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Clément Dell'Aiera

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Clément Dell'Aiera. Controlled K-theory for groupoids and applications. K-Theory and Homology [math.KT]. Université de Lorraine, 2017. English. NNT : 2017LORR0114 . tel-01661420

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# Controlled $K$ -theory for groupoids and applications

## THÈSE

présentée et soutenue publiquement le 12 Juillet 2017

pour l'obtention du

Doctorat de l'Université de Lorraine  
(Mathématiques)

par

Clément Dell'Aiera

### Composition du jury

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# Résumé

Dans leur article de 2015 intitulé ” *On quantitative operator  $K$ -theory*”, H. Oyono-Oyono et G. Yu introduisent un raffinement de la  $K$ -théorie opératorielle adapté au cadre des  $C^*$ -algèbres filtrées, appelé  $K$ -théorie quantitative ou contrôlée. Dans cette thèse, nous généralisons la notion de filtration de  $C^*$ -algèbres. Nous montrons ensuite que ce cadre contient celui déjà traité par G. Yu et H. Oyono-Oyono, tout en se révélant assez souple pour traiter les produits croisés de groupoïdes étales et de groupes quantiques discrets. Nous construisons ensuite des applications d’assemblage à valeurs dans les groupes de  $K$ -théorie contrôlée associés, pour les  $C^*$ -algèbres de Roe à coefficients et les produits croisés de groupoïdes étales. Nous montrons que ces applications factorisent les applications d’assemblage usuelles de Baum-Connes. Nous prouvons ensuite ce que nous appelons des énoncés quantitatifs, et nous montrons qu’une version contrôlée de la conjecture de Baum-Connes est vérifiée pour une large classe de groupoïdes étales. La fin de la thèse est consacrée à plusieurs applications de ces résultats. Nous montrons que l’application d’assemblage contrôlée coarse est équivalente à son analogue à coefficients pour le groupoïde coarse introduit par G. Skandalis, J-L. Tu et G. Yu. Nous donnons ensuite une preuve que les espaces coarses qui admettent un plongement hilbertien fibré vérifient la version maximale de la conjecture de Baum-Connes coarse contrôlée. Enfin nous étudions les groupoïdes étales dont toutes les actions propres sont localement induites par des sous-groupoïdes compacts ouverts, dont un exemple est donné par les groupoïdes amples introduits par J. Renault. Nous développons un principe de restriction pour cette classe de groupoïdes, et prouvons que, sous des hypothèses raisonnables, leurs produits croisés vérifient la formule de Künneth en  $K$ -théorie contrôlée.

## Abstract

In their paper entitled ” *On quantitative operator  $K$ -theory*”, H. Oyono-Oyono and G. Yu introduced a refinement of operator  $K$ -theory, called quantitative or controlled  $K$ -theory, adapted to the setting of filtered  $C^*$ -algebras. In this thesis, we generalize filtration of  $C^*$ -algebras. We show that this setting contains the theory developed by H. Oyono-Oyono and G. Yu, and is general enough to be applied to the setting of crossed products by étale groupoids and discrete quantum groups. We construct controlled assembly maps with values into this controlled  $K$ -groups, for Roe  $C^*$ -algebras and crossed products by étale groupoids. We show that these controlled assembly maps factorize the usual Baum-Connes and coarse Baum-Connes assembly maps. We prove statements called quantitative statements, and we show that a controlled version of the Baum-Connes conjecture is satisfied for a large class of étale groupoids. The end of the thesis is devoted to several applications of these results. We show that the controlled coarse assembly map is equivalent to its analog with coefficients for the coarse groupoid introduced by G. Skandalis, J-L. Tu and G. Yu. We give a proof that coarse spaces which admit a fibred coarse embedding into Hilbert space satisfy the maximal controlled coarse Baum-Connes conjecture. Finally, we study étale groupoids whose proper actions are locally induced by compact open subgroupoids, e.g. ample groupoids introduced by J. Renault. We develop a restriction principle for these groupoids, and prove that under suitable assumptions, their crossed products satisfy the controlled Künneth formula.

## Remerciements

Je voudrais remercier mon directeur de thèse, Hervé Oyono-Oyono, de m'avoir proposé ce sujet de thèse et d'avoir accepté de m'encadrer pendant ces trois années. C'est avec plaisir que je lui exprime ma gratitude. La générosité et la disponibilité dont il a fait preuve à mon égard m'ont permis d'apprendre beaucoup de mathématiques.

Je voudrais aussi remercier mon co-directeur de thèse, Andrzej Zuk, pour son soutien, et son enseignement que j'ai suivi dès le master. J'ai régulièrement eu des discussions très enrichissantes avec lui.

Je remercie Jacek Brodzki et Guoliang Yu d'avoir accepté de rapporter cette thèse, ainsi que les autres membres du jury, Paulo Carillo-Rouse, Emmanuel Germain, Maria-Paula Gomez-Aparicio, Jean Renault, et Jean Louis-Tu.

Un grand merci à l'équipe du septième étage de Sophie Germain, d'Orsay et aux participants du groupe de travail sur la  $KK$ -théorie, pour leurs conseils et discussions, en particulier Pierre Fima, Maria Paula Gomez Aparicio, et Stéphane Vassout. J'ai pu aussi apprendre beaucoup au contact des doctorants, Kevin Boucher, Aurélien Sagnier et Vito Zenobi.

Je voudrais remercier Siegfried Echterhoff et son étudiant Christian Bönicke, pour les différentes invitations à Münster et leur accueil chaleureux.

C'est avec regret que je vais quitter le laboratoire de l'IECL. Je voudrais remercier Victor Nistor, Nicolas Prudhon et Salah Mehdi. Leur bonne humeur et leur disponibilité m'ont été d'un grand secours. Merci à Matthieu Brachet et Benjamin Alvarez. La thèse aurait été beaucoup plus rude sans la légèreté et l'humour qu'ils ont apporté au laboratoire.

Je voudrais remercier mon ami de longue date, Pierre Cagne, qui m'a profondément influencé. Discuter avec lui de mathématiques et de sciences en général se révèle toujours un plaisir. Un grand merci à Rubén Martos et à son enthousiasme constant depuis maintenant quatre années.

Enfin, un grand merci à Ariane Constantin. Ses efforts resteront à jamais méconnus. S'il m'a fallu de la patience pour finir cette thèse, ce n'est rien comparé aux longues heures qu'elle a passé à m'écouter en parler.

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# Résumé français de la thèse

Cette thèse porte sur la  $K$ -théorie contrôlée, ou  $K$ -théorie quantitative, ainsi que ses applications en géométrie non-commutative, en particulier sur la conjecture de Baum-Connes.

La  $K$ -théorie contrôlée, introduite par H. Oyono-Oyono et G. Yu dans [40], est un raffinement de la  $K$ -théorie opératorielle. Elle permet de prendre en compte les phénomènes de propagation en  $K$ -théorie. Détaillons un peu notre propos.

Dans le cadre des  $C^*$ -algèbres, on dispose d'un foncteur à valeurs dans la catégorie des groupes abéliens  $A \mapsto K_*(A)$ , qui à une  $C^*$ -algèbre  $A$  associe ses groupes de  $K$ -théorie  $K_0(A)$  et  $K_1(A)$ . Calculer la  $K$ -théorie d'une  $C^*$ -algèbre est un problème important mais souvent difficile. Dans les années 90, P. Baum, A. Connes et N. Higson proposèrent dans [3] de calculer la  $K$ -théorie de la  $C^*$ -algèbre réduite d'un groupe localement compact  $G$  grâce à une application d'assemblage  $\mu_G : K^{top}(G) \rightarrow K(C_r^*(G))$ . Le membre de gauche est appelé la  $K$ -théorie topologique de  $G$ , et se calcule par des méthodes de topologie algébrique classiques. La conjecture de Baum-Connes affirme que l'application d'assemblage est un isomorphisme.

S'inspirant de cette conjecture, une application d'assemblage fut définie dans le cadre de la géométrie coarse. La géométrie coarse s'intéresse à une certaine classe d'espaces métriques, dont elle cherche à déterminer les propriétés à grande échelle. Si l'on se donne un espace coarse  $X$ , on peut construire la  $C^*$ -algèbre de Roe  $C^*(X)$ . La  $K$ -théorie de  $C^*(X)$  est le réceptacle pour les hautes signatures associées à des variétés complètes non nécessairement compactes quasi-isométriques à  $X$ . Pour la calculer, il est alors possible de définir une application d'assemblage coarse  $\mu_X : KX(X) \rightarrow K(C^*(X))$ . Le membre de gauche est appelé  $K$ -homologie coarse de  $X$ . La conjecture de Baum-Connes coarse affirme que cette application est un isomorphisme pour tout espace coarse à géométrie bornée.

Bien qu'ouverte dans le cas général, la conjecture de Baum-Connes a été démontrée pour une large classe de groupes. Un contre-exemple a été donné à la conjecture de Baum-Connes coarse, mais elle est vérifiée pour une grande famille d'espaces coarses. L'idée sous-jacente à la  $K$ -théorie contrôlée est déjà contenue dans la preuve de la conjecture de Baum-Connes coarse que donne G. Yu dans [59] pour les espaces coarses de dimension asymptotique finie. L'un des intérêts de cette démonstration est qu'elle évite des arguments analytiques de type Dirac-Dual-Dirac. Un objectif de la  $K$ -théorie contrôlée est d'extraire de cette preuve des méthodes générales qui permettent de calculer la  $K$ -théorie de  $C^*$ -algèbres autres que les algèbres de Roe d'espaces coarses de dimension asymptotique finie.

Par exemple, de larges classes de  $C^*$ -algèbres sont construites à partir :

- d'espaces métrique  $(X, d)$  : les algèbres de Roe  $C^*(X)$ ,
- de groupes localement compacts  $G$  : les  $C^*$ -algèbres réduite  $C_r^*(G)$  et maximale  $C_{max}^*(G)$ , et les produits croisés  $A \rtimes G$ . Plus généralement, les mêmes notions existent pour les groupoïdes localement compacts avec système de Haar,

- de groupes quantiques compacts  $\mathbb{G}$  et de leur duals, les groupes quantiques discrets  $\hat{\mathbb{G}}$  : à nouveau leurs  $C^*$ -algèbres réduites, maximales et leurs produits croisés.

Or, dans la plupart des cas, ces  $C^*$ -algèbres possèdent une structure supplémentaire appelée filtration. La première étape de notre travail a consisté en une généralisation de la notion de filtration, ce qui permet d'appliquer les techniques développées dans [40] et [39] à de nouveaux exemples. Dans ces articles, les auteurs définissent une  $C^*$ -algèbre filtrée de la manière suivante.

**Définition.** Une  $C^*$ -algèbre  $A$  est filtrée s'il existe une famille de sous-espaces vectoriels fermés auto-adjoints  $\{A_R\}_{R>0}$  de  $A$  telle que :

- si  $R \leq R'$ , alors  $A_R \subseteq A_{R'}$ ,
- pour tout  $R, R' > 0$ ,  $A_R \cdot A_{R'} \subseteq A_{R+R'}$ ,
- $A$  est l'adhérence de l'union des sous-espaces  $A_R$ , i.e.  $\overline{\cup_{R>0} A_R} = A$ .
- si  $A$  est unitale, on a de plus  $1 \in A_R, \forall R > 0$ .

Nous proposons alors d'étendre cette définition en considérant des familles des sous-espaces vectoriels indexés par des structures plus générales que nous appelons structures coarses.

**Définition.** Une structure coarse  $\mathcal{E}$  est un semi-groupe abélien qui est aussi un treillis. Rappelons qu'un treillis est un ensemble partiellement ordonné tel que toute paire  $(E, E')$  admette un supremum  $E \vee E'$  et un infimum  $E \wedge E'$ .

Si une structure coarse  $\mathcal{E}$  est donnée, une  $C^*$ -algèbre  $\mathcal{E}$ -filtrée est définie comme dans la définition précédente, en remplaçant les nombres positifs par des éléments de la structure coarse, et l'addition par la composition. On montre alors comment la  $K$ -théorie contrôlée de [40] s'adapte à ce cadre plus général.

De nouvelles  $C^*$ -algèbres peuvent alors être vues comme des  $C^*$ -algèbres filtrées. Voici des exemples qui sont détaillés par la suite.

- Soit  $(X, d)$  un espace métrique discret à géométrie bornée. Alors les ensembles  $E \subseteq X \times X$  symétriques tels que  $\sup d|_E < \infty$ , sont munis d'un ordre partiel  $\subseteq$  et d'une loi de composition donnée par

$$E \circ E = EE' \cup E'E,$$

où  $EE' = \{(x, y) \in X \times X \text{ t.q. } \exists z \in X / (x, z) \in E \text{ et } (z, y) \in E'\}$ . Cela définit une structure coarse  $\mathcal{E}_X$ .

- Soit  $G$  un groupoïde étale  $\sigma$ -compact. Alors les ensembles  $E \subseteq G$  symétriques compacts, sont munis d'un ordre partiel  $\subseteq$  et d'une loi de composition donnée par

$$E \circ E = EE' \cup E'E,$$

où  $EE' = \{gg'; (g, g') \in G^{(2)}\}$ . Cela définit une structure coarse  $\mathcal{E}_G$ .

- Soit  $\mathbb{G}$  un groupe quantique compact. Alors l'ensemble des classes d'équivalence de représentations unitaires symétriques de dimension finie de  $\mathbb{G}$  est muni de l'ordre suivant :  $\pi \leq \pi'$  ssi  $\pi$  est unitairement équivalente à une sous-représentation de  $\pi'$ . De plus, le produit tensoriel symétrisé fournit une loi de composition, et cela définit une structure coarse  $\mathcal{E}_{\mathbb{G}}$ .

Il s'avère que les algèbres de Roe  $C^*(X, B)$ , les produits croisés par  $G$ , et les produits croisés par  $\hat{\mathbb{G}}$  sont filtrés par  $\mathcal{E}_X$ ,  $\mathcal{E}_G$  et  $\mathcal{E}_{\mathbb{G}}$  respectivement.

Dans un premier temps, nous traitons la construction d'applications d'assemblage à valeurs dans la  $K$ -théorie contrôlée, dans le cas des espaces coarses et dans celui des groupoïdes étales. Une propriété importante est que ces applications factorisent les applications d'assemblage classiques. Cela permet de relier les conjectures de Baum-Connes et de Baum-Connes coarse aux propriétés de nos applications. Ces dernières sont définies en deux étapes. La première est de définir ce que l'on appelle un foncteur de descente. Puis, on définit le complexe de Rips associé à  $X$  et à  $G$ , et un certain projecteur canonique associé.

Plus précisément, soit  $(X, d)$  un espace métrique dénombrable discret, que l'on suppose à géométrie bornée; i.e.  $\forall R > 0, \sup |\{(x, y) \in X \times X \text{ t.q. } d(x, y) < R\}| < \infty$ . Nous construisons d'abord la transformation de Roe contrôlée  $\hat{\sigma}_X(z)$  pour tout élément  $z \in KK(A, B)$ . Les propositions 4.2.5 et 4.2.6, qui décrivent ses propriétés, peuvent être résumées dans la proposition suivante :

**Proposition.** Soient  $A$  et  $B$  deux  $C^*$ -algèbres. Pour tout  $z \in KK_*(A, B)$ , il existe une paire de contrôle  $(\alpha_X, k_X)$  et un morphisme  $(\alpha_X, k_X)$ -contrôlé

$$\hat{\sigma}_X(z) : \hat{K}(C^*(X, A)) \rightarrow \hat{K}(C^*(X, B))$$

de même degré que  $z$ , tels que :

(i)  $\hat{\sigma}_X(z)$  induit la multiplication à droite par  $\sigma_X(z)$  en  $K$ -théorie;

(ii)  $\hat{\sigma}_X$  est additif, i.e.

$$\hat{\sigma}_X(z + z') = \hat{\sigma}_X(z) + \hat{\sigma}_X(z').$$

(iii) Pour tout  $*$ -homomorphisme  $f : A_1 \rightarrow A_2$ ,

$$\hat{\sigma}_X(f^*(z)) = \hat{\sigma}_X(z) \circ f_{X,*}$$

pour tout  $z \in KK_*(A_2, B)$ .

(iv) Pour tout  $*$ -homomorphisme  $g : B_1 \rightarrow B_2$ ,

$$\hat{\sigma}_X(g_*(z)) = g_{X,*} \circ \hat{\sigma}_X(z)$$

pour tout  $z \in KK_*(A, B_1)$ .

(v) Soient  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  une extension semi-scindée de  $C^*$ -algèbres et  $[\partial_J] \in KK_1(A/J, J)$  son bord en  $KK$ -théorie. Alors :

$$\hat{\sigma}_X([\partial_{J,A}]) = D_{C^*(X,J), C^*(X,A)}.$$

(vi)  $\hat{\sigma}_X([id_A]) \sim_{(\alpha_X, k_X)} id_{\hat{K}(C^*(X, A))}$

Une fois la transformation de Roe contrôlée construite, l'étape suivante naturelle est la construction de l'application d'assemblage contrôlée  $\hat{\mu}_{X, B}$ , pour toute  $C^*$ -algèbre  $B$ . Pour cela, pour tout entourage  $E$ , on définit une projection canonique  $P_E \in C^*(X, C_0(P_E(X)))$  de propagation finie. Elle définit alors une classe paire de  $K$ -théorie contrôlée.

**Définition.** Soient  $B$  une  $C^*$ -algèbre,  $\varepsilon \in (0, \frac{1}{4})$  et  $E, F \in \mathcal{E}_X$  des entourages tels que  $k_X(\varepsilon).E \subseteq F$ . L'application d'assemblage contrôlée  $\hat{\mu}_{X, B} = (\mu_{X, B}^{\varepsilon, E, F})_{\varepsilon, E}$  est définie comme la famille d'applications

$$\hat{\mu}_{X, B}^{\varepsilon, E, F} : \begin{cases} KK(C_0(P_E(X)), B) & \rightarrow K^{\varepsilon, F}(C^*(X, B)) \\ z & \mapsto \iota_{\alpha_X \varepsilon', k_X(\varepsilon').F'} \circ \hat{\sigma}_X(z)[P_E, 0]_{\varepsilon', F'} \end{cases}$$

où  $\varepsilon'$  et  $F'$  vérifient :

- $\varepsilon' \in (0, \frac{1}{4})$  tel que  $\alpha_X \varepsilon' \leq \varepsilon$ ,
- et  $F' \in \mathcal{E}$  tel que  $E \subseteq F'$  et  $k_X(\varepsilon').F' \subseteq F$ .

Nous définissons des applications analogues pour les groupoïdes étales. Plus précisément, soit  $G$  un groupoïde étale  $\sigma$ -compact. Nous montrons que les ensembles  $E \subseteq G$  compacts symétriques forment une structure coarse  $\mathcal{E}$ , et que les produits croisés réduits de  $G$  sont filtrés par  $\mathcal{E}$ . La première étape est à nouveau de construire un analogue du foncteur de descente en  $K$ -théorie contrôlée. Les propositions 4.3.3 et 4.3.4 peuvent être résumées dans la proposition suivante.

**Proposition.** Soient  $A$  et  $B$  deux  $G$ -algèbres. Pour tout  $z \in KK_*^G(A, B)$ , il existe une paire de contrôle  $(\alpha_J, k_J)$  et un morphisme  $(\alpha_J, k_J)$ -contrôlé

$$J_{red, G}(z) : \hat{K}(A \rtimes_r G) \rightarrow \hat{K}(B \rtimes_r G)$$

de même degré que  $z$ , tels que

- $J_{red, G}(z)$  induit la multiplication à droite par  $j_{red, G}(z)$  en  $K$ -théorie;
- $J_{red, G}$  est additif, i.e.

$$J_{red, G}(z + z') = J_{red, G}(z) + J_{red, G}(z').$$

- Pour tout  $G$ -morphisme  $f : A_1 \rightarrow A_2$ ,

$$J_{red, G}(f^*(z)) = J_{red, G}(z) \circ f_{G, red, *}$$

pour tout  $z \in KK_*^G(A_2, B)$ .

- Pour tout  $G$ -morphisme  $g : B_1 \rightarrow B_2$ ,

$$J_{red, G}(g_*(z)) = g_{G, red, *} \circ J_{red, G}(z)$$

pour tout  $z \in KK_*^G(A, B_1)$ .

(v) Soient  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  une extension semi-scindée  $G$ -équivariante de  $G$ -algèbres et  $[\partial_J] \in KK_1^G(A/J, J)$  son bord en  $KK^G$ -théorie. Alors

$$J_G([\partial_J]) = D_{J \rtimes_r G, A \rtimes_r G}.$$

(vi)  $J_{red, G}([id_A]) \sim_{(\alpha_J, k_J)} id_{\hat{K}(A \rtimes G)}$

Cette transformation, que nous appelons transformation de Kasparov contrôlée, permet de construire l'application d'assemblage contrôlée  $\hat{\mu}_{G, B}$  pour toute  $G$ -algèbre  $B$ . Pour tout ensemble contrôlé  $E \subseteq G$ , on dispose d'un projecteur  $\mathcal{L}_E \in C_0(P_E(G)) \rtimes_r G$  qui est de propagation finie. Il définit donc une classe de  $K$ -théorie contrôlée.

**Définition.** Soient  $B$  une  $G$ -algèbre,  $\varepsilon \in (0, \frac{1}{4})$ , et  $E \in \mathcal{E}$ . Soit  $F \in \mathcal{E}$  tel que  $k_J(\varepsilon).E \subseteq F$ . L'application d'assemblage contrôlée pour  $G$  est définie comme la famille d'applications :

$$\mu_{G, B}^{\varepsilon, E, F} \begin{cases} RK^G(P_E(G), B) & \rightarrow K_*^{\varepsilon, F}(B \rtimes_r G) \\ z & \mapsto \iota_{\alpha_J \varepsilon', k_J(\varepsilon').F'}^{\varepsilon, F} \circ J_G^{\varepsilon', F'}(z)([\mathcal{L}_E, 0]_{\varepsilon', F'}) \end{cases}$$

où  $\varepsilon'$  et  $F'$  vérifient :

- $\varepsilon' \in (0, \frac{1}{4})$  tel que  $\alpha_J \varepsilon' \leq \varepsilon$ ,
- et  $F' \in \mathcal{E}$  tel que  $E \subseteq F'$  et  $k_J(\varepsilon').F' \subseteq F$ .

L'application d'assemblage admet aussi une version maximale à valeurs dans  $\hat{K}(A \rtimes_{max} G) = \{K^{\varepsilon, E}(A \rtimes_{max} G)\}_{\varepsilon \in (0, \frac{1}{4}), E \in \mathcal{E}}$ .

La deuxième étape du travail consiste à formuler une version contrôlée de la conjecture de Baum-Connes. Nous montrons ensuite comment relier l'application d'assemblage contrôlée à l'application d'assemblage classique. Les théorèmes qui réalisent ce programme sont les théorèmes 4.3.18 et 4.3.19, que nous appelons énoncés quantitatifs. Pour cela, il nous faut d'abord introduire les propriétés suivantes :

- $QI_{G, B}(E, E', F, \varepsilon)$  : pour tout  $x \in RK^G(P_E(G), B)$ , alors  $\mu_{G, B}^{\varepsilon, E, F}(x) = 0$  implique que  $q_E^{E'}(x) = 0$  dans  $RK^G(P_{E'}(G), B)$ .
- $QS_{G, B}(E, F, F', \varepsilon, \varepsilon')$  : pour tout  $y \in K^{\varepsilon, F}(B \rtimes G)$ , il existe  $x \in RK^G(P_E(G), B)$  tel que  $\mu_{G, B}^{\varepsilon', E, F'}(x) = \iota_{\varepsilon, F'}^{\varepsilon', F'}(y)$ .

Les énoncés quantitatifs forment le résultat central de la thèse. Ils prennent la forme suivante. Soit  $G$  un groupoïde étale  $\sigma$ -compact à base compacte.

**Théorème.** Soient  $B$  une  $G$ -algèbre, et  $\tilde{B} = l^\infty(\mathbb{N}, B \otimes \mathfrak{K})$ . Alors  $\mu_{G, \tilde{B}}$  est injective ssi pour tout  $E \in \mathcal{E}$ ,  $\varepsilon \in (0, \frac{1}{4})$  et  $F$  tel que  $k_J(\varepsilon).E \subseteq F$ , il existe  $E' \in \mathcal{E}$  tel que  $E \subseteq E'$  et  $QI_{G, B}(E, E', \varepsilon, F)$  soit vérifiée.

**Théorème.** Soient  $B$  une  $G$ -algèbre, et  $\tilde{B} = l^\infty(\mathbb{N}, B \otimes \mathfrak{K})$ . Alors il existe  $\lambda \geq 1$  tel que  $\mu_{G, \tilde{B}}$  est surjective ssi pour tout  $0 < \varepsilon < \frac{1}{4\lambda}$  et  $F \in \mathcal{E}$ , il existe  $E \in \mathcal{E}$  et  $F'$  tel que  $k_J.E \subseteq F$  tel que  $QS_{G, B}(E, F, F', \varepsilon, \lambda\varepsilon)$  soit vérifiée.

Ils admettent une version uniforme :

**Théorème.** Soit  $G$  un groupoïde étale  $\sigma$ -compact avec une base compacte.

- Supposons que pour toute  $G$ -algèbre  $B$ ,  $\mu_{G,B}$  soit injective. Alors, pour tout  $\varepsilon \in (0, \frac{1}{4})$  et tout  $E, F \in \mathcal{E}$  tels que  $k_J(\varepsilon).E \subseteq F$ , il existe  $E' \in \mathcal{E}$  tel que  $E \subseteq E'$  et tel que  $QI_{G,B}(E, E', \varepsilon, F)$  soit satisfait pour toute  $G$ -algèbre  $B$ .
- Supposons que pour toute  $G$ -algèbre  $B$ ,  $\mu_{G,B}$  soit surjective. Alors, pour un certain  $\lambda \geq 1$  et pour tout  $\varepsilon \in (0, \frac{1}{4\lambda})$  et tout  $F \in \mathcal{E}$ , il existe  $E, F' \in \mathcal{E}$  tels que  $k_J(\varepsilon).E \subseteq F'$  et  $F \subseteq F'$  tel que, pour toute  $G$ -algèbre  $B$ ,  $QS_{G,B}(E, F, F', \varepsilon, \lambda\varepsilon)$  soit satisfait.

Le reste du chapitre présente deux applications. La première concerne un résultat dit d'Approximation Persistante, et le second donne une version contrôlée d'un résultat de  $K$ -moyennabilité de groupes quantiques discrets.

- Rappelons la définition de la propriété d'approximation persistante (PAP), donnée dans [39].

**Définition.** Soit  $(B, \mathcal{E})$  une  $C^*$ -algèbre filtrée,  $\lambda > 0$  et soient  $\varepsilon, \varepsilon'$  des nombres positifs tels que  $0 < \varepsilon < \varepsilon' < \frac{1}{4}$  et  $F, F' \in \mathcal{E}$  tels que  $F \subseteq F'$ .

- $PA_B(\varepsilon, \varepsilon', F, F')$  : pour tout  $x \in K_*^{\varepsilon, F}(B)$  tel que  $\iota_{\varepsilon, F}(x) = 0$  dans  $K_*(B)$ , alors  $\iota_{\varepsilon', F'}(x) = 0$  dans  $K_*^{\varepsilon', F'}(B)$ .
- $B$  satisfait la propriété d'approximation persistante  $(PAP)_\lambda$  si pour tout  $F \in \mathcal{E}$  et  $\varepsilon \in (0, \frac{1}{4})$ , il existe  $F' \in \mathcal{E}$  tel que  $PA_B(\varepsilon, \lambda\varepsilon, F, F')$  soit vérifiée.

La propriété (PAP) est satisfaite lorsque la  $K$ -théorie est bien approximée, de manière uniforme, par la  $K$ -théorie contrôlée. Le théorème 4.3.22 permet de relier l'application d'assemblage et la propriété d'approximation persistante.

**Théorème.** Soit  $G$  un groupoïde étale à base compacte qui admet un classifiant des actions propres cocompact, et  $A$  une  $G$ -algèbre. Si  $\mu_{G, l^\infty(\mathbb{N}, K_A)}$  est surjective et  $\mu_{G,A}$  est injective, alors, pour une constante universelle  $\lambda_{PA} \geq 1$ , pour tout  $\varepsilon \in (0, \frac{1}{4\lambda_{PA}})$  et  $F \in \mathcal{E}$ , il existe  $F' \in \mathcal{E}$  tel que  $F \subseteq F'$  et tel que  $PA_{A \times G}(\varepsilon, \lambda_{PA}\varepsilon, F, F')$  soit vérifiée.

- En remarque, nous expliquons comment la transformation de Kasparov contrôlée  $J_G$  peut se définir pour les produits croisés réduits et maximaux de groupes quantiques discrets. Il suffit en effet de procéder de la même manière que dans le cas des groupoïdes étales. Pour ce faire, il suffit de remarquer que les propriétés importantes que vérifie le produit croisé, et qui permettent de construire  $J_G$ , restent vraies dans le cas quantique. Les preuves écrites dans le cadre des groupoïdes étales se transposent

alors sans modification.

Nous donnons une application à la  $K$ -moyennabilité des groupes quantiques discrets, sous la forme de la proposition 4.4.4. Notons, pour tout  $\hat{\mathbb{G}}$ -algèbre  $A$ ,  $\lambda_A$  l'application canonique  $A \rtimes_{max} \hat{\mathbb{G}} \rightarrow A \rtimes_r \hat{\mathbb{G}}$ .

**Proposition.** Soit  $\hat{\mathbb{G}}$  un groupe quantique discret  $K$ -moyennable. Alors, il existe une paire de contrôle  $\rho$  telle que, pour toute  $\hat{\mathbb{G}}$ -algèbre  $A$ ,

$$(\lambda_A)_* : \hat{K}(A \rtimes_{max} \hat{\mathbb{G}}) \rightarrow \hat{K}(A \rtimes_r \hat{\mathbb{G}})$$

est un isomorphisme  $\rho$ -contrôlé.

La dernière partie de la thèse se concentre sur deux applications.

- La première application concerne la géométrie coarse. Un moyen pour ce faire est d'utiliser le groupoïde coarse  $G(X)$ , introduit dans [47], qui est un groupoïde étale associé à tout espace coarse. Notamment, nous démontrons que l'application d'assemblage coarse contrôlée pour  $X$  à valeurs dans  $B$  est équivalente à l'application d'assemblage contrôlée pour  $G(X)$  à valeurs dans  $l^\infty(X, B \otimes \mathfrak{K})$ . Le théorème 5.1.4 est le suivant.

**Théorème.** Soient  $B$  une  $C^*$ -algèbre,  $E \in \mathcal{E}_X$  un entourage et  $\tilde{B}$  la  $G$ -algèbre  $l^\infty(X, B \otimes \mathfrak{K})$ . Alors, pour tout  $z \in RK^G(P_{\tilde{E}}(G), \tilde{B})$  et tout  $\varepsilon \in (0, \frac{1}{4})$ , l'égalité suivante est vérifiée :

$$(\Psi_B)_* \circ \mu_{G, \tilde{B}}^{\varepsilon, \tilde{E}}(z) = \mu_{X, B}^{\varepsilon, E}(l^*(z)).$$

Ce théorème induit en  $K$ -théorie un résultat de G. Skandalis, J-L. Tu et G. Yu [47], qui établit l'équivalence entre la conjecture de Baum-Connes coarse pour  $X$  à coefficients dans  $B$  et la conjecture de Baum-Connes pour  $G(X)$  à coefficients dans  $l^\infty(X, B \otimes \mathfrak{K})$ . Il permet notamment de donner une version contrôlée d'un résultat de M. Finn-Sell [15].

**Corollaire.** Soit  $X$  un espace coarse qui admet un plongement fibré dans l'espace de Hilbert. Alors  $\hat{\mu}_X^{max}$  est un isomorphisme contrôlé, i.e.  $X$  vérifie la version maximale de la conjecture de Baum-Connes coarse contrôlée.

- La dernière application concerne une formule de Künneth en  $K$ -théorie quantitative pour les  $C^*$ -algèbres de groupoïdes, qui constitue le théorème 5.2.13. Le cas des groupes discrets a été couvert dans un article de H. Oyono-Oyono et G. Yu [38]. Pour établir la formule de Künneth, nous montrons qu'un certain morphisme  $\alpha_{A \rtimes_r G, B}$  est un isomorphisme. La stratégie consiste en une généralisation des techniques dites de "Going Down Principle" développées par J. Chabert, S. Echterhoff

et H. Oyono-Oyono dans [6]. Elle peut se résumer ainsi : nous définissons une version topologique  $\alpha_{A,B}^{G,Z} : RK^G(Z, A) \otimes K_*(B) \rightarrow RK^G(Z, A \otimes B)$  de  $\alpha_{A \rtimes_r G, B}$ , puis démontrons que l'application d'assemblage les entrelace. Pour cela, nous avons besoin de définir le foncteur  $Ind_H^G$  d'induction pour un sous-groupe compact ouvert  $H$  de  $G$  en  $KK$ -théorie équivariante, ainsi que d'introduire une propriété sur les actions de groupoïdes, appelée propriété forte. Nous définissons ensuite une classe  $\mathcal{C}$  de groupoïdes, dont toutes les actions propres sont fortement propres, i.e. localement induites par des sous-groupe compacts ouverts. Nous prouvons que les groupoïdes amples sont dans la classe  $\mathcal{C}$ , ce qui donne une large classe d'exemples. Le résultat suivant permet alors de relier la classe  $\mathcal{C}$  à la formule de Künneth.

**Théorème.** Soit  $G$  un groupoïde étale de la classe  $\mathcal{C}$ , et soient  $E \in \mathcal{E}$  un ensemble contrôlé de  $G$  et  $P_E(G)$  le complexe de Rips associé. Si, pour tout sous-groupe compact ouvert  $H$  de  $G$  et tout  $H$ -espace  $V$  tel que l'application moment  $p : V \rightarrow H^{(0)}$  soit localement injective,  $\alpha_{Res_H^G(A), B}^{H, V}$  est un isomorphisme, alors  $\alpha_{A, B}^{G, P_E(G)}$  est un isomorphisme pour toute  $C^*$ -algèbre  $B$  telle que  $K_*(B)$  est un groupe abélien libre.

Voici le contenu du théorème principal.

**Théorème.** Soit  $G$  un groupoïde  $\sigma$ -compact étale et  $A$  une  $G$ -algèbre. Si

- $G$  vérifie la conjecture de Baum-Connes à coefficients,
- $G$  est un groupoïde fortement propre,
- pour tout sous-groupe compact ouvert  $H$  de  $G$  et tout  $H$ -espace  $V$  tels que l'application moment  $p : V \rightarrow H^{(0)}$  soit localement injective,  $\alpha_{A, B}^{G, V}$  est un isomorphisme.

Alors  $A \rtimes_r G$  vérifie la formule de Künneth contrôlée.



## Background and historical perspective

After the pioneering work of Grothendieck on the Riemann-Roch theorem,  $K$ -theory was generalized to the topological setting by Atiyah and Hirzebruch. This construction was crucial for the proof of the index theorem by Atiyah and Singer. From another perspective,  $C^*$ -algebras define a natural extension of classical topology to the noncommutative setting. A  $K$ -theory for  $C^*$ -algebras has been defined, which generalizes the topological  $K$ -theory by the Serre-Swan theorem. This operator  $K$ -theory led to a huge amount of work and interest, based on the wide range of its applications. From classification of  $C^*$ -algebras to index theorems and representation theory, even appearing in some areas of theoretical physics,  $K$ -theory has been the subject of increasing interest. Describing the  $K$ -theory groups of a  $C^*$ -algebra is a difficult and interesting problem. It is thus crucial to develop a strategy for its computation.

During the eighties, Alain Connes and Paul Baum conjectured that one could describe the  $K$ -theory of the reduced  $C^*$ -algebra of a discrete group with a geometrical object, what is now called the analytical  $K$ -homology of the classifying space for proper action. They gave a relation between the two objects, embodied in the assembly map. Since then, the assembly map has been defined for locally compact group, for actions of groups by automorphisms on  $C^*$ -algebras, for topological groupoids, for coarse spaces and, in some cases, for quantum groups. The Baum-Connes conjecture, and its variants, state that the assembly map is an isomorphism, thus providing an algorithm to compute the  $K$ -theory of the  $C^*$ -algebra under consideration.

More precisely, for any locally compact group  $G$ , there exists a homomorphism of  $\mathbb{Z}_2$ -graded abelian groups  $\mu_G : K^{top}(G) \rightarrow K(C_r^*(G))$  where :

- $K^{top}(G)$  is the topological  $K$ -theory group of  $G$ ,
- $C_r^*(G)$  is the reduced  $C^*$ -algebra of  $G$  and  $K(C_r^*(G))$  is its (operator)  $K$ -theory group.

The Baum-Connes conjecture is the following claim.

**Conjecture 0.0.1** (Baum-Connes). For every locally compact group  $G$ ,  $\mu_G$  is an isomorphism.

There exist more general versions of the conjecture : first, one can take  $G$  to be a locally compact  $\sigma$ -compact groupoid with a Haar system. Let  $A$  be a  $G$ -algebra, which is a  $C^*$ -algebra endowed with an action of  $G$ . It is possible to build a  $C^*$ -algebra  $A \rtimes_r G$  out of the action, called the reduced crossed product, such that the reduced crossed product with  $\mathbb{C}$  (endowed with the trivial  $G$ -action) coincides with  $C_r^*(G)$ . Then, we can consider a coefficient version of the assembly map  $\mu_{G,A} : K^{top}(G, A) \rightarrow K(A \rtimes_r G)$ . The Baum-Connes conjecture for  $G$  with coefficients in  $A$  is the claim that  $\mu_{G,A}$  is an isomorphism. We say that  $G$  satisfies the Baum-Connes conjecture if the Baum-Connes conjecture for  $G$  with coefficients in  $A$  holds, for every  $G$ -algebra  $A$ .

Inspired by these ideas, a coarse assembly map was defined. For every coarse space  $X$  and every  $C^*$ -algebra  $A$ , there exists a homomorphism of  $\mathbb{Z}_2$ -graded abelian groups

$\mu_{X,A} : KX(X, A) \rightarrow K(C^*(X, A))$  where :

- $KX(X, A)$  is the coarse  $K$ -homology group of  $X$  with coefficients in  $A$ ,
- $C^*(X, A)$  is the Roe algebra of  $X$  with coefficients in  $A$ .

**Conjecture 0.0.2** (Coarse Baum-Connes). For every coarse space  $X$  with bounded geometry,  $\mu_{X,\mathbb{C}}$  is an isomorphism.

The coarse Baum-Connes conjecture admits a version with coefficients as in the group setting.

Great success was achieved in the study of the conjecture. It is known to hold :

- with coefficients for groups with the Haagerup property, by work of N. Higson and G. Kasparov [18],
- with coefficients for hyperbolic groups (which may satisfy Kazhdan's property T), by work of V. Lafforgue [26],
- for almost connected groups, by work of J. Chabert, S. Echterhoff and R. Nest [5],
- with coefficients for groupoids which admit a proper action by isometries on a field of Hilbert spaces, by work of J-L. Tu [48],
- for coarse spaces which satisfy property A, by work of G. Yu [60].

Even if these classes of examples is very large, counterexamples have been exhibited by N. Higson, V. Lafforgue, and G. Skandalis in [19].

The main interest of the Baum-Connes conjecture lies in its connection with the Novikov conjecture. Let us remind the reader of the statement of the Novikov conjecture. The reader is referred to [24] for details. Let  $\Gamma$  be a discrete group. Then there exists a unique, up to homotopy, connected CW-complex  $B\Gamma$  such that  $\pi_1(B\Gamma) \cong \Gamma$  and such that its universal covering  $\tilde{B}\Gamma$  is contractible [34]. This space  $B\Gamma$  is called the classifying space. For any smooth oriented closed manifold  $M$ , denote by

$$\mathcal{L}_M \in \bigoplus_{k \geq 0} H^{4k}(M, \mathbb{Q}).$$

the  $L$ -class of  $M$ . Recall (see [24] for precise definitions) that each components  $(\mathcal{L}_M)_k \in H^{4k}(M, \mathbb{Q})$  is a homogeneous polynomial of degree  $k$  in the rational Pontrjagin classes. For any map  $f : M \rightarrow B\Gamma$ , define the higher signature

$$\sigma_x(M, f) = \langle \mathcal{L}_M \cup f^*(x), [M] \rangle \quad \forall x \in H(B\Gamma, \mathbb{Q}).$$

Here  $[M] \in H_{\dim M}(M, \mathbb{Q})$  is the fundamental class of the closed oriented manifold  $M$ , and  $\langle \cdot, \cdot \rangle$  is the pairing between cohomology and homology, given by Kronecker product. Recall that, by the classifying theorem for principal bundles, any Galois  $\Gamma$ -covering  $\tilde{M} \rightarrow M$  is

classified by continuous maps  $M \rightarrow B\Gamma$ . An interesting example is given by the universal cover of  $\tilde{M} \rightarrow M$ .

The Novikov conjecture claims that higher signatures are homotopy invariant.

**Conjecture 0.0.3** (Novikov). Let  $M$  and  $N$  two smooth oriented closed manifolds,  $f : N \rightarrow B\Gamma$  a map and  $\phi : M \rightarrow N$  an orientation preserving homotopy equivalence. Then

$$\sigma_x(N, f) = \sigma_x(M, f \circ \phi) \quad \forall x \in H^*(B\Gamma, \mathbb{Q}).$$

The relation between the Novikov conjecture and the Baum-Connes conjecture is contained in the following result ([24], Corollary 23.15). The study of the relations between the Novikov conjecture and operator algebras was initiated by Lusztig [32].

**Theorem 0.0.4.** Let  $\Gamma$  be a discrete group. Then the Baum-Connes conjecture for  $\Gamma$ , or more generally the rational injectivity of the Baum-Connes assembly map  $\mu_\Gamma$ , implies the Novikov conjecture for  $\Gamma$ .

Any finitely generated group  $\Gamma$  can be endowed with a left invariant metric, and let  $|\Gamma|$  be the corresponding metric space. The coarse Baum-Connes conjecture for  $\Gamma$  implies the rational injectivity of  $\mu_\Gamma$ . The previous theorem implies that one can prove the Novikov conjecture by proving that  $\mu_{|\Gamma|}$  is an isomorphism.

Another connection between the coarse Baum-Connes conjecture and the Baum-Connes conjecture was revealed in the article of G. Skandalis, J.L. Tu and G.Yu [47]. They built, out of every coarse space with bounded geometry  $X$ , an étale groupoid  $G(X)$  such that :

- $l_B^\infty \rtimes_r G(X)$  and  $C^*(X, B)$  are isomorphic as  $C^*$ -algebras,
- $\mu_{X, B}$  and  $\mu_{G(X), l_B^\infty}$  are equivalent,

where  $l_B^\infty$  is the  $G$ -algebra  $l^\infty(X, B \otimes \mathfrak{K})$ . This result gives a general setting for the Dirac-Dual-Dirac argument. Indeed,  $X$  satisfies property A iff  $G(X)$  is a-T-menable. The proof of the coarse Baum-Connes conjecture for property A spaces hence reduces to an instance of the Baum-Connes conjecture for a-T-menable groupoid. When  $X = |\Gamma|$ , one can show that  $G(X) \cong \beta|\Gamma| \rtimes \Gamma$ , hence the coarse assembly map for  $|\Gamma|$  is equivalent to  $\mu_{\Gamma, l^\infty(\Gamma, \mathfrak{K})}$  [47] ( $\beta X$  denotes the Stone-Cech compactification). These relations are surprising in that the coarse space  $|\Gamma|$  forgets the algebraic structure of  $\Gamma$ , replacing it by metric properties of  $|\Gamma|$ . These methods use infinite dimensional analysis, hence are hard to extend to other setting.

The first occurrence of controlled  $K$ -theory appeared in G. Yu's celebrated proof of the Baum-Connes conjecture for coarse spaces with finite asymptotic dimension [59]. It is worth noticing that any coarse space with finite asymptotic dimension satisfies property A, hence this proof is less general than the one in [60]. Nonetheless, the idea is very different. To simplify, we could see this proof as a "controlled cutting and pasting" : the finite asymptotic-dimensional coarse space is cut into smaller pieces, which satisfy the coarse Baum-Connes conjecture in a uniform way. One needs then to paste the different pieces in a way compatible with the assembly maps. The benefit of this method is that

only geometric arguments are used, without infinite dimensional analysis.

In order to apply this strategy to more general situations, controlled  $K$ -theory was developed in [40] by H. Oyono-Oyono and G. Yu in the setting of filtered  $C^*$ -algebras. It is a refinement of operator  $K$ -theory, which keeps track of the filtration by replacing the  $K$ -groups  $K_*(A)$  of a  $C^*$ -algebra  $A$  by a family of groups  $\hat{K}_*(A) = \{K_*^{\varepsilon,R}(A)\}_{\varepsilon \in (0, \frac{1}{4}), R > 0}$ . These controlled groups are naturally related to the usual  $K$ -theory, and are compatible when one increases the propagation. They are defined using almost projections and almost unitaries, and a new parameter appears,  $\varepsilon \in (0, \frac{1}{4})$ , which is the defect for an almost projection (or unitary) to be a genuine projection (or unitary). Roe algebras of metric spaces, crossed products of  $C^*$ -algebras by discrete groups, or by groupoids endowed with a proper length, are examples of filtered  $C^*$ -algebras. The filtration encodes propagation of operators. A good example to understand propagation is the case of the algebra of pseudo differential operators which are pseudo local.

The aim of this thesis is, first, to extend the setting of controlled  $K$ -theory to more general filtrations that naturally arise in geometry. Then, we define controlled assembly maps with values in these controlled  $K$ -theory groups. We develop a formulation general enough to include the study of the Roe algebra of coarse spaces and crossed products of  $C^*$ -algebras by étale groupoids and discrete quantum groups. Also, the filtration comes directly from the geometry and does not depend on the choice of a particular metric in the case of coarse spaces. We treat mainly the case of étale groupoids and discrete coarse spaces with bounded geometry. We define Kasparov transform in the case of discrete quantum groups, but the lack of geometric point of view does not allow such powerful results nor the definition of an assembly map.

We begin by the definition of a coarse structure.

**Definition 0.0.5.** A coarse structure  $\mathcal{E}$  is a lattice which is an abelian semi-group. Recall that a lattice is a poset for which every pair  $(E, E')$  admits a supremum  $E \vee E'$  and an infimum  $E \wedge E'$ .

A coarse structure  $\mathcal{E}$  allows to define the notion of a  $\mathcal{E}$ -filtered  $C^*$ -algebra.

**Definition 0.0.6.** A  $C^*$ -algebra  $A$  is said to be  $\mathcal{E}$ -filtered if there exists a coarse structure  $\mathcal{E}$  and, for every  $E \in \mathcal{E}$ , linear subspaces  $A_E$  of  $A$  such that :

- if  $E \leq E'$ , then  $A_E \subseteq A_{E'}$ , and the inclusion  $\phi_E^{E'} : A_E \hookrightarrow A_{E'}$  induces an inductive system of linear spaces,
- $A_E$  is stable by involution,
- for all  $E, E' \in \mathcal{E}$ ,  $A_E \cdot A_{E'} \subseteq A_{E \vee E'}$ ,
- the union of subspaces is dense in  $A$ , i.e.  $\overline{\cup_{E \in \mathcal{E}} A_E} = \varinjlim A_E = A$ .
- if  $A$  is unital, we impose that  $1 \in A_E, \forall E \in \mathcal{E}$ .

New examples of  $C^*$ -algebras can then be seen as filtered  $C^*$ -algebras in this new setting.

- Let  $(X, d)$  a discrete metric space with bounded geometry. Then, the symmetric non-empty subsets  $E \subseteq X \times X$  such that  $\sup d|_E < \infty$ , are naturally ordered by  $\subseteq$  and we can define a composition law by:

$$E \circ E' = EE' \cup E'E,$$

where  $EE' = \{(x, y) \in X \times X \text{ s.t. } \exists z \in X / (x, z) \in E \text{ and } (z, y) \in E'\}$ . This defines a coarse structure  $\mathcal{E}_X$ .

- Let  $G$  be a  $\sigma$ -compact étale groupoid. Then the non-empty symmetric compact subsets  $E \subseteq G$  are naturally ordered by  $\subseteq$  and we can define a composition law by:

$$E \circ E' = EE' \cup E'E,$$

where  $EE' = \{gg'; (g, g') \in G^{(2)}\}$ . This defines a coarse structure  $\mathcal{E}_G$ .

- Let  $\mathbb{G}$  a compact quantum group. Then the set of finite dimensional equivalence classes of symmetric unitary representations of  $\mathbb{G}$  is ordered by  $: \pi \leq \pi'$  iff  $\pi$  is unitarily equivalent to a subrepresentation of  $\pi'$ . Moreover, the symmetric tensor product defines a composition law

$$\pi \circ \pi' = (\pi \otimes \pi') \oplus (\pi' \otimes \pi) \quad \forall \pi, \pi'.$$

This defines a coarse structure  $\mathcal{E}_{\mathbb{G}}$ .

It turns out that Roe algebras  $C^*(X, B)$ , crossed products by  $G$ , and crossed products by  $\hat{\mathbb{G}}$  are filtered by  $\mathcal{E}_X$ ,  $\mathcal{E}_G$  and  $\mathcal{E}_{\mathbb{G}}$  respectively.

We then construct controlled assembly maps with values in the controlled  $K$ -theory in the case of discrete coarse spaces with bounded geometry and étale groupoids. A crucial property is that these applications factorize the usual assembly maps. This allows to relate the Baum-Connes and the coarse Baum-Connes conjectures to the properties of our controlled assembly maps. The definitions of these assembly maps is performed in two steps. The first is to define a descent functor. We then define the Rips complex of  $X$  and  $G$ , and a canonical projection. The assembly map is the evaluation of the descent functor at this projection.

More precisely, let  $(X, d)$  be a countable discrete metric space with bounded geometry; i.e.  $\forall R > 0, \sup |\{(x, y) \in X \times X \text{ s.t. } d(x, y) < R\}| < \infty$ . We first build the controlled Roe transformation  $\hat{\sigma}_X(z)$  for every  $z \in KK(A, B)$ . Propositions 4.2.5 and 4.2.6, which describe its properties, can be summarized in the following proposition :

**Proposition 0.0.7.** Let  $A$  and  $B$  two  $C^*$ -algebras. For every  $z \in KK_*(A, B)$ , there exists a control pair  $(\alpha_X, k_X)$  and a  $(\alpha_X, k_X)$ -controlled morphism

$$\hat{\sigma}_X(z) : \hat{K}(C^*(X, A)) \rightarrow \hat{K}(C^*(X, B))$$

of the same degree as  $z$ , such that

- (i)  $\hat{\sigma}_X(z)$  induces right multiplication by  $\sigma_X(z)$  in  $K$ -theory ;

(ii)  $\hat{\sigma}_X$  is additive, i.e.

$$\hat{\sigma}_X(z + z') = \hat{\sigma}_X(z) + \hat{\sigma}_X(z').$$

(iii) For every  $*$ -homomorphism  $f : A_1 \rightarrow A_2$ ,

$$\hat{\sigma}_X(f^*(z)) = \hat{\sigma}_X(z) \circ f_{X,*}$$

for all  $z \in KK_*(A_2, B)$ .

(iv) For every  $*$ -homomorphism  $g : B_1 \rightarrow B_2$ ,

$$\hat{\sigma}_X(g_*(z)) = g_{X,*} \circ \hat{\sigma}_X(z)$$

for all  $z \in KK_*(A, B_1)$ .

(v) Let  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  be a semi-split extension of  $C^*$ -algebras and  $[\partial_J] \in KK_1(A/J, J)$  be its boundary element. Then

$$\hat{\sigma}_X([\partial_{J,A}]) = D_{C^*(X,J), C^*(X,A)}.$$

(vi)  $\hat{\sigma}_X([id_A]) \sim_{(\alpha_X, k_X)} id_{\hat{K}(C^*(X,A))}$

The controlled Roe transformation is a descent functor, and the natural following step is the definition of the controlled assembly map  $\hat{\mu}_{X,B}$ , for any  $C^*$ -algebra  $B$ . For every entourage  $E$ , we define a canonical projection  $P_E \in C^*(X, C_0(P_E(X)))$  with finite propagation. This defines a controlled  $K$ -theory class.

**Definition 0.0.8.** Let  $B$  a  $C^*$ -algebra,  $\varepsilon \in (0, \frac{1}{4})$  and  $E, F \in \mathcal{E}_X$  controlled subsets such that  $k_X(\varepsilon).E \subseteq F$ . The controlled coarse assembly map  $\hat{\mu}_{X,B} = (\mu_{X,B}^{\varepsilon,E,F})_{\varepsilon,E}$  is defined as the family of maps

$$\hat{\mu}_{X,B}^{\varepsilon,E,F} : \begin{cases} KK(C_0(P_E(X)), B) & \rightarrow & K^{\varepsilon,F}(C^*(X, B)) \\ z & \mapsto & \iota_{\alpha_X \varepsilon', k_X(\varepsilon').F'}^{\varepsilon,F} \circ \hat{\sigma}_X(z)[P_E, 0]_{\varepsilon', F'} \end{cases}$$

where  $\varepsilon'$  and  $F'$  satisfy :

- $\varepsilon' \in (0, \frac{1}{4})$  such that  $\alpha_X \varepsilon' \leq \varepsilon$ ,
- and  $F' \in \mathcal{E}$  such that  $E \subseteq F'$  and  $k_X(\varepsilon').F' \subseteq F$ .

We then define corresponding assembly maps in the setting of étale groupoid. More precisely, let  $G$  be a  $\sigma$ -compact étale groupoid. We prove that compact symmetric subsets  $E \subseteq G$  define a coarse structure  $\mathcal{E}$ , and that reduced crossed products by  $G$  are filtered by  $\mathcal{E}$ . The first step is again to build a controlled descent functor in controlled  $K$ -theory. Propositions 4.3.3 and 4.3.4 can be summarized in the following proposition :

**Proposition 0.0.9.** Let  $A$  and  $B$  two  $G$ - $C^*$ -algebras. For every  $z \in KK_*^G(A, B)$ , there exists a control pair  $(\alpha_J, k_J)$  and a  $(\alpha_J, k_J)$ -controlled morphism

$$J_{red,G}(z) : \hat{K}(A \rtimes_r G) \rightarrow \hat{K}(B \rtimes_r G)$$

of the same degree as  $z$ , such that

- (i)  $J_{red,G}(z)$  induces right multiplication by  $j_{red,G}(z)$  in  $K$ -theory ;

(ii)  $J_{red,G}$  is additive, i.e.

$$J_{red,G}(z + z') = J_{red,G}(z) + J_{red,G}(z').$$

(iii) For every  $G$ -morphism  $f : A_1 \rightarrow A_2$ ,

$$J_{red,G}(f^*(z)) = J_{red,G}(z) \circ f_{G,red,*}$$

for all  $z \in KK_*^G(A_2, B)$ .

(iv) For every  $G$ -morphism  $g : B_1 \rightarrow B_2$ ,

$$J_{red,G}(g_*(z)) = g_{G,red,*} \circ J_{red,G}(z)$$

for all  $z \in KK_*^G(A, B_1)$ .

