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Stability for the models of neuronal network and chemotaxis

Qilong Weng

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THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres
PSL Research University

Préparée à l'Université Paris-Dauphine

Stabilité pour des modèles de réseaux de neurones et de
chimiotaxie

École Doctorale de Dauphine — ED 543

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Stabilité pour des modèles de réseaux de neurones et de chimiotaxie

Qilong WENG

Stabilité pour des modèles de réseaux de neurones et de chimiotaxie

Résumé : Cette thèse vise à étudier certains modèles biologiques dans le réseau neuronal et dans la chimiotaxie avec la méthode d'analyse spectrale. Afin de traiter les principaux problèmes, tels que l'existence et l'unicité des solutions et des états stationnaires ainsi que les comportements asymptotiques, le modèle linéaire ou linéarisé associé est considéré par l'aspect du spectre et des semi-groupes dans les espaces appropriés, puis la stabilité de modèle non linéaire suit. Plus précisément, nous commençons par une équation de courses-et-chutes linéaire dans la dimension $d \geq 1$ pour établir l'existence d'un état stationnaire unique, positif et normalisé et la stabilité exponentielle asymptotique dans l'espace L^1 pondéré basé sur la théorie de Krein-Rutman avec quelques estimations du moment de la théorie cinétique. Ensuite, nous considérons le modèle du temps écoulé sous les hypothèses générales sur le taux de tir et nous prouvons l'unicité de l'état stationnaire et sa stabilité exponentielle non linéaire en cas sans ou avec délai au régime de connectivité faible de la théorie de l'analyse spectrale pour les semi-groupes. Enfin, nous étudions le modèle sous une hypothèse de régularité plus faible sur le taux de tir et l'existence de la solution ainsi que la même stabilité exponentielle sont généralement établies n'importe la prise en compte du délai ou non, au régime de connectivité faible ou forte.

Mots-clés : théorie de l'analyse spectrale, modèle de courses-et-chutes, équations cinétiques, processus de vitesse-saut, chimiotaxie, état stationnaire, stabilité asymptotique, hypocoercivité, réseau de neurones, dynamique du temps écoulé, connectivité faible, connectivité forte, hypodissipativité, convergence exponentielle.

Stability for the models of neuron networks and chemotaxis

Abstract: This thesis is aimed to study some biological models in neuronal network and chemotaxis with the spectral analysis method. In order to deal with the main concerning problems, such as the existence and uniqueness of the solutions and steady states as well as the asymptotic behaviors, the associated linear or linearized model is considered from the aspect of spectrum and semigroups in appropriate spaces then the nonlinear stability follows. More precisely, we start with a linear runs-and-tumbles equation in dimension $d \geq 1$ to establish the existence of a unique positive and normalized steady state and the exponential asymptotic stability in weighted L^1 space based on the Krein-Rutman theory together with some moment estimates from kinetic theory. Then, we consider time elapsed model under general assumptions on the firing rate and prove the uniqueness of the steady state and its nonlinear exponential stability in case without or with delay in the weak connectivity regime from the spectral analysis theory for semigroups. Finally, we study the model under weaker regularity assumption on the firing rate and the existence of the solution as well as the same exponential stability are established generally no matter taking delay into account or not and no matter in weak or strong connectivity regime.

Keywords: spectral analysis theory, runs-and-tumbles model, kinetic equations, velocity-jump processes, chemotaxis, stationary state, asymptotic stability, hypocoercivity, neuron network, time elapsed dynamics, weak connectivity, strong connectivity, hypodissipativity, exponential convergence.

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Introduction Générale

1 La théorie de l'analyse spectrale

La théorie de l'analyse spectrale a d'abord été introduite par Hilbert et été formulée initialement en théorie d'espace de Hilbert, qui a été capable d'expliquer la découverte ultérieure sur les caractéristiques des spectres atomiques en mécanique quantique. La théorie a étendu les résultats de décomposition bien connus en algèbre linéaire de dimension finie, tels que la diagonalisation et la triangularisation des matrices, à des situations analogues pour les opérateurs autoadjoints en dimension infinie. Ensuite, le développement ultérieur de l'espace abstrait de Hilbert ainsi que la théorie d'un seul endomorphisme normal étaient requis sous l'aspect de la physique et ont été construits par von Neumann.

Une autre formulation de la théorie ultérieure plus abstraite a été construite pour inclure les algèbres de Banach, qui dépassaient la restriction sur les opérateurs autoadjoints ni la nécessité d'agir dans le cadre des espace de Hilbert. Un tel développement a récemment été appliqué à plusieurs classes d'EDP pour la théorie cinétique des gaz et des modèles biologiques afin d'étudier l'évolution du comportement asymptotique ou même de décrire la convergence vers l'équilibre des solutions. Principalement, une telle approche vise à traiter les propriétés spectrales des opérateurs linéaires bornés ainsi que les propriétés de désintégration du semi-groupe continu associé dans le cadre de Banach. En particulier, la théorie réussit à établir les propriétés suivantes.

- (1) Le taux de convergence des équations linéaires d'évolutions dissipatives ou hypodissipatives sans structure autoadjointe dans les espaces pondérés de Banach, tels que les applications à certaines équations linéaires de Boltzmann, l'équation cinétique de Fokker-Planck et l'équation cinétique de courses-et-chutes dans la chimiotaxie.
- (2) Le comportement asymptotique des équations non linéaire d'évolution selon l'analyse spectrale sur leurs équations linéarisées relatives dans les espaces physiques naturels, par exemple l'équation homogène de Boltzmann et l'équation parabolique-elliptique de Keller-Segel.
- (3) L'existence, l'unicité ainsi que la stabilité de l'équilibre dans grands espaces sous les petits régimes de perturbation des équations positives strictes, par exemple l'équation inélastique de Boltzmann, l'équation parabolique-parabolique de Keller-Segel, le modèle temporel écoulé et d'autres dans réseau de neurones.

L'approche est basée sur une structure appropriée de fractionnement des générateurs du semi-groupe associés aux équations linéaires. Profitant de la structure appropriée de fractionnement de l'opérateur associé, nous pouvons utiliser la méthode de factorisation au niveau du résolvant et du semi-groupe pour analyser la structure du spectre de l'opéra-

teur et l'estimation asymptotique du semi-groupe associé. Ici, nous présentons les éléments principaux nécessaires de la théorie.

1.1 Semi-groupe d'opérateur linéaire

Nous commençons par plusieurs notations. Pour deux espaces de Banach $(X, \|\cdot\|_X)$ et $(Y, \|\cdot\|_Y)$, nous désignons $\mathcal{B}(X, Y)$ comme l'espace d'opérateurs linéaires bornés de X à Y avec $\|\cdot\|_{\mathcal{B}(X, Y)}$ ou $\|\cdot\|_{X \rightarrow Y}$ comme la norme associée tandis que $\mathcal{C}(X, Y)$ comme l'ensemble des opérateurs linéaires fermés non bornés et $\mathcal{K}(X, Y)$ comme l'espace des opérateurs linéaires compacts de X à Y . En particulier, nous simplifions $\mathcal{B}(X) := \mathcal{B}(X, X)$, $\mathcal{C}(X) := \mathcal{C}(X, X)$ and $\mathcal{K}(X) := \mathcal{K}(X, X)$. Pour un générateur $\Lambda \in \mathcal{C}(X)$ de certain semi-groupe associé $S_\Lambda(t)$, nous désignons son domaine par $D(\Lambda)$ et nous présentons les définitions élémentaires.

- (1) Nous disons que $z \in \mathbb{C}$ appartient à l'ensemble résolvant $\rho(\Lambda)$ si $\Lambda - z : D(\Lambda) \rightarrow X$ est bijective et son opérateur inverse appartient à $\mathcal{B}(\Lambda)$.
- (2) Pour tout $z \in \rho(\Lambda)$, l'opérateur résolvant $R_\Lambda(z)$ est donné par

$$\forall z \in \rho(\Lambda), \quad R_\Lambda(z) := (\Lambda - z)^{-1}.$$

- (3) Nous définissons l'ensemble du spectre $\Sigma(\Lambda)$ comme le complémentaire de l'ensemble résolvant, i.e.

$$\Sigma(\Lambda) := \mathbb{C} \setminus \rho(\Lambda).$$

Nous appelons $\xi \in \Sigma(\Lambda)$ une valeur propre isolée s'il existe $r > 0$, tel que

$$\Sigma(\Lambda) \cap B(\xi, r) = \{\xi\}.$$

D'ailleurs, le projecteur propre associé est défini par l'intégrale de Dunford

$$\Pi_{\Lambda, \xi} := \frac{i}{2\pi} \int_{\Gamma} R_\Lambda(z) dz,$$

où $\Gamma = \partial B(\xi, r/2)$. Le résolvant peut également être obtenu de l'opposé de la transformation de Laplace par

$$R_\Lambda(z) = - \int_0^\infty S_\Lambda(t) e^{-zt} dt,$$

tandis qu'une fois que le spectre de l'opérateur $\Sigma(\Lambda)$ est déterminé, nous pouvons revenir au semi-groupe S_Λ à travers la transformation inverse de Laplace

$$S_\Lambda(t)f = \frac{i}{2\pi} \int_{\uparrow_\alpha} R_\Lambda(z) f e^{zt} dz, \quad f \in X,$$

où $\uparrow_\alpha = \{z = \alpha + iy, y \in \mathbb{R}\}$ pour α supérieur que la borne spectrale $s(\Lambda)$, définie comme

$$s(\Lambda) := \sup\{\Re \lambda; \lambda \in \Sigma(\Lambda)\}.$$

Nous disons que $\Lambda \in \mathcal{C}(X)$ avec domaine dense $D(\Lambda)$ est α -hypodissipatif s'il existe une norme équivalente $\|\!\| \cdot \|\!\|$ sur X satisfaisant

$$\forall f \in D(\Lambda), \exists \varphi \in F_{\|\!\|, \|\!\|}(f) \text{ tel que } \Re \langle \varphi, (\Lambda - \alpha)f \rangle \leq 0,$$

où, pour tout $f \in X$, l'ensemble dual associé $F_{\|\cdot\|}(f) \subset X'$ est défini par

$$F_{\|\cdot\|}(f) := \{\varphi \in X'; \langle \varphi, f \rangle = \|f\|_X^2 = \|\varphi\|_{X'}^2\}.$$

Nous disons aussi que le générateur Λ d'un semi-groupe d'opérateurs bornés est α hypodissipatif de type (α, M) for $\alpha \in \mathbb{R}$ et $M \geq 1$, si Λ est α -hypodissipatif, et la norme équivalente $\|\cdot\|$ satisfait

$$\|f\| \leq \|f\| \leq M\|f\|, \quad \forall f \in X;$$

ou également si le semi-groupe associé S_Λ sur X remplit l'estimation de croissance

$$(1.1) \quad \|S_\Lambda(t)\|_{\mathcal{B}(X)} \leq M e^{\alpha t}, \quad \forall t \geq 0.$$

Nous définissons ensuite la borne de croissance

$$\omega(\Lambda) = \omega(S_\Lambda) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|S_\Lambda(t)\| = \inf\{\alpha \in \mathbb{R}; (1.1) \text{ est satisfaite}\}.$$

Nous allons simplifier quelquefois en disant que S_Λ is α -hypodissipatif lorsque l'estimation de croissance ci-dessus est valable.

1.2 Méthode de décomposition de semi-groupe

Il est nécessaire d'introduire la convolution des semi-groupes. Pour $T \in (0, \infty)$, étant donné certains espaces de Banach X_1 , X_2 et X_3 et deux semi-groupes agissant dans les espaces ci-dessus $S_1(t) \in L^1(0, T; \mathcal{B}(X_1, X_2))$ et $S_2(t) \in L^1(0, T; \mathcal{B}(X_2, X_3))$, nous définissons la convolution $S_2 * S_1(t) \in L^1(0, T; \mathcal{B}(X_1, X_3))$ par

$$(S_2 * S_1)(t) := \int_0^t S_2(t-s)S_1(s) ds, \quad \forall t \geq 0.$$

Particulièrement, nous désignons récursivement $S^{(*n)} := S * S^{(*(n-1))}$ avec $S^{(*1)} = S$.

Pour approfondir l'approche, nous considérons un opérateur Λ sur un espace de Banach X avec la structure de fractionnement

$$(1.2) \quad \Lambda = \mathcal{A} + \mathcal{B},$$

où \mathcal{B} a une propriété dissipative, ce qui conduit à une bonne localisation de son spectre et \mathcal{A} est beaucoup plus régulier que \mathcal{B} . Le choix de la décomposition de l'opérateur varie selon les différents modèles. Dans de nombreuses œuvres, la théorie de l'analyse spectrale a été appliquée avec l'aide de la fractionnement (1.2) satisfaisant que \mathcal{B} est dissipatif et \mathcal{A} est \mathcal{B} -compact en considérant l'opérateur Λ comme une perturbation compacte de \mathcal{B} . Cependant, la régularité est beaucoup plus nécessaire dans certaines situations, ce qui nous encourage à ajuster la structure de fractionnement de la manière suivante. Nous divisons \mathcal{A} de plus comme $\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^c$ tel que $\tilde{\mathcal{A}}$ est lisse dans une certaine mesure et $\tilde{\mathcal{A}}^c$ est petit, ce qui génère une nouvelle décomposition appropriée

$$\Lambda = \tilde{\mathcal{A}} + \tilde{\mathcal{B}}, \quad \tilde{\mathcal{B}} := \tilde{\mathcal{A}}^c + \mathcal{B}$$

au lieu de l'habituel. Par rapport à \mathcal{A} , $\tilde{\mathcal{A}}$ possède plus de propriété de régularité, tandis que $\tilde{\mathcal{B}}$ conserve encore suffisamment de dissipativité. Pour être plus précis, les propriétés suivantes sont nécessaires dans l'analyse spectrale. Pour un $a \in \mathbb{R}$ donné et un opérateur Λ avec la structure de fractionnement (1.2) définie sur un espace approprié de Banach \mathcal{X} ,

- \mathcal{B} s'appelle \mathcal{A} -puissance α -dissipatif dans \mathcal{X} , si pour tout $\alpha > a$ et pour tout $\ell \in \mathbb{N}$, il satisfait

$$S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t)e^{-\alpha t} \in L^\infty(\mathbb{R}_+; \mathcal{B}(\mathcal{X}));$$

- \mathcal{A} s'appelle droite \mathcal{B} -puissance régulier dans $(\mathcal{X}, \mathcal{Y})$ avec $\mathcal{Y} \subset \mathcal{X}$, si pour tout $\alpha > a$, il existe un entier $n \geq 1$, tel que

$$(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)e^{-\alpha t} \in L^1(\mathbb{R}_+; \mathcal{B}(\mathcal{X}, \mathcal{Y})),$$

ou \mathcal{A} s'appelle gauche \mathcal{B} -puissance régulier si

$$(S_{\mathcal{B}}\mathcal{A})^{(*n)}(t)e^{-\alpha t} \in L^1(\mathbb{R}_+; \mathcal{B}(\mathcal{X}, \mathcal{Y}));$$

- \mathcal{A} s'appelle gauche $R_{\mathcal{B}}$ -puissance régulier dans $(\mathcal{X}, \mathcal{Y})$ avec $\mathcal{Y} \subset \mathcal{X}$, si pour tout $\alpha > a$, $\mathcal{Y} \neq \mathcal{X}$ ou $\alpha \neq 0$, il existe $n \geq 1$ et $C > 0$, tel que

$$\|(\mathcal{A}R_{\mathcal{B}}(z))^n\|_{\mathcal{B}(\mathcal{X}, \mathcal{Y})} \leq C\langle x \rangle^{-\alpha}, \quad z \in \Delta_a,$$

ou \mathcal{A} s'appelle droite $R_{\mathcal{B}}$ -puissance régulier si

$$\|(R_{\mathcal{B}}(z)\mathcal{A})^n\|_{\mathcal{B}(\mathcal{X}, \mathcal{Y})} \leq C\langle x \rangle^{-\alpha}, \quad z \in \Delta_a,$$

où $\Delta_a := \{z \in \mathbb{C}; \Re z > a\}$;

- \mathcal{A} s'appelle gauche ou droite \mathcal{B} -puissance compact dans $(\mathcal{X}, \mathcal{Y})$ si \mathcal{A} est gauche ou droite \mathcal{B} -puissance régulier et $\mathcal{Y} \subset \mathcal{X}$ avec plongement compact.

Outre les estimations de croissance ci-dessus, il est également nécessaire de représenter le semi-groupe S_Λ et le générateur associé Λ avec la structure de fractionnement. Rappelant la formule basique de Duhamel avec factorisation droite ou gauche,

$$S_\Lambda = S_{\mathcal{B}} + S_\Lambda * \mathcal{A}S_{\mathcal{B}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_\Lambda,$$

il est plus pratique de l'itérer finiment, par exemple avec la droite factorisation, comme

$$S_\Lambda = \sum_{\ell=0}^{n-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)} + S_\Lambda * (\mathcal{A}S_{\mathcal{B}})^{(*n)}$$

ou de remplacer en plus S_Λ dans le dernier terme par la transformation inversée de Laplace, afin d'appliquer les propriétés ci-dessus puisque la convergence de l'itération infinie de la formule de Duhamel ne peut pas être habituellement atteinte. De la même façon, S_Λ peut également être représenté par l'itération finie de la factorisation gauche. Quant au générateur Λ , l'approche est effectuée de l'identité de factorisation résolvante.

$$R_\Lambda(z) = R_{\mathcal{B}}(z) - R_{\mathcal{B}}(z)\mathcal{A}R_\Lambda(z) = R_{\mathcal{B}}(z) - R_\Lambda(z)\mathcal{A}R_{\mathcal{B}}(z)$$

à la formule itérée de la première identité

$$R_\Lambda(z) = \sum_{\ell=0}^{n-1} (-R_{\mathcal{B}}(z)\mathcal{A})^\ell R_{\mathcal{B}}(z) + (-R_{\mathcal{B}}(z)\mathcal{A})^n R_\Lambda(z)$$

ou de la deuxième

$$R_\Lambda(z) = \sum_{\ell=0}^{n-1} R_{\mathcal{B}}(z)(-\mathcal{A}R_{\mathcal{B}}(z))^\ell + R_\Lambda(z)(-\mathcal{A}R_{\mathcal{B}}(z))^n.$$

1.3 Théorèmes spectraux

Le but de l'analyse spectrale des semigroups S_Λ avec le générateur Λ sur un espace de Banach X est de décrire le spectre $\Sigma(\Lambda)$ et l'écart spectral de $\Sigma(S_\Lambda)$, ce qui implique le comportement asymptotique des trajectoires associées au semi-groupe. Bénéficiant des hypothèses de croissance ci-dessus ainsi que des identités itérées de Duhamel, les théorèmes suivants ont été obtenus ces dernières années.

Théorème de l'application spectrale. Pour un semi-groupe continu $S_\Lambda(t) = e^{t\Lambda}$, le théorème de l'application spectrale vise à montrer

$$\Sigma(e^{t\Lambda}) \setminus \{0\} = e^{t\Sigma(\Lambda)}, \quad \forall t \geq 0,$$

ce qui nécessite des conditions sévères sur l'opérateur Λ et l'espace associé X . Toutefois, avec l'aide d'une structure de fractionnement appropriée (1.2) satisfaisant de plus les hypothèses ci-dessus, une autre version moins précise du théorème de l'application spectrale a été développée dans [50]. Pour tout $\alpha > a$, il satisfait

$$(1.3) \quad \Sigma(e^{t\Lambda}) \setminus B(0, e^{\alpha t}) = e^{t\Sigma(\Lambda) \cap \Delta_\alpha}, \quad t \geq 0,$$

ce qui est moins précis que l'intention initiale, mais qui est déjà capable de décrire l'évolution du semi-groupe au premier ordre dans de nombreuses situations. En particulier, lorsque $\Sigma(\Lambda) \cap \Delta_\alpha \neq \emptyset$, le théorème partiel de l'application spectrale ci-dessus (1.3) présente aussi des informations asymptotiques plus précises du semi-groupe à savoir que la borne spectrale $s(\Lambda)$ co-incide avec la borne de croissance $\omega(\Lambda)$ à la droite de a , i.e.

$$\max\{s(\Lambda), a\} = \max\{\omega(\Lambda), a\}.$$

Entre-temps, le semi-groupe associé satisfait l'estimation de croissance

$$(1.4) \quad \|S_\Lambda(t) - S_\Lambda(t)\Pi\|_{\mathcal{B}(X)} \leq C_\alpha e^{\alpha t}$$

pour une constante positive C_α avec un projecteur Π commutant avec S_Λ , qui implique (1.3) en particulier.

Le théorème partiel de l'application spectrale réussit à réduire l'analyse du spectre de Λ dans le demi plan complexe Δ_α , pour tout $\alpha > a$. Pour une classe appropriée d'opérateurs Λ avec la structure de fractionnement (1.2), le spectre $\Sigma(\Lambda) \cap \Delta_\alpha$ est invariant sous le changement de X , ce qui assure que telle analyse spectrale de Λ et l'estimation de croissance sur S_Λ sont capable d'être étendues à un autre espace grâce à l'élargissement ou à la rétrécissement.

Théorème de Weyl. Le type classique du théorème de Weyl concernant la perturbation compacte de l'opérateur affirme à peu près que

$$\forall \mathcal{A} \prec\prec \mathcal{B}, \quad \Sigma_{\text{ess}}(\mathcal{A} + \mathcal{B}) = \Sigma(\mathcal{B}),$$

où $\mathcal{A} \prec\prec \mathcal{B}$ ssi $D(\mathcal{B}) \subset D(\mathcal{A})$ et $\mathcal{A} \in \mathcal{K}(D(\mathcal{B}), X)$ tandis que Σ_{ess} désigne le spectre essentiel d'un opérateur comme l'ensemble complémentaire de spectre du spectre discret (l'ensemble de toutes les valeurs propres isolées avec multiplicité algébrique finie). Dans l'esprit du théorème de l'application spectrale précédent et conjointement avec l'hypothèse de compacité ci-dessus sur la structure de fractionnement (1.2), nous présentons

une version améliorée du théorème de Weyl aux perturbations régulières et compactes de puissance comme

$$\Sigma(\Lambda) \cap \Delta_\alpha = \{\xi_1, \dots, \xi_J\} \subset \Sigma_d(\Lambda),$$

pour un entier $J \in \mathbb{N}$ et des nombres complexes finis distincts ξ_1, \dots, ξ_J , ce qui permet une estimation sur la distribution du spectre, telles que le nombre et la localisation des valeurs propres ainsi que la dimension des espaces propres algébriques associés dans le demi plan complexe Δ_α . De plus, il existe aussi quelques projecteurs de rang fini Π_1, \dots, Π_J communiquant avec Λ et certains opérateurs T_j satisfaisant $\Lambda\Pi_j = T_j\Pi_j$, $\Sigma(T_j) = \{\xi_j\}$, $1 \leq j \leq J$, tel que la partie de projection $S_\Lambda(t)\Pi$ dans l'estimation de croissance précédente (1.4) n'est rien que la somme des projections finies à travers les projecteurs propres de toutes les valeurs propres isolées dans le demi plan complexe Δ_α , autrement dit :

$$S_\Lambda(t)\Pi = \sum_{j=1}^J S_{T_j}(t)\Pi_j.$$

La stratégie des perturbations. Etant donné deux opérateurs Λ_ε et Λ , nous pouvons souhaiter confirmer la similitude entre les spectres correspondants $\Sigma(\Lambda_\varepsilon)$ et $\Sigma(\Lambda)$ lorsque $\Lambda_\varepsilon - \Lambda$ est assez petit dans une certaine mesure. Une telle attente engendre la création de la stratégie des perturbations paraphée par le travail de R.S. Phillips dans [65] et beaucoup développée par Kato dans [36], ce qui établit la stabilité du spectre ainsi que les espaces propres associés sous une petite perturbation appropriée. Sous le même encadre de la structure de fractionnement que précédemment, la stratégie adaptée implique la structure du spectre de Λ_ε et les estimations de croissance sur S_{Λ_ε} associé sous une petite perturbation appropriée par rapport à celles de Λ et S_Λ qui sont beaucoup plus faciles à analyser et à généraliser.

Théorème de Krein-Rutman. Dans l'algèbre linéaire, le théorème de Perron-Frobenius affirme que la valeur propre avec la plus grande partie réelle d'une matrice strictement positive dans un espace de Banach de dimension finie est réelle, unique et simple. Pour un semi-groupe positif S_Λ agissant dans un treillis de Banach X , un espace de Banach doté d'une ordre partielle \geq or \leq , le théorème de Kerin-Rutman dans [41] atteint la même conclusion sur la valeur propre d'un opérateur Λ sous des hypothèses de positivité strictes et de compacité. Nous appelons un treillis de Banach X "standard" si la fonction signe est bien définie et il existe un élément strictement positif dans X . Dans [50], une version générale du théorème de Kerin-Rutman a été adaptée pour l'opérateur Λ avec une structure de fractionnement susmentionnée, ce qui satisfait un principe du maximum faible et fort en plus et qui est défini sur un treillis "standard" de Banach X . Cette version du théorème montre que

$$\Sigma(\Lambda) \cap \Delta_\alpha = \{\lambda\},$$

pour certains $\alpha \in \mathbb{R}$ et $\lambda \in \mathbb{R}$ qui est une valeur propre simple. Le résultat sur le spectre est bénéfique pour l'analyse spectrale et l'estimation asymptotique pour les semi-groupes positifs car la partie principale est une valeur propre simple réelle.

Dans l'application, nous nous intéressons à la construction d'un semi-groupe fortement continu, ce qui, cependant, est assez difficile parfois pour de nombreux opérateurs, tel que le modèle du temps écoulé dans les réseaux de neurones. En conséquence, afin d'effectuer la théorie de l'analyse spectrale ci-dessus dans ces situations, nous pouvons soit étendre

la théorie au semi-groupe faiblement continu avec des propriétés faiblement dissipatives dans un espace plus large, soit appliquer l'approche sur le semi-groupe dual continu dans l'espace dual.

1.4 Outils complémentaires de la théorie cinétique

Généralement, le problème de l'équation cinétique sur les quantités macroscopiques subit une dynamique complexe en raison de la non-linéarité découlant de l'intégrale vers la vitesse v . Chaque variable t, x, v produit différents effets, par exemple, la dispersion et la régularité par la moyenne de la vitesse, qui sont puissantes pour souligner l'existence d'états stationnaires ainsi que de les solutions faibles et le comportement asymptotique. Des progrès substantiels ont été réalisés récemment par ces effets, tels que la solution globale de l'équation de Boltzmann, la justification complète de la limite hydrodynamique incompressible de l'équation de Boltzmann et des applications en biologie.

Les effets de dispersion. Les effets de dispersion nous permettent d'obtenir une estimation sur le déclin de temps dans la intégrabilité L^p d'espace-temps. L'estimation précédente a été introduite pour la première fois par Bardos et Degond considérant la dispersion dans l'espace entier dans [6], puis elle est largement utilisée comme un outil de base pour étudier l'existence de petites solutions globales dans le temps $f(t, x, v)$ de problèmes non linéaires. Pour être plus précis, nous prenons une solution f à la plus simple et pure équation de transport cinétique.

$$\begin{cases} \partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = 0, \\ f(t = 0, x, v) = f_0(x, v), \end{cases}$$

comme un exemple. La densité macroscopique définie par

$$\rho(t, x) := \int_{\mathbb{R}^d} f(t, x, v) dv$$

bénéficie d'une estimation sur le déclin de temps liée à la dispersion dans l'espace entier, qui écrit

$$|\rho(t, x)| \leq \frac{1}{t^d} \|f_0\|_{L_x^1 L_v^\infty}, \quad \forall t > 0, x \in \mathbb{R}^d.$$

D'autre part, une approche plus sophistiquée, appelée les inégalités de type Strichartz, a été conçue par Castella et Perthame dans [13] pour exprimer la dispersion dans l'espace entier pour les équations cinétiques et pour obtenir l'intégrabilité de la densité macroscopique, à savoir

$$\|\rho(t, x)\|_{L_t^q L_x^p} \leq C_d \|f_0\|_{L_{x,v}^\beta},$$

pout tout nombres réels p, q et β satisfaisant

$$1 \leq p < \frac{d}{d-1}, \quad \frac{2}{q} = d\left(1 - \frac{1}{p}\right), \quad 1 \leq \beta := \frac{2p}{p+1}.$$

Par ailleurs, une meilleure intégrabilité peut être obtenue des injections de Sobolev ainsi que de la régularité de Sobolev prouvée sur des quantités macroscopiques, si nous abandonnons l'avantage des effets de dispersion pour effectuer les données initiales dans les espaces

de Lebesgue classiques, voir [61]. Cependant, cette approche est simplement appliquée pour les vitesses bornées sans la capacité d'indiquer l'information asymptotique.

La méthode des multiplicateurs. Dans [42], la méthode des multiplicateurs a été conçue par Lions et Perthame pour une meilleure intégrabilité de vitesse, qui peut être transformée dans l'intégrabilité L^p d'espace pour les intégrales de vitesse sous des contrôles macroscopiques. Les résultats sont conclus en intégrant l'équation cinétique avec différents multiplicateurs $M(x, v)$, c'est-à-dire

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^{m \times n}} M(x, v) f(t, x, v) dt dx dv \lesssim \|f_0 M'\|_{L^1(\mathbb{R}^{m \times n})},$$

où $M'(x, v)$ dépend du choix du multiplicateur $M(x, v)$. Une illustration typique est l'application aux états stationnaires des équations cinétiques sans absorption dans l'espace entier, découlant de la limite de haute fréquence des équations dispersives. Les difficultés sont générées directement en raison de l'absence de l'unicité ainsi que d'une estimation à priori sans absorption.

Le lemme de moyenne de vitesse. Afin d'acquérir la régularité, le lemme de moyenne pour les solutions des équations cinétiques a été introduit dans [29] et été développé dans de nombreux travaux pendant des dernières décennies, en particulier dans [42, 7, 64], comme un outil puissant pour traiter le problème de la compacité. Pour une solution $f(t, x, v)$ à une équation cinétique, la version classique du lemme de moyenne montre que la quantité macroscopique

$$\rho_\psi(t, x) := \int f(t, x, v) \psi(v) dv,$$

avec ψ une fonction test lisse, possède une meilleure régularité par rapport à (t, x) que la quantité microscopique $f(t, x, v)$. Des diverses versions du lemme proviennent de certaines extensions à la droite de la définition ci-dessus, à partir des dérivées de fonctions L^2 dans [18] à un cadre L^p général dans [20], ce qui ont été largement appliquées aux équations de Vlasov pour les limites du champ moyen, aux équations de Boltzmann pour les flux dilués des collisions et aux équations de la diffusion pour le mouvement cellulaire. La preuve du lemme repose sur des arguments de Fourier dans L^2 , des intégrales singulières en L^p pour éviter les singularités pour l'opérateur $f_0 \mapsto f$ et l'inégalité d'interpolation pour les espaces de Sobolev et de Besov ou simplement sur des arguments de dualité comme récemment montré par Jabin et Vega dans [34].

2 Le contexte biomathématiques

Pour l'intégralité du texte, avant de passer à la présentation des principaux résultats mathématiques, nous revoyons certains faits basiques sur le contexte biologique. Le but de la discussion présentée ici n'est pas de donner une description complète de structures si complexes telles que le cerveau ou le problème de la morphogenèse. Néanmoins, nous voulons fournir des concepts de base qui expliqueront les motivations derrière nos modèles et leur intérêt biologique. Quoique, ces informations ne sont pas nécessaires à la compréhension des développements mathématiques et à des contributions de la thèse.

2.1 Chimiotaxie

Le processus “taxie” montre le mouvement des cellules, influencé par la lumière, la concentration d’oxygène et par autres facteurs, y compris les gradients chimiques, en s’approchant de l’environnement avantageux et en évitant les inconvénients. Donc la chimiotaxie joue un rôle central pour décrire les phénomènes des cellules se déplaçant vers une concentration de chimioattracteurs plus élevée et elle est très commune dans de nombreux processus biologiques. En immunologie, par exemple, les leucocytes migrent de la circulation sanguine après des antigènes étrangers comme la réponse immunitaire entraînant l’inflammation et la réparation des tissus est régulée par des stimuli externes pendant la guérison des plaies. Les cellules souches migrent pour former des tissus puis des organes complexes tels que l’embryologie et l’angiogenèse dans la biologie de développement. Un autre exemple se concentre sur les bactéries dans leurs colonies auto-organisées, où la chimiotaxie est également responsable de divers types de modèles d’agrégation tels que les impulsions itinérantes de colonies et les grappes stables à cause de l’action comme des signes parmi les microorganismes.

Au niveau macroscopique, la chimiotaxie est habituellement modélisée par un système d’équations paraboliques, dont le plus connu est le système de Keller-Segel. Puisque les bactéries ne recherchent pas seulement les conditions optimales pour croître mais elles se déplacent pour agréger, le modèle a considéré l’agrégation des amibes de moisissures *Dictyostelium discoideum* entraînées par la substance chimique attrayante et a été étudié dans [38, 39, 40], résumée Dans [37]. Le mécanisme est donné par

$$(2.5) \quad \begin{cases} \partial_t \rho = D_1 \Delta \rho - \chi \nabla \cdot (\rho \nabla S), \\ \partial_t S = D_2 \Delta S + \phi(\rho, S), \end{cases}$$

où $\rho(t, x) \geq 0$ représente la densité des cellules et $S(t, x) \geq 0$ est la concentration de la substance chimique. Les constantes positives D_1 et D_2 sont les diffusivités de l’agent chimique et des cellules respectivement tandis que $\chi \geq 0$ désigne la sensibilité chimio-tactique. La fonction $\phi(\rho, S)$ décrit l’interaction entre le substrat et les microorganismes puisque S peut également être produit par les cellules elles-mêmes en plus de diffuser dans le substrat. Un simple choix de ϕ est donné par

$$\phi(\rho, S) = \alpha \rho - \beta S,$$

où nous supposons que les bactéries produisent le chimioattractant à un taux constant $\alpha \geq 0$ et la substance chimique se désintègre à un taux constant $\beta \geq 0$. L’équation à ρ dans le système (2.5) peut être dérivée d’une limite hydrodynamique macroscopique d’un système stochastique d’une grande quantité de particules et le premier terme à la droite représente la diffusion sous l’intention des microorganismes à travers leur propre mouvement brownien tandis que le second montre leur tendance à l’agrégation grâce à la présence du chimioattractant.

Le problème principal dans l’étude du système ci-dessus est de savoir si les solutions vont exploser en temps fini. Lorsque la dimension spatiale $n \geq 3$, la réponse est affirmative et a été prouvée dans [32]. A l’inverse, l’explosion ne se produira jamais pour $n = 1$. Quant à $n = 2$, l’apparition d’une explosion dépend de la condition initiale, voir [33]. Cependant, nous pouvons empêcher une telle explosion après une modification appropriée du modèle

de Keller-Segel (2.5), par exemple, par la perte d'efficacité chimiotactique supérieure à une densité de saturation.

Au niveau microscopique, on observe dans les expériences que les bactéries nageant comme *Escherichia coli* subissent un mouvement irrégulier composé d'une série de deux différentes périodes alternatives, "course" et "chute". Une phase de course consiste en un mouvement dirigé en ligne droite qui sera long dans une direction favorable mais sera court dans un défavorable. Après une course, la bactérie est réorientée en raison d'une phase de chute et elle suit une nouvelle course dans la nouvelle direction. En ce qui concerne *Escherichia coli*, représentant typiquement les bactéries flagellées, ce genre de mouvement de chute est réalisé par transmission de signal entre les complexes de récepteurs et les complexes flagellaires-moteurs, voir [3]. Sur une cellule, les premiers sont situés dans les pôles afin de détecter la présence de substances chimiques tandis que les derniers sont intégrés dans la membrane et ils sont distribués de manière aléatoire.

Ce mécanisme allonge beaucoup la phase de course par rapport à la phase invariante de chute, ce qui entraîne une marche aléatoire biaisée dans le sens de gradient de la substance chimique, voir [70]. Quant aux leucocytes ou aux amibes, le mode de direction de virage et la vitesse changeant dans la phase de chute est également déterminé par la substance chimique lors de l'observation d'expérimentation. Généralement, la phase de chute est beaucoup plus courte que la phase de course (voir [9]), ce qui nous inspire à modéliser la chimiotaxie avec un processus instantané de saut de vitesse dans de nombreux modèles. Bien que la taille des bactéries soit trop petite pour mesurer le gradient spatial de la substance chimique le long de la longueur individuelle, ce processus stochastique est encore biaisé sous le gradient avec le bénéfice de la fréquence augmentée de chute ainsi que de la capacité de mesurer le gradient dans le temps le long de son parcours après réduisant la concentration de l'agent chimique. Un tel biais permet le mouvement désiré malgré la vitesse impartiale vers l'extérieur après la chute.

Au niveau mésoscopique, le modèle de chimiotaxie consiste en une équation cinétique de la densité d'espace de phase $f(t, x, v)$ des organismes avec une équation elliptique ou parabolique présentant une réaction-diffusion sur la concentration $S(t, x)$ de la substance chimique, qui a d'abord été introduite par Alt [2, 3] et été étudiée par Othmer, Dunbar et Alt [54], donnée par

$$(2.6) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f = Q[f, S] \\ \qquad \qquad \qquad := \int_{\mathcal{V}} (K[S](t, x, v', v) f(t, x, v') - K[S](t, x, v, v') f) dv', \\ S - \Delta S = \rho := \int_{\mathcal{V}} f(t, x, v) dv, \end{cases}$$

où $f = f(t, x, v) \geq 0$ indique la densité d'espace de phase des cellules et l'ensemble de distribution de vitesse \mathcal{V} est compact. On récupère le modèle de Keller-Segel (2.5) par la limite à l'échelle de la diffusion macroscopique à partir du modèle cinétique ci-dessus (également appelé le modèle de saut de vitesse). La limite produit une équation de convection-diffusion, dont les diffusivités et les vitesses de convection différentes sont avec le même comportement qualitatif. L'équation (2.5) peut également être dérivée des modèles cinétiques par l'expansion de Hilbert en raison de l'ensemble fini dépendant d'espace-temps de la distribution de la vitesse. En fait, l'équation intégro-différentielle (2.6) est également une sorte d'équation de type Boltzman, qui a d'abord été conçue pour décrire un système

thermodynamique, tel que les gaz moyennement raréfiés.

En plus de l'accélération du champ de force dérivé d'un potentiel de confinement sur la randomisation de vitesse sans biais causée par le mouvement brownien ou par un processus de saut, le champ d'accélération peut également être remplacé par un processus biaisé de saut de vitesse pour obtenir le confinement. Sous la restriction suffisamment forte pour couvrir l'effet dispersif, le mouvement évoluera vers une mesure de probabilité invariante comme l'équilibre. La convergence vers l'équilibre pour ce type de modèles de transport cinétique a été étudiée et certaines convergences fortes ont été obtenues, voir [21] pour un taux exponentiel et [10] pour un taux algébrique. Toutefois, le modèle (2.6) est encore si complexe que les résultats sur la stabilité ont simplement été obtenus sous des hypothèses strictes sur la concentration chimiotactique S et avec le bénéfice de différents choix de $K[S]$ avec la dépendance appropriée de S .

2.2 Réseau de neurones

Au cours du dernier siècle, l'étude sur la structure et sur la fonction du cerveau s'est accumulée dans la recherche biologique et une grande quantité de connaissances détaillées a été obtenue. Dans le système nerveux central, les composants élémentaires du noyau sont les neurones pour traiter et transmettre des informations pas des signaux électriques et chimiques, qui sont reliés l'un à l'autre pour former un réseau de neurones dans un mode compliqué. Par exemple, plus de 104 neurones corticaux connectés ainsi que plusieurs kilomètres de "fils" sont emballés dans un réseau dense par millimètre cube dans le cortex. Dans tous les domaines, la forme différente et la diverse taille des neurones jouent un rôle dans le système. Outre les différents types de neurones dans le cortex, il existe également une grande quantité de cellules comme supporteurs, appelés cellules gliales ou parfois nevroglia, qui protègent la stabilisation structurale et qui assurent l'approvisionnement d'énergie dans le système nerveux central. Généralement, les nevroglia sont négligées dans l'étude du réseau de neurones en raison d'un manque d'implication directe pendant le traitement de l'information.

Il existe plusieurs types de neurones spécialisés. Par exemple, les neurones sensoriels répondent aux stimuli affectant les cellules des organes sensoriels, tels que la lumière, le son ou le toucher, puis transmettent des signaux à la moelle épinière et au cerveau, qui stimulent les neurones activateurs à produire des contractions de muscle ou une sécrétion glandulaire ; Les interneurons relient les neurones aux autres et forment des circuits neuronaux dans la moelle épinière ou le cerveau, permettant une communication entre les neurones sensoriels ou de moteur et le système nerveux central. Un neurone typique consiste en trois parties distinctes, les dendrites, le soma ainsi que l'axone. Les dendrites sont des structures minces s'étendant et se ramifiant plusieurs fois, qui transmettent les signaux recueillis d'autres neurones au soma. Le soma est le corps de la cellule comme le déterminateur dans un traitement non linéaire : si l'entrée totale dépasse un certain seuil, une sortie sera produite et sera livrée à d'autres neurones par un axone, qui est une extension cellulaire spéciale découlant du soma.

La transmission des signaux entre les neurones se compose d'impulsions électriques courtes d'une amplitude d'environ 100 mV et d'une durée typique de 1-2 ms, appelées clous ou potentiels d'action, dont la forme ne changera pas pendant la propagation le long de l'axone. En conséquence, l'information est portée par le nombre et le moment des clous plutôt que par la forme car tous les clous émis par un neurone donné sont similaires. Une

séquence de potentiels d'action envoyés par un seul neurone s'appelle un train de clous, où les clous sont habituellement bien séparés car il est impossible d'exciter un clou à nouveau pendant ou immédiatement après un premier clou, même stimulé par un signal d'entrée assez fort. Nous définissons la période réfractaire absolue d'un neurone comme la pause minimale entre deux clous.

La structure permettant aux signaux électriques ou chimiques de passer d'un terminal axone aux dendrites d'un autre neurone s'appelle synapse. Il existe deux types fondamentaux différents de synapses, le plus commun dans le cerveau du vertébré est chimique. Dans une synapse chimique, il n'y a qu'un petit écart appelé la fente synaptique entre le présynaptique, le terminal axone et les postsynaptiques, les dendrites. L'activité électrique dans le neurone présynaptique génère un processus biochimique complexe qui aboutit à la libération d'un produit chimique appelé neurotransmetteur dans la fente synaptique, qui sera détecté par les récepteurs situés dans la membrane plasmique du neurone postsynaptique. Le neurotransmetteur déclenchera alors une réponse électrique qui peut soit exciter ou soit inhiber le neurone postsynaptique. Les effets complexes sont dérivés de la complexité de la transduction de signal du récepteur. L'autre type est une synapse électrique, où les membranes des neurones présynaptiques et postsynaptiques sont reliées par des canaux spécialisés de protéines membranaires, appelés jonctions d'écart. Tels canaux sont capables de transmettre un courant conduisant à des changements de voltage des neurones présynaptiques aux neurones postsynaptiques, ce qui permet un transfert bien plus rapide par rapport à la synapse chimique. On pense que les jonctions d'écart sont impliquées dans la synchronisation des neurones, bien que leurs aspects fonctionnels ne soient pas complètement clairs.

L'effet d'un clou sur le neurone postsynaptique peut être mesuré par la différence entre l'intérieur et l'extérieur du neurone, appelé le potentiel de membrane, désigné par $u(t)$. Le neurone est à l'état de repos avec un potentiel de membrane constant u_r avant qu'un clou se produisant et il sera stimulé pour provoquer le changement de potentiel $u(t)$ après l'arrivée d'un clou puis se calme de retour à u_r . Nous appelons la synapse excitatrice avec un changement positif sur $u(t)$ ou inhibitrice avec un changement négatif. Un exemple classique dans les modèles phénoménologiques de la dynamique neuronale est le modèle simplifié de réponse de clou, voir [27], qui décrit la valeur momentanée du potentiel membranaire du neurone individuel i et est donné par

$$(2.7) \quad u_i(t) = \eta(t - \hat{t}_i) + \sum_i \sum_j \epsilon_{i,j}(t - t_j) + u_r,$$

où t_j indique le temps de tir comme le moment du dépassement du seuil ϑ , si $u_j(t)$ du neurone j atteint ϑ par le bas, ce qui signifie que le neurone déclenche un clou. Pour le neurone i , \hat{t}_i donne le dernier temps de tir, c'est-à-dire $\hat{t}_i := \max\{t_i; t_i < t\}$. Le terme η décrit la forme du clou tandis que le terme $\epsilon_{i,j}$ exprime la réponse du neurone i aux clous d'un neurone présynaptique j . Le modèle (2.7) suit le fait que les potentiels d'action sont toujours à peu près dans la même forme.

Le comportement bruyant des neurones centraux existe omniprésément dans des expériences *in vivo*. Par exemple, le train de clou du cortex visuel est détecté de varier d'un essai à l'autre même dans les mêmes plusieurs répétitions de simulations d'objets externes et même spontanément actif sans aucun stimulus externe appliqué. Le phénomène implique que le neurone cortical reçoit des entrées non seulement de la rétine mais des

autres neurones du cerveau dont l'efficacité est essentiellement inconnue. Donc, bien que le stimulus externe ne change que légèrement, le courant d'entrée fluctue encore fortement dans les neurones corticaux. En réalité, les neurones semblent se conduire de manière déterministe dans une certaine mesure avec une stimulation externe qui change rapidement en étant entraînée par ce genre de bruit de courant d'entrée intracellulaire dépendant du temps. Généralement, si \mathcal{Z} désigne l'ensemble des variables de l'état neuronal, nous pouvons modéliser l'évolution du neurone individuel selon un processus général de Markov à travers une variable aléatoire dépendante du temps $Z_t \in \mathcal{Z}$ satisfaisant la EDS

$$(2.8) \quad dZ_t = F(Z_t, M_t, d\mathcal{L}_t),$$

avec une activité donnée de réseau de neurones M_t et un processus de bruit $d\mathcal{L}_t$ soit brownien ou soit de Poisson.

Le problème dans le codage des neurones est l'énorme quantité de neurones densément emballés connectés l'un à l'autre dans un réseau complexe, par exemple, le cerveau des mammifères contient habituellement plus de 10^{10} neurones, même si dans une petite partie du cortex, Il y a des milliers de clous émis par milliseconde. Pour étudier l'information dans un tel mode spatio-temporel de pulsations, le taux de tir moyen est introduit. Le concept remonte au travail pionnier d'Adrian [1] et il a été appliqué avec succès au cours des dernières décennies simultanément avec trois définitions en considérant différentes procédures de la moyenne : soit au temps, soit à plusieurs répétitions de l'expérience, ou à une population de neurones .

La première et la plus couramment utilisée définition du taux de tir moyen est essentiellement le compte de clous dans une petite période de fenêtre de temps définie par l'expérimentateur, qui fonctionne bien lorsque le stimulus est constant ou lentement variable et qu'une réaction rapide de l'organisme n'est pas requise. Cependant, de plus en plus d'expériences se sont accumulées récemment, impliquant qu'une approche simple basée sur une moyenne temporelle est trop simpliste car toutes les informations éventuellement contenues dans l'heure exacte des clous ont souvent été négligées. En fait, les temps de réaction dans les expériences de comportement sont généralement trop courts pour effectuer une moyenne longuement temporelle, sinon, les neurones doivent attendre pour lire le message des neurones présynaptiques dans chaque processus, ce qui conduit à un temps de réaction beaucoup plus long.

La deuxième définition du taux est appropriée pour les stimuli stationnaires ou dépendants du temps. La réponse d'un neurone est enregistrée après avoir été stimulée par une certaine séquence d'entrée et le processus sera répété plusieurs fois avec la même séquence de stimulation. Le taux de tir moyen est donné par le nombre moyen d'occurrences de clous sur toutes les répétitions dans une plage de temps minimale appropriée, ce qui est impossible pour certaine expérience de comportement et qui limite l'application de la méthode évidemment. Néanmoins, une telle mesure de densité de clous expérimentale est toujours logique lorsqu'une grande population de neurones indépendants reçoit le même stimulus en faisant la moyenne sur les répétitions d'enregistrement d'un seul neurone au lieu de faire la moyenne sur la population à un seul tour. L'idée est dérivée de l'hypothèse implicite sur les grandes populations de neurones qui nous inspirent une troisième façon de calculer la moyenne du taux.

La dernière définition est effectuée à la moyenne de la population. En général, une grande quantité de neurones possède des propriétés similaires et répond aux mêmes stimuli

dans un système neuronal complexe. Il est naturel d'idéaliser la population des neurones avec des propriétés identiques et de les considérer comme dans le même mode de transmission. Le taux de tir moyen est alors défini par la somme de tous les clous qui se produisent dans un intervalle de temps minuscule à la moyenne de la taille de la population sous l'hypothèse d'une population homogène de neurones avec des connexions identiques. En outre, le taux peut être ajusté à la moyenne pondérée pour la population non homogène. Bien que la population neuronale réelle est toujours hétérogène dans une certaine mesure dans son mode de connectivité et dans ses paramètres internes, le taux évite les lacunes de la première définition de la moyenne temporelle au niveau individuel, car une activité de la population répond aux changements de stimulus presque immédiatement et elle aussi varie rapidement.

Dans le système nerveux, certaines activités électriques neuronales omniprésentes sont spontanées et synchrones sans signaux externes, telles que les fonctions physiologiques basiques comme la cognition et la respiration et les symptômes pathologiques comme l'épilepsie, qui reflètent la répétition et la simultanéité dans les décharges d'assemblages spécifiques de neurones à cause de l'interaction des propriétés neuronales intrinsèques par des connexions excitatrices. Au cours des dernières décennies, des expériences ont montré l'apparition généralisée de telles activités dans diverses régions, telles que la rétine, le cortex, la moelle épinière et l'hippocampe, en particulier pendant la maturation du réseau de neurones. L'une des recherches théoriques sur la cohérence globale est le mécanisme de l'interconnexion des unités d'auto-oscillation dont les fréquences déterminent fortement la synchronicité. Dans la période d'état de repos individuel, il y a encore des décharges irrégulières occasionnelles en tant que le bruit, qui seront transformées en un rythme synchrone régulier global grâce aux interactions. L'émergence d'oscillations dans les populations avec du bruit persiste avec une connectivité diverse, par exemple, dans des réseaux connectés de manière aléatoire ou dans le réseau homogène entièrement connecté.

La simulation précoce des études numériques sur les expériences enregistrant l'activité spontanée de la tranche du tronc cérébral a révélé que tels rythmes réguliers réservent dans les modèles avec certaine plage de paramètres, où la relation entre le temps d'entrée et le temps de récupération du neurone joue un rôle clé et le transporteur principal de l'information est des temps de décharge ou certaines statistiques des temps de décharge. L'observation généralisée du phénomène ci-dessus dans différents modèles nous inspire un modèle simple sur la densité des neurones f avec des hypothèses minimales partagées par tous les systèmes excitables. Le modèle, appelé modèle du temps écoulé, a été introduit et étudié dans [28, 56, 57], qui décrit la récupération post-décharge des membranes neuronales avec une quantité de p par un taux de tir instantané ressemble au seuil qui dépend du temps écoulé depuis la dernière décharge et les entrées par des neurones d'un montant de m , donné par

$$(2.9) \quad \begin{cases} \partial_t f + \partial_x f + \mathbf{1}_{x > \sigma(m(t))} f = 0, \\ p(t) := f(t, 0) = \int_0^\infty \mathbf{1}_{x > \sigma(m(t))} f(t, x) dx, \\ m(t) = \varepsilon \int_0^t p(t-y)b(dy), \end{cases}$$

avec le paramètre de la force d'interactions ε . La stabilité de l'état stationnaire a été exposée dans [56] lorsque ε est petit ou grand. Puisqu'il existe de la désynchronisation dans

une population de neurones individuels lorsque $\varepsilon = 0$ et pareil lorsque ε petit, l'interprétation de ε large implique que beaucoup d'assemblages neuronaux connectés subissent la propriété de la désynchronisation.

Le modèle de population de neurones du temps écoulé (2.9) contient à peine un squelette de propriétés neuronales physiologiques visant à souligner et à extraire les mécanismes clés en négligeant les mécanismes de la génération de clous sous-jacents et en se concentrant uniquement sur la description de la dynamique neuronale en termes de temps de décharge, néanmoins, il génère toujours l'activité synchrone. En outre, il appartient à la classe de modèles structurés avec une large gamme d'applications, telles que [43, 62]. Comme la limite de champ moyen du nombre fini de modèles de réseau de neurones comme la somme de (2.8), l'observation des effets dans les modèles stochastiques de taille finie apparaît également dans le modèle moyen de la population de taille infinie.

3 Cadre général de la thèse

Dans cette thèse, la théorie de l'analyse spectrale présentée dans la section 1 est appliquée à l'étude des problèmes biomathématiques dans la chimiotaxie et dans les réseaux de neurones brièvement introduits dans la section 2 précédente. Nous nous concentrons sur le modèle de courses-et-chutes pour la chimiotaxie et le modèle du temps écoulé dans les réseaux de neurones motivés par les phénomènes biologiques ci-dessus, qui seront décrits en détail dans la section suivante 4 avec les résultats principaux établis, tels que l'existence de l'état stationnaire et des solutions ainsi que la stabilité asymptotique, y compris l'esquisse de la preuve correspondante. L'essence de la mise en œuvre de la stratégie dans l'étude de ces équations est d'obtenir l'écart spectral pour les opérateurs associés à l'équation linéaire ou linéarisée dans divers espaces appropriés, laquelle sera pleinement expliquée dans la partie suivante de cette thèse dans plusieurs chapitres.

Le prochain chapitre 1 traite d'une équation linéaire de courses et de chutes dans la dimension $d \geq 1$ pour laquelle nous établissons l'existence d'un état stable unique, positif et normalisé ainsi que sa stabilité asymptotique dans [53], améliorant les résultats similaires obtenus par Calvez et al. [12] dans la dimension $d = 1$. Notre analyse est basée sur la théorie de Kerin-Rutman revisitée dans la section 1.3 ainsi que quelques nouvelles estimations de moment pour prouver le mécanisme de confinement ainsi que les arguments de la dispersion, des multiplicateurs et le lemme de moyenne pour prouver certaine propriété de régularité de quantités de moyenne itérées appropriées sous des fractionnements appropriés de l'opérateur dans l'espace pondéré approprié. L'écart spectral est réalisé dans un processus d'agrandissement de l'espace pondéré et de la structure de fractionnement ajustée de l'opérateur.

Dans le chapitre 2, un modèle du temps écoulé a été pris en compte pour décrire l'activité de tir d'un assemblage homogène de neurones par leur densité de probabilité structurée par la distribution des temps écoulés depuis la dernière décharge. L'unicité de l'état stationnaire et sa stabilité exponentielle non linéaire dans le régime de connectivité faible sont établies dans [52] sous l'hypothèse générale sur le taux de tir, qui généralise certains résultats similaires obtenus dans [56, 57] dans le cas sans délai. Notre approche suit la théorie de l'analyse spectrale et l'écart spectral est particulièrement déterminé par un argument de perturbation où l'opérateur linéarisé en régime de connectivité faible est considéré comme une perturbation d'un opérateur limité décrivant une population de

neurones isolés sans interaction, lequel est déjà connu qu'il admet un écart spectral.

Le chapitre suivant 3 considère à nouveau le modèle du temps écoulé pour les grands réseaux de neurones entièrement connectés sous des hypothèses plus générales sur le taux de tir, y compris celles dans des travaux précédents [56, 57, 52]. Après avoir établi l'existence totale de la solution faible au cas avec et sans délai, nous améliorons les résultats du comportement asymptotique dans le régime de connectivité faible dans [49], n'importe la prise en compte du délai ou non, par rapport à l'estimation moins précise dans [56, 57] et l'exigence de lissage sur le taux de tir dans [52]. Les résultats se fondent dans plusieurs arguments ci-dessus dans le Chapitre 2 précédent avec des versions raffinée du théorème de Wely et du théorème d'application spectrale présentés dans la section 1.3.

Enfin, dans le chapitre 4, nous nous concentrons sur le modèle du temps écoulé dans les réseaux de neurones comme ci-dessus dans les autres cas dans [77]. D'une part, l'existence et l'unicité partielle de la solution faible sont prouvées grâce au théorème de point fixe. D'autre part, nous obtenons les résultats parallèles sur la stabilité asymptotique obtenue dans [52] dans le régime de connectivité forte dans le même cadre, ce qui étend les résultats similaires pour l'équation (2.9) avec un taux de tir particulier d'un seuil dans [56, 57] dans le cas sans délai au cas prenant le délai en compte dans les réseaux de neurones faiblement connectés et qui affirme la stabilité comme exponentielle dans le régime de connectivité forte.

4 Présentation des modèles étudiés

4.1 Le modèle de courses-et-chutes

Dans le papier [53], nous nous concentrons sur une équation cinétique d'évolution qui s'appelle le modèle de *course-et-chutes*, qui décrit le mouvement des cellules en présence de substance chimique chimiotactique. A un niveau microscopique, le modèle révèle l'évolution d'une densité de microorganismes $f = f(t, x, v)$ à l'instant t et à la position $x \in \mathbb{R}^d$ se déplaçant en ligne droite avec la vitesse $v \in \mathcal{V}$ satisfaisant

$$(4.10) \quad \begin{cases} \partial_t f = \mathcal{L}f = -v \cdot \nabla_x f + \int_{\mathcal{V}} (K'[S]f' - K[S]f) dv', \\ -\Delta S + S = \varrho := \int_{\mathcal{V}} f dv, \\ f(0, x, v) = f_0(x, v), \end{cases}$$

où $K = K[S](x, v)$ s'appelle le noyau tournant en tant que le paramètre du processus de saut en raison du changement de vitesse dans le mouvement de microorganismes ainsi que de l'influence du chimio-attirant à cause de x -dépendance. A partir de l'efficacité des stratégies de recherche chimiotactique, la distribution de la vitesse sortante est biaisée par l'entrée, ce qui conduit à une hypothèse raisonnable sur l'indépendance de la distribution de la vitesse sortante à partir du gradient chimioattirant. Ainsi, le noyau tournant dépendant simplement de la vitesse entrante montre une randomisation totale pendant les chutes. Pour simplifier, nous désignons $f' = f(t, x, v')$ et $K' = K[S](x, v')$ et supposons que $\mathcal{V} \subset \mathbb{R}^d$ soit la boule centrée avec le volume de l'unité, c'est-à-dire $\mathcal{V} := B(0, V_0)$ avec V_0 selon la dimension d tel que $|\mathcal{V}| = 1$.

Précisément, le noyau tournant est déterminé par la concentration $S = S(t, x)$ d'un agent chimique produit par les microorganismes eux-mêmes, car les microorganismes tendent à se déplacer vers une concentration chimique plus élevée et il est donné par

$$K = K[S](v) := 1 - \chi \Phi(\partial_t S + v \cdot \nabla_x S),$$

où $\chi \in (0, 1)$ et $\Phi(\cdot)$ est la fonction signe qui implique que K prend la valeur de $1 + \chi$ ou $1 - \chi$ dépendant de la direction de la vitesse de l'individu ainsi que le gradient de la concentration chimique.

D'après la conservation de la masse, nous normalisons la masse totale comme l'unité, à savoir

$$\langle\langle f(t, \cdot) \rangle\rangle = \langle\langle f_0 \rangle\rangle = 1, \quad t \geq 0,$$

où pour les fonctions $g = g(x, v)$ et $h = h(v)$, nous définissons

$$\langle h \rangle = \int_{\mathcal{V}} h \, dv, \quad \langle\langle g \rangle\rangle = \int_{\mathbb{R}^d} \langle g(x, \cdot) \rangle \, dx.$$

Nous observons que pour une solution donnée (f, S) à (4.10) et pour toute rotation $R \in SO(d)$, le couple $(f(Rx, Rv), S(Rx))$ est toujours une solution des mêmes équations, ce qui implique que la solution (f, S) à l'équation (4.10) est invariante par des rotations sous la condition que la donnée initiale f_0 a la même propriété.

Dans ce travail, nous nous concentrons uniquement sur le cas lorsque S est indépendant du temps, radialement symétrique et strictement décroissant par rapport à la position, alors nous avons

$$-\Phi(\partial_t S + v \cdot \nabla_x S) = -\Phi(-v \cdot x) = \text{sign}(x \cdot v).$$

Pour des raisons purement mathématiques, nous négligeons l'équation non linéaire d'accouplement pour l'agent chimique S . Sous cette restriction, nous nous intéressons à l'équation linéaire de "courses et chutes"

$$(4.11) \quad \begin{cases} \partial_t f = \mathcal{L}f = -v \cdot \nabla_x f + \int_{\mathcal{V}} (K' f' - K f) \, dv', \\ f(0, x, v) = f_0(x, v), \end{cases}$$

avec le noyau tournant associé

$$K = K(x, v) = 1 + \chi \text{sign}(x \cdot v),$$

qui prend la valeur de $1 + \chi$ si les cellules s'éloignent de $x = 0$ et de $1 - \chi$ si elles se déplacent vers $x = 0$. Par conséquent, toute distribution d'équilibre de l'opérateur tournant ne peut pas être la solution de (4.11) en raison du saut à $x = 0$. Tout état stationnaire à l'équation linéaire (4.11) doit équilibrer l'opérateur tournant ainsi que l'opérateur de transport $v \cdot \nabla_x$.

Un tel noyau tournant a été introduit dans [12], où il a déjà été prouvé qu'il existe un état stationnaire unique positif et normalisé et la solution à l'équation linéaire d'évolution (4.11) est exponentielle asymptotique stable dans la dimension $d = 1$. Comme souligné dans [12], la principale nouveauté et l'intérêt mathématique du modèle résident dans le fait que le confinement est réalisé par un processus de saut de la vitesse biaisé, où le biais remplace le champ d'accélération de confinement qui est le mécanisme de confinement

classique pour les modèles de Boltzmann et de Fokker-Planck, voir par exemple [74, 47, 21] et les références citées à ceux-ci.

Le résultat principal que nous obtenons dans [53] généralise l'analyse de ([12, Theorem 2.1] et [12, Proposition 1]) de la dimension $d = 1$ à toute dimension $d \geq 1$ par une approche alternative pour étudier l'équation linéaire de "courses et chutes" (4.11).

Avant de présenter notre conclusion principale, il est nécessaire d'introduire des notations et le cadre fonctionnel associé.

Tout d'abord, nous désignons m comme une fonction de poids satisfaisant l'une des conditions suivantes

$$(4.12) \quad \begin{aligned} &\bullet \text{ le poids polynomial : } m(x) = \langle x \rangle^k, \quad k > 0; \\ &\bullet \text{ le poids exponentiel : } m(x) = e^{\gamma \langle x \rangle}, \quad \gamma \in (0, \gamma^*), \end{aligned}$$

où $\langle x \rangle^2 = 1 + |x|^2$ et γ^* est une constante positive à déterminer plus tard. Pour un poids donné m , nous définissons la fonction de taux associée Θ_m et la fonction de poids ω par

- $\Theta_m(t) := \langle t \rangle^{-\ell}$, $\forall \ell \in (0, k)$ et $\omega = 1$ pour le poids polynomial m ;
- $\Theta_m(t) := e^{at}$, $\forall a \in (a^*, 0)$ et $\omega = m$ pour le poids exponentiel m ,

avec un taux optimal $a^* = a^*(\gamma) < 0$ également défini plus tard.

Enfin, pour une fonction de poids donnée $m = m(x) : \mathbb{R}^d \rightarrow \mathbb{R}_+$ et l'exposant $1 \leq p \leq \infty$, nous définissons l'espace de Lebesgue pondéré associé $L^p(m)$ et l'espace de Sobolev pondéré associé $W^{1,p}(m)$, selon leurs normes

$$\|f\|_{L^p(m)} := \|mf\|_{L^p}, \quad \|f\|_{W^{1,p}(m)} := \|mf\|_{W^{1,p}}.$$

Notre résultat principal commence comme suit.

Théorème 4.1. *Il existe $\gamma^* > 0$ et il existe un état stationnaire unique positif, invariant par rotations et normalisé*

$$\begin{aligned} 0 < G \in L^\infty(m_0), \quad \langle\langle G \rangle\rangle &= 1, \\ -v \cdot \nabla_x G + \int_{\mathcal{V}} (K'G' - KG) dv' &= 0, \end{aligned}$$

où m_0 représente la fonction de poids exponentielle $m_0(x) := \exp(\gamma_* \langle x \rangle)$. De plus, pour toute fonction de poids m satisfaisant (4.12) et pour tout $0 \leq f_0 \in L^1(m)$, il existe une solution unique $f \in C([0, \infty); L^1(m))$ à l'équation (4.11) satisfaisant

$$(4.13) \quad \|f(t) - \langle\langle f_0 \rangle\rangle G\|_{L^1(\omega)} \leq \Theta_m(t) \|f_0\|_{L^1(m)}, \quad \forall t \geq 0,$$

où ω et Θ_m sont définis juste au-dessus.

Notre stratégie est radicalement différente de celle utilisée dans [12] qui emploie la méthode d'entropie modifiée pour les opérateurs hypothécaires abstraits introduit dans [21] mais elle est similaire à l'approche de [50] où une version adaptée de la théorie de Kerin-Rutman a été appliquée pour des semi-groupes positifs sans l'hypothèse de compacité classique sur le résolvant associé, mais avec une structure de fractionnement agréable au lieu de cela. Bénéficiant de la loi de conservation de la masse, nous sommes capables de

simplifier la ligne de preuve dans [50] plutôt que d'appliquer directement la théorie Kerin-Rutman [50, Theorem 5.3]. La difficulté principale de la preuve est de trouver un moyen approprié de la décomposition de l'opérateur afin d'obtenir des estimations appropriées sur les opérateurs et les semi-groupes concernés dans l'espace correspondant avec un poids approprié. Nous présentons ici notre approche à plusieurs étapes.

- **L^1 estimation exponentielle pondérée** Pour commencer, nous établissons une nouvelle estimation exponentielle pondérée de L^1 borne uniformément dans le temps montrant clairement le mécanisme de confinement. Notre estimation est la nouvelle borne principale qui est dans l'esprit des L^p estimations pondérées dans [31, 50, 47] pour les modèles de Boltzmann, de la fragmentation de la croissance et de Fokker-Planck cinétique.
- **La décomposition du semi-groupe.** Pour effectuer l'analyse plus loin, nous présentons la décomposition de l'opérateur comme

$$\mathcal{L} = A + B,$$

pour un opérateur B suffisamment dissipatif et un opérateur A B -borné satisfaisant que AS_B possède des propriétés de régularisation, où S_B représente le semi-groupe associé au générateur B . Afin d'établir la régularisation sur AS_B , nous présentons deux choix de la décomposition de \mathcal{L} expliqués comme suit.

