact reproduced for  $L_{(\theta)}^p(\mathbb{G})$  with the similar idea, and we omit the details.

#### 4.2.4 Existence of $\Lambda(p)$ -sets

In this short subsection we discuss the existence of  $\Lambda(p)$ -sets for compact quantum groups. We refer to the following result proved in the appendix: let  $M$  be a von Neumann algebra with a normal faithful state  $\varphi$  and  $B = \{x_k : k \geq 1\} \subset M$  be an orthonormal system with respect to  $\varphi$  such that  $\sup_k \|x_k\|_\infty < \infty$ , then there exists an infinite subset  $Y \subset B$  and a constant  $C_p$  such that  $\|x\|_p \leq C_p \|x\|_2$  for all  $x \in \text{span}(Y)$ . Immediately we deduce the existence of  $\Lambda(p)$ -sets with uniform dimension assumptions.

**Theorem 4.2.15.** *Let  $\mathbb{G}$  be a compact quantum group. Let  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  be an infinite subset with  $\sup_{\pi \in \mathbf{E}} d_\pi < \infty$ . Then for each  $1 < p < \infty$ , there exists an infinite subset  $\mathbf{F} \subset \mathbf{E}$  which is a  $\Lambda(p)$ -set for  $\mathbb{G}$ .*

*Proof.* Denote  $D_0 = \sup_{\pi \in \mathbf{E}} d_\pi < \infty$  and fix  $2 \leq p < \infty$ . Then also  $n_\pi = \dim(H_\pi) \leq d_\pi \leq D_0$ . Choose an appropriate basis of  $H_\pi$  such that the matrix  $Q_\pi$  is diagonal under this basis. For each  $\pi \in \mathbf{E}$ ,  $1 \leq i, j \leq n_\pi$ , write  $v_{ij}^{(\pi)} = \|u_{ij}^{(\pi)}\|_2^{-1} u_{ij}^{(\pi)}$ . By (1.2),  $\|v_{ij}^{(\pi)}\|_\infty \leq \|u_{ij}^{(\pi)}\|_2^{-1} \leq d_\pi^2 \leq D_0^2$ . Consider  $B_0^{(\pi)} = \{v_{ij}^{(\pi)} : 1 \leq i, j \leq n_\pi\}$  for  $\pi \in \mathbf{E}$ , and  $B_0 = \cup_{\pi \in \mathbf{E}} B_0^{(\pi)}$ . Then  $B_0$  is an orthogonal system since  $Q_\pi$  is chosen diagonal. Write  $\mathbf{E}_0 = \mathbf{E}$ . According to the theorem in the appendix, we can find an infinite subset  $B_1 \subset B_0$  with constant  $C_1 > 0$  such that for all finitely many  $c_1, \dots, c_n \in \mathbb{C}$ ,  $x_1, \dots, x_n \in B_1$ ,

$$\left\| \sum_{l=1}^n c_l x_l \right\|_p \leq C_1 \left\| \sum_{l=1}^n c_l x_l \right\|_2.$$

Set  $\mathbf{E}_1 = \{\pi \in \mathbf{E}_0 : \exists v_{ij}^{(\pi)} \in B_1\}$  and let  $B_1^{(\pi)} = \{v_{ij}^{(\pi)} \in B_1 : 1 \leq i, j \leq n_\pi\}$  for  $\pi \in \mathbf{E}_1$ . The last set is non-empty. Then

$$B_1 = \cup_{\pi \in \mathbf{E}_1} B_1^{(\pi)}, \quad \text{Card}(B_0^{(\pi)} \setminus B_1^{(\pi)}) \leq n_\pi^2 - 1 \leq D_0^2 - 1, \quad \pi \in \mathbf{E}_1.$$

Repeating inductively the above procedures, if  $k \geq 1$  and if  $\exists \pi \in \mathbf{E}_k$ ,  $B_0^{(\pi)} \setminus (\cup_{l=1}^k B_l^{(\pi)}) \neq \emptyset$ , we construct the proper subsets  $\mathbf{E}_{k+1} \subset \mathbf{E}_k$ ,  $B_{k+1}^{(\pi)} \subset B_0^{(\pi)} \setminus (\cup_{l=1}^k B_l^{(\pi)})$ ,  $B_{k+1} = \cup_{\pi \in \mathbf{E}_{k+1}} B_{k+1}^{(\pi)} \subset B_0 \setminus (\cup_{l=1}^k B_l)$  and a constant  $C_{k+1} > 0$  such that for all finitely many  $c_1, \dots, c_n \in \mathbb{C}$ ,  $x_1, \dots, x_n \in B_{k+1}$ ,

$$\left\| \sum_{l=1}^n c_l x_l \right\|_p \leq C_{k+1} \left\| \sum_{l=1}^n c_l x_l \right\|_2$$

and such that

$$\text{Card}(\mathbf{E}_{k+1}) = \infty, \quad \text{Card}(B_0^{(\pi)} \setminus (\cup_{l=1}^{k+1} B_l^{(\pi)})) \leq D_0^2 - (k+1), \quad \pi \in \mathbf{E}_{k+1}.$$

Since  $D_0$  is finite, the above inequality shows that there exists  $\tilde{k} \leq D_0^2$  such that  $B_0^{(\pi)} = \cup_{l=1}^{\tilde{k}} B_l^{(\pi)}$  for  $\pi \in \mathbf{E}_{\tilde{k}}$ . Let  $\mathbf{F} = \mathbf{E}_{\tilde{k}}$  and  $\tilde{B}_l = \cup_{\pi \in \mathbf{F}} B_l^{(\pi)} \subset B_l$ . Then for each  $1 \leq l \leq \tilde{k}$ ,

$$\|x\|_p \leq C_l \|x\|_2, \quad x \in \text{span}(\tilde{B}_l).$$

Since  $\cup_{l=1}^{\tilde{k}} \tilde{B}_l \subset B_0$  is an orthonormal system and

$$\text{Pol}_{\mathbf{F}}(\mathbb{G}) = \text{span}(\cup_{\pi \in \mathbf{F}} B_0^{(\pi)}) = \text{span}(\cup_{\pi \in \mathbf{F}} \cup_{l=1}^{\tilde{k}} B_l^{(\pi)}) = \text{span}(\cup_{l=1}^{\tilde{k}} \tilde{B}_l),$$

we obtain

$$\|x\|_p \leq D_0^2 \max\{C_l : 1 \leq l \leq \tilde{k}\} \|x\|_2, \quad x \in \text{Pol}_{\mathbf{F}}(\mathbb{G}).$$

Hence  $\mathbf{F} \subset \text{Irr}(\mathbb{G})$  is the desired infinite  $\Lambda(p)$ -set.  $\square$

The existence of  $\Lambda(p)$ -sets without the assumption  $\sup_{\pi \in \mathbf{E}} d_\pi < \infty$  in the above theorem is in general not true, which can be seen from the non-existence of central  $\Lambda(4)$ -sets for the classical  $SU(2)$ , as well as from the quantum non-tracial example 4.2.13.(2).

#### 4.2.5 Remarks on the lacunarity for $SU_q(2)$

In [HR70], a classical version of Theorem 4.2.8 was used to prove the fact that the special unitary group  $SU(2)$  does not admit any Sidon set. The key observation therein is that  $SU(2)$  does not admit any central  $\Lambda(4)$ -set (see also [Cec72]). Here we want to show that, in strong contrast, the  $q$ -deformed quantum group  $SU_q(2)$  with  $0 < q < 1$  *does* admit a central  $\Lambda(4)$ -set. The non-traciality of the Haar state on  $SU_q(2)$  plays an essential role for this result. We recall that  $\text{Irr}(SU_q(2))$  can be identified with  $\mathbb{N} \cup \{0\}$  and follow the notation in the preliminary part.

**Proposition 4.2.16.** *Let  $0 < q < 1$  and let  $\mathbf{E} = \{n_k \in \mathbb{N} \cup \{0\} : k \geq 0\} \subset \text{Irr}(SU_q(2))$  be such that  $n_k = n_{k-1} + k$  for  $k \geq 1$ . Then  $\mathbf{E}$  is a central  $\Lambda(4)$ -set for  $SU_q(2)$ . More precisely, there exists  $K_q \geq 0$  such that for any finitely supported sequence  $(c_n)_{n \in \mathbf{E}} \in \mathbb{C}$ ,*

$$\left\| \sum_{n \in \mathbf{E}} c_n \chi_n \right\|_4 \leq K_q \left\| \sum_{n \in \mathbf{E}} c_n \chi_n \right\|_2. \quad (4.21)$$

*Proof.* Recall the formulae (1.3) and (1.11). Then for each  $m \in \mathbb{N} \cup \{0\}$ ,

$$\begin{aligned} h(\chi_m \sigma_{-\frac{i}{2}}(\chi_m)) &= \sum_{i,j=1}^{m+1} h((u_{ii}^{(m)})^* \sigma_{-\frac{i}{2}}(u_{jj}^{(m)})) = \sum_{i,j=1}^{m+1} h((u_{ii}^{(m)})^* (Q_m^{1/2})_{jj} u_{jj}^{(m)} (Q_m^{1/2})_{jj}) \\ &= \sum_{i=1}^{m+1} (Q_m)_{ii} h((u_{ii}^{(m)})^* u_{ii}^{(m)}) = \sum_{i=1}^{m+1} (Q_m)_{ii} (Q_m^{-1})_{ii} / \text{Tr}(Q_m) \\ &= \frac{m+1}{q^{-m} + q^{-m+2} + \dots + q^{m-2} + q^m}. \end{aligned} \quad (4.22)$$