(v) Let  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  be a semi-split equivariant extension of  $G$ -algebras and  $[\partial_J] \in KK_1^G(A/J, J)$  be its boundary element. Then

$$J_G([\partial_J]) = D_{J \rtimes_r G, A \rtimes_r G}.$$

(vi)  $J_{red,G}([id_A]) \sim_{(\alpha_J, k_J)} id_{\hat{K}(A \rtimes G)}$

This transformation, that we call controlled Kasparov transformation, allows to define the controlled assembly map  $\hat{\mu}_{G,B}$  for any  $G$ -algebra  $B$ . For any controlled subset  $E \subseteq G$ , we also have a canonical projection  $\mathcal{L}_E \in C_0(P_E(G)) \rtimes_r G$  with finite propagation. It defines a controlled  $K$ -theory class.

**Definition 0.0.10.** Let  $B$  be a  $G$ -algebra,  $\varepsilon \in (0, \frac{1}{4})$ , and  $E \in \mathcal{E}$ . Let  $F \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F$ . The controlled assembly map for  $G$  is defined as the family of maps :

$$\mu_{G,B}^{\varepsilon,E,F} \begin{cases} RK^G(P_E(G), B) & \rightarrow K_*^{\varepsilon,F}(B \rtimes_r G) \\ z & \mapsto \iota_{\alpha_J \varepsilon', k_J(\varepsilon').F'}^{\varepsilon,F} \circ J_G^{\varepsilon',F'}(z)([\mathcal{L}_E, 0]_{\varepsilon',F'}) \end{cases}$$

where  $\varepsilon'$  and  $F'$  satisfy :

- $\varepsilon' \in (0, \frac{1}{4})$  such that  $\alpha_J \varepsilon' \leq \varepsilon$ ,
- and  $F' \in \mathcal{E}$  such that  $E \subseteq F'$  and  $k_J(\varepsilon').F' \subseteq F$ .

The assembly map also admits a maximal version with values in  $\hat{K}(A \rtimes_{max} G)$ .

The next step consists in the formulation of a controlled Baum-Connes conjecture. We then show how the controlled assembly maps relates to the usual assembly map in the Theorems 4.3.18 et 4.3.19, which we call quantitative statements. In order to state them, let us introduce the following properties :

- $QI_{G,B}(E, E', F, \varepsilon)$  : for any  $x \in RK^G(P_E(G), B)$ , then  $\mu_{G,B}^{\varepsilon,E,F}(x) = 0$  implies  $q_E^{E'}(x) = 0$  in  $RK^G(P_{E'}(G), B)$ .
- $QS_{G,B}(E, F, F', \varepsilon, \varepsilon')$  : for any  $y \in K^{\varepsilon,F}(B \rtimes_r G)$ , there exists  $x \in RK^G(P_E(G), B)$  such that  $\mu_{G,B}^{\varepsilon',E,F'}(x) = \iota_{\varepsilon,F}^{\varepsilon',F'}(y)$ .

The quantitative statements are the main result of the thesis, and can be stated as follows. Let  $G$  be a  $\sigma$ -compact étale groupoid with compact base space  $G^{(0)}$ .

**Theorem 0.0.11.** Let  $B$  a  $G$ -algebra, and  $\tilde{B} = l^\infty(\mathbb{N}, B \otimes \mathfrak{K})$ . Then  $\mu_{G, \tilde{B}}$  is injective if and only if for every  $E \in \mathcal{E}$ ,  $\varepsilon \in (0, \frac{1}{4})$  and  $F$  such that  $k_J(\varepsilon).E \subseteq F$ , there exists  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and  $QI_{G,B}(E, E', \varepsilon, F)$  holds.

**Theorem 0.0.12.** Let  $B$  a  $G$ -algebra, and  $\tilde{B} = l^\infty(\mathbb{N}, B \otimes \mathfrak{K})$ . Then there exists  $\lambda > 1$  such that  $\mu_{G, \tilde{B}}$  is onto if and only if for any  $\varepsilon \in (0, \frac{1}{4\lambda})$  and every nonempty  $F \in \mathcal{E}$ , there exist  $E, F' \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F$ ,  $F \subseteq F'$  and such that  $QS_{G,B}(E, F, F', \varepsilon, \lambda\varepsilon)$  holds.

We also give a uniform version :

**Theorem 0.0.13.** Let  $G$  be an étale groupoid with compact base space.

- Assume that for any  $G$ -algebra  $B$ ,  $\mu_{G,B}$  is one-to-one. Then, for any  $\varepsilon \in (0, \frac{1}{4})$  and every  $E, F \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F$ , there exists  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and such that  $QI_{G,A}(E, E', \varepsilon, F)$  holds for any  $G$ -algebra  $B$ .
- Assume that for any  $G$ -algebra  $B$ ,  $\mu_{G,B}$  is onto. Then, for some  $\lambda \geq 1$  and for any  $\varepsilon \in (0, \frac{1}{4\lambda})$  and every  $F \in \mathcal{E}$ , there exists  $E, F' \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F'$  and  $F \subseteq F'$  such that, for any  $G$ -algebra  $B$ ,  $QS_{G,A}(E, F, F', \varepsilon, \lambda\varepsilon)$  holds.

The remainder of the chapter presents two applications. One deals with what is called Persistent Approximation Property, and the second one gives a controlled version of a result on  $K$ -amenability of discrete quantum groups.

- Let us first recall the definition of the Persistent Approximation Property (PAP), given in [39].

**Definition 0.0.14.** Let  $B$  be a  $\mathcal{E}$ -filtered  $C^*$ -algebra,  $\lambda \geq 1$ ,  $\varepsilon, \varepsilon'$  be positive numbers such that  $0 < \varepsilon < \varepsilon' < \frac{1}{4}$  and  $F, F' \in \mathcal{E}$  be controlled subsets such that  $F \subseteq F'$ . The following property is called Persistence Approximation Property :

- $PA_B(\varepsilon, \varepsilon', F, F')$  : for every  $x \in K_*^{\varepsilon, F}(B)$  such that  $\iota_{\varepsilon, F}(x) = 0$  in  $K_*(B)$ , then  $\iota_{\varepsilon, F}^{\varepsilon', F'}(x) = 0$  in  $K_*^{\varepsilon', F'}(B)$ .
- $B$  is said to satisfy the Persistence Approximation Property  $(PAP)_\lambda$  if for every  $F \in \mathcal{E}$  and every  $\varepsilon \in (0, \frac{1}{4})$ , there exists  $F' \in \mathcal{E}$  nonempty such that  $PA_B(\varepsilon, \lambda\varepsilon, F, F')$  holds.

The property (PAP) is satisfied when controlled  $K$ -theory approximates uniformly  $K$ -theory. Theorem 4.3.22 gives a relation between the assembly map and the Persistent Approximation Property.



**Theorem 0.0.15.** Let  $G$  be an étale groupoid with compact base space. Assume that  $G$  admits a cocompact universal space for proper actions. Then there exists a universal constant  $\lambda_{PA} \geq 1$  such that, for every  $G$ -algebra  $A$ , if  $\mu_{G, l^\infty(\mathbb{N}, A \otimes \mathfrak{K})}$  is onto and  $\mu_{G, A}$  is one-to-one, then for every  $\varepsilon \in (0, \frac{1}{4\lambda_{PA}})$  and every nonempty  $F \in \mathcal{E}$ , there exists  $F' \in \mathcal{E}$  such that  $F \subseteq F'$  and  $PA_{A \rtimes_r G}(\varepsilon, \lambda_{PA}\varepsilon, F, F')$  holds.

- As a final remark, we explain how the controlled Kasparov transformation  $J_G$  can be defined for reduced and maximal crossed products of discrete quantum groups by using the same method as in the groupoid setting. It is indeed sufficient to notice that crucial properties of the crossed product are still valid for quantum groups, and the construction of  $J_G$  can be carried out without modification in the quantum case.

We give an application to  $K$ -amenability of discrete quantum groups in the proposition 4.4.4. Let us denote by  $\lambda_A$  the canonical map  $A \rtimes_{max} \hat{G} \rightarrow A \rtimes_r \hat{G}$  for any  $\hat{G}$ -algebra  $A$ .

**Proposition 0.0.16.** Let  $\hat{G}$  be a  $K$ -amenable discrete quantum group. Then, there exists a control pair  $\rho$  such that, for every  $\hat{G}$ -algebra  $A$ ,

$$(\lambda_A)_* : \hat{K}(A \rtimes_{max} G) \rightarrow \hat{K}(A \rtimes_r G)$$

is a  $\rho$ -controlled isomorphism.

The last part focuses on two applications.

- The first application deals with coarse geometry. We follow the route of [47] by using the coarse groupoid  $G(X)$  of a coarse space  $X$ , which is étale. We prove that the controlled coarse assembly map for  $X$  with values in  $B$  is equivalent to the controlled assembly map for  $G(X)$  with values in  $l^\infty(X, B \otimes \mathfrak{K})$ . The theorem 5.1.4 is the following.

**Theorem 0.0.17.** Let  $B$  be a  $C^*$ -algebra,  $E$  be an entourage and  $\bar{E}$  the corresponding compact open subset of  $G(X)$ . With the above notations, for all  $z \in RK^G(P_{\bar{E}}(G), \tilde{B})$  and all  $\varepsilon \in (0, \frac{1}{4})$ , the following equality holds :

$$(\Psi_B)_* \circ \mu_{G, \tilde{B}}^{\varepsilon, \bar{E}}(z) = \mu_{X, B}^{\varepsilon, E}(l^*(z)).$$

This theorem induces in  $K$ -theory a result of G. Skandalis, J-L. Tu and G. Yu [47], which asserts the equivalence between the coarse Baum-Connes conjecture for  $X$  with coefficients in  $B$  and the Baum-Connes conjecture for  $G(X)$  with coefficients in  $l^\infty(X, B \otimes \mathfrak{K})$ . This allows in particular to prove a controlled version of a result of M. Finn-Sell in [15]. In order to state the result, we define the controlled coarse Baum-Connes conjecture as follows.

**Definition 0.0.18.** Let  $X$  be a discrete metric space with bounded geometry and  $B$  a  $C^*$ -algebra. We introduce the following properties.

- $QI_{X,B}(E, E', F, \varepsilon)$  : for any  $x \in KK(C_0(P_E(X)), B)$ , then  $\mu_{X,B}^{\varepsilon,E,F}(x) = 0$  implies  $q_E^{E'}(x) = 0$  in  $KK^G(C_0(P_{E'}(X)), B)$ .
- $QS_{X,B}(E, F, F', \varepsilon, \varepsilon')$  : for any  $y \in K^{\varepsilon,F}(C^*(X, B))$ , there exists  $x \in KK(C_0(P_E(G)), B)$  such that  $\mu_{X,B}^{\varepsilon',E,F'}(x) = \iota_{\varepsilon,F'}^{\varepsilon,F}(y)$ .

Let  $\lambda \geq 1$  be a positive number. We say that  $X$  satisfies the controlled Baum-Connes conjecture with coefficients in  $B$  with rescaling  $\lambda$  if :

- for every  $\varepsilon \in (0, \frac{1}{4\lambda})$ , every  $E, F \in \mathcal{E}$  such that  $k_X(\varepsilon).E \subseteq F$ , there exists  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and  $QI_{X,B}(E, E', F, \varepsilon)$  holds;
- for every  $\varepsilon \in (0, \frac{1}{4\lambda})$ , every  $F \in \mathcal{E}$ , there exists  $E, F' \in \mathcal{E}$  such that  $k_X(\varepsilon).E \subseteq F'$  and  $F \subseteq F'$  and  $QS_{X,B}(E, F, F', \varepsilon, \lambda\varepsilon)$  holds.

If  $\hat{\mu}_X$  is replaced by  $\hat{\mu}_X^{max}$ , we will say that  $X$  satisfies the maximal controlled Baum-Connes conjecture with rescaling  $\lambda$ .

**Corollary 0.0.19.** Let  $X$  be a coarse space that admits a fibred coarse embedding into Hilbert space. Then  $X$  satisfies the controlled Coarse Baum-Connes conjecture.

- The last application deals with a Künneth formula in controlled  $K$ -theory for groupoid  $C^*$ -algebras, in theorem 5.2.13. The case of discrete groups was covered in [38] by H. Oyono-Oyono and G. Yu. To prove the Künneth formula, we show that a certain morphism

$$\alpha_{A \rtimes_r G, B} : K_*(A \rtimes_r G) \otimes K_*(B) \rightarrow K_*((A \otimes B) \rtimes_r G)$$

is an isomorphism for any  $C^*$ -algebra  $B$  such that  $K(B)$  is a free abelian group. The strategy is a generalization of the "Going Down Principle" developed by J. Chabert, S. Echterhoff and H. Oyono-Oyono in [6] to the setting of étale groupoids. We can summarize it as follows : first define a topological version

$$\alpha_{A,B}^{G,Z} : RK^G(Z, A) \otimes K_*(B) \rightarrow RK^G(Z, A \otimes B)$$

of  $\alpha_{A \rtimes_r G, B}$ , then show that the assembly maps intertwines them. This requires the definition of a restriction functor  $Res_H^G$  in equivariant  $KK$ -theory, for any compact subgroupoid  $H$  of  $G$ , and the definition of a property on groupoid proper actions, called strong properness (which is to be locally induced by a compact open subgroupoid). We then define the class  $\mathcal{C}$  as the class of groupoids of which every proper actions are strongly proper. We prove that ample groupoids are in class  $\mathcal{C}$ , which gives a large class of examples. The following result describes the relation between class  $\mathcal{C}$  and the Künneth formula.

**Theorem 0.0.20.** Let  $G$  be an étale groupoid in the class  $\mathcal{C}$ , and let  $E \in \mathcal{E}$  be a controlled subset of  $G$  and  $P_E(G)$  be the corresponding Rips complex. If, for all compact open subgroupoids  $H$  of  $G$  and every  $H$ -space  $V$  such that the anchor map  $p : V \rightarrow H^{(0)}$  is locally injective,  $\alpha_{Res_H^G(A), B}^{H, V}$  is an isomorphism, then  $\alpha_{A, B}^{G, P_E(G)}$  is an isomorphism for all  $C^*$ -algebras  $B$  such that  $K_*(B)$  is a free abelian group.

The main theorem can then be stated as follows.

**Theorem 0.0.21.** Let  $G$  be a  $\sigma$ -compact étale groupoid and  $A$  a  $G$ -algebra. Suppose that

- $G$  satisfies the Baum-Connes conjecture with coefficients,
- $G$  is in class  $\mathcal{C}$ ,
- for every compact open subgroupoid  $H$  of  $G$  and every  $H$ -space  $V$  such that the anchor map  $p : V \rightarrow H^{(0)}$  is locally injective,  $\alpha_{Res_H^G(A), B}^{H, V}$  is an isomorphism.

Then  $A \rtimes_r G$  satisfies the quantitative Künneth formula.

Let us detail the plan of the thesis.

In the first part we recall the definitions and results we will need. After a quick introduction to  $C^*$ -algebras and to Hilbert modules, the reader is guided through étale groupoids, coarse spaces and their relations. We give details on an important construction of [47], the (étale) coarse groupoid  $G(X)$  associated to a coarse space  $X$ . We also give a detailed proof of the  $*$ -isomorphism between the Roe algebra of  $X$  with coefficients in a  $C^*$ -algebra  $B$  and the reduced crossed product of  $l^\infty(X, B \otimes \mathfrak{K})$  with the natural action of  $G(X)$ . We end this section with a survey of the construction of crossed products in the setting of discrete quantum groups.

The second part is devoted to the study of controlled  $K$ -theory.

The chapter 3 begins with a definition of a filtered  $C^*$ -algebra with respect to what we call a coarse structure. An exposition of the controlled  $K$ -theory in this setting follows. All the results obtained by H. Oyono-Oyono and G. Yu in [40] and [39] extend to this setting. The interesting part is the definition of a coarse structure and the examples which cannot fit into the former formalism adopted in [40]. These examples are given by the coarse structure  $\mathcal{E}_X$  generated by symmetric entourages of a coarse space  $X$ , the coarse structure  $\mathcal{E}_G$  generated by symmetric compact subsets of an étale groupoid  $G$ , and the coarse structure  $\mathcal{E}_{\mathbb{G}}$  generated by symmetric unitary finite dimensional representations of a discrete quantum group  $\mathbb{G}$ . We show that these structures define filtrations on the Roe algebras of  $X$ , and crossed products of  $C^*$ -algebras endowed with an action of  $G$  and  $\hat{\mathbb{G}}$  respectively.

The chapter 4 is divided into four sections. The first contains definitions of the usual assembly maps for étale groupoids and coarse spaces. It is followed by two very similar

parts, so as to underline the similarity between the two situations, in which the controlled assembly maps are built. At the end of the third part, we state what we called quantitative statements, relating the behaviour of usual assembly maps to controlled assembly maps. Doing so requires the computation of the  $K$ -homology groups of the Rips complex of  $G$  with respect to infinite products of stable  $C^*$ -algebras, i.e. the content of the lemma 4.3.14. We end the chapter by stating that the descent transformation can be defined in a similar way for quantum groups and give an application to  $K$ -amenability for discrete quantum groups.

The chapter 5 gives two applications of the controlled assembly maps. The first application deals with a controlled version of the main result of [47] : the coarse Baum-Connes conjecture for  $X$  with coefficients in  $B$  and the Baum-Connes conjecture for  $G(X)$  with coefficients in  $l^\infty(X, B \otimes \mathfrak{K})$  are equivalent. We prove in theorem 5.1.4 that the two corresponding controlled assembly maps are equivalent. This result induces the classical one. We apply the equivalence between the two controlled assembly maps to obtain a sufficient condition for the maximal Baum-Connes conjecture to hold. Namely, if  $X$  admits a fibred coarse embedding into Hilbert space, then it satisfies the maximal version of the controlled coarse Baum-Connes conjecture. The last application deals with a controlled Künneth formula for étale groupoids and its connection with the Baum-Connes conjecture.

# Part I

## Introduction



# Chapter 1

## Basics

### 1.1 $C^*$ -algebras

We recall in this section basic results on  $C^*$ -algebras. Most of the proofs are omitted, and the reader can find them in many textbooks (see [36] or [57] for instance).

An involutive algebra is a  $\mathbb{C}$ -algebra  $A$  endowed with an involution  $*$  :  $A \rightarrow A$  such that

$$(x + y)^* = x^* + y^* \quad (x^*)^* = x \quad (xy)^* = y^* x^* \quad (\lambda x)^* = \bar{\lambda} x^*,$$

for all  $x, y \in A$  and  $\lambda \in \mathbb{C}$ .

**Definition 1.1.1.** A  $C^*$ -algebra  $A$  is an involutive algebra which has a Banach norm with respect to which it is complete and which satisfies the following equality :

$$\|x^*x\| = \|x\|^2 \quad \forall x \in A.$$

Actually, the norm on a  $C^*$ -algebra is unique, and is entirely determined by the relation

$$\|x\|^2 = \sup\{t \in \mathbb{R}_+ : x^*x - t^2 \notin A^\times\}$$

where  $A^\times$  denotes the set of invertible elements of  $A$ .

**Example 1.1.2.** For any complex Hilbert space  $H$ , the set of bounded linear operators  $\mathcal{L}(H)$  with involution given by the adjoint, and norm given by the operator norm, is a  $C^*$ -algebra. When  $H$  is of finite dimension  $d$ , we get  $\mathfrak{M}_d(\mathbb{C})$  as a particular case.

All finite dimensional (as  $\mathbb{C}$ -vector spaces)  $C^*$ -algebras are of the form

$$A \cong \bigoplus_{j=1}^N \mathfrak{M}_{d_j}(\mathbb{C}),$$

for some integers  $N$  and  $d_1, \dots, d_N$ .

**Example 1.1.3.** If  $X$  is a locally compact Hausdorff topological space, the set of continuous functions vanishing at infinity  $C_0(X)$  from  $X$  to  $\mathbb{C}$  is a  $C^*$ -algebra, with  $\|f\| = \sup\{|f(x)| : x \in X\}$  and  $f^*(x) = \bar{f}(x)$ .

The Gelfand-Naimark theorem asserts that all commutative  $C^*$ -algebras are this form. More specifically, the Gelfand-Naimark correspondence associates to any commutative  $C^*$ -algebra  $A$  the set  $X = \text{Spec}(A)$  of its maximal ideals. It is a locally compact Hausdorff space, which can be identified with the set of algebra homomorphisms  $\phi : A \rightarrow \mathbb{C}$  with the weak- $*$  topology (seen as a subspace of the dual  $A^*$ ). The Gelfand transform  $A \rightarrow C_0(X); a \mapsto [\phi \mapsto \phi(a)]$  is a  $*$ -isomorphism.

**Definition 1.1.4.** Let  $A$  and  $B$  be two  $C^*$ -algebras and  $\phi : A \rightarrow B$  a linear map.

- $\phi$  is said to be a  $*$ -homomorphism if  $\phi(a^*) = \phi(a)^*$  and  $\phi(ab) = \phi(a)\phi(b)$ , for every  $a, b \in A$ ,
- $\phi$  is said to be a positive map if it sends positive elements of  $A$  to positive elements of  $B$ ,
- $\phi$  is said to be a completely positive map if, for every  $n \in \mathbb{N}^*$ , the map  $\phi_n : \mathfrak{M}_n(A) \rightarrow \mathfrak{M}_n(A)$  defined by  $\phi_n([a_{ij}]_{ij}) = [\phi(a_{ij})]_{ij}$  is positive.

We recall some useful constructions. Let  $A$  and  $B$  be  $C^*$ -algebras. Then :

- $SA$  denotes its suspension defined by  $A(0, 1) = A \otimes C_0(0, 1) = C_0((0, 1), A)$ ,
- $CA$  denotes its cone defined by  $A[0, 1) = A \otimes C_0[0, 1) = C_0([0, 1), A)$ .
- More generally, let  $\phi : A \rightarrow B$  be a  $*$ -homomorphism. The cone of  $\phi$ , denoted  $C_\phi$ , is defined as

$$C_\phi = \{(f, a) \in CB \oplus A \text{ s.t. } f(0) = \phi(a)\}.$$

## 1.2 Hilbert modules

### 1.2.1 Definitions

The  $KK$ -theory of Kasparov can be defined using the theory of Hilbert modules. We present in this section the basic facts about Hilbert modules that we will need.

**Definition 1.2.1.** Let  $A$  be a  $C^*$ -algebra. A pre-Hilbert  $A$ -module is a  $\mathbb{C}$ -linear space which is a right  $A$ -module  $E$  endowed with a sesquilinear function  $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$  satisfying :

- $(\lambda x)a = \lambda(xa) = x(\lambda a)$  for all  $\lambda \in \mathbb{C}$ ,  $x \in E$  and  $a \in A$ ,
- $\langle x, ya \rangle = \langle x, y \rangle a$  for all  $x, y \in E$  and  $a \in A$ ,
- $\langle x, y \rangle^* = \langle y, x \rangle$  for all  $x, y \in E$ ,
- $\langle x, x \rangle \geq 0$  and if  $\langle x, x \rangle = 0$ , then  $x = 0$ , for all  $x \in E$ .

For  $x \in E$ , we define  $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ . It is a norm on  $E$  [30].

A  $A$ -Hilbert module  $(E, \langle \cdot, \cdot \rangle)$  is a pre-Hilbert module which is complete for the norm  $\|\cdot\|$  defined above.  $E$  is called a full Hilbert module if the linear span of  $\{\langle x, y \rangle : x, y \in E\}$  is dense in  $A$ .



The same definition exists for left-Hilbert modules. Usually, we will omit to specify right or left, as most of our modules will be right modules. Through misuse of language, we will call the sesquilinear function  $\langle \cdot, \cdot \rangle$  an inner product. If the inner product satisfies all the conditions for  $E$  to be a pre-Hilbert module but the last (if  $\langle x, x \rangle = 0$ , then  $x = 0$ , for all  $x \in E$ ), we call  $E$  a semi-pre-Hilbert module.

**Example 1.2.2.** If  $J$  is any closed right ideal in  $A$ ,  $J$  is naturally a  $A$ -Hilbert module with respect to the inner product  $\langle a, b \rangle = a^*b$ . In particular,  $A$  is a  $A$ -Hilbert module.

**Example 1.2.3.** If  $\{E_j\}_{j \in J}$  is a finite or countable family of  $A$ -Hilbert modules, define the direct sum as

$$\bigoplus_{j \in J} E_j = \{(x_j) : x_j \in E_j \text{ s.t. } \sum_j \langle x_j, x_j \rangle_{E_j} \text{ converges in } A\}$$

with the inner product  $\langle x, y \rangle = \sum_{j \in J} \langle x_j, y_j \rangle_{E_j}$ . It is a  $A$ -Hilbert module. As a particular case, we get a central example  $H_A$  called the **standard  $A$ -Hilbert module**. It is obtained as the direct sum of the family  $\{A\}_{j \in \mathbb{N}}$  where each  $A$  is seen as a  $A$ -Hilbert module.

Let  $E$  be a semi-pre-Hilbert  $A$ -module. If the sesquilinear function  $\langle \cdot, \cdot \rangle$  has isotropy, i.e. there exists a non-zero  $x \in E$  such that  $\langle x, x \rangle = 0$ , we can divide out the null vectors. Indeed,  $N = \{x \in E : \langle x, x \rangle = 0\}$  is a sub- $A$ -module of  $A$ , and  $\langle x + N, y + N \rangle := \langle x, y \rangle$  defines an inner product on  $E/N$ . The separation-completion of  $E$  will refer to the Hilbert  $A$ -module obtained as the completion of  $E/N$  with respect to this inner product.

**Definition 1.2.4.** Let  $E$  and  $F$  be two  $A$ -Hilbert modules. The set of adjointable operators  $\mathcal{L}_A(E, F)$  is defined as the set of maps  $T : E \rightarrow F$  such that there exists a map  $T^* : F \rightarrow E$  satisfying

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \forall x \in E, \forall y \in F.$$

The operator  $T^*$  is called the adjoint of  $T$ .

**Remark 1.2.5.** Notice that any adjointable operator is automatically a module map, and a linear map. Indeed,  $\langle T(xa), y \rangle = \langle xa, T^*y \rangle = a^* \langle x, T^*y \rangle = a^* \langle Tx, y \rangle = \langle a(Tx), y \rangle$ . It is also automatically bounded

It turns out that  $\mathcal{L}_A(E, F)$  with the operator norm is a  $C^*$ -algebra [30]. For  $x \in E, y \in F$ , define  $\theta_{x,y} \in \mathcal{L}_A(F, E)$  as  $\theta_{x,y}(z) = x \langle y, z \rangle$ . It is indeed adjointable as  $\theta_{x,y}^* = \theta_{y,x}$ .

**Definition 1.2.6.** Let  $E$  and  $F$  be two  $A$ -Hilbert modules. The set of compact operators  $\mathfrak{K}_A(F, E)$  is defined as the closure for the operator norm of the linear span of  $\{\theta_{x,y} : x \in E, y \in F\}$ . It is a closed ideal in  $\mathcal{L}_A(F, E)$ . [30]

**Example 1.2.7.** (a) If  $E = A$ ,  $\mathcal{L}_A(A) \cong \mathcal{M}(A)$  and  $\mathfrak{K}_A(A) \cong A$ .

(b) If  $E = H_A$ ,  $\mathfrak{K}_A(H_A) \cong A \otimes \mathfrak{K}$ .

(c) For any Hilbert  $A$ -module,  $\mathcal{L}_A(E) \cong \mathcal{M}(\mathfrak{K}_A(E))$ .

The following result states that any countably generated Hilbert module can be embedded into the standard Hilbert module.

**Theorem 1.2.8** (Stabilization). Let  $E$  be a countably generated Hilbert  $A$ -module, then there is an isomorphism of Hilbert  $A$ -module  $E \oplus H_A \cong H_A$ .

As a corollary, we get that for any countably generated Hilbert  $A$ -module, there exists a projection  $p \in \mathcal{L}_A(H_A)$  such that  $E \cong pH_A$ .

## 1.2.2 Tensor product of Hilbert modules

Let  $E$  a Hilbert  $A$ -module,  $F$  a Hilbert  $B$ -module and  $\phi : A \rightarrow \mathcal{L}_B(F)$  a  $*$ -homomorphism. The following section recalls the construction of the tensor product of  $E$  and  $F$  over  $\phi$ , denoted  $E \otimes_\phi F$ .

We can provide  $F$  with the structure of a left  $A$ -module via  $\phi : a \cdot \eta := \phi(a)\eta$  for  $a \in A, \eta \in F$ . Consider the algebraic tensor product of  $A$ -modules  $E \odot F$ , with the structure of right  $B$  module given by  $x \otimes y \cdot b = x \otimes (yb)$ . Define on  $E \odot F$  the following sesquilinear function :  $\langle x \otimes y, x' \otimes y' \rangle := \langle y, \phi(\langle x, x' \rangle)y' \rangle_F$ . It satisfies the condition for  $(E \odot F, \langle \cdot, \cdot \rangle)$  to be a semi-pre-Hilbert  $B$ -module.

**Definition 1.2.9.** The Hilbert  $B$ -module  $E \otimes_\phi F$  is defined as the completion-separation of the semi-pre-Hilbert  $B$ -module  $E \odot F$  with respect to  $\langle \cdot, \cdot \rangle$ . It is called the interior tensor product of  $E$  and  $F$ . When  $F = B$ , it can be called the push forward of  $E$  along  $\phi$ .

**Example 1.2.10.** The standard Hilbert  $A$ -module satisfies  $H_A \cong H \otimes_\phi A$ , which can be used to prove that  $\mathfrak{K}_A(H_A) \cong A \otimes \mathfrak{K}$ . Here  $\phi : \mathbb{C} \rightarrow \mathcal{M}(A)$  maps  $z \in \mathbb{C}$  to  $z1_{\mathcal{M}(A)}$ .

This tensor product operation enjoys functorial properties that will be useful for us.

**Proposition 1.2.11.** [30] Let  $T \in \mathcal{L}_A(E)$ . The map  $x \otimes y \mapsto (Tx) \otimes y$  extends to a adjointable operator  $\phi_*(T) \in \mathcal{L}_B(E \otimes_\phi F)$ , and this defines a unital  $*$ -homomorphism

$$\phi_* \begin{cases} \mathcal{L}_A(E) & \rightarrow \mathcal{L}_B(E \otimes_\phi F) \\ T & \mapsto \phi_*(T) \end{cases}$$

which is injective if  $\phi$  is. We will write  $\phi_*(T) := T \otimes_\phi 1$  or even  $T \otimes 1$ .

**Remark 1.2.12.** The condition  $\phi_*(\mathfrak{K}_A(E)) \subseteq \mathfrak{K}_B(E \otimes_\phi F)$  holds when  $\phi(A) \subseteq \mathfrak{K}_B(F)$  [30], and this condition will be satisfied when we will consider the push-forward with respect to  $\phi : A \rightarrow B$  as  $\mathfrak{K}_B(B) \cong B$ .

**Remark 1.2.13.** When we look at a  $*$ -homomorphism  $\phi : A \rightarrow B$ , we always have an isometry  $V \in \mathcal{L}_B(H_A \otimes_\phi B, H_B)$  defined on simple tensors as  $(x_j)_j \otimes b \mapsto (\phi(x_j)b)_j$ . Indeed, this map extends linearly to  $H_A \odot B$ , and if  $x = (x_j)$  and  $x' = (x'_j)$  are in  $H_A$  :

$$\langle V(x \otimes b), V(x' \otimes b') \rangle = b^* \sum_j \phi(x_j)^* \phi(x'_j) b' = b^* \phi(\langle x, x' \rangle) b' = \langle x \otimes b, x' \otimes b' \rangle.$$

This isometry can be used to explicitly describe the projection and the isomorphism appearing in the stabilization theorem in this particular example. Indeed  $p = VV^* \in \mathcal{L}_B(H_B)$  is a projection such that  $pH_B \cong H_A \otimes_\phi B$ .

Moreover, for  $T \in \mathcal{L}_B(H_A \otimes_\phi B)$ ,  $Ad_V(T) = VTV^*$  defines a  $*$ -homomorphism  $Ad_V : \mathcal{L}_B(H_A \otimes_\phi B) \rightarrow \mathcal{L}_B(H_B)$  such that  $Ad_V(\mathfrak{K}_B(H_A \otimes_\phi B)) \subseteq \mathfrak{K}_B(H_B)$ . Indeed, notice that

$$V\theta_{\xi, \xi'}V^* = \theta_{V\xi, V\xi'}$$

for every  $\xi, \xi' \in H_A \otimes_\phi B$ .

Composing with  $\phi_*$ , we get a  $*$ -homomorphism  $\mathcal{L}_A(H_A) \rightarrow \mathcal{L}_B(H_B); T \mapsto V(T \otimes_\phi 1)V^*$  respecting compact operators in a natural way : if  $\theta = (a_{ij}) \in A \otimes \mathfrak{K}$ , then  $V(T \otimes_\phi 1)V^* = (\phi(a_{ij})) \in B \otimes \mathfrak{K}$ .

We recall the following generalized theorem of Stinespring in the setting of Hilbert module. It was proven by G. Kasparov in [22].

**Theorem 1.2.14** (Kasparov-Stinespring [22]). Let  $A$  and  $B$  be  $C^*$ -algebras with  $A$  separable and  $B$  unital. Let  $\Psi : A \rightarrow B$  be a completely positive map. Then, there exists a countably generated Hilbert  $B$ -module  $E$  and a  $*$ -homomorphism  $\pi : A \rightarrow \mathcal{L}_B(B \oplus E)$  such that

$$\Psi(a) = P\pi(a)P \quad \forall a \in A,$$

where  $P \in \mathcal{L}_B(B \oplus E)$  is the projection on the  $B$  factor. The statement remains true when  $B$  is non unital by replacing  $\mathcal{L}_B(B \oplus E)$  by  $\mathcal{L}_{\tilde{B}}(\tilde{B} \oplus E)$ .



# Chapter 2

## Groupoids and Coarse Geometry

### 2.1 Preliminaries on topological groupoids

**Definition 2.1.1.** A topological groupoid is the data of

- two topological spaces  $G$  and  $G^{(0)}$ , called respectively the space of arrows, and the unit space,
- a topological embedding  $e : G^{(0)} \rightarrow G; x \mapsto e_x$  called the unit map,
- source and range maps  $r, s : G \rightarrow G^{(0)}$  which are continuous, surjective, and satisfy  $r \circ e = s \circ e = Id_{G^{(0)}}$ ,
- a continuous multiplication map  $m : G^{(2)} = G \times_{s,r} G \rightarrow G; (g, g') \mapsto gg'$  such that  $(gg')g'' = g(g'g'')$  for all triples  $g, g', g''$  that are composable, and  $ge_{s(g)} = e_{r(g)}g = g$ ,
- an involutive map  $\text{inv} : G \rightarrow G : g \mapsto g^{-1}$  which is continuous and such that  $s \circ \text{inv} = r, r \circ \text{inv} = s$  and  $\forall g \in G, gg^{-1} = e_{s(g)}$  and  $g^{-1}g = e_{r(g)}$ . This map is called the inverse map.

**Definition 2.1.2.** Let  $G$  be a locally compact groupoid, i.e. a topological groupoid locally compact as a topological space.

$G$  is said to be étale if the range map  $r : G \rightarrow G^{(0)}$  is a local homeomorphism.

$G$  is said to be principal if  $r \times s : G \rightarrow G^{(0)} \times G^{(0)}$  is injective, transitive if  $r \times s$  is surjective, and proper if  $r \times s$  is a proper map.

If  $G$  is étale, then any  $g \in G$  has an open neighborhood  $U$  such that  $r$  and  $s$  are injective on  $U$ . The element  $g$  defines in this way a partial homeomorphism  $\alpha_U = r \circ (s|_U)^{-1} : s(U) \rightarrow r(U)$ . Such open sets  $U$  where  $r$  and  $s$  are injective are called bisections.

For  $U, V \subseteq G^{(0)}$ , we will use the notations  $G_U = s^{-1}(U)$ ,  $G^V = r^{-1}(V)$  and  $G_U^V = s^{-1}(U) \cap r^{-1}(V)$ . If one of the subsets is reduced to a single point, we will use  $G_x = G_{\{x\}}$ , etc. We will write  $G|_U$  for  $G_U^U$ . It is a subgroupoid of  $G$ .

A Haar system on  $G$  is a family of borelian measures  $(\lambda^x)_{x \in G^{(0)}}$  on  $G$  such that

- the support of  $\lambda^x$  is  $G^x$ ,
- the map  $G^{(0)} \rightarrow \mathbb{C}; x \mapsto \int_{G^x} f d\lambda^x$  is continuous for all  $f \in C_c(G)$ ,

- for all  $f \in C_c(G)$  and  $g \in G$ ,  $\int_{G^{r(g)}} f(h) d\lambda^{r(g)}(dh) = \int_{G^{s(g)}} f(gh) \lambda^{s(g)}(dh)$  holds.

If the groupoid is étale, a natural Haar system is the counting measure on the discrete fibers, i.e.  $\lambda^x(A) = |A \cap G^x|$  for all  $A \subseteq G$ .

Here are classical examples of groupoids.

**Example 2.1.3** (Groups and spaces). Any group  $G$  is a groupoid over a point  $G^{(0)} = *$ .  $G$  is étale as a groupoid iff it is a discrete group. A topological space is also a groupoid over itself, the only arrows being the units over the points :  $G^{(0)} = X = G$ ,  $s = r = id_X$ , it is always étale.

**Example 2.1.4** (Pair groupoid). Let  $X$  be a set, and consider  $G = X \times X$  with base space  $G^{(0)} = X$  and projection maps as target and source  $s = pr_1, r = pr_2$ . Composition is given by  $(x, y)(y, z) = (x, z)$ , and one can easily check that  $e_x = (x, x)$  and  $(x, y)^{-1} = (y, x)$ .  $X \times X$  is étale if  $X$  is discrete.

**Example 2.1.5** (Equivalence relations). More generally, consider an equivalence relation  $R \subseteq X \times X$  on a topological space  $X$ . It naturally has a structure of topological groupoid, as a subgroupoid of  $X \times X$ .

**Example 2.1.6** (Pull-back). Let  $X$  be a topological space and  $f : X \rightarrow G^{(0)}$  a continuous map. The pull-back groupoid of  $G$  along  $f$ , denoted  $G[X]$ , is defined as  $\{(x, g, y) \in X \times G \times X \text{ s.t. } g \in G_{f(x)}^{f(y)}\}$ , with base space  $X$ . A particular example is given by considering an open cover  $\mathcal{U} = (U_j)$  of  $G^{(0)}$ . Let  $p : \coprod_j U_j \rightarrow G^{(0)}$  be the natural projection. Then  $G[\mathcal{U}]$  denotes the pull-back of  $G$  along  $p$ .

**Example 2.1.7** (Action groupoid). Let  $G$  be a group acting on a topological space  $X$  by homeomorphisms. Define the groupoid  $X \rtimes G$  as the topological space  $X \times G$  over the base space  $X$ , endowed with the maps  $s(x, g) = x$  and  $r(x, g) = g.x$ . The unit map will be given by  $x \mapsto (x, e_G)$  and the inverse by  $(x, g)^{-1} = (g.x, g^{-1})$ . This groupoid is called the action groupoid. It is étale when  $G$  is a discrete group.

These examples show how the notion of topological groupoid interpolates between a group and a space. However, other examples of interesting groupoids abound in analysis and geometry. For example, we will define later a so called Coarse Groupoid associated to any Coarse Space, after [47], whose importance cannot be emphasized enough to our approach to the Coarse Baum-Connes conjecture.

Other examples include the holonomy groupoid of a foliation [10], Lie groupoids [41], etc. One of the reasons groupoids are such famous objects in noncommutative geometry is that we can still do analysis on it. According to the general philosophy of Connes [11], groupoids encode geometric situations and remain available to analysis, even when the situation is so pathological that classical tools become trivial. For example, when confronted with a group action, considering the convolution algebra of the principal action groupoid can be a substitute to the studying the quotient space. It turns out that, when the action is free and proper, the envelopping  $C^*$ -algebra is Morita equivalent to  $C_0(X/G)$ , which represents the quotient space by Gelfand duality. However, when the quotient is singular, the envelopping  $C^*$ -algebra still carries useful information, even if the quotient is such that  $C_0(X/G) \cong \mathbb{C}$ .

### 2.1.1 Actions of groupoids

**Definition 2.1.8.** A left action of  $G$  on a topological space  $Z$  is given by a continuous map  $p : Z \rightarrow G^{(0)}$ , called the anchor map, and a map  $\alpha : G \times_{s,p} Z \rightarrow Z$  such that :

- $\alpha(g', \alpha(g, z)) = \alpha(g'g, z)$  whenever  $(g, g') \in G^{(2)}$  and  $p(z) = s(g)$ ,
- $p(\alpha(g, z)) = r(g)$
- $\alpha(e_{p(z)}, z) = z$

We will use the notation  $\alpha(g, z) = g.z$  when the action is clear from the context, and  $Z$  will then be referred as a (left)  $G$ -space.

A right action of  $G$  on a topological space  $Z$  is given by a continuous map  $p : Z \rightarrow G^{(0)}$ , again called the anchor map, and a map  $\alpha : Z \times_{p,r} G \rightarrow Z$  such that :

- $\alpha(\alpha(z, g), g') = \alpha(z, gg')$  whenever  $(g, g') \in G^{(2)}$  and  $p(z) = r(g)$ ,
- $p(\alpha(z, g)) = s(g)$
- $\alpha(z, e_{p(z)}) = z$

We will use the notation  $\alpha(g, z) = z.g$  when the action is clear from the context.

A left action  $(Z, p, \alpha)$  is said to be

- proper if  $id_Z \times \alpha : (g, z) \mapsto (z, g.z)$  is proper as a continuous map,
- free if  $\alpha(g, z) = z \Rightarrow g = e_{p(z)}$ .

For any subset  $V \subseteq Z$ , we use the notation  $G.V$  for the orbit of  $V$  under the action of  $G$ , i.e.  $G.V = \{g.v : v \in V, g \in G_{p(v)}\}$ . Similarly, we define these notions for right actions.

Let  $G$  be a locally compact  $\sigma$ -compact Hausdorff groupoid. A cutoff function for  $G$  is a continuous function  $c : G^{(0)} \rightarrow \mathbb{R}_+$  such that :

- for all compact subsets  $K \subseteq G^{(0)}$ ,  $\text{supp}(c) \cap s(G^K)$  is compact,
- for all  $x \in G^{(0)}$ ,  $\int_{g \in G^x} c(s(g)) \lambda^x(dg) = 1$ .

We recall the following proposition.

**Proposition 2.1.9** ([49]). A locally compact  $\sigma$ -compact Hausdorff groupoid is proper iff there exists a cutoff function for  $G$ .

Let  $Z$  be a right  $G$ -space. Define :

- $(Z \rtimes G)^{(0)} = Z$ ,  $Z \rtimes G = Z \times_{r,p} G$  as a topological space,
- $u_z = (z, e_{p(z)})$ ,  $s(z, g) = z$  and  $r(z, g) = z.g$ ,
- if  $y = x.g$ ,  $(x, g)(y, g') = (x, gg')$  and  $(x, g)^{-1} = (x.g, g^{-1})$ .

These maps define a structure of topological groupoid on  $Z \rtimes G$ . It is called the crossed product groupoid of  $Z$  by  $G$ . It is étale if  $G$  is.

We present now an important class of  $G$ -spaces called  $G$ -simplicial complexes. The reader is referred to [52] for details.

**Definition 2.1.10.** A map between two topological spaces  $f : X \rightarrow Y$  is said to be locally injective if there exists an open cover  $\mathcal{U}$  of  $X$  such that, for all  $U \in \mathcal{U}$ ,  $f|_U$  is injective.

**Definition 2.1.11.** Let  $n \in \mathbb{N}$ . A  $G$ -simplicial complex of dimension  $\leq n$  is a pair  $(X, \Delta)$  where :

- $X$  is a locally compact proper  $G$ -space, called the space of vertices, such that the anchor map  $p : X \rightarrow G^{(0)}$  is locally injective;
- $\Delta$  is a closed  $G$ -invariant subset of the space of measures on  $X$ , denoted  $M_X$ , endowed with the weak  $*$ -topology. Moreover,  $\Delta$  contains only probability measures and satisfies :
  - for all  $\eta \in \Delta$ , there exists  $x \in G^{(0)}$  such that  $\text{supp } \eta \subseteq p^{-1}(x)$  and  $|\text{supp } \eta| \leq n + 1$ ,
  - if  $\eta' \in \Delta$  and  $\eta \in M_X$  such that  $\text{supp } \eta \subseteq \text{supp } \eta'$ , then  $\eta \in \Delta$ .

For  $\eta \in \Delta$ ,  $\text{supp } \eta$  is called a simplex, or a  $j$ -simplex when  $|\text{supp } \eta| = j$ .

The complex is said to be typed if there exists a finite space  $T$  and a  $G$ -invariant continuous map  $\tau : X \rightarrow T$  such that, for every simplex  $S$ ,  $\tau|_S$  is injective.

For any typed proper  $G$ -compact  $G$ -simplicial complex  $(X, \Delta)$  of dimension  $\leq n$ , one can decompose it into its  $n$ -skeleton  $Z_0 \subseteq Z_1 \subseteq \dots \subseteq Z_n$ , where each  $Z_j$  is a closed  $G$ -invariant subset of  $\Delta$  such that :

- for all  $\eta \in Z_j \setminus Z_{j-1}$ ,  $|\text{supp } \eta| = j$ ,
- $Z_j \setminus Z_{j-1}$  is  $G$ -equivariantly homeomorphic to  $\mathring{\sigma}_j \times \Sigma_j$ , where  $\mathring{\sigma}_j$  is the interior of the standard simplex of dimension  $j$ , and  $\Sigma_j$  is the subspace of centers of  $j$ -simplices.

**Definition 2.1.12.** Let  $(X, \Delta)$  be a  $G$ -simplicial complex of dimension  $\leq n$ . Its barycentric subdivision is the  $G$ -simplicial complex  $(S, \Delta_S)$  of dimension  $\leq n$  defined by :

$$S = \left\{ \frac{1}{|\text{supp } \eta|} \sum_{x \in \text{supp } \eta} \delta_x, \eta \in \Delta \right\} \subseteq \Delta,$$

i.e. the space of vertices  $S$  contains  $X$  plus the barycentric center of any simplex of  $\Delta$ , and  $\Delta_S$  satisfies that  $\sigma = \{\eta_0, \dots, \eta_k\}$  is a simplex in  $\Delta_S$  iff  $\{\text{supp } (\eta_0), \dots, \text{supp } (\eta_k)\}$  is totally ordered for the inclusion.

This construction shows that, up to replacing a  $G$ -simplicial complex of dimension  $\leq n$  by its barycentric subdivision, we can always suppose that it is a typed  $G$ -simplicial complex of dimension  $\leq n$ .



### 2.1.2 $G$ -algebras and crossed products

This paragraph describes what is an action of a topological groupoid on a  $C^*$ -algebra, and how to construct the reduced crossed-product out of the action.

**Definition 2.1.13.** Let  $X$  be a topological space. A  $C(X)$ -algebra is a  $C^*$ -algebra  $A$  together with a  $*$ -homomorphism  $\theta : C_0(X) \rightarrow Z(\mathcal{M}(A))$  such that  $\overline{\theta(C_0(X))A} = A$ . The couple  $(A, \theta)$  is then called a  $C(X)$ -structure.

If  $f : X \rightarrow Y$  is a continuous map between locally compact spaces, and  $A$  a  $C(Y)$ -algebra, then  $f$  induces a  $C(X)$ -structure on  $A$  by  $f^* : C(Y) \rightarrow C_b(X); g \mapsto g \circ f$ . Define  $f^*A = A \otimes_{f^*} C_0(X)$  the resulting  $C(X)$ -algebra. For details on tensor products of  $C(X)$ -algebras, see [31] for instance. As a particular case, one gets the notion of a fiber of a  $C(X)$ -algebra : for  $x \in X$ ,  $A_x = ev_x^* A$  is called the fiber over  $x$ , where  $ev_x$  is the inclusion of  $x$  in  $X$ . It is convenient to see  $A_x$  as  $A/(I_x \cdot A)$  where  $I_x$  is the ideal  $\ker(ev_x : C_0(X) \rightarrow \mathbb{C})$ . Moreover we can describe the fibers of  $f^*A$  by a canonical isomorphism:

$$(f^*A)_x \cong A_{f(x)}$$

for all  $x \in X$ .

A homomorphism between two  $C(X)$ -algebras  $\alpha : A \rightarrow A'$  is a  $*$ -homomorphism which commutes with the action of  $C_0(X)$ , i.e.  $\alpha(\theta(f)a) = \theta'(f)\alpha(a)$  for all  $a \in A, f \in C_0(X)$ . Its fibers are defined unambiguously by  $\alpha_x = id_{\mathbb{C}} \otimes_{ev_x} \alpha$ .

Notice that, if  $A$  is a  $C(G^{(0)})$ -algebra, then  $s^*A = A \otimes_s C_0(G)$  and  $r^*A = A \otimes_r C_0(G)$  are  $C(G)$ -algebra with fibers  $(s^*A)_g \cong A_{s(g)}$  and  $(r^*A)_g \cong A_{r(g)}$  respectively, for all  $g \in G$ .

**Definition 2.1.14.** An action of a groupoid  $G$  on a  $C^*$ -algebra  $A$  is a triple  $(A, \theta, \alpha)$  where  $(A, \theta)$  is a  $C(G^{(0)})$ -structure and  $\alpha : s^*A \rightarrow r^*A$  is an isomorphism of  $C(G)$ -algebras such that  $\alpha_{gg'} = \alpha_g \circ \alpha_{g'}$  for all  $(g, g') \in G^{(2)}$ . Such an action will be called a  $G$ -algebra.

**Definition 2.1.15.** Let  $A$  and  $B$  be two  $G$ -algebras, with action  $\alpha$  and  $\beta$  respectively. A  $G$ -equivariant  $*$ -homomorphism  $\phi : A \rightarrow B$  is a morphism of  $C(G^{(0)})$ -algebra which intertwines the action, i.e. such that  $(r^*\phi)(\alpha(a)) = \beta((s^*\phi)(a))$  for every  $a \in s^*A$ .

If  $A$  is a  $G$ -algebra, and  $U \subseteq G^{(0)}$  is an open subset, recall from [31] that the restriction  $A_U$  is defined as  $\eta^*A = C_0(U) \otimes_{\eta} A$  where  $\eta : C_0(U) \hookrightarrow C_0(G^{(0)})$  is the canonical inclusion. If  $U \subseteq G$  is an open subset, we can define in a similar way  $A_U$  as  $C_0(U) \otimes_r A$ .

The space of continuous sections with compact support is defined as :

$$C_c(G, A) = \cup_{U \in \mathcal{E}} A_U,$$

where  $U$  runs through the open relatively compact subsets of  $G$ . It is naturally an involutive algebra with the convolution product

$$\phi * \psi(g) = \int_{G^{r(g)}} \phi(h) \alpha_h(\psi(h^{-1}g)) \lambda^{r(g)}(h).$$

**Remark 2.1.16.** Another possible definition of  $C_c(G, A)$  would be to take a tensor product of Fréchet algebras  $C_c(G) \otimes A$ , which gives a Fréchet algebra. Even if this definition seems more elegant, this would require more definitions on Fréchet  $G$ -algebras, which we do not want to dive into.

Let  $g \in G$  and  $f \in C_c(G, A)$ . Choose an open relatively compact subset  $U \subseteq G$  which contains  $\text{supp } f$ . We can define  $f(g)$  as  $1 \otimes_{\text{ev}_g} f \in \mathbb{C} \otimes_{\text{ev}_g} C_0(U) \otimes_{\eta} A \cong A_{s(g)}$ . This justifies the name "continuous sections" for  $C_c(G, A)$ .

Let  $A$  be a  $G$ -algebra and  $x \in G^{(0)}$ . As the groupoid  $G$  is étale,  $G_x$  is discrete and  $l^2(G_x)$  is a separable Hilbert space with dimension the cardinality of  $G_x$ . Define  $l^2(G_x, A)$  as the  $A_x$ -Hilbert module  $l^2(G_x) \otimes A_x$ . We denote by  $\|\cdot\|_2$  the norm induced by the inner product on  $l^2(G_x, A)$  i.e.

$$\begin{aligned} \langle \xi, \eta \rangle_{A_x} &= \sum_{g \in G_x} \xi(g)^* \eta(g), \\ \|\xi\|_2 &= \|\langle \xi, \xi \rangle\|_2^{\frac{1}{2}} \end{aligned}$$

for all  $\xi, \eta \in l^2(G_x, A)$ .

Define, for  $f \in C_c(G, A)$ , the reduced norm as

$$\|f\|_r = \sup_{x \in G^{(0)}} \{\|f * \xi\|_2 : \xi \in l^2(G_x, A) \text{ s.t. } \|\xi\|_2 \leq 1\}.$$

**Definition 2.1.17.** The reduced crossed product  $A \rtimes_r G$  is defined as the completion of  $C_c(G, A)$  under the reduced norm.

When  $G$  is étale,  $G^{(0)}$  is open in  $G$ , hence  $A$  is a sub- $C^*$ -algebra of  $A \rtimes_r G$ . For every  $G$ -equivariant  $*$ -homomorphism  $\phi : A \rightarrow B$ , we denote by  $\phi_G$  the  $*$ -homomorphism  $\text{id}_{C_0(G)} \otimes \phi : A \rtimes_r G \rightarrow B \rtimes_r G$ . It is the unique  $*$ -homomorphism extending  $\phi$ . Moreover, the rule  $\phi \mapsto \phi_G$  is functorial.

### 2.1.3 Equivariant Hilbert modules

Let  $B$  be a  $G$ -algebra, with action  $\beta$ , and  $E$  a  $B$ -Hilbert module. As earlier, we can define the fiber  $E_x$  of  $E$  over  $x \in G^{(0)}$ . Define  $s^*E = E \otimes_s C_0(G)$  and  $r^*E = E \otimes_r C_0(G)$  which are  $s^*B$  and  $r^*B$ -Hilbert modules respectively, with fibers  $(s^*E)_g \cong E_{s(g)}$  and  $(r^*E)_g \cong E_{r(g)}$  for all  $g \in G$ . We can endow  $r^*E$  with a structure of  $s^*B$ -Hilbert module with the help of the isomorphism  $\beta^{-1} : r^*B \rightarrow s^*B$ .

**Definition 2.1.18.** An equivariant Hilbert  $B$ -module is a Hilbert  $B$ -module  $E$  together with an unitary  $V \in \mathcal{L}_{s^*B}(s^*E, r^*E)$  such that  $V_g V_{g'} = V_{gg'}$  for all  $(g, g') \in G^{(2)}$ . If the underlying  $G$ -algebra is clear, we will often abbreviate  $G$ -equivariant Hilbert  $B$ -module by  $G$ -module.