- **L'argument de dispersion et l'état stationnaire.** Nous commençons par une décomposition assez simple de l'opérateur \mathcal{L} et nous effectuons le semi-groupe associé $S_{\mathcal{L}}$ dans l'espace de Lebesgue pondéré $X := L^1(m) \cap L^p(m)$. Nous pouvons obtenir une meilleure intégrabilité du semi-groupe S_B dans la variable de position par la transférant de la variable de vitesse à la variable de position sous un argument de dispersion comme introduit par Bardos et Degond dans [6] et présenté dans la section 1.4 avec un argument de rétrécissement de l'espace fonctionnel dans [47]. Iterating la formule de Duhamel, nous avons

$$S_{\mathcal{L}} = S_B + \dots + S_B * (AS_B)^{(*n-1)} + (S_B A)^{(*n)} * S_{\mathcal{L}}, \quad \forall n \in \mathbb{N}^*.$$

A partir d'une estimation exponentielle de déclin pour les termes $S_B * (AS_B)^{(*k)}$ dans $\mathcal{B}(X)$, une estimation exponentielle de déclin pour $(S_B A)^{(*n)}$ dans $\mathcal{B}(L^1, X)$ ainsi que le fait que $S_{\mathcal{L}}$ est borné dans $\mathcal{B}(L^1)$ en tant qu'une conséquence immédiate de la conservation de la masse et de la propriété de positivité, nous pouvons déduire que $S_{\mathcal{L}}$ est borné dans $\mathcal{B}(X)$. Grâce à un argument standard de point fixe de type Brouwer, il existe un état stationnaire G uniformément non négatif et normalisé. Il est également unique d'après des principes classiques du maximum faible et fort. Plus précisément, 0 est une valeur propre simple avec l'espace propre $\text{span}(G)$ et c'est la seule valeur propre non négative.

- **Le lemme de moyenne et la stabilité asymptotique.** Pour avancer plus loin, nous utilisons une méthode de multiplicateur introduite dans la section 1.4 dans l'esprit du multiplicateur de Lions-Perthame dans [60, 42, 63] à tronquer \mathcal{A} de manière appropriée afin que l'estimation d'intégrabilité dans la variable de position près de l'origine puisse être améliorée à nouveau. Nous puis présentons une telle décomposition sophistiquée de l'opérateur \mathcal{L} avec un \mathcal{A} plus régulier et l'opérateur correspondant \mathcal{B} toujours correctement dissipatif pour une norme équivalente appropriée. En outre, AS_B possède une

bonne propriété de compacité et de régularité en conséquence d'un lemme de moyenne en variable spatiale dans [16, 7], initié par le lemme de moyenne classique de temps et d'espace de Golse et al. [30, 29]. Ensuite, en bénéficiant du Théorème de Weyl ainsi que du Théorème de l'application spectrale quantitatif dans [50], nous concluons un écart spectral sur le spectre de l'opérateur \mathcal{L} et l'estimation correspondante sur le semi-groupe $S_{\mathcal{L}}$ comme indiquée dans Théorème 4.1

Il convient de souligner que l'estimation de la régularité nécessaire en tant que la pierre angulaire de la preuve découle du lemme de moyenne plus classique et plus robuste, au lieu d'utiliser un opérateur d'hypocoercivité abstrait comme dans [21, 12] ni d'un lemme de moyenne itéré comme dans [31]. Car la dernière méthode nous permet de transférer la régularité de la variable de vitesse à la variable de position à l'aide d'un commutateur approprié et des "normes glissantes" associées.

4.2 Le modèle du temps écoulé dans les réseaux de neurones

Nous considérons maintenant le modèle du temps écoulé qui décrit le nombre de densité des neurones $f = f(t, x) \geq 0$ à l'instant $t \geq 0$ et à l'heure locale $x \geq 0$ correspondant au temps écoulé depuis la dernière décharge. La dynamique est donnée par l'équation non linéaire suivante

$$(4.14) \quad \begin{cases} \partial_t f = -\partial_x f - a(x, \varepsilon m(t))f =: \mathcal{L}_{\varepsilon m(t)} f, \\ f(t, 0) = p(t) := \int_0^\infty a(x, \varepsilon m(t))f(x) dx, \quad f(0, x) = f_0(x) \\ m(t) := \int_0^\infty p(t-y)b(dy), \end{cases}$$

avec un taux de tir raisonnable et instantané flexible $a \geq 0$ afin de montrer la réponse à la récupération des membranes neuronales après chaque décharge ainsi qu'un paramètre de régime de connectivité de réseau $\varepsilon \geq 0$. Dans ce modèle, nous désignons la densité totale des neurones en cours de décharge à l'instant $t \geq 0$ par la fonction $p(t)$, tandis que la fonction $m(t)$ représente l'activité du réseau à l'instant $t \geq 0$ en tant que la conséquence des décharges antérieures. Nous appelons b la distribution de délai comme une mesure de probabilité en tenant compte de la persistance de l'activité électrique dans le réseau résultant des décharges. Donc, en fonction de divers degrés de la persistance, le modèle est habituellement considéré dans les deux situations différentes suivantes :

- Le cas sans délai par le choix de $b = \delta_0$ puis $m(t) = p(t)$.
- Le cas avec délai par le choix de $b \geq 0$ comme une fonction lisse et une mesure de probabilité.

La non-linéarité de ce modèle provient du paramètre de régime de connectivité ε du réseau, qui décrit la force de l'influence des interactions du réseau neuronal. Par conséquent, dans le cas limite $\varepsilon = 0$, l'équation (4.14) devient linéaire car le neurone individuel évolue simplement par sa propre dynamique alors le réseau a la propriété de désynchronisation. Ce type de propriété est également trouvé dans le régime de connectivité faible lorsque ε est assez petit à cause de la faible non linéarité. Alors que $\varepsilon > 0$, la dynamique d'un neurone donné est affectée par tous les autres neurones à travers l'activité globale de l'assemblage. Cependant, lorsque ε est assez grand, désigné comme étant dans un régime

de connectivité forte en conséquence, cette désynchronisation du réseau sera restaurée résultant d'une grande étendue de connexions.

En raison de la conservation du nombre de densité totale de neurones dans les deux cas, nous pouvons normaliser la masse initiale comme 1, puis la solution f de l'équation du temps écoulé (4.14) satisfait

$$\int_0^\infty f(t, x) dx = 1, \quad \forall t \geq 0.$$

Nous disons qu'un couple du nombre de densités de neurones F_ε et de l'activité du réseau $M_\varepsilon \geq 0$ est l'état stationnaire pour le système d'évolution du temps écoulé (4.14) si

$$(4.15) \quad \begin{cases} 0 = -\partial_x F_\varepsilon - a(x, \varepsilon M_\varepsilon) F_\varepsilon =: \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon, \\ F_\varepsilon(0) = M_\varepsilon, \quad \int_0^\infty F_\varepsilon(x) dx = 1. \end{cases}$$

Dans notre travail, nous faisons les hypothèses physiquement raisonnables sur le taux de tir a et la distribution de délai b .

$$(4.16) \quad a \geq 0, \quad \partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0,$$

$$(4.17) \quad 0 < a_0 := \lim_{x \rightarrow \infty} a(x, 0) \leq \lim_{x, \mu \rightarrow \infty} a(x, \mu) =: a_1 < \infty,$$

$$(4.18) \quad A(x, \cdot) := \int_0^x a(y, \cdot) dy \in C^0(\mathbb{R}_+), \quad \forall x > 0,$$

associée avec certaine hypothèse de continuité

$$(4.19) \quad a \in W^{2, \infty}(\mathbb{R}_+^2).$$

Dans le régime de connectivité forte, nous considérons l'hypothèse de déclin comme complément, pour p.p. $x \geq 0$,

$$(4.20) \quad \varepsilon \sup_{x \geq 0} \partial_\mu a(x, \varepsilon \mu) \rightarrow 0, \quad \text{lorsque } \varepsilon \rightarrow \infty.$$

Dans le cas de délai, nous supposons que $b(dy) = b(y) dy$ satisfait la borne exponentielle et la condition de lissage

$$(4.21) \quad \exists \delta > 0, \quad \int_0^\infty e^{\delta y} (b(y) + |b'(y)|) dy < \infty.$$

Grâce au Théorème de point fixe de Schauder-Brouwer et au Théorème de point fixe de Banach, pour une donnée initiale donnée $0 \leq f_0 \in L^1(\mathbb{R}_+)$, l'existence de la solution faible à l'équation du temps écoulé (4.14) dans le sens distributionnel est établie dans le document [49, 77].

Théorème 4.2. *Supposons (4.16)-(4.17)-(4.18), puis pour toute $f_0 \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ et pour tout $\varepsilon > 0$, il existe une solution faible f non négative et conservée en masse à l'équation du temps écoulée (4.14) dans le cas avec délai, pour certaines fonctions $m, p \in C(\mathbb{R}_+)$, satisfaisant $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$. Tandis qu'au cas sans délai, l'existence et l'unicité de la solution faible dans le régime de connectivité faible et forte sont réalisées sous les hypothèses supplémentaires (4.19)-(4.20).*

L'existence de l'état stationnaire satisfaisant (4.45) est prouvée pour tout $\varepsilon > 0$ dans les documents [52] et [77], en outre, nous pouvons obtenir l'unicité de plus dans le régime de connectivité faible et forte dans les deux papiers respectivement.

Théorème 4.3. *D'une part, pour un taux de tir général satisfaisant (4.16)-(4.17)-(4.18), pour tout $\varepsilon \geq 0$, il existe au moins une paire $(F_\varepsilon(x), M_\varepsilon) \in W^{1,\infty}(\mathbb{R}_+) \times \mathbb{R}_+$ comme la solution de problème stationnaire (4.15) tel que*

$$(4.22) \quad 0 \leq F_\varepsilon(x) \lesssim e^{-\frac{\alpha_0}{2}x}, \quad |F'_\varepsilon(x)| \lesssim e^{-\frac{\alpha_0}{2}x}, \quad x \geq 0.$$

D'autre part, en supposant (4.19)-(4.20) de plus, il existe $\varepsilon_0 > 0$ assez petit et $\varepsilon_1 > 0$ assez grand, tels que l'état stationnaire ci-dessus soit unique pour tout $\varepsilon \in [0, \varepsilon_0) \cup (\varepsilon_1, +\infty)$.

Le but principal est d'étudier le comportement asymptotique de la solution au modèle du temps écoulé (4.14). Dans les œuvres [56, 57, 58], Pakdaman, Perthame et Salort ont étudié ce modèle avec un taux de tir particulier en type de *fonction échelon*

$$(4.23) \quad a(x, \mu) = \mathbf{1}_{x > \sigma(\mu)}, \quad \sigma, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+), \quad \sigma' \leq 0.$$

Avec un avantage de ce taux de tir particulier, ils ont prouvé la stabilité asymptotique de la solution dans le régime de connectivité faible ou forte. En outre, ils ont établi la convergence exponentielle vers l'équilibre seulement dans le cas sans délai et dans le régime de connectivité faible. Bénéficiant de la méthode d'analyse spectrale, nous sommes capables d'améliorer le résultat de la stabilité. Plus précisément, nous améliorons la stabilité comme exponentielle, avec ou sans délai dans le régime de connectivité faible (voir [52]) pour des taux de tir de lissage sous les hypothèses ci-dessus ou de connectivité forte (voir [77]) pour un taux de tir plus général sous les hypothèses plus généraux. En d'autres termes, nous concluons que le déclenchement asynchrone total des neurones semble exponentiellement rapide dans le temps considérable asymptotique comme notre résultat principal.

Théorème 4.4. *Nous supposons que le taux de tir a satisfait (4.16)-(4.17)-(4.19)-(4.20). Nous supposons également que la distribution de délai b satisfait $b = \delta_0$ ou (4.21). Il existe $\varepsilon_0 > 0$ ($\varepsilon_1 > 0$), assez petit (grand), tel que pour tout $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon \in (\varepsilon_1, +\infty)$) l'état stationnaire $(F_\varepsilon, M_\varepsilon)$ est unique. Il existe également certaines constantes $\alpha < 0$, $C \geq 1$ and $\eta > 0$ (en plus $\zeta_\varepsilon \rightarrow 0$ lorsque $\varepsilon \rightarrow \infty$) telles que pour tout paramètre de connectivité $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon \in (\varepsilon_1, +\infty)$) et pour toute donnée initiale de masse unitaire $0 \leq f_0 \in L^1 \cap L^\infty$, telle que $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\varepsilon$ ($\leq \eta/\zeta_\varepsilon$), puis la solution (unique, positive et conservée en masse) f à l'équation d'évolution (4.14) satisfait*

$$\|f(t, \cdot) - F_\varepsilon\|_{L^1} \leq C e^{\alpha t}, \quad \forall t \geq 0.$$

Notre approche n'est pas seulement différente de la façon habituelle de traiter les équations de délai, telles qu'introduites par I. Fredholm [26] and V. Voltera [76], consistant à utiliser le cadre spécifique de "l'espace de mémoire morte", qui remonte au moins à Coleman & Mizel [14], ou utiliser la théorie des "systèmes différentiels abstraits de délai algébrique" développés par O. Diekmann et co-auteurs [17]. Elle est également différente des travaux précédents [56, 57, 58] où l'analyse de la stabilité asymptotique a été effectuée avec un avantage d'un taux de tir particulier (4.23), dont la structure de la fonction échelon permet d'exposer explicitement une norme appropriée telle que certain opérateur du temps écoulé linéaire lié est dissipatif. Sur la base d'une approche plus abstraite et quelque peu plus souple sans obligation d'exposer explicitement une telle norme ci-dessus, notre preuve suit une stratégie de "la perturbation du semi-groupe" initiée dans [46] pour étudier la convergence à long terme vers l'équilibre pour l'équation de Boltzmann homogène inélastique et utilisée récemment dans [48] pour une équation de réseau de neurones basée sur une perturbation brownienne (hypoélliptique) de la célèbre dynamique de FitzHugh-Nagumo. Plus précisément, nous présentons ici notre stratégie dans le régime de connectivité faible et nous la séparons en deux étapes.

- **L'équation linéarisée et la décomposition du semi-groupe.** Nous présentons l'opérateur linéarisé Λ_ε pour les fonctions de variation $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$ autour d'un état stationnaire $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ défini par

$$(4.24) \quad \begin{cases} \partial_t g = -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon n(t), \\ g(t, 0) = q(t) \int_0^\infty a_\varepsilon g \, dx + n(t) \int_0^\infty a'_\varepsilon F_\varepsilon \, dx, \\ n(t) := \int_0^\infty q(t-y) b(dy), \\ g(0, x) = g_0(x), \end{cases}$$

où nous désignons $a_\varepsilon = a(x, \varepsilon M_\varepsilon)$ et $a'_\varepsilon = \varepsilon(\partial_\mu a)(x, \varepsilon M_\varepsilon)$ pour la simplicité. Nous associons le générateur linéaire Λ_ε agissant dans des espaces appropriés dans les différents cas avec ou sans délai et dans le régime de connectivité faible ou forte à son semi-groupe S_{Λ_ε} . Il s'avère que nous pouvons diviser l'opérateur Λ_ε comme

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon,$$

pour un opérateur \mathcal{B}_ε α -hypodissipatif, $\alpha < 0$, et un opérateur \mathcal{A}_ε borné et \mathcal{B}_ε -puissance régulier comme défini dans la section 1.2. Les versions adaptées du Théorème de l'application spectrale dans [50, 45, 44] et du Théorème de Weyl dans [75, 31, 50, 45, 44] impliquent que le semi-groupe S_{Λ_ε} a une partie dominante dimensionnelle finie et un écart spectral. Plus précisément, nous obtenons le décrement du semi-groupe dans l'espace des mesures de Radon $\mathcal{X} := M^1(\mathbb{R}_+)$. Pour tout $t \geq 0$, il existe des constantes $C > 0$ et $a < 0$, tel que

$$(4.25) \quad \|S_{\Lambda_\varepsilon}(I - \Pi_{\Lambda_\varepsilon})\|_{\mathcal{B}(\mathcal{X})} \leq C e^{at},$$

où $\Pi_{\Lambda_\varepsilon}$ est le projecteur de rang fini.

- **L'argument de perturbation et le problème non linéaire.** De plus, dans le cas limite lorsque $\varepsilon = 0$, le terme $n(t)$ disparaît dans l'équation linéarisée (4.24), ce

qui conduit à la positivité du semi-groupe limité S_{Λ_0} . Cela nous permet d'utiliser le Théorème de Kerin-Rutman établi dans [50, 45, 44] afin d'obtenir l'unicité et la stabilité exponentielle de l'état stationnaire (F_0, M_0) ainsi que la structure de spectre

$$\Sigma(\Lambda_0) = \{0\}.$$

En suivant un argument perturbatif présenté dans la section 1.3, nous étendons cet unicité, cette structure de spectre de Λ_ε ainsi que cette stabilité exponentielle à l'état stationnaire $(F_\varepsilon, M_\varepsilon)$ dans le régime de connectivité faible. Bénéficiant de l'estimation quadratique du terme non linéaire et de la stabilité exponentielle de l'équation linéarisée, nous pouvons conclure notre résultat principal pour le problème non linéaire (4.14) dans Théorème 4.4.

La même stratégie s'applique au cas avec délai comme ci-dessus, nous considérons toujours la condition de limite dans l'équation du temps écoulé comme une terme de source et nous remplaçons l'équation de délai par une équation simple d'âge comme une fonction auxiliaire, de sorte que l'équation linéarisée résultante sur les fonctions de variation peut être écrite comme un système autonome de deux équations d'évolution afin de générer un semi-groupe pour suivre l'approche similaire.

En notant que le taux de tir de type de la fonction échelon (4.23) ne relève pas de la classe de taux de tir ci-dessus en raison de l'échec de la condition (4.19), nous ne pouvons plus linéariser le modèle du temp écoulé comme (4.24). Cependant, nous sommes toujours capables d'appliquer la méthode d'analyse spectrale à une autre équation linéaire plus concise sur les fonctions de variation $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$

$$(4.26) \quad \begin{cases} \partial_t g + \partial_x g + a(x, \varepsilon M_\varepsilon)g = 0, \\ g(t, 0) = q(t), \quad g(0, x) = g_0(x), \\ q(t) = \int_0^\infty a(x, \varepsilon M_\varepsilon)g \, dx, \\ n(t) := \int_0^\infty q(t-y)b(dy). \end{cases}$$

Au lieu de l'hypothèse de régularité forte(4.19) sur a , des hypothèses plus faibles sont nécessaires, telles que le taux de tir en type de "fonction échelon" (4.23) aussi remplit des conditions. D'abord, nous assumons

$$(4.27) \quad A(x, \cdot) := \int_0^x a(y, \cdot) \, dy \in C^0(\mathbb{R}_+), \quad \forall x > 0,$$

et $a \in L_x^1 \text{Lip}_\mu$. Pour le régime de connectivité faible, nous assumons que pour certain $\xi > 0$ assez petit et pour tout $\mu_0 > 0$, il existe $\varepsilon_0 > 0$ assez petit tel que pour tout $\varepsilon \in (0, \varepsilon_0)$, il satisfait

$$(4.28) \quad \int_0^\infty |a(x, \varepsilon \mu_2) - a(x, \varepsilon \mu_1)| \, dx \leq \xi |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (0, \mu_0).$$

Tandis que au régime de connectivité fort, nous assumons que pour certain le même $\xi > 0$ que celui dans l'hypothèse (4.28) et pour tout $\mu_\infty > 0$, il existe $\varepsilon_\infty > 0$ assez grand tels que pour tout $\varepsilon \in (\varepsilon_\infty, \infty)$, il satisfait

$$(4.29) \quad \int_0^\infty |a(x, \varepsilon \mu_2) - a(x, \varepsilon \mu_1)| \, dx \leq \xi |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (\mu_\infty, \infty).$$

Avec la structure de décomposition similaire (4.25), la version adaptée du Théorème de l'application spectrale, du Théorème de Wely et le Théorème de Kerin-Rutman permettent également une estimation exponentielle du semi-groupe linéaire associé

$$\|S_{\Lambda_\varepsilon}\|_{\mathcal{B}(X)} \lesssim e^{\alpha t},$$

pour certaine constante $\alpha < 0$ ainsi que la description de spectre comme

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}.$$

Ensuite, grâce à l'argument de perturbation et à l'estimation du terme non linéaire, nous sommes capables de généraliser également les résultats de stabilité exponentielle asymptotique dans le Théorème 4.4, qui sont obtenus dans [56, 57, 58] dans le cas sans délai au cas en tenant compte du délai, au-delà de la restriction du taux de tir de type de la "fonction échelon", soit au régime de connectivité faible, soit au régime de celle forte dans les documents [49, 77].

5 Les perspectives et problèmes ouverts

Enfin, nous présentons les problèmes complémentaires qui méritent d'être considérés après cette thèse pour terminer le chapitre introductif.

1. **Le modèle de courses-et-chutes non linéaire.** L'étude dans le document [53] ne s'est concentrée que sur une version linéaire de l'équation de courses-et-chutes (4.11). Cependant, nous devons souligner qu'il n'est encore pas clair comment appliquer l'approche d'analyse spectrale expliquée dans la section 4.1 au modèle de courses-et-chutes non linéaire (4.10) afin de faire des progrès pour une meilleure compréhension. En particulier, nous n'avons pas pu prouver l'hypothèse utilisée dans l'équation linéaire (4.11) sur la densité chimioattirant S telle que S diminue par rapport à la variable radiale $|x|$ pour éviter la relation entre le noyau tournant K avec S . D'autre part, nous nous demandons si un état stationnaire \bar{G} pour l'équation linéaire (4.11) peut également devenir un état stationnaire de l'équation non linéaire (4.10), comme par exemple, on peut s'attendre en prenant une analogie avec le cas de dimension une et lorsque l'ensemble de la vitesse \mathcal{V} est remplacé par $\bar{\mathcal{V}} := \{-1, 1\}$. En effet, dans ce cas, on peut observer que pour $\bar{G}(x, v) = g(|x|) = C e^{-\chi|x|}$, nous avons

$$\begin{aligned} v \cdot \nabla_x \bar{G} &= v \cdot \frac{x}{|x|} g'(|x|) = -\chi \text{sign}(x \cdot v) g(|x|) \\ &= \int_{\bar{\mathcal{V}}} (K' \bar{G} - K \bar{G}) dv', \end{aligned}$$

afin que nous avons exposé un état stationnaire explicite (unique, positif et en masse unitaire) \bar{G} . La densité macroscopique associée $\bar{\rho}$ est alors en déclin et donc aussi la densité d'agent chimique associée \bar{S} (grâce au principe du maximal appliqué à l'équation elliptique sur la concentration d'agent chimique S). Il s'avère alors que \bar{G} est également un état stationnaire de l'équation non linéaire (4.10). L'existence de solutions d'onde de la propagation pour un modèle non linéaire similaire a été établie dans l'œuvre récente [11], qui peut nous inspirer d'une autre façon d'étudier le modèle non linéaire (4.10).

2. **La solution périodique de l'équation du temps écoulé.** Dans les assemblages de neurones, les rythmes collectifs sont produits par des oscillations périodiques affichées par des neurones individuels qui surviennent à un blocage de phase globale générant des rythmes synchrones au niveau de la population en raison des interactions. Alors que dans les systèmes excitants bruyants interactifs couplés, les neurones individuels se calment en état de repos sans oscillation périodique intrinsèque, néanmoins, les décharges irrégulières provoquées par le bruit peuvent être transformées et être émergées dans un rythme global régulier et cohérent sous l'interaction entre les neurones. L'activité synchrone rythmique ci-dessus nous encourage à essayer de construire des solutions périodiques à l'équation du temps écoulé (4.14) avec la fonction périodique $p(t)$ et $m(t)$. Certains exemples explicites d'un tel type de solutions périodiques correspondant à une activité rythmique et synchrone dans les réseaux de neurones ainsi que l'analyse numérique relative ont été affichés dans [56, 57]. Les simulations numériques garantissent l'existence de diverses solutions périodiques en fonction des données initiales qui, cependant, ne sont pas toujours stables. La conclusion sur l'existence générale et la stabilité des solutions périodiques au modèle du temps écoulé (4.14) attend pour être déterminée, et nous pourrions nous tourner vers la théorie de Floquet pour l'améliorer.
3. **La stabilité de modèle du temps écoulé d'autre type.** L'équation du temps écoulé (4.14) que nous avons étudié dans la section 4.2 pour mieux comprendre la synchronisation ou la désynchronisation des réseaux de neurones dépend de la force d'interconnexion. Dans le papier [58], une façon d'extension est d'ajouter un terme de fragmentation dans le modèle sans délai, par exemple, comme suit

$$(5.30) \quad \begin{cases} \partial_t f = -\partial_x f - a(x, p(t))f = \int_0^\infty K(x, y)a(y, p(t))f(y, t) dy, \\ p(t) := \int_0^\infty a(x, p(t))f(x) dx, \\ f(0, t) = 0, \quad f(0, x) = f_0(x) \geq 0, \quad \int_0^\infty f_0(x) dx = 1, \end{cases}$$

où le noyau $K(x, y) \in M^1(\mathbb{R}_+ \times \mathbb{R}_+)$ montre la distribution des neurones à l'état x après l'apparition d'une décharge avec un temps écoulé y depuis la dernière décharge. Le nouveau type de modèle (5.30) intègre la propriété des neurones qui précise que leur dynamique dépend également de leur activité passée, ce qui atténue progressivement leur tendance à tirer avec la stimulation d'un courant maintenu d'étape. Tels phénomènes sont connus comme l'adaptation et la fatigue des neurones, qui est également l'une des propriétés les plus communes des neurones indiquant la cohérence de temps de tir. Nous espérons que notre stratégie d'analyse spectrale présentée dans la section 4.2 peut être adaptée au modèle d'adaptation et de fatigue (5.30) et ainsi étendre les résultats de stabilité exponentielle établis dans [58] dans réseau faiblement connecté dans le cas sans délai au cas général avec un délai aussi bien que dans un réseau fortement connecté.

General Introduction

1 The theory of spectral analysis

The theory of spectral analysis was first introduced by Hilbert and formulated initially in Hilbert space theory, which was able to explain the later discovery on features of atomic spectra in quantum mechanics. The theory extended the well-known decomposition results in finite dimensional linear algebra, such as the diagonalization and triangularization of matrices, to analogous situations for self-adjoint operators in infinite dimension. Then, the further development of abstract Hilbert space as well as the theory of a single normal operator were required on the aspect of physics and built by von Neumann.

Another formulation of further more abstract theory was built to include Banach algebras, which exceeded the restriction on self-adjoint operators nor necessity of acting in Hilbert space framework. Such development has been recently applied to several classes of PDE for the kinetic theory of gases and biological models in order to study the long time asymptotic behavior of the solutions or even describe the convergence to the equilibrium. Primarily, such approach aims to deal with the spectral properties of bounded linear operators as well as the decay properties for the associated continuous semigroup in Banach framework. In particular, the theory succeeds in establishing the following properties.

- (1) The convergence rate of linear dissipative or hypodissipative evolution equations without self-adjoint structure in weighted Banach spaces, such as the applications to some linear Boltzmann equations, the kinetic Fokker-Planck equation and the kinetic run-and-tumbles equation in chemotaxis.
- (2) The asymptotic behavior of nonlinear evolution equations according to the spectral analysis on their relative linearized equations in natural physical spaces, for instances, the space homogeneous Boltzmann equation and the parabolic-elliptic Keller-Segel equation.
- (3) The existence, uniqueness as well as the stability of the equilibrium in large spaces in small perturbation regimes of strict positive equations, for example, the inelastic Boltzmann equation, the parabolic-parabolic Keller-Segel equation, the time-elapsd model and some others in neuronal network.

The approach is based on a suitable splitting structure of the generators of the semigroup associated to linear equations. Benefitting from the appropriate splitting structure of the associated operator, we are able to use the factorization method at the level on the resolvent and the semigroup to analysis the spectrum structure of the operator and the asymptotic estimates of the associated semigroup. Here, we present the necessary principle elements of the theory.

1.1 Semigroup of linear operator

We start with several notations. For two given Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we denote $\mathcal{B}(X, Y)$ as the space of bounded linear operators from X to Y with $\|\cdot\|_{\mathcal{B}(X, Y)}$ or $\|\cdot\|_{X \rightarrow Y}$ as the associated norm while $\mathcal{C}(X, Y)$ as the set of unbounded closed linear operators and $\mathcal{K}(X, Y)$ as the space of compact linear operators from X to Y . Particularly, we simplify $\mathcal{B}(X) := \mathcal{B}(X, X)$, $\mathcal{C}(X) := \mathcal{C}(X, X)$ and $\mathcal{K}(X) := \mathcal{K}(X, X)$. For a generator $\Lambda \in \mathcal{C}(X)$ to some associated semigroup $S_\Lambda(t)$, we denote its domain by $D(\Lambda)$ and present the elementary definitions.

- (1) We say that $z \in \mathbb{C}$ belongs to the resolvent set $\rho(\Lambda)$ if $\Lambda - z: D(\Lambda) \rightarrow X$ is bijective and its inverse operator belongs to $\mathcal{B}(X)$.
- (2) For any $z \in \rho(\Lambda)$, the resolvent operator $R_\Lambda(z)$ is given by

$$\forall z \in \rho(\Lambda), \quad R_\Lambda(z) := (\Lambda - z)^{-1}.$$

- (3) We define the spectrum set $\Sigma(\Lambda)$ as the complement of the resolvent set, i.e.

$$\Sigma(\Lambda) := \mathbb{C} \setminus \rho(\Lambda).$$

We call $\xi \in \Sigma(\Lambda)$ an isolated eigenvalue if there exists $r > 0$, such that

$$\Sigma(\Lambda) \cap B(\xi, r) = \{\xi\}.$$

Moreover, the associated eigenprojector is defined by the Dunford integral

$$\Pi_{\Lambda, \xi} := \frac{i}{2\pi} \int_{\Gamma} R_\Lambda(z) dz,$$

where $\Gamma = \partial B(\xi, r/2)$. The resolvent can also be obtained from the opposite of Laplace transform by

$$R_\Lambda(z) = - \int_0^\infty S_\Lambda(t) e^{-zt} dt,$$

while once the spectrum of the operator $\Sigma(\Lambda)$ is determined, we are able to turn back to the semigroup S_Λ through the inverse Laplace transform

$$S_\Lambda(t)f = \frac{i}{2\pi} \int_{\uparrow_\alpha} R_\Lambda(z) f e^{zt} dz, \quad f \in X,$$

where $\uparrow_\alpha = \{z = \alpha + iy, y \in \mathbb{R}\}$ for α larger than the spectral bound $s(\Lambda)$, defined as

$$s(\Lambda) := \sup\{\Re e \lambda; \lambda \in \Sigma(\Lambda)\}.$$

We say that $\Lambda \in \mathcal{C}(X)$ with dense domain $D(\Lambda)$ is α -hypodissipative if there exists an equivalent norm $\|\!\| \cdot \|\!\|$ on X satisfying

$$\forall f \in D(\Lambda), \exists \varphi \in F_{\|\!\|, \|\!\|}(f) \text{ such that } \Re e \langle \varphi, (\Lambda - \alpha)f \rangle \leq 0,$$

where, for any $f \in X$, the associated dual set $F_{\|\!\|, \|\!\|}(f) \subset X'$ is defined by

$$F_{\|\!\|, \|\!\|}(f) := \{\varphi \in X'; \langle \varphi, f \rangle = \|\!\|f\|\!\|_X^2 = \|\!\|\varphi\|\!\|_{X'}^2\}.$$

We also say that the generator Λ of a semigroup of bounded operators is α -hypodissipative of type (α, M) for $\alpha \in \mathbb{R}$ and $M \geq 1$, if Λ is α -hypodissipative, and the equivalent norm $\| \cdot \|$ satisfies

$$\|f\| \leq \| \|f\| \| \leq M \|f\|, \quad \forall f \in X;$$

or equivalently if the associated semigroup S_Λ on X fulfills the growth estimate

$$(1.31) \quad \|S_\Lambda(t)\|_{\mathcal{B}(X)} \leq M e^{\alpha t}, \quad \forall t \geq 0.$$

We then define the growth bound

$$\omega(\Lambda) = \omega(S_\Lambda) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|S_\Lambda(t)\| = \inf\{\alpha \in \mathbb{R}; (1.31) \text{ holds}\}.$$

We will sometime abuse by saying that S_Λ is α -hypodissipative when the above growth estimate holds.

1.2 Semigroup decomposition method

It is necessary to introduce the convolution of semigroups. For $T \in (0, \infty)$, considering some Banach spaces X_1, X_2 and X_3 and two semigroups acting on the above spaces $S_1(t) \in L^1(0, T; \mathcal{B}(X_1, X_2))$ and $S_2(t) \in L^1(0, T; \mathcal{B}(X_2, X_3))$, we define the convolution $S_2 * S_1(t) \in L^1(0, T; \mathcal{B}(X_1, X_3))$ by

$$(S_2 * S_1)(t) := \int_0^t S_2(t-s)S_1(s) ds, \quad \forall t \geq 0.$$

Particularly, we denote $S^{(*n)} := S * S^{(*(n-1))}$ recursively with $S^{(*1)} = S$.

To perform the approach further, we consider an operator Λ on a Banach space X with the splitting structure

$$(1.32) \quad \Lambda = \mathcal{A} + \mathcal{B},$$

where \mathcal{B} has some dissipative property, which leads to a good localization of its spectrum and \mathcal{A} is much more regular than \mathcal{B} . The choice of the decomposition of the operator varies on different models. In plenty of works, the spectral analysis theory has been applied with the help of the splitting (1.32) satisfying that \mathcal{B} is dissipative and \mathcal{A} is \mathcal{B} -compact by regarding the operator Λ as a compact perturbation of \mathcal{B} . However, the regularity is much more required in some situations, which encourages us to adjust the splitting structure in the following way. We split \mathcal{A} furthermore as $\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^c$ such that $\tilde{\mathcal{A}}$ is smooth in some extent and $\tilde{\mathcal{A}}^c$ is small, which generates a new suitable decomposition

$$\Lambda = \tilde{\mathcal{A}} + \tilde{\mathcal{B}}, \quad \tilde{\mathcal{B}} := \tilde{\mathcal{A}}^c + \mathcal{B}$$

instead of the usual one. Compared to \mathcal{A} , $\tilde{\mathcal{A}}$ possesses more regularity smoothness property while $\tilde{\mathcal{B}}$ still keeps enough dissipativity. To be detailed, the following properties are necessary in the spectral analysis. For a given $a \in \mathbb{R}$ and an operator Λ with the splitting structure (1.32) defined on an appropriate Banach space \mathcal{X} ,

- \mathcal{B} is called \mathcal{A} -power α -dissipative in \mathcal{X} , if for any $\alpha > a$, there holds

$$\forall \ell \in \mathbb{N}, \quad S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t)e^{-\alpha t} \in L^\infty(\mathbb{R}_+; \mathcal{B}(\mathcal{X}));$$

- \mathcal{A} is called right \mathcal{B} -power regular in $(\mathcal{X}, \mathcal{Y})$ with $\mathcal{Y} \subset \mathcal{X}$, if for any $\alpha > a$, there exists an integer $n \geq 1$, such that

$$(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)e^{-\alpha t} \in L^1(\mathbb{R}_+; \mathcal{B}(\mathcal{X}, \mathcal{Y})),$$

or called left \mathcal{B} -power regular if

$$(S_{\mathcal{B}}\mathcal{A})^{(*n)}(t)e^{-\alpha t} \in L^1(\mathbb{R}_+; \mathcal{B}(\mathcal{X}, \mathcal{Y}));$$

- \mathcal{A} is called left $R_{\mathcal{B}}$ -power regular in $(\mathcal{X}, \mathcal{Y})$ with $\mathcal{Y} \subset \mathcal{X}$, if for any $\alpha > a$, $\mathcal{Y} \neq \mathcal{X}$ or $\alpha \neq 0$, there exists $n \geq 1$ and $C > 0$, such that

$$\|(\mathcal{A}R_{\mathcal{B}}(z))^n\|_{\mathcal{B}(\mathcal{X}, \mathcal{Y})} \leq C\langle x \rangle^{-\alpha}, \quad z \in \Delta_a,$$

or called right $R_{\mathcal{B}}$ -power regular if

$$\|(R_{\mathcal{B}}(z)\mathcal{A})^n\|_{\mathcal{B}(\mathcal{X}, \mathcal{Y})} \leq C\langle x \rangle^{-\alpha}, \quad z \in \Delta_a,$$

where $\Delta_a := \{z \in \mathbb{C}; \Re z > a\}$;

- \mathcal{A} is called left or right \mathcal{B} -power compact in $(\mathcal{X}, \mathcal{Y})$ if \mathcal{A} is left or right \mathcal{B} -power regular and $\mathcal{Y} \subset \mathcal{X}$ with compact embedding.

Besides the above growth estimates, it is also necessary to represent the semigroup S_{Λ} and the associated generator Λ with the splitting structure. Recalling the basic Duhamel formula with right or left factorization,

$$S_{\Lambda} = S_{\mathcal{B}} + S_{\Lambda} * \mathcal{A}S_{\mathcal{B}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\Lambda},$$

it is more practical to iterate it finitely, for instance with right factorization, as

$$S_{\Lambda} = \sum_{\ell=0}^{n-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)} + S_{\Lambda} * (\mathcal{A}S_{\mathcal{B}})^{(*n)}$$

or replace furthermore S_{Λ} in the last term through the inverse Laplace transform, in order to apply the above properties since the convergence of infinite iteration of Duhamel formula cannot be usually achieved. Similarly, S_{Λ} can also be represented by the finite iteration from the left factorization. As for the generator Λ , the approach is performed from the resolvent factorization identity

$$R_{\Lambda}(z) = R_{\mathcal{B}}(z) - R_{\mathcal{B}}(z)\mathcal{A}R_{\Lambda}(z) = R_{\mathcal{B}}(z) - R_{\Lambda}(z)\mathcal{A}R_{\mathcal{B}}(z)$$

to the iterated formula from the first identity

$$R_{\Lambda}(z) = \sum_{\ell=0}^{n-1} (-R_{\mathcal{B}}(z)\mathcal{A})^{\ell} R_{\mathcal{B}}(z) + (-R_{\mathcal{B}}(z)\mathcal{A})^n R_{\Lambda}(z)$$

or from the second one

$$R_{\Lambda}(z) = \sum_{\ell=0}^{n-1} R_{\mathcal{B}}(z)(-\mathcal{A}R_{\mathcal{B}}(z))^{\ell} + R_{\Lambda}(z)(-\mathcal{A}R_{\mathcal{B}}(z))^n.$$

1.3 Spectral theorems

The aim of the spectral analysis of semigroups S_Λ with the generator Λ on a Banach space X is to describe the spectrum set $\Sigma(\Lambda)$ and the spectral gap of $\Sigma(S_\Lambda)$, which imply the asymptotic behavior of the associated trajectories to the semigroup. Benefiting from the above growth assumptions as well as the iterated Duhamel identities, the following theorems have been obtained in recent years.

Spectral mapping theorem. For a continuous semigroup $S_\Lambda(t) = e^{t\Lambda}$, the spectral mapping theorem aims to show

$$\Sigma(e^{t\Lambda}) \setminus \{0\} = e^{t\Sigma(\Lambda)}, \quad \forall t \geq 0,$$

which requires severe conditions on the operator Λ and the associated space X . However, with the help of appropriate splitting structure (1.32) possessing the above assumptions additionally, another less accurate version of the spectral mapping theorem has been developed in [50]. For any $\alpha > a$, there hold

$$(1.33) \quad \Sigma(e^{t\Lambda}) \setminus B(0, e^{\alpha t}) = e^{t\Sigma(\Lambda) \cap \Delta_\alpha}, \quad t \geq 0,$$

which is less accurate than the original intention but is already capable to describe the evolution of semigroup at first order in many situations. In particular, when $\Sigma(\Lambda) \cap \Delta_\alpha \neq \emptyset$, the above partial spectral mapping theorem (1.33) also exhibits more precise asymptotic information of the semigroup as the spectral bound $s(\Lambda)$ coincides with the growth bound $\omega(\Lambda)$ at the right hand side of a , i.e.

$$\max\{s(\Lambda), a\} = \max\{\omega(\Lambda), a\}.$$

Meanwhile, the associated semigroup satisfies the growth estimate

$$(1.34) \quad \|S_\Lambda(t) - S_\Lambda(t)\Pi\|_{\mathcal{B}(X)} \leq C_\alpha e^{\alpha t}$$

for some positive constant C_α with some projector Π commuting with S_Λ , which implies (1.33) particularly.

The partial spectral mapping theorem succeeds in reducing to the analysis of the spectrum of Λ in the half complex plane Δ_α , for any $\alpha > a$. For some suitable class of operators Λ with the splitting structure (1.32), the spectrum set $\Sigma(\Lambda) \cap \Delta_\alpha$ is invariant under the change of X , which ensures that such spectral analysis of Λ and the growth estimate on S_Λ are able to be extended to another space through enlargement or shrinkage.

Weyl's Theorem. The classical type of Weyl's Theorem about compact perturbation of operator roughly asserts that

$$\forall \mathcal{A} \prec\prec \mathcal{B}, \quad \Sigma_{\text{ess}}(\mathcal{A} + \mathcal{B}) = \Sigma(\mathcal{B}),$$

where $\mathcal{A} \prec\prec \mathcal{B}$ iff $D(\mathcal{B}) \subset D(\mathcal{A})$ and $\mathcal{A} \in \mathcal{K}(D(\mathcal{B}), X)$ while Σ_{ess} denotes the essential spectrum of an operator as the complementary spectrum set of the discrete spectrum (the set of all the isolated eigenvalues with finite algebraical multiplicity). In the spirit of the previous spectral mapping theorem and together with the above compactness assumption on the splitting structure (1.32), we present an improved version of Weyl's Theorem to power regular and compact perturbations as

$$\Sigma(\Lambda) \cap \Delta_\alpha = \{\xi_1, \dots, \xi_J\} \subset \Sigma_d(\Lambda),$$

for some integer $J \in \mathbb{N}$ and finite distinct complex numbers ξ_1, \dots, ξ_J , which permits some estimate on the distribution of the spectrum, such as the number and localization of eigenvalues as well as the dimension of the associated algebraic eigenspaces in the half complex plane Δ_α . In addition, there also exist some finite rank projectors Π_1, \dots, Π_J communicating with Λ and some operators T_j satisfying $\Lambda\Pi_j = T_j\Pi_j$, $\Sigma(T_j) = \{\xi_j\}$, $1 \leq j \leq J$, such that the projection part $S_\Lambda(t)\Pi$ in the preceding growth estimate (1.34) is nothing but the sum of the finite projections through the eigenprojectors of all the isolated eigenvalues in the half complex plane Δ_α , in other words,

$$S_\Lambda(t)\Pi = \sum_{j=1}^J S_{T_j}(t)\Pi_j.$$

Perturbation strategy. Given two operators Λ_ε and Λ , we may desire to confirm the similarity between the corresponding spectrums $\Sigma(\Lambda_\varepsilon)$ and $\Sigma(\Lambda)$ when $\Lambda_\varepsilon - \Lambda$ is small enough in some extent. Such an expectation breeds the creation of the perturbation strategy initial by R.S. Phillips' work in [65] and much developed by Kato in [36], which establishes the stability of the spectrum as well as the associated eigenspaces under a suitable small perturbation. Under the same splitting structure framework as previously, the adapted strategy implies the spectrum structure of Λ_ε and growth estimates on the associated S_{Λ_ε} under appropriate small perturbation from those of Λ and S_Λ which are much easier to analyze and generalize.

Krein-Rutman Theorem. In linear algebra, the Perron-Frobenius Theorem claims that the eigenvalue with largest real part of a strictly positive matrix on a finite dimensional Banach space is real, unique and simple. For a positive semigroup S_Λ acting on a Banach lattice X , a Banach space endowed with a partial order \geq or \leq , the Krein-Rutman Theorem in [41] achieves the same conclusion on the eigenvalue of an operator Λ under strict positivity and compactness assumptions. We call a Banach lattice X "standard" if the sign function is well-defined and there exists a strictly positive element in X . In [50], a general version of the Krein-Rutman Theorem adapted for the operator Λ with above splitting structure, which satisfies a weak and strong maximum principle additionally and is defined on a "standard" Banach lattice X , shows that

$$\Sigma(\Lambda) \cap \Delta_\alpha = \{\lambda\},$$

for some $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ a simple eigenvalue. The result on the spectrum is beneficial to the spectral analysis and asymptotic estimate for the positive semigroups as the principal part is a simple real eigenvalue.

In the application, we are concerned with constructing a strong continuous semigroup, which, however, is quite difficult sometimes for many operators, such as the time elapsed model in neuron networks. Therefore, in order to perform the above spectral analysis theory in those situation, we may either extend the theory to the weak continuous semigroup with weakly dissipative properties in larger space or apply the approach on the dual continuous semigroup in the dual space.

1.4 Complementary tools from kinetic theory

Generally, the problem of kinetic equation about macroscopic quantities undergo complex dynamics because of the nonlinearity arising from the integral towards the velocity v .

Each variable t, x, v produces various effects, for instance, the dispersion and regularity by velocity averaging, which are powerful to point out the existence of steady states as well as weak solutions and the asymptotic behavior. Substantial progress has been achieved recently by those effects, such as the global solution of the Boltzmann equation, the complete justification of the incompressible hydrodynamic limit of the Boltzmann equation and applications in biology.

Dispersion effects. Dispersive effects permit us to obtain time decay estimate in space-time L^p integrability. The former estimate has been first introduced by Bardos and Degond considering the dispersion in the full space in [6] and then widely applied as a basic tool to study the existence of small and global in time solutions $f(t, x, v)$ of nonlinear problems. To be more accurate, we take a solution f to the simplest pure kinetic transport equation

$$\begin{cases} \partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = 0, \\ f(t = 0, x, v) = f_0(x, v), \end{cases}$$

as an instance. The macroscopic density defined by

$$\rho(t, x) := \int_{\mathbb{R}^d} f(t, x, v) dv$$

enjoys a time decay estimate related to dispersion in the full space, which writes

$$|\rho(t, x)| \leq \frac{1}{t^d} \|f_0\|_{L_x^1 L_v^\infty}, \quad \forall t > 0, x \in \mathbb{R}^d.$$

On the other hand, a more sophisticated approach, called Strichartz-type inequalities, was devised by Castella and Perthame in [13] to express dispersion in the full space for kinetic equations and obtain the integrability of the macroscopic density, namely

$$\|\rho(t, x)\|_{L_t^q L_x^p} \leq C_d \|f_0\|_{L_{x,v}^\beta},$$

for any real numbers p, q and β satisfying

$$1 \leq p < \frac{d}{d-1}, \quad \frac{2}{q} = d\left(1 - \frac{1}{p}\right), \quad 1 \leq \beta := \frac{2p}{p+1}.$$

Furthermore, better integrability can be obtained from Sobolev injections as well as the Sobolev regularity proved on macroscopic quantities, if we abandon the advantage of the dispersion effects to perform the initial data in classical Lebesgue spaces, see [61]. However, this approach is merely applied for bounded velocities without the capacity to indicate the asymptotic information.

Multipliers Method. In [42], the method of multipliers has been designed by Lions and Perthame for higher velocity integrability, which is able to be transformed into space L^p integrability for velocity integrals under macroscopic controls. The results are concluded by integrating the kinetic equation with different multipliers $M(x, v)$, namely

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^{m \times n}} M(x, v) f(t, x, v) dt dx dv \lesssim \|f_0 M'\|_{L^1(\mathbb{R}^{m \times n})},$$

where $M'(x, v)$ depends on the choice of the multiplier $M(x, v)$. One typical illustration is the application to the steady states of kinetic equations without absorption in the full

space, arising from the high frequency limit of dispersive equations. The difficulties are generated directly from lacking the uniqueness as well as a priori bound estimate with no absorption.

Velocity averaging lemma. In order to gain the regularity, the averaging lemma for solutions of kinetic equations has been introduced in [29] and developed in many works during the past few decades, particularly in [42, 7, 64], as a powerful tool for dealing with the compactness issue. For a solution $f(t, x, v)$ to a kinetic equation, the classical version of averaging lemma shows that the macroscopic quantity

$$\rho_\psi(t, x) := \int f(t, x, v)\psi(v) dv,$$

with ψ a smooth test function, possesses better regularity with respect to (t, x) than the microscopic quantity $f(t, x, v)$. Various versions of the lemma are originated from some extensions with the right hand side of the above definition, from the derivatives of L^2 functions in [18] to a general L^p framework in [20], which have been widely applied to the Vlasov type equations for mean field limits, to the Boltzmann equations for collisional dilute flows and to the scattering equations for cell movement. The proof of the lemma relies on Fourier arguments in L^2 , singular integrals in L^p to avoid singularities for the operator $f_0 \mapsto f$ and the interpolation inequality for Sobolev and Besov spaces or merely on duality arguments as recently shown by Jabin and Vega in [34].

2 Biomathematics background

For completeness of the text, before passing to the presentation of the main mathematical results, we review some basic facts about the biological background. The aim of the discussion presented here is not to give a comprehensive description of such complex structures such as the brain or of the problem of morphogenesis. Nevertheless, we want to provide some basic concepts that will explain the motivations behind our models and their biological interest. Notwithstanding, this information is not necessary to the understanding of the mathematical developments and contributions of the thesis.

2.1 Chemotaxis

The process “taxis” shows the movement of cells, influenced by light, oxygen concentration and other factors including chemical gradients, as approaching to the advantageous environment and avoiding from the disadvantageous one. Therefore, the chemotaxis plays a pivotal role to describe the phenomena of cells moving towards higher concentration of chemoattractants and is very common in many biological processes. In immunology, for instance, the leukocytes migrate from the bloodstream after foreign antigens as the immune response resulting in the inflammation and the tissue repair is regulated through external stimuli during wound healing. The stem cells migrate to form tissue then complex organs such as the embryology and angiogenesis in the developmental biology. Another example focuses on the bacteria in their self-organized colonies, where chemotaxis is also responsible for various types of aggregation pattern such as travelling pulses of colonies and stable clusters on account of acting as signals among microorganisms.

At the macroscopic level, chemotaxis is usually modeled by a system of parabolic equations, the best known of which is the Keller-Segel system. Since the bacteria not only

seek the optimal conditions to grow but move to aggregate, the model has considered the aggregation of slime mold amoebae *Dictyostelium discoideum* driven by the attractive chemical substance and has been studied in [38, 39, 40], summarized in [37]. The mechanism is given by

$$(2.35) \quad \begin{cases} \partial_t \rho = D_1 \Delta \rho - \chi \nabla \cdot (\rho \nabla S), \\ \partial_t S = D_2 \Delta S + \phi(\rho, S), \end{cases}$$

where $\rho(t, x) \geq 0$ represents the density of the cells and $S(t, x) \geq 0$ is the concentration of the chemical substance. The positive constants D_1 and D_2 are the diffusivities of the chemical agent and the cells respectively while $\chi \geq 0$ denotes the chemotactic sensitivity. The function $\phi(\rho, S)$ describes the interaction between the substrate and the microorganisms since S can also be produced by cells themselves besides diffusing in the substrate. One simple choice of ϕ is given by

$$\phi(\rho, S) = \alpha \rho - \beta S,$$

where we assume that the bacteria produce the chemoattractant at a constant rate $\alpha \geq 0$ and the chemical substance decay at a constant rate $\beta \geq 0$. The equation about ρ in the system (2.35) can be derived from a macroscopic hydrodynamic limit of a stochastic quantity particles system and the first term on the right hand side represents the diffusion under the intent of microorganisms through their own Brownian motion while the second one shows their tendency of aggregation due to the presence of the chemoattractant.

The main issue in the study of the above system is whether the solutions will blow up in finite time. When the space dimension $n \geq 3$, the answer is affirmative and has been proved in [32]. Oppositely, blow up will never happens for $n = 1$. As for $n = 2$, the occurrence of a blow-up depends on the initial condition, see [33]. However, we can prevent such a blow-up after a suitable modification of the Keller-Segel model (2.35), for instance, by loosing chemotactic efficacy above a saturation density.

At microscopic level, it is observed in the experiments that swimming bacteria such as *Escherichia coli* undergo an irregular motion composed by a series of two alternate different periods, “run” and “tumble”. A run phase consists of directed movement in a straight line which will be long in a favorable direction while short in an unfavorable one. After a run, the bacterium is reoriented because of a tumble phase and follows a new run in the new direction. As regards *Escherichia coli*, typically representing the flagellated bacteria, such kind of tumble movement is accomplished by signal transmission between the receptor complexes and the flagellar-motor complexes, see [3]. On a cell, the former are located in the poles in order to detect the presence of chemical substances while the latter are embedded within the membrane and distributed randomly. This mechanism much lengthens the run phase compared to the invariant tumble phase, which results in a biased random walk in the gradient direction of the chemical substance, see [70]. As for leukocytes or amoebae, the pattern of turning direction and changing velocity in the tumble phase is also determined by the chemical substance during experiment observation. Generally, the tumble phase is much shorter than the run phase (see [9]), which inspires us to model the chemotaxis with an instantaneous velocity jump process in many models. Although the size of bacteria is too small to measure the spatial chemical substance gradient along the individual length, this stochastic process is still biased under the gradient with the

benefit of increased frequency of tumble as well as the ability to measure the gradient in time along its path after decreasing chemical agent concentration. Such a bias permits the desired drift in spite of the unbiased outward velocity after tumble.

At the mesoscopic level, the chemotaxis model consists of a kinetic equation for the phase space density $f(t, x, v)$ of the organisms together with an elliptic or parabolic equation exhibiting reaction-diffusion on the concentration $S(t, x)$ of the chemical substance, which has been first introduced by Alt [2, 3] and studied by Othmer, Dunbar and Alt [54], given by

$$(2.36) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f = Q[f, S] \\ \qquad \qquad \qquad := \int_{\mathcal{V}} (K[S](t, x, v', v) f(t, x, v') - K[S](t, x, v, v') f) dv', \\ S - \Delta S = \rho := \int_{\mathcal{V}} f(t, x, v) dv, \end{cases}$$

where $f = f(t, x, v) \geq 0$ indicates the phase space density of the cells and the velocity distribution set \mathcal{V} is compact. One recovers the the Keller-Segel model (2.35) by macroscopic diffusion scaling limit from the above kinetic model (also called velocity jump model). The limit produces convection-diffusion equation, whose different diffusivities and convection velocities are with the same qualitative behavior. Equation (2.35) can also be derived from kinetic models by the Hilbert expansion because of the finite space-time dependent set of velocity distribution. In fact, the integro-differential equation (2.36) is also a kind of Boltzman-type equation, which has originally been devised to describe a thermodynamic system, such as moderately rarefied gases.

Besides the acceleration of the force field derived from a confining potential on unbiased velocity randomization caused by Brownian motion or a jump process, the acceleration field can also be replaced by a biased velocity jump process to achieve the confinement. Under the strong enough restriction to cover the dispersive effect, the movement will evolve to an invariant probability measure as equilibrium. The convergence to equilibrium for this kind of kinetic transport models has been studied and some strong convergences have been obtained, see [21] for an exponential rate and [10] for an algebraic one. However, the model (2.36) is still so complex that the results about the stability have been merely obtained under strict assumptions on the chemotactic concentration S and with the benefit of different choices of $K[S]$ with suitable dependence on S .

2.2 Neuron network

Over the past century, the study about the structure and function of the brain has accumulated in biological research and a huge amount of detailed knowledge has been obtained. In the central nervous system, the elementary core components are the neurons to process and transmit information through electrical and chemical signals, which are connected to each other to form a neuron network in a complicated pattern. For instance, more than 104 connected cortical neurons as well as several kilometers of “wires” are packed into a dense network per cubic millimeter in the cortex. In all areas, different shape and various size of neurons play a role in the system. Besides the various type of neurons in the cortex, there are also a large quantity of cells as supporters, called glial cells or sometimes neuroglia, that protect structural stabilization and ensure the energy supply in the central

nervous system. Generally, the neuroglia are neglected in the study of neuron network because of lacking direct involvement during information processing.

There are several types of specialized neurons. For example, sensory neurons respond to stimuli affecting the cells of the sensory organs, such as light, sound or touch, then transmit signals to the spinal cord and brain, which stimulate motor neurons to produce muscle contractions or glandular secretion; Interneurons connect neurons to others and form neural circuits in the spinal cord or brain, enabling communication between sensory or motor neurons and the central nervous system. A typical neuron consists three distinct parts, dendrites, soma as well as axon. Dendrites are thin structures extending and branching multiple times, which transmit signals collected from other neurons to soma. The soma is the cell body as the determinator in a non-linear processing: if the total input rises beyond a certain threshold, an output will be produced and delivered to other neurons by an axon, which is a special cellular extension arising from soma.

The signals transmission among the neurons consist of short electrical pulses with an amplitude of around 100 mV and a typical duration of 1-2 ms, called spikes or action potentials, whose form will not change during the propagation along the axon. Therefore, the information is carried by the number and the timing of spikes rather than the form as all spikes emitted by a given neuron are alike. A sequence of action potentials sent by a single neuron is called a spike train, where the spikes are usually well separated since it is impossible to excite a spike again during or immediately after a first one even stimulated by a quite strong input signal. We define the absolute refractory period of a neuron as the minimal pause between two spikes.

The structure permitting the electrical or chemical signals passing from an axon terminal to the dendrites of another neuron is called synapse. There are two fundamentally different types of synapse, the most common one in the brain of the vertebrate is chemical. In a chemical synapse, there is only a tiny gap called synaptic cleft between the presynaptic, the axon terminal, and the postsynaptic, the dendrites. The electrical activity in the presynaptic neuron generates a complex bio-chemical process resulting in the release of a chemical called neurotransmitter into the synaptic cleft, which will be detected by the receptors located in the plasma membrane of the postsynaptic neuron. The neurotransmitter will then initiate an electrical response that may either excite or inhibit the postsynaptic neuron. The complex effects are derived from the complexity of receptor signal transduction. The other type is an electrical synapse, where the presynaptic and postsynaptic neuron membranes are connected by specialized membrane proteins channels, called gap junctions. Such channels are capable to pass a current leading to voltage changes from presynaptic neurons to postsynaptic neurons, which permits a much rapider transfer compared to the chemical synapse. The gap junctions are thought to be involved in the synchronization of the neurons, even though their functional aspects are not completely clear.

The effect of a spike on the postsynaptic neuron can be measured by the difference between the interior and exterior of the neuron, called the membrane potential, denoted by $u(t)$. The neuron is at the rest state with a constant membrane potential u_r before any spike occurs and will be stimulated to cause the change of potential $u(t)$ after the arrival of a spike then calm down back to u_r . We call the synapse excitatory with a positive change on $u(t)$ or inhibitory with negative change. One classical example in the phenomenological models of neuronal dynamics is the simplified Spike Response Model,

see [27], which describes the momentary value of the membrane potential of individual neuron i and is given by

$$(2.37) \quad u_i(t) = \eta(t - \hat{t}_i) + \sum_i \sum_j \epsilon_{i,j}(t - t_j) + u_r,$$

where t_j denotes the firing time as the moment of beyond the threshold ϑ , if $u_j(t)$ of the neuron j reaches ϑ from below, which is called that the neuron fires a spike. For neuron i , \hat{t}_i gives the last firing time, i.e. $\hat{t}_i := \max\{t_i; t_i < t\}$. The term η describes the form of the spike while the term $\epsilon_{i,j}$ expresses the response of neuron i to spikes of a presynaptic neuron j . The model (2.37) follows of the fact that action potentials are always roughly in the same form.

The noisy behavior of central neurons exists ubiquitously in vivo experiments. For example, the spike train of the visual cortex is detected to vary from trial to trial even within the same several repetitions of simulations of external object and even spontaneously active without any external stimulus applied. The phenomenon implies that the cortical neuron receives inputs not only from the retina but from the other neurons in brain, whose efficiency is basically unknown. Therefore, although the external stimulus merely change slightly, the input current still fluctuates strongly in cortical neurons. Actually, neurons seem to behave deterministically in some extent with rapidly changing external stimulus as driven by such kind of time-dependent intracellular input current noise. Generally, if \mathcal{Z} denotes the set of neuronal state variable, we are able to modeling the evolution of individual neuron according to a general Markov process through a time dependent random variable $Z_t \in \mathcal{Z}$ satisfying the SDE

$$(2.38) \quad dZ_t = F(Z_t, M_t, d\mathcal{L}_t),$$

with some given neuron network activity M_t and a noise process $d\mathcal{L}_t$ either Brownian or Poissonian.

The problem in the neuron coding is the huge quantity of densely packed neurons connected to each other in an intricate network, for instance, the mammalian brain usually contains more than 10^{10} neurons, even if in a small part of cortex, there are thousands of spikes emitted per millisecond. In order to study the information in such a spatio-temporal pattern of pulses, the mean firing rate is introduced. The concept dates back to pioneering work of Adrian [1] and has been successfully applied during the past decades simultaneously with three definitions considering different averaging procedures: either on time, or on several repetitions of the experiment, or on a population of neurons.

The first and most commonly used definition of mean firing rate is essentially the spike count in a small time window period set by the experimenter, which works well when the stimulus is constant or slowly varying and a fast reaction of the organism is not required. However, more and more experiments have accumulated recently implying that a straightforward approach based on a temporal average is too simplistic because of the neglect of all the information possibly contained in the exact timing of the spikes. Actually, reaction times in behavior experiments are usually too short to perform long temporal average, otherwise, the neurons have to wait to read the message from the presynaptic neurons in each process, which leads to a much longer reaction time.

The second definition of the rate is appropriate for stationary or time-dependent stimuli. The response of a neuron is recorded after stimulated by some input sequence and

process will be repeated for several times with the same stimulation sequence. The mean firing rate is given by the average number of occurrences of spikes over all repetitions in a suitable tiny time range, which is impossible for some behavior experiment and limits the application of the method obviously. Nevertheless, such an experimental spike density measure still makes sense when a large populations of independent neurons receive the same stimulus by averaging over repetitions of recording a single neuron instead of averaging over population in a single run. The idea is derived from the implicit assumption on the large populations of neurons which inspiring us the third way to average the rate.

The last definition is performed on population average. In general, a huge quantity of neurons possess the similar properties and respond to the same stimuli in a complex neuronal system. It is natural to idealize the population of neurons with identical properties and consider them as in the same pattern of transmission. The mean firing rate is then defined by the sum of all the spikes occurring within a tiny time interval averaged over the size of population under the assumption of homogeneous population of neurons with identical connections. Moreover, the rate can be adjusted by weighted average for inhomogeneous population. Although the real neuronal population is always heterogeneous to some extent both in their connectivity pattern and internal parameters, the rate avoids the shortcomings of the first definition by temporal average at the individual level since a population activity responds to the changes of stimulus almost immediately and also varies rapidly.

In nervous system, some ubiquitous neuronal electrical activities are spontaneous and synchronous without external signals, such as the basic physiological functions like cognition and respiration and the pathological symptoms like epilepsy, which reflect the repetition and simultaneity in the discharges of specific neuron assemblies because of the interaction of the intrinsic neuronal properties through excitatory connections. During the past few decades, experiments have shown the widespread occurrence of such activities in various region, such as the retina, the cortex, the spinal cord and the hippocampus, especially during the maturation of the neuron network. One of theoretical investigations on the global coherence is the mechanism of the interconnection of self oscillation units whose frequencies strongly determine the synchrony. In the period of individual rest state, their are still occasional irregular discharges as noise, which will be transformed into a global regular synchronous rhythm thanks to interactions. The emergence of oscillations in populations with noise persists with diverse connectivity, for instance, in randomly connected networks or the fully connected homogenous one.

The early simulation of numerical studies on the experiments recording the spontaneous activity of brainstem slice revealed that such regular rhythms hold in the models with some parameter range, where the relation between the input time and the recovery time of the neuron plays a key role and the main carrier of information is the discharge times or some statistics of the discharge times. The widespread observation of the above phenomenon in various models inspires us a simple model on the neuron density f with minimal assumptions shared by all the excitable system. The model, called time elapsed model, has been introduced and studied in [28, 56, 57], which describes the post-discharge recovery of neuronal membranes with amount of p through an instantaneous threshold-like firing rate that depends on the time elapsed since the last discharge and the inputs by

neurons with amount of m , given by

$$(2.39) \quad \begin{cases} \partial_t f + \partial_x f + \mathbf{1}_{x > \sigma(m(t))} f = 0, \\ p(t) := f(t, 0) = \int_0^\infty \mathbf{1}_{x > \sigma(m(t))} f(t, x) dx, \\ m(t) = \varepsilon \int_0^t p(t-y)b(dy), \end{cases}$$

with the strength of the interactions parameter ε . The stability of the steady state has been exhibited in [56] when ε is small or large. Since there is desynchronization in a population of individual neuron as $\varepsilon = 0$ and so is there when ε small, the interpretation for ε large implies that much connected neuronal assembly sustains the property of desynchronization.

The time elapsed neuron population model (2.39) barely contains a skeleton of physiological neuronal properties aiming to emphasize and extract the key mechanisms by neglecting the mechanisms of underlying spike generation and concentrating merely on describing the neuronal dynamics in terms of discharge times, nevertheless, it still generates the synchronous activity. Besides, it belongs to the class of structured models with a broad range of applications, such as [43, 62]. As the mean field limit of finite number of neuron network models as the sum of (2.38), the observation of effects in finite size stochastic models also appears in the averaged model of infinite size of population.

3 Plan of the thesis

In this thesis, the spectral analysis theory presented in section 1 is applied to the study of biomathematics problems in the chemotaxis and neuron networks briefly introduced in section 2 separately. We concentrate on the runs-and-tumbles model for chemotaxis and time-elapsed model in neuron networks motivated by the above biological phenomena, which will be detailedly described in the following section 4 together with the established main results, such as the existence of the steady state and of the solutions as well as the asymptotic stability, including the corresponding sketch of the proof. The essence to implement the strategy in the study of these equations is to obtain the spectral gap for the operators associated to the linear or linearized equation in various appropriate spaces, which will be fully explained in the rest part of this thesis within several chapters.

The next Chapter 1 deals with a linear runs and tumbles equation in dimension $d \geq 1$ for which we establish the existence of a unique positive and normalized steady state as well as its asymptotic stability in [53], improving similar results obtained by Calvez et al. [12] in dimension $d = 1$. Our analysis is based on the Krein-Rutman theory revisited in section 1.3 together with some new moment estimates for proving confinement mechanism as well as dispersion, multipliers and averaging lemma arguments for proving some regularity property of suitable iterated averaging quantities under suitable splits of the operator in appropriate weighted space. The spectral gap is realized in a process of enlargement of the weighted space and adjusted split structure of the operator.