On the other hand, it is easy to see

$$\|\chi_m\|_2^2 = h(\chi_m^2) = 1, \quad \left\| \sum_{n \in \mathbf{E}} c_n \chi_n \right\|_2^2 = \sum_{n \in \mathbf{E}} |c_n|^2. \quad (4.23)$$

By (1.2) and (1.3), we also have

$$h(\chi_m \sigma_{-\frac{i}{2}} \chi_n) = 0, \quad m \neq n. \quad (4.24)$$

Let  $\text{tr}$  be the trace on the Haagerup  $L^1$ -space  $L^{1,\text{H}}(\overline{SU}_q(2))$ . Recall Proposition 1.1.2 - we have for a finitely supported sequence  $(c_n)_{n \in \mathbf{E}}$  with  $\max_n |c_n| = 1$  and  $f = \sum_{n \in \mathbf{E}} c_n \chi_n$ ,

$$\begin{aligned} \|f\|_4^4 &= \text{tr}(|f D^{1/4}|^4) = \text{tr}(D^{1/4} f^* f D^{1/2} f^* f D^{1/4}) \\ &= \text{tr}(f^* f (D^{1/2} f^* f D^{-1/2}) D) = h(f^* f \sigma_{-\frac{i}{2}}(f^* f)) \\ &= \sum_{i,j,r,s \in \mathbf{E}} \bar{c}_i c_j \bar{c}_r c_s h(\chi_i \chi_j \sigma_{-\frac{i}{2}}(\chi_r \chi_s)). \end{aligned}$$

Then using (4.22), (4.24) and the fact  $\chi_m \chi_{m'} = \chi_{|m-m'|} + \chi_{|m-m'|+1} + \dots + \chi_{m+m'}$  for any  $m, m' \in \mathbb{N} \cup \{0\}$ , we get

$$\|f\|_4^4 \leq \sum_{i,j,r,s \in \mathbf{E}} |\bar{c}_i| |c_j| |\bar{c}_r| |c_s| \sum_{m=\max\{|i-j|, |r-s|\}}^{\min\{i+j, r+s\}} |h(\chi_m \sigma_{-\frac{i}{2}}(\chi_m))|$$

$$\begin{aligned}
&= \sum_{i,j,r,s \in \mathbf{E}} |\bar{c}_i| |c_j| |\bar{c}_r| |c_s| \sum_{m=\max\{|i-j|, |r-s|\}}^{\min\{i+j, r+s\}} \frac{m+1}{q^{-m} + q^{-m+2} + \dots + q^{m-2} + q^m} \\
&\leq \sum_{i,j,r,s \in \mathbf{E}} |\bar{c}_i| |c_j| |\bar{c}_r| |c_s| \sum_{m=\max\{|i-j|, |r-s|\}}^{\min\{i+j, r+s\}} \frac{m+1}{q^{-m}},
\end{aligned}$$

and therefore

$$\begin{aligned}
\|f\|_4^4 &\leq K \sum_{i,j,r,s \in \mathbf{E}} |\bar{c}_i| |c_j| |\bar{c}_r| |c_s| \sum_{m=\max\{|i-j|, |r-s|\}}^{\min\{i+j, r+s\}} q^{m/2} \leq K \sum_{i,j,r,s \in \mathbf{E}} |\bar{c}_i| |c_j| |\bar{c}_r| |c_s| \frac{q^{\frac{\max\{|i-j|, |r-s|\}}{2}}}{1 - q^{1/2}} \\
&\leq \frac{K}{1 - q^{1/2}} \sum_{i,j,r,s \in \mathbf{E}} |\bar{c}_i| |c_j| |\bar{c}_r| |c_s| q^{\frac{|i-j|+|r-s|}{4}} = \frac{K}{1 - q^{1/2}} \left( \sum_{i,j \in \mathbf{E}} |\bar{c}_i| |c_j| q^{\frac{|i-j|}{4}} \right)^2 \\
&\leq \frac{2K}{1 - q^{1/2}} \left( \left( \sum_{i \in \mathbf{E}, i=j} |\bar{c}_i|^2 \right)^2 + \left( 2 \sum_{i,j \in \mathbf{E}, i < j} |c_i| |c_j| q^{\frac{j-i}{4}} \right)^2 \right)
\end{aligned}$$

where  $K \geq 1$  is the constant such that  $x + 1 \leq Kq^{-x/2}$  for  $x \geq 1$ . Recall the assumption on  $\mathbf{E} = \{n_k \in \mathbb{N} \cup \{0\} : k \geq 0\}$  that  $n_k = n_{k-1} + k$  for  $k \geq 1$  and also the assumption  $\max_n |c_n| = 1$ , so that  $1 \leq \|f\|_2$ . We have

$$\begin{aligned}
\sum_{i,j \in \mathbf{E}, i < j} |c_i| |c_j| q^{\frac{j-i}{4}} &= \sum_{k \geq 0} |c_{n_k}| \sum_{l \geq 1} |c_{n_k+(k+1)+\dots+(k+l)}| q^{\frac{(k+1)+\dots+(k+l)}{4}} \\
&\leq \sum_{k \geq 0} \sum_{l \geq 1} q^{\frac{(2k+l+1)l}{8}} \leq \sum_{k \geq 0} \sum_{l \geq 1} q^{\frac{2k+l+1}{8}} \\
&= \sum_{k \geq 0} q^{\frac{2k+1}{8}} \sum_{l \geq 1} q^{\frac{l}{8}} \leq q^{1/8} \cdot \frac{1}{1 - q^{1/4}} \cdot \frac{q^{1/8}}{1 - q^{1/8}} := K'.
\end{aligned}$$

Therefore together with (4.23) we get

$$\|f\|_4^4 \leq \frac{2K}{1 - q^{1/2}} \left( \left( \sum_{i \in \mathbf{E}, i=j} |\bar{c}_i|^2 \right)^2 + 4K'^2 \right) \leq \frac{2K(4K'^2 + 1)}{1 - q^{1/2}} \|f\|_2^4.$$

Take  $K_q = \left( \frac{2K(4K'^2 + 1)}{1 - q^{1/2}} \right)^{1/4}$  and we get the desired inequality (4.21).  $\square$

In the end we remark that for the case  $0 < q < 1$ , we may also state below the non-existence of infinite  $\Lambda(p)$ -sets for  $\text{SU}_q(2)$ , which directly follows from Proposition 4.2.5.

**Proposition 4.2.17.** *Let  $0 < q < 1$  and  $2 < p < \infty$ . There exist no infinite  $\Lambda(p)$ -sets or infinite Sidon sets for  $\text{SU}_q(2)$ .*

*Proof.* Recall that the irreducible representations of  $\text{Irr}(\text{SU}_q(2))$  indexed by  $\mathbb{N} \cup \{0\}$ , and for each  $n \in \mathbb{N} \cup \{0\}$  we have  $\|Q_n\| = \|Q_n^{-1}\| = q^{-n}$ . So for any infinite subset  $\mathbf{E} \subset \text{Irr}(\text{SU}_q(2))$ , we have

$$\sup_{n \in \mathbf{E}} \{\|Q_n\|, \|Q_n^{-1}\|\} = \infty.$$

So according to Proposition 4.2.5,  $\text{SU}_q(2)$  does not admit any infinite interpolation set for  $M(L^p(\mathbb{G}))$ , or equivalently, it does not admit any infinite  $\Lambda(p)$ -set by Theorem 4.2.7. And by Corollary 4.2.8,  $\text{SU}_q(2)$  does not admit any infinite Sidon set.  $\square$

In the next section we will use another method to show in Corollary 4.3.9 that a large class of quantum deformations of semi-simple Lie groups do not admit any infinite Sidon set.

### 4.3 Central Sidon sets with examples

In this final section we briefly discuss some properties of central Sidon sets.

**Definition 4.3.1.** (1) We say that a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a *central Sidon set* if there exists  $K > 0$  such that for any finite sequence  $(c_\pi) \subset \mathbb{C}$  and  $x = \sum_\pi c_\pi \chi_\pi$ , we have  $\|\hat{x}\|_1 \leq K \|x\|_\infty$ .

(2) A linear functional on  $\text{Pol}(\mathbb{G})$  is said to be *central* if there exists numbers  $(\omega_\pi : \pi \in \text{Irr}(\mathbb{G}))$  such that

$$\hat{\omega}(\pi) = \omega_\pi \text{Id}_\pi, \quad \pi \in \text{Irr}(\mathbb{G}).$$

Denote by  $\text{Pol}^z(\mathbb{G}) = \{x = \sum_{\pi \in \text{Irr}(\mathbb{G})} c_\pi \chi_\pi \in \text{Pol}(\mathbb{G}) : c_\pi \in \mathbb{C}\}$  the subspace of central polynomials and let  $C_r^z(\mathbb{G})$  be the norm closure of  $\text{Pol}^z(\mathbb{G})$  in  $C_r(\mathbb{G})$ .

*Remark 4.3.2.* Any Sidon set  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a central Sidon set for  $\mathbb{G}$ . In fact, note that for  $x = \sum_\pi c_\pi \chi_\pi$  as in the above definition, we have  $\hat{x}(\pi) = \dim(\pi)^{-1} c_\pi Q_\pi^{-1}$  and then by definition,

$$\|\hat{x}\|_1 = \sum_{\pi \in \text{Irr}(\mathbb{G})} d_\pi \text{Tr}(|p_\pi \hat{x}(\pi) Q_\pi|) = \sum_{\pi \in \text{Irr}(\mathbb{G})} \dim(\pi) |c_\pi|. \quad (4.25)$$

Hence  $\mathbf{E}$  is a central Sidon set.