**Example 2.1.19.** Define  $L^2(G)$  as the completion of  $C_c(G)$  seen as a pre-Hilbert  $C_0(G^{(0)})$ -module with the  $C_0(G^{(0)})$ -valued inner product

$$\langle \xi, \eta \rangle(x) = \sum_{g \in G^x} \xi(g^{-1})^* \eta(g^{-1}) \quad \forall \xi, \eta \in C_c(G).$$

Its fibers are isomorphic to  $l^2(G^x)$ , and the action  $\lambda : s^*L^2(G) \rightarrow r^*L^2(G)$  is given by left translation, i.e. if  $\xi \in l^2(G^{s(g)})$ , then

$$(\lambda_g \xi)(h) = \xi(g^{-1}h) \quad \forall h \in G^{r(g)}.$$

**Example 2.1.20.** Let  $B$  be a  $G$ -algebra, the  $C(G^{(0)})$ -structure being given by  $\theta : C_0(G^{(0)}) \rightarrow Z(\mathcal{M}(B)) \subseteq \mathcal{L}_B(B)$ . The  $G$ -module  $L^2(G, B)$  is defined as the tensor product  $L^2(G) \otimes_{\theta} B$ . Its fibers are  $l^2(G^x, B)$  and the action is given by  $\lambda \otimes \text{id}_B$ .

**Example 2.1.21.** More generally, let  $B$  be a  $G$ -algebra and  $E$  be a  $B$ -module, with  $C(G^{(0)})$ -structure given by  $\theta : C_0(G^{(0)}) \rightarrow Z(\mathcal{M}(B)) \subseteq \mathcal{L}_B(B)$ . The  $G$ -equivariant  $B$ -Hilbert module  $L^2(G, E)$  is defined as the tensor product  $L^2(G) \otimes_\theta E$ .

**Remark 2.1.22.** We could alternatively have defined, for  $A$  a  $G$ -algebra,  $A \rtimes_r G$  as a subalgebra of  $\mathcal{L}_A(L^2(G, A))$ . Indeed, any  $f \in C_c(G, A)$  acts as a convolution operator on  $L^2(G, A)$  :

$$(\lambda(f)\xi)(h) = \sum_{g \in G^x} f(g)\alpha_g(\xi(g^{-1}h)) \quad \forall \xi \in L^2(G, A), g, h \in G^x,$$

and  $\|f\|_r = \|\lambda(f)\|_{\mathcal{L}_A(L^2(G, A))}$ .

**Lemma 2.1.23** (Equivariant stabilization lemma [31]). Let  $E$  be a countably generated  $G$ -equivariant  $B$ -Hilbert module, then there exists an  $G$ -equivariant isomorphism of Hilbert module  $E \oplus H_B \cong H_B$ .

There exists an equivariant version of the generalized theorem of Stinespring in the setting of groupoids. It was proven by JL. Tu in [49].

**Theorem 2.1.24** (Kasparov-Stinespring [49]). Let  $A$  and  $B$  be  $G$ -algebras with  $A$  separable and  $B$  unital. Let  $\Psi : A \rightarrow B$  be a completely positive  $G$ -equivariant map. Then, there exists a countably generated  $G$ -equivariant Hilbert  $B$ -module  $E$  and a  $G$ -equivariant  $*$ -homomorphism  $\pi : A \rightarrow \mathcal{L}_B(B \oplus E)$  such that

$$\Psi(a) = P\pi(a)P \quad \forall a \in A,$$

where  $P \in \mathcal{L}_B(B \oplus E)$  is the  $G$ -equivariant projection on the  $B$  factor. The statement remains true when  $B$  is non unital by replacing  $\mathcal{L}_B(B \oplus E)$  by  $\mathcal{L}_{\tilde{B}}(\tilde{B} \oplus E)$ .

## 2.1.4 Equivariant $KK$ -theory

From now on, all the groupoids considered are locally compact and  $\sigma$ -compact.

In [31], Le Gall defined, for any locally compact groupoid  $G$ , and  $G$ -algebras  $A$  and  $B$ ,  $\mathbb{Z}_2$ -graded abelian groups  $KK^G(A, B)$  generalizing Kasparov's equivariant  $KK$ -theory for a locally compact group. Elements of  $KK^G(A, B)$  are defined as equivalence classes of triples  $(E, \pi, T)$  that we will call  $K$ -cycles.

**Definition 2.1.25.** A  $K$ -cycle is a triple  $(E, \pi, T)$  where :

- $E$  is a right  $\mathbb{Z}_2$ -graded  $B$ -Hilbert module, equipped with a unitary action of  $G$ , i.e. an even unitary  $V \in \mathcal{L}_B(s^*E, r^*E)$  such that  $V_g V_{g'} = V_{gg'}$  for all  $(g, g') \in G^{(2)}$ ,
- $\pi : A \rightarrow \mathcal{L}_B(E)$  is a  $G$ -equivariant  $*$ -homomorphism whose image is contained in even operators, and we will often write  $a$  instead of  $\pi(a)$ ,
- $T \in \mathcal{L}_B(E)$  is an odd bounded  $B$ -linear operator satisfying the  $K$ -cycle relations i.e.  $[a, T]$ ,  $a(T - T^*)$ ,  $a(T^2 - 1)$  are compact operators in  $\mathfrak{K}_B(E)$  and  $a(r^*T - Vs^*TV^*)$  is a compact operator in  $\mathfrak{K}_{r^*B}(r^*E)$  for all  $a \in A$ .

The set of such  $K$ -cycles is denoted by  $\mathbb{E}^G(A, B)$ .

**Remark 2.1.26.** The  $*$ -homomorphism  $\pi$  is  $G$ -equivariant in the following sense :

$$(r^*\pi)(\alpha(s^*a)) = V (r^*\pi)(r^*a) V^*$$

for every  $a \in A$ . By convention, for every  $G$ -equivariant Hilbert module  $E$  with action  $V$ ,  $\mathfrak{K}_B(E)$  will be endowed with a  $G$ -algebra structure given by  $Ad_V(T) = VTV^*$ .

**Definition 2.1.27.** Two  $K$ -cycles  $(E, \pi, T)$  and  $(E', \pi', T')$  in  $\mathbb{E}^G(A, B)$  are homotopic if there exists  $(\tilde{E}, \tilde{\pi}, \tilde{T}) \in \mathbb{E}^G(A, B[0, 1])$  such that  $(\tilde{E}_0, \tilde{\pi}_0, \tilde{T}_0) \simeq (E, \pi, T)$  and  $(\tilde{E}_1, \tilde{\pi}_1, \tilde{T}_1) \simeq (E', \pi', T')$ . The action on  $B[0, 1]$  is the action obtained as the tensor product of the action on  $B$  and of the trivial action on  $C[0, 1]$ .

**Definition 2.1.28.**  $KK_0^G(A, B)$  is defined as the quotient of  $\mathbb{E}^G(A, B)$  by the equivalence relation given by homotopy. Let  $\mathbb{C}_1$  denotes the trivial  $G$ -Hilbert module  $\mathbb{C}^2$  with the standard grading given by  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .  $KK_1^G(A, B)$  is defined as  $KK_0^G(A, B \otimes \mathbb{C}_1)$ .

Doing computations, we will often use different equivalence relations which turn out to coincide when  $A$  is separable and  $B$  is  $\sigma$ -unital. See [4] for details.

**Definition 2.1.29.** Unitary and operator homotopy equivalences.

- Unitary equivalence is defined as  $(E, \pi, T) \sim_u (E', \pi', T')$  if there exists a unitary  $u \in \mathcal{L}_B(E, E')$  such that  $T = u^*T'u$  and  $\pi(a) = u^*\pi'(a)u, \forall a \in A$ .
- Let  $\mathbb{D}^G(A, B)$  be the set of  $(E, \pi, T) \in \mathbb{E}^G(A, B)$  s.t.  $\pi(a)(F^2 - 1) = 0$ ,  $[\pi(a), F] = 0$  and  $V(s^*T)V^* = r^*T, \forall a \in A$ . It is called the set of degenerate  $K$ -cycles.
- An operator homotopy of  $K$ -cycles is a family  $[0, 1] \rightarrow \mathbb{E}^G(A, B); s \mapsto (E, \pi, T_s)$ , where  $s \mapsto T_s$  is norm-continuous.
- $\sim_{oh}$  is the equivalence relation on  $\mathbb{E}^G(A, B)$  generated by operator homotopy and addition of degenerate elements, i.e.  $(E, \pi, T) \sim_{oh} (E', \pi', T')$  if there exists  $(E'', \pi'', T'') \in \mathbb{E}^G(A, B)$  such that  $(E, \pi, T) \oplus (E'', \pi'', T'')$  and  $(E', \pi', T') \oplus (E'', \pi'', T'')$  are operator homotopic, up to unitary equivalence.

The following proposition summarizes propositions 17.2.3 and 17.2.5 from [4].

**Proposition 2.1.30** ([4]). If  $(E, \pi, T) \in \mathbb{D}^G(A, B)$ , then  $(E, \pi, T)$  is homotopic to the 0  $K$ -cycle. If  $(E, \pi, T')$  is a compact perturbation of  $(E, \pi, T)$  in  $\mathbb{E}^G(A, B)$ , i.e.  $T' - T \in \mathfrak{K}_B(E)$ , then  $(E, \pi, T')$  is operator homotopic to  $(E, \pi, T)$ .

Moreover, if  $A$  is separable and  $B$  is  $\sigma$ -unital, then  $\sim_{oh}$  coincides with homotopy equivalence of  $K$ -cycles.

One of the properties that makes equivariant  $KK$ -theory so powerful is the existence of a product, called the Kasparov product. Its existence is highly non trivial, and it fails to exist in the non-separable case or in the case of the  $KK$ -theory for Banach algebras developed by V. Lafforgue [26] for example.

**Proposition 2.1.31.** [31] Let  $A, B$  and  $C$  be  $G$ -algebras such that  $A$  is separable. There exists a map

$$\otimes_B : \begin{cases} KK^G(A, B) \times KK^G(B, C) & \rightarrow & KK^G(A, C) \\ (z, z') & \mapsto & z \otimes_B z' \end{cases}$$

which is associative and distributive w.r.t. the addition of  $K$ -cycles. The Kasparov product is covariant w.r.t.  $C$  and contravariant w.r.t.  $A$ .

**Remark 2.1.32.** By the equivariant stabilization lemma 2.1.23, any  $B$ -Hilbert module  $E$  sits as a complemented  $G$ -submodule of the standard module  $H_B$ , and as  $[E, \pi, T] = [E \oplus H_B, \pi \oplus 0, T \oplus 0]$ , we can suppose  $E = H_B$ . Notice that the action can be different from the one given by the action on  $B$ .

Here are some examples of  $K$ -cycles. For detailed proofs of the various constructions, we send the reader to [4] and [31].

**Example 2.1.33.** If  $\phi : A \rightarrow B$  is a  $G$ -equivariant  $*$ -homomorphism, it induces a  $K$ -cycle  $(B, \phi, 0) \in \mathbb{E}^G(A, B)$  whose class is denoted by  $[\phi] \in KK_0^G(A, B)$ . This induces a covariant functor  $\phi_* = - \otimes_A [\phi] : KK^G(D, A) \rightarrow KK^G(D, B)$  and a contravariant functor  $\phi^* = [\phi] \otimes_B - : KK^G(B, D) \rightarrow KK^G(A, D)$  for any  $G$ -algebra  $D$ . Moreover, if  $\phi' : B \rightarrow C$ , the following relation holds

$$[\phi' \circ \phi] = [\phi] \otimes_B [\phi'].$$

This example allows one to think of  $KK(A, B)$  as generalized morphisms between  $A$  and  $B$  : every  $*$ -homomorphism induces a  $K$ -cycle such that composition is preserved. Actually, it turns out that generalized morphisms are not so far from coming from genuine  $*$ -homomorphisms. Indeed, H. Oyono-Oyono has shown in the appendice of [28] that every element of  $KK^G(A, B)$  can be written as the Kasparov product of at most  $d$  elements, each one coming either from a  $*$ -homomorphism or from a  $KK$ -inverse of a  $*$ -homomorphism.

More precisely, let us define decomposition property ( $d$ ).

**Definition 2.1.34.** Let  $d$  be a positive integer. An element  $z \in KK^G(A, B)$  is said to satisfy decomposition property ( $d$ ) if

- there exist  $G$ -algebras  $A_0, A_1, \dots, A_d$  such that  $A_0 = A$  and  $A_d = B$ ,
- there exist elements  $z_j \in KK^G(A_j, A_{j+1})$  for  $j \in \{0, \dots, d-1\}$  such that, either  $z_j$  is induced by a  $G$ -morphism  $A_j \rightarrow A_{j+1}$ , or there exists a  $G$ -morphism  $\phi_j : A_{j+1} \rightarrow A_j$  such that  $z_j \otimes_{A_{j+1}} [\phi_j] = 1_{A_j}$  and  $[\phi_j] \otimes_{A_j} z_j = 1_{A_{j+1}}$ ,

such that  $z = z_1 \otimes_{A_1} \dots \otimes_{A_{d-1}} z_{d-1}$  holds.

Then, the following theorem is true for a universal constant  $d$ , which does not depend on the groupoid. It will be crucial to prove that the controlled Kasparov and Roe transforms, applications to be defined later, respect the Kasparov product.

**Theorem 2.1.35** ([28]). Let  $G$  be a locally compact groupoid with Haar system. Then, there exists a universal constant  $d$  such that every element  $z \in KK^G(A, B)$  has decomposition property ( $d$ ).

We saw that  $KK^G(A, B)$  is a bifunctor w.r.t.  $*$ -homomorphisms, contravariant in the  $A$  variable and covariant in the  $B$  variable. Let us describe how  $KK^G$  transforms under morphisms of groupoids. See [31] for a proof.

**Proposition 2.1.36.** Let  $G$  and  $G'$  be étale groupoids and  $\phi : G \rightarrow G'$  a strict morphism of topological groupoids, i.e.  $\phi$  is continuous such that  $\phi(e_x) = e_{\phi(0)(x)}$  and  $\phi(g)\phi(g') = \phi(gg')$  for all  $(g, g') \in G^{(2)}$ . Let  $A$  and  $B$  be  $G'$ -algebras, and  $\phi^*A = A \otimes_{\phi} C_0(G)$  and

$\phi^*B = B \otimes_{\phi} C_0(G)$  the pull back  $G$ -algebras. Then  $\phi$  induces a homomorphism of  $\mathbb{Z}_2$ -graded abelian groups

$$\phi^* : KK^{G'}(A, B) \rightarrow KK^G(\phi^*A, \phi^*B),$$

such that, if  $\phi' : G' \rightarrow G''$  is another strict morphism, then  $(\phi' \circ \phi)^* = \phi^* \circ (\phi')^*$ . In other words,  $KK^G$  is contravariant in the  $G$  variable.

The next two propositions reminds the reader about crucial natural morphisms built by P.Y. Le Gall in his thesis that we will need later on. We only state results and remind the reader about the constructions, without proofs. The reader can consult [31] for details.

**Proposition 2.1.37.** [31] Let  $A, B$ , and  $D$  be  $G$ -algebras. Then

$$\tau_D : \begin{cases} KK^G(A, B) & \rightarrow KK^G(A \otimes D, B \otimes D) \\ [E, \pi, T] & \mapsto [E \otimes_{C_0(G^{(0)})} D, \pi \otimes_{C_0(G^{(0)})} id_D, T \otimes_{C_0(G^{(0)})} id_D] \end{cases}$$

defines a homomorphism of  $\mathbb{Z}_2$ -graded abelian groups which respects the Kasparov product, i.e.

$$\tau_D(x \otimes_B y) = \tau_D(x) \otimes_{B \otimes D} \tau_D(y) \quad \forall x \in KK^G(A, B), y \in KK^G(B, C).$$

The next transformation is called the descent functor, or the Kasparov transform. Let  $(E, \pi, T) \in \mathbb{E}^G(A, B)$ . Set  $E_G = E \otimes_B (B \rtimes_r G)$ ,  $T_G = T \otimes id_{B \rtimes_r G} \in \mathcal{L}_{B \rtimes_r G}(E_G)$ , and  $\pi_G : A \rtimes_r G \rightarrow \mathcal{L}_{B \rtimes_r G}(E_G)$ .

**Proposition 2.1.38.** [31] Let  $A, B, C$  and  $D$  be  $G$ -algebras. Then

$$j_G : \begin{cases} KK^G(A, B) & \rightarrow KK^G(A \rtimes_r G, B \rtimes_r G) \\ [E, \pi, T] & \mapsto [E_G, \pi_G, T_G] \end{cases}$$

defines a homomorphism of  $\mathbb{Z}_2$ -graded abelian groups which respects the Kasparov product, i.e.

$$j_G(x \otimes_B y) = j_G(x) \otimes_{B \rtimes_r G} j_G(y) \quad \forall x \in KK^G(A, B), y \in KK^G(B, C).$$

The remainder of the section is devoted to compile some useful facts that we will use.

We will use the following result.[4][12]

**Proposition 2.1.39.** Let  $z \in KK_1^G(A, B)$  and choose any  $K$ -cycle  $(H_B, \pi, T)$  representing  $z$ . Let  $P = \frac{1+T}{2}$ . The induced homomorphism

$$- \otimes_A z : K_*(A) \rightarrow K_*(B)$$

can be realized, up to Morita equivalence, as the boundary of the following semi-split  $G$ -equivariant extension of  $G$ -algebras

$$0 \longrightarrow \mathfrak{K} \otimes B \longrightarrow E^{(\pi, T)} \longrightarrow A \longrightarrow 0$$

where  $E^{\pi, T} = \{(a, P\pi(a)P + y) : a \in A, y \in \mathfrak{K} \otimes B\}$ , the arrows are the obvious inclusion and projection, and the completely positive section is  $a \mapsto (a, P\pi(a)P)$ . The fact that the choice of the  $K$ -cycle does not affect the boundary will be proven in the upcoming sections.

The following result will be constantly used in the upcoming sections. In short, it explains how to construct the boundary map associated to any semi-split extension  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  as the Kasparov product of an element  $[\partial_{J,A}] \in KK_1^G(A/J, J)$ . For references, see [4].

Recall that, for any  $G$ -equivariant  $*$ -homomorphism  $\phi : A \rightarrow B$ , we can define its cone denoted  $C_\phi$  by

$$C_\phi = \{(a, f) \in A \times CB \text{ s.t. } \phi(a) = f(0)\}.$$

If any  $G$ -algebra  $A$  is given,  $0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0$  is a  $G$ -equivariant semi-split extension, and defines an element  $[\partial_A] = \tau_A([\partial_C]) \in KK_1^G(A, SA)$  which is invertible. Now, give yourself a semi-split extension

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{q} A/J \longrightarrow 0$$

and define  $\alpha : J \rightarrow C_q; x \mapsto (x, 0)$  and the canonical inclusion  $\beta : S(A/J) \hookrightarrow C_q$ . It turns out that  $[\alpha] \in KK_0^G(J, C_q)$  is invertible.

**Proposition 2.1.40.** The boundary map  $\partial_{J,A} : K_*(A/J) \rightarrow K_{*+1}(J)$  is given by the Kasparov product by an element  $[\partial_{J,A}] \in KK_1^G(J, A/J)$  which satisfies

$$[\partial_{J,A}] = [\partial_{A/J}] \otimes \beta^*([\alpha]^{-1}).$$

The next lemma will be used during the proof of the functorial properties of the controlled Roe and Kasparov transformations that will be defined later. These transformations are defined on  $K$ -cycle which are put in standard form, i.e.  $[H_B, \pi, T]$ . Changing the coefficients leads to  $K$ -cycles which are not of this form, hence the lemma. It states that we can always describe explicitly the standard form of a  $K$ -cycle pushed forward.

**Lemma 2.1.41.** Let  $\phi : B \rightarrow B'$  be a  $G$ -equivariant homomorphism, and  $z = [H_B, \pi, T] \in KK^G(A, B)$ . Let  $V \in \mathcal{L}_{B'}(H_B \otimes B', H_{B'})$  be the isometry of the remark 1.2.13 and  $p = VV^* \in \mathcal{L}_{B'}(H_{B'})$ . Define  $\pi' : A \rightarrow \mathcal{L}_{B'}(H_{B'})$  as  $\pi'(a) = V\pi(a)V^*$  and  $T' = V(T \otimes_\phi 1)V^* + 1 - p \in \mathcal{L}_{B'}(H_{B'})$ . Then  $(H_{B'}, \pi', T')$  is a  $K$ -cycle and

$$g^*(z) = [H_B \otimes B', \pi \otimes_\phi 1, \phi_*(T)] = [H'_{B'}, \pi', T'] \text{ in } KK^G(A, B').$$

**Proof.** As a simple computation shows :

$$\begin{aligned} [\pi'(a), T'] &= V[\pi(a), T]V^*, & T'^* - T' &= V(T^* \otimes_\phi 1 - T \otimes_\phi 1)V^* \\ T'^2 - T' &= V(T^2 \otimes_\phi 1 - T \otimes_\phi 1)V^* & \text{and } r^*T' - Us^*T'U^* &= V((r^*(T) - Us^*(T)U^*) \otimes_\phi 1)V^* \end{aligned}$$

(where  $U$  is the action of  $G$  on  $E$ ). Hence  $[H_{B'}, \pi', T']$  satisfies the relations of  $K$ -cycle.

Remark 1.2.13 shows that  $(H_B \otimes_\phi B') \oplus (1 - p)H_{B'} \cong H_{B'}$ .

Put  $E = H_{B'} \oplus (H_B \otimes_\phi B')$  and  $R_s = \begin{pmatrix} \cos(s)id_{H_{B'}} & -\sin(s)V \\ \sin(s)V^* & \cos(s)V^*V \end{pmatrix}$ , and define

$$\begin{aligned} T_s &= R_s \begin{pmatrix} V(T \otimes_\phi 1)V^* + 1 - VV^* & 0 \\ 0 & -T \otimes_\phi 1 \end{pmatrix} R_s^*, \\ \pi_s(a) &= \begin{pmatrix} V(\pi(a) \otimes_\phi 1)V^* & 0 \\ 0 & \pi(a) \otimes_\phi 1 \end{pmatrix}. \end{aligned}$$

Then  $s \mapsto [E, \pi_s, T_s]$  is an homotopy from  $[H_{B'}, \pi', T'] - [H_B \otimes B', \pi \otimes_\phi 1, \phi_*(T)]$  to the cycle given by

$$\begin{pmatrix} V(T \otimes_\phi 1)V^* & 0 \\ 0 & -T \otimes_\phi 1 \end{pmatrix}$$

which is  $0 \in KK^G(A, B)$ . □

## 2.2 Coarse Geometry

Some notations are in order. Let  $X$  be a set. For any subsets  $A$  and  $B$  of  $X \times X$ ,

- let  $A \circ B = \{(x, z) \in X \times X \text{ s.t. } \exists y \in X \text{ s.t. } (x, y) \in A, (y, z) \in B\}$ ,
- and  $A^{-1} = \{(x, y) \in X \times X \text{ s.t. } (y, x) \in A\}$ .

### 2.2.1 Definitions and examples

**Definition 2.2.1.** [45] A coarse structure on a set  $X$  is a subset  $\mathcal{E}$  of  $\mathcal{P}(X \times X)$  such that :

- if  $A$  and  $B$  are in  $\mathcal{E}$ , so are  $A^{-1}$ ,  $A \cup B$  and  $A \circ B$ ,
- every finite subset of  $X \times X$  is in  $\mathcal{E}$ ,
- if  $A \in \mathcal{E}$  and  $B \subseteq A$ , then  $B \in \mathcal{E}$ .

**Example 2.2.2.** When  $X$  is a discrete proper metric space, we define, for all positive number  $R > 0$ , the  $R$ -diagonal as

$$\Delta_R = \{(x, y) \in X \times X \text{ s.t. } d(x, y) \leq R\}.$$

A set  $E \subseteq X \times X$  is called an entourage if there exists  $R > 0$  such that  $E \subseteq \Delta_R$ . The set of entourages  $\mathcal{E}_X$  is a coarse structure on  $X$ .

**Example 2.2.3.** Let  $\Gamma$  be a finitely generated group, and  $S$  be a symmetric (i.e.  $S = S^{-1}$ ) generating subset. Define the word length associated to  $S$  as

$$l(g) = \inf\{k \in \mathbb{N} : \exists s_1, \dots, s_k \in S \text{ s.t. } g = s_1 \cdots s_k\}.$$

This defines a proper length on  $\Gamma$  which does depend on  $S$ , but the coarse structure associated to the left invariant metric  $d(\gamma_1, \gamma_2) = l(\gamma_1^{-1}\gamma_2)$  does not. It is common in the literature to denote  $|\Gamma|$  the associated coarse space.

In order to give another example, we recall the definition of the coarse disjoint union.

**Definition 2.2.4.** Let  $(X_j, d_j)$  be a countable family of metric spaces. The coarse disjoint union  $\coprod X_j$  is the coarse space obtained as the usual disjoint union with a metric  $d$  such that

- the restriction of  $d$  to any  $X_j$  coincides with  $d_j$ ,
- $\lim d(X_i, X_j) = +\infty$  when  $|i + j|$  goes to  $+\infty$ .



Any such metric defines the same coarse structure on, so that the coarse equivalence class of  $\coprod X_j$  is well defined.

**Example 2.2.5.** Let  $\Gamma$  be a finitely generated group which is residually finite w.r.t.  $\mathcal{N} = \{\Gamma_j\}$ , a nested family of normal subgroups  $\Gamma_0 > \Gamma_1 > \dots$  with trivial intersection. Notice that the image of  $S$  in the group  $\Gamma/\Gamma_j$  is a generating subset. The box space of  $\Gamma$  w.r.t.  $\mathcal{N}$  is the coarse disjoint union of the metric spaces  $\Gamma/\Gamma_j$ ,

$$X_{\mathcal{N}}(\Gamma) = \coprod_j |\Gamma/\Gamma_j|.$$

This last example gives, under suitable conditions, an example of coarse space which is not coarsely embeddable into Hilbert space.

**Definition 2.2.6.** A metric space  $(X, d_X)$  coarsely embeds into Hilbert space if there exists two increasing functions  $\rho_{+/-} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\lim_{\infty} \rho(R) = +\infty$ , and a map  $\phi : X \rightarrow H$  such that :

$$\rho_-(d_X(x, y)) \leq \|\phi(x) - \phi(y)\|_H \leq \rho_+(d_X(x, y)) \quad , \forall x, y \in X,$$

where  $H$  is the separable Hilbert space.

Recall that a graph is a couple  $(V, E)$ ,  $V$  being the set of vertices, and  $E \subseteq V \times V$  the set of edges. Define, for  $v \in V$ , the set of neighbours of  $v$  as  $N_v = \{w \in V / (v, w) \in E\}$  and the degree of  $x$  as  $\text{deg}(x) = |N_x|$ . The length of a path in a graph is defined as the number of edges it contains. A graph is naturally a metric space with the path distance. The Laplacian of the graph is the positive operator  $\Delta \in \mathcal{L}(l^2(V))$  defined by :

$$(\Delta f)(x) = f(x) - \frac{1}{\text{deg}(x)} \sum_{y \in N_x} f(y).$$

Let  $X_j = (V_j, E_j)$  be a countable family of finite graphs. The Laplacian of the coarse disjoint union  $X = \coprod X_j$  is defined as  $\Delta_X = \bigoplus \Delta_j \in \mathcal{L}(l^2(X))$ .

**Definition 2.2.7.** An expander is a family  $X = (X_j)_j$  of finite graphs  $(X_j, E_j)$  such that

- $\lim_j |X_j| = \infty$
- the degree of the graphs is constant :  $\exists k, \text{deg}(X_j) = k, \forall j$ ,
- the second eigenvalue of the Laplacian is bounded above :  $\text{sp}(\Delta_j) \subset \{0\} \cup [\epsilon, 1]$ .

When speaking of expanders, we often confuse them with the metric space consisting of the coarse disjoint union of all the graphs, which is just a metric space with the distance induced by the length on the graph when restricted to one of the graph, and such that  $\lim_{j+k \rightarrow \infty} d(X_j, X_k)$ .

**Proposition 2.2.8.** [37] Let  $(X_j)$  be a countable family of metric spaces such that  $\coprod X_j$  is an expander. Then  $\coprod X_j$  does not coarsely embeds into Hilbert space.

We can construct expanders using groups satisfying some representation theoretic properties that we now recall. If  $G$  is a locally compact group, let  $\hat{G}$  be its dual, i.e. the topological space of equivalence classes of all unitary continuous representations  $(\pi, H)$  of  $G$  on a Hilbert space  $H$ . For such a representation  $(\pi, H)$ , denote by  $[\pi, H]$  its equivalence class. Let  $(\pi, H)$  be a unitary representation of  $G$ . For all  $\varepsilon > 0$ , compact subset  $K \subseteq G$ , and unit vector  $v \in H$ ,

$$\mathcal{V}_{\varepsilon, K, v} = \{[\pi', H'] \in \hat{G} \text{ s.t. } \exists v' \in H', \|v'\| = 1 \text{ and } |\langle v, \pi(g)v \rangle - \langle v', \pi'(g)v' \rangle| < \varepsilon, \forall g \in K\}$$

defines a basis of neighborhoods for  $[\pi, H]$ . The topology generated by these subsets is called the Fell topology on  $\hat{G}$ .

Property (T) was introduced by Kazhdan in 1967 in [23]. Property  $\tau$  is a weaker version of property (T). Margulis was the first to construct expanders in [33] using the following fact : property (T) implies property  $\tau$ . This was not stated with this terminology at the time.

**Definition 2.2.9.** Let  $G$  be a locally compact group, and let  $\hat{G}_0$  denote the space of equivalence classes of unitary continuous representations of  $G$  without invariant vectors in  $\hat{G}$ . Then,  $G$  has property (T) if the trivial representation is isolated from  $\hat{G}_0$  in  $\hat{G}$ .

**Definition 2.2.10.** Let  $\Gamma$  be a finitely generated group and  $\mathcal{N} = \{\Gamma_j\}$  be a family of finite index normal subgroups of  $\Gamma$ . Let  $\hat{\mathcal{N}}$  be the set of equivalence classes of unitary continuous representations of  $\Gamma$  without invariant vectors which factorizes through  $\Gamma/\Gamma_j$  for some  $j$ . Then,  $\Gamma$  has property  $\tau$  with respect to  $\mathcal{N}$  if the trivial representation is isolated from  $\hat{\mathcal{N}}$  in  $\hat{G}$ .

**Theorem 2.2.11.** [37] If  $\Gamma$  has property  $\tau$  w.r.t.  $\mathcal{N}$ , then  $X_{\mathcal{N}}(\Gamma)$  is an expander. In particular, Kazhdan's property (T) implies property  $\tau$ , so that property (T) groups satisfy this obstruction. For example, you can take  $SL(n, \mathbb{Z})$  for  $n \geq 3$ .

The following property was introduced by X. Chen, Q. Wang and G. Yu in [7]. It is a generalization of coarse embeddability.

**Definition 2.2.12.** A discrete metric space  $X$  is said to admit a fibred coarse embedding into Hilbert space if there exist

- a field of Hilbert spaces  $\{H_x\}_{x \in X}$  over  $X$ ,
- a section  $s : X \rightarrow \coprod H_x$ , i.e.  $s(x) \in H_x$ ,
- two non-decreasing functions  $\rho_{+/-} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\lim_{\infty} \rho(R) = +\infty$ ,
- a reference Hilbert space  $H$

such that, for any  $R > 0$ , there exists a bounded subset  $E_R \subseteq X$  and a trivialization

$$t_C : \coprod_{x \in C} H_x \rightarrow C \times H$$

for all  $C \subseteq X \setminus E_R$  of diameter less than  $R$ . Moreover, for all  $x \in X$ , the map  $t_C(x)$  is an affine isometry satisfying

- for all  $x, y \in C$ ,  $\rho_1(d(x, y)) \leq \|t_C(x)(s(x)) - t_C(y)(s(y))\| \leq \rho_2(d(x, y))$
- for all  $C_1, C_2$  in  $X - E_R$  of diameter less than  $R$  with nonempty intersection  $C_1 \cap C_2$ , there exists an affine isometry  $t_{C_1, C_2} : H \rightarrow H$  such that  $t_{C_1}(x)t_{C_2}(x)^{-1} = t_{C_1, C_2}(x)$  for all  $x \in C_1 \cap C_2$ .

**Remark 2.2.13.** If  $X$  admits a coarse embedding into Hilbert space, then it admits a fibred coarse embedding into Hilbert space. However, spaces that admit a fibred coarse embedding into Hilbert space which do not coarsely embed into Hilbert space exists. Indeed, let  $\Gamma$  be a finitely generated group residually finite w.r.t. a nested family  $\mathcal{N}$  of normal subgroups with trivial intersection. Then  $X_{\mathcal{N}}(\Gamma)$  admits a fibred coarse embedding into Hilbert space iff  $\Gamma$  is a-T-menable, see [15] for reference. For example, let  $\Gamma = SL(2, \mathbb{Z})$  and  $\mathcal{N} = \{SL(2, p^k \mathbb{Z})\}_{k>1}$  for a prime  $p$ . Then  $\Gamma$  satisfies property  $\tau$ , so that  $X_{\mathcal{N}}(\Gamma)$  is an expander and does not embed into Hilbert space. But  $\Gamma$  is a-T-menable.

## 2.2.2 Geometric modules and Roe algebras

From now on,  $(X, d)$  will be a discrete metric space with bounded geometry, i.e. such that, for all  $R > 0$ ,  $\sup_{x \in X} |B(x, R)|$  is finite. In this section, we define the Roe algebras of  $X$ . Notice that the proofs which are given are more general and can be carried out for non discrete proper metric space with bounded geometry.

**Definition 2.2.14.** A  $X$ -module is a pair  $(H_X, \phi)$  where :

- $H_X$  is a Hilbert space,
- $\phi : C_0(X) \rightarrow \mathcal{L}(H_X)$  is a  $*$ -homomorphism.

The module is said to be non-degenerate if the closure of the subspace generated by  $\{\phi(f)\xi : f \in C_0(X), \xi \in H_X\}$  is dense in  $H_X$ , and it is called standard if no non-zero function acts as a compact operator, i.e.  $\phi(f) \in \mathfrak{K}(H_X)$  implies  $f = 0$ .

We will sometime allow ourself a slight misuse of notation by writing  $f\xi$  for  $\phi(f)\xi$  when  $f \in C_0(X)$  and  $\xi \in H_X$ .

**Example 2.2.15.** Let  $H$  denotes the separable Hilbert space. Continuous functions  $C_0(X)$  act on  $H_X = l^2(X) \otimes H$  as bounded operators, which gives  $H_X$  the structure of a  $X$ -module, called the standard  $X$ -module. It is standard non-degenerate (s.n.d. for short).

**Definition 2.2.16.** Let  $X$  and  $Y$  be two discrete proper metric spaces with bounded geometry,  $H_X$  and  $H_Y$   $X$  and  $Y$ -modules respectively, and  $T \in \mathcal{L}(H_X, H_Y)$  be a bounded operator.

- $T$  is said to be locally compact if  $\phi(g)T$  and  $T\phi(f)$  are compact operators for all  $f \in C_0(X), g \in C_0(Y)$ .
- The support of  $T$  is the complement of the set of points  $(x, y) \in X \times X$  such that there exist  $f_x, f_y \in C_0(X), C_0(Y)$  such that  $f_x(x) \neq 0, f_y(y) \neq 0$  and  $\phi(f_y)T\phi(f_x) = 0$ .

- When  $X = Y$ , the propagation of  $T$ , denoted  $\text{prop}(T)$ , is the smallest  $R > 0$  such that  $\text{supp } T \subseteq \Delta_R$ .

The following proposition says that there is actually one s.n.d.  $X$ -module, up to isomorphism. For  $V \in \mathcal{L}(E, E')$ , we denote  $Ad_V : \mathcal{L}(E) \rightarrow \mathcal{L}(E'); T \mapsto VTV^*$ .

**Proposition 2.2.17.** Let  $H_X$  and  $H'_X$  two s.n.d.  $X$ -modules. Then, for any  $\varepsilon > 0$ , there exists an unitary  $V \in \mathcal{L}(H_X, H'_X)$  such that  $\text{supp } V \subseteq \Delta_\varepsilon$ .

**Proof.** We denote by  $\mathcal{L}^\infty(X)$  the algebra of bounded Borel functions. By bounded functional calculus, extend the representations  $\phi : C_0(X) \rightarrow \mathcal{L}(H_X)$  and  $\phi' : C_0(X) \rightarrow \mathcal{L}(H'_X)$  to  $\tilde{\phi} : \mathcal{L}^\infty(X) \rightarrow \mathcal{L}(H_X)$  and  $\tilde{\phi}' : \mathcal{L}^\infty(X) \rightarrow \mathcal{L}(H'_X)$ . Choose a Borel partition  $\mathcal{U}$  of  $X$  such that each Borel subset  $U \in \mathcal{U}$  satisfies  $\text{diam}(U) \leq \varepsilon$  and has non-empty interior. If  $\chi_U$  denotes the characteristic function of  $U$ ,  $p_U = \tilde{\phi}(\chi_U)$  and  $p'_U = \tilde{\phi}'(\chi_U)$  define projections in  $H_X$  and  $H'_X$  respectively. Hence, if  $H_U = p_U H_X$  and  $H'_U = p'_U H'_X$ , we have an orthogonal decomposition

$$H_X = \bigoplus_{U \in \mathcal{U}} H_U \quad \text{and} \quad H'_X = \bigoplus_{U \in \mathcal{U}} H'_U.$$

But, the representations being standard,  $H_U$  and  $H'_U$  are separable infinite dimensional Hilbert spaces, hence there exists an isometry  $V_U : H_U \rightarrow H'_U$ . Define

$$V = \bigoplus_{U \in \mathcal{U}} V_U : H_X = \bigoplus_{U \in \mathcal{U}} H_U \rightarrow \bigoplus_{U \in \mathcal{U}} H'_U = H'_X.$$

$V$  is an isometry, and  $\text{supp}(V) \subseteq \coprod U \times U \subseteq \Delta_\varepsilon$ . □

We now define the Roe algebra  $C^*(X, H_X)$  of  $X$  when we have fixed a standard non-degenerate  $X$ -module  $H_X$ . We will prove that it is unique up to unnatural isomorphism, and that this  $*$ -isomorphism induces a natural isomorphism in  $K$ -theory.

Define  $C_R[X, H_X]$  as the following subspace of  $\mathcal{L}(H_X)$  :

$$C_R[X, H_X] = \{T \in \mathcal{L}(H_X) \text{ locally compact s.t. } \text{supp } T \subseteq \Delta_R\},$$

namely,  $C_R[X, H_X]$  consists of locally compact operators with propagation less than  $R$ . It is a closed self-adjoint subspace of  $\mathcal{L}(H_X)$  which satisfies  $C_R[X, H_X] \cdot C_S[X, H_X] \subseteq C_{R+S}[X, H_X]$  for any positive numbers  $R > 0, S > 0$ .

Let  $C[X, H_X]$  be the subspace

$$\bigcup_{R>0} C_R[X, H_X] \subseteq \mathcal{L}(H_X).$$

It is actually an involutive algebra naturally represented in a Hilbert space. Completion with respect to the operator norm will naturally give rise to a  $C^*$ -algebra called the Roe algebra of  $X$ . They are the object of the following definition. Let us first define another norm in order to define the maximal Roe algebra of  $X$ .

**Lemma 2.2.18.** Let  $(\pi, E_\pi)$  be a  $*$ -representation  $\pi : C[X, H_X] \rightarrow \mathcal{L}(E_\pi)$  where  $E_\pi$  is a Hilbert space. Let  $T \in C_R[H_X]$ , and define  $N_R = \sup_{x \in X} |B(x, R)| < \infty$  and  $T_{xy} = \chi_y T \chi_x$ . Then  $M = \sup_{x, y \in X} |T_{xy}|$  is finite and the following inequality holds :

$$\|\pi(T)\| \leq N_R M.$$

**Proof.**  $X$  being discrete,  $T = \sum_{x, y \in X} T_{xy}$ , and

$$\|\pi(T)\| \leq \sup_{x \in X} \left\| \sum_{y \in X} T_{xy} \right\| \leq M \sup_{x \in X} |B(x, R)|.$$

□

The maximal norm is defined, for any  $T \in C[X, H_X]$ , as

$$\|T\|_{max} = \sup_{\pi \in \mathcal{F}} \|\pi(T)\|$$

where  $\pi$  runs in the family  $\mathcal{F}$  of  $*$ -representations  $\pi : C[X, H_X] \rightarrow \mathcal{L}(E_\pi)$ . By the previous lemma, it is finite.

**Definition 2.2.19.** Let  $X$  be a discrete metric space with bounded geometry and fix a s.n.d.  $X$ -module. Let  $C[X, H_X]$  be the subspace

$$\bigcup_{R>0} C_R[X, H_X] \subseteq \mathcal{L}(H_X).$$

- The Roe algebra of  $X$ , denoted  $C^*(X)$ , is defined as the closure of  $C[X, H_X]$  under the operator norm.
- The uniform Roe algebra of  $X$ , denoted  $C_u^*(X)$ , is defined as the closure of  $C[X, l^2(X)]$  in the operator norm.
- The maximal Roe algebra of  $X$ , denoted  $C_{max}^*(X)$ , is defined as the closure of  $C[X, H_X]$  under the maximal norm.

Let us show that the definition of  $C^*(X)$  does not depend, up to non canonical  $*$ -isomorphism, on the choice of the s.n.d.  $X$ -module. Before doing so, we denote  $C^*(X, H_X)$  the completion of  $C[X, H_X]$  in order to keep track of the s.n.d.  $X$ -module.

**Proposition 2.2.20.** Let  $H_X$  and  $H'_X$  be two s.n.d.  $X$ -modules. Then there exists a non canonical  $*$ -isomorphism  $C^*(X, H_X) \cong C^*(X, H'_X)$  which is natural in  $K$ -theory.

**Proof.** Let  $H_X$  and  $H'_X$  be two s.n.d.  $X$ -modules and  $\varepsilon > 0$ . By proposition 2.2.17, there exists an isometry  $V \in \mathcal{L}(H_X, H'_X)$  with propagation less than  $\varepsilon$ . Hence  $Ad_V(T) = VTV^*$  defines a bounded linear map  $C_R[X, H_X] \rightarrow C_{R+2\varepsilon}[X, H'_X]$  which extends to a  $*$ -isomorphism  $Ad_V : C^*(X, H_X) \rightarrow C^*(X, H'_X)$ .

Let  $V_1$  and  $V_2$  two isometries in  $\mathcal{L}(H_X, H'_X)$  with propagation less than  $\varepsilon$ . Define  $P : H_X \rightarrow H_X \oplus H'_X; \xi \mapsto \xi \oplus 0$  and, for  $t \in [0, \frac{\pi}{2}]$ ,  $W_t : H_X \rightarrow H_X \oplus H'_X$  by

$$W_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}^* P.$$

Then  $Ad_{W_t}$  is a homotopy between  $Ad_{V_1}$  and  $Ad_{V_2}$ , so that

$$(Ad_{V_1})_* = (Ad_{V_2})_* : K_*(C^*(X, H'_X)) \rightarrow K_*(C^*(X, H'_X)).$$

□

Let us define the Roe algebra of  $X$  with coefficients in a  $C^*$ -algebra  $B$ . Define  $C_R[X, B]$  as the following subspace of  $\mathcal{L}_B(H \otimes l^2(X) \otimes B)$  :

$$C_R[X, B] = \{T \in \mathcal{L}_B(H \otimes l^2(X) \otimes B) \text{ locally compact s.t. } \text{supp } T \subseteq \Delta_R\}.$$

It is a subspace of  $\mathcal{L}_B(H \otimes l^2(X) \otimes B)$  which satisfies  $C_R[X, B] \cdot C_S[X, B] \subseteq C_{R+S}[X, B]$ . It is easy to see that

$$C[X, B] = \bigcup_{R>0} C_R[X, B]$$

is an involutive sub-algebra of  $\mathcal{L}_B(H \otimes l^2(X) \otimes B)$ .

**Definition 2.2.21.** Let  $B$  be a  $C^*$ -algebra. The Roe algebra of  $X$  with coefficients in  $B$  is the completion of  $C[X, B]$  under the operator norm of  $\mathcal{L}_B(H \otimes l^2(X) \otimes B)$ .

In the next section, we will define the coarse groupoid  $G(X)$ , which is an étale groupoid, and express  $C^*(X, B)$  as the reduced crossed-product of a well chosen  $G(X)$ -algebra.

## 2.3 Coarse Groupoid

Recall that  $\beta X$  denotes the Stone-Cech compactification of  $X$ . Let us first define the coarse groupoid of  $X$ . It is defined as the smallest topological groupoid  $G(X)$  with unit space  $\beta X$  extending the pair groupoid  $X \times X$  over  $X$ . It was defined by G. Skandalis, J.L. Tu and G. Yu in [47]. It allows to translate coarse properties of  $X$  into topological or dynamical properties of  $G(X)$ . Moreover, it turns out that Roe algebras with coefficients of  $X$  can be expressed as crossed products of  $G(X)$ , and, as we will see in another chapter, that the coarse Baum-Connes conjecture for  $X$  coincides with the Baum-Connes conjecture for this crossed-product.

For any entourage  $E$ , let  $\overline{E}$  denotes the closure of  $E$  in  $\beta(X \times X)$ . Define  $G(X) = \cup_{E \in \mathcal{E}_X} \overline{E}$ . By universal property of the Stone-Cech compactification, the first and second projections  $X \times X \rightarrow X$  extend to continuous maps  $G(X) \rightarrow \beta X$  denoted  $s$  and  $r$  respectively, as shown in this commutative diagramm :

$$\begin{array}{ccc} X \times X & \longrightarrow & X \xrightarrow{\iota_X} \beta X \\ \downarrow \iota_{X \times X} & & \nearrow \text{dotted} \\ \beta(X \times X) & & \end{array} .$$

The same remains true for the inverse map  $(x, y) \mapsto (y, x)$  and the unit map  $x \mapsto (x, x)$ . The following lemma defines the groupoid structure on  $G(X)$ . The reader can check [45] for a proof.

**Lemma 2.3.1.** [45] For any entourage, the map  $(s, r) : E \rightarrow X \times X$  extends to a topological embedding  $\overline{E} \hookrightarrow \beta X \times \beta X$ .

Using the lemma, we get a topological embedding  $G(X) \hookrightarrow \beta X \times \beta X$ , and the multiplication map is defined as conjugation by this topological embedding of the multiplication map on the pair groupoid  $\beta X \times \beta X$ .

If  $E$  is an entourage,  $E$  is contained in a finite union of entourages  $E_1 \cup \dots \cup E_n$  such that  $s : X \times X \rightarrow X$  and  $r : X \times X \rightarrow X$  are injective when restricted to each  $E_k$  [45]. We call such entourages **partial translations**. Every partial translation  $E$  acts as a partial bijection  $s(E) \rightarrow r(E)$  on  $X$  in the following way :  $E.x = r \circ (s|_E)^{-1}(x)$  if  $x \in s(E)$ . The composition of two partial translations remains a partial translation, and if  $E$  and  $E'$  are partial translations, then  $E'.(E.x) = (E' \circ E).x$  for all  $x \in s(E)$ . In other words, partial translations of  $X$  are a semigroup which acts on  $X$ .

**Proposition 2.3.2.** [47]  $G(X) \rightrightarrows \beta X$  is an étale groupoid such that  $G(X)|_X$  is the pair groupoid  $X \times X \rightrightarrows X$ .

**Example 2.3.3.** If  $\Gamma$  is a finitely generated group, then the action of  $\Gamma$  on  $|\Gamma|$  extend to  $\beta|\Gamma|$ . In [47], the reader can find a proof that  $G(|\Gamma|) \cong \beta|\Gamma| \rtimes \Gamma$ .

**Example 2.3.4.** Let  $\Gamma$  be a finitely generated group, residually finite w.r.t. a nested family  $\mathcal{N}$  of normal subgroups with trivial intersection, and  $X = X_{\mathcal{N}}(\Gamma)$ . Then the restriction  $G(X)|_{\partial\beta X}$  of the coarse groupoid is isomorphic to  $\partial\beta X \rtimes \Gamma$ . See [16] for a proof.

We compile in the next theorem a list of some known relations between topological properties of  $G(X)$  and coarse properties of  $X$ .

**Theorem 2.3.5.** Let  $X$  be a discrete metric space with bounded geometry. Then :

- $X$  has property  $A$  iff  $G(X)$  is amenable [47];
- $X$  admits a coarse embedding into Hilbert space iff  $G(X)$  is a-T-menable [47];
- $\partial\beta X$  is a  $G$ -invariant closed subset of  $\beta X$  hence  $G(X)|_{\partial\beta X}$  is a subgroupoid of  $G(X)$ . Moreover  $X$  admits a fibred coarse embedding into Hilbert space iff  $G(X)|_{\partial\beta X}$  is a-T-menable [15].

The remaining of the section is devoted to prove that the Roe algebra of  $X$  is isomorphic to a reduced cross-product of a well chosen  $G(X)$ -algebra by  $G(X)$ . For the reader's convenience, we will abbreviate  $G(X)$  as  $G$ .

For any  $C^*$ -algebra  $B$ , let  $\tilde{B}$  denotes the  $C^*$ -algebra  $l^\infty(X, B \otimes \mathfrak{K})$ . Multiplication by  $l^\infty(X) \cong C(\beta X)$  provides a  $C(\beta X)$ -structure on  $\tilde{B}$ . Recall that  $s^*\tilde{B} = C_0(G) \otimes_s \tilde{B}$  and  $r^*\tilde{B} = C_0(G) \otimes_r \tilde{B}$ . As  $C_0(G) \subseteq C(\beta(X \times X)) \cong l^\infty(X \times X)$ , these algebras are generated by elementary tensors  $\chi_E \otimes_s f$  and  $\chi_E \otimes_r f$  where  $f \in \tilde{B}$  and  $E$  is a partial translation. Let us first define the action of  $G$  on  $\tilde{B}$ , which is an isomorphism of  $C(\beta X)$ -algebras  $V : s^*\tilde{B} \rightarrow r^*\tilde{B}$ .

Suppose  $E$  is a partial translation and  $f \in \tilde{B}$ . Define

$$E.f(x) = \begin{cases} f(E^{-1}.x) & \text{if } x \in r(E) \\ 0 & \text{otherwise.} \end{cases}$$

Of course  $E(E'.f) = (E \circ E').f$ . Define the action on elementary tensors as follows :

$$V(\chi_E \otimes_s f) = \chi_E \otimes_r (E.f).$$

**Lemma 2.3.6.**  $V$  defines an action of  $G$  on  $\tilde{B}$ .

**Proof.** Let us show that  $V$  extends to a isomorphism of  $C(\beta X)$ -algebras from  $s^*\tilde{B}$  to  $r^*\tilde{B}$ .

$G$  is totally disconnected, and the set of the closure in  $\beta(X \times X)$  of controlled subsets is a basis for the topology. Hence  $C_0(G)$  is the closure of the algebra of locally constant functions with finite value set. We can suppose these functions are of the form  $f = \sum_{j=1}^k \lambda_j \chi_{E_j}$ , the  $E_j$  being mutually disjoint partial translations, and  $\lambda_j \in \mathbb{C}$ . The  $C(\beta X)$ -algebras  $s^*\tilde{B} = C_0(G) \otimes_s \tilde{B}$  and  $r^*\tilde{B} = C_0(G) \otimes_r \tilde{B}$  are similarly generated by functions of the form  $h = \sum_{j=1}^k \lambda_j \chi_{E_j} \otimes f_j$  with the  $E_j$  and  $\lambda_j$  as before, and  $f_j \in \tilde{B}$ . Let  $h$  be such a function. Then the expression  $\sum_{j=1}^k \lambda_j \chi_{E_j} \otimes_r (E_j \cdot f_j)$  does not depend on the decomposition. Indeed, if  $E_1$  and  $E_2$  are partial translations such that  $E_1 \subseteq E_2$ , then  $\chi_{E_1} \otimes_r E_2 \cdot f = \chi_{E_1} \otimes_r E_1 \cdot f$ . Consider two decompositions  $\sum_i \alpha_i \chi_{E_i} \otimes f_i = \sum_j \beta_j \chi_{F_j} \otimes g_j$ . Notice that  $\alpha_i \chi_{E_i \cap F_j} \otimes f_i = \beta_j \chi_{E_i \cap F_j} \otimes g_j$ . Then :

$$\begin{aligned} \sum_i \alpha_i \chi_{E_i} \otimes (E_i \cdot f_i) &= \sum_{i,j} \alpha_j \chi_{E_i \cap F_j} \otimes (E_i \cdot f_i) \\ &= \sum_{i,j} \alpha_j \chi_{E_i \cap F_j} \otimes (E_i \cap F_j) \cdot f_i \\ &= \sum_{i,j} \beta_j \chi_{E_i \cap F_j} \otimes (E_i \cap F_j) \cdot g_j \\ &= \sum_j \beta_j \chi_{F_j} \otimes (F_j \cdot g_j) \end{aligned}$$

Moreover :

$$\begin{aligned} \|\sum_j \lambda_j \chi_{E_j} \otimes (E_j \cdot f)\| &\leq \sum_j |\lambda_j| \|f_j\| \\ &\leq \sup\{\sum_j |\lambda'_j| \times \|f'_j\| \text{ s.t. } \sum \lambda'_j \chi_{E'_j} \otimes f'_j = h\}, \\ &\leq \|h\|, \end{aligned}$$

so  $V$  extends by continuity to a continuous linear map  $C_0(G) \otimes \tilde{B} \rightarrow r^*\tilde{B}$ . This map being  $C(\beta X)$ -linear,  $V$  extends to a  $C(\beta X)$ -algebra morphism  $V : s^*\tilde{B} \rightarrow r^*\tilde{B}$ . The rule  $\chi_E \otimes_r f \mapsto \chi_E \otimes_r (E^{-1} \cdot f)$  defines similarly a  $C(\beta X)$ -algebra morphism between  $r^*\tilde{B} \rightarrow s^*\tilde{B}$ . It is the inverse of  $V$ , hence  $V$  is a isomorphism of  $C(\beta X)$ -algebras.

Let us show  $V$  is an action. For  $g \in G$ , denote by  $V_g : (\tilde{B})_{s(g)} \rightarrow (\tilde{B})_{r(g)}$  the induced map between the fibers of  $s^*\tilde{B}$  and  $r^*\tilde{B}$ . Let  $(g, g') \in G^{(2)}$ , and choose partial translations  $E, E' \subseteq X \times X$  such that  $g \in \overline{E}$  and  $g' \in \overline{E'}$ . Then  $g' \circ g \in \overline{E' \circ E}$  and  $E' \circ E$  is a partial translation. Then if  $f \in s^*\tilde{B}$ ,  $V_g(f_{s(g)}) = (E \cdot f)_{r(g)}$ . Then :

$$\begin{aligned} V_{g'g}(f_{s(g)}) &= ((E' \circ E) \cdot f)_{r(g')} \\ &= V_{g'}((E \cdot f)_{s(g')}) \\ &= V_{g'}((E \cdot f)_{r(g)}) \\ &= V_{g'}V_g(f_{s(g)}) \end{aligned}$$

so that  $V_{g'g} = V_{g'}V_g$ , and  $V$  defines an action of  $G$  on  $\tilde{B}$ . □

**Theorem 2.3.7.** Let  $X$  be a discrete metric space with bounded geometry, and  $B$  a  $C^*$ -algebra. There exists a natural isomorphism

$$\Psi_B : l^\infty(X, B \otimes \mathfrak{K}) \rtimes_r G(X) \xrightarrow{\cong} C^*(X, B) .$$

**Proof.** The proof is a detailed version of the proof of [47]. The idea is to represent faithfully the two  $C^*$ -algebras on the same Hilbert module, and to show that they are equal



in this representation.