In Chapter 2, a time-elapsed model has been consider in order to describe the firing activity of a homogenous assembly of neurons by their probability density structured by the distribution of times elapsed since the last discharge. The uniqueness of the steady state and its nonlinear exponential stability in the weak connectivity regime are established in [52] under general assumption on the firing rate, which generalizes some similar results

obtained in [56, 57] in the case without delay. Our approach follows the spectral analysis theory and the spectral gap is particularly determined through a perturbation argument where the linearized operator in weak connectivity regime is regarded as a perturbation of a limited operator describing a population of isolated neurons without interplay, which is already known as admitting a spectral gap.

The following Chapter 3 considers again the time-elapsd model for large fully connected neuron networks under more general assumptions on the firing rate including the ones in the previous works [56, 57, 52]. After establishing the total existence of the weak solution both in the case with and without delay, we improve the results of the asymptotic behavior in the weak connectivity regime in [49], no matter taking delay into account or not, compared to the less accurate estimate in [56, 57] and requirement of smoothness on the firing rate in [52]. The results root in several above arguments in the previous Chapter 2 together with refined versions of the Wely's theorem and spectral mapping theorem presented in section 1.3.

Finally, in Chapter 4, we focus on the time-elapsd model in neuron networks as above in the other cases in [77]. On the one hand, the existence and part of the uniqueness of the weak solution are proven thanks to the fixed point Theorem. On the other hand, we obtain the parallel results on the asymptotic stability obtained in [52] in the strong connectivity regime within the same framework, which extend the similar results for the equation (2.39) with a particular threshold firing rate in [56, 57] in the case without delay to the case taking delay into account in the weakly connected neuron networks and assert the stability as exponential in the strong connectivity regime.

4 Presentation of studied models

4.1 Runs-and-tumbles model

In the paper [53], we concentrate on a kinetic evolution equation called *run-and-tumble* model, which describes the cells' movement in the presence of chemotactic chemical substance. At a microscopic level, the model reveals the evolution of a microorganisms density $f = f(t, x, v)$ at time t and at position $x \in \mathbb{R}^d$ moving in straight line with the velocity $v \in \mathcal{V}$ satisfying

$$(4.40) \quad \begin{cases} \partial_t f = \mathcal{L}f = -v \cdot \nabla_x f + \int_{\mathcal{V}} (K'[S]f' - K[S]f) dv', \\ -\Delta S + S = \varrho := \int_{\mathcal{V}} f dv, \\ f(0, x, v) = f_0(x, v), \end{cases}$$

where $K = K[S](x, v)$ is called turning kernel as the parameter of jump process because of the change of velocity in the microorganisms movement as well as the influence of the chemo-attractant because of x -dependence. From the efficiency of chemotactic foraging strategies, the outgoing velocity distribution is biased by the incoming one, which leads to the reasonable assumption on the independence of the outgoing velocity distribution from the chemoattractant gradient. Thus, the turing kernel depending merely on the incoming velocity shows a total randomization during the tumbles. For simplicity, we denote $f' =$

$f(t, x, v')$ and $K' = K[S](x, v')$ and assume that $\mathcal{V} \subset \mathbb{R}^d$ is the centered ball with unit volume, i.e. $\mathcal{V} := B(0, V_0)$ with V_0 depending on dimension d such that $|\mathcal{V}| = 1$.

Precisely, the turning kernel is determined by the concentration $S = S(t, x)$ of a chemical agent produced by the microorganisms themselves since microorganisms tend to move towards higher chemical concentration and is given by

$$K = K[S](v) := 1 - \chi \Phi(\partial_t S + v \cdot \nabla_x S),$$

where $\chi \in (0, 1)$ and $\Phi(\cdot)$ is the sign function, which implies that K takes either $1 + \chi$ or $1 - \chi$ as value depending on the velocity direction of the individual as well as the gradient of the chemical concentration.

From the mass conservation, we normalize the total mass as unit, namely

$$\langle\langle f(t, \cdot) \rangle\rangle = \langle\langle f_0 \rangle\rangle = 1, \quad t \geq 0,$$

where for functions $g = g(x, v)$ and $h = h(v)$, we define

$$\langle h \rangle = \int_{\mathcal{V}} h \, dv, \quad \langle\langle g \rangle\rangle = \int_{\mathbb{R}^d} \langle g(x, \cdot) \rangle \, dx.$$

We observe that for a given solution (f, S) to (4.40) and for any rotation $R \in SO(d)$, the couple $(f(Rx, Rv), S(Rx))$ is still a solution of the same equations, which implies that the solution (f, S) to the equation (4.40) is invariant by rotations under the condition that the initial datum f_0 has the same property.

In this work, we only focus on the case when S is time independent, radially symmetric and strictly decreasing with respect to the position, then we have

$$-\Phi(\partial_t S + v \cdot \nabla_x S) = -\Phi(-v \cdot x) = \text{sign}(x \cdot v).$$

For purely mathematical reasons, we neglect the nonlinear coupling equation for the chemical agent S . Under this restriction, we are interested in the linear “runs and tumbles” equation

$$(4.41) \quad \begin{cases} \partial_t f = \mathcal{L}f = -v \cdot \nabla_x f + \int_{\mathcal{V}} (K' f' - K f) \, dv', \\ f(0, x, v) = f_0(x, v), \end{cases}$$

with the associated turning kernel

$$K = K(x, v) = 1 + \chi \text{sign}(x \cdot v),$$

which takes the value of $1 + \chi$ if cells move away from $x = 0$ and $1 - \chi$ if they move towards $x = 0$. Therefore, any equilibrium distribution of the turning operator cannot be the solution of (4.41) because of the jump at $x = 0$. Any steady state to the linear equation (4.41) has to balance the turning operator as well as the transport operator $v \cdot \nabla_x$.

Such a turning kernel has been introduced in [12], where it has already been proven that there exists a unique positive and normalized steady state and the solution to the linear evolution equation (4.41) is exponential asymptotic stable in dimension $d = 1$. As pointed out in [12], the main novelty and mathematical interest of the model lie in the fact that

the confinement is achieved by a biased velocity jump process, where the bias replaces the confining acceleration field which is the classical confinement mechanism for Boltzmann and Fokker-Planck models, see for instance [74, 47, 21] and the references quoted therein.

The main result we obtain in [53] generalizes the analysis of ([12, Theorem 2.1] and [12, Proposition 1]) from dimension $d = 1$ to any dimension $d \geq 1$ through an alternative approach to study the linear “runs and tumbles” equation (4.41).

Before presenting our main conclusion, it is necessary to introduce some notations and the associated functional framework.

First, we denote m as some weight function satisfying one of the following conditions

$$(4.42) \quad \begin{aligned} &\bullet \text{ polynomial weight : } m(x) = \langle x \rangle^k, \quad k > 0; \\ &\bullet \text{ exponential weight : } m(x) = e^{\gamma \langle x \rangle}, \quad \gamma \in (0, \gamma^*), \end{aligned}$$

where $\langle x \rangle^2 = 1 + |x|^2$ and γ^* is a positive constant to be determined later. For a given weight m we define the associated rate function Θ_m and weight function ω by

- $\Theta_m(t) := \langle t \rangle^{-\ell}$, $\forall \ell \in (0, k)$ and $\omega = 1$ for polynomial weight m ;
- $\Theta_m(t) := e^{at}$, $\forall a \in (a^*, 0)$ and $\omega = m$ for exponential weight m ,

with an optimal rate $a^* = a^*(\gamma) < 0$ also defined later.

Finally for given weight function $m = m(x): \mathbb{R}^d \rightarrow \mathbb{R}_+$ and exponent $1 \leq p \leq \infty$, we define the associated weighted Lebesgue space $L^p(m)$ and weighted Sobolev space $W^{1,p}(m)$, through their norms

$$\|f\|_{L^p(m)} := \|mf\|_{L^p}, \quad \|f\|_{W^{1,p}(m)} := \|mf\|_{W^{1,p}}.$$

Our main result starts as follows.

Theorem 4.1. *There exist $\gamma^* > 0$ and a unique positive, invariant by rotations and normalized stationary state*

$$\begin{aligned} 0 < G \in L^\infty(m_0), \quad \langle\langle G \rangle\rangle &= 1, \\ -v \cdot \nabla_x G + \int_{\mathcal{V}} (K'G' - KG) dv' &= 0, \end{aligned}$$

where m_0 stands for the exponential weight function $m_0(x) := \exp(\gamma_* \langle x \rangle)$. Moreover, for any weight function m satisfying (4.42) and for any $0 \leq f_0 \in L^1(m)$, there exists a unique solution $f \in C([0, \infty); L^1(m))$ to the equation (4.41) satisfying

$$(4.43) \quad \|f(t) - \langle\langle f_0 \rangle\rangle G\|_{L^1(\omega)} \leq \Theta_m(t) \|f_0\|_{L^1(m)}, \quad \forall t \geq 0,$$

where ω and Θ_m are defined just above.

Our strategy is drastically different from the one used in [12] which employs the modified entropy method for abstract hypocoercive operators introduced in [21] but similar to the approach of [50] where an adapted version of Krein-Rutman theory has been applied for positive semigroups without the classical compactness assumption on the associated resolvent but with a nice splitting structure instead. Benefiting from the mass conservation law, we are able to simplify the line of proof in [50] rather than applying the Krein-Rutman

theory [50, Theorem 5.3] directly. The main difficulty in the proof is to find appropriate way of decomposition of the operator in order to get suitable estimates on the related operators and semigroups in the corresponding space with proper weight. We present our approach here into several steps.

- **Exponential weighted L^1 estimate** To begin with, we establish a new exponential weighted L^1 estimate uniformly in time exhibiting the confinement mechanism. Our estimate is the main new bound which is in the spirit of the weighted L^p estimates in [31, 50, 47] for Boltzmann, growth-fragmentation and kinetic Fokker-Planck models.
- **Semigroup decomposition.** To perform the analysis further, we introduce the decomposition of the operator as

$$\mathcal{L} = A + B,$$

for some dissipative enough operator B and some B -bounded operator A satisfying that AS_B possesses some regularization properties, where S_B represents the semigroup associated to the generator B . In order to establish the regularization on AS_B , we introduce two choices of decomposition of \mathcal{L} explained as follows.

- **Dispersion argument and steady state.** We start with a rather simple splitting of the operator \mathcal{L} and perform the associated semigroup $S_{\mathcal{L}}$ in the weighted Lebesgue space $X := L^1(m) \cap L^p(m)$. We are able to obtain a better integrability of the semigroup S_B in the position variable by transferring it from the velocity variable to the position variable under a dispersion argument as introduced by Bardos and Degond in [6] and presented in section 1.4 together with a shrinkage of the functional space argument in [47]. Iterating the Duhamel formula, we have

$$S_{\mathcal{L}} = S_B + \dots + S_B * (AS_B)^{*(n-1)} + (S_B A)^{*(n)} * S_{\mathcal{L}}, \quad \forall n \in \mathbb{N}^*.$$

From an exponential decay estimate for the terms $S_B * (AS_B)^{*(k)}$ in $\mathcal{B}(X)$, an exponential decay estimate for $(S_B A)^{*(n)}$ in $\mathcal{B}(L^1, X)$ as well as the fact that $S_{\mathcal{L}}$ is bounded in $\mathcal{B}(L^1)$ as an immediate consequence of the mass conservation and the positivity property, we are able to deduce that $S_{\mathcal{L}}$ is bounded in $\mathcal{B}(X)$. Thanks to a standard Brouwer type fixed point argument, there exists a uniformly nonnegative and normalized steady state G . It is also unique from the classical weak and strong maximum principles. More precisely, 0 is a simple eigenvalue with eigenspace $\text{span}(G)$ and is the only nonnegative eigenvalue.

- **Averaging lemma and asymptotic stability.** To go further, we use a multiplier method introduced in section 1.4 in the spirit of Lions-Perthame multiplier in [60, 42, 63] to truncate A appropriately so that the integrability estimate in the position variable near the origin can be improved again. We introduce then such a new sophisticated splitting of the operator \mathcal{L} with a more regular \mathcal{A} and then a still properly dissipative corresponding operator \mathcal{B} for a suitable equivalent norm. Besides, $\mathcal{A}S_{\mathcal{B}}$ possesses nice compactness and regularity property as a consequence of a space variable averaging lemma in [16, 7] initial by the classical time and space averaging lemma of Golse et al. [30, 29]. Next, benefiting from the Weyl's Theorem as well as the quantitative spectral mapping theorem in [50], we conclude to a spectral gap on the spectrum of the operator \mathcal{L} and the related estimate on the semigroup $S_{\mathcal{L}}$ as stated in Theorem 4.1

It is worth emphasizing that the needed regularity estimate as the cornerstone in the proof derive from the more classical and more robust averaging lemma, instead of using

an abstract hypocoercivity operator as in [21, 12] nor an iterated averaging lemma as in [31]. Since the latter method permits us to transfer regularity from the velocity variable to the position variable with the help of a suitable commutator and the associated “gliding norms”.

4.2 Time elapsed model in neuron networks

We consider now the time elapsed model which describes the neuron density number $f = f(t, x) \geq 0$ at time $t \geq 0$ and in the local time $x \geq 0$ corresponding to the elapsed time since the last discharge. The dynamic is given by the following nonlinear equation

$$(4.44) \quad \begin{cases} \partial_t f = -\partial_x f - a(x, \varepsilon m(t))f =: \mathcal{L}_{\varepsilon m(t)} f, \\ f(t, 0) = p(t) := \int_0^\infty a(x, \varepsilon m(t))f(x) dx, \quad f(0, x) = f_0(x) \\ m(t) := \int_0^\infty p(t-y)b(dy), \end{cases}$$

with a reasonable and flexible instantaneous firing rate $a \geq 0$ in order to show the response to the recovery of the neuronal membranes after each discharge as well as a network connectivity regime parameter $\varepsilon \geq 0$. In this model, we denote the total density of neurons undergoing discharge at time $t \geq 0$ as the function $p(t)$, while the function $m(t)$ represents the network activity at time $t \geq 0$ as a result of the earlier discharges. We call b the delay distribution as a probability measure taking into account the persistence of the electric activity in the network resulting from discharges. Therefore, depending on the different extent of the persistence, the model is usually considered in two following different situations:

- The case without delay by choosing $b = \delta_0$ then $m(t) = p(t)$.
- The case with delay by choosing $b \geq 0$ as a smooth function and probability measure.

The nonlinearity of this model arises from the connectivity regime parameter ε of the network, which describes the strength of the influence of the neuronal network interactions. Therefore, in the limit case $\varepsilon = 0$, the equation (4.44) becomes linear since individual neuron evolves merely by its own dynamic then the network has the property of desynchronization. Such kind of property is also possessed in the weak connectivity regime as ε small enough because of the weak nonlinearity. While $\varepsilon > 0$, the dynamic of any given neuron is affected by all the other neurons through the global activity of the assembly. However, when ε is large enough, denoted as in strong connectivity regime correspondingly, that desynchronization of the network will be restored resulting from a large extent of connections.

Because of the conservation of the total density number of neurons in both two cases, we may normalize the initial mass as 1 and then the solution f of the time elapsed equation (4.44) satisfies

$$\int_0^\infty f(t, x) dx = 1, \quad \forall t \geq 0.$$

We say that a couple of density number of neurons F_ε and network activity $M_\varepsilon \geq 0$ is the

steady state for the time elapsed evolution system (4.44) if

$$(4.45) \quad \begin{cases} 0 = -\partial_x F_\varepsilon - a(x, \varepsilon M_\varepsilon) F_\varepsilon =: \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon, \\ F_\varepsilon(0) = M_\varepsilon, \quad \int_0^\infty F_\varepsilon(x) dx = 1. \end{cases}$$

In our work, we make the physically reasonable assumptions on the firing rate a and delay distribution b .

$$(4.46) \quad a \geq 0, \quad \partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0,$$

$$(4.47) \quad 0 < a_0 := \lim_{x \rightarrow \infty} a(x, 0) \leq \lim_{x, \mu \rightarrow \infty} a(x, \mu) =: a_1 < \infty,$$

$$(4.48) \quad A(x, \cdot) := \int_0^x a(y, \cdot) dy \in C^0(\mathbb{R}_+), \quad \forall x > 0,$$

associated with some continuity assumption

$$(4.49) \quad a \in W^{2, \infty}(\mathbb{R}_+^2).$$

In the strong connectivity regime, we consider the decay assumption as complement, for a.e. $x \geq 0$,

$$(4.50) \quad \varepsilon \sup_{x \geq 0} \partial_\mu a(x, \varepsilon \mu) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow \infty.$$

In the delay case, we assume that $b(dy) = b(y) dy$ satisfies the exponential bound and smoothness condition

$$(4.51) \quad \exists \delta > 0, \quad \int_0^\infty e^{\delta y} (b(y) + |b'(y)|) dy < \infty.$$

Thanks to the Schauder-Brouwer fixed point Theorem and Banach fixed point Theorem, for a given initial datum $0 \leq f_0 \in L^1(\mathbb{R}_+)$, the existence of the weak solution to the time elapsed equation (4.44) in the distributional sense is established in the paper [49, 77].

Theorem 4.2. *Assume (4.46)-(4.47)-(4.48), then for any $f_0 \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and any $\varepsilon > 0$, there exists a nonnegative and mass conserved weak solution f to the time-elapsed equation (4.44) in the case with delay for some functions $m, p \in C(\mathbb{R}_+)$, satisfying $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$. While in the case without delay, the existence and uniqueness of the weak solution in the weak and strong connectivity regime are established under the additional assumption (4.49)-(4.50).*

The existence of the steady state satisfying (4.45) is proven for any $\varepsilon > 0$ both in the paper [52] and [77], furthermore, we are able to obtain the uniqueness in the weak and strong connectivity regime in the two papers respectively.

Theorem 4.3. *On the one hand, for a general firing rate satisfying (4.46)-(4.47)-(4.48), for any $\varepsilon \geq 0$, there exists at least one pair $(F_\varepsilon(x), M_\varepsilon) \in W^{1,\infty}(\mathbb{R}_+) \times \mathbb{R}_+$ as solution to the stationary problem (4.45) such that*

$$(4.52) \quad 0 \leq F_\varepsilon(x) \lesssim e^{-\frac{\alpha_0}{2}x}, \quad |F'_\varepsilon(x)| \lesssim e^{-\frac{\alpha_0}{2}x}, \quad x \geq 0.$$

On the other hand, assuming (4.49)-(4.50) additionally, there exists $\varepsilon_0 > 0$ small enough and $\varepsilon_1 > 0$ large enough, such that the above steady state is unique for any $\varepsilon \in [0, \varepsilon_0) \cup (\varepsilon_1, +\infty]$.

The principal purpose is to study the asymptotic behavior of the solution to the time elapsed model (4.44). In the paper [56, 57, 58], Pakdaman, Perthame and Salort have studied this model with a particular *step function* firing rate

$$(4.53) \quad a(x, \mu) = \mathbf{1}_{x > \sigma(\mu)}, \quad \sigma, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+), \quad \sigma' \leq 0.$$

By taking advantage of this particular firing rate, they have proved the asymptotic stability of the solution in the weak or strong connectivity regime. Besides, they have established the exponential convergence to the equilibrium merely in the case without delay and in the weak connectivity regime. Benefiting from the spectral analysis method, we are able to improve the stability result. More precisely, we ameliorate it as exponential no matter with or without delay in the weak connectivity regime (see [52]) for smooth firing rates under the above assumptions or in the strong one (see [77]) for a more general firing rate under more general assumptions. In other words, we conclude that the total asynchronous firing of neurons appears exponentially fast in the large time asymptotic as our main result.

Theorem 4.4. *We assume that the firing rate a satisfies (4.46)-(4.47)-(4.49)-(4.50). We also assume that the delay distribution b satisfies $b = \delta_0$ or (4.51). There exists $\varepsilon_0 > 0$ ($\varepsilon_1 > 0$), small (large) enough, such that for any $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon \in (\varepsilon_1, +\infty)$) the steady state $(F_\varepsilon, M_\varepsilon)$ is unique. There also exist some constants $\alpha < 0$, $C \geq 1$ and $\eta > 0$ (besides $\zeta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow \infty$) such that for any connectivity parameter $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon \in (\varepsilon_1, +\infty)$) and for any unit mass initial datum $0 \leq f_0 \in L^1 \cap L^\infty$, such that $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\varepsilon$ ($\leq \eta/\zeta_\varepsilon$), then the positive and mass conserving solution f to the evolution equation (4.44) satisfies*

$$\|f(t, \cdot) - F_\varepsilon\|_{L^1} \leq C e^{\alpha t}, \quad \forall t \geq 0.$$

Our approach is not only different from the usual way to deal with delay equations, as introduced by I. Fredholm [26] and V. Volterra [76], consisting in using the specific framework of “fading memory space”, which goes back at least to Coleman & Mizel [14], or the theory of “abstract algebraic-delay differential systems” developed by O. Diekmann and co-authors [17]. It is also different from the previous works [56, 57, 58] where the asymptotic stability analysis were performed with benefit of the particular firing rate (4.53), whose step function structure makes it possible to explicitly exhibit a suitable norm such that some related linear time elapsed operator is dissipative. Based on a more abstract and somewhat more flexible approach without requirement to explicitly exhibit such an above norm, our proof follows a strategy of “perturbation of semigroup” initiated in [46] for studying long time convergence to the equilibrium for the homogeneous inelastic Boltzmann equation and used recently in [48] for a neuron network equation based on a

brownian (hypoelliptic) perturbation of the well-known FitzHugh-Nagumo dynamic. More precisely, we present here our strategy in the weak connectivity regime and separate it into two steps.

- **Linearized equation and semigroup decomposition.** We introduce the linearized operator Λ_ε for the variation functions $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$ around a stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ defined by

$$(4.54) \quad \begin{cases} \partial_t g = -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon n(t), \\ g(t, 0) = q(t) \int_0^\infty a_\varepsilon g \, dx + n(t) \int_0^\infty a'_\varepsilon F_\varepsilon \, dx, \\ n(t) := \int_0^\infty q(t-y)b(dy), \\ g(0, x) = g_0(x), \end{cases}$$

where we denote $a_\varepsilon = a(x, \varepsilon M_\varepsilon)$ and $a'_\varepsilon = \varepsilon(\partial_\mu a)(x, \varepsilon M_\varepsilon)$ for simplicity. We associate the linearized generator Λ_ε acting on appropriate spaces in different cases with delay or not and in the weak or strong connectivity regime to its semigroup S_{Λ_ε} . It turns out that we may split the operator Λ_ε as

$$(4.55) \quad \Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon,$$

for some α -hypodissipative operator \mathcal{B}_ε , $\alpha < 0$, and some bounded and \mathcal{B}_ε -power regular operator \mathcal{A}_ε as defined in section 1.2. Adapted versions of the Spectral Mapping Theorem in [50, 45, 44] and the Weyl's Theorem in [75, 31, 50, 45, 44] imply that the semigroup S_{Λ_ε} has a finite dimensional dominant part and a spectral gap. More precisely, we obtain the decrement of the semigroup in the space of Radon measures $\mathcal{X} := M^1(\mathbb{R}_+)$. For any $t \geq 0$, there exist some constants $C > 0$ and $a < 0$, such that

$$(4.56) \quad \|S_{\Lambda_\varepsilon}(I - \Pi_{\Lambda_\varepsilon})\|_{\mathcal{B}(\mathcal{X})} \leq C e^{at},$$

where $\Pi_{\Lambda_\varepsilon}$ is the finite rank projector.

- **Perturbation argument and nonlinear problem.** Moreover, in the limit case when $\varepsilon = 0$, the term $n(t)$ disappears in the linearized equation (4.54), which leads to the positivity of the limited semigroup S_{Λ_0} . That allows us to use the Krein-Rutman Theorem established in [50, 45, 44] in order to get the uniqueness and exponential stability of the stationary state (F_0, M_0) as well as the spectrum structure

$$\Sigma(\Lambda_0) \cap \Delta_\alpha = \{0\}.$$

Using next a perturbative argument presented in section 1.3, we extend that uniqueness, spectrum structure of Λ_ε as well as exponential stability to the stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ in the weak connectivity regime. Benefiting from the quadratic estimate of the nonlinear term and the exponential stability of the linearized equation, we are able to conclude our main result for the nonlinear problem (4.44) in Theorem 4.4.

The same strategy applies to the case with delay as above, we still regard the boundary condition in the time elapsed equation as a source term and replace the delay equation by a simple age equation on an auxiliary function, so that the resulting linearized equation

on variation functions can be written as an autonomous system of two evolution equations in order to generate a semigroup to follow the similar approach.

Notice that the step function firing rate (4.53) does not fall in the above class of firing rates because of the failure in meeting condition (4.49), we are not able to linearize the time elapsed model as (4.54). However, we are still able to apply the spectral analysis method to another more concise linear equation on variation functions $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$

$$(4.57) \quad \begin{cases} \partial_t g + \partial_x g + a(x, \varepsilon M_\varepsilon)g = 0, \\ g(t, 0) = q(t), \quad g(0, x) = g_0(x), \\ q(t) = \int_0^\infty a(x, \varepsilon M_\varepsilon)g \, dx, \\ n(t) := \int_0^\infty q(t-y)b(dy). \end{cases}$$

Instead of the strong regularity assumption (4.49) on a , weaker assumptions are required, such that the “step function firing rate” (4.53) also fulfills the conditions. First, we assume

$$(4.58) \quad A(x, \cdot) := \int_0^x a(y, \cdot) \, dy \in C^0(\mathbb{R}_+), \quad \forall x > 0,$$

and $a \in L_x^1 \text{Lip}_\mu$. For the weak connectivity regime, we assume that for some $\xi > 0$ small enough and for any $\mu_0 > 0$, there exists $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_0)$, there holds

$$(4.59) \quad \int_0^\infty |a(x, \varepsilon \mu_2) - a(x, \varepsilon \mu_1)| \, dx \leq \xi |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (0, \mu_0).$$

While in the strong connectivity regime, we assume that for some the same $\xi > 0$ as in the assumption (4.59) and for any $\mu_\infty > 0$, there exists $\varepsilon_\infty > 0$ large enough such that for any $\varepsilon \in (\varepsilon_\infty, \infty)$, there holds

$$(4.60) \quad \int_0^\infty |a(x, \varepsilon \mu_2) - a(x, \varepsilon \mu_1)| \, dx \leq \xi |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (\mu_\infty, \infty).$$

With the similar splitting structure (4.55), the adapted version of the Spectral Mapping Theorem, Wely’s Theorem and the Krein-Rutman Theorem also permit the exponential estimate of the associated linear semigroup

$$\|S_{\Lambda_\varepsilon}\|_{\mathcal{B}(X)} \lesssim e^{\alpha t},$$

for some constant $\alpha < 0$ as well as the description of the spectrum as

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}.$$

Then thanks to the perturbation argument and the estimate of the nonlinear term, we are also able to generalize the exponential asymptotic stability results in the Theorem 4.4, which are obtained in [56, 57, 58] in the case without delay to the case considering of the delay term, beyond the restriction of the “step function” on the firing rate, no matter in the weak or the strong connectivity regime respectively in the papers [49, 77].

5 Perspectives and open problems

Finally, we present the complementary problems which is worth considering after this thesis to finish the introductory chapter.

1. **Nonlinear run and tumble model.** The study in the paper [53] has merely concentrated on a linear version of run and tumble equation (4.41). However, we have to point out that it is still not clear how to apply the spectral analysis approach explained in section 4.1 to the nonlinear run and tumble model (4.40) in order to make any progress for better understanding. In particular, we have not been able to prove the assumption used in the linear equation (4.41) on the chemoattractant density S such that S decays with respect to the radial variable $|x|$ to avoid the relation between the turning kernel K with S . On the other hand, we are wondering if a stationary state \bar{G} to the linear equation (4.41) can also become a steady state of the nonlinear equation (4.40), as for instance, one can expect by making an analogy with the one dimension case and when the velocity set \mathcal{V} is replaced by $\bar{\mathcal{V}} := \{-1, 1\}$. Indeed, in that case, we may observe that for $\bar{G}(x, v) = g(|x|) = C e^{-\chi|x|}$, we have

$$\begin{aligned} v \cdot \nabla_x \bar{G} &= v \cdot \frac{x}{|x|} g'(|x|) = -\chi \operatorname{sign}(x \cdot v) g(|x|) \\ &= \int_{\bar{\mathcal{V}}} (K' \bar{G} - K \bar{G}) dv', \end{aligned}$$

so that we have exhibited an explicit (unique, positive and unit mass) stationary state \bar{G} . The associated macroscopic density $\bar{\rho}$ is then decaying and thus also the associated chemical agent density \bar{S} (thanks to the maximum principle applied to the elliptic equation on the concentration of chemical agent S). It turns out then that \bar{G} is also a stationary state of the nonlinear equation (4.40). The existence of traveling wave solutions for a similar nonlinear model has been established in the recent paper [11], which may inspire us another way to study the nonlinear model (4.40).

2. **Periodic solution of time-elapsd equation.** In the neuron assemblies, collective rhythms are produced by periodic oscillations displayed by single neurons which arise global phase locking yielding synchronous rhythms at the population level because of the interactions. While in the coupled interacting noisy excitable systems, individual neurons calm at a resting state without intrinsic periodic oscillations, nevertheless, the irregular discharges caused by noise can be transformed and emerged into a global regular and coherent rhythm under the interplay among neurons. The above rhythmic synchronous activity encourages us the attempt to construct periodic solutions to the time-elapsd equation (4.44) with periodic function $p(t)$ and $m(t)$. Some explicit examples of such kind of periodic solutions corresponding to rhythmic and synchronous activity in the neuron networks together with the relative numerical analysis have been displayed in [56, 57]. The numerical simulations ensure the existence of various periodic solutions depending on the initial data which is, however, not always stable. The conclusion about general existence and stability of the periodic solutions to the time-elapsd model (4.44) waits to be determined, and we might turn to the Floquet Theory for amelioration.
3. **Stability of other type of time-elapsd model.** The time-elapsd equation (4.44) we studied in section 4.2 to understand better of synchronization or desynchronization

of the neuron networks depending on the interconnection strength. In paper [58], one way of extension is to add a fragmentation term in the model without delay term, for instance, as following

$$(5.61) \quad \begin{cases} \partial_t f = -\partial_x f - a(x, p(t))f = \int_0^\infty K(x, y)a(y, p(t))f(y, t) dy, \\ p(t) := \int_0^\infty a(x, p(t))f(x) dx, \\ f(0, t) = 0, \quad f(0, x) = f_0(x) \geq 0, \quad \int_0^\infty f_0(x) dx = 1, \end{cases}$$

where the kernel $K(x, y) \in M^1(\mathbb{R}_+ \times \mathbb{R}_+)$ exhibits the distribution of neurons in the state x after the occurrence of a discharge with an elapsed time y since the last discharge. The new type model (5.61) incorporates the property of neurons whose dynamics also depends on their past activity, which progressively attenuate their tendency to fire with the stimulation from a step maintained current. Such phenomenon is known as adaptation and fatigue of neurons that is also one of the most common properties of neurons indicating the coherence in firing times. We hope that our spectral analysis strategy presented in section 4.2 can be adapted to the adaptation and fatigue model (5.61) and thus extend the exponential stability results established in [58] in weakly connected network in the case without delay to the general case with a delay term as well as in strongly connected network.

First Part
Chemotaxis

Chapter 1

Runs-and-tumbles model

Résumé

Nous considérons une équation linéaire de courses-et-chutes en dimension $d \geq 1$ pour laquelle nous établissons l'existence d'un état stationnaire unique positif et normalisé ainsi que sa stabilité asymptotique, en améliorant les résultats similaires obtenus par Calvez et al. [12] dans la dimension $d = 1$. Notre analyse est basée sur la théorie de Krein-Rutman revisitée dans [50] avec des nouvelles estimations de moment pour prouver le mécanisme de confinement ainsi que les arguments des effets de dispersion, de la méthode des multiplicateurs et du lemme de moyenne pour prouver une certaine propriété de régularité de quantités de moyenne itérées appropriées.

Abstract

We consider a linear runs and tumbles equation in dimension $d \geq 1$ for which we establish the existence of a unique positive and normalized steady state as well as its asymptotic stability, improving similar results obtained by Calvez et al. [12] in dimension $d = 1$. Our analysis is based on the Krein-Rutman theory revisited in [50] together with some new moment estimates for proving confinement mechanism as well as dispersion, multiplier and averaging lemma arguments for proving some regularity property of suitable iterated averaging quantities.

1.1 Introduction and main result

1.1.1 The “runs and tumbles” equation in chemotaxis

In the present paper we are interested in a kinetic evolution PDE coming from the modeling of cells movement in the presence of a chemotactic chemical substance. The so-called *run-and-tumble* model introduced by Stroock [71] and Alt [2], and studied further in [54, 55, 23],

$$(1.1.1) \quad \partial_t f = \mathcal{L}f = -v \cdot \nabla_x f + \int_{\mathcal{V}} (K' f' - K f) dv'$$

describes the evolution of the distribution function of a microorganisms density $f = f(t, x, v) \geq 0$ which at time $t \geq 0$ and at position $x \in \mathbb{R}^d$ move with the velocity $v \in \mathcal{V}$. At a microscopic description level, microorganisms move in straight line with their own velocity v which changes accordingly to a jump process of parameter $K = K(x, v, v') \geq 0$. Here and below, we used the shorthands $f' = f(t, x, v')$ and $K' = K(x, v', v)$. For the sake of simplicity, we assume that $\mathcal{V} \subset \mathbb{R}^d$ is the centered ball with unit volume ($\mathcal{V} := B(0, V_0)$) with V_0 chosen such that $|\mathcal{V}| = 1$). We complement the evolution PDE (1.1.1) with an initial condition

$$(1.1.2) \quad f(0, \cdot) = f_0 \quad \text{on} \quad \mathbb{R}^d \times \mathcal{V}.$$

At least formally, for any multiplier $\varphi = \varphi(x, v)$, we have

$$\frac{d}{dt} \int f \varphi = \int f v \cdot \nabla_x \varphi + \int f K \int_{\mathcal{V}} (\varphi' - \varphi).$$

In particular, choosing $\varphi \equiv 1$ in the above identity, we see that the total mass is conserved and we may assume that it is normalized to the unit, namely

$$(1.1.3) \quad \langle\langle f(t, \cdot) \rangle\rangle = \langle\langle f_0 \rangle\rangle = 1, \quad t \geq 0,$$

where for functions $g = g(x, v)$ and $h = h(v)$, we define

$$\langle h \rangle = \int_{\mathcal{V}} h dv, \quad \langle\langle g \rangle\rangle = \int_{\mathbb{R}^d} \langle g(x, \cdot) \rangle dx.$$

The precise form of the turning kernel K depends upon the possibly time and space dependent concentration $S = S(t, x)$ of a chemical agent: microorganisms have the tendency to move to where the chemical concentration is higher. More specifically, we assume that the turning kernel is given by

$$(1.1.4) \quad K = K[S](v) := 1 - \chi \Phi(\partial_t S + v \cdot \nabla_x S), \quad \chi \in (0, 1), \quad \Phi(y) = \text{sign}(y),$$

where the sign function is defined by $\Phi(y) = -1$ if $y < 0$ and $\Phi(y) = 1$ if $y > 0$. In other words, the turning kernel K takes the two values $1 \pm \chi$ depending on the velocity direction of the microorganism with respect to the gradient of the chemical concentration.

When the chemical agent is produced by the microorganisms themselves, it is usually assumed to be given as the solution to the damped Poisson equation

$$(1.1.5) \quad -\Delta S + S = \varrho := \int_{\mathcal{V}} f dv,$$

so that the evolution of the microorganisms density f is given by the coupled system of equations (1.1.1)-(1.1.4)-(1.1.5). We refer the reader interested by the well-posedness issue for related models to the review paper [8] and the references quoted therein. We also refer to [35] for related modeling considerations. Concerning the qualitative behaviour of the solutions it seems that the unique available information is the mass conservation (1.1.3). One of the main difficulty comes from the fact that both equations (1.1.1) and (1.1.5) are invariant by translations so that the expected confinement mechanism seems to be hard to prove.

On the other way round, one can see that for a given solution (f, S) of (1.1.1)-(1.1.4)-(1.1.5) and for any rotation $R \in SO(d)$ the couple (f_R, S_R) is also a solution of the same equations, where we have set $f_R(x, v) := f(Rx, Rv)$ and $S_R(x) := S(Rx)$. In particular, an invariant by rotations initial datum f_0 gives rise to an invariant by rotations solution (f, S) .

More specifically, we may observe that in the case when S does not depend of time and it is radially symmetric and strictly decreasing in the position variable (which is the case if the density ϱ satisfies the same properties thanks to the maximum principle), we have

$$-\Phi(\partial_t S + v \cdot \nabla_x S) = -\Phi(-v \cdot x) = \text{sign}(x \cdot v),$$

and thus the associated turning kernel writes

$$(1.1.6) \quad K = K(x, v) := 1 + \chi \zeta, \quad \chi \in (0, 1),$$

where

$$\zeta = \zeta(x, v) = \text{sign}(x \cdot v).$$

Such a kernel has been introduced in [12] and the associated (now linear!) evolution equation (1.1.1)-(1.1.6) has then been analyzed in dimension $d = 1$: the existence of a unique (positive and normalized) steady state has been established and its asymptotic exponential stability has been proved.

1.1.2 The linear “runs and tumbles” equation

The main purpose of the present work is to provide an alternative approach to study the linear “runs and tumbles” (linear RT) equation (1.1.1)-(1.1.6) which makes possible to generalize the analysis of [12] to any dimension $d \geq 1$. In order to state our main result, we introduce some notations and the functional framework we will work with.

First, we denote by m some weight function which is either a polynomial or an exponential

$$(1.1.7) \quad m(x) = \langle x \rangle^k, \quad k > 0, \quad \text{or} \quad m(x) = \exp(\gamma \langle x \rangle), \quad \gamma \in (0, \gamma^*),$$

for some positive constant γ^* which will be defined latter, and where $\langle x \rangle^2 = 1 + |x|^2$. To a given weight m we define the associated rate function Θ_m and weight function ω by

- $\Theta_m(t) := \langle t \rangle^{-\ell}, \quad \forall \ell \in (0, k), \quad \omega = 1, \quad \text{if } m = \langle x \rangle^k;$
- $\Theta_m(t) := e^{at}, \quad \forall a \in (a^*, 0), \quad \omega = m, \quad \text{if } m = e^{\gamma \langle x \rangle},$

for an optimal rate $a^* = a^*(\gamma) < 0$ which will be also defined later. Finally for given weight function $m = m(x) : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and exponent $1 \leq p \leq \infty$, we define the associated weighted Lebesgue space $L^p(m)$ and weighted Sobolev space $W^{1,p}(m)$, through their norms

$$(1.1.8) \quad \|f\|_{L^p(m)} := \|mf\|_{L^p}, \quad \|f\|_{W^{1,p}(m)} := \|mf\|_{W^{1,p}}.$$

We use the shorthands $L_k^p = L^p(m)$, when $m = \langle x \rangle^k$, and $H^1(m) = W^{1,2}(m)$. We write $a \lesssim b$ if there exists a positive constant C such that $a \leq Cb$.

Theorem 1.1.1. *There exists $\gamma^* > 0$ and there exists a unique positive, invariant by rotations and normalized stationary state*

$$\begin{aligned} 0 < G \in L^\infty(m_0), \quad \langle\langle G \rangle\rangle = 1, \\ -v \cdot \nabla_x G + \int_{\mathcal{V}} (K'G' - KG) dv' = 0, \end{aligned}$$

where m_0 stands for the exponential weight function $m_0(x) := \exp(\gamma_* \langle x \rangle)$. Moreover, for any weight function m satisfying (1.1.7) and for any $0 \leq f_0 \in L^1(m)$, there exists a unique solution $f \in C([0, \infty); L^1(m))$ to the equation (1.1.1)-(1.1.6) associated to the initial datum f_0 and

$$(1.1.9) \quad \|f(t) - \langle\langle f_0 \rangle\rangle G\|_{L^1(\omega)} \leq \Theta_m(t) \|f_0\|_{L^1(m)}, \quad \forall t \geq 0,$$

where ω and Θ_m are defined just above.

The present result generalizes to any dimension $d \geq 1$ similar results ([12, Theorem 2.1] and [12, Proposition 1]) obtained by Calvez et al. in dimension $d = 1$. As pointed out in [12], the main novelty and mathematical interest of the model lie in the fact that the confinement is achieved by a biased velocity jump process, where the bias replaces the confining acceleration field which is the classical confinement mechanism for Boltzmann and Fokker-Planck models, see for instance [74, 47, 21] and the references quoted therein.

Our strategy is drastically different from the one used in [12] but similar to the approach of [50] which develops a Krein-Rutman theory for positive semigroups which do not fulfill the classical compactness assumption on the associated resolvent but have a nice splitting structure.

However, instead of applying directly the Krein-Rutman Theorem [50, Theorem 5.3] and in order to be a bit more self-consistent and pedagogical, we rather follow the same line of proof as in [50] but we perform some simplifications by taking advantage of the mass conservation law (or equivalently, that the dual operator has an explicit positive eigenvector). The main difficulty is then to get suitable estimates on some related operators and semigroups.

In section 1.2, the first step consists in proving a weighted L^1 bound which brings out the confinement mechanism. That is the main new bound which is in the spirit of weighted L^p estimates obtained for performing similar spectral analysis in [31, 50, 47] for Boltzmann, growth-fragmentation and kinetic Fokker-Planck models.

Next, in order to go further in the analysis, we introduce suitable decompositions

$$\mathcal{L} = A + B,$$

with several choices of operators A and B such that B is adequately dissipative, A is B -bounded and AS_B enjoys some regularization properties, where S_B stands here for the semigroup associated to the generator B . For establishing these regularization properties on AS_B , we successively use a dispersion argument as introduced by Bardos and Degond [6] for providing better integrability in the position variable (transfer of integrability from the velocity variable to the position variable), next a multiplier method in the spirit of Lions-Perthame multiplier (see [60, 42, 61]) for improving again the integrability estimate in the position variable near the origin and finally a space variable averaging lemma in the spirit of the variant [16, 7] of the classical time and space averaging lemma of Golse et al [30, 29]. It is worth emphasizing that the needed regularity estimate is not obtained using an abstract hypocoercivity operator as in [21, 12] nor using an iterated averaging lemma as in [31] (which allows to transfer regularity from the velocity variable to the position variable thanks to a suitable commutator and the associated “gliding norms”) but using the more classical (and more robust) averaging lemma.

More precisely, in Section 1.3, we make a first rather simple choice for the splitting of the operator \mathcal{L} and we obtain that the associated semigroup $S_{\mathcal{L}}$ is bounded in the weighted Lebesgue space $X := L^1(m) \cap L^p(m)$ by gathering the above dispersion argument and a shrinkage of the functional space argument as in [47]. A flavor of the argument reads as follows. We write the iterative version of the Duhamel formula

$$S_{\mathcal{L}} = S_B + \dots + S_B * (AS_B)^{*(n-1)} + (S_B A)^{*(n)} * S_{\mathcal{L}}, \quad \forall n \in \mathbb{N}^*,$$

where $*$ stands for the usual convolution operator on \mathbb{R}_+ . We then deduce that $S_{\mathcal{L}}$ is bounded in $\mathcal{B}(X)$, the space of bounded linear maps from X into itself, by using an exponential decay estimate in $\mathcal{B}(X)$ for the terms $S_B * (AS_B)^{*(k)}$, an exponential decay estimate in $\mathcal{B}(L^1, X)$ for $(S_B A)^{*(n)}$ and exploiting that $S_{\mathcal{L}}$ is bounded in $\mathcal{B}(L^1)$ as an immediate consequence of the mass conservation and the positivity property. We then deduce the existence of a weighted uniformly bounded nonnegative and normalized steady state thanks to a standard Brouwer type fixed point argument. Its uniqueness follows by classical weak and strong maximum principles. As a matter of fact, in the same way one deduces that 0 is a simple eigenvalue and the only nonnegative eigenvalue.

In section 1.4, we introduce a more sophisticated surgical truncation A of the kernel operator involved in \mathcal{L} and thus a second splitting. In such a way, using the above mentioned multiplier method and space averaging lemma, we obtain that the new operator A is more regular, the corresponding operator B is still appropriately dissipative for a suitable equivalent norm and finally AS_B has nice compactness and regularity property. With the help of [50], we conclude to a spectral gap on the spectrum of the operator \mathcal{L} (Weyl theorem) and its translation into an estimate on the semigroup $S_{\mathcal{L}}$ (quantitative spectral mapping theorem) as stated in Theorem 1.1.1.

Finally, it is worth pointing out that it is not clear how to use the above analysis in order to make any progress in the understanding of the nonlinear equation (1.1.1)-(1.1.4)-(1.1.5). In particular, we have not been able to prove that the chemical agent density S is decaying with respect to the radial variable $|x|$ and thus that G is also a stationary state of the nonlinear equation (1.1.1)-(1.1.4)-(1.1.5), as one can expect by making an analogy with the one dimension case and when the velocity set \mathcal{V} is replaced by $\bar{\mathcal{V}} := \{-1, 1\}$.

Indeed, in that case, we may observe that for $\bar{G}(x, v) = g(|x|) = C e^{-\chi|x|}$, we have

$$\begin{aligned} v \cdot \nabla_x \bar{G} &= v \cdot \frac{x}{|x|} g'(|x|) = -\chi \operatorname{sign}(x \cdot v) g(|x|) \\ &= \int_{\bar{\mathcal{V}}} (K' \bar{G} - K \bar{G}) \, dv', \end{aligned}$$

so that we have exhibited an explicit (unique, positive and unit mass) stationary state \bar{G} . The associated macroscopic density $\bar{\rho}$ is then decaying and thus also the associated chemical agent density \bar{S} (thanks to the maximum principle applied to the elliptic equation (1.1.5)). It turns out then that \bar{G} is also a stationary state of the nonlinear equation (1.1.1)-(1.1.4)-(1.1.5). We refer the interested reader to the recent paper by Calvez [11] who establishes the existence of traveling wave solutions for a similar nonlinear model.

Let us end the introduction by describing again the plan of the paper. In Section 1.2, we mainly present the weighted L^1 estimate which highlights the confinement mechanism of the model. In Section 1.3, we introduce a first splitting of the generator in order to prove the existence (and next the uniqueness) of a positive stationary state. Finally, in Section 1.4, we introduce a second splitting of the generator which enjoys better smoothness properties and for which we can use the Krein-Rutman theory revisited in [50] and conclude to the asymptotic stability of the stationary state.

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1.2 Well-posedness and exponential weighted L^1 estimate

We first state a well-posedness result concerning the linear RT equation (1.1.1)-(1.1.6), whose very classical proof is skipped, and we recall some notations and definition.

Lemma 1.2.1. *For any $f_0 \in L^p(m)$, $1 \leq p \leq \infty$, there exists a unique weak (distributional) solution $f \in C([0, T]; L^1_{loc}) \cap L^\infty(0, T; L^p(m))$, $\forall T \in (0, \infty)$, to the linear RT equation (1.1.1)-(1.1.6) which furthermore satisfies:*

- (1) $f(t, \cdot) \geq 0$ for any $t \geq 0$ if $f_0 \geq 0$ (preservation of positivity);
- (2) $\langle\langle f(t, \cdot) \rangle\rangle = \langle\langle f_0 \rangle\rangle$ for any $t \geq 0$ if $L^p(m) \subset L^1$ (mass conservation);
- (3) $f \in C(\mathbb{R}_+; L^p(m))$ and we may associate to \mathcal{L} a continuous semigroup $S_{\mathcal{L}}$ in $L^p(m)$ by setting $S_{\mathcal{L}}(t)f_0 := f(t, \cdot)$ for any $t \geq 0$ and $f_0 \in L^p(m)$. Here the continuity has to be understood in the strong (norm) topology sense when $p \in [1, \infty)$ and in the weak $\sigma(L^\infty, L^1)^*$ topology sense when $p = \infty$.

In the sequel, we will thus associate to the generator \mathcal{L} (and next to the related generators $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}, \dots$) a semigroup $S_{\mathcal{L}}$ and to any initial datum $f_0 \in L^p(m)$ we will denote by $f(t) = f_{\mathcal{L}}(t) = S_{\mathcal{L}}(t)f_0$ the solution associated to the related abstract evolution equation. Thanks to by-now standard results (due in particular to Ball [5] and to DiPerna-Lions [19]) this solution is equivalently a distributional, weak or renormalized solution to

the corresponding PDE equation. More precisely, the above function f satisfies the linear RT equation (1.1.1) in the following renormalized sense

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \mathcal{V}} \beta(f) \varphi = \int_{\mathbb{R}^d \times \mathcal{V}} \beta(f) v \cdot \nabla_x \varphi + \int_{\mathbb{R}^d \times \mathcal{V}} f K \int_{\mathcal{V}} (\varphi' \beta(f') - \varphi \beta'(f))$$

for any (renormalizing) Lipschitz function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ and for any (test) function $\varphi \in C_c^1(\mathbb{R}^d \times \mathcal{V})$. It is worth mentioning that we deduce the preservation of positivity property by just choosing $\beta(s) = s_-$, $\varphi = 1$, in the above identity and next using the Grönwall's lemma. When $L^p(m) \subset L^1$, the uniqueness result follows from choosing $\beta(s) = |s|$ and $\varphi = 1$ in the above identity. The existence part can be achieved by combining the characteristics method for the free transport equation and a perturbation by bounded operators argument. In the sequel, we will denote $\mathcal{B}(X, Y)$ the space of bounded linear operators from a Banach space X into a Banach space Y , and we write $\mathcal{B}(X) = \mathcal{B}(X, X)$. Another way to prove existence and uniqueness consists in using the Hille-Yosida theorem for maximal dissipative unbounded operators. We recall for futur references that we say that an unbounded operator \mathcal{L} with dense domain $D(\mathcal{L}) \subset X$ is a -dissipative, $a \in \mathbb{R}$, if

$$\forall f \in D(\mathcal{L}), \exists f^* \in J_f \quad \langle f^*, \mathcal{L}f \rangle_{X', X} \leq a \|f\|_X^2,$$

where J_f denotes the (nonempty) dual set

$$J_f := \{g \in X'; \langle g, f \rangle_{X', X} = \|g\|_{X'}^2 = \|f\|_X^2\}.$$

We now establish a uniform in time exponential weighted L^1 estimate, which is one of the cornerstone arguments of the proof of our main theorem.

Lemma 1.2.2. *There exists a constant $\gamma^* > 0$ and for any $\gamma \in (0, \gamma^*)$ there exist a weight function \tilde{m} such that $\tilde{m}(x) \sim e^{\gamma\langle x \rangle}$ as $x \rightarrow \infty$ and a constant $C \in (0, \infty)$ such that the solution f to the linear RT equation with initial datum $f_0 \in L^1(m)$ satisfies*

$$(1.2.1) \quad \int |f(t)| \tilde{m} \leq \max\left(C \int |f_0|, \int |f_0| \tilde{m}\right), \quad \forall t \geq 0.$$

In particular, the semigroup $S_{\mathcal{L}}$ is bounded in $L^1(m)$.

Proof of Lemma 1.2.2. We define the dual operator \mathcal{L}^* by

$$\int (\mathcal{L}^* \varphi) f = \int (\mathcal{L}f) \varphi, \quad \forall \varphi \in W^{1, \infty}(\mathbb{R}^d \times \mathcal{V}), f \in C_c(\mathbb{R}^d \times \mathcal{V}),$$

so that

$$(\mathcal{L}^* \varphi)(x, v) := v \cdot \nabla_x \varphi + K \int_{\mathcal{V}} (\varphi' - \varphi) dv'.$$

For a given $\gamma > 0$, we compute

$$\mathcal{L}^* e^{\gamma\langle x \rangle} = \gamma \left(v \cdot \frac{x}{\langle x \rangle}\right) e^{\gamma\langle x \rangle},$$

and next

$$\begin{aligned} \mathcal{L}^* \left[\left(v \cdot \frac{x}{\langle x \rangle}\right) e^{\gamma\langle x \rangle} \right] &= v \cdot \nabla_x \left(\left(v \cdot \frac{x}{\langle x \rangle}\right) e^{\gamma\langle x \rangle} \right) - K \left(v \cdot \frac{x}{\langle x \rangle}\right) e^{\gamma\langle x \rangle} \\ &= \left(\frac{|v|^2}{\langle x \rangle} - \frac{(v \cdot x)^2}{\langle x \rangle^3} + \gamma \frac{(v \cdot x)^2}{\langle x \rangle^2} - \left(v \cdot \frac{x}{\langle x \rangle}\right) - \chi \frac{|v \cdot x|}{\langle x \rangle} \right) e^{\gamma\langle x \rangle}. \end{aligned}$$

Defining

$$(1.2.2) \quad V_1 := \int_{\mathcal{V}} |v'_1| \, dv',$$

and recalling that ζ is defined in (1.1.6), we finally have

$$\begin{aligned} \mathcal{L}^* \left[\frac{|v \cdot x|}{\langle x \rangle} e^{\gamma \langle x \rangle} \right] &= v \cdot \nabla_x \left(\frac{|v \cdot x|}{\langle x \rangle} e^{\gamma \langle x \rangle} \right) + (1 + \chi \zeta) \left(V_1 \frac{|x|}{\langle x \rangle} - \frac{|v \cdot x|}{\langle x \rangle} \right) e^{\gamma \langle x \rangle} \\ &= \zeta \left(\frac{|v|^2}{\langle x \rangle} - \frac{|v \cdot x|^2}{\langle x \rangle^3} + \gamma \frac{|v \cdot x|^2}{\langle x \rangle^2} \right) e^{\gamma \langle x \rangle} \\ &\quad + \left((1 + \chi \zeta) V_1 \frac{|x|}{\langle x \rangle} - (1 + \chi \zeta) \frac{|v \cdot x|}{\langle x \rangle} \right) e^{\gamma \langle x \rangle}. \end{aligned}$$

Defining the weight function

$$(1.2.3) \quad \tilde{m} := \left(1 + \gamma \left(v \cdot \frac{x}{\langle x \rangle} \right) - \beta \frac{|v \cdot x|}{\langle x \rangle} \right) e^{\gamma \langle x \rangle},$$

for $\beta, \gamma \in (0, 1)$ to be fixed precisely later, we observe that \tilde{m} satisfies

$$(1.2.4) \quad \exists \delta \in (0, 1), \quad e^{\gamma \langle x \rangle} (1 - \delta) \leq \tilde{m} \leq (1 + \delta) e^{\gamma \langle x \rangle},$$

by choosing β, γ small enough and because $\mathcal{V} := B(0, V_0)$ is a bounded set. Gathering the previous identities, we find

$$\begin{aligned} (\mathcal{L}^* \tilde{m}) e^{-\gamma \langle x \rangle} &= \gamma \left(\frac{|v|^2}{\langle x \rangle} - \frac{(v \cdot x)^2}{\langle x \rangle^3} + \gamma \frac{(v \cdot x)^2}{\langle x \rangle^2} - \chi \frac{|v \cdot x|}{\langle x \rangle} \right) \\ &\quad - \beta \zeta \left(\frac{|v|^2}{\langle x \rangle} - \frac{|v \cdot x|^2}{\langle x \rangle^3} + \gamma \frac{|v \cdot x|^2}{\langle x \rangle^2} \right) \\ &\quad - \beta \left((1 + \chi \zeta) V_1 \frac{|x|}{\langle x \rangle} - (1 + \chi \zeta) \frac{|v \cdot x|}{\langle x \rangle} \right) \\ &\leq \gamma V_0^2 \left(\frac{1}{\langle x \rangle} + \gamma \right) + \beta V_0^2 \left(\frac{2}{\langle x \rangle} + \gamma \right) + \beta (1 + \chi) \frac{|v \cdot x|}{\langle x \rangle} \\ &\quad - \gamma \chi \frac{|v \cdot x|}{\langle x \rangle} - \beta (1 - \chi) V_1 \frac{|x|}{\langle x \rangle} \\ &\leq \left(V_0^2 (\gamma + 2\beta) + \beta (1 - \chi) V_1 \right) \frac{1}{\langle x \rangle} + \left(\gamma^2 V_0^2 + \beta \gamma V_0^2 - \beta (1 - \chi) V_1 \right). \end{aligned}$$

We thus deduce that

$$(1.2.5) \quad (\mathcal{L}^* \tilde{m}) e^{-\gamma \langle x \rangle} \leq \frac{C}{\langle x \rangle} - 2\alpha,$$

by choosing $\beta(1 + \chi) = \gamma \chi$ and $\gamma > 0$ small enough in such a way that

$$2\alpha := \beta(1 - \chi) V_1 - \gamma^2 V_0^2 - \beta \gamma V_0^2 > 0.$$

Observing that

$$\left(\frac{C}{\langle x \rangle} - 2\alpha \right) e^{\gamma \langle x \rangle} \leq C e^{\gamma \langle R \rangle} \mathbf{1}_{B(0, R)} - \alpha (1 + \delta) e^{\gamma \langle x \rangle} \leq A - \alpha \tilde{m},$$

for some constant $A > 0$, we have proved

$$\mathcal{L}^* \tilde{m} \leq A - \alpha \tilde{m}, \quad \alpha < 0.$$

We consider now f the solution to the linear RT equation (1.1.1) associated to $f_0 \in L^1(m)$. Denoting $g := |f|$, we deduce from the above inequality that

$$\frac{d}{dt} \int g \tilde{m} = \int (\mathcal{L}f) (\text{sign} f) \tilde{m} \leq \int (\mathcal{L}g) \tilde{m} \leq A \int g - \alpha \int g \tilde{m}.$$

As a consequence, (1.2.1) holds with $C := A/\alpha$. \square

1.3 The stationary state problem

We introduce two generators \mathcal{B}_0 and next \mathcal{B}_1 in the following paragraphs and we study the gain of integrability properties of the associated semigroups. Introducing the splitting $\mathcal{L} = \mathcal{A}_1 + \mathcal{B}_1$, we then use these estimates in order to prove that $S_{\mathcal{L}}$ is a bounded semigroup in $L^1(m) \cap L^p(m)$, from what we deduce the existence of a stationary state. Uniqueness of this one is finally proved thanks to classical weak and strong maximum principles.

1.3.1 The operator \mathcal{B}_0 and the associated semigroup $S_{\mathcal{B}_0}$.

We define \mathcal{B}_0 by

$$\mathcal{B}_0 f := -v \cdot \nabla_x f - K f.$$

Lemma 1.3.1. *There exist $\gamma^* > 0$ and $a^* < 0$ such that for any $1 \leq p \leq \infty$, $m = e^{\gamma(x)}$, $\gamma \in [0, \gamma^*)$, there holds*

$$\|S_{\mathcal{B}_0}(t)\|_{L^p(m) \rightarrow L^p(m)} \lesssim e^{at}, \quad \forall a > a^*.$$

Proof of Lemma 1.3.1. We consider a solution $f = S_{\mathcal{B}_0}(t)f_0$ to the evolution equation associated to \mathcal{B}_0 and we compute

$$\begin{aligned} \frac{d}{dt} \int |f|^p m^p &= \int |f|^p p (v \cdot \nabla_x m - K m) m^{p-1} \\ &\leq p (V_0 \gamma + \chi - 1) \int |f|^p m^p, \end{aligned}$$

from which we easily conclude thanks to the Grönwall's lemma. \square

For any $\varphi \in L_{xv}^\infty$, we define $A = A_\varphi : L_x^q L_v^1 \rightarrow L_x^q$ by

$$A f = A_\varphi f(x) := \int_{\mathcal{V}} \varphi(x, v') f(x, v') dv'.$$

Given some Banach spaces X_i , $i \in \{1, 2, 3\}$ and two functions $u \in L^1(\mathbb{R}_+; \mathcal{B}(X_2, X_3))$, $v \in L^1(\mathbb{R}_+; \mathcal{B}(X_1, X_2))$, we define the convolution function $u * v \in L^1(\mathbb{R}_+; \mathcal{B}(X_1, X_3))$ by

$$(u * v)(t) := \int_0^t u(t-s) v(s) ds.$$

We also define $u^{(*n)}$ by $u^{(*1)} = u$ and $u^{(*n)} = u^{(*n-1)} * u$ for $n \geq 2$.

Lemma 1.3.2. *There exists $\gamma^* > 0$ and for any $\gamma \in [0, \gamma^*)$ there exists $a^* < 0$ such that for $m = e^{\gamma\langle x \rangle}$ and for any $\varphi \in L_{xv}^\infty$, there holds*

$$(1.3.1) \quad \|A_\varphi S_{\mathcal{B}_0}(t)\|_{L_x^1 L_v^\infty(m) \rightarrow L_{xv}^\infty(m)} \lesssim t^{-d} e^{at}, \quad \forall t > 0, \forall a > a^*.$$

As a consequence, there exists $n \in \mathbb{N}^*$ ($n = d + 2$ is suitable) such that

$$(1.3.2) \quad \|(A_\varphi S_{\mathcal{B}_0})^{(*n)}(t)\|_{L_{xv}^1(m) \rightarrow L_{xv}^\infty(m)} \lesssim e^{at}, \quad \forall t \geq 0, \forall a > a^*.$$

Proof of Lemma 1.3.2. We split the proof into two steps.

Step 1. We adapt the classical dispersion result of Bardos and Degond [6]. We denote $f(t) := S_{\mathcal{B}_0}(t)f_0$ the solution to the damped transport equation

$$\partial_t f + v \cdot \nabla_x f + Kf = 0, \quad f(0) = f_0.$$

The characteristics method gives the representation formula

$$(S_{\mathcal{B}_0}(t)f_0)(x, v) = f_0(x - vt, v) e^{-\int_0^t K(x - vs, v) ds}.$$

We then have

$$\begin{aligned} \rho(t, x) &:= (AS_{\mathcal{B}_0}(t)f_0)(x) \\ &= \int_{\mathcal{V}} \varphi(x, v_*) f_0(x - v_* t, v_*) e^{-\int_0^t K(x - v_* s, v_*) ds} dv_*. \end{aligned}$$

Using that $K(x, v) \geq 1 - \chi$ and $\langle x \rangle \leq \langle x - v_* t \rangle + |v_*|t + 1$, we deduce

$$\begin{aligned} |\rho(t, x)| &\leq e^{-(1-\chi)t} \int_{\mathcal{V}} |\varphi(x, v_*)| \left(\sup_{w \in \mathcal{V}} |f_0|(x - v_* t, w) \right) e^{\gamma\langle x - v_* t \rangle} dv_* e^{\gamma(V_0 t + 1 - \langle x \rangle)} \\ &\leq e^{(\chi + \gamma V_0 - 1)t + \gamma} \|\varphi\|_{L^\infty} \int_{\mathbb{R}^d} \left(\sup_{w \in \mathcal{V}} |f_0|(y, w) \right) e^{\gamma\langle y \rangle} \frac{dy}{t^d} e^{-\gamma\langle x \rangle}. \end{aligned}$$

Defining $\gamma^* := (1 - \chi)/V_0$ and for any $\gamma \in [0, \gamma^*)$ defining $a^* := \chi + \gamma V_0 - 1 < 0$, we conclude with

$$\|AS_{\mathcal{B}_0}(t)f_0\|_{L_{xv}^\infty(m)} \lesssim \frac{e^{a^* t}}{t^d} \|f_0\|_{L_x^1 L_v^\infty(m)},$$

which in particular implies (1.3.1).

Step 2. From Lemma 1.3.1, for $r = 1$ and $r = \infty$, we clearly have

$$\|AS_{\mathcal{B}_0}(t)\|_{L_x^r L_v^\infty(m) \rightarrow L_x^r L_v^\infty(m)} \lesssim e^{at}.$$

Gathering that estimate with (1.3.1), we may repeat the proof of [48, Proposition 2.2] (see also [47, Lemma 2.4] or [44]) and we get

$$(1.3.3) \quad \|(AS_{\mathcal{B}_0})^{*(d+1)}(t)\|_{L_x^1 L_v^\infty(m) \rightarrow L_x^\infty L_v^\infty(m)} \lesssim e^{at}.$$

Observing that

$$(1.3.4) \quad \|(AS_{\mathcal{B}_0})(t)\|_{L_x^1 L_v^1(m) \rightarrow L_x^1 L_v^\infty(m)} \lesssim e^{at},$$

thanks to Lemma 1.3.1 and $A : L_{xv}^1(m) \rightarrow L_x^1 L_v^\infty(m)$, we conclude to (1.3.2) by taking $n = d + 2$. \square

1.3.2 The operator \mathcal{B}_1 and the associated semigroup $S_{\mathcal{B}_1}$.

We define \mathcal{B}_1 by

$$(1.3.5) \quad \mathcal{B}_1 f := -v \cdot \nabla_x f - K f + (1 - \phi_R) \int_{\mathcal{V}} K' f' \, dv',$$

where we have defined the truncation functions $\phi_R(x) := \phi(x/R)$ for a given $\phi \in \mathcal{D}(\mathbb{R}^d)$ which is radially symmetric and satisfies $\mathbf{1}_{B(0,1)} \leq \phi \leq \mathbf{1}_{B(0,2)}$.