**Lemma 4.3.3.** *Let  $\mathbb{G}$  be a compact quantum group. The following assertions are equivalent:*

- (1)  $\mathbb{G}$  is of Kac type;
- (2) any central functional  $\omega$  is bounded on  $\text{Pol}(\mathbb{G})$  (with respect to  $\|\cdot\|_\infty$ ) if and only if it is bounded on  $\text{Pol}^z(\mathbb{G})$  with the same norm;
- (3) there exists a conditional expectation  $\mathcal{E}$  from  $C_r(\mathbb{G})$  onto  $C_r^z(\mathbb{G})$  such that  $h \circ \mathcal{E} = h$ .

*Proof.* (1)  $\Rightarrow$  (2). Assume that  $\mathbb{G}$  is of Kac type. Let  $\omega$  be a central functional which is bounded on  $\text{Pol}^z(\mathbb{G})$ . Let  $\tilde{\omega} \in L^\infty(\mathbb{G})^*$  be its Hahn-Banach extension to  $L^\infty(\mathbb{G})$ . Denote by  $\mathcal{E}$  the  $h$ -preserving conditional expectation from  $\mathcal{M} = L^\infty(\mathbb{G})$  onto the von Neumann subalgebra  $\mathcal{N}$  generated by  $\text{Pol}^z(\mathbb{G})$  in  $L^\infty(\mathbb{G})$ . Recall that for  $\pi, \beta \in \text{Irr}(\mathbb{G})$ , we have  $\chi_\pi^* = \chi_{\bar{\pi}}$  and  $\chi_\pi \chi_{\pi'} = \chi_{\pi \otimes \pi'}$ , so the subspace  $\text{Pol}^z(\mathbb{G})$  spanned by characters is ultraweakly dense in  $\mathcal{N}$ . Note also that  $\mathcal{E}$  is the adjoint map of the embedding  $\iota : L^1(\mathcal{N}) \rightarrow L^1(\mathcal{M})$ , so for any  $\pi, \pi' \in \text{Irr}(\mathbb{G})$  and any  $i, j$ ,

$$\begin{aligned} \langle \mathcal{E}(u_{ij}^{(\pi)}), \chi_{\pi'} \rangle_{L^1(\mathcal{N})^*, L^1(\mathcal{N})} &= \langle u_{ij}^{(\pi)}, \iota(\chi_{\pi'}) \rangle_{L^1(\mathcal{M})^*, L^1(\mathcal{M})} = h((u_{ij}^{(\pi)})^* \chi_{\pi'}) = \dim(\pi)^{-1} \delta_{ij} \delta_{\pi \pi'} \\ &= \dim(\pi)^{-1} \delta_{ij} h(\chi_\pi^* \chi_{\pi'}) = \langle \dim(\pi)^{-1} \delta_{ij} \chi_\pi, \chi_{\pi'} \rangle_{L^1(\mathcal{N})^*, L^1(\mathcal{N})}, \end{aligned}$$

which means that  $E(u_{ij}^{(\pi)}) = \delta_{ij} \dim(\pi)^{-1} \chi_\pi$ . Consequently,  $\omega = \tilde{\omega} \circ E$  on  $\text{Pol}(\mathbb{G})$ , and therefore  $\omega$  is bounded on  $\text{Pol}(\mathbb{G})$  with the same norm.

(2)  $\Rightarrow$  (3). Define the linear map  $\mathcal{E} : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}^z(\mathbb{G})$  by

$$\mathcal{E}(u_{ij}^{(\pi)}) = \delta_{ij} \dim(\pi)^{-1} \chi_\pi, \quad \pi \in \text{Irr}(\mathbb{G}).$$

It is easy to see that  $h \circ \mathcal{E} = h$  on  $\text{Pol}(\mathbb{G})$ . Also for any central functional  $\omega$  on  $\text{Pol}(\mathbb{G})$ , we have  $\omega \circ \mathcal{E} = \omega$ . Now for any  $x \in \text{Pol}(\mathbb{G})$ , by the assertion (2), we have

$$\|\mathcal{E}(x)\| = \sup_{\omega \in \text{Pol}^z(\mathbb{G})^*, \|\omega\|=1} |\omega(\mathcal{E}(x))| = \sup_{\omega \in \text{Pol}(\mathbb{G})^*, \|\omega\|=1} |\omega(x)| = \|x\|.$$

So  $\mathcal{E}$  is contractive on  $\text{Pol}(\mathbb{G})$  and can be extended to a conditional expectation from  $C_r(\mathbb{G})$  onto  $C_r^z(\mathbb{G})$  preserving the Haar state.



(3)  $\Rightarrow$  (1). Assume (3) holds. It is a standard argument that  $\sigma_t(\text{Pol}^z(\mathbb{G})) \subset \text{Pol}^z(\mathbb{G})$ , see e.g. the proof of Theorem 4.2 in [Tak03, Chap.IX]. In fact, let  $(H, \Lambda, \pi)$  be the faithful GNS construction of  $\text{Pol}(\mathbb{G})$  with respect to the Haar state  $h$  and denote by  $H_0$  the completion of  $\Lambda(\text{Pol}^z(\mathbb{G}))$  in  $H$ . Let  $E$  be the orthogonal projection from  $H$  onto  $H_0$ . Then for  $x \in \text{Pol}(\mathbb{G}), y \in \text{Pol}^z(\mathbb{G})$ , we have

$$\langle \Lambda(x), \Lambda(y) \rangle = h(x^*y) = h(\mathcal{E}(x^*y)) = h(\mathcal{E}(x)^*y) = \langle \Lambda(\mathcal{E}(x)), \Lambda(y) \rangle,$$

so

$$E(\Lambda(x)) = \Lambda(\mathcal{E}(x)), \quad x \in \text{Pol}(\mathbb{G}).$$

As usual denote by  $\mathcal{S} : \Lambda(x) \mapsto \Lambda(x^*)$  the operator on  $H$  induced by involution, and also denote by  $\mathfrak{d}$  the modular operator on  $\text{Pol}(\mathbb{G})$  associated to the Haar state  $h$ . Since  $\mathcal{E}$  preserves the  $*$ -operation, we see that  $E\mathcal{S} = \mathcal{S}E$  on the subspace  $\Lambda(\text{Pol}(\mathbb{G}))$ . Taking adjoint we see that  $\mathcal{S}^*$  also commutes with  $E$  on  $\Lambda(\text{Pol}(\mathbb{G}))$ , hence so it is for  $\mathfrak{d} = \mathcal{S}^*\mathcal{S}$ . Therefore  $\mathfrak{d}^{it}$  leaves  $\Lambda(\text{Pol}^z(\mathbb{G}))$  invariant for all  $t \in \mathbb{R}$ , which yields that  $\sigma_t(\text{Pol}^z(\mathbb{G})) \subset \text{Pol}^z(\mathbb{G})$ . However we recall that the modular automorphism group acts on  $\text{Pol}(\mathbb{G})$  as

$$\sigma_t(u_{ij}^{(\pi)}) = \sum_{k,l} (Q_\pi^{it})_{ik} u_{kl}^{(\pi)} (Q_\pi^{it})_{lj}, \quad \pi \in \text{Irr}(\mathbb{G}), \quad t \in \mathbb{R}, \quad 1 \leq i, j \leq n_\pi$$

and hence

$$\sigma_t(\chi_\pi) = \sigma_t\left(\sum_i u_{ii}^{(\pi)}\right) = \sum_{k,l} \left(\sum_i (Q_\pi^{it})_{li} (Q_\pi^{it})_{ik}\right) u_{kl}^{(\pi)} = \sum_{k,l} (Q_\pi^{2it})_{lk} u_{kl}^{(\pi)}, \quad \pi \in \text{Irr}(\mathbb{G}) \quad t \in \mathbb{R}.$$

So the invariance  $\sigma_t(\text{Pol}^z(\mathbb{G})) \subset \text{Pol}^z(\mathbb{G})$  yields that  $Q_\pi = \text{Id}_\pi$  for all  $\pi \in \text{Irr}(\mathbb{G})$ , that is,  $\mathbb{G}$  is of Kac type.  $\square$

**Proposition 4.3.4.** *Let  $\mathbb{G}$  be a compact quantum group of Kac type. Then  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a central Sidon set if and only if for all bounded sequences  $(a_\pi)_{\pi \in \mathbf{E}} \subset \mathbb{C}$ , there exists a bounded central functional  $\varphi \in C_r(\mathbb{G})^*$  such that  $\hat{\varphi}(\pi) = a_\pi \text{Id}_\pi$  for  $\pi \in \mathbf{E}$ .*

*Proof.* The proof is an analogue of that of Theorem 4.1.3 (1)  $\Leftrightarrow$  (2). We replace  $C_r(\mathbb{G})$  by  $C_r^z(\mathbb{G})$  and  $\ell^\infty(\mathbf{E})$  by  $(\ell^\infty(\mathbf{E}))^z = \{(a_\pi) \in \ell^\infty(\mathbf{E}) : a_\pi \in \mathbb{C} \text{Id}_\pi, \pi \in \text{Irr}(\mathbb{G})\}$ , etc., and argue as before, thanks to the previous lemma.  $\square$

*Remark 4.3.5.* There have been some suggestions on the definition of unconditional Sidon sets for a compact (quantum) group which would be different from that in Definition 4.1.13. More precisely, for a compact quantum group  $\mathbb{G}$  and a subset  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$ , we may consider the following lacunary condition: there exists a constant  $K > 0$  such that

$$\forall (\varepsilon_\pi)_{\pi \in \mathbf{E}} \subset \{-1, 1\}, \quad \left\| \sum_{\pi \in \mathbf{E}} d_\pi \varepsilon_\pi (\iota \otimes \text{Tr})((1 \otimes \hat{x}(\pi) Q_\pi) u^{(\pi)}) \right\|_\infty \leq K \|x\|_\infty, \quad x \in \text{Pol}_{\mathbf{E}}(\mathbb{G}). \quad (4.26)$$

And it was ever unclear on the relations between (4.26) and the Sidon sets even if  $\mathbb{G}$  is a compact group  $G$ . Here we remark that the above two notions of lacunarity are in fact totally different if  $G$  is non-abelian. Indeed, more generally, if the compact quantum group  $\mathbb{G}$  is coamenable and of Kac type, we may follow the same idea as in Theorem 4.1.15 and the above proposition to see that (4.26) holds if and only if  $\mathbf{E} \subset \text{Irr}(\mathbb{G})$  is a central Sidon set for  $\mathbb{G}$ , which is well-known to be different from being a Sidon set.