By definition,  $\tilde{B} \rtimes_r G$  is faithfully represented on  $L^2(G, \tilde{B})$ . Consider the  $G$ -invariant ideal  $J = C_0(X, B \otimes \mathfrak{K})$  and the Hilbert  $\tilde{B}$ -module  $L^2(G, J) = L^2(G, \tilde{B}) \otimes J$ . The  $*$ -homomorphism  $\mathcal{L}_{\tilde{B}}(L^2(G, \tilde{B})) \rightarrow \mathcal{L}_{\tilde{B}}(L^2(G, J)) ; T \mapsto T \otimes 1$  is isometric, so we obtain a faithful representation  $\tilde{B} \rtimes_r G \rightarrow \mathcal{L}_{\tilde{B}}(L^2(G, J))$ .

But  $L^2(G, J) \cong C_0(X) \otimes L^2(G) \otimes B \otimes \mathfrak{K}$ , where the  $G$  action is diagonal, i.e. trivial on  $C_0(X)$ . Hence  $C_0(X) \otimes l^2(X) \otimes B \otimes \mathfrak{K}$  is a submodule of  $L^2(G, J)$ . Any  $T \in C_R[X, B]$  is canonically a multiplier in  $\mathcal{M}(B \otimes \mathfrak{K}(H \otimes l^2(X))) \cong \mathcal{L}_{B \otimes \mathfrak{K}}(l^2(X) \otimes B \otimes \mathfrak{K})$ , hence  $C^*(X, B)$  is faithfully represented on  $L^2(G, J)$ , acting trivially on the  $C_0(X)$  factor.

Recall that  $\tilde{B} \rtimes_r G$  is generated by elementary tensors  $\chi_E \otimes f$ , where  $E$  is a partial translation and  $f \in \tilde{B}$ . To such a  $\chi_E \otimes f$ , associate the following operator  $T$  on  $l^2(X) \otimes H \otimes B$  :

$$T_{xy} = \begin{cases} f(x) & \text{if } (x, y) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Then, if  $R = \sup_E d$ ,  $T \in C_R[X, B]$ , and this defines a  $*$ -map  $C_c(G, \tilde{B}) \rightarrow \cup_{R>0} C_R[X, B]$ . Let  $T \in C_R[X, B]$  such that  $\text{supp}(T) \subseteq E$  where  $E$  is a partial translation. Then  $f(x) = T_{x, Ex} \in B \otimes \mathfrak{K}$  defines an element  $f$  in  $\tilde{B}$ , and  $T \mapsto \chi_E \otimes f$  defines an inverse for the previous map.

The images of  $\cup_{R>0} C_R[X, B]$  and  $C_c(G, \tilde{B})$  coincide under the faithful representations described above, so that  $\chi_E \otimes f \rightarrow T$  extends to an isomorphism  $\Psi_B$ .

If  $\phi : A \rightarrow B$  is a  $*$ -homomorphism, it induces  $\tilde{\phi} : \tilde{A} \rightarrow \tilde{B}$  and  $\phi_X : C^*(X, A) \rightarrow C^*(X, B)$ , see the theorem 4.1.12. Naturality of  $\Psi$  follows then from functoriality of the Hilbert module tensor product and functoriality of  $l^\infty(X, - \otimes \mathfrak{K})$  and  $C^*(X, -)$ . □

**Remark 2.3.8.** A minor modification of this proof yields that  $C_r^*(G(X)) \cong C_u^*(X)$ . Let  $\Gamma$  be a finitely generated group. Then  $C_u^*(|\Gamma|) \cong l^\infty(\Gamma) \rtimes_r \Gamma$ .

## 2.4 Discrete and Compact Quantum groups

We recall in this section basic facts about the theory of compact quantum groups. This theory was developed by Woronowicz in the 1980's. The reader can find more details in [58]. Our goal is to present reduced cross products by discrete quantum group. We define discrete quantum groups as duals of compact quantum groups. The ultimate goal of this section is to notice that reduced cross products by discrete quantum group are filtered, in a sense to be defined later, by a subset of the set of equivalence classes of unitary representations of the compact quantum group.

Our presentation is strongly inspired by the article [53]. We recall that  $\otimes$  denotes the minimal or spatial tensor product of  $C^*$ -algebras. If  $x \in A \otimes A$ , we will use the notation  $a_{ij}$  for the obvious element in  $A \otimes A \otimes A$ , e.g.  $a_{12} = a \otimes 1$ . For every Hilbert space  $H$  and every  $\xi, \eta \in H$ ,  $\omega_{\xi, \eta}$  denotes the linear form on  $\mathcal{L}(H)$  defined by  $\omega_{\xi, \eta}(T) = \langle T\eta, \xi \rangle$ .

**Definition 2.4.1.** A compact quantum group is a pair  $\mathbb{G} = (C(\mathbb{G}), \Delta)$  where  $C(\mathbb{G})$  is a unital  $C^*$ -algebra and  $\Delta : C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$  is a unital  $*$ -homomorphism satisfying :

- the coassociativity condition, i.e.  $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$  ;
- the left and right cancellation property, i.e. the linear spans of  $\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))$  and  $\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)$  are dense in  $C(\mathbb{G}) \otimes C(\mathbb{G})$ .

**Example 2.4.2.** Let  $G$  be a compact group, and define  $\Delta : C(G) \rightarrow C(G) \otimes C(G)$  by  $(\Delta f)(g, g') = f(gg')$ , where  $C(G) \otimes C(G)$  is identified with  $C(G \times G)$ . Then  $(C(G), \Delta)$  is a compact quantum group.

The notation  $C(\mathbb{G})$  was adopted to suggest an analogy with this example. In general though, the  $C^*$ -algebra is noncommutative, and there is no topological space underlying the quantum group. Still, one often refers to  $C(\mathbb{G})$  as "the algebra of continuous functions" on  $\mathbb{G}$ .

**Example 2.4.3.** Let  $\Gamma$  be a discrete group and define the reduced  $C^*$ -algebra  $C_r^*(\Gamma)$  of  $\Gamma$ . Recall that it is the  $C^*$ -algebra generated in  $\mathcal{L}(l^2(\Gamma))$  by the left translation operators  $\lambda_\gamma$ , for every  $\gamma \in \Gamma$ . The map  $\lambda_\gamma \mapsto \lambda_\gamma \otimes \lambda_\gamma$  extends to a unital  $*$ -homomorphism  $\Delta : C_r^*(\Gamma) \rightarrow C_r^*(\Gamma) \otimes C_r^*(\Gamma)$ . Then  $(C_r^*(\Gamma), \Delta)$  is a compact quantum group. It can be seen as a candidate for what should be the continuous functions on the reduced dual  $\hat{\Gamma}$ . By Fourier transform, it is exactly the case when  $\Gamma$  is commutative.

**Definition 2.4.4.** A unitary representation  $(H, U)$  of  $\mathbb{G}$  on a Hilbert space  $H$  is a unitary element  $U \in \mathcal{M}(\mathfrak{K}(H) \otimes C(\mathbb{G}))$  such that  $(id \otimes \Delta)(U) = U_{12}U_{13}$ . Let  $(H, U)$  and  $(H', U')$  be two unitary representations of  $\mathbb{G}$ . The space of intertwiners between  $U$  and  $U'$  is defined as

$$Hom(U, U') = \{T \in \mathcal{L}(H', H) \text{ s.t. } U(T \otimes 1) = (T \otimes 1)U'\}.$$

We denote by  $End(U)$  the intertwiners of  $Hom(U, U)$ . A unitary representation is called irreducible if  $End(U) = \mathbb{C}1$ . The trivial representation is denoted by  $\varepsilon$ . If  $(H, U)$  is a unitary representation of  $\mathbb{G}$ , and  $\eta, \xi \in H$ , then  $U_{\xi, \eta} = (\omega_{\xi, \eta} \otimes 1)(U)$  is called a coefficient of  $U$ .

**Example 2.4.5.** Let  $\Gamma$  be a discrete group and  $\mathbb{G} = (C_r^*(\Gamma), \Delta)$  be the corresponding compact quantum group. Then a finite dimensional unitary representation  $U \in$

$\mathcal{M}(\mathfrak{K}(H) \otimes C(\mathbb{G})) \cong \mathfrak{M}_n(C_r^*(\Gamma))$  of  $\mathbb{G}$  is given by a family of orthogonal projections  $P_\gamma \in \mathfrak{M}_n(\mathbb{C})$  for every  $\gamma \in \Gamma$  such that

$$P_\gamma P_{\gamma'} = \delta_{\gamma, \gamma'} P_\gamma \quad \text{and} \quad \sum_{\gamma \in \Gamma} P_\gamma = I_n.$$

We define sum and tensor product of representations as follows. Let  $(H, U)$  and  $(H', U')$  be two unitary representations. Then the sum is defined as  $(H \oplus H', U \oplus U')$ . The tensor product is defined as  $(H \otimes H', W)$  where  $W = U_{13}U'_{23} \in \mathcal{M}(\mathfrak{K}(H \otimes H') \otimes C(\mathbb{G}))$ .

**Definition 2.4.6.** As in the case of compact groups, we define an equivalence relation on unitary representations of  $\mathbb{G}$  by

$$(H, U) \sim (H', U') \text{ iff } \text{Hom}(U, U') \text{ contains a non trivial unitary operator.}$$

The set of equivalence classes of irreducible unitary representations is denoted by  $\text{Irrep}(\mathbb{G})$ . We choose for each  $\pi \in \text{Irrep}(\mathbb{G})$  a unitary representation  $(H_\pi, U^\pi)$ .

The following theorem is crucial, and is a generalization of the classical case.

**Theorem 2.4.7** ([53], section 1). Every irreducible representation of a compact quantum group is finite dimensional. Every unitary representation of a compact quantum group is equivalent to a direct sum of irreducible unitary representations.

We also have the analog of the Haar measure, and of the regular representations.

**Proposition 2.4.8** ([53], section 1). For every compact quantum group  $\mathbb{G}$ , there exists a unique state  $h$  on  $C(\mathbb{G})$ , called the Haar state on  $\mathbb{G}$ , satisfying  $(id \otimes h)\Delta(a) = h(a)1 = (h \otimes id)\Delta(a)$  for every  $a \in C(\mathbb{G})$ .

Let us denote by  $L^2(\mathbb{G})$  the Hilbert space obtained as the GNS construction with respect to the Haar state, i.e.  $L^2(\mathbb{G}) = L^2(C(\mathbb{G}), h)$ . We denote by  $\xi_h \in L^2(\mathbb{G})$  and  $\pi_h : C(\mathbb{G}) \rightarrow \mathcal{L}(L^2(\mathbb{G}))$  the corresponding cyclic vector and representation. According to [58], there exists a unique unitary  $W \in \mathcal{L}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$  satisfying

$$W^*(\xi \otimes \pi_h(a)\xi_h) = (\pi_h \otimes \pi_h) \circ \Delta(a)(\xi \otimes \xi_h) \quad \forall \xi \in L^2(\mathbb{G}), a \in C(\mathbb{G}).$$

**Definition 2.4.9.** The reduced form  $C_r^*(\mathbb{G})$  of  $\mathbb{G}$  is defined as the  $C^*$ -algebra  $\pi_h(C(\mathbb{G}))$ . The unitary  $W$  defines an element of  $W \in \mathcal{M}(C_r^*(\mathbb{G}) \otimes \mathfrak{K}(L^2(\mathbb{G})))$  which is a unitary representation called the regular representation of  $\mathbb{G}$ .

According to [58] (section 6), for every  $\pi \in \text{Irrep}(\mathbb{G})$ , there exists a unique  $\pi^* \in \text{Irrep}(\mathbb{G})$  such that  $\text{Hom}(\pi \otimes \pi^*, \varepsilon) \neq 0 \neq \text{Hom}(\varepsilon, \pi \otimes \pi^*)$ . For a representative  $(H_\pi, U^\pi)$ ,  $(H_{\pi^*}, U^{\pi^*})$  is called the contragredient representation.

We now turn to the case of discrete quantum groups. We will define a discrete quantum group as the dual of a compact quantum group. An intrinsic definition (without reference to a given compact quantum group) exists and can be found in [54].

Define the following  $C^*$ -algebras :

$$\begin{aligned} c_0(\hat{\mathbb{G}}) &= \bigoplus_{\pi \in \text{Irred}(\mathbb{G})} \mathcal{L}(H_\pi), \\ l^\infty(\hat{\mathbb{G}}) &= \prod_{\pi \in \text{Irred}(\mathbb{G})} \mathcal{L}(H_\pi). \end{aligned}$$

Up to our choice of representations  $\{(H_\pi, U^\pi)\}_{\pi \in \text{Irrep}(\mathbb{G})}$ , we can define a unitary  $U = \bigoplus_{\pi \in \text{Irrep}(\mathbb{G})} U^\pi \in \mathcal{M}(c_0(\hat{\mathbb{G}}) \otimes C(\mathbb{G}))$  which is a unitary representation of  $\mathbb{G}$ .

**Proposition 2.4.10** ([42], Theorem 2.1). There exists a unique  $*$ -homomorphism  $\hat{\Delta} : c_0(\hat{\mathbb{G}}) \rightarrow \mathcal{M}(c_0(\hat{\mathbb{G}}) \otimes c_0(\hat{\mathbb{G}}))$  such that

$$(\hat{\Delta} \otimes id)U = U_{13}U_{23}.$$

Moreover,  $\hat{\Delta}$  is coassociative, i.e.  $(\hat{\Delta} \otimes id)\hat{\Delta} = (id \otimes \hat{\Delta})\hat{\Delta}$ .

**Definition 2.4.11.** Let  $\mathbb{G}$  be a compact quantum group. With the above notations, its dual is defined as the discrete quantum group  $\hat{\mathbb{G}} = (c_0(\hat{\mathbb{G}}), \hat{\Delta})$ .

**Example 2.4.12.** Let  $\Gamma$  be a discrete group and consider the compact quantum group  $\mathbb{G} = (C^*(\Gamma), \Delta)$ . Then,  $\gamma \mapsto \lambda_\gamma$  gives a bijection  $Irrep(\mathbb{G}) \cong \Gamma$ , and  $\hat{\mathbb{G}}$  is  $c_0(\Gamma)$  with comultiplication  $\hat{\Delta}(f)(\gamma, \gamma') = f(\gamma\gamma')$  for every  $f \in c_0(\Gamma)$  and  $\gamma, \gamma' \in \Gamma$ .

The dual of a compact quantum group is also represented on  $L^2(\mathbb{G})$  via the regular representations.

**Proposition 2.4.13.** For every compact quantum group  $\mathbb{G}$ , there exists a unique state  $h$  on  $c_0(\hat{\mathbb{G}})$ , called the left Haar state on  $\hat{\mathbb{G}}$ , satisfying  $(id \otimes \hat{h})\hat{\Delta}(a) = \hat{h}(a)1$  for every  $a \in c_0(\hat{\mathbb{G}})$ . The GNS Hilbert space is canonically isomorphic to  $L^2(\mathbb{G})$ . We will denote by  $\hat{\pi}_h : c_0(\hat{\mathbb{G}}) \rightarrow \mathcal{L}(L^2(\mathbb{G}))$  and  $\hat{\xi}_h$  the corresponding representation and cyclic vector respectively.

**Definition 2.4.14.** An action of a discrete quantum group  $\hat{\mathbb{G}}$  on a  $C^*$ -algebra  $A$  is a nondegenerate  $*$ -homomorphism  $\alpha : A \rightarrow \mathcal{M}(A \otimes c_0(\hat{\mathbb{G}}))$  such that :

- $(\alpha \otimes id)\alpha = (id \otimes \hat{\Delta})\alpha$
- for every  $a \in A$ ,  $(\hat{\varepsilon} \otimes id)\alpha(a) = a$ .

Such a pair  $(A, \alpha)$  is called a  $\hat{\mathbb{G}}$ -algebra.

Let  $\alpha : A \rightarrow \mathcal{M}(A \otimes c_0(\hat{\mathbb{G}}))$  be an action of  $\hat{\mathbb{G}}$ , and  $\pi \in Irrep(\mathbb{G})$  a unitary representation of  $\mathbb{G}$ . Let  $p_\pi \in c_0(\hat{\mathbb{G}})$  be the orthogonal projection on  $H_\pi$ . We define the coefficients of  $\alpha$  as the elements  $\alpha_{\xi, \eta}^\pi(a) = (id_A \otimes \omega_{\xi, \eta})(\alpha(a)(id_A \otimes p_\pi)) \in A$  for any  $\xi, \eta \in H_\pi$ .

Let  $(A, \alpha)$  be a  $\hat{\mathbb{G}}$ -algebra. Define  $W = id_A \otimes (id \otimes \pi_h)(U) \in \mathcal{M}(A \otimes c_0(\hat{\mathbb{G}}) \otimes C_r^*(\mathbb{G}))$ . Let  $\pi \in Irrep(\mathbb{G})$ , and  $\xi, \eta \in H_\pi$ . Then the coefficient  $W_{\xi, \eta}^\pi := (id \otimes \omega_{\xi, \eta} \otimes id_{C_r^*(\mathbb{G})})(W)$  is an element of  $\mathcal{M}(A \otimes C_r^*(\mathbb{G})) \subseteq \mathcal{L}_A(A \otimes L^2(\mathbb{G}))$ .

**Definition 2.4.15.** Let  $(A, \alpha)$  be a  $\hat{\mathbb{G}}$ -algebra. The reduced crossed product  $A \rtimes_r \hat{\mathbb{G}}$  is the  $C^*$ -subalgebra of  $\mathcal{L}_A(A \otimes L^2(\mathbb{G}))$  generated by

$$\{ \theta(a)W_{\xi, \eta}^\pi, a \in A, \pi \in Irrep(\mathbb{G}), \xi, \eta \in H_\pi \},$$

where  $\theta = (id \otimes \hat{\pi}_h)\alpha$ .

**Remark 2.4.16.** There is also a notion of maximal (or full) crossed product for  $\hat{\mathbb{G}}$ -algebras.

**Remark 2.4.17.** As in the classical case, the reduced and the maximal crossed products respect the semi-split exact sequences of  $\hat{\mathbb{G}}$ -algebras.

Denote by  $\mathcal{E}_{\mathbb{G}}$  the set of finite dimensional unitary representations  $\pi$  of  $\mathbb{G}$  such that  $\pi$  and  $\pi^*$  are unitarily equivalent. We endow  $\mathcal{E}_{\mathbb{G}}$  with a semigroup structure with the following product

$$\pi \circ \pi' = (\pi \otimes \pi') \oplus (\pi' \otimes \pi) \quad \forall \pi, \pi' \in \mathcal{E}_{\mathbb{G}}.$$

For every  $\pi \in \mathcal{E}_{\mathbb{G}}$ , denote by  $C_{\pi}(\hat{\mathbb{G}}, A)$  the closure of the linear spans of

$$\{ \theta(a)W_{\xi, \eta}^{\pi}, a \in A, \xi, \eta \in H_{\pi} \}.$$

**Proposition 2.4.18.** For every  $\hat{\mathbb{G}}$ -algebra  $A$ ,  $A \rtimes_r \hat{\mathbb{G}}$  is the closure of  $\cup_{\pi \in \mathcal{E}_{\mathbb{G}}} C_{\pi}(\hat{\mathbb{G}}, A)$ . Moreover,

$$C_{\pi}(\hat{\mathbb{G}}, A).C_{\pi'}(\hat{\mathbb{G}}, A) \subseteq C_{\pi \circ \pi'}(\hat{\mathbb{G}}, A).$$

**Proof.** Let  $\{\xi_1, \dots, \xi_{\dim \pi}\}$  be an orthonormal basis of  $H_{\pi}$  for every  $\pi \in \mathcal{E}_{\mathbb{G}}$ . Then the coefficients  $W_{\xi, \eta}^{\pi}$  are uniquely determined by the coefficients  $W_{i, j}^{\pi} := W_{\xi_i, \xi_j}^{\pi}$ . Moreover, the following relation holds

$$W_{i, j}^{\pi} \theta(a) = \sum_k \theta(\alpha_{i, k}^{\pi}(a)) W_{k, j}^{\pi} \quad \forall a \in A,$$

hence, for  $\pi$  and  $\pi'$  in  $\mathcal{E}_{\mathbb{G}}$ , and every  $a, a' \in A$ ,

$$\theta(a)W_{i, j}^{\pi} \theta(a')W_{p, q}^{\pi'} = \sum_k \theta(a \alpha_{i, k}^{\pi}(a')) W_{k, j}^{\pi} W_{p, q}^{\pi'} = \sum_k \theta(a \alpha_{i, k}^{\pi \otimes \pi'}(a')) W_{kp, jq}^{\pi \otimes \pi'},$$

where, if  $\{\xi_k\}$  and  $\{\xi'_p\}$  are orthonormal basis of  $H_{\pi}$  and  $H_{\pi'}$  respectively,  $W_{kp, jq}^{\pi \otimes \pi'}$  denotes the coefficient  $W_{\xi_k \otimes \xi'_p, \xi_j \otimes \xi'_q}^{\pi \otimes \pi'}$ . □

As in the case of locally compact groups or locally compact  $\sigma$ -compact groupoids with Haar system, equivariant  $KK$ -theory was developed by S. Baaq and G. Skandalis in [1] and R. Vergnioux in [55]. Details are beyond the scope of this thesis, and we refer to [55] for details.

**Proposition 2.4.19** ([55]). Let  $A$  and  $B$  be two  $\hat{\mathbb{G}}$ -algebras. Then there exists a  $\mathbb{Z}_2$ -graded abelian group  $KK^{\hat{\mathbb{G}}}(A, B)$  such that :

- every  $\hat{\mathbb{G}}$ -equivariant  $*$ -homomorphism  $\phi : A \rightarrow B$  defines a class  $[\phi] \in KK^{\hat{\mathbb{G}}}(A, B)$ ,
- $KK^{\hat{\mathbb{G}}}(A, B)$  is a contravariant functor in the  $A$  variable w.r.t.  $\hat{\mathbb{G}}$ -equivariant  $*$ -homomorphisms,
- $KK^{\hat{\mathbb{G}}}(A, B)$  is a covariant functor in the  $B$  variable w.r.t.  $\hat{\mathbb{G}}$ -equivariant  $*$ -homomorphisms,
- for every  $\hat{\mathbb{G}}$ -algebra  $D$ , and if  $A$  is separable, there exists a bilinear map called the Kasparov product

$$\otimes_D : KK^{\hat{\mathbb{G}}}(A, D) \times KK^{\hat{\mathbb{G}}}(D, B) \rightarrow KK^{\hat{\mathbb{G}}}(A, B)$$

such that

$$[\phi] \otimes_D z' = \phi^*(z') \text{ and } z \otimes_D [\phi'] = \phi'_*(z)$$

for every  $z \in KK^{\hat{\mathbb{G}}}(A, D)$ ,  $z' \in KK^{\hat{\mathbb{G}}}(D, B)$ , and every  $\hat{\mathbb{G}}$ -equivariant  $*$ -homomorphisms  $\phi : A \rightarrow D$  and  $\phi' : D \rightarrow B$ .

Moreover, one can construct a descent morphism

$$j_{\hat{\mathbb{G}}} : KK^{\hat{\mathbb{G}}}(A, B) \rightarrow KK(A \rtimes_r \hat{\mathbb{G}}, B \rtimes_r \hat{\mathbb{G}})$$

compatible with Kasparov products, i.e.

$$j_{\hat{\mathbb{G}}}(z \otimes_D z') = j_{\hat{\mathbb{G}}}(z) \otimes_{D \rtimes \hat{\mathbb{G}}} j_{\hat{\mathbb{G}}}(z') \quad \forall z \in KK^{\hat{\mathbb{G}}}(A, D), z' \in KK^{\hat{\mathbb{G}}}(D, B)$$

Notice that every  $KK^{\hat{\mathbb{G}}}$ -element satisfies property (d) for some universal  $d > 0$  independent of  $\mathbb{G}$  as well (see the definition 2.1.34). Indeed, the proof of [28] can be carried out without modification in the setting of discrete quantum groups.

## Part II

# Controlled $K$ -theory and assembly maps





# Chapter 3

## Controlled $K$ -theory

### 3.1 $K$ -theory controlled by a coarse structure

In this section, we define controlled  $K$ -theory in more generality than the functor defined in [40] and in the previous section. The goal is to develop controlled  $K$ -theory to a broader setting than what is used until now. Notably, we give a definition that allows to state that the controlled  $K$ -theory is a genuine functor between appropriately chosen categories. The price to pay is a little more generality, and functorial definitions, but the reward is a larger range of applications of the theory. We will show how it is possible to apply controlled-lingo to compact quantum groups in the sense of Woronowicz, foliations groupoids and twisted representations of locally compact groups.

**Definition 3.1.1.** A coarse structure  $\mathcal{E}$  is a lattice which is an abelian semi-group. Recall that a lattice is a poset for which every pair  $(E, E')$  admits a supremum  $E \vee E'$  and an infimum  $E \wedge E'$ .

**Definition 3.1.2.** A  $C^*$ -algebra  $A$  is said to be  $\mathcal{E}$ -filtered if there exists a coarse structure  $\mathcal{E}$  and, for every  $E \in \mathcal{E}$ , linear subspaces  $A_E$  of  $A$  such that :

- if  $E \leq E'$ , then  $A_E \subseteq A_{E'}$ , and the inclusion  $\phi_E^{E'} : A_E \hookrightarrow A_{E'}$  induces an inductive system of linear spaces,
- $A_E$  is stable by involution,
- for all  $E, E' \in \mathcal{E}$ ,  $A_E \cdot A_{E'} \subseteq A_{EE'}$ ,
- the union of subspaces is dense in  $A$ , i.e.  $\overline{\cup_{E \in \mathcal{E}} A_E} = \varinjlim A_E = A$ .
- if  $A$  is unital, we impose that  $1 \in A_E, \forall E \in \mathcal{E}$ .

If  $A$  is a non-unital filtered  $C^*$ -algebra, we will by default endow it with its unitalization  $\tilde{A}$  with the filtration  $\tilde{A}_E = A_E + \mathbb{C}$ . A  $*$ -homomorphism  $\phi : A \rightarrow B$  is said to be filtered if  $\phi(A_E) \subseteq B_E$  for all  $E \in \mathcal{E}$ .

If  $(A, \mathcal{E})$  and  $(B, \mathcal{E}')$  are two filtered  $C^*$ -algebras, a filtered morphism  $(\phi, \rho) : (A, \mathcal{E}) \rightarrow (B, \mathcal{E}')$  is :

- a non-decreasing map  $\rho : \mathcal{E} \rightarrow \mathcal{E}'$ ,

- a  $*$ -homomorphism  $\phi : A \rightarrow B$  such that  $\phi(A_E) \subseteq B_{\rho(E)}$  for all  $E \in \mathcal{E}$ .

A crucial example for us will be crossed products of  $G$ -algebras by an étale groupoid  $G$ . Note that this definition generalizes that of [40]. Indeed, as will be recalled later, the Roe algebras can be expressed as a crossed product by the so-called coarse groupoid, which is étale [47], and the definition given here, applied to this groupoid, gives back the filtration in the sense of [40]. The second example will be that of crossed products by compact quantum groups of Woronowicz.[58]

### 3.1.1 Almost unitaries and almost projections

**Definition 3.1.3.** Let  $(A, \mathcal{E})$  be a unital filtered  $C^*$ -algebra. Let  $\varepsilon \in (0, \frac{1}{4})$  and  $E \in \mathcal{E}$  a controlled subset. The set of  $\varepsilon$ - $E$ -unitaries is the set

$$U^{\varepsilon, E}(A) = \{u \in A_E \text{ s.t. } \|u^*u - 1\| < \varepsilon \text{ and } \|uu^* - 1\| < \varepsilon\}$$

and the set  $\varepsilon$ - $E$ -projections is the set

$$P^{\varepsilon, E}(A) = \{p \in A_E \text{ s.t. } p = p^* \text{ and } \|p^2 - p\| < \varepsilon\}.$$

We will use the notation  $P_n^{\varepsilon, E}(A)$  for  $P^{\varepsilon, E}(M_n(A))$ , and  $U_n^{\varepsilon, E}(A)$  for  $U^{\varepsilon, E}(M_n(A))$ . Also,  $P_\infty^{\varepsilon, E}(A)$  is the algebraic inductive limit of the  $P_n^{\varepsilon, E}(A)$  under the natural inclusions

$$\left\{ \begin{array}{l} P_n^{\varepsilon, E}(A) \rightarrow P_{n+1}^{\varepsilon, E}(A) \\ p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \end{array} \right.$$

and  $U_\infty^{\varepsilon, E}(A)$  is the algebraic inductive limit of the  $U_n^{\varepsilon, E}(A)$  under the natural inclusions

$$\left\{ \begin{array}{l} U_n^{\varepsilon, E}(A) \rightarrow U_{n+1}^{\varepsilon, E}(A) \\ u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right. .$$

**Remark 3.1.4.** Let  $\varepsilon \in (0, \frac{1}{4})$  and  $E \in \mathcal{E}$ .

- If  $p \in P^{\varepsilon, E}(A)$ , then  $p$  has a spectral gap around  $\frac{1}{2}$ , and functional calculus allows one to define a genuine projection  $\kappa_0(p)$ , as in [40], by taking  $\kappa_0$  to be a continuous function that vanishes inside the spectral gap and that is respectively 0 and 1 on the left and right part of the spectrum of  $p$ . As this projection does not depend on  $\kappa_0$ , we will always denote it the same, even if the function depends on  $p$ .
- If  $u \in U^{\varepsilon, E}(A)$ , then  $u^*u$  is invertible, and  $u(u^*u)^{-1}$  defines a unitary, that we will denote  $\kappa_1(u)$ .

In order to define controlled  $K$ -groups, define the following equivalence relations on  $P_\infty^{\varepsilon, E}(A) \times \mathbb{N}$  and  $U_n^{\varepsilon, E}(A)$ .

- $(p, l) \sim (q, l')$  if there exists a homotopy of almost projections  $h \in P_\infty^{\varepsilon, E}(A[0, 1])$  and an integer  $k$  such that

$$h(0) = \begin{pmatrix} p & 0 \\ 0 & 1_{k+l'} \end{pmatrix} \text{ and } h(1) = \begin{pmatrix} q & 0 \\ 0 & 1_{k+l} \end{pmatrix}$$

- $u \sim v$  if there exists a homotopy of almost unitaries  $h \in U_\infty^{3\varepsilon, E \circ E}(A[0, 1])$  and an integer  $k$  such that  $h(0) = u$  and  $h(1) = v$ .

The following fact will be useful for future purposes. The reader can look at Proposition 1.30 of [39] for references. Recall the following definition.

**Definition 3.1.5.** Let  $C > 0$  and  $A$  be a  $C^*$ -algebra. A map  $h : [0, 1] \rightarrow A$  is called  $C$ -Lipschitz if  $\|h(s) - h(t)\| \leq C|s - t|$  for all  $s, t \in [0, 1]$ .

**Proposition 3.1.6.** There exists a universal constant  $L > 0$  such that, for any unital filtered  $C^*$ -algebra  $(A, \mathcal{E})$ , any  $\varepsilon \in (0, \frac{1}{4})$  and any  $E \in \mathcal{E}$ , if  $u_0$  and  $u_1$  are homotopic in  $U_n^{\varepsilon, E}(A)$ , then there exists an integer  $k$  and a  $L$ -lipschitz homotopy in  $U_{n+k}^{3\varepsilon, E \circ E}$  connecting  $\begin{pmatrix} u_0 & 0 \\ 0 & 1_k \end{pmatrix}$  and  $\begin{pmatrix} u_1 & 0 \\ 0 & 1_k \end{pmatrix}$ .

Denote  $[(p, l)]_{\varepsilon, E}$  and  $[u]_{\varepsilon, E}$  for the equivalence classes of almost-projections and almost-unitaries. Then, the same proof as [40] shows that  $[p, l]_{\varepsilon, E} + [q, l']_{\varepsilon, E} = [\text{diag}(p, q), l + l']_{\varepsilon, E}$  and  $[u]_{\varepsilon, E} + [v]_{\varepsilon, E} = [\text{diag}(u, v)]_{\varepsilon, E}$  induces a group structure on the equivalence classes, that we denote  $K_0^{\varepsilon, E}(A) = P_\infty^{\varepsilon, E}(A) \times \mathbb{N} / \sim$  and  $K_1^{\varepsilon, E}(A) = U_\infty^{\varepsilon, E}(A) / \sim$ .

If  $A$  is not unital,  $K_0^{\varepsilon, E}(A)$  is defined as

$$\{[p, l]_{\varepsilon, E} : p \in P_\infty^{\varepsilon, E}(\tilde{A}), l \in \mathbb{N} \text{ s.t. } \text{rank}(\kappa_0(\rho_A(p))) = l\}$$

and  $K_1^{\varepsilon, E}(A)$  is defined as  $U_\infty^{\varepsilon, E}(\tilde{A}) / \sim_{\varepsilon, E}$ .

**Definition 3.1.7.** The controlled  $K$ -theory of a filtered  $C^*$ -algebra  $(A, \mathcal{E})$  is the family of abelian groups  $\hat{K}_0(A) = (K_0^{\varepsilon, E}(A))_{\varepsilon \in (0, \frac{1}{4}), E \in \mathcal{E}}$  and  $\hat{K}_1(A) = (K_1^{\varepsilon, E}(A))_{\varepsilon \in (0, \frac{1}{4}), E \in \mathcal{E}}$  defined above.

We define canonical morphisms : if  $\varepsilon, \varepsilon' \in (0, \frac{1}{4})$  and  $E, E' \in \mathcal{E}$  such that  $\varepsilon \leq \varepsilon'$  and  $E \subseteq E'$ , the natural homomorphism  $K_*^{\varepsilon, E}(A) \hookrightarrow K_*^{\varepsilon', E'}(A)$  is denoted by  $\iota_{\varepsilon, E}^{\varepsilon', E'}$ . Notice that  $\iota_{\varepsilon', E'}^{\varepsilon'', E''} \circ \iota_{\varepsilon, E}^{\varepsilon', E'} = \iota_{\varepsilon, E}^{\varepsilon'', E''}$  when this expression makes sense.

One has also forgetful morphisms  $\iota_{\varepsilon, E} : K_*^{\varepsilon, E}(A) \rightarrow K_*(A)$  given by  $[p, l]_{\varepsilon, E} \mapsto [\kappa_0(p)] - [1_l]$  and  $[u]_{\varepsilon, E} \mapsto [\kappa_1(u)]$ , and  $\iota_{\varepsilon', E'} \circ \iota_{\varepsilon, E}^{\varepsilon', E'} = \iota_{\varepsilon, E}$  holds. The controlled  $K$ -theory groups approximate the usual  $K$ -groups in the sense of the following remarks. The reader is referred to [40], remark 1.17, for a proof.

**Remark 3.1.8.** For every filtered  $C^*$ -algebra  $(A, \mathcal{E})$ , any  $y \in K(A)$  and any  $\varepsilon \in (0, \frac{1}{4})$ , there exists  $E \in \mathcal{E}$  and  $x \in K^{\varepsilon, E}(A)$  such that  $\iota_{\varepsilon, E}(x) = y$ .

**Remark 3.1.9.** There exists a universal constant  $\lambda \geq 1$  such that, for every filtered  $C^*$ -algebra  $(A, \mathcal{E})$ , every  $\varepsilon \in (0, \frac{1}{4\lambda})$  and every  $E \in \mathcal{E}$ , the following holds :

Let  $x \in K^{\varepsilon, E}(A)$  such that  $\iota_{\varepsilon, E}(x) = 0$  in  $K(A)$ . Then, there exists  $E' \in \mathcal{E}$  such that  $E \leq E'$  and  $\iota_{\varepsilon, E'}^{\lambda\varepsilon, E'}(x) = 0$  in  $K^{\lambda\varepsilon, E'}(A)$ .

We list some examples that we will use, and some others that shall hopefully be developed in future work.

**Example 3.1.10.** Let  $(X, \mathcal{E})$  be a coarse space. The set of symmetric controlled subsets  $\mathcal{E}_X$  is our prominent example. Recall that it is the set of subsets  $E \subseteq X \times X$  such that  $\sup\{d(x, y) : (x, y) \in E\}$  is finite and  $E = E^{-1}$ . Let  $E, E'$  be such sets, their composition is given by :

$$E \circ E' = EE' \cup E'E \text{ where } EE' = \{(x, y) \text{ s.t. } \exists z \in X / (x, z) \in E \text{ and } (z, y) \in E'\}.$$

**Example 3.1.11.** Let  $G$  be a  $\sigma$ -compact étale groupoid. Then the set of symmetric compact subsets  $\mathcal{E}_G$  of  $G$  is a coarse structure. It is the set of compact subsets  $E \subseteq G$  such that  $E = E^{-1}$ , where  $E^{-1} = \{g^{-1} : g \in E\}$ . For  $E$  and  $E'$  in  $\mathcal{E}_G$ , their composition is given by :

$$E \circ E' = EE' \cup E'E \text{ where } EE' = \{gg' : (g, g') \in G^{(2)}\}.$$

If  $G$  is  $\sigma$ -compact, and  $A$  is a  $G$ -algebra,  $A \rtimes_r G$  is naturally filtered by  $\mathcal{E}_G$  : if  $E \subseteq G$  is a compact subset, define  $(A \rtimes_r G)_E = \{f \in C_c(G, A) : \text{supp}(f) \subseteq E\}$ . This situation has important particular cases :

- Let  $G$  be the coarse groupoid of a coarse space  $(X, \mathcal{E})$ , which is étale. Then  $\mathcal{E}_G$  is given by

$$\{\overline{E} : E \in \mathcal{E}_X\} \cong \mathcal{E}_X.$$

- Let  $\Gamma$  be a finitely generated group acting by homeomorphism on a topological space  $X$ . Recall that the word length  $l$  defines a proper metric on  $\Gamma$ . Define, for  $R > 0$  and  $K \subseteq X$  compact,

$$\Delta_{R, K} = \{(x, g) \in G \text{ s.t. } l(g) \leq R \text{ and } x \in K\}$$

and  $\mathcal{E}_G$  as the set of  $E \subseteq G$  such that  $\exists R > 0, E \subseteq \Delta_R$  and  $E = E^{-1}$ . It provides the étale action groupoid  $G = X \rtimes \Gamma$  with a coarse structure.

**Example 3.1.12.** Let  $\mathbb{G}$  be a compact quantum group in the sense of Woronowicz. By Proposition 2.4.18, the set  $\mathcal{E}_{\mathbb{G}}$  of unitary representations of  $\mathbb{G}$  is a coarse structure w.r.t. the composition  $\pi \circ \pi' = (\pi \otimes \pi') \oplus (\pi \otimes \pi')$ , and  $\pi \leq \pi'$  if  $\pi$  is equivalent to a subrepresentation of  $\pi'$ . Moreover, for every  $\mathbb{G}$ -algebra  $A$ ,  $A \rtimes_r \hat{\mathbb{G}}$  is  $\mathcal{E}_{\mathbb{G}}$ -filtered.

### 3.1.2 Quantitative objects

In order to study functorial properties of controlled  $K$ -theory, we will adapt and study the notion of quantitative object defined in [40].

**Definition 3.1.13.** A quantitative object is a family of abelian groups  $\hat{\mathcal{O}} = \{\mathcal{O}_{\varepsilon, E}\}_{\varepsilon \in (0, \frac{1}{4}), E \in \mathcal{E}}$  endowed with a family of group homomorphisms  $\phi_{\varepsilon, E}^{\varepsilon', E'} : \mathcal{O}_E \rightarrow \mathcal{O}_{E'}$  for any  $E, E' \in \mathcal{E}$  and  $0 < \varepsilon \leq \varepsilon' < \frac{1}{4}$  such that  $E \subseteq E'$ , satisfying  $\phi_{\varepsilon, E}^{\varepsilon, E} = id_{\mathcal{O}_E}$  and  $\phi_{\varepsilon', E'}^{\varepsilon'', E''} \circ \phi_{\varepsilon, E}^{\varepsilon', E'} = \phi_{\varepsilon, E}^{\varepsilon'', E''}$  whenever  $E \subseteq E' \subseteq E''$  and  $\varepsilon < \varepsilon' < \varepsilon''$ .

We need to define controlled morphisms between quantitative objects. We first define control pairs, which are essentially what ensures that the controlled morphisms do not distort too much the propagation.

**Definition 3.1.14.** A control pair is a couple  $\rho = (a, h)$  where  $a \in (0, \frac{1}{4})$  and  $h : (0, \frac{1}{4a}) \rightarrow \mathbb{N}^*$  is a non-decreasing function.

Control pairs can be naturally composed, and if  $(a, h)$  and  $(b, k)$  are two control pairs, then their composition, denoted by  $(b, k) * (a, h)$ , is defined by  $(ab, k * h)$ , where  $k * h : (0, \frac{1}{4ab}) \rightarrow \mathbb{N}^*$ ;  $\varepsilon \mapsto k_{a\varepsilon} h_\varepsilon$ .

Control pairs naturally act on the index subset of the controlled  $K$ -groups. Indeed, if  $\varepsilon \in (0, \frac{1}{4a})$  and  $E \in \mathcal{E}$ ,  $(a, h).(\varepsilon, E) = (a\varepsilon, E^{h_\varepsilon})$  is in  $(0, \frac{1}{4}) \times \mathcal{E}$ . This allows to define controlled morphisms.

We can also compare control pairs. Indeed, define the following partial order :  $(a, h) \leq (b, k)$  if  $a \leq b$  and  $h_\varepsilon \leq k_\varepsilon$  for all  $\varepsilon \in (0, \frac{1}{4a})$ .

**Definition 3.1.15.** Let  $\hat{\mathcal{O}}$  and  $\hat{\mathcal{O}}'$  be two quantitative objects and  $\rho = (a, h)$  a control pair. A  $\rho$ -controlled morphism is a family of groups homomorphisms  $F_{\varepsilon, E} : \mathcal{O}_{\varepsilon, E} \rightarrow \mathcal{O}'_{a\varepsilon, E^{h_\varepsilon}}$  for any  $\varepsilon \in (0, \frac{1}{4a})$  and  $E \in \mathcal{E}$ , such that

$$\phi_{a\varepsilon, E^{h_\varepsilon}}^{a\varepsilon', E'^{h'_\varepsilon}} \circ F_{\varepsilon, E} = F_{\varepsilon', E'} \circ \phi_{\varepsilon, E}^{\varepsilon', E'}$$

for any  $0 < \varepsilon < \varepsilon' < \frac{1}{4a}$  and  $E \subseteq E'$ .

**Remark 3.1.16.** When not specified, the control pair is evident from the context. For example, we will often refer to a controlled morphism, meaning a  $\alpha$ -controlled morphism for some control pair  $\alpha$ . For a controlled morphism  $\hat{F} : \hat{K}(A) \rightarrow \hat{K}(B)$ , we will denote  $F : K(A) \rightarrow K(B)$  the unique homomorphism it induces in  $K$ -theory. We will always try to indicate an analogy with the classical case (as opposed to the controlled or quantitative case) by putting a hat on top of controlled objects that are hopefully inducing a well known object. For example, controlled  $K$ -theory is  $\hat{K}$ , the controlled assembly map will be denoted  $\hat{\mu}_{G, A}$ , etc.

Let  $\rho = (\lambda, h), \alpha, \beta$  be control pairs, and  $F : \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}'$  and  $G : \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}'$  be  $\alpha$ - and  $\beta$ -controlled morphisms respectively. We write  $F \sim_\rho G$  if :

- $\alpha \leq \rho$  and  $\beta \leq \rho$ ,

- for any  $\varepsilon \in (0, \frac{1}{4\lambda})$  and  $E \in \mathcal{E}$ , the following diagram commutes :

$$\begin{array}{ccc}
 \mathcal{O}_{\varepsilon,E} & \xrightarrow{F_{\varepsilon,E}} & \mathcal{O}'_{\alpha(\varepsilon,E)} \\
 \downarrow G_{\varepsilon,E} & & \downarrow \iota_{\beta(\varepsilon,E)}^{\rho(\varepsilon,E)} \\
 \mathcal{O}'_{\beta(\varepsilon,E)} & \xrightarrow{\iota_{\alpha(\varepsilon,E)}^{\rho(\varepsilon,E)}} & \mathcal{O}'_{\rho(\varepsilon,E)}
 \end{array}$$

**Remark 3.1.17.** The following statement can be found in [40] (remark 2.5). Let  $F, F' : \mathcal{O} \rightarrow \mathcal{O}'$ ,  $G : \mathcal{O}' \rightarrow \mathcal{O}''$  and  $G' : \mathcal{O}''' \rightarrow \mathcal{O}$  be  $\alpha$ -,  $\alpha'$ -,  $\beta$ - and  $\beta'$ -controlled morphisms respectively. Let  $\rho$  be a control pair such that  $F \sim_{\rho} F'$ . Then  $G \circ F \sim_{\beta * \rho} G \circ F'$  and  $F \circ G' \sim_{\rho * \beta'} F' \circ G'$ .

**Definition 3.1.18.** Let  $\alpha$  and  $\rho$  be control pairs satisfying  $\alpha \leq \rho$  and  $F : \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}'$  a  $\alpha$ -controlled morphism. We say that  $F$  is  $\rho$ -invertible if there exists a controlled morphism  $G : \hat{\mathcal{O}}' \rightarrow \hat{\mathcal{O}}$  such that  $G \circ F \sim_{\rho} Id_{\hat{\mathcal{O}}}$  and  $F \circ G \sim_{\rho} Id_{\hat{\mathcal{O}}'}$ .  $G$  is said to be a  $\rho$ -inverse for  $F$ .

As we will see for controlled assembly maps, the correct notions of injectivity and surjectivity for controlled morphisms need to be adapted in the following way.

**Definition 3.1.19.** Let  $\rho = (\lambda, h)$  and  $\alpha$  be controlled pairs, and  $F : \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}'$  a  $\alpha$ -controlled morphism.

- $F$  is  $\rho$ -injective if, given any  $\varepsilon \in (0, \frac{1}{4\lambda})$  and  $E \in \mathcal{E}$ ,  $\alpha \leq \rho$  and, for all  $x \in \mathcal{O}_{\varepsilon,E}$  such that  $F_{\varepsilon,E}(x) = 0$ , then  $\iota_{\varepsilon,E}^{\lambda\varepsilon, h\varepsilon E}(x) = 0$ ,
- $F$  is  $\rho$ -surjective if, given any  $\varepsilon \in (0, \frac{1}{4\lambda})$  and  $E \in \mathcal{E}$ , for any  $y \in \mathcal{O}'_{\varepsilon,E}$ , there exists  $x \in \mathcal{O}_{\rho(\varepsilon,E)}$  such that  $F_{\rho(\varepsilon,E)}(x) = \iota_{\varepsilon,E}^{\rho(\varepsilon,E)}(y)$ ,

**Remark 3.1.20.** If  $F$  is a  $\rho$ -isomorphism, then there exists a control pair  $\rho'$  only depending on  $\rho$  such that  $F$  is both  $\rho'$ -injective and  $\rho'$ -surjective.

### 3.1.3 Controlled exact sequences

**Definition 3.1.21.** Let  $F : \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}'$  and  $G : \hat{\mathcal{O}}' \rightarrow \hat{\mathcal{O}}''$  be  $(\alpha, h)$ -controlled and a  $(\beta, k)$ -controlled morphisms respectively. The sequence

$$\hat{\mathcal{O}} \xrightarrow{F} \hat{\mathcal{O}}' \xrightarrow{G} \hat{\mathcal{O}}''$$

is called  $\rho$ -exact at  $\hat{\mathcal{O}}'$  if  $G \circ F = 0$  and if for all  $\varepsilon \in (0, \frac{1}{4\max(\lambda\alpha, \beta)})$ ,  $E \in \mathcal{E}$  and any  $y \in \mathcal{O}'_{\varepsilon,E}$  such that  $G_{\varepsilon,E}(y) = 0$ , then there exists  $x \in \mathcal{O}_{\rho(\varepsilon,E)}$  such that  $F_{\rho(\varepsilon,E)}(x) = \iota_{\varepsilon,E}^{\rho(\varepsilon,E)}(y)$ . A sequence of controlled morphisms

$$\dots \xrightarrow{F_{k-2}} \hat{\mathcal{O}}_{k-2} \xrightarrow{F_{k-1}} \hat{\mathcal{O}}_{k-1} \xrightarrow{F_k} \hat{\mathcal{O}}_k \xrightarrow{F_{k+1}} \dots$$

is said to be  $\rho$ -exact if the sequence

$$\hat{\mathcal{O}}_{j-1} \xrightarrow{F_j} \hat{\mathcal{O}}_j \xrightarrow{F_{j+1}} \hat{\mathcal{O}}_{j+1}$$

is  $\rho$ -exact at  $\hat{\mathcal{O}}_j$  for all  $j$ .

### 3.1.4 Morita equivalence

A controlled version of the Morita equivalence exists. Indeed, the classical Morita equivalence states that, if  $e$  is a rank 1 projection in  $\mathcal{L}(H)$ , the map  $A \rightarrow A \otimes \mathfrak{K}(H); a \mapsto a \otimes e$  induces an isomorphism in  $K$ -theory. But this map preserves propagation, hence the same proof as in [40] gives the following proposition.

**Proposition 3.1.22.** Let  $A$  be a  $\mathcal{E}$ -filtered  $C^*$ -algebra and  $H$  a separable Hilbert space. Then the  $*$ -homomorphism

$$A \rightarrow A \otimes \mathfrak{K}(H); \quad a \mapsto \begin{pmatrix} a & & \\ & 0 & \\ & & \dots \end{pmatrix}$$

induces a group isomorphism

$$\mathcal{M}_A^{\varepsilon, E} : K^{\varepsilon, E}(A) \rightarrow K^{\varepsilon, E}(A \otimes \mathfrak{K}(H))$$

for every  $\varepsilon \in (0, \frac{1}{4})$  and  $E \in \mathcal{E}$ . The family  $\mathcal{M}_A = (\mathcal{M}_A^{\varepsilon, E})_{\varepsilon \in (0, \frac{1}{4}), E \in \mathcal{E}}$  is called the controlled Morita equivalence and is a controlled morphism. It induces the usual Morita equivalence  $M_A : K(A) \rightarrow K(A \otimes \mathfrak{K}(H))$  in  $K$ -theory.

### 3.1.5 Controlled 6-term exact sequences

We will describe the 6-term controlled exact sequence associated to a completely filtered extension of  $C^*$ -algebras. For any extension of  $C^*$ -algebras

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0 \quad ,$$

we denote  $\partial_{J,A}$  the boundary map  $K_*(A/J) \rightarrow K_{*+1}(J)$ . The reader can find all the proofs and properties of the following results in [40].

We fix a coarse structure  $\mathcal{E}$ , and we will consider  $\mathcal{E}$ -filtered  $C^*$ -algebras.

**Definition 3.1.23.** Let  $A$  a filtered  $C^*$ -algebra and  $J \subseteq A$  an ideal. If  $J_E = A_E \cap J$ , the extension

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$$

is said to be completely filtered if the continuous linear bijection  $A_E/J_E \hookrightarrow (A_E + J)/J$  induced by the inclusion  $A_E \hookrightarrow A$  is a complete isometry, i.e.

$$\inf_{y \in M_n(J_E)} \|x + y\| = \inf_{y \in M_n(J)} \|x + y\| \quad , \forall n \in \mathbb{N}, x \in M_n(A_E), E \in \mathcal{E}.$$

**Proposition 3.1.24.** There exists a control pair  $(\alpha_D, k_D)$  such that for any completely filtered extension of  $C^*$ -algebras

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$$

there exists a  $(\alpha_D, k_D)$ -controlled morphism of odd degree

$$D_{J,A} : \hat{K}(A/J) \rightarrow \hat{K}(J)$$

which induces  $\partial_{J,A}$  in  $K$ -theory.

We will denote

- $D_A$  the controlled boundary map associated to the completely filtered extension of  $C^*$ -algebras  $0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0$ ,
- $D_{J,A}^j$  the restriction of  $D_{J,A}$  to  $\hat{K}_j(A/J) \rightarrow \hat{K}_{j+1}(J)$ .

**Theorem 3.1.25.** There exists a universal control pair  $(\lambda, h)$ , which does not depend on  $\mathcal{E}$ , such that for any completely filtered extension of  $C^*$ -algebras

$$0 \rightarrow J \xrightarrow{\iota} A \xrightarrow{q} A/J \rightarrow 0$$

the following 6-term exact sequence is  $(\lambda, h)$ -exact

$$\begin{array}{ccccc} \hat{K}_1(J) & \xrightarrow{\iota_*} & \hat{K}_1(A) & \xrightarrow{q_*} & \hat{K}_1(A/J) \\ D_{J,A}^0 \uparrow & & & & \downarrow D_{J,A}^1 \\ \hat{K}_0(A/J) & \xleftarrow{q_*} & \hat{K}_0(A) & \xleftarrow{\iota_*} & \hat{K}_0(J) \end{array} \cdot$$

The following remark can be found in [40] (remark 3.8) and will be used to prove functorial properties of the controlled Roe and Kasparov transformations.

**Remark 3.1.26.** Let  $A$  and  $B$  two  $\mathcal{E}$ -filtered  $C^*$ -algebras, and  $\phi : A \rightarrow B$  a filtered  $*$ -homomorphism. Let  $I$  and  $J$  be respectively ideals in  $A$  and  $B$  and assume that :

- $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$  and  $0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$  are completely filtered extensions of  $C^*$ -algebras,
- $\phi(I) \subseteq J$ ,

then  $D_{J,B} \circ \tilde{\phi}_* = \phi_* \circ D_{I,A}$ .

### 3.1.6 Tensorisation in $KK$ -theory

If  $B$  is a filtered  $C^*$ -algebra and  $A$  any  $C^*$ -algebra, and if  $A \otimes B$  is the spatial tensor product, for all  $E \in \mathcal{E}$  define  $A \otimes B_E$  as the closure of the algebraic tensor product in  $A \otimes B$ . Then  $(A \otimes B_E)_{E \in \mathcal{E}}$  defines a filtration of  $A \otimes B$ . If  $\phi : A_1 \rightarrow A_2$  is a  $*$ -homomorphism, we use the notation  $\phi_B$  for the induced  $*$ -homomorphism  $A_1 \otimes B \rightarrow A_2 \otimes B$ .

In [21], G. Kasparov defined a map

$$\tau_B : KK(A_1, A_2) \rightarrow KK(A_1 \otimes B, A_2 \otimes B)$$

for any  $C^*$ -algebras  $A_1$  and  $A_2$ , which is compatible with the Kasparov product. Any  $z \in KK(A_1, A_2)$  defines a morphism

$$K(A_1 \otimes B) \rightarrow K(A_2 \otimes B)$$

which is proved in [40] to be induced from a controlled morphism. The following theorem is borrowed from [40].



