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Thibault De Poyferré

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THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres
PSL Research University

Préparée à l'École normale supérieure

On the Cauchy Problem for the Water Waves Equations

Sur le problème de CAUCHY pour l'équation des vagues

École doctorale N° 574

ÉCOLE DOCTORALE DE MATHÉMATIQUES HADAMARD

Spécialité Mathématiques

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Introduction

Les vagues sont des ondulations de la surface d'un fluide, provoquées par des perturbations localisées de celle-ci, et se propageant très loin de leur lieu d'origine. Ce sont les ondes les plus facilement visualisables, et leur étude forme par conséquent un des plus anciens champs de la physique. Malheureusement, faute d'outils mathématiques suffisants il a longtemps fallu se limiter à des observations pratiques. La première dérivation d'un modèle rigoureux fut permise par la découverte par EULER [41, 42, 43] des équations régissant le mouvement des fluides. Les premiers travaux sur le sujet furent ceux DE LAPLACE [72] et de LAGRANGE [69, 68], suivi par le mémoire magistral de CAUCHY [28]. Ces premiers travaux s'intéressaient à la théorie linéaire de ces équations, c'est-à-dire régissant les ondes proches de l'équilibre, dont le mémoire de CAUCHY notamment donna une description très précise grâce à l'analyse de FOURIER. Plus tard, ces études furent reprises par KELLAND [61], RUSSELL [89], AIRY [2], et d'autres encore ; on pourra consulter pour un historique plus complet de ces commencements l'article de CRAIK [38].

Les équations

Décrivons ces équations, dans leur formulation moderne. Nous travaillons en dimension $n \geq 2$, les dimensions physiques étant $n = 2$ ou 3 . Un contenant nous est donné ; mathématiquement c'est un ouvert connexe \mathcal{O} , dont le bord \mathcal{M} est imperméable. Un fluide remplit partiellement ce contenant : il occupe un domaine Ω_t qui change avec le temps. Le bord de ce domaine peut être décomposé en deux parties : la partie correspondant au fond immergé \mathcal{B}_t est incluse dans \mathcal{M} , tandis que la partie séparant le liquide de l'air, nommée frontière ou surface libre, sera notée \mathcal{S}_t .

Deux cas de figure sont possibles. Dans le premier, le plus traditionnellement étudié, le domaine du fluide est supposé infini latéralement ; le fond \mathcal{B} est donc indépendant du temps, et il est séparé de la surface \mathcal{S}_t par une distance toujours plus grande qu'une quantité $h > 0$ fixée à l'avance. Ce cas, celui de la figure 1, décrit les vagues en haute mer.

Dans le second cas, dont l'étude constitue la deuxième partie de cet ouvrage, on permet au fond d'émerger. Le fond \mathcal{B}_t et la surface \mathcal{S}_t se coupent alors transversalement suivant la ligne d'eau \mathcal{L}_t , comme sur la figure 2. En dimension 2, représentée sur le dessin, cette ligne d'eau est composée de points isolés ; en dimension 3, c'est un lacet. Plus généralement, en dimension n , c'est une variété de dimension $n - 2$. Le domaine Ω_t sera dans ce cas borné.

L'air est supposé inerte et de pression constante ; il ne reste donc à décrire que le fluide. Choissant la description la plus simple, nous le prendrons parfait, incompressible, non visqueux, et de densité constante. Il sera alors entièrement décrit par son champ de vitesse v , défini sur Ω_t et à valeurs dans \mathbf{R}^n . Ces hypothèses simplificatrices sur la nature du fluide sont d'assez bonnes approximations pour l'eau. Les équations d'EULER nous apprennent alors que le champ de vitesse

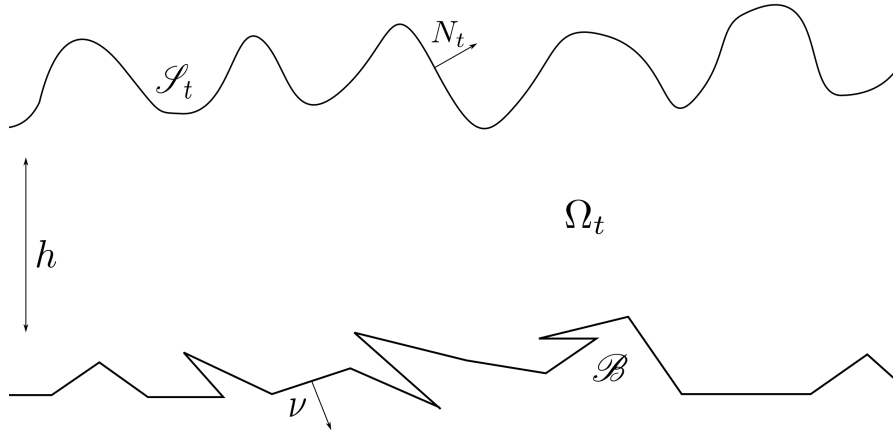


FIGURE 1 – Cas non borné

suit les relations

$$(E) \quad \begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p - g e^n, \\ \nabla \cdot v = 0, \end{cases}$$

dans le domaine Ω_t . Ici, la pression p est une fonction scalaire définie sur Ω_t . Nous supposons la présence d'un champ de gravité constant, dirigé selon $-e^n$ et dont la force est mesurée par la constante $g > 0$. La première équation est simplement la loi de Newton, tandis que la seconde assure l'incompressibilité du fluide. Comme nous le verrons plus tard, une conséquence de ces équations est que la pression est solution d'une équation elliptique dans le domaine. Si on connaît le champ de vitesse, elle sera alors entièrement déterminée par sa valeur sur la surface \mathcal{S}_t . La pression extérieure étant supposée constante, elle peut être choisie égale à 0 (p est définie à une constante près). Il s'agit alors de décrire le saut de pression à travers cette interface. Ce saut, dont l'origine physique est la tension de surface (ou capillarité), est proportionnel à la courbure moyenne de l'interface. Le coefficient de proportionnalité dépend de la taille caractéristique des vagues étudiées. Il est important pour des vagues de petite longueur d'onde, et ces vagues (ou vaguelettes), sont dites de capillarité. Pour des vagues de plus grande longueur d'onde, il est si petit qu'on peut le supposer nul, si bien que la pression est nulle sur la surface. Les vagues correspondantes sont alors dites de gravité. Ce sont ces dernières que l'on observe en mer. Il est important de noter que la prise en compte ou non de la tension de surface mène à des comportements très différents, ce qui est reflété par des théories différentes pour chacun des deux cas.

Il nous reste à décrire le mouvement du domaine Ω_t . Il est pour tout temps composé précisément des particules de fluides, si bien que si l'on suit la trajectoire de chacune de ces particules, trajectoire obtenue en intégrant le champ de vitesse au court du temps, on obtient le domaine au temps t comme image par le flot du domaine au temps 0. Géométriquement, cela signifie que le champ de vecteurs $\partial_t + v \cdot \nabla$ est tangent à la variété $\cup_t \Omega_t$. Si le champ v est suffisamment lisse, cela entraîne

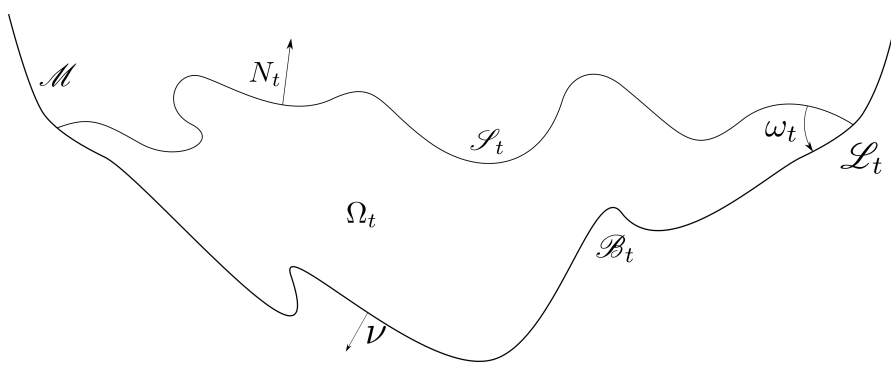


FIGURE 2 – Cas borné

notamment que les particules composant la surface libre restent toujours les mêmes (c'est à dire que la surface libre est transportée par le flot), et que le champ de vecteur v est tangent au fond. Si la ligne d'eau L_t est présente, elle est alors elle-aussi transportée par le flot, et ne se décolle pas du bord \mathcal{M} .

Le système ainsi décrit se rapproche de ce que l'on appelle une équation aux dérivées partielles. Il s'en distingue cependant par la mobilité du domaine sur lequel il est posé. Il est en outre très éloigné, formellement comme qualitativement, des équations dites linéaires, à propos desquelles la connaissance mathématique est la plus complète. Il est donc peu surprenant qu'excepté les travaux déjà cités, assez peu de mathématiciens se soient attaqués au sujet durant la majeure partie des XIX^e et XX^e siècles. La plupart des efforts étaient concentrés sur l'étude de modèles approchés, valables seulement dans des régimes bien particuliers. Les deux situations les plus étudiées étaient celle des vagues de grande amplitude, régie par l'équation linéarisée qu'avaient déjà étudiée LAPLACE, LAGRANGE et CAUCHY, et celle des eaux peu profondes, décrite par de nombreux modèles comme celui découvert par KORTEWEG et DE VRIES [67] et qui porte leurs noms.

Ces modèles étaient dérivés formellement à partir des équations totales, et puisqu'on ne savait rien du modèle original, pas même s'il avait des solutions, ils n'avaient aucune justification théorique. Cependant, ils se montrèrent si aptes à prédire le comportement des vagues dans de si nombreuses situations, qu'ils sont toujours à la base de l'océanographie moderne. Ils ont pour la plupart été rigoureusement justifiés depuis. On en trouvera les références dans le livre de LANNES [71].

Parallèlement, l'étude des équations aux dérivées partielles (où EDP) se développait fortement. Les équations étudiées peuvent être grossièrement séparées en trois catégories. Les équations elliptiques tout d'abord, dont le prototype est l'équation de LAPLACE, sont indépendantes du temps et décrivent l'équilibre d'un milieu continu. On a vu que la pression satisfait une équation de ce type dans le domaine fluide Ω_t . Plus généralement, la théorie de ces équations est fondamentale dans l'étude des vagues. Ensuite, les équations paraboliques, dont il ne sera pas question ici, et dont le prototype est l'équation de la chaleur, et enfin les équations hyperboliques, qui décrivent fréquemment un phénomène ondulatoire, et auxquelles se rattachent le plus nos équations des vagues. Leur prototype est l'équation qui régit la propagation des ondes sonores, et c'est elle qui concentra la majorité des efforts des mathématiciens dans ce domaine, durant toute cette période.

Problème de CAUCHY

Dans l'étude d'une équation aux dérivées partielles donnée, le premier problème à se poser est nommé problème de CAUCHY. Il s'agit de montrer que, étant donné l'état du système à un temps donné (que l'on choisit comme origine $t = 0$), on peut trouver une solution de l'équation continuant cet état. Dans de nombreuses situations, la solution ne sera pas nécessairement globale, c'est-à-dire qu'on ne peut pas espérer la continuer pour tout temps. On cherche donc cette solution sur un intervalle de temps fini $[0, T]$, et il est important que le temps T puisse être choisi indépendamment de la forme de la donnée initiale, en ce sens qu'une petite variation de celle-ci n'entraîne pas l'écroulement du temps T vers 0. Additionnellement, on veut que cette solution soit uniquement déterminée par sa donnée initiale, et enfin qu'une petite variation de celle-ci n'entraîne qu'une petite variation de la solution. Si l'équation vérifie ces trois conditions, le problème de CAUCHY est dit bien posé au sens d'HADAMARD.

Pour pouvoir utiliser efficacement cette notion, il est important de choisir dans quel espace fonctionnel travailler, c'est-à-dire à quel niveau de régularité se trouvent la donnée initiale et la solution attendue. C'est en effet en choisissant un tel espace, et en considérant sa topologie, que l'on peut préciser ce que l'on entend par « petite variation » ; en outre une EDP n'est généralement pas bien posée dans tous les espaces. Plus précisément, on s'attend à ce que l'équation soit bien posée pour des données très régulières, et cesse de l'être à basse régularité. De plus, les méthodes utilisées sont généralement plus simples pour des données plus régulières. Comme on le verra cependant dans la deuxième partie de ce mémoire, on s'attend à ce que le système décrivant les vagues lorsque la ligne d'eau est présente ne soit bien posé qu'aux régularités intermédiaires. Dans le cas des équations ondulatoires, les espaces choisis pour mesurer la régularité sont les espaces de SOBOLEV H^s , $s \in \mathbf{R}$, basés sur l'espace de Lebesgue $L^2 = H^0$.

Les raisons pour s'intéresser aux problèmes de CAUCHY sont multiples. Tout d'abord, les mathématiciens n'aiment pas raisonner sur des objets qui n'existent pas. On pourrait à bon droit se questionner sur la pertinence physique d'un modèle qui n'aurait pas de solution ! Ainsi, cette question est toujours la première à traiter dans l'étude d'un tel modèle.

Pour comprendre la deuxième raison, il faut observer ce que sera ensuite la stratégie d'étude d'une telle équation. On cherchera en général à identifier des solutions assez explicites, et dont le comportement semblera typique, comme par exemple des solutions stationnaires, des ondes solitaires se propageant sans changer de forme (les solitons), ou des solutions cessant d'exister en temps finis, soit en concentrant leur énergie ou leur masse en un point, soit en formant des discontinuités (les chocs). Une fois ces solutions particulières identifiées, on voudra montrer leur stabilité, c'est à dire que des solutions proches de ces solutions particulières se comporteront similairement, si bien qu'avec suffisamment de solutions particulières, on espérera décrire toutes les solutions. Il s'agira donc de développer une théorie perturbative robuste, c'est à dire une manière de comprendre le comportement d'une petite variation d'un objet donné. Il se trouve que les méthodes mises en œuvre pour résoudre le problème de CAUCHY sont du même ordre, en considérant la solution sur un temps assez court comme une petite variation de sa donnée initiale. Le problème de CAUCHY fonctionne ainsi comme un laboratoire d'essai, permettant de développer une théorie perturbative simple. Ainsi, moins les espaces dans lesquels les solutions sont recherchées seront réguliers, plus la théorie sera efficace. Cet argument est donc aussi une raison de s'intéresser au problème de CAUCHY pour des régularités aussi basses que possible, et de s'approcher ainsi du seuil où l'équation devient mal posée.

La dernière raison est liée à la structure hamiltonienne que l'on retrouve dans de nombreuses équations d'évolution. Cette structure fait dériver toute la dynamique d'une unique fonctionnelle, le hamiltonien ; une conséquence de cette structure est que la fonctionnelle en question, qui est fréquemment l'énergie physique, est conservée le long des solutions. Cette quantité permet souvent

de contrôler un certain niveau de régularité de la solution, et une théorie de CAUCHY au même niveau d'existence permet alors d'obtenir des solutions globales. Comme ce niveau de régularité est assez bas, cela justifie encore de travailler à une théorie de CAUCHY à faible régularité. Un des résultats les plus importants de la théorie des vagues est la découverte par ZAKHAROV [111] de leur structure hamiltonienne. La régularité en est cependant très basse, correspondant à une surface et un champ de vitesse L^2 . À ce niveau de régularité, le sens à donner aux équations n'est pas clair.

Revenons maintenant à l'historique du problème. Les premiers résultats d'existence locale pour les équations des vagues, dans le cas d'un océan infini latéralement et d'un fluide irrotationnel, furent ceux de NALIMOV [83], SHINBROT [93], KANO et NISHIDA [60], YOSIHARA [109], et CRAIG [34]. Afin de simplifier la situation, tous ces résultats utilisaient des hypothèses, soit de petitesse de la donnée initiale, soit de régularité analytique, en conséquence de quoi ils ne représentaient pas encore une théorie de CAUCHY complète. Pour obtenir une telle théorie, il fallut attendre les résultats de WU [104, 105], en 2D puis 3D.

Système de coordonnées

Comme nous l'avons précisé plus haut, l'une des difficultés majeures dans l'étude de ce problème provient de la mobilité du domaine. Une manière simple de la contourner est de choisir un système de coordonnées globales pour le domaine Ω_t , c'est à dire une famille de difféomorphismes Φ_t depuis un domaine fixe vers Ω_t . On obtient alors un système de deux équations couplées, une concernant l'évolution du difféomorphisme Φ_t et traduisant l'évolution du domaine Ω_t , et une correspondant aux équations d'EULER sur le champ de vitesse, mais tirées en arrière sur le domaine fixe. L'avantage est que ce système est posé sur un domaine fixe ; l'inconvénient, nous y reviendrons, est que la structure algébrique de l'équation est rendue plus complexe. Il y a de nombreuses manières de choisir ces coordonnées, et les équations obtenues sont assez différentes, quoiqu'équivalentes pour des solutions assez régulières. Présentons rapidement les principales, en détaillant celles qui seront utilisées dans ce mémoire.

La première manière consiste à utiliser l'application lagrangienne, qui n'est autre que le flot du champ de vitesse : ainsi à un point du domaine initial, on associe sa position au temps t . Le domaine fixe est alors ce domaine initial. Cette formulation très physique met en lumière la structure variationnelle de l'équation, comme nous le verrons au chapitre 5 ; malheureusement la régularité de l'application lagrangienne est trop faible pour tirer en arrière l'équation d'EULER sans perdre de régularité. Il y a néanmoins des moyens de passer outre, employés notamment par CHRISTODOULOU et LINDBLAD [31], LINDBLAD [74, 75], COUTAND et SHKOLLER [32], et CHENG, COUTAND et SHKOLLER [29].

La deuxième méthode emploie en 2D le théorème de l'application conforme pour ramener Ω_t à un domaine fixe, comme par exemple un demi-plan ou une bande. C'est la méthode employée par la plupart des articles originaux sur le problème de CAUCHY cités à la section précédente, parfois combinée avec l'usage de l'application lagrangienne, comme dans le premier article de Sijue Wu, car elle simplifie fortement l'étude du problème elliptique dans le domaine fluide. Le désavantage de cette méthode est qu'elle est intrinsèquement bi-dimensionnelle. Sijue Wu [105] a cependant étendu cette formulation au cas tri-dimensionnel grâce aux algèbres de Clifford.

À l'opposé de la formulation lagrangienne, la formulation dite eulérienne ne suit pas les trajectoires, et redresse chaque domaine à temps fixé indépendamment de son passé. La méthode précédente en est déjà un exemple dans sa formulation originale, bien qu'on la classe généralement à part en raison de l'emploi qu'on y fait de l'analyse complexe. On parle plutôt de formulation eulérienne lorsqu'on emploie un difféomorphisme non nécessairement conforme pour redresser le

domaine, ce qui peut se faire en dimension arbitraire. Une manière d'implémenter cette méthode est celle utilisée par ZAKHAROV [111] pour exhiber la structure hamiltonienne des équations. Elle fut ensuite développée par CRAIG, SULEM et SULEM [36] puis CRAIG et SULEM [35], et porte le nom de système de ZAKHAROV-CRAIG-SULEM (ZCS). Comme cette formulation sera utilisée dans toute la première partie de ce travail, nous la décrivons en détail.

Tout d'abord, nous sommes dans la situation de la figure 1, où le fond est séparé de la surface par une distance plus grande qu'un $h > 0$, choisit une fois pour toutes. Nous ajoutons deux hypothèses simplificatrices. Tout d'abord, nous supposons que le fluide est potentiel, c'est-à-dire qu'il existe une fonction scalaire ϕ telle que $v = \nabla\phi$: c'est notamment le cas pour un fluide irrotationnel (c'est-à-dire tel que $\text{rot}v = 0$, condition propagée par le flot) dans un domaine simplement connexe. Cette hypothèse est classique en océanographie, au-delà même de la sphère mathématique : c'est en effet une excellente approximation pour le cas d'un océan près de l'équateur (ce qui permet de négliger les forces non conservatives, comme celle de Coriolis). Tous les articles précédemment cités utilisaient d'ailleurs ces hypothèses. Comme le champ de vitesse est à divergence nulle, cela entraîne que $\Delta\phi = 0$ dans Ω_t . Comme en outre la dérivée normale $\partial_n\phi$ au fond est nulle (v est tangent au fond), la théorie elliptique standard nous dit que la fonction ϕ est entièrement déterminée par sa valeur à la surface libre \mathcal{S}_t . Ainsi dans le cas potentiel, toute la dynamique est concentrée sur la surface. On fait encore l'hypothèse que la surface libre est le graphe d'une fonction η au-dessus de l'hyperplan $x_n = 0$. Dans ce cas, on peut noter $\psi(t, x) := \phi(t, x, \eta(x))$, où $x = x_1, \dots, x_{n-1}$, et les équations s'écrivent

$$(ZCS) \quad \begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + g\eta - \kappa H(\eta) + \frac{1}{2}|\nabla_x\psi|^2 - \frac{1}{2} \frac{(\nabla_x\eta \cdot \nabla_x\psi + G(\eta)\psi)^2}{1 + |\nabla_x\eta|^2} = 0. \end{cases}$$

L'opérateur $G(\eta)\psi$ est l'opérateur de DIRICHLET vers NEUMANN renormalisé : il est défini par

$$G(\eta)\psi = \sqrt{1 + |\nabla_x\eta|^2} (\partial_n\phi|_{y=\eta}),$$

où ϕ est la solution du problème elliptique

$$\begin{cases} \phi|_{y=\eta} = \psi, \\ \Delta\phi = 0 \text{ dans } \Omega_t, \\ \partial_n\phi|_{\mathcal{B}} = 0. \end{cases}$$

L'expression $H(\eta) = \nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}} \right)$ est celle de la courbure moyenne de la surface, et le nombre $\kappa \geq 0$ mesure l'importance de la capillarité. Pour simplifier les notations, on écrira $d = n - 1$ lorsqu'on se servira de cette formulation, si bien que les inconnues sont des fonctions de $(t, x) \in \mathbf{R} \times \mathbf{R}^d$. La variable x_n sera notée y en raison de son rôle particulier.

L'avantage principal de cette méthode est que les équations y sont posées sur \mathbf{R}^n , ce qui permet l'usage de nombreuses techniques d'analyse harmonique. L'inconvénient est que le problème elliptique, ici encapsulé dans G , est plus complexe. Cette formulation permit à LANNES [70] de résoudre le problème de CAUCHY, puis de donner un cadre unifié pour la justification des modèles asymptotiques. Sur ces sujets, et pour une bibliographie plus complète, on consultera son livre [71]. Cette formulation est aussi le point de départ du programme d'ALAZARD, BURQ et ZUILY, dont il sera question à la section suivante.

La dernière formulation, qui sera celle employée dans la deuxième partie du mémoire, se passe de coordonnées. On y considère le domaine Ω_t comme une variété à bord plongée dans \mathbf{R}^n , et les

divers calculs sont fait sur des quantités géométriques naturelles, comme le vecteur normal, la seconde forme fondamentale, et la courbure moyenne. Cette méthode fut employée par BEYER et GÜNTHER [19, 20] et par SHATAH et ZENG [90, 91, 92] pour étudier des situations où le domaine est une goutte d'eau, quoiqu'elle s'applique aussi bien aux situations que nous nous proposons d'étudier. Ses avantages sont multiples : elle s'adapte très bien à des géométries compliquées, comme celle de la figure 2 ; elle permet d'éviter certaines complications dues à l'usage de coordonnées et qui, dans les autres formulations, obligent à utiliser une quantité nommée « bonne inconnue d'Alinhac », sur laquelle nous reviendrons dans les preuves de la première partie ; elle s'adapte très bien à la présence d'un rotationnel, comme dans les articles de Shatah et Zeng ; et enfin toutes les quantités qui y apparaissent ont un sens géométrique, et donc physique. Elle a néanmoins plusieurs inconvénients : elle se prête mal aux techniques d'analyse harmonique ; la compréhension des calculs nécessite quelques connaissances en géométrie riemannienne ; et enfin, quoiqu'elle soit suffisante pour prouver des inégalités comme les estimations a priori dont il sera question dans la deuxième partie, la résolution du problème de CAUCHY nécessite de faire converger des fonctions posées sur un domaine fixe, et donc on ne peut plus faire l'économie des coordonnées. En ne choisissant celles-ci qu'à la toute fin de l'opération, on peut cependant conserver les avantages de cette méthode, comme on peut le voir par exemple chez SHATAH et ZENG [92].

Description des résultats

La première partie de ce mémoire s'inscrit dans le programme d'ALAZARD, BURQ et ZUILY, dont l'objet était l'étude des équations de vagues par le calcul paradifférentiel. Durant les trente dernières années, de nombreuses techniques assez sophistiquées ont été développées pour l'étude des équations aux dérivées partielles non linéaires. Ces méthodes reposent en partie sur une décomposition des solutions en fonctions dont le spectre de FOURIER est très localisé, que l'on fait ensuite évoluer par l'équation pour en étudier l'interaction. La plus simple de ces décompositions est celle dite de LITTLEWOOD-PALEY, dans laquelle les fréquences sont localisées dans des couronnes dyadiques. Le calcul paradifférentiel de BONY [21, 22, 23] est une manière d'organiser les interactions des différentes pièces en une unique équation, que l'on appelle le paralinéarisé, et qui présente une analogie structurelle forte avec le linéarisé de l'équation originale, ce qui permet de transférer de nombreux résultats de ce dernier, plus simple d'étude, vers le problème non linéaire. On retrouvera les principaux résultats de cette théorie à l'annexe A. Pour plus de détail, on pourra consulter le livre de MÉTIVIER [80].

Les équations des vagues, par exemple sous la forme du système de ZAKHAROV-CRAIG-SULEM, ont une structure fortement non-linéaire, ainsi que non locale, principalement à cause de l'opérateur de DIRICHLET-NEUMANN. Cet opérateur était le premier obstacle à la paralinéarisation du système ; il fut dépassé par ALAZARD et MÉTIVIER [11]. La première partie du programme d'ALAZARD, BURQ et ZUILY [3, 7, 9] fut consacrée à la paralinéarisation du système complet, et à son étude pour résoudre le problème de CAUCHY par des méthodes d'énergie. La résolution du problème de CAUCHY pour des équations non linéaires s'effectue classiquement par la convergence d'une certaine suite de fonctions vers la solution. La structure de l'équation permet généralement, par une simple intégration par parties, d'obtenir les bornes permettant cette convergence (on les appelle estimations d'énergie a priori). Pour des équations complètement non linéaires, comme le sont celles des vagues, la structure de l'équation ne permet cependant ni d'identifier la suite, ni de prouver les estimations. C'est pourquoi une transformation de ces équations est nécessaire ; classiquement on dérive l'équation en espace, ce qui dans les méthodes employées précédemment avait le défaut de demander beaucoup de régularité de la donnée initiale. La paralinéarisation

remplace cette dérivation, tout en étant essentiellement optimale en matière de régularité. Les papiers précédemment cités traitaient les cas avec, puis sans tension de surface, prouvant l'existence locale dans les espaces de SOBOLEV correspondant à un champ de vitesse spatialement Lipschitz, puis le cas sans tension de surface dans les espaces de SOBOLEV uniformément locaux, ce qui permet de s'affranchir de la décroissance vers zéro des solutions à l'infini, associée aux espaces de SOBOLEV classiques.