*Remark 4.3.6.* For a central functional  $\omega$  on  $\text{Pol}(\mathbb{G})$ , the associated multiplier map  $T_\omega = (\omega \otimes \text{id}) \circ \Delta$  acts as  $u_{ij}^{(\pi)} \mapsto \omega_\pi u_{ij}^{(\pi)}$  on  $\text{Pol}(\mathbb{G})$ . In [DCFY14, Sect.2] it is proved that if  $\Phi : \text{Irr}(\mathbb{G}_1) \rightarrow \text{Irr}(\mathbb{G}_2)$  is a monoidal equivalence between two compact quantum groups  $\mathbb{G}_1$  and  $\mathbb{G}_2$  and if  $\omega^{(1)}, \omega^{(2)}$  are the central functionals on  $\text{Pol}(\mathbb{G}_1), \text{Pol}(\mathbb{G}_2)$  respectively, such that  $\omega_\pi^{(1)} = \omega_{\Phi(\pi)}^{(2)}$ , then  $T_{\omega^{(1)}}$  and  $T_{\omega^{(2)}}$  have the same complete bounded norms. Note that if additionally  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are coamenable, then  $\|\omega^{(1)}\| = \|T_{\omega^{(1)}}\| = \|T_{\omega^{(2)}}\| = \|\omega^{(2)}\|$  since  $\omega = \omega \star \epsilon = (\text{id} \otimes \epsilon) \circ T_\omega$  for any functional  $\omega$  on  $\text{Pol}(\mathbb{G})$ . Then according to the previous proposition, any two coamenable compact quantum groups of Kac type which are monoidally equivalent, have a one-to-one correspondence of their central Sidon sets via the monoidal equivalence map. The following result generalizes this fact.

**Proposition 4.3.7.** *Let  $\mathbb{G}_1, \mathbb{G}_2$  be two compact quantum groups. Assume that there is an injective map  $\Phi : \text{Rep}(\mathbb{G}_1) \rightarrow \text{Rep}(\mathbb{G}_2)$  preserving the fusion rules, that is, for all  $\pi, \pi' \in \text{Rep}(\mathbb{G}_1)$  we have*

$$\Phi(\pi \otimes \pi') = \Phi(\pi) \otimes \Phi(\pi'), \quad \Phi(\oplus_{i=1}^n \pi_i) = \oplus_{i=1}^n \Phi(\pi_i), \quad \pi, \pi', \pi_i \in \text{Rep}(\mathbb{G}_1), n \geq 1.$$

*Then for any finite sequence  $(c_\pi) \subset \mathbb{C}$  and  $x = \sum_{\pi \in \text{Irr}(\mathbb{G}_1)} c_\pi \chi_\pi \in \text{Pol}(\mathbb{G}_1)$ , we have*

$$\left\| \sum_{\pi \in \text{Irr}(\mathbb{G}_1)} c_\pi \chi_\pi \right\|_{L^\infty(\mathbb{G}_1)} = \left\| \sum_{\pi \in \text{Irr}(\mathbb{G}_1)} c_\pi \chi_{\Phi(\pi)} \right\|_{L^\infty(\mathbb{G}_2)}. \quad (4.27)$$

*Consequently, for any central Sidon set  $\mathbf{E} \subset \text{Irr}(\mathbb{G}_1)$ , if additionally there exists  $C > 0$  satisfying  $\dim(\Phi(\pi)) \leq C \dim(\pi)$  for all  $\pi \in \mathbf{E}$ , then  $\Phi(\mathbf{E}) \subset \text{Irr}(\mathbb{G}_2)$  is a central Sidon set for  $\mathbb{G}_2$ .*

*Proof.* The isometry (4.27) has been mentioned in [Ban99]. In fact, as remarked in [Ban99], the map  $\Phi$  automatically satisfies  $\Phi(\bar{\pi}) = \overline{\Phi(\pi)}$  for all  $\pi \in \text{Rep}(\mathbb{G})$  once we note that for an irreducible representation  $\pi$  its adjoint  $\bar{\pi}$  is the unique irreducible representation such that 1 is the subrepresentation of  $\pi \otimes \bar{\pi}$ . Denote by  $A$  the  $*$ -algebra generated by  $\{\chi_\pi : \pi \in \text{Irr}(\mathbb{G}_1)\}$ . By the linear independence of  $\{\chi_\pi : \pi \in \text{Irr}(\mathbb{G}_1)\}$ , the injection  $\Phi$  induces an injective  $*$ -homomorphism

$$\tilde{\Phi} : A \rightarrow \text{Pol}(\mathbb{G}_2), \quad \chi_\pi \mapsto \chi_{\Phi(\pi)}, \quad \pi \in \text{Rep}(\mathbb{G}_1).$$

To see that  $\tilde{\Phi}$  well defines a  $*$ -homomorphism, it suffices to notice that for  $\pi \in \text{Irr}(\mathbb{G}_1)$ , we have  $\chi_\pi^* = \chi_{\bar{\pi}}$  and

$$\tilde{\Phi}(\chi_\pi^*) = \tilde{\Phi}(\chi_{\bar{\pi}}) = \chi_{\Phi(\bar{\pi})} = \chi_{\overline{\Phi(\pi)}} = \tilde{\Phi}(\chi_\pi)^*$$

and for  $\pi, \pi' \in \text{Rep}(\mathbb{G}_1)$  satisfying the decomposition formula  $\pi \otimes \pi' = \sum_k \pi_k$  with each  $\pi_k \in \text{Irr}(\mathbb{G}_1)$ , we have

$$\tilde{\Phi}(\chi_\pi \chi_{\pi'}) = \sum_k \tilde{\Phi}(\chi_{\pi_k}) = \sum_k \chi_{\Phi(\pi_k)} = \chi_{\Phi(\pi) \Phi(\pi')} = \tilde{\Phi}(\chi_\pi) \tilde{\Phi}(\chi_{\pi'}).$$

Note that  $\tilde{\Phi}$  preserves the restriction of Haar states on  $\text{Pol}^z(\mathbb{G}_1)$  and  $\text{Pol}^z(\mathbb{G}_2)$ . So  $\tilde{\Phi}$  gives rise to an equivalence between the faithful sub-GNS-representations of  $\text{Pol}^z(\mathbb{G}_1) \subset \text{Pol}(\mathbb{G}_1)$  and  $\text{Pol}^z(\mathbb{G}_2) \subset \text{Pol}(\mathbb{G}_2)$  with respect to the Haar states. As a result  $\tilde{\Phi}$  is an isometry, which gives (4.27). The assertion regarding the central Sidon sets then follows directly from the definition and (4.25).  $\square$

**Example 4.3.8.** Let  $q \in [-1, 1] \setminus \{0\}$  and consider the quantum groups  $SU_q(n)$ ,  $n \geq 2$ .  $SU_q(n)$  is a compact matrix quantum group in the sense of [Wor87a] and we denote by  $\pi_n^{(q)}$  the fundamental representation of  $SU_q(n)$ . It is easy to see that for the classical case  $q = 1$ , the subset  $\{\pi_n^{(1)} : n \geq 2\}$  is a Sidon set for the compact group  $\prod_{n \geq 2} SU(n)$  (see [CM81, p.308-310]). Now by Proposition 1.2.7 for  $0 < q < 1$  we may find a map  $\Phi : \text{Irr}(SU_q(n)) \rightarrow \text{Irr}(SU(n))$  satisfying the assumptions of the above proposition. As a result, for any  $0 < q < 1$  the subset  $\{\pi_n^{(q)} : n \geq 2\}$  is a central Sidon set for the compact quantum group  $\prod_{n \geq 2} SU_q(n)$ .

In [Rid72] it is shown that for a connected compact group  $G$ ,  $G$  has an infinite central Sidon set if and only if  $G$  is not a semi-simple Lie group. Combined with Proposition 1.2.7 and Proposition 4.3.7 we get the following observation, which in particular shows that the quantum  $SU_q(2)$  ( $0 < q < 1$ ) does not admit any infinite Sidon set, as mentioned in the last section after Proposition 4.2.16.

**Corollary 4.3.9.** *For all  $0 < q < 1$  and all simply connected compact semi-simple Lie group  $G$ , the compact quantum group  $G_q$  given by the Drinfeld-Jimbo deformation does not admit any infinite central Sidon set.*

## 4.4 Appendix: Existence of $\Lambda(p)$ -sets in orthogonal systems

In this appendix we present a method of constructing  $\Lambda(p)$ -sets in orthogonal systems for noncommutative  $L^p$ -spaces. The main result is the following theorem. It is due to Marek Bożejko in the tracial case.