Lemma 1.3.3. *There exist $\gamma^* > 0$, $a^* < 0$ and $C \geq 1$ such that for R large enough and $m = e^{\gamma \langle x \rangle}$, $\gamma \in (0, \gamma^*)$, the semigroup $S_{\mathcal{B}_1}$ satisfies the following growth estimate*

$$\|S_{\mathcal{B}_1}(t)\|_{L^1(m) \rightarrow L^1(m)} \leq C e^{a^* t}, \quad \forall t \geq 0.$$

Proof of Lemma 1.3.3. We observe that the dual operator \mathcal{B}_1^* writes

$$\mathcal{B}_1^* \varphi = \mathcal{L}^* \varphi - \phi_R K \varphi.$$

Defining the modified weight function \tilde{m} as in (1.2.3) and using the inequalities (1.2.4) and (1.2.5), we get

$$\begin{aligned} \mathcal{B}_1^* \tilde{m} &\leq \left(\frac{C}{\langle x \rangle} - 2\alpha \right) e^{-\gamma \langle x \rangle} - \phi_R K \tilde{m} \\ &\leq \left(\frac{C}{\langle x \rangle} - 2\alpha \right) e^{-\gamma \langle x \rangle} - \mathbf{1}_{B_R(0)} (1 - \chi) (1 - \delta) e^{-\gamma \langle x \rangle} \\ &\leq -\alpha \tilde{m}, \end{aligned}$$

for $\gamma > 0$ small enough and $R > 1$ large enough, because $C = \mathcal{O}(\gamma)$ and $\delta = \mathcal{O}(\gamma)$. We then have proved that \mathcal{B}_1 is dissipative in $L^1(\tilde{m})$ and we immediately conclude. \square

Lemma 1.3.4. *For the same constants $\gamma^* > 0$ and $a^* < 0$ as defined in Lemma 1.3.3, for any $1 \leq p \leq \infty$ and $m = e^{\gamma \langle x \rangle}$, $\gamma \in (0, \gamma^*)$, the semigroup $S_{\mathcal{B}_1}$ satisfies the growth estimate*

$$(1.3.6) \quad \|S_{\mathcal{B}_1}(t)\|_{L^1(m) \cap L^p(m) \rightarrow L^1(m) \cap L^p(m)} \leq C e^{a t}, \quad \forall t \geq 0, \quad \forall a > a^*.$$

Proof of Lemma 1.3.4. We write

$$\mathcal{B}_1 = \mathcal{B}_0 + \mathcal{A}_0^c,$$

with $\mathcal{A}_0^c = A_\psi$, $\psi := (1 - \phi_R)K(x, v)$, and then the iterated Duhamel formula

$$S_{\mathcal{B}_1} = (S_{\mathcal{B}_0} + \dots + S_{\mathcal{B}_0} * (\mathcal{A}_0^c S_{\mathcal{B}_0})^{(*n)}) + S_{\mathcal{B}_0} * (\mathcal{A}_0^c S_{\mathcal{B}_0})^{(*n)} * \mathcal{A}_0^c S_{\mathcal{B}_1} =: U_1 + U_2,$$

with $n = d + 2$. From Lemma 1.3.1, (1.3.2) in Lemma 1.3.2 and Lemma 1.3.3, we easily deduce

$$\|U_2\|_{L^1 \rightarrow L^\infty} \leq \|S_{\mathcal{B}_0}\|_{L^\infty \rightarrow L^\infty} * \|(\mathcal{A}_0^c S_{\mathcal{B}_0})^{(*n)}\|_{L^1 \rightarrow L^\infty} * \|\mathcal{A}_0^c S_{\mathcal{B}_1}\|_{L^1 \rightarrow L^1} \lesssim e^{a t},$$

for any $a > a^*$, where we have removed the weight dependence to shorten notations. We have similarly the same decay estimate on the remainder term U_1 in $\mathcal{B}(X)$ by just using Lemma 1.3.1. We deduce that (1.3.6) holds for $p = \infty$. We conclude that (1.3.6) holds for any $1 \leq p \leq \infty$ by interpolating that first estimate in L^∞ together with the estimate established in Lemma 1.3.3. \square

Lemma 1.3.5. *For the same constants $\gamma^* > 0$ and $a^* < 0$ as defined in Lemma 1.3.3, for any $\varphi \in L_{xv}^\infty$ and $m = e^{\gamma\langle x \rangle}$, $\gamma \in [0, \gamma^*)$, there holds*

$$(1.3.7) \quad \|A_\varphi S_{\mathcal{B}_1}(t)\|_{L_x^1 L_v^\infty(m) \rightarrow L_{xv}^\infty(m)} \lesssim t^{-d} e^{at}, \quad \forall t > 0, \quad \forall a > a^*.$$

As a consequence, for $n \in \mathbb{N}^*$ large enough ($n = d + 2$ is suitable), there holds

$$(1.3.8) \quad \|(A_\varphi S_{\mathcal{B}_1})^{(*n)}(t)\|_{L^1(m) \rightarrow L^\infty(m)} \lesssim e^{at}, \quad \forall t \geq 0, \quad \forall a > a^*.$$

Proof of Lemma 1.3.5. With the notation of the proof of Lemma 1.3.4, we write

$$A_\varphi S_{\mathcal{B}_1} = A_\varphi S_{\mathcal{B}_0} + A_\varphi S_{\mathcal{B}_0} * \mathcal{A}_0^c S_{\mathcal{B}_1},$$

and we immediately conclude that (1.3.7) holds putting together (1.3.1) and the fact that $\mathcal{A}_0^c S_{\mathcal{B}_1}$ has the appropriate decay rate in $\mathcal{B}(L^1(m); L_x^1 L_v^\infty(m))$ thanks to Lemma 1.3.3. Introducing the notations $X_1 := L_x^1 L_v^\infty(m)$ and $X_\infty := L_x^1 L_v^\infty(m) \cap L_{xv}^\infty(m)$, we then easily see from (1.3.6) and (1.3.7) that

$$\|A_\varphi S_{\mathcal{B}_1}\|_{X_p \rightarrow X_q} \lesssim \Theta_{p,q}(t) e^{at}, \quad \forall a > a^*,$$

for $(p, q) = (1, 1), (1, \infty), (\infty, \infty)$, with $\Theta_{1,1} = \Theta_{\infty, \infty} = 1$ and $\Theta_{1, \infty} = t^{-d}$. Repeating again the proof of [48, Proposition 2.2], we deduce

$$\|(A_\varphi S_{\mathcal{B}_1})^{*(n-1)}(t)\|_{X_1 \rightarrow X_\infty} \lesssim e^{at}.$$

We conclude by using that $A_\varphi S_{\mathcal{B}_1}$ has the appropriate decay rate in $\mathcal{B}(L^1(m); L_x^1 L_v^\infty(m))$ thanks to Lemma 1.3.3. \square

1.3.3 Existence of a steady state

We establish now the existence of a steady state thanks to a fixed point argument. We fix an exponential weight $m := e^{\gamma\langle x \rangle}$, $\gamma \in (0, \gamma^*)$. We define the Banach space $X := L^1(m) \cap L^\infty(m)$ as well as

$$\forall f \in X, \quad \|f\| := \sup_{t \geq 0} \|S_{\mathcal{L}}(t)f\|_X.$$

Lemma 1.3.6. *The semigroup $S_{\mathcal{L}}$ is bounded in X . As a consequence, there exists at least one nonnegative, invariant by rotation and normalized stationary state $G \in X$ to the linear RT equation (1.1.1).*

Proof of Lemma 1.3.6. We split the proof into two steps.

Step 1. We split the operator \mathcal{L} as

$$\mathcal{L} = \mathcal{A}_1 + \mathcal{B}_1, \quad \mathcal{A}_1 := A_\psi,$$

with

$$\psi := \phi_R K(x, v).$$

With the same integer n as in Lemma 1.3.5, we write the iterated Duhamel formula

$$\begin{aligned} S_{\mathcal{L}} &= (S_{\mathcal{B}_1} + \dots + S_{\mathcal{B}_1} * (\mathcal{A}_1 S_{\mathcal{B}_1})^{(*n)}) + S_{\mathcal{B}_1} * (\mathcal{A}_1 S_{\mathcal{B}_1})^{(*n)} * \mathcal{A}_1 S_{\mathcal{L}} \\ &=: V_1 + V_2. \end{aligned}$$

For the first term and thanks to Lemma 1.3.4, for some constant $K_1 \geq 1$, we have

$$\|V_1(t)f_0\|_X \leq \sum_{\ell=0}^n \|S_{\mathcal{B}_1} * (\mathcal{A}_1 S_{\mathcal{B}_1})^{(*\ell)} f_0\|_X \leq K_1 \|f_0\|_X.$$

For the second term, for some constant $K_2 \geq 1$, we have

$$\|V_2(t)\|_{L^1(m) \rightarrow X} \leq \|S_{\mathcal{B}_1}\|_{X \rightarrow X} * \|(\mathcal{A}_1 S_{\mathcal{B}_1})^{(*n)}\|_{L^1(m) \rightarrow X} * \|\mathcal{A}_1 S_{\mathcal{L}}\|_{L^1(m) \rightarrow L^1(m)} \leq K_2,$$

using both (1.3.6) and (1.3.8). All together, we have proved

$$\|S_{\mathcal{L}}(t)f_0\|_X \lesssim \|f_0\|_X, \quad \forall t \geq 0.$$

As a consequence, the quantity $\| \cdot \|$ defines a norm on X which is equivalent to its usual norm $\| \cdot \|_X$.

Step 2. For a given $g_0 \in X$ which is also invariant by rotation and a probability measure, we define $C := \|g_0\|$ and next the set

$$\mathcal{C} := \left\{ f \in X; f \geq 0, \langle f \rangle = 1, f_R = f, \forall R \in SO(d), \|f\| \leq C \right\},$$

which is not empty (e.g. $g_0 \in \mathcal{C}$), convex and compact for the weak $*$ topology of X . Moreover, thanks to Lemma 1.2.1, the flow is continuous for the L^1 norm and preserves positivity and total mass. By construction, we see that for any $f_0 \in \mathcal{C}$ and $t \geq 0$, we have

$$\begin{aligned} \|S_{\mathcal{L}}(t)f_0\| &= \sup_{s \geq t} \|S_{\mathcal{L}}(s)f_0\|_{L^p(m)} \\ &\leq \sup_{s \geq 0} \|S_{\mathcal{L}}(s)f_0\|_{L^p(m)} = \|f_0\| \leq C. \end{aligned}$$

All together, the set \mathcal{C} is clearly invariant by the flow $S_{\mathcal{L}}$. Thanks to a standard variant of the Brouwer-Schauder-Tychonoff fixed point theorem (see for instance [24, Theorem 1.2]), we obtain the existence of an invariant element G for the linear RT flow which furthermore belongs to \mathcal{C} , in other words

$$\exists G \in \mathcal{C} \text{ such that } S_{\mathcal{L}}(t)G = G, \quad \forall t \geq 0.$$

As a consequence, we have $G \in D(\mathcal{L}) \setminus \{0\}$ and $\mathcal{L}G = 0$, so that G is a stationary state for the linear RT equation which fulfills all the properties listed in the statement of Theorem 1.1.1. \square

1.3.4 Uniqueness of the stationary state

In this section we prove a weak and a strong maximal principle on the operator $-\mathcal{L}$. The uniqueness of the normalized and positive steady state then follows using classical arguments. We skip the proof of that last one and we refer for instance to [50, Step 4, proof of Theorem 5.3] for details.

Lemma 1.3.7. *The operator \mathcal{L} satisfies the following Kato's inequality*

$$(1.3.9) \quad (\text{sign} f) \mathcal{L}f := \frac{1}{2|f|} (f \mathcal{L} \bar{f} + \bar{f} \mathcal{L} f) \leq \mathcal{L}|f|,$$

for any complex valued function $f \in X + iX$. As a consequence, the operator $-\mathcal{L}$ satisfies the weak maximum principle.

Proof of Lemma 1.3.7. For $f \in X + iX$, $f \neq 0$, we just compute

$$\begin{aligned} \frac{1}{2|f|}(f\mathcal{L}\bar{f} + \bar{f}\mathcal{L}f) &= -v \cdot \nabla|f| - K|f| + \frac{1}{2|f|} \left(f \int_{\mathcal{V}} K' \bar{f}' + \bar{f} \int_{\mathcal{V}} K' f' \right) \\ &\leq -v \cdot \nabla|f| - K|f| + \int_{\mathcal{V}} K' |f'| = \mathcal{L}|f|. \end{aligned}$$

The weak maximum principle follows from Kato's inequality (1.3.9) and the mass conservation $\mathcal{L}^*1 = 0$ thanks to classical characterizations of positive semigroups in [4, 69]. \square

Lemma 1.3.8. *The operator $-\mathcal{L}$ satisfies the following version of the strong maximum principle: for any given $0 \leq g \in L^2(m)$ and $\lambda \in \mathbb{R}$, there holds*

$$g \in D(\mathcal{L}) \setminus \{0\} \text{ and } (-\mathcal{L} + \lambda)g \geq 0 \text{ imply } g > 0$$

Proof of Lemma 1.3.8. We consider g as in the above statement and we prove that it is a positive function in several steps.

Step 1. Defining $M := 1 + \chi + \lambda \in \mathbb{R}$ and $m := 1 - \chi > 0$, we see that

$$\begin{aligned} v \cdot \nabla_x g + M g &\geq v \cdot \nabla_x g + K g + \lambda g \\ &\geq \int_{\mathcal{V}} K' g' dv' \geq m \varrho, \quad \varrho := \int_{\mathcal{V}} g' dv'. \end{aligned}$$

Integrating the above inequality along the free transport characteristics, we get

$$(1.3.10) \quad g(x, v) \geq m \int_0^t \varrho(x - v s) e^{-Ms} ds + g(x - vt, v) e^{-Mt}, \quad \forall t \geq 0.$$

Step 2. In particular, because the second term at the RHS is nonnegative, we may keep only the contribution of the first term and we get

$$\begin{aligned} \varrho(x) &\geq m \int_{1/2}^1 \int_{\mathcal{V}} \varrho(x - v s) e^{-Ms} ds dv \\ &\geq \kappa \int_{\mathcal{V}/2} \varrho(x + w) dw, \quad \kappa > 0. \end{aligned}$$

Since $g \geq 0$ and $g \not\equiv 0$, we also have $\varrho \geq 0$ and $\varrho \not\equiv 0$, and there exists $x_0 \in \mathbb{R}^d$ and $r > 0$, small enough, such that $\langle \varrho \mathbf{1}_{B(x_0, r)} \rangle = \alpha > 0$ and $B(0, 2r) \subset \mathcal{V}$, or in other words $B(x_0, r) \subset x + \mathcal{V}/2$ for any $x \in B(x_0, r)$. As a consequence, we have

$$(1.3.11) \quad \varrho \geq \alpha_0 \mathbf{1}_{B(x_0, r)}, \quad \alpha_0 := \kappa \alpha.$$

Step 3. Observing that for any $x \in \mathbb{R}^d$, there exists a small ball $B \subset \mathcal{V}$ and some times $\tau_1 > \tau_0 > 0$ such that $x - sv \in B(x_0, r/2)$ for any $v \in B$ and $s \in (\tau_0, \tau_1)$, we may argue as above and we get

$$\begin{aligned} \varrho(x) &\geq m \int_{\tau_0}^{\tau_1} \int_B \varrho(x - v s) e^{-Ms} ds dv \\ &\geq \alpha_x := m|B|(\tau_1 - \tau_0) e^{-M\tau_1} \alpha_0 > 0. \end{aligned}$$

Finally, using (1.3.10) again, we deduce

$$g(x, v) \geq m \int_0^1 \varrho(x - v s) e^{-M s} ds > 0,$$

which concludes the proof. \square

1.4 Asymptotic stability of the stationary state

1.4.1 A new splitting

In all this section, excepted in paragraph 1.4.5, we fix an exponential weight function $m = e^{\gamma(x)}$, with $\gamma \in (0, \gamma^*)$ and $\gamma^* > 0$ defined in Lemma 1.3.3, and we define the Banach space $X := L^1(m) \cap L^2(m)$. We also introduce the second splitting

$$(1.4.1) \quad \mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A}f := \int_{\mathcal{V}} K'_{R, \delta_i} f' dv',$$

where

$$K_{R, \delta_i}(x, v) = \phi_{\delta_2, R}(x) \psi_{\delta_1}(v) K_{\delta_3}(x, v),$$

with

$$K_{\delta_3}(x, v) = 1 + \chi \zeta_{\delta_3}(x \cdot v),$$

for some real numbers $R > 1$, $\delta_1, \delta_2, \delta_3 \in (0, 1)$ to be fixed, and where we have defined the truncation functions $\phi_\lambda(z) := \phi(z/\lambda)$, $\phi \in \mathcal{D}(\mathbb{R}^d)$ radially symmetric, $\mathbf{1}_{B(0,1)} \leq \phi \leq \mathbf{1}_{B(0,2)}$, and then

$$\phi_{\delta, R}(x) := \phi_R(x) - \phi_\delta(x), \quad \psi_\delta(v) := 1 - \phi_\delta(V_0 - |v|) - \phi_\delta(v),$$

as well as a regularized sign function $\zeta_\delta \in C^\infty(\mathbb{R})$ which is odd, increasing and satisfies $\zeta_\delta(s) = 1$ for any $s \geq \delta$. It is worth mentioning that the kernel $K_{R, \delta_i} \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with support included in $B(0, 2R) \cap B(0, V'_0)$, $V'_0 \in (V_0 - \delta_1, V_0)$, what will be a cornerstone used property during the proofs of Proposition 1.4.1 and Proposition 1.4.2. That smoothness property contracts with the non-smoothness property of the kernel $\phi_R K$ associated to the operator \mathcal{A}_1 and that is the main reason for introducing that new splitting.

We establish that $S_{\mathcal{B}}$ and $\mathcal{A}S_{\mathcal{B}}$ enjoy suitable decay estimate (Section 1.4.2) and regularity estimate (Section 1.4.3) from which we deduce the asymptotic stability in X thanks to a semigroup version of the Krein-Rutman theorem (Section 1.4.4) and next the asymptotic stability in any exponential and polynomial weighted L^1 space by using an extension argument (Section 1.4.5).

1.4.2 Decay estimates for the semigroup $S_{\mathcal{B}}$

Proposition 1.4.1. *For the same constant $a^* < 0$ as defined in Lemma 1.3.3, there holds*

$$(1.4.2) \quad \|S_{\mathcal{B}}(t)\|_{X \rightarrow X} \lesssim e^{at}, \quad \forall t \geq 0, \quad \forall a > a^*.$$

Proof of Proposition 1.4.1. We split the proof into five steps.

Step 1. Norms and splitting. Inspired by [31, Proposition 5.15] and the moment trick introduced in [42], we define the three norms $\|\cdot\|_X$, $\|\!\| \cdot \|\!\|$ and $N(\cdot)$ in the following way

$$\begin{aligned} \|f\|_X^2 &:= \|f\|_{L^1(m)}^2 + \|f\|_{L^2(m)}^2, \\ \|\!\| f \|\!\|^2 &:= \eta_2 \|f\|_X^2 + \int_0^\infty \|S_{\mathcal{B}_1}(\tau)f\|_X^2 d\tau, \\ N(f)^2 &:= \eta_1 \|f\|_{L^2(\mu^{1/2})}^2 + \|\!\| f \|\!\|^2, \end{aligned}$$

for some constants $\eta_1, \eta_2 \in (0, 1)$ to be fixed and where μ is the weight function

$$\mu := \left(1 - \frac{x}{|x|^{1/2}} \cdot \frac{v}{|v|}\right) \phi_{1/2}(x),$$

so that $0 \leq \mu \leq 2\phi_1$. Thanks to the decay estimate of Lemma 1.3.4, one easily sees that these three norms are equivalent. In particular, there exists $a^* := a^*(m, \eta_i) < 0$ such that

$$(1.4.3) \quad \forall f \in X, \quad \|f\|_X \leq 4a^* N(f).$$

We fix $f_0 \in X$ and we define $f(t) = f_{\mathcal{B}}(t) = S_{\mathcal{B}}(t)f_0$ the associated trajectory along the action of the semigroup $S_{\mathcal{B}}$. Our goal is to establish (1.4.2) by proving that \mathcal{B} is suitably dissipative for the norm $N(\cdot)$. In order to do so, we write

$$(1.4.4) \quad T := \frac{1}{2} \frac{d}{dt} N(f_{\mathcal{B}}(t))^2 = \eta_1 T_1 + \eta_2 T_2 + T_3,$$

where T_i are the contributions of the terms involved in the definition of the norm $N(\cdot)$ that we compute separately. For latter references, we introduce the splitting of \mathcal{B} as

$$\mathcal{B} := \mathcal{B}_1 + \mathcal{A}_1^c + \mathcal{A}_2^c + \mathcal{A}_3^c,$$

where \mathcal{B}_1 is defined in section 1.3.2 with $R \geq 1$ large enough so that Lemma 1.3.3 and Lemma 1.3.4 hold true, and we have set

$$\begin{aligned} \mathcal{A}_1^c f &= \phi_R(x) \int_{\mathcal{V}} K' f' \psi_{\delta_1}^c(v') dv' \\ \mathcal{A}_2^c f &= \phi_{\delta_2}(x) \int_{\mathcal{V}} K' f' \psi_{\delta_1}(v') dv' \\ \mathcal{A}_3^c f &= \phi_{\delta_2, R}(x) \int_{\mathcal{V}} K_{\delta_3}^c(x \cdot v') f' \psi_{\delta_1}(v') dv', \end{aligned}$$

with $\psi_{\delta_1}^c := 1 - \psi_{\delta_1}$, $\phi_R^c := 1 - \phi_R$, $K_{\delta}^c = \chi \zeta_{\delta}^c$, $\zeta_{\delta}^c = \zeta - \zeta_{\delta}$. We shall also denote

$$\begin{aligned} \mathcal{A}_{123}^c &:= \mathcal{A}_1^c + \mathcal{A}_1^c + \mathcal{A}_2^c, \\ \mathcal{A}_{0123}^c &:= \mathcal{A}_0^c + \mathcal{A}_1^c + \mathcal{A}_1^c + \mathcal{A}_2^c, \end{aligned}$$

where we recall that \mathcal{A}_0^c has been defined during the proof of Lemma 1.3.4.

Step 2. Contribution of the term T_1 . We prove that

$$(1.4.5) \quad T_1 := \frac{1}{2} \frac{d}{dt} \|f_{\mathcal{B}}(t)\|_{L^2(\mu^{1/2})}^2 \leq -\frac{1}{4} \|f_{\mathcal{B}}(t)\|_{L^2(\nu^{1/2})}^2 + C_1 \|f_{\mathcal{B}}(t)\|_X^2,$$

for some positive constant C_1 and where ν is the weight function defined by

$$\nu(x, v) := \frac{|v|}{|x|^{1/2}} \phi_{1/2}(x).$$

In order to prove (1.4.5), we first observe that

$$T_1 = (\mathcal{B} f_{\mathcal{B}}(t), f_{\mathcal{B}}(t))_{L^2(\mu^{1/2})},$$

and next we compute the RHS by splitting it in several pieces. For any $f \in X$, we have

$$\begin{aligned} (\mathcal{B}f, f)_{L^2(\mu^{1/2})} &= (-v \cdot \nabla_x f, f)_{L^2(\mu^{1/2})} + (\mathcal{A}_{0123}^c f - Kf, f)_{L^2(\mu^{1/2})} \\ &=: T_{1,1} + T_{1,2}. \end{aligned}$$

We compute the first key term. Performing one integration by parts, we find

$$\begin{aligned} T_{1,1} &= \int (-v \cdot \nabla_x f) f \left(1 - \frac{x}{|x|^{1/2}} \cdot \frac{v}{|v|}\right) \phi_{1/2}(x) \, dx dv \\ &= \frac{1}{2} \int f^2 v \cdot \nabla_x \left[\left(1 - \frac{x}{|x|^{1/2}} \cdot \frac{v}{|v|}\right) \phi_{1/2}(x) \right] \, dx dv \\ &= \frac{1}{2} \int f^2 \left[\frac{1}{2} \left(\frac{x}{|x|} \cdot \frac{v}{|v|} \right)^2 - 1 \right] \frac{|v|}{|x|^{1/2}} \phi_{1/2} \, dx dv \\ &\quad + \frac{1}{2} \int f^2 \left(1 - \frac{x}{|x|^{1/2}} \cdot \frac{v}{|v|}\right) (v \cdot \nabla_x \phi_{1/2}) \, dx dv \\ &\leq -\frac{1}{4} \int f^2 \frac{|v|}{|x|^{1/2}} \phi_{1/2} \, dx dv + \tilde{T}_{1,1} \end{aligned}$$

with $\tilde{T}_{1,1} \lesssim \|f\|_{L^2}^2$. For the remainder term, we also easily have $|T_{1,2}| \lesssim \|f\|_{L^2}^2$ from what (1.4.5) immediately follows.

Step 3. Contribution of the term T_2 . We prove that

$$(1.4.6) \quad T_2 := \frac{1}{2} \frac{d}{dt} \|f_{\mathcal{B}}\|_X^2 \leq C_2 \|f_{\mathcal{B}}\|_X^2,$$

for some positive constant C_2 . We have

$$\begin{aligned} T_2 &= \frac{1}{2} \frac{d}{dt} \|f_{\mathcal{B}}\|_{L^2(m)}^2 + \frac{1}{2} \frac{d}{dt} \|f_{\mathcal{B}}\|_{L^1(m)}^2 \\ &= (\mathcal{B} f_{\mathcal{B}}, f_{\mathcal{B}})_{L^2(m)} + \langle \mathcal{B} f_{\mathcal{B}}, \text{sign} f_{\mathcal{B}} \rangle_{L^1(m), L^\infty} \|f_{\mathcal{B}}\|_{L^1(m)} \\ &=: T_{2,1} + T_{2,2}, \end{aligned}$$

with

$$\begin{aligned} T_{2,1} &:= (\mathcal{B}_0 f_{\mathcal{B}}, f_{\mathcal{B}})_{L^2(m)} + \langle \mathcal{B}_0 f_{\mathcal{B}}, \text{sign} f_{\mathcal{B}} \rangle_{L^1(m), L^\infty} \|f_{\mathcal{B}}\|_{L^1(m)} \\ &\leq 0, \end{aligned}$$

because \mathcal{B}_0 is dissipative in both $L^1(m)$ and $L^2(m)$ from Lemma 1.3.1, and with

$$\begin{aligned} T_{2,2} &:= (\mathcal{A}_{0123}^c f_{\mathcal{B}}, f_{\mathcal{B}})_{L^2(m)} + \langle \mathcal{A}_{0123}^c f_{\mathcal{B}}, \text{sign} f_{\mathcal{B}} \rangle_{L^1(m), L^\infty} \|f_{\mathcal{B}}\|_{L^1(m)}^2 \\ &\leq \|\mathcal{A}_{0123}^c f_{\mathcal{B}}\|_{L^2(m)} \|f_{\mathcal{B}}\|_{L^2(m)} + \|\mathcal{A}_{0123}^c f_{\mathcal{B}}\|_{L^1(m)} \|f_{\mathcal{B}}\|_{L^1(m)}^2 \\ &\lesssim \|f_{\mathcal{B}}\|_X^2, \end{aligned}$$

because $|\mathcal{A}_i^c f| \leq A_{1+\chi}|f|$ for any $i \in \{0, \dots, 3\}$ with $A_{1+\chi} \in \mathcal{B}(L^p(m), L^p(m))$ for any $p \in \{1, 2\}$.

Step 4. Contribution of the term T_3 . We prove that for any $\eta_1 \in (0, 1)$, we can find $\delta_1, \delta_2, \delta_3 \in (0, 1)$ and $R \geq 1$ such that the associated operator \mathcal{B} satisfies

$$(1.4.7) \quad \begin{aligned} T_3 &:= \frac{1}{2} \frac{d}{dt} \int_0^\infty \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_X^2 d\tau \\ &\leq -\frac{3}{8} \|f_{\mathcal{B}}(t)\|_X^2 + \frac{\eta_1}{4} \|f_{\mathcal{B}}(t)\|_{L^2(\nu^{1/2})}^2. \end{aligned}$$

We split the term T_3 as

$$\begin{aligned} T_3 &= \frac{1}{2} \frac{d}{dt} \int_0^\infty \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_{L^1(m)}^2 d\tau + \frac{1}{2} \frac{d}{dt} \int_0^\infty \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_{L^2(m)}^2 d\tau \\ &=: T_{3,1} + T_{3,2}. \end{aligned}$$

For the first term, we compute

$$\begin{aligned} T_{3,1} &= \int_0^\infty \frac{1}{2} \frac{d}{dt} \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_{L^1(m)}^2 d\tau \\ &= \int_0^\infty \langle S_{\mathcal{B}_1}(\tau) \mathcal{B} f_{\mathcal{B}}(t), \text{sign}(S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)) \rangle_{L^1(m), L^\infty} \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_{L^1(m)} d\tau \\ &= \int_0^\infty \langle \mathcal{B}_1 S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t), \text{sign}(S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)) \rangle_{L^1(m), L^\infty} \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_{L^1(m)} d\tau \\ &\quad + \int_0^\infty \langle S_{\mathcal{B}_1}(\tau) \mathcal{A}_{123}^c f_{\mathcal{B}}(t), \text{sign}(S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)) \rangle_{L^1(m), L^\infty} \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_{L^1(m)} d\tau \\ &=: T_{3,1,1} + T_{3,1,2}. \end{aligned}$$

On the one hand, we observe that

$$T_{3,1,1} = \int_0^\infty \frac{1}{2} \frac{d}{d\tau} \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_{L^1(m)}^2 d\tau = -\frac{1}{2} \|f_{\mathcal{B}}(t)\|_{L^1(m)}^2,$$

where in the last line we have use that $S_{\mathcal{B}_1} f_0$ has the nice decay estimate (1.3.6) in the space $L^1(m)$ because $f_0 \in X$. On the other hand, using again the decay estimate (1.3.6), for any $\varepsilon > 0$, we have

$$\begin{aligned} T_{3,1,2} &= \int_0^\infty \|S_{\mathcal{B}_1}(\tau) \mathcal{A}_{123}^c f_{\mathcal{B}}(t)\|_{L^1(m)} \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_{L^1(m)} d\tau \\ &\leq \frac{1}{2\varepsilon} \int_0^\infty \|S_{\mathcal{B}_1}(\tau) \mathcal{A}_{123}^c f_{\mathcal{B}}(t)\|_{L^1(m)}^2 d\tau \\ &\quad + \frac{\varepsilon}{2} \int_0^\infty \|S_{\mathcal{B}_1}(\tau) f_{\mathcal{B}}(t)\|_{L^1(m)}^2 d\tau \\ &\lesssim \frac{1}{\varepsilon} \|\mathcal{A}_{123}^c f_{\mathcal{B}}(t)\|_X^2 + \varepsilon \|f_{\mathcal{B}}(t)\|_X^2. \end{aligned}$$

We may treat the second term in a similar way, using in particular the fact that $S_{\mathcal{B}_1} f_0$ has the nice decay estimate (1.3.6) in the space $L^2(m)$ because $f_0 \in X$. Gathering the two resulting estimates and taking $\varepsilon > 0$ small enough, we get

$$(1.4.8) \quad \begin{aligned} T_3 &\leq -\frac{7}{16} \|f_{\mathcal{B}}(t)\|_X^2 + \widetilde{T}_3, \\ \widetilde{T}_3 &\lesssim \|\mathcal{A}_{123}^c f_{\mathcal{B}}(t)\|_X^2. \end{aligned}$$

In order to conclude, we compute the contributions $\|\mathcal{A}_i^c f_{\mathcal{B}}(t)\|_X^2$ for any $i \in \{1, 2, 3\}$.

On the one hand, using the Cauchy-Schwarz inequality, for any $p \in \{1, 2\}$, $R > 2$ and $\delta_1 \in (0, 1)$, we have

$$\begin{aligned} \|\mathcal{A}_1^c f\|_{L^p}^p(m) &= \int \left| \phi_R(x) \int_{\mathcal{V}} K' f' \psi_{\delta_1}^c(v') dv' \right|^p m^p dx dv \\ &\leq 2^p m (2R)^p \int \left(\int_{\mathcal{V}} |f'|^2 dv' \right)^{p/2} \left(\int_{\mathcal{V}} (\mathbf{1}_{|v'| \leq 2\delta_1} + \mathbf{1}_{V_0 - |v'| \leq 2\delta_1}) dv' \right)^{p/2} dx dv \\ &\lesssim m (2R)^p \delta_1^{p/2} \|f\|_{L^2}^{p/2}. \end{aligned}$$

Similarly, when furthermore $\delta_2 \in (0, 1/4)$, we have

$$\begin{aligned} \|\mathcal{A}_2^c f\|_{L^p}^p(m) &= \int \left| \phi_{\delta_2}(x) \int_{\mathcal{V}} K' f' \psi_{\delta_1}(v') dv' \right|^p m^p dx dv \\ &\leq 2^p m (2)^p \left(\int_{B(0,1) \times \mathcal{V}} \mathbf{1}_{|x| \leq 2\delta_2} |f'|^2 \mathbf{1}_{|v'| \geq \delta_1} dv' dx \right)^{p/2} \\ &\lesssim \left(\frac{\delta_2^{1/2}}{\delta_1} \right)^{p/2} \|f\|_{L^2(\nu^{1/2})}^p. \end{aligned}$$

Finally and similarly again, when furthermore $\delta_3 \in (0, 1/2)$, observing that

$$\begin{aligned} 0 \leq K_{\delta_3}^c(x, v') &= \chi(\zeta - \zeta_{\delta_3})(x \cdot v') \\ &\leq \chi \mathbf{1}_{|x \cdot v'| \leq \delta_3}, \end{aligned}$$

we have

$$\begin{aligned} \|\mathcal{A}_3^c f\|_{L^p}^p(m) &= \int \left| \phi_{\delta_2, R}(x) \int K_{\delta_3}^c(x \cdot v') f' \psi_{\delta_1}(v') dv' \right|^p m^p dx dv \\ &\leq m (2R)^p \chi^p \int \left(\int_{\mathcal{V}} |f'|^2 dv' \right)^{p/2} \left(\int_{\mathcal{V}} \mathbf{1}_{|x \cdot v'| \leq 2\delta_3} dv' \right)^{p/2} \mathbf{1}_{|x| \geq \delta_2} dx dv \\ &\lesssim m (2R)^p \left(\int f^2 dv dx \right)^{p/2} \left(\text{meas}\{v \in \mathcal{V}; |v_1| \leq \delta_3/\delta_2\} \right)^{p/2} \\ &\lesssim m (2R)^p \frac{\delta_3^{p/2}}{\delta_2^{p/2}} \|f\|_{L^2}^p. \end{aligned}$$

All these estimates together, we get

$$(1.4.9) \quad \|\mathcal{A}_{123}^c f_{\mathcal{B}}(t)\|_X^2 \lesssim m (2R)^p \delta_1 \|f\|_X^2 + \frac{\delta_2^{1/2}}{\delta_1} \|f\|_{L^2(\nu^{1/2})}^2 + m (2R)^p \frac{\delta_3}{\delta_2} \|f\|_X^2.$$

We thus obtain (1.4.7) by just gathering (1.4.8) and (1.4.9) and by choosing $\delta_1, \delta_2, \delta_3 > 0$ adequately.

Step 5. Conclusion. From estimates (1.4.3), (1.4.4), (1.4.5), (1.4.6) and (1.4.7), we have

$$\begin{aligned} T &\leq \eta_1 C_1 \|f_{\mathcal{B}}\|_X^2 + \eta_2 C_2 \|f_{\mathcal{B}}\|_X^2 - \frac{3}{8} \|f_{\mathcal{B}}\|_X^2 \\ &\leq -\frac{1}{4} \|f_{\mathcal{B}}\|_X^2 \leq a^* N(f_{\mathcal{B}})^2, \end{aligned}$$

by choosing $\eta_1, \eta_2 > 0$ small enough. We have proved that $\mathcal{B} - a^*$ is dissipative for the norm $N(\cdot)$ and thus (1.4.2) follows. \square

1.4.3 Some regularity associated to $\mathcal{AS}_{\mathcal{B}}$

In this section we show that the family of operators $\mathcal{AS}_{\mathcal{B}}$ satisfies a regularity and growth estimate that we express in terms of the abstract Sobolev space $X_{\mathcal{B}}^{1/2}$ defined as the usual 1/2 interpolated space between X and the domain

$$X_{\mathcal{B}}^1 = D(\mathcal{B}) := \{f \in X; \mathcal{B}f \in X\}$$

endowed with the graph norm.

Proposition 1.4.2. *For the same constant $a^* < 0$ as defined in Lemma 1.3.3, for any $a > a^*$ there exists $C_a \in (0, \infty)$ such that the family of operators $\mathcal{AS}_{\mathcal{B}}$ satisfies*

$$(1.4.10) \quad \int_0^\infty \|\mathcal{AS}_{\mathcal{B}}(t) f\|_Y^2 e^{-2at} dt \leq C_a \|f\|_X^2, \quad \forall f \in X,$$

with

$$Y := \{f \in L^2(\mathbb{R}^d \times \mathcal{V}); \text{supp } f \subset B(0, R) \times \mathcal{V}, f \in H^{1/2}\}.$$

The proof is mainly a consequence of Bouchut-Desvillettes' version [7, Theorem 2.1] (see also [16] for a related discrete version) of the classical averaging Lemma initiated in the famous articles of Golse et al. [30, 29]. We give in step 1 below a simpler, more accurate and more adapted version of [7, Theorem 2.1] for which we sketch the proof for the sake of completeness. During the proof, we will use the following classical trace result.

Lemma 1.4.3. *There exists a constant $C_d \in (0, \infty)$ such that for any $\phi \in H^{d/2}(\mathbb{R}^d)$ and any $u \in \mathbb{R}^d$, $|u| = 1$, the real function ϕ_u , defined by $\phi_u(s) := \phi(su)$ for any $s \in \mathbb{R}$, satisfies*

$$\|\phi_u\|_{L^2(\mathbb{R})} \leq C_d \|\phi\|_{H^{d/2}(\mathbb{R}^d)} = C_d \left(\int_{\mathbb{R}^d} |\check{F}\phi|^2(w) \langle w \rangle^d dw \right)^{1/2},$$

where \check{F} stands for the (inverse) Fourier transform operator.

Proof of Proposition 1.4.2. We split the proof into two steps.

Step 1. We consider the damped free transport equation

$$(1.4.11) \quad \partial_t f = \mathcal{T}f := -v \cdot \nabla_x f - f, \quad f|_{t=0} = f_0,$$

and we denote by $S_{\mathcal{T}}(t)$ the associated semigroup defined through the characteristics formula

$$(1.4.12) \quad [S_{\mathcal{T}}(t)f_0](x, v) := f(t, x, v) = f_0(x - vt, v) e^{-t}.$$

We claim that for any $\varphi \in L^2(\mathcal{V})$, there holds

$$(1.4.13) \quad \int_0^\infty \|A_\varphi S_{\mathcal{T}}(t)\varphi\|_{H_x^{1/2}}^2 e^{2t} dt \lesssim \|\varphi\|_{L^2(\mathcal{V})}.$$

For a given function h which depends on the x variable or on the (x, v) variable, we denote by \hat{h} its Fourier transform on the x variable and by $\mathcal{F}h$ its Fourier transform on both

variables x and v . We fix $f_0 \in L^2(\mathbb{R}^d \times \mathcal{V})$ and $\varphi \in L^\infty(\mathbb{R}^d)$, we denote by f the solution to the free transport equation (1.4.11) and by ρ the average function

$$\rho(t, x) := \int_{\mathbb{R}^d} f(t, x, v) \varphi(v) dv = [A_\varphi S_{\mathcal{T}}(t) f_0](x).$$

In Fourier variables, the free transport equation (1.4.11) writes

$$\partial_t \hat{f} + iv \cdot \xi \hat{f} - \hat{f} = 0, \quad \hat{f}|_{t=0} = \hat{f}_0,$$

so that

$$\hat{f}(t, \xi, v) = e^{iv \cdot \xi t - t} \hat{f}_0(\xi, v)$$

and

$$\hat{\rho}(t, \xi) = \int_{\mathbb{R}^d} e^{iv \cdot \xi t - t} \hat{f}_0(\xi, v) \varphi(v) dv = \mathcal{F}(f_0 \varphi)(\xi, t\xi) e^{-t}.$$

We deduce

$$\int_0^\infty |\hat{\rho}(t, \xi)|^2 e^{2t} dt \leq \int_{\mathbb{R}} |\mathcal{F}(f_0 \varphi)(\xi, t\xi)|^2 dt.$$

Performing one change of variable, introducing the notation $\sigma_\xi = \xi/|\xi|$ and using Lemma 1.4.3, we deduce

$$\begin{aligned} \int_{\mathbb{R}} |\mathcal{F}(f_0 \varphi)(\xi, t\xi)|^2 dt &= \frac{1}{|\xi|} \int_{\mathbb{R}} |\mathcal{F}(f_0 \varphi)(\xi, s\sigma_\xi)|^2 ds \\ &\lesssim \frac{1}{|\xi|} \int_{\mathbb{R}^d} |(\hat{f}_0 \varphi)(\xi, w)|^2 \langle w \rangle^d dw. \end{aligned}$$

Thanks to Plancherel identity, we then obtain

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} |\xi| |\hat{\rho}(t, \xi)|^2 d\xi e^{2t} dt &\lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(f_0 \varphi)(x, w)|^2 \langle w \rangle^d dw dx \\ &= \|\varphi\|_{L_{d/2}^2}^2 \|f_0\|_{L_{xv}^2}^2, \end{aligned}$$

which ends the proof (1.4.13).

Step 2. We show a similar estimate on $\mathcal{A}S_{\mathcal{T}}(t)$. Using that $K_{R, \delta_i} \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $\text{supp } K_{R, \delta_i} \subset B(0, 2R) \cap B(0, V'_0)$, $V'_0 \in (0, V_0)$, we may expand it as a Fourier series

$$K_{R, \delta_i}(x, v) = \sum_{k, \ell \in \mathbb{Z}^d} a_{k, \ell} e^{ix \cdot k} e^{iv \cdot \ell} \vartheta(v), \quad \forall (x, v) \in \mathcal{Q},$$

$\mathcal{Q} := \{x \in \mathbb{R}^d, v \in \mathbb{R}^d; |x_i| \leq 2R, |v_i| \leq V_0, \forall i = 1, \dots, d\}$, for a truncation function $\vartheta \in C^\infty(\mathbb{R}^d)$, $\text{supp } \vartheta \subset B(0, V_0)$, $\vartheta \equiv 1$ on $B(0, V'_0)$ and with fast decaying Fourier coefficients

$$|a_{k, \ell}| \lesssim \langle k \rangle^{-2d-4} \langle \ell \rangle^{-2d-2}.$$

From the definition of \mathcal{A} and denoting $f(t) = S_{\mathcal{T}}(t) f_0$ for some $f_0 \in L^2(\mathbb{R}^d \times \mathcal{V})$, we may then write

$$\begin{aligned} (\mathcal{A}S_{\mathcal{T}}(t) f_0)(x) &= \sum_{k, \ell \in \mathbb{Z}^d} a_{k, \ell} e^{ix \cdot k} \rho_\ell(t, x), \\ \rho_\ell(t, x) &:= \int_{\mathcal{V}} f(t, x, v) e^{iv \cdot \ell} \vartheta(v) dv. \end{aligned}$$

On the one hand, from Step 1, we have

$$(1.4.14) \quad \sup_{\ell \in \mathbb{Z}^d} \int_0^\infty \|\rho_\ell(t, \cdot)\|_{H^{1/2}}^2 e^{2t} dt \lesssim \|f_0\|_{L^2}^2.$$

On the other hand, we denote $e_k(x) := e^{ix \cdot k}$ and we define the mapping

$$U(\rho_\ell) := \sum_{k, \ell \in \mathbb{Z}^d} a_{k, \ell} e_k \rho_\ell.$$

From Cauchy-Schwarz inequality and Fubini Theorem and denoting

$$I = \int_0^\infty \|U(\rho_\ell)(t, \cdot)\|_{L^2(B_{2R})}^2 e^{2t} dt,$$

we have

$$\begin{aligned} I &\leq \int_0^\infty \int_{B_{2R}} \left(\sum_{k, \ell} |a_{k, \ell}|^2 \langle k \rangle^{d+1} \langle \ell \rangle^{d+1} \right) \left(\sum_{k, \ell} |\rho_\ell|^2 \langle k \rangle^{-d-1} \langle \ell \rangle^{-d-1} \right) e^{2t} dx dt \\ &\lesssim \sum_{k, \ell} \langle k \rangle^{-d-1} \langle \ell \rangle^{-d-1} \int_0^\infty \int_{B_{2R}} |\rho_\ell|^2 e^{2t} dx dt \\ &\lesssim \sup_{\ell \in \mathbb{Z}^d} \int_0^\infty \|\rho_\ell(t, \cdot)\|_{L^2(B_R)}^2 e^{2t} dt. \end{aligned}$$

Using furthermore that

$$\nabla_x U(\rho_\ell) = \sum_{k, \ell \in \mathbb{Z}^d} a_{k, \ell} (ik) e_k \rho_\ell + \sum_{k, \ell \in \mathbb{Z}^d} a_{k, \ell} e_k \nabla_x \rho_\ell,$$

we find similarly

$$\int_0^\infty \|\nabla_x U(\rho_\ell)(t, \cdot)\|_{L^2(B_{2R})}^2 e^{2t} dt \lesssim \sup_{\ell \in \mathbb{Z}^d} \int_0^\infty \|\rho_\ell(t, \cdot)\|_{H^1(B_R)}^2 e^{2t} dt.$$

Observing that

$$\{g \in L^2(\mathbb{R}^d \times \mathcal{V}); \text{supp } g \subset B(0, 2R) \times \mathcal{V}, \nabla_x g \in L^2\} \subset X_{\mathcal{B}}^1,$$

both estimates together and an interpolation argument yield

$$(1.4.15) \quad \int_0^\infty \|U(\rho_\ell)(t, \cdot)\|_{X_{\mathcal{B}}^{1/2}}^2 e^{2t} dt \lesssim \sup_{\ell \in \mathbb{Z}^d} \int_0^\infty \|\rho_\ell(t, \cdot)\|_{H^{1/2}(B_R)}^2 e^{2t} dt.$$

Gathering estimates (1.4.14) and (1.4.15), we have established

$$(1.4.16) \quad \int_0^\infty \|\mathcal{A}S_{\mathcal{T}}(t)f_0\|_{X_{\mathcal{B}}^{1/2}}^2 e^{2t} dt \lesssim \|f_0\|_{L^2}^2.$$

Step 3. Conclusion. We split \mathcal{B} as $\mathcal{B} = \mathcal{T} + \mathcal{C}$. The Duhamel formula writes

$$S_{\mathcal{B}} = S_{\mathcal{T}} + S_{\mathcal{T}} * \mathcal{C}S_{\mathcal{B}},$$

from which we deduce

$$\mathcal{A}S_{\mathcal{B}} = \mathcal{A}S_{\mathcal{T}} + \mathcal{A}S_{\mathcal{T}} * \mathcal{C}S_{\mathcal{B}}.$$

We just have to bound the last term in order to establish (1.4.10). For that purpose, we fix $f \in X$, $a > \alpha > a^*$, and we compute

$$\begin{aligned} \int_0^\infty \|\mathcal{A}S_{\mathcal{T}} * \mathcal{C}S_{\mathcal{B}}(t)f\|_Y^2 e^{-2at} dt &\leq \int_0^\infty \int_0^t \|\mathcal{A}S_{\mathcal{T}}(t-s)\mathcal{C}S_{\mathcal{B}}(s)f\|_Y^2 ds t e^{-2at} dt \\ &\leq \int_0^\infty \int_0^\infty \|\mathcal{A}S_{\mathcal{T}}(\tau)\mathcal{C}S_{\mathcal{B}}(s)f\|_Y^2 e^{-2\alpha\tau} d\tau e^{-2\alpha s} ds \\ &\lesssim \int_0^\infty \|\mathcal{C}S_{\mathcal{B}}(s)f\|_X^2 e^{-2\alpha s} ds \lesssim \|f\|_X^2, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality, estimates (1.4.16) and (1.4.2). \square

1.4.4 A first asymptotic stability estimate in X

In order to apply the semigroup version [50, Theorem 5.3] and [45] of the Krein-Rutman theorem, we list below some properties satisfied by the operators \mathcal{L} , \mathcal{A} and \mathcal{B} .

Fact 1. There exists $a^* < 0$ such that for any $a > a^*$ and $\ell \in \mathbb{N}$, the following growth estimate holds

$$t \mapsto \|(\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t)\|_{\mathcal{B}(X)} e^{-at} \in L^\infty(\mathbb{R}_+).$$

That is an immediate consequence of Proposition 1.4.1 and $\mathcal{A} \in \mathcal{B}(X)$.

Fact 2. We define the resolvent operator

$$R_{\mathcal{B}}(z) := (\mathcal{B} - z)^{-1} = - \int_0^\infty S_{\mathcal{B}}(t) e^{-zt} dt$$

for $z \in \Delta_a := \{\zeta \in \mathbb{C}; \Re \zeta > a\}$ and a large enough. For the same value $a^* < 0$ as above, there exists $Y \subset X_{\mathcal{L}}^s$, $s \in (0, 1/2)$, with compact embedding such that for any $a > a^*$ the following estimate holds

$$\|\mathcal{A}R_{\mathcal{B}}(z)\|_{\mathcal{B}(X,Y)} \leq C_a, \quad \forall z \in \Delta_a.$$

That is an immediate consequence of Proposition 1.4.2, which readily implies

$$\begin{aligned} \|\mathcal{A}R_{\mathcal{B}}(z)f\|_Y^2 &\leq \int_0^\infty \|\mathcal{A}S_{\mathcal{B}}(t)f\|_Y^2 \langle t \rangle^2 e^{-2at} dt \int_0^\infty \langle t \rangle^{-2} dt \\ &\lesssim \|f\|_X^2, \quad \forall f \in X, \end{aligned}$$

together with the fact that

$$\{f \in L^2(\mathbb{R}^d \times \mathcal{V}); \text{supp } f \subset B(0, R) \times \mathcal{V}, f \in H^1\} \subset X_{\mathcal{L}}^1$$

and an interpolation argument.

Fact 3. The semigroup $S_{\mathcal{L}}$ is positive, the operator $-\mathcal{L}$ satisfies the strong maximum principle as stated in Lemma 1.3.8 and \mathcal{L} satisfies Kato's inequality (1.3.9).

Fact 4. The mass conservation property writes $\mathcal{L}^*1 = 0$, so that $0 > a^*$ and 0 is an eigenvalue for the dual problem associated to a positive dual eigenfunction.

Gathering these above facts, we may then apply [50, Theorem 5.3], or more exactly we may repeat the proof of [50, Theorem 5.3] with minor and straightforward adaptations (we refer to [44] where these slight modifications are performed), in order to obtain that 0 is a (algebraically) simple eigenvalue, that there exists a spectral gap between this largest eigenvalue 0 and the remainder part of the spectrum and that a quantitative (partial but principal) spectral mapping theorem holds true. More precisely, we have the following asymptotic estimate: there exists $\alpha \in (a_*, 0)$ such that

$$(1.4.17) \quad \|S_{\mathcal{L}}(t)\Pi^\perp f_0\|_X \lesssim e^{at} \|f_0\|_X, \quad \forall f_0 \in X, \forall a > \alpha, \forall t \geq 0,$$

where we have set $\Pi^\perp := I - \Pi$ and $\Pi f_0 := \langle\langle f_0 \rangle\rangle G$.

1.4.5 Asymptotic stability estimate in weighted L^1 spaces

We first consider the exponential weight $m(x) := \exp(\gamma\langle x \rangle)$ with $\gamma \in (0, \gamma^*)$ and $\gamma^* > 0$ identified in Lemma 1.2.2. Iterating the Duhamel formula, we may write

$$S_{\mathcal{L}}\Pi^\perp = \Pi^\perp \{ \mathcal{S}_{\mathcal{B}_1} + \dots + \mathcal{S}_{\mathcal{B}_1} * (\mathcal{A}_1 \mathcal{S}_{\mathcal{B}_1})^{*(N-1)} \} + (\mathcal{S}_{\mathcal{L}}\Pi^\perp) * (\mathcal{A}_1 \mathcal{S}_{\mathcal{B}_1})^{*(N)},$$

with $N = d+2$. From Lemma 1.3.3 and (1.3.8) we have $\mathcal{S}_{\mathcal{B}_1} * (\mathcal{A}_1 \mathcal{S}_{\mathcal{B}_1})^{*(\ell)} : L^1(m) \rightarrow L^1(m)$ with rate e^{at} for any $\ell \in \{0, \dots, N-1\}$ and $(\mathcal{A}_1 \mathcal{S}_{\mathcal{B}_1})^{*(N)} : L^1(m) \rightarrow X$ with rate e^{at} . Using that $\mathcal{S}_{\mathcal{L}}\Pi^\perp : X \rightarrow X \subset L^1(m)$ with rate e^{at} from (1.4.17) and gathering all the preceding decay estimates, we conclude that (1.1.9) holds in $L^1(m)$.

We next consider the polynomial weight $m(x) := \langle x \rangle^k$ with $k \in (0, \infty)$. We begin with a decay estimate on the semigroup $\mathcal{S}_{\mathcal{B}_1}$.

Lemma 1.4.4. *For any $k > \ell > 0$, the semigroup $\mathcal{S}_{\mathcal{B}_1}$ satisfies the following growth estimate*

$$\|\mathcal{S}_{\mathcal{B}_1}\|_{L_k^1 \rightarrow L_\ell^1} \lesssim \langle t \rangle^{-(k-\ell)}, \quad \forall t \geq 0.$$

Proof of Lemma 1.4.4. Recalling that the dual operator \mathcal{L}^* has been defined in the proof of Lemma 1.2.2, for any $q > 0$, we compute

$$\mathcal{L}^* \langle \gamma x \rangle^q = q\gamma(v \cdot x) \langle \gamma x \rangle^{q-2},$$

$$\begin{aligned} \mathcal{L}^*(v \cdot x) \langle \gamma x \rangle^{q-2} &= v \cdot \nabla_x ((v \cdot x) \langle \gamma x \rangle^{q-2}) - q(v \cdot x) \langle \gamma x \rangle^{q-2} \\ &= \left(|v|^2 - (v \cdot x) - \chi |v \cdot x| \right) \langle \gamma x \rangle^{q-2} \\ &\quad + (q-2)\gamma(v \cdot x)^2 \langle \gamma x \rangle^{q-4}. \end{aligned}$$

We then compute

$$\begin{aligned} \mathcal{L}^* |v \cdot x| \langle \gamma x \rangle^{q-2} &= v \cdot \nabla_x (|v \cdot x| \langle \gamma x \rangle^{q-2}) \\ &\quad + (1 + \chi \zeta) \left(V_1 |x| \langle \gamma x \rangle^{q-2} - |v \cdot x| \langle \gamma x \rangle^{q-2} \right) \\ &= \left(|v|^2 \frac{v \cdot x}{|v \cdot x|} + (1 + \chi \zeta) V_1 |x| - (1 + \chi \zeta) |v \cdot x| \right) \langle \gamma x \rangle^{q-2} \\ &\quad + (q-2)\gamma |v \cdot x| (v \cdot x) \langle \gamma x \rangle^{q-4}, \end{aligned}$$

where we recall that V_1 has been defined in (1.2.2). We consider $\beta, \gamma \in (0, 1)$ to be fixed later such that the weight function

$$\tilde{m}_q := \langle \gamma x \rangle^q + q\gamma(v \cdot x)\langle \gamma x \rangle^{q-2} - q\beta|v \cdot x|\langle \gamma x \rangle^{q-2}$$

satisfies

$$(1 - \delta)\langle \gamma x \rangle^q \leq \tilde{m}_q \leq (1 + \delta)\langle \gamma x \rangle^q,$$

for some $\delta \in (0, 1)$. Gathering the previous estimates, there holds

$$\begin{aligned} \mathcal{B}_1^* \tilde{m}_q &= \mathcal{L}^* \tilde{m}_q - \mathcal{A}_1^* \tilde{m}_q = \mathcal{L}^* \tilde{m}_q - (1 + \chi \zeta) \phi_R \int_{\mathcal{V}} \tilde{m}_q \, dv \\ &= q\gamma|v|^2 \langle \gamma x \rangle^{q-2} + q(q-2)\gamma(v \cdot x)^2 \langle \gamma x \rangle^{q-4} - q\gamma \chi |v \cdot x| \langle \gamma x \rangle^{q-2} \\ &\quad - q\beta|v|^2 \frac{v \cdot x}{|v \cdot x|} \langle \gamma x \rangle^{q-2} - q(q-2)\beta\gamma|v \cdot x|(v \cdot x) \langle \gamma x \rangle^{q-4} \\ &\quad - q\beta(1 + \chi \zeta) V_1 |x| \langle \gamma x \rangle^{q-2} \phi_R^c + q\beta(1 + \chi \zeta) |v \cdot x| \langle \gamma x \rangle^{q-2} \\ &\quad - (1 + \chi \zeta) \langle \gamma x \rangle^q \phi_R, \end{aligned}$$

and then

$$\begin{aligned} \mathcal{B}_1^* \tilde{m}_q &\leq q\gamma V_0^2 \langle \gamma x \rangle^{q-2} + q|q-2|\gamma V_0^2 |x|^2 \langle \gamma x \rangle^{q-4} - q\gamma \chi |v \cdot x| \langle \gamma x \rangle^{q-2} \\ &\quad + q\beta V_0^2 \langle \gamma x \rangle^{q-2} + q|q-2|\beta\gamma V_0^2 |x|^2 \langle \gamma x \rangle^{q-2} \\ &\quad - q\beta(1 - \chi) V_1 |x| \langle \gamma x \rangle^{q-2} + q\beta(1 + \chi) |v \cdot x| \langle \gamma x \rangle^{q-2} \\ &\quad - (1 - \chi) \langle \gamma x \rangle^{q-1} \phi_R + q\beta(1 - \chi) V_1 \langle \gamma x \rangle^{q-1} \phi_R \\ &\leq \left(qV_0^2(\gamma + \beta) + q\beta(1 - \chi) V_1 \right) \langle \gamma x \rangle^{q-2} + q|q-2| V_0^2 \gamma(1 + \beta) |x|^2 \langle \gamma x \rangle^{q-4} \\ &\quad - (1 - \chi)(1 - q\beta V_1) \langle \gamma x \rangle^q \phi_R - q\beta(1 - \chi) V_1 \langle \gamma x \rangle^{q-1} \\ &\leq \left(\frac{C_1}{\langle \gamma x \rangle} - C_2 \phi_R - q\beta(1 - \chi) V_1 \right) \langle \gamma x \rangle^{q-1}. \end{aligned}$$

Choosing $\beta(1 + \chi) = \gamma\chi$ with $\gamma > 0$ small enough and $R \geq 1$ large enough, and observing that $C_1 = O(\gamma)$, $C_2 \geq (1 - \chi)/2$ as $\gamma \rightarrow 0$, we deduce

$$\mathcal{B}_1^* \tilde{m}_q \leq -\frac{q\beta(1 - \chi) V_1}{2} \langle \gamma x \rangle^{q-1} \lesssim -\langle x \rangle^{q-1}.$$

We denote $f_{\mathcal{B}_1}(t) := S_{\mathcal{B}_1}(t)f_0$ for some $0 \leq f_0 \in L_k^1$ and then $\widetilde{M}_q = \langle\langle f_{\mathcal{B}_1} \tilde{m}_q \rangle\rangle$, $M_q = \langle\langle f_{\mathcal{B}_1} \langle x \rangle^q \rangle\rangle$, so that

$$(1.4.18) \quad \widetilde{M}_q \lesssim M_q \lesssim \widetilde{M}_q.$$

From the above inequality, we get

$$(1.4.19) \quad \frac{d}{dt} \widetilde{M}_q = \int f(\mathcal{B}_1^* \tilde{m}_q) \lesssim -M_{q-1},$$

and in particular

$$(1.4.20) \quad \widetilde{M}_k(t) \leq \widetilde{M}_k(0), \quad \forall t \geq 0.$$

A classical interpolation inequality together with (1.4.18) and (1.4.20) give

$$M_\ell(t) \leq M_{\ell-1}(t)^\theta M_k(t)^{1-\theta} \lesssim M_{\ell-1}(t)^\theta M_k(0)^{1-\theta},$$

with $\theta \in (0, 1)$ such that $\ell = \theta(\ell - 1) + (1 - \theta)k$. Coming back to (1.4.19), we get

$$\frac{d}{dt} \widetilde{M}_\ell \lesssim -M_k(0)^{-1/\alpha} \widetilde{M}_\ell^{1+1/\alpha}$$

where

$$\alpha := \frac{1}{(1/\theta) - 1} = k - \ell.$$

Integrating the above differential inequality, we obtain

$$M_\ell(t) \lesssim \frac{M_k(0)}{t^\alpha}, \quad \forall t > 0,$$

and we conclude gathering that last inequality with (1.4.20). \square

In order to establish the asymptotic stability in $L^1(m)$ for a polynomial weight m , we write

$$S_{\mathcal{L}}\Pi^\perp = \Pi^\perp \mathcal{S}_{\mathcal{B}_1} + (\mathcal{S}_{\mathcal{L}}\Pi^\perp) * (\mathcal{A}_1 \mathcal{S}_{\mathcal{B}_1}).$$

Introducing the exponential weight $m_0 := e^{\langle x \rangle}$, we observe that $\Pi^\perp \mathcal{S}_{\mathcal{B}_1} : L^1(m) \rightarrow L^1$ and $\mathcal{A}_1 \mathcal{S}_{\mathcal{B}_1} : L^1(m) \rightarrow L^1(m_0)$, with rate $\langle t \rangle^{-\ell}$ for any $\ell \in (0, k)$ from Lemma 1.4.4. Because we have already established that $\mathcal{S}_{\mathcal{L}}\Pi^\perp : L^1(m_0) \rightarrow L^1$ with rate e^{at} for any $a \in (a^*, 0)$, we immediately conclude that (1.1.9) holds in $L^1(m)$.

Second Part
Neuron networks

Chapter 2

Time elapsed model in weak connectivity regime

Résumé

Afin de décrire l'activité de tir d'un assemblage homogène de neurones, nous considérons le modèle du temps écoulés, qui donnent des descriptions mathématiques de la densité de probabilité des neurones structurés par la distribution du temps écoulé depuis la dernière décharge. Sous l'hypothèse générale sur le taux de tir et la distribution de délai, nous prouvons l'unicité de l'état stationnaire et sa stabilité exponentielle non linéaire dans le régime de connectivité faible. En d'autres termes, le déclenchement asynchrone total des neurones apparaît de manière asymptotique en grand temps. Le résultat génère des résultats similaires obtenus dans [56, 57] dans le cas sans délai. Notre approche utilise la théorie de l'analyse spectrale pour les semi-groupes dans les espaces de Banach développée récemment par S. Mishler et les collaborateurs.

Abstract

In order to describe the firing activity of a homogenous assembly of neurons, we consider time elapsed models, which give mathematical descriptions of the probability density of neurons structured by the distribution of times elapsed since the last discharge. Under general assumption on the firing rate and the delay distribution, we prove the uniqueness of the steady state and its nonlinear exponential stability in the weak connectivity regime. In other words, total asynchronous firing of neurons appears asymptotically in large time. The result generalizes some similar results obtained in [56, 57] in the case without delay. Our approach uses the spectral analysis theory for semigroups in Banach spaces developed recently by S. Mischler and collaborators.

2.1 Introduction

In nervous systems, neuronal circuits carry out tasks of information transmission and processing. Many neurons generate trains of stereotyped electrical pulses in response to incoming stimulations. Following each discharge, the neuron undergoes a period of refractoriness during which it is less responsive to inputs, before recovering its excitability [57]. The main carrier of information is the discharge times or some statistics of the discharge times. In this work, we consider a simple neuronal model which neglects the mechanisms underlying spike generation and focusses on describing the neuronal dynamics in terms of discharge times. More precisely, we consider a model which has been introduced and studied in [28, 56, 57] and which describes the post-discharge recovery of neuronal membranes through an instantaneous firing rate that depends on the time elapsed since the last discharge and the inputs by neurons. We refer to these papers for biologic motivation and discussions. We also refer to [15, 25, 68, 66] where these models (or similar ones) are obtained as a mean field limit of finite number of neuron network models.

The neuronal network is described here by the density number of neurons $f = f(t, x) \geq 0$ which at time $t \geq 0$ are in the state $x \geq 0$. The state of a neuron is a local time (or internal clock) which corresponds to the elapsed time since the last discharge. The dynamic of the neuron network is given by the following nonlinear time elapsed (or of age structured type) evolution equation

$$(2.1.1) \quad \begin{cases} \partial_t f = -\partial_x f - a(x, \varepsilon m(t))f =: \mathcal{L}_{\varepsilon m(t)} f, \\ f(t, 0) = p(t), \quad f(0, x) = f_0(x). \end{cases}$$

Here $a(x, \varepsilon \mu) \geq 0$ represents the firing rate of a neuron in the state x for a network activity $\mu \geq 0$ and a network connectivity parameter $\varepsilon \geq 0$. The function $p(t)$ represents the total density of neurons which undergo a discharge at time t and is defined through

$$(2.1.2) \quad p(t) := \mathcal{P}[f(t); m(t)],$$

where

$$(2.1.3) \quad \mathcal{P}[g, \mu] = \mathcal{P}_\varepsilon[g, \mu] := \int_0^\infty a(x, \varepsilon \mu) g(x) dx.$$

The function $m(t)$ represents the network activity at time $t \geq 0$ resulting from earlier discharges and is defined by

$$(2.1.4) \quad m(t) := \int_0^\infty p(t-y)b(dy),$$

where the delay distribution b is a probability measure which takes into account the persistence of the electric activity in the network resulting from discharges (synaptic integration).

In the sequel, we will consider the two following situations :

- The *case without delay*, when $b = \delta_0$ and then $m(t) = p(t)$.
- The *case with delay*, when b is a smooth function.

We observe that in both cases, the solution f of the time elapsed equation (2.1.1)–(2.1.3) satisfies

$$\frac{d}{dt} \int_0^\infty f(t, x) dx = f(t, 0) - \int_0^\infty a(x, \varepsilon m(t)) f(t, x) dx = 0.$$

As a consequence, the total density number of neurons (also called *mass* in the sequel) is conserved and we can normalize that mass to be 1. In other words, we may always assume

$$\langle f(t, \cdot) \rangle = \langle f_0 \rangle = 1, \quad \forall t \geq 0, \quad \langle g \rangle := \int_0^\infty g(x) dx.$$

A (normalized) steady state for the time elapsed evolution system of equations (2.1.1)–(2.1.3) is a couple $(F_\varepsilon, M_\varepsilon)$ of a density number of neurons $F_\varepsilon = F_\varepsilon(x) \geq 0$ and a network activity $M_\varepsilon \geq 0$ such that

$$(2.1.5) \quad \begin{cases} 0 = -\partial_x F_\varepsilon - a(x, \varepsilon M_\varepsilon) F_\varepsilon = \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon, \\ F_\varepsilon(0) = M_\varepsilon, \quad \langle F_\varepsilon \rangle = 1. \end{cases}$$

It is worth emphasizing that for a steady state the associated network activity and discharge activity are two equal constants because of the normalization of the delay distribution, i.e. $\langle b \rangle = 1$.

In equations (2.1.1)–(2.1.3) and (2.1.5), the connectivity parameter $\varepsilon \geq 0$ corresponds to the strength of the influence of the neuronal network activity on each neuron through the functions $m(t)$ and $f(t, x)$ respectively. In the limit case $\varepsilon = 0$, equation (2.1.1)–(2.1.3) is linear which means that each neuron evolves accordingly to its own dynamic. In the other hand, when $\varepsilon > 0$, equation (2.1.1)–(2.1.3) is nonlinear and the dynamic of any given neuron is affected by the state (or the past states in the case of the model with delay) of all the other neurons through the global activity of the neuronal network. Finally, the weak connectivity regime, about which we are mainly concerned in the present paper, corresponds to a range of connectivity parameter $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 > 0$ small enough, such that the nonlinearity of equations (2.1.1)–(2.1.3) and (2.1.5) is not too strong.