Critère d'explosion

Le premier travail accompli dans cette thèse était une extension du résultat d'existence locale pour les vagues de gravité [7]. En conséquence des preuves d'existence locale par méthodes d'énergie, on peut généralement obtenir un critère d'explosion. Il s'agit de dire que si la solution dont on a prouvé l'existence cesse d'exister à un temps T , une certaine norme de cette solution doit nécessairement tendre vers l'infini lorsqu'on s'approche de ce temps d'explosion. L'intérêt d'une telle proposition réside dans sa contraposée : si on contrôle la norme en question au temps T , la solution se doit d'exister plus longtemps. Il s'agissait ici de prendre appui sur la formulation paradifférentielle pour prouver un tel critère. WANG et ZHANG [102] avaient déjà proposé un résultat dans cette direction ; j'ai prouvé une amélioration de leur résultat.

Rappelons pour commencer le résultat d'ALAZARD, BURQ et ZUILY ; nous noterons V la partie horizontale de la vitesse à la surface libre, et B sa partie verticale ; c'est-à-dire

$$B := (\partial_y \phi)|_{y=\eta} = \frac{\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi}{1 + |\nabla_x \eta|^2}, \quad V = (\nabla_x \phi)|_{y=\eta} = \nabla_x \psi - B \nabla_x \eta.$$

On notera $a = -(\nabla_y p)|_{y=\eta}$ le coefficient de TAYLOR, qui ne dépend lui aussi que de η et ψ .

Théorème (ALAZARD, BURQ et ZUILY ; théorème 2.1 de [7]).

Soient $d \geq 1$, $s > 1 + d/2$ et des fonctions (η_0, ψ_0) telles que

1. $\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $\psi_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $V_0 \in H^s(\mathbf{R}^d)$, $B_0 \in H^s(\mathbf{R}^d)$,
2. il existe $h > 0$ tel que la distance entre le graphe de η_0 et le fond \mathcal{B} soit plus grande que h à $t = 0$,
3. il existe une constante a_0 telle que, pour $x \in \mathbf{R}^d$, $a(0, x) \geq a_0$.

Alors il existe $T > 0$ tel que le problème de CAUCHY pour (ZCS) de donnée initiale (η_0, ψ_0) ait une unique solution (η, ψ) dans $C^0\left([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d)\right)$, qui est telle que

1. $(V, B) \in C^0\left([0, T]; H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d)\right)$,
2. la distance entre le graphe de $\eta(t)$, pour $0 \leq t \leq T$, et le fond \mathcal{B} , est plus grande que $h/2$,
3. pour $0 \leq t \leq T$ et $x \in \mathbf{R}^d$, $a(t, x) \geq a_0/2$.

Mon critère d'explosion est le suivant :

Théorème (P.).

Soient $d \geq 1$, $s > 1 + d/2$, $s > s_0 > 1/2 + d/2$ et $s_0 - 1/2 - d/2 > \varepsilon > 0$, et soient (η_0, ψ_0) des fonctions de même nature qu'au précédent théorème. Si T est le temps maximal d'existence de la solution associée, soit $T = +\infty$, soit l'une des quantités suivantes est infinie :

- $\sup_{0 \leq t < T} \frac{1}{h(t)}$,
- $\sup_{0 \leq t < T} \frac{1}{a_0(t)}$,
- $\sup_{0 \leq t < T} \|(\eta, \psi, V, B)(t)\|_{H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d)}$,

-
- $\sup_{0 \leq t < T} \|a(t)\|_{W^{\varepsilon, \infty}(\mathbf{R}^d)}$,
 - $\int_0^T \|(\partial_t a + V \cdot \nabla a)(t)\|_{L^\infty(\mathbf{R}^d)} dt$,
 - $\int_0^T \|a(t)\|_{W^{\frac{1}{2}, \infty}(\mathbf{R}^d)} dt$,
 - $\int_0^T \|\nabla \eta(t)\|_{W^{\frac{1}{2}, \infty}(\mathbf{R}^d)} dt$,
 - $\int_0^T \|(V, B)(t)\|_{W^{1+\varepsilon, \infty}(\mathbf{R}^d)} dt$.

Ici $h(t)$ est la distance entre le fond \mathcal{B} et le graphe de la fonction η au temps t , et $a_0(t)$ est l'infimum de $a(t, x)$.

On trouvera au chapitre 1 d'autres critères similaires, dont un n'utilisant que des normes hölderiennes, ainsi que leur preuve.

Inégalités de STRICHARTZ

Une des propriétés les plus remarquables des vagues est leur dispersion. Il s'agit ici de cette propriété physique bien connue, qui veut qu'une perturbation de la surface de l'eau, causée par exemple par la chute d'un corps ou par le vent, et pouvant être de forme initiale très irrégulière, se propage ensuite loin de sa source en des ondes très régulières. Ce phénomène s'explique aisément par l'étude du linéarisé des équations, dont on rappelle qu'il est une bonne approximation pour les vagues de faible amplitude : on y voit que des solutions de spectre localisé à des valeurs distinctes voyageront à des vitesses de groupe très différentes ; en conséquence de quoi des données initiales très irrégulières se transforment rapidement en une superposition d'ondes planes de fréquences très étalées, au fur et à mesure que les composantes de son spectre se dispersent. Le lien entre le nombre d'onde k et la fréquence ω d'une telle solution suit la relation de dispersion

$$\omega^2 = \kappa |k|^3 + g |k|,$$

dans le cas d'une profondeur indéfinie. On y observe immédiatement une différence fondamentale entre les vagues de gravité, pour lesquelles les hautes fréquences voyagent moins vite que les basses, et les vagues de capillarité, pour lesquelles l'inverse est vrai.

Cette relation, observée par CAUCHY pour le linéarisé autour de l'équilibre, se retrouve en réalité en linéarisant autour d'une solution arbitraire. On s'attend donc à ce que quelque chose de ces propriétés subsiste pour le système non linéaire. Cependant, comme les ondes planes ne peuvent plus être des solutions du problème complet, ce phénomène se manifeste nécessairement de manière plus subtile. Les vagues ne sont pas les seuls phénomènes physiques ayant ces propriétés ; la classe des équations non linéaires dispersives est large, et ses représentants les plus étudiés sont l'équation des ondes sonores et l'équation de Schrödinger non linéaire, qui apparaît dans certains phénomènes quantiques. Les manifestations non linéaires de la dispersion sont souvent divisées entre celles de temps court, et celle de temps long. La première catégorie, qui est celle qui nous intéresse ici, suit l'argument heuristique suivant : puisque le spectre des solutions semble avoir tendance à s'étaler, des solutions initialement très peu régulières, c'est-à-dire de spectre très concentrés, devraient se régulariser instantanément. On peut en pratique mesurer cet effet régularisant, par des inégalités dites de STRICHARTZ. Il semble naturel que cette régularisation instantanée permette de résoudre le problème de CAUCHY à basse régularité ; et on peut en effet combiner ces inégalités avec les méthodes d'énergie classique pour améliorer le seuil de régularité. Leur preuve repose usuellement sur la construction d'une solution approchée, ou paramétrice, sous la forme d'une intégrale oscillante. Ces intégrales peuvent être vues comme une généralisation des solutions planes précédemment évoquées, superposées pour approcher des solutions arbitraires du problème non linéaire. On peut alors très aisément dériver les estimations de STRICHARTZ

pour ces intégrales, par la méthode de la phase stationnaire, combinée avec un argument dû à STEIN, l'argument TT*.

Les modèles classiques d'équations dispersives, l'équation des ondes et celle de Schrödinger, ont une non-linéarité assez faible : ces équations sont dites semi-linéaire. Ici, on peut en prendre comme définition que la partie principale de leur linéarisé ne dépend pas de la solution autour de laquelle on linéarise. Le résultat est que le chemin pris par les paquets d'ondes, c'est-à-dire par des solutions localisées en espace et en fréquence, est indépendant de la solution elle-même. La structure de l'intégrale oscillante est alors assez simple, et l'obtention des estimations de STRICHARTZ pour ce type d'équation est maintenant classique et très bien compris. On consultera par exemple le livre de TAO [96], et les références qu'il contient.

Les équations des vagues ont cependant une non-linéarité plus forte ; on les dit quasilinearaires. Pour ce type d'équation, le linéarisé dépend dans sa partie principale de la solution autour de laquelle on linéarise : le chemin des paquets d'ondes est bien plus compliqué. La paramétrice est alors plus complexe qu'une simple intégrale oscillante, et doit être rangée dans la classe des opérateurs de FOURIER intégraux. Ces paramétrices sont bien plus difficiles à exhiber à basse régularité, et les estimations de STRICHARTZ obtenues sont typiquement moins bonnes qu'à haute régularité. Le jeu, pour descendre aussi bas que possible pour le problème de CAUCHY, est alors de minimiser cette perte. Le modèle de ce type d'équations est celui des ondes quasilinearaires, qui apparaît notamment en relativité générale ; c'est pour celui-ci que la théorie a été initiée, par BAHOURI et CHEMIN [16] et par TATARU [97, 98, 99] à partir de travaux de SMITH [94] sur les équations linéaires à coefficients peu réguliers. Des méthodes plus fines, essentiellement optimale, furent mises en place par la suite par SMITH et TATARU [95], puis par KLAINERMAN, RODNIANSKI et SZEFTEL [63].

Au vu de ces remarques, il paraissait naturel de rechercher des inégalités de STRICHARTZ pour les équations des vagues, la formulation paradifférentielle permettant d'organiser l'équation d'une manière adaptée à la recherche d'une paramétrice. C'était la partie suivante du programme d'ALAZARD, BURQ et ZUILY [4, 8]. Dans ces articles, ils les prouvèrent tout d'abord pour les vagues de capillarité, mais en dimension 2 et à haute régularité seulement ; puis pour les vagues de gravité, en dimension arbitraire et avec l'application au problème de CAUCHY à basse régularité. Comme on l'a remarqué précédemment, les dispersions des cas de gravité et de capillarité sont très différentes ; ce qui justifie la séparation des cas.

Le deuxième travail de cette thèse, accompli en collaboration avec Quang Huy NGUYEN, visait à prouver les estimations de STRICHARTZ pour les vagues de capillarité, en dimension quelconque et à basse régularité, puis à les appliquer au problème de CAUCHY, c'est-à-dire à étendre le résultat de [4], parallèlement à [8].

Avant de citer nos résultats, je rajouterai que les applications de la dispersion à l'existence en temps arbitrairement long des EDP sont un sujet particulièrement riche, et que de nombreux résultats ont été obtenus pour les vagues, par WU [106, 107], GERMAIN, MASMOUDI et SHATAH [45, 46], puis par ALAZARD et DELORT [10] et IONESCU et PUSATERI [58, 59], et enfin par HUNTER, IFRIM, TATARU et WONG [53], HUNTER, IFRIM et TATARU [52], IFRIM et TATARU [55, 56], IONESCU et PUSATERI [59], et DENG, IONESCU, PAUSADER et PUSATERI [40].

Notre travail sur ce problème se décompose en trois étapes. Tout d'abord, nous améliorons la formulation paradifférentielle de [3] pour la rendre valable à basse régularité ; ceci nous permet de prouver des estimations d'énergie et un critère d'explosions. Nous prouvons ensuite les inégalités de STRICHARTZ pour ce paralinéarisé. Enfin nous combinons ces ingrédients, ainsi que des estimations de contraction, pour prouver l'existence locale à basse régularité.

Dans leur article sur le problème de CAUCHY pour les vagues de capillarité, ALAZARD, BURQ et ZUILY [3] montrent l'existence locale pour des données initiales $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)$, avec une régularité $s > 2 + \frac{d}{2}$.

Introduisons les quantités suivantes

$$\text{Norme SOBOLEV : } M_{\sigma,T} = \|(\eta, \psi)\|_{L^\infty([0,T]; H^{\sigma+\frac{1}{2}} \times H^\sigma)}, \quad M_{\sigma,0} = \|(\eta_0, \psi_0)\|_{H^{\sigma+\frac{1}{2}} \times H^\sigma},$$

$$\text{Norme STRICHARTZ : } Z_{\sigma,T} = \|(\eta, \psi)\|_{L^p([0,T]; C^{\sigma+\frac{1}{2}} \times C^\sigma)}.$$

Alors nos estimations d'énergie sont décrites par le théorème suivant.

Théorème (NGUYEN, P.).

Soient $d \geq 1$, $h > 0$, $r > 2$, $p = 1$ et $s > \frac{3}{2} + \frac{d}{2}$. Alors il existe une fonction croissante $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$, dépendante uniquement de (d, s, r, h) , telle que pour tout $T \in (0, 1]$ et toute solution (η, ψ) de (ZCS) sur $[0, T]$ telle que

$$\begin{aligned} (\eta, \psi) &\in L^\infty\left([0, T]; H^{s+\frac{1}{2}} \times H^s\right), \\ (\eta, \psi) &\in L^1\left([0, T]; C^{r+\frac{1}{2}} \times C^r\right), \\ \inf_{t \in [0, T]} \text{dist}(\eta(t), \mathcal{B}) &> h, \end{aligned}$$

nous ayons

$$M_{s,T} \leq \mathcal{F}(M_{s,0} + T\mathcal{F}(M_{s,T}) + Z_{r,T}).$$

On remarquera que ces estimations ne sont pas fermées, car elles impliquent la norme de type STRICHARTZ. À plus haute régularité, $s > 2 + \frac{d}{2}$, on pourrait contrôler cette norme par la norme SOBOLEV, en utilisant les inégalités de SOBOLEV pour la norme spatiale et la borne triviale sur la norme de Lebesgue en temps. C'est là l'esprit de l'article d'ALAZARD, BURQ et ZUILY [3]. Pour résoudre le problème de CAUCHY à plus basse régularité cependant, il est nécessaire de compléter les inégalités d'énergie par une borne sur la norme STRICHARTZ du même type. C'est l'objet du théorème suivant.

Théorème (NGUYEN, P.). Soient $d \geq 1$, $h > 0$ et $(s, r) \in \mathbf{R}^2$ tels que

$$s > 2 + \frac{d}{2} - \mu, \quad 2 < r < s - \frac{d}{2} + \mu,$$

avec μ et p vérifiant

$$\mu \in \left(0, \frac{3}{20}\right), \quad p = 4 \quad \text{si } d = 1; \quad \mu \in \left(0, \frac{3}{10}\right), \quad p = 2 \quad \text{si } d \geq 2.$$

Alors il existe une fonction croissante $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ telle que pour $T \in (0, 1]$, pour (η, ψ) une solution de (ZCS) sur $[0, T]$ telle que $\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma) > h$, on ait

$$Z_r(T) \leq \mathcal{F}(T\mathcal{F}(M_s(T) + Z_r(T))).$$

Ce théorème appelle quelques commentaires. Tout d'abord, il s'agit bien d'un effet régularisant, car la régularité spatiale est un peu meilleure que ce que donneraient les injections de SOBOLEV, ce qui est mesuré par la quantité μ . En revanche, cet effet régularisant n'est pas vrai pour tout temps t , mais uniquement en une version moyennée, mesurée par la norme L^p .

Ensuite, le même type d'estimations est valable pour le problème linéaire, avec cependant un meilleur gain de régularité

$$\begin{cases} \mu_{opt} = \frac{3}{8}, \quad p = 4 & \text{si } d = 1, \\ \mu_{opt} = \frac{3}{4} - \varepsilon, \quad p = 2 & \text{si } d \geq 2. \end{cases}$$

Cette perte par rapport au cas linéaire a deux raisons. La première, assez fondamentale, est que les hautes fréquences se déplacent plus vite que les basses. Deux paquets d'ondes différant uniquement par l'amplitude de leur fréquence suivent un même trajet, mais celui de fréquence plus élevé va plus vite. Leur chemin, pour le problème non linéaire, peut devenir singulier à un certain point dépendant de la solution, et nous ne serons plus capable de les suivre passé ce point. Pour une fréquence arbitrairement grande, le paquet s'y retrouvera en temps arbitrairement court, et nous ne pourrions pas construire notre paramétrice. La solution, due à LEBEAU [73] puis à BURQ, GÉRARD et TZVETKOV [25], est de travailler sur une échelle de temps dépendant de la fréquence. Comme on le verra dans la preuve, cela entraîne une perte dans les estimations de STRICHARTZ; cet artifice est inutile pour le problème linéarisé car le chemin ne dépend pas de la solution. Comme ces inégalités de STRICHARTZ sont un phénomène concernant les hautes fréquences, cette difficulté est spécifique au cas des vagues de capillarité, et n'apparaît pas dans l'étude des vagues de gravité. La deuxième partie de la perte est due à la méthode utilisée; elle est purement technique. De meilleures méthodes, comme celle de SMITH et TATARU [95], permettraient sans doute de l'éviter.

La stratégie de preuve peut être résumée en quelques grandes étapes. Tout d'abord, l'équation paralinéarisée peut être vue comme une équation d'évolution pseudodifférentielle, de symbole dans la classe $S_{1,1}^{\frac{3}{2}}$ d'HÖRMANDER, avec un reste satisfaisant déjà les estimations attendues. Quoique ce symbole soit lisse, son comportement dépend de la régularité de la solution. Ce comportement se trouve être trop singulier pour mener la stratégie jusqu'au bout; la solution est ici de régulariser les coefficients de cet opérateur. On obtient ainsi une équation satisfaisant de meilleures estimations, au prix d'un reste qui en satisfait de pires. L'équilibre entre ces deux facteurs mènera à une perte de régularité dans les inégalités finales, perte dont la valeur est assez arbitraire.

En décomposant la solution en bandes de fréquences, on peut voir le problème comme une famille d'équations pseudodifférentielles, que l'on indexera par h (h petit correspondant aux hautes fréquences). Il s'agit alors de construire une paramétrice à haute fréquence (c'est-à-dire à petit h). Avant que cela soit possible, il faut appliquer plusieurs réductions à l'équation. Tout d'abord, la partie sous-principale du symbole, d'ordre $1/2$, peut être rejetée dans le reste. Ensuite, il reste à l'ordre 1 un transport par le champ de vitesse; les oscillations de ce terme pourrait parasiter le terme principal, d'ordre $3/2$. Heureusement on peut le supprimer, en suivant les caractéristiques de ce champ de vitesse aussi loin que sa régularité le permet, ce qui n'apporte aucune limitation aux inégalités finales.

L'étape suivante, spécifique aux vagues de capillarité, consiste en une renormalisation du temps à une taille dépendant de h , pour que la vitesse de déplacement des solutions soit bornée quand h approche 0. On s'est alors ramené à une équation d'évolution semiclassique pour laquelle on peut facilement construire une paramétrice. Celle-ci est une intégrale oscillante. Sa phase est solution d'une équation de HAMILTON-JACOBI associée au symbole de l'équation; la renormalisation du temps permet de la trouver sur un temps semiclassique uniforme (donc le temps d'existence réel tend vers 0 à haute fréquence). La régularisation des coefficients lui permet d'avoir suffisamment de régularité pour être utilisable. L'amplitude de l'intégrale oscillante est la solution d'une simple équation de transport.

On peut alors montrer les inégalités de STRICHARTZ pour cette paramétrice, par la méthode TT^* de STEIN, qui repose elle-même sur des estimations de noyau obtenue par la méthode de la phase stationnaire. Ces estimations passent alors à la solution originale, puisque l'intégrale oscillante est une solution approchée. Elles ne sont cependant valables, à fréquence h , que sur un petit intervalle de temps de taille dépendant de h . On peut montrer les estimations sur un temps d'ordre 1 en recollant un grand nombre de ces estimations, ce qui amène une puissance de h dans les inégalités, et donc une perte de dérivée lorsqu'on prouvera les inégalités pour l'équation

originale.

Ces estimations jouent un rôle central dans notre théorème d'existence locale à basse régularité, dont voici l'énoncé.

Théorème (NGUYEN, P.).

Soient $d \geq 1$ et trois réels r, s, μ tels que

$$2 < r < s - \frac{d}{2} + \mu, \quad \mu = \begin{cases} \frac{3}{20} & \text{si } d = 1, \\ \frac{3}{10} & \text{si } d \geq 2. \end{cases}$$

Soit $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}} \times H^s$ tel que $\text{dist}(\eta_0, \Gamma) > h > 0$. Alors il existe un temps $T > 0$ tel que le problème de CAUCHY pour (ZCS) ait une unique solution

$$(\eta, \psi) \in L^\infty \left([0, T]; H^{s+\frac{1}{2}} \times H^s \right) \cap L^p \left([0, T]; C^{r+\frac{1}{2}} \times C^r \right)$$

où $p = 4$ si $d = 1$ et $p = 2$ si $d \geq 2$. En outre,

$$(\eta, \psi) \in C^0 \left([0, T]; H^{s'+\frac{1}{2}} \times H^{s'} \right), \quad \forall s' < s$$

et

$$\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma) > h/2.$$

La preuve de ces résultats est aux chapitres 2 et 3.

Je présente ensuite une application de ces résultats au problème de CAUCHY dans un canal. Il s'agit ici d'étudier, en deux ou trois dimensions, une situation du type de la figure 2, mais où le fond ressort à la verticale, formant des murs droits. ALAZARD, BURQ et ZUILY [5] ont pu prouver que la condition de TAYLOR, dans ce cas, imposait un contact à angle droit avec la surface de l'eau. Utilisant une procédure due à BOUSSINESQ [24] consistant en une symétrisation suivie d'une périodisation de la géométrie, ils ont su ramener ce cas à celui du système de ZAKHAROV-CRAIG-SULEM sur le Tore, pour lequel la théorie de CAUCHY par méthodes d'énergie fonctionne identiquement à celle sur l'espace Euclidien. Le problème est que cette procédure limite la régularité des fonctions, même si on commence par des fonctions régulières. Les méthodes d'énergie sont cependant suffisantes dans le cas des vagues de gravité, comme prouvé dans [5].

Dans le cas des vagues de capillarité en trois dimensions cependant, la régularité est trop basse, et seules les inégalités de STRICHARTZ permettent de prouver l'existence locale. Il s'agit donc de prouver ces inégalités dans le cas du Tore, ce qui repose simplement sur une observation de BURQ, GÉRARD et TZVETKOV [25] : grâce au choix d'une échelle temporelle dépendant de la fréquence, chaque bande de fréquence se déplace à la même vitesse. On peut ainsi travailler dans une carte du Tore pour laquelle la paramétrice déjà construite est suffisante. On trouvera ce théorème d'existence locale et sa démonstration au chapitre 4.

Fond émergent

Quoique ces derniers résultats soient déjà dans le cadre d'un fond réémergent, ils utilisent une astuce qui ne se généralise pas à des géométries plus complexes. L'étude de celles-ci constitue la dernière partie de ce mémoire. J'y prouve des inégalités d'énergie a priori, qui sont la première étape vers la résolution du problème de CAUCHY. Je ne parle ici que des vagues de gravité, la modélisation des vagues de capillarité étant moins claire.

Il y a pour cela plusieurs obstacles à franchir. Le premier est la complexité de la géométrie, à cause de laquelle les systèmes de coordonnées évoqués plus haut deviennent d'usage peu naturel,

en introduisant des problèmes artificiels. Par exemple, étendre le système de ZAKHAROV-CRAIG-SULEM obligerait à considérer la surface comme un graphe au-dessus d'un domaine lui aussi mobile, perdant l'avantage du domaine fixe qui nous avait fait choisir ce modèle. Pour contrer cette difficulté, nous travaillons sans coordonnées, selon la méthode géométrique de SHATAH et ZENG [90]. L'adoption de ce système permet en outre de ne pas se restreindre à des surfaces formant un graphe, et de traiter le cas d'un fluide de tourbillon non nul très naturellement.

Il faut ensuite évoquer l'étude du problème elliptique dans le domaine singulier Ω_t . Ce problème est présent initialement dans l'équation satisfaite par la pression p , et apparaît ensuite de manière prééminente dans l'analyse, afin de réduire l'essentiel de la dynamique à la surface libre. La présence des coins en deux dimensions, et de l'arrête en dimension plus grande que trois, rend le problème singulier. Il n'est pas difficile de montrer l'existence d'une solution peu régulière par des méthodes variationnelles classiques, ni de prouver que cette solution est régulière loin de la ligne triple \mathcal{L}_t ; cependant près de celle-ci, cette régularité dégénère. La théorie en est heureusement assez développée, du moins dans le cas d'un domaine lisse en-dehors de la singularité. On consultera le livre de DAUGE [39] pour un développement de la théorie; les contributions majeures étant dues à KONDRAT'EV [64, 65, 66] et à MAZ'JA et PLAMENEVSKIĬ [79, 78, 77]. On trouvera une bibliographie plus complète dans le livre de GRISVARD [49].

Bien que la régularité elliptique dégénère près d'un point singulier comme un coin, cette perte de régularité est quantifiée : en deux dimensions, il apparaît une série discrète de singularités, chacune limitant la régularité en terme d'espaces de SOBOLEV à un exposant de la forme $(k+1/2)\frac{\pi}{\omega}$ (en choisissant des conditions au bord mixtes de type DIRICHLET d'un côté du coin, et NEUMANN de l'autre), où ω est la valeur de l'angle. Ainsi la régularité elliptique fonctionne comme dans un domaine lisse, tant qu'on se limite à des espaces de SOBOLEV de régularité plus petite que celle de la première singularité (ici $\frac{\pi}{2\omega}$). Le cas des dimensions plus grandes, lorsqu'on généralise le coin en une arrête d'angle variable comme dans notre problème, est un peu plus complexe; les singularités ne sont plus aussi simples à décrire. Néanmoins, la régularité elliptique en dessous de $\frac{\pi}{2\bar{\omega}}$ reste vraie, où $\bar{\omega}$ est la plus grande valeur de l'angle.

Pour éviter des complications, c'est cette régularité en-dessous de la première singularité qui nous intéresse. Ici, les difficultés par rapport à la théorie classique que je viens d'évoquer sont que l'on travaille avec des domaines dont la frontière est de régularité limitée, mesurée dans les espaces de SOBOLEV, et que l'on voudra obtenir des constantes uniformes pour une faible variation du domaine, afin que celles-ci ne changent pas au cours du temps. Ainsi, pour éviter les singularités, on travaille à une régularité basse dépendant du plus grand angle possible. Comme on a d'un autre côté besoin d'un champ de vitesse de régularité Lipschitz pour résoudre le problème de CAUCHY par des méthodes d'énergie, on doit travailler à régularité intermédiaire. Ce compromis n'est possible que pour des angles plus petits que $\pi/(n+1)$.