**Theorem 4.4.1.** *Let  $\mathcal{M}$  be a von Neumann algebra equipped with a normal faithful state  $\varphi$  and consider the associated  $L^p$ -spaces  $L^p(\mathcal{M}, \varphi)$ . Let  $B = \{x_i \in \mathcal{M} : i \geq 1\}$  be an orthogonal system with respect to  $\varphi$  (i.e.,  $\varphi(x_i^* x_j) = 0$  for all  $i \neq j$ ) such that  $\sup_i \|x_i\|_\infty < \infty$ . Then for each  $2 < p < \infty$ , there exists an infinite subset  $\{x_{i_k} : k \geq 1\} \subset B$  and a constant  $C > 0$  such that for all finitely supported sequences  $(c_k) \subset \mathbb{C}$  we have*

$$\left\| \sum_{k \geq 1} c_k x_{i_k} \right\|_p \leq C \left( \sum_{k \geq 1} |c_k|^2 \right)^{\frac{1}{2}}.$$

This result was first proved by [KS36] in the commutative case and then by [Boż79] in the case where  $\varphi$  is tracial. The same idea also applies to the general setting, and for the sake of completeness we include a detailed proof below. We refer to [Pic73, Boż73, Boż75b] for different approaches.

To establish the theorem, let us show the following slightly stronger result.

**Theorem 4.4.2.** *Let  $\mathcal{M}$  and  $\varphi$  be given as in Theorem 4.4.1 and  $n \geq 2$ . Let  $B = \{x_i \in L^{2n}(\mathcal{M}, \varphi) : i \geq 1\}$  be an orthogonal system with respect to  $\varphi$  such that  $\sup_i \|x_i\|_{2n} < \infty$ . Then there exists an infinite subset  $\{x_{i_k} : k \geq 1\} \subset B$  and a constant  $C > 0$  such that for all finitely supported sequences  $(c_k) \subset \mathbb{C}$  we have*

$$\left\| \sum_{k \geq 1} c_k x_{i_k} \right\|_{2n} \leq C \left( \sum_{k \geq 1} |c_k|^2 \right)^{\frac{1}{2}}. \quad (4.28)$$

*Proof.* Write  $K = \max\{1, \sup_i \|x_i\|_{2n}\} < \infty$ . Denote by  $\mathcal{M}_a$  the subalgebra of all analytic elements in  $\mathcal{M}$ . Recall that  $\mathcal{M}_a$  a ultraweakly dense subspace of  $\mathcal{M}$ , and for all  $x \in \mathcal{M}_a$ ,

the analytic extension  $\sigma_z(x) \in \mathcal{M}$  for  $z \in \mathbb{C}$  is well-defined (see [Tak03]). In particular  $\mathcal{M}_a$  is also a dense subspace of  $L^p(\mathcal{M}, \varphi)$  for  $1 \leq p < \infty$  according to Lemma 1.1.1.

(1) Firstly, assume that  $B \subset \mathcal{M}_a$ . Note that for any  $y \in \mathcal{M}_a$ ,

$$\varphi(yx_k) \rightarrow 0, \quad \varphi(x_k y) \rightarrow 0, \quad k \rightarrow \infty.$$

We choose a subset  $\{x_{i_k} : k \geq 1\} \subset B$  inductively as follows. Let  $x_{i_1} = x_1$  and if  $\{x_{i_j} : 1 \leq j \leq k\}$  for  $k \geq 1$  is chosen, we take an  $x_{i_{k+1}} \in B$  such that for all  $1 \leq k_0, k_1, \dots, k_{n-1}, l_1, \dots, l_{n-1} \leq k$ ,

$$\left| \varphi \left( \sigma_{\frac{(n-1)i}{n}}(x_{i_{k_1}}^* x_{i_{l_1}}) \sigma_{\frac{(n-2)i}{n}}(x_{i_{k_2}}^* x_{i_{l_2}}) \cdots \sigma_{\frac{i}{n}}(x_{i_{k_{n-1}}}^* x_{i_{l_{n-1}}}) x_{i_{k_0}}^* x_{i_{k+1}} \right) \right| \leq \frac{1}{2k^{2n-1}(k+1)}$$

and

$$\left| \varphi \left( \sigma_i(x_{i_{k_0}}) \sigma_{\frac{(n-1)i}{n}}(x_{i_{k_1}}^* x_{i_{l_1}}) \sigma_{\frac{(n-2)i}{n}}(x_{i_{k_2}}^* x_{i_{l_2}}) \cdots \sigma_{\frac{i}{n}}(x_{i_{k_{n-1}}}^* x_{i_{l_{n-1}}}) x_{i_{k+1}}^* \right) \right| \leq \frac{1}{2k^{2n-1}(k+1)}.$$

This can always be done by the orthogonality of  $B$ . In fact, since the elements in  $B$  are assumed to be analytic, we have

$$\sigma_{\frac{(n-1)i}{n}}(x_{i_{k_1}}^* x_{i_{l_1}}) \sigma_{\frac{(n-2)i}{n}}(x_{i_{k_2}}^* x_{i_{l_2}}) \cdots \sigma_{\frac{i}{n}}(x_{i_{k_{n-1}}}^* x_{i_{l_{n-1}}}) x_{i_{k_0}}^* \in \mathcal{M}(\subset L^2(\mathcal{M}, \varphi)),$$

$$\sigma_i(x_{i_{k_0}}) \sigma_{\frac{(n-1)i}{n}}(x_{i_{k_1}}^* x_{i_{l_1}}) \sigma_{\frac{(n-2)i}{n}}(x_{i_{k_2}}^* x_{i_{l_2}}) \cdots \sigma_{\frac{i}{n}}(x_{i_{k_{n-1}}}^* x_{i_{l_{n-1}}}) \in \mathcal{M}(\subset L^2(\mathcal{M}, \varphi)).$$

Note that there exist only finitely many elements of the above forms for each given  $k \geq 1$ . So the element  $x_{i_{k+1}}$  can be well chosen. With the same notation we deduce the following inequalities,

$$\begin{aligned} & \left| \text{tr} \left( D^{\frac{1}{2n}} x_{i_{k_1}}^* x_{i_{l_1}} D^{\frac{1}{2n}} D^{\frac{1}{2n}} x_{i_{k_2}}^* x_{i_{l_2}} D^{\frac{1}{2n}} \cdots D^{\frac{1}{2n}} x_{i_{k_{n-1}}}^* x_{i_{l_{n-1}}} D^{\frac{1}{2n}} D^{\frac{1}{2n}} x_{i_{k_0}}^* x_{i_{k+1}} D^{\frac{1}{2n}} \right) \right| \\ &= \left| \text{tr} \left( D \sigma_{\frac{(n-1)i}{n}}(x_{i_{k_1}}^* x_{i_{l_1}}) \sigma_{\frac{(n-2)i}{n}}(x_{i_{k_2}}^* x_{i_{l_2}}) \cdots \sigma_{\frac{i}{n}}(x_{i_{k_{n-1}}}^* x_{i_{l_{n-1}}}) x_{i_{k_0}}^* x_{i_{k+1}} \right) \right| \\ &= \left| \varphi \left( \sigma_{\frac{(n-1)i}{n}}(x_{i_{k_1}}^* x_{i_{l_1}}) \sigma_{\frac{(n-2)i}{n}}(x_{i_{k_2}}^* x_{i_{l_2}}) \cdots \sigma_{\frac{i}{n}}(x_{i_{k_{n-1}}}^* x_{i_{l_{n-1}}}) x_{i_{k_0}}^* x_{i_{k+1}} \right) \right| \leq \frac{1}{2k^{2n-1}(k+1)}, \end{aligned}$$

and similarly

$$\left| \text{tr} \left( D^{\frac{1}{2n}} x_{i_{k_1}}^* x_{i_{l_1}} D^{\frac{1}{2n}} \cdots D^{\frac{1}{2n}} x_{i_{k_{n-1}}}^* x_{i_{l_{n-1}}} D^{\frac{1}{2n}} D^{\frac{1}{2n}} x_{i_{k+1}}^* x_{i_{k_0}} D^{\frac{1}{2n}} \right) \right| \leq \frac{1}{2k^{2n-1}(k+1)}.$$

Fix a sequence  $(c_k) \subset \mathbb{C}$  such that  $\sum_{k \geq 1} |c_k|^2 = 1$ . Take  $r \geq 1$  and write

$$g_r = \sum_{k=1}^r c_k x_{i_k}.$$

Since then each  $|c_k| \leq 1$  for  $k \geq 1$ , the above inequalities yield that

$$\left| \text{tr} \left( (D^{\frac{1}{2n}} g_r^* g_r D^{\frac{1}{2n}})^{n-1} D^{\frac{1}{2n}} g_r^* x_{i_{r+1}} D^{\frac{1}{2n}} \right) \right| \leq r^{2n-1} \cdot \frac{1}{2(r+1)r^{2n-1}} = \frac{1}{2(r+1)}, \quad (4.29)$$

and similarly

$$\left| \text{tr} \left( (D^{\frac{1}{2n}} g_r^* g_r D^{\frac{1}{2n}})^{n-1} D^{\frac{1}{2n}} x_{i_{r+1}}^* g_r D^{\frac{1}{2n}} \right) \right| \leq \frac{1}{2(r+1)}. \quad (4.30)$$