**Theorem 3.1.27.** There exists a control pair  $(\alpha_\tau, k_\tau)$  such that, for any filtered  $C^*$ -algebra  $B$ , any  $C^*$ -algebras  $A_1$  and  $A_2$  and any  $K$ -cycle  $z \in KK(A_1, A_2)$ , there exists a  $(\alpha_\tau, k_\tau)$ -controlled morphism  $\hat{\tau}_B : \hat{K}(A_1 \otimes B) \rightarrow \hat{K}(A_2 \otimes B)$  such that :

- $\hat{\tau}_B(z)$  induces right-multiplication by  $\tau_B(z)$  in  $K$ -theory,
- for any  $K$ -cycles  $z, z' \in KK(A_1, A_2)$ ,  $\hat{\tau}_B(z + z') = \hat{\tau}_B(z) + \hat{\tau}_B(z')$ ,
- if  $\phi : A_1 \rightarrow A'_1$  is a  $*$ -homomorphism, then  $\hat{\tau}_B(\phi^*(z)) = \hat{\tau}_B(z) \circ (\phi_B)_*$  for any  $z \in KK(A'_1, A_2)$ ,
- if  $\phi : A'_2 \rightarrow A_2$  is a  $*$ -homomorphism, then  $\hat{\tau}_B(\phi_*(z)) = (\phi_B)_* \circ \hat{\tau}_B(z)$  for any  $z \in KK(A_1, A'_2)$ ,
- $\hat{\tau}_B([Id_A]) \sim_{(\alpha_\tau, k_\tau)} Id_{\hat{K}(A \otimes B)}$ ,
- for any  $C^*$ -algebra  $D$ , any  $K$ -cycle  $z \in KK(A_1, A_2)$ ,  $\hat{\tau}_B(\tau_D(z)) = \hat{\tau}_{B \otimes D}(z)$ .
- for any semi-split extension  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  with boundary element  $[\partial_{J,A}] \in KK_1(A/J, J)$ ,  $\hat{\tau}_B([\partial_{J,A}]) = D_{J \otimes B, A \otimes B}$ .

This controlled tensorisation map respects Kasparov product. See [40] for a proof.

**Theorem 3.1.28.** There exists a control pair  $\lambda$  such that, for any separable  $C^*$ -algebras  $A_1$  and  $A_2$ , any filtered  $C^*$ -algebra  $B$ , the following holds : for any  $z \in KK(A_1, A_2)$  and  $z' \in KK(A_2, A_3)$ ,

$$\hat{\tau}_B(z \otimes z') \sim_\lambda \hat{\tau}_B(z') \circ \hat{\tau}_B(z)$$

### 3.1.7 Controlled Bott periodicity

We recall in this section how to construct controlled Bott maps.

Recall the definition of the Toeplitz extension. Let  $v \in \mathcal{L}(l^2(\mathbb{N}))$  be the unilateral shift on  $l^2(\mathbb{N})$ , i.e.  $v(e_n) = e_{n+1}$  if  $(e_n)_{n \in \mathbb{N}}$  is the canonical basis. Then  $v$  is an isometry, i.e.  $v^*v = 1$ . Let us denote  $\mathcal{T}$  the  $C^*$ -algebra of  $\mathcal{L}(l^2(\mathbb{N}))$  generated by  $v$ . Then the sequence of  $C^*$ -algebras

$$0 \rightarrow \mathfrak{K}(l^2(\mathbb{N})) \rightarrow \mathcal{T} \rightarrow C(\mathbb{S}) \rightarrow 0$$

is exact. Set  $\mathcal{T}_0$  the preimage of  $C_0(0, 1)$  under the last arrow, so that the sequence of  $C^*$ -algebras

$$0 \rightarrow \mathfrak{K}(l^2(\mathbb{N})) \rightarrow \mathcal{T}_0 \rightarrow C_0(0, 1) \rightarrow 0$$

is exact. This last sequence is called the Toeplitz extension, let  $T \in KK_1(S, \mathfrak{K}(l^2(\mathbb{N})))$  denotes its boundary.

Let us put  $S = C_0(0, 1)$  and  $C = C_0(0, 1]$ , so that evaluation at 1 induces the following exact sequence of  $C^*$ -algebras  $0 \rightarrow S \rightarrow C \rightarrow \mathbb{C} \rightarrow 0$ . Let us denote by  $[\partial] \in KK_1(C, S)$  the class in  $KK$ -theory of this extension. It is called the Bott element, and is invertible in  $KK_1(C, S)$ , its inverse being the boundary of the Toeplitz extension, up to Morita equivalence (see [40], section 4.2). Recall that the boundary of this extension is given by right Kasparov product by  $[\partial]$ .

For any  $C^*$ -algebra  $A$ , we can tensorize the previous extension by  $A$  to have an exact sequence  $0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0$ . Its class in  $KK$ -theory, denoted  $[\partial_A] \in KK_1(A, SA)$ , actually satisfies  $\tau_A([\partial]) = [\partial_A]$ . Let us denote  $\partial_A : K_*(A) \rightarrow K_*(SA)$  the odd degree boundary homomorphism of the extension, it is induced by right multiplication by  $[\partial_A]$ . It is also invertible, as we can tensorize the Toeplitz extension to get the following exact sequence of  $C^*$ -algebras

$$0 \rightarrow \mathfrak{K}(l^2(\mathbb{N})) \otimes A \rightarrow \mathcal{T}_0 \otimes A \rightarrow SA \rightarrow 0$$

whose boundary element  $T_A = [\partial_{\mathcal{T}_0 \otimes A}] \in KK_1(SA, \mathfrak{K}(l^2(\mathbb{N})) \otimes A)$  satisfies  $\tau_A(T) = T_A$ . As  $\tau_A$  respects Kasparov product, one can see that  $[\partial_A]$  and  $T_A$  are mutual inverse up to Morita equivalence.

Tensorization by  $S$  and  $C$  is functorial with respect to  $*$ -homomorphisms, and if  $g : A \rightarrow B$  is a  $*$ -homomorphism, we denote  $Sg : SA \rightarrow SB$  and  $Cg : CA \rightarrow CB$  the induced  $*$ -homomorphisms. By naturality of boundary maps w.r.t. extensions, we get the following relation :

$$(Sg)_*[\partial_{SA}] = [\partial_{SA}] \otimes_{SA} [Sg] = [g] \otimes_B [\partial_{SB}] = g^*[\partial_{SB}].$$

Recall that, if  $A$  is unital and  $p$  is a projection in  $A$ ,  $\partial_A[p]$  is given by the homotopy class of the path of unitaries

$$\begin{cases} [0, 1] & \rightarrow U(\tilde{S}A) \\ t & \mapsto e^{2i\pi t}p + 1 - p \end{cases}$$

One can perform a similar construction in term of almost-projections. Let  $q \in P_n^{\varepsilon, E}(A)$  and  $l$  an integer, define

$$V_{q,l}(t) = \text{diag} (e^{-2i\pi t}, 1, 1, \dots, 1) (e^{2i\pi t}q + 1 - q) \in U_n^{5\varepsilon, E}(\tilde{S}A),$$

then

$$\begin{cases} K_0^{\varepsilon, E}(A) & \rightarrow K_1^{\varepsilon, E}(SA) \\ [q, l]_{\varepsilon, E} & \mapsto [V_{q,l}]_{5\varepsilon, E} \end{cases}$$

defines a  $(5, \mathbf{1})$ -controlled morphism  $Z_A : \hat{K}_0(A) \rightarrow \hat{K}_1(SA)$ , where  $\mathbf{1}$  is the constant function  $\mathbf{1}_\varepsilon(E) = 1, \forall E \in \mathcal{E}$ .

**Proposition 3.1.29** ([40] Prop. 3.9 and 3.10). Let  $A$  be a unital  $C^*$ -algebra. There exist universal control pairs  $(\lambda, h)$  and  $(\lambda', h')$  such that

$$D_A^0 \sim_{(\lambda, h)} Z_A \quad \text{and} \\ D_{\mathfrak{K}(l^2(\mathbb{N})) \otimes A, \mathcal{T}_0 \otimes A}^1 \circ Z_A \sim_{(\lambda', h')} \mathcal{M}_A$$

where  $\mathcal{M}_A$  is the controlled Morita equivalence.

# Chapter 4

## Assembly maps

### 4.1 Assembly Maps

In this section, all groupoids  $G \rightrightarrows G^{(0)}$  are étale.

#### 4.1.1 Proper groupoids and proper actions

Recall that the introduction contains a survey on groupoid actions, in particular the definition of proper actions for groupoids.

**Definition 4.1.1.** A  $G$ -space  $\underline{E}G$  is called universal if, given any proper  $G$ -space  $Z$ , there exists a continuous  $G$ -equivariant map  $Z \rightarrow \underline{E}G$ .

We now construct a model for universal spaces of  $G$ , based on J.-L.Tu's work [52]. For any compact subset  $K \subseteq G$ , define  $P_K(G)$  to be the space of probability measures  $\nu$  with support contained in one and only one fiber  $G^x$  for some  $x \in G^{(0)}$ , and such that if  $g, g' \in \text{supp}(\nu)$ , then  $g'g^{-1} \in K$ . We endow  $P_K(G)$  with the weak-\* topology.

Every element  $\eta$  is a finite probability measure on a fiber  $G^x$ , for some  $x \in G^{(0)}$ , hence can be written as a finite convex combination  $\eta = \sum_{g \in G^x} \lambda_g(\eta) \delta_g$ , where  $\lambda_g(\eta) \in [0, 1]$  for every  $g$  and  $\delta_g$  is the Dirac probability measure at  $g \in G^x$ .

Let us define a left action of  $G$  on  $P_K(G)$ . The momentum map  $p : P_K(G) \rightarrow G^{(0)}$  is the map associating to  $\nu$  the only  $x$  such that  $\text{supp}(\nu) \subseteq G^x$ . Let us show that  $p$  is continuous. Let  $U \subseteq G^{(0)}$  be an open set. Let  $x \in U$  and  $\eta \in p^{-1}(x)$ . There exists an element  $g \in G^x$  such that  $\lambda_g(\eta) \neq 0$ . Let  $S \subseteq G$  be an open bisection such that  $g \in S$ . Up to replacing  $U$  by a smaller neighborhood, we can suppose  $r(S) = U$ . Let  $\phi : G \rightarrow [0, 1]$  be a continuous function with support in  $S$  such that  $\phi(g) = 1$ . Then  $\langle \eta, \phi \rangle \neq 0$  holds. Then,  $\{\eta' \in P_K(G) \text{ s.t. } \langle \eta', \phi \rangle \neq 0\}$  is open for the weak-\* topology and its image through  $p$  is contained in  $U$ .

The action is defined by left translation, i.e. for every  $(g, \eta) \in G \times_{s,p} P_K(G)$  :

$$(g.\eta)(h) = \eta(g^{-1}h) \quad \forall h \in G^{r(g)}.$$

**Lemma 4.1.2** (Tu,[52]). The action of  $G$  on  $P_K(G)$  is proper and cocompact.

**Lemma 4.1.3** (Tu,[52]). Let  $Z$  be a proper  $G$ -compact  $G$ -space. Then there exists a compact subset  $K \subseteq G$  and a  $G$ -equivariant continuous map  $Z \rightarrow P_K(G)$ .

### 4.1.2 Topological $K$ -theory for groupoids

We will use the equivariant  $KK$ -theory developed by Le Gall in his thesis [31], which is an extension of the usual equivariant  $KK$ -theory of Kasparov. Recall that, if  $A$  and  $B$  are two  $G$ -algebras, elements of  $KK^G(A, B)$  are homotopy classes of triple  $(E, \pi, T)$  where :

- $E$  is a  $G$ -module, the action of  $G$  being implemented by a unitary  $V \in \mathcal{L}_{s^*B}(s^*E, r^*E)$ ,
- $\pi : A \rightarrow \mathcal{L}_B(E)$  is a  $G$ -equivariant  $*$ -homomorphism,
- $T \in \mathcal{L}_B(E)$  is an adjointable operator such that the triple satisfies the  $K$ -cycle conditions :  $\pi(a)(T^2 - T), \pi(a)(T^* - T), [\pi(a), T]$  are compact operators in  $\mathfrak{K}(E)$ , and  $\pi(a)(r^*T - Vs^*TV^*)$  are compact operators in  $\mathfrak{K}_{r^*B}(r^*E)$  for all  $a \in A, g \in G$ .

**Definition 4.1.4.** Let  $Y$  be a proper  $G$ -space and  $B$  a  $G$ -algebra. Then the analytic  $K$ -homology of  $Y$  with coefficients in  $B$  is defined by

$$RK^G(Y, B) = \varinjlim_{Z \subseteq Y} KK^G(C_0(Z), B),$$

the inductive limit being taken on proper  $G$ -compact  $G$ -subspaces  $Z$  of  $Y$ . The topological  $K$ -theory of  $G$  is defined as the analytic  $K$ -homology of  $\underline{EG}$ . In the literature, the notation  $K^{top}(G, B)$  is common to denote  $RK^G(\underline{EG}, B)$ .

The previous lemmas ensure that the topological  $K$ -theory of  $G$  can be computed as an inductive limit of the  $K$ -homology of the spaces  $P_K(G)$ :

**Lemma 4.1.5.** Let  $B$  be a  $G$ -algebra. Then :

$$RK^G(\underline{EG}, B) = \varinjlim_{K \subseteq G \text{ compact}} KK^G(C_0(P_K(G)), B).$$

**Proof.** Let us denote by  $\mathcal{A}$  the inductive limit

$$\varinjlim_{K \subseteq G} KK^G(C_0(P_K(G)), B).$$

By the universal property of the inductive limit, there exists a unique homomorphism

$$\phi : \mathcal{A} \rightarrow RK^G(\underline{EG}, B).$$

Let  $Z$  and  $Z'$  be two proper  $G$ -compact  $G$ -spaces such that  $Z \subseteq Z'$ . By lemma 4.1.3, there exist compact subsets  $K$  and  $K'$  of  $G$  and  $G$ -equivariant continuous maps  $h : Z \rightarrow P_K(G)$  and  $h' : Z' \rightarrow P_{K'}(G)$ . Up to enlarge  $K'$ , we can suppose  $h|_Z$  and  $h'$  are homotopic. By the universal property of the inductive limit, there exists a unique homomorphism

$$\psi : RK^G(\underline{EG}, B) \rightarrow \mathcal{A}.$$

Then  $\psi \circ \phi = id_{\mathcal{A}}$  is obvious. The universal property of  $\underline{EG}$  ensures that there exist  $G$ -equivariant continuous maps, unique up to homotopy,  $P_K(G) \rightarrow \underline{EG}$  and  $P_{K'}(G) \rightarrow \underline{EG}$ . The composition of  $Z \rightarrow P_K(G)$  by  $P_{K'}(G)$  is, up to enlarge  $K'$ , homotopic to the inclusion  $Z \rightarrow \underline{EG}$ . Hence  $\phi \circ \psi = id$  and

$$RK^G(\underline{EG}, B) \cong \mathcal{A}.$$

□

### 4.1.3 Descent functor

There exists a natural transformation  $KK^G(A, B) \rightarrow KK(A \rtimes_r G, B \rtimes_r G)$  respecting the Kasparov product, which is called the descent functor. The same statement remains true for maximal crossed products. We recall its construction, which was first stated in [31].

Let  $A$  and  $B$  be two  $G$ -algebras and  $(E, \pi, T) \in E^G(A, B)$  be a  $K$ -cycle. Define :

- $E_G$  to be the  $B \rtimes_r G$ -Hilbert module  $E \otimes_B (B \rtimes_r G)$ ,
- $\pi_G = \pi \otimes_A id_{A \rtimes_r G}$  to be the image of  $\pi : A \rightarrow \mathcal{L}_B(E)$  under the reduced crossed-product functor,
- $T_G = T \otimes_B 1$  to be the image of  $T \in \mathcal{L}(E)$  under the reduced crossed-product functor.

Then  $(E_G, \pi_G, T_G) \in E(A \rtimes_r G, B \rtimes_r G)$  and the map  $(E, \pi, T) \mapsto (E_G, \pi_G, T_G)$  induces a homomorphism of abelian groups

$$j_G : KK^G(A, B) \rightarrow KK(A \rtimes_r G, B \rtimes_r G)$$

satisfying  $j_G(z \otimes_D z') = j_G(z) \otimes_{D \rtimes_r G} j_G(z')$  for any  $z \in KK^G(A, D), z' \in KK^G(D, B)$ .

### 4.1.4 The assembly map

If  $Z$  is a proper  $G$ -compact  $G$ -space,  $Z \rtimes G$  is a proper groupoid by definition, hence there exists a cutoff function  $c : Z \rightarrow [0, 1]$  such that  $\sum_{g \in G^p(z)} c(zg) = 1$ . The function  $g \mapsto c(r(g))^{\frac{1}{2}} c(s(g))^{\frac{1}{2}}$  defines a projection in  $C_0(Z) \rtimes_r G$  which we denote by  $\mathcal{L}_Z$ . If  $Z = P_K(G)$ , we will denote  $\mathcal{L}_Z$  by  $\mathcal{L}_K$ . Notice that we will give a more explicit construction in this case in the following section. Indeed, for the action of  $G$  on  $P_K(G)$ , the cutoff function can be described using well defined coordinate functions.

**Definition 4.1.6.** The assembly map for  $G$  with coefficients in  $B$  is defined as the inductive limit of the maps  $\mu_{G,B}^{(Z)} : KK^G(C_0(Z), B) \rightarrow K(B \rtimes_r B)$  given by

$$\mu_{G,B}^{(Z)}(z) = [\mathcal{L}_Z] \otimes_{C_0(Z) \rtimes_r G} j_G(z),$$

that is  $\mu_{G,B} = \varinjlim \mu_{G,B}^{(Z)}$  (one has to check that these maps respects the inductive systems, which they do).

**Remark 4.1.7.** By lemma 4.1.5, one can restrict to  $Z$  of the form  $P_E(G)$  for  $E \subseteq G$  compact. We will denote by  $\mu_{G,B}^E$  the assembly map  $\mu_{G,B}^{P_E(G)}$ , and  $\mu_{G,B} = \varinjlim \mu_{G,B}^E$  still holds.

### 4.1.5 The Baum-Connes conjecture

**Conjecture 4.1.8** (Baum-Connes conjecture). Let  $G$  be an étale groupoid, and  $A$  a  $G$ -algebra.

The Baum-Connes conjecture for  $G$  with coefficients in  $A$  is the following claim :

$\mathbf{BC}_{G,A} : \mu_{G,A}$  is an isomorphism of abelian groups.

The Baum-Connes conjecture with coefficients is : for all  $G$ -algebras  $A$ ,  $\mu_{G,A}$  is an isomorphism of abelian groups.

The conjecture was first stated for groups in [2] and in [3]. The statement is a generalization of the Connes-Kasparov conjecture, which is simply, when one knows the Baum-Connes conjecture, the latter for almost connected groups (i.e. locally compact groups  $G$  such that  $G/G_0$  is compact, where  $G_0$  is the connected component of the identity). J.-L. Tu stated the conjecture for general locally compact,  $\sigma$ -compact, Hausdorff groupoids with Haar systems.[50]

Here is the status of the conjecture :

- N. Higson and G. Kasparov proved that the conjecture with coefficients holds for groups having Haagerup property [18].
- V. Lafforgue proved the conjecture with coefficients for hyperbolic groups [29].
- V. Lafforgue showed the conjecture for every semi-simple Lie group, real or  $p$ -adic reductive, and discrete cocompact subgroup of real rank 1 Lie groups or of  $SL(3, k)$  for any local field  $k$  [25].
- The Connes-Kasparov conjecture was established by J. Chabert, S. Echterhoff and R. Nest, who proved that  $\mu_G$  is an isomorphism for every second countable almost connected group  $G$ , and every group of  $k$ -rational points of a linear algebraic group over a local field of characteristic 0 [5].

Recall that a function  $h : G \rightarrow \mathbb{R}$  is said to be of negative type if :

- $h|_{G^{(0)}} = 0$ ,
- $h(g) = h(g^{-1})$  for all  $g \in G$ ,
- for all  $t_i \geq 0, i = 1, \dots, n$  such that  $\sum t_i = 0$  and all  $g_1, \dots, g_n \in G$  having the same range,  $\sum_i t_i t_j h(g_i^{-1} g_j) \leq 0$ .

The function  $h$  is said to be locally proper if, for every compact  $K \subseteq G^{(0)}$ ,  $h|_{G_K^K}$  is proper.

**Definition 4.1.9.** A topological groupoid  $G$  is said to be a-T-menable if it satisfies one of the following equivalent properties :

- there exists a continuous field of Hilbert spaces over  $G^{(0)}$  with a proper affine action of  $G$ ,
- there exists a locally proper negative type function on  $G$ .

For groupoids, the conjecture is known to be false in full generality [19], but the following result was shown by J-L. Tu in [48]

**Theorem 4.1.10.** Let  $G$  be a locally compact  $\sigma$ -compact Hausdorff groupoid. Then if  $G$  is a-T-menable, the Baum-Connes conjecture with coefficients holds for  $G$ .

### 4.1.6 The Coarse Assembly map

The construction of the coarse assembly map is very similar to the assembly map for groupoids, but relies on different functors.

The metric space  $X$  is discrete, hence  $l^2(X)$  is generated by  $e_x = (\delta_{xy})_{y \in X} \in l^2(X)$ . For  $x, y \in X$  and  $T \in \mathcal{L}_B(l^2(X) \otimes H_B)$ ,  $T_{x,y}$  is defined as the unique operator in  $\mathcal{L}_B(H_B)$  satisfying  $\langle T_{xy}\xi, \eta \rangle = \langle T(e_x \otimes \xi), e_y \otimes \eta \rangle$  for all  $\xi, \eta \in H_B$ .

Recall how to define the Roe algebra  $C^*(X, B)$  of a discrete metric space with bounded geometry  $X$  with coefficients in an arbitrary  $C^*$ -algebra  $B$ . For  $R > 0$ , define  $C_R[X, B]$  to be the involutive subspace of operators  $T \in \mathcal{L}_B(l^2(X) \otimes H \otimes B)$  such that  $T_{xy} \in \mathfrak{K}_B(H \otimes B)$  and  $T_{xy} = 0$  as soon as  $d(x, y) > R$ , and  $\mathbb{C}[X, B] = \cup_{R>0} C_R[X, B]$ .

**Definition 4.1.11.** The Roe algebra  $C^*(X, B)$  is defined as the Hilbert  $B$ -module obtained after the completion of  $\mathbb{C}[X, B]$  with respect to the operator norm in  $\mathcal{L}_B(l^2(X) \otimes H \otimes B)$ .

This construction is functorial : any  $*$ -homomorphism  $\phi : A \rightarrow B$  gives rise to a  $*$ -homomorphism  $\phi_X : C^*(X, A) \rightarrow C^*(X, B)$ . For the reader's convenience, we give details for the construction of  $\phi \mapsto \phi_X$ , which is a standard fact in Coarse Geometry.

**Theorem 4.1.12.** Let  $X$  be a discrete metric space with bounded geometry and  $\phi : A \rightarrow B$  a  $*$ -homomorphism. Then there exists a  $*$ -homomorphism  $\phi_X : C^*(X, A) \rightarrow C^*(X, B)$  extending  $\phi$ . Moreover,  $\phi \mapsto \phi_X$  respects composition of  $*$ -homomorphisms.

**Proof.** Recall that any  $*$ -morphism  $\phi : A \rightarrow B$  induces, for any  $A$ -Hilbert module  $E$ , a  $*$ -morphism  $\phi_* : \mathcal{L}_A(E) \rightarrow \mathcal{L}_B(E \otimes_A B)$ . Now take  $E$  to be  $l^2(X) \otimes A$ . Then, according to remark 1.2.13,  $\eta \otimes a \otimes b \mapsto \eta \otimes \phi(a)b$  extends to an isometry  $V \in \mathcal{L}_B(E \otimes_A B, l^2(X) \otimes B)$  which respects compact operators. The composition  $Ad_V \circ \phi_*$  sends a compact operator  $(T_{xy})_{x,y}$  to  $(\phi(T_{xy}))_{x,y}$ .

Hence, the linear map  $T \mapsto V\phi_*(T)V^*$  maps  $C_R[X, A]$  into  $C_R[X, B]$ , and so extends to a  $*$ -morphism  $C^*(X, A) \rightarrow C^*(X, B)$ . The composition property is clear from the construction. □

**Remark 4.1.13.** Notice that  $\phi \mapsto \phi_X$  can be defined in the same way for completely positive maps : for every completely positive map  $\phi : A \rightarrow B$ , there exists a completely positive map  $\phi_X : C^*(X, A) \rightarrow C^*(X, B)$  extending  $\phi$ .

This functoriality extends to  $KK$ -theory.

**Theorem 4.1.14.** There exists a natural transformation

$$\sigma_X : KK(A, B) \rightarrow KK(C^*(X, A), C^*(X, B))$$

which transforms the Kasparov product into composition, i.e.

$$\sigma_X(z \otimes_D z') = \sigma_X(z') \circ \sigma_X(z) \quad \forall z \in KK(A, D), z' \in KK(D, B),$$

where the  $KK$ -elements  $\sigma_X(z \otimes_D z')$ ,  $\sigma_X(z)$  and  $\sigma_X(z')$  are seen as the maps they induce in  $K$ -theory.

**Remark 4.1.15.** This relation has to be thought as if  $\sigma_X$  respects the Kasparov product. The problem is that  $C^*(X, A)$  is not a separable  $C^*$ -algebra in general, a condition which is required for the domain of the left element in the Kasparov product. As  $K(A) \cong KK(\mathbb{C}, A)$  for every  $C^*$ -algebra  $A$ , if  $x \in K(C^*(X, A))$ ,  $x \otimes_{C^*(X, A)} \sigma_X(z) \in KK(\mathbb{C}, C^*(X, B)) \cong K(C^*(X, B))$  is well defined. The map  $x \mapsto x \otimes_{C^*(X, A)} \sigma_X(z)$  is what we called the map induced by  $\sigma_X(z)$  in  $K$ -theory.

Let us recall the construction of  $\sigma_X$ . Let us first notice that  $C^*(X; B \otimes \mathfrak{K}) \cong C^*(X, B)$ . Indeed,  $H$  being the separable Hilbert space, any linear isomorphism induces a  $*$ -isomorphism  $\mathfrak{K} \otimes \mathfrak{K} \cong \mathfrak{K}$ . An element  $T \in C_R[X, B \otimes \mathfrak{K}]$  is then an operator such that  $T_{xy} \in \mathfrak{K}_{B \otimes \mathfrak{K}} = \mathfrak{K} \otimes \mathfrak{K} \otimes B \cong \mathfrak{K} \otimes B$  for every  $x, y \in X$ . As the propagation is on the  $l^2(X)$  factor, this induces the claimed isomorphism.

Let  $(H_B, \pi, T) \in \mathbb{E}(A, B)$  be a  $K$ -cycle. Put  $E = l^2(X) \otimes H \otimes A$ . We can construct the internal tensor product  $E \otimes_\pi H_B$  with the help of the  $*$ -homomorphism  $\pi : A \rightarrow \mathcal{L}_B(H_B)$ . We also dispose of a  $*$ -homomorphism  $\mathcal{L}_A(E) \rightarrow \mathcal{L}_B(E \otimes_\pi H_B)$ . But the map

$$(\xi \otimes a) \otimes_\pi \eta \in E \otimes_\pi H_B \mapsto \xi \otimes (\pi(a)\eta) \in l^2(X) \otimes H \otimes H_B$$

extends to an isometry  $E \otimes_\pi H_B \rightarrow l^2(X) \otimes H \otimes H_B$ . As in remark 1.2.13, the conjugation by this isometry sends compact operators to compact operators. Moreover, it does not alter propagation, hence  $\mathcal{L}_A(E) \rightarrow \mathcal{L}_B(E \otimes_\pi H_B) \subseteq \mathcal{L}_B(l^2(X) \otimes H \otimes H_B)$  induces a map  $\tilde{\pi} : C^*(X, A) \rightarrow M(C^*(X, B \otimes \mathfrak{K})) \cong \mathcal{L}_{C^*(X, B)}(C^*(X, B))$ . Define  $\tilde{T}$  as  $1 \otimes T$  acting on  $l^2(X) \otimes H \otimes H_B$ . Then, according to remark 1.2.13, the  $K$ -cycle relations are satisfied for  $\tilde{T}$  and  $\tilde{\pi}$ . Hence  $(l^2(X) \otimes H \otimes B, \tilde{\pi}, \tilde{T}) \in \mathbb{E}(C^*(X, A), C^*(X, B))$  and we put  $\sigma_X([H_B, \pi, T]) = [l^2(X) \otimes H \otimes B, \tilde{\pi}, \tilde{T}] \in KK(C^*(X, A), C^*(X, B))$ .

A proof of the compatibility with Kasparov products can be found in [39]. Notice than it can be deduced from Proposition 4.2.6, where we construct a controlled morphism which induces right-multiplication by  $\sigma_X(z)$  in  $K$ -theory and respects Kasparov products.

Let  $X$  be a coarse space with bounded geometry. The Rips complex is the inductive system of finite dimensional simplicial complexes

$$P_E(X) = \{\eta \in \text{Prob}(X) \text{ s.t. } \text{supp } \eta \subseteq E\} \quad , \forall E \in \mathcal{E}_X,$$

endowed with the weak- $*$  topology. It is an inductive system with respect to inclusion of entourages.

Let us recall the definition of the left-hand side of the coarse assembly map. The coarse homology group of  $X$  with coefficients in an arbitrary  $C^*$ -algebra  $B$  is defined as

$$KX_*(X, B) = \varinjlim_{E \in \mathcal{E}_X} KK(C_0(P_E(X)), B)$$



where the limit is taken along the inductive system  $\{P_E(X)\}_{E \in \mathcal{E}_X}$ .

Let  $E \in \mathcal{E}_X$ . Any simplex  $\eta \in P_E(X)$  can be written as a finite sum  $\eta = \sum_x \lambda_x(\eta) \delta_x$  by boundedness of the geometry. Here,  $\delta_x$  is the Dirac measure at  $x \in X$ . The functions  $\lambda_x : \eta \mapsto \lambda_x(\eta)$  are continuous and satisfy  $\sum \lambda_x(\eta) = 1, \forall \eta \in P_E(X)$ , so that

$$V_0 \begin{cases} C_0(P_E(X)) & \rightarrow l^2(X) \otimes C_0(P_E(X)) \\ f & \mapsto (\lambda_x^{\frac{1}{2}} f)_{x \in X} \end{cases} \quad (1)$$

is an isometry of  $C_0(P_E(X))$ -modules. Let  $e$  be a rank one projection in  $\mathfrak{K}$  and set  $\mathcal{L}_E = V_0 V_0^* \otimes e$ . This defines a locally compact projection in  $l^2(X) \otimes H \otimes C_0(P_E(X))$  with finite propagation (less than  $R = \sup_E d$ , because  $\text{supp}(V_0 V_0^*) \subseteq E$ ), so a class  $[\mathcal{L}_E] \in K_0(C^*(X, C_0(P_E(X))))$ .

**Definition 4.1.16.** The coarse assembly map  $\mu : KX_*(X, B) \rightarrow K_*(C^*(X, B))$  for  $X$  with coefficients in  $B$  is defined as

$$\forall z \in RK(P_E(X), B), \mu_{X,B}(z) = [\mathcal{L}_E] \otimes_{C^*(X, C_0(P_E(X)))} \sigma_X(z).$$

The coarse Baum-Connes conjecture is the following claim.

**Conjecture 4.1.17** (Coarse Baum-Connes conjecture). For any coarse space  $X$  with bounded geometry, the coarse assembly map  $\mu_{X,C}$  is an isomorphism.

One of the interesting properties of the coarse Baum-Connes conjecture is the following application to topology, which is called the descent principle. The reader is referred to J. Roe's book [46] for details. Recall that if  $\Gamma$  is a finitely generated group,  $|\Gamma|$  denotes the metric space obtained by choosing a set of generators and endowing  $\Gamma$  with the associated word-length metric. Also,  $B\Gamma$  denotes the classifying space of  $\Gamma$ , which is defined in any standard textbook on algebraic topology, see [34] for instance. Definitions and statements of the Novikov conjectures can be found in [13]. A short survey can be found in the introduction at 0.0.3.

**Theorem 4.1.18** (Descent principle). Let  $\Gamma$  be a finitely generated group such that the classifying space  $B\Gamma$  has the homotopy type of a finite CW-complex. Then, the coarse Baum-Connes conjecture for  $|\Gamma|$  implies the strong Novikov conjecture for  $\Gamma$ .

Here are some remarks about the status of the conjecture :

- the conjecture is known to hold for any coarse space that admits a coarse embedding into a separable Hilbert space [60].
- counterexamples have been constructed [19].

Here is another description of the coarse assembly map which will be of some interest for us.

Let  $E \in \mathcal{E}_X$  be an entourage. Take a cycle  $(H, \pi, T) \in E(C_0(P_E(X)), \mathbb{C})$ , and equip the finite dimensional simplicial complex  $P_E(X)$  with a metric which restricts to the spherical

metric on simplices, so that  $H$  is a  $P_E(X)$ -module that we can suppose standard non-degenerate. Then

$$T' = \sum \lambda_x^{\frac{1}{2}} T \lambda_x^{\frac{1}{2}}$$

is a compact perturbation of  $T$  in  $\mathcal{L}(H)$  with bounded propagation, and is invertible modulo  $C^*(P_E(X), H)$ , so defines a class  $[T'] \in K_0(C^*(P_E(X), H))$ . The same formula for  $V_0$  than formula 1, i.e.  $V_0 f = (\lambda_x^{\frac{1}{2}} f)_x$  defines an isometry of Hilbert spaces, still denoted  $V_0$ , from  $l^2(P_E(X))$  to  $l^2(X) \otimes l^2(P_E(X))$ . Multiplication by  $C_0(P_E(X))$  on the  $l^2(P_E(X))$  factor endows these Hilbert spaces with a structure of  $P_E(X)$ -module, with respect to which the propagation of  $V_0$  is bounded by  $s = 2 \sup_E d$ . Put  $V = 1 \otimes V_0$  which defines a  $*$ -homomorphism

$$Ad_V : \begin{cases} \mathcal{L}(H \otimes l^2(X) \otimes l^2(P_E(X))) & \rightarrow \mathcal{L}(H \otimes l^2(X) \otimes l^2(P_E(X))) \\ T & \mapsto VTV^* \end{cases}$$

which sends operators of finite propagation to operators of finite propagation and respects local compactness. Hence, it induces a  $*$ -homomorphism  $C^*(P_E(X)) \rightarrow C^*(X)$  and a homomorphism

$$(Ad_V)_* : K_*(C^*(P_E(X))) \rightarrow K_*(C^*(X)).$$

With that in mind, one can show that  $\mu_X([H, \pi, T]) = (Ad_{V_0})_*[T']$ .

Indeed, with the same notations and if  $z = [H, \pi, T] \in KK(C_0(P_E(X)), \mathbb{C})$ , we want to compute the Kasparov product  $[\mathcal{L}_E] \otimes \sigma_X(z)$ . According to Proposition 8.7.2 of [20], this is equal to the Fredholm index of  $P(1 \otimes T)P$  where  $P = VV^*$ . A simple computation shows that  $V^*(1 \otimes T)V = T'$  so that  $P(1 \otimes T)P = Ad_V(T')$ , hence the equality holds.

The point of the article [47] is to show that the coarse assembly map is equivalent to the assembly map for the coarse groupoid  $G = G(X)$  with coefficients in the  $G$ -algebra  $l^\infty = l^\infty(X, \mathfrak{K})$ .

More precisely, let  $x \in X$ . Then  $Z = P_E(G)$  is a proper cocompact  $G$ -space with fiber  $Z_x \simeq P_E(X)$ , and  $l_x^\infty \simeq \mathfrak{K}$ , so that the inclusion of groupoid  $\{x\} \hookrightarrow G$  induces a morphism  $\iota_Z : RK^G(Z, l^\infty) \rightarrow KK(Z_x, \mathfrak{K})$  which is actually an isomorphism. Recall that one can construct a  $*$ -isomorphism  $\Psi_X : l^\infty \rtimes_r G \rightarrow C^*(X)$ , and  $\mu_X^{(Z_x)} \circ \iota_Z = (\Psi_X)_* \circ \mu_{G, l^\infty}^{(Z)}$  holds and respects inductive limits over  $Z$  to give the following commutative diagram :

$$\begin{array}{ccc} K_*^{top}(\underline{E}G, l^\infty) & \xrightarrow{\mu_{G, l^\infty}} & K_*(l^\infty \rtimes_r G) \\ \downarrow \lim \iota_Z & & \downarrow (\Psi_X)_* \\ KX_*(X) & \xrightarrow{\mu_X} & K_*(C^*(X)) \end{array}$$

with vertical arrows being isomorphism. We will give another proof of this result, based on controlled  $K$ -theory. Indeed, we actually prove a stronger result in theorem 5.1.4, which induces the previous result.

A key point is to use the correspondence between analytical properties of  $G$  and coarse properties of  $X$ . In particular, the results of theorem 2.3.5, combined with theorem 4.1.10, give examples of coarse spaces satisfying the coarse Baum-Connes conjecture and the boundary coarse Baum-Connes conjecture.

## 4.2 Controlled assembly maps for coarse spaces

We will first construct a controlled assembly map for coarse space  $(X, \mathcal{E})$ . In this section,  $X$  will be a discrete metric space with bounded geometry, and  $\mathcal{E}$  is the coarse structure generated by its controlled subsets. We also fix a separable Hilbert space  $H$ . For  $R > 0$ ,  $\Delta_R$  is  $\{(x, y) \in X \times X \text{ s.t. } d(x, y) < R\}$ .

$H_B$  denotes the standard  $B$ -Hilbert module  $H \otimes B$ . Recall that for every  $x, y \in X$ , and  $T \in \mathcal{L}_B(l^2(X) \otimes H_B)$ , we put  $T_{xy} \in \mathcal{L}_B(H_B)$  to be the unique operator such that  $\langle T_{xy}\xi, \eta \rangle = \langle T(e_x \otimes \xi), e_y \otimes \eta \rangle$  for every  $x, y \in X$  and every  $\xi, \eta \in H_B$ .

Remark that the  $C^*$ -algebra  $C^*(X, B)$  is filtered by  $\mathcal{E}$ , and also by  $\mathbb{R}_+^*$ , seen as a coarse structure. Indeed, the composition law  $R \circ R' = R + R'$  provides  $\mathbb{R}_+^*$  with a coarse structure and  $C_R[X, B]C_{R'}[X, B] \subseteq C_{R+R'}[X, B]$ . For the  $\mathcal{E}$ -filtration, one can put :

$$C_E[X, B] = \{T \in \mathcal{L}(l^2(X) \otimes H_B) \text{ s.t. } T_{xy} \in \mathfrak{K}(H_B) \text{ and } T_{xy} = 0 \text{ for } (x, y) \notin E\} \quad \forall E \in \mathcal{E}.$$

To construct the coarse assembly map, we will need the following proposition.

Let  $A$  be a  $C^*$ -algebra. The theorem 4.1.12 allows us to take the image of the exact sequence  $0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0$  under the functor  $C^*(X, \cdot)$  to get the following filtered semi-split exact sequence

$$0 \rightarrow C^*(X, SA) \rightarrow C^*(X, CA) \rightarrow C^*(X, A) \rightarrow 0.$$

Let  $D_{X,A} : \hat{K}_*(C^*(X, A)) \rightarrow \hat{K}_*(C^*(X, SA))$  be the controlled boundary morphism associated to this last extension.

**Proposition 4.2.1.** Let  $A$  be a  $C^*$ -algebra. Then there exists a control pair  $(\lambda, h)$ , independent of  $X$  and  $A$ , such that  $D_{X,A}$  is  $(\lambda, h)$ -invertible.

**Proof.** Recall that  $0 \rightarrow \mathfrak{K}(l^2(\mathbb{N})) \rightarrow \mathcal{T}_0 \rightarrow S \rightarrow 0$  is the Toeplitz extension. Let  $\Psi$  be the obvious  $*$ -homomorphism  $SC^*(X, A) \rightarrow C^*(X, SA)$ . The following diagram has exact rows and commutes

$$\begin{array}{ccccc} SC^*(X, A) & \longrightarrow & CC^*(X, A) & \longrightarrow & C^*(X, A) \\ \downarrow \Psi & & \downarrow & & \downarrow \\ C^*(X, SA) & \longrightarrow & C^*(X, CA) & \longrightarrow & C^*(X, A) \end{array}$$

where vertical arrows are obvious inclusions. Remark 3.1.26 implies that

$$D_{X,A} = \Psi_* \circ D_{C^*(X,A)}.$$

The following diagram also has exact rows and commutes

$$\begin{array}{ccccc} \mathfrak{K}(l^2(\mathbb{N})) \otimes C^*(X, A) & \longrightarrow & \mathcal{T}_0 \otimes C^*(X, A) & \longrightarrow & SC^*(X, A) \\ \downarrow & & \downarrow & & \downarrow \Psi \\ C^*(X, \mathfrak{K}(l^2(\mathbb{N})) \otimes A) & \longrightarrow & C^*(X, \mathcal{T}_0 \otimes A) & \longrightarrow & C^*(X, SA) \end{array}$$

where vertical arrows are obvious inclusions. Remark 3.1.26 implies that

$$D_{\mathfrak{K}(l^2(\mathbb{N})) \otimes C^*(X,A), \mathcal{T}_0 \otimes C^*(X,A)} = D_{C^*(X,A), C^*(X, \mathcal{T}_0 \otimes A)} \circ \Psi_*.$$

A simple computation shows that

$$D_{C^*(X,A), C^*(X, \mathcal{T}_0 \otimes A)} \circ D_{X,A} \sim D_{\mathfrak{K}(l^2(\mathbb{N})) \otimes C^*(X,A), \mathcal{T}_0 \otimes C^*(X,A)} \circ D_{C^*(X,A)} \sim \mathcal{M}_{C^*(X,A)}$$

□

**Remark 4.2.2.** This result induces a similar statement in  $K$ -theory. Namely, the boundary maps of the extensions  $0 \rightarrow C^*(X, SA) \rightarrow C^*(X, CA) \rightarrow C^*(X, A) \rightarrow 0$  and  $0 \rightarrow \mathfrak{K}(l^2(\mathbb{N})) \otimes C^*(X, A) \rightarrow \mathcal{T}_0 \otimes C^*(X, A) \rightarrow SC^*(X, A) \rightarrow 0$  are inverse of each other in  $K$ -theory.

### 4.2.1 Controlled Roe transform

Every  $K$ -cycle  $z \in KK(A, B)$  can be represented as a triplet  $(H_B, \pi, T)$  where :

- $\pi : A \rightarrow \mathcal{L}_B(H_B)$  is a  $*$ -representation of  $A$  on  $H_B$ .
- $T \in \mathcal{L}_B(H_B)$  is a self-adjoint operator.
- $T$  and  $\pi$  satisfy the  $K$ -cycle condition, i.e.  $[T, \pi(a)]$ ,  $\pi(a)(T^* - T)$  and  $\pi(a)(T^2 - id_{H_B})$  are compact operators in  $\mathfrak{K}_B(H_B) \cong \mathfrak{K} \otimes B$  for all  $a \in A$ .

We first define a controlled morphism  $\hat{\sigma}_X(z) : \hat{K}(A) \rightarrow \hat{K}(B)$  of odd degree for all  $z \in KK(A, B)$ , which we name the controlled Roe transform. It induces  $-\otimes \sigma_X(z)$  in  $K$ -theory, and will be needed in the definition of the Controlled Coarse Assembly map. Recall that if  $\phi : A \rightarrow B$  is a  $*$ -homomorphism, we denote by  $\phi_X : C^*(X, A) \rightarrow C^*(X, B)$  the induced  $*$ -homomorphism.

#### Odd case

For  $z \in KK_1(A, B)$ , represented by  $(H_B, \pi, T) \in E(A, B)$ , define  $P = (\frac{1+T}{2}) \in \mathcal{L}_B(H_B)$  and

$$E^{(\pi, T)} = \{(a, P\pi(a)P + y) : a \in A, y \in B \otimes \mathfrak{K}\}$$

which is a  $C^*$ -algebra such that the following sequence :

$$0 \rightarrow B \otimes \mathfrak{K} \rightarrow E^{(\pi, T)} \rightarrow A \rightarrow 0 .$$

is exact and semi-split by the completely positive section  $s : A \rightarrow B \otimes \mathfrak{K}; a \mapsto P\pi(a)P$ . Define  $E_X = C^*(X, E^{(\pi, T)})$ . Up to the  $*$ -isomorphism  $C^*(X, B \otimes \mathfrak{K}) \cong C^*(X, B)$ , the following sequence

$$0 \rightarrow C^*(X, B) \rightarrow E_X^{(\pi, T)} \rightarrow C^*(X, A) \rightarrow 0 .$$

is exact and semi-split by the completely positive section  $s_X : C^*(X, A) \rightarrow E_X^{(\pi, T)}$ .

**Proposition 4.2.3.** The controlled boundary map  $D^{(\pi,T)} = D_{C^*(X,B),E_X^{(\pi,T)}}$  of the extension  $E_X^{(\pi,T)}$  only depends on the class  $z$ .

**Proof.** Let  $(H_B, \pi_j, T_j), j = 0, 1$  two  $K$ -cycles which are homotopic via  $(H_{B[0,1]}, \pi, T)$ . We denote by  $e_t$  the evaluation at  $t \in [0, 1]$  for an element of  $B[0, 1]$ , and set  $y_t = (e_t)_X(y)$  for every  $y \in C^*(X, B[0, 1])$ . The  $*$ -homomorphism

$$\phi : \begin{cases} E_X^{(\pi,T)} & \rightarrow E_X^{(\pi_t, T_t)} \\ (x, y) & \mapsto (x, y_t) \end{cases}$$

satisfies  $\phi(C^*(X, B[0, 1])) \subseteq C^*(X, B)$  and makes the following diagram commute

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^*(X, B[0, 1]) & \longrightarrow & E_X^{(\pi,T)} & \longrightarrow & C^*(X, A) \longrightarrow 0 \\ & & \downarrow \phi|_{C^*(X, B[0, 1])} & & \downarrow \phi & & \downarrow = \\ 0 & \longrightarrow & C^*(X, B) & \longrightarrow & E_X^{(\pi_t, T_t)} & \longrightarrow & C^*(X, A) \longrightarrow 0 \end{array} .$$

According to remark 3.1.26, the following holds

$$D_{C^*(X,B),E_X^{(\pi_t, T_t)}} = \phi_* \circ D_{C^*(X, B[0, 1]), E_X^{(\pi, T)}} .$$

As  $id \otimes (e_t)_X$  gives a homotopy between  $id \otimes (e_0)_X$  and  $id \otimes (e_1)_X$ , and as if two  $*$ -homomorphisms are homotopic, then they are equal in controlled  $K$ -theory,

$$D_{C^*(X,B),E_X^{(\pi_0, T_0)}} = D_{C^*(X,B),E_X^{(\pi_1, T_1)}}$$

holds, and the boundary of the extension  $E_X^{(\pi,T)}$  depends only on  $z$ . □

**Definition 4.2.4.** For every  $z = [H_B, \pi, T] \in KK_1(A, B)$ , we define the Roe transformation  $\hat{\sigma}_X$  as

$$\hat{\sigma}_X(z) = D_{C^*(X,B),E_X^{(\pi,T)}} .$$

It is a  $(\alpha_D, k_D)$ -controlled morphism  $\hat{K}(C^*(X, A)) \rightarrow \hat{K}(C^*(X, B))$  of odd degree.

**Proposition 4.2.5.** Let  $A$  and  $B$  two  $C^*$ -algebras. There exists a control pair  $(\alpha_X, k_X)$  such that for every  $z \in KK_1(A, B)$ , there exists a  $(\alpha_X, k_X)$ -controlled morphism

$$\hat{\sigma}_X(z) : \hat{K}_*(C^*(X, A)) \rightarrow \hat{K}_{*+1}(C^*(X, B))$$

such that

- (i)  $\hat{\sigma}_X(z)$  induces right multiplication by  $\sigma_X(z)$  in  $K$ -theory ;
- (ii)  $\hat{\sigma}_X$  is additive, i.e.

$$\hat{\sigma}_X(z + z') = \hat{\sigma}_X(z) + \hat{\sigma}_X(z') .$$

- (iii) For every  $*$ -homomorphism  $f : A_1 \rightarrow A_2$ ,

$$\hat{\sigma}_X(f^*(z)) = \hat{\sigma}_X(z) \circ f_{X,*}$$

for every  $z \in KK_1(A_2, B)$ .

(iv) For every  $*$ -homomorphism  $g : B_1 \rightarrow B_2$ ,

$$\hat{\sigma}_X(g_*(z)) = g_{X,*} \circ \hat{\sigma}_X(z)$$

for every  $z \in KK_1(A, B_1)$ .

(v) Let  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  be a semi-split extension of  $C^*$ -algebras and  $[\partial_J] \in KK_1(A/J, J)$  be its boundary element. Then

$$\hat{\sigma}_X([\partial_{J,A}]) = D_{C^*(X,J), C^*(X,A)}.$$

**Proof.** (i) The  $K$ -cycle  $[\partial_{C^*(X,B), E_X^{(\pi,T)}}] \in KK_1(C^*(X, A), C^*(X, B))$  implementing the boundary of the extension  $E^{(\pi,T)}$  induces the map  $\sigma_X(z)$  by definition which immediately gives the first point.

(ii) If  $z, z'$  are elements of  $KK_1(A, B)$ , represented by two  $K$ -cycles  $(H_B, \pi_j, T_j)$ , let  $(\tilde{H}_B, \pi, T)$  be  $(H_B \oplus H_B, \pi_0 \oplus \pi_1, T_0 \oplus T_1)$ , which is a  $K$ -cycle representing the sum  $z + z'$ . Then  $E_X^{(\pi,T)}$  is naturally isomorphic to the extension sum of the  $E_X^{(j)} := E_X^{(\pi_j, T_j)}$ , namely

$$0 \rightarrow \mathfrak{M}_2(C^*(X, B)) \rightarrow D \rightarrow C^*(X, A) \rightarrow 0$$

where

$$D = \left\{ \begin{pmatrix} x_1 & k_{12} \\ k_{21} & x_2 \end{pmatrix} : x_j \in E_X^{(j)}, p_1(x_1) = p_2(x_2), k_{ij} \in C^*(X, B) \right\},$$

with  $p_j : E_X^{(j)} \rightarrow C^*(X, A)$  the  $*$ -homomorphisms of the extensions. Naturality of the controlled boundary maps 3.1.26 ensures that the boundary of the sum of two extensions is the sum of the boundary of each, thus the result holds.

(iii) Let  $z \in KK_1(A_2, B)$ , represented by a cycle  $(H_B, \pi, T)$ . Representing  $f^*(z)$  is  $(H_B, f^*\pi, T)$  with  $f^*\pi = \pi \circ f$ . The map

$$\phi : \begin{cases} E_X^{f^*(\pi,T)} & \rightarrow E_X^{(\pi,T)} \\ (x, P_X(f^*\pi)(x)P_X + y) & \rightarrow (f_X(x), P_X(f^*\pi)_X(x)P_X + y) \end{cases}$$

satisfies

- $\phi(C^*(X, B)) \subseteq C^*(X, B)$ , and makes the following diagram commutes

$$\begin{array}{ccccccc} 0 & \rightarrow & C^*(X, B) & \rightarrow & E^{f^*(\pi,T)} & \rightarrow & C^*(X, A_1) \rightarrow 0 \\ & & \downarrow = & & \downarrow \phi & & \downarrow f_X \\ 0 & \rightarrow & C^*(X, B) & \rightarrow & E^{(\pi,T)} & \rightarrow & C^*(X, A_2) \rightarrow 0 \end{array} \quad .$$

- It intertwines the sections of the two extensions.

Remark 3.1.26 ensures that

$$D_{C^*(X,B),E_X^{f^*(\pi,T)}} = D_{C^*(X,B),E_X^{(\pi,T)}} \circ f_{X,*},$$

and the claim is clear.

- (iv) Let  $z \in KK(A, B_1)$  be represented by the  $K$ -cycle  $(H_{B_1}, \pi, T)$ . Let  $V \in \mathcal{L}_{B_2}(H_{B_1} \otimes_g B_2)$  be the isometry of remark 1.2.13. According to Lemma 2.1.41,

$$g_*(z) = [H_{B_1} \otimes_g B_2, \pi \otimes_g 1, T \otimes_g 1] \in KK^G(A, B_2)$$

is also represented by  $[H_{B_2}, \pi', T']$  where  $\pi' = Ad_V \circ (\pi \otimes_g 1)$  and  $T' = V(T \otimes_g 1)V^* + 1 - VV^*$ . Let  $\psi$  be given by the composition  $Ad_{V_X} \circ g_X$ .

The map  $\Psi : (x, y) \mapsto (x, \psi(y))$  defines a  $*$ -homomorphism  $E_X^{(\pi,T)} \rightarrow E_X^{(\pi',T')}$  such that

$$\Psi(x, P_X \pi_X(x) P_X + y) = (x, P'_X \pi'_X(x) P'_X + \psi(y)).$$

Indeed, the functor  $C^*(X, -)$  commutes with pull-back by  $*$ -homomorphisms, and  $Ad_{V_X} \circ g_X \circ \pi_X = (Ad_V \circ g \circ \pi)_X = \pi'_X$  and  $\psi(P_X) = V_X(P_X \otimes_{g_X} 1)V_X^* = (V(P \otimes_g 1)V^*)_X = (P')_X$  so that

$$\psi(P_X \pi_X(x) P_X) = P'_X \pi'_X(x) P'_X.$$

This gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^*(X, B_1) & \longrightarrow & E_X^{(\pi,T)} & \longrightarrow & C^*(X, A) \longrightarrow 0 \\ & & \downarrow \psi & & \downarrow \Psi & & \downarrow = \\ 0 & \longrightarrow & C^*(X, B_2) & \longrightarrow & E_X^{(\pi',T')} & \longrightarrow & C^*(X, A) \longrightarrow 0 \end{array} .$$

and  $\Psi$  intertwines the two filtered sections by the previous relation. Moreover,  $\Psi(C^*(X, B_1)) \subseteq C^*(X, B_2)$ , so that we can again apply the remark 3.1.26 to state

$$D_{C^*(X,B_2),E_X^{(\pi',T')}} = \psi_* \circ D_{C^*(X,B_1),E_X^{(\pi,T)}} .$$

Under the identification  $C^*(X, B \otimes \mathfrak{K}) \cong C^*(X, B)$ ,  $\psi = g_X$ , hence

$$\hat{\sigma}_X(g_*(z)) = g_{X,*} \circ D_{C^*(X,B_1),E_X^{(\pi,T)}} = g_{X,*} \circ \hat{\sigma}_X(z).$$

- (v) We can suppose  $A$  unital. Let  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  be a semi-split extension of  $C^*$ -algebras with  $q : A \rightarrow A/J$  the quotient map.. Let us denote by  $s : A/J \rightarrow A$  the completely positive cross section. According to Kasparov-Stinespring theorem 1.2.14, there exists a  $A$ -Hilbert module  $E$  and a  $*$ -homomorphism  $\pi : A/J \rightarrow \mathcal{L}_A(A \oplus E)$  such that  $s(x) = P\pi(x)P$ , where  $P \in \mathcal{L}_A(A \oplus E)$  is the projection on the  $A$  factor. Consider the  $J$ -Hilbert module  $E' = (A \oplus E) \otimes_J J \cong J \oplus (E \otimes_J J)$ , and the natural map  $\tilde{\pi} = \pi \otimes_J 1 : A/J \rightarrow \mathcal{L}_J(E')$ . Put  $\tilde{T} = (2P - 1) \otimes_J id_J$ . By the stabilization theorem, we can suppose that  $E'$  is a standard  $J$ -Hilbert module. Put

$$\psi_0(x)(y \oplus \xi) = (xy) \oplus 0 \quad \forall \xi \in E \otimes J, \forall y \in J,$$

for every  $x \in A$ . This defines a  $*$ -homomorphism  $\psi_0 : A \rightarrow \mathcal{L}_J(E')$  such that  $\psi_0(x) \in \mathfrak{K}_J(E')$  when  $x \in J$ . Set  $\psi(a) = (q(a), \psi_0(a))$ . As  $P\pi(q(a))P - \psi_0(a) = \psi_0(s(q(a)) - a) \in \mathfrak{K}_J(E')$ ,  $\psi(a) \in E^{(\tilde{\pi}, \tilde{T})}$  holds, and the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & A/J \longrightarrow 0 \\ & & \downarrow \psi_0 & & \downarrow \psi & & \downarrow = \\ 0 & \longrightarrow & \mathfrak{K}_J(E') & \longrightarrow & E^{(\tilde{\pi}, \tilde{T})} & \longrightarrow & A/J \longrightarrow 0 \end{array},$$

Hence, by functoriality and semi-split exactness of the Roe algebra, the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^*(X, J) & \longrightarrow & C^*(X, A) & \longrightarrow & C^*(X, A/J) \longrightarrow 0 \\ & & \downarrow (\psi_0)_X & & \downarrow \psi_X & & \downarrow = \\ 0 & \longrightarrow & C^*(X, J) & \longrightarrow & E_X^{(\tilde{\pi}, \tilde{T})} & \longrightarrow & C^*(X, A/J) \longrightarrow 0 \end{array},$$

and remark 3.1.26 ensures that  $\hat{\sigma}_X([\partial_J]) = D_{C^*(X, J), C^*(X, A)}$ .