La dernière difficulté dans cette étude apparaît dans la quasilinearisation de l'équation. Il s'agit de cette étape, évoquée plus haut, révélant dans l'équation une structure adaptée aux estimations d'énergie et à la construction d'une suite d'approximations de la solution. Ce rôle, dans le cas latéralement infini, est généralement rempli soit par une dérivation spatiale des équations, soit comme au-dessus par une paralinéarisation. Ici, à défaut d'une théorie paradifférentielle satisfaisante sur les domaines à bords, cette dernière méthode semble difficile à mettre en œuvre. Pour ce qui est de la dérivation spatiale, il a fallu que je me rende compte de son inefficacité dans ce problème : j'interprète ce défaut comme dû à l'absence d'une symétrie de translation dans cette géométrie. Comme le problème reste cependant invariant par translation temporelle, cette remarque pousse à essayer de dériver l'équation en temps. Au vu des équations d'EULER, cela revient à chercher une équation sur le gradient spatial de la pression, ou de manière équivalente sur le coefficient de TAYLOR $a = -\nabla_N p|_{\mathcal{S}}$. (Attention au changement de définition de celui-ci par rapport à la partie précédente; au vu de la nullité de p sur la surface, ces deux quantités

sont équivalentes.) Grâce à cette approche, on peut effectivement prouver les estimations a priori résumées dans le théorème suivant.

Théorème (P.).

Soient \mathcal{S}_t une famille C^2 en temps d'hypersurfaces H^s , et $v \in C^2(H^s(\Omega_t))$, solutions des équations des vagues de gravité dans le domaine défini par le container \mathcal{O} .

Ici $s > 1 + \frac{n}{2}$, et $s < \frac{1}{2} + \frac{\pi}{2\bar{\omega}}$, où $\bar{\omega} > 0$ est un nombre tel que pour tout temps t , pour tout $x \in \mathcal{L}_t$, $\omega(x) \leq \bar{\omega}$. Remarquons que cela implique $\omega < \frac{\pi}{n+1}$.

Supposons aussi l'existence d'un nombre $a_0 > 0$ tel que le coefficient de TAYLOR $a := -\nabla_{Np} \geq a_0 > 0$ pour tout t , et d'un nombre $\underline{\omega} > 0$ tel que $\omega \geq \underline{\omega}$ pour tout t .

Alors, pour une certaine énergie $E(t) = E(\Omega_t, v(t, \cdot))$ à définir, contrôlant \mathcal{S}_t dans H^s et $v \in H^s(\Omega_t)$, il existe un temps $T > 0$, dépendant seulement des normes des données initiales, tel que $\forall t \in [0, T]$,

$$E(t) \leq E(0) + \int_0^t \mathcal{F}(E(t')) dt',$$

où \mathcal{F} est une fonction croissante dépendant seulement de ω , s , a_0 , et d'un voisinage des données initiales dans la topologie moins régulière $H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}}(\Omega_t)$.

On trouvera la preuve au chapitre 5. Elle peut être séparée en deux parties. La première partie est entièrement consacrée au développement d'outils analytiques pour donner un sens à la régularité d'une surface, à la topologie sur l'ensemble des surfaces, et à l'étude des opérateurs elliptiques rencontrés. La partie principale de cette étude est celle de la régularité elliptique au niveau de l'arête (ou du coin en deux dimensions). Elle se fait localement près d'un point de l'arête, en étudiant le problème modèle du laplacien à coefficients constants dans un secteur $\mathbf{R}^{n-2} \times \Gamma$, où Γ est le secteur angulaire de même ouverture que celle au point considéré. On supprime alors les éventuelles variables non bornés grâce à la transformée de FOURIER, ce qui nous ramène à étudier le laplacien dans le domaine Γ . On utilise alors la transformée de MELLIN, qui en polaire transforme la dérivation $r\partial_r$ en multiplication par $\lambda \in \mathbf{C}$, nous ramenant ainsi à inverser une famille holomorphe sur $[0, \omega]$ de même partie principale. Les valeurs propres de cette partie principale sont des obstructions à cette inversion; l'inverse est cependant méromorphe, et on peut retrouver la solution initiale comme somme d'une fonction régulière et des résidus de cette famille, qui sont les singularités. La difficulté est ici de développer cette analyse pour un domaine susceptible de varier en topologie Lipschitz.

La deuxième partie de la preuve est plus spécifique au problème d'évolution. Il s'agit tout d'abord de trouver l'équation, d'ordre deux en temps, satisfaite par le coefficient de TAYLOR, ce qui demande quelques calculs. On peut alors s'apercevoir que celle-ci est, aux termes d'ordre nul près, l'équation linéarisée autour de la solution. On prend alors comme énergie celle associée au problème linéarisé, à laquelle on rajoute la norme appropriée du tourbillon. Cette dernière satisfait de bonnes estimations car le tourbillon suit essentiellement une équation de transport. Il s'agit pour finir de montrer que le coefficient de TAYLOR, sa dérivée en temps, et le tourbillon contrôlent la solution complète. Le premier contrôle la régularité de la surface, puisqu'il est le gradient transverse de la pression à son lieu d'annulation, la surface libre; les deux autres variables contrôlent la vitesse par un lemme divergence-rotationnel dans le domaine fluide, qui est une conséquence de la régularité elliptique pour le laplacien.

Il me reste à mentionner que plusieurs travaux ont déjà été accomplis dans cette direction : KINSEY et WU [62], puis WU [108] ont cherché à étendre la stratégie de périodisation utilisée pour le cas du canal au cas d'un angle non-droit, ce qui les amène à considérer des vagues présentant des crêtes à angles aigus; MING et WANG [81] ont étudié le problème elliptique dans le cas de la dimension deux, et ont donné une description complète des singularités. Enfin GUO et TICE

[50] et ZHENG et TICE [112] ont étudié la situation analogue pour le problème de Stokes en deux dimensions.

Première partie

Critère d'explosion

Critère d'explosion

Cet Article est disponible sur ArXiv ; voir [84].

1.1 Introduction

In this paper, we derive a blow-up criterion for the water-waves system, without surface tension and with arbitrary bottom. The water-waves problem is the study of the motion under the influence of gravity of a homogeneous, inviscid fluid, typically water, inside a laterally infinite container, and separated from the atmosphere by a free interface.

We will assume the presence of a constant gravity field acting along the e_y axis, distinguishing it from the horizontal plane. This horizontal plane will be of dimension $d \geq 1$, with in applications $d = 1$ or 2 . Positions will be expressed in coordinates $(x, y) \in \mathbf{R}^d \times \mathbf{R}$. We write $\nabla = \nabla_x = (\partial_{x_1}, \dots, \partial_{x_d})$ and $\nabla_{x,y} = (\nabla_x, \partial_y)$.

At each time $t \in \mathbf{R}^+$, the fluid will occupy a domain $\Omega(t)$. We suppose that the free surface, which will be denoted $\Sigma(t)$ is the graph of a continuous function $y = \eta(t, x)$ representing the variation of the water surface from its rest level. In order to account for a wide variety of bottoms, we will consider a simply connected open subset \mathcal{O} of \mathbf{R}^{d+1} , such that

$$\Omega(t) = \left\{ (x, y) \in \mathbf{R}^d \times \mathbf{R}; (x, y) \in \mathcal{O}, y < \eta(t, x) \right\}.$$

We suppose that there exists $h > 0$ such that, for all times, the domain $\Omega(t)$ contains a horizontal strip of width h ,

$$(1.1.1) \quad \Omega_h(t) := \left\{ (x, y) \in \mathbf{R}^d \times \mathbf{R}; \eta(t, x) - h < y < \eta(t, x) \right\} \subset \Omega(t).$$

This means that the bottom, denoted by Γ , is nowhere emerging (which precludes islands and beaches).

The velocity $v(t, x, y) \in \mathbf{R}^{d+1}$ of the fluid occupying $\Omega(t)$ follows the incompressible Euler equations

$$\begin{cases} \partial_t v + (v \cdot \nabla_{x,y}) v + \nabla_{x,y} x, y P = -g e_y, \\ \operatorname{div}_{x,y} v = 0, \end{cases}$$

where g is the acceleration of gravity, supposed constant and positive, and where $P(t, x, y) \in \mathbf{R}$ is the pressure of the fluid. It is customary in oceanography to impose in addition for the fluid to be curl-free, so that $\operatorname{curl}_{x,y} v = 0$ in $\Omega(t)$.

In addition, we need to impose boundary conditions on $\Sigma(t)$ and Γ . First there are the kinematic conditions that the fluid does not cross or leaves those boundaries, so that

$$\begin{aligned} v \cdot n &= 0 \quad \text{on } \Gamma, \\ \partial_t \eta &= \sqrt{1 + |\nabla_{x,y} x, y l a \eta|^2} v \cdot \nu \quad \text{on } \Sigma, \end{aligned}$$

where n and $\nu(t)$ are the exterior unit normals respectively to Γ and $\Sigma(t)$. At last there is a dynamic boundary condition on the pressure. We suppose that there is no surface tension at the surface, which implies that there is no pressure jump between the fluid and the atmosphere. We assume this atmospheric pressure to be constant, and we can change the definition of P by an additive constant to take $P_{\text{atm}} = 0$. Then

$$P|_{y=\eta} = 0.$$

Remark. — By imposing for the surface $\Sigma(t)$ to be a graph, we implicitly assumed that our solutions will blow up when this ceases to be the case. It has been proved by Castro, Córdoba, Fefferman, Gancedo and Gómez-Serrano [26] (see also Coutand-Shkoller [33]) that some cases of a non-graph smooth surface can evolve to a self-intersecting surface, the so-called splash singularities, where this physical model does not make sense anymore. This shows that any study of blow-up without the graph hypothesis should involve some geometric quantities.

- The curl-free hypothesis is a good approximation for most deep-ocean applications, however it ceases to apply near a coast or when we take the Coriolis effect into account. See Castro and Lannes ([27]) for a formulation and some results with vorticity.
- Since Γ is not always smooth, its normal may not be defined. We will later give a variational meaning to this condition, coinciding with the strong sense when the normal exists.

For more on those hypotheses and on this model see the book by Lannes [70].

Now from the simple connectedness of \mathcal{O} , and therefore of $\Omega(t)$, and because $\text{div}_{x,y} v = 0$ and $\text{curl}_{x,y} v = 0$, we see that there exists a scalar function ϕ defined on the fluid domain such that

$$\begin{aligned} \nabla_{x,y} x, y \phi &= v \quad \text{in } \Omega, \\ \Delta_{x,y} \phi &= 0 \quad \text{in } \Omega. \end{aligned}$$

Now the Euler equation (1.1) and the boundary conditions (1.1) and (1.1) can be recast for this velocity potential, becoming —up to a harmless change of the definition of ϕ by a time-dependent constant— the Bernoulli formulation

$$\left\{ \begin{array}{l} \partial_t \phi + \frac{1}{2} |\nabla_{x,y} x, y \phi|^2 + gy = -P \quad \text{in } \Omega(t), \\ \partial_t \eta = \partial_y \phi - \nabla_{x,y} x, y l a \eta \cdot \nabla_{x,y} x, y l a \phi \quad \text{on } \Sigma(t), \\ \partial_n \phi = 0 \quad \text{on } \Gamma, \\ P = 0 \quad \text{on } \Sigma(t). \end{array} \right.$$

The Cauchy problem for this system has been widely studied, starting from the works of Nalimov ([83]), Shinbrot ([93]), Yosihara ([109]) and Craig ([34]). The first results for Sobolev spaces and without smallness assumptions are due to Wu ([104, 105]). A recent extension with rougher data, essentially Hölder with exponent 3/2 has been proposed by Alazard, Burq and Zuily in [3], with another extension using Strichartz estimates in [8]. More Recently, Kinsey and

Wu have obtained in [62] a priori estimates covering the case with angled crests. The next natural objective is to find a blow-up criterion for the system. Christodoulou and Lindblad ([31]) proved such a criterion involving geometric quantities for the case without bottom. They showed that the solutions can be extended as long as the curvature of the surface and the derivative of the velocity remain bounded. More recently, Wang and Zhang ([102]) used some of the methods of [3] to prove that as long as

$$\sup_{0 \leq t < T} \|\kappa(t)\|_{L^2 \cap L^p} + \int_0^T \|(\nabla_{x,y} x, ylaV, \nabla_{x,y} x, ylaB)\|_{W^{1,\infty}}^6 dt$$

is bounded, the solution can be extended after the time T . Here κ is the curvature of Σ , V and B are respectively the horizontal and vertical traces of the velocity v at Σ and $p > 2d$. We will prove three blow-up criterions which extend this result. The results proved in this paper involve less regular norms of the free surface and are valid for the case with rough bottom. More importantly, we will prove two results which involve only L^1 norms in time of the highest-order norms. Notice that one of the results below (see Theorem 1.1.4) is used in [8](see Section 5.4) to deduce an existence result from a priori Sobolev and Strichartz estimates. Since Strichartz estimates involve L^2 norms in time (in dimension $d \geq 2$), it is crucial to have a blow-up result which involves only L^p norms for $p \leq 2$. In this direction, we will obtain sharp results involving only L^1 norms (see Theorem 1.1.3 and Theorem 1.1.4). In the case of 2D water-waves ($d = 1$), Hunter, Ifrim and Tataru have obtained in [52] a blow-up criterion in holomorphic coordinates, corresponding to ours but sharpened to BMO norms instead of Hölder norms in space.

An important quantity appears in the analysis of the system (1.1), the so-called Rayleigh-Taylor coefficient

$$a := -\partial_y P|_{y=\eta}.$$

In order to solve the Cauchy problem, we need to make a positivity hypothesis on a . One of the important contributions of Wu's articles [104, 105] is that this condition is always true when the depth is infinite, which corresponds to the case $\Gamma = \emptyset$. Lannes then proved the same result for a small regular perturbation of a flat bottom ([70]).

Inspired by Craig [34] and Lannes [70], we will use the eulerian formulation of the equations in connection with elliptic estimates and microlocal or harmonic analysis. In particular, we use the Craig-Sulem-Zakharov formulation of the equations ([35, 111]). Notice that since the potential ϕ is harmonic, it is entirely determined by its value at the surface. We define

$$\psi(t, x) = \phi(t, x, \eta(x)).$$

The equation can then be recast in terms of η and ψ , which are functions defined on \mathbf{R}^d . In order to simplify the presentation, Craig and Sulem introduced the use of the Dirichlet-Neumann operator in [35]. This operator is defined as associating to a function defined on Σ the exterior normal of its harmonic extension to $\Omega(t)$. Here for convenience we re-normalize it to get

$$G(\eta)\psi = \sqrt{1 + |\nabla_{x,y} x, yla\eta|^2} \partial_n \phi|_{z=\eta}.$$

With this operator, we get a closed system of equations, known as the Craig-Sulem-Zakharov System

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta + \frac{1}{2} |\nabla_{x,y} x, yla\psi|^2 - \frac{1}{2} \frac{(\nabla_{x,y} x, yla\eta \cdot \nabla_{x,y} x, yla\psi + G(\eta)\psi)^2}{1 + |\nabla_{x,y} x, yla\eta|^2} = 0. \end{cases}$$

Remark. — Under this formulation, the system is Hamiltonian. This is what motivated the original idea of Zakharov. The Hamiltonian is

$$\frac{1}{2} \int_{\mathbf{R}^d} \psi G(\eta) \psi \, dx + \frac{1}{2} \int_{\mathbf{R}^d} g \eta^2 \, dx,$$

and is conserved by the evolution (see e.g. [71]).

— The formal equivalence of this system to the original one is clear and we refer to [6] for a rigorous proof.

This work is based upon the paper [7] by Alazard, Burq and Zuily. To recall their main result, we introduce the vertical and horizontal parts of the velocity at the surface,

$$B := (\partial_y \phi)|_{y=\eta}, \quad V = (\nabla_{x,y} x, y l a_x \phi)|_{y=\eta}.$$

Those quantities can be computed from knowing only η and ψ . Then

Theorem 1.1.1 (Theorem 2.1 of [7]). *Let $d \geq 1$, $s > 1 + d/2$ and consider (η_0, ψ_0) such that*

1. $\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $\psi_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $V_0 \in H^s(\mathbf{R}^d)$, $B_0 \in H^s(\mathbf{R}^d)$,
2. *there is $h > 0$ such that condition (1.1.1) holds for $t = 0$,*
3. *there is a positive constant c such that, for any $x \in \mathbf{R}^d$, $a_0(x) \geq c$.*

Then there exists $T > 0$ such that the Cauchy problem for (1.1) with initial data (η_0, ψ_0) has a unique solution (η, ψ) in $C^0([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d))$, such that

1. *we have $(V, B) \in C^0([0, T]; H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d))$,*
2. *the condition (1.1.1) holds for $0 \leq t \leq T$, with h replaced by $h/2$,*
3. *for any $0 \leq t \leq T$ and any $x \in \mathbf{R}^d$, $a(t, x) \geq c/2$.*

The proof of this theorem relies on paradifferential calculus to reduce the equations to a quasi-linear system, and then use classical energy methods for hyperbolic symmetrizable quasi-linear systems. Some notions about paradifferential calculus are recalled in appendix A. It has the advantage of yielding tame estimates of the various nonlinearities, meaning that those estimates are linear with respect to the higher order norm. This will enable us to derive new a priori energy estimates for the paradifferential reduction of the system, from which we will derive a blow-up criterion complementing Theorem 1.1.1.

Our main result will be derived in three different flavors, which we believe are all equally interesting. The first one controls the dynamic using only Hölder norms of the quantities.

Theorem 1.1.2. *Let $d \geq 1$, $s > 1 + d/2$, $\varepsilon > 0$ and consider (η_0, ψ_0) satisfying the assumptions of Theorem 1.1.1. If T is the maximum existence time of the solution given by this theorem, then either $T = +\infty$ or one of the following quantities is infinite*

- $\sup_{0 \leq t < T} \frac{1}{h(t)}$,
- $\sup_{0 \leq t < T} \frac{1}{c(t)}$,
- $\sup_{0 \leq t < T} \|\eta(t)\|_{W^{1+\varepsilon, \infty}(\mathbf{R}^d)}$,
- $\sup_{0 \leq t < T} \|(V, B)(t)\|_{W^{\varepsilon, \infty}(\mathbf{R}^d)}$,
- $\sup_{0 \leq t < T} \|a(t)\|_{W^{\varepsilon, \infty}(\mathbf{R}^d)}$,
- $\int_0^T \|(\partial_t a + V \cdot \nabla_{x,y} x, y l a a)(t)\|_{L^\infty(\mathbf{R}^d)} \, dt$,
- $\int_0^T \|a(t)\|_{W^{\frac{1}{2}, \infty}(\mathbf{R}^d)} \, dt$,

$$\begin{aligned} & - \int_0^T \|\nabla_{x,y} x, yla\eta(t)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)}^3 dt, \\ & - \int_0^T \|(V, B)(t)\|_{W^{1+\varepsilon,\infty}(\mathbf{R}^d)}^3 dt. \end{aligned}$$

Here $h(t)$ is the largest number h satisfying condition (1.1) at time t and $c(t)$ the largest number c such that $a(t, x) \geq c$ for all $x \in \mathbf{R}^d$.

Before introducing the second criterion, we observe that in the case where the domain is infinitely deep (that is $\Gamma = \emptyset$), the equation enjoys a scaling invariance. The critical space corresponds to the index $s = d/2 + 1/2$. We expect to find a better criterion by authorizing a control of a Sobolev norm of a fixed reference index s_0 close to the scaling. This corresponds to our second result

Theorem 1.1.3. *Let $d \geq 1$, $s > 1 + d/2$, $s > s_0 > 1/2 + d/2$ and $s_0 - 1/2 - d/2 > \varepsilon > 0$, and consider (η_0, ψ_0) satisfying the assumptions of Theorem 1.1.1. If T is the maximum existence time of the solution given by this theorem, then either $T = +\infty$ or one of the following quantities is infinite*

$$\begin{aligned} & - \sup_{0 \leq t < T} \frac{1}{h(t)}, \\ & - \sup_{0 \leq t < T} \frac{1}{c(t)}, \\ & - \sup_{0 \leq t < T} \|(\eta, \psi, V, B)(t)\|_{H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d)}, \\ & - \sup_{0 \leq t < T} \|a(t)\|_{W^{\varepsilon,\infty}(\mathbf{R}^d)}, \\ & - \int_0^T \|(\partial_t a + V \cdot \nabla_{x,y} x, ylaa)(t)\|_{L^\infty(\mathbf{R}^d)} dt, \\ & - \int_0^T \|a(t)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)} dt, \\ & - \int_0^T \|\nabla_{x,y} x, yla\eta(t)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)} dt, \\ & - \int_0^T \|(V, B)(t)\|_{W^{1+\varepsilon,\infty}(\mathbf{R}^d)} dt. \end{aligned}$$

Here $h(t)$ is the largest number h satisfying condition (1.1) at time t and $c(t)$ the largest number c such that $a(t, x) \geq c$ for all $x \in \mathbf{R}^d$.

Here the main improvement to the preceding theorem is that we only need to control the L^1 -norm in time of the higher order quantities, rather than L^3 norms. The proofs of those two theorems will be parallel, however one can not be deduced from the other.

The last criterion is a simplification of the preceding one, and is the most compact of the three. It trades a higher reference Sobolev index $s_0 > 3/4 + d/2$ against control of the Taylor coefficient.

Theorem 1.1.4. *Let $d \geq 1$, $s > 1 + d/2$, $s > s_0 > 3/4 + d/2$ and $1/4 > \varepsilon > 0$, and consider (η_0, ψ_0) satisfying the assumptions of Theorem 1.1.1. If T is the maximum existence time of the solution given by this theorem, then either $T = +\infty$ or one of the following quantities is infinite*

$$\begin{aligned} & - \sup_{0 \leq t < T} \frac{1}{h(t)}, \\ & - \sup_{0 \leq t < T} \frac{1}{c(t)}, \\ & - \sup_{0 \leq t < T} \|(\eta, \psi, V, B)(t)\|_{H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d)}, \\ & - \int_0^T \|\nabla_{x,y} x, yla\eta(t)\|_{W^{\frac{1}{2},\infty}(\mathbf{R}^d)} dt, \\ & - \int_0^T \|(V, B)(t)\|_{W^{1+\varepsilon,\infty}(\mathbf{R}^d)} dt. \end{aligned}$$

Here $h(t)$ is the largest number h satisfying condition (1.1) at time t and $c(t)$ the largest number c such that $a(t, x) \geq c$ for all $x \in \mathbf{R}^d$.

Let us explain why the index $3/4 + d/2$ enters into the analysis. As already mentioned, in the recent paper [8], Alazard, Burq and Zuily used Strichartz estimates to deduce existence

for data with regularity associated to $s = 11/12 + d/2$. The theoretical limit of this method is at $s > 3/4 + d/2$ for $d = 2$ (since Strichartz estimate gains only $1/4$ derivative), and even for such solutions we expect the quantities in this last theorem to be finite on the existence time interval. This would not be the case of the quantities $\sup_{0 \leq t < T} \|\eta(t)\|_{W^{2,\infty}(\mathbf{R}^d)}$, corresponding to the curvature of the surface, or $\int_0^T \|(V, B)(t)\|_{W^{1+\varepsilon,\infty}(\mathbf{R}^d)}^3 dt$, since solutions can be found for which those quantities would be infinite.

Section 1.2 will start with a rigorous definition of the harmonic extension ϕ and of the Dirichlet-Neumann, adapted to the case with rough bottom. It also contains in subsection 1.2.2 a maximum principle adapted to this framework, that we believe is of independent interest and ends with results on the elliptic regularity of this problem and their uses to control the Dirichlet-Neumann. In section 1.3 we will perform the reduction of the system to a symmetric quasilinear hyperbolic equation. This imposes to change the variables we work with; in section 1.4 we will construct a parametrix to control the new variables with the originals. Section 1.5 contains the a priori energy estimates of the new system, and section 1.6 completes the proofs of the theorems. Appendix A recalls some notions on paradifferential calculus, the main technical tool of this analysis.

1.2 Elliptic Regularity

Following the general strategy of [7], the first step of the proof is to estimate solutions of the Laplace equation near the free surface. The method is essentially the same, but we look for tame estimates whose constants depend on the Hölder norm of the surface rather than on its Sobolev norm. This requires some new techniques, and in particular we shall prove a maximum principle adapted to this setting. This analysis being valid at fixed time, we will drop the dependence in t for this whole section.

1.2.1 Variational solution

We have to give a suitable sense to quantities defined in Ω , from data defined only on the free surface. Here, we recall this construction.

Those quantities need to be, in a suitable sense, solutions of

$$\Delta_{x,y}v = 0, \quad v|_{\Sigma} = f, \quad \partial_n v|_{\Gamma} = 0.$$

This definition will come from variational theory.

Notation 1.2.1. Let \mathcal{D} be the space of functions $u \in C^\infty(\Omega)$ such that $\nabla_{x,y}u \in L^2(\Omega)$.

Let then \mathcal{D}_0 be the subspace of \mathcal{D} whose elements are equal to 0 near the top boundary Σ .

Proposition 1.2.2 (Proposition 2.2 of [7]). *There exists a positive weight $g \in L^\infty_{loc}(\Omega)$, equal to 1 near the top boundary of Ω , and a constant $C > 0$ such that for all $u \in \mathcal{D}_0$,*

$$\int_{\Omega} g(x,y)|u(x,y)|^2 dx dy \leq C \int_{\Omega} |\nabla_{x,y}u(x,y)|^2 dx dy.$$

Definition 1.2.3. Let $\mathcal{H}^{1,0}$ be the space of functions u on Ω such that there exists a sequence $(u_n)_{n \in \mathbf{N}}$, with $u_n \in \mathcal{D}_0$, satisfying

$$\nabla_{x,y}u_n \rightarrow \nabla_{x,y}u \text{ in } L^2(\Omega, dx dy), \quad u_n \rightarrow u \text{ in } L^2(\Omega, g(x,y) dx dy).$$

We see from Proposition 1.2.2 that $\mathcal{H}^{1,0}$ can be equipped with the norm

$$\|u\|_{\mathcal{H}^{1,0}} = \|\nabla_{x,y} u\|_{L^2(\Omega)}.$$

As seen in [3], it is a Hilbert space. By regularizing the function η , we can construct $\eta_* \in C_b^\infty(\mathbf{R}^d)$ such that $\eta - h/20 > \eta_*$ and

$$\left\{ (x, y) \in \mathbf{R}^d \times \mathbf{R}; \eta_*(x) < y < \eta(x) \right\} \subset \Omega.$$

Recall that \mathcal{O} denotes the fixed container in which the fluid is located.

Definition 1.2.4. We denote by $\mathcal{H}^1(\mathcal{O})$ the space of functions \tilde{u} on \mathcal{O} such that there exists a sequence $(u_n) \in C^\infty(\mathcal{O})$ such that

$$\nabla_{x,y} u_n \rightarrow \nabla_{x,y} \tilde{u} \text{ in } L^2(\mathcal{O}, dx dy), \quad u_n \rightarrow \tilde{u} \text{ in } L^2(\mathcal{O}, \hat{g}(x, y) dx dy),$$

where \hat{g} is the extension of g by 1 to \mathcal{O} .

Lemma 1.2.5. *Let w be measurable on Ω . Then $w \in \mathcal{H}^{1,0}(\Omega)$ if and only if the zero extension of w to \mathcal{O} is in $\mathcal{H}^1(\mathcal{O})$.*

Proof. We follow the proof for the classical Sobolev setting, found for example in [1]. It is routine to show that $\widetilde{\nabla_{x,y} w} = \nabla_{x,y} \tilde{w}$, from which the direct part is immediate.