Our main purpose in this paper is to prove that solutions to the time elapsed evolution equation (2.1.1)–(2.1.3) converge to a stationary state with an exponential rate in the weak connectivity regime under more general assumptions on the firing rate a . Before stating that result, let us present the precise mathematical assumptions we will need on the firing rate a and on the delay distribution b .

We make the physically reasonable assumption

$$(2.1.6) \quad \partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0,$$

$$(2.1.7) \quad 0 < a_0 := \lim_{x \rightarrow \infty} a(x, 0) \leq \lim_{x, \mu \rightarrow \infty} a(x, \mu) =: a_1 < \infty,$$

as well as the smoothness assumption

$$(2.1.8) \quad a \in W^{2, \infty}(\mathbb{R}_+^2).$$

In the delay case, we assume that $b(dy) = b(y) dy$ satisfies the exponential bound and smoothness condition

$$(2.1.9) \quad \exists \delta > 0, \quad \int_0^\infty e^{\delta y} (b(y) + |b'(y)|) dy < \infty.$$

We begin by stating our main result about the stationary problem (2.1.5).

Theorem 2.1.1. *Assume (2.1.6)-(2.1.7)-(2.1.8). For any $\varepsilon \geq 0$, there exists at least one solution $(F_\varepsilon, M_\varepsilon) \in W^{1,\infty}(\mathbb{R}_+) \times \mathbb{R}_+$ to the stationary problem (2.1.5) such that*

$$(2.1.10) \quad 0 \leq F_\varepsilon(x) \leq C e^{-\frac{a_0}{2}x}, \quad |F'_\varepsilon(x)| \leq C e^{-\frac{a_0}{2}x}, \quad \forall x \geq 0,$$

for a constant $C \in (0, \infty)$. Moreover, there exists $\varepsilon_0 > 0$, small enough, such that the above solution is unique for any $\varepsilon \in [0, \varepsilon_0)$.

For a given initial datum $0 \leq f_0 \in L^1(\mathbb{R}_+)$, we say that a function f is a weak (positive and mass conserving) solution to (2.1.1)–(2.1.3) if

$$0 \leq f \in C([0, \infty); L^1(\mathbb{R})), \quad \langle f(t) \rangle = \langle f_0 \rangle, \quad \forall t \geq 0,$$

and f satisfies (2.1.1) in the distributional sense $\mathcal{D}'([0, \infty) \times [0, \infty))$ for some functions $m, p \in C([0, \infty))$ which fulfilled the constrains (2.1.2) and (2.1.3).

Under the above assumptions, existence of weak solutions are established in [77, Theorem 1.1]. The main concern of the present work is the following long-time asymptotic result on the solutions.

Theorem 2.1.2. *We assume that the firing rate a satisfies (2.1.6), (2.1.7) and (2.1.8). We also assume that the delay distribution b satisfies $b = \delta_0$ or (2.1.9). There exists $\varepsilon_0 > 0$, small enough, and there exist some constants $\alpha < 0$, $C \geq 1$ and $\eta > 0$ such that for any connectivity parameter $\varepsilon \in (0, \varepsilon_0)$ and any initial datum $0 \leq f_0 \in L^1$ with mass 1 and such that $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\varepsilon$, the positive and mass conserving solution f to the evolution equation (2.1.1)–(2.1.3) satisfies*

$$\|f(t, \cdot) - F_\varepsilon\|_{L^1} \leq C e^{\alpha t}, \quad \forall t \geq 0.$$

In other words, in that weak connectivity regime, we prove that the total asynchronous firing of neurons appears exponentially fast in the large time asymptotic. Theorem 2.1.2 extends to firing rates a satisfying (2.1.6)–(2.1.8) some similar exponential stability results obtained in [56, 57] in the case without delay and for a *step function* firing rate a given by

$$(2.1.11) \quad a(x, \mu) = \mathbf{1}_{x > \sigma(\mu)}, \quad \sigma, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+), \quad \sigma' \leq 0.$$

It is worth mentioning that the above firing rate does not fall in the class of rates considered in the present paper because condition (2.1.8) is not met. On the other hand, we are able to tackle the case without and with delay in the same time, what it was not the case in [56, 57]. In the delay case, stability results were established in [56], but not exponential stability.

Our proof follows a strategy of “perturbation of semigroup” initiated in [46] for studying long time convergence to the equilibrium for the homogeneous inelastic Boltzmann equation and used recently in [48] for a neuron network equation based on a brownian (hypoelliptic) perturbation of the well-known FitzHugh-Nagumo dynamic. More precisely, we introduce the linearized equation for the variation functions $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$ around a stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$, which writes

$$(2.1.12) \quad \begin{cases} \partial_t g = -\partial_x g - a(x, \varepsilon M_\varepsilon)g - n(t) \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon)F_\varepsilon, \\ g(t, 0) = q(t), \quad g(0, x) = g_0(x), \end{cases}$$

with

$$(2.1.13) \quad q(t) = \int_0^\infty a(x, \varepsilon M_\varepsilon) g \, dx + n(t) \varepsilon \int_0^\infty (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon \, dx$$

and

$$(2.1.14) \quad n(t) := \int_0^\infty q(t-y) b(dy).$$

We associate to that linear evolution equation a generator Λ_ε (which acts on an appropriate space to be specified in the two cases without and with delay) and its semigroup S_{Λ_ε} . It turns out that we may split the operator Λ_ε as

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon,$$

for some α -hypodissipative operator \mathcal{B}_ε , $\alpha < 0$, and some bounded and \mathcal{B}_ε -power regular operator \mathcal{A}_ε as defined in [75, 31, 50, 45, 44]. In particular, adapted versions of the Spectral Mapping Theorem in [50, 45, 44] and the Weyl's Theorem in [75, 31, 50, 45, 44] imply that the semigroup S_{Λ_ε} has a finite dimensional dominant part. Moreover, in the limit case when $\varepsilon = 0$, the term $n(t)$ disappears from equation (2.1.12) and the resulting semigroup S_{Λ_0} becomes positive. That allows us to use the Krein-Rutman Theorem established in [50, 45, 44] in order to get that the stationary state (F_0, M_0, M_0) is unique and exponentially stable. Using next a perturbative argument developed in [46, 72, 44, 51], we get that the unique stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ is also exponentially stable in the weak connectivity regime. We conclude the proof of Theorem 2.1.2 by a somewhat classical nonlinear exponential stability argument.

The same strategy applies to the case without delay and with delay. In both cases, the boundary condition in the age structure equation is treated as a source term and, in the delay case, the delay equation (2.1.4) is replaced by a simple age equation on an auxiliary function, so that the resulting linearized equation writes as an autonomous system of two PDEs and falls in the classical framework of linear evolution equations generating a semigroup.

Our approach is thus quite different from the usual way to deal with delay equations, as introduced by I. Fredholm [26] and V. Volterra [76], which consists in using the specific framework of “fading memory space”, which goes back at least to Coleman & Mizel [14], or the theory of “abstract algebraic-delay differential systems” developed by O. Diekmann and co-authors [17].

Our approach is also different from the previous works [56, 57, 58] where the asymptotic stability analysis were performed by taking advantage of the step function structure (2.1.11) of the firing rate. That one makes possible to explicitly exhibit a suitable norm (related to the W_1 Monge-Kantorovich-Wasserstein optimal transport distance) such that some related linear age structure operator is dissipative. The present method is based on a more abstract approach but in the other hand it is somewhat more flexible because it does not require to explicitly exhibit a norm for which the underlying linear(ized) operators are dissipative. In particular, we hope that our strategy can be adapted to the large connectivity regime, to the step function firing rate (2.1.11), which are achieved in [77], as well as to models including fragmentation term to describe neuronal networks with

adaptation and fatigue, and thus generalize to the case with a delay term all the stability results established in [56, 57, 58] in the case without delay.

Let us end the introduction by describing the plan of the paper. In Section 2.2, we introduce the strategy, we prove the stationary state result and we establish Theorem 2.1.2 in the case without delay. In Section 2.3, we establish Theorem 2.1.2 in the case with delay. As we mentioned above, the strategy of proof for the case without and with delay is rather the same. For pedagogical reason, we start presenting the method on the simplest “*without delay case*” in Section 2.2, where we prove the stationary problem result Theorem 2.1.1 as well as Theorem 2.1.2 in that case. Next, in Section 2.3, we only explain how the proof must be modified in order to treat the more complicated “*with delay case*” and thus establish Theorem 2.1.2 in all generality.

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2.2 Case without delay

The present section is devoted to the proof of our main result Theorem 2.1.2 in the without delay case.

2.2.1 The stationary problem

We first deal with the stationary problem and we prove the existence of steady state as well as its uniqueness in the small connectivity regime.

Proof of Theorem 2.1.1. Step 1. The existence of a steady state. We set

$$A(x, m) := \int_0^x a(y, m) \, dy, \quad \forall x, m \geq 0.$$

For any $m \geq 0$, we can solve the equation (2.1.5), by writing

$$(2.2.15) \quad F_{\varepsilon, m}(x) := T_m e^{-A(x, \varepsilon m)},$$

where $T_m \geq 0$ is chosen in order that $F_{\varepsilon, m}$ satisfies the mass normalized condition, namely

$$T_m^{-1} = \int_0^\infty e^{-A(x, \varepsilon m)} \, dx.$$

In order to conclude the existence of a solution, we just have to find a real number $m = M_\varepsilon$ such that $m = F_{\varepsilon, m}(0) = T_m$. Equivalently, we need to find $M_\varepsilon \geq 0$ such that

$$(2.2.16) \quad \Phi(\varepsilon, M_\varepsilon) = 1,$$

where

$$\Phi(\varepsilon, m) = m T_m^{-1} := m \int_0^\infty e^{-A(x, \varepsilon m)} \, dx.$$

From the assumption (2.1.7) of a , there exists $x_0 \in [0, \infty)$ such that $a(x, \mu) \geq \frac{a_0}{2}$, for any $x \geq x_0$, $\mu \geq 0$, and therefore

$$(2.2.17) \quad \frac{a_0}{2}(x - x_0)_+ \leq A(x, \mu) \leq a_1 x, \quad \forall x \geq 0, \quad \forall \mu \geq 0,$$

which implies

$$\frac{m}{a_1} = m \int_0^\infty e^{-a_1 x} dx \leq \Phi(\varepsilon, m) \leq m \int_0^\infty e^{-\frac{a_0}{2}(x-x_0)_+} dx = m \left(x_0 + \frac{2}{a_0} \right).$$

Thanks to the Lebesgue dominated convergence theorem, we deduce that $\Phi(\varepsilon, \cdot)$ is a continuous function. Because $\Phi(\varepsilon, 0) = 0$, $\Phi(\varepsilon, \infty) = \infty$ and thanks to the intermediate value theorem, we conclude to the existence of at least one real number $M_\varepsilon \in (0, \infty)$ such that (2.2.16) holds. Estimates (2.1.10) immediately follow from the identity (2.2.15) and the estimate (2.2.17).

Step 2. The uniqueness of the steady state in the weak connectivity regime. Obviously, there exists a unique

$$M_0 := \left(\int_0^\infty e^{-A(x,0)} dx \right)^{-1} \in (0, \infty)$$

satisfying $\Phi(0, M_0) = 1$. Moreover, we compute

$$\frac{\partial}{\partial m} \Phi(\varepsilon, m) = \int_0^\infty e^{-A(x, \varepsilon m)} \left(1 - m \varepsilon \frac{\partial A}{\partial m}(x, \varepsilon m) \right) dx,$$

which is continuous as a function of the two variables because of (2.1.8). We then easily obtain that $\Phi \in C^1$. Since moreover

$$\partial_m \Phi(\varepsilon, m)|_{\varepsilon=0} = \int_0^\infty e^{-A(x,0)} dx > 0,$$

the implicit function theorem implies that there exists $\varepsilon_0 > 0$, small enough, such that the equation (2.2.16) has a unique solution for any $\varepsilon \in [0, \varepsilon_0)$. \square

Remark 2.2.1. *In the above proof, we do not need (2.1.8) but only the weaker smoothness assumption that A and $\partial_m A$ are continuous.*

2.2.2 Linearized equation and structure of the spectrum

To go one step further, we introduce the linearized equation around the stationary solution $(F_\varepsilon, M_\varepsilon)$. On the variation (g, n) , the linearized equation writes

$$\begin{cases} \partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t) = 0, \\ g(t, 0) = n(t) = \int_0^\infty (a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t)) dx, \quad g(0, x) = g_0(x), \end{cases}$$

with $a_\varepsilon := a(x, \varepsilon M_\varepsilon)$, $a'_\varepsilon := \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon)$. Since there exists $\varepsilon_0 > 0$, small enough, such that

$$\forall \varepsilon \in (0, \varepsilon_0) \quad \kappa := \int_0^\infty a'_\varepsilon F_\varepsilon dx < 1,$$

we may define

$$(2.2.18) \quad \mathcal{M}_\varepsilon[g] := (1 - \kappa)^{-1} \int_0^\infty a_\varepsilon g \, dx,$$

and the linearized equation is then equivalent to

$$(2.2.19a) \quad \partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g(t, \cdot)] = 0,$$

$$(2.2.19b) \quad g(t, 0) = \mathcal{M}_\varepsilon[g(t, \cdot)], \quad g(0, x) = g_0(x).$$

To the above linear evolution equation, one can classically associate a semigroup $S_{L_\varepsilon}(t)$ acting on $X := L^1(\mathbb{R}_+)$, with generator

$$L_\varepsilon g := -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g]$$

and domain

$$D(L_\varepsilon) := \{g \in W^{1,1}(\mathbb{R}_+); g(0) = \mathcal{M}_\varepsilon[g]\}.$$

Here we also use another approach by considering the boundary term as a source term, and then rewriting the equation as

$$(2.2.20) \quad \partial_t g = \Lambda_\varepsilon g := -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g] + \delta_{x=0} \mathcal{M}_\varepsilon[g],$$

acting on the space of bounded Radon measures, given by

$$\mathcal{X} := M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))'; \text{supp } g \subset \mathbb{R}_+\},$$

endowed with the weak $*$ topology $\sigma(M^1, C_0)$. Here and below, $C_0(I)$ denotes the space of continuous functions on the closed interval $I = \mathbb{R}$ or $I = \mathbb{R}_+$ which goes to 0 at infinity. We then denote by $S_{\Lambda_\varepsilon}(t)$ the semigroup on \mathcal{X} generated by Λ_ε .

It is worth emphasizing that for any $g_0 \in X$, the function $g(t) = S_{L_\varepsilon}(t)g_0 \in C([0, \infty); X)$ is a weak solution to equations (2.1.12)-(2.1.13), and more precisely clearly satisfies

$$-\int_0^\infty \varphi(0) g_0 \, dx + \int_0^\infty \int_0^\infty g (-\partial_t \varphi + \Lambda_\varepsilon^* \varphi) \, dx dt = 0,$$

for any $\varphi \in C_c^1([0, \infty); C_0(\mathbb{R}_+)) \cap C_c([0, \infty); C_0^1(\mathbb{R}_+))$. Here, we have defined

$$\Lambda_\varepsilon^* \psi := \partial_x \psi - a_\varepsilon \psi - a_\varepsilon (1 - \kappa)^{-1} \left(\int_0^\infty a'_\varepsilon(y) F_\varepsilon(y) \psi(y) \, dy - \psi(0) \right),$$

for any $\psi \in C_0^1(\mathbb{R}_+)$, the space of C^1 functions which goes to 0 at infinity as well as their first derivative. As a consequence, the semigroup S_{Λ_ε} being defined by duality from the semigroup $S_{\Lambda_\varepsilon^*}$, we have $S_{\Lambda_\varepsilon}|_X = S_{L_\varepsilon}$.

For a generator L , we denote by $\Sigma(L)$ its spectrum and by S_L the associated semigroup. We refer to the classical textbooks [36, 59, 22] for an introduction to the spectral analysis of operators and the semigroup theory. Our next result deals with the structure of the spectral set $\Sigma(\Lambda_\varepsilon)$ of Λ_ε and the splitting structure of the associated semigroup S_{Λ_ε} .

Theorem 2.2.2. *Assume (2.1.6)-(2.1.7)-(2.1.8) and define $a^* := -a_0/2 < 0$. The operator Λ_ε is the generator of a weakly $*$ continuous semigroup S_{Λ_ε} acting on \mathcal{X} endowed with the weak $*$ topology $\sigma(M^1, C_0)$. Moreover, there exists a finite rank projector $\Pi_{\Lambda_\varepsilon, a^*}$ which commutes with S_{Λ_ε} , an integer $j \geq 0$ and some complex numbers*

$$\xi_1, \dots, \xi_j \in \Delta_{a^*} := \{z \in \mathbb{C}, \Re z > a^*\},$$

such that on $E_1 := \Pi_{\Lambda_\varepsilon, a^*} \mathcal{X}$ the restricted operator satisfies

$$(2.2.21) \quad \Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_{a^*} = \{\xi_1, \dots, \xi_j\}$$

(with the convention $\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_{a^*} = \emptyset$ when $j = 0$) and for any $a > a^*$ there exists a constant C_a such that the remainder semigroup satisfies

$$(2.2.22) \quad \|S_{\Lambda_\varepsilon}(I - \Pi_{\Lambda_\varepsilon, a^*})\|_{\mathcal{B}(\mathcal{X})} \leq C_a e^{at}, \quad \forall t \geq 0.$$

The proof of the result is a direct consequence of the fact that the operator Λ_ε splits as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ where \mathcal{A}_ε and \mathcal{B}_ε are defined on \mathcal{X} by

$$(2.2.23a) \quad \mathcal{A}_\varepsilon g := \gamma_\varepsilon \mathcal{M}_\varepsilon[g], \quad \gamma_\varepsilon := \delta_0 - a'_\varepsilon F_\varepsilon,$$

$$(2.2.23b) \quad \mathcal{B}_\varepsilon g := -\partial_x g - a_\varepsilon g,$$

for which we can adapt Weyl's Theorem of [75, 31, 50, 45, 44] and the Spectral Mapping Theorem of [50, 45, 44]. The picture may seem not that simple, because Λ_ε does not generate a strongly continuous semigroup on \mathcal{X} and then apparently does not fit the framework developed in [50, 45]. However, we may probably circumvent that issue in the following ways:

- (1) We may observe that the strong continuity is little used in [50, 45] and then the results stated therein extend to weakly $*$ continuous semigroups. In other words, on the finite dimensional eigenspace associated to the principal part of the spectrum (2.2.21) continuity and weak $*$ continuity are equivalent, while in the remainder part (2.2.22), we just use a decay bound which does not require the strong continuity property, see [59, Chapter 1] as well as [77, 44] where such a weak $*$ continuous framework is also discussed.
- (2) We may apply the theory developed in [50, 45] to the adjoint operator Λ_ε^* and the associated semigroup acting on $C_0(\mathbb{R})$ and then deduce the result by duality.
- (3) We may probably reduce the space X by “sun duality”, see [22, Chapter II.2.6], apply the theory developed in [50, 45] to the resulting strongly continuous semigroup and then conclude by a density argument.

We rather follow another strategy, which has the advantage to be more self-contained and pedagogical, by adapting to our context the proofs from [50, 45]. It is worth emphasizing that we introduce the two spaces $X \subset \mathcal{X}$ because the splitting $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ has some clearer meaning in the space \mathcal{X} while the Banach structure of X is used when we establish the adequate version of Weyl's theorem.

We recall the definition of hypodissipativity introduced in [31]. We say that the closed operator L on a Banach space E with dense domain $D(L)$ is α -hypodissipative if there exists an equivalent norm $\|\!\| \cdot \|\!\|$ on E such that

$$\forall f \in D(L), \exists \varphi \in F_{\|\!\| \cdot \|\!\|}(f) \quad \Re \langle \varphi, (L - \alpha) f \rangle \leq 0,$$

where, for any $f \in E$, the associated dual set $F_{\|\cdot\|}(f) \subset E'$ is defined by

$$F_{\|\cdot\|}(f) := \{\varphi \in E'; \langle \varphi, f \rangle = \|f\|_X^2 = \|\varphi\|_{E'}^2\}.$$

We also recall that the generator L of a semigroup of bounded operators is α -hypodissipative if, and only if, there exists a constant $M \geq 1$ such that the associated semigroup S_L on E satisfies the growth estimate

$$\|S_L(t)\|_{\mathcal{B}(E)} \leq M e^{\alpha t}, \quad \forall t \geq 0,$$

where $\mathcal{B}(E)$ denotes the space of linear and bounded operators on E . We will sometime abuse by saying that S_L is α -hypodissipative when it satisfies the above growth estimate. We refer to [59, 31, 50, 44, 36, 59, 22] for details.

We start with the properties of the two auxiliary operators.

Lemma 2.2.3. *Assume that a satisfies conditions (2.1.6)–(2.1.8). For any $\varepsilon \geq 0$, the operators \mathcal{A}_ε and \mathcal{B}_ε satisfy:*

- (i) $\mathcal{A}_\varepsilon \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathcal{X})$.
- (ii) $S_{\mathcal{B}_\varepsilon}$ is a^* -hypodissipative in both X and \mathcal{X} .
- (iii) The family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{\mathcal{B}(\mathcal{X}, Y)} \leq C_a e^{a^* t}, \quad \forall a > a^*,$$

for some constant $C_a \in (0, \infty)$ and with $Y := BV(\mathbb{R}_+) \cap L_1^1(\mathbb{R}_+)$.

Here $BV(\mathbb{R}_+)$ stands for the space of functions with bounded variation and $L_1^1(\mathbb{R}_+)$ stands for the weighted Lebesgue space associated to the weight function $x \mapsto \langle x \rangle$.

Proof of Lemma 2.2.3. In order to shorten notation we skip the ε dependency, we write $a(x) = a(x, \varepsilon M_\varepsilon)$, $A(x) = A(x, \varepsilon M_\varepsilon)$, $S_\Lambda = S_{\Lambda_\varepsilon}$, $S_{\mathcal{A}} = S_{\mathcal{A}_\varepsilon}$, $S_{\mathcal{B}} = S_{\mathcal{B}_\varepsilon}$ and so on.

Step 1. Proof of (i). We have $\mathcal{A} \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathcal{X})$ from the fact that

$$\mathcal{M}_\varepsilon[\cdot] \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathbb{R})$$

since $\|a\|_{W^{1,\infty}} < \infty$ as assumed in (2.1.8).

Step 2. Proof of (ii). We use the same notation \mathcal{B} and $S_{\mathcal{B}}$ for the generator and its associated semigroup defined in the spaces X and \mathcal{X} . We write $S_{\mathcal{B}}$ with the explicit formula

$$(2.2.24) \quad S_{\mathcal{B}}(t)g(x) = e^{A(x-t)-A(x)}g(x-t)\mathbf{1}_{x-t \geq 0} =: S(t, x).$$

From the inequality

$$a(z, 0) \geq \frac{3}{4}a_0(1 - \mathbf{1}_{0 \leq z \leq x_1})$$

for some $x_1 \in (0, \infty)$ coming from (2.1.7), we deduce

$$A(x-t) - A(x) \leq \frac{3}{4}a_0x_1 - \frac{3}{4}a_0t$$

for any $x \geq t \geq 0$, next

$$(2.2.25) \quad e^{A(x-t)-A(x)} \leq C e^{3\beta t}, \quad \forall x \geq t \geq 0,$$

with $C := e^{\frac{3a_0x_1}{4}} > 0$, $\beta := -a_0/4 < 0$, and finally

$$\begin{aligned} \|S_{\mathcal{B}}(t)g\|_X &= \|e^{A(x)-A(x+t)}g(x)\|_X \\ &\leq Ce^{3\beta t}\|g\|_X, \quad \forall t \geq 0, \forall g \in X. \end{aligned}$$

We conclude by observing that $3\beta < a^*$ and that the same estimate holds with X replaced by \mathcal{X} thanks to a (weakly $*$) density argument.

Step 3. Proof of (iii). For $g \in \mathcal{X}$ and with the notation of Step 2, we have

$$\mathcal{A}S_{\mathcal{B}}(t)g = \gamma_{\varepsilon} N(t),$$

where γ_{ε} is defined in (2.2.23a) and

$$\begin{aligned} N(t) &:= \mathcal{M}_{\varepsilon}[S(t, \cdot)] \\ &= (1 - \kappa)^{-1} \int_0^{\infty} a(x)e^{A(x-t)-A(x)}g(x-t)\mathbf{1}_{x-t \geq 0} dx. \end{aligned}$$

It is worth noticing that $N \in C_b(\mathbb{R}_+)$, because $(x, t) \mapsto a(x)e^{A(x-t)-A(x)}$ is a bounded and continuous function on the set $\mathbb{T} := \{(x, t); x \geq t \geq 0\}$. Moreover, as in Step 2, we have

$$\begin{aligned} |N(t)| &\leq Ca_1 \int_0^{\infty} e^{3\beta t} |g(x-t)|\mathbf{1}_{x-t \geq 0} dx \\ (2.2.26) \quad &\leq Ca_1 e^{3\beta t} \|g\|_{\mathcal{X}}, \end{aligned}$$

for any $t \geq 0$. We deduce

$$\begin{aligned} (S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g(x) &= \int_0^t (S_{\mathcal{B}}(s)\gamma_{\varepsilon})(x)N(t-s) ds \\ (2.2.27) \quad &= \int_0^t e^{A(x-s)-A(x)} \gamma_{\varepsilon}(x-s)N(t-s)\mathbf{1}_{x-s \geq 0} ds \end{aligned}$$

$$(2.2.28) \quad = e^{-A(x)} (\nu_{\varepsilon} * \check{N}_t)(x),$$

with $\nu_{\varepsilon} := \gamma_{\varepsilon} e^A$ and the classical notation $\check{N}_t(s) = N(t-s)$. Starting from identity (2.2.27) and denoting $e_{-\beta}(x) := e^{-\beta x}$, we have

$$\begin{aligned} |(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g(x)| e_{-\beta}(x) &\leq e^{-\beta x} \int_0^t C e^{3\beta t} |\gamma_{\varepsilon}(x-s)| |N(t-s)| ds \\ &= C e^{a^*t} \int_0^t e^{2\beta(t-s)} e^{-\beta(x-s)} |\gamma_{\varepsilon}(x-s)| e^{-\beta(t-s)} |N(t-s)| ds \\ &\leq C e^{a^*t} \|(|\gamma_{\varepsilon}| e_{-\beta}) * (|N| e_{-\beta})\|_{C_b} \\ &\leq C e^{a^*t} \|\gamma_{\varepsilon} e_{-\beta}\|_{\mathcal{X}} \|N e_{-\beta}\|_{C_b}. \end{aligned}$$

Using the definition (2.2.23a) and the estimates (2.1.10) and (2.2.26), we deduce

$$(2.2.29) \quad \|(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g e_{-\beta}\|_{L^{\infty}} \leq C' e^{a^*t} \|g\|_{\mathcal{X}}, \quad \forall t \geq 0,$$

for some constant $C' \in (0, \infty)$.

Next, differentiating the functions in both sides of identity (2.2.28), we get

$$\begin{aligned}\partial_x[(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g](x) &= -a(x)e^{-A(x)}(\nu_{\varepsilon} * \check{N}'_t)(x) - e^{-A(x)}(\nu_{\varepsilon} * \check{N}'_t)(x) \\ &\quad - e^{-A(x)}\nu_{\varepsilon}(x-t)N(0)\mathbf{1}_{x-t \geq 0} \\ &\quad + e^{-A(x)}\nu_{\varepsilon}(x)N(t),\end{aligned}$$

with $\check{N}'_t(s) = N'(t-s)$. In order to estimate the second term, we compute

$$\begin{aligned}N'(t) &= (1-\kappa)^{-1} \int_0^{\infty} \partial_x(a(x)e^{-A(x)})e^{A(x-t)}g(x-t)\mathbf{1}_{x-t \geq 0} dx \\ &= (1-\kappa)^{-1} \int_0^{\infty} (a'(x) - a(x)^2)e^{A(x-t)-A(x)}g(x-t)\mathbf{1}_{x-t \geq 0} dx.\end{aligned}$$

Using the inequality (2.2.25), we deduce

$$\begin{aligned}|N'(t)| &\leq C' \int_0^{\infty} (|a'(x)| + a(x)^2)e^{3\beta t}g(x-t)\mathbf{1}_{x-t \geq 0} dx \\ &\lesssim e^{a^*t}\|g\|_{\mathcal{X}}.\end{aligned}$$

As a consequence, using again (2.2.25), we get

$$\begin{aligned}\|e^{-A}(\nu_{\varepsilon} * \check{N}'_t)\|_{\mathcal{X}} &\lesssim \int_0^t \int_0^{\infty} e^{3\beta s}|\gamma_{\varepsilon}(x-s)||N'(t-s)| dx ds \\ &\lesssim e^{a^*t}\|\gamma_{\varepsilon}\|_{\mathcal{X}}\|N'\|_{\mathcal{X}} \\ &\lesssim e^{a^*t}\|g\|_{\mathcal{X}}.\end{aligned}$$

Treating in a similar way the two other terms involved in the expression of $\partial_x[(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g]$, we finally obtain

$$\|\partial_x(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g\|_{\mathcal{X}} \lesssim e^{a^*t}\|g\|_{\mathcal{X}}.$$

Together with (2.2.29) that concludes the proof of (iii). \square

Proof of Theorem 2.2.2. The main idea is to apply or adapt the versions of Weyl's Theorem [50, Theorem 3.1], [45, Theorem E.3.1] and of the Spectral Mapping Theorem [50, Theorem 2.1], [45, Theorem E.2.1] (see also the variant results in [44]). We start collecting the three key properties satisfied by the operators $\mathcal{A}_{\varepsilon}$ and $\mathcal{B}_{\varepsilon}$ involved in the splitting of Λ_{ε}

We denote by X_{ζ} the abstract Sobolev space associated to L_{ε} for $\zeta \in \mathbb{R}$, see e.g. [22, Section II.5], so that in particular $X_{\zeta} \subset W^{\zeta,1}(\mathbb{R}_+)$, with equality when $\zeta \leq 0$. We recall that for the generator L of a semigroup S_L , we define the resolvent set $\rho(L)$ by

$$\rho(L) := \{z \in \mathbb{C}; L - z \text{ is a bijection}\} = \mathbb{C} \setminus \Sigma(L),$$

as well as the resolvent (operator) $R_L(z) := (L - z)^{-1}$ for any $z \in \rho(L)$. We finally define the half plane $\Delta_a := \{z \in \mathbb{C}; \Re z > a\}$ for any $a \in \mathbb{R}$.

(A1) For any $a > a^*$ and $\ell \in \mathbb{N}$, there exists a positive constant $C_{a,\ell}$ such that the following growth estimate holds

$$(2.2.30) \quad \|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)}(t)\|_{\mathcal{B}(X)} \leq C_{a,\ell} e^{at}, \quad \forall t \geq 0.$$

It is obvious that **(A1)** is true for $\ell = 0$ from Lemma 2.2.3-(ii) and for $\ell = 1$ from (2.2.29). We then deduce **(A1)** for any $\ell \geq 2$ and $a > a^*$ by writing $S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)} = [S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})] * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*(\ell-1))}$ and by using that $(\mathcal{B}_\varepsilon - a^*)$ is hypodissipative in \mathcal{X} and $\mathcal{A}_\varepsilon \in \mathcal{B}(\mathcal{X})$.

(A2) There holds $\mathcal{A}_\varepsilon \in \mathcal{B}(X_{-1}, X_\zeta)$ for any $\zeta \in (-1, 0)$ and the family of operators $(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*2)}(t)$ satisfies the estimate

$$(2.2.31) \quad \|(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*2)}(t)\|_{\mathcal{B}(X_{-1}, X)} \leq C'_{a^*,1} e^{a^*t}, \quad \forall t \geq 0,$$

for a positive constant $C'_{a^*,1}$. The first claim is a consequence of the continuous embedding $\mathcal{X} \subset W^{\zeta,1}$ and of Lemma 2.2.3-(i). The second estimate is obtained by putting together the properties (i) and (ii) in Lemma 2.2.3.

(A3) The family of operators $(R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^3(z)$ satisfies the compactness estimate

$$\|(R_{\mathcal{B}_\varepsilon}(z) \mathcal{A}_\varepsilon)^3\|_{\mathcal{B}(\mathcal{X}, Y)} \leq C''_{2,a}, \quad \forall z \in \Delta_a, \quad \forall a > a^*,$$

for some positive constant $C''_{2,a}$. Observing that $z \mapsto -(R_{\mathcal{B}_\varepsilon}(z) \mathcal{A}_\varepsilon)^3$ is nothing but the Laplace transform of the function $t \mapsto (S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*3)}(t)$, that clearly holds true thanks to Lemma 2.2.3.

We now briefly explain how the proof goes on, referring to [50, 45, 44] for more details and developments. In order to shorten notation we skip again the ε dependency.

Step 1. On the one hand, because of (2.2.31) and the fact that $R_{\mathcal{B}}$ is nothing but the opposite of the Laplace transform of $S_{\mathcal{B}}$, we have

$$(2.2.32) \quad \|(R_{\mathcal{B}}(z) \mathcal{A})^2\|_{X_{-1} \rightarrow X} \leq C, \quad \forall z \in \Delta_a.$$

Recalling that the negative abstract Sobolev norm is defined by

$$\|g\|_{X_{-1}} := \|R_{\mathcal{B}}(a)g\|_X,$$

and using the dissipativity property of \mathcal{B} , we immediately have

$$\forall z \in \Delta_a, \quad \forall g \in X, \quad \|R_{\mathcal{B}}(z)g\|_{X_{-1}} \leq C \|g\|_{X_{-1}}.$$

Moreover, using the resolvent identity

$$\begin{aligned} R_{\mathcal{B}}(a)R_{\mathcal{B}}(z) &= z^{-1} (R_{\mathcal{B}}(a)\mathcal{B}R_{\mathcal{B}}(z) - R_{\mathcal{B}}(a)), \\ &= z^{-1} (R_{\mathcal{B}}(z) - R_{\mathcal{B}}(a) + aR_{\mathcal{B}}(z)R_{\mathcal{B}}(a)), \end{aligned}$$

we also have

$$\begin{aligned} \|R_{\mathcal{B}}(z)g\|_{X_{-1}} &\leq \frac{1}{|z|} \left(\|R_{\mathcal{B}}(z)g\|_X + \|R_{\mathcal{B}}(a)g\|_X + |a| \|R_{\mathcal{B}}(z)R_{\mathcal{B}}(a)g\|_X \right) \\ &\leq \frac{C}{|z|} \|g\|_X, \end{aligned}$$

for any $g \in X$ and $z \in \Delta_a$. Finally, using both estimates and an interpolation argument, we deduce

$$(2.2.33) \quad \forall z \in \Delta_a, \quad \|R_{\mathcal{B}}(z)\|_{X_\zeta \rightarrow X_{-1}} \leq \frac{C}{\langle z \rangle^{1+\zeta}}, \quad \forall \zeta \in (-1, 0).$$

We conclude with

$$(2.2.34) \quad \forall z \in \Delta_a, \quad \|(R_{\mathcal{B}}(z)\mathcal{A})^3\|_{X \rightarrow X} \leq \frac{C}{\langle z \rangle^{3/4}},$$

by writing

$$(R_{\mathcal{B}}(z)\mathcal{A})^3 = (R_{\mathcal{B}}(z)\mathcal{A})^2 R_{\mathcal{B}}(z)\mathcal{A}$$

and by using estimates (i) in Lemma 2.2.3 with $\zeta = -1/4$, (2.2.33) with $\zeta = -1/4$ and (2.2.32).

Step 2. Here, we follow [50, Proof of Theorem 2.1] and [75] by taking advantage of the analysis of degenerate-meromorphic functions performed in [67]. Iterating the factorization identity

$$R_\Lambda = R_{\mathcal{B}} - R_{\mathcal{B}}\mathcal{A}R_\Lambda,$$

we obtain

$$(2.2.35) \quad (I - \mathcal{V})R_\Lambda = \mathcal{U} \quad \text{in} \quad \mathcal{B}(X)$$

with

$$\begin{aligned} \mathcal{U} &:= R_{\mathcal{B}} - R_{\mathcal{B}}\mathcal{A}R_{\mathcal{B}} + (R_{\mathcal{B}}\mathcal{A})^2 R_{\mathcal{B}}, \\ \mathcal{V} &:= -(R_{\mathcal{B}}\mathcal{A})^3. \end{aligned}$$

Because of (2.2.34) and (2.2.30) as well as of the identity, which comes from the opposite of the Laplace transform,

$$R_{\mathcal{B}}(\mathcal{A}R_{\mathcal{B}})^\ell(z) = (-1)^{\ell+1} \int_0^\infty e^{-zt} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t) dt \in \mathcal{B}(X)$$

for any $z \in \Delta_a$, we see that all the terms involved in (2.2.35) act in $\mathcal{B}(X)$, so that we also have

$$(2.2.36) \quad (I - \mathcal{V})R_L = \mathcal{U} \quad \text{in} \quad \mathcal{B}(X).$$

Because of (2.2.34), we see that $I - \mathcal{V}$ is invertible on $\Delta_a \setminus B(0, M)$ for $M > 0$ large enough and (2.2.36) tells us that $R_L = (I - \mathcal{V})^{-1}\mathcal{U}$ is uniformly bounded in $\Delta_a \setminus B(0, M)$, in particular $\Sigma(L) \subset \Delta_a^c \cup B(0, M)$.

On the other hand, because of **(A3)**, there holds

$$\|\mathcal{V}(z)\|_{X \rightarrow Y} \leq C,$$

with compact embedding $Y \subset X$. As a consequence of the fact that \mathcal{V} is the opposite of the Laplace transform of the nice function $\mathbb{R}_+ \rightarrow X$, $t \mapsto (S_{\mathcal{B}}\mathcal{A})^{(*3)}(t)$, we also deduce that \mathcal{V} is holomorphic in $\mathcal{B}(X)$ on Δ_a . We define $\Psi(z) := I - \mathcal{V}(z)$. Because Ψ is an holomorphic function on Δ_a , $\Psi(z)$ is invertible for (some) $z \in \Delta_a \setminus B(0, M)$ and $R\mathcal{V}(z) \subset Y$ for

any $z \in \Delta_a$, [67, Corollary II] asserts that the function $z \mapsto \Psi(z)^{-1}$ is a degenerate-meromorphic operators valued function, that is Ψ is holomorphic on $\Delta_a \setminus D$, $D \subset \Delta_a$ is discrete, any point $\xi \in D$ is an isolate pole and the coefficients of the principal part in the Laurent series are finite rank operators. Next, from [67] and the identity

$$R_L = \Psi^{-1}\mathcal{U},$$

we deduce that R_L is also degenerate-meromorphic on Δ_a , in particular $\Sigma(L) \cap \Delta_a$ is a finite set of discrete eigenvalues, which means isolated eigenvalues associated to an algebraically finite eigenspace. That is nothing but (2.2.21). We define $\Pi := I - \Pi_{\Lambda, a}$.

Step 3. We claim that for any $n \geq 2$, there holds

$$(2.2.37) \quad \begin{aligned} S_L(t)\Pi &= \left(S_{\mathcal{B}}(t) + \dots + (S_{\mathcal{B}\mathcal{A}})^{*(n-1)} * S_{\mathcal{B}}(t) \right) \Pi \\ &\quad + (-1)^{n+1} \frac{i}{2\pi} \int_{\uparrow_a} (R_{\mathcal{B}\mathcal{A}})^n(z) R_L(z) \Pi e^{zt} dz \end{aligned}$$

in $\mathcal{B}(X)$ where $\uparrow_c := \{c + iy, y \in \mathbb{R}\}$ is the complex line of abscissa $c \in \mathbb{R}$. In order to prove that identity, we iterate the Duhamel formula

$$S_{\Lambda}(t) = S_{\mathcal{B}}(t) + S_{\mathcal{B}\mathcal{A}} * S_{\Lambda}(t),$$

in $\mathcal{B}(X)$ and for any $f \in X$ and $n \geq 2$. We then obtain

$$S_L(t)\Pi f = \left(S_{\mathcal{B}}(t) + \dots + (S_{\mathcal{B}\mathcal{A}})^{*(n-1)} * S_{\mathcal{B}}(t) \right) \Pi f + g_n(t),$$

with

$$g_n(t) := (S_{\mathcal{B}\mathcal{A}})^{*(n)} * S_L(t)\Pi f.$$

Because $t \mapsto g'_n(t)e^{-bt} = \mathcal{B}g_n(t)e^{-bt} + \mathcal{A}g_{n-1}(t)e^{-bt} \in L^1(\mathbb{R}_+, X)$ for $b > 0$ large enough from **(A1)**-**(A2)** and the fact that $t \mapsto S_L(t)\Pi f e^{-bt} \in L^\infty(X)$, the following inverse Laplace transform formula holds

$$g_n(t) = (-1)^{n+1} \frac{i}{2\pi} \int_{\uparrow_b} (R_{\mathcal{B}\mathcal{A}})^n(z) R_L(z) \Pi f e^{zt} dz, \quad \forall t \geq 0.$$

From the preceding steps, the function under the integral sign is holomorphic on Δ_a^* and we may move the line of integration from \uparrow_b to \uparrow_a . Observing that every term in the resulting identity belongs to X , we conclude to (2.2.37). More precisely and importantly, taking $n = 6$, using (2.2.34) and the fact that $\|R_L(z)\Pi\|_{\mathcal{B}(X)}$ is uniformly bounded on Δ_a from Step 2, we have

$$\begin{aligned} \|g_n(t)\|_X &\leq \frac{e^{at}}{2\pi} \int_{\uparrow_a} \|(R_{\mathcal{B}\mathcal{A}})^6(z)\|_{\mathcal{B}(X)} \|R_L(z)\Pi\|_{\mathcal{B}(X)} dz \|f\|_X \\ &\lesssim e^{at} \|f\|_X, \end{aligned}$$

for any $t \geq 0$. We conclude to (2.2.22) using that the other terms involved in (2.2.37) are similarly bounded thanks to **(A1)** and using a weakly $*$ density argument. \square

2.2.3 The vanishing connectivity regime

When the network connectivity parameter vanishes, $\varepsilon = 0$, the linearized time elapsed operator simplifies into

$$(2.2.38) \quad \Lambda_0 g = -\partial_x g - a(x, 0)g + \delta_{x=0} \mathcal{M}_0[g],$$

where

$$\mathcal{M}_0[g] = \int_0^\infty a(x, 0)g(x) dx.$$

The associated semigroup is then positive and the following version of the Krein-Rutman theorem holds.

Theorem 2.2.4. *There exist some constants $\alpha_0 < 0$ and $C > 0$ such that $\Sigma(\Lambda_0) \cap \Delta_{\alpha_0} = \{0\}$ and for any $g_0 \in \mathcal{X}$, $\langle g_0 \rangle = 0$, there holds*

$$(2.2.39) \quad \|S_{\Lambda_0}(t)g_0\|_{\mathcal{X}} \leq C e^{\alpha_0 t} \|g_0\|_{\mathcal{X}}, \quad \forall t \geq 0.$$

We denote

$$\mathcal{X}_+ := \{g \in M^1(\mathbb{R}_+); g \geq 0\},$$

the space of bounded and nonnegative Radon measures.

We start with two elementary auxiliary results.

Lemma 2.2.5. *S_{Λ_0} is positive: $S_{\Lambda_0}(t)g \in \mathcal{X}_+$ for any $g \in \mathcal{X}_+$ and any $t \geq 0$.*

Proof of Lemma 2.2.5. We introduce a dual problem of (2.2.38) defined on the space $C_0(\mathbb{R})$ by

$$(2.2.40) \quad \partial_t \varphi = \tilde{\Lambda} \varphi = \tilde{\mathcal{B}} \varphi + \tilde{\mathcal{A}} \varphi, \quad \varphi(0, \cdot) = \varphi_0,$$

where the operators $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are defined by

$$\begin{aligned} \tilde{\mathcal{B}} \varphi &= \partial_x \varphi - a(x, 0)\varphi, \\ \tilde{\mathcal{A}} \varphi &= a(x, 0)\varphi(0). \end{aligned}$$

A solution φ to (2.2.40) then satisfies

$$\varphi(t) = S_{\tilde{\mathcal{B}}}(t)\varphi_0 + (S_{\tilde{\mathcal{B}}} * \tilde{\mathcal{A}}\varphi)(t).$$

Let us fix $\varphi_0 \in C_0(\mathbb{R})$ such that $\varphi_0 \leq 0$ and let us prove that $\varphi(t) \leq 0$ for any $t \geq 0$. We obviously have that $S_{\tilde{\mathcal{B}}}$ is a positive operator and it is a contraction in $C_0(\mathbb{R})$. Taking the positive part in (2.2.40), we get

$$\begin{aligned} \varphi_+(t) &\leq S_{\tilde{\mathcal{B}}}(t)\varphi_{0+} + (S_{\tilde{\mathcal{B}}} * \tilde{\mathcal{A}}\varphi_+)(t) \\ &\leq a_1 \int_0^t S_{\tilde{\mathcal{B}}}(t-s)\varphi_+(0) ds, \end{aligned}$$

so that

$$\|\varphi_+(t)\|_{L^\infty} \leq C \int_0^t \|\varphi_+(s)\|_{L^\infty} ds.$$

From Grönwall's lemma, we deduce that $\varphi_+(t) = 0$ for any $t \geq 0$ and then $\varphi \leq 0$. We conclude by observing that S_{Λ_0} is the dual of $S_{\tilde{\Lambda}}$. \square

Lemma 2.2.6. $-\Lambda_0$ satisfies the following version of the strong maximum principle: for any given $g \in \mathcal{X}_+$ and $\mu \in \mathbb{R}$, there holds

$$g \in D(\Lambda_0) \setminus \{0\} \text{ and } (-\Lambda_0 + \mu)g \geq 0 \text{ imply } g > 0.$$

Proof of Lemma 2.2.6. Suppose that there holds

$$(-\Lambda_0 + \mu)g \geq 0$$

for g satisfying the above conditions. It is only necessary to prove that g does not vanish in \mathbb{R}_+ . Since $g \not\equiv 0$, there exists $x^* \in \mathbb{R}_+$ such that $g(x^*) > 0$. Rewriting the assumption as

$$\partial_x g + (a(x, 0) + \mu)g \geq \delta_{x=0} \int_0^\infty a(x, 0)g \, dx,$$

we observe that

$$(2.2.41) \quad \partial_x (e^{A(x,0)+\mu x} g) = e^{A(x,0)+\mu x} (\partial_x g + (a(x, 0) + \mu)g) \geq 0.$$

(i) For $x \in (x^*, \infty)$, we have

$$e^{A(x,0)+\mu x} g \geq e^{A(x^*,0)+\mu x^*} g(x^*) > 0.$$

(ii) For $x \in (0, x^*)$, by integrating the same equation on $(0, x)$, we obtain

$$\begin{aligned} e^{A(x,0)+\mu x} g &\geq \int_0^x \delta_{u=0} e^{A(u,0)+\mu u} \int_0^\infty a(y, 0)g(y) \, dy \, du + g(0) \\ &\geq \int_0^\infty a(y, 0)g(y) \, dy. \end{aligned}$$

From the positivity assumption (2.1.7) on a and step (i), we have

$$\int_0^\infty a(y, 0)g(y) \, dy > \frac{a_0}{2} \int_{\max\{x_0, x^*\}}^\infty g(y) \, dy > 0.$$

Therefore, g does not vanish on $(0, \infty)$. □

Proof of Theorem 2.2.4. First, we know from Theorem 2.1.1 that there exists at least one nonnegative and non-vanishing solution F_0 to the eigenvalue problem $\Lambda_0 F_0 = 0$ and the associated dual eigenvector is $\psi = 1$.

Next, we observe that, defining the sign f operator for $f \in D(\Lambda_0^2)$ by

$$((\text{sign } f)^* \psi)(x) := \frac{1}{2|f(x)|} \left(\bar{f}(x)\psi(x) + f(x)\bar{\psi}(x) \right), \quad \forall \psi \in C_0(\mathbb{R}),$$

we have, for any $\psi \in C_0(\mathbb{R})_+$,

$$\begin{aligned} \Re \langle (\text{sign } f) \mathcal{M}_0[f], \psi \rangle &= \Re \langle \mathcal{M}_0[f], (\text{sign } f)^* \psi \rangle \\ &= \Re \left(\left(\int a_0 f \right) \frac{\bar{f}(0)\psi(0) + f(0)\bar{\psi}(0)}{2|f(0)|} \right) \\ &= \Re \left(\int a_0 f \right) \frac{\Re f(0)}{|f(0)|} \psi(0) \\ &\leq \left(\int a_0 |f| \right) \psi(0) = \langle \mathcal{M}_0[|f|], \psi \rangle, \end{aligned}$$

which is nothing but the complex Kato's inequality

$$(2.2.42) \quad \forall f \in D(\Lambda_0^2), \quad \Re e(\text{sign}f) \Lambda_0 f \leq \Lambda_0 |f|.$$

We also observe that $D(\Lambda_0^2) \subset C_b(0, \infty)$, and, as a consequence, $g \in D(\Lambda_0^2)$ and $|g| > 0$ implies $g > 0$ or $g < 0$. We then may use exactly the same argument as in [50, 45, Proof of Theorem 5.3] (see also [44]):

- Kato's inequality (2.2.42) and the strong maximum principle imply that the eigenvalue $\lambda = 0$ is simple and the associated eigenspace is $\text{Vect}(F_0)$;
- together with the fact that S_{Λ_0} is a positive semigroup, one deduces that $\lambda = 0$ is the only eigenvalue with nonnegative real part.

We conclude to the spectral gap estimate (2.2.39) for some $\alpha_0 \in (a^*, 0)$ with the help of Theorem 2.2.2. \square

2.2.4 Weak connectivity regime - exponential stability of the linearized equation

We extend the exponential stability property which holds for a vanishing connectivity to the weak connectivity regime thanks to a perturbation argument.

Theorem 2.2.7. *There exist some constants $\varepsilon_0 > 0$, $\alpha < 0$ and $C > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ there hold $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

$$(2.2.43) \quad \|S_{\Lambda_\varepsilon}(t)g_0\|_{\mathcal{X}} \leq C e^{\alpha t} \|g_0\|_{\mathcal{X}}, \quad \forall t \geq 0,$$

for any $g_0 \in \mathcal{X}$, $\langle g_0 \rangle = 0$.

We introduce first the following lemma in order to establish the conclusion.

Lemma 2.2.8. *Consider two projectors P, Q acting on a Banach space X and in $\mathcal{B}(X)$ satisfying*

$$\|P - Q\|_{\mathcal{B}(X)} \leq 1.$$

Then the range $R(P)$ is isomorphic to the one $R(Q)$. In particular,

$$\dim(R(P)) = \dim(R(Q)).$$

Proof of Lemma 2.2.8. For two projectors P and Q , we set

$$(2.2.44) \quad R = (P - Q)^2 = P + Q - PQ - QP,$$

which commutes with P and Q , since

$$PR = P - PQP = RP$$

and similarly for Q . And so does $(1 - P - Q)^2$. From direct computations, we actually have

$$(P - Q)^2 + (1 - P - Q)^2 = I$$

as well as

$$\begin{aligned} (PQ - QP)^2 &= \left((P - Q)(I - P - Q) \right)^2 = (P - Q)^2 (I - P - Q)^2 \\ &= (P - Q)^2 (1 - (P - Q)^2) \\ &= (P - Q)^2 - (P - Q)^4 = R - R^2. \end{aligned}$$

Furthermore, we set

$$\begin{aligned} U' &= QP + (I - Q)(I - P) \\ V' &= PQ + (I - P)(I - Q). \end{aligned}$$

Obviously, there holds

$$\begin{aligned} R(P) &\xrightarrow{U'} R(Q), & R(I - P) &\xrightarrow{U'} R(I - Q), \\ R(Q) &\xrightarrow{V'} R(P), & R(I - Q) &\xrightarrow{V'} R(I - P), \end{aligned}$$

and

$$V'U' = U'V' = I - R.$$

When $\|R\|_{\mathcal{B}(X)} < 1$, we are able to define

$$(I - R)^{-1/2} := \sum_{n=0}^{\infty} \binom{-1/2}{n} (-R)^n$$

as the inverse square root of $I - R$. From the fact that R commutes with P and Q , therefore also with U' and V' , we define a couple of mutually inverse operators as

$$\begin{aligned} U &:= U'(I - R)^{-1/2} = (I - R)^{-1/2}U' \\ V &:= V'(I - R)^{-1/2} = (I - R)^{-1/2}V', \end{aligned}$$

satisfying

$$UV = VU = 1.$$

From the definition of U' and V' , we have

$$\begin{cases} U'P = QP = QU', \\ PV' = PQ = V'Q, \end{cases}$$

which implies

$$UP = QU, \quad PV = VQ$$

thanks to the commutativity of all the operators considered. Thus, we finally obtain

$$Q = UPU^{-1}, \quad P = VQV^{-1},$$

which implies that P and Q are similar to each other and isomorphic in the sense of linear application. In particular, their ranges $R(P)$ and $R(Q)$ are isomorphic and there holds additionally

$$\dim P = \dim Q.$$

□

Now, we come back to the proof of Theorem 2.2.7, which follows the stability theory for semigroups developed in Kato's book [36] and revisited in [46, 72, 44, 51].

Proof of Theorem 2.2.7. With the definitions (2.2.18), (2.2.23a) and (2.2.23b) of \mathcal{M}_ε , \mathcal{A}_ε and \mathcal{B}_ε , we have

$$(\mathcal{B}_\varepsilon - \mathcal{B}_0)g = (a(x, 0) - a_\varepsilon)g$$

and

$$(\mathcal{A}_\varepsilon - \mathcal{A}_0)g = \delta_0(\mathcal{M}_\varepsilon[g] - \mathcal{M}_0[g]) - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g],$$

where we recall $a_\varepsilon = a(x, \varepsilon M_\varepsilon)$ and $a'_\varepsilon = \varepsilon(\partial_\mu a)(x, \varepsilon M_\varepsilon)$. Together with the smoothness assumption (2.1.8), we deduce

$$\begin{aligned} \|(\mathcal{B}_\varepsilon - \mathcal{B}_0)g\|_{\mathcal{X}} &= \|(a(x, 0) - a_\varepsilon)g\|_{\mathcal{X}} \\ &\leq \varepsilon \|a'\|_{L^\infty} \|g\|_{\mathcal{X}}, \\ \|(\mathcal{A}_\varepsilon - \mathcal{A}_0)g\|_{\mathcal{X}} &= |\mathcal{M}_\varepsilon[g] - \mathcal{M}_0[g]| + \|a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g]\|_{\mathcal{X}} \\ &\leq \|(a_\varepsilon - a(x, 0))g\|_{\mathcal{X}} + 2\|a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g]\|_{\mathcal{X}} \\ &\leq \varepsilon \|a'\|_{L^\infty} \|g\|_{\mathcal{X}} + 2\varepsilon a_1 \|a'\|_{L^\infty} (1 - \kappa)^{-1} \|F_\varepsilon\|_{\mathcal{X}} \|g\|_{\mathcal{X}} \\ &= C \varepsilon \|g\|_{\mathcal{X}}, \end{aligned}$$

which implies

$$(2.2.45) \quad \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{\mathcal{B}(\mathcal{X})} + \|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{\mathcal{B}(\mathcal{X})} \leq C \varepsilon, \quad \forall \varepsilon \geq 0.$$

We then argue similarly as in the proof of [72, Theorem 2.15] (see also [36, 46, 44, 51]) and therefore just sketch the proof. We now define

$$K_\varepsilon(z) := (R_{\mathcal{B}_\varepsilon}(z)\mathcal{A}_\varepsilon)^2 R_{\Lambda_0}(z)(\Lambda_\varepsilon - \Lambda_0) \in \mathcal{B}(\mathcal{X}, X)$$

and we take any $\alpha \in (\alpha_0, 0)$, recalling that $a^* < \alpha_0 < 0$. We deduce from (2.2.45) and the estimates (i) and (ii) in Lemma 2.2.3 that for some $\eta < |\alpha|$, $\varepsilon_0 > 0$, small enough, $C > 0$, and for any $z \in \Delta_\alpha \setminus B(0, \eta)$, $\varepsilon \in [0, \varepsilon_0)$, we have

$$(2.2.46) \quad \|K_\varepsilon(z)\|_{\mathcal{B}(X)} \leq C \varepsilon < C \varepsilon_0 < 1,$$

and thus $(1 - K_\varepsilon(z))^{-1}$ is well defined in $\mathcal{B}(X)$. Using the elementary identities

$$(2.2.47) \quad \begin{aligned} R_{\Lambda_\varepsilon} &= R_{\mathcal{B}_\varepsilon} - R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon R_{\mathcal{B}_\varepsilon} + (R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2 R_{\Lambda_\varepsilon} \\ &=: \mathcal{U}_\varepsilon + (R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2 R_{\Lambda_\varepsilon}, \end{aligned}$$

and

$$R_{\Lambda_\varepsilon} = R_{\Lambda_0} + R_{\Lambda_0}(\Lambda_\varepsilon - \Lambda_0)R_{\Lambda_\varepsilon},$$

we get

$$(I - K_\varepsilon)R_{\Lambda_\varepsilon} = \mathcal{U}_\varepsilon + (R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2 R_{\Lambda_0}.$$

Because both terms in the above identity are well defined in $\mathcal{B}(X)$ from Lemma 2.2.3, we have

$$\mathcal{U}_\varepsilon \in \mathcal{B}(\mathcal{X}) \cap \mathcal{B}(X), \quad (R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2 \in \mathcal{B}(\mathcal{X}, X).$$

Thanks to the link between resolvent and semigroup, we also have $R_{\Lambda_\varepsilon}(b)|_X = R_{L_\varepsilon}(b)$ for $b \in \mathbb{R}$ large enough, which implies furthermore

$$R_{\Lambda_\varepsilon}(z)|_X = R_{L_\varepsilon}(z), \quad \forall \Delta_\alpha \setminus B(0, \eta)$$

by unique continuity principle for holomorphic functions. All together, we deduce

$$(I - K_\varepsilon)R_{L_\varepsilon} = \mathcal{U}_\varepsilon + (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{L_0},$$

from what we deduce

$$R_{L_\varepsilon} = (I - K_\varepsilon)^{-1} \left(\mathcal{U}_\varepsilon + (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{L_0} \right).$$

Since the RHS expression is clearly uniformly bounded in $\mathcal{B}(X)$ on the complex set $\Delta_\alpha \setminus B(0, \eta)$ for any $\varepsilon \in [0, \varepsilon_0)$, we have $\Sigma(L_\varepsilon) \cap \Delta_\alpha \subset B(0, \eta)$.

Using the definition of the eigenprojector Π_0 on the eigenspace associated to the spectral values of Λ_0 lying in $B(0, \eta)$ by mean of Dunford integral (see [36, Section III.6.4] or [31, 44]) and the analogous of (2.2.47) in $\mathcal{B}(X)$, we have

$$\begin{aligned} \Pi_0 &= \frac{i}{2\pi} \int_{|z|=\eta} R_{L_0}(z) dz \\ &= \frac{i}{2\pi} \int_{|z|=\eta} (R_{\mathcal{B}_0}\mathcal{A}_0)^2 R_{L_0} dz, \end{aligned}$$

by using that the contribution of holomorphic functions vanish. In a similar way, we have

$$\begin{aligned} \Pi_\varepsilon &= \frac{i}{2\pi} \int_{|z|=\eta} (I - K_\varepsilon + K_\varepsilon) R_{L_\varepsilon} dz \\ &= \frac{i}{2\pi} \int_{|z|=\eta} (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{L_0} dz + \frac{i}{2\pi} \int_{|z|=\eta} K_\varepsilon R_{L_\varepsilon} dz. \end{aligned}$$

For $g \in X$, we next compute

$$\begin{aligned} \|(\Pi_\varepsilon - \Pi_0)g\|_X &\leq \frac{1}{2\pi} \int_{|z|=\eta} \|((R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 - (R_{\mathcal{B}_0}\mathcal{A}_0)^2) R_{\Lambda_0} g\|_X dz \\ &\quad + \frac{1}{2\pi} \int_{|z|=\eta} \|K_\varepsilon R_{L_\varepsilon} g\|_X dz \\ &\leq C \varepsilon \|g\|_X, \end{aligned}$$

where we have used the identity

$$\begin{aligned} (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 - (R_{\mathcal{B}_0}\mathcal{A}_0)^2 &= R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon R_{\mathcal{B}_\varepsilon} \left((\mathcal{A}_\varepsilon - \mathcal{A}_0) + (\mathcal{B}_0 - \mathcal{B}_\varepsilon) R_{\mathcal{B}_0}\mathcal{A}_0 \right) \\ &\quad + R_{\mathcal{B}_\varepsilon} \left((\mathcal{A}_\varepsilon - \mathcal{A}_0) + (\mathcal{B}_0 - \mathcal{B}_\varepsilon) R_{\mathcal{B}_0}\mathcal{A}_0 \right) R_{\mathcal{B}_0}\mathcal{A}_0 \end{aligned}$$

and the estimates (2.2.46) and (2.2.45). As a consequence, we deduce

$$\|\Pi_\varepsilon - \Pi_0\|_{\mathcal{B}(X)} < 1.$$

for any $\varepsilon \in (0, \varepsilon_0)$, up to take a smaller real number $\varepsilon_0 > 0$. From Lemma 2.2.8, we deduce that there exists $\xi_\varepsilon \in \Delta_\alpha$ such that

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\}, \quad \xi_\varepsilon \text{ is a simple eigenvalue,}$$

for any $\varepsilon \in [0, \varepsilon_0]$. We conclude by observing that $\xi_\varepsilon = 0$ because $1 \in X'$ and $\Lambda_\varepsilon^* 1 = 0$ (which is nothing but the mass conservation). \square

2.2.5 Weak connectivity regime - nonlinear exponential stability

Now, we focus on the nonlinear exponential stability of the solution to the evolution equation (2.1.1)–(2.1.3) in the case without delay. We start with an auxiliary result. We define the function $\Phi : L^1(\mathbb{R}_+) \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Phi[g, \mu] := \int_0^\infty a(x, \varepsilon \mu) g(x) dx - \mu.$$

We denote by W_1 the optimal transportation Monge-Kantorovich-Wasserstein distance on the probability measures set $\mathbf{P}(\mathbb{R}_+)$ associated to the distance $d(x, y) = |x - y| \wedge 1$, or equivalently defined by

$$\forall f, g \in \mathbf{P}(\mathbb{R}_+), \quad W_1(f, g) := \sup_{\varphi, \|\varphi\|_{W^{1,\infty}} \leq 1} \int_0^\infty (f - g) \varphi.$$

Lemma 2.2.9. *Assume (2.1.8). There exists $\varepsilon_0 > 0$ and for any $\varepsilon \in (0, \varepsilon_0)$ there exists a function $\varphi_\varepsilon : \mathbf{P}(\mathbb{R}) \rightarrow \mathbb{R}$ which is Lipschitz continuous for the weak topology of probability measures and such that $\mu = \varphi_\varepsilon[g]$ is the unique solution to the equation*

$$\mu \in \mathbb{R}_+, \quad \Phi(g, \mu) = 0.$$

Proof of Lemma 2.2.9. Step 1. Existence. For any $g \in \mathbf{P}(\mathbb{R})$ we have $\Phi(g, 0) > 0$, and for any $g \in \mathbf{P}(\mathbb{R})$ and $\mu \geq 0$, we have

$$\Phi(g, \mu) \leq \|a\|_{L^\infty} - \mu,$$

so that $\Phi(g, \mu) < 0$ for $\mu > \|a\|_{L^\infty}$. By the intermediate value theorem and the continuity property of Φ , for any fixed $g \in \mathbf{P}(\mathbb{R}_+)$ and $\varepsilon \geq 0$, there exists at least one solution $\mu \in (0, \|a\|_{L^\infty}]$ to the equation $\Phi(g, \mu) = 0$.

Step 2. Uniqueness and Lipschitz continuity. Fix $f, g \in \mathbf{P}(\mathbb{R}_+)$ and consider $\mu, \nu \in \mathbb{R}_+$ such that

$$\Phi(f, \mu) = \Phi(g, \nu) = 0.$$

We have

$$\nu - \mu = \int_0^\infty a(x, \varepsilon \nu)(g - f) + \int_0^\infty (a(x, \varepsilon \nu) - a(x, \varepsilon \mu))f,$$

with

$$\left| \int_0^\infty a(x, \varepsilon \nu)(g - f) \right| \leq \|a(\cdot, \varepsilon \nu)\|_{W^{1,\infty}} W_1(g, f),$$

and

$$\begin{aligned} \left| \int_0^\infty (a(x, \varepsilon \nu) - a(x, \varepsilon \mu))f \right| &\leq \|a(\cdot, \varepsilon \nu) - a(\cdot, \varepsilon \mu)\|_{L^\infty} \\ &\leq \varepsilon \|\partial_\mu a\|_{L^\infty} |\mu - \nu|. \end{aligned}$$

We then obtain

$$(2.2.48) \quad |\mu - \nu| (1 - \varepsilon \|\partial_\mu a\|_{L^\infty}) \leq \|a(\cdot, \varepsilon \nu)\|_{W^{1,\infty}} W_1(g, f),$$

and we may fix $\varepsilon_0 > 0$ such that $1 - \varepsilon_0 \|\partial_\mu a\|_{L^\infty} \in (0, 1)$, $\varepsilon \in [0, \varepsilon_0]$. On the one hand, for $f = g$, we deduce that $\mu = \nu$ and that uniquely defines the mapping $\varphi_\varepsilon[g] := \mu$. On the other hand, the function is Lipschitz continuous because of (2.2.48). \square

We also recall the following classical Grönwall's type lemma.