□

### Even case

We can now define  $\hat{\sigma}_X$  for even  $K$ -cycles. Let  $A$  and  $B$  be two  $C^*$ -algebras and  $z \in KK(A, B)$ . Recall that  $[\partial_{SB}] \in KK_1(B, SB)$  is the  $K$ -cycle implementing the boundary of the extension  $0 \rightarrow SB \rightarrow CB \rightarrow B \rightarrow 0$ , and  $[\partial] \in KK_1(\mathbb{C}, S)$  is the Bott generator. Recall from proposition 4.2.1 that  $D_{X, A}$  and  $D_{C^*(X, A), C^*(X, \mathcal{T}_0 \otimes A)}$  are controlled inverse of each other. We will denote  $D_{C^*(X, A), C^*(X, \mathcal{T}_0 \otimes A)}$  by  $T_{X, A}$ .

As  $z \otimes_B [\partial_{SB}]$  is an odd  $K$ -cycle, we can define

$$\hat{\sigma}_X(z) := T_{X, B} \circ \hat{\sigma}_X(z \otimes [\partial_{SB}]).$$

**Proposition 4.2.6.** Let  $A$  and  $B$  two  $C^*$ -algebras. For every  $z \in KK_*(A, B)$ , there exists a control pair  $(\alpha_X, k_X)$  and a  $(\alpha_X, k_X)$ -controlled morphism

$$\hat{\sigma}_X(z) : \hat{K}(C^*(X, A)) \rightarrow \hat{K}(C^*(X, B))$$

of the same degree as  $z$ , such that

(i)  $\hat{\sigma}_X(z)$  induces right multiplication by  $\sigma_X(z)$  in  $K$ -theory ;

(ii)  $\hat{\sigma}_X$  is additive, i.e.

$$\hat{\sigma}_X(z + z') = \hat{\sigma}_X(z) + \hat{\sigma}_X(z').$$

(iii) For every  $*$ -homomorphism  $f : A_1 \rightarrow A_2$ ,

$$\hat{\sigma}_X(f^*(z)) = \hat{\sigma}_X(z) \circ f_{X, *}$$

for all  $z \in KK_*(A_2, B)$ .



(iv) For every  $*$ -homomorphism  $g : B_1 \rightarrow B_2$ ,

$$\hat{\sigma}_X(g_*(z)) = g_{X,*} \circ \hat{\sigma}_X(z)$$

for all  $z \in KK_*(A, B_1)$ .

(v)  $\hat{\sigma}_X([id_A]) \sim_{(\alpha_X, k_X)} id_{\hat{K}(C^*(X, A))}$

**Proof.** The work is already done for odd  $KK$ -elements. Let  $z \in KK_0(A, B)$ .

(i) Proposition 4.2.5 ensures that  $\hat{\sigma}_X(z \otimes [\partial_{SB}])$  induces right multiplication by  $\sigma_X(z \otimes [\partial_{SB}])$  in  $K$ -theory. Proposition 4.2.1 implies that  $T_{X,B}$  induces multiplication by  $[\partial_{C^*(X,B), C^*(X,B \otimes \mathcal{T}_0)}]^{-1}$  in  $K$ -theory. Hence, according to theorem 4.1.14, their composition induces  $\sigma_X([\partial_{A, \mathcal{T}_0 \otimes A}]^{-1}) \circ \sigma_X([\partial_{SB}]) \circ \sigma_X(z)$  where we identify  $KK$ -elements with the maps they induce in  $K$ -theory. But this element is just  $\sigma_X(z)$ , because, by remark 4.2.2,  $\sigma_X([\partial_{A, \mathcal{T}_0 \otimes A}]^{-1}) \circ \sigma_X([\partial_{SB}]) = id_{K(C^*(X, B))}$ .

(ii) This is clear by 4.2.5.

(iii) It is a consequence of the previous proposition 4.2.5, and of the equality  $f^*(x) \otimes_D y = f^*(x \otimes_D y)$  for every  $*$ -homomorphism  $f : A' \rightarrow A$  and for every  $x \in KK(A, D), y \in KK(D, B)$ .

(iv) By naturality of the boundary element, we have  $g_*(z) \otimes_{B_2} [\partial_{SB_2}] = z \otimes_{B_1} g^*[\partial_{SB_2}] = (Sg)_*(z \otimes_{B_1} [\partial_{SB_1}])$ , hence, by proposition 4.2.5,

$$\hat{\sigma}_X(g_*(z)) = T_{X, B_2} \circ ((Sg)_X)_* \circ \hat{\sigma}_X(z \otimes_{B_1} [\partial_{SB_1}]).$$

Controlled boundaries are natural, hence  $T_{X, B_2} \circ ((Sg)_X)_* = (g_X)_* \circ T_{X, B_1}$ , and  $\hat{\sigma}_X(g_*(z)) = (g_X)_* \circ \hat{\sigma}_X(z)$ .

(v) The same kind of argument we used for the first point concludes. By definition,  $\hat{\sigma}_X([id_A]) = T_{X, A} \circ \hat{\sigma}_X([\partial_{SA}])$ , and, by Proposition 4.2.1,  $T_{X, A}$  is a controlled inverse of  $D_{C^*(X, SA), C^*(X, CA)} = D_{X, A}$  which is equal to  $\hat{\sigma}_X([\partial_{SA}])$  by point (v) of 4.2.5.

□

We now show that the Roe transform respects in a quantitative way the Kasparov product. Let us recall the following result from [27]. It states that every  $KK$ -element comes from the product of an element coming from a  $*$ -homomorphism and an element coming from the inverse in  $KK$ -theory of a  $*$ -homomorphism. The following lemma is a particular case of decomposition property  $d$ , defined in 2.1.34.

**Lemma 4.2.7** ([27], lemma 1.6.11). Let  $A$  and  $B$  be two  $C^*$ -algebras and  $z \in KK_0(A, B)$ . Then, there exists a  $C^*$ -algebra  $A_1$ , an element  $\alpha \in KK(A, A_1)$  and  $*$ -homomorphisms  $\theta : A_1 \rightarrow A$  and  $\eta : A_1 \rightarrow B$  such that  $\theta^*(\alpha) = id_{A_1}$ ,  $\theta_*(\alpha) = id_A$  and  $\theta^*(z) = \eta$ .

**Proposition 4.2.8.** There exists a control pair  $(\alpha_R, k_R)$  such that for every  $C^*$ -algebras  $A, B$  and  $C$ , and every  $z \in KK(A, B), z' \in KK(B, C)$ , the controlled equality

$$\hat{\sigma}_X(z \otimes_B z') \sim_{\alpha_R, k_R} \hat{\sigma}_X(z') \circ \hat{\sigma}_X(z)$$

holds.

**Proof.** Assume  $\alpha \in KK_0(A, B)$ . By naturality, the previous lemma reduces the proof to the special case of  $\alpha$  being the inverse of a  $*$ -homomorphism  $\theta : B \rightarrow A$  in  $KK$ -theory :  $\alpha \otimes_B [\theta] = 1_A$ . Let  $z \in KK(B, C)$  :

$$\begin{aligned} \hat{\sigma}_X(\alpha \otimes z) &\sim_{\alpha_{\alpha, k_J^* k_J}^2} \hat{\sigma}_X(\alpha \otimes z) \circ \hat{\sigma}_X(\alpha \otimes [\theta]) \\ &\sim \hat{\sigma}_X(\alpha \otimes z) \circ \hat{\sigma}_X(\theta_*(\alpha)) \\ &\sim \hat{\sigma}_X(\alpha \otimes z) \circ \theta_{X,*} \circ \hat{\sigma}_X(\alpha) \\ &\sim \hat{\sigma}_X(\theta^*(\alpha \otimes z)) \circ \hat{\sigma}_X(\alpha) \\ &\sim \hat{\sigma}_X(z) \circ \hat{\sigma}_X(\alpha) \end{aligned}$$

because  $\theta^*(\alpha \otimes z) = \theta^*(\alpha) \otimes z = 1 \otimes z = z$ . The control on the propagation of the first line follows from remark 3.1.17 and point (v), the other lines are equal by points (iii) and (iv), hence  $(\alpha_R, k_R)$  can be taken to be  $(2\alpha_X^4, (k_X)^{*2})$ . If  $z'$  is even, we can apply the same argument.

Let  $z$  and  $z'$  be odd  $KK$ -elements. Then :

$$\begin{aligned} \hat{\sigma}_X(z \otimes z') &= \hat{\sigma}_X(z \otimes_B [\partial_B] \otimes_{SB} [\partial_B]^{-1} \otimes_B z') \\ &\sim \hat{\sigma}_X([\partial_B]^{-1} \otimes_B z') \circ \hat{\sigma}_X(z \otimes_B [\partial_B]) \\ &\sim \hat{\sigma}_X([\partial_B]^{-1} \otimes_B z') \circ \hat{\sigma}_X([\partial_B]) \circ \hat{\sigma}_X([\partial_B]^{-1}) \circ \hat{\sigma}_X(z \otimes_B [\partial_B]) \\ &\sim \hat{\sigma}_X(z') \circ \hat{\sigma}_X(z), \end{aligned}$$

where we used the previous case for the second line, Lemma 4.2.5 for the third line, and Proposition 4.2.1 for the last one.  $\square$

## 4.2.2 Controlled coarse assembly maps

Let  $E \in \mathcal{E}$  be a controlled subset. Then any probability  $\eta$  of the Rips complex  $P_E(X)$  can be written as  $\eta = \sum_{x \in X} \lambda_x(\eta) \delta_x$ , where  $\delta_x$  is the Dirac probability at  $x$ , and  $\lambda_x : P_E(X) \rightarrow [0, 1]$  is a continuous function. Set :

$$h_E : \begin{cases} X \times X &\rightarrow C_0(P_E(X)) \\ (x, y) &\mapsto \lambda_x^{\frac{1}{2}} \lambda_y^{\frac{1}{2}} \end{cases}$$

Let  $(e_x)_{x \in X}$  be the canonical basis of  $l^2(X)$ ,  $e$  be a rank-one projection in  $H$  and  $P_E$  be defined as the extension by linearity and continuity of

$$P_E(e_x \otimes \xi \otimes f) = \sum_{y \in X} e_y \otimes (e\xi) \otimes (h(x, y)f)$$

for every  $x \in X$ ,  $\xi \in H$  and  $f \in C_0(P_E(X))$ . As  $\sum_{x \in X} \lambda_x = 1$ ,  $P_E$  is a projection of  $\mathfrak{K}(l^2(X)) \otimes C_0(P_E(X))$  of controlled support :  $\text{supp } P_E \subseteq E$ . Indeed,  $\lambda_x^{\frac{1}{2}} \lambda_y^{\frac{1}{2}} = 0$  as soon as  $(x, y) \notin E$ . Hence  $P_E$  defines a class  $[P_E, 0]_{\varepsilon, E'} \in K_0^{\varepsilon, E'}(C^*(X, C_0(P_E(X))))$  for any  $\varepsilon \in (0, \frac{1}{4})$  and any  $E' \in \mathcal{E}$  satisfying  $E \subseteq E'$ .

For every  $C^*$ -algebra  $B$  and every controlled subsets  $E, E' \in \mathcal{E}$  such that  $E \subseteq E'$ , the canonical inclusion  $P_E(X) \hookrightarrow P_{E'}(X)$  induces a  $*$ -homomorphism  $q_E^{E'} : C_0(P_{E'}(X)) \rightarrow C_0(P_E(X))$ , hence a map  $(q_E^{E'})^* : KK(C_0(P_E(X)), B) \rightarrow KK(C_0(P_{E'}(X)), B)$  in  $KK$ -theory. It induces another map  $((q_E^{E'})_X)^* : K(C^*(X, C_0(P_{E'}(X)))) \rightarrow K(C^*(X, C_0(P_E(X))))$  in  $K$ -theory. The family of projections  $P_E$  are compatible with the morphisms  $q_E^{E'}$ , i.e.  $((q_E^{E'})_X)^*[P_{E'}, 0]_{\varepsilon, E'} = [P_E, 0]_{\varepsilon, E}$ , for every  $\varepsilon \in (0, \frac{1}{4})$ .

**Definition 4.2.9.** Let  $B$  a  $C^*$ -algebra,  $\varepsilon \in (0, \frac{1}{4})$  and  $E, F \in \mathcal{E}_X$  controlled subsets such that  $k_X(\varepsilon).E \subseteq F$ . The controlled coarse assembly map  $\hat{\mu}_{X,B} = (\mu_{X,B}^{\varepsilon,E,F})_{\varepsilon,E}$  is defined as the family of maps

$$\hat{\mu}_{X,B}^{\varepsilon,E,F} : \begin{cases} KK(C_0(P_E(X)), B) & \rightarrow K^{\varepsilon,F}(C^*(X, B)) \\ z & \mapsto \iota_{\alpha_X \varepsilon', k_X(\varepsilon').F'}^{\varepsilon,F} \circ \hat{\sigma}_X(z)[P_E, 0]_{\varepsilon',F'} \end{cases}$$

where  $\varepsilon'$  and  $F'$  satisfy :

- $\varepsilon' \in (0, \frac{1}{4})$  such that  $\alpha_X \varepsilon' \leq \varepsilon$ ,
- and  $F' \in \mathcal{E}$  such that  $E \subseteq F'$  and  $k_X(\varepsilon').F' \subseteq F$ .

**Remark 4.2.10.** The controlled coarse assembly map is compatible with the structure morphisms  $q_E^{E'}$ . Indeed, for every  $E, E' \in \mathcal{E}$  such that  $E \subseteq E'$ , by proposition 4.2.6,

$$\hat{\sigma}_X((q_E^{E'})^*(z))[P_{E'}, 0]_{\varepsilon,E'} = \hat{\sigma}_X(z) \circ ((q_E^{E'})_X)^*[P_{E'}, 0]_{\varepsilon,E'} = \hat{\sigma}_X(z)[P_E, 0]_{\varepsilon,E}.$$

Hence  $\hat{\mu}_{X,B}^{\varepsilon,E,F} \circ (q_E^{E'})^* = \hat{\mu}_{X,B}^{\varepsilon,E',F}$ .

**Remark 4.2.11.** The controlled coarse assembly map is also compatible with the structure morphisms  $\iota_{\varepsilon,E}^{\varepsilon',E'}$ , i.e.  $\iota_{\varepsilon,E}^{\varepsilon',E'} \circ \hat{\mu}_{X,B}^{\varepsilon,E,F} = \hat{\mu}_{X,B}^{\varepsilon',E',F}$  for every  $F \subseteq F'$  and  $\varepsilon \leq \varepsilon'$  such that this equality is defined.

**Remark 4.2.12.** According to Proposition 4.2.6,  $\hat{\sigma}_X(z)$  induces right-multiplication by  $\sigma_X(z)$ . Hence, the controlled coarse assembly map  $\hat{\mu}_{X,B}$  induces the coarse assembly map  $\mu_{X,B}$  in  $K$ -theory.

**Remark 4.2.13.** This assembly map is defined for the usual Roe algebra of  $X$ , but could be defined for any "nice" completion of the algebraic Roe algebra  $\cup_{E \in \mathcal{E}_X} C_E[X]$ . In particular, we can define an assembly map with values in the controlled  $K$ -theory of the maximal Roe algebra  $C_{max}^*(X)$ , that we will denote by  $\hat{\mu}_X^{max}$ .

## 4.3 Controlled assembly maps for étale groupoids

In this section, we will always use a coarse structure  $\mathcal{E}$  of a locally compact  $\sigma$ -compact étale groupoid generated by a countable subset of its compact symmetric subsets such that for every compact subset  $K \subseteq G$ , there exists a  $E \in \mathcal{E}$  such that  $K \subseteq E$ .

### 4.3.1 Controlled Kasparov transform

In this section, for every  $G$ -algebra  $B$ , we use the notation  $K_{B \rtimes_r G}$  for  $\mathfrak{K}_{B \rtimes_r G}(B \rtimes_r G) \cong \mathfrak{K} \otimes (B \rtimes_r G)$ .

Let  $A$  and  $B$  be two  $G$ - $C^*$ -algebras, with actions denoted by  $\alpha : s^*A \rightarrow r^*A$  and  $\beta : s^*B \rightarrow r^*B$ . Recall that every  $K$ -cycle  $z \in KK^G(A, B)$  can be represented as a triple  $(H_B, \pi, T)$  where :

- $H_B$  is equipped with an action  $V \in \mathcal{L}_{s^*B}(s^*H_B, r^*H_B)$  of  $G$ .
- $\pi : A \rightarrow \mathcal{L}_B(H_B)$  is a  $G$ -equivariant  $*$ -homomorphism.

- $T \in \mathcal{L}_B(H_B)$  is a self-adjoint operator.
- $T$  and  $\pi$  satisfy the  $K$ -cycle conditions, i.e.  $[T, \pi(a)]$ ,  $\pi(a)(T^2 - id_{H_B})$  are compact operators in  $\mathfrak{K}_B(H_B)$  and  $\pi(a)(r^*T - Vs^*TV^*)$  are compact operators in  $\mathfrak{K}_{r^*B}(r^*H_B) \cong \mathfrak{K}_{r^*B}(H_{r^*B})$  for all  $a \in A, g \in G$ .

Set  $T_G = T \otimes id_{B \rtimes_r G} \in \mathcal{L}_{B \rtimes_r G}(H_B \otimes (B \rtimes_r G)) \simeq \mathcal{L}_{B \rtimes_r G}(H_{B \rtimes_r G})$ , and  $\pi_G : A \rtimes_r G \rightarrow \mathcal{L}_{B \rtimes_r G}(H_{B \rtimes_r G})$ . Then, according to Le Gall [31],  $(H_{B \rtimes_r G}, \pi_G, T_G)$  represents the  $K$ -cycle  $j_G(z) \in KK(A \rtimes_r G, B \rtimes_r G)$ . Let us construct a controlled morphism associated to  $z$ ,

$$J_G(z) : \hat{K}(A \rtimes_r G) \rightarrow \hat{K}(B \rtimes_r G),$$

which induces right multiplication by  $j_G(z)$  in  $K$ -theory.

### Odd case

Let us start with an odd element  $z \in KK_1^G(A, B)$ . Let  $(H_B, \pi, T)$  be a  $K$ -cycle representing  $z$ . Set  $P = \frac{1+T}{2}$  and  $P_G = P \otimes id_{B \rtimes_r G}$ . We define

$$E^{(\pi, T)} = \{(x, P_G \pi_G(x) P_G + y) : x \in A \rtimes_r G, y \in K_{B \rtimes_r G}\}$$

a  $C^*$ -algebra which is filtered by

$$E_U^{(\pi, T)} = \{(x, P_G \pi_G(x) P_G + y) : x \in (A \rtimes_r G)_U, y \in K \otimes (B \rtimes_r G)_U\}$$

for all  $U \in \mathcal{E}$ . This  $C^*$ -algebra fits into the filtered extension

$$0 \longrightarrow K_{B \rtimes_r G} \longrightarrow E^{(\pi, T)} \longrightarrow A \rtimes_r G \longrightarrow 0$$

which is semi split by  $s : \begin{cases} A \rtimes_r G & \rightarrow E^{(\pi, T)} \\ x & \mapsto (x, P_G \pi_G(x) P_G) \end{cases}$ .

Let us show that the controlled boundary map of this extension does not depend on the representant chosen, but only on the class  $z$ .

**Lemma 4.3.1.** With the above notations, the controlled boundary map  $D_{K_{B \rtimes_r G}, E^{(\pi, T)}}$  only depends on the class  $z$ .

**Proof.** Let  $(H_B, \pi_j, T_j), j = 0, 1$  two  $K$ -cycles which are homotopic via  $(H_{B[0,1]}, \pi, T)$ . We denote  $e_t$  the evaluation at  $t \in [0, 1]$  for an element of  $B[0, 1]$ , and set  $y_t = e_t(y)$  for such a  $y$ . The  $*$ -morphism

$$\phi : \begin{cases} E^{(\pi, T)} & \rightarrow E^{(\pi_t, T_t)} \\ (x, y) & \mapsto (x, y_t) \end{cases}$$

satisfies  $\phi(K_{B[0,1] \rtimes_r G}) \subseteq K_{B \rtimes_r G}$  and makes the following diagram commute

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{B[0,1] \rtimes_r G} & \longrightarrow & E^{(\pi, T)} & \longrightarrow & A \rtimes_r G \longrightarrow 0 \\ & & \downarrow \phi|_{K_{B[0,1] \rtimes_r G}} & & \downarrow \phi & & \downarrow = \\ 0 & \longrightarrow & K_{B \rtimes_r G} & \longrightarrow & E^{(\pi_t, T_t)} & \longrightarrow & A \rtimes_r G \longrightarrow 0 \end{array} .$$

According to remark 3.1.26, the following holds

$$D_{K_{B \rtimes_r G}, E(\pi_t, T_t)} = \phi_* \circ D_{K_{B[0,1] \rtimes_r G}, E(\pi, T)}.$$

As  $id \otimes e_t$  gives a homotopy between  $id \otimes e_0$  and  $id \otimes e_1$ , and as if two  $*$ -morphisms are homotopic, then they are equal in controlled  $K$ -theory,

$$D_{K_{B \rtimes_r G}, E(\pi_0, T_0)} = D_{K_{B \rtimes_r G}, E(\pi_1, T_1)}$$

holds, and the boundary of the extension  $E^{(\pi, T)}$  depends only on  $z$ . □

**Definition 4.3.2.** The controlled Kasparov transform of an element  $z \in KK_1^G(A, B)$  is defined as the composition

$$J_{red, G}(z) = \mathcal{M}_{B \rtimes_r G}^{-1} \circ D_{K_{B \rtimes_r G}, E(\pi, T)}.$$

As the boundary map is a  $(\alpha_D, k_D)$ -controlled morphism and the Morita equivalence preserves the filtration,  $J_{red, G}(z)$  is  $(\alpha_D, k_D)$ -controlled.

**Proposition 4.3.3.** Let  $A$  and  $B$  two  $G$ - $C^*$ -algebras. There exists a control pair  $(\alpha_J, k_J)$  such that for every  $z \in KK_1^G(A, B)$ , there exists a  $(\alpha_J, k_J)$ -controlled morphism

$$J_{red, G}(z) : \hat{K}_*(A \rtimes_r G) \rightarrow \hat{K}_{*+1}(B \rtimes_r G)$$

such that

- (i)  $J_{red, G}(z)$  induces right multiplication by  $j_{red, G}(z)$  in  $K$ -theory ;
- (ii)  $J_{red, G}$  is additive, i.e.

$$J_{red, G}(z + z') = J_{red, G}(z) + J_{red, G}(z').$$

- (iii) For every  $G$ -morphism  $f : A_1 \rightarrow A_2$ ,

$$J_{red, G}(f^*(z)) = J_{red, G}(z) \circ f_{G, red, *}$$

for all  $z \in KK_1^G(A_2, B)$ .

- (iv) For every  $G$ -morphism  $g : B_1 \rightarrow B_2$ ,

$$J_{red, G}(g_*(z)) = g_{G, red, *} \circ J_{red, G}(z)$$

for all  $z \in KK_1^G(A, B_1)$ .

- (v) Let  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  be a semi-split equivariant extension of  $G$ -algebras and  $[\partial_J] \in KK_1^G(A/J, J)$  be its boundary element. Then

$$J_G([\partial_J]) = D_{J \rtimes_r G, A \rtimes_r G}.$$

**Proof.** (i) The  $K$ -cycle  $[\partial_{K_{B \rtimes_r G}, E(\pi, T)}] \in KK_1(A \rtimes_r G, B \rtimes_r G)$  implementing the boundary of the extension  $E^{(\pi, T)}$  induces the map  $j_{red, G}$  by definition, and modulo Morita equivalence, which immediately gives the first point.

- (ii) If  $z, z'$  are elements of  $KK_1^G(A, B)$ , represented by two  $K$ -cycles  $(H_B, \pi_j, T_j)$ , let  $(\tilde{H}_B, \pi, T)$  be  $(H_B \oplus H_B, \pi_0 \oplus \pi_1, T_0 \oplus T_1)$  which is a  $K$ -cycle representing the sum  $z + z'$ . Then  $E^{(\pi, T)}$  is naturally isomorphic to the extension sum of the  $E_j := E^{(\pi_j, T_j)}$ , namely

$$0 \longrightarrow \mathfrak{M}_2(K_{B \rtimes_r G}) \longrightarrow D \longrightarrow A \rtimes_r G \longrightarrow 0$$

where

$$D = \left\{ \begin{pmatrix} x_1 & k_{12} \\ k_{21} & x_2 \end{pmatrix} : x_j \in E_j, p_1(x_1) = p_2(x_2), k_{ij} \in K_{B \rtimes_r G} \right\},$$

with  $p_j : E_j \rightarrow A \rtimes_r G$  the  $*$ -homomorphisms of the extensions. Naturality of the controlled boundary maps [40] ensures that the boundary of the sum of two extensions is the sum of the boundary of each, thus the result holds.

- (iii) Let  $z \in KK_1^G(A_2, B)$ , represented by a cycle  $(H_B, \pi, T)$ . Representing  $f^*(z)$  is  $(H_B, f^*\pi, T)$  with  $f^*\pi = \pi \circ f$ . The map

$$\phi : \begin{cases} E^{f^*(\pi, T)} & \rightarrow E^{(\pi, T)} \\ (x, P_G(f^*\pi)(x)P_G + y) & \rightarrow (f_G(x), P_G(f^*\pi)(x)P_G + y) \end{cases}$$

satisfies

- $\phi(K_{B \rtimes_r G}) \subseteq K_{B \rtimes_r G}$ , and makes the following diagram commute

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{B \rtimes_r G} & \longrightarrow & E^{f^*(\pi, T)} & \longrightarrow & A_1 \rtimes_r G \longrightarrow 0 \\ & & \downarrow = & & \downarrow \phi & & \downarrow f_G \\ 0 & \longrightarrow & K_{B \rtimes_r G} & \longrightarrow & E^{(\pi, T)} & \longrightarrow & A_2 \rtimes_r G \longrightarrow 0 \end{array} .$$

- It intertwines the sections of the two extensions.

Remark 3.1.26 ensures that

$$D_{K_{B \rtimes_r G}, E^{f^*(\pi, T)}} = D_{K_{B \rtimes_r G}, E^{(\pi, T)}} \circ f_{G, *},$$

and the claim is clear from composition by  $\mathcal{M}_{B \rtimes_r G}^{-1}$ .

- (iv) Let  $z \in KK^G(A, B_1)$  be represented by the  $K$ -cycle  $(H_{B_1}, \pi, T)$ . Let  $V \in \mathcal{L}_{B_2}(H_{B_1} \otimes_g B_2, H_{B_2})$  be the isometry of remark 1.2.13. Notice that  $V$  intertwines the actions of  $G$  on  $H_{B_1} \otimes B_2$  and  $H_{B_2}$ . According to Lemma 2.1.41,

$$g_*(z) = [H_{B_1} \otimes_g B_2, \pi \otimes_g 1, T \otimes_g 1] \in KK^G(A, B_2)$$

is also represented by  $[H_{B_2}, \pi', T']$  where  $\pi' = Ad_V \circ (\pi \otimes_g 1)$  and  $T' = V(T \otimes_g 1)V^* + 1 - VV^*$ . Let  $\psi$  be given by the composition  $Ad_{V_G} \circ g_G$ , and  $P' = \frac{1+T'}{2}$ . The map  $\Psi : (x, y) \mapsto (x, \psi(y))$  defines a  $*$ -homomorphism  $E^{(\pi, T)} \rightarrow E^{(\pi', T')}$  such that

$$\Psi(x, P_G \pi_G(x)P_G + y) = (x, P'_G \pi'_G(x)P'_G + \psi(y)).$$

Indeed, the crossed-product functor commutes with pull-back by  $G$ -morphisms, and  $Ad_{V_G} \circ g_G \circ \pi_G = (Ad_V \circ g_* \circ \pi)_G = \pi'_G$  and  $\psi(P_G) = V_G(P_G \otimes_{g_G} 1)V_G^* = (V(P \otimes_g 1)V^*)_G = (P')_G$  so that

$$\psi(P_G \pi_G(x)P_G) = P'_G \pi'_G(x)P'_G.$$

This gives a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_{B_1 \rtimes G} & \longrightarrow & E^{(\pi, T)} & \longrightarrow & A \rtimes_r G \longrightarrow 0 \\
 & & \downarrow \psi & & \downarrow \Psi & & \downarrow = \\
 0 & \longrightarrow & K_{B_2 \rtimes G} & \longrightarrow & E^{(\pi', T')} & \longrightarrow & A \rtimes G \longrightarrow 0
 \end{array}$$

and  $\Psi$  intertwines the two filtered sections by the previous relation. Moreover, recall from remark 1.2.13 that  $\Psi(K_{B_1 \rtimes G}) \subseteq K_{B_2 \rtimes G}$ , hence we can again apply the remark 3.1.26 to state

$$D_{K_{B_2 \rtimes G}, E^{(\pi', T')}} = \psi_* \circ D_{K_{B_1 \rtimes G}, E^{(\pi, T)}},$$

which we compose by the Morita equivalence on the left  $M_{B_2 \rtimes G}^{-1}$

$$J_G(g_*(z)) = M_{B_2 \rtimes G}^{-1} \circ g_{G,*} \circ D_{K_{B_1 \rtimes G}, E^{(\pi, T)}}.$$

Notice that  $\psi$  is the  $*$ -homomorphism induced by  $g_G$  on the compact operators, hence the homomorphisms inducing the Morita equivalence make the following diagram commutes,

$$\begin{array}{ccc}
 B_1 \rtimes G & \xrightarrow{g_G} & B_2 \rtimes G \\
 \downarrow & & \downarrow \\
 K_{B_1 \rtimes G} & \xrightarrow{\psi} & K_{B_2 \rtimes G}
 \end{array},$$

and  $J_G(g_*(z)) = g_{G,*} \circ M_{B_1 \rtimes G}^{-1} \circ D_{K_{B_1 \rtimes G}, E^{(\pi, T)}} = g_{G,*} \circ J_G(z)$ .

- (v) We can suppose  $A$  unital. Let  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  be a  $G$ -equivariant semi-split extension of  $G$ -algebras with  $q : A \rightarrow A/J$  the quotient map.. Let us denote by  $s : A/J \rightarrow A$  the  $G$ -equivariant completely positive cross section. According to the equivariant Kasparov-Stinespring theorem 2.1.24, there exists an equivariant  $A$ -Hilbert module  $E$  and a  $G$ -equivariant  $*$ -homomorphism  $\pi : A/J \rightarrow \mathcal{L}_A(A \oplus E)$  such that  $s(x) = P\pi(x)P$ , where  $P \in \mathcal{L}_A(A \oplus E)$  is the projection on the  $A$  factor. Consider the  $J$ -Hilbert module  $E' = (A \oplus E) \otimes_J J \cong J \oplus (E \otimes_J J)$ , and the natural map  $\tilde{\pi} = \pi \otimes_J 1 : A/J \rightarrow \mathcal{L}_J(E')$ . Put  $\tilde{T} = (2P - 1) \otimes_J id_J$ . By the stabilization theorem, we can suppose that  $E'$  is a standard  $G$ -equivariant  $J$ -Hilbert module. Put

$$\psi_0(x)(y \oplus \xi) = (xy) \oplus 0 \quad \forall \xi \in E \otimes J, \forall y \in J,$$

for every  $x \in A$ . This defines a  $G$ -equivariant  $*$ -homomorphism  $\psi_0 : A \rightarrow \mathcal{L}_J(E')$  such that  $\psi_0(x) \in \mathfrak{K}_J(E')$  when  $x \in J$ . Put  $\psi(a) = (q(a), \psi_0(a))$ . As  $P\tilde{\pi}(q(a))P - \psi_0(a) = \psi_0(s(q(a)) - a) \in \mathfrak{K}_J(E')$ ,  $\psi(a) \in E^{(\tilde{\pi}, \tilde{T})}$  holds, and the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & A/J \longrightarrow 0 \\
 & & \downarrow \psi_0 & & \downarrow \psi & & \downarrow = \\
 0 & \longrightarrow & \mathfrak{K}_J(E') & \longrightarrow & E^{(\tilde{\pi}, \tilde{T})} & \longrightarrow & A/J \longrightarrow 0
 \end{array},$$

Hence, by functoriality and semi-split exactness of the reduced cross product, the following diagram commutes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & J \rtimes_r G & \longrightarrow & A \rtimes_r G & \longrightarrow & A/J \rtimes_r G \longrightarrow 0 \\
 & & \downarrow (\psi_0)_G & & \downarrow \psi_G & & \downarrow = \\
 0 & \longrightarrow & \mathfrak{K} \otimes J \rtimes_r G & \longrightarrow & E^{(\tilde{\pi}, \tilde{T})} & \longrightarrow & A/J \rtimes_r G \longrightarrow 0
 \end{array},$$

and remark 3.1.26 ensures that  $J_G([\partial_J]) = D_{J \rtimes_r G, A \rtimes_r G}$ .

□

### Even case

We can now define  $J_G$  for even  $K$ -cycles. Let  $A$  and  $B$  be two  $G$ -algebras. Let  $[\partial_{SB}] \in KK_1(B, SB)$  be the  $K$ -cycle implementing the boundary of the extension  $0 \rightarrow SB \rightarrow CB \rightarrow B \rightarrow 0$ , and  $[\partial] \in KK_1(\mathbb{C}, S)$  be the Bott generator. As  $z \otimes_B [\partial_{SB}]$  is an odd  $K$ -cycle, we can define

$$J_G(z) := \hat{\tau}_{B \rtimes G}([\partial]^{-1}) \circ J_G(z \otimes [\partial_{SB}]).$$

Here  $\hat{\tau}_D$  refers, for any  $C^*$ -algebras  $D, A_1, A_2$  and  $z \in KK_*(A_1, A_2)$ , to the  $(\alpha_\tau, k_\tau)$ -controlled map  $\hat{K}(A_1 \otimes D) \rightarrow \hat{K}(A_2 \otimes D)$  of theorem 3.1.27. We can see that, if we set  $\alpha_J = \alpha_\tau \alpha_D$  and  $k_J = k_\tau * k_D$ ,  $J_G(z)$  is  $(\alpha_J, k_J)$ -controlled.

**Proposition 4.3.4.** Let  $A$  and  $B$  two  $G$ - $C^*$ -algebras. For every  $z \in KK_*^G(A, B)$ , there exists a control pair  $(\alpha_J, k_J)$  and a  $(\alpha_J, k_J)$ -controlled morphism

$$J_{red,G}(z) : \hat{K}(A \rtimes_r G) \rightarrow \hat{K}(B \rtimes_r G)$$

of the same degree as  $z$ , such that

(i)  $J_{red,G}(z)$  induces right multiplication by  $j_{red,G}(z)$  in  $K$ -theory ;

(ii)  $J_{red,G}$  is additive, i.e.

$$J_{red,G}(z + z') = J_{red,G}(z) + J_{red,G}(z').$$

(iii) For every  $G$ -morphism  $f : A_1 \rightarrow A_2$ ,

$$J_{red,G}(f^*(z)) = J_{red,G}(z) \circ f_{G,red,*}$$

for all  $z \in KK_*^G(A_2, B)$ .

(iv) For every  $G$ -morphism  $g : B_1 \rightarrow B_2$ ,

$$J_{red,G}(g_*(z)) = g_{G,red,*} \circ J_{red,G}(z)$$

for all  $z \in KK_*^G(A, B_1)$ .

(v)  $J_{red,G}([id_A]) \sim_{(\alpha_J, k_J)} id_{\hat{K}(A \rtimes_r G)}$



**Proof.** Let  $z \in KK_0^G(A, B)$ .

(i)  $J_{red,G}(z)$  induces in  $K$ -theory right multiplication by  $j_{red,G}(z \otimes [\partial_{SB}]) \otimes_{\tau_{B \rtimes_r G}}([\partial]^{-1})$ . But  $j_{red,G}$  respects Kasparov products, and by [31],  $j_{red,G}([\partial_{SB}]) = [\partial_{SB \rtimes_r G}]$  and  $\tau_{B \rtimes_r G}([\partial]^{-1}) = [\partial_{SB \rtimes_r G}]^{-1}$  are inverse of each others.

(ii) Additivity follows from additivity of the Kasparov product and of proposition 4.3.3.

(iii) Recall the equality  $f^*(x) \otimes y = f^*(x \otimes y)$ . As a consequence of the previous proposition 4.3.3,

$$\begin{aligned} J_{red,G}(f^*(z)) &= \hat{\tau}_{B \rtimes_r G}([\partial]^{-1}) \circ J_{red,G}(f^*(z) \otimes [\partial_{SB}]) \\ &= \hat{\tau}_{B \rtimes_r G}([\partial]^{-1}) \circ J_{red,G}(f^*(z \otimes [\partial_{SB}])) \\ &= \hat{\tau}_{B \rtimes_r G}([\partial]^{-1}) \circ J_{red,G}(z \otimes [\partial_{SB}]) \circ f_{G,red,*} \\ &= J_{red,G}(z) \circ f_{G,red,*} \end{aligned}$$

(iv) We have  $g_*(z) \otimes_{B'} [\partial_{SB'}] = z \otimes_B g^*[\partial_{SB'}] = (Sg)_*(z \otimes_B [\partial_{SB}])$ , hence, by proposition 4.3.3,

$$J_{red,G}(g_*(z)) = \hat{\tau}_{B' \rtimes_r G}([\partial]^{-1}) \circ (Sg)_{G,red,*} \circ J_{red,G}(z \otimes_B [\partial_{SB}]),$$

but, using properties of  $\hat{\tau}_{SB}$ , see 3.1.27,  $\tau_{B' \rtimes_r G}([\partial]^{-1}) \circ (Sg)_{*,red,G} = g_{G,red,*} \circ \tau_{B \rtimes_r G}([\partial]^{-1})$ .

(v) By definition,  $J_G([id_A]) = \hat{\tau}_{A \rtimes_r G}([\partial]^{-1}) \circ J_G([\partial_{SA}])$ . By point (v) of 4.3.3,  $J_G([\partial_{SA}]) = D_{A \rtimes_r G, SA \rtimes_r G}$  and by property of  $\hat{\tau}$ ,  $\hat{\tau}_{A \rtimes_r G}([\partial]^{-1})$  is a controlled left inverse of  $\hat{\tau}_{A \rtimes_r G}([\partial]) = D_{A \rtimes_r G, SA \rtimes_r G}$ , hence  $J_G([id_A]) \sim id_{\hat{K}(A \rtimes_r G)}$ .

□

We now show that the controlled Kasparov transform respects in a quantitative way the Kasparov product.

**Proposition 4.3.5.** There exists a control pair  $(\alpha_K, k_K)$  such that for every  $G$ - $C^*$ -algebras  $A, B$  and  $C$ , and every  $z \in KK^G(A, B), z' \in KK^G(B, C)$ , the controlled equality

$$J_G(z \otimes_B z') \sim_{\alpha_K, k_K} J_G(z') \circ J_G(z)$$

holds.

**Proof.** Recall from 2.1.35 that every  $\alpha \in KK_0^G(A, B)$  satisfies property (d) for a universal  $d$ . By naturality, this property reduces the proof to the special case of  $\alpha \in KK^G(A, B)$  being the inverse of a  $*$ -homomorphism  $\theta : B \rightarrow A$  in  $KK^G$ -theory :  $\alpha \otimes_B [\theta] = 1_A$ . Let  $z \in KK^G(B, C)$  :

$$\begin{aligned} J_G(\alpha \otimes z) &\sim_{\alpha_J^2, k_J * k_J} J_G(\alpha \otimes z) \circ J_G(\alpha \otimes [\theta]) \\ &\sim J_G(\alpha \otimes z) \circ J_G(\theta_*(\alpha)) \\ &\sim J_G(\alpha \otimes z) \circ \theta_{G,*} \circ J_G(\alpha) \\ &\sim J_G(\theta^*(\alpha \otimes z)) \circ J_G(\alpha) \\ &\sim J_G(z) \circ J_G(\alpha) \end{aligned}$$

because  $\theta^*(\alpha \otimes z) = \theta^*(\alpha) \otimes z = 1 \otimes z = z$ . The control on the propagation of the first line follows from remark 2.5 of [40] and point (v), the other lines are equal by points (iii) and (iv). As  $d$  is uniform for all locally compact groupoids with Haar systems, a simple induction concludes, and  $(\alpha_K, k_K)$  can be taken to be  $(\alpha_J^{2d}, (k_J)^{*2d})$ . The same argument

works if  $z'$  is an even  $KK$ -element.

Let  $z$  and  $z'$  be odd  $KK$ -elements. Then, as for  $\hat{\sigma}_X$ , we have :

$$\begin{aligned} J_G(z \otimes z') &= J_G(z \otimes_B [\partial_B] \otimes_{SB} [\partial_B]^{-1} \otimes_B z') \\ &\sim J_G([\partial_B]^{-1} \otimes_B z') \circ J_G(z \otimes_B [\partial_B]) \\ &\sim J_G([\partial_B]^{-1} \otimes_B z') \circ J_G([\partial_B]) \circ J_G([\partial_B]^{-1}) \circ J_G(z \otimes_B [\partial_B]) \\ &\sim J_G(z') \circ J_G(z), \end{aligned}$$

where we used the previous case for the second line, Lemma 4.3.3 for the third line, and Proposition 4.2.1 for the last one.  $\square$

### 4.3.2 Quantitative assembly maps

Let  $E \in \mathcal{E}$ . Recall that every  $\eta \in P_E(G)$  is a finite probability measure in  $G^{p(\eta)}$ , hence can be written as a finite sum

$$\eta = \sum_{g \in G^{p(\eta)}} \lambda_g(\eta) \delta_g,$$

where  $\delta_g$  is the Dirac probability at  $g \in G^{p(\eta)}$ . This defines a map

$$\lambda : \begin{cases} G \times_{s,p} P_E(G) & \rightarrow [0, 1] \\ (g, \eta) & \mapsto \lambda_g(\eta) \end{cases}.$$

**Lemma 4.3.6.** The function  $\lambda : G \times_{s,p} P_E(G) \rightarrow [0, 1]$  is continuous.

**Proof.** Let  $\{(g_j, \eta_j)\}_{j \in J} \subseteq G \times_{s,p} P_E(G)$  be a generalized sequence converging to  $(g, \eta) \in G \times_{s,p} P_E(G)$ . Let  $x = p(\eta) = s(g)$  and  $x_j = p(\eta_j) = s(g_j)$ . There exists an open neighborhood  $U \subseteq G^{(0)}$  of  $x$  such that  $s$  is injective on  $s^{-1}(U)$  and such that  $x_j$  belongs ultimately to  $U$ . We can suppose  $x_j \in U$  for every  $j$ . Then we can decompose  $s^{-1}(U) = \coprod_{g \in G^x} S_g$  with, for each  $g \in G^x$ , an open bisection  $S_g \subseteq G$ . Let, for every  $g \in G^x$ ,  $\phi_g : G \rightarrow [0, 1]$  be a continuous function equal to 1 on  $S_g$  and vanishing outside a small neighborhood of  $S_g$ . Then  $\lambda_g(\eta) = \langle \eta, \phi_g \rangle$  and  $\lambda_{g_j}(\eta_j) = \langle \eta_j, \phi_g \rangle$  for every  $j \in J$ . It is now clear, by definition of the weak-\* topology, that  $\lambda_{g_j}(\eta_j)$  converges to  $\lambda_g(\eta)$ .  $\square$

Notice that  $\lambda_{g'}(g, \eta) = \lambda_{g^{-1}g'}(\eta)$  for every  $\eta \in P_E(G)$ ,  $g \in G_{p(\eta)}$  and  $g' \in G^{r(g)}$ . Define  $h$  as the restriction of  $\lambda$  to  $G^{(0)} \times_{s,p} P_E(G)$ , i.e.  $h(x, \eta) = \lambda_{e_x}(\eta)$ , for every  $x \in G^{(0)}$  and  $\eta \in P_E(G)$  such that  $p(\eta) = x$ . Notice that  $h(x, g^{-1} \cdot \eta) = \lambda(g, \eta) = \lambda_g(\eta)$  for every  $(g, \eta) \in G \times_{s,p} P_E(G)$ . Let us set

$$\mathcal{L}_E(g, \eta) = h^{\frac{1}{2}}(x, \eta) h^{\frac{1}{2}}(x, g^{-1} \cdot \eta) = \lambda_{e_x}^{\frac{1}{2}}(\eta) \lambda_g^{\frac{1}{2}}(\eta),$$

where  $x = r(g)$ . Notice that  $\text{supp } \mathcal{L}_E \subseteq E \times_{s,p} p^{-1}(r(E))$ , which is compact. Hence  $\mathcal{L}_E$  is an element of  $C_c(G, C_0(P_E(G)))$ . Because  $\sum_{g \in G^{p(\eta)}} \lambda_g(\eta) = 1, \forall (g, \eta) \in G \times_{s,p} P_E(G)$ , this defines a projection in  $C_0(P_E(G)) \rtimes_r G$  of propagation less than  $E$ . Moreover  $\mathcal{L}_E$  defines a controlled  $K$ -theory class  $[\mathcal{L}_E, 0]_{\varepsilon, F}$  for any  $\varepsilon \in (0, \frac{1}{4})$  and any  $F \in \mathcal{E}$  such that  $E \subseteq F$ .

For every  $G$ -algebra  $B$  and every controlled subsets  $E, E' \in \mathcal{E}$  such that  $E \subseteq E'$ , the canonical inclusion  $P_E(G) \hookrightarrow P_{E'}(G)$  induces a \*-homomorphism  $q_E^{E'} : C_0(P_{E'}(G)) \rightarrow$

$C_0(P_E(G))$ , hence a map  $(q_E^{E'})^* : RK^G(P_E(G), B) \rightarrow RK^G(P_{E'}(G), B)$  in  $K$ -homology and a map  $((q_E^{E'})_G)_* : K(C_0(P_{E'}(G)) \rtimes_r G) \rightarrow K(C_0(P_E(G)) \rtimes_r G)$  in  $K$ -theory. The family of projections  $\mathcal{L}_E$  are compatible with the morphisms  $q_E^{E'}$ , i.e.  $((q_E^{E'})_G)_*[\mathcal{L}_{E'}, 0]_{\varepsilon, E'} = [\mathcal{L}_E, 0]_{\varepsilon, E}$ , for every  $\varepsilon \in (0, \frac{1}{4})$ .

**Definition 4.3.7.** Let  $B$  be a  $G$ -algebra,  $\varepsilon \in (0, \frac{1}{4})$ , and  $E \in \mathcal{E}$ . Let  $F \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F$ . The controlled assembly map for  $G$  is defined as the family of maps :

$$\mu_{G,B}^{\varepsilon, E, F} \begin{cases} RK^G(P_E(G), B) & \rightarrow K_*^{\varepsilon, F}(B \rtimes_r G) \\ z & \mapsto \iota_{\alpha_J \varepsilon', k_J(\varepsilon'), F'}^{\varepsilon, F} \circ J_G^{\varepsilon', F'}(z)([\mathcal{L}_E, 0]_{\varepsilon', F'}) \end{cases}$$

where  $\varepsilon'$  and  $F'$  satisfy :

- $\varepsilon' \in (0, \frac{1}{4})$  such that  $\alpha_J \varepsilon' \leq \varepsilon$ ,
- and  $F' \in \mathcal{E}$  such that  $E \subseteq F'$  and  $k_J(\varepsilon').F' \subseteq F$ .

**Remark 4.3.8.** The assembly map is defined for any reasonable crossed-products by  $G$ . In particular for the reduced one and the maximal one, so that we have two different assembly maps, which we shall distinguish writing  $\mu_{G,r}$  and  $\mu_{G,max}$  if necessary.

**Remark 4.3.9.** As for the controlled coarse assembly map, the controlled assembly map for an étale groupoid is compatible with the structure morphisms  $q_E^{E'}$ , and  $\iota_{\varepsilon, E}^{\varepsilon', F'}$ .

**Remark 4.3.10.** The family of assembly maps  $\mu_{G,B}^{\varepsilon, E, F}$  induces the Baum-Connes assembly map for  $G$  in  $K$ -theory. Notice that  $h$  is a cutoff function for the action on  $P_E(G)$ , hence  $\mathcal{L}_E$  coincides with  $\mathcal{L}_{P_E(G)}$ . Moreover, the following diagram commutes

$$\begin{array}{ccc} RK_*^G(P_E(G), B) & \xrightarrow{\mu_{G,B}^{\varepsilon, E, F}} & K_*^{\varepsilon, F}(B \rtimes_r G) \\ & \searrow \mu_{G,B}^E & \downarrow \iota_{\varepsilon, F} \\ & & K_*(B \rtimes_r G) \end{array}$$

because  $J_G(z)$  induces the right multiplication by  $j_G(z)$  and also  $\mu_G^E(z) = [\mathcal{L}_E] \otimes j_G(z)$  by remark 4.1.7. But, as  $(q_E^{E'})_*[\mathcal{L}_E, 0]_{\varepsilon, E} = [\mathcal{L}_{E'}, 0]_{\varepsilon, E'}$  as soon as  $E \subseteq E'$ , this diagram commutes with inductive limit over  $E$ .

### 4.3.3 Quantitative statements

We generalize in this subsection to the setting of étale groupoids the results obtained by H. Oyono-Oyono and G. Yu in [40]. We will define a controlled version of the Baum-Connes conjecture, and give conditions that ensure it is satisfied. In particular, when  $G$  satisfies the Baum-Connes conjecture with coefficients, it satisfies the controlled Baum-Connes conjecture. To this end, we will have to compute the  $K$ -homology of a finite typed  $G$ -simplicial complex with coefficients in the infinite product of stable  $G$ -algebras.

In this subsection, the groupoid  $G$  must have a compact unit space  $G^{(0)}$ . Then, if  $\{B_j\}$  is a countable family of  $G$ -algebras, the diagonal action of  $G$  on the product provides  $\prod_j B_j$

with the structure of a  $G$ -algebra.

Let  $A$  be a  $G$ -algebra. We will say that :

- (Quantitative Injectivity)  $\mu_{G,A}$  is quantitatively injective if, for every  $E \in \mathcal{E}$ , there exists  $\varepsilon \in (0, \frac{1}{4})$  such that, for every  $F \in \mathcal{E}$  satisfying  $k_J(\varepsilon).E \subseteq F$ , there exists  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and

$$\forall z \in RK^G(P_E(G), A), \quad \mu_{G,A}^{\varepsilon,E,F}(z) = 0 \text{ implies } q_E^{E'}(z) = 0.$$

- (Quantitative Surjectivity)  $\mu_{G,A}$  is quantitatively surjective if there exists  $\varepsilon$  such that, for every  $F \in \mathcal{E}$  such that, there exists  $\varepsilon' \in (\varepsilon, \frac{1}{4})$  and  $E, F' \in \mathcal{E}$  such that  $F \subseteq F'$  and  $k_J(\varepsilon').E \subseteq F'$ ,

$$\forall y \in K^{\varepsilon,F}(A \rtimes_r G), \exists z \in RK^G(P_E(G), A) \text{ such that } \mu_{G,A}^{\varepsilon',E,F'}(z) = \iota_{\varepsilon,F}^{\varepsilon',F'}(y).$$

**Proposition 4.3.11.** Let  $G$  be an étale groupoid with compact base space, and let  $B$  be a  $G$ -algebra.

- (Quantitative Injectivity) If  $\mu_{G,A}$  is quantitatively injective then  $\mu_{G,A}$  is one-to-one.
- (Quantitative Surjectivity) If  $\mu_{G,A}$  is quantitatively surjective then  $\mu_{G,A}$  is onto.

**Proof.** Let  $E \in \mathcal{E}$  and  $x \in RK(P_E(G), A)$  such that  $\mu_{G,A}^E(x) = 0$ . Then, for every  $\varepsilon \in (0, \frac{1}{4})$  and every  $F \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F$ ,  $\iota_{\varepsilon,F} \circ \mu_{G,A}^{\varepsilon,E,F}(x) = 0$ . Let  $\varepsilon'' > 0$  and  $F'' \in \mathcal{E}$  satisfying  $\alpha_J \varepsilon'' \leq \varepsilon$  and  $k_J(\varepsilon'').F'' \subseteq F$ . By remark 3.1.9, there exists a universal  $\lambda \geq 1$  and a controlled subset  $F' \in \mathcal{E}$  such that  $F \subseteq F'$  and

$$\begin{aligned} 0 &= \iota_{\varepsilon,F}^{\lambda\varepsilon,F'} \circ \mu_{G,A}^{\varepsilon,E,F}(x) \\ &= \iota_{\varepsilon,F}^{\lambda\varepsilon,F'} \circ \iota_{\alpha_J \varepsilon'', k_J(\varepsilon'').F''}^{\varepsilon,F} (J_G^{\varepsilon'',F''}(x)([\mathcal{L}_E, 0]_{\varepsilon'',F''})) \\ &= J_G^{\lambda\varepsilon,F'}(x)([\mathcal{L}_E, 0]_{\lambda\varepsilon,F'}) \\ &= \mu_{G,A}^{\lambda\varepsilon,E,F'}(x). \end{aligned}$$

But then the quantitative injectivity condition ensures that  $q_E^{E'}(x) = 0$  in  $RK^G(P_{E'}(G), A)$  and  $x = 0$  in  $K^{top}(G, A)$ , which is an inductive limit over  $E$ .

Let us prove the second point. Let  $y \in K(B \rtimes_r G)$ , and let  $\varepsilon \in (0, \frac{1}{4})$ ,  $F \in \mathcal{E}$  and  $x \in K^{\varepsilon,F}(B \rtimes_r G)$  such that  $\iota_{\varepsilon,F}(x) = y$ . The quantitative surjectivity condition implies that there exist  $\varepsilon' \in (0, \frac{1}{4})$ ,  $E, F' \in \mathcal{E}$ , and  $z \in RK^G(P_E(G), B)$  satisfying  $\varepsilon \leq \varepsilon'$ ,  $k_J(\varepsilon').E \subseteq F'$ ,  $F \subseteq F'$  and  $\mu_{G,B}^{\varepsilon',E,F'}(z) = \iota_{\varepsilon,F}^{\varepsilon',F'}(x)$ , hence  $\mu_{G,B}^E(z) = y$ . □

This kind of statement leads us to define the following properties, following [39].