For the indirect part, suppose $\tilde{w} \in \mathcal{H}^1(\mathcal{O})$. Now we can cover Ω with V_0 which does not intersect Σ and V_1 which does not intersect Γ . Then using a partition of unity, we can split w between w_0 supported in V_0 , which by definition is already in $\mathcal{H}^{1,0}(\Omega)$, and w_1 supported in V_1 . Then we consider $\tilde{w}_1(x, y + t)$, which is in $\mathcal{H}^{1,0}(\Omega)$ and converge to w_1 as t goes to 0^+ , since the translation in L^2 is continuous. This proves that w_2 , and then w is in $\mathcal{H}^{1,0}(\Omega)$. \square

Let $f \in H^{1/2}(\mathbf{R}^d)$. We define ψ an H^1 lifting of f in Ω . Let $\chi_0(z) \in C^\infty(\mathbf{R})$ be such that $\chi_0(z) = 1$ if $z \geq -1/2$ and $\chi_0(z) = 0$ if $z \leq -1$. Set

$$\psi_1(x, z) := \chi_0(z) e^{z|D_x|} f(x), \quad x \in \mathbf{R}^d, z \leq 0.$$

Then set

$$\psi(x, y) := \psi_1 \left(x, \frac{y - \eta(x)}{h} \right), \quad (x, y) \in \Omega,$$

which is well defined and vanishes near the bottom Γ .

>From the usual properties of the Poisson kernel, we have

$$\|\psi\|_{H^1(\Omega)} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \|f\|_{H^{1/2}(\mathbf{R}^d)},$$

and

$$\|\psi\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\mathbf{R}^d)}.$$

We can now use this framework to define $u \in \mathcal{H}^{1,0}$ as a variational solution of the problem

$$-\Delta_{x,y} u = \Delta_{x,y} \psi, \quad u|_\Sigma = 0, \quad \partial_n u|_\Gamma = 0.$$

We then define

$$v := u + \psi.$$

We see from lemma 3.5 of [7] that this is independent of the lifting function ψ vanishing near the bottom and we freely get the estimate

$$\int_{\Omega} |\nabla_{x,y} v|^2 dx dy \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \|f\|_{H^{1/2}(\mathbf{R}^d)}^2.$$

1.2.2 Maximum principle

In studying equation (1.2.1) we will need a weak maximum principle adapted to our variational formulation. Adapting the proof from [101], we get the following comparison principle.

Proposition 1.2.6. *If ϕ is a weakly differentiable function such that:*

1. $\phi^+ = \max(\phi, 0) \in \mathcal{H}^{1,0}$;
2. $\int_{\Omega} \nabla_{x,y} \phi \cdot \nabla_{x,y} \sigma dx dy \leq 0$ for all $\sigma \geq 0$ in $\mathcal{H}^{1,0}$;

then $\phi \leq 0$ in Ω .

Remark. Condition 1 is the adapted way to say that $\phi|_{\Sigma} \leq 0$ for the variational space $\mathcal{H}^{1,0}$.

Proof. Since $\phi^+ \geq 0$, and $\phi^+ \in \mathcal{H}^{1,0}$, we have from condition 2, taking $\sigma = \phi^+$

$$\|\phi^+\|_{\mathcal{H}^{1,0}}^2 = \int_{\Omega} \nabla_{x,y} \phi^+ \cdot \nabla_{x,y} \phi^+ dx dy \leq 0$$

so that $\phi^+ = 0$, which is the desired conclusion. \square

We can now extend this comparison principle to get the following maximum principle.

Proposition 1.2.7. *Let $\eta \in W^{1,\infty}(\mathbf{R}^d)$, $f \in H^{1/2}(\mathbf{R}^d)$. If v is the solution of Laplace equation defined in (1.2.1), and if f is bounded, then*

$$\|v\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\mathbf{R}^d)}.$$

Proof. Keeping in mind the preceding theorem, the only thing we need to prove is that $(v - (1 + \varepsilon)\|f\|_{L^\infty(\mathbf{R}^d)})^+ \in \mathcal{H}^{1,0}$. Replacing v with $-v$ and letting ε go to 0^+ will then complete the proof.

To prove this claim, we will use Lemma 1.2.5. Since $v \in \mathcal{H}^{1,0}(\Omega)$, the zero extension \tilde{v} is in $\mathcal{H}^1(\mathcal{O})$. As in subsection 1.2.1, we can extend f to \mathcal{O} using the Poisson kernel $e^{\varepsilon z \langle D_x \rangle}$. This extension \tilde{f} is bounded by $(1 + \varepsilon)\|f\|_{L^\infty}$ (see Lemma 1.2.9), so that $(\tilde{v} + \tilde{f} - (1 + \varepsilon)\|f\|_{L^\infty})^+$ is in $\mathcal{H}^1(\mathcal{O})$ by elementary properties of this space, is zero on $\mathcal{O} \setminus \Gamma$, and so by Lemma 1.2.5 $(\tilde{v} + \tilde{f} - (1 + \varepsilon)\|f\|_{L^\infty})^+ \in \mathcal{H}^{1,0}(\Omega)$. \square

We will mainly use the following classical consequence of the maximum principle:

Proposition 1.2.8. *If $0 < h' < h$, and $\Omega_{h'} = \{(x, y) \in \mathbf{R}^d \times \mathbf{R}, \eta(x) - h' < y < \eta(x)\}$, there exists a constant $C_{h'} > 0$ such that if $f \in C^{1+\varepsilon}(\mathbf{R}^d) \cap H^{\frac{1}{2}}(\mathbf{R}^d)$ and v is a variational solution of (1.2.1),*

$$\|v\|_{C^{1+\varepsilon}(\Omega_{h'})} \leq C_{h'} \|f\|_{C^{1+\varepsilon}(\mathbf{R}^d)}.$$

Proof. Noticing that v is an H^1 variational solution of $\Delta_{x,y} v = 0$ in Ω_h , Corollary 8.36 of [47] gives this on compact sub-domains of $\Omega_{h'}$, and the constant being uniform, we can deduce the result on the full $\Omega_{h'}$. \square

1.2.3 Straightening the boundary

In order to study further regularity of those solutions, it is convenient to straighten the domain, transforming an equation with constant coefficients on a variable domain to an equation with variable coefficients on a fixed domain. As seen in [7], there exists a function η_* such that

$$\begin{cases} \eta_* + \frac{h}{4} \in H^\infty(\mathbf{R}^d), \\ \eta(x) - \eta_*(x) = \frac{h}{4} + g, \quad \|g\|_{L^\infty(\mathbf{R}^d)} \leq \frac{h}{5}, \\ \Gamma \subset \{(x, y) \in \mathcal{O} : y < \eta_*(x)\}. \end{cases}$$

We can take for example

$$\eta_*(x) = -\frac{h}{4} + e^{-\nu\langle D_x \rangle} \eta(x),$$

where $\nu > 0$ is such that $\nu \|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \leq \frac{h}{5}$.

This gives

$$\|g\|_{L^\infty(\mathbf{R}^d)} = \left\| e^{-\nu\langle D_x \rangle} \eta - \eta \right\|_{L^\infty(\mathbf{R}^d)} \leq \nu \|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \leq \frac{h}{5},$$

thanks to the following classical lemma.

Lemma 1.2.9. *Let $f \in W^{1,\infty}(\mathbf{R}^d)$, $c > 0$, and $t > 0$. Then $e^{-bt\langle D_x \rangle} f \in L^\infty(\mathbf{R}^d)$ and*

$$\left\| e^{-bt\langle D_x \rangle} f - f \right\|_{L^\infty(\mathbf{R}^d)} \leq C \times bt \|f\|_{W^{1,\infty}(\mathbf{R}^d)},$$

with $C > 0$ a constant.

Now define

$$\begin{cases} \Omega_1 := \{(x, y) : x \in \mathbf{R}^d, \eta_*(x) < y < \eta(x)\}, \\ \Omega_2 := \{(x, y) \in \mathcal{O} : y \leq \eta_*(x)\}, \\ \Omega := \Omega_1 \cup \Omega_2, \end{cases}$$

and

$$\begin{cases} \tilde{\Omega}_1 := \{(x, z) : x \in \mathbf{R}^d, z \in I\}, \quad I = (-1, 0), \\ \tilde{\Omega}_2 := \{(x, z) \in \mathbf{R}^d \times (-\infty, -1] : (x, z+1+\eta_*(x)) \in \Omega_2\}, \\ \tilde{\Omega} := \tilde{\Omega}_1 \cup \tilde{\Omega}_2. \end{cases}$$

Following Lannes ([70]), we consider the map ρ from $\tilde{\Omega}$ to \mathbf{R}^d defined as

$$\begin{cases} \rho(x, z) := (1+z)e^{\delta z\langle D_x \rangle} \eta(x) - z\eta_*(x) & \text{if } (x, z) \in \tilde{\Omega}_1, \\ \rho(x, z) := z+1+\eta_*(x) & \text{if } (x, z) \in \tilde{\Omega}_2, \end{cases}$$

with $\delta = \delta(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) > 0$ small.

Using lemma 1.2.9, we have

$$\left\| \partial_z \rho - \frac{h}{4} \right\|_{L^\infty(\tilde{\Omega}_1)} \leq \delta C \|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} + \frac{h}{5},$$

which, taking δ small enough, gives

$$\begin{cases} \partial_z \rho(x, z) \geq \min\left(1, \frac{h}{5}\right), & \forall (x, z) \in \tilde{\Omega}, \\ \|\nabla_{x,z} \rho\|_{L^\infty(\tilde{\Omega})} \leq \mathcal{F}\left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}\right). \end{cases}$$

This proves that the map $(x, z) \mapsto (x, \rho(x, z))$ is a C^1 -diffeomorphism from $\tilde{\Omega}$ to Ω .

Lemma 1.2.10. *Let $I = [-1, 0]$. We have in Sobolev-type spaces, for any $\sigma > 1/2 + d/2$,*

$$\begin{aligned} \left\| \partial_z \rho - \frac{h}{4} \right\|_{X^{\sigma-\frac{1}{2}}(I)} + \|\nabla_x \rho\|_{X^{\sigma-\frac{1}{2}}(I)} &\leq \mathcal{F}\left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}\right) \|\eta\|_{H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)}, \\ \left\| \partial_z \rho - \frac{h}{4} \right\|_{L^1(I; H^{\sigma-\frac{1}{2}}(\mathbf{R}^d))} + \|\nabla_x \rho\|_{L^1(I; H^{\sigma-\frac{1}{2}}(\mathbf{R}^d))} &\leq C \|\eta\|_{H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)}. \end{aligned}$$

With $0 \leq r \leq 1/2$, we have in Hölder-type spaces

$$\begin{aligned} \left\| \partial_z \rho - \frac{h}{4} \right\|_{C^0(I; C_*^r(\mathbf{R}^d))} + \|\nabla_x \rho\|_{C^0(I; C_*^r(\mathbf{R}^d))} &\leq \mathcal{F}\left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}\right) \|\eta\|_{C_*^{1+r}(\mathbf{R}^d)}, \\ \left\| \partial_z \rho - \frac{h}{4} \right\|_{\tilde{L}^2(I; C_*^{\frac{1}{2}+r}(\mathbf{R}^d))} + \|\nabla_x \rho\|_{\tilde{L}^2(I; C_*^{\frac{1}{2}+r}(\mathbf{R}^d))} &\leq C[\|\eta\|_{L^2(\mathbf{R}^d)} + \|\eta\|_{C_*^{1+r}(\mathbf{R}^d)}], \\ \|\nabla_{x,z}^2 \rho\|_{\tilde{L}^2(I; C_*^{-\frac{1}{2}+r}(\mathbf{R}^d))} &\leq C[\|\eta\|_{L^2(\mathbf{R}^d)} + \|\eta\|_{C_*^{1+r}(\mathbf{R}^d)}]. \end{aligned}$$

Proof. (1.2.10) is proved in [7], and the first part of (1.2.10) is a straightforward consequence of Lemma 1.2.9. The second part of (1.2.10) will be proved with Littlewood-Paley decomposition, following [102]. We have $\|\nabla_x \rho\|_{\tilde{L}^2(I; C_*^{\frac{1}{2}+r}(\mathbf{R}^d))} \lesssim \|\langle D_x \rangle \rho\|_{\tilde{L}^2(I; C_*^{\frac{1}{2}+r}(\mathbf{R}^d))}$, so that we only need to control this last norm. We can split $\langle D_x \rangle \rho$ in two parts, $(1+z)e^{\delta z \langle D_x \rangle} \langle D_x \rangle \eta$ and $-ze^{-\nu \langle D_x \rangle} \langle D_x \rangle \eta$, whose norms will be respectively A and B . For the first part, we have

$$\begin{aligned} A &\leq \sup_{j>0} 2^{j(\frac{1}{2}+r)} \left\| \Delta_j e^{\delta z \langle D_x \rangle} \langle D_x \rangle \eta \right\|_{L^2(I; L^\infty(\mathbf{R}^d))} + \left\| \Delta_0 e^{\delta z \langle D_x \rangle} \langle D_x \rangle \eta \right\|_{L^1(I; L^\infty(\mathbf{R}^d))} \\ &\leq C \sup_{j>0} 2^{j(\frac{3}{2}+r)} \left\| e^{c\delta z 2^j} \right\|_{L^2(I)} \|\Delta_j \eta\|_{L^\infty(\mathbf{R}^d)} + \left\| e^{\delta z \langle D_x \rangle} \Delta_0 \eta \right\|_{L^2(I \times \mathbf{R}^d)} \\ &\leq C \sup_{j \geq 0} 2^{j(1+r)} \|\Delta_j \eta\|_{L^\infty(\mathbf{R}^d)} + \|\Delta_0 \eta\|_{L^2(\mathbf{R}^d)}, \end{aligned}$$

where we have used extensively Bernstein estimates (A.0.20) and the smoothing effect (A.0.21). The same method applies for B , taking $z = -1$ and bounding $e^{-c\nu 2^j}$ by $1/2^j$. This gives the expected result, and computations for $\partial_z \rho - h/4$ and the second-order terms are identical. \square

Now for a function v defined on Ω , put

$$\tilde{v}(x, z) := v(x, \rho(x, z)).$$

We then have

$$\begin{cases} (\partial_y v)(x, \rho(x, z)) = \Lambda_1 \tilde{v}(x, z), & (\nabla_x v)(x, \rho(x, z)) = \Lambda_2 \tilde{v}(x, z), \\ \Lambda_1 := \frac{1}{\partial_z \rho} \partial_z, & \Lambda_2 := \nabla_x - \frac{\nabla_x \rho}{\partial_z \rho} \partial_z. \end{cases}$$

If v is a solution of $\Delta_{x,y}v = F_0$ in Ω then \tilde{v} satisfies

$$(\Lambda_1^2 + \Lambda_2^2)\tilde{v} = \tilde{F}_0 \quad \text{in } \tilde{\Omega}.$$

This equation can be expanded to

$$\left\{ \begin{array}{l} (\partial_z^2 + \alpha\Delta_x + \beta \cdot \nabla_x \partial_z - \gamma \partial_z)\tilde{v} = \tilde{F}_0, \\ \tilde{v}(z=0) = f, \\ \alpha := \frac{(\partial_z \rho)^2}{1 + |\nabla_x \rho|^2}, \quad \beta := -2 \frac{\partial_z \rho \nabla_x \rho}{1 + |\nabla_x \rho|^2}, \quad \gamma := \frac{1}{\partial_z \rho} (\partial_z^2 \rho + \alpha \Delta_x \rho + \beta \cdot \nabla_x \partial_z \rho). \end{array} \right.$$

We also remark that

$$(\Lambda_1^2 + \Lambda_2^2)\rho = 0 \quad \text{in } \tilde{\Omega},$$

and that

$$[\Lambda_1, \Lambda_2] = 0.$$

We now derive some estimates on the new coefficients.

Lemma 1.2.11. *Let $I = [-1, 0]$. We have for $\sigma > 1/2 + d/2$*

$$(1.2.1) \quad \left\| \alpha - \frac{h^2}{16} \right\|_{X^{\sigma-\frac{1}{2}}(I)} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \|\eta\|_{H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)},$$

$$(1.2.2) \quad \|\beta\|_{X^{\sigma-\frac{1}{2}}(I)} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \|\eta\|_{H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)},$$

$$(1.2.3) \quad \|\gamma\|_{X^{\sigma-\frac{3}{2}}(I)} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \left[1 + \|\eta\|_{H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)} \right],$$

and for any $0 \leq r \leq 1/2$, we have

$$(1.2.4) \quad \left\| \alpha - \frac{h^2}{16} \right\|_{C^0(I; C_*^r(\mathbf{R}^d))} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \|\eta\|_{C_*^{1+r}(\mathbf{R}^d)},$$

$$(1.2.5) \quad \|\beta\|_{C^0(I; C_*^r(\mathbf{R}^d))} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \|\eta\|_{C_*^{1+r}(\mathbf{R}^d)}.$$

Proof. We see from (1.2.10) that we can write

$$(\partial_z \rho)^2 = \frac{h^2}{16} + G \quad \text{with} \quad \|G\|_{X^{\sigma-\frac{1}{2}}(I)} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \|\eta\|_{H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)}.$$

We can now decompose

$$\alpha - \frac{h^2}{16} = -\frac{h^2}{16} \times \frac{|\nabla_x \rho|^2}{1 + |\nabla_x \rho|^2} + G - G \frac{|\nabla_x \rho|^2}{1 + |\nabla_x \rho|^2},$$

and we use the tame estimates of (A.0.25) to conclude. The other inequalities are all proved with the same method, using the other estimates of (1.2.10) and (1.2.10). \square

1.2.4 Elliptic regularity in the new domain

We now study the new equation (1.2.3), following the method from [7], with tame estimates at every step. Recall from (A.0.4) the definition of the spaces

$$\begin{aligned} X^\mu(I) &= L_z^\infty(I; H^\mu(\mathbf{R}^d)) \cap L_z^2(I; H^{\mu+\frac{1}{2}}(\mathbf{R}^d)), \\ Y^\mu(I) &= L_z^1(I; H^\mu(\mathbf{R}^d)) + L_z^2(I; H^{\mu-\frac{1}{2}}(\mathbf{R}^d)). \end{aligned}$$

We suppose \tilde{v} to be a solution of (1.2.3) in $I \times \mathbf{R}^d$, $I = [-1, 0]$ with the additional assumption

$$\|\tilde{v}\|_{X^{-\frac{1}{2}}([-1,0])} < \infty.$$

We know from [7] that this estimation holds for our variational solutions of (1.2.1), with

$$\|\tilde{v}\|_{X^{-\frac{1}{2}}([-1,0])} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

The main result of the section will be stated in two versions, corresponding to our two main theorems.

Proposition 1.2.12. *Let $\epsilon > 0$, $s_0 > 1/2 + d/2$, and $s > 1 + d/2$. Let*

$$-\frac{1}{2} \leq \sigma \leq s - \frac{1}{2}.$$

Consider $f \in H^{\sigma+1}(\mathbf{R}^d)$, $\tilde{F} \in Y^\sigma(I)$ and \tilde{v} satisfying the hypothesis (1.2.4), solution to (1.2.3). Then for any $z_0 \in (-1, 0)$,

$$(1.2.6) \quad \|\nabla_{x,z}\tilde{v}\|_{X^\sigma([z_0,0])} \leq \mathcal{F} \left(\|\eta\|_{(C_*^{1+\epsilon} \cap L^2)(\mathbf{R}^d)} \right) \left[\|\nabla_x f\|_{H^\sigma(\mathbf{R}^d)} + \|\tilde{F}_0\|_{Y^\sigma(I)} \right. \\ \left. + \|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \|\nabla_{x,z}\tilde{v}\|_{L^\infty(I \times \mathbf{R}^d)} + \|\nabla_{x,z}\tilde{v}\|_{X^{-\frac{1}{2}}([-1,0])} \right],$$

and

$$(1.2.7) \quad \|\nabla_{x,z}\tilde{v}\|_{X^\sigma([z_0,0])} \leq \mathcal{F} \left(\|\eta\|_{H^{s_0+\frac{1}{2}}(\mathbf{R}^d)} \right) \left[\|\nabla_x f\|_{H^\sigma(\mathbf{R}^d)} + \|\tilde{F}_0\|_{Y^\sigma(I)} + \|\nabla_{x,z}\tilde{v}\|_{X^{-\frac{1}{2}}([-1,0])} \right].$$

Remark. The proof will show that the function \mathcal{F} depends on z_0 and on σ . On every application, those parameters will be fixed, and independent of time.

As in [7], this will be proved by induction on the regularity σ . The property that inequalities (1.2.6) and (1.2.7) hold for σ , for all admissible z_0 , will be denoted by \mathcal{H}_σ . In this notation, hypothesis (1.2.4) means that $\mathcal{H}_{-1/2}$ is satisfied.

Proposition 1.2.12 is then a consequence of

Proposition 1.2.13. *For any δ such that*

$$0 < \delta \leq \inf \left(\epsilon, s_0 - \frac{1}{2} - \frac{d}{2} \right) \leq \frac{1}{2},$$

if \mathcal{H}_σ is satisfied for some $-1/2 \leq \sigma \leq s - 1/2 - \delta$, then $\mathcal{H}_{\sigma+\delta}$ is satisfied.

To prove this, we first estimate the lower order term.

Lemma 1.2.14. *For all $J \subset I$,*

$$\|\tilde{F}_1\|_{Y^{\sigma+\delta}(J)} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty} \cap L^2} \right) \left(1 + \|\eta\|_{C_*^{1+\delta}} \right) \left[\|\eta\|_{H^{s+\frac{1}{2}}} \|\partial_z \tilde{v}\|_{L^\infty(\mathbf{R}^d \times J)} + \|\partial_z \tilde{v}\|_{X^\sigma(J)} \right],$$

and

$$\|\tilde{F}_1\|_{Y^{\sigma+\delta}(J)} \leq \mathcal{F} \left(\|\eta\|_{H^{s_0+\frac{1}{2}}} \right) \|\partial_z \tilde{v}\|_{X^\sigma(J)},$$

with $\tilde{F}_1 = \gamma \partial_z \tilde{v}$.

Proof. For the first inequality, we decompose \tilde{F}_1 as

$$\tilde{F}_1 = \partial_z^2 \rho \frac{1}{\partial_z \rho} \partial_z \tilde{v} + \Delta_x \rho \frac{\alpha}{\partial_z \rho} \partial_z \tilde{v} + \nabla_x \partial_z \rho \cdot \frac{\beta}{\partial_z \rho} \partial_z \tilde{v} := A + B + C.$$

We then use the paraproduct estimates (A.0.13), (A.0.18), and (A.0.19) to get

$$\begin{aligned} \|A\|_{Y^{\sigma+\delta}} &\leq \|A\|_{L^2(J; H^{s+\delta-\frac{1}{2}})} \\ &\leq \|\partial_z^2 \rho\|_{\tilde{L}^2(J; C^{\delta-\frac{1}{2}})} \left\| \frac{1}{\partial_z \rho} \partial_z \tilde{v} \right\|_{X^\sigma} + \left\| \frac{1}{\partial_z \rho} \partial_z \tilde{v} \right\|_{L^\infty(J \times \mathbf{R}^d)} \|\partial_z^2 \rho\|_{L^2(J; H^{s-1})}, \end{aligned}$$

and using the estimates (1.2.10) and tame estimates gives the majoration for A . Estimates on B and C follows along the same lines. For the second one, paraproduct estimates give immediately

$$\|\gamma \partial_z \tilde{v}\|_{L^1(J; H^{\sigma+\delta})} \leq \|\partial_z \tilde{v}\|_{L^2(J; H^{\sigma+\frac{1}{2}})} \|\gamma\|_{L^2(J; H^{s_0-1})},$$

and using the estimate on γ from Lemma 1.2.11 enable us to conclude. \square

We now replace multiplication by α (resp. β) with paramultiplication by T_α (resp. T_β).

Lemma 1.2.15. *\tilde{v} satisfies the paradifferential equation*

$$\partial_z^2 \tilde{v} + T_\alpha \Delta_x \tilde{v} + T_\beta \cdot \nabla_x \partial_z \tilde{v} = \tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2,$$

for some remainder

$$\tilde{F}_2 = (T_\alpha - \alpha) \Delta_x \tilde{v} + (T_\beta - \beta) \cdot \nabla_x \partial_z \tilde{v}$$

satisfying for all $J \subset I$

$$\left\| \tilde{F}_2 \right\|_{Y^{\sigma+\delta}(J)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \left[1 + \|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \right] \|\nabla_{x,z} \tilde{v}\|_{L^\infty(\mathbf{R}^d \times J)},$$

or

$$\left\| \tilde{F}_2 \right\|_{Y^{\sigma+\delta}(J)} \leq \mathcal{F}(\|\eta\|_{H^{s_0}(\mathbf{R}^d)}) \|\nabla_{x,z} \tilde{v}\|_{X^\sigma(J)}.$$

Proof. First, we have

$$\tilde{F}_2 = \left(T_{\alpha - \frac{h^2}{16}} - \left(\alpha - \frac{h^2}{16} \right) \right) \Delta_x \tilde{v} - \frac{h^2}{16} (T_1 - 1) \Delta_x \tilde{v} + (T_\beta - \beta) \cdot \nabla_x \partial_z \tilde{v},$$

which, according to (A.0.19) and (A.0.15) gives

$$\begin{aligned} \left\| \tilde{F}_2 \right\|_{Y^{\sigma+\delta}} &\leq C \left[\|\Delta_x \tilde{v}\|_{L^\infty C_*^{-1}} \left(\left\| \alpha - \frac{h^2}{16} \right\|_{L^2 H^{\sigma+\delta+\frac{1}{2}}} + 1 \right) \right. \\ &\quad \left. + \|\nabla_x \partial_z \tilde{v}\|_{L^\infty C_*^{-1}} \|\beta\|_{L^2 H^{\sigma+\delta+\frac{1}{2}}} \right]. \end{aligned}$$

As $\sigma + \delta + 1/2 \leq s$, we have

$$\left\| \tilde{F}_2 \right\|_{Y^{\sigma+\delta}} \leq C \left[1 + \left\| \alpha - \frac{h^2}{16} \right\|_{X^{s-\frac{1}{2}}} + \|\beta\|_{X^{s-\frac{1}{2}}} \right] \|\nabla_{x,z} \tilde{v}\|_{L^\infty}.$$

Using the estimates (1.2.1) and (1.2.2) completes the proof of the first inequality.