Lemma 2.2.10. *Assume that $u \in C([0, \infty); \mathbb{R}_+)$ satisfies the integral inequality*

$$u(t) \leq C_1 e^{at} u_0 + C_2 \int_0^t e^{a(t-s)} u(s)^2 ds, \quad \forall t > 0,$$

for some constants $C_1 \geq 1$, $C_2, u_0 \geq 0$ and $a < 0$. Under the smallness assumption

$$a + 2C_2 u_0 < 0,$$

there holds

$$u(t) \leq \left(1 + \frac{C_1 u_0 C_2}{|a + 2C_2 u_0|}\right) C_1 e^{at} u_0, \quad \forall t \geq 0.$$

Proof of Lemma 2.2.10. We fix $A \in (C_1 u_0, 2C_1 u_0)$, so that $C_1 u(t) \leq A$ at least on a small interval, that is for any $t \in [0, \tau]$, $\tau > 0$ small enough, and then the integral inequality implies on the same interval

$$u(t) \leq C_1 e^{at} u_0 + C_2 C_1^{-1} A \int_0^t e^{a(t-s)} u(s) ds.$$

The classical Grönwall's lemma (for linear integral inequality) and the smallness assumption $a + C_2 C_1^{-1} A \leq 0$ imply

$$\begin{aligned} u(t) &\leq C_1 u_0 e^{(a+C_2 C_1^{-1} A)t} \\ &\leq C_1 u_0 < A \end{aligned}$$

on that interval. By a continuity argument, the first above inequality holds on \mathbb{R}_+ and then with $A := C_1 u_0$. Next, replacing that first estimate in the integral inequality we started with, we get

$$u(t) \leq C_1 e^{at} u_0 + C_2 C_1^2 u_0^2 e^{at} \int_0^t e^{(a+2C_2 u_0)s} ds, \quad \forall t > 0,$$

from which we immediately conclude. \square

We come to the proof of our main result Theorem 2.1.2 in the case without delay in the weak connectivity regime.

Proof of Theorem 2.1.2 in the case without delay in the weak connectivity regime. We split the proof into two steps.

Step 1. New formulation. We start giving a new formulation of the solutions to the evolution and stationary equations in the weak connectivity regime $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is defined in Lemma 2.2.9. For a given initial datum $0 \leq f_0 \in L^1(\mathbb{R}_+)$ with unit mass the solution $f \in C([0, \infty); L^1(\mathbb{R}_+))$ to the evolution equation (2.1.1) and the solution F_ε to the stationary equation (2.1.5) clearly satisfy

$$\begin{aligned} \partial_t f + \partial_x f + a(\varepsilon \varphi[f])f &= 0, & f(t, 0) &= \varphi[f(t, \cdot)], \\ \partial_x F + a(\varepsilon M)F &= 0, & F(0) &= M = \varphi[F], \end{aligned}$$

where here and below the ε and x dependency is often removed without risk of misleading.

We introduce the variation function $g := f - F$ which satisfies the PDE

$$\begin{aligned}\partial_t g &= -\partial_x g - a(\varepsilon \varphi[f])f + a(\varepsilon \varphi[F]) \\ &= -\partial_x g - a(\varepsilon M)g - \varepsilon a'(\varepsilon M)F \mathcal{M}[g] - Q[g]\end{aligned}$$

with

$$Q[g] := a(\varepsilon \varphi[f])f - a(\varepsilon \varphi[F])F - a(\varepsilon \varphi[F])g - \varepsilon a'(\varepsilon \varphi[F])F \mathcal{M}[g],$$

where $\mathcal{M} = \mathcal{M}_\varepsilon$ is defined in (2.2.18). The above PDE is complemented with the boundary condition

$$g(t, 0) = \varphi[f(t, \cdot)] - \varphi[F],$$

and we may write again

$$\begin{aligned}\varphi[f] - \varphi[F] &= \int_0^\infty a(\varepsilon \varphi[f])f - \int_0^\infty a(\varepsilon \varphi[F])F \\ &= \int_0^\infty (a(\varepsilon M)g + \varepsilon a'(\varepsilon M)F \mathcal{M}[g]) + \int_0^\infty Q[g] \\ &= \mathcal{M}[g] + \mathcal{Q}[g],\end{aligned}$$

with

$$\mathcal{Q}[g] := \langle Q[g] \rangle.$$

As a consequence, we have proved that the variation function g satisfies the equation

$$(2.2.49) \quad \partial_t g = \Lambda_\varepsilon g + Z[g],$$

with the nonlinear term

$$Z[g] := -Q[g] + \delta_0 \mathcal{Q}[g].$$

Step 2. The nonlinear term. On the one hand, we obviously have

$$(2.2.50) \quad \langle Z[g] \rangle = 0, \quad \forall g \in M^1(\mathbb{R}_+).$$

On the other hand, in order to get an estimate on the nonlinear term $Z[g]$, we introduce the notation

$$\psi(u) = a(x, \varepsilon m_u) f_u,$$

where, we set

$$\begin{cases} f_u := u f + (1 - u) F, \\ m_u := \varphi[f_u]. \end{cases}$$

We first notice that

$$\psi(0) = a(\varepsilon \varphi[F])F, \quad \psi(1) = a(\varepsilon \varphi[f])f.$$

Second, we have

$$(2.2.51) \quad \psi'(u) = a'_\varepsilon(m_u) f_u m'_u + a_\varepsilon(m_u) g.$$

In order to compute m'_u , we differentiate with respect to u the identity

$$m_u = \int_0^\infty a_\varepsilon(m_u) f_u \, dx,$$

and we have

$$\begin{aligned} m'_u &= \int_0^\infty a'_\varepsilon(m_u) f_u \, dx \, m'_u + \int_0^\infty a_\varepsilon(m_u) f_u \, dx \\ &= \int_0^\infty a'_\varepsilon(m_u) f_u \, dx \, m'_u + \int_0^\infty a_\varepsilon(m_u) g \, dx, \end{aligned}$$

which implies

$$(2.2.52) \quad m'_u = \left(1 - \int_0^\infty a'_\varepsilon(m_u) f_u \, dx\right)^{-1} \int_0^\infty a_\varepsilon(m_u) g \, dx.$$

We may thus observe that $m'_0 = \mathcal{M}[g]$, so that

$$\psi'(0) = a'_\varepsilon(M) F \mathcal{M}_\varepsilon[g] + a_\varepsilon(M) g.$$

Therefore, for any $g \in \mathbf{P}(\mathbb{R}_+)$, $\langle g \rangle = 0$, we have

$$Q[g] = \psi(1) - \psi(0) - \psi'(0).$$

Third, from (2.2.51), we have

$$\psi''(u) = a''_\varepsilon(m_u) f_u (m'_u)^2 + 2a'_\varepsilon(m_u) g m'_u + a'_\varepsilon(m_u) f_u m''_u,$$

and from (2.2.52), we have

$$\begin{aligned} m''(u) &= 2 \left(1 - \int_0^\infty a'_\varepsilon f_u\right)^{-2} \int_0^\infty a_\varepsilon g \int_0^\infty a'_\varepsilon g \\ &\quad + 2 \left(1 - \int_0^\infty a'_\varepsilon f_u\right)^{-3} \int_0^\infty a''_\varepsilon f \left(\int_0^\infty a_\varepsilon g\right)^2. \end{aligned}$$

In the small connectivity regime $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \|a'\|_{L^\infty} < 1$, we get the bound

$$\begin{aligned} \|\psi''(u)\|_X &\leq \|a''_\varepsilon\|_{L^\infty} |m'_u|^2 + 2\|a'_\varepsilon\|_{L^\infty} \|g\|_X |m'_u| \\ &\quad + \|a'_\varepsilon\|_{L^\infty} |m''_u| \\ &\leq \varepsilon^2 \frac{\|a''\|_{L^\infty} \|a\|_{L^\infty}^2}{(1 - \varepsilon \|a'\|_{L^\infty})^2} \|g\|_X^2 + 2\varepsilon \frac{\|a'\|_{L^\infty} \|a\|_{L^\infty}}{1 - \varepsilon \|a'\|_{L^\infty}} \|g\|_X^2 \\ &\quad + 2\varepsilon^3 \frac{\|a''\|_{L^\infty} \|a'\|_\infty \|a\|_{L^\infty}}{(1 - \varepsilon \|a'\|_{L^\infty})^3} \|g\|_X^2 \\ &\quad + 2\varepsilon^2 \frac{\|a'\|_{L^\infty}^2 \|a\|_{L^\infty}}{(1 - \varepsilon \|a'\|_{L^\infty})^2} \|g\|_X^2 \\ &\leq \varepsilon K \|g\|_X^2, \end{aligned}$$

for some constant $K \in (0, \infty)$. Using the Taylor expansion

$$Q[g] = \psi(1) - \psi(0) - \psi'(0) = \int_0^1 (1-u) \psi''(u) \, du,$$

we then obtain

$$\begin{aligned}\|Z[g]\|_X &\leq 2\|Q[g]\|_X \\ &\leq \int_0^1 (1-u)\|\psi''(u)\|_X du \\ &\leq C\varepsilon\|g\|_X^2.\end{aligned}$$

Step 3. Decay estimate. Thanks to the Duhamel formula, the solution g to the evolution equation (2.2.49) satisfies

$$g(t) = S_{\Lambda_\varepsilon}(t)(f_0 - F) + \int_0^t S_{\Lambda_\varepsilon}(t-s)Z[g(s)] ds.$$

Using Theorem 2.2.7 and the second step, we deduce

$$\begin{aligned}\|g(t)\|_X &\leq C e^{\alpha t} \|g_0\|_X + \int_0^t C e^{\alpha(t-s)} \|Z[g(s)]\|_X ds \\ &\leq C e^{\alpha t} \|g_0\|_X + C\varepsilon K \int_0^t e^{\alpha(t-s)} \|g(s)\|_X^2 ds,\end{aligned}$$

for any $t \geq 0$ and for some constant $C \geq 1$, $\alpha < 0$, independent of $\varepsilon \in (0, \varepsilon_0]$. Observing that $\|g(t)\|_X \in C([0, \infty))$, we conclude thanks to Lemma 2.2.10. \square

2.3 Case with delay

This section is devoted to the proof of our main result, Theorem 2.1.2, in the case with delay by following the same strategy as in the case without delay but adapting the functional framework. The main difference comes from the boundary term and it will be explained in the first subsection. We have already proved in Theorem 2.1.1 the existence of a unique stationary solution $(F_\varepsilon, M_\varepsilon)$ in the weak connectivity regime and we may then focus on the evolution equation.

2.3.1 Linearized equation and structure of the spectrum

In order to write as a time autonomous equation the linearized equation (2.1.12)-(2.1.13)-(2.1.14), we introduce the following intermediate evolution equation on a function $v = v(t, y)$

$$(2.3.53) \quad \partial_t v + \partial_y v = 0, \quad v(t, 0) = q(t), \quad v(0, y) = 0,$$

where $y \geq 0$ represents the local time for the network activity. That last equation can be solved with the characteristics method

$$v(t, y) = q(t-y)\mathbf{1}_{0 \leq y \leq t}.$$

Therefore, equation (2.1.14) on the variation $n(t)$ of network activity writes

$$n(t) = \mathcal{D}[v(t)], \quad \mathcal{D}[v] := \int_0^\infty v(y)b(dy),$$

and then equation (2.1.13) on the variation $q(t)$ of discharging neurons writes

$$q(t) = \mathcal{O}_\varepsilon[g(t), v(t)],$$

with

$$\mathcal{O}_\varepsilon[g, v] := \mathcal{N}_\varepsilon[g] + \kappa_\varepsilon \mathcal{D}[v],$$

where

$$\begin{aligned} \mathcal{N}_\varepsilon[g] &:= \int_0^\infty a_\varepsilon(M_\varepsilon)g \, dx, \\ \kappa_\varepsilon &:= \int_0^\infty a'_\varepsilon(M_\varepsilon)F_\varepsilon \, dx. \end{aligned}$$

All together, we may rewrite the linearized system (2.1.12)-(2.1.13)-(2.1.14), as the autonomous system

$$(2.3.54) \quad \partial_t(g, v) = \mathcal{L}_\varepsilon(g, v),$$

where the operator \mathcal{L}_ε is defined by

$$\mathcal{L}_\varepsilon \begin{pmatrix} g \\ v \end{pmatrix} := \begin{pmatrix} -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] \\ -\partial_y v \end{pmatrix},$$

with domain

$$D(\mathcal{L}_\varepsilon) := \{(g, v) \in W^{1,1}(\mathbb{R}_+) \times W^{1,1}(\mathbb{R}_+, \omega); g(0) = v(0) = \mathcal{O}_\varepsilon[g, v]\},$$

where $\omega(x) := e^{-\delta x}$ with $\delta > 0$ defining in (2.1.9). The associated semigroup $S_{\mathcal{L}_\varepsilon}(t)$ acts on

$$X := X_1 \times X_2 := L^1(\mathbb{R}_+) \times L^1(\mathbb{R}_+, \omega).$$

Considering the boundary condition as a source term, we also introduce the semigroup $S_{A_\varepsilon}(t)$ acting on

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 := M^1(\mathbb{R}_+) \times M^1(\mathbb{R}_+, \omega)$$

with the generator $A_\varepsilon = (A_\varepsilon^1, A_\varepsilon^2)$ given by

$$\begin{aligned} A_\varepsilon^1(g, v) &:= -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v], \\ A_\varepsilon^2(g, v) &:= -\partial_y v + \delta_{y=0} \mathcal{O}_\varepsilon[g, v]. \end{aligned}$$

In a similar way as in section 2.2.2, we have $S_{A_\varepsilon}|_X = S_{\mathcal{L}_\varepsilon}$.

As a first step, we establish that the semigroup S_{A_ε} has a nice decomposition structure with finite dimensional principal modes and a fast decaying remainder term.

Theorem 2.3.1. *Assume (2.1.6)-(2.1.7)-(2.1.8) and (2.1.9). The conclusions of Theorem 2.2.2 holds true with $a^\sharp := \max\{a^*, -\delta\} < 0$.*

The result is obtained as a consequence of the Spectral Mapping and Weyl's theory developed in [50, 45, 44] and taken over in section 2.2.2. For that purpose, we introduce the convenient splitting of the operator A_ε on \mathcal{X} as $A_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ defined by

$$\mathcal{B}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{B}_\varepsilon^1(g, v) \\ \mathcal{B}_\varepsilon^2(g, v) \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g \\ -\partial_y v \end{pmatrix}$$

and

$$\mathcal{A}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{A}_\varepsilon^1(g, v) \\ \mathcal{A}_\varepsilon^2(g, v) \end{pmatrix} = \begin{pmatrix} -a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v] \\ \delta_{y=0} \mathcal{O}_\varepsilon[g, v] \end{pmatrix}.$$

It is only necessary to establish the following adequate properties of the operators \mathcal{A}_ε and \mathcal{B}_ε . We skip the rest of the proof and refer to the proof of Theorem 2.2.2 for more details.

Lemma 2.3.2. *Assume that a satisfies conditions (2.1.6)–(2.1.8) and that b satisfies (2.1.9). For any $\varepsilon \geq 0$, the operators \mathcal{A}_ε and \mathcal{B}_ε satisfy :*

- (i) $\mathcal{A}_\varepsilon \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+) \times W^{-1,1}(\mathbb{R}_+, \omega), \mathcal{X})$;
- (ii) $S_{\mathcal{B}_\varepsilon}(t)$ is a^\sharp -hypodissipative in both X and \mathcal{X} ;
- (iii) the family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{\mathcal{B}(\mathcal{X}, Y)} \leq C_\alpha e^{\alpha t}, \quad \forall \alpha > a^\sharp,$$

for some constant $C_\alpha > 0$ and with $Y := Y_1 \times Y_2$, where $Y_1 = BV(\mathbb{R}_+) \cap L^1_1(\mathbb{R}_+)$ and $Y_2 = BV(\mathbb{R}_+, \omega) \cap L^1_1(\mathbb{R}_+, \omega)$.

Proof of Lemma 2.3.2. For simplicity, we ignore the ε dependency without risk of ambiguity and write $a(x) = a(x, \varepsilon M_\varepsilon)$, $A(x) = A(x, \varepsilon M_\varepsilon)$, $S_{\mathcal{A}} = S_{\mathcal{A}_\varepsilon}$, $S_{\mathcal{B}} = S_{\mathcal{B}_\varepsilon}$, $\mathcal{N}[g] = \mathcal{N}_\varepsilon[g]$ and so on.

Step 1. Proof of (i). It is an immediate consequence of the fact that $\mathcal{D} \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+, \omega), \mathbb{R})$ and $\mathcal{N} \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathbb{R})$ because of (2.1.8) and (2.1.9).

Step 2. Proof of (ii). Since $S_{\mathcal{B}^1}$ is nothing but the semigroup $S_{\mathcal{B}}$ defined in (2.2.24) which is a^* -dissipative thanks to Lemma 2.2.3-(ii), we just have to prove the dissipativity of the translation semigroup $S_{\mathcal{B}^2}$ which is defined through the explicit formula

$$(S_{\mathcal{B}^2}(t)v)(y) = v(y-t)\mathbf{1}_{y-t \geq 0}.$$

That follows from

$$\begin{aligned} \|S_{\mathcal{B}^2}(t)v\|_{X_2} &\leq \int_0^\infty |v(y-t)|\mathbf{1}_{y-t \geq 0} e^{-\delta y} dy \\ &\leq e^{-\delta t} \|v\|_{X_2}, \end{aligned}$$

for any $v \in X_2$ and any $t \geq 0$.

Step 3. Proof of (iii). For $(g, v) \in \mathcal{X}$, we have

$$\begin{cases} \mathcal{A}^1 S_{\mathcal{B}}(t)(g, v)(x) = \gamma(x)D(t) + \delta_{x=0}N(t), \\ \mathcal{A}^2 S_{\mathcal{B}}(t)(g, v)(y) = \delta_{y=0}\kappa D(t) + \delta_{y=0}N(t), \end{cases}$$

with

$$\begin{aligned} \gamma(x) &:= \kappa \delta_{x=0} - a'(x)F_\varepsilon(x), \\ N(t) &:= \mathcal{N}[S_{\mathcal{B}^1}(t)g] \\ &= \int_0^\infty a(x)e^{A(x-t)-A(x)}g(x-t)\mathbf{1}_{x-t \geq 0} dx, \\ D(t) &:= \mathcal{D}[S_{\mathcal{B}^2}(t)v] \\ &= \int_0^\infty v(y-t)\mathbf{1}_{y-t \geq 0} b(dy). \end{aligned}$$

We observe that

$$\begin{aligned} |N(t)| &\leq C a_1 e^{3\beta t} \|g\|_{\mathcal{X}_1}, \\ |D(t)| &\leq C e^{\delta t} \|v\|_{\mathcal{X}_2}, \end{aligned}$$

for any $t \geq 0$. We then compute

$$\begin{aligned} N'(t) &= \int_0^\infty \partial_x (a(x) e^{-A(x)}) e^{A(x-t)} g(x-t) \mathbf{1}_{x-t \geq 0} dx \\ &= \int_0^\infty (a'(x) - a(x)^2) e^{A(x-t)-A(x)} g(x-t) \mathbf{1}_{x-t \geq 0} dx, \\ D'(t) &= \int_0^\infty v(y-t) \mathbf{1}_{y-t \geq 0} b'(dy) = b' * \check{v}(t), \end{aligned}$$

from what we get the estimates

$$\begin{aligned} |N'(t)| &\lesssim e^{a^* t} \|g\|_{\mathcal{X}_1}, \\ |D'(t)| &\lesssim e^{-\delta t} \|v\|_{\mathcal{X}_2}. \end{aligned}$$

Denoting

$$\begin{cases} T_1(t)(g, v)(x) := (S_{\mathcal{B}^1} * \mathcal{A}^1 S_{\mathcal{B}})(t)(g, v)(x), \\ T_2(t)(g, v)(y) := (S_{\mathcal{B}^2} * \mathcal{A}^2 S_{\mathcal{B}})(t)(g, v)(y), \end{cases}$$

we compute

$$\begin{aligned} T_1(t)(g, v)(x) &= \int_0^t S_{\mathcal{B}^1}(s) (\gamma(x) D(t-s) + \delta_{x=0} N(t-s)) ds \\ &= \int_0^t e^{A(x-s)-A(x)} (\gamma(x-s) D(t-s) + \delta_{x-s=0} N(t-s) \mathbf{1}_{x-s \geq 0}) ds \\ &= e^{-A(x)} (\nu * \check{D}_t)(x) + e^{-A(x)} N(t-x) \mathbf{1}_{x \leq t}, \\ T_2(t)(g, v)(y) &= \int_0^t S_{\mathcal{B}^2}(s) \delta_{y=0} (\kappa D(t-s) + N(t-s)) ds \\ &= \int_0^t \delta_{y-s=0} (\kappa D(t-s) + N(t-s)) ds \\ &= (\kappa D(t-y) + N(t-y)) \mathbf{1}_{y \leq t}, \end{aligned}$$

where we use the notation $\nu := \gamma e^A$. We next differentiate the above identity and get

$$\begin{aligned} \partial_x T_1(t)(g, v)(x) &= -a(x) e^{-A(x)} ((\nu * \check{D}_t)(x) + N(t-x) \mathbf{1}_{x \leq t}) \\ &\quad - e^{-A(x)} (\nu * \check{D}'_t)(x) + N'(t-x) \mathbf{1}_{x \leq t} \\ &\quad - e^{-A(x)} \nu(x-t) D(0) \mathbf{1}_{x-t \geq 0} - e^{-A(x)} N(0) \delta_{x=t} \\ &\quad + e^{-A(x)} \nu(x) D(t), \\ \partial_y T_2(t)(g, v)(y) &= -(\kappa D' + N')(t-y) \mathbf{1}_{y \leq t} - (\kappa D + N)(0) \delta_{y=t}. \end{aligned}$$

All together, we deduce

$$\begin{cases} \|\partial_x T_1(t)(g, v)(x)\|_{\mathcal{X}_1} \lesssim e^{a^* t} \|(g, v)\|_{\mathcal{X}}, \\ \|\partial_y T_2(t)(g, v)(y)\|_{\mathcal{X}_2} \lesssim e^{a^* t} \|(g, v)\|_{\mathcal{X}}, \end{cases}$$

and the similar estimate for $\|(S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{B}})(t)(g, v)\|_{\mathcal{X}}$. As a consequence, the announced estimate holds for the family of operators $S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{B}}$. \square

2.3.2 The vanishing connectivity regime

When the network connectivity parameter vanishes, $\varepsilon = 0$, the linearized operator simplifies as

$$(2.3.55) \quad \Lambda_0 \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a(x, 0)g + \delta_{x=0} \mathcal{O}_0[g, v] \\ -\partial_y v + \delta_{y=0} \mathcal{O}_0[g, v] \end{pmatrix},$$

where

$$\mathcal{O}_0[g, v] = \mathcal{N}_0[g] = \int_0^\infty a(x, 0)g(x) dx.$$

The associated semigroup is exponentially stable as shown in the following theorem.

Theorem 2.3.3. *There exist some constants $\alpha < 0$ and $C > 0$ such that $\Sigma(\Lambda_0) \cap \Delta_\alpha = \{0\}$ and for any $(g_0, v_0) \in \mathcal{X}$, $\langle g_0 \rangle = 0$, there holds*

$$(2.3.56) \quad \|S_{\Lambda_0}(t)(g_0, v_0)\|_{\mathcal{X}} \leq C e^{\alpha t} \|(g_0, v_0)\|_{\mathcal{X}}, \quad \forall t \geq 0.$$

Proof of Theorem 2.3.3. Since $\Lambda_0^1 = \Lambda_0$, from Theorem 2.2.4 we have already proved that $g(t) := S_{\Lambda_0^1}(t)g_0$ satisfies $\|g(t)\|_{\mathcal{X}_1} \leq C e^{at} \|g_0\|_{\mathcal{X}_1}$ for any $t \geq 0$ and any $a \in (a^*, 0)$. We then focus on Λ_0^2 . The Duhamel formula associated to the equation $\partial_t v = \Lambda_0^2(g, v)$ writes

$$v(t) = S_{\mathcal{B}_0^2}(t)v_0 + \int_0^t S_{\mathcal{B}_0^2}(t-s)\mathcal{A}_0^2(g(s), v(s)) ds.$$

Using the already known estimate on $g(t)$, we deduce

$$\begin{aligned} \|S_{\Lambda_0^2}v_0(t)\|_{\mathcal{X}_2} &= \|v(t)\|_{\mathcal{X}_2} \leq \|S_{\mathcal{B}_0^2}(t)v_0\|_{\mathcal{X}_2} + \int_0^t \|S_{\mathcal{B}_0^2}(t-s)\delta_0 \mathcal{N}_0[g(s)]\|_{\mathcal{X}_2} ds \\ &\leq e^{-\delta t} \|v_0\|_{\mathcal{X}_2} + \int_0^t e^{-\delta(t-s)} C e^{as} \|g_0\|_{\mathcal{X}_1} ds \\ &\leq C e^{\alpha t} \|(g_0, v_0)\|_{\mathcal{X}}, \end{aligned}$$

for $\max\{a, -\delta\} < \alpha < 0$, which yields our conclusion. \square

2.3.3 Weak connectivity regime - exponential stability of the linearized equation

In this part, we shall discuss the geometry structure of the spectrum of the linearized time elapsed equation in weak connectivity regime taking delay into account and using again a perturbation argument.

Theorem 2.3.4. *There exists some constants $\varepsilon_0 > 0$, $C \geq 1$ and $\alpha < 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ there holds $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

$$(2.3.57) \quad \|S_{\Lambda_\varepsilon}(t)(g_0, v_0)\|_{\mathcal{X}} \leq C e^{\alpha t} \|(g_0, v_0)\|_{\mathcal{X}},$$

for any $(g_0, v_0) \in \mathcal{X}$ such that $\langle g_0 \rangle = 0$.

We start presenting a technical result needed in the proof below.

Lemma 2.3.5. *The operator A_ε is continuous with respect to ε , and more precisely*

$$(2.3.58) \quad \|A_\varepsilon - A_0\|_{\mathcal{B}(\mathcal{X})} \leq O(\varepsilon).$$

Proof of Lemma 2.3.5. For all $(g, v) \in \mathcal{X}$, we have

$$(2.3.59a) \quad A_\varepsilon \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v] \\ -\partial_y v + \delta_{y=0} \mathcal{O}_\varepsilon[g, v] \end{pmatrix},$$

$$(2.3.59b) \quad A_0 \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a(x, 0)g + \delta_{x=0} \mathcal{O}_0[g, v] \\ -\partial_y v + \delta_{y=0} \mathcal{O}_0[g, v] \end{pmatrix}.$$

From (2.3.59a)-(2.3.59b), we deduce

$$(A_\varepsilon - A_0) \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} (a(x, 0) - a_\varepsilon)g - a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v] + \delta_{x=0} (\mathcal{O}_\varepsilon[g, v] - \mathcal{O}_0[g, v]) \\ \delta_{y=0} (\mathcal{O}_\varepsilon[g, v] - \mathcal{O}_0[g, v]) \end{pmatrix}.$$

We then compute

$$\begin{aligned} \|(A_\varepsilon - A_0)(g, v)\|_{\mathcal{X}} &= \|(a(x, 0) - a_\varepsilon)g\|_{\mathcal{X}_1} + \|a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v]\|_{\mathcal{X}_1} + 2|\mathcal{O}_\varepsilon[g, v] - \mathcal{O}_0[g, v]| \\ &\leq 3\|(a(x, 0) - a_\varepsilon)g\|_{\mathcal{X}_1} + 2\|a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v]\|_{\mathcal{X}_2} \\ &\leq 3\varepsilon \|a'\|_{L^\infty} \|g\|_{\mathcal{X}_1} + 2\varepsilon a_1 \|a'\|_{L^\infty} (1 - \varepsilon \|a'\|_{L^\infty}) \|F_\varepsilon\|_{\mathcal{X}_1} \|v\|_{\mathcal{X}_2} \\ &= C \varepsilon \|(g, v)\|_{\mathcal{X}}, \end{aligned}$$

which is nothing but (2.3.58). \square

Proof of Theorem 2.3.4. With the help of Lemma 2.3.5, we may proceed similarly as in the proof of Theorem 2.3.4 (see also again [72, 51]) and we conclude that

$$\Sigma(A_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\},$$

with $|\xi_\varepsilon| \leq O(\varepsilon)$ and ξ_ε is algebraically simple. We observe that

$$A_\varepsilon^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_x \varphi - a_\varepsilon \varphi + a_\varepsilon (\varphi(0) + \psi(0)) \\ \partial_y \psi + \kappa_\varepsilon b \psi(0) + \kappa_\varepsilon b \varphi(0) - b \int a'_\varepsilon F_\varepsilon \varphi \end{pmatrix},$$

from which we deduce that $A_\varepsilon^*(1, 0) = 0$. Then $0 \in \Sigma(A_\varepsilon^*)$ and $\xi_\varepsilon = 0$. Moreover, the orthogonality condition

$$\left\langle \begin{pmatrix} g_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle_{\mathcal{X}, \mathcal{X}'} = \left(\begin{matrix} \langle g_0 \rangle \\ 0 \end{matrix} \right) = 0$$

implies that the exponential estimate (2.3.57) holds. \square

2.3.4 Weak connectivity regime - nonlinear exponential stability

We finally come back on the nonlinear problem and we present the proof of the second part of our main result for the case with delay in the weak connectivity regime.

Proof of Theorem 2.1.2 in case with delay in the weak connectivity regime. We rewrite the nonlinear equation (2.1.1) with the simple age equation as a system in the following way:

$$\begin{cases} \partial_t f = -\partial_x f - a_\varepsilon(\mathcal{D}[u])f + \delta_0 \mathcal{P}[f, \mathcal{D}[u]], \\ \partial_t u = -\partial_y u + \delta_0 \mathcal{P}[f, \mathcal{D}[u]], \end{cases}$$

with

$$\mathcal{P}[f, m] = \int a(m)f, \quad \mathcal{D}[u] = \int b u.$$

We recall that the steady state (F, U) , $U := M\mathbf{1}_{y \geq 0}$, satisfies

$$\begin{cases} 0 = -\partial_x F - a_\varepsilon(M)F + \delta_0 M, \\ 0 = -\partial_y U + \delta_0 M, \end{cases}$$

where M satisfies

$$M = \mathcal{D}[U] = \mathcal{P}[F, \mathcal{D}[U]].$$

We introduce the variation $g := f - F$ and $v = u - U$. The equation on g is

$$\begin{aligned} \partial_t g &= -\partial_x g - a_\varepsilon(\mathcal{D}[u])f + a_\varepsilon(M)F + \delta_0(\mathcal{P}[f, \mathcal{D}[u]] - \mathcal{P}[F, \mathcal{D}[U]]) \\ &= -\partial_x g - a_\varepsilon(M)g - a'_\varepsilon F \mathcal{D}[v] - Q[g, v] + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \Lambda_\varepsilon^1(g, v) + \mathcal{Z}^1[g, v], \end{aligned}$$

where

$$\begin{aligned} \mathcal{Q}[g, v] &:= \langle Q[g, v] \rangle, \\ \mathcal{Z}^1[g, v] &:= -Q[g, v] + \delta_0 \mathcal{Q}[g, v], \end{aligned}$$

with $Q[g, v]$ denoting that

$$\mathcal{Q}[g, v] := a_\varepsilon(\mathcal{D}[u])f - a_\varepsilon(M)F - a_\varepsilon(M)g - a'_\varepsilon F \mathcal{D}[v].$$

The equation on v is

$$\begin{aligned} \partial_t v &= -\partial_y v + \delta_0(\mathcal{P}[f, \mathcal{D}[u]] - \mathcal{P}[F, \mathcal{D}[U]]) \\ &= -\partial_y v + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \Lambda_\varepsilon^2(g, v) + \mathcal{Z}^2[g, v], \end{aligned}$$

where

$$\mathcal{Z}^2[g, v] := \delta_0 \mathcal{Q}[g, v].$$

Consider the function

$$\Phi(\theta) = a_\varepsilon(\mathcal{D}[\theta u + (1 - \theta)U])(\theta f + (1 - \theta)F).$$

Differentiate it directly, we have

$$\begin{aligned} \Phi'(\theta) &= a'_\varepsilon(\mathcal{D}[\theta u + (1 - \theta)U])(\theta f + (1 - \theta)F)\mathcal{D}[v] \\ &\quad + a_\varepsilon(\mathcal{D}[\theta u + (1 - \theta)U])g, \end{aligned}$$

which implies

$$\begin{aligned}\Phi'(0) &= a_\varepsilon(\mathcal{D}[U])g + a'_\varepsilon(\mathcal{D}[U])F\mathcal{D}[v] \\ &= a_\varepsilon(M)g + a'_\varepsilon(M)F\mathcal{D}[v].\end{aligned}$$

Observing

$$\Phi(0) = a_\varepsilon(M)F, \quad \Phi(1) = a_\varepsilon(\mathcal{D}[u])f,$$

we rewrite the term $Q[g, v]$ as

$$Q[g, v] = \Phi(0) - \Phi(1) + \Phi'(0).$$

Differentiate $\Phi'(\theta)$ again, we compute

$$\begin{aligned}\Phi''(\theta) &= a''_\varepsilon(\mathcal{D}[\theta u + (1 - \theta)U])(\theta f + (1 - \theta)F)\mathcal{D}^2[v] \\ &\quad + 2a'_\varepsilon(\mathcal{D}[\theta u + (1 - \theta)U])g\mathcal{D}[v],\end{aligned}$$

satisfying the estimate

$$\begin{aligned}\|\Phi''(\theta)\|_{X_1} &\leq \|a''_\varepsilon\|_{L^\infty}(\theta\|f\|_{X_1} + (1 - \theta)\|F\|_{X_1})|\mathcal{D}[v]| \\ &\quad + 2\|a'_\varepsilon\|_{L^\infty}\|g\|_{X_1}|\mathcal{D}[v]| \\ &\leq \varepsilon^2\|a''\|_{L^\infty}\|v\|_{X_2}^2 + 2\varepsilon\|a'\|_{L^\infty}\|g\|_{X_1}\|v\|_{X_2} \\ &\leq \varepsilon C\|(g, v)\|_X^2.\end{aligned}$$

Thanks to the Taylor expansion, we have

$$\begin{aligned}\|Q[g, v]\|_{X_1} &= \|\psi(1) - \psi(0) - \psi'(0)\|_{X_1} \\ &= \int_0^1 (1 - u)\|\psi''(u)\|_{X_1} du \\ &\leq \varepsilon C\|(g, v)\|_X^2,\end{aligned}$$

which implies

$$\begin{aligned}\|\mathcal{Z}[g, v]\|_X &= \left(\begin{array}{l} \|\mathcal{Z}^1[g, v]\|_{X_1} \\ \|\mathcal{Z}^2[g, v]\|_{X_2} \end{array} \right) \\ &\leq \left(\begin{array}{l} 2\|Q[g, v]\|_{X_1} \\ \|Q[g, v]\|_{X_1} \end{array} \right) \lesssim \varepsilon\|(g, v)\|_X^2.\end{aligned}$$

Referring to Step 3 in the proof of Theorem 2.1.2, we then conclude the rest part of Theorem 2.1.2 in the case with delay in the weak connectivity regime. \square

Chapter 3

General time elapsed model in weak connectivity regime

Résumé

Pour les grands réseaux de neurones entièrement connectés, nous étudions la dynamique des assemblages homogènes de neurones interactifs décrits par les modèles du temps écoulé, indiquant comment le temps écoulé depuis la dernière décharge construit la densité de probabilité des neurones. Sous les hypothèses générales sur le taux de tir qui inclut celles proposées dans les travaux antérieurs [56, 57, 52], nous établissons une estimation précise sur le comportement asymptotique des solutions dans le régime de connectivité faible dans le cas avec et sans délai. Nos résultats améliorent [56, 57] où une estimation moins précise a été établie et [52] où seuls les taux de tir de lissage ont été considérés. Notre approche combine plusieurs arguments introduits dans les travaux précédents ci-dessus ainsi qu'une version légèrement raffinée du théorème de Weyl et du théorème d'application spectrale présentés dans [50].

Abstract

For large fully connected neuron networks, we study the dynamics of homogenous assemblies of interacting neurons described by time elapsed models, indicating how the time elapsed since the last discharge constructs the probability density of neurons. Under general assumptions on the firing rate which include the ones made in previous works [56, 57, 52], we establish accurate estimate on the long time behavior of the solutions in the weak connectivity regime both in the case with and without delay. Our results improve [56, 57] where a less accurate estimate was established and [52] where only smooth firing rates were considered. Our approach combines several arguments introduced in the above previous works as well as a slightly refined version of the Weyl's and spectral mapping theorems presented in [50].

3.1 Introduction

The information transmission and processing mechanism in the nervous systems relies on the quantity of electrical pulses as the reflect to incoming stimulations, during which the neurons experience a period of recalcitrance called discharge time before reactive. In this work, we shall focus on the model describing the neuronal dynamics in accordance with this kind of discharge time which has been introduced and studied in [28, 56, 57]. In order to show the response to the recovery of the neuronal membranes after each discharge, the model consider an instantaneous firing rate depending on the time elapsed since last discharge as well as the inputs of neurons. This sort of models are also regarded as a mean field limit of finite number of neuron network models referred to [15, 25, 68, 66].

For a local time (or internal clock) $x \geq 0$ corresponding to the elapsed time since the last discharge, we consider the dynamic of the neuronal network with the density number of neurons $f = f(t, x) \geq 0$ in state $x \geq 0$ at time $t \geq 0$, given by the following nonlinear time elapsed (or of age structured type) evolution equation

$$(3.1.1a) \quad \partial_t f = -\partial_x f - k(x, \varepsilon m(t))f =: \mathcal{L}_{\varepsilon m(t)} f,$$

$$(3.1.1b) \quad f(t, 0) = p(t), \quad f(0, x) = f_0(x),$$

where $k(x, \varepsilon \mu) \geq 0$ denotes the firing rate of a neuron in state x and in an environment $\mu \geq 0$ formed by the global neuronal activity with a network connectivity parameter $\varepsilon \geq 0$ corresponding to the strength of the interactions. The total density of neurons $p(t)$ undergoing a discharge at time t is defined through

$$(3.1.2a) \quad p(t) := \mathcal{P}[f(t); m(t)],$$

where

$$(3.1.2b) \quad \mathcal{P}[g, \mu] = \mathcal{P}_\varepsilon[g, \mu] := \int_0^\infty k(x, \varepsilon \mu)g(x) dx,$$

while the global neuronal activity $m(t)$ at time $t \geq 0$ taking into account the interactions among the neurons resulting from earlier discharges is given by

$$(3.1.3) \quad m(t) := \int_0^\infty p(t-y)b(dy).$$

Here the delay distribution b is a probability measure taking into account the persistence of the electric activity to those discharges in the network. In the sequel, we will consider the two following situations respectively:

- The *case without delay* when $b = \delta_0$, and then $m(t) = p(t)$.
- The *case with delay* when b is a smooth function.

Observe that the solution f of the time elapsed equation (3.1.1) satisfies

$$\frac{d}{dt} \int_0^\infty f(t, x) dx = f(t, 0) - \int_0^\infty k(x, \varepsilon m(t))f(t, x) dx = 0,$$

in both cases. That implies the conservation of the total density number of neurons (also called *mass* in the sequel) which can be thus normalized to 1. As a consequence, we may assume in the sequel

$$\langle f(t, \cdot) \rangle = \langle f_0 \rangle = 1, \quad \forall t \geq 0, \quad \langle g \rangle := \int_0^\infty g(x) dx.$$

We call steady state a couple $(F_\varepsilon, M_\varepsilon)$ of a nonnegative function and a positive real number which satisfies

$$(3.1.4a) \quad 0 = -\partial_x F_\varepsilon - k(x, \varepsilon M_\varepsilon) F_\varepsilon = \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon,$$

$$(3.1.4b) \quad F_\varepsilon(0) = M_\varepsilon, \quad \langle F_\varepsilon \rangle = 1.$$

Noticing that the associated network activity and the discharge activity are equal constants for a steady state because $\langle b \rangle = 1$.

Our main purpose in this paper is to prove existence, uniqueness and exponential asymptotic stability of solutions to the time elapsed evolution equation (3.1.1) at least in a weak connectivity regime, which is a range of connectivity parameter $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 > 0$ small enough, chosen in such a way that the nonlinear term in equations (3.1.1) and (3.1.4) is not too strong.

These results are obtained for a rather large class of firing rate. More precisely, we make the physically reasonable assumptions

$$(3.1.5) \quad k \geq 0, \quad \partial_x k \geq 0, \quad k' = \partial_\mu k \geq 0,$$

$$(3.1.6) \quad 0 < k_0 := \lim_{x \rightarrow \infty} k(x, 0) \leq \lim_{x, \mu \rightarrow \infty} k(x, \mu) =: k_1 < \infty,$$

as well as the regularity assumption

$$(3.1.7) \quad K(x, \cdot) := \int_0^x k(y, \cdot) dy \in C^0(\mathbb{R}_+), \quad \forall x > 0.$$

We will also need the stronger regularity assumption $k \in L_x^1 \text{Lip}_\mu$, meaning that there exists a constant $C_k \in (0, \infty)$ such that

$$(3.1.8) \quad \int_0^\infty |k(x, \mu_2) - k(x, \mu_1)| dx \leq C_k |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (0, \infty).$$

One example of firing rate which fulfills the above condition is the “step function” rate considered in [56, 57] which is given by

$$(3.1.9) \quad k(x, \mu) = \mathbf{1}_{x > \sigma(\mu)}, \quad \sigma' \leq 0,$$

$$(3.1.10) \quad \sigma(0) = \sigma_+, \quad \sigma(\infty) = \sigma_- < \sigma_+ < 1,$$

where σ satisfies the regularity condition

$$(3.1.11) \quad \sigma, \sigma^{-1} \in W^{1, \infty}(\mathbb{R}_+).$$

In the case with delay, we assume that the delay distribution $b(dy) = b(y)dy$ has the exponential bound and satisfies the smoothness condition

$$(3.1.12) \quad \exists \delta > 0, \quad \int_0^\infty e^{\delta y} (b(y) + |b'(y)|) dy < \infty.$$

Our first result establishes the existence of weak solution to the evolution problem (3.1.1). We call weak solution a function $0 \leq f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)_w) \cap L^\infty(\mathbb{R}_+^2)$ such that

$$\int_0^T \int_0^\infty f (\partial_t \varphi + \partial_x \varphi) dx dt = \int_0^T \int_0^\infty k(x, \varepsilon m) f \varphi dx dt - \int_0^T p \varphi(t, 0) dt$$

for any $\varphi \in C_c^1(\mathbb{R}_+^2)$, where p and m satisfy (3.1.2)-(3.1.3). Here and below L_w^1 denotes the $L^1(\mathbb{R}_+)$ space endowed with the weak topology $\sigma(L^1, L^\infty)$.

Theorem 3.1.1. *We assume that the firing rate k satisfies (3.1.5)-(3.1.6)-(3.1.7) and that the delay distribution b satisfies (3.1.12) or satisfies $b = \delta_0$ with $\varepsilon \in (0, \varepsilon_0)$. There exists $\varepsilon_0 > 0$ small enough such that for any unit mass initial datum $0 \leq f_0 \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and any $\varepsilon > 0$ (in the delay case) or any $\varepsilon \in (0, \varepsilon_0)$ (in the case without delay), there exists a nonnegative and mass conserving weak solution $0 \leq f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)_w) \cap L^\infty(\mathbb{R}_+^2)$ to the evolution equation (3.1.1)-(3.1.2)-(3.1.3) for some functions $m, p \in C(\mathbb{R}_+)$. Moreover, there hold*

$$(3.1.13) \quad \|f_t\|_{L^1} = \|f_0\|_{L^1} = 1 \quad \text{and} \quad \|f_t\|_{L^\infty} \leq \|f_0\|_{L^\infty} + k_1.$$

The proof follows the same strategy as in [56, 57] and makes use of a Schauder fixed point theorem.

As a second step, we state an existence of solution to the stationary problem (3.1.4) and the uniqueness of that solution in the weak connectivity regime.

Theorem 3.1.2. *Under the above assumption (3.1.5)-(3.1.6)-(3.1.7) on the firing rate, for any $\varepsilon \geq 0$, there exists at least one couple $(F_\varepsilon(x), M_\varepsilon) \in W^{1,\infty}(\mathbb{R}_+) \times \mathbb{R}_+$ solutions to the stationary problem (3.1.4), and such that*

$$(3.1.14) \quad 0 \leq F_\varepsilon(x) \lesssim e^{-\frac{k_0}{2}x}, \quad |F'_\varepsilon(x)| \lesssim e^{-\frac{k_0}{2}x}, \quad x \geq 0.$$

Moreover, there exists $\varepsilon_0 > 0$ small enough, such that the above steady state is unique for any $\varepsilon \in [0, \varepsilon_0)$.

The proof being identical to the ones presented in [56] and [52, Theorem 2.1], it will be skipped.

Finally our third and main result in the present paper states the exponential nonlinear stability of the above stationary state in the weak connectivity regime.

Theorem 3.1.3. *We assume that the firing rate a satisfies (3.1.5)-(3.1.6)-(3.1.7)-(3.1.8) and that the delay distribution b satisfies $b = \delta_0$ or (3.1.12). There exists $\varepsilon_0 > 0$, small enough, such that for any $\varepsilon \in (0, \varepsilon_0)$ there exist some constants $\alpha < 0$, $C \geq 1$ and $\eta > 0$ such that for any unit mass initial datum $0 \leq f_0 \in L^1 \cap L^\infty$ satisfying $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta$, the (unique, positive and mass conserving) solution f to the evolution equation (3.1.1) satisfies*

$$(3.1.15) \quad \|f(t, \cdot) - F_\varepsilon\|_{L^1} \leq C e^{\alpha t}, \quad \forall t \geq 0.$$

Roughly speaking this theorem generalizes to the delay case the similar results obtained in [56, 57] and it generalizes the similar result obtained in [52] to a more general firing rate including the step function rate considered in [56, 57].

The proof is mainly based on an extension of the abstract semigroup theory developed in [50, 52] which has probably its own interest. It uses an auxiliary linear problem introduced in [56, 57] instead of the linearized equation considered in [52]. Both arguments together make possible to get ride of the smoothness assumption needed in [52].

This paper is organized as follows. In Section 3.2, we establish the existence and uniqueness results for the evolution equation as stated in Theorem 3.1.1. The estimate on the long time behavior of solutions as formulated in Theorem 3.1.3 is established in Section 3.4 in the case without delay. The case with delay is tackled in Section 3.5.

3.2 Existence of solutions

3.2.1 Delay case

In order to establish the existence of a solution to (3.1.1)-(3.1.2)-(3.1.3), we will apply a Schauder fixed point argument. To begin with, we analyze the continuity property of the functional \mathcal{P} defined in (3.1.2b).

Lemma 3.2.1. *Assume (3.1.5)-(3.1.6)-(3.1.7). Consider a sequence (m_n) of nonnegative real numbers converging to a limit m in \mathbb{R} as well as a sequence of functions (f_n) which converges to f in the sense of the weak topology $\sigma(L^1, L^\infty)$ and is uniformly bounded in L^∞ . We then have*

$$\mathcal{P}[m_n, f_n] \rightarrow \mathcal{P}[m, f], \quad \text{as } n \rightarrow \infty.$$

Proof of Lemma 3.2.1. Step 1. Continuity of k . We are going to show that

$$(3.2.16) \quad k(\cdot, m_n) \rightarrow k(\cdot, m), \quad \text{a.e. as } n \rightarrow \infty.$$

If (m_n) is increasing, the sequence $(k(\cdot, m_n))$ is also increasing because of assumption (3.1.6), and there exists some $\bar{k}(x)$ such that

$$k(x, m_n) \rightarrow \bar{k}(x), \quad \text{as } n \rightarrow \infty, \quad \text{for any } x \geq 0,$$

which in turn implies

$$K(x, m_n) = \int_0^x k(y, m_n) dy \rightarrow \int_0^x \bar{k}(y) dy.$$

From assumption (3.1.7), we deduce

$$\int_0^x k(y, m) dy = K(x, m) = \int_0^x \bar{k}(y) dy, \quad \forall x \geq 0.$$

Thus, we clearly have

$$\bar{k}(x) = k(x, m), \quad \text{for a.e. } x \geq 0.$$

The same holds in the case when (m_n) is a decreasing sequence. In the general case, we may define two monotonous sequences (m_n^i) , $i = 1, 2$, such that $m_n^1 \leq m_n \leq m_n^2$ for any $n \geq 1$ and such that $m_n^i \rightarrow m$ as $n \rightarrow \infty$ for $i = 1, 2$. Then, there holds

$$k(x, m_n^1) \leq k(x, m_n) \leq k(x, m_n^2), \quad \forall n \geq 1, x \geq 0,$$

which implies

$$k(x, m_n^i) \rightarrow k(x, m), \quad i = \{1, 2\},$$

as $k \rightarrow \infty$ for a.e. $x \geq 0$. We immediately conclude that (3.2.16) holds.

Step 2. Continuity of the functional \mathcal{P} . We compute

$$\begin{aligned} \mathcal{P}[m_n, f_n] - \mathcal{P}[m, f] &= \int k(\cdot, m_n) f_n - \int k(\cdot, m) f \\ &= \int (k(\cdot, m_n) - k(\cdot, m)) f_n + \int k(\cdot, m) (f_n - f) \\ &:= I_1 + I_2. \end{aligned}$$

From the assumption (3.1.5) and the weak convergence of f_n in L^1 , we have $I_2 \rightarrow 0$, as $n \rightarrow \infty$. We write

$$I_1 = \int_0^R (k(x, m_n) - k(x, m)) f_n(x) dx + \int_R^\infty (k(x, m_n) - k(x, m)) f_n(x) dx.$$

From Step 1 and the assumption that (f_n) is bounded in L^∞ and uniformly integrable at the infinity (as a consequence of its weak $\sigma(L^1, L^\infty)$ convergence and the Dunford-Pettis theorem), we deduce that

$$|I_1| \leq \|f_n\|_{L^\infty} \int_0^R |k(x, m_n) - k(x, m)| dx + 2k_1 \int_R^\infty f_n(x) dx \rightarrow 0,$$

as $R \rightarrow \infty$ and $n \rightarrow \infty$. The two above estimates together imply the conclusion. \square

In a next step, we fix $T > 0$ and we analyse the linear mapping which associates to a given function $m \in C([0, T])$ the solution $f \in C([0, T]; L^1) \cap L^\infty([0, T]; L^\infty)$ to the transport equation

$$(3.2.17) \quad \begin{aligned} \partial_t f + \partial_x f + k(x, m(t)) f &= 0 \\ f(t, 0) = \mathcal{P}[m(t), f], \quad f(0, x) &= f_0(x). \end{aligned}$$

The following lemma gives the continuity of this mapping.

Lemma 3.2.2. *Fix $T > 0$. Consider a sequence (m_n) such that $m_n \rightarrow \bar{m}$ in $C([0, T])$, as $n \rightarrow \infty$. There exists then a sequence (f_n) of solutions to the linear transport equation (3.2.17) associated to (m_n) and this one satisfies*

$$f_n \rightarrow \bar{f} \quad \text{in } C([0, T]; L_w^1) \cap L^\infty([0, T]; L^\infty), \quad \text{as } n \rightarrow \infty,$$

where \bar{f} stands for the solution to the linear transport equation (3.2.17) associated to \bar{m} .

Proof of Lemma 3.2.2. Step 1. Existence of f_n . For any $m \in C([0, T])$ and any $0 \leq g_1 \in X_T := C([0, T]; L^1) \cap L^\infty(0, T; L^\infty)$ we may associate $0 \leq g_2 \in X_T$ as the solution to the equation

$$(3.2.18) \quad \begin{aligned} \partial_t g_2 + \partial_x g_2 + k(x, m(t))g_2 &= 0 \\ g_2(t, 0) = p_1(t) &:= \mathcal{P}[m(t), g_1], \quad g_2(0, x) = f_0(x), \end{aligned}$$

which is classically defined through the characteristic method. More precisely, we introduce the space

$$\mathcal{C} := \{0 \leq g \in X_T; \|g\|_{L^1} \leq e^{k_1 t}, \|g\|_{L^\infty} \leq \|f_0\|_{L^\infty} + k_1\},$$

and we consider $f_1 \in \mathcal{C}$. Integrating the equation (3.2.18) on x , we find

$$\frac{d}{dt} \int_0^\infty g_2(t, x) dx = \int_0^\infty (g_1(t, x) - g_2(t, x))k(x, m(t)) dx \leq k_1 \|g_1\|_{L^1} \leq k_1 e^{k_1 t},$$

which implies

$$\|g_2\|_{L^1} \leq \int_0^t k_1 e^{k_1 s} ds + \|f_0\|_{L^1} = e^{k_1 t}.$$

Thanks to the method of characteristics the solution $g_2(t, x)$ to equation (3.2.18) can be expressed as

$$g_2(t, x) = \begin{cases} f_0(x-t)e^{-\int_0^t k(s+x-t, m(s)) ds}, & \forall x \geq t, \\ p_1(t-x)e^{-\int_0^x k(s, m(s+t-x)) ds}, & \forall x \leq t, \end{cases}$$

which implies

$$\|g_2\|_{L^\infty} \leq \|f_0\|_{L^\infty} + \|p_1\|_{L^\infty} \leq \|f_0\|_{L^\infty} + k_1.$$

Denoting $g_2 := \mathcal{I}(g_1)$, we have proved $\mathcal{I}(\mathcal{C}) \subset \mathcal{C}$. On the other hand, denoting $h_1 = g_1 - \tilde{g}_1$ and $h_2 := \mathcal{I}(g_1) - \mathcal{I}(\tilde{g}_1)$ for $g_1, \tilde{g}_1 \in \mathcal{C}$, a similar computation as above leads to

$$\sup_{t \in [0, T]} \|h_2(t)\|_{L^1} \leq \frac{1}{k_1} (e^{k_1 T} - 1) \sup_{t \in [0, T]} \|h_1(t)\|_{L^1},$$

from what we conclude to the existence of a unique function $f \in \mathcal{C}$ such that $\mathcal{I}(f) = f$ by a classical contraction fixed point Theorem for $T > 0$ small enough. We get $T > 0$ arbitrary by iterating the argument. By integrating the transport equation (3.2.17), we obtain that the solution is mass preserving. We thus get the existence of the sequence (f_n) and the possible limit \bar{f} by applying the above construction with $m = m_n$ and $m = \bar{m}$.

Step 2. Continuity of the mapping. From equation (3.2.17), we observe that

$$\partial_t f_n \leq -\partial_x f_n, \quad f_n(t, 0) \leq k_1, \quad f_n(0, x) = f_0,$$

which implies

$$f_n(t, x) \leq k_1 \mathbf{1}_{x \leq t} + f_0(x-t) \mathbf{1}_{x \geq t},$$

for any $t \leq T$. From this upper bound, one gets

$$\int_R^\infty f_n \leq \int_R^\infty f_0(x-t) dx,$$

for any $R \geq T$, and in particular

$$\int_R^\infty f_n \leq \int_{R-T}^\infty f_0 \rightarrow 0,$$

as $R \rightarrow \infty$ and uniformly in $n \geq 1$. From equation (3.2.18), for any $\varphi \in C_c^1((0, \infty))$, we also have

$$\frac{d}{dt} \int_0^\infty f_n \varphi dx = A_n^\varphi,$$

with A_n^φ bounded in $L^\infty(0, T)$ uniformly in $n \geq 1$. Together with the fact that $f_n \in \mathcal{C}$ for any $n \geq 1$, we may use the de la Vallée-Poussin Theorem and the Dunford-Pettis Lemma to conclude that there exists $f \in C([0, T]; L_w^1) \cap L^\infty([0, T]; L^\infty)$ and a subsequence $f_{n'}$ of the sequence f_n such that $f_{n'} \rightharpoonup f$ weakly. We deduce $\mathcal{P}[m_{n'}, f_{n'}] \rightarrow \mathcal{P}[m, f]$ as $n' \rightarrow \infty$, from Lemma 3.2.1. We finally conclude by passing n' to the limit in the equation (3.2.17) with $m_{n'}$. \square

Proof of Theorem 3.1.1 - The delay case. We recall that $b \in L^1(\mathbb{R}_+)$ in that case. We consider the application $\mathcal{J} : C([0, T]) \rightarrow C([0, T])$, defined as

$$\mathcal{J}(m)(t) := \int_0^t p(t-y)b(dy), \quad \forall m \in C([0, T]), \forall t \in [0, T],$$

where $p(t) = \mathcal{P}[m(t), f(t, \cdot)]$ and f is a solution to (3.2.17) which existence has been established during the proof of Lemma 3.2.2. From Lemma 3.2.1, we deduce that the application $m \rightarrow \mathcal{P}[m, f]$ is continuous and so is \mathcal{J} . Define

$$\mathcal{K} := \{m \in C([0, T]), \|m\|_{L^\infty} \leq k_1\}.$$

Obviously, \mathcal{K} is a convex subset of $C([0, T])$ and, for any $m \in \mathcal{K}$, we have

$$\|\mathcal{J}(m)\|_{L^\infty} \leq \int_0^t |b(y)| \|p\|_{L^\infty} dy \leq k_1,$$

so that $\mathcal{J} : \mathcal{K} \rightarrow \mathcal{K}$. On the other hand, for any $\epsilon > 0$, there exists $\theta > 0$ such that for any $t, s \in [0, T]$ satisfying $|t - s| < \theta$, we have

$$\begin{aligned} |\mathcal{J}(m)(t) - \mathcal{J}(m)(s)| &\leq \int_0^\infty |b(t-y) - b(s-y)| \|p\|_{L^\infty} dy \\ &\leq \|b(t-\cdot) - b(s-\cdot)\|_{L^1(\mathbb{R}_+)} k_1 \|f_0\|_{L^1} \\ &\leq k_1 \|\tau_\theta b - b\|_{L^1(\mathbb{R}_+)} < \epsilon, \end{aligned}$$

where $\tau_\theta b := b(\cdot + \theta)$, which implies that \mathcal{J} is equicontinuous. Thanks to the Arzela-Ascoli Theorem, we deduce that $\mathcal{J}(\mathcal{K})$ is compactly embedded into \mathcal{K} . Using the Schauder-Brouwer fixed point Theorem, the application \mathcal{J} admits a fixed point $m \in \mathcal{K}$. The corresponding solution f to the equation (3.2.17) is also a solution to the equation (3.1.1)-(3.1.2)-(3.1.3). Iterating on T , we deduce the existence of a global solution (f, m, p) to equation (3.1.1)-(3.1.3). \square

3.2.2 Without delay case

We will need the following auxilliary result. We define the function $\Phi : L^1(\mathbb{R}_+) \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Phi[g, \mu] := \int_0^\infty k(x, \varepsilon \mu) g(x) dx - \mu.$$

Lemma 3.2.3. *Assume (3.1.5)-(3.1.6)-(3.1.7)-(3.1.8). There exists $\varepsilon_0 > 0$ and for any $\varepsilon \in (0, \varepsilon_0)$ there exists a function $\varphi_\varepsilon : L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \rightarrow \mathbb{R}$ which is Lipschitz continuous in any bounded set in the sense of the L^1 norm and such that $\mu = \varphi_\varepsilon[g]$ is the unique solution to the equation*

$$\mu \in \mathbb{R}_+, \quad \Phi(g, \mu) = 0.$$

Proof of Lemma 3.2.3. The proof is similar to the one of [52, Lemma 2.8], thus we skip the existence part and merely present the uniqueness here. Fix $R > 0$ and take $f, g \in L^1 \cap L^\infty$ and $\mu, \nu \in \mathbb{R}_+$ such that

$$\|f\|_{L^\infty} \leq R \quad \text{and} \quad \Phi(f, \mu) = \Phi(g, \nu) = 0.$$

We have

$$\nu - \mu = \int_0^\infty a(x, \varepsilon \nu)(g - f) dx + \int_0^\infty (a(x, \varepsilon \nu) - a(x, \varepsilon \mu)) f dx,$$

with

$$\left| \int_0^\infty a(x, \varepsilon \nu)(g - f) dx \right| \leq k_1 \|f - g\|_{L^1}$$

and

$$\left| \int_0^\infty (a(x, \varepsilon \nu) - a(x, \varepsilon \mu)) f dx \right| \leq R C_k \varepsilon |\mu - \nu|.$$

We then deduce

$$|\mu - \nu| \leq 2k_1 \|f - g\|_{L^1},$$

for any $\varepsilon \in [0, \varepsilon_0]$, with $\varepsilon_0 = \varepsilon_0(R) > 0$ small enough. That implies the uniqueness of the solution $\mu = \varphi_\varepsilon(f) \in \mathbb{R}$ to the constraint problem $\Phi(f, \mu) = 0$ for any given $f \in L^1 \cap L^\infty$ and the Lipschitz continuity of φ_ε .

Proof of Theorem 3.1.1 - The case without delay. We fix $\varepsilon_0 > 0$ as defined in Lemma 3.2.3 and we take $\varepsilon \in (0, \varepsilon_0)$. For a given function $m \in C([0, T])$, we define $M(t) := \varphi_\varepsilon(f(t, \cdot)) \in C([0, T])$, where $f \in C([0, T]; L^1(\mathbb{R}_+)) \cap L^\infty([0, T] \times \mathbb{R}_+)$ is the solution of (3.2.17) associated to m . We denote $\mathcal{K}(m) := M$. For two given $m_1, m_2 \in C([0, T])$, we denote $f_1, f_2 \in C([0, T]; L^1(\mathbb{R}_+)) \cap L^\infty([0, T] \times \mathbb{R}_+)$ the associated solutions to (3.2.17) and we easily compute

$$\begin{aligned} \frac{d}{dt} \int |f_2 - f_1| &\leq 2 \int |k(m_2) f_2 - k(m_1) f_1| \\ &\leq 2 \|f_0\|_{L^\infty} C_k \varepsilon_0 |m_2 - m_1| + 2k_1 \int |f_2 - f_1|. \end{aligned}$$

We deduce that $m \mapsto f$ is Lipschitz from $C([0, T])$ to $C([0, T]; L^1(\mathbb{R}_+))$ with constant CT . As a consequence, \mathcal{K} is Lipschitz from $C([0, T])$ into itself with constant $C'T$. Choosing

$T > 0$ small enough, the mapping \mathcal{K} is a contraction and admits a unique fixed point thanks to the Banach fixed point theorem. Iterating on T , we deduce the existence and uniqueness of a global solution (f, m) to equation (3.1.1)-(3.1.2) in the case without delay and the weak connectivity regime. \square

3.3 A Weyl's and spectral mapping theorem

In this section we establish a simple version of Weyl's and spectral mapping theorem for semigroup in an abstract setting which slightly generalizes the versions of the same theorems established in [50, 52]. More precisely, we consider the generators \mathcal{L} and \mathcal{B} of two semigroups $S_{\mathcal{L}}$ and $S_{\mathcal{B}}$ in a Banach space \mathcal{X} . We denote $\mathcal{A} := \mathcal{L} - \mathcal{B}$ as well as

$$\begin{aligned} R_{\mathcal{L}}(z) &:= (\mathcal{L} - z)^{-1}, \\ R_{\mathcal{B}}(z) &:= (\mathcal{B} - z)^{-1}, \end{aligned}$$

the resolvent operators defined in the corresponding resolvent sets. We assume that for some fixed $a^* \in \mathbb{R}$ the following growth and regularizing estimates hold true for any $a > a^*$:

(H1) \mathcal{B} is \mathcal{A} -power dissipative in X , in the sense that

$$(3.3.19) \quad \forall \ell \geq 0, \quad t \mapsto \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t)\|_{\mathcal{B}(\mathcal{X})} e^{-at} \in L^\infty(0, \infty)$$

and $u := S_{\mathcal{B}}\mathcal{A}$ satisfies

$$(3.3.20) \quad \exists n \geq 1, C \in (0, \infty), \forall f \in \mathcal{X}, \quad \int_0^\infty \|u^{(*n)}(t)f\|_{\mathcal{X}} e^{-at} dt \leq C \|f\|_{\mathcal{X}}.$$

(H2) For the same integer $n \geq 1$, the operator $\mathcal{U} := -R_{\mathcal{B}}\mathcal{A}$ is power regular in the sense that

$$(3.3.21) \quad \exists \alpha > 0, C \in (0, \infty), \quad \|\mathcal{U}(z)^n\|_{\mathcal{B}(\mathcal{X})} \leq C \langle x \rangle^{-\alpha}, \quad \forall z \in \Delta_a$$

and

$$(3.3.22) \quad \forall M > 0, \exists C \in (0, \infty), \quad \|\mathcal{U}(z)^n\|_{\mathcal{B}(\mathcal{X}, \mathcal{Y})} \leq C, \quad \forall z \in \Delta_a \cap B(0, M),$$

for some linear space \mathcal{Y} such that the embedding $\mathcal{Y} \subset \mathcal{X}$ is compact.

For a given operator L we denote $\Sigma(L)$ its spectral set and we define $\Sigma_d(L)$ the discrete spectrum as the set of isolated eigenvalues with finite dimensional associated eigenspace. We also denote $D(L)$ the domain and RL the range.

Theorem 3.3.1. *We make the above growth and regularizing assumptions **(H1)** and **(H2)** on \mathcal{A} and \mathcal{B} for some $a^* < 0$ and we assume furthermore that*

$$\Sigma(\mathcal{L}) \cap \bar{\Delta}_0 = \{0\} \subset \Sigma_d(\mathcal{L}),$$

so that there exist a finite rank projector $\Pi_0 \in \mathcal{B}(\mathcal{X})$ and an operator $T_0 \in \mathcal{B}(R\Pi_0)$ satisfying $\mathcal{L}\Pi_0 = \Pi_0\mathcal{L} = T_0\Pi_0$, $\Sigma(T_0) = \{0\}$. There exist $a < 0$ and $C \geq 1$ such that

$$(3.3.23) \quad \forall t \geq 0, \quad \left\| e^{t\mathcal{L}} - e^{tT_0} \Pi_0 \right\|_{\mathcal{B}(\mathcal{X})} \leq C e^{at}.$$

For two given time dependent operators valued functions U and V , we define the convolution product

$$(U * V)(t) := \int_0^t U(t-s)V(s) ds.$$

We also denote $V^{(*1)} = V$ and $V^{(*\ell)} := V * V^{(*(\ell-1))}$ for any $\ell \geq 2$.

Proof of Theorem 3.3.1. Step 1. We define

$$\mathcal{V}(z) := \mathcal{R}_{\mathcal{B}}(z) - \cdots + (-1)^{n-1} \mathcal{R}_{\mathcal{B}}(z) (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{n-1}$$

and

$$\mathcal{W}(z) := (-1)^n (\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^n,$$

where $n \geq 1$ is the integer given by assumption **(H1)**. From the definition $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we immediately have

$$R_{\mathcal{L}} = R_{\mathcal{B}} - R_{\mathcal{B}}\mathcal{A}R_{\mathcal{L}},$$

and by iterating that relation, we deduce

$$R_{\mathcal{L}} = \mathcal{V} + \mathcal{W}R_{\mathcal{L}},$$

or equivalently

$$(I - \mathcal{W})R_{\mathcal{L}} = \mathcal{V}.$$

Thanks to (3.3.21), for M large enough, we have

$$z \in \Delta_a, |z| \geq M \quad \Rightarrow \quad \|\mathcal{W}(z)\|_{\mathcal{B}(X)} \leq \frac{1}{2}.$$

We get that $R_{\mathcal{L}}(z) = (I - \mathcal{W}(z))^{-1}\mathcal{V}(z)$ is well defined and uniformly bounded in the region $\Delta_a \setminus B(0, M)$, or in other word $\Sigma(\mathcal{L}) \cap \Delta_a \subset B(0, M)$.