Now for each  $r \geq 1$  write  $I_r = \|\sum_{k=1}^r c_k x_{i_k}\|_{2n}^{2n}$ . Then

$$I_{r+1} = \left\| \sum_{k=1}^{r+1} c_k x_{i_k} \right\|_{2n}^{2n} = \|g_r + c_{r+1} x_{i_{r+1}}\|_{2n}^{2n} = \text{tr} \left( (D^{\frac{1}{2n}} (g_r + c_{r+1} x_{i_{r+1}}))^* (g_r + c_{r+1} x_{i_{r+1}}) D^{\frac{1}{2n}} \right)^n.$$

We may write

$$I_{r+1} = J_1 + J_2 + J_3 + J_4, \quad (4.31)$$

where

$$J_1 = \text{tr} \left( (D^{\frac{1}{2n}} g_r^* g_r D^{\frac{1}{2n}})^n \right) = I_r, \quad J_2 = \text{tr} \left( (D^{\frac{1}{2n}} (c_{r+1} x_{i_{r+1}})^* (c_{r+1} x_{i_{r+1}}) D^{\frac{1}{2n}})^n \right),$$

$$J_3 = \sum_{\underline{y} \in A_1} \text{tr} \left( D^{\frac{1}{2n}} y_1^* y_1' D^{\frac{1}{2n}} D^{\frac{1}{2n}} y_2^* y_2' D^{\frac{1}{2n}} \cdots D^{\frac{1}{2n}} y_n^* y_n' D^{\frac{1}{2n}} \right),$$

$$J_4 = \sum_{j=2}^{n-1} \sum_{\underline{y} \in A_j} \text{tr} \left( D^{\frac{1}{2n}} y_1^* y_1' D^{\frac{1}{2n}} D^{\frac{1}{2n}} y_2^* y_2' D^{\frac{1}{2n}} \cdots D^{\frac{1}{2n}} y_n^* y_n' D^{\frac{1}{2n}} \right),$$

with

$$A_j = \{ \underline{y} = (y_1, \dots, y_n, y_1', \dots, y_n') : y_k, y_k' \in \{g_r, c_{r+1} x_{i_{r+1}}\}, 1 \leq k \leq n, \\ \text{Card}\{k : y_k = x_{i_{r+1}}\} + \text{Card}\{k : y_k' = x_{i_{r+1}}\} = j \}.$$

We see that

$$|J_2| = |c_{r+1}|^2 \|x_{i_{r+1}}\|_{2n}^{2n} \leq K^{2n} |c_{r+1}|^2. \quad (4.32)$$

By traciality of  $\text{tr}$  and by (4.29)-(4.30) we have

$$|J_3| = \left| n c_{r+1} \text{tr} \left( (D^{\frac{1}{2n}} g_r^* g_r D^{\frac{1}{2n}})^{n-1} D^{\frac{1}{2n}} g_r^* x_{i_{r+1}} D^{\frac{1}{2n}} \right) \right. \\ \left. + n c_{r+1} \text{tr} \left( (D^{\frac{1}{2n}} g_r^* g_r D^{\frac{1}{2n}})^{n-1} D^{\frac{1}{2n}} x_{i_{r+1}}^* g_r D^{\frac{1}{2n}} \right) \right| \\ \leq \frac{n |c_{r+1}|}{r+1}. \quad (4.33)$$

By the Hölder inequality

$$\left| \text{tr} \left( D^{\frac{1}{2n}} y_1^* y_1' D^{\frac{1}{2n}} D^{\frac{1}{2n}} y_2^* y_2' D^{\frac{1}{2n}} \cdots D^{\frac{1}{2n}} y_n^* y_n' D^{\frac{1}{2n}} \right) \right| \leq \prod_{k=1}^n \|y_k\|_{2n} \|y_k'\|_{2n},$$

so

$$|J_4| \leq \sum_{j=2}^n 2^{2n} \|g_r\|_{2n}^{n-j} \|c_{r+1} x_{i_{r+1}}\|_{2n}^j \leq K^n 4^n n |c_{r+1}|^2 (I_r + 1). \quad (4.34)$$

Combining (4.31)-(4.34) we have

$$I_{r+1} \leq (1 + K^n 4^n n |c_{r+1}|^2) I_r + (K^{2n} + K^n 4^n n) |c_{r+1}|^2 + \frac{n |c_{r+1}|}{r+1}.$$

Iterating the above inequality and denoting  $a_k = K^n 4^n n |c_k|^2$ ,  $b_k = (K^{2n} + K^n 4^n n) |c_k|^2 + \frac{n |c_k|}{k}$  for each  $k \geq 1$ , we obtain for  $r \geq 2$

$$I_r \leq \prod_{k=2}^r (1 + a_k) I_1 + \sum_{l=2}^{r-1} b_l \prod_{j=l+1}^r (1 + a_j) + b_r$$

$$\leq e^{\sum_{k \geq 1} a_k} I_1 + e^{\sum_{k \geq 1} a_k} \sum_{l \geq 1} b_l \leq (2K^{2n} + K^n 4^n n + n \sum_{l \geq 1} l^{-2}) e^{K^n 4^n n}.$$

The above right hand side is a constant only depending on  $K$  and  $n$ , so the desired inequality (4.28) is proved.

(2) Now choose an arbitrary family  $B = \{x_i \in L^{2n}(\mathcal{M}, \varphi) : i \geq 1\}$  satisfying the assumption. Write  $p = 2n$ . Without loss of generality we take  $K \geq 1$  in the sequel. By the density of  $\mathcal{M}_a$  in  $L^p(\mathcal{M}, \varphi)$ , we choose the sequences  $(x'_k), (y_k) \subset \mathcal{M}_a$  and  $(j_k)_{k \geq 1} \subset \mathbb{N}$  inductively as follows. Let  $j_1 = 1$  and  $x'_1 = y_1 \in \mathcal{M}_a$  such that

$$\|x_1 - x'_1\|_p \leq 2^{-1}.$$

If  $x'_l, y_l, j_l$  for  $1 \leq l \leq k$  are chosen, take by orthogonality a  $j_{k+1} \geq j_k$  such that for all  $1 \leq l \leq k$ ,

$$|\varphi(y_l^* x_{j_{k+1}})| \leq 2^{-k-l-2} \|y_l\|_2^2 \|y_l\|_p^{-1}$$

and by the density property choose  $x'_{k+1} \in \mathcal{M}_a$  such that

$$\|x_{j_{k+1}} - x'_{k+1}\|_p \leq 2^{-k-2} \left(1 + \sum_{l=1}^k \frac{\|y_l\|_p}{\|y_l\|_2}\right)^{-1},$$

where we always make the standard convention in this proof that the terms such as  $\|y_l\|_2^2 / \|y_l\|_p^{-1}$  and  $\|y_l\|_p / \|y_l\|_2^{-1}$  are regarded as zero if  $y_l = 0$ . Also set

$$y_{k+1} = x'_{k+1} - \sum_{l=1}^k \varphi(y_l^* y_l)^{-1} \varphi(y_l^* x'_{k+1}) y_l.$$

We see immediately that  $(y_k)$  is an orthogonal system. Moreover, when we recall the orthogonality of  $(x_k)$ , by the choice of the above sequences we have

$$\begin{aligned} \|x_{j_{k+1}} - y_{k+1}\|_p &\leq \|x_{j_{k+1}} - x'_{k+1}\|_p + \sum_{l=1}^k \frac{\|y_l\|_p}{\|y_l\|_2} |\varphi(y_l^* x'_{k+1})| \\ &= \|x_{j_{k+1}} - x'_{k+1}\|_p + \sum_{l=1}^k \frac{\|y_l\|_p}{\|y_l\|_2} |\varphi(y_l^* (x'_{k+1} - x_{j_{k+1}})) + \varphi(y_l^* x_{j_{k+1}})| \\ &\leq \|x_{j_{k+1}} - x'_{k+1}\|_p + \sum_{l=1}^k \frac{\|y_l\|_p}{\|y_l\|_2} \|y_l\|_2 \|x_{j_{k+1}} - x'_{k+1}\|_2 + \sum_{l=1}^k 2^{-k-l-2} \\ &\leq \|x_{j_{k+1}} - x'_{k+1}\|_p \left(1 + \sum_{l=1}^k \frac{\|y_l\|_p}{\|y_l\|_2}\right) + 2^{-k-2} \leq 2^{-k-1}. \end{aligned} \quad (4.35)$$

By the argument in (1), the orthogonal system  $\{y_k : k \geq 1\} \subset \mathcal{M}_a$  admits a subset  $\{y_{k_r} : r \geq 1\}$  satisfying the estimation of the type (4.28). Now it is easy to observe that  $\{x_{j_{k_r}} : r \geq 1\} \subset B$  is the desired subset. To see this, it suffices to note that, there exists a constant  $C > 0$  such that for all sequences  $(c_r)_{r \geq 1} \subset \mathbb{C}$  with  $\sum_r |c_r|^2 = 1$  and for all  $s \geq 1$ ,

$$\left\| \sum_{r=1}^s c_r y_{k_r} \right\|_p \leq C$$

and hence by the estimates (4.35),

$$\left\| \sum_{r=1}^s c_r x_{j_{k_r}} \right\|_p \leq \left\| \sum_{r=1}^s c_r y_{k_r} \right\|_p + \sum_{r=1}^s |c_r| \|x_{j_{k_r}} - y_{k_r}\|_p \leq C + \sum_{r=1}^s |c_r| 2^{-r} \leq C + 1.$$

So the proof is complete.  $\square$

*Remark 4.4.3.* In general, the noncommutative  $L^p$ -spaces introduced by [Kos84] depend on different interpolation parameters. Following the same line of the proof one can see that the above result still holds for those types of  $L^p$ -spaces.