Using again (A.0.19) and (A.0.15) gives

$$\left\| \left(T_{\alpha - \frac{h^2}{16}} - \left(\alpha - \frac{h^2}{16} \right) \right) \Delta_x \tilde{v} \right\|_{L^1 H^{\sigma+\delta}} \leq C \|\Delta_x \tilde{v}\|_{L^2 H^{\sigma-\frac{1}{2}}} \left(\left\| \alpha - \frac{h^2}{16} \right\|_{L^2 H^{\sigma_0+\frac{1}{2}}} + 1 \right),$$

and

$$\|(T_\beta - \beta) \cdot \nabla_x \partial_z \tilde{v}\|_{L^1 H^{\sigma+\delta}} \leq C \|\nabla_x \partial_z \tilde{v}\|_{L^2 H^{\sigma-\frac{1}{2}}} \|\beta\|_{L^2 H^{\sigma_0}},$$

so that

$$\|\tilde{F}_2\|_{Y^{\sigma+\delta}} \leq C \left[1 + \left\| \alpha - \frac{h^2}{16} \right\|_{X^{\sigma_0-\frac{1}{2}}} + \|\beta\|_{X^{\sigma_0-\frac{1}{2}}} \right] \|\nabla_{x,z} \tilde{v}\|_{X^\sigma}.$$

The estimates (1.2.1) and (1.2.2) enable us to conclude once again. \square

We can now decouple the equation into a forward and a backward parabolic evolution equation.

Lemma 1.2.16. *There exists two symbols a, A in $\Gamma_\varepsilon^1(\mathbf{R}^d \times I)$ and a remainder \tilde{F}_3 such that*

$$(\partial_z - T_a)(\partial_z - T_A)\tilde{v} = \tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3,$$

with

$$\mathcal{M}_\varepsilon^1(a) + \mathcal{M}_\varepsilon^1(A) \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|\eta\|_{C_*^{1+\varepsilon}},$$

and for all $J \subset I$

$$\|\tilde{F}_3\|_{Y^{\sigma+\delta}(J)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|\eta\|_{C_*^{1+\varepsilon}} \|\nabla_{x,z} \tilde{v}\|_{X^\sigma(I)}.$$

Proof. We see from [7] that this holds true with

$$a = \frac{1}{2}(i\beta \cdot \xi - \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}), \quad A = \frac{1}{2}(-i\beta \cdot \xi + \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}).$$

Using the Hölder estimates (1.2.4) and (1.2.5) gives (1.2.16). A straightforward estimate of the remainder along the lines of [7] ends the proof. \square

Proof of Proposition 1.2.13. Still following [7], we apply proposition A.0.18 twice. We will prove only inequality (1.2.6), since the proof of inequality (1.2.7) is along the same lines.

Suppose first that \mathcal{H}_σ is satisfied. This means that for $J_0 = [\zeta_0, 0]$, we have

$$\begin{aligned} \|\nabla_{x,z} \tilde{v}\|_{X^\sigma([\zeta_0, 0])} &\leq \mathcal{F} \left(\|\eta\|_{(C_*^{1+\varepsilon} \cap L^2)(\mathbf{R}^d)} \right) \left[\|f\|_{H^{\sigma+1}(\mathbf{R}^d)} + \|\tilde{F}_0\|_{Y^\sigma(I)} \right. \\ &\quad \left. + \|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \|\nabla_{x,z} \tilde{v}\|_{L^\infty(I \times \mathbf{R}^d)} + \|\tilde{v}\|_{X^{-\frac{1}{2}}([-1, 0])} \right]. \end{aligned}$$

We will then prove that, for any $\zeta_1 > \zeta_0$, we have

$$\begin{aligned} \|\nabla_{x,z} \tilde{v}\|_{X^{\sigma+\delta}([\zeta_1, 0])} &\leq \mathcal{F}(\|\eta\|_{(C_*^{1+\varepsilon} \cap L^2)(\mathbf{R}^d)}) \left[\|f\|_{H^{\sigma+1+\delta}(\mathbf{R}^d)} + \|\tilde{F}_0\|_{Y^{\sigma+\delta}(J_0)} \right. \\ &\quad \left. + \|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \|\nabla_{x,z} \tilde{v}\|_{L^\infty(I \times \mathbf{R}^d)} + \|\nabla_{x,z} \tilde{v}\|_{X^\sigma(J_0)} \right], \end{aligned}$$

which will finish the proof of the proposition.

Take χ a cutoff function such that $\chi(\zeta_0) = 0$ and $\chi(z) = 1$ for $z \geq \zeta_1$. We then put $\tilde{w} := \chi(z)(\partial_z - T_A)\tilde{v}$. We see from (1.2.16) that

$$\partial_z \tilde{w} - T_a \tilde{w} = \tilde{F}',$$

with

$$\tilde{F}' = \chi(z)(\tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3) + \chi'(z)(\partial_z - T_A)\tilde{v}.$$

All those terms, the last one excepted, have already been estimated. Since $\delta \leq 1/2$, a simple computation gives

$$\|(\partial_z - T_A)\tilde{v}\|_{Y^{\sigma+\delta}(J_0)} \leq \|(\partial_z - T_A)\tilde{v}\|_{X^\sigma(J_0)} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} \right) \|\eta\|_{C_*^{1+\varepsilon}} \|\nabla_{x,z}\tilde{v}\|_{X^\sigma(J_0)},$$

where we have used the fact that A is a symbol of order 1, whose norm has already been estimated in (1.2.16).

We see from the definition of a that it is elliptic, with an ellipticity constant depending only on h . Since we have $w|_{z=z_0} = 0$, Proposition A.0.18 gives the estimate

$$\begin{aligned} \|\tilde{w}\|_{X^{\sigma+\delta}(J_0)} &\leq \mathcal{F} \left(\|\eta\|_{C^{1+\varepsilon}(\mathbf{R}^d)} \right) \left\| \tilde{F}' \right\|_{Y^{\sigma+\delta}(J_0)} \\ &\leq \mathcal{F} \left(\|\eta\|_{(C_*^{1+\varepsilon} \cap L^2)(\mathbf{R}^d)} \right) \left[\left\| \tilde{F}_0 \right\|_{Y^{\sigma+\delta}(J_0)} \right. \\ &\quad \left. + \|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \|\nabla_{x,z}\tilde{v}\|_{L^\infty(I \times \mathbf{R}^d)} + \|\nabla_{x,z}\tilde{v}\|_{X^\sigma(J_0)} \right], \end{aligned}$$

where we have used the estimates from Lemma 1.2.14, 1.2.15 and 1.2.16. Notice that on $J_1 := [\zeta_1, 0]$, we have $\chi = 1$, so that

$$\partial_z \tilde{v} - T_A \tilde{v} = \tilde{w}.$$

In fact, we have

$$\partial_z \nabla_x \tilde{v} - T_A \nabla_x \tilde{v} = \nabla_x \tilde{w} + T_{\nabla_x A} \tilde{v}.$$

Changing z to $-z$, and using again Proposition A.0.18, we have

$$\|\nabla_x \tilde{v}\|_{X^{\sigma+\delta}(J_1)} \leq \mathcal{F} \left(\|\eta\|_{C^{1+\varepsilon}(\mathbf{R}^d)} \right) \left[\|\nabla_x f\|_{H^{\sigma+\delta}(\mathbf{R}^d)} + \|\nabla_x \tilde{w}\|_{Y^{\sigma+\delta}(J_1)} + \|\nabla_x \tilde{v}\|_{X^\sigma(J_0)} \right].$$

Since $\|\nabla_x \tilde{w}\|_{Y^{\sigma+\delta}(J_1)} \leq \|\tilde{w}\|_{X^{\sigma+\delta}(J_1)}$, we obtain the estimate

$$\begin{aligned} \|\nabla_x \tilde{v}\|_{X^{\sigma+\delta}([\zeta_1, 0])} &\leq \mathcal{F} \left(\|\eta\|_{(C_*^{1+\varepsilon} \cap L^2)(\mathbf{R}^d)} \right) \left[\|\nabla_x f\|_{H^{\sigma+1+\delta}(\mathbf{R}^d)} + \left\| \tilde{F}_0 \right\|_{Y^{\sigma+\delta}(J_0)} \right. \\ &\quad \left. + \|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \|\nabla_{x,z}\tilde{v}\|_{L^\infty(I \times \mathbf{R}^d)} + \|\nabla_{x,z}\tilde{v}\|_{X^\sigma(J_0)} \right]. \end{aligned}$$

The estimate for $\partial_z \tilde{v}$ follows from $\partial_z \tilde{v} = T_A \tilde{v} + \tilde{w}$, the estimate for w and the fact that A is of order 1. This completes the proof of Proposition 1.2.13, and hence of Proposition 1.2.12. \square

1.2.5 Applications

In this section we apply the previous elliptic estimates to study the Dirichlet-Neumann operator and its parilinearization.

Proposition 1.2.17. *Let $d \geq 1$, $s > 1 + d/2$, $s_0 > 1/2 + d/2$, and $1/2 \leq \sigma \leq s + 1/2$. Then*

$$\|G(\eta)f\|_{H^{\sigma-1}} \leq \mathcal{F} \left(\|\eta\|_{C_*^{1+\varepsilon}} \right) \left[\|f\|_{H^\sigma} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z}v\|_{L^\infty} \right],$$

and

$$\|G(\eta)f\|_{H^{\sigma-1}} \leq \mathcal{F} \left(\|\eta\|_{H^{s_0+\frac{1}{2}}} \right) \|f\|_{H^\sigma},$$

where v is the harmonic extension of f . We also have

$$\|G(\eta)f\|_{L^\infty} \leq \mathcal{F} \left(\|\eta\|_{W^{1,\infty}} \right) \|\nabla_{x,z}v\|_{L^\infty}.$$

Remark. The term $\|\nabla_{x,z}v\|_{L^\infty}$ can generally be expressed only with terms defined on Σ , using the maximum principle of Proposition 1.2.7 and its consequence, Proposition 1.2.8.

Proof. As seen in [7], the Dirichlet-Neumann can be expressed by

$$G(\eta)f = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} \partial_z v - \nabla_x \rho \cdot \nabla_x v \Big|_{z=0}.$$

Now using Proposition 1.2.12, the estimates on ρ from (1.2.10), and the tame estimates (A.0.25), the first two estimates of the proposition immediately follow. The last one is a straightforward consequence of (1.2.5). \square

We know from [7] that the Dirichlet-Neumann can be expressed as

$$G(\eta) = T_\lambda + R(\eta),$$

where

$$\lambda(x, \xi) := \sqrt{(1 + |\nabla \eta(x)|^2) |\xi|^2 - (\nabla \eta(x) \cdot \xi)^2},$$

and $R(\eta)$ is a smoothing operator. Using tame estimates, we obtain

Proposition 1.2.18. *Let $d \geq 1$, $s > 1 + d/2$, $0 < \varepsilon \leq \varepsilon' \leq 1/2$ and $1/2 \leq \sigma \leq s + 1/2$. Then*

$$\|R(\eta)f\|_{H^{\sigma-1+\varepsilon'}} \leq \mathcal{F} \left(\|\eta\|_{C_*^{1+\varepsilon}} \right) \left(1 + \|\eta\|_{C_*^{1+\varepsilon'}} \right) \left[\|f\|_{H^\sigma} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z}v\|_{L^\infty} \right],$$

and if $s_0 > 1/2 + d/2$ and $\varepsilon' \leq s_0 - d/2 - 1/2$,

$$\|R(\eta)f\|_{H^{\sigma-1+\varepsilon'}} \leq \mathcal{F} \left(\|\eta\|_{H^{s_0+\frac{1}{2}}} \right) \|f\|_{H^\sigma}.$$

Proof. As in the proof of Proposition 1.2.13, we use Proposition A.0.18 to get

$$\|(\partial_z - T_A)\tilde{v}\|_{X^{\sigma-1+\varepsilon'}(J_0)} \leq \mathcal{F} \left(\|\eta\|_{C^{1+\varepsilon}(\mathbf{R}^d)} \right) \left\| \tilde{F}' \right\|_{Y^{\sigma-1+\varepsilon'}(J_0)}$$

Now since

$$G(\eta)f = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho} \partial_z v - \nabla_x \rho \cdot \nabla_x v \Big|_{z=0},$$

we set

$$\zeta_1 := \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho}, \quad \zeta_2 := \nabla_x \rho.$$

According to (1.2.10),

$$\left\| \zeta_1 - \frac{4}{h} \right\|_{C([-1,0]; H^{s-\frac{1}{2}})} + \|\zeta_2\|_{C([-1,0]; H^{s-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|\eta\|_{H^{s+\frac{1}{2}}}.$$

Let

$$R' := \zeta_1 \partial_z v - \zeta_2 \cdot \nabla_x v - (T_{\zeta_1} \partial_z v - T_{\zeta_2} \nabla_x v).$$

Using the tame estimates of (A.0.25), we verify that

$$\|R'\|_{C^0(I; H^{\sigma-1+\varepsilon'})} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|\partial_z v\|_{L^\infty} \|\eta\|_{H^{s+\frac{1}{2}}}.$$

Furthermore,

$$T_{\zeta_1} \partial_z v - T_{\zeta_2} \partial_x v|_{z=0} - (T_{\zeta_1} T_A v - T_{i\zeta_2 \cdot \xi} v)|_{z=0} := R'',$$

with

$$\|R''\|_{H^{\sigma-1+\varepsilon'}} \leq \mathcal{F} \left(\|\eta\|_{C_*^{1+\varepsilon}} \right) \left(1 + \|\eta\|_{C_*^{1+\varepsilon'}} \right) \left[\|f\|_{H^\sigma} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z} v\|_{L^\infty} \right].$$

Finally, we have

$$\|T_{\zeta_1(z)} T_A(z) - T_{\zeta_1(z)A(z)}\|_{H^\sigma \rightarrow H^{\sigma-1+\varepsilon'}} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|\eta\|_{C_*^{1+\varepsilon'}},$$

and hence

$$G(\eta)f = T_{\zeta_1 A} v - T_{i\zeta_2 \cdot \xi} v|_{z=0} + R(\eta)f,$$

where

$$\|R(\eta)f\|_{H^{\sigma-1+\varepsilon'}} \leq \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}}) \left(1 + \|\eta\|_{C_*^{1+\varepsilon'}} \right) \left[\|f\|_{H^\sigma} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z} v\|_{L^\infty} \right].$$

Let

$$\lambda := \frac{1 + |\partial_x \rho|^2}{\partial_z \rho} A - i \partial_x \rho \cdot \xi|_{z=0} = \sqrt{(1 + |\partial_x \eta(x)|^2) |\xi|^2 - (\partial_x \eta(x) \cdot \xi)^2}.$$

Then

$$G(\eta)f = T_\lambda f + R(\eta)f,$$

which concludes the proof of the first inequality. The second one is proved along the same lines. \square

1.3 Paralinearization of the system

We still follow [7] to reduce the equations to a paradifferential hyperbolic system. This process yields remainders that we need to estimate. To simplify the expression of these estimates, we will denote by K a constant of the form

$$K(t) := \mathcal{F} \left(\frac{1}{h}, \frac{1}{c}, \|\eta(t)\|_{W^{1+\varepsilon,\infty}(\mathbf{R}^d)}, \|a(t)\|_{W^{\varepsilon,\infty}(\mathbf{R}^d)}, \|\eta\|_{L^2(\mathbf{R}^d)}, \|(V, B)\|_{W^\varepsilon(\mathbf{R}^d)} \right),$$

with \mathcal{F} positive nondecreasing. It will appear in the proof of the first version of our main theorem, Theorem 1.1.2, involving only Hölder norms. For the proof of the second version, Theorem 1.1.3, involving the reference Sobolev norm of index $s_0 > 1/2 + d/2$, we will use a constant

$$K_0(t) := \mathcal{F} \left(\frac{1}{h}, \frac{1}{c}, \|a(t)\|_{W^{\varepsilon,\infty}(\mathbf{R}^d)}, \|(\eta, \psi, V, B)(t)\|_{H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d)} \right).$$

Using Sobolev embeddings, we see that we can take $K \leq K_0$.

We still denote by C a generic constant. To get the optimal regularity, we need to change the unknowns we are working with, using instead

$$\zeta = \nabla \eta, \quad B = \partial_y \phi|_{y=\eta}, \quad V = \nabla_x \phi|_{y=\eta}, \quad a = -\partial_y P|_{y=\eta},$$

where ϕ is the velocity potential and P the pressure, uniquely determined by the equation

$$-P = \partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + gy.$$

Those follow the following evolution equations.

Proposition 1.3.1. *We have*

$$(1.3.1) \quad (\partial_t + V \cdot \nabla) B = a - g,$$

$$(1.3.2) \quad (\partial_t + V \cdot \nabla) V = -a\zeta,$$

$$(1.3.3) \quad (\partial_t + V \cdot \nabla) \zeta = G(\eta)V + \zeta G(\eta)B + \gamma,$$

where the remainder γ satisfies the estimates

$$\|\gamma\|_{H^{s-\frac{1}{2}}} \leq K \left[\|(V, B, \psi)\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s+\frac{1}{2}}} + \|\eta\|_{H^{s+\frac{1}{2}}} \|(V, B)\|_{C_*^{1+\varepsilon}} \right],$$

and

$$\|\gamma\|_{H^{s-\frac{1}{2}}} \leq K_0 \left[\|(V, B, \psi)\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s+\frac{1}{2}}} \right].$$

Proof. The first two equations are proved in [7]. For the estimations of γ , we start from the equation

$$\partial_t \eta = B - V \cdot \nabla \eta.$$

Differentiating this with respect to x_i yields for $i = 1, \dots, d$

$$(\partial_t + V \cdot \nabla) \partial_i \eta = \partial_i B - \sum_{j=1}^d \partial_i V_j \partial_j \eta,$$

and using the definitions of V and B and the chain rule,

$$(\partial_t + V \cdot \nabla) \partial_i \eta = [\partial_y \partial_i \phi - \nabla \eta \cdot \nabla \partial_i \phi]|_{y=\eta} + \partial_i \eta [\partial_y (\partial_y \phi) - \nabla \eta \cdot \nabla \partial_y \phi]|_{y=\eta}.$$

We now introduce θ_i , the variational solution of the problem

$$\Delta_{x,y} \theta_i = 0 \text{ in } \Omega, \quad \theta_i|_{\Sigma} = V_i, \quad \partial_n \theta_i = 0 \text{ on } \Gamma.$$

Then

$$G(\eta)V_i = (\partial_y \theta_i - \nabla \eta \cdot \nabla \theta_i)|_{\Sigma}.$$

We can now write

$$(\partial_y - \nabla \eta \cdot \nabla) \partial_i \phi|_{y=\eta} = G(\eta)V_i + R_i, \quad \text{where } R_i = (\partial_y - \nabla \eta \cdot \nabla) (\partial_i \phi - \theta_i)|_{\Sigma}.$$

If there was no bottom, we would see that at least formally, R_i would be 0. Then, in our setting, we expect a control of this remainder, and to obtain it, we continue to follow [7] and localize the problem near Σ .

Let $\chi_0 \in C^\infty(\mathbf{R})$, $\eta_1 \in H^\infty(\mathbf{R}^d)$ be such that $\chi_0(z) = 1$ if $z \geq 0$, $\chi_0(z) = 0$ if $z \leq -1/4$, and

$$\eta(x) - \frac{h}{4} \leq \eta_1(x) \leq \eta(x) - \frac{h}{5}.$$

Set

$$U_i(x, y) = \chi_0 \left(\frac{y - \eta_1(x)}{h} \right) (\partial_i \phi - \theta_i)(x, y).$$

We see that

$$R_i = (\partial_y - \nabla \eta \cdot \nabla) U_i|_{\Sigma}.$$

And U_i satisfies the equation

$$\Delta_{x,y} U_i = \left[\Delta_{x,y}, \chi_0 \left(\frac{y - \eta_1(x)}{h} \right) \right] (\partial_i \phi - \theta_i) := F_i$$

with

$$\text{supp} F_i \subset S_{\frac{1}{2}, \frac{1}{5}} := \left\{ (x, y) : x \in \mathbf{R}^d, \eta(x) - \frac{h}{2} \leq y \leq \eta(x) - \frac{h}{5} \right\}.$$

We can then control the right hand term of this equation, using lemma 3.16 of [7], which gives for all $\alpha \in \mathbf{N}^{d+1}$,

$$\|D_{x,y}^\alpha F_i\|_{L^\infty(S_{\frac{1}{2}, \frac{1}{5}}) \cap L^2(S_{\frac{1}{2}, \frac{1}{5}})} \leq C_\alpha \|(V, B)\|_{H^{\frac{1}{2}} \times H^{\frac{1}{2}}}.$$

Then changing variables, we get on the domain $\tilde{\Omega}$ that

$$(\partial_z^2 + \alpha \Delta + \beta \cdot \nabla \partial_z - \gamma \partial_z) \tilde{U}_i = \frac{(\partial_z \rho)^2}{1 + |\nabla \rho|^2} \tilde{F}_i.$$

We can then apply the estimate (1.2.6), with $f = 0$.

$$\|\nabla_{x,z} \tilde{U}_i\|_{C^0([z_0, 0]; H^{s-\frac{1}{2}})} \leq K \left[\|\tilde{F}_i\|_{Y^{s-\frac{1}{2}}} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z} \tilde{U}_i\|_{L^\infty} + \|\nabla_{x,z} \tilde{U}_i\|_{X^{-\frac{1}{2}}} \right].$$

Using equation 1.3, the control on the $X^{-\frac{1}{2}}$ norm of a variational solution, and the maximum principle for gradients of Proposition 1.2.8, this yields

$$\|\nabla_{x,z} \tilde{U}_i\|_{C^0([z_0, 0]; H^{s-\frac{1}{2}})} \leq K \left[\|(V, B, \psi)\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s+\frac{1}{2}}} + \|\eta\|_{H^{s+\frac{1}{2}}} \|(V, B)\|_{C_*^{1+\varepsilon}} \right].$$

Since

$$R_i = \left[\left(\frac{1 + |\nabla \eta|^2}{1 + \delta \langle D_x \rangle \eta} \partial_z - \nabla \eta \cdot \nabla \right) \tilde{U}_i \right] \Big|_{z=0},$$

we have

$$\|R_i\|_{H^{s-\frac{1}{2}}} \leq K \left[\|(V, B, \psi)\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s+\frac{1}{2}}} + \|\eta\|_{H^{s+\frac{1}{2}}} \|(V, B)\|_{C_*^{1+\varepsilon}} \right].$$

The same argument shows that

$$(\partial_y - \nabla \eta \cdot \nabla) \partial_y \phi = G(\eta)B + R_0,$$

where R_0 satisfies the same estimate as R_i . This proves the first estimate. The second one follows exactly the same scheme, using (1.2.7) instead. \square

Using the same method, and following proposition 4.5 of [7], it is possible to prove the following relation.

Proposition 1.3.2. *We have $G(\eta)B = -\text{div} V + \gamma'$, where*

$$\|\gamma'\|_{H^{s-\frac{1}{2}}} \leq K \left[\|(V, B)\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}}} + \|\eta\|_{H^{s+\frac{1}{2}}} \|(V, B)\|_{C_*^{1+\varepsilon}} \right],$$

and

$$\|\gamma'\|_{H^{s-\frac{1}{2}}} \leq K_0 \left[\|(V, B)\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}}} \right].$$

We will now perform a parilinearization of the system. We will start with the estimate for the first theorem. We introduce as a new unknown

$$U := V + T_\zeta B.$$

Rather than estimating U and ζ in Sobolev spaces, it will be easier to estimate

$$\begin{aligned} U_s &:= \langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B, \\ \zeta_s &:= \langle D_x \rangle^s \zeta. \end{aligned}$$

Proposition 1.3.3. *We have*

$$(1.3.4) \quad \begin{cases} (\partial_t + T_V \cdot \nabla) U_s + T_a \zeta_s = f_1, \\ (\partial_t + T_V \cdot \nabla) \zeta_s = T_\lambda U_s + f_2, \end{cases}$$

where λ is the symbol

$$\lambda(t; x, \xi) := \sqrt{\left(1 + |\nabla \eta(t, x)|^2\right) |\xi|^2 - (\nabla \eta(t, x) \cdot \xi)^2},$$

and where

$$(1.3.5) \quad \begin{aligned} \|f_1\|_{L^2} \leq K & \left\{ \left[\|\nabla \eta\|_{C_*^{\frac{1}{2}}} \|\nabla B\|_{C_*^{-\frac{1}{2}}} + \|\nabla B\|_{L^\infty} + \|V\|_{W^{1,\infty}} \right] \|V\|_{H^s} \right. \\ & + \left[1 + \|a\|_{C_*^{\frac{1}{2}}} \right] \|\zeta\|_{H^{s-\frac{1}{2}}} \\ & + \left[\|V\|_{C_*^{1+\varepsilon}} + \|\nabla B\|_{L^\infty} \right] \|B\|_{H^s} \\ & \left. + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \|a - g\|_{H^{s-\frac{1}{2}}} \right\}, \end{aligned}$$

and

$$(1.3.6) \quad \begin{aligned} \|f_2\|_{H^{-\frac{1}{2}}} \leq K & \left[\left(1 + \|\nabla \eta\|_{C_*^{\frac{1}{2}}}\right) (\|B\|_{H^s} + \|V\|_{H^s}) \right. \\ & \left. + \left(\|B\|_{C_*^{1+\varepsilon}} + \|V\|_{C_*^{1+\varepsilon}}\right) \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^{s+\frac{1}{2}}} \right]. \end{aligned}$$

Proof. The computations are long, however they still mirror the ones of [7]. First we parilinearize the equation

$$(\partial_t + V \cdot \nabla) V + a\zeta = 0.$$

We will prove the identity

$$(\partial_t + T_V \cdot \nabla) V + T_a \zeta + T_\zeta (\partial_t + T_V \cdot \nabla) B = h_1$$

with a remainder h_1 satisfying

$$\|h_1\|_{H^s} \leq K \left[\|\nabla V\|_{L^\infty} \|V\|_{H^s} + \left(1 + \|a\|_{C_*^{\frac{1}{2}}}\right) \|\zeta\|_{H^{s-\frac{1}{2}}} + \|\nabla B\|_{L^\infty} \|V\|_{H^s} \right].$$

First we have $V \cdot \nabla V = T_V \cdot \nabla V + A_1$ with

$$\|A_1\|_{H^s} \leq C \|\nabla V\|_{L^\infty} \|V\|_{H^s}.$$

We also have $(a - g)\zeta = T_{a-g}\zeta + T_\zeta(a - g) + R(\zeta, a - g)$ where

$$\|R(\zeta, a - g)\|_{H^s} \leq C \|\zeta\|_{H^{s-\frac{1}{2}}} \|a\|_{C_*^{\frac{1}{2}}}.$$

We can now replace a by $g + (\partial_t B + V \cdot \nabla B)$ to get

$$T_\zeta a = T_\zeta (\partial_t B + T_V \cdot \nabla B) + T_\zeta (V - T_V) \cdot \nabla B,$$

with

$$\|T_\zeta (V - T_V) \cdot \nabla B\|_{H^s} \leq C \|\nabla \eta\|_{L^\infty} \|\nabla B\|_{L^\infty} \|V\|_{H^s}.$$

We then commute our identity with $\langle D_x \rangle^s$. Using the product estimates of Theorem A.0.4, we have the estimates

$$\begin{aligned} \|[T_V \cdot \nabla, \langle D_x \rangle^s] V\|_{L^2} &\leq C \|V\|_{W^{1,\infty}} \|V\|_{H^s}, \\ \|[T_a, \langle D_x \rangle^s] \zeta\|_{L^2} &\leq C \|a\|_{C_*^{\frac{1}{2}}} \|\zeta\|_{H^{s-\frac{1}{2}}}, \end{aligned}$$

$$\begin{aligned} \|[T_\zeta, \langle D_x \rangle^s] (\partial_t + T_V \cdot \nabla) B\|_{L^2} &\leq C \|\zeta\|_{C_*^{\frac{1}{2}}} \|(\partial_t + T_V \cdot \nabla) B\|_{H^{s-\frac{1}{2}}} \\ &\leq C \|\zeta\|_{C_*^{\frac{1}{2}}} \left[\|(\partial_t + V \cdot \nabla) B\|_{H^{s-\frac{1}{2}}} + \|(V - T_V) \cdot \nabla B\|_{H^{s-\frac{1}{2}}} \right] \\ &\leq C \|\zeta\|_{C_*^{\frac{1}{2}}} \left[\|a - g\|_{H^{s-\frac{1}{2}}} + \|\nabla B\|_{C_*^{-\frac{1}{2}}} \|V\|_{H^s} \right], \end{aligned}$$

$$\|T_\zeta [T_V \cdot \nabla, \langle D_x \rangle^s] B\|_{L^2} \leq C \|\zeta\|_{L^\infty} \|V\|_{W^{1,\infty}} \|B\|_{H^s},$$

$$\|[T_\zeta, \partial_t + T_V \cdot \nabla] \langle D_x \rangle^s B\|_{L^2} \leq C \left[\|\nabla B\|_{L^\infty} + \|\nabla \eta\|_{L^\infty} \|V\|_{C_*^{1+\varepsilon}} \right] \|B\|_{H^s}.$$

Those commutators estimates prove that

$$(\partial_t + T_V \cdot \nabla) (\langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B) + T_a \langle D_x \rangle^s \zeta = f_1,$$

where f_1 satisfies (1.3.5).