- $QI_{G,B}(E, E', F, \varepsilon)$  : for any  $x \in RK^G(P_E(G), B)$ , then  $\mu_{G,B}^{\varepsilon,E,F}(x) = 0$  implies  $q_E^{E'}(x) = 0$  in  $RK^G(P_{E'}(G), B)$ .
- $QS_{G,B}(E, F, F', \varepsilon, \varepsilon')$  : for any  $y \in K^{\varepsilon,F}(B \rtimes_r G)$ , there exists  $x \in RK^G(P_E(G), B)$  such that  $\mu_{G,B}^{\varepsilon',E,F'}(x) = \iota_{\varepsilon,F}^{\varepsilon',F'}(y)$ .

To prove the main theorems of this section, what we called quantitative statements, we will need a serie of lemmas. The first part is devoted to a study of the behaviour of  $K$ -homology under product of the coefficients. The strategy is very similar to the proofs of the part 3 of [52].

Recall first that the Haar system  $(\lambda^x)_{x \in G^{(0)}}$  on  $G$  defines, for every  $f \in C_c(G)$ , a continuous function  $x \mapsto \int_{g \in G^x} f(g) d\lambda^x(g)$ . We denote by  $\int f d\lambda$  the element of  $C_0(X)$  obtained when integrating a compactly supported function  $f$  w.r.t. the Haar system. If  $A$  is a  $G$ -algebra and  $E$  is a  $A$ -Hilbert module,  $f \mapsto \int f d\lambda$  extends by linearity and continuity to  $C_c(G, A) \subseteq r^*A$  and  $C_c(G, E) \subseteq r^*E$ . We will still denote by  $\int f d\lambda$  the element in  $A$  or  $E$  if  $f$  is in  $r^*A$  or  $r^*E$  respectively.

**Lemma 4.3.12** (lemma 3.6,[52]). Let  $X$  be a  $G$ -compact proper  $G$ -space such that the anchor map  $p : X \rightarrow G^{(0)}$  is locally injective, and let  $B$  be a  $G$ -algebra. Then for every  $z \in RK^G(X, B)$  there exists a  $G$ -proper  $G$ -compact space  $Z$  and a  $K$ -cycle  $(H_B, \pi, T) \in \mathbb{E}^G(C_0(Z), B)$  representing  $z$  such that :

- $T$  is self-adjoint and  $-1 \leq T \leq 1$ ,
- $T$  is  $G$ -equivariant, i.e.  $r^*T = Vs^*TV^*$ ,
- $T$  commutes with the action of  $X$ , i.e.  $[\pi(a), T] = 0$  for all  $a \in C_0(Z)$ .

**Proof.** Let  $(E, \pi, T) \in \mathbb{E}(C_0(X), B)$  be a  $K$ -cycle. Denote by  $\alpha : s^*C_0(Z) \rightarrow r^*C_0(Z)$  the action of  $G$ . Let  $K$  be a compact fundamental domain for the action of  $G$  on  $Z$ . By local injectivity of  $p$ , let  $(U_j)_j$  be a finite open cover of  $K$  such that  $p|_{U_j}$  is injective for every  $U_j \in \mathcal{U}$ . There exist compactly supported continuous functions  $\phi_j : Z \rightarrow [0, 1]$  such that

$$\text{supp } \phi_j \subseteq U_j \quad \text{and} \quad K \subseteq \cup_j \phi_j^{-1}(0, +\infty).$$

Up to replacing  $\phi_j$  by the continuous function  $\phi_j(z) / \sum_{k,g} \phi_k(z.g)$ , we can assume

$$\sum_{j,g \in G^{p(z)}} \phi_U(z.g) = 1, \forall z \in Z.$$

The latter is indeed continuous, as it can be expressed as  $\frac{\phi_j}{(\sum_j \int \phi_j d\lambda)}$ . The condition  $\sum_{j,g \in G^{p(z)}} \phi_j(z.g) = 1, \forall z \in Z$  implies that

$$\sum_j \int V(s^*\pi(\phi_j))V^* d\lambda = id_E,$$

in the sense of a weak integral, i.e. the equality holds when evaluated on elements of  $E$ .

Indeed,  $\pi$  is  $G$ -equivariant, and by invariance of the Haar system,  $\sum_j \int \alpha(s^*\phi_j) d\lambda = 1_{C_0(Z)}$ . Composing with  $\pi$  gives the equality by continuity and linearity of  $\pi$ .

Define  $F_j = \pi(\phi_j^{\frac{1}{2}})F\pi(\phi_j^{\frac{1}{2}}) \in \mathcal{L}_B(E)$ . Then  $s^*F_j \in C_c(G, \mathcal{L}_{s^*B}(s^*E))$  and set

$$F' = \sum_j \int V(s^*F_j)V^* d\lambda.$$

By invariance of the Haar system, the operator  $F'$  is  $G$ -invariant. Moreover it commutes with the action of  $C_0(Z)$ . Indeed, we can compute its fibers : for every  $x \in G^{(0)}$ ,

$$F'_x = \sum_{j,g \in G^x} V_g \pi(\phi_j^{\frac{1}{2}}) F_{s(g)} \pi(\phi_j^{\frac{1}{2}}) V_g^*.$$

Hence, by local injectivity, for all  $g \in G$ , there exists  $z_g \in Z$  such that  $Z_{s(g)} \cap U = \{z_g\}$ , and for all  $f \in C_0(Z_{s(g)})$ , we have that  $\phi_U^{\frac{1}{2}} f = f(z_g) \phi_U^{\frac{1}{2}}$ , hence  $[\phi_U^{\frac{1}{2}} T_{s(g)} \phi_U^{\frac{1}{2}}, f] = 0$ .

Moreover,  $F'$  is a compact perturbation of  $F$  as the following computation shows.

$$\begin{aligned} F - F' &= (\sum_j \int V(s^* \phi_j) V^* d\lambda) F - F' \\ &= \sum_j \int V(s^* \phi_j^{\frac{1}{2}}) \left( (s^* \phi_j^{\frac{1}{2}}) V^* (r^* F) V - (s^* F) (s^* \phi_j^{\frac{1}{2}}) \right) V^* d\lambda \\ &= \sum_j \int V(s^* \phi_j^{\frac{1}{2}}) \left( (s^* \phi_j^{\frac{1}{2}}) (V^* (r^* F) V - (s^* F)) + s^* (\phi_j^{\frac{1}{2}} [\phi_j^{\frac{1}{2}}, F]) \right) V^* d\lambda \end{aligned}$$

Each of the summand is compact, hence  $[H_B, \pi, T] = [H_B, \pi, T']$ . □

**Lemma 4.3.13.** Let  $X$  be a  $G$ -compact proper  $G$ -space such that the anchor map  $p : X \rightarrow G^{(0)}$  is locally injective, and let  $(B_j)_j$  be a countable family of  $G$ -algebras. Then the projection  $\prod_j B_j \otimes K \rightarrow B_j \otimes K$  induces an isomorphism

$$\Theta : RK^G(X, \prod_j B_j \otimes \mathfrak{K}) \rightarrow \prod_j RK^G(X, B_j \otimes \mathfrak{K}) \cong \prod_j RK^G(X, B_j).$$

**Proof.** Set  $B_\infty = \prod_j B_j \otimes \mathfrak{K}$ . Let us define a  $Z_2$ -graded homomorphism

$$\eta : \prod_j RK^G(X, B_j \otimes \mathfrak{K}) \rightarrow RK^G(X, \prod_j B_j \otimes \mathfrak{K}).$$

Let  $Z \subseteq X$  be a  $G$ -proper  $G$ -compact subspace, and, for all  $j$ , let  $z_j \in KK^G(C_0(Z), B_j)$  be represented by a standard  $K$ -cycle  $(H_{B_j}, \pi_j, T_j)$ . By lemma 4.3.12, we can suppose that  $T_j$  is  $G$ -equivariant self-adjoint, commutes with the action of  $C_0(Z)$  and  $-1 \leq T_j \leq 1$ .

Define  $E_\infty = \prod_j (E_j \otimes \mathfrak{K})$ . It is a  $B_\infty$ -Hilbert module with respect to the scalar product

$$\langle \xi, \eta \rangle = \prod_j (\langle \xi_j, \eta_j \rangle \otimes F_j^* F_j) \in B_\infty$$

for every  $\xi = (\xi_j \otimes F_j)$  and  $\eta = (\eta_j \otimes F_j')$  in  $E_\infty$ . Define  $T = (T_j \otimes E_{11} + id_{E_j} \otimes (1 - E_{11}))_j$ , where  $E_{11}(x) = \langle e_1, x \rangle e_1$  is the rank-one operator projecting on  $e_1$ . ( $\{e_j\}$  denote the canonical orthonormal basis of  $l^2(\mathbb{N})$ ). These conditions ensure that  $T$  defines an operator in  $\mathcal{L}_{B_\infty}(E_\infty)$ .

Define  $\pi(a) = (\pi_j(a) \otimes id_{\mathfrak{K}})_j \in \mathcal{L}_{B_\infty}(E_\infty)$ , as  $\sup \|\pi_j(a)\| \leq \|a\|$ . Then,  $[\pi(a), T] = 0$ ,  $T_j^* = T_j$  and  $r^* T = V s^* T V^*$ . Let us show that  $T^2 - 1$  is in  $\mathfrak{K}_{B_\infty}(E_\infty)$ .

Indeed, each  $F_j = T_j^2 - 1 \in \mathfrak{K} \otimes B_j$  can be approximated by a finite rank operator, hence there exist  $N_j > 0$ , and  $\xi_k^{(j)}, \eta_k^{(j)} \in H_{B_j}$  such that  $F_j = \sum_{k=1}^{N_j} \theta_{\xi_k^{(j)}, \eta_k^{(j)}}$  for every

$j$ . A computation shows that  $(F_j \otimes E_{11})_j = \theta_{\sum_k \tilde{\xi}_k, \sum_k \tilde{\eta}_k}$  where  $\tilde{\xi}_k = (\xi_k^{(j)} \otimes E_{j1})$  and  $\tilde{\eta}_k = (\eta_k^{(j)} \otimes E_{j1})$ , which is a rank one operator, hence  $T^2 - 1$  is compact since a product of rank one operators is of rank one.

This ensures that  $[E_\infty, \pi, T] \in KK^G(C_0(Z), B_\infty)$ . Let  $\eta((z_j)_j) = [H_{B_\infty}, \pi, T]$ . It is clear that  $\Theta \circ \eta = id_{\prod_j RK^G(X, B_j \otimes K)}$ , and  $\Theta$  is onto.

Let us prove that  $\Theta$  is one to one. Let  $z \in KK^G(C_0(Z), B_\infty)$  such that  $\Theta(z) = 0$ . Let us denote  $\Theta(z) = (z_j)_j$  where each  $z_j \in KK^G(C_0(Z), B_j)$  is represented by  $K$ -cycles  $(H_{B_j}, \pi_j, T_j)$  homotopic to a degenerate  $K$ -cycle by an operator homotopy

$$\begin{cases} [0, 1] & \rightarrow \mathbb{E}^G(C_0(Z), B_\infty) \\ s & \mapsto (H_{B_j}, \pi_j, T_j(s)) \end{cases}$$

such that  $T_j(s)$  is  $G$ -equivariant self-adjoint, commutes with the action of  $C_0(Z)$  and  $-1 \leq T_j \leq 1$  for every  $s \in [0, 1]$ .

Set

$$\tilde{\mathcal{C}}_j = \{T \in \mathcal{L}_{B_j}(H_{B_j}) \text{ s.t. } [\pi(a), T] = 0 \forall a \in C_0(Z)\},$$

and let  $\mathcal{C}_j$  be the closed ideal  $\{T \in \tilde{\mathcal{C}}_j \text{ s.t. } \pi(a)T \in \mathfrak{K}_{B_j}(H_{B_j}) \forall a \in C_0(Z)\}$ . Similarly for  $\mathcal{L}_{B_\infty}(E_\infty)$  and  $\mathfrak{K}_{B_\infty}(E_\infty)$ , define  $\tilde{\mathcal{C}}$  and the closed ideal  $\mathcal{C}$ . Our goal is to show that the family of operator homotopies can be lifted to a global one.

Denote by  $\bar{T}_j$  the class of  $T_j$  in  $\tilde{\mathcal{C}}_j/\mathcal{C}_j$ . For every  $j$ , let  $t_0 = 0 < t_1 < \dots < t_{l_j} = 1$  be a partition of  $[0, 1]$  such that  $\|\bar{T}_j(t_{k+1}) - \bar{T}_j(t_k)\| < 1$  for every  $k \in \{0, \dots, l_j - 1\}$ . Put  $T_j(t_k) = T_{j,k}$ . Then

$$diag(\bar{T}_{j,0}, 1, 1, 1, \dots) = diag(1, \bar{T}_{j,1}^*, \dots, \bar{T}_{j,l}^*, 1, 1, 1, \dots) \quad diag(\bar{T}_{j,0}, \bar{T}_{j,1}, \dots, \bar{T}_{j,l}, 1, 1, 1, \dots)$$

is 2-Lipschitz homotopic to

$$\begin{aligned} &diag(\bar{T}_{j,1}^*, \dots, \bar{T}_{j,l}^*, 1, 1, 1, \dots) \quad diag(\bar{T}_{j,0}, \bar{T}_{j,1}, \dots, \bar{T}_{j,l}, 1, 1, 1, \dots) \\ &= diag(\bar{T}_{j,1}^* \bar{T}_{j,0}, \dots, \bar{T}_{j,l}^* \bar{T}_{j,l-1}, \bar{T}_{j,l}, 1, 1, 1, \dots) \end{aligned}$$

by rotations. We obtained that

$$diag(\bar{T}_{j,0}, 1, 1, 1, \dots)$$

is 2-Lipschitz homotopic to  $diag(\bar{T}_{j,l}, \bar{T}_{j,1}^* \bar{T}_{j,0}, \dots, \bar{T}_{j,l}^* \bar{T}_{j,l-1}, 1, 1, 1, \dots)$ .

By ([57], Proposition 4.2.4) for every  $j$  and  $k$ , there exists  $a_j^{(k)} \in \tilde{\mathcal{C}}_j/\mathcal{C}_j$  such that :

$$\bar{T}_{j,k+1}^* \bar{T}_{j,k} = e^{a_j^{(k)}} \text{ and such that } \|a_j^{(k)}\| < 1.$$

Define

$$\tilde{T}_j(s) = \begin{pmatrix} \bar{T}_{j,l} & 0 \\ 0 & \exp(s \, diag(a_j^{(1)}, \dots, a_j^{(l)}, 0, 0, 0, \dots)) \end{pmatrix}.$$

Then  $s \mapsto \tilde{T}_j(s)$  composed with the first homotopy by rotation and permutation is  $L$ -Lipschitz for every  $j$  for some constant  $L$  independent of  $j$ , and it provides an element of  $C([0, 1], \tilde{\mathcal{C}}/\mathcal{C})$ . We can lift it to  $T \in C([0, 1], \tilde{\mathcal{C}})$ , which gives a homotopy

$$\begin{cases} [0, 1] & \rightarrow \mathbb{E}^G(C_0(Z), B_\infty) \\ s & \mapsto (E_\infty, \pi, T(s)) \end{cases}$$

between 0 and  $z$ .

□

**Lemma 4.3.14.** Let  $G$  be a locally compact,  $\sigma$ -compact étale groupoid,  $\{B_j\}_{j \geq 0}$  a family of  $G$ -algebras and  $\mathfrak{K}$  the algebra of compact operators over a separable Hilbert space. Then, for every finite dimensional proper  $G$ -compact  $G$ -simplicial complex  $\Delta$ , we have an  $\mathbb{Z}_2$ -graded isomorphism of abelian groups

$$RK^G(\Delta, \prod_j B_j \otimes \mathfrak{K}) \cong \prod_j RK^G(\Delta, B_j)$$

**Proof.** For all  $j$  and any locally compact  $G$ -space  $X$ , the projection  $\prod_j B_j \otimes K \rightarrow B_j \otimes K$  induces a morphism

$$\Theta^X : KK^G(C_0(X), \prod_j B_j \otimes K) \rightarrow \prod_j KK^G(C_0(X), B_j \otimes K).$$

Let  $X_0 \subseteq X_1 \subseteq \dots \subseteq X_n$  be the  $n$ -skeleton decomposition associated to the simplicial structure of typed  $G$ -simplicial complex  $\Delta$  and let  $Z_j = C_0(X_j)$ ,  $Z_{j-1}^j = C_0(X_j \setminus X_{j-1})$  and  $\Theta_j = \Theta^{X_j}$ . We will show the claim by induction on the dimension of  $\Delta$ .

By naturality of the boundary element, the extension of  $G$ -algebras  $0 \rightarrow Z_{j-1}^j \rightarrow Z_j \rightarrow Z_{j-1} \rightarrow 0$  gives a commutative diagram with exact columns :

$$\begin{array}{ccc} KK_*^G(Z_{j-1}^j, \prod_j B_j \otimes K) & \xrightarrow{\Theta_{j-1}^j} & \prod_j KK_*^G(Z_{j-1}^j, B_j \otimes K) \\ \downarrow \partial & & \downarrow \partial \\ KK_*^G(Z_{j-1}, \prod_j B_j \otimes K) & \xrightarrow{\Theta_{j-1}} & \prod_j KK_*^G(Z_{j-1}, B_j \otimes K) \\ \downarrow & & \downarrow \\ KK_*^G(Z_j, \prod_j B_j \otimes K) & \xrightarrow{\Theta_j} & \prod_j KK_*^G(Z_j, B_j \otimes K) \\ \downarrow & & \downarrow \\ KK_*^G(Z_{j-1}^j, \prod_j B_j \otimes K) & \xrightarrow{\Theta_{j-1}^j} & \prod_j KK_*^G(Z_{j-1}^j, B_j \otimes K) \\ \downarrow \partial & & \downarrow \partial \\ KK_*^G(Z_{j-1}, \prod_j B_j \otimes K) & \xrightarrow{\Theta_{j-1}} & \prod_j KK_*^G(Z_{j-1}, B_j \otimes K) \end{array}$$

The five lemma ensures that if  $\Theta_{j-1}$  and  $\Theta_{j-1}^j$  are isomorphisms, then so is  $\Theta_j$ . Moreover, because  $\Delta$  is a typed  $G$ -simplicial complex (see 2.1.11),  $X_j \setminus X_{j-1}$  is  $G$ -equivariantly



homeomorphic to  $\mathring{\sigma}_j \times \Sigma_j$ , where  $\mathring{\sigma}_j$  denotes the interior of the standard simplex,  $\Sigma_j$  is the set of centers of  $j$ -simplices of  $X_j$ , and where  $G$  acts trivially on  $\mathring{\sigma}_j$ . Bott periodicity ensures then that the following diagram commutes :

$$\begin{array}{ccc} KK^G(Z_{j-1}^j, \prod_k B_k \otimes K) & \xrightarrow{\Theta_{j-1}^j} & \prod_k KK^G(Z_{j-1}^j, B_k) \\ \downarrow \cong & & \downarrow \cong \\ KK^G(\Sigma_j, \prod_k B_k \otimes K) & \xrightarrow{\Theta^{\Sigma_j}} & \prod_k KK^G(\Sigma_j, B_k) \end{array}$$

with vertical arrows being isomorphisms given by Bott periodicity. By lemma 4.3.13,  $\Theta^{\Sigma_j}$  is an isomorphism, hence  $\Theta_{j-1}^j$  is an isomorphism. We proved that if  $\Theta_{j-1}$  is an isomorphism, then so is  $\Theta_j$ . By induction, proving that  $\Theta_0$  is an isomorphism concludes the proof, which is essentially the content of lemma 4.3.13 :  $X_0$  is a  $G$ -compact proper  $G$ -space, and its anchor map is just the target map  $r : G \rightarrow G^{(0)}$ , which is supposed to be étale, so locally injective.  $\square$

We now turn our attention to the behaviour of controlled  $K$ -theory with respect to products. Let  $\mathcal{B} = (B_j)_j$  a countable family of  $\mathcal{E}$ -filtered  $C^*$ -algebras. For every  $E \in \mathcal{E}$ , put

$$\mathcal{B}_E = \prod_j (B_j)_E \otimes \mathfrak{K}.$$

Define  $\mathcal{B}_\infty$  as the closure of  $\cup_{E \in \mathcal{E}} \mathcal{B}_E$  in  $\prod_j (B_j \otimes \mathfrak{K})$ . This construction is called the controlled product of a family of filtered  $C^*$ -algebras, and is also a  $\mathcal{E}$ -filtered  $C^*$ -algebra. The following lemma ([39], Lemma 1.14) relates the controlled  $K$ -theory of  $\mathcal{B}_\infty$  and the product of the controlled  $K$ -theory groups of  $B_j$ .

**Lemma 4.3.15** ([39]). Let  $\mathcal{B}$  be a countable family of  $\mathcal{E}$ -filtered  $C^*$ -algebras. With the previous notations, and for every  $\varepsilon \in (0, \frac{1}{4})$  and every  $E \in \mathcal{E}$ , there exists a control pair  $(\alpha, h)$ , independent of the family  $\mathcal{B}$ , such that the family of maps

$$K^{\varepsilon, E}(\mathcal{B}_\infty) \rightarrow \prod K^{\varepsilon, E}(B_j)$$

induced by the composition  $\prod B_j \otimes \mathfrak{K} \rightarrow B_k \otimes \mathfrak{K}$  and the controlled Morita equivalence gives a  $(\alpha, h)$ -controlled isomorphism

$$\hat{K}(\mathcal{B}_\infty) \rightarrow \prod \hat{K}(B_j).$$

**Lemma 4.3.16.** Let  $G$  be an étale groupoid with compact base space  $G^{(0)}$ . Let  $(B_j)_j$  be a countable family of  $G$ -algebras, and  $\mathfrak{K}$  the  $G$ -algebra of compact operators on a separable Hilbert space with trivial  $G$ -action. Then,  $\mathcal{B}^G = ((B_j \otimes \mathfrak{K}) \rtimes_r G)_j$  is a countable family of  $\mathcal{E}$ -filtered  $C^*$ -algebra, and there exists a filtered  $*$ -isomorphism :

$$\left( \prod_j B_j \otimes \mathfrak{K} \right) \rtimes_r G \rightarrow \mathcal{B}_\infty^G.$$

**Proof.** Notice that, as  $\mathfrak{K}$  is endowed with the trivial action,  $(B_j \rtimes_r G) \otimes \mathfrak{K} \cong (B_j \otimes \mathfrak{K}) \rtimes_r G$  for every  $j$ . By definition, for every  $E \in \mathcal{E}$ ,  $((\prod B_j \otimes \mathfrak{K}) \rtimes_r G)_E = \prod_j ((B_j \otimes \mathfrak{K}) \rtimes_r G)_E \cong \prod_j ((B_j \rtimes_r G)_E \otimes \mathfrak{K}) = (\mathcal{B}_\infty^G)_E$ . □

The two previous lemmas entail the following result.

**Corollary 4.3.17.** Let  $G$  be an étale groupoid with compact base space  $G^{(0)}$ . Let  $(B_j)_j$  be a countable family of  $G$ -algebras, and  $\mathfrak{K}$  the  $G$ -algebra of compact operators on a separable Hilbert space with trivial  $G$ -action. Then, there exists a control pair  $(\alpha, h)$  and a  $(\alpha, h)$ -controlled isomorphism

$$\hat{K}(\mathcal{B}_\infty^G) \rightarrow \prod \hat{K}(B_j \rtimes_r G).$$

We can now prove the quantitative statements.

**Theorem 4.3.18.** Let  $B$  a  $G$ -algebra, and  $\tilde{B} = l^\infty(\mathbb{N}, B \otimes \mathfrak{K})$ . Then  $\mu_{G, \tilde{B}}$  is injective if and only if for every  $E \in \mathcal{E}$ ,  $\varepsilon \in (0, \frac{1}{4})$  and  $F$  such that  $k_J(\varepsilon).E \subseteq F$ , there exists  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and  $QI_{G,B}(E, E', \varepsilon, F)$  holds.

**Proof.** Let  $x \in RK^G(P_E(G), \tilde{B})$  such that  $\mu_{G, \tilde{B}}^E(x) = 0$ . Then, as the quantitative assembly maps factorize  $\mu_{G, \tilde{B}}$ , there exist  $\varepsilon > 0$  and  $F$  such that  $k_J(\varepsilon).E \subseteq F$ , satisfying  $\mu_{G, \tilde{B}}^{\varepsilon, E, F}(x) = 0$ . Let us denote by  $(x_j)_{j \in \mathbb{N}}$  the element of  $\prod_j RK^G(P_E(G), A)$  corresponding to  $x$  under the isomorphism of lemma 4.3.14. Now let  $F'$  be in  $\mathcal{E}$  such that  $F \subseteq F'$  and such that  $QI_A(E, E', \varepsilon, F)$  holds. That ensures that  $q_E^{E'}(x_j) = 0$  in  $RK^G(P_{E'}(G), B)$ , and  $q_E^{E'}(x) = 0$  hence  $\mu_{G, \tilde{B}}$  is injective.

For the converse, suppose one can find  $E, \varepsilon, F$  such that  $QI_{G,B}(E, E', \varepsilon, F)$  is NOT true for all  $F'$  such that  $F \subseteq F'$ . Then, by  $\sigma$ -compactness, one can extract an increasing exhausting sequence  $E_j \in \mathcal{E}$  such that  $\cup E_j = G$ ,  $E \subseteq E_j$  and  $x_j \in RK^G(P_{E_j}(G), B)$  such that  $\mu_{G, \tilde{B}}^{\varepsilon, E, F}(x_j) = 0$  and  $q_{E_j}^{E_j}(x_j) \neq 0$  in  $RK^G(P_{E_j}(G), B)$ . Let  $x$  be the image of  $(x_j) \in \prod RK^G(P_{E_j}(G), B)$  in  $RK^G(P_E(G), \tilde{B})$  under the isomorphism of lemma 4.3.14. By corollary 4.3.17, and up to a rescaling, we have  $\mu_{G, \tilde{B}}^E(x) = 0$ , and  $q_E^{E'}(x) \neq 0$  in  $RK^G(P_{E'}(G), \tilde{A})$  for at least one  $E'$  such that  $E \subseteq E'$  by exhaustivity, so  $\mu_{G, \tilde{B}}$  is not injective. □

We also have a theorem relating quantitative surjectivity for  $\hat{\mu}_{G,B}$  and surjectivity of  $\mu_{G, \tilde{B}}$ .

**Theorem 4.3.19.** Let  $B$  a  $G$ -algebra, and  $\tilde{B} = l^\infty(\mathbb{N}, B \otimes \mathfrak{K})$ . Then there exists  $\lambda > 1$  such that  $\mu_{G, \tilde{B}}$  is onto if and only if for any  $\varepsilon \in (0, \frac{1}{4\lambda})$  and every  $F \in \mathcal{E}$ , there exist  $E, F' \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F$ ,  $F \subseteq F'$  and such that  $QS_{B,G}(E, F, F', \varepsilon, \lambda\varepsilon)$  holds.

**Proof.** Let  $\lambda \geq 1$  the universal constant of remark 3.1.9 : for any  $C^*$ -algebra and any  $x \in K^{\varepsilon, F}(A)$  such that  $\iota_{\varepsilon, F}(x) = 0$ , there exists  $F'$  such that  $F \subseteq F'$  and  $\iota_{\varepsilon, F'}^{\lambda\varepsilon, F'}(x) = 0$ .

Let  $y \in K_*(\tilde{B} \rtimes_r G)$ . By remark 3.1.9, there exist  $\varepsilon \in (0, \frac{1}{4})$ ,  $F \in \mathcal{E}$  and  $z \in K^{\varepsilon, F}(\tilde{B} \rtimes_r G)$  such that  $\iota_{\varepsilon, F}(z) = y$ . Up to a rescaling of the parameters, let  $(z_j)$  the element of  $\prod_j K^{\varepsilon, F}(B \rtimes_r G)$  corresponding to  $z$  under the controlled isomorphism of corollary 4.3.17.

Let  $E$  and  $F'$  such that  $k_J(\varepsilon).E \subseteq F$  and  $QS(E, F, F', \varepsilon, \lambda\varepsilon)$  : for every  $j$ , there exists  $x_j \in RK^G(P_E(G), B)$  such that  $\mu_{G, \tilde{B}}^{\lambda\varepsilon, E, F'}(x_j) = \iota_{\varepsilon, F'}^{\lambda\varepsilon, F'}(z_j)$ . Let  $x \in RK^G(P_E(G), \tilde{B})$  be the element of  $RK^G(P_E(G), \tilde{B})$  corresponding to  $(x_j) \in \prod_j RK^G(P_E(G), B)$  under the isomorphism of lemma 4.3.14. Naturality of the assembly maps, and compatibility of the controlled assembly maps with the usual one ensures that  $\mu_{G, \tilde{B}}^E(x) = z$ , whereby  $\mu_{G, \tilde{B}}$  is onto.

Assume that there exist  $\varepsilon \in (0, \frac{1}{4\lambda})$  and a nonempty controlled subset  $F \in \mathcal{E}$  such that for every  $E, F' \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F$ ,  $QS_{G, B}(E, F, F', \varepsilon, \lambda\varepsilon)$  does not hold. Let  $(E_j)$  and  $(F_j)$  be unbounded increasing sequences of controlled subsets such that  $(E_j)$  is an exhausting family,  $F \subseteq F_j$  and  $k_J(\varepsilon).E_j \subseteq F_j$ . Let  $y_j \in K^{\varepsilon, F'}(B \rtimes_r G)$  such that  $\iota_{\varepsilon, F'}^{\lambda\varepsilon, F_j}(y_j)$  is not in the range of  $\mu_{G, \tilde{B}}^{\lambda\varepsilon, E_j, F_j}$ . Let  $y \in K^{\varepsilon, F'}(\tilde{B} \rtimes_r G)$  be the element of corresponding to  $(y_j)_j$  under the isomorphism of corollary 4.3.17, up to a rescaling of the parameters. If there exists  $x \in RK^G(P_{E'}(G), \tilde{B})$  for a  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and  $\iota_{\varepsilon, F'}(y) = \mu_{G, \tilde{B}}^{E'}(x)$  then there would exist a  $F' \in \mathcal{E}$  such that  $F \subseteq F'$  and

$$\iota_{\varepsilon, F'}^{\lambda\varepsilon, F'}(y) = \mu_{G, \tilde{B}}^{\lambda\varepsilon, E', F'}(x) = \iota_{\varepsilon, F'}^{\lambda\varepsilon, F'} \circ \mu_{G, \tilde{B}}^{\varepsilon, E', F'}(x).$$

Now choose  $j$  such that  $E' \subseteq E_j$  and  $F' \subseteq F_j$ , and compose the previous equality with  $\iota_{\lambda\varepsilon, F'}^{\lambda\varepsilon, F_j}$  and  $q_{E'}^{E_j}$  to obtain  $\iota_{\varepsilon, F'}^{\lambda\varepsilon, F_j}(y_j) = \mu_{G, \tilde{B}}^{\lambda\varepsilon, E_j, F_j}(x_j)$  which contradicts our assumption. Hence  $\mu_{G, \tilde{B}}$  is not onto. □

Using the only if part of the proofs of the quantitative statements, we can easily prove the following theorem by replacing  $l^\infty(\mathbb{N}, B \otimes \mathfrak{K})$  by  $\prod(B_j \otimes \mathfrak{K})$ .

**Theorem 4.3.20.** Let  $G$  be an étale groupoid with compact base space.

- Assume that for any  $G$ -algebra  $B$ ,  $\mu_{G, B}$  is one-to-one. Then, for any  $\varepsilon \in (0, \frac{1}{4})$  and every  $E, F \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F$ , there exists  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and such that  $QI_{G, A}(E, E', \varepsilon, F)$  holds for any  $G$ -algebra  $B$ .
- Assume that for any  $G$ -algebra  $B$ ,  $\mu_{G, B}$  is onto. Then, for some  $\lambda \geq 1$  and for any  $\varepsilon \in (0, \frac{1}{4\lambda})$  and every  $F \in \mathcal{E}$ , there exists  $E, F' \in \mathcal{E}$  such that  $k_J(\varepsilon).E \subseteq F'$  and  $F \subseteq F'$  such that, for any  $G$ -algebra  $B$ ,  $QS_{G, A}(E, F, F', \varepsilon, \lambda\varepsilon)$  holds.

### 4.3.4 Persistence approximation property

We recall the following definition from [39].

**Definition 4.3.21.** Let  $B$  be a  $\mathcal{E}$ -filtered  $C^*$ -algebra,  $\lambda > 0$ ,  $\varepsilon, \varepsilon'$  be positive numbers such that  $0 < \varepsilon < \varepsilon' < \frac{1}{4}$  and  $F, F' \in \mathcal{E}$  be nonempty controlled subsets such that  $F \subseteq F'$ . The following property is called Persistence Approximation Property :

- $PA_B(\varepsilon, \varepsilon', F, F')$  : for every  $x \in K_*^{\varepsilon, F'}(B)$  such that  $\iota_{\varepsilon, F'}(x) = 0$  in  $K_*(B)$ , then  $\iota_{\varepsilon, F'}^{\varepsilon', F'}(x) = 0$  in  $K_*^{\varepsilon', F'}(B)$ .
- $B$  is said to satisfy the Persistence Approximation Property  $(PAP)_\lambda$  if for every nonempty  $F \in \mathcal{E}$  and every  $\varepsilon \in (0, \frac{1}{4\lambda})$ , there exists  $F' \in \mathcal{E}$  nonempty such that  $PA_B(\varepsilon, \lambda\varepsilon, F, F')$  holds.

The following result gives a sufficient condition for  $(PAP)_\lambda$  to be satisfied for a large class of  $C^*$ -algebras.

**Theorem 4.3.22.** Let  $G$  be an étale groupoid such that :

- $G^{(0)}$  is compact,
- $G$  admits a cocompact example for universal space for proper actions.

Then there exists a universal constant  $\lambda_{PA} \geq 1$  such that, for every  $G$ -algebra  $A$ , if  $\mu_{G, l^\infty(\mathbb{N}, A \otimes \mathfrak{K})}$  is onto and  $\mu_{G, A}$  is one-to-one, then for every  $\varepsilon \in (0, \frac{1}{4\lambda_{PA}})$  and every nonempty  $F \in \mathcal{E}$ , there exists  $F' \in \mathcal{E}$  such that  $F \subseteq F'$  and  $PA_{A \rtimes_r G}(\varepsilon, \lambda_{PA}\varepsilon, F, F')$  holds.

**Proof.** Let us denote  $l^\infty(\mathbb{N}, A \otimes \mathfrak{K})$  by  $\tilde{A}$ . Let  $(\alpha, h)$  be the control pair of the controlled isomorphism of corollary 4.3.17, and  $\lambda \geq 1$  be the constant of remark 3.1.9. Set  $\lambda_{PA} = \lambda\alpha$ .

Assume the statement does not hold : there exists  $\varepsilon$  and  $F$  such that  $PA_{A \rtimes_r G}(\varepsilon, \lambda_{PA}\varepsilon, F, F')$  is not true for every  $F' \in \mathcal{E}$  such that  $F \subseteq F'$ . Then we can extract an increasing exhausting sequence of controlled subsets  $F_j$  such that  $F \subseteq F_j$  and elements  $x_j \in K^{\varepsilon, F_j}(A \rtimes_r G)$  such that  $\iota_{\varepsilon, F}(x_j) = 0$  and  $\iota_{\varepsilon, F}^{\lambda_{PA}\varepsilon, F_j}(x_j) \neq 0$ .

Let  $x$  be the element of  $K^{\alpha\varepsilon, h_\varepsilon F}(\tilde{A} \rtimes_r G)$  corresponding to

$$(x_j) \in \prod_j K^{\varepsilon, F_j}(A \rtimes_r G)$$

under the controlled isomorphism of corollary 4.3.17. Recall that the following diagram commutes

$$\begin{array}{ccc} RK^G(P_E(G), \tilde{A}) & \xrightarrow{\mu_{G, \tilde{A}}^{\alpha\varepsilon, E, h_\varepsilon F}} & K^{\alpha\varepsilon, h_\varepsilon F}(\tilde{A} \rtimes_r G) \\ & \searrow \mu_{G, \tilde{A}}^E & \downarrow \iota_{\alpha\varepsilon, h_\varepsilon F} \\ & & K(\tilde{A} \rtimes_r G) \end{array}$$

If  $\iota_{\alpha\varepsilon, h_\varepsilon F}(x)$  is in the range of  $\mu_{G, \tilde{A}}^E$ , there exists  $E \in \mathcal{E}$  and  $z \in RK^G(P_E(G), \tilde{A})$  such that  $\mu_{G, \tilde{A}}^E(z) = \iota_{\alpha\varepsilon, h_\varepsilon F}(x)$ . Denote by  $(z_j)$  the element of  $\prod_j RK^G(P_E(G), A)$  corresponding to  $z$  under the isomorphism  $RK^G(P_E(G), \tilde{A}) \cong \prod_j RK^G(P_E(G), A)$  of lemma 4.3.14. By remark 3.1.9, there exists  $F'' \in \mathcal{E}$  such that  $h_\varepsilon F \subseteq F''$  and such that  $\mu_{G, \tilde{A}}^{\lambda\alpha\varepsilon, E, F''}(z) = \iota_{\alpha\varepsilon, h_\varepsilon F}^{\lambda\alpha\varepsilon, F''}(x)$ .

By naturality,  $\mu_{G, A}^E(z_j) = 0$ . As  $G$  admits a cocompact example for  $\underline{E}G$ , there exists  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and  $q_E^{E'}(z) = 0$ . Since

$$\mu_{G, \tilde{A}}^{\lambda\alpha\varepsilon, E, F''}(z) = \mu_{G, \tilde{A}}^{\lambda\alpha\varepsilon, E', F''} \circ q_E^{E'}(z),$$

we have that  $\iota_{\alpha\varepsilon, h_\varepsilon F}^{\lambda\alpha\varepsilon, F''}(x) = 0$  in  $K^{\lambda\alpha\varepsilon, F''}(\tilde{A} \rtimes_r G)$ . By naturality, for any  $i$  such that  $F'' \subseteq F_i$ ,  $\iota_{\varepsilon, F}^{\lambda\alpha\varepsilon, F_i}(x_i) = 0$  is satisfied, which contradicts our assumption.  $\square$

**Remark 4.3.23.** The theorems 4.3.18 and 4.3.19 provide examples that satisfy (PAP). Recall that every a-T-menable groupoid satisfies the Baum-Connes conjecture with coefficients, hence, if  $G^{(0)}$  is compact and  $G$  admits a cocompact example for  $\underline{EG}$ ,  $G$  satisfies the hypothesis of the theorem 4.3.22.

**Remark 4.3.24.** The theorem 4.3.22 provides an obstruction for the Baum-Connes conjecture to be satisfied.

## 4.4 Controlled Kasparov transform for discrete Quantum Groups

We end this chapter by the construction of the controlled Kasparov transform for discrete quantum groups, and an application to  $K$ -amenability.

Let  $\mathbb{G}$  be a compact quantum group, and  $\mathcal{E}_{\mathbb{G}}$  be the coarse structure defined by its finite dimensional representations as defined in 3.1.12. Then, for every  $\hat{\mathbb{G}}$ -algebra  $A$ , the reduced cross product  $A \rtimes_r \hat{\mathbb{G}}$  and the maximal cross product  $A \rtimes_{max} \hat{\mathbb{G}}$  are filtered by the family of subspaces of coefficients of finite dimensional representations, namely the closure of the linear spans  $C_{\pi}(\hat{\mathbb{G}}, A)$  of

$$\{ \theta(a)W_{\xi,\eta}^{\pi}, a \in A, \xi, \eta \in H_{\pi} \},$$

for  $\pi$  a finite dimensional unitary representation of  $\mathbb{G}$ .

The constructions we performed for coarse spaces and étale groupoids can be carried out for functors which preserve filtered semi-split extension of filtered  $C^*$ -algebras. In particular, the reduced and maximal cross products of a discrete quantum group do. We thus get the following proposition :

**Proposition 4.4.1.** Let  $A$  and  $B$  two  $\hat{\mathbb{G}}$ -algebras. For every  $z \in KK_{*}^{\hat{\mathbb{G}}}(A, B)$ , there exists a control pair  $(\alpha_J, k_J)$  and a  $(\alpha_J, k_J)$ -controlled morphism

$$J_{red,\hat{\mathbb{G}}}(z) : \hat{K}(A \rtimes_r \hat{\mathbb{G}}) \rightarrow \hat{K}(B \rtimes_r \hat{\mathbb{G}})$$

of the same degree as  $z$ , such that

- (i)  $J_{red,\hat{\mathbb{G}}}(z)$  induces right multiplication by  $j_{red,\hat{\mathbb{G}}}(z)$  in  $K$ -theory ;
- (ii)  $J_{red,\hat{\mathbb{G}}}$  is additive, i.e.

$$J_{red,\hat{\mathbb{G}}}(z + z') = J_{red,\hat{\mathbb{G}}}(z) + J_{red,\hat{\mathbb{G}}}(z').$$

- (iii) For every  $\hat{\mathbb{G}}$ -equivariant morphism  $f : A_1 \rightarrow A_2$ ,

$$J_{red,\hat{\mathbb{G}}}(f^{*}(z)) = J_{red,\hat{\mathbb{G}}}(z) \circ f_{\hat{\mathbb{G}},red,*}$$

for all  $z \in KK_{*}^{\hat{\mathbb{G}}}(A_2, B)$ .

- (iv) For every  $\hat{\mathbb{G}}$ -morphism  $g : B_1 \rightarrow B_2$ ,

$$J_{red,\hat{\mathbb{G}}}(g_{*}(z)) = g_{\hat{\mathbb{G}},red,*} \circ J_{red,\hat{\mathbb{G}}}(z)$$

for all  $z \in KK_{*}^{\hat{\mathbb{G}}}(A, B_1)$ .

- (v) Let  $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$  be a semi-split equivariant extension of  $\hat{\mathbb{G}}$ -algebras and  $[\partial_J] \in KK_1^{\hat{\mathbb{G}}}(A/J, J)$  be its boundary element. Then

$$J_{red, \hat{\mathbb{G}}}([\partial_J]) = D_{J \rtimes_r \hat{\mathbb{G}}, A \rtimes_r \hat{\mathbb{G}}}.$$

- (vi)  $J_{red, \hat{\mathbb{G}}}([id_A]) \sim_{(\alpha_J, k_J)} id_{\hat{K}(A \rtimes \hat{\mathbb{G}})}$

As every  $KK^{\hat{\mathbb{G}}}$ -element satisfies decomposition property (d) (see definition 2.1.34), we can prove, using the same proof as the groupoid case, that  $J_{red, \hat{\mathbb{G}}}$  respects Kasparov products.

**Proposition 4.4.2.** There exists a control pair  $(\alpha_K, k_K)$  such that for every  $\hat{\mathbb{G}}$ -algebras  $A, B$  and  $C$ , and every  $z \in KK^{\hat{\mathbb{G}}}(A, B), z' \in KK^{\hat{\mathbb{G}}}(B, C)$ , the controlled equality

$$J_{\hat{\mathbb{G}}}(z \otimes_B z') \sim_{\alpha_K, k_K} J_{\hat{\mathbb{G}}}(z') \circ J_{\hat{\mathbb{G}}}(z)$$

holds.

The problem to define a controlled assembly map for  $\hat{\mathbb{G}}$  is that, so far, there does not exist any construction similar to the Rips complex nor to the universal space for proper action. The only construction of assembly maps for quantum groups the author is aware of relies on the work of R. Meyer and R. Nest in [35]. It is based on localisation of functors in triangulated categories. In the case of locally compact group, they first define a triangulated categories  $\mathcal{KK}^G$  based on  $KK^G$ -theory. Let  $F$  be the functor  $A \rightarrow K(A \rtimes_r G)$ . It is triangulated on  $\mathcal{KK}^G$  and the natural transformation  $\mathbb{L}F(A) \rightarrow F(A)$  can be shown to be isomorphic to the assembly map. The natural path taken by the authors is to define the assembly map as  $\mathbb{L}F(A) \rightarrow F(A)$  in the case of quantum groups.

We give the following application of controlled  $K$ -theory to  $K$ -amenability of quantum groups. The following definition of  $K$ -amenability can be found in [56] (section 1). Recall that, by universal property, for every  $\hat{\mathbb{G}}$ -algebra  $A$ , there exists a  $*$ -homomorphism  $\lambda_A : A \rtimes_{max} G \rightarrow A \rtimes_r G$ , and it is onto and filtered.

**Definition 4.4.3** ([56]). A discrete quantum group  $\hat{\mathbb{G}}$  is  $K$ -amenable if the  $KK$ -element  $1_{\hat{\mathbb{G}}} = [\mathbb{C}, 0, id_{\mathbb{C}}] \in KK^{\hat{\mathbb{G}}}(\mathbb{C}, \mathbb{C})$  ( $\mathbb{C}$  is endowed with the trivial action) can be represented by a  $K$ -cycle  $(E, \pi, T)$  such that the representation of the quantum group on  $E$  is weakly contained in the regular representation.

Amenable quantum groups are  $K$ -amenable. In [56], R. Vergnioux proved that amalgamated free products of amenable discrete quantum groups are  $K$ -amenable. In [14], P. Fima and A. Freslon proved that the fundamental quantum group of a graph of amenable discrete quantum groups is  $K$ -amenable. We refer to [56] and [14] for definitions. The proof of [40] (theorem 5.10) can be reproduce and give the following result.

**Proposition 4.4.4.** Let  $\hat{\mathbb{G}}$  be a  $K$ -amenable discrete quantum group. Then, there exists a control pair  $\rho$  such that, for every  $\hat{\mathbb{G}}$ -algebra  $A$ ,

$$(\lambda_A)_* : \hat{K}(A \rtimes_{max} G) \rightarrow \hat{K}(A \rtimes_r G)$$

is a  $\rho$ -controlled isomorphism.

# Chapter 5

## Applications

### 5.1 Applications to Coarse Geometry

We present in this section a result on the equivalence between the controlled assembly map for a discrete metric space with bounded geometry  $X$  with coefficients in a  $C^*$ -algebra  $B$  and the controlled assembly map for the coarse groupoid  $G(X)$  with coefficients in the  $G(X)$ -algebra  $l^\infty(X, B \otimes \mathfrak{K})$ . This result is applied to show that any such space that admits a fibred coarse embedding into Hilbert space satisfies the maximal controlled Baum-Connes conjecture.

#### 5.1.1 Equivalence between the controlled coarse assembly map for $X$ and the controlled assembly map for $G$ with coefficients in $l^\infty(X, \mathfrak{K})$

In this section, we prove how the result of G. Skandalis, J.-L. Tu and G. Yu [47] extends to the setting of controlled  $K$ -theory.

Recall from theorem 2.3.7 that, for every  $C^*$ -algebra  $B$ , there exists a natural isomorphism of  $C^*$ -algebras

$$\Psi_B : l^\infty(X, B \otimes \mathfrak{K}) \rtimes_r G(X) \rightarrow C^*(X, B).$$

Moreover, it is filtered in the strong sense : for all  $R > 0$ ,  $\Psi_B(C_{\Delta_R}^-(G, B)) = C_R[X, B]$ , where  $R = \sup_E d$ .

The following theorem is proved in [47]. It states the equivalence between the coarse Baum-Connes conjecture with coefficients in  $B$  and the Baum-Connes conjecture for  $G(X)$  with coefficients in  $l^\infty(X, B \otimes \mathfrak{K})$ .

**Theorem 5.1.1** ([47]). Let  $X$  be a discrete metric space with bounded geometry. Let  $\Psi_B$  be the isomorphism of theorem 2.3.7,  $x \in X$  and  $\iota : \{x\} \rightarrow G(X)$  be the natural inclusion of groupoids. Denote by  $G = G(X)$  the coarse groupoid of  $X$  and by  $\tilde{B}$  the  $G$ -algebra  $l^\infty(X, B \otimes \mathfrak{K})$ . Then, for every controlled subset  $E \subseteq X \times X$ , the following diagram is

commutative with vertical arrows being isomorphisms :

$$\begin{array}{ccc} RK_*^G(P_{\bar{E}}(G), \tilde{B}) & \xrightarrow{\mu_{G, \tilde{B}}^{\bar{E}}} & K_*(\tilde{B} \rtimes_r G) \\ \downarrow \iota^* & & \downarrow (\Psi_B)_* \\ RK_*(P_E(X), B) & \xrightarrow{\mu_{X, B}^E} & K_*(C^*(X, B)) \end{array} ,$$

where  $\iota^*$  is the natural transformation induced by  $\iota$  and  $d = \sup_E d$ .

We shall prove a controlled analogue of this result which induces it in  $K$ -theory. We need the following lemmas.

**Lemma 5.1.2** ([47]). Let  $x \in X$  and  $\iota : \{x\} \hookrightarrow G$  the natural inclusion of groupoids. Then

$$\iota^* : KK^G(C_0(P_{\bar{E}}(G), \tilde{B}) \rightarrow KK(C_0(P_E(X), B))$$

is an isomorphism of  $\mathbb{Z}_2$ -graded abelian groups.

The reader can find a proof in [47] (Lemma 4.7). We recall the explicit construction of the inverse

$$j : KK(C_0(P_E(X), B) \rightarrow KK^G(C_0(P_{\bar{E}}(G), \tilde{B}))$$

of  $\iota^*$ . Let  $(H_B, \pi, T) \in \mathbb{E}(C_0(P_E(X), B))$  be a standard  $K$ -cycle. Let  $\tilde{B} = l^\infty(X, B \otimes \mathfrak{K})$  seen as Hilbert module over itself,  $(\tilde{\pi}(a)\xi)(x) = \pi(a(x))\xi(x)$  and  $(\tilde{T}\xi)(x) = T\xi(x)$ , for every  $x \in X$  and  $\xi \in E$ . Then  $j([H_B, \pi, T]) = [\tilde{B}, \tilde{\pi}, \tilde{T}]$ .

**Lemma 5.1.3.** Let  $E \subseteq X \times X$  be controlled subset and  $B$  be  $C^*$ -algebra. Denote  $C_0(P_E(X))$  by  $A$ ,  $C_0(P_{\bar{E}}(G))$  by  $\tilde{A}$  and  $\tilde{B} = l^\infty(X, B \otimes \mathfrak{K})$ . Then, for every  $z \in KK^G(\tilde{A}, \tilde{B})$ , the following equality of controlled morphisms holds :

$$\hat{\sigma}_X(\iota^*(z)) \circ (\Psi_A)_* = (\Psi_B)_* \circ \hat{J}_G(z).$$

**Proof.** Let  $z \in KK_1^G(\tilde{A}, \tilde{B})$ . Let  $\iota^*(z)$  be represented by the  $K$ -cycle  $[H_B, \pi, T] \in \mathbb{E}(A, B)$ , and let  $P = \frac{1+T}{2}$ . Denote by  $(\tilde{B}, \tilde{\pi}, \tilde{T}) \in \mathbb{E}^G(\tilde{A}, \tilde{B})$  the representative of  $j(\iota^*(z)) = z$  constructed as in lemma 5.1.2 and  $\tilde{P} = \frac{1+\tilde{T}}{2}$ . Recall that

$$E^{(\pi, T)} = \{(x, P\pi(x)P + y) : x \in A, y \in B \otimes \mathfrak{K}\},$$

and  $E_X^{(\pi, T)} = C^*(X, E^{(\pi, T)})$ .

First, notice that  $z$  is the boundary element in  $KK^G(\tilde{A}, \tilde{B})$  of the following extension

$$0 \rightarrow \tilde{B} \rightarrow E' \rightarrow \tilde{A} \rightarrow 0$$

where  $E'$  is the  $G$ -algebra  $\{(a, \tilde{P}\tilde{\pi}(a)\tilde{P} + y) : a \in \tilde{A}, y \in \tilde{B}\} \subseteq \tilde{A} \oplus \mathcal{M}(\tilde{B})$ , and the  $*$ -homomorphisms are the obvious ones. Set

$$E'_G = \{(a, \tilde{P}_G\tilde{\pi}_G(a)\tilde{P}_G + y) : a \in \tilde{A} \rtimes_r G, y \in \tilde{B} \rtimes_r G\}.$$

We take the previous extension under the reduced crossed product to get the following extension

$$0 \rightarrow \tilde{B} \rtimes_r G \rightarrow E'_G \rightarrow \tilde{A} \rtimes_r G \rightarrow 0.$$



By 4.3.3,  $J_G(z)$  is given by the controlled boundary of  $E'_G$ .

We shall define a  $*$ -homomorphism from  $E' \rtimes G$  to  $E_X^{(\pi, T)}$  that intertwines the two extensions. Extend the  $*$ -isomorphism  $\Psi_B : \tilde{B} \rtimes_r G \rightarrow C^*(X, B)$  to  $\tilde{\Psi}_B : \mathcal{M}(\tilde{B} \rtimes_r G) \rightarrow \mathcal{M}(C^*(X, B))$ . Set  $\Psi_{E'}(a, y) = (\Psi_A(a), \tilde{\Psi}_B(y))$  for every  $(a, y) \in E'_G$ . This map makes the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{B} \rtimes_r G & \longrightarrow & E'_G & \longrightarrow & \tilde{A} \rtimes_r G \longrightarrow 0 \\ & & \downarrow \Psi_B & & \downarrow \Psi_{E'} & & \downarrow \Psi_A \\ 0 & \longrightarrow & C^*(X, B) & \longrightarrow & E_X^{(\pi, T)} & \longrightarrow & C^*(X, A) \longrightarrow 0 \end{array} .$$

By remark 3.1.26, we get

$$(\Psi_B)_* \circ D_{\tilde{B} \rtimes_r G, E'_G} = D_{C^*(X, B), E_X^{(\pi, T)}} \circ (\Psi_A)_*,$$

hence,

$$(\Psi_B)_* \circ \hat{J}_G(z) = \hat{\sigma}_X(\iota^*(z)) \circ (\Psi_A)_*.$$

□

**Theorem 5.1.4.** Let  $B$  be a  $C^*$ -algebra,  $E \in \mathcal{E}_X$  an entourage and  $\bar{E} \in \mathcal{E}_G$  the corresponding compact open subset of  $G$ . With the above notations, for all  $z \in RK^G(P_{\bar{E}}(G), \tilde{B})$  and all  $\varepsilon \in (0, \frac{1}{4})$ , the following equality holds :

$$(\Psi_B)_* \circ \mu_{G, \tilde{B}}^{\varepsilon, \bar{E}}(z) = \mu_{X, B}^{\varepsilon, E}(\iota^*(z)).$$

**Proof.** By the previous lemma, we only need to check that  $(\Psi_A)_*[\mathcal{L}_{\bar{E}}, 0]_{\varepsilon, \bar{E}} = [P_E, 0]_{\varepsilon, E}$ , which is trivial.