On the other hand, $\Phi := I - \mathcal{W}$ is holomorphic on Δ_a^* and $R(\mathcal{W}) \subset \mathcal{Y} \subset \mathcal{X}$ because of (3.3.19)-(3.3.22). Together with $\Phi(M)$ is invertible, we may use Ribarič-Vidav-Voigt theory [73, 76] and deduce that $R_{\mathcal{L}} = (I - \mathcal{W})^{-1}\mathcal{V}$ is a degenerate-meromorphic operator and next that $\Sigma(\mathcal{L}) \cap \Delta_a^*$ is discrete. All together, we have proved that there exists $a < 0$ such that $\bar{\Delta}_a \cap \Sigma(\mathcal{L}) = \{0\}$.

Step 2. For any integer $N \geq 1$ and iterating the Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{L}},$$

we have

$$S_{\mathcal{L}}(I - \Pi_0) = \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(I - \Pi_0) + (S_{\mathcal{B}}\mathcal{A})^{(*N)} * (S_{\mathcal{L}}(I - \Pi_0)).$$

For $b > \omega(\mathcal{L})$, we may use the inverse Laplace formula

$$\begin{aligned} \mathcal{T}(t)f &:= (\mathcal{A}S_{\mathcal{B}})^{(*N)} * (S_{\mathcal{L}}(I - \Pi_0))(t)f \\ &= \lim_{M' \rightarrow \infty} \frac{i}{2\pi} \int_{b-iM'}^{b+iM'} e^{zt} (-1)^{N+1} (\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N (I - \Pi_0)\mathcal{R}_{\mathcal{L}}(z) f dz, \end{aligned}$$

for any $f \in D(\mathcal{L})$ and $t \geq 0$, and we emphasize that the term $\mathcal{T}(t)f$ might be only defined as a semi-convergent integral. Because $z \mapsto (\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N(I - \Pi_0)\mathcal{R}_{\mathcal{L}}(z)$ is a bounded analytic function on a neighborhood of $\bar{\Delta}_a$, we may move the segment on which the integral is performed, and we obtain

$$(3.3.24) \quad \mathcal{T}(t)f = \lim_{M' \rightarrow \infty} \frac{i}{2\Pi_0} \int_{a-iM'}^{a+iM'} e^{zt} (-1)^{N+1} (\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N \mathcal{R}_{\mathcal{L}}(z) (I - \Pi_0) f \, dz,$$

for any $f \in D(\mathcal{L})$ and $t \geq 0$. In order to conclude we only have to explain why the RHS term in (3.3.24) is appropriately bounded for N large enough. We define

$$\mathcal{W}(z) := \mathcal{R}_{\mathcal{L}}(z) (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N$$

for $z \in \Delta_a \setminus B(0, M)$, $N := ([1/\alpha] + 1)n$. From Step 1 and (3.3.21), we deduce

$$(3.3.25) \quad \|\mathcal{W}(z)\|_{\mathcal{B}(X)} \leq \frac{C}{|y|^\beta}, \quad \forall z = a + y, |y| \geq M,$$

with $\beta := ([1/\alpha] + 1)\alpha > 1$. We then have

$$\begin{aligned} \|\mathcal{T}(t)f\|_{\mathcal{B}(X)} &\leq \frac{e^{at}}{2\pi} \int_{a-iM}^{a+iM} \|(\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N\|_{\mathcal{B}(X)} \|(I - \Pi_0)\mathcal{R}_{\mathcal{L}}(z)\|_{\mathcal{B}(X)} \, dy \\ &\quad + \frac{e^{at}}{2\pi} \int_{\mathbb{R} \setminus [-M, M]} \|\mathcal{W}(a + iy)(I - \Pi_0)\|_{\mathcal{B}(X)} \, dy, \end{aligned}$$

where the first integral is finite thanks to $\Sigma(\mathcal{L}(I - \Pi_0)) \cap [a - iM, a + iM] = \emptyset$ and (3.3.21), while the second integral is finite because of (3.3.25). \square

3.4 Case without delay

In this section, we present the proof of our main result Theorem 3.1.3 in the case without delay.

3.4.1 An auxiliary linear equation

We introduce the auxiliary linear equation on the variation g given by

$$(3.4.26) \quad \begin{aligned} \partial_t g + \partial_x g + k_\varepsilon g &= 0, \\ g(t, 0) &= \mathcal{M}_\varepsilon[g(t, \cdot)], \quad g(0, x) = g_0(x), \end{aligned}$$

with the notations

$$(3.4.27) \quad \mathcal{M}_\varepsilon[h] := \int_0^\infty k_\varepsilon h \, dx, \quad k_\varepsilon := k(x, \varepsilon M_\varepsilon),$$

and where M_ε is defined in Theorem 3.1.2. The corresponding linear operator \mathcal{L} is

$$\mathcal{L}g := -\partial_x g - k_\varepsilon g$$

in the domain

$$D(\mathcal{L}) := \{g \in W^{1,1}(\mathbb{R}_+), g(0) = \mathcal{M}_\varepsilon[g]\}$$

generating the semigroup $S_{\mathcal{L}}$ in the Lebesgue space $X := L^1(\nu)$ for some polynomial weight function $\nu := 1 + x^q$, $q > 0$. For any initial datum $g_0 \in X$, the weak solution of the linearized equation is given by $g(t) = S_{\mathcal{L}}(t)g_0$. By regarding the boundary condition as a source term, we may rewrite the above equation as

$$(3.4.28) \quad \partial_t g = \Lambda g := -\partial_x g - k_\varepsilon g + \delta_{x=0} \mathcal{M}_\varepsilon[g],$$

with the associated semigroup S_{Λ_ε} , acting on the space of bounded Radon measures

$$\mathcal{X} := M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))'; \text{supp } g \subset \mathbb{R}_+\},$$

endowed with the weak $*$ topology $\sigma(M^1, C_0)$, where C_0 represents the space of continuous functions converging to 0 at infinity.

Theorem 3.4.1. *For any $\varepsilon \in (0, \varepsilon_0)$, there exist $\alpha < 0$ and $C > 0$ such that $\Sigma(\mathcal{L}) \cap \Delta_\alpha = \{0\}$ and*

$$(3.4.29) \quad \|S_{\mathcal{L}}(t)g_0\|_X \leq C e^{\alpha t} \|g_0\|_X, \quad \forall t \geq 0,$$

for any $g_0 \in X$, $\langle g_0 \rangle = 0$.

We proceed in several steps.

Lemma 3.4.2. *The semigroup $S_{\mathcal{L}}$ is well defined in L^1 and it is positive in the sense that $S_{\mathcal{L}}(t)f_0 \geq 0$ for any $f_0 \in L^1$, $f_0 \geq 0$ and any $t \geq 0$.*

The proof being exactly the same as for [52, Lemma 2.5] it is skipped.

Lemma 3.4.3. *$-\mathcal{L}$ satisfies the following version of the strong maximum principle: for any given $g \in \mathcal{X}_+$ and $\mu \in \mathbb{R}$, there holds*

$$g \in D(\mathcal{L}) \setminus \{0\} \text{ and } (-\mathcal{L} + \mu)g \geq 0 \text{ imply } g > 0.$$

The proof being exactly the same as for [52, Lemma 2.6] it is skipped. As an immediate consequence of Theorem 3.1.2, Lemma 3.4.2 and Lemma 3.4.3, we obtain the following result about the first eigenvalue and eigenspace associated to \mathcal{L} . We refer to [52, Proof of Theorem 2.4.] or [50, 45, Proof of Theorem 5.3] where similar results are established.

Corollary 3.4.4. *There hold $\Sigma(\mathcal{L}) \cap \bar{\Delta}_0 = \{0\}$ and $N(\mathcal{L}) = \text{span}(F_\varepsilon)$.*

We come to the slightly new argument we need in order to generalize the proof presented in [52] to the non smooth firing rate we are considered here. Because of (3.1.5)-(3.1.6), we have

$$k_\varepsilon \in L^\infty(\mathbb{R}_+), \quad k_\varepsilon(x) \geq k_0 \mathbf{1}_{x \geq x_0},$$

for some $x_0 \in [0, \infty)$, $k_0 > 0$, and we set

$$a^* := -k_0 < 0.$$

We rewrite the evolution equation as

$$(3.4.30) \quad \partial_t f = \mathcal{L}f = \mathcal{A}f + \mathcal{B}f,$$

with \mathcal{A} and \mathcal{B} defined by

$$\begin{aligned} (\mathcal{A}f)(x) &:= \delta_{x=0}\mathcal{K}[f], \quad \mathcal{K}[f] := \int_0^\infty k_\varepsilon(y) f(y) dy, \\ (\mathcal{B}f)(x) &:= -\partial_x f(x) - k_\varepsilon(x)f(x), \end{aligned}$$

and we emphasize that the boundary condition in (3.4.26) has been equivalently replaced by the term $\mathcal{A}f$ involving a Dirac mass $\delta_{x=0}$.

Lemma 3.4.5. *For any $a > a^*$ and any $R > 0$, the operators \mathcal{A} and \mathcal{B} satisfy :*

- (i) *the operator $\mathcal{B} - a$ is dissipative in $L^1(\tilde{\nu})$, with norm equivalent to the norm of X ;*
- (ii) *the operators valued function of time $t \mapsto (S_{\mathcal{B}}\mathcal{A})^{(\ell)} * S_{\mathcal{B}}(t) e^{-at}$ is bounded in $L^\infty(0, \infty; \mathcal{B}(X))$;*
- (iii) *the operators valued holomorphic function $z \mapsto \langle z \rangle (R_{\mathcal{B}}(z)\mathcal{A})^2(z)$ is bounded in $\mathcal{B}(X)$ uniformly in $z \in \Delta_a$;*
- (iv) *$z \mapsto (R_{\mathcal{B}}(z)\mathcal{A})^2(z)$ is bounded in $\mathcal{B}(X, Y)$ uniformly in $z \in \Delta_a \cap B(0, R)$, with $Y := BV \cap L^1_{q+1}$.*

Proof of Lemma 3.4.5. During the proof we write $k = k_\varepsilon$.

Step 1. In order to prove the first point, we fix $a > a^*$ and we introduce the modified weight function

$$\tilde{\nu}(x) := \frac{e^{ax}}{e^{ax_1}} \mathbf{1}_{x \leq x_1} + \frac{x^q}{x_1^q} \mathbf{1}_{x > x_1},$$

with $x_1 > \max(1, q/(a - a^*))$. We compute

$$\begin{aligned} \mathcal{B}^* \tilde{\nu} &= \partial_x \tilde{\nu} - k \tilde{\nu} \leq a \tilde{\nu} \quad \text{on } [0, x_1] \\ \mathcal{B}^* \tilde{\nu} &\leq \left(\frac{q}{x} - k \right) \tilde{\nu} \leq a \tilde{\nu} \quad \text{on } (x_1, \infty), \end{aligned}$$

from what we deduce

$$\int_0^\infty (\mathcal{B}f)f/|f|\tilde{\nu} = \int_0^\infty (\mathcal{B}^* \tilde{\nu})|f| \leq a \int_0^\infty |f|\tilde{\nu}$$

and $\mathcal{B} - a$ is dissipative in $L^1(\tilde{\nu})$.

Step 2. From the first step we have $\|S_{\mathcal{B}}(t)\|_{\mathcal{B}(X)} = \mathcal{O}(e^{at})$. We deduce (ii) recursively.

Step 3. We have

$$S_{\mathcal{B}}(t)f(x) = f(x-t) \exp(K(x-t) - K(x)),$$

with

$$K(x) := \int_0^x k(u) du.$$

We deduce successively

$$S_{\mathcal{B}}(t)\mathcal{A}f(x) = \delta_{x-t=0} \exp(K(x-t) - K(x)) \mathcal{K}[f],$$

next

$$\begin{aligned}
g_{t-s}(x) &:= \mathcal{A}S_{\mathcal{B}}(t-s)\mathcal{A}f(x) \\
&= \delta_{x=0} \int_0^\infty k(y) \delta_{y-(t-s)=0} \exp(K(y-s) - K(y)) \, dy \mathcal{K}[f] \\
&= \delta_{x=0} k(t-s) \exp(-K(t-s)) \mathcal{K}[f],
\end{aligned}$$

and finally

$$\begin{aligned}
(S_{\mathcal{B}}\mathcal{A})^{(*2)}(t)f(x) &= \int_0^t (S_{\mathcal{B}}(s)g_{t-s})(x) \, ds \\
&= \int_0^t g_{t-s}(x-s) \exp(K(x-s) - K(x)) \, ds \\
&= \int_0^t \delta_{x-s=0} k(t-s) \exp(-K(t-s)) \exp(K(x-s) - K(x)) \, ds \mathcal{K}[f] \\
&= \mathbf{1}_{t \geq x} k(t-x) \exp(-K(t-x) - K(x)) \mathcal{K}[f] \\
&=: \varphi_t(x) \mathcal{K}[f].
\end{aligned}$$

Summarizing, we have

$$(S_{\mathcal{B}}\mathcal{A})^{(*2)}(t)f = \varphi_t \mathcal{K}[f],$$

with

$$\varphi_t(x) = \psi(t-x) \exp(-K(x)), \quad \psi(u) := \mathbf{1}_{u \geq 0} k(u) e^{-K(u)}.$$

We then compute

$$\hat{\varphi}_z(x) = \hat{\psi}(z) e^{-zx} e^{-K(x)}.$$

On the one hand, using that $K(u) \geq k_0 u - k_1$, for any $a > a^* = -k_0$ and some $k_1 \in \mathbb{R}$, we have

$$\begin{aligned}
\|e^{-zx} e^{-K(x)}\|_{L^1(\nu)} &\leq \int_0^\infty e^{-\Re z x - K(x)} \langle x \rangle^q \, dx \\
&\leq \int_0^\infty C e^{-(\Re z - a)x} \, dx \\
&\leq \frac{C}{\Re z - a},
\end{aligned}$$

for any $z \in \Delta_a$ and some constant $C \in (0, \infty)$. On the other hand, when furthermore $k' \in L^1(0, \infty)$, we may perform one integration by parts and we get

$$\begin{aligned}
\hat{\psi}(z) &= \int_0^\infty k(u) e^{-K(u)} e^{-zu} \, du \\
&= \frac{1}{z} \left(k(0) - \int_0^\infty (k^2(u) - k'(u)) e^{-K(u)} e^{-zu} \, du \right).
\end{aligned}$$

As a consequence and similarly as above, we have

$$|\hat{\psi}(z)| \leq \frac{1}{|z|} \left(\|k\|_{L^\infty} + C_1 \|k'\|_{M^1} + C_2 \frac{\|k\|_{L^\infty}^2}{\Re z - a} \right),$$

for any $z \in \Delta_a$, $a > a^*$. By a standard regularization argument, we get the same estimate in the general case when $k' \in M^1([0, \infty))$. All together, we obtain

$$\begin{aligned} \|(R_{\mathcal{B}}(z)\mathcal{A})^2(z)f\|_X &= \|\hat{\varphi}_z \mathcal{K}[f]\| \\ &= \|e^{-zx-K(x)}\|_X |\hat{\psi}(z)| |\mathcal{K}[f]| \\ &\leq \frac{C_a}{|z|} \|f\|_X, \end{aligned}$$

for any $z \in \Delta_a$, $a > a^*$ and a constant C_a depending of a , $L^1(\nu)$ and k (through the quantities $\|k\|_{L^\infty}$, $\|k'\|_{M^1}$ and k_1).

Step 4. We observe that

$$\|\hat{\psi}_z\|_Y := \int_0^\infty \left(|\psi'_z(x)| + |\psi_z(x)|(1+x^{q+1}) \right) dx \leq C,$$

uniformly in $z \in B(0, R)$, from what (iv) immediately follows. \square

Proof of Theorem 3.4.1. Collecting the information obtained on the \mathcal{L} in Corollary 3.4.4, Lemma 3.4.5 and using Theorem 3.3.1, we immediately deduce that (3.4.29) holds. \square

3.4.2 Proof of Theorem 3.1.3 in the case without delay

We present the proof of our main result Theorem 3.1.3 in the case without delay.

Proof of Theorem 3.1.3 in the case without delay. We split the proof into three steps.

Step 1. A new formulation. From Lemma 3.2.3, for a given unit mass initial datum $0 \leq f_0 \in X$, we may write the solution $f \in C([0, \infty); X)$ to the evolution equation (3.1.1) and the solution F_ε to the stationary problem (3.1.4) as

$$\begin{aligned} \partial_t f + \partial_x f + k(\varepsilon \varphi[f])f &= 0, & f(t, 0) &= \varphi[f(t, \cdot)], \\ \partial_x F + k(\varepsilon M)F &= 0, & F(0) &= M = \varphi[F], \end{aligned}$$

where here and below the ε and x dependency is often removed without any confusion.

Next, we consider the variation function $g := f - F$ which satisfies

$$\partial_t g = -\partial_x g - k(\varepsilon M)g + (k_\varepsilon(m) - k_\varepsilon(M))f$$

complemented with the boundary condition

$$\begin{aligned} g(t, 0) &= \varphi[f(t, \cdot)] - \varphi[F] \\ &= \int_0^\infty k(\varepsilon \varphi[f])f - \int_0^\infty k(\varepsilon \varphi[F])F \\ &= \mathcal{M}[g] + (\mathcal{P}_\varepsilon(f, m) - \mathcal{P}_\varepsilon(f, M)), \end{aligned}$$

with $\mathcal{M} = \mathcal{M}_\varepsilon$ defined in (3.4.27). Considering the boundary condition as a source term again, we deduce that the variation function g satisfies the equation

$$(3.4.31) \quad \partial_t g = \mathcal{L}g + Z[g],$$

with the nonlinear term $Z[g] := -Q[g] + \delta_0 \mathcal{Q}[g]$ and where

$$\begin{aligned} Q[g] &:= (k_\varepsilon(\varphi[f]) - k_\varepsilon(M))f \\ \mathcal{Q}[g] &:= \mathcal{P}_\varepsilon(f, \varphi[f]) - \mathcal{P}_\varepsilon(f, M). \end{aligned}$$

Step 2. The nonlinear term. Using the properties (3.1.13), the assumption (3.1.8) and Lemma 3.2.3, we estimate

$$\begin{aligned} \|Q[g]\|_X &= \|k(\varepsilon \varphi[f])f - k(\varepsilon \varphi[F])f\|_{L^1} \\ &\leq \|f\|_{L^\infty} C_k \varepsilon |\varphi[f] - \varphi[F]| \\ &\lesssim \varepsilon \|g\|_{L^1}. \end{aligned}$$

Similarly, for the boundary term, we have

$$|\mathcal{P}_\varepsilon(f, m) - \mathcal{P}_\varepsilon(f, M)| \lesssim \varepsilon \|g\|_{L^1}.$$

Step 3. Decay estimate. Thanks to the Duhamel formula, the solution g to the evolution equation (3.4.31) satisfies

$$g(t) = S_{\mathcal{L}}(t)g_0 + \int_0^t S_{\mathcal{L}}(t-s)Z[g(s)] \, ds.$$

From Theorem 3.4.1 and the second step, we deduce

$$\begin{aligned} \|g(t)\|_X &\leq C e^{\alpha t} \|g_0\|_X + \int_0^t C e^{\alpha(t-s)} \|Z[g(s)]\|_X \, ds \\ &\lesssim e^{\alpha t} \|g_0\|_X + \varepsilon \int_0^t e^{\alpha(t-s)} \|g(s)\|_X \, ds, \end{aligned}$$

for any $t \geq 0$ and for some constant $\alpha < 0$. Thanks to the Grönwall's lemma, we have

$$\begin{aligned} \|g(t)\|_X &\lesssim e^{\alpha t} \|g_0\|_X + \varepsilon \|g_0\|_X \int_0^t e^{\alpha t} \exp\left(\int_s^t e^{\alpha(t-r)} \, dr\right) \, ds \\ &\lesssim e^{\alpha t} \|g_0\|_X + \varepsilon t e^{\alpha t} \|g_0\|_X \\ &\lesssim e^{\alpha' t} \|g_0\|_X, \end{aligned}$$

for some constant $\alpha' \in (\alpha, 0)$. That concludes the proof of Theorem 3.1.3 in the case without delay. \square

3.5 Case with delay

This section is dedicated to the proof of Theorem 3.1.3 in the case with delay. Because the arguments are very similar to those of the previous section, they are only briefly explained. Following [52], we get ride of the delay formulation by writing the problem as a system of PDEs. We recall or introduce the notations

$$\begin{aligned} \mathcal{P}[h, \mu] &:= \int_0^\infty k_\varepsilon(\mu) h \, dx, \quad k_\varepsilon(\mu) = k(x, \varepsilon \mu), \\ \mathcal{D}[w] &:= \int_0^\infty w(y) b(dy). \end{aligned}$$

The evolution equation (3.1.1) writes

$$\partial_t f = -\partial_x f - k_\varepsilon(m)f + \delta_0 p,$$

with

$$p(t) = \mathcal{P}[f(t), m(t)], \quad m(t) = \int_0^\infty p(t-y)b(dy).$$

Introducing the auxiliary unknown u and the auxiliary equation

$$\partial_t u = -\partial_y u + \delta_0 p,$$

the value of m is given by $m(t) = \mathcal{D}[u(t)]$. All together, the evolution equation rewrites

$$\begin{aligned} \partial_t f &= -\partial_x f - k_\varepsilon(\mathcal{D}[u])f + \delta_0 p \\ \partial_t u &= -\partial_y u + \delta_0 p, \end{aligned}$$

with $p = \mathcal{P}(f, \mathcal{D}[u])$.

On the other hand, the stationary equation (3.1.4) writes

$$0 = -\partial_x F - k_\varepsilon(M)F + \delta_0 M$$

with $M = \mathcal{P}[F, M]$. Introducing the auxiliary unknown U and equation

$$0 = -\partial_y U + \delta_0 M,$$

the value of M is given by $M = \mathcal{D}[U]$. All together, the stationary equation rewrites

$$\begin{aligned} 0 &= -\partial_x F - k_\varepsilon(M)F + \delta_0 M \\ 0 &= -\partial_y U + \delta_0 M, \end{aligned}$$

with $M = \mathcal{P}(F, M)$, $M = \mathcal{D}[U]$.

We define now $g := f - F$, $v = u - U$ and we compute

$$\begin{aligned} \partial_t g &= -\partial_x g - k_\varepsilon(M)g + (k_\varepsilon(M) - k_\varepsilon(m))f + \delta_0(p(t) - M) \\ \partial_t v &= -\partial_x v + \delta_0(p(t) - M) \end{aligned}$$

with

$$p(t) - M = \mathcal{P}[g, M] + \int_0^\infty f(k_\varepsilon(m) - k_\varepsilon(M)) dx.$$

We first consider the linear system of equations

$$\begin{aligned} \partial_t g &= -\partial_x g - k_\varepsilon(M)g + \delta_0 \mathcal{P}[g, M] \\ \partial_t v &= -\partial_x v + \delta_0 \mathcal{P}[g, M] \end{aligned}$$

and the associated semigroup S_Λ , where Λ stands for the operator

$$\Lambda \begin{pmatrix} g \\ v \end{pmatrix} := \begin{pmatrix} -\partial_x g - k_\varepsilon(M)g + \delta_0 \mathcal{P}[g, M] \\ -\partial_x v + \delta_0 \mathcal{P}[g, M] \end{pmatrix}.$$

We introduce the space $X := L^1(\nu) \times L^1(\mu)$ with $\mu := e^{-\delta x} dx$ and $\delta > 0$ is defined in condition (3.1.12).

Lemma 3.5.1. *The semigroup S_Λ associated to Λ satisfies*

$$S_\Lambda(t) = \mathcal{O}(e^{a't}), \quad a' < 0,$$

in X .

Proof of Lemma 3.5.1. We write $\Lambda = (\Lambda_1, \Lambda_2)$ and we observe that $\Lambda_1 = \mathcal{L}$, where \mathcal{L} has been defined in the previous section. Because of Theorem 3.4.1, we already know that

$$\|S_{\Lambda_1}(t)(g_0, v_0)\|_{L^1_q} \lesssim e^{a't} \|g_0\|_{L^1_q}, \quad \forall t \geq 0,$$

for any $(g_0, v_0) \in X$, with $\langle g_0 \rangle = 0$. Next, we denote $g(t) := S_{\Lambda_1}(t)(g_0, v_0)$, $v(t) := S_{\Lambda_2}(t)(g_0, v_0)$, and we compute

$$\begin{aligned} \frac{d}{dt} \int |v(t)| e^{-\delta x} &= -\delta \int |v(t)| e^{-\delta x} + |\mathcal{P}[g(t), M]| \\ &\leq -\delta \int |v(t)| e^{-\delta x} + C e^{a't} \|g_0\|_{L^1_q}. \end{aligned}$$

Setting $a' := \max(-\delta, a)$, we conclude by integrating the above differential inequality. \square

The last step consists in proving the nonlinear stability by generalizing again the estimates established in the previous section and in [56, 52]. We write the nonlinear equation as

$$\partial_t(g, v) = \Lambda(g, v) + \mathcal{Z}$$

for a nonlinearity source term $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2)$ with

$$\mathcal{Z}_1 = (k_\varepsilon(M) - k_\varepsilon(m))f + \delta_0 \int_0^\infty f (k_\varepsilon(m) - k_\varepsilon(M)) dx$$

and

$$\mathcal{Z}_2 = \delta_0 \int_0^\infty f (k_\varepsilon(m) - k_\varepsilon(M)) dx.$$

We have

$$|\mathcal{Z}_2| \leq \delta_0 \|f\|_\infty \int_0^\infty |k(x, \varepsilon m) - k(x, \varepsilon M)| dx \leq \delta_0 C \varepsilon |m - M|$$

and similarly

$$\|\mathcal{Z}_1\|_{L^1} \leq C \varepsilon |m - M|.$$

From the definition of \mathcal{D} , we have

$$|m - M| = |\mathcal{D}[u] - \mathcal{D}[U]| \leq C \|v\|_{L^1}.$$

We finally use Duhamel formula

$$(g, v)(t) = S_\Lambda(t)(g_0, v_0) + (S_\Lambda * \mathcal{Z})(t)$$

and then obtain

$$\|(g, v)(t)\| \leq C_0 e^{a't} + C_0 C \int_0^t e^{a(t-s)} \varepsilon \|(g, v)(s)\| ds.$$

Thanks to the Grönwall's lemma, we have

$$\varphi(t) \leq C_0 e^{at} + C_1 \varepsilon \int_0^t e^{a(t-s)} \varphi(s) \, ds =: \psi(t),$$

which implies

$$\psi'(t) = a \psi(t) + C_1 \varepsilon \varphi(t) \leq (a + C_1 \varepsilon) \psi.$$

We deduce

$$\psi(t) \leq \psi(0) e^{(a+C_1\varepsilon)t},$$

and then finally obtain

$$\varphi(t) \leq C_0 e^{(a+C_1\varepsilon)t}.$$

We conclude by taking $\varepsilon > 0$ small enough.

Chapter 4

Time elapsed model in general case

Résumé

Pour le grand réseau de neurones entièrement connecté, nous étudions la dynamique des assemblages homogènes de neurones interactifs décrits par le modèle du temps écoulé, qui indique comment le temps écoulé depuis la dernière décharge construit la densité de probabilité des neurones. D'une part, nous prouvons l'existence et l'unicité partielle de la solution faible dans l'ensemble du régime de connectivité, grâce au théorème du point fixe de Schauder dans le cas de prendre en compte de délai et au théorème du point fixe de Banach au cas sans délai. D'autre part, d'après la théorie de l'analyse spectrale pour des semi-groupes dans les espaces de Banach développé récemment dans [44, 50], nous établissons les résultats parallèles sur le comportement à long temps des solutions obtenues dans [52] sous des hypothèses générales sur le taux de tir et la distribution de délai.

Abstract

For large fully connected neuron networks, we study the dynamics of homogenous assemblies of interacting neurons described by time elapsed models, indicating how the time elapsed since the last discharge constructs the probability density of neurons. On the one hand, we prove the existence and part of the uniqueness of the weak solution in the whole connectivity regime, thanks to the Schauder fixed point theorem in the case with delay and the Banach fixed point theorem in the case without delay. On the other hand, through the spectral analysis theory for semigroups in Banach spaces developed recently in [44, 50], we establish the parallel results on the long time behavior of solutions obtained in [52] under general assumptions on the firing rate and the delay distribution.

4.1 Introduction

The information transmission and processing mechanism in the nervous systems relies on the quantity of electrical pulses as the reflect to incoming stimulations, during which the neurons experience a period of recalcitrance called discharge time before reactive. In this work, we shall focus on the model describing the neuronal dynamics in accordance with this kind of discharge time which has been introduced and studied in [28, 56, 57]. In order to show the response to the recovery of the neuronal membranes after each discharge, the model consider an instantaneous firing rate depending on the time elapsed since last discharge as well as the inputs of neurons. This sort of models are also regarded as a mean field limit of finite number of neuron network models referred to [15, 25, 68, 66].

For a local time (or internal clock) $x \geq 0$ corresponding to the elapsed time since the last discharge, we consider the dynamic of the neuronal network with the density number of neurons $f = f(t, x) \geq 0$ in state $x \geq 0$ at time $t \geq 0$, given by the following nonlinear time elapsed (or of age structured type) evolution equation

$$(4.1.1) \quad \begin{cases} \partial_t f = -\partial_x f - a(x, \varepsilon m(t))f =: \mathcal{L}_{\varepsilon m(t)} f, \\ f(t, 0) = p(t), \\ f(0, x) = f_0(x), \end{cases}$$

where $a(x, \varepsilon \mu) \geq 0$ denotes the firing rate of a neuron in state x and in an environment $\mu \geq 0$ formed by the global neuronal activity with a network connectivity parameter $\varepsilon \geq 0$ corresponding to the strength of the interactions. The total density of neurons $p(t)$ undergoing a discharge at time t is defined through

$$(4.1.2a) \quad p(t) := \mathcal{P}[f(t); m(t)],$$

where

$$(4.1.2b) \quad \mathcal{P}[g, \mu] = \mathcal{P}_\varepsilon[g, \mu] := \int_0^\infty a(x, \varepsilon \mu) g(x) dx,$$

while the global neuronal activity $m(t)$ at time $t \geq 0$ taking into account the interactions among the neurons resulting from earlier discharges is given by

$$(4.1.3) \quad m(t) := \int_0^\infty p(t-y)b(dy),$$

where the delay distribution b is a probability measure considering the persistence of the electric activity to those discharges in the network. In the sequel, we will consider the two following situations respectively:

- The *case without delay*, when $b = \delta_0$ then $m(t) = p(t)$.
- The *case with delay*, when b is smooth.

Observe that the solution f of the time elapsed equation (4.1.1) satisfies

$$\frac{d}{dt} \int_0^\infty f(t, x) dx = f(t, 0) - \int_0^\infty a(x, \varepsilon m(t)) f(t, x) dx = 0,$$

in both the cases, which implies the conservation of the total density number of neurons (also called *mass* in the sequel) permitting us to normalize it to be 1. Then we assume in the sequel

$$\langle f(t, \cdot) \rangle = \langle f_0 \rangle = 1, \quad \forall t \geq 0, \quad \langle g \rangle := \int_0^\infty g(x) dx.$$

We define a couple $(F_\varepsilon, M_\varepsilon)$ as a corresponding steady state, which satisfy

$$(4.1.4) \quad \begin{cases} 0 = -\partial_x F_\varepsilon - a(x, \varepsilon M_\varepsilon) F_\varepsilon = \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon, \\ F_\varepsilon(0) = M_\varepsilon, \quad \langle F_\varepsilon \rangle = 1. \end{cases}$$

Noticing that the associated network activity and the discharge activity are equal constants for a steady state as $\langle b \rangle = 1$.

Our main purpose in this paper is to prove the existence of the weak solution to the time elapsed evolution equation (4.1.1) no matter which $\varepsilon > 0$ in the case with delay as well as the existence and uniqueness in the weak and strong connectivity regime, which is a range of connectivity parameter $\varepsilon \in (0, \varepsilon_0) \cup (\varepsilon_\infty, \infty)$, with ε_0 small enough and ε_∞ large enough, in the case without delay. Furthermore, we obtain the exponential asymptotic stability in weak and strong connectivity regime, where the equations (4.1.1) and (4.1.4) do not possess intense nonlinearity. The results extend the result in [56, 57] in the case without delay for a *step function* firing rate a to the case with delay rather than the indescribable stability. In order to conclude those results, it is necessary to give the following mathematical assumptions on the firing rate a and on the delay distribution b .

We make the physically reasonable assumptions

$$(4.1.5) \quad a \geq 0, \quad \partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0,$$

$$(4.1.6) \quad 0 < a_0 := \lim_{x \rightarrow \infty} a(x, 0) \leq \lim_{x, \mu \rightarrow \infty} a(x, \mu) =: a_1 < \infty,$$

as well as the regularity assumption

$$(4.1.7) \quad A(x, \cdot) := \int_0^x a(y, \cdot) dy \in C^0(\mathbb{R}_+), \quad \forall x > 0.$$

We will also need the stronger regularity assumption $a \in L_x^1 \text{Lip}_\mu$. For the weak connectivity regime, we assume that for some $\xi > 0$ small enough and for any $\mu_0 > 0$, there exists $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_0)$, there holds

$$(4.1.8) \quad \int_0^\infty |a(x, \varepsilon \mu_2) - a(x, \varepsilon \mu_1)| dx \leq \xi |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (0, \mu_0).$$

While in the strong connectivity regime, we assume that for some the same $\xi > 0$ as in the assumption (4.1.8) and for any $\mu_\infty > 0$, there exists $\varepsilon_\infty > 0$ large enough such that for any $\varepsilon \in (\varepsilon_\infty, \infty)$, there holds

$$(4.1.9) \quad \int_0^\infty |a(x, \varepsilon \mu_2) - a(x, \varepsilon \mu_1)| dx \leq \xi |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (\mu_\infty, \infty).$$

A possible example of firing rate which fulfills the above condition (4.1.8) for the weak connectivity regime is the “step function firing rate” considered in [56, 57] which is given by

$$(4.1.10) \quad a(x, \mu) = \mathbf{1}_{x > \sigma(\mu)}, \quad \sigma' \leq 0,$$

$$(4.1.11) \quad \sigma(0) = \sigma_+, \quad \sigma(\infty) = \sigma_- < \sigma_+ < 1,$$

where σ satisfies the regularity condition

$$(4.1.12) \quad \sigma, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+),$$

Similarly, the above condition (4.1.9) for the strong connectivity regime is met for a “step function firing rate” introduced in [56, 57] given by the same function as above which additionally fulfills

$$\sup_{\varepsilon \geq \varepsilon'_\infty} \sup_{\mu \in [\varepsilon(1-\sigma_+), \varepsilon]} \varepsilon |\sigma'(\mu)| \leq \tilde{\sigma},$$

for some $\tilde{\sigma} > 0$ small enough and $\varepsilon'_\infty > 0$ large enough. In the case with delay, we assume that the delay distribution $b(dy) = b(y)dy$ has the exponential bound and satisfies the smoothness condition

$$(4.1.13) \quad \exists \delta > 0, \quad \int_0^\infty e^{\delta y} (b(y) + |b'(y)|) dy < \infty.$$

The above assumptions permit the existence and part of the uniqueness of the solution to the nonlinear problem (4.1.1) thanks to the Schauder fixed point theorem in the case with delay and the Banach fixed point theorem in the case without delay.

Theorem 4.1.1. *We assume that the firing rate a satisfies (4.1.5)-(4.1.6)-(4.1.7) and that the delay distribution b satisfies (4.1.13) or satisfies $b = \delta_0$. In the latter “without delay” case, we also assume that (4.1.8) and (4.1.9) hold. There exist $\varepsilon_0 > 0$ small enough and $\varepsilon_\infty > 0$ large enough such that for any unit mass initial datum $0 \leq f_0 \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ satisfying*

$$(4.1.14) \quad k[f_0] := \int_0^\infty a(x, 0) f_0(x) dx > 0,$$

and any $\varepsilon > 0$ (in the delay case) or any $\varepsilon \in (0, \varepsilon_0) \cup (\varepsilon_\infty, \infty)$ (in the case without delay), there exists a nonnegative and mass conserving weak solution $0 \leq f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)_w) \cap L^\infty(\mathbb{R}_+^2)$ to the evolution equation (4.1.1)-(4.1.2)-(4.1.3) for some functions $m, p \in C(\mathbb{R}_+)$, which is unique additionally in the case without delay. Moreover, there hold

$$(4.1.15) \quad \begin{aligned} \|f_t\|_{L^1} &= \|f_0\|_{L^1} = 1, \\ \|f_t\|_{L^1_q} &\leq \|f_0\|_{L^1_q}, \quad \|f_t\|_{L^\infty} \leq \|f_0\|_{L^\infty} + a_1. \end{aligned}$$

$$(4.1.15) \quad e^{-a_1 \bar{x}} \kappa[f_0] \leq \|m\|_{L^\infty} \leq a_1, \quad \kappa[f_0] := \min\{k[f_0], a_0/2\},$$

for some \bar{x} large enough.

For any $\varepsilon > 0$, there also exists a corresponding steady state, which is unique additionally in the weak and strong connectivity regime.

Theorem 4.1.2. *Under the assumption (4.1.5)-(4.1.6)-(4.1.7) on the firing rate, for any $\varepsilon \geq 0$, there exists at least one pair of solutions $(F_\varepsilon(x), M_\varepsilon) \in W^{1,\infty}(\mathbb{R}_+) \times \mathbb{R}_+$ to the stationary problem (4.1.4) such that*

$$(4.1.16) \quad 0 \leq F_\varepsilon(x) \lesssim e^{-\frac{a_0}{2}x}, \quad |F'_\varepsilon(x)| \lesssim e^{-\frac{a_0}{2}x}, \quad x \geq 0.$$

Moreover, when we assume additionally that (4.1.8) and (4.1.9) hold, there exist $\varepsilon_0 > 0$ small enough and $\varepsilon_\infty > 0$ large enough, such that the above steady state is unique for any $\varepsilon \in [0, \varepsilon_0) \cup (\varepsilon_\infty, +\infty]$.

We conclude the exponential long time stability in the weak and strong connectivity regime as our main result under more general assumptions on the firing rate a compared to the regular assumption (4.1.8) in Chapter 2 considered in [52].

Theorem 4.1.3. *We assume that the firing rate a satisfies (4.1.5)-(4.1.6)-(4.1.7)-(4.1.8)-(4.1.9). We also assume that the delay distribution b satisfies $b = \delta_0$ or (4.1.13). There exists $\varepsilon_0 > 0$ small enough and ε_∞ large enough, such that for any $\varepsilon \in (0, \varepsilon_0) \cup (\varepsilon_\infty, +\infty)$, there exist some constants $\alpha < 0$, $C \geq 1$ and $\eta > 0$ such that for any unit mass initial datum $0 \leq f_0 \in L^1 \cap L^\infty$ satisfying (4.1.15) and $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\xi$, then the positive and mass conserving solution f to the evolution equation (4.1.1) satisfies*

$$\|f(t, \cdot) - F_\varepsilon\|_{L^1} \leq C e^{\alpha t}, \quad \forall t \geq 0.$$

In order to study the asymptotic convergence to an equilibrium for the homogeneous inelastic Boltzman equation, the strategy of ‘‘perturbation of semigroup spectral gap’’ is first introduced in [46]. Inspired by its recent application to a neuron network equation in [48], we consider a linear equation on the variation functions $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$ around a stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ such that

$$(4.1.17) \quad \begin{cases} \partial_t g = -\partial_x g - a(x, \varepsilon M_\varepsilon)g, \\ g(t, 0) = q(t), \quad g(0, x) = g_0(x), \end{cases}$$

with

$$(4.1.18) \quad q(t) = \int_0^\infty a(x, \varepsilon M_\varepsilon)g \, dx$$

and

$$(4.1.19) \quad n(t) := \int_0^\infty q(t-y)b(dy).$$

By regarding the boundary condition as a source term, we construct the linear generator Λ_ε and the associated semigroup S_{Λ_ε} respectively from the above linear equations to apply the spectral analysis. As in [75, 31, 50, 44], we split the operator Λ_ε into two parts, one of which is α -hypodissipative, $\alpha < 0$, denoted by \mathcal{B}_ε while the other one is bounded and \mathcal{B}_ε -power regular, denoted as \mathcal{A}_ε . Benefiting from this split, the semigroup S_{Λ_ε} admits a finite dimensional dominant part, thanks to a particular version of the Spectral

Mapping Theorem in [50, 44] and the Weyl's Theorem in [75, 31, 50, 44]. As $\varepsilon \rightarrow \infty$, the limited semigroup S_∞ becomes positive because of the vanishment of the items with $n(t)$ in (4.1.17) and (4.1.18), which permits the Krein-Rutman Theorem established in [50, 44] to imply that the steady state (F_∞, M_∞) possesses the exponential stability. And so does the stationary state $(F_\varepsilon, M_\varepsilon)$ in the strong connectivity regime after a perturbative argument developed in [46, 72, 44]. Then we extend the exponential stability to our main result Theorem 4.1.3 in the case without delay by the analysis on the rest term of the linear equation (4.1.17) and (4.1.18) compared to the original nonlinear equation. As for the delay case, we replace the delay equation (4.1.3) by a simple age equation to form an autonomous system with the linear equation (4.1.17) and (4.1.17) to generate a semigroup and follow the same strategy.

Actually, the previous works [56, 57] shows the asymptotic stability with simpler explicitly expression and appropriate norm benefiting from the choice of the step function firing rate (4.1.10). Different from those approach, we take consideration of more realistic and flexible firing rate with more abstract method, which allows us to obtain the dissipativity of the corresponding linear operators without the explicitly exhibited norm. In particular, we are able to establish the existence and part of the uniqueness of weak solutions first as the complement of the results in [52] in the weak connectivity regime and adapt our approach to the more general firing rate with less regularity restriction in the weak and strong connectivity regime to generalize the stability results obtained in [56, 57] in the case without delay to the case considering of the delay term.

This article is organized by the following plan. In Section 4.2, we demonstrate the existence and part of the uniqueness of the solution and the stationary state result. Both in the weak and strong connectivity regime, we introduce the strategy and establish Theorem 4.1.3 in the case without delay in Section 4.3, meanwhile the case with delay in section 4.4.

4.2 Existence and the steady state

4.2.1 Existence of the solution

We start from taking the delay into account. To establish the existence of a solution to (4.1.1)-(4.1.2)-(4.1.3), we are going to apply a fixed point argument with the benefit of the following lemmas. To begin with, we analyse the continuity property of the functional \mathcal{P} defined in (4.1.2b).

Lemma 4.2.1. *Assume (4.1.5)-(4.1.6)-(4.1.7). Consider a sequence m_k converging to m in \mathbb{R}_+ as well as a sequence of bounded functions f_k which converges weakly to f in the weak topology $\sigma(L^1, L^\infty)$ and bounded in L^∞ , then we have*

$$\mathcal{P}[m_k, f_k] \rightarrow \mathcal{P}[m, f], \quad \text{as } k \rightarrow \infty.$$

Proof. Step 1. Continuity of a. We are going to show that

$$a(\cdot, m_k) \rightarrow a(\cdot, m), \quad \text{a.e. as } k \rightarrow \infty.$$

If m_k is increasing, then the assumption (4.1.6) permits that there exists some $\bar{a}(x)$, such that for any $x \geq 0$,

$$a(x, m_k) \rightarrow \bar{a}(x),$$

which implies that

$$A(x, m_k) = \int_0^x a(y, m_k) dy \rightarrow \int_0^x \bar{a}(y) dy.$$

While from the assumption (4.1.7), we deduce that for any $x \geq 0$,

$$\int_0^x a(y, m) dy = A(x, m) = \int_0^x \bar{a}(y) dy, \quad \forall x \geq 0.$$

Thus, we clearly have

$$a(x, m) = \bar{a}(x), \quad a.e. \ x,$$

which also holds for the case m_k decreasing since $a \geq 0$. Now, consider the general case when $m_k \rightarrow m$ in \mathbb{R}_+ as $k \rightarrow \infty$. we may extract a monotonous subsequence $m_{k'}$ in order to pass to the limit as in the above argument. Therefore, we conclude that $a(x, \mu)$ possesses the almost everywhere continuity towards μ .

Step 2. Continuity of the functional \mathcal{P} . We compute

$$\begin{aligned} \mathcal{P}[m_k, f_k] - \mathcal{P}[m, f] &= \int a(\cdot, m_k) f_k - \int a(\cdot, m) f \\ &= \int (a(\cdot, m_k) - a(\cdot, m)) f_k + \int a(\cdot, m) (f_k - f) \\ &:= I_1 + I_2, \end{aligned}$$

From the assumption (4.1.5) and the weak convergence of f_k in L^1 , we have $I_2 \rightarrow 0$, as $k \rightarrow \infty$. From Step 1 and the assumption that f_k is bounded, we deduce that

$$\begin{aligned} I_1 &= \int_0^R (a(x, m_k) - a(x, m)) f_k(x) dx \\ &\quad + \int_R^\infty (a(x, m_k) - a(x, m)) f_k(x) dx \\ &\leq \|f_k\|_{L^\infty} \int_0^R |a(x, m_k) - a(x, m)| dx \\ &\quad + 2a_1 \int_R^\infty f_k(x) dx \rightarrow 0, \end{aligned}$$

as $R \rightarrow \infty$ and $k \rightarrow \infty$, which implies the conclusion. \square

Next, we analyse the linear mapping from a function $m \in C([0, T])$ to the associated solution $f \in C([0, T]; L^1) \cap L^\infty([0, T]; L^\infty)$ to the transport equation (4.1.1)-(4.1.2) as

$$(4.2.20) \quad \begin{cases} \partial_t f + \partial_x f + a(x, m(t)) f = 0 \\ f(t, 0) = \mathcal{P}[m(t), f], \\ f(0, x) = f_0(x), \end{cases}$$

with $T > 0$ fixed. The following lemma gives the continuity of this mapping.

Lemma 4.2.2. *Fix $T > 0$. Suppose a sequence $m_k \rightarrow m$ in $C([0, T])$, as $k \rightarrow \infty$, then the corresponding solution to the linear transport equation (4.2.20) satisfies that*

$$f_k \rightarrow f \quad \text{in } C([0, T]; L_w^1) \cap L^\infty([0, T]; L^\infty), \quad \text{as } k \rightarrow \infty$$

Proof. Step 1. Existence of f_k . For any $m \in C([0, T])$, consider the mapping $\mathcal{I}: f_1 \mapsto f_2$ satisfying

$$(4.2.21) \quad \begin{cases} \partial_t f_2 + \partial_x f_2 + a(x, m(t))f_2 = 0 \\ f_2(t, 0) = p_1(t) := \mathcal{P}[m(t), f_1], \\ f_2(0, x) = f_0(x), \end{cases}$$

in a set

$$\mathcal{C} := \{0 \leq f \in C([0, T]; L^1) \cap L^\infty; \|f\|_{L^1} \leq e^{a_1 t}, \|f\|_{L^\infty} \leq \|f_0\|_{L^\infty} + a_1\}.$$

Integrating the equation (4.2.21) towards x , we get

$$\begin{aligned} \frac{d}{dt} \int_0^\infty f_2(x) dx &= \int_0^\infty (f_1(x) - f_2(x))a(x, m(t)) dx \\ &\leq a_1 \|f_1\|_{L^1} \leq a_1 e^{a_1 t}, \end{aligned}$$

which implies that

$$\|f_2\|_{L^1} \leq \int_0^t a_1 e^{a_1 s} ds + \|f_0\|_{L^1} = e^{a_1 t}.$$

Thanks to the method of characteristics with respect to x or t , a solution $f_2(t, x)$ to the above equation (4.2.21) can be expressed as

$$f_2(t, x) = \begin{cases} f_0(x-t)e^{-\int_0^t a(s+x-t, m(s)) ds}, & \forall x \geq t, \\ p_1(t-x)e^{-\int_0^x a(s, m(s+t-x)) ds}, & \forall x \leq t, \end{cases}$$

which implies

$$\begin{aligned} \|f_2\|_{L^\infty} &\leq \|f_0\|_{L^\infty} + \|p_1\|_{L^\infty} \\ &\leq \|f_0\|_{L^\infty} + a_1. \end{aligned}$$

Thus, $\mathcal{I}(\mathcal{C}) \subset \mathcal{C}$, which permits us to obtain f_k by a classical contraction fixed point Theorem.

Step 2. Continuity of the mapping. From the equation (4.2.20), observe that f_k naturally satisfies

$$\begin{cases} \partial_t f_k \leq -\partial_x f_k \\ f_k(t, 0) \leq a_1, \\ f_k(0, x) = f_0, \end{cases}$$

which implies that

$$f_k(t, x) \leq a_1 \mathbf{1}_{x \leq t} + f_0(x-t) \mathbf{1}_{x \geq t},$$

for any $t \leq T$. Benefiting with this estimate, we have

$$\begin{aligned} \int_0^R f_k &\leq \int_0^R a_1 \mathbf{1}_{x \leq T} dx + \int_0^R f_0(x-t) \mathbf{1}_{x \geq t} dx \\ &\leq a_1 T + \int_0^R f_0, \end{aligned}$$

while for any $R \geq T$, there holds

$$\int_R^\infty f_k \leq \int_R^\infty f_0(x-t) dx \leq \int_{R-T}^\infty f_0 \rightarrow 0,$$

as $R \rightarrow \infty$. Together with the fact that

$$(4.2.22) \quad \|f_k\|_{L^\infty} \leq a_1 + \|f_0\|_{L^\infty}, \quad \int f_k = \int f_0 = 1.$$

We use de la Vallee-Poussin Theorem and Dunford-Pettis Lemma to conclude that there exists $f \in C([0, T]; L^1) \cap L^\infty([0, T]; L^\infty)$ and a subsequence $f_{k'}$ of the sequence f_k such that $f_{k'} \rightarrow f$, which implies $\mathcal{P}[m_{k'}, f_{k'}] \rightarrow \mathcal{P}[m, f]$ as $k' \rightarrow \infty$, from Lemma 4.2.1. Then, we conclude by passing k' to the limit in the equation (4.2.20) with $m_{k'}$. \square

Proof of Theorem 4.1.1 in the case with delay. First, we consider the application $\mathcal{J}(m) : C([0, T]) \rightarrow C([0, T])$, defined as

$$\mathcal{J}(m)(t) := \int_0^t p(t-y)b(dy),$$

where $b \in L^1(\mathbb{R}_+)$ and $p(t) = \mathcal{P}[m(t), f(t, x)]$. From Lemma 4.2.1, we deduce that the application $m \rightarrow \mathcal{P}[m, f]$ is continuous and so is $\mathcal{J}(m)$ by the definition. Define

$$\mathcal{K} := \{m \in C([0, T]), \|m\|_{L^\infty} \leq K\},$$

where $K > 0$ is a constant to be determined. Obviously, \mathcal{K} is a convex subset of $C([0, T])$. Then, for any $m \in \mathcal{K}$, we have

$$\|\mathcal{J}(m)\|_{L^\infty} \leq \int_0^t |b(y)| \|p\|_{L^\infty} dy \leq a_1 \|b\|_{L^1(\mathbb{R}_+)}$$

Next, we take $K := a_1 \|b\|_{L^1(\mathbb{R}_+)}$. For any $\epsilon > 0$, there exists $\theta > 0$ such that for any $t, s \in [0, T]$ satisfying $|t - s| < \theta$, we have

$$\begin{aligned} |\mathcal{J}(m)(t) - \mathcal{J}(m)(s)| &\leq \int_0^\infty |b(t-y) - b(s-y)| \|p\|_{L^\infty} dy \\ &\leq \|b(t-\cdot) - b(s-\cdot)\|_{L^1(\mathbb{R}_+)} a_1 \|f_0\|_{L^1} \\ &\leq a_1 \|\tau_\theta b - b\|_{L^1(\mathbb{R}_+)} < \epsilon, \end{aligned}$$

where $\tau_\theta b := b(\cdot + \theta)$, which implies that \mathcal{J} is equicontinuous. By Arzela-Ascoli Theorem, we deduce that $\mathcal{J}(\mathcal{K})$ is compactly embedded into \mathcal{K} . Using Schauder-Brouwer fixed point Theorem, the application \mathcal{J} admits a fixed point $m \in \mathcal{K}$. The corresponding solution $f(t, x)$ to the equation (4.2.20) is also a solution to the equation (4.1.1)-(4.1.2)-(4.1.3). Iterating on T , we deduce the global existence of the solution (f, m) to equation (4.1.1). As for the lower bound of m in (4.1.15), from assumption (4.1.6)-(4.1.7), there clearly exists $x_0 > 0$ large enough, such that

$$a(x, \mu) \geq \frac{a_0}{2} \mathbf{1}_{x \geq x_0}, \quad \forall \mu > 0.$$

For $t \leq x_0$, thanks to the characteristics expression, we have

$$\begin{aligned} p(t) &\geq \int_0^\infty a(x,0) f_0(x-t) e^{-\int_0^t a(s+x-t, m(s)) ds} dx \\ &\geq \int_0^\infty a(x,0) f_0(x) e^{-a_1 t} dx \\ &\geq e^{-a_1 x_0} \kappa[f_0]. \end{aligned}$$

While for $t > x_0$, from (4.1.1), we directly deduce that

$$\partial_t f \geq \partial_x f - a_1 f,$$

which implies that

$$f(t, x) \geq e^{-a_1 y} f(t-y, x-y).$$

Then, we have

$$\begin{aligned} p(t) &\geq e^{-a_1 x_0} \int_0^\infty \frac{a_0}{2} \mathbf{1}_{x \geq x_0} f(t-x_0, x-x_0) dx \\ &\geq \frac{a_0}{2} e^{-a_1 x_0} \int_0^\infty f(t-x_0, x) dx \\ &= \frac{a_0}{2} e^{-a_1 x_0} \geq \kappa[f_0]. \end{aligned}$$

All together, we deduce the same lower bound for $m(t)$ from the definition (4.1.3). \square

As for the case without delay, it is necessary to show that $m(t)$ is well defined. We consider $\Phi : L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$\Phi[g, \mu] := \int_0^\infty a(x, \varepsilon \mu) g(x) dx - \mu.$$

Lemma 4.2.3. *Assume (4.1.5)-(4.1.6)-(4.1.7)-(4.1.8)-(4.1.9). For any bounded set $\mathcal{C} \subset \mathbf{P} \cap L^\infty$, there exists $\varepsilon_0 = \varepsilon_0(\mathcal{C}) > 0$ and $\varepsilon_\infty = \varepsilon_\infty(\mathcal{C}) > 0$, such that for any $\mu_0 > 0$ and $\mu_\infty > 0$, there exists a function $\varphi_\varepsilon : \mathcal{C} \rightarrow \mathbb{R}$ which is Lipschitz continuous in the sense of the L^1 norm and such that $\mu = \varphi_\lambda[g]$ is the unique solution to the equation*

$$\begin{aligned} \mu \in (0, \mu_0), \quad \Phi(g, \mu) &= 0, \quad \forall \varepsilon \in (0, \varepsilon_0), \\ \mu \in (\mu_\infty, a_1), \quad \Phi(g, \mu) &= 0, \quad \forall \varepsilon \in (\varepsilon_\infty, \infty). \end{aligned}$$

Proof of Lemma 4.2.3. Step 1. Existence. For any $g \in L^1(\mathbb{R}_+)$, we obviously have $\Phi(g, 0) > 0$ while for any $g \in L^1(\mathbb{R}_+)$ and $\mu > a_1$, we have

$$\Phi(g, \mu) \leq a_1 - \mu < 0.$$

Thanks to the intermediate value theorem, for any fixed $g \in \mathbf{P}(\mathbb{R}_+)$ and any $\varepsilon \geq 0$, there exists at least one solution $\mu \in (0, a_1]$ to the equation $\Phi(g, \mu) = 0$ from the continuity property of Φ .

Step 2. Uniqueness and Lipschitz continuity. For any $f, g \in (L^1 \cap L^\infty)(\mathbb{R}_+)$, from Step 1, we are able to consider $\mu, \nu \in (0, \mu_0)$ or $\mu, \nu \in (\mu_\infty, a_1)$ such that

$$\Phi(f, \mu) = \Phi(g, \nu) = 0,$$

which implies that

$$\nu - \mu = \int_0^\infty a(x, \varepsilon \nu)(g - f) dx + \int_0^\infty (a(x, \varepsilon \nu) - a(x, \varepsilon \mu))f dx,$$

with

$$\left| \int_0^\infty a(x, \varepsilon \nu)(g - f) dx \right| \leq a_1 \|g - f\|_{L^1}.$$

From the assumption (4.1.8) and (4.1.9), there holds

$$\left| \int_0^\infty (a(x, \varepsilon \nu) - a(x, \varepsilon \mu))f dx \right| \leq \|f\|_{L^\infty} \xi |\mu - \nu|.$$

We obtain

$$(4.2.23) \quad |\mu - \nu| \leq 2a_1 \|f - g\|_{L^1},$$

for any $\varepsilon \in [0, \varepsilon_0] \cup [\varepsilon_\infty, \infty]$ with $\varepsilon_0 = \varepsilon_0(\|f\|_{L^\infty})$ small enough and $\varepsilon_\infty = \varepsilon_\infty(\|f\|_{L^\infty})$ large enough. On the one hand, when $f = g$, we immediately deduce that $\mu = \nu$, the uniqueness of the definition of the mapping $\varphi_\varepsilon[g] := \mu$. On the other hand, we get the Lipschitz continuity of the function φ_ε directly from the inequation (4.2.23). \square

Proof of Theorem 4.1.1 in the case without delay. We fix $\varepsilon_0 > 0$ and $\varepsilon_\infty > 0$ as defined in Lemma 4.2.3. For a given function $m \in C([0, T])$, we define $M(t) := \varphi_\varepsilon(f(t, \cdot)) \in C([0, T])$, where $f \in C([0, T]; L^1(\mathbb{R}_+)) \cap L^\infty([0, T] \times \mathbb{R}_+)$ is the solution of (4.2.20) associated to m . We denote $\mathcal{K}(m) := M$. For two given $m_1, m_2 \in C([0, T])$, we denote $f_1, f_2 \in C([0, T]; L^1(\mathbb{R}_+)) \cap L^\infty([0, T] \times \mathbb{R}_+)$ the associated solutions to (4.2.20) and we easily compute

$$\begin{aligned} \frac{d}{dt} \int |f_2 - f_1| &\leq 2 \int |a(m_2)f_2 - a(m_1)f_1| \\ &\leq 2\|f_0\|_{L^\infty} \xi |m_2 - m_1| + 2a_1 \int |f_2 - f_1|, \end{aligned}$$

for any $\varepsilon \in (0, \varepsilon_0) \cup (\varepsilon_\infty, \infty)$. We deduce that $m \mapsto f$ is Lipschitz from $C([0, T])$ to $C([0, T]; L^1(\mathbb{R}_+))$ with constant CT . As a consequence, \mathcal{K} is Lipschitz from $C([0, T])$ into itself with constant $C'T$. Choosing $T > 0$ small enough, the mapping \mathcal{K} is a contraction and admits a unique fixed point thanks to the Banach fixed point theorem. Iterating on T , we deduce the existence and uniqueness of a global solution (f, m) to equation (4.1.1)-(4.1.2) in the case without delay and in the weak and strong connectivity regime. \square

4.2.2 The stationary problem

Now we present the proof of the steady state in the strong connectivity regime, since the case of weak connectivity regime has already been proven in [52, Theorem 1.1].

Proof of Theorem 4.1.2. Step 1. Existence. We introduce the function

$$A(x, \mu) := \int_0^x a(y, \mu) dy, \quad \forall x, \mu \geq 0.$$

From assumption (4.1.6), we deduce

$$(4.2.24) \quad \frac{a_0}{2}(x - x_0)_+ \leq A(x, \mu) \leq a_1 x, \quad \forall x \geq 0, \mu \geq 0.$$

From assumption (4.1.7), we also deduce $A(x, \cdot) \in W_\mu^{1, \infty}$ for any fixed $x \geq 0$. For any $m \geq 0$, the equation (4.1.4) can be solved by

$$F_{\varepsilon, m}(x) := T_m e^{-A(x, \varepsilon m)},$$

whose mass conservation gives

$$T_m^{-1} = \int_0^\infty e^{-A(x, \varepsilon m)} dx.$$

Then the existence of the solution is equivalent to find $m = M_\varepsilon$ satisfying

$$m = F_{\varepsilon, m}(0) = T_m.$$

Considering

$$\Psi(\varepsilon, m) = m T_m^{-1} := m \int_0^\infty e^{-A(x, \varepsilon m)} dx,$$

it is merely necessary to find $M_\varepsilon \geq 0$ such that

$$(4.2.25) \quad \Psi(\varepsilon, M_\varepsilon) = 1.$$

From the Lebesgue dominated convergence theorem, the function $\Psi(\varepsilon, \cdot)$ is continuous. In addition to the fact that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$, the intermediate value theorem implies the existence immediately. The inequality (4.2.24) shows the estimates (4.1.16) clearly.

Step 2. Uniqueness in the strong connectivity regime. Obviously,

$$M_\infty := \left(\int_0^\infty e^{-A(x, \infty)} dx \right)^{-1} \in (0, \infty)$$

is the unique solution to $\Psi(\infty, M_\infty) = 1$. It is clear that

$$\frac{\partial}{\partial m} \Psi(\varepsilon, m) = \int_0^\infty e^{-A(x, \varepsilon m)} \left(1 - m \int_0^x \varepsilon \partial_\mu a(y, \varepsilon m) dy \right) dx,$$

is continuous with respect to the two variables because of (4.1.9), which implies that $\Psi \in C^1$. Coupled with that

$$\frac{\partial}{\partial m} \Psi(\varepsilon, m)|_{\varepsilon=\infty} = \int_0^\infty e^{-A(x, \infty)} dx > 0,$$

we conclude from the implicit function theorem that there exists $\varepsilon_\infty > 0$, large enough, such that the equation (4.2.25) has a unique solution for any $\varepsilon \in (\varepsilon_\infty, +\infty]$. \square

4.3 Case without delay

In this section, we conclude our main result Theorem 4.1.3 gradually in the case without delay.

4.3.1 An auxiliary linear equation

We introduce the auxiliary linear equation on the variation functions $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$ around the steady state

$$(4.3.26) \quad \begin{cases} \partial_t g + \partial_x g + a_\varepsilon g = 0, \\ g(t, 0) = \mathcal{M}_\varepsilon[g(t, \cdot)], \\ g(0, x) = g_0(x), \end{cases}$$

with the notations

$$(4.3.27) \quad \mathcal{M}_\varepsilon[h] := \int_0^\infty a_\varepsilon h \, dx, \quad a_\varepsilon := a(x, \varepsilon M_\varepsilon).$$

The corresponding linear operator L_ε is

$$L_\varepsilon g := -\partial_x g - a_\varepsilon g$$

in the domain

$$D(L_\varepsilon) := \{g \in W^{1,1}(\mathbb{R}_+), g(0) = \mathcal{M}_\varepsilon[g]\}$$

generating the semigroup S_{L_ε} on space $X := L^1(\mathbb{R}_+)$. Then for any initial datum $g_0 \in X$, the weak solution of the linearized equation is given by $g(t) = S_{L_\varepsilon}(t)g_0$. By regarding the boundary condition as a source term, we may rewrite the above equation as

$$(4.3.28) \quad \partial_t g = \Lambda_\varepsilon g := -\partial_x g - a_\varepsilon g + \delta_{x=0} \mathcal{M}_\varepsilon[g],$$

with the associated semigroup S_{Λ_ε} , acting on the space of bounded Radon measures

$$\mathcal{X} := M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))'; \text{supp } g \subset \mathbb{R}_+\},$$

endowed with the weak * topology $\sigma(M^1, C_0)$, where C_0 represents the space of continuous functions converging to 0 at infinity. From the duality of $S_{\Lambda_\varepsilon^*}$, we deduce $S_{\Lambda_\varepsilon}|_X = S_{L_\varepsilon}$.

4.3.2 Steady state problem

To begin with, we are going to prove that S_{L_ε} is a bounded semigroup in $X' := L^1(\mathbb{R}_+) \cap L^p(\mathbb{R}_+)$, $1 \leq p \leq \infty$, from what we are able to deduce the existence of the steady state.

Lemma 4.3.1. *The semigroup S_{L_ε} is bounded in X' . Furthermore, there exists at least nonnegative and normalized stationary state G_ε to the linear equation (4.3.26).*

Proof of Lemma 4.3.1. We split the proof into two steps.

Step 1. Preservation of positivity. We start with the positivity of S_{L_ε} . For any $0 \leq g \in X'$, the explicit formula gives

$$(4.3.29) \quad S_{L_\varepsilon} g := e^{A_\varepsilon(x-t) - A_\varepsilon(x)} g(x-t) \mathbf{1}_{x-t \geq 0}$$

where we denote $A_\varepsilon(\cdot) := A(\cdot, \varepsilon M_\varepsilon)$ for simplicity, which clearly implies the positivity of L_ε .