We now parilinearize the equation

$$(\partial_t + V \cdot \nabla) \zeta = G(\eta)V + \zeta G(\eta)B + \gamma.$$

We use the parilinearization of the Dirichlet-Neumann (1.2.5) to get

$$(\partial_t + T_V \cdot \nabla) \zeta = T_\lambda U + h_2$$

with

$$h_2 = -(V - T_V) \cdot \nabla \zeta + [T_\zeta, T_\lambda] B + R(\eta)V + \zeta R(\eta)B + (\zeta - T_\zeta) T_\lambda B + \gamma.$$

From Theorem A.0.8, we get

$$\|(V - T_V) \cdot \nabla \zeta\|_{H^{s-\frac{1}{2}}} \leq C \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \|V\|_{H^s}.$$

Theorem A.0.4 and simple estimates on the symbol λ give

$$\begin{aligned} \|[T_\zeta, T_\lambda] B\|_{H^{s-\frac{1}{2}}} &\leq C \left(\mathcal{M}_0^0(\zeta) \mathcal{M}_{\frac{1}{2}}^1(\lambda) + \mathcal{M}_{\frac{1}{2}}^0(\zeta) \mathcal{M}_0^1(\lambda) \right) \|B\|_{H^s} \\ &\leq K \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \|B\|_{H^s}. \end{aligned}$$

Then with the estimates of Proposition 1.2.18 and the maximum principle (1.2.8), we have

$$\|R(\eta)V\|_{H^{s-\frac{1}{2}}} \leq K \left(\|V\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla V\|_{C_*^{1+\varepsilon}} \right),$$

and using paraproduct from Theorem A.0.8, a rough estimate of $\|R(\eta)B\|_{L^\infty}$, and the maximum principle (1.2.8),

$$\begin{aligned} \|\zeta R(\eta)B\|_{H^{s-\frac{1}{2}}} &\leq C \left(\|\zeta\|_{H^{s-\frac{1}{2}}} \|R(\eta)B\|_{L^\infty} + \|\zeta\|_{L^\infty} \|R(\eta)B\|_{H^{s-\frac{1}{2}}} \right) \\ &\leq K \left(\|\eta\|_{H^{s+\frac{1}{2}}} \|B\|_{C_*^{1+\varepsilon}} + \|B\|_{H^s} \right). \end{aligned}$$

At last, we see thanks to the estimates of Theorem A.0.4 that

$$\|(\zeta - T_\zeta) T_\lambda B\|_{H^{s-\frac{1}{2}}} \leq K \|B\|_{C_*^{1+\varepsilon}} \|\zeta\|_{H^{s-\frac{1}{2}}}.$$

Then as in the previous part, commuting the equation with $\langle D_x \rangle^s$ and using (1.3.1) yields (1.3.6). \square

In order to obtain a closed inequality system, we need an estimate of $\|a(t) - g\|_{H^{s-\frac{1}{2}}}$ in terms of the Sobolev norms of the unknowns η, ψ, V, B . This is the object of the following proposition.

Proposition 1.3.4. *The Taylor coefficient satisfies the estimates*

$$\|a - g\|_{H^{s-\frac{1}{2}}} \leq K \left[\|\eta\|_{H^{s+\frac{1}{2}}} \left(1 + \|(V, B)\|_{C_*^{1+\varepsilon}} \right) + \|(V, B)\|_{C_*^{1+\varepsilon}} \|\psi\|_{H^{s+\frac{1}{2}}} \right],$$

and

$$\|a - g\|_{H^{s-\frac{1}{2}}} \leq K_0 \|\eta, \psi, V, B\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}$$

Proof. The pressure is defined by

$$P = - \left(\partial_t \phi + \frac{1}{2} |\nabla_x \phi|^2 + \frac{1}{2} (\partial_y \phi)^2 + gy \right),$$

where ϕ is the harmonic extension of ψ . This means that P satisfies the elliptic equation

$$\Delta_{x,y} P = - |\nabla_{x,y} \phi|^2,$$

with $P = 0$ on the free surface Σ . We change variables using the transformation ρ from (1.2.3), and set

$$\varphi(x, z) = \phi(x, \rho(x, z)), \quad \mathfrak{P}(x, z) = P(x, \rho(x, z)) + g\rho(x, z),$$

with

$$a - g = - \frac{1}{\partial_z \rho} \partial_z \mathfrak{P} \Big|_{z=0}.$$

The elliptic equation on P becomes

$$\begin{aligned} \partial_z^2 \mathfrak{P} + \alpha \Delta_x \mathfrak{P} + \beta \cdot \nabla_x \partial_z \mathfrak{P} - \gamma \partial_z \mathfrak{P} &= -\alpha |\Lambda^2 \varphi|^2 && \text{for } z < 0, \\ \mathfrak{P} &= g\eta && \text{on } z = 0, \end{aligned}$$

where $\Lambda = (\Lambda_1, \Lambda_2)$ is defined in (1.2.3).

We first need to study the right-hand term of the equation. Since ϕ is harmonic, we recover from Proposition 1.2.12 and the variational estimate of $\|\nabla_{x,y} z \varphi\|_{X^{-\frac{1}{2}}}$ the inequality

$$\|\nabla_{x,y} z \varphi\|_{X^{s-\frac{1}{2}}} \leq K \left[\|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^{s+\frac{1}{2}}} \right].$$

Now using the fact that $(\Lambda_1^2 + \Lambda_2^2)\varphi = 0$, we can recover estimates on $\partial_z^2 \varphi$ from estimates on $\nabla_x \nabla_{x,y} z \varphi$, so that

$$\|\Lambda^2 \varphi\|_{X^{s-\frac{3}{2}}} \leq K \left[\|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^{s+\frac{1}{2}}} \right].$$

At last, using the paraproduct rules, and the estimates on α from Lemma 1.2.11, we find

$$\begin{aligned} \left\| -\alpha |\Lambda^2 \varphi|^2 \right\|_{Y^{s-\frac{1}{2}}} &\leq K \|\Lambda^2 \varphi\|_{L^\infty} \left[\|\eta\|_{H^{s+\frac{1}{2}}} \|\Lambda^2 \varphi\|_{L^\infty([-1,0]; C_*^{-1})} + \|\Lambda^2 \varphi\|_{X^{s-\frac{3}{2}}} \right] \\ &\leq K \|(V, B)\|_{C_*^{1+\varepsilon}} \left[\|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^{s+\frac{1}{2}}} \right], \end{aligned}$$

where $\|\Lambda^2 \varphi\|_{L^\infty([-1,0]; C_*^{-1})}$ has been estimated from $\|\Lambda \varphi\|_{L^\infty}$ using $(\Lambda_1^2 + \Lambda_2^2)\varphi = 0$ once again.

For the version with a reference Sobolev index, we recall from the proof of Proposition 1.3.1 that if θ_i is the harmonic extension of V_i , and if τ_i is its straightening by the diffeomorphism ρ to the strip, then up to a harmless restriction of the interval J close to the boundary we have

$$\|\nabla_{x,y} z (\tau_i - \Lambda_i \varphi)\|_{X^{s-\frac{1}{2}}(J)} \leq K_0 \|(\psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}}}.$$

Also, we have from (1.2.7)

$$\|\nabla_{x,y} z \tau_i\|_{X^{s-1}} \leq K_0 \|V\|_{H^s}.$$

Combining those two results and doing the same for B and $\partial_z \varphi$ gives

$$\|\Lambda^2 \varphi\|_{X^{s-1}} \leq K_0 \|\eta, \psi, V, B\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}.$$

At last, using paraproduct estimates, we gain

$$(1.3.7) \quad \left\| -\alpha |\Lambda^2 \varphi|^2 \right\|_{Y^{s-\frac{1}{2}}} \leq K_0 \|\eta, \psi, V, B\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}.$$

We then take from [7] the estimate

$$\begin{aligned} \|\nabla_{x,y} z \blacklozenge\|_{X^{-\frac{1}{2}}} &\leq K \left[\|\eta\|_{H^{\frac{1}{2}}} + \left\| |\nabla \varphi|^2 \right\|_{X^{\frac{1}{2}}} \right] \\ &\leq K \left[1 + \|(V, B)\|_{C_*^{1+\varepsilon}} \right] \left[\|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^{s+\frac{1}{2}}} \right]. \end{aligned}$$

A last application of the elliptic regularity of Proposition 1.2.12 gives the estimate

$$\begin{aligned} \|\partial_z \blacklozenge\|_{X^{s-\frac{1}{2}}} &\leq K \left[\|\eta\|_{H^{s+\frac{1}{2}}} (1 + \|\partial_z \blacklozenge\|_{L^\infty}) + \left\| -\alpha |\Lambda^2 \varphi|^2 \right\|_{Y^{s-\frac{1}{2}}} + \|\nabla_{x,y} z \blacklozenge\|_{X^{-\frac{1}{2}}} \right] \\ &\leq K \left[\|\eta\|_{H^{s+\frac{1}{2}}} \left(1 + \|(V, B)\|_{C_*^{1+\varepsilon}} \right) + \|(V, B)\|_{C_*^{1+\varepsilon}} \|\psi\|_{H^{s+\frac{1}{2}}} \right], \end{aligned}$$

and a last use of the paraproduct gives the first result. The second one follows along the same lines, using (1.2.7) and (1.3.7) instead. \square

We can now perform a symmetrization of the system as follows

Proposition 1.3.5. *We introduce the symbols*

$$\gamma := \sqrt{a\lambda}, \quad q := \sqrt{\frac{a}{\lambda}},$$

and the new variable

$$\theta_s := T_q \zeta_s.$$

Then we have the equations

$$(1.3.8) \quad \begin{cases} \partial_t U_s + T_V \cdot \nabla U_s + T_\gamma \theta_s = F_1, \\ \partial_t \theta_s + T_V \cdot \nabla \theta_s - T_\gamma U_s = F_2, \end{cases}$$

where the source terms F_1 and F_2 satisfy

$$\begin{aligned} \|F_1\|_{L^2} \leq K & \left[\left(1 + \|\nabla \eta\|_{C_*^{\frac{1}{2}}}\right) \left(1 + \|(V, B)\|_{C_*^{1+\varepsilon}}\right) \|(V, B)\|_{H^s} \right. \\ & + \left. \left(1 + \|a\|_{C_*^{\frac{1}{2}}} + \left(1 + \|(V, B)\|_{C_*^{1+\varepsilon}}\right) \left(1 + \|\nabla \eta\|_{C_*^{\frac{1}{2}}}\right)\right) \left(1 + \|\zeta\|_{H^{s-\frac{1}{2}}}\right) \right. \\ & \left. + \left(1 + \|(V, B)\|_{C_*^{1+\varepsilon}}\right) \left(1 + \|\nabla \eta\|_{C_*^{\frac{1}{2}}}\right) \|\psi\|_{H^{s+\frac{1}{2}}} \right], \end{aligned}$$

and

$$\begin{aligned} \|F_2\|_{L^2} \leq K & \left[\left(1 + \|\nabla \eta\|_{C_*^{\frac{1}{2}}}\right) \|(V, B)\|_{H^s} \right. \\ & + \left. \left(1 + \|(V, B)\|_{C_*^{1+\varepsilon}} + \|\partial_t a + V \cdot \nabla a\|_{L^\infty}\right) \left(1 + \|\zeta\|_{H^{s-\frac{1}{2}}}\right) \right. \\ & + \|\psi\|_{H^{s+\frac{1}{2}}} \\ & \left. + \left(1 + \|a\|_{C_*^{\frac{1}{2}}} + \|\nabla \eta\|_{C_*^{\frac{1}{2}}}\right) \|U_s\|_{L^2} \right]. \end{aligned}$$

Proof. We have from the preceding system (1.3.4) that (1.3.8) is satisfied for

$$F_1 = f_1 + (T_\gamma T_q - T_a) \zeta_s,$$

$$F_2 = T_q f_2 + (T_q T_\lambda - T_\gamma) U_s - [T_q, \partial_t + T_V \cdot \nabla] \zeta_s.$$

Thanks to Lemma A.0.12, we have

$$\|[T_q, \partial_t + T_V \cdot \nabla] \zeta_s\|_{L^2} \leq C \left[\mathcal{M}_0^{-\frac{1}{2}}(q) \|V\|_{C_*^{1+\varepsilon}} + \mathcal{M}_0^{-\frac{1}{2}}(\partial_t q + V \cdot \nabla q) \right] \|\zeta_s\|_{H^{-\frac{1}{2}}}.$$

It can be computed that

$$\mathcal{M}_0^{-\frac{1}{2}}(q) \leq K,$$

and that

$$\mathcal{M}_0^{-\frac{1}{2}}(\partial_t q + V \cdot \nabla q) \leq K (1 + \|\partial_t a + V \cdot \nabla a\|_{L^\infty} + \|\partial_t \nabla \eta + V \cdot \nabla \nabla \eta\|_{L^\infty}).$$

A differentiation of the identity

$$(\partial_t + V \cdot \nabla) \eta = B$$

gives the estimate

$$\|(\partial_t + V \cdot \nabla) \partial_{x_i} \eta\|_{L^\infty} \leq \|\nabla B\|_{L^\infty} + \|\nabla V\|_{L^\infty} \|\nabla \eta\|_{L^\infty},$$

so that

$$\|[T_q, \partial_t + T_V \cdot \nabla] \zeta_s\|_{L^2} \leq K \left(\|(V, B)\|_{C_*^{1+\varepsilon}} + \|\partial_t a + V \cdot \nabla a\|_{L^\infty} \right) \|\zeta_s\|_{H^{-\frac{1}{2}}}.$$

The estimates of the other terms give respectively

$$\begin{aligned} \|(T_\gamma T_q - T_a) \zeta_s\|_{L^2} &\leq C \left[\mathcal{M}_{\frac{1}{2}}^{-\frac{1}{2}}(\gamma) \mathcal{M}_0^{-\frac{1}{2}}(q) + \mathcal{M}_0^{\frac{1}{2}}(\gamma) \mathcal{M}_{\frac{1}{2}}^{-\frac{1}{2}}(q) \right] \|\zeta_s\|_{H^{-\frac{1}{2}}} \\ &\leq K \left[1 + \|a\|_{C_*^{\frac{1}{2}}} + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \right] \|\zeta_s\|_{H^{-\frac{1}{2}}}, \end{aligned}$$

$$\begin{aligned} \|(T_q T_\lambda - T_\gamma) U_s\|_{L^2} &\leq C \left[\mathcal{M}_{\frac{1}{2}}^{-\frac{1}{2}}(q) \mathcal{M}_0^1(\lambda) + \mathcal{M}_0^{-\frac{1}{2}}(q) \mathcal{M}_{\frac{1}{2}}^1(\lambda) \right] \|U_s\|_{L^2} \\ &\leq K \left[1 + \|a\|_{C_*^{\frac{1}{2}}} + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \right] \|U_s\|_{L^2}, \end{aligned}$$

and lastly

$$\begin{aligned} \|T_q f_2\|_{L^2} &\leq C \mathcal{M}_0^{-\frac{1}{2}}(q) \|f_2\|_{H^{-\frac{1}{2}}} \\ &\leq K \|f_2\|_{H^{-\frac{1}{2}}}. \end{aligned}$$

This, together with the previous estimates, give the expected result. \square

The analogous result with the reference Sobolev index is

Proposition 1.3.6. *The source terms F_1 and F_2 from the preceding proposition satisfy*

$$\begin{aligned} \|F_1\|_{L^2} &\leq K_0 \left[\left(1 + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} + \|(V, B)\|_{C_*^{1+\varepsilon}} \right) \|(V, B)\|_{H^s} \right. \\ &\quad + \left(1 + \|a\|_{C_*^{\frac{1}{2}}} + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \right) \left(1 + \|\zeta\|_{H^{s-\frac{1}{2}}} \right) \\ &\quad \left. + \left(1 + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \right) \|\psi\|_{H^{s+\frac{1}{2}}} \right], \end{aligned}$$

and

$$\begin{aligned} \|F_2\|_{L^2} &\leq K_0 \left[\left(1 + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \right) \|(V, B)\|_{H^s} \right. \\ &\quad + \left(1 + \|(V, B)\|_{C_*^{1+\varepsilon}} + \|\partial_t a + V \cdot \nabla a\|_{L^\infty} \right) \left(1 + \|\zeta\|_{H^{s-\frac{1}{2}}} \right) \\ &\quad + \|\psi\|_{H^{s+\frac{1}{2}}} \\ &\quad \left. + \left(1 + \|a\|_{C_*^{\frac{1}{2}}} + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \right) \|U_s\|_{L^2} \right]. \end{aligned}$$

1.4 Estimates of the original unknowns

In order to obtain a closed system of energy estimates, we need a control of the norms of η, ψ, V, B , in terms of lower order norms and of norms of U_s, θ_s . The formers will then be studied using transport equations on the various unknowns, and for the latters we will use the parilinearized system of Proposition 1.3.5.

Proposition 1.4.1. *There holds*

$$\|\eta\|_{H^{s+\frac{1}{2}}} \leq K (\|\theta_s\|_{L^2} + \|\zeta_s\|_{H^{-1}}),$$

$$\begin{aligned} \|(V, B)\|_{H^s} \leq K \left[1 + \|U_s\|_{L^2} + \left(1 + \|\eta\|_{C_*^{\frac{3}{2}}} \right) \|(V, B)\|_{H^{s-\frac{1}{2}}} \right. \\ \left. + \|(V, B)\|_{C_*^{1+\varepsilon}} (\|\theta_s\|_{L^2} + \|\zeta_s\|_{H^{-1}}) \right], \end{aligned}$$

and

$$\begin{aligned} \|\psi\|_{H^{s+\frac{1}{2}}} \leq K \left[1 + \|U_s\|_{L^2} + \left(1 + \|\eta\|_{C_*^{\frac{3}{2}}} \right) \|(V, B)\|_{H^{s-\frac{1}{2}}} \right. \\ \left. + \|(V, B)\|_{C_*^{1+\varepsilon}} (\|\theta_s\|_{L^2} + \|\zeta_s\|_{H^{-1}}) + \|\psi\|_{L^2} \right]. \end{aligned}$$

Proof. We start with the estimate on η . First we remark that

$$\begin{aligned} \|\eta\|_{H^{s+\frac{1}{2}}} &\leq \|\eta\|_{L^2} + \|\nabla\eta\|_{H^{s-\frac{1}{2}}} \\ &\leq K \left(1 + \|\zeta_s\|_{H^{-\frac{1}{2}}} \right), \end{aligned}$$

since $\zeta_s = \langle D_x \rangle^s \nabla \eta$. We then construct and use a parametrix to go back from $\theta_s = T_q \zeta_s$ to ζ_s . If our ε is small enough, typically $0 < \varepsilon < s - d/2 - 1$, we choose N an integer such that $(N+1)\varepsilon > 1/2$, and we take $R = I - T_{1/q} T_q$, keeping in mind that $q = \sqrt{a/\lambda}$. Then

$$\begin{aligned} \zeta_s &= T_{1/q} T_q \zeta_s + R \zeta_s \\ &= (I + R + \dots + R^N) T_{1/q} T_q \zeta_s + R^{N+1} \zeta_s. \end{aligned}$$

Then from the composition formula for paradifferential operators in Theorem A.0.4, we have

$$\begin{aligned} \|R\|_{H^\mu \rightarrow H^{\mu+\varepsilon}} &\leq C \left(\mathcal{M}_\varepsilon^{-\frac{1}{2}}(q) \mathcal{M}_0^{\frac{1}{2}}(1/q) + \mathcal{M}_0^{-\frac{1}{2}}(q) \mathcal{M}_\varepsilon^{\frac{1}{2}}(1/q) \right) \\ &\leq K, \end{aligned}$$

and from the definition of q , we see that

$$\|T_{1/q}\|_{L^2 \rightarrow H^{-\frac{1}{2}}} \leq \mathcal{M}_0^{\frac{1}{2}}(1/q) \leq K.$$

Those estimations put together give

$$\|\eta\|_{H^{s+\frac{1}{2}}} \leq K (\|\theta_s\|_{L^2} + \|\zeta_s\|_{H^{-1}}).$$

To simplify the equations, we worked with the unknown $U_s = \langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B$. We first show how to go back from this to $U = V + T_\zeta B$. We have

$$(1.4.1) \quad \langle D_x \rangle^s U = U_s + [\langle D_x \rangle^s, T_\zeta] B,$$

and Theorem A.0.4 gives

$$\|[\langle D_x \rangle^s, T_\zeta] B\|_{L^2} \leq \|\zeta\|_{C_*^{\frac{1}{2}}} \|B\|_{H^{s-\frac{1}{2}}}.$$

Putting those two identities together gives

$$\|U\|_{H^s} \leq \|U_s\|_{L^2} + \|\nabla\eta\|_{C_*^{\frac{1}{2}}} \|B\|_{H^{s-\frac{1}{2}}}.$$

Then to get back to B from this, we take the divergence of U and use Proposition 1.3.2 to link V and B and the parilinearization of the Dirichlet-Neumann (1.2.5), so that

$$\begin{aligned} \operatorname{div} U &= \operatorname{div} T_\zeta B \\ &= G(\eta)B + \gamma' + T_{\operatorname{div} \zeta} B + T_\zeta \cdot \nabla B \\ &= T_p B + R(\eta)B + T_{\operatorname{div} \zeta} B + \gamma', \end{aligned}$$

where

$$p := -\lambda + i\zeta \cdot \xi.$$

Now p is a symbol of order 1 and $1/p$ of order -1 , with

$$(1.4.2) \quad \mathcal{M}_r^1(p) + \mathcal{M}_r^{-1}(1/p) \leq K \|\nabla\eta\|_{C_*^r}.$$

Now we use a new parametrix from $T_q B$ to B , giving

$$(1.4.3) \quad \begin{aligned} B &= T_{1/p} T_p B + (I - T_{1/p} T_p) B \\ &= T_{1/p} \operatorname{div} U - T_{1/p} \gamma' + T_{1/p} (-T_{\operatorname{div} \zeta} - R(\eta)) B + (I - T_{1/p} T_p) B. \end{aligned}$$

This gives, using (1.4.2), Proposition 1.3.2, Proposition 1.2.18, and the maximum principle 1.2.8,

$$\begin{aligned} \|B\|_{H^s} &\leq K \left(\|U\|_{H^s} + \|\gamma'\|_{H^{s-1}} + \|\nabla\eta\|_{C_*^{\frac{1}{2}}} \|B\|_{H^{s-\frac{1}{2}}} + \|R(\eta)B\|_{H^{s-1}} \right) \\ &\leq K \left(\|U_s\|_{L^2} + \left(1 + \|\eta\|_{C_*^{\frac{3}{2}}} \right) \|(V, B)\|_{H^{s-\frac{1}{2}}} + \|(V, B)\|_{C_*^{1+\varepsilon}} \|\eta\|_{H^{s+\frac{1}{2}}} \right), \end{aligned}$$

which combined with the previous estimate on η gives the expected result. Using the relation

$$U = V + T_\zeta B$$

gives the same estimation on V .

At last, we have the identity

$$\nabla\psi = V + B\nabla\eta,$$

and the quantities in the right side have all been estimated, so that a tame estimate on $B\nabla\eta$ concludes the proof. \square

For the version with a reference Sobolev norm, we have a simpler proposition.

Proposition 1.4.2. *There holds*

$$\|\eta\|_{H^{s+\frac{1}{2}}} \leq K_0 (\|\theta_s\|_{L^2} + \|\zeta_s\|_{H^{-1}}),$$

$$\|(V, B)\|_{H^s} \leq K_0 \left[\|U_s\|_{L^2} + \|(V, B)\|_{H^{s-\frac{1}{2}}} \right],$$

and

$$\|\psi\|_{H^{s+\frac{1}{2}}} \leq K_0 \left[\|U_s\|_{L^2} + \|(V, B)\|_{H^{s-\frac{1}{2}}} \right].$$

Proof. The first estimate is a simple consequence of the previous proposition and of our hypothesis that $K \leq K_0$.

For the second one, we combine (1.4.1) and (1.4.3) to get

$$\begin{aligned} B &= T_{\frac{1}{p}} [\operatorname{div}\langle D_x \rangle^{-s} U_s + \gamma'] + \left[T_{\frac{1}{p}} (\operatorname{div}\langle D_x \rangle^{-s} [\langle D_x \rangle^s, T_\zeta] - R(\eta) - T_{\operatorname{div}\zeta}) + (I - T_{\frac{1}{p}} T_p) \right] B, \\ &:= M + RB. \end{aligned}$$

As before, this gives

$$B = (I + R + \dots + R^N) M + R^{N+1} B,$$

where again $(N+1)\varepsilon > 1/2$ with $0 < \varepsilon < s_0 - 1/2 - d/2$. Then we see using Proposition 1.3.2 that

$$\|M\|_{H^s} \leq K_0 \left[\|U_s\|_{L^2} + \|(V, B)\|_{H^{s-\frac{1}{2}}} \right].$$

We see using (1.4.2) and Proposition 1.2.18 that R is of order $-\varepsilon$, with

$$\|R\|_{H^\sigma \rightarrow H^{\sigma+\varepsilon}} \leq K_0$$

when $s - \frac{1}{2} \leq \sigma \leq s$. This gives the estimate on B , and the estimates on V and ψ are deduced from it as in the previous proposition. \square

1.5 Energy estimates

We start with a standard energy estimate on the now symmetric quasilinear system (1.3.8).