# Bibliography

- [Ban99] T. Banica. Fusion rules for representations of compact quantum groups. *Exposition. Math.*, 17(4):313–337, 1999.
- [Ban16a] T. Banica. Deformed Fourier models with formal parameters (preprint). Available at arXiv:1603.09433.
- [Ban16b] T. Banica. Quantum groups from stationary matrix models (preprint). Available at arXiv:1604.03756.
- [BB10] T. Banica and J. Bichon. Hopf images and inner faithful representations. *Glasg. Math. J.*, 52(3):677–703, 2010.
- [BBN12] T. Banica, J. Bichon, and S. Natale. Finite quantum groups and quantum permutation groups. *Adv. Math.*, 229(6):3320–3338, 2012.
- [BF84] M. Bożejko and G. Fendler. Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. *Boll. Un. Mat. Ital. A (6)*, 3(2):297–302, 1984.
- [BFS12] T. Banica, U. Franz, and A. Skalski. Idempotent states and the inner linearity property. *Bull. Pol. Acad. Sci. Math.*, 60(2):123–132, 2012.
- [BFZ16] M. Brannan, B. Forrest, and C. Zwarich. Leinert sets and complemented ideals in Fourier algebras (preprint). Available at arXiv:1602.04778.
- [BGS11] J. Bhowmick, D. Goswami, and A. Skalski. Quantum isometry groups of 0-dimensional manifolds. *Trans. Amer. Math. Soc.*, 363(2):901–921, 2011.
- [BL76] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.
- [BLS96] M. Bożejko, M. Leinert, and R. Speicher. Convolution and limit theorems for conditionally free random variables. *Pacific J. Math.*, 175(2):357–388, 1996.
- [BM13] T. Blendek and J. Michaliček.  $L^1$ -norm estimates of character sums defined by a Sidon set in the dual of a compact Kac algebra. *J. Operator Theory*, 70(2):375–399, 2013.
- [BN16] T. Banica and I. Nechita. Universal models for quantum permutation groups (preprint). Available at arXiv:1602.04456.
- [Bož73] M. Bożejko. The existence of  $\Lambda(p)$  sets in discrete noncommutative groups. *Boll. Un. Mat. Ital. (4)*, 8:579–582, 1973.



- [Bož74] M. Bożejko. Sidon sets in dual objects of compact groups. *Colloq. Math.*, 30:137–141, 1974.
- [Bož75a] M. Bożejko. On  $\Lambda(p)$  sets with minimal constant in discrete noncommutative groups. *Proc. Amer. Math. Soc.*, 51:407–412, 1975.
- [Bož75b] M. Bożejko. A remark to my paper: “The existence of  $\Lambda(p)$  sets in discrete noncommutative groups” (Boll. Un. Mat. Ital. (4) 8 (1973), 579–582). *Boll. Un. Mat. Ital. (4)*, 11(1):43, 1975.
- [Bož79] M. Bożejko. Lacunary sets in finite von Neumann algebras. Unpublished manuscript, 1979.
- [Bož81] M. Bożejko. A new group algebra and lacunary sets in discrete noncommutative groups. *Studia Math.*, 70(2):165–175, 1981.
- [Bož85] M. Bożejko. Positive definite bounded matrices and a characterization of amenable groups. *Proc. Amer. Math. Soc.*, 95(3):357–360, 1985.
- [Bra12] M. Brannan. Quantum symmetries and strong Haagerup inequalities. *Comm. Math. Phys.*, 311(1):21–53, 2012.
- [BS09] T. Banica and R. Speicher. Liberation of orthogonal Lie groups. *Adv. Math.*, 222(4):1461–1501, 2009.
- [BT03] E. Bédos and L. Tuset. Amenability and co-amenability for locally compact quantum groups. *Internat. J. Math.*, 14(8):865–884, 2003.
- [Cas13] M. Caspers. The  $L^p$ -Fourier transform on locally compact quantum groups. *J. Operator Theory*, 69(1):161–193, 2013.
- [Cec72] C. Cecchini. Lacunary Fourier series on compact Lie groups. *J. Funct. Anal.*, 11:191–203, 1972.
- [CFK14] F. Cipriani, U. Franz, and A. Kula. Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory. *J. Funct. Anal.*, 266(5):2789–2844, 2014.
- [CH89] M. Cowling and U. Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.*, 96(3):507–549, 1989.
- [CM81] D. I. Cartwright and J. R. McMullen. A structural criterion for the existence of infinite Sidon sets. *Pacific J. Math.*, 96(2):301–317, 1981.
- [Con94] A. Connes. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.
- [Coo10] T. Cooney. A Hausdorff-Young inequality for locally compact quantum groups. *Internat. J. Math.*, 21(12):1619–1632, 2010.
- [CPPR15] M. Caspers, J. Parcet, M. Perrin, and É. Ricard. Noncommutative de Leeuw theorems. *Forum Math. Sigma*, 3:e21, 59, 2015.

- [CW71] R. R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes*. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971. Étude de certaines intégrales singulières.
- [Daw10] M. Daws. Multipliers, self-induced and dual Banach algebras. *Dissertationes Math. (Rozprawy Mat.)*, 470:62, 2010.
- [Daw12] M. Daws. Completely positive multipliers of quantum groups. *Internat. J. Math.*, 23(12):1250132, 23, 2012.
- [DCFY14] K. De Commer, A. Freslon, and M. Yamashita. CCAP for universal discrete quantum groups. *Comm. Math. Phys.*, 331(2):677–701, 2014, With an appendix by Stefaan Vaes.
- [DPR16] A. D’Andrea, C. Pinzari, and S. Rossi. Polynomial growth for compact quantum groups, topological dimension and  $*$ -regularity of the Fourier algebra (preprint). Available at arXiv:1602.07496.
- [Dru70] S. W. Drury. Sur les ensembles de Sidon. *C. R. Acad. Sci. Paris Sér. A-B*, 271:A162–A163, 1970.
- [ER00] E. G. Effros and Z.-J. Ruan. *Operator spaces*, volume 23 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 2000.
- [Eym64] P. Eymard. L’algèbre de Fourier d’un groupe localement compact. *Bull. Soc. Math. France*, 92:181–236, 1964.
- [FKS15] U. Franz, A. Kula, and A. Skalski. Lévy Processes on Quantum Permutation Groups. In *Proceedings of the Conference ‘Noncommutative Analysis, Operator Theory, and Applications’, Milano, 2014*. To appear. Available at arXiv:1510.08321.
- [Fol95] G. B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.
- [FT77] A. Figà-Talamanca. Lacunary sets in noncommutative groups. *Rend. Sem. Mat. Fis. Milano*, 47:45–59 (1979), 1977.
- [FTP83] A. Figà-Talamanca and M. A. Picardello. *Harmonic analysis on free groups*, volume 87 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1983.
- [FTR66] A. Figà-Talamanca and D. Rider. A theorem of Littlewood and lacunary series for compact groups. *Pacific J. Math.*, 16:505–514, 1966.
- [Gro06] L. Gross. Hypercontractivity, logarithmic Sobolev inequalities, and applications: a survey of surveys. In *Diffusion, quantum theory, and radically elementary mathematics*, volume 47 of *Math. Notes*, pages 45–73. Princeton Univ. Press, Princeton, NJ, 2006.
- [Haa79] U. Haagerup. An example of a nonnuclear  $C^*$ -algebra, which has the metric approximation property. *Invent. Math.*, 50(3):279–293, 1978/79.

- [Har99] A. Harcharras. Fourier analysis, Schur multipliers on  $S^p$  and non-commutative  $\Lambda(p)$ -sets. *Studia Math.*, 137(3):203–260, 1999.
- [Hel58] S. Helgason. Lacunary Fourier series on noncommutative groups. *Proc. Amer. Math. Soc.*, 9:782–790, 1958.
- [HJX10] U. Haagerup, M. Junge, and Q. Xu. A reduction method for noncommutative  $L_p$ -spaces and applications. *Trans. Amer. Math. Soc.*, 362(4):2125–2165, 2010.
- [HR70] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups.* Die Grundlehren der mathematischen Wissenschaften, Band 152. Springer-Verlag, New York-Berlin, 1970.
- [JMP14a] M. Junge, T. Mei, and J. Parcet. Noncommutative Riesz transforms – Dimension free bounds and Fourier multipliers. *J. Eur. Math. Soc.*, to appear. Available at arXiv:1407.2475.
- [JMP14b] M. Junge, T. Mei, and J. Parcet. Smooth Fourier multipliers on group von Neumann algebras. *Geom. Funct. Anal.*, 24(6):1913–1980, 2014.
- [JNR09] M. Junge, M. Neufang, and Z.-J. Ruan. A representation theorem for locally compact quantum groups. *Internat. J. Math.*, 20(3):377–400, 2009.
- [Jol90] P. Jolissaint. Rapidly decreasing functions in reduced  $C^*$ -algebras of groups. *Trans. Amer. Math. Soc.*, 317(1):167–196, 1990.
- [JR03] M. Junge and Z.-J. Ruan. Approximation properties for noncommutative  $L_p$ -spaces associated with discrete groups. *Duke Math. J.*, 117(2):313–341, 2003.
- [Jun02] M. Junge. Doob’s inequality for non-commutative martingales. *J. Reine Angew. Math.*, 549:149–190, 2002.
- [Jun05] M. Junge. Embedding of the operator space  $OH$  and the logarithmic ‘little Grothendieck inequality’. *Invent. Math.*, 161(2):225–286, 2005.
- [JV91] P. Jolissaint and A. Valette. Normes de Sobolev et convoluteurs bornés sur  $L^2(G)$ . *Ann. Inst. Fourier (Grenoble)*, 41(4):797–822, 1991.
- [JX03] M. Junge and Q. Xu. Noncommutative Burkholder/Rosenthal inequalities. *Ann. Probab.*, 31(2):948–995, 2003.
- [Kah10] B.-J. Kahng. Fourier transform on locally compact quantum groups. *J. Operator Theory*, 64(1):69–87, 2010.
- [Kos84] H. Kosaki. Applications of the complex interpolation method to a von Neumann algebra: noncommutative  $L^p$ -spaces. *J. Funct. Anal.*, 56(1):29–78, 1984.
- [KS36] S. Kaczmarz and H. Steinhaus. *Theorie der Orthogonalreihen.* Monografie Matematyczne. Instytut Matematyczny Polskiej Akademi Nauk, Warszawa-Lwów, 1936.
- [KS97] A. Klimyk and K. Schmüdgen. *Quantum groups and their representations.* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