Step 2. Conservative. For any $g \in X'$, we compute

$$\begin{aligned} \frac{d}{dt} \int |g| &= \int (L_\varepsilon^1 g) \text{sign } g \leq \int L_\varepsilon^1 |g| \\ &= |g(0)| - \int a_\varepsilon |g| = 0, \end{aligned}$$

which gives

$$(4.3.30) \quad \|g(t)\|_{L^1} \lesssim \|g_0\|_{L^1}.$$

On the other hand, we have

$$\begin{aligned} \frac{d}{dt} \int |g|^p &= p \int (L_\varepsilon^1 g) \text{sign } g |g|^{p-1} \leq p \int (L_\varepsilon^1 |g|) |g|^{p-1} \\ &= - \int \partial_x (|g|^p) - p \int a_\varepsilon |g|^p \\ &= |g(0)|^p - p \int a_\varepsilon |g|^p \\ &\leq a_1 \|g\|_{L^1}^p + \frac{a_0}{2} \int_0^{x_0} |g|^p - \frac{a_0 p}{2} \|g\|_{L^p}^p \\ &\leq (a_1 + \frac{a_0}{2} x_0^{1-1/p}) \|g\|_{L^1}^p - \frac{a_0 p}{2} \|g\|_{L^p}^p, \end{aligned}$$

which implies

$$(4.3.31) \quad \|g(t)\|_{L^p} \leq \max\{C \|g_0\|_{L^1}, \|g_0\|_{L^p}\}.$$

Thanks to the interpolation of the above two inequalities (4.3.30) and (4.3.31), we deduce

$$\|g(t)\|_{L^1 \cap L^p} \lesssim \|g_0\|_{L^1 \cap L^p}.$$

Step 3. Steady State. From the proven fact $S_{L_\varepsilon}(t) \in \mathcal{B}(X')$ for any $t \geq 0$, we are able to define a new norm on X' , namely

$$\| \|h\|_\varepsilon := \sup_{t \geq 0} \|S_{L_\varepsilon}(t)h\|_{X'}, \quad \forall h \in X',$$

which is equivalent to the usual norm $\|\cdot\|_{X'}$. Then for any $g_0 \in X'$, we denote $C := \| \|g_0\|$ and define the set

$$\mathcal{C} := \{h \in X'; h \geq 0, \langle\langle h \rangle\rangle = 1, \| \|h\| \leq C\},$$

which is well-defined (e.g. $g_0 \in \mathcal{C}$), convex and compact for the weak $*$ topology of X' . Moreover, for any $h_0 \in \mathcal{C}$ and $t \geq 0$, the new norm satisfies

$$\| \|S_{L_\varepsilon}(t)h_0\| = \sup_{\tau \geq t} \|S_{L_\varepsilon}(\tau)h_0\|_{X'} \leq \sup_{\tau \geq 0} \|S_{L_\varepsilon}(\tau)h_0\|_{X'} = \| \|h_0\| \leq C,$$

which implies that the set \mathcal{C} is invariant by the flow S_{L_ε} . Thanks to the Brouwer-Schauder-Tychono type of fixed point theorem, there exists an element $G_\varepsilon \in \mathcal{C}$ invariant by S_{L_ε} , i.e. satisfying

$$S_{L_\varepsilon}(t)G_\varepsilon = G_\varepsilon, \quad t \geq 0,$$

which implies $L_\varepsilon G_\varepsilon = 0$. Therefore, G_ε is a stationary state for the linear equation (4.3.26) since $G_\varepsilon \in D(L_\varepsilon) \setminus \{0\}$, which also happens to be the steady state of the equation (4.1.4). \square

It is worth emphasizing that G_ε is nothing but F_ε related to any given M_ε , which is furthermore unique under the condition of positivity and normalization.

Lemma 4.3.2. *The kernel of L_ε is the linear span of G_ε . Particularly, the positive and normalized steady state G_ε is unique.*

The uniqueness of the steady state derives from the classical weak and strong maximum principle.

Lemma 4.3.3. *The operator L_ε satisfies the Kato's inequality, i.e.*

$$(\text{sign } h)L_\varepsilon h := \frac{1}{2|h|}(h L_\varepsilon \bar{h} + \bar{h} L_\varepsilon h) \leq L_\varepsilon |h|$$

for any complexed valued function $h \in X + iX$. Furthermore, the weak maximum principle holds for the operator $-L_\varepsilon$.

Proof of Lemma 4.3.3. For any $h = g \in X + iX$, we compute directly

$$\begin{aligned} \frac{1}{2|g|}(g L_\varepsilon \bar{g} + \bar{g} L_\varepsilon g) &= \frac{1}{2|g|}(-\partial_x(g\bar{g}) - 2a_\varepsilon|g|^2) \\ &= -\partial_x|g| - a_\varepsilon|g| = L_\varepsilon|g|, \end{aligned}$$

which is nothing but the complex kato's inequality. Together with the positivity of S_{L_ε} as well as the mass conservation, we conclude the weak maximum principle. \square

Lemma 4.3.4. *The operator $-L_\varepsilon$ satisfies the following version of strong maximum principle, for any $0 \leq h \in X$ and $\lambda \in \mathbb{R}$, there holds*

$$h \in D(L_\varepsilon) \setminus \{0\} \text{ and } (-L_\varepsilon + \lambda)g \geq 0 \text{ imply } h > 0.$$

Proof of Lemma 4.3.4. Consider $0 \leq h \in X$. Suppose that there holds $(-L_\varepsilon + \lambda)h \geq 0$ for h satisfying the above conditions. It is only necessary to prove that h does not vanish in \mathbb{R}_+ . Since $h \not\equiv 0$, there exists $x^* \in \mathbb{R}_+$ such that $h(x^*) > 0$. Rewriting the assumption as

$$\partial_x h + (a_\varepsilon + \lambda)h \geq 0,$$

we observe that

$$\partial_x(e^{A_\varepsilon(x)+\lambda x} h) = e^{A_\varepsilon(x)+\lambda x}(\partial_x h + (a_\varepsilon + \lambda)h) \geq 0.$$

(i) For $x \in (x^*, \infty)$, we have

$$e^{A_\varepsilon(x)+\lambda x} h \geq e^{A_\varepsilon(x^*)+\lambda x^*} h(x^*) > 0.$$

(ii) For $x \in (0, x^*)$, by integrating the same equation on $(0, x)$, we obtain

$$e^{A_\varepsilon(x)+\lambda x} h \geq h(0) = \int_0^\infty a_\varepsilon(y)h(y) dy.$$

From the positivity assumption (4.1.7) on a and step (i), we have

$$\int_0^\infty a_\varepsilon(y)h(y) dy > \frac{a_0}{2} \int_{\max\{x_0, x^*\}}^\infty h(y) dy > 0.$$

Therefore, h does not vanish on $(0, \infty)$, which implies the strong maximum principle for $-L_\varepsilon$. \square

Proof of Lemma 4.3.2. Suppose that there exists another normalized positive steady state G'_ε , then $\tilde{G} := G_\varepsilon - G'_\varepsilon \neq 0$ is also a steady state to the linear equation (4.3.26). The Kato's inequality (4.3.3) implies that

$$L_\varepsilon \tilde{G}_+ \geq (\text{sign}_+ \tilde{G}) L_\varepsilon \tilde{G} \geq 0.$$

Together with the fact

$$\int L_\varepsilon \tilde{G}_+ = \int \tilde{G}_+ L_\varepsilon^* 1 = 0,$$

we have $L_\varepsilon \tilde{G}_+ = 0$. From the strong maximum principle (4.3.4), we deduce that $\tilde{G}_+ = 0$ or $\tilde{G}_+ > 0$. In the second case, we get $\tilde{G} = G_\varepsilon - G'_\varepsilon > 0$, which implies the contradiction

$$1 = \langle G_\varepsilon \rangle > \langle G'_\varepsilon \rangle = 1.$$

Similarly, we have $\tilde{G}_- = 0$ and we conclude $\tilde{G} = G_\varepsilon - G'_\varepsilon = 0$. \square

4.3.3 Decomposition of the semigroup

In this section, we establish that the semigroup S_{Λ_ε} has a nice decomposition structure with finite dimensional principal part and a fast decaying remainder term.

Theorem 4.3.5. *Assume (4.1.6)-(4.1.7)-(4.1.8). In the case without delay, there exist some constant $C > 0$, $a^* < \alpha < 0$ such that for any $\varepsilon \geq 0$, there hold $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

$$(4.3.32) \quad \|S_{\Lambda_\varepsilon}(t)g_0\|_X \leq C e^{a^*t} \|g_0\|_X, \quad \forall t \geq 0,$$

for any $g_0 \in X$, $\langle g_0 \rangle = 0$.

The result is obtained as a consequence of the Spectral Mapping and Weyl's theory developed in [50, 45, 44]. For that purpose, we introduce the convenient splitting of the operator Λ_ε on \mathcal{X} or on X' as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ defined by

$$\begin{cases} \mathcal{B}_\varepsilon g := -\partial_x g - a_\varepsilon g \\ \mathcal{A}_\varepsilon g := \delta_{x=0} \mathcal{M}_\varepsilon[g] \end{cases}$$

which possess the following properties, where we perform the spectral analysis of \mathcal{L}_ε in the space \mathcal{X} or X' , denoting as X .

Lemma 4.3.6. *Assume that a satisfies conditions (4.1.6)-(4.1.7)-(4.1.8), then for any $\varepsilon \geq 0$, the operators \mathcal{A}_ε and \mathcal{B}_ε satisfy:*

- (i) $S_{\mathcal{B}_\varepsilon}(t)$ is α -hypodissipative in X ;
- (ii) the family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{\mathcal{B}(X)} \leq C e^{a^*t}, \quad \forall a > \alpha;$$

- (iii) the family of operators $(R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2(z)$ satisfies the compactness estimate

$$\|(R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2(z)\|_{\mathcal{B}(X)} = \mathcal{O}(\langle z \rangle^{-1})$$

for any $z \in \Delta_{a^*}$, $a^* > \alpha$.

Proof of Lemma 4.3.6. We ignore the ε in the proof without any ambiguity.

Proof of (i). Recalling that the explicit formula gives (4.3.29), we deduce

$$\begin{aligned}\|S_{\mathcal{B}}(t)g\|_{X'} &= \|e^{A(x)-A(x+t)}g(x)\|_{X'} \\ &\leq e^{-\frac{\alpha_0}{2}t}\|g\|_{X'},\end{aligned}$$

for any $g \in L^p$ and $t \geq 0$. As a consequence, the hypodissipativity of $S_{\mathcal{B}}(t)$ is established by taking $\alpha := -\frac{\alpha_0}{2}$. Furthermore, the same estimate holds by \mathcal{X} replacing X' with the help of a weakly * density argument.

Proof of (ii). From the fact that $\mathcal{M} \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathbb{R})$ because of (4.1.8), the operator \mathcal{A} satisfies that $\mathcal{A} \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathcal{X})$, then $\mathcal{A} \in \mathcal{B}(X)$ is bounded. For any $g \in \mathcal{X}$, we have

$$\mathcal{A}S_{\mathcal{B}}(t)g(x) = \delta_{x=0}N(t)[g],$$

with

$$\begin{aligned}(4.3.33) \quad N(t)[g] &:= \mathcal{M}[S_{\mathcal{B}}(t)g] \\ &= \int_0^\infty a(x)e^{A(x-t)-A(x)}g(x-t)\mathbf{1}_{x-t \geq 0} dx.\end{aligned}$$

Observe that

$$|N(t)[g]| \leq Ca_1 e^{\alpha t} \|g\|_{\mathcal{X}},$$

for any $t \geq 0$. We then compute

$$\begin{aligned}N'(t)[g] &= \int_0^\infty \partial_x(a(x)e^{-A(x)})e^{A(x-t)}g(x-t)\mathbf{1}_{x-t \geq 0} dx \\ &= \int_0^\infty (a'(x) - a(x)^2)e^{A(x-t)-A(x)}g(x-t)\mathbf{1}_{x-t \geq 0} dx,\end{aligned}$$

or in the case of “step function”

$$\begin{aligned}N'(t)[g] &= \int_{\sigma_\varepsilon}^\infty \partial_x(e^{-A(x)})e^{A(x-t)}g(x-t)\mathbf{1}_{x-t \geq 0} dx \\ &= \int_{\sigma_\varepsilon}^\infty -e^{A(x-t)-A(x)}g(x-t)\mathbf{1}_{x-t \geq 0} dx,\end{aligned}$$

where $\sigma_\varepsilon := \sigma(\varepsilon M_\varepsilon)$. From them, we get the estimate

$$|N'(t)[g]| \lesssim e^{\alpha t} \|g\|_{\mathcal{X}}.$$

Denoting

$$T(t)g(x) := (S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g(x),$$

we compute

$$\begin{aligned}T(t)g(x) &= \int_0^t S_{\mathcal{B}^1}(s)(\delta_{x=0}N(t-s)[g]) ds \\ &= \int_0^t e^{A(x-s)-A(x)}\delta_{x-s=0}N(t-s)[g]\mathbf{1}_{x-s \geq 0} ds \\ &= e^{-A(x)}N(t-x)[g]\mathbf{1}_{t \geq x},\end{aligned}$$

with the estimate

$$\begin{aligned} \|T(t)g(x)\|_{X'} &\lesssim \|e^{\alpha(t-x)-A(x)}\|_{X'} \|g\|_{X'} \\ &\lesssim e^{\alpha t} \|g\|_{X'}. \end{aligned}$$

We next differentiate the above identity, and we get

$$\begin{aligned} \partial_x T(t)g(x) &= -a(x)e^{-A(x)}N(t-x)[g]\mathbf{1}_{x \leq t} - e^{-A(x)}N(0)[g]\delta_{x=t} \\ &\quad - e^{-A(x)}N'(t-x)[g]\mathbf{1}_{x \leq t}. \end{aligned}$$

All together, we deduce

$$\|\partial_x T(t)g(x)\|_{\mathcal{X}} \lesssim e^{\alpha t} \|g\|_{\mathcal{X}},$$

and the similar estimate for $\|(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g\|_{\mathcal{X}}$. As a consequence, the announced estimate holds for the family of operators $S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}}(t)$.

Proof of (iii) We have

$$\begin{aligned} N(t)[\mathcal{A}g](x) &= \int_0^\infty a(x)e^{A(x-t)-A(x)}\delta_{x=t}\mathcal{M}[g] dx \\ &= a(t)e^{-A(t)}\mathcal{M}[g]. \end{aligned}$$

We deduce

$$\begin{aligned} (S_{\mathcal{B}}\mathcal{A})^{(*2)}(t)g &= \int_0^t S_{\mathcal{B}}(s)(\mathcal{A}S_{\mathcal{B}}(t-s)\mathcal{A})g ds \\ &= T(t)\mathcal{A}g(x) = e^{-A(x)}N(t-x)[\mathcal{A}g]\mathbf{1}_{t \geq x} \\ &= \mathbf{1}_{t \geq x} a(t-x) e^{-A(x)-A(t-x)}\mathcal{M}[g] \\ &= \varphi_t(x)\mathcal{M}[g]. \end{aligned}$$

with

$$(4.3.34) \quad \varphi_t(x) = \psi(t-x)e^{-A(x)}, \quad \psi(u) := \mathbf{1}_{u \geq 0} a(u) e^{-A(u)}.$$

We then compute

$$\hat{\varphi}_z(x) = \hat{\psi}(z) e^{-zx} e^{-A(x)}.$$

From the assumption (4.1.6) on a , we have

$$A(x) \geq -\alpha x - \frac{a_0}{2}x_0,$$

which implies

$$\begin{aligned} \|e^{-zx} e^{-A(x)}\|_{L^p}^p &\leq \int_0^\infty \left(e^{-\Re z x - A(x)} \right)^p dx \\ &\leq \int_0^\infty C^p e^{-p(\Re z + \alpha)x} dx \\ &\leq \frac{C^p}{p(\Re z - \alpha)}, \end{aligned}$$

for any $z \in \Delta_\alpha$ and some constant $C \in (0, \infty)$. The similar estimate holds in L^1 , therefore, we have

$$\|e^{-zx} e^{-A(x)}\|_{X'} \lesssim \frac{1}{\Re z - \alpha}, \quad \forall z \in \Delta_\alpha.$$

On the one hand, we have

$$\begin{aligned} |\hat{\psi}(z)| &= \int_0^\infty a(u) e^{-A(u)} e^{-zu} du \\ &\leq a_1 \|e^{-zx} e^{-A(x)}\|_{L^1} \\ &\lesssim \frac{1}{\Re z - \alpha}, \end{aligned}$$

for any $z \in \Delta_\alpha$. On the other hand, thanks to the regularity assumption $a \in W_x^{1,\infty}$, we may perform one integration by parts and we get

$$\begin{aligned} \hat{\psi}(z) &= \int_0^\infty a(u) e^{-A(u)} e^{-zu} du \\ &= \frac{1}{z} \left(a(0) + \int_0^\infty (a^2(u) - a'(u)) e^{-A(u)} e^{-zu} du \right), \end{aligned}$$

while a is the ‘‘step function’’ case, we have

$$\begin{aligned} \hat{\psi}(z) &= \int_{\sigma_\varepsilon}^\infty e^{-A(u)} e^{-zu} du \\ &= \frac{1}{z} \left(e^{-(\Re z - \alpha)\sigma_\varepsilon} + \int_{\sigma_\varepsilon}^\infty e^{-A(u)} e^{-zu} du \right), \end{aligned}$$

both of which imply

$$(4.3.35) \quad |\hat{\psi}(z)| \leq \frac{1}{|z|} \left(C_1 + \frac{C_2}{\Re z - \alpha} \right),$$

for any $z \in \Delta_\alpha$.

All together, we obtain

$$\begin{aligned} \|(R_{\mathcal{B}}(z)\mathcal{A})^2(z)g\|_{X'} &= \|\hat{\varphi}_z \mathcal{M}[g]\|_{X'} \\ &= \|e^{-zx-A(x)}\|_{X'} |\hat{\psi}(z)| |\mathcal{M}[g]| \\ &\leq \frac{C_{a^*}}{\langle z \rangle} \|g\|_{X'}, \end{aligned}$$

for any $z \in \Delta_{a^*}$, $a^* > \alpha$ and a constant C_{a^*} depending of a . \square

Proof of Theorem 4.3.5. Step 1. The structure of the spectrum. Observe that the positive steady state F_ε is also the solution to the eigenvalue problem $\Lambda F_\varepsilon = 0$ and the associated dual eigenvector is $\omega = 1$. According to the Kato’s inequality in Lemma 4.3.3 and the strong maximum principle in Lemma 4.3.4, we deduce that the eigenvalue $\lambda = 0$ is simple with the associated eigenspace $\text{Span}(F_\varepsilon)$ from the Krein-Rutman Theorem. Furthermore, the fact that S_{Λ_ε} is positive implies that $\lambda = 0$ is the only eigenvalue with nonnegative real part.

Step 2. The exponential decay. We write

$$R_\Lambda = R_B - R_B \mathcal{A} R_B + (R_B \mathcal{A})^2 R_\Lambda = \mathcal{U} + \mathcal{V} R_\Lambda$$

and next

$$(I - \mathcal{V}) R_\Lambda = \mathcal{U}.$$

We have $\mathcal{U} := R_B - R_B \mathcal{A} R_B : \Delta_a \rightarrow \mathcal{B}(X)$ as well as $\mathcal{V}(z) := (R_B(z) \mathcal{A})^2 : X \rightarrow X$ with bound of order $\mathcal{O}(\langle z \rangle^{-1})$ and $(R_B(z) \mathcal{A})^2 : X \rightarrow Y$ with $Y \subset \subset X$ thanks to Lemma 4.3.6. Arguing exactly as in [50, 45] we deduce that $R_\Lambda = (I - \mathcal{V})^{-1} \mathcal{U}$ is degenerate-meromorphic on Δ_{a^*} and $\Sigma(\Lambda) \cap \Delta_{a^*} = \Sigma_a(\Lambda) \cap B(0, R)$. Together with $\Sigma(\Lambda) \cap \bar{\Delta}_0 = \{0\}$ that implies that there exists $\alpha < 0$ such that $\Sigma(\Lambda) \cap \bar{\Delta}_\alpha = \{0\}$.

Next, we define $\Pi^\perp := I - \Pi_{\Lambda, a^*}$. From the iterated Duhamel formula and the inverse Laplace transform formula, we write

$$S_{\mathcal{L}} \Pi^\perp = S_B \Pi^\perp + \dots + (S_B \mathcal{A})^{(3)} S_B \Pi^\perp + (-1)^{n+1} \frac{i}{2\pi} \int_{\uparrow_{a^*}} (R_B \mathcal{A})^4 R_{\mathcal{L}} \Pi^\perp e^{zt} dz$$

in $\mathcal{B}(X)$ where $\uparrow_c := \{c + iy, y \in \mathbb{R}\}$ is the complex line of abscissa $c \in \mathbb{R}$. For the last term, we have

$$\begin{aligned} \left\| \int_{\uparrow_{a^*}} (R_B \mathcal{A})^4 R_{\mathcal{L}} \Pi^\perp e^{zt} dz \right\| &\leq e^{a^* t} \int_{\uparrow_{a^*}} \|(R_B(z) \mathcal{A})^4\| \|R_{\mathcal{L}}(z) \Pi^\perp\| dz \\ &\lesssim e^{a^* t} \int_{\uparrow_{a^*}} \frac{dz}{\langle z \rangle} \lesssim e^{a^* t} \end{aligned}$$

because of Lemma 4.3.6-(iii) and the fact that $\|R_\Lambda \Pi^\perp\|$ is uniformly bounded on Δ_{a^*} . \square

4.3.4 Nonlinear exponential stability

Now, we come back to the nonlinear problem (4.1.1) and present the proof of our main result Theorem 4.1.3 in the case without delay

$$m(t) = p(t) = \int_0^\infty a(x, \varepsilon m(t)) f(x) dx.$$

Proof of Theorem 4.1.3 in the case without delay. We split the proof into three steps.

Step 1. New formulation. Benefiting from Lemma 4.2.3, in the weak connectivity regime $\varepsilon \in [0, \varepsilon_0]$ or in the strong connectivity regime $\varepsilon \in [\varepsilon_\infty, \infty]$, where ε_0 and ε_∞ is the same as that in Lemma 4.2.3, we introduce a new formulation of the solution $f \in C([0, \infty); X)$ to the evolution equation (4.1.1) and the solution F_ε to the stationary problem (4.1.4) satisfying

$$\begin{aligned} \partial_t f + \partial_x f + a(\varepsilon \varphi[f]) f &= 0, & f(t, 0) &= \varphi[f(t, \cdot)], \\ \partial_x F + a(\varepsilon M) F &= 0, & F(0) &= M = \varphi[F], \end{aligned}$$

for a given unit mass initial datum $0 \leq f_0 \in X$, where here and below the ε and x dependency is often removed without any confusion.

Next, we consider the variation function $g := f - F$ satisfying

$$\begin{aligned}\partial_t g &= -\partial_x g - a(\varepsilon M)g + (a(\varepsilon M) - a(\varepsilon \varphi[f]))f \\ &= \Lambda g + Q[g]\end{aligned}$$

with $\mathcal{M} = \mathcal{M}_\varepsilon$ defined in (4.3.27), complemented with the boundary condition given by

$$\begin{aligned}g(t, 0) &= \varphi[f(t, \cdot)] - \varphi[F] \\ &= \int_0^\infty a(\varepsilon \varphi[f])f - \int_0^\infty a(\varepsilon \varphi[F])F \\ &= \mathcal{M}[g] - \langle Q[g] \rangle\end{aligned}$$

Regarding the boundary condition as a source term again, we deduce that the variation function g satisfies the equation

$$(4.3.36) \quad \partial_t g = \Lambda_\varepsilon g + Z[g],$$

with the nonlinear term $Z[g] := Q[g] - \delta_0 \mathcal{Q}[g]$.

Step 2. The nonlinear term. With the fact that $f \in L^\infty$ in Theorem 4.1.1, φ is Lipschitz continuous in Lemma 4.2.3 and the assumption (4.1.8), we estimate

$$\begin{aligned}\|Q[g]\|_X &= \|(a(\varepsilon \varphi[F]) - a(\varepsilon \varphi[f]))f\|_{L^1} \\ &\leq \|f\|_{L^\infty} \xi |\varphi[f] - \varphi[F]| \\ &\lesssim \xi \|g\|_{L^1}.\end{aligned}$$

Similarly, for the boundary term, we have

$$|\langle Q[g] \rangle| \lesssim \xi \|g\|_{L^1}.$$

Step 3. Decay estimate. Thanks to the Duhamel formula, the solution g to the evolution equation (4.3.36) satisfies

$$g(t) = S_{\Lambda_\varepsilon}(t)(g_0) + \int_0^t S_{\Lambda_\varepsilon}(t-s)Z[g(s)]ds.$$

Benefiting from Theorem 4.3.5 and the second step, we deduce

$$\begin{aligned}\|g(t)\|_X &\leq C e^{\alpha t} \|g_0\|_X + \int_0^t C e^{\alpha(t-s)} \|Z[g(s)]\|_X ds \\ &\lesssim e^{\alpha t} \|g_0\|_X + \xi \int_0^t e^{\alpha(t-s)} \|g(s)\|_X ds,\end{aligned}$$

for any $t \geq 0$ and for some constant $\alpha < 0$, independent of $\varepsilon \in [0, \varepsilon_0] \cup [\varepsilon_\infty, \infty]$. Thanks to the Grönwall's lemma, we have

$$\begin{aligned}\|g(t)\|_X &\lesssim e^{\alpha t} \|g_0\|_X + \xi \|g_0\|_X \int_0^t e^{\alpha t} \exp\left(\int_s^t e^{\alpha(t-r)} dr\right) ds \\ &\lesssim e^{\alpha t} \|g_0\|_X + \xi t e^{\alpha t} \|g_0\|_X \\ &\lesssim e^{\alpha' t} \|g_0\|_X,\end{aligned}$$

for some constant $\alpha < \alpha' < 0$. □

4.4 Case with delay

In this section, we conclude our main result Theorem 4.1.3, in the case with delay by following the same strategy as section 4.3 but with appropriate adaptation towards the boundary term. In order to express the relationship between $q(t)$ and $n(t)$ more clearly, we introduce the following intermediate evolution equation on a function $v = v(t, y)$

$$(4.4.37) \quad \partial_t v + \partial_y v = 0, \quad v(t, 0) = q(t), \quad v(0, y) = 0,$$

where $y \geq 0$ represent the local time for the network activity. Clearly, the above equation can be solved by the characteristics method as

$$v(t, y) = q(t - y) \mathbf{1}_{0 \leq y \leq t},$$

which simplifies the expression of the variation $n(t)$ of network activity in the equation (4.1.19), given by

$$(4.4.38) \quad n(t) = \mathcal{D}[v(t)], \quad \mathcal{D}[v] := \int_0^\infty v(y) b(dy),$$

while the variation $q(t)$ of discharging neurons in the equation (4.1.18) is rewritten as

$$q(t) = \mathcal{O}_\varepsilon[g(t), v(t)] := \mathcal{M}_\varepsilon[g(t)].$$

Therefore, we may rewrite the linear system (4.1.17)-(4.1.18)-(4.1.19) together with the evolution equation (4.4.37) as an autonomous system

$$(4.4.39) \quad \partial_t \begin{pmatrix} g \\ v \end{pmatrix} = \mathcal{L}_\varepsilon \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{L}_\varepsilon^1 g \\ \mathcal{L}_\varepsilon^2 v \end{pmatrix} := \begin{pmatrix} -\partial_x g - a_\varepsilon g \\ -\partial_y v \end{pmatrix},$$

with the associated semigroup $S_{\mathcal{L}_\varepsilon}(t)$ acting on

$$X := X_1 \times X_2 := L^1(\mathbb{R}_+) \times L^1(\mathbb{R}_+, \nu),$$

where the measure $\nu(y) = e^{-\delta y}$ with the same $\delta > 0$ in the condition (4.1.13). The domain of the operator \mathcal{L}_ε is given by

$$D(\mathcal{L}_\varepsilon) := \{(g, v) \in W^{1,1}(\mathbb{R}_+) \times W^{1,1}(\mathbb{R}_+, \nu); g(0) = v(0) = \mathcal{O}_\varepsilon[g, v]\}.$$

Then for any initial datum $g_0 \in X$, the weak solution of the linear autonomous system is given by

$$\partial_t \begin{pmatrix} g(t) \\ v(t) \end{pmatrix} = S_{\mathcal{L}_\varepsilon}(t) \begin{pmatrix} g_0 \\ v_0 \end{pmatrix}.$$

By regarding the boundary condition as a source term, we may rewrite the autonomous system as

$$\partial_t \begin{pmatrix} g(t) \\ v(t) \end{pmatrix} = \Lambda_\varepsilon \begin{pmatrix} g(t) \\ v(t) \end{pmatrix},$$

where the generator $\Lambda_\varepsilon = (\Lambda_\varepsilon^1, \Lambda_\varepsilon^2)$ is given by

$$\begin{pmatrix} \Lambda_\varepsilon^1(g, v) \\ \Lambda_\varepsilon^2(g, v) \end{pmatrix} := \begin{pmatrix} -\partial_x g - a_\varepsilon g + \delta_{x=0} \mathcal{O}_\varepsilon[g, v] \\ -\partial_y v + \delta_{y=0} \mathcal{O}_\varepsilon[g, v] \end{pmatrix},$$

and the associated semigroup $S_{\Lambda_\varepsilon}(t)$ acting on

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 := M^1(\mathbb{R}_+) \times M^1(\mathbb{R}_+, \nu),$$

where

$$M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))'; \text{supp } g \subset \mathbb{R}_+\}$$

is the bounded Radon measures endowed with the weak $*$ topology $\sigma(M^1, C_0)$ with C_0 representing the space of continuous functions converging to 0 at infinity. From the duality of S_{Λ_ε} , we deduce $S_{\Lambda_\varepsilon}|_X = S_{\mathcal{L}_\varepsilon}$. Next, we are going to show that $S_{\mathcal{L}_\varepsilon}$ possesses a finite dimensional principle part together with an exponential decaying remainder.

4.4.1 Steady state problem

We are going to prove that $S_{\mathcal{L}_\varepsilon}$ is a bounded semigroup in $X' := X \cap (L^p \times L^p(\nu))$, $1 \leq p \leq \infty$, from what we are able to deduce the existence of the steady state. On the other hand, the uniqueness derives from the classical weak and strong maximum principle.

Lemma 4.4.1. *The semigroup $S_{\mathcal{L}_\varepsilon}$ is bounded in X' . Furthermore, there exists at least nonnegative and normalized stationary state G_ε to the linear autonomous system (4.4.39)*

Proof of Lemma 4.4.1. Step 1. Conservative. In Lemma 4.3.1, we have already concluded for $\mathcal{L}_\varepsilon^1$. It is only necessary to focus on the operator \mathcal{L}_ε on $X_2 \cap L^p(\nu)$. For any $0 \leq v \in X_2 \cap L^p(\nu)$, the explicit formula gives

$$(4.4.40) \quad S_{L_\varepsilon^2} v = v(y-t) \mathbf{1}_{y-t \geq 0},$$

which clearly implies the positivity of $\mathcal{L}_\varepsilon^2$. Next, we compute

$$\begin{aligned} \frac{d}{dt} \int |v|^p \nu^p &= p \int (L_\varepsilon^2 v) \text{sign } v |v|^{p-1} \nu^p \leq p \int (L_\varepsilon^2 |v|) |v|^{p-1} \nu^p \\ &= - \int \partial_y (|v|^p) \nu^p \\ &= |v(0)|^p - p \delta \int |v|^p \nu^p \\ &\leq a_1 \|g\|_{L^1}^p - p \delta \|v\|_{L^p(\nu)}^p, \end{aligned}$$

which implies

$$(4.4.41) \quad \|v(t)\|_{L^p(\nu)} \leq \max\{C \|g_0\|_{L^1}, \|v_0\|_{L^p(\nu)}\}.$$

On the other hand, we have

$$\begin{aligned} \frac{d}{dt} \int |v| \nu &= \int (L_\varepsilon^2 v) \text{sign } v \leq \int L_\varepsilon^2 |v| \\ &= |v(0)| - \delta \int |v| \nu \\ &\leq a_1 \|g\|_{L^1} - \delta \|v\|_{L^1(\nu)}, \end{aligned}$$

which gives

$$(4.4.42) \quad \|v(t)\|_{L^1(\nu)} \leq \max\{C \|g_0\|_{L^1}, \|v_0\|_{L^1(\nu)}\}.$$

Interpolating the above two inequalities (4.4.41) and (4.4.42), we have

$$\|v(t)\|_{L^1(\nu) \cap L^p(\nu)} \lesssim \|(g_0, v_0)\|_{X'},$$

which implies

$$\|(g(t), v(t))\|_{X'} \lesssim \|(g_0, v_0)\|_{X'}.$$

Step 2. Steady State. Similarly to the Step 3 in the proof of Lemma 4.3.1, we define the equivalent new norm on X' as

$$\| \|h\|_\varepsilon := \sup_{t \geq 0} \|S_{\mathcal{L}_\varepsilon}(t)h\|_{X'}, \quad \forall h \in X',$$

since $S_{\mathcal{L}_\varepsilon}(t) \in \mathcal{B}(X')$ for any $t \geq 0$. Then for any $(g_0, v_0) \in X'$, we define the set

$$\tilde{\mathcal{C}} := \{h \in X'; h \geq 0, \langle\langle h \rangle\rangle = 1, \| \|h\| \| \leq \| \| (g_0, v_0) \| \| \},$$

which is invariant by the flow $S_{\mathcal{L}_\varepsilon}$. The Brouwer-Schauder-Tychono type of fixed point theorem gives the existence of stationary state for the linear autonomous system (4.4.39), which also happens to be the couple of $(F_\varepsilon, M_\varepsilon)$ as the steady state of the equation (4.1.4). \square

The uniqueness of the positive and normalized steady state $G = (F_\varepsilon, U_\varepsilon)$ to any given M_ε obtained above is realized from the weak and strong maximum principle.

Lemma 4.4.2. *The operator \mathcal{L}_ε satisfies the Kato's inequality, i.e.*

$$(\text{sign } h)\mathcal{L}_\varepsilon h := \frac{1}{2|h|}(h\mathcal{L}_\varepsilon \bar{h} + \bar{h}\mathcal{L}_\varepsilon h) \leq \mathcal{L}_\varepsilon |h|$$

for any complexed valued function $h \in X + iX$.

Proof of Lemma 4.4.2. For any $h = (g, v) \in X + iX$, we have already proven the Kato's inequality for $\mathcal{L}_\varepsilon^1$ in Lemma 4.3.3, thus we compute for $\mathcal{L}_\varepsilon^2$

$$\begin{aligned} \frac{1}{2|v|}(v\mathcal{L}_\varepsilon^2 \bar{v} + \bar{v}\mathcal{L}_\varepsilon^2 v) &= -\frac{1}{2|v|}\partial_y(v\bar{v}) \\ &= -\partial_y|v| = \mathcal{L}_\varepsilon|v|. \end{aligned}$$

Therefore, the weak maximum principle holds for the operator $-\mathcal{L}_\varepsilon$. \square

Lemma 4.4.3. *The operator $-\mathcal{L}_\varepsilon$ satisfies the following version of strong maximum principle, for any $0 \leq h \in X$ and $\lambda \in \mathbb{R}$, there holds*

$$h \in D(\mathcal{L}_\varepsilon) \setminus \{0\} \text{ and } (-\mathcal{L}_\varepsilon + \lambda)g \geq 0 \text{ imply } h > 0.$$

Proof of Lemma 4.4.3. Consider $0 \leq h = (g, v) \in X$. Suppose that there holds $(-\mathcal{L}_\varepsilon^2 + \lambda)v \geq 0$ for v satisfying the above conditions. It is only necessary to prove that v does not vanish in \mathbb{R}_+ . Since $v \not\equiv 0$, there exists $y^* \in \mathbb{R}_+$ such that $v(y^*) > 0$. Rewriting the assumption as

$$\partial_y v + \lambda v \geq 0,$$

we observe that

$$\partial_y(e^{\lambda y}v) = e^{\lambda y}(\partial_y v + \lambda v) \geq 0.$$

(i) For $y \in (y^*, \infty)$, we have

$$e^{\lambda y} v \geq e^{\lambda y^*} v(y^*) > 0.$$

(ii) For $y \in (0, y^*)$, by integrating the same equation on $(0, y)$, we obtain

$$e^{\lambda y} v \geq v(0) = \int_0^\infty a_\varepsilon(y) g(y) dy > 0,$$

thanks to the positivity assumption (4.1.7) on a and $g > 0$ because of the strong maximum principal on $-\mathcal{L}_\varepsilon^1$ in Lemma 4.3.4.

Therefore, v does not vanish on $(0, \infty)$, which implies the strong maximum principle for $-\mathcal{L}_\varepsilon$. \square

4.4.2 Decomposition of the semigroup

In this section, we establish that the semigroup S_{Λ_ε} is exponential stable and possesses a nice decomposition structure on the spectrum.

Theorem 4.4.4. *Assume (4.1.6)-(4.1.7)-(4.1.8) and (4.1.13). There exist some constants $\alpha < 0$ and $C > 0$ such that $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ and for any $\varepsilon \geq 0$ and any $(g_0, v_0) \in \mathcal{X}$, $\langle g_0 \rangle = 0$, there holds*

$$(4.4.43) \quad \|S_{\Lambda_\varepsilon}(t)(g_0, v_0)\|_{\mathcal{X}} \leq C e^{\alpha t} \|(g_0, v_0)\|_{\mathcal{X}}, \quad \forall t \geq 0.$$

Similarly, we introduce the convenient splitting of the operator Λ_ε on \mathcal{X} as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ defined by

$$\mathcal{B}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{B}_\varepsilon^1(g, v) \\ \mathcal{B}_\varepsilon^2(g, v) \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g \\ -\partial_y v \end{pmatrix}$$

and

$$\mathcal{A}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{A}_\varepsilon^1(g, v) \\ \mathcal{A}_\varepsilon^2(g, v) \end{pmatrix} = \begin{pmatrix} \delta_{x=0} \mathcal{O}_\varepsilon[g, v] \\ \delta_{y=0} \mathcal{O}_\varepsilon[g, v] \end{pmatrix},$$

which possess the following properties.

Lemma 4.4.5. *Assume that a satisfies conditions (4.1.6)-(4.1.8) and that b satisfies (4.1.13). Denoting $X = X_1 \times X_2$ representing \mathcal{X} or X' , for any $\varepsilon \geq 0$, the operators \mathcal{A}_ε and \mathcal{B}_ε satisfy :*

- (i) $S_{\mathcal{B}_\varepsilon}(t)$ is α -hypodissipative in X ;
- (ii) the family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{\mathcal{B}(X)} \leq C e^{a^* t}, \quad \forall a^* > \alpha;$$

- (iii) the family of operators $(R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2(z)$ satisfies the compactness estimate

$$\|(R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2(z)\|_{\mathcal{B}(X)} = \mathcal{O}(\langle z \rangle^{-1})$$

for any $z \in \Delta_{a^*}$, $a^* > \alpha$.

Proof of Lemma 4.4.5. From here, we remove the ε in the following proof for simplicity.

Proof of (i). The explicit formula gives (4.4.40) shows

$$\begin{aligned} \|S_{\mathcal{B}^2}(t)v\|_{X_2}^p &= \int_0^\infty |v(y-t)|^p \mathbf{1}_{y-t \geq 0} e^{-\delta t} dy \\ &\leq e^{-\delta t} \|v\|_{X_2}^p, \end{aligned}$$

for any $v \in X_2$ and $t \geq 0$. As a consequence, the hypodissipativity of $S_{\mathcal{B}}(t)$ is established by taking $\alpha := \min\{\frac{\alpha_0}{2}, \delta\}$ from Lemma 4.3.6-(i).

Proof of (ii). Recall the fact that $\mathcal{A} \in \mathcal{B}(X)$ in the proof of Lemma 4.3.6-(ii) and for any $(g, v) \in \mathcal{X}$, we have

$$\begin{cases} \mathcal{A}^1 S_{\mathcal{B}}(t)(g, v)(x) = \delta_{x=0} N(t)[g], \\ \mathcal{A}^2 S_{\mathcal{B}}(t)(g, v)(y) = \delta_{y=0} N(t)[g], \end{cases}$$

with the same $N(t)[g]$ as defined in (4.3.33), satisfying

$$\begin{aligned} |N(t)| &\lesssim e^{\alpha t} \|g\|_{\mathcal{X}_1}, \\ |N'(t)| &\lesssim e^{\alpha t} \|g\|_{\mathcal{X}_1}, \end{aligned}$$

Denoting

$$\begin{cases} T_1(t)(g, v)(x) := (S_{\mathcal{B}^1} * \mathcal{A}^1 S_{\mathcal{B}})(t)(g, v)(x), \\ T_2(t)(g, v)(y) := (S_{\mathcal{B}^2} * \mathcal{A}^2 S_{\mathcal{B}})(t)(g, v)(y), \end{cases}$$

we compute

$$\begin{aligned} T_2(t)(g, v)(y) &= \int_0^t S_{\mathcal{B}^2}(s) \delta_{y=0} (N(t-s)[g]) ds \\ &= \int_0^t \delta_{y-s=0} N(t-s)[g] ds \\ &= N(t-y)[g] \mathbf{1}_{y \leq t} \end{aligned}$$

as well as the differentiation

$$\partial_y T_2(t)(g, v)(y) = -N'(t-y)[g] \mathbf{1}_{y \leq t} - N(0) \delta_{y=t}.$$

All together, we deduce

$$\|\partial_y T_2(t)(g, v)(y)\|_{X_2} \lesssim e^{\alpha t} \|(g, v)\|_{\mathcal{X}},$$

and the similar estimate for $\|(S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{B}})(t)(g, v)\|_{\mathcal{X}}$.

Proof of (iii). We compute

$$\mathcal{A} S_{\mathcal{B}}(t-s) \mathcal{A} \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} \delta_{x=0} a(t-s) e^{-A(t-s)} \mathcal{M}[g] \\ \delta_{y=0} a(t-s) e^{-A(t-s)} \mathcal{M}[g] \end{pmatrix}$$

next

$$\begin{aligned}
(S_{\mathcal{B}}\mathcal{A})^{(*2)}(t) \begin{pmatrix} g \\ v \end{pmatrix} &= \int_0^t S_{\mathcal{B}}(s) (\mathcal{A}S_{\mathcal{B}}(t-s)\mathcal{A}) \begin{pmatrix} g \\ v \end{pmatrix} ds \\
&= \begin{pmatrix} T_1(t) \\ T_2(t) \end{pmatrix} \mathcal{A} \begin{pmatrix} g \\ v \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1}_{t \geq x} a(t-x) e^{-A(x)-A(t-x)} \mathcal{M}[g] \\ \mathbf{1}_{t \geq y} a(t-y) e^{-A(t-y)} \mathcal{M}[g] \end{pmatrix} \\
&= \begin{pmatrix} \varphi_t(x) \mathcal{M}[g] \\ \phi_t(y) \mathcal{M}[g] \end{pmatrix},
\end{aligned}$$

with $\varphi_t(x)$ is the same as defined in (4.3.34) and $\phi_t(y) = \psi(t-y)$. There holds the similar estimate

$$\begin{aligned}
\|e^{-zy}\|_{X_2} &= \|e^{-(z+\delta)y}\|_{X_1} \\
&\lesssim \frac{1}{\Re z - \alpha}, \quad \forall z \in \Delta_\alpha.
\end{aligned}$$

Together with the estimate of $|\hat{\psi}(z)|$ in (4.3.35), we finally conclude

$$\|(R_{\mathcal{B}}\mathcal{A})^2(z)\|_{\mathcal{B}(X)} \lesssim \langle z \rangle^{-1}.$$

Proof of Theorem 4.4.4. Step 1. The structure of the spectrum. From Lemma 4.4.5 and with the benefit of Spectral Mapping Theorem and Weyl's Theorem, we conclude that the spectrum of Λ_ε merely contains several isolated eigenvalues in the half complex plane Δ_α . Since the steady state is the eigenvector of 0, then the Kato's inequality in Lemma 4.4.2 and the strong maximum principle in Lemma 4.4.3 together with the Krein-Rutman Theorem imply that the eigenvalue $\lambda = 0$ is simple and the associated eigenspace is $\text{Span}(G)$. Moreover, $\lambda = 0$ is the only eigenvalue with nonnegative real part thanks to the fact that S_{Λ_ε} is positive.

Step 2. The exponential decay. From Theorem 4.3.5, we have already proved that $g(t) := S_{\Lambda_\varepsilon^1}(t)g_0 = \mathcal{L}_\varepsilon(t)g_0$ satisfies

$$\|g(t)\|_{X_1} \leq C e^{\beta t} \|g_0\|_{X_1}$$

for any $t \geq 0$ and any $\beta \in (a^*, 0)$. We then focus merely on Λ_ε^2 . Thanks to the Duhamel formula associated to the equation $\partial_t v = \Lambda_\varepsilon^2(g, v)$, we have

$$v(t) = S_{\mathcal{B}_\varepsilon^2}(t)v_0 + \int_0^t S_{\mathcal{B}_\varepsilon^2}(t-s)\mathcal{A}_\varepsilon^2(g(s), v(s)) ds.$$

The proven estimate (4.3.32) on $g(t)$ implies

$$\begin{aligned}
\|S_{\Lambda_\varepsilon^2}v_0(t)\|_{X_2} &= \|v(t)\|_{X_2} \\
&\leq \|S_{\mathcal{B}_\varepsilon^2}(t)v_0\|_{X_2} + \int_0^t \|S_{\mathcal{B}_\varepsilon^2}(t-s)\delta_0\mathcal{M}_\varepsilon[g(s)]\|_{X_2} ds \\
&\leq e^{-\delta t}\|v_0\|_{X_2} + \int_0^t e^{-\delta(t-s)}C e^{\beta s}\|g_0\|_{X_1} ds \\
&\leq C e^{\alpha t}\|(g_0, v_0)\|_{\mathcal{X}}
\end{aligned}$$

for $\max\{\beta, -\delta\} < \alpha < 0$, which yields our conclusion. \square

4.4.3 Nonlinear exponential stability

Proof of Theorem 4.1.3 in the case with delay. First, we recall the notation (4.1.2) and introduce a new notation

$$a_x(\mu) = a(x, \varepsilon \mu),$$

from which the evolution equation (4.1.1) writes

$$\partial_t f = -\partial_x f - a_x(m)f + \delta_0 p$$

with

$$p(t) = \mathcal{P}[f(t), m(t)], \quad m(t) = \int_0^\infty p(t-y)b(dy).$$

Introducing an auxiliary intermediate evolution equation

$$\partial_t u = -\partial_y u + \delta_0 p,$$

then the value of m is given by $m(t) = \mathcal{D}[u(t)]$ from the notation (4.4.38). All together, the evolution equation system rewrites

$$\begin{cases} \partial_t f = -\partial_x f - a_x(\mathcal{D}[u])f + \delta_0 p \\ \partial_t u = -\partial_y u + \delta_0 p \\ p = \mathcal{P}(f, \mathcal{D}[u]). \end{cases}$$

The stationary equation writes

$$0 = -\partial_x F - a_x(M)F + \delta_0 M$$

with $M = \mathcal{P}[F, M]$, while the corresponding stationary problem to the auxiliary equation is given by

$$0 = -\partial_y U + \delta_0 M$$

the value of M is given by $M = \mathcal{D}[U]$, with the notation (4.4.38). Then the stationary equation system rewrites

$$\begin{cases} 0 = -\partial_x F - a_x(M)F + \delta_0 M \\ 0 = -\partial_y U + \delta_0 M \\ M = \mathcal{P}(F, M), \quad M = \mathcal{D}[U]. \end{cases}$$

Consider the variation functions $g := f - F$, $v = u - U$ and we compute

$$\begin{cases} \partial_t g = -\partial_x g - a_x(M)g + (a_x(M) - a_x(m))f + \delta_0(p(t) - M) \\ \partial_t v = -\partial_y v + \delta_0(p(t) - M) \end{cases}$$

with

$$p(t) - M = \mathcal{P}[g, M] + \int_0^\infty (a_x(m) - a_x(M))f \, dx.$$

Denoting $Q[g, v] := (a_x(\mathcal{D}[U]) - a_x(\mathcal{D}[u]))f$, then the above system goes to

$$\begin{aligned}\partial_t \begin{pmatrix} g \\ v \end{pmatrix} &= \Lambda \begin{pmatrix} g \\ v \end{pmatrix} + \begin{pmatrix} Q[g, v] + \delta_{x=0} \langle Q[g, v] \rangle \\ \delta_{y=0} \langle Q[g, v] \rangle \end{pmatrix} \\ &:= \Lambda \begin{pmatrix} g \\ v \end{pmatrix} + \begin{pmatrix} \mathcal{Z}^1[g, v] \\ \mathcal{Z}^2[g, v] \end{pmatrix} \\ &:= \Lambda \begin{pmatrix} g \\ v \end{pmatrix} + \mathcal{Z}[g, v],\end{aligned}$$

with a linear system $\partial_t \begin{pmatrix} g \\ v \end{pmatrix} = \Lambda \begin{pmatrix} g \\ v \end{pmatrix}$ and the nonlinear term $\mathcal{Z}[g, v]$. Under the assumption (4.1.8) on a and from the definition of \mathcal{D} (4.4.38), we observe that

$$\begin{aligned}|\langle Q[g, v] \rangle| &\leq \|f\|_\infty \int_0^\infty |a_x(\mathcal{D}[U]) - a_x(\mathcal{D}[u])| dx \\ &\leq C \xi |\mathcal{D}[u] - \mathcal{D}[U]| \\ &\lesssim \xi \|v\|_{L^1},\end{aligned}$$

which implies that

$$\|\mathcal{Z}[g, v]\|_{\mathcal{X}} \leq C \varepsilon(\zeta) \|g, v\|_{\mathcal{X}}.$$

We finally use the Duhamel formula

$$(g, v) = S_\Lambda(t)(g_0, v_0) + S_\Lambda * \mathcal{Z}[g, v].$$

In Theorem 4.4.4, we establish the linear stability of the semigroup $S_\Lambda(t)$ in $\mathcal{B}(X)$,

$$S_\Lambda(t) = \mathcal{O}(e^{\alpha t}), \quad \alpha < 0,$$

therefore, we have

$$\|(g, v)(t)\|_X \leq C_0 e^{\alpha t} + C_0 C \int_0^t e^{\alpha(t-s)} \xi \|(g, v)(s)\|_X ds.$$

With the help of the Grönwall's lemma, denoting $\eta(t) := \|(g, v)(t)\|_X$, there indeed holds

$$\eta(t) \leq C_0 e^{\alpha t} + C_1 \xi \int_0^t e^{\alpha(t-s)} \eta(s) ds =: \theta(t),$$

which implies

$$\begin{aligned}\theta'(t) &= \alpha \theta(t) + C_1 \xi \eta(t) \\ &\leq (\alpha + C_1 \xi) \theta(t),\end{aligned}$$

so that

$$\theta(t) \leq \theta(0) e^{(\alpha + C_1 \xi)t} \leq \theta(0) e^{\alpha^* t},$$

with $\alpha^* < 0$ for $\varepsilon > 0$ small or large enough. Thus, we conclude that

$$\|(g, v)(t)\|_X = \eta(t) \leq C_0 e^{\alpha^* t} \|(g_0, v_0)\|_X,$$

for any $\varepsilon \in (0, \varepsilon_0) \cup (\varepsilon_\infty, \infty)$.

Bibliography

- [1] ADRIAN, E. D. The impulses produced by sensory nerve endings. *The Journal of physiology* 61, 1 (1926), 49–72.
- [2] ALT, W. Biased random walk models for chemotaxis and related diffusion approximations. *Journal of mathematical biology* 9, 2 (1980), 147–177.
- [3] ALT, W. Orientation of cells migrating in a chemotactic gradient. In *Biological growth and spread*. Springer, 1980, pp. 353–366.
- [4] ARENDT, W. Kato’s inequality: a characterisation of generators of positive semigroups. In *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences* (1984), JSTOR, pp. 155–174.
- [5] BALL, J. M. Strongly continuous semigroups, weak solutions, and the variation of constants formula. *Proceedings of the American Mathematical Society* 63, 2 (1977), 370–373.
- [6] BARDOS, C., AND DEGOND, P. Global existence for the vlasov-poisson equation in 3 space variables with small initial data. In *Annales de l’IHP Analyse non linéaire* (1985), vol. 2, pp. 101–118.
- [7] BOUCHUT, F., AND DESVILLETES, L. Averaging lemmas without time fourier transform and application to discretized kinetic equations. In *Proceedings of the Royal Society of Edinburgh-A-Mathematics* (1999), vol. 129, Edinburgh: The Society, 1974-, pp. 19–36.
- [8] BOURNAVEAS, N., AND CALVEZ, V. A review of recent existence and blow-up results for kinetic models of chemotaxis. *Can. Appl. Math. Q* 18, 3 (2010), 253–265.
- [9] BREN, A., AND EISENBACH, M. How signals are heard during bacterial chemotaxis: protein-protein interactions in sensory signal propagation. *Journal of bacteriology* 182, 24 (2000), 6865–6873.
- [10] CÁCERES, M. J., CARRILLO, J. A., AND GOUDON, T. Equilibration rate for the linear inhomogeneous relaxation-time boltzmann equation for charged particles. *Communications in Partial Differential Equations* 28, 5-6 (2003), 969–989.
- [11] CALVEZ, V. Chemotactic waves of bacteria at the mesoscale. *arXiv preprint arXiv:1607.00429* (2016).
- [12] CALVEZ, V., RAOUL, G., AND SCHMEISER, C. Confinement by biased velocity jumps: aggregation of escherichia coli. *Kinetic and Related Models* 8, 4 (2015), 651–666.
- [13] CASTELLA, F., AND PERTHAME, B. Strichartz’ estimates for kinetic transport equations. *Comptes rendus de l’Académie des sciences. Série 1, Mathématique* 322, 6 (1996), 535–540.

- [14] COLEMAN, B. D., AND MIZEL, V. J. Norms and semi-groups in the theory of fading memory. *Archive for Rational Mechanics and Analysis* 23, 2 (1966), 87–123.
- [15] DE MASI, A., GALVES, A., LÖCHERBACH, E., AND PRESUTTI, E. Hydrodynamic limit for interacting neurons. *Journal of Statistical Physics* 158, 4 (2015), 866–902.
- [16] DESVILLETES, L., AND MISCHLER, S. About the splitting algorithm for boltzmann and bgk equations. *Mathematical Models and Methods in Applied Sciences* 6, 08 (1996), 1079–1101.
- [17] DIEKMANN, O., VAN GILS, S. A., VERDUYN LUNEL, S. M., AND WALTHER, H.-O. Delay equations, vol. 110 of applied mathematical sciences, 1995.
- [18] DiPERNA, R. J., AND LIONS, P.-L. Global weak solutions of vlasov-maxwell systems. *Communications on Pure and Applied Mathematics* 42, 6 (1989), 729–757.
- [19] DiPERNA, R. J., AND LIONS, P.-L. Ordinary differential equations, transport theory and sobolev spaces. *Inventiones mathematicae* 98, 3 (1989), 511–547.
- [20] DiPERNA, R. J., LIONS, P.-L., AND MEYER, Y. l^p regularity of velocity averages. In *Annales de l'IHP Analyse non linéaire* (1991), vol. 8, pp. 271–287.
- [21] DOLBEAULT, J., MOUHOT, C., AND SCHMEISER, C. Hypocoercivity for linear kinetic equations conserving mass. *Transactions of the American Mathematical Society* 367, 6 (2015), 3807–3828.
- [22] ENGEL, K.-J., AND NAGEL, R. One-parameter semigroups for linear evolution equations, vol. 194 of graduate texts in mathematics. In *Semigroup Forum* (2001), vol. 63, Springer, pp. 278–280.
- [23] ERBAN, R., AND OTHMER, H. G. From individual to collective behavior in bacterial chemotaxis. *SIAM Journal on Applied Mathematics* 65, 2 (2004), 361–391.
- [24] ESCOBEDO, M., MISCHLER, S., AND RODRIGUEZ RICARD, M. On self-similarity and stationary problem for fragmentation and coagulation models. In *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis* (2005), vol. 22, Elsevier, pp. 99–125.
- [25] FOURNIER, N., AND LÖCHERBACH, E. On a toy model of interacting neurons. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* (2016), vol. 52, Institut Henri Poincaré, pp. 1844–1876.
- [26] FREDHOLM, I. Sur une classe d'équations fonctionnelles. *Acta mathematica* 27, 1 (1903), 365–390.
- [27] GERSTNER, W. Time structure of the activity in neural network models. *Physical review E* 51, 1 (1995), 738.
- [28] GERSTNER, W., AND KISTLER, W. M. *Spiking neuron models: Single neurons, populations, plasticity*. Cambridge university press, 2002.
- [29] GOLSE, F., LIONS, P.-L., PERTHAME, B., AND SENTIS, R. Regularity of the moments of the solution of a transport equation. *Journal of functional analysis* 76, 1 (1988), 110–125.
- [30] GOLSE, F., PERTHAME, B., AND SENTIS, R. Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre principale d'un opérateur de transport. *CR Acad. Sci. Paris Sér. I Math* 301, 7 (1985), 341–344.

- [31] GUALDANI, M. P., MISCHLER, S., AND MOUTHOT, C. Factorization for non-symmetric operators and exponential h-theorem. *arXiv preprint arXiv:1006.5523* (2010).
- [32] HERRERO, M. A., AND L., V. J. J. Chemotactic collapse for the keller-segel model. *Journal of Mathematical Biology* 35, 2 (1996), 177–194.
- [33] HERRERO, M. A., MEDINA, E., AND VELÁZQUEZ., J. J. L. Finite-time aggregation into a single point in a reaction-diffusion system. *Nonlinearity* 10, 6 (1997), 1739.
- [34] JABIN, P. E., AND VEGA, L. A real space method for averaging lemmas. *Journal de mathématiques pures et appliquées* 83, 11 (2004), 1309–1351.
- [35] JAMES, F., AND VAUCHELET, N. Chemotaxis: from kinetic equations to aggregate dynamics. *NoDEA: Nonlinear Differential Equations and Applications* (2013), 1–27.
- [36] KATO, T. *Perturbation theory for linear operators*, vol. 132. Springer Science & Business Media, 2013.
- [37] KELLER, E. F. Assessing the keller-segel model: how has it fared? In *Biological growth and spread*. Springer, 1980, pp. 379–387.
- [38] KELLER, E. F., AND SEGEL, L. A. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology* 26, 3 (1970), 399–415.
- [39] KELLER, E. F., AND SEGEL, L. A. Model for chemotaxis. *Journal of theoretical biology* 30, 2 (1971), 225–234.
- [40] KELLER, E. F., AND SEGEL, L. A. Traveling bands of chemotactic bacteria: a theoretical analysis. *Journal of theoretical biology* 30, 2 (1971), 235–248.
- [41] KREIN, M. G., AND RUTMAN, M. A. Linear operators leaving invariant a cone in a banach space. *Uspekhi Matematicheskikh Nauk* 3, 1 (1948), 3–95.
- [42] LIONS, P.-L., AND PERTHAME, B. Lemmes de moments, de moyenne et de dispersion. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique* 314, 11 (1992), 801–806.
- [43] METZ, J. A., AND DIEKMANN, O. *The dynamics of physiologically structured populations*, vol. 68. Springer, 2014.
- [44] MISCHLER, S. Semigroups in banach spaces-factorization approach for spectral analysis and asymptotic estimates.
- [45] MISCHLER, S. Erratum: Spectral analysis of semigroups and growth-fragmentation equations.
- [46] MISCHLER, S., AND MOUTHOT, C. Stability, convergence to self-similarity and elastic limit for the boltzmann equation for inelastic hard spheres. *Communications in Mathematical Physics* 288, 2 (2009), 431–502.
- [47] MISCHLER, S., AND MOUTHOT, C. Exponential stability of slowly decaying solutions to the kinetic-fokker-planck equation. *Archive for rational mechanics and analysis* 221, 2 (2016), 677–723.
- [48] MISCHLER, S., QUIÑINAO, C., AND TOUBOUL, J. On a kinetic fitzhugh-nagumo model of neuronal network. *Communications in mathematical physics* 342, 3 (2016), 1001–1042.
- [49] MISCHLER, S., QUIÑINAO, C., AND WENG, Q. Weak connectivity regime for a general time elapsed neuron network model. *In preparation* (2017).

- [50] MISCHLER, S., AND SCHER, J. Spectral analysis of semigroups and growth-fragmentation equations. In *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis* (2016), vol. 33, Elsevier, pp. 849–898.
- [51] MISCHLER, S., AND TRISTANI, I. Uniform semigroup spectral analysis of the discrete, fractional & classical fokker-planck equations. *arXiv preprint arXiv:1507.04861* (2015).
- [52] MISCHLER, S., AND WENG, Q. Relaxation in time elapsed neuron network models in the weak connectivity regime. *arXiv preprint arXiv:1505.06097* (2015).
- [53] MISCHLER, S., AND WENG, Q. On a linear runs and tumbles equation. *Kinetic and Related Models* 10, 3 (2017).
- [54] OTHMER, H. G., DUNBAR, S. R., AND ALT, W. Models of dispersal in biological systems. *Journal of mathematical biology* 26, 3 (1988), 263–298.
- [55] OTHMER, H. G., AND STEVENS, A. Aggregation, blowup, and collapse: the abc's of taxis in reinforced random walks. *SIAM Journal on Applied Mathematics* 57, 4 (1997), 1044–1081.
- [56] PAKDAMAN, K., PERTHAME, B., AND SALORT, D. Dynamics of a structured neuron population. *Nonlinearity* 23, 1 (2009), 55.
- [57] PAKDAMAN, K., PERTHAME, B., AND SALORT, D. Relaxation and self-sustained oscillations in the time elapsed neuron network model. *SIAM Journal on Applied Mathematics* 73, 3 (2013), 1260–1279.
- [58] PAKDAMAN, K., PERTHAME, B., AND SALORT, D. Adaptation and fatigue model for neuron networks and large time asymptotics in a nonlinear fragmentation equation. *The Journal of Mathematical Neuroscience* 4, 1 (2014), 1–26.
- [59] PAZY, A. Semigroups of linear operators and applications to partial differential equations, vol. 44 of applied mathematical sciences.
- [60] PERTHAME, B. Global existence to the bgk model of boltzmann equation. *Journal of Differential equations* 82, 1 (1989), 191–205.
- [61] PERTHAME, B. Mathematical tools for kinetic equations. *Bulletin of the American Mathematical Society* 41, 2 (2004), 205–244.
- [62] PERTHAME, B. *Transport equations in biology*. Springer Science & Business Media, 2006.
- [63] PERTHAME, B., AND LIONS, P.-L. Time decay, propagation of low moments and dispersive. *Communications in Partial Differential Equations* 21, 3-4 (1996), 801–806.
- [64] PERTHAME, B., AND SOUGANIDIS, P. E. A limiting case for velocity averaging. In *Annales scientifiques de l'École normale supérieure* (1998), vol. 31, pp. 591–598.
- [65] PHILLIPS, R. S. Perturbation theory for semi-groups of linear operators. *Transactions of the American Mathematical Society* 74, 2 (1953), 199–221.
- [66] QUÍÑANO, C. A microscopic spiking neuronal network for the age-structured model. *Acta Applicandae Mathematicae* 146, 1 (2016), 29–55.
- [67] RIBARIČ, M., AND VIDAV, I. Analytic properties of the inverse a (z)- 1 of an analytic linear operator valued function a (z). *Archive for Rational Mechanics and Analysis* 32, 4 (1969), 298–310.

- [68] ROBERT, P., AND TOUBOUL, J. D. On the dynamics of random neuronal networks. *Journal of Statistical Physics* 165, 3 (2016), 545–584.
- [69] SCHEP, A. R. Weak kato-inequalities and positive semigroups. *Mathematische Zeitschrift* 190, 3 (1985), 305–314.
- [70] STOCK, A. A nonlinear stimulus-response relation in bacterial chemotaxis. *Proceedings of the National Academy of Sciences* 96, 20 (1999), 10945–10947.
- [71] STROOCK, D. W. Some stochastic processes which arise from a model of the motion of a bacterium. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 28, 4 (1974), 305–315.
- [72] TRISTANI, I. Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting. *Journal of Functional Analysis* 270, 5 (2016), 1922–1970.
- [73] VIDAV, I. Spectra of perturbed semigroups with applications to transport theory. *Journal of Mathematical Analysis and Applications* 30, 2 (1970), 264–279.
- [74] VILLANI, C. *Hypocoercivity*. No. 949-951. American Mathematical Soc., 2009.
- [75] VOIGT, J. A perturbation theorem for the essential spectral radius of strongly continuous semigroups. *Monatshefte für Mathematik* 90, 2 (1980), 153–161.
- [76] VOLTERRA, V. Theory of functionals and of integral and integro. *Differential Equations* 61 (1930).
- [77] WENG, Q. General time elapsed neuron network model: well-posedness and strong connectivity regime. *arXiv preprint arXiv:1512.07112* (2016).

Résumé

Cette thèse vise à étudier certains modèles biologiques dans le réseau neuronal et dans la chimiotaxie avec la méthode d'analyse spectrale. Afin de traiter les principaux problèmes, tels que l'existence et l'unicité des solutions et des états stationnaires ainsi que les comportements asymptotiques, le modèle linéaire ou linéarisé associé est considéré par l'aspect du spectre et des semi-groupes dans les espaces appropriés, puis la stabilité de modèle non linéaire suit. Plus précisément, nous commençons par une équation de courses-et-chutes linéaire dans la dimension $d \geq 1$ pour établir l'existence d'un état stationnaire unique, positif et normalisé et la stabilité exponentielle asymptotique dans l'espace L^1 pondéré basé sur la théorie de Krein-Rutman avec quelques estimations du moment de la théorie cinétique. Ensuite, nous considérons le modèle du temps écoulé sous les hypothèses générales sur le taux de tir et nous prouvons l'unicité de l'état stationnaire et sa stabilité exponentielle non linéaire en cas sans ou avec délai au régime de connectivité faible de la théorie de l'analyse spectrale pour les semi-groupes. Enfin, nous étudions le modèle sous une hypothèse de régularité plus faible sur le taux de tir et l'existence de la solution ainsi que la même stabilité exponentielle sont généralement établies n'importe la prise en compte du délai ou non, au régime de connectivité faible ou forte.

Mots Clés

théorie de l'analyse spectrale, modèle de courses-et-chutes, équations cinétiques, processus de vitesse-saut, chimiotaxie, état stationnaire, stabilité asymptotique, hypocoercivité, réseau de neurones, dynamique du temps écoulé, connectivité faible, connectivité forte, hypodissipativité, convergence exponentielle.

Abstract

This thesis is aimed to study some biological models in neuronal network and chemotaxis with the spectral analysis method. In order to deal with the main concerning problems, such as the existence and uniqueness of the solutions and steady states as well as the asymptotic behaviors, the associated linear or linearized model is considered from the aspect of spectrum and semigroups in appropriate spaces then the nonlinear stability follows. More precisely, we start with a linear runs-and-tumbles equation in dimension $d \geq 1$ to establish the existence of a unique positive and normalized steady state and the exponential asymptotic stability in weighted L^1 space based on the Krein-Rutman theory together with some moment estimates from kinetic theory. Then, we consider time elapsed model under general assumptions on the firing rate and prove the uniqueness of the steady state and its nonlinear exponential stability in case without or with delay in the weak connectivity regime from the spectral analysis theory for semigroups. Finally, we study the model under weaker regularity assumption on the firing rate and the existence of the solution as well as the same exponential stability are established generally no matter taking delay into account or not and no matter in weak or strong connectivity regime.

Keywords

spectral analysis theory, runs-and-tumbles model, kinetic equations, velocity-jump processes, chemotaxis, stationary state, asymptotic stability, hypocoercivity, neuron network, time elapsed dynamics, weak connectivity, strong connectivity, hypodissipativity, exponential convergence.