Proposition 1.5.1. *The following estimates hold punctually in time*

$$(1.5.1) \quad \begin{aligned} \frac{d}{dt} \|(U_s, \theta_s)\|_{L^2} &\leq K \left(\|a\|_{C_*^{\frac{1}{2}}} + \|(\partial_t + V \cdot \nabla) a\|_{L^\infty} + Q \left(\|\nabla \eta\|_{C_*^{\frac{1}{2}}}, \|(V, B)\|_{C_*^{1+\varepsilon}} \right) \right) \\ &\quad \times \left(\|(U_s, \theta_s)\|_{L^2} + \|\psi\|_{L^2} + \|\zeta_s\|_{H^{-1}} + \|(V, B)\|_{H^{s-\frac{1}{2}}} \right), \end{aligned}$$

where Q is an explicit polynomial of degree 3, and

$$(1.5.2) \quad \begin{aligned} \frac{d}{dt} \|(U_s, \theta_s)\|_{L^2} &\leq K_0 \left(\|a\|_{C_*^{\frac{1}{2}}} + \|(\partial_t + V \cdot \nabla) a\|_{L^\infty} + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} + \|(V, B)\|_{C_*^{1+\varepsilon}} \right) \\ &\quad \times \left(\|(U_s, \theta_s)\|_{L^2} + \|\psi\|_{L^2} + \|\zeta_s\|_{H^{-1}} + \|(V, B)\|_{H^{s-\frac{1}{2}}} \right). \end{aligned}$$

Proof. Multiplication of the equations by U_s and θ_s respectively, followed by integration in space gives

$$\frac{d}{dt} [\|U_s\|_{L^2} + \|\theta_s\|_{L^2}] \leq A + B + C,$$

with

$$\begin{aligned} A &:= \langle T_V \cdot \nabla U_s, U_s \rangle + \langle T_V \cdot \nabla \theta_s, \theta_s \rangle, \\ B &:= \langle T_\gamma \theta_s, U_s \rangle - \langle T_\gamma U_s, \theta_s \rangle, \\ C &:= \langle F_1, U_s \rangle + \langle F_2, \theta_s \rangle. \end{aligned}$$

Now using Theorem A.0.4, we see that

$$\|(T_V \cdot \nabla)^* + (T_V \cdot \nabla)\|_{L^2 \rightarrow L^2} \leq C \|V\|_{W^{1,\infty}},$$

and

$$\|T_\gamma - T_\gamma^*\|_{L^2 \rightarrow L^2} \leq C \mathcal{M}_{\frac{1}{2}}(\gamma),$$

and the estimates on F_1 and F_2 of Proposition 1.3.5 and Proposition 1.3.6 complete the proof. \square

The next proposition exploits the transport equations available on the remaining variables to close the system of estimates.

Proposition 1.5.2. *With the same Q as in the previous proposition, there holds, for $A = \|\psi\|_{L^2}$, $A = \|\zeta_s\|_{H^{-1}}$ or $A = \|(V, B)\|_{H^{s-\frac{1}{2}}}$,*

$$\begin{aligned} \frac{d}{dt} A \leq & K \left(\|a\|_{C_*^{\frac{1}{2}}} + \|(\partial_t + V \cdot \nabla) a\|_{L^\infty} + Q \left(\|\nabla \eta\|_{C_*^{\frac{1}{2}}}, \|(V, B)\|_{C_*^{1+\varepsilon}} \right) \right) \\ & \times \left(\|(U_s, \theta_s)\|_{L^2} + \|\psi\|_{L^2} + \|\zeta_s\|_{H^{-1}} + \|(V, B)\|_{H^{s-\frac{1}{2}}} \right), \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} A \leq & K_0 \left(\|a\|_{C_*^{\frac{1}{2}}} + \|(\partial_t + V \cdot \nabla) a\|_{L^\infty} + \|\nabla \eta\|_{C_*^{\frac{1}{2}}} + \|(V, B)\|_{C_*^{1+\varepsilon}} \right) \\ & \times \left(\|(U_s, \theta_s)\|_{L^2} + \|\psi\|_{L^2} + \|\zeta_s\|_{H^{-1}} + \|(V, B)\|_{H^{s-\frac{1}{2}}} \right). \end{aligned}$$

We will need the following lemma on transport equations

Lemma 1.5.3. *If $\sigma > 0$, and if u solves*

$$(\partial_t + V \cdot \nabla) u = f,$$

Then

$$\frac{d}{dt} \|u\|_{L^2} \lesssim \|V\|_{W^{1,\infty}} \|u\|_{L^2} + \|f\|_{L^2},$$

and

$$\frac{d}{dt} \|u\|_{H^\sigma} \lesssim \|V\|_{W^{1,\infty}} \|u\|_{H^\sigma} + \|(V - T_V) \cdot \nabla u\|_{H^\sigma} + \|(V - T_V) \cdot \langle D_x \rangle^\sigma \nabla u\|_{L^2} + \|f\|_{H^\sigma}.$$

Proof. The L^2 energy estimate is standard, and the Sobolev estimate follows from commuting the equation with $\langle D_x \rangle^\sigma$, using the L^2 estimate, and observing that

$$\begin{aligned} \|[\langle D_x \rangle^\sigma, V] \nabla u\|_{L^2} & \leq \|[\langle D_x \rangle^\sigma, T_V] \cdot \nabla u\|_{L^2} + \|[\langle D_x \rangle^\sigma, V - T_V] \cdot \nabla u\|_{L^2} \\ & \leq \|V\|_{W^{1,\infty}} \|u\|_{H^\sigma} + \|(V - T_V) \cdot \nabla u\|_{H^\sigma} + \|(V - T_V) \cdot \langle D_x \rangle^\sigma \nabla u\|_{L^2}. \end{aligned}$$

\square

Proof of Proposition 1.5.2. First, from the equation on ψ and the definitions of V and B we have the transport equation

$$(\partial_t + V \cdot \nabla) \psi = -g\eta + \frac{1}{2}V^2 + \frac{1}{2}B^2.$$

The previous L^2 estimate and a simple tame estimate on the L^2 norms of V^2 and B^2 give the estimate on ψ .

We then recall equation (1.3.3),

$$(\partial_t + V \cdot \nabla) \zeta = G(\eta)V + \zeta G(\eta)B + \gamma,$$

and use the previous Sobolev estimate with $\sigma = s - 1$ to get

$$\begin{aligned} \frac{d}{dt} \|\zeta\|_{H^{s-1}} &\lesssim \|V\|_{W^{1,\infty}} \|\zeta\|_{H^{s-1}} + \|(V - T_V) \cdot \nabla \zeta\|_{H^{s-1}} + \|(V - T_V) \cdot \langle D_x \rangle^{s-1} \nabla \zeta\|_{L^2} \\ &\quad + \|G(\eta)V\|_{H^{s-1}} + \|\zeta\|_{L^\infty} \|G(\eta)B\|_{H^{s-1}} + \|\zeta\|_{H^{s-1}} \|G(\eta)B\|_{L^\infty} \\ &\quad + \|\gamma\|_{H^{s-1}}. \end{aligned}$$

Using the parproduct rules from Theorem A.0.8 gives

$$\begin{aligned} \|(V - T_V) \cdot \nabla \zeta\|_{H^{s-1}} &\leq \|\nabla \zeta\|_{C_*^{-\frac{1}{2}}} \|V\|_{H^{s-\frac{1}{2}}} \\ &\leq \|\nabla \eta\|_{C_*^{\frac{1}{2}}} \|V\|_{H^{s-\frac{1}{2}}} \end{aligned}$$

and

$$\begin{aligned} \|(V - T_V) \cdot \langle D_x \rangle^{s-1} \nabla \zeta\|_{L^2} &\leq \|\langle D_x \rangle^{s-1} \nabla \zeta\|_{C_*^{1-s+\varepsilon}} \|V\|_{H^s} \\ &\leq \|\nabla \eta\|_{C_*^\varepsilon} \|V\|_{H^s}. \end{aligned}$$

To estimate the Dirichlet-Neumann operators, we use Proposition 1.2.17, and γ is estimated using (1.3.1) or (1.3.1).

Recall also that B follows equation (1.3.1),

$$(\partial_t + V \cdot \nabla) B = a - g.$$

The Sobolev estimate gives

$$\begin{aligned} \frac{d}{dt} \|B\|_{H^{s-\frac{1}{2}}} &\lesssim \|V\|_{W^{1,\infty}} \|B\|_{H^{s-\frac{1}{2}}} + \|(V - T_V) \cdot \nabla B\|_{H^{s-\frac{1}{2}}} \\ &\quad + \|(V - T_V) \cdot \langle D_x \rangle^{s-\frac{1}{2}} \nabla B\|_{L^2} + \|a - g\|_{H^{s-\frac{1}{2}}}, \end{aligned}$$

and as for ζ , we have

$$\|(V - T_V) \cdot \nabla B\|_{H^{s-\frac{1}{2}}} \leq \|\nabla B\|_{L^\infty} \|V\|_{H^{s-\frac{1}{2}}}$$

and

$$\begin{aligned} \|(V - T_V) \cdot \langle D_x \rangle^{s-\frac{1}{2}} \nabla B\|_{L^2} &\leq \|\langle D_x \rangle^{s-\frac{1}{2}} \nabla B\|_{C_*^{\frac{1}{2}-s+\varepsilon}} \|V\|_{H^{s-\frac{1}{2}}} \\ &\leq \|B\|_{C_*^{1+\varepsilon}} \|V\|_{H^{s-\frac{1}{2}}}. \end{aligned}$$

$a - g$ is estimated using Proposition 1.3.4.

At last V follows equation (1.3.2),

$$(\partial_t + V \cdot \nabla) V = -a\zeta,$$

so we use the same bound for the commutator that we used for B and remark that

$$\|a\zeta\|_{H^{s-\frac{1}{2}}} \lesssim \|a - g\|_{H^{s-\frac{1}{2}}} \|\zeta\|_{L^\infty} + (g + \|a\|_{L^\infty}) \|\zeta\|_{H^{s-\frac{1}{2}}}$$

from which the proposition follows. \square

1.6 Proof of the main results

The main theorems will follow as usual from the expression of the energy estimates. The one with only Hölder components is

Proposition 1.6.1. *Let $d \geq 1$, $s > 1 + d/2$, and $\varepsilon > 0$. Let (η, ψ) be a solution of the water-waves system (1.1) on $[0, T]$ from theorem 1.1.1, and define V, B, h, a, c in the same way. Then*

$$(1.6.1) \quad \sup_{0 \leq t \leq T} \|(\eta, \psi, V, B)(t)\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d)} \\ \leq \mathcal{F} \left(T, \sup_{0 \leq t \leq T} \frac{1}{h}, \sup_{0 \leq t \leq T} \frac{1}{c}, \sup_{0 \leq t \leq T} \|\eta(t)\|_{W^{1+\varepsilon, \infty}(\mathbf{R}^d)}, \sup_{0 \leq t \leq T} \|(V, B)(t)\|_{W^{\varepsilon, \infty}(\mathbf{R}^d)}, \right. \\ \left. \sup_{0 \leq t \leq T} \|a\|_{W^{\varepsilon, \infty}(\mathbf{R}^d)}, \int_0^T \|(\partial_t a + V \cdot \nabla a)(t)\|_{L^\infty(\mathbf{R}^d)} dt, \int_0^T \|a(t)\|_{W^{\frac{1}{2}, \infty}(\mathbf{R}^d)} dt, \right. \\ \left. \int_0^T \|\nabla \eta(t)\|_{W^{\frac{1}{2}, \infty}(\mathbf{R}^d)}^3 dt, \int_0^T \|(V, B)(t)\|_{W^{1+\varepsilon, \infty}(\mathbf{R}^d)}^3 dt \right),$$

with \mathcal{F} a positive, strictly increasing function of each of its variable, depending only on d, s, ε , the bottom Γ , and on $\|\eta_0, \psi_0, V_0, B_0\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d)}$.

Proof. We see from propositions 1.5.1 and 1.5.2 that if

$$\mathcal{A} := \|(U_s, \theta_s)\|_{L^2} + \|\psi\|_{L^2} + \|\zeta_s\|_{H^{-1}} + \|(V, B)\|_{H^{s-\frac{1}{2}}},$$

and if

$$\mathcal{B} := \left(\|a\|_{C_*^{\frac{1}{2}}} + \|(\partial_t + V \cdot \nabla) a\|_{L^\infty} + Q \left(\|\nabla \eta\|_{C_*^{\frac{1}{2}}}, \|(V, B)\|_{C_*^{1+\varepsilon}} \right) \right),$$

with Q the polynomial of degree 3 of those propositions, then for all $t \in [0, T]$,

$$\frac{d}{dt} \mathcal{A}(t) \leq K(t) \times \mathcal{B}(t) \times \mathcal{A}(t) \leq \mathcal{K} \times \mathcal{B}(t) \times \mathcal{A}(t),$$

where

$$\mathcal{K} = \sup_{0 \leq t \leq T} K(t).$$

Using Grönwall's lemma gives

$$\sup_{0 \leq t \leq T} \mathcal{A}(t) \leq \mathcal{K} \mathcal{A}(0) \exp \left(\int_0^T \mathcal{B}(t) dt \right).$$

Using Hölder inequality to bound

$$\int_0^T Q \left(\|\nabla \eta\|_{C_*^{\frac{1}{2}}}, \|(V, B)\|_{C_*^{1+\varepsilon}} \right) dt$$

gives

$$\sup_{0 \leq t \leq T} \mathcal{A}(t) \leq \mathcal{F} \left(T, \sup_{0 \leq t \leq T} \frac{1}{h}, \sup_{0 \leq t \leq T} \frac{1}{c}, \sup_{0 \leq t \leq T} \|\eta(t)\|_{W^{1+\varepsilon, \infty}(\mathbf{R}^d)}, \sup_{0 \leq t \leq T} \|(V, B)(t)\|_{W^{\varepsilon, \infty}(\mathbf{R}^d)}, \right. \\ \left. \sup_{0 \leq t \leq T} \|a\|_{W^{\varepsilon, \infty}(\mathbf{R}^d)}, \int_0^T \|(\partial_t a + V \cdot \nabla a)(t)\|_{L^\infty(\mathbf{R}^d)} dt, \int_0^T \|a(t)\|_{W^{\frac{1}{2}, \infty}(\mathbf{R}^d)} dt, \right. \\ \left. \int_0^T \|\nabla \eta(t)\|_{W^{\frac{1}{2}, \infty}(\mathbf{R}^d)}^3 dt, \int_0^T \|(V, B)(t)\|_{W^{1+\varepsilon, \infty}(\mathbf{R}^d)}^3 dt \right),$$

with \mathcal{F} a positive, strictly increasing function of each of its variable, depending only on d , s , ε , the bottom Γ , and on $\mathcal{A}(0)$, which is easily seen to be controlled by the initial data $\|\eta_0, \psi_0, V_0, B_0\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}$.

The water-waves system 1.1 is Hamiltonian, and the Hamiltonian (1.1) controls the L^2 norm of η , so that

$$\sup_{0 \leq t \leq T} \|\eta(t)\|_{L^2} \leq \|\eta_0\|_{H^{s+\frac{1}{2}}}$$

Now to finish the proof of the estimate, we remark that for any $\nu > 0$ there exists a constant $C_\nu > 0$ such that

$$\|(\eta, V, B)\|_{C_*^{\frac{3}{2}} \times C_*^{1+\varepsilon} \times C_*^{1+\varepsilon}} \leq C_\nu \|(\eta, V, B)\|_{L^2} + \nu \|(\eta, V, B)\|_{H^{s+\frac{1}{2}} \times H^s \times H^s},$$

which, combined with Proposition 1.4.1 gives that $\|(\eta, V, B)\|_{H^{s+\frac{1}{2}} \times H^s \times H^s}$ is finite as soon as the right side of (1.6.1) is bounded. \square

The energy estimate with reference Sobolev index is proved along the same lines, using the corresponding estimates.

Proposition 1.6.2. *Let $d \geq 1$, $s > 1 + d/2$, $s > s_0 > 1/2 + d/2$ and $s_0 - 1/2 - d/2 > \varepsilon > 0$. Let (η, ψ) be a solution of the water-waves system (1.1) on $[0, T]$ from theorem 1.1.1, and define V, B, h, a, c in the same way. Then*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|(\eta, \psi, V, B)(t)\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d)} \\ & \leq \mathcal{F} \left(T, \sup_{0 \leq t \leq T} \frac{1}{h}, \sup_{0 \leq t \leq T} \frac{1}{c}, \sup_{0 \leq t \leq T} \|(\eta, \psi, V, B)(t)\|_{H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0+\frac{1}{2}}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d) \times H^{s_0}(\mathbf{R}^d)}, \right. \\ & \quad \left. \sup_{0 \leq t \leq T} \|a\|_{W^{\varepsilon, \infty}(\mathbf{R}^d)}, \int_0^T \|(\partial_t a + V \cdot \nabla a)(t)\|_{L^\infty(\mathbf{R}^d)} dt, \int_0^T \|a(t)\|_{W^{\frac{1}{2}, \infty}(\mathbf{R}^d)} dt, \right. \\ & \quad \left. \int_0^T \|\nabla \eta(t)\|_{W^{\frac{1}{2}, \infty}(\mathbf{R}^d)} dt, \int_0^T \|(V, B)(t)\|_{W^{1+\varepsilon, \infty}(\mathbf{R}^d)} dt \right), \end{aligned}$$

with \mathcal{F} a positive, strictly increasing function of each of its variable, depending only on d , s , s_0 , ε , the bottom Γ , and on $\|\eta_0, \psi_0, V_0, B_0\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d)}$.

At last Theorem 1.1.4 is a consequence of the following proposition from [8],

Proposition 1.6.3 (Proposition 3.6 of [8]). *For $s_0 > 3/4 + d/2$, and $0 < \varepsilon < s_0 - 3/4 - d/2$,*

$$\begin{aligned} & \|a\|_{C_*^{\frac{1}{2}}} + \|(\partial_t + V \cdot \nabla) a\|_{L^\infty} \\ & \leq \mathcal{F} \left(\|(\eta, \psi, V, B)(t)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0+\frac{1}{2}} \times H^{s_0} \times H^{s_0}} \right) \left[1 + \|\eta\|_{C_*^{\frac{1}{2}+\varepsilon}} + \|(V, B)\|_{C_*^{1+\varepsilon}} \right]. \end{aligned}$$

Proof. The estimate on $\|a\|_{C_*^{\frac{1}{2}}}$ is proved in details in [8]. We will record here a proof of the estimate on $\|(\partial_t + V \cdot \nabla) a\|_{L^\infty}$. First, observe that

$$\begin{aligned} (\partial_t + V \cdot \nabla) a &= (\partial_t + \nabla_{x,y} \phi \cdot \nabla_{x,y}) (-\partial_y P)|_{y=\eta} \\ &= -\partial_y (\partial_t + \nabla_{x,y} \phi \cdot \nabla_{x,y}) P|_{y=\eta} - [(\partial_t + \nabla_{x,y} \phi \cdot \nabla_{x,y}), \partial_y] P|_{y=\eta}. \end{aligned}$$

The second term on the right hand side is

$$\begin{aligned} - [(\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y}), \partial_y] P|_{y=\eta} &= \nabla_x \partial_y \phi \cdot \nabla_x P|_{y=\eta} + \partial_y^2 \phi \partial_y P|_{y=\eta} \\ &= a \nabla \eta \cdot \nabla_x \partial_y \phi|_{y=\eta} + a \Delta_x \phi|_{y=\eta} \\ &= a \operatorname{div} V, \end{aligned}$$

where we have used that since $P|_{y=\eta} = 0$,

$$0 = \nabla(P|_{y=\eta}) = (\nabla_x P)|_{y=\eta} + \nabla \eta (\partial_y P)|_{y=\eta},$$

and that

$$\operatorname{div} V = \Delta_x \phi|_{y=\eta} + \nabla \eta \cdot (\nabla_x \partial_y \phi)|_{y=\eta}.$$

The proposition will then be proved once we have shown that

$$(1.6.2) \quad \|\partial_y (\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y}) P\|_{X^{s_0 - \frac{3}{4}}} \leq \mathcal{F} \left(\|(\eta, \psi, V, B)(t)\|_{H^{s_0 + \frac{1}{2}} \times H^{s_0 + \frac{1}{2}} \times H^{s_0} \times H^{s_0}} \right).$$

This is a consequence of the following elliptic equation

$$\begin{aligned} \Delta_{x,y} (\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y}) P &= (\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y}) \Delta_{x,y} P + [\Delta_{x,y}, (\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y})] P \\ &= - (\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y}) |\nabla_{x,y}^2 \phi|^2 + 2 \nabla_{x,y}^2 \phi \cdot \nabla_{x,y}^2 P \\ &= -2 \nabla_{x,y}^2 \phi \cdot \nabla_{x,y}^2 (-\partial_t \phi - \frac{1}{2} |\nabla_{x,y}\phi|^2 - gy) + 2 \nabla_{x,y}^2 \phi \cdot \nabla_{x,y}^2 P \\ &= 4 \nabla_{x,y}^2 \phi \cdot \nabla_{x,y}^2 P. \end{aligned}$$

Now (1.6.2) follow along the same lines as in Proposition 1.3.4, using the regularity on P already established in this proposition. \square

Deuxième partie

Existence à basse régularité

Réduction paradifférentielle

Cet article est une collaboration avec Quang Huy NGUYEN. Il a été accepté dans le Bulletin de la Société Mathématique de France ; voir [86].

2.1 Introduction

We consider the system of gravity-capillary waves describing the motion of a fluid interface under the effect of both gravity and surface tension. From the well-posedness result in Sobolev spaces of Yosihara [109] (see also Wu [104, 105] for pure gravity waves) it is known that the system is quasilinear in nature. In the more recent work [3], Alazard-Burq-Zuily showed explicitly this quasilinearity by using a paradifferential approach (see Appendix A) to symmetrize the system into the following paradifferential equation

$$(2.1.1) \quad (\partial_t + T_{V(t,x)} \cdot \nabla + iT_{\gamma(t,x,\xi)})u(t, x) = f(t, x)$$

where V is the horizontal component of the trace of the velocity field on the free surface, γ is an elliptic symbol of order $3/2$, depending only on the free surface. In other words, the transport part comes from the fluid and the dispersive part comes from the free boundary. The reduction (2.1.1) was implemented for

$$(2.1.2) \quad u \in L_t^\infty H_x^s \quad s > 2 + \frac{d}{2},$$

d being the dimension of the free surface. It has many consequences, among them are the local well-posedness and smoothing effect in [3], Strichartz estimates in [4]. As remarked in [3], $s > 2 + d/2$ is the minimal Sobolev index (in term of Sobolev's embedding) to ensure that the velocity field is Lipschitz up to the boundary, without taking into account the dispersive property. From the works of Alazard-Burq-Zuily [7, 8], Hunter-Ifrim-Tataru [52] for pure gravity waves, it seems natural to require that *the velocity is Lipschitz* so that the particles flow is well-defined, in view of the Cauchy-Lipschitz theorem. On the other hand, from *the standard theory of quasilinear pdes*, it is natural to ask if the reduction (2.1.1) holds at the Sobolev threshold $s > 3/2 + d/2$ and then, if a local-wellposedness theory holds at the same level of regularity? The two observations above motivate us to study the gravity-capillary system at the following regularity level:

$$(2.1.3) \quad u \in \mathcal{X} := L_t^\infty H_x^s \cap L_t^p W_x^{2,\infty} \quad \text{with } s > \frac{3}{2} + \frac{d}{2},$$

which exhibits a gap of $1/2$ derivative that may be filled up by Strichartz estimates. (2.1.3) means that on the one hand, the Sobolev regularity is that of quasilinear equations of order $3/2$; on the other hand, the $L_t^p W_x^{2,\infty}$ -norm ensures that the velocity is still Lipschitz for *a.e.* $t \in [0, T]$ (which is the threshold (2.1.2) after applying Sobolev's embedding).

By sharpening the analysis in [3], we shall perform the reduction (2.1.1) assuming merely the regularity \mathcal{X} of the solution. In order to do so, the main difficulty, compared to [3], is that further studies of the Dirichlet-Neumann operator in Besov spaces are demanded. Moreover, we have to keep all the estimates in the analysis to be *tame*, *i.e.*, linear with respect to the highest norm which is the Hölder norm in this case.

From this reduction, we deduce several consequences. The first one will be an *a priori estimate for the Sobolev norm* $L_t^\infty H_x^s$ using in addition the Strichartz norm $L_t^p W_x^{2,\infty}$ (see Theorem 2.1.1 below for an exact statement). This is an expected result, which follows the pattern established for other quasilinear equations. However, for water waves, it requires much more care due to the fact that the system is nonlocal and highly nonlinear. This problem has been addressed by Alazard-Burq-Zuily [8] for pure gravity water waves. In the case with surface tension, though the regularity level is higher, it requires a more precise analysis of the Dirichlet-Neumann operator in that lower order terms in the expansion of this operator need to be taken into consideration (see Proposition 2.3.5 below).

Another consequence will be a *blow-up criterion* (see Theorem 2.1.2), which implies that the solution can be continued as long as the \mathcal{X} -norm of u remained bounded (at least in the infinite depth case) with $p = 1$, *i.e.*, merely integrable in time. It also implies that, starting from a smooth datum, the solution remains smooth provided its C^{2+} -norm is bounded in time.

For more precise discussions, let us recall the Zakharov/Craig-Sulem formulation of water waves.