- [KV00] J. Kustermans and S. Vaes. Locally compact quantum groups. *Ann. Sci. École Norm. Sup. (4)*, 33(6):837–934, 2000.
- [LDIS11] V. Lafforgue and M. De la Salle. Noncommutative  $L^p$ -spaces without the completely bounded approximation property. *Duke Math. J.*, 160(1):71–116, 2011.
- [Lei74] M. Leinert. Faltungsoperatoren auf gewissen diskreten Gruppen. *Studia Math.*, 52:149–158, 1974.
- [LS91] S. Levendorskii and Y. Soibelman. Algebras of functions on compact quantum groups, Schubert cells and quantum tori. *Comm. Math. Phys.*, 139(1):141–170, 1991.
- [LS08] J. M. Lindsay and A. Skalski. Quantum stochastic convolution cocycles. II. *Comm. Math. Phys.*, 280(3):575–610, 2008.
- [LS11] J. M. Lindsay and A. Skalski. Convolution semigroups of states. *Math. Z.*, 267(1-2):325–339, 2011.
- [Lun09] A. Lunardi. *Interpolation theory*. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, second edition, 2009.
- [LWW16] Z. Liu, Simeng Wang, and J. Wu. Young’s inequality for locally compact quantum groups. In progress.
- [Meg98] R. E. Megginson. *An introduction to Banach space theory*, volume 183 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.
- [MP81] M. B. Marcus and G. Pisier. *Random Fourier series with applications to harmonic analysis*, volume 101 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1981.
- [MVD98] A. Maes and A. Van Daele. Notes on compact quantum groups. *Nieuw Arch. Wisk. (4)*, 16(1-2):73–112, 1998.
- [NS06] A. Nica and R. Speicher. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.
- [NT13] S. Neshveyev and L. Tuset. *Compact quantum groups and their representation categories*, volume 20 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2013.
- [Obe82] D. M. Oberlin. A convolution property of the Cantor-Lebesgue measure. *Colloq. Math.*, 47(1):113–117, 1982.
- [Pic73] M. A. Picardello. Lacunary sets in discrete noncommutative groups. *Boll. Un. Mat. Ital. (4)*, 8:494–508, 1973.
- [Pis78a] G. Pisier. Ensembles de Sidon et espaces de cotype 2. In *Séminaire sur la Géométrie des Espaces de Banach (1977–1978)*, pages Exp. No. 14, 12. École Polytech., Palaiseau, 1978.

- [Pis78b] G. Pisier. Sur l'espace de Banach des séries de Fourier aléatoires presque sûrement continues. In *Séminaire sur la Géométrie des Espaces de Banach (1977–1978)*, pages Exp. No. 17–18, 33. École Polytech., Palaiseau, 1978.
- [Pis83a] G. Pisier. Conditions d'entropie et caractérisations arithmétiques des ensembles de Sidon. In *Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982)*, pages 911–944. Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983.
- [Pis83b] G. Pisier. Some applications of the metric entropy condition to harmonic analysis. In *Banach spaces, harmonic analysis, and probability theory (Storrs, Conn., 1980/1981)*, volume 995 of *Lecture Notes in Math.*, pages 123–154. Springer, Berlin, 1983.
- [Pis95] G. Pisier. Multipliers and lacunary sets in non-amenable groups. *Amer. J. Math.*, 117(2):337–376, 1995.
- [PW90] P. Podleś and S. L. Woronowicz. Quantum deformation of Lorentz group. *Comm. Math. Phys.*, 130(2):381–431, 1990.
- [PX03] G. Pisier and Q. Xu. Non-commutative  $L^p$ -spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1459–1517. North-Holland, Amsterdam, 2003.
- [Rid72] D. Rider. Central lacunary sets. *Monatsh. Math.*, 76:328–338, 1972.
- [Rid75] D. Rider. Randomly continuous functions and Sidon sets. *Duke Math. J.*, 42(4):759–764, 1975.
- [Rit84] D. L. Ritter. A convolution theorem for probability measures on finite groups. *Illinois J. Math.*, 28(3):472–479, 1984.
- [Rud60] W. Rudin. Trigonometric series with gaps. *J. Math. Mech.*, 9:203–227, 1960.
- [Rud90] W. Rudin. *Fourier analysis on groups*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1990. Reprint of the 1962 original, A Wiley-Interscience Publication.
- [RX06] É. Ricard and Q. Xu. Khintchine type inequalities for reduced free products and applications. *J. Reine Angew. Math.*, 599:27–59, 2006.
- [RX16] É. Ricard and Q. Xu. A noncommutative martingale convexity inequality. *Ann. Probab.*, 44(2):867–882, 2016.
- [Soi90] Y. Soibelman. Algebra of functions on a compact quantum group and its representations. *Algebra i Analiz*, 2(1):190–212, 1990.
- [Sol05] P. M. Sołtan. Quantum Bohr compactification. *Illinois J. Math.*, 49(4):1245–1270, 2005.
- [SS16] A. Skalski and P. M. Sołtan. Quantum families of invertible maps and related problems. *Canad. J. Math.*, 68(3):698–720, 2016.
- [Ste70] E. M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory*. Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970.

- [Tak02] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.
- [Tak03] M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 6.
- [Ter81] M. Terp.  $L_p$  spaces associated with von neumann algebras. In *Notes, Report No. 3a + 3b*. Københavns Universitets Matematiske Institut, 1981.
- [Tim08] T. Timmermann. *An invitation to quantum groups and duality*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. From Hopf algebras to multiplicative unitaries and beyond.
- [VD94] A. Van Daele. Multiplier Hopf algebras. *Trans. Amer. Math. Soc.*, 342(2):917–932, 1994.
- [VD96] A. Van Daele. Discrete quantum groups. *J. Algebra*, 180(2):431–444, 1996.
- [VD97] A. Van Daele. The Haar measure on finite quantum groups. *Proc. Amer. Math. Soc.*, 125(12):3489–3500, 1997.
- [VD07] A. Van Daele. The Fourier transform in quantum group theory. In *New techniques in Hopf algebras and graded ring theory*, pages 187–196. K. Vlaam. Acad. Belgie Wet. Kunsten (KVAB), Brussels, 2007.
- [VDN92] D.-V. Voiculescu, K. J. Dykema, and A. Nica. *Free random variables*, volume 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.
- [VDW96] A. Van Daele and Shuzhou Wang. Universal quantum groups. *Internat. J. Math.*, 7(2):255–263, 1996.
- [Ver07] R. Vergnioux. The property of rapid decay for discrete quantum groups. *J. Operator Theory*, 57(2):303–324, 2007.
- [Voi98] D. Voiculescu. A strengthened asymptotic freeness result for random matrices with applications to free entropy. *Internat. Math. Res. Notices*, (1):41–63, 1998.
- [Wan95a] Shuzhou Wang. Free products of compact quantum groups. *Comm. Math. Phys.*, 167(3):671–692, 1995.
- [Wan95b] Shuzhou Wang. Tensor products and crossed products of compact quantum groups. *Proc. London Math. Soc. (3)*, 71(3):695–720, 1995.
- [Wan98] Shuzhou Wang. Quantum symmetry groups of finite spaces. *Comm. Math. Phys.*, 195(1):195–211, 1998.
- [Wan14] Simeng Wang.  $L_p$ -improving convolution operators on finite quantum groups. *Indiana Univ. Math. J.*, to appear. Available at arXiv:1412.2085.

- 
- [Wan16] Simeng Wang. Lacunary Fourier series for compact quantum groups. *Comm. Math. Phys.*, Available online, 2016, <http://dx.doi.org/10.1007/s00220-016-2670-3>.
- [Wor87a] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987.
- [Wor87b] S. L. Woronowicz. Twisted  $SU(2)$  group. An example of a noncommutative differential calculus. *Publ. Res. Inst. Math. Sci.*, 23(1):117–181, 1987.
- [Wor88] S. L. Woronowicz. Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted  $SU(N)$  groups. *Invent. Math.*, 93(1):35–76, 1988.
- [Wor89] S. L. Woronowicz. Differential calculus on compact matrix pseudogroups (quantum groups). *Comm. Math. Phys.*, 122(1):125–170, 1989.
- [Wor98] S. L. Woronowicz. Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, pages 845–884. North-Holland, Amsterdam, 1998.