□

**Remark 5.1.5.** This theorem remains true for the maximal version of the assembly map when  $B = \mathbb{C}$ . One then has to replace  $\hat{\mu}_{G, \tilde{C}}$  and  $\hat{\mu}_X$  by  $\hat{\mu}_{G, \tilde{C}}^{max}$  and  $\hat{\mu}_X^{max}$  respectively.

This result induces the result of [47] in  $K$ -theory. It also implies interesting consequences for Coarse Geometry. Recall that if the groupoid  $G$  satisfies the Baum-Connes conjecture with coefficients, it satisfies the controlled Baum-Connes conjecture. Interesting examples follow from the result of J-L. Tu [48] that  $a$ - $T$ -menable groupoids satisfy the Baum-Connes conjecture with coefficients. In particular,

- amenable groupoids are  $a$ - $T$ -menable.
- Let  $X$  be a uniformly discrete metric space with bounded geometry. Then, if  $X$  is coarsely embeddable into a separable Hilbert space,  $G(X)$  is  $a$ - $T$ -menable [47].

### 5.1.2 Fibred coarse embedding

We now present an application to fibred coarse embedding.

**Definition 5.1.6.** Let  $X$  be a discrete metric space with bounded geometry and  $B$  a  $C^*$ -algebra. We introduce the following properties.

- $QI_{X,B}(E, E', F, \varepsilon)$  : for any  $x \in KK(C_0(P_E(X)), B)$ , then  $\mu_{X,B}^{\varepsilon, E, F}(x) = 0$  implies  $q_E^{E'}(x) = 0$  in  $KK^G(C_0(P_{E'}(X)), B)$ .
- $QS_{X,B}(E, F, F', \varepsilon, \varepsilon')$  : for any  $y \in K^{\varepsilon, F}(C^*(X, B))$ , there exists  $x \in KK(C_0(P_E(G)), B)$  such that  $\mu_{X,B}^{\varepsilon', E, F'}(x) = \iota_{\varepsilon, F}^{\varepsilon', F'}(y)$ .

Let  $\lambda \geq 1$  be a positive number. We say that  $X$  satisfies the controlled Baum-Connes conjecture with coefficients in  $B$  with rescaling  $\lambda$  if :

- for every  $\varepsilon \in (0, \frac{1}{4\lambda})$ , every  $E, F \in \mathcal{E}$  such that  $k_X(\varepsilon).E \subseteq F$ , there exists  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and  $QI_{X,B}(E, E', F, \varepsilon)$  holds;
- for every  $\varepsilon \in (0, \frac{1}{4\lambda})$ , every  $F \in \mathcal{E}$ , there exists  $E, F' \in \mathcal{E}$  such that  $k_X(\varepsilon).E \subseteq F'$  and  $F \subseteq F'$  and  $QS_{X,B}(E, F, F', \varepsilon, \lambda\varepsilon)$  holds.

If  $\hat{\mu}_X$ , is replaced by  $\hat{\mu}_X^{max}$ , we will say that  $X$  satisfies the maximal controlled Baum-Connes conjecture with rescaling  $\lambda$ .

Recall from theorem 2.3.5 that if  $X$  admits a fibred coarse embedding into Hilbert space, then  $G(X)|_{\partial\beta X}$  is a-T-menable. For interesting examples of this type, recall the definition of a box space. Let  $\Gamma$  be a finitely generated group, and  $\mathcal{N}$  a family of nested normal subgroups with trivial intersection, which have finite index in  $\Gamma$ . Take the coarse union of the quotients to construct a coarse space  $X_{\mathcal{N}}(\Gamma) = \cup_{H \in \mathcal{N}} \Gamma/H$ . Then,  $X_{\mathcal{N}}(\Gamma)$  admits a fibred coarse embedding if and only if  $\Gamma$  is a-T-menable. But if  $X_{\mathcal{N}}$  is an expander, it cannot be coarsely embedded into a Hilbert space, so just take an a-T-menable group which has a box space  $X$  which is an expander to get a coarse space that is not coarsely embeddable into Hilbert space ( $SL(2, \mathbb{Z})$  for instance), but admits a fibred coarse embedding.

The last example gives the following corollary.

**Corollary 5.1.7.** Let  $X$  be a coarse space that admits a fibred coarse embedding into Hilbert space. Then  $X$  satisfies the maximal controlled Coarse Baum-Connes conjecture.

**Proof.** By theorem 5.1.4, it is sufficient to show that  $G(X)$  satisfies the maximal controlled Baum-Connes conjecture with coefficients in  $l^\infty(X, \mathfrak{K})$ . We will denote  $l^\infty(X, \mathfrak{K})$  by  $l^\infty$ .

The maximal crossed product turns restriction of a groupoid to invariant open subsets into exact sequences of  $C^*$ -algebras, hence

$$0 \rightarrow l^\infty \rtimes_{max} G|_U \rightarrow l^\infty \rtimes_{max} G \rightarrow l^\infty \rtimes_{max} G|_Y \rightarrow 0$$

is an exact sequence, with  $Y = \partial\beta X$  and  $U = Y^c$ . Moreover

$$l^\infty \rtimes_{max} G|_U \cong l^\infty_U \rtimes_{max} G \quad \text{and} \quad l^\infty \rtimes_{max} G|_Y \cong (l^\infty/l^\infty_U) \rtimes_{max} G,$$

hence  $[\partial_{l^\infty \rtimes_{G|_U, l^\infty \rtimes_r G}}] = j_G([\partial_{l^\infty_U, l^\infty}])$ . Recall from Proposition 4.3.3 that  $J_G([\partial_{l^\infty_U, l^\infty}]) = D_{l^\infty_U \rtimes_r G, l^\infty \rtimes_r G}$ , hence there exists a control pair  $(\alpha, k)$  such that for every  $z \in RK^G(P_E(G), l^\infty/l^\infty_U)$ ,

$$\mu_G^{\varepsilon, E, F}(z \otimes [\partial_{l^\infty_U, l^\infty}]) \sim_{\alpha, k} D_{l^\infty_U \rtimes_r G, l^\infty \rtimes_r G} \circ \mu_G^{\varepsilon, E, F}(z)$$

Hence the following diagram commutes :

$$\begin{array}{ccc} RK^G(P_E(G), l^\infty_Y) & \xrightarrow{\mu_G^{\varepsilon, E, F}} & K_*^{\varepsilon, F}(l^\infty_Y \rtimes_{max} G) \\ \downarrow \otimes [\partial_{l^\infty_U, l^\infty}] & & \downarrow D_{l^\infty_U \rtimes_r G, l^\infty \rtimes_r G} \\ RK^G(P_E(G), l^\infty_U) & \xrightarrow{\mu_G^{\alpha\varepsilon, E, k(\varepsilon), F}} & K_*^{\alpha\varepsilon, k(\varepsilon), F}(l^\infty_U \rtimes_{max} G) \\ \downarrow & & \downarrow \\ RK^G(P_E(G), l^\infty) & \xrightarrow{\mu_G^{\alpha\varepsilon, E, k(\varepsilon), F}} & K_*^{\alpha\varepsilon, k(\varepsilon), F}(l^\infty \rtimes_{max} G) \\ \downarrow & & \downarrow \\ RK^G(P_E(G), l^\infty_Y) & \xrightarrow{\mu_G^{\alpha\varepsilon, E, k(\varepsilon), F}} & K_*^{\alpha\varepsilon, k(\varepsilon), F}(l^\infty_Y \rtimes_{max} G) \\ \downarrow \otimes [\partial_{l^\infty_U, l^\infty}] & & \downarrow D_{l^\infty_U \rtimes_r G, l^\infty \rtimes_r G} \\ RK^G(P_E(G), l^\infty) & \xrightarrow{\mu_G^{\alpha\varepsilon, E, k(\varepsilon), F}} & K_*^{\alpha\varepsilon, k(\varepsilon), F}(l^\infty \rtimes_{max} G) \end{array} .$$

Now,  $G|_Y$  being a-T-menable and  $G|_U$  being proper,  $\mu_{G|_Y, B}$  and  $\mu_{G|_U, B}$  are isomorphisms for any  $G$ -algebra  $B$ . By theorems 4.3.18 and 4.3.19, the families of the four exterior horizontal maps satisfies the controlled Baum-Connes conjecture, and the controlled version of the five lemma concludes the proof.  $\square$

## 5.2 A quantitative Künneth formula

In this section, we present an application of the previous results to a Künneth formula for crossed products by an étale groupoid. Controlled Künneth formulas are more stable than Künneth formulas in  $K$ -theory. In [38], it was proved that they are stable by controlled Mayer-Vietoris decomposition. The reader is referred to the conclusion for details (in particular Definition 6.0.15).

Recall that for  $A$  and  $B$  two  $C^*$ -algebras, one can define a homomorphism

$$\alpha_{A, B} : K_*(A) \otimes K_*(B) \rightarrow K_*(A \otimes B) \quad ; \quad (x, y) \mapsto x \otimes \tau_A(y),$$

where  $\tau_A$  is the external Kasparov product.

Recall that, when  $A$  and  $B$  are unital  $C^*$ -algebras, if  $p$  and  $q$  are projections in  $\mathfrak{M}_r(A)$  and  $\mathfrak{M}_s(B)$  and  $u$  and  $v$  are unitaries in  $\mathfrak{M}_r(A)$  and  $\mathfrak{M}_s(B)$ , then :

$$\begin{aligned}\alpha_{A,B}([p] \otimes [q]) &= [p \otimes q], \\ \alpha_{A,B}([u] \otimes [q]) &= [u \otimes q + 1 \otimes (1 - q)], \\ \alpha_{A,B}([p] \otimes [v]) &= [p \otimes v + (1 - p) \otimes v].\end{aligned}$$

**Definition 5.2.1.** A  $C^*$ -algebra  $A$  is said to satisfy the Künneth formula if, for every  $C^*$ -algebra  $B$  such that  $K_*(B)$  is a free abelian group,  $\alpha_{A,B}$  is an isomorphism.

If  $A$  satisfies the Künneth formula, then, for any  $C^*$ -algebra  $B$ , one has the following exact sequence

$$0 \longrightarrow K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B) \longrightarrow \text{Tor}(K_*(A), K_*(B)) \longrightarrow 0$$

If  $(A, \mathcal{E})$  is now a  $\mathcal{E}$ -filtered  $C^*$ -algebra,  $(A \otimes B, \mathcal{E})$  is also filtered, and one can define a controlled morphism

$$\hat{\alpha}_{A,B} : \hat{K}_*(A) \otimes K_*(B) \rightarrow \hat{K}_*(A \otimes B) \quad ; \quad (x, y) \mapsto \hat{\tau}_A(y)(x),$$

which induces  $\alpha_{A,B}$  in  $K$ -theory.

We will say that  $\alpha_{A,B}$  is quantitatively injective if there exists  $\lambda \geq 1$  such that, for every  $\varepsilon \in (0, \frac{1}{4\alpha_\tau\lambda})$  and every  $F \in \mathcal{E}$ , there exists  $F' \in \mathcal{E}$ ,  $F \subseteq F'$ , such that :

$$\forall x \in K^{\varepsilon, F}(A) \otimes K(B) \text{ such that } \alpha_{A,B}^{\varepsilon, F}(x) = 0 \text{ then } (\iota_{\varepsilon, F}^{\lambda\varepsilon, F'} \otimes id)(x) = 0.$$

We will say that  $\alpha_{A,B}$  is quantitatively surjective if there exists  $\lambda \geq 1$  such that, for every  $\varepsilon \in (0, \frac{1}{4\alpha_\tau})$  and every  $F \in \mathcal{E}$ , there exist  $F' \in \mathcal{E}$ ,  $F \subseteq h_{\tau, \lambda\varepsilon} \cdot F'$ , such that:

$$\forall y \in K^{\varepsilon, F}(A \otimes B), \exists x \in K^{\lambda\varepsilon, F'}(A) \otimes K(B) \text{ such that } \alpha_{A,B}^{\lambda\varepsilon, F'}(x) = \iota_{\varepsilon, F}^{\alpha\lambda\varepsilon, h_{\tau, \lambda\varepsilon} \cdot F'}(y).$$

**Definition 5.2.2.** A filtered  $C^*$ -algebra  $(A, \mathcal{E})$  is said to satisfy the quantitative Künneth formula if there exists  $\lambda \geq 1$  such that, for every  $C^*$ -algebra  $B$  such that  $K_*(B)$  is a free abelian group,  $\hat{\alpha}_{A,B}$  is quantitatively injective and quantitatively surjective.

The remainder of the section is devoted to a proof of Theorem 5.2.13. We first recall the induction and restriction machinery in the context of equivariant  $KK$ -theory. The important result is lemma 5.2.6, which allows to restrict  $KK^G$ -elements to compact open subgroupoids under some suitable conditions. We then define a class  $\mathcal{C}$  of groupoids which satisfies these suitable conditions, and give instances of this class. The end of the section is devoted to the proof of the main theorem.

## 5.2.1 Induction and Restriction transformations

We develop a restriction principle in order to apply the "Going Down" technique developed in [6]. A restriction principle for groupoids has been studied in great details by Christian Bönicke in his PhD thesis (unpublished so far). I would like to thank him for useful discussions and comments.

**Definition 5.2.3.** A subset  $H \subseteq G$  is called a subgroupoid if :

- for every  $x \in H^{(0)}$ ,  $e_x \in H$ ,
- for all  $h, h' \in H$  such that  $s(h') = r(h)$ ,  $h'h \in H$ ,
- if  $h \in H$ ,  $h^{-1} \in H$ .

Then, the restriction of the multiplication, inverse, unit, target and source maps on  $H$  defines a structure of groupoid on  $H$  over  $H^{(0)} = s(H)$ . If  $G$  is étale,  $H$  is also étale. We will write  $H < G$  to indicate that  $H$  is a subgroupoid of  $G$ .

In this section, we define for all compact open subgroupoids  $H < G$  induction and restriction transformations. Let  $G$  be an étale groupoid and  $H < G$ . The action of  $G$  by left translation on the  $G$ -algebra  $C_0(G)$  is the action  $C_0(G) \otimes_{s,r} C_0(G) \rightarrow C_0(G) \otimes_{r,r} C_0(G)$  induced by

$$\begin{cases} G \times_{r,r} G & \rightarrow G \times_{s,r} G \\ (g, g') & \mapsto (g, g^{-1}g') \end{cases}$$

Let  $H < G$  be a compact open subgroupoid.

Let  $A$  be a  $H$ -algebra, with action given by  $\alpha : s^*A \rightarrow r^*A$ . Let  $E$  be a Hilbert  $A$ -module, endowed with an action  $V \in \mathcal{L}_A(s^*E, r^*E)$  of  $H$ . Define the induced Hilbert module as follows. The space  $C_0(G|_{H^{(0)}}) \otimes_s E$  is endowed with an action of  $H$  by left translation on the  $C_0(G|_{H^{(0)}})$  factor. Let  $E_0$  be the space of compactly supported  $H$ -invariant elements of  $C_0(G|_{H^{(0)}}) \otimes E$ , and  $h \in C_0(H^{(0)})$  be the function  $x \mapsto \frac{1}{|H^x|}$ . Define on  $E_0$  the following  $A$ -valued inner product :

$$\langle\langle \xi, \eta \rangle\rangle = h \int \alpha(s^*\langle \xi, \eta \rangle) d\lambda \quad \forall \xi, \eta \in E_0,$$

which gives fiberwise that

$$\langle\langle \xi, \eta \rangle\rangle_x = \frac{1}{|H^x|} \sum_{g \in G^x} \alpha_g(\langle \xi(g), \eta(g) \rangle_{s(g)}) \quad \forall x \in H^{(0)}.$$

Then, there exists a unique action of  $A$  on  $E_0$  such that

$$(\xi \cdot a)(g)_{s(g)} = \xi(g)_{s(g)} \cdot \alpha_{g^{-1}}(a_{r(g)}) \quad \forall \xi \in E_0, a \in A, g \in G.$$

Indeed, let  $U$  be a relatively compact bisection. The target map  $r$  being a homeomorphism when restricted to  $U$ , we have an isomorphism

$$\Psi_U : A_{r(U)} \cong C_0(U) \otimes_r A \rightarrow_{\alpha^{-1}} C_0(U) \otimes_s A.$$

Then, if  $f \otimes \xi \in C_0(U) \otimes_s E$ ,  $\Psi_U(a)$  acts naturally on  $f \otimes \xi$ . Let  $\mathcal{U}$  be an open cover of  $G|_{H^{(0)}}$  by relatively compact bisections, and let  $\{\phi_U\}$  be a corresponding partition of unity. Then, if  $f \otimes \xi \in E_0$  and  $a \in A$ ,  $(f \otimes \xi) \cdot a$  is defined as the  $H$ -invariant element obtained by averaging

$$\sum_U (\phi_U \cdot f \otimes \xi) \Psi_U(a).$$

over the  $H$ -action.

It is easy to check that if  $\xi, \eta \in E_0$  and  $a \in A$ , then  $\xi \cdot a \in E_0$ ,  $\langle\langle \xi, \eta \cdot a \rangle\rangle = \langle\langle \xi, \eta \rangle\rangle \cdot a$  and that left translation, denoted by  $V \in \mathcal{L}_{s^*B}(s^*E, r^*E)$ , is unitary. Hence this defines a

structure of  $G$ -equivariant pre-Hilbert  $A$ -module. The action of  $A$  extends to the completion of  $E_0$  with respect to the previous inner product  $\langle\langle \cdot, \cdot \rangle\rangle$ , hence the completion is a  $G$ -equivariant Hilbert  $A$ -module, called the induced module and denoted by  $\text{Ind}_H^G(E)$ .

Let  $Z$  be a right  $H$ -space. Define on  $Z \times_{p,r} G$  the following equivalence relation :

$$(z, g) \sim_H (z.h, h^{-1}g) \quad \forall z \in Z, h \in H^{p(z)}, g \in G^{p(z)}.$$

Let  $Z$  be a left  $H$ -space. Define on  $G \times_{s,p} Z$  the following equivalence relation :

$$(g, z) \sim_H (gh^{-1}, h.z) \quad \forall z \in Z, h \in H_{p(z)}, g \in G_{p(z)}.$$

**Definition 5.2.4.** The induced  $G$ -space of a left  $H$ -space  $Z$  is defined as  $G \times_H Z = (G \times_{s,p} Z) / \sim_H$ .

Let  $V$  be a  $H$ -space such that the anchor map  $p : V \rightarrow H^{(0)}$  is locally injective and  $B$  a  $G$ -algebra.

Let  $(E, \pi, T) \in \mathbb{E}^H(C_0(V), B)$  such that  $T$  is  $H$ -equivariant self-adjoint and commutes with the action of  $C_0(V)$ . Define  $\tilde{\pi}$  and  $\tilde{T}$  as the map and the operator induced by  $id \otimes \pi$  and  $1 \otimes T$  on  $\text{Ind}_H^G(E)$ . We have that

$$(\tilde{T}\xi)_{s(g)} = T(\xi(g)_{s(g)}) \quad \forall \xi \in \text{Ind}_H^G(E),$$

hence  $H$ -equivariance of  $T$  implies  $\tilde{T} \in \mathcal{L}_B(\text{Ind}_H^G(E))$ . Also, remark that

$$(\tilde{\pi}(a)\xi)(g) = \pi_{s(g)}(a(g, \cdot)|_{V_{s(g)}})\xi(g)_{s(g)}$$

holds for every  $a \in C_0(G \times_H V)$ ,  $\xi \in \text{Ind}_H^G(E)$ ,  $g \in G$ . Then  $\tilde{T}$  is  $G$ -equivariant self-adjoint, and  $[\tilde{\pi}(a), \tilde{T}] = 0$  for every  $a \in C_0(G \times_H V)$ .

**Lemma 5.2.5.** With the previous notations, if  $\tilde{E} = \text{Ind}_H^G(E)$ , then

$$(\tilde{E}, \tilde{\pi}, \tilde{T}) \in \mathbb{E}^G(C_0(G \times_H V), B).$$

**Proof.** Let us show that  $\tilde{\pi}(a).(\tilde{T}^2 - 1)$  are compact operators, for any  $a \in C_0(G \times_H V)$ . Recall that, for every Hilbert  $B$ -module  $E$  and any  $\xi, \eta \in E$ ,  $\theta_{\xi, \eta} \in \mathfrak{K}_B(E)$  denotes the rank-one operator  $s \mapsto \xi \langle \eta, s \rangle$ .

Let  $\{G_i\}_i$  be a cover of  $G|_{H^{(0)}}$  by relatively compact open bisections. For each  $i$ , let  $\{H_{ij}\}_j$  be an open cover of  $r^{-1}(s(G_i)) \cap H$  by relatively compact open bisections  $H_{ij} \subseteq H$ . Up to taking a subcover, we can suppose  $s(G_i) = r(H_{ij})$ , for every  $i$  and  $j$ . Put  $\tilde{G}_i = \coprod H_{ij} \circ G_i$ , which is a  $H$ -invariant open subset of  $G$ , when taking right-translation of  $H$  on  $G$ . Let  $\{\phi_i\}_i$  be continuous functions  $\phi_i : G^{(0)} \rightarrow [0, 1]$  such that

$$\sum_i \phi_i(x) = 1 \quad \forall x \in r(G|_{H^{(0)}}) \quad \text{and} \quad \text{supp } \phi_i \subseteq r(G_i).$$

For  $\theta_{\xi, \eta}$ , we will denote by  $T_{\xi, \eta}$  the  $H$ -invariant operator obtained by averaging  $\theta_{\xi, \eta}$  over the compact subgroupoid  $H$ . The operator  $\tilde{\pi}(a)(\tilde{T}^2 - 1)$  is in the closure of the algebra generated by the  $T_{\xi, \eta}$ .

It is sufficient to prove that  $\tilde{T}_{\xi,\eta} \in \mathfrak{K}_B(\tilde{E})$  for every  $i$ .

For  $x$  in  $H^{(0)}$ , let  $g_x$  the only element of  $G_i$  such that  $r(g_x) = x$ . If  $\xi \in E$ , define  $\tilde{\xi} \in \tilde{E}$  the  $H$ -invariant element obtained from  $\phi_i \circ r \otimes_s \xi \in C_0(G_i) \otimes_s E$ . Then, if  $s \in \tilde{E}$ :

$$\begin{aligned} \theta_{\tilde{\xi},\tilde{\eta}}(s) &= \tilde{\xi} \cdot \frac{1}{|H^x|} \sum_{g \in G^x} \alpha_g(\langle \tilde{\eta}(g), s(g) \rangle) \\ &= \tilde{\xi} \cdot \frac{1}{|H^x|} \sum_{h \in G_i^{s(g_x)}} \alpha_{g_x h}(\langle \tilde{\eta}(g_x h), s(g_x h) \rangle) \\ &= \tilde{\xi} \cdot \alpha_{g_x}(\langle \eta, s(g_x) \rangle) \quad \text{by } H\text{-invariance.} \end{aligned}$$

Hence,  $\theta_{\tilde{\xi},\tilde{\eta}}(s)(g) = 0$  if  $g \notin \tilde{G}_i$  and  $\tilde{T}_{\xi,\eta}$  is equal to the  $H$ -invariant operator obtained by averaging (by a finite sum)  $\theta_{\tilde{\xi},\tilde{\eta}}$  over the action of  $H$  by left translation. The proof is thus complete.  $\square$

Set

$$\text{Ind}_H^G : \begin{cases} RK^H(Z, B) & \rightarrow RK^G(G \times_H Z, B) \\ [E, \pi, T] & \mapsto [\text{Ind}_H^G(E), \tilde{\pi}, \tilde{T}] \end{cases}$$

Let  $(E, \pi, T) \in \mathbb{E}^G(C_0(G \times_H V), B)$  such that  $T$  is  $G$ -equivariant and commutes with the action of  $C_0(G \times_H V)$ . By lemma 4.3.12, every  $KK$ -element of  $KK^G(C_0(G \times_H Z), B)$  is represented by a  $K$ -cycle of this type. As  $H$  is open in  $G$ ,  $x \mapsto (e_{p(x)}, x)$  is a topological embedding and  $V$  can be seen as a  $H$ -invariant open subset in  $G \times_H V$ . Denote by  $E_V$  the Hilbert  $H$ -invariant  $B$ -submodule of  $E$  generated by

$$\{\pi(f)\xi, f \in C_0(V), \xi \in E\}.$$

Then  $E_V$  is a  $H$ -equivariant Hilbert  $B$ -module. Moreover,  $\pi$  is  $G$ -equivariant, and

$$[T, \pi(a)] = 0$$

for every  $a \in C_0(G \times_H V)$ . This ensures that  $(E_V, \pi|_{C_0(V)}, T|_{E_V}) \in \mathbb{E}^H(C_0(V), B)$ . We set:

$$\text{Res}_H^G : \begin{cases} RK^G(G \times_H V, B) & \rightarrow RK^H(V, B) \\ [E, \pi, T] & \mapsto [E_V, \pi|_{C_0(V)}, T|_{E_V}] \end{cases}$$

We will use the notation  $(C_0(G) \otimes_s E)^H$  for the  $H$ -invariant elements of  $C_0(G) \otimes_s E$ .

**Lemma 5.2.6.** Let  $H$  be a compact open subgroupoid of  $G$ , and  $V$  a  $H$ -space such that the anchor map  $p : V \rightarrow H^{(0)}$  is locally injective. Then, for every  $G$ -algebra  $B$ , we have an isomorphism of  $\mathbb{Z}_2$ -graded abelian groups:

$$RK^G(G \times_H V, B) \cong RK^H(V, B).$$

**Proof.** It is clear that induction followed by restriction is the identity.

For the converse, let  $(E, \pi, T) \in \mathbb{E}^G(C_0(G \times_H V), B)$ . The moment map is locally injective, hence, by lemma 4.3.12, we can suppose that  $T$  is self-adjoint  $G$ -equivariant and commutes with the action of  $C_0(G \times_H V)$ .

Let  $E_V$  be the  $H$ -equivariant Hilbert  $B$ -module previously defined. Let us show that  $E_V$  satisfies  $E \cong \text{Ind}_H^G(E_V)$ . Indeed, let  $\{G_i\}_i$  be a cover of  $G_{|H^{(0)}}$  by relatively compact open bisections. For each  $i$ , let  $\{H_{ij}\}_j$  be a disjoint open cover of  $r^{-1}(s(G_i)) \cap H$  by

relatively compact open bisections  $H_{ij} \subseteq H$ . Up to taking a subcover, we can suppose  $s(G_i) = r(H_{ij})$ , for every  $i$  and  $j$ . Put  $\tilde{G}_i = \coprod G_i \circ H_{ij}$ , which is a  $H$ -invariant open subset of  $G$ , when taking right-translation of  $H$  on  $G$ . Let  $\{\phi_i\}_i$  be continuous functions  $\phi_i : G^{(0)} \rightarrow [0, 1]$  such that

$$\sum_i \phi_i(x) = 1 \quad \forall x \in r(G|_{H^{(0)}}) \quad \text{and} \quad \text{supp } \phi_i \subseteq r(G_i).$$

$C_0(\tilde{G}_i)$  is stable by the action of  $H$  by left translation, hence  $f \mapsto (\phi_i \circ r).f$  induces a homomorphism

$$\text{Ind}_H^G(E_V) \rightarrow \left( C_0(\tilde{G}_i) \otimes_s E_V \right)^H.$$

Moreover  $C_0(\tilde{G}_i) = \bigoplus_j C_0(G_i \circ H_{ij})$ , and the action  $V \in \mathcal{L}_{s^*B}(s^*E, r^*E)$  induces an isomorphism

$$E_{s(H_{ij})} = C_0(G_i \circ H_{ij}) \otimes_s E \rightarrow C_0(G_i \circ H_{ij}) \otimes_r E = E_{r(G_i)}.$$

Combining these, we get an application

$$\left( C_0(\tilde{G}_i) \otimes_s E_V \right)^H \rightarrow (C_0(\tilde{G}_i) \otimes_r E)^H \cong C_0(\tilde{G}_i/H) \otimes_r E$$

for every  $i$ . Composing with the map  $f \otimes \xi \mapsto f\xi$  induces

$$\Psi_i : \text{Ind}_H^G(E_V)|_{s(\tilde{G}_i)} \rightarrow E_{r(G_i)}$$

for every  $i$ . Let us compute  $\Psi_i$  fiberwise. For every  $x \in r(\tilde{G}_i)$ , there exists  $g_x \in \tilde{G}_i$  such that  $r(g_x) = x$ . Hence the evaluation map induces an isomorphism

$$\left( C_0(\tilde{G}_i/H) \otimes_r E \right)_x \cong_{\text{ev}_{g_x}} E_x.$$

Let  $x \in H^{(0)}$ , and let  $g_x \in \tilde{G}_i$  such that  $r(g_x) = x$  as before. We identify elements of  $\text{Ind}_H^G(E_V)|_{s(\tilde{G}_i)}$  with sections  $f : \tilde{G}_i \rightarrow E_V$  such that  $V_h(f(gh)_{s(h)}) = f(g)_{r(g)}$  for any  $g \in \tilde{G}_i$  and any  $h \in H^{s(g)}$ . Then

$$(\Psi_i)_x(f_x) = V_{g_x}(f(g_x)_{s(g_x)}) \quad \forall f \in \text{Ind}_H^G(E_V)|_{s(\tilde{G}_i)},$$

which is an isomorphism. Notice that, by the  $H$ -equivariance of  $f$ , this last identity does not depend on the  $g_x$  chosen, because they all differ by a right translation by an element in  $H$ .

Define  $\Psi : \text{Ind}_H^G(E_V) \rightarrow E$  by  $\sum_{i, g \in G_x} (\phi_i \circ r)\Psi_i$ . Then

$$(\Psi)_x(f_x) = \sum_i \phi_i(x) V_{g_x}(f(g_x)_{s(g_x)}) = V_{g_x}(f(g_x)_{s(g_x)})$$

for every  $f \in (C_0(G) \otimes_{C_0(H^{(0)})} E_V)^H$  and any  $x \in H^{(0)}$ . Hence  $\Psi$  is an isomorphism.

Moreover,  $\pi$  is  $G$ -equivariant, hence  $\pi(a) = \tilde{\pi}(a)|_{E_V}$ . As  $[T, \pi(a)] = 0$  for every  $a \in C_0(G \times_H V)$ ,  $T(E_V) \subseteq E_V$ , and by  $G$ -equivariance,  $T|_{E_V}$  determines  $T$ . Hence  $T = \tilde{T}|_{E_V}$ . Hence, if  $z = [E, \pi, T]$  and  $z_H = [E_V, \pi(a)|_{E_V}, T|_{E_V}]$ , we proved that  $z = \text{Ind}_H^G(z_H)$ , hence  $\text{Ind}_H^G \circ \text{Res}_H^G(z) = z$ . □



## 5.2.2 Strongly proper groupoids

We now introduce a property on groupoids that will entail a nice result on induction and restriction transformations at the level of  $K$ -homology.

**Definition 5.2.7.** A groupoid  $G$  is said to be strongly proper if there exists an open cover  $\mathcal{U}$  of  $G^{(0)}$  such that, for all  $U \in \mathcal{U}$ , there exists a compact open subgroupoid  $H_U$  of  $G$  and a  $H_U$ -space  $Z_U$  together with a  $G$ -equivariant homeomorphism

$$\psi_U : U \rightarrow G \times_{H_U} Z_U.$$

An action of  $G$  on a space  $Z$  is said to be strongly proper if the groupoid  $Z \rtimes G$  is strongly proper. A groupoid is said to be in the class  $\mathcal{C}$  if every proper action of  $G$  is strongly proper.

**Remark 5.2.8.** For any strongly proper action of  $G$  on a space  $Z$ , there exists an open cover of  $Z$  by subsets of the type  $V = G \times_H U$ , where  $H$  is a compact open subgroupoid and  $U$  is a  $H$ -space. Then, by the previous section, we have an isomorphism

$$RK^G(V, B) \cong RK^H(U, B)$$

for every  $G$ -algebra  $B$ .

Let us give examples of groupoids in class  $\mathcal{C}$ . Recall the following definition from [44] (page 20).

**Definition 5.2.9.** A topological groupoid is said to be ample if it has a basis  $G^a$  of neighborhoods consisting of compact open subsets.

In [41] (page 17) is stated the following property. An étale groupoid  $G$  is ample iff  $G^{(0)}$  is totally disconnected. Hence the coarse groupoid of every coarse space  $X$  is ample, its basis being  $\beta X$ .

**Proposition 5.2.10.** Every ample groupoid is in class  $\mathcal{C}$ .

**Proof.** The following argument is an adaptation of the lemmas 2.41 and 2.42 of [51].

Let  $G$  be an étale ample groupoid and  $Z$  a  $G$ -space with proper action of  $G$ . Let  $z_0 \in Z$  and let  $W$  be a compact open neighborhood of  $x_0 = p(z_0) \in G^{(0)}$ . Let  $F$  be the stabilizer of  $z_0$ . By properness, it is a finite group. We can suppose  $W$  small enough to satisfy that, for any  $g \in F$ , there exist bisections  $U_g \subseteq G$  such that  $W \subseteq s(U_g)$ . Denote by  $\rho_g : W \rightarrow$  corresponding local sections of  $s$  such that  $\rho_g(x_0) = g$ . We denote by  $\alpha_g = r \circ \rho_g$  the corresponding partial homeomorphisms. By continuity of the product, we can suppose  $W$  small enough to satisfy

$$\rho_{g'}(\alpha_g(x))\rho_g(x) = \rho_{g'g}(x) \quad \forall x \in W$$

for all  $g$  and  $g'$  composable in  $F$ . Set  $W_0 = \bigcap_{g \in F} \alpha_g(W)$ , which is a  $F$ -invariant neighborhood of  $x_0$ . It is endowed with an action of  $F$  by  $g.w = \alpha_g(w)$ , and

$$\phi : \begin{cases} W_0 \rtimes F & \rightarrow G|_{W_0} \\ (w, g) & \mapsto \alpha_g(w) \end{cases}$$

defines a morphism of groupoids. Then  $H = \phi(W_0 \rtimes F)$  is a compact open subgroupoid of  $G$ .

There exists a neighborhood of  $z_0$  such that  $V \cap \alpha_g(V) \neq \emptyset \Rightarrow g \in H$ . Indeed, denote by  $\alpha : G \times_{s,p} Z \rightarrow Z$  the action and let  $C$  be  $(G \times U) \setminus (H \times U)$ . Hence  $(z_0, z_0) \notin \alpha(C)$ , hence there exists a neighborhood  $V$  of  $z_0$  such that  $V \times V \subseteq \alpha(C)$ . This  $V$  satisfies the previous condition.

Let  $K \subseteq V$  be a compact  $H$ -invariant neighborhood of  $z_0$ . Then

$$\Psi : \begin{cases} G \times_H K & \rightarrow G.K \\ (g, y) & \mapsto g.y \end{cases}$$

is well defined, continuous and  $G$ -equivariant. If  $gy = g'y'$ , then  $g^{-1}g' \in H$ , hence  $\Psi$  is bijective. As the action on  $K$  being proper,  $G \times_{s,p} K \rightarrow K \times GK$  is closed. Moreover,  $K$  being compact,  $pr_2 : K \times GK \rightarrow GK$  is closed. Hence  $\Psi$  is closed by composition, hence a  $G$ -equivariant homeomorphism.  $\square$

### 5.2.3 Baum-Connes and the Künneth formula

The first step in proving theorem 5.2.13 is to define an analytical version of  $\alpha_{A,B}$  when an étale groupoid  $G$  is given. More precisely, we first construct a homomorphism  $\alpha_{A,B}^G : K_*^{top}(G, A) \otimes K_*(B) \rightarrow K_*^{top}(G, A \otimes B)$ , inductive limit of  $\alpha_{A,B}^{G,Z} : RK^G(Z, A) \otimes K_*(B) \rightarrow RK^G(Z, A \otimes B)$  where  $Z$  runs through  $G$ -proper  $G$ -compact spaces. Then we show that the controlled assembly map intertwines  $\alpha_{A,B}^{G, P_E(G)}$  and  $\alpha_{A \rtimes_r G, B}$  uniformly.

If  $A$  is a  $G$ -algebra, and  $B$  a  $C^*$ -algebra,  $A \otimes B$  naturally inherits a  $G$ -algebra structure with trivial action of  $G$  on the  $B$  factor. Then  $(A \rtimes_r G) \otimes B \cong (A \otimes B) \rtimes_r G$ .

Let  $Z$  be a  $G$ -proper space. Define the homomorphism :

$$\alpha_{A,B}^{G,Z} : RK_*^G(Z, A) \otimes K_*(B) \rightarrow RK_*^G(Z, A \otimes B) \quad ; \quad (x, y) \mapsto x \otimes \tau_A(y),$$

which respects inductive limits w.r.t. inclusions of  $G$ -proper spaces, so that it induces

$$\alpha_{A,B}^G : K_*^{top}(G, A) \otimes K_*(B) \rightarrow K_*^{top}(G, A \otimes B).$$

To prove theorem 5.2.13, we will need the following result.

**Theorem 5.2.11.** Let  $G$  be an étale groupoid in the class  $\mathcal{C}$ , and let  $E \in \mathcal{E}$  be a controlled subset of  $G$  and  $P_E(G)$  be the corresponding Rips complex. If, for all compact open subgroupoids  $H$  of  $G$  and every  $H$ -space  $V$  such that the anchor map  $p : V \rightarrow H^{(0)}$  is locally injective,  $\alpha_{A,B}^{H,V}$  is an isomorphism, then  $\alpha_{A,B}^{G, P_E(G)}$  is an isomorphism for all  $C^*$ -algebras  $B$  such that  $K_*(B)$  is a free abelian group.

**Proof.** Let  $Z_0 \subseteq Z_1 \subseteq \dots \subseteq Z_n$  be the skeleton decomposition of  $P_E(G)$ .

Let us prove by induction that  $\alpha_j = \alpha_{A,B}^{G, Z_j}$  is an isomorphism. By a standard argument similar to the proof of theorem 4.3.14, it is sufficient to prove the statement for  $j = 0$ .

Let  $U \subseteq Z_0$  be a  $G$ -compact  $G$ -proper space. By strong properness,  $U$  can be finitely covered by open subsets of the type  $G \times_H V$ . By a standard Mayer-Vietoris argument, we can suppose that there exists a compact open subgroupoid  $H$  of  $G$  and a compact  $H$ -space  $V$  such that  $U = G \times_H V$ . The following diagram is commutative :

$$\begin{array}{ccc} RK_*^G(U, A) \otimes K_*(B) & \xrightarrow{\alpha_{A,B}^{G,U}} & RK_*^G(U, A \otimes B) \\ \downarrow \text{Res}_H^G & & \downarrow \text{Res}_H^G \\ RK_*^H(V, A) \otimes K_*(B) & \xrightarrow{\alpha_{A,B}^{H,V}} & RK_*^H(V, A \otimes B) \end{array}$$

The first vertical arrows are isomorphisms by Proposition 5.2.6, the last horizontal one is by hypothesis. Hence  $\alpha_{A,B}^{G,U}$  is an isomorphism. By taking the inductive limit on  $G$ -proper  $G$ -compact spaces  $U \subseteq Z_0$ , we get that  $\alpha_{A,B}^{G,Z_0}$  is an isomorphism.

**Lemma 5.2.12.** There exists a positive number  $\kappa \geq 1$  and a function  $c : (0, \frac{1}{4\kappa}) \times \mathcal{E} \rightarrow \mathcal{E}$ , decreasing in  $E$  and non increasing in  $\varepsilon$  such that, for any étale groupoid  $G$ , any filtered  $G$ -algebra  $(A, \mathcal{E})$ , and any  $C^*$ -algebra  $B$ , the following diagram commutes :

$$\begin{array}{ccc} RK_*^G(P_E(G), A) \otimes K_*(B) & \xrightarrow{\alpha_{A,B}^{G,P_E(G)}} & RK_*^G(P_E(G), A \otimes B) \\ \downarrow \mu_{G,A}^{\varepsilon,E,F} \otimes \text{id}_{K_*(B)} & & \downarrow \mu_{G,A}^{\alpha_{\tau\varepsilon,E,h\varepsilon F}} \\ K_*^{\varepsilon,F}(A \rtimes_r G) \otimes K_*(B) & \xrightarrow{\alpha_{A \rtimes_r G, B}^{\varepsilon,F}} & K_*^{\alpha_{\tau\varepsilon,E,h\varepsilon F}}((A \otimes B) \rtimes_r G) \end{array}$$

for all  $\varepsilon \in (0, \frac{1}{4\kappa})$  and  $E, F \in \mathcal{E}$  such that  $c(\varepsilon, E) \subseteq F$ . □

**Proof.** Let  $z \in RK_*^G(P_E(G), A)$  and  $y \in K_*(B)$ . As the action of  $G$  on  $B$  is trivial,  $(A \rtimes_r G) \otimes B \simeq (A \otimes B) \rtimes_r G$  so that  $\hat{\tau}_A(y) \circ \hat{J}_G(z) \sim \hat{J}_G(z) \circ \hat{\tau}_A(y)$  holds. This entails

$$\hat{J}_G^{\alpha_{\tau\varepsilon,E,h\varepsilon F}}(z) ([\mathcal{L}_E, 0]_{\varepsilon,E}) \otimes \tau_{A \rtimes_r G}(y) = \left( \hat{\tau}_A(y) \circ \hat{J}_G(z) \right)^{\varepsilon,E} ([\mathcal{L}_E, 0]_{\varepsilon,E}),$$

which is just the statement of the lemma. □

We can now prove the main theorem of the section.

**Theorem 5.2.13.** Let  $G$  be a  $\sigma$ -compact étale groupoid and  $A$  a  $G$ -algebra. Suppose that

- $G$  satisfies the Baum-Connes conjecture with coefficients,
- $G$  is in class  $\mathcal{C}$ ,
- for every compact open subgroupoid  $H$  of  $G$  and every  $H$ -space  $V$  such that the anchor map  $p : V \rightarrow H^{(0)}$  is locally injective,  $\alpha_{A,B}^{H,V}$  is an isomorphism.

Then  $A \rtimes_r G$  satisfies the quantitative Künneth formula.

**Proof.** Let  $\kappa \geq 1$  and  $c : (0, \frac{1}{\kappa}) \times \mathcal{E} \rightarrow \mathcal{E}$  as in lemma 5.2.12. Let  $\lambda \geq 1$  the universal constant of theorem 4.3.20.

Let  $A$  be a  $G$ -algebra and  $B$  a  $C^*$ -algebra, seen as a  $G$ -algebra with trivial action. As  $(A \otimes B) \rtimes_r G \cong (A \rtimes_r G) \otimes B$ , we will identify them in the remaining of the proof. Let us prove that  $\hat{\alpha}_{A \rtimes_r G, B}$  is a controlled isomorphism.

Let us show the controlled surjectivity. Let  $F \in \mathcal{E}$  and  $\varepsilon \in (0, \frac{1}{4\kappa})$ , and let

$$y \in K_*^{\varepsilon, F}((A \otimes B) \rtimes_r G).$$

By controlled surjectivity of  $\hat{\mu}_{G, A \otimes B}$ , there exist  $E, F' \in \mathcal{E}$  and  $z \in RK_*^G(P_E(G), A \otimes B)$  such that  $k_J(\varepsilon).E \subseteq F'$ ,  $F \subseteq F'$  and

$$\mu_{G, A \otimes B}^{\lambda\varepsilon, E, F'}(z) = \iota_{\varepsilon, F'}^{\lambda\varepsilon, F'}(y).$$

Theorem 5.2.11 entails that there exists  $z_1 \in RK_*^G(P_E(G), A) \otimes K_*(B)$  such that  $\alpha_{A, B}^{G, P_E(G)}(z_1) = z$ . Let  $F'' \in \mathcal{E}$  such that  $c(\varepsilon, E) \subseteq F''$  and  $F' \subseteq F''$ . By lemma 5.2.12, if we set  $z_2 = (\mu_{G, A}^{\varepsilon, E, F''} \otimes id_{K_*(B)})(z_1)$ , then

$$\alpha_{A \rtimes_r G, B}^{\varepsilon, F''}(z_2) = \mu_{G, A \otimes B}^{\alpha_\tau \varepsilon, E, h_\varepsilon \cdot F''}(z).$$

Hence

$$\iota_{\varepsilon, F}^{\lambda\alpha\varepsilon, h_\varepsilon F''}(y) = \iota_{\alpha_\tau \varepsilon, h_\varepsilon F''}^{\lambda\alpha\varepsilon, h_\varepsilon F''} \circ \alpha_{A \rtimes_r G, B}^{\varepsilon, F''}(z_2) = \alpha_{A \rtimes_r G, B}^{\lambda\varepsilon, F''}(z_2)$$

and  $\hat{\alpha}_{A \rtimes_r G, B}$  is quantitatively surjective.

Let us show the controlled injectivity. Let  $\varepsilon \in (0, \frac{1}{4\alpha_\tau})$  and  $F \in \mathcal{E}$ .

Let  $x \in K_*^{\varepsilon, F}(A \rtimes_r G) \otimes K_*(B)$  such that  $\alpha_{A \rtimes_r G, B}^{\varepsilon, F}(x) = 0$  in  $K^{\alpha_\tau \varepsilon, h_\tau \cdot \varepsilon F}((A \otimes B) \rtimes_r G)$ . By controlled surjectivity of  $\hat{\mu}_{G, A}$ , there exist  $E, F' \in \mathcal{E}$  such that  $F \subseteq F'$ ,  $k_J(\varepsilon).E \subseteq F'$  and  $z \in RK(P_E(G), A) \otimes K_*(B)$  such that

$$(\mu_{G, A}^{\lambda\varepsilon, E, F'} \otimes id_{K_*(B)})(z) = (\iota_{\varepsilon, F'}^{\lambda\varepsilon, F'} \otimes id)(x).$$

If  $z' = \alpha_{A, B}^{G, P_E(G)}(z)$ , pick any  $F'' \in \mathcal{E}$  such that  $c(\lambda\varepsilon, E) \subseteq F''$  and  $F' \subseteq F''$ . By lemma 5.2.12,

$$\mu_{G, A \otimes B}^{\alpha_\tau \lambda\varepsilon, E, F''}(z') = \alpha_{A \rtimes_r G, B}^{\lambda\varepsilon, F''} \circ (\mu_{G, A}^{\lambda\varepsilon, E, F'} \otimes id_{K(B)})(z) = \alpha_{A \rtimes_r G, B}^{\lambda\varepsilon, F''} \circ (\iota_{\varepsilon, F'}^{\lambda\varepsilon, F'} \otimes id)(x) = 0.$$

By controlled injectivity, there exists  $E' \in \mathcal{E}$  such that  $E \subseteq E'$  and  $q_E^{E'}(z') = 0$  in  $RK^G(P_{E'}(G), A \otimes B)$ . As  $\alpha_{A, B}^{G, P_E(G)}$  is an isomorphism and is compatible with inclusion,  $(q_E^{E'} \otimes id_{K(B)})(z) = 0$  in  $RK^G(P_{E'}(G), A) \otimes K(B)$  and

$$(\iota_{\varepsilon, F}^{\lambda\varepsilon, F''} \otimes id)(x) = (\mu_{G, A}^{\lambda\varepsilon, E', F''} \otimes id_{K_*(B)}) \circ (q_E^{E'} \otimes id_{K(B)})(z) = 0,$$

hence  $\hat{\alpha}_{A \rtimes_r G, B}$  is quantitatively injective.  $\square$

**Remark 5.2.14.** For principal groupoids, the third condition in theorem 5.2.13, "  $\alpha_{Res_H^G(A), B}^{H, V}$  is an isomorphism", reduces to "  $A$  satisfies the Kunneth formula". Indeed, the only non trivial subgroupoid is the trivial one.

# Chapter 6

## Conclusion and Perspectives

This thesis contains a generalization of controlled  $K$ -theory and constructions of controlled assembly maps, and some applications to computability in  $K$ -theory. We give now some natural questions that arise for future research.

One should look at the real advantage of controlled  $K$ -theory and controlled assembly maps : stability. Indeed, one should expect the controlled assembly map to respect weak notions of decomposition. The original goal of H. Oyono-Oyono and G. Yu was to extend the proof of the coarse Baum-Connes conjecture in the finite asymptotic dimension case to the finite decomposition complexity setting. We suggest the following program.

In a first step, study controlled Mayer-Vietoris decompositions associated to étale groupoids. The idea is to consider a decomposition of the base space into open subsets, which are not supposed to be invariant. This leads to a Mayer-Vietoris exact sequence in topological  $K$ -theory for the groupoid, but also to a controlled Mayer-Vietoris exact sequence in controlled  $K$ -theory.

Then, one could develop a theory of decomposition complexity for étale groupoids, and, if properly defined, the technique using Mayer-Vietoris decomposition should really work for this notion. More precisely, one should be able to prove that groupoids with finite decomposition complexity satisfy the Baum-Connes conjecture. The work would include showing explicit examples of groupoids which are not of finite dynamic asymptotic dimension, but are of finite decomposition complexity. The point of using both this kind of decomposition and controlled  $K$ -theory is that it is more algebraic in nature and could be adapted to other assembly conjectures, such as the Farrell-Jones conjecture or the algebraic Novikov conjecture. [43]

This leads to stability results of the controlled Baum-Connes Conjecture, which could be used to show the Baum-Connes conjecture for some kind of groupoids. The highlight of this method is that a proof actually gives you an algorithm to compute the  $K$ -theory of the crossed-product  $C^*$ -algebra of the groupoid, which is far to be obvious in the usual cases, e.g. for a-T-menable groupoids. This could be applied to a new proof of the Baum-Connes conjecture for étale groupoids with finite dynamic asymptotic dimension as defined by E. Guentner, R. Willett, and G. Yu [17]. This last step should be the ultimate generalization of the "controlled cutting and pasting" of G. Yu's proof [59] in the setting of étale groupoids.

The key notion for these techniques was introduced by H. Oyono-Oyono and G. Yu in [38] : Mayer-Vietoris pairs in a filtered  $C^*$ -algebra. Let us give more details.

We introduce the notation for a controlled neighborhood as follows. Let  $(A, \mathcal{E})$  be a filtered  $C^*$ -algebra,  $E \in \mathcal{E}$  and  $\Delta$  be a closed linear subspace of  $A_E$  stable by involution. Then the  $E$ -controlled neighborhood of  $\Delta$ , denoted by  $C^*N_{\Delta}^{(E, E')}$ , is defined as the  $C^*$ -algebra generated by  $N_{\Delta}^{(E, E')} = \Delta + A_{E'} \cdot \Delta + \Delta \cdot A_{E'} + A_{E'} \cdot \Delta \cdot A_{E'}$  for every  $E' \in \mathcal{E}$ .

**Definition 6.0.15.** Let  $(A, \mathcal{E})$  be a filtered  $C^*$ -algebra and  $E \in \mathcal{E}$ . A  $E$ -controlled Mayer-Vietoris pair for  $A$  is a quadruple  $(\Delta_0, \Delta_1, A_0, A_1)$  such that for some constant  $c > 0$ :

- $\Delta_j$  are closed linear subspaces of  $A$ ,
- $A_j$  are sub- $C^*$ -algebras of  $A$  such that  $A_j$  is  $\mathcal{E}$ -filtered by  $(A_{E'} \cap A_j)_{E' \in \mathcal{E}}$  and  $C^*N_{\Delta_j}^{(E, 5.E)} \subseteq A_j$ ,
- if  $x \in M_n(A_{E'})$  for  $E' \in \mathcal{E}$  such that  $E' \subseteq E$  and any  $n > 0$ , there exists  $x_j \in M_n(\Delta_j \cap A_{E'})$  such that  $x = x_0 + x_1$  and  $\|x_j\| \leq c\|x\|$ ,
- for every  $\varepsilon > 0$  and every  $E' \in \mathcal{E}$ , for any  $x \in M_n(A_{0, E'})$ ,  $y \in M_n(A_{1, E'})$  such that  $\|x - y\| < \varepsilon$ , there exists  $z \in M_n(A_{0, E'} \cap A_{1, E'})$  with  $\|z - x\| < \varepsilon$  and  $\|z - y\| < \varepsilon$ .

**Example 6.0.16.** Let  $G$  be an étale groupoid with compact base space and let  $V \subseteq G^{(0)}$  be an open subset. For  $E \in \mathcal{E}$ , put :

- $V^E = \{r(g) : g \in G_V \cap E\}$ ,
- $\Delta_V = C_0(G_V \cap E)$ ,
- $G_V^{V, (E)}$  the subgroupoid generated by  $G_V^V \cap E$ .

Let  $V_0$  and  $V_1$  be open subsets such that  $G^{(0)} = V_0 \cup V_1$ . Denote  $G_{V_j}^{V_j^E, (E)}$  by  $G_j$ . Then, under the existence of almost invariant partition of unity,

$$(\Delta_{V_0}, \Delta_{V_1}, C_r^*(G_0), C_r^*(G_1))$$

is a  $E$ -controlled Mayer-Vietoris pair for  $C_r^*(G)$ .

The existence of a controlled Mayer-Vietoris pair is nice, because even if the  $C^*$ -algebra is simple, it can possess such a decomposition, and the following result gives a way to compute the  $K$ -theory analogous to the situation of a classical Mayer-Vietoris decomposition:

**Theorem 6.0.17.** For every positive  $c$ , there exists a control pair  $\rho_{MV}$  such that for any  $\mathcal{E}$ -filtered  $C^*$ -algebra  $A$  and any  $E \in \mathcal{E}$ , if there exists a  $E$ -controlled Mayer-Vietoris pair

$(\Delta_0, \Delta_1, A_0, A_1)$ , then there exists a controlled morphism  $D : \hat{K}_*(A) \rightarrow \hat{K}(A_0 \cap A_1)$  such that the following sequence

$$\begin{array}{ccccc}
 \hat{K}_0(A_0 \cap A_1) & \longrightarrow & \hat{K}_0(A_0) \oplus \hat{K}_0(A_1) & \longrightarrow & \hat{K}_0(A) \\
 \uparrow D & & & & \downarrow D \\
 \hat{K}_1(A) & \longleftarrow & \hat{K}_1(A_0) \oplus \hat{K}_1(A_1) & \longleftarrow & \hat{K}_1(A_0 \cap A_1)
 \end{array}$$

is  $\rho$ -exact at order  $E$ .

To go back to our example, if  $G^{(0)} = V_0 \cup V_1$ , and with the same notations, we simultaneously have two sequences :

- the exact sequence in topological  $K$ -theory arising from the decomposition of the Rips complex into two open sets

$$P_E(G) = P_E(G_{V_0^E}^{V_0^E, (E)}) \cup P_E(G_{V_1^E}^{V_1^E, (E)}),$$

- the controlled exact sequence in controlled  $K$ -theory arising from the controlled Mayer Vietoris pair  $(\Delta_{V_0}, \Delta_{V_1}, C_r^*(G_0), C_r^*(G_1))$ .

The next step is to show that the controlled assembly map intertwines these two sequences.

As a conclusion, a lot of questions stem from the use of controlled  $K$ -theory for the study of groupoids. Controlled  $K$ -theory for filtered Banach algebras, developed in [9] by Yeong Chyuan Chung, would also hopefully lead to interesting ideas. One can off course always ask why the  $K$ -theory of the reduced  $C^*$ -algebra is chosen to be the range of the assembly map. Some situations require to consider other Banach algebras, possibly lacking the  $C^*$ -property. In this area,  $L^p$ -version of the Baum-Connes conjecture are heavily studied, and a controlled  $K$ -theory could be of strong interest, as shown by the work of Yeong Chyuan Chung in [8].





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