2.1.1 The Zakharov/Craig-Sulem formulation

We consider an incompressible, irrotational, inviscid fluid with unit density moving in a time-dependent domain

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : (x, y) \in \Omega_t\}$$

where each Ω_t is a domain located underneath a free surface

$$\Sigma_t = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : y = \eta(t, x)\}$$

and above a fixed bottom $\Gamma = \partial\Omega_t \setminus \Sigma_t$. We make the following separation assumption (H_t) on the domain at time t :

Ω_t is the intersection of the half space

$$\Omega_{1,t} = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : y < \eta(t, x)\}$$

and an open connected set \mathcal{O} containing a fixed strip around Σ_t , *i.e.*, there exists $h > 0$ such that

$$(2.1.4) \quad \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : \eta(x) - h \leq y \leq \eta(t, x)\} \subset \mathcal{O}.$$

The velocity field v admits a harmonic potential $\phi : \Omega \rightarrow \mathbf{R}$, *i.e.*, $v = \nabla\phi$ and $\Delta\phi = 0$. Using the idea of Zakharov, we introduce the trace of ϕ on the free surface

$$\psi(t, x) = \phi(t, x, \eta(t, x)).$$

Then $\phi(t, x, y)$ is the unique variational solution to the problem

$$(2.1.5) \quad \Delta\phi = 0 \text{ in } \Omega_t, \quad \phi(t, x, \eta(t, x)) = \psi(t, x), \quad \partial_n\phi(t)|_\Gamma = 0.$$

The Dirichlet-Neumann operator is then defined by

$$(2.1.6) \quad \begin{aligned} G(\eta)\psi &= \sqrt{1 + |\nabla_x \eta|^2} \mathcal{B}ig \left(\frac{\partial \phi}{\partial n} \mathcal{B}ig|_{\Sigma} \mathcal{B}ig \right) \\ &= (\partial_y \phi)(t, x, \eta(t, x)) - \nabla_x \eta(t, x) \cdot (\nabla_x \phi)(t, x, \eta(t, x)). \end{aligned}$$

The gravity-capillary water waves problem with surface tension consists in solving the following so-called Zakharov-Craig-Sulem system of (η, ψ)

$$(2.1.7) \quad \begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi = -g\eta - H(\eta) - \frac{1}{2} |\nabla_x \psi|^2 + \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1 + |\nabla_x \eta|^2}. \end{cases}$$

Here, $H(\eta)$ denotes the mean curvature of the free surface:

$$H(\eta) = -\operatorname{div} \mathcal{B}ig \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \mathcal{B}ig \right).$$

The vertical and horizontal components of the velocity on Σ can be expressed in terms of η and ψ as

$$(2.1.8) \quad B = (v_y)|_{\Sigma} = \frac{\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi}{1 + |\nabla_x \eta|^2}, \quad V = (v_x)|_{\Sigma} = \nabla_x \psi - B \nabla_x \eta.$$

As observed by Zakharov (see [111] and the references therein), (2.1.7) has a Hamiltonian canonical Hamiltonian structure

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \eta},$$

where the Hamiltonian \mathcal{H} is the total energy given by

$$(2.1.9) \quad \mathcal{H} = \frac{1}{2} \int_{\mathbf{R}^d} \psi G(\eta)\psi \, dx + \frac{g}{2} \int_{\mathbf{R}^d} \eta^2 \, dx + \int_{\mathbf{R}^d} (\sqrt{1 + |\nabla \eta|^2} - 1) \, dx.$$

2.1.2 Main results

The Cauchy problem has been extensively studied, for example in Nalimov [83], Yosihara [109], Coutand-Shkoller [32], Craig [34], Shatah-Zeng [90, 91, 92], Ming-Zhang [82], Lannes [71]: for sufficiently smooth solutions and Alazard-Burq-Zuily [3] for solutions at the energy threshold. See also Craig [34], Wu [104, 105], Lannes [70] for the studies on gravity waves. Observe that the linearized system of (2.1.7) about the rest state $(\eta = 0, \psi = 0)$ (modulo a lower order term, taking $g = 0$) reads

$$\begin{cases} \partial_t \eta - |D_x| \psi = 0, \\ \partial_t \psi - \Delta \eta = 0. \end{cases}$$

Put $\Phi = |D_x|^{\frac{1}{2}} \eta + i\psi$, this becomes

$$(2.1.10) \quad \partial_t \Phi + i |D_x|^{\frac{3}{2}} \Phi = 0.$$

Therefore, it is natural to study (2.1.7) at the following algebraic scaling

$$(\eta, \psi) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d).$$

From the formula (2.1.8) for the velocity trace, we have that the Lipschitz threshold in [3] corresponds to $s > 2 + d/2$. On the other hand, the threshold $s > 3/2 + d/2$ suggested by the quasilinear nature (2.1.1) is also *the minimal Sobolev index to ensure that the mean curvature $H(\eta)$ is bounded*. The question we are concerned with is the following:

(Q) Is the Cauchy problem for (2.1.7) solvable for initial data

$$(2.1.11) \quad (\eta_0, \psi_0) \in H^{s+\frac{1}{2}} \times H^s, \quad s > \frac{3}{2} + \frac{d}{2}?$$

Assume now that

$$(2.1.12) \quad (\eta, \psi) \in L^\infty \left([0, T]; H^{s+\frac{1}{2}} \times H^s \right) \cap L^p \left([0, T]; W^{r+\frac{1}{2}, \infty} \times W^{r, \infty} \right)$$

with

$$(2.1.13) \quad s > \frac{3}{2} + \frac{d}{2}, \quad r > 2$$

is a solution with prescribed data as in (2.1.11). We shall prove in Proposition 2.4.1 that the quasilinear reduction (2.1.1) of system (2.1.7) still holds with the right-hand-side term $f(t, x)$ satisfying a tame estimate, meaning that it is linear with respect to the Hölder norm. To be concise in the following statements, let us define the quantities that control the system (see Definition A.0.1 for the definitions of functional spaces):

$$\begin{aligned} \text{Sobolev norms : } M_{\sigma, T} &= \|(\eta, \psi)\|_{L^\infty([0, T]; H^{\sigma+\frac{1}{2}} \times H^\sigma)}, \quad M_{\sigma, 0} = \|(\eta_0, \psi_0)\|_{H^{\sigma+\frac{1}{2}} \times H^\sigma}, \\ \text{“Strichartz norm” : } N_{\sigma, T} &= \|(\eta, \nabla \psi)\|_{L^1([0, T]; W^{\sigma+\frac{1}{2}, \infty} \times B_{\infty, 1}^1)}. \end{aligned}$$

Our first result concerns an a priori estimate for the Sobolev norm $M_{s, T}$ in terms of itself and the Strichartz norm $N_{r, T}$.

Theorem 2.1.1. *Let $d \geq 1$, $h > 0$, $r > 2$ and $s > \frac{3}{2} + \frac{d}{2}$. Then there exists a nondecreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$, depending only on (d, s, r, h) , such that: for all $T \in (0, 1]$ and all (η, ψ) solution to (2.1.7) on $[0, T]$ with*

$$\begin{aligned} (\eta, \psi) &\in L^\infty \left([0, T]; H^{s+\frac{1}{2}} \times H^s \right), \\ (\eta, \nabla \psi) &\in L^1 \left([0, T]; W^{r+\frac{1}{2}, \infty} \times B_{\infty, 1}^1 \right), \\ \inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma) &> h, \end{aligned}$$

there holds

$$M_{s, T} \leq \mathcal{F}(M_{s, 0} + T\mathcal{F}(M_{s, T}) + N_{r, T}).$$

Remark. Some comments are in order with respect to the preceding a priori estimate.

1. We require only $\nabla \psi \in B_{\infty, 1}^1$ instead of $\psi \in W^{r, \infty}$.
2. The function \mathcal{F} above can be highly nonlinear. It is not simply a straightforward outcome of a Grönwall inequality but also comes from estimates of the Dirichlet-Neumann operator in Sobolev spaces and Besov spaces (see the proof of Theorem 2.4.5).
3. When $s > 2 + d/2$ one can take $r = s - \frac{d}{2}$ and retrieves by Sobolev embeddings the a priori estimate of [3] (see Proposition 5.2 there).

Our second result provides a blow-up criterion for solutions at the energy threshold constructed in [3]. Let C_*^r denote the Zygmund space of order r (see Definition A.0.1). Note that $C_*^r = W^{r,\infty}$ if $r \in (0, \infty) \setminus \{1, 2, 3, \dots\}$ while $W^{r,\infty} \subsetneq C_*^r$ if $r \in \{0, 1, 2, \dots\}$.

Theorem 2.1.2. *Let $d \geq 1$, $h > 0$ and $\sigma > 2 + \frac{d}{2}$. Let*

$$(\eta_0, \psi_0) \in H^{\sigma+\frac{1}{2}} \times H^\sigma, \quad \text{dist}(\eta_0, \Gamma) > h > 0.$$

Let $T^* = T^*(\eta_0, \psi_0, \sigma, h)$ be the maximal time of existence defined by (2.4.17) and

$$(\eta, \psi) \in L^\infty\left([0, T^*]; H^{\sigma+\frac{1}{2}} \times H^\sigma\right)$$

be the maximal solution of (2.1.7) with prescribed data (η_0, ψ_0) . If T^* is finite, then for all $\varepsilon > 0$,

$$(2.1.14) \quad P_\varepsilon(T^*) + \int_0^{T^*} Q_\varepsilon(t) dt + \frac{1}{h(T^*)} = +\infty,$$

where

$$\begin{aligned} P_\varepsilon(T^*) &= \sup_{t \in [0, T^*]} \|\eta(t)\|_{C_*^{2+\varepsilon}} + \|\nabla\psi(t)\|_{B_{\infty,1}^0}, \\ Q_\varepsilon(t) &= \|\eta(t)\|_{C_*^{\frac{5}{2}+\varepsilon}} + \|\nabla\psi(t)\|_{C_*^1}, \\ h(T^*) &= \inf_{t \in [0, T^*]} \text{dist}(\eta(t), \Gamma). \end{aligned}$$

Consequently, if T^* is finite then for all $\varepsilon > 0$,

$$(2.1.15) \quad P_\varepsilon^0(T^*) + \int_0^{T^*} Q_\varepsilon^0(t) dt + \frac{1}{h(T^*)} = +\infty,$$

where

$$\begin{aligned} P_\varepsilon^0(T^*) &= \sup_{t \in [0, T^*]} \|\eta(t)\|_{C_*^{2+\varepsilon}} + \|(V, B)(t)\|_{B_{\infty,1}^0}, \\ Q_\varepsilon^0(t) &= \|\eta(t)\|_{C_*^{\frac{5}{2}+\varepsilon}} + \|(V, B)(t)\|_{C_*^1}. \end{aligned}$$

Remark. 1. We shall prove in Proposition 2.4.7 below that the Sobolev norm of the solution, $\|(\eta, \psi)\|_{L^\infty([0, T]; H^{\sigma+\frac{1}{2}} \times H^\sigma)}$, $\sigma > 2 + \frac{d}{2}$, is bounded by a double exponential

$$\exp\left(e^{C(T) \int_0^T Q_\varepsilon(t) dt}\right)$$

where $C(T)$ depends only on the lower norm $P_\varepsilon(T)$. In the preceding estimate, Q_ε can be replaced by Q_ε^0 by virtue of (2.4.23). These bounds are reminiscent of the well-known result due to Beale-Kato-Majda [18] for the incompressible Euler equations in the whole space, where the C_*^1 -norm of the velocity was sharpened to the L^∞ -norm of the vorticity. An analogous result in bounded, simply connected domains was obtained by Ferrari [44].

2. If in Q_ε the Zygmund norm $\|\nabla\psi\|_{C_*^1}$ is replaced by the stronger norm $\|\nabla\psi\|_{B_{\infty,1}^1}$, then one obtains the following exponential bound (see Remark 2.4.2)

$$\|(\eta, \psi)\|_{L^\infty([0, T]; H^{\sigma+\frac{1}{2}} \times H^\sigma)} \leq C(T) \|(\eta(0), \psi(0))\|_{H^{\sigma+\frac{1}{2}} \times H^\sigma} \exp\left(C(T) \int_0^T Q_\varepsilon(t) dt\right),$$

where $C(T)$ depends only on the lower norm $P_\varepsilon(T)$ and $\sigma > \frac{3}{2} + \frac{d}{2}$. The same remarks applies to Q_ε^0 and (V, B) .

In the survey paper [37] Craig-Wayne posed (see Problem 3 there) the following questions on *How do solutions break down?*:

(Q1) For which α is it true that, if one knows a priori that $\sup_{[-T, T]} \|(\eta, \psi)\|_{C^\alpha} < +\infty$ then C^∞ data (η_0, ψ_0) implies that the solution is C^∞ over the time interval $[-T, T]$?

(Q2) It would be more satisfying to say that the solution fails to exist because the "curvature of the surface has diverged at some point", or a related geometrical and/or physical statement.

With regard to question **(Q1)**, we deduce from Theorem 2.1.2 (more precisely, from (2.1.14)) the following persistence of Sobolev regularity.

Corollary 2.1.3. *Let $T \in (0, +\infty)$ and (η, ψ) be a distributional solution to (2.1.7) on the time interval $[0, T]$ such that $\inf_{[0, T]} \text{dist}(\eta(t), \Gamma) > 0$. Then the following property holds: if one knows a priori that for some $\varepsilon_0 > 0$*

$$(2.1.16) \quad \sup_{[0, T]} \|(\eta(t), \nabla \psi(t))\|_{C_*^{\frac{5}{2} + \varepsilon_0} \times C_*^1} < +\infty,$$

then $(\eta(0), \psi(0)) \in H^\infty(\mathbf{R}^d)^2$ implies that $(\eta, \psi) \in L^\infty([0, T]; H^\infty(\mathbf{R}^d))^2$.

Theorem 2.1.2 gives a partial answer to **(Q2)**. Indeed, the criterion (2.1.15) implies that the solution fails to exist if

- the Lipschitz norm of the velocity trace explodes, *i.e.*, $\sup_{[0, T^*]} \|(V, B)\|_{W^{1, \infty}} = +\infty$, or
- the bottom rises to the surface, *i.e.*, $h(T^*) = 0$.

Some results are known about blow-up criteria for pure gravity water waves (without surface tension). Wang-Zhang [102] obtained a result stated in terms of the curvature $H(\eta)$ and the gradient of the velocity trace

$$(2.1.17) \quad \int_0^{T^*} \|(\nabla V, \nabla B)(t)\|_{L^\infty}^6 dt + \sup_{t \in [0, T^*]} \|H(\eta(t))\|_{L^2 \cap L^p} = +\infty, \quad p > 2d.$$

The second author [84] showed, for highest regularities,

$$\int_0^{T^*} (\|\eta\|_{C^{\frac{3}{2}+}} + \|(V, B)\|_{C^{1+}}) dt = +\infty;$$

the temporal integrability was thus improved. In two space dimensions, using holomorphic coordinates, Hunter-Ifrim-Tataru [52] obtained a sharpened criterion with $\|(V, B)\|_{C^{1+}}$ replaced by $\|(\nabla V, \nabla B)\|_{BMO}$. Also in two space dimensions, Wu [108] proved a blow-up criterion using the energy constructed by Kinsey-Wu [62], which concerns water waves with angled crests, hence the surface is even not Lipschitz. Remark that all the above results but [84] consider the bottomless case. In a more recent paper, [103] considered rotational fluids and obtained

$$\sup_{t \in [0, T^*]} (\|v(t)\|_{W^{1, \infty}} + \|H(\eta(t))\|_{L^2 \cap L^p}) = +\infty \quad p > 2d,$$

v being the Eulerian velocity. In order to obtain the sharp regularity for $\nabla \psi$ and (V, B) in Theorem 2.1.2, we shall use a technical idea from [103]: deriving elliptic estimates in Chemin-Lerner type spaces.

Finally, we observe that the relation (2.1.13) exhibits a gap of $1/2$ derivative from H^s to $W^{2, \infty}$ in terms of Sobolev's embedding. To fill up this gap we need to take into account the dispersive property of water waves to prove a Strichartz estimate with a gain of $1/2$ derivative. As remarked in [87] this gain can be achieved for the 3D linearized system (*i.e.* $d = 2$) and corresponds to the so called *semiclassical Strichartz estimate*. The proof of Theorem 2.5.9 on the Lipschitz continuity

of the solution map shows that if the semiclassical Strichartz estimate were proved, this theorem would hold with the gain $\mu = \frac{1}{2}$ in (2.5.31) (see Remark 2.5.3). Then, applying Theorems 2.1.1, 2.1.2 one would end up with an affirmative answer for **(Q)** by implementing the standard method of regularizing initial data. Therefore, the problem boils down to studying Strichartz estimates for (2.1.7). As a first effort in this direction, we prove in the companion paper [87] Strichartz estimates with an intermediate gain $0 < \mu < 1/2$ which yields a Cauchy theory (see Theorem 1.6, [87]) in which the initial velocity may fail to be Lipschitz (up to the boundary) but becomes Lipschitz at almost all later time; this is an analogue of the result in [8] for pure gravity waves. The article is organized as follows. Section 2.2 is devoted to the study of the Dirichlet-Neumann operator in Sobolev spaces, Besov spaces and Zygmund spaces. Next, in Section 2.3 we adapt the method in [3] to parilinearize and then symmetrize system (2.1.7) at our level of regularity (2.1.13). With this reduction, we use the standard energy method to derive an a priori estimate and a blow-up criterion in Section 2.4. Section 2.5 is devoted to contraction estimates; more precisely, we establish the Lipschitz continuity of the solution map in weaker norms. Finally, we gather some basic features of the paradifferential calculus and some technical results in Appendix A.

2.2 Elliptic estimates and the Dirichlet-Neumann operator

2.2.1 Construction of the Dirichlet-Neumann operator

Let $\eta \in W^{1,\infty}(\mathbf{R}^d)$ and $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$. In order to define the Dirichlet-Neumann operator $G(\eta)f$, we consider the boundary value problem

$$(2.2.1) \quad \Delta_{x,y}\phi = 0 \text{ in } \Omega, \quad \phi|_{\Sigma} = f, \quad \partial_n\phi|_{\Gamma} = 0.$$

For any $h' \in (0, h]$, define the curved strip of width h' below the free surface

$$(2.2.2) \quad \Omega_{h'} := \{(x, y) : x \in \mathbf{R}^d, \eta(x) - h' < y < \eta(x)\}.$$

We recall here the construction of the variational solution to (2.2.1) in [7].

Notation 2.2.1. Denote by \mathcal{D} the space of functions $u \in C^\infty(\Omega)$ such that $\nabla_{x,y}u \in L^2(\Omega)$. We then define \mathcal{D}_0 as the subspace of functions $u \in \mathcal{D}$ such that u is equal to 0 in a neighborhood of the top boundary Σ .

Proposition 2.2.2 (see [3, Proposition 2.2]). *There exists a positive weight $g \in L_{loc}^\infty(\Omega)$, locally bounded from below, equal to 1 near the top boundary of Ω , say in Ω_h , and a constant $C > 0$ such that for all $u \in \mathcal{D}_0$,*

$$(2.2.3) \quad \iint_{\Omega} g(x, y)|u(x, y)|^2 dx dy \leq C \iint_{\Omega} |\nabla_{x,y}u(x, y)|^2 dx dy.$$

Definition 2.2.3. Denote by $H^{1,0}(\Omega)$ the completion of \mathcal{D}_0 under the norm

$$\|u\|_* := \|u\|_{L^2(\Omega, g(x,y)dx dy)} + \|\nabla_{x,y}u\|_{L^2(\Omega, dx dy)}.$$

Owing to the Poincaré inequality (2.2.3), $H^{1,0}(\Omega)$ endowed with the norm $\|u\| = \|\nabla_{x,y}u\|_{L^2(\Omega)}$ is a Hilbert space, see Definition 2.6 [3].

Now, let $\chi_0 \in C^\infty(\mathbf{R})$ be such that $\chi_0(z) = 1$ if $z \geq -\frac{1}{4}$, $\chi_0(z) = 0$ if $z \leq -\frac{1}{2}$. Then with $f \in H^{\frac{1}{2}}$, define

$$f_1(x, z) = \chi_0(z)e^{z\langle D_x \rangle} f(x), \quad x \in \mathbf{R}^d, \quad z \leq 0.$$

Next, define

$$(2.2.4) \quad \underline{f}(x, y) = f_1\left(x, \frac{y - \eta(x)}{h}\right), \quad (x, y) \in \Omega.$$

This "lifting" function satisfies $\underline{f}|_{y=\eta(x)} = f(x)$, $\underline{f} \equiv 0$ in $\Omega \setminus \Omega_{h/2}$ and

$$(2.2.5) \quad \|\underline{f}\|_{H^1(\Omega)} \leq K(1 + \|\eta\|_{W^{1,\infty}}) \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

The map

$$\varphi \mapsto - \int_{\Omega} \nabla_{x,y} \underline{f} \cdot \nabla_{x,y} \varphi \, dx dy$$

is thus a bounded linear form on $H^{1,0}(\Omega)$. The Riesz theorem then provides a unique $u \in H^{1,0}(\Omega)$ such that

$$(2.2.6) \quad \forall \varphi \in H^{1,0}(\Omega), \quad \iint_{\Omega} \nabla_{x,y} u \cdot \nabla_{x,y} \varphi \, dx dy = - \iint_{\Omega} \nabla_{x,y} \underline{f} \cdot \nabla_{x,y} \varphi \, dx dy.$$

Definition 2.2.4. With \underline{f} and u constructed as above, the function $\phi := u + \underline{f}$ is defined to be the variational solution of the problem (2.2.1). The Dirichlet-Neumann operator is defined formally by

$$(2.2.7) \quad G(\eta)\psi = \sqrt{1 + |\nabla\eta|^2} \partial_n \phi \Big|_{y=\eta(x)} = [\partial_y \phi - \nabla\eta \cdot \nabla\phi] \Big|_{y=\eta(x)}.$$

As a consequence of (2.2.5) and (2.2.6), the variational solution ϕ satisfies

$$(2.2.8) \quad \|\nabla_{x,y} \phi\|_{L^2(\Omega)} \leq K(1 + \|\eta\|_{W^{1,\infty}}) \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

Moreover, it was proved in [84] the following maximum principle.

Proposition 2.2.5 (see [84, Proposition 2.7]). *Let $\eta \in W^{1,\infty}(\mathbf{R}^d)$ and $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$. There exists a constant $C > 0$ independent of η , ψ such that*

$$\|\phi\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\mathbf{R}^d)}.$$

The continuity of $G(\eta)$ in Sobolev spaces is given in the next theorem.

Theorem 2.2.6 (see [7, Theorem 3.12]). *Let $d \geq 1$, $s > \frac{1}{2} + \frac{d}{2}$ and $\frac{1}{2} \leq \sigma \leq s + \frac{1}{2}$. For all $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, the operator*

$$G(\eta) : H^\sigma \rightarrow H^{\sigma-1}$$

is continuous. Moreover, there exists a nondecreasing function $\mathcal{F} : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ such that, for all $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ and all $f \in H^\sigma(\mathbf{R}^d)$, there holds

$$(2.2.9) \quad \|G(\eta)f\|_{H^{\sigma-1}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^\sigma}.$$

2.2.2 Elliptic estimates

The Dirichlet-Neumann requires the regularity of $\nabla_{x,y} \phi$ at the free surface. We follow [70] and [7] straightening out Ω_h using the map

$$(2.2.10) \quad \begin{aligned} \rho(x, z) &= (1+z)e^{\delta z \langle D_x \rangle} \eta(x) - z \left\{ e^{-(1+z)\delta \langle D_x \rangle} \eta(x) - h \right\} \\ (x, z) &\in S := \mathbf{R}^d \times (-1, 0). \end{aligned}$$

According to Lemma 3.6, [7], there exists an absolute constant $K > 0$ such that if $\delta\|\eta\|_{W^{1,\infty}} \leq K$ then

$$(2.2.11) \quad \partial_z \rho \geq \frac{h}{2}$$

and the map $(x, z) \mapsto (x, \rho(x, z))$ is thus a Lipschitz diffeomorphism from S to Ω_h . Then if we call

$$(2.2.12) \quad v(x, z) = \phi(x, \rho(x, z)) \quad \forall (x, z) \in S$$

the image of ϕ via this diffeomorphism, it solves

$$(2.2.13) \quad \mathcal{L}v := (\partial_z^2 + \alpha\Delta_x + \beta \cdot \nabla_x \partial_z - \gamma \partial_z)v = 0 \quad \text{in } S$$

where

$$\alpha := \frac{(\partial_z \rho)^2}{1 + |\nabla_x \rho|^2}, \quad \beta := -2 \frac{\partial_z \rho \nabla_x \rho}{1 + |\nabla_x \rho|^2}, \quad \gamma := \frac{1}{\partial_z \rho} (\partial_z^2 \rho + \alpha \Delta_x \rho + \beta \cdot \nabla_x \partial_z \rho).$$

Sobolev estimates

Define the following interpolation spaces

$$(2.2.14) \quad \begin{aligned} X^\mu(J) &= C_z(I; H^\mu(\mathbf{R}^d)) \cap L_z^2(J; H^{\mu+\frac{1}{2}}(\mathbf{R}^d)), \\ Y^\mu(J) &= L_z^1(I; H^\mu(\mathbf{R}^d)) + L_z^2(J; H^{\mu-\frac{1}{2}}(\mathbf{R}^d)). \end{aligned}$$

Remark that $\|\cdot\|_{Y^\mu(J)} \leq \|\cdot\|_{X^{\mu-1}(J)}$ for any $\mu \in \mathbf{R}$. We get started by providing estimates for the coefficients α, β, γ . We refer the reader to Appendix A for a review of the paradifferential calculus and notations of functional spaces.

Notation 2.2.7. We will denote \mathcal{F} any nondecreasing function from \mathbf{R}^+ to \mathbf{R}^+ . \mathcal{F} may change from line to line but is independent of relevant parameters.

Lemma 2.2.8. Denote $I = [-1, 0]$.

1. For any $\sigma > \frac{1}{2} + \frac{d}{2}$ and $\varepsilon > 0$, there holds

$$(2.2.15) \quad \|\alpha - h^2\|_{X^{\sigma-\frac{1}{2}}(I)} + \|\beta\|_{X^{\sigma-\frac{1}{2}}(I)} + \|\gamma\|_{X^{\sigma-\frac{3}{2}}(I)} \leq \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}}) \|\eta\|_{H^{\sigma+\frac{1}{2}}}.$$

2. If $\mu > \frac{3}{2}$ then

$$(2.2.16) \quad \|\alpha\|_{C(I; C_*^{\mu-\frac{1}{2}})} + \|\beta\|_{C(I; C_*^{\mu-\frac{1}{2}})} + \|\gamma\|_{C(I; C_*^{\mu-\frac{3}{2}})} \leq \mathcal{F}(\|\eta\|_{C_*^{\mu+\frac{1}{2}}}),$$

$$(2.2.17) \quad \|\alpha\|_{\tilde{L}^2(I; C_*^\mu)} + \|\beta\|_{\tilde{L}^2(I; C_*^\mu)} + \|\gamma\|_{\tilde{L}^2(I; C_*^{\mu-1})} \leq \mathcal{F}(\|\eta\|_{C_*^{\mu+\frac{1}{2}}} + \|\eta\|_{L^2}).$$

Proof. These estimates stem from estimates for derivatives of ρ . For the proof of (2.2.15) we refer the reader to Lemmas 3.7 and 3.19 in [7]. Concerning (2.2.16) we remark that α and β involve merely derivatives up to order 1 of η while γ involves second order derivatives of η . Finally, for (2.2.17) we use the following smoothing property of the Poisson kernel in the high frequency regime (see Lemma 2.4, [17] and Lemma 3.2, [102]): for all $\kappa > 0$ and $p \in [1, \infty]$, there exists $C > 0$ such that for all $j \geq 1$,

$$\|e^{-\kappa \langle D_x \rangle} \Delta_j u\|_{L^p(\mathbf{R}^d)} \leq C e^{-C2^j} \|\Delta_j u\|_{L^p(\mathbf{R}^d)},$$

where, we recall the dyadic partition of unity in Definition A.0.1: $\text{Id} = \sum_{j=0}^{\infty} \Delta_j$. The low frequency part Δ_0 can be trivially bounded by the L^2 -norm using Bernstein's inequalities. \square

We first use the variational estimate (2.2.8) to derive a regularity for $\nabla_{x,z}v$.

Lemma 2.2.9. *Let $f \in H^{\frac{1}{2}}$. Set*

$$(2.2.18) \quad E(\eta, f) = \|\nabla_{x,y}\phi\|_{L^2(\Omega_h)}.$$

1. *If $\eta \in C_*^{\frac{3}{2}+\varepsilon}$ with $\varepsilon > 0$ then $\nabla_{x,z}v \in C([-1, 0]; H^{-\frac{1}{2}})$ and*

$$(2.2.19) \quad \|\nabla_x v\|_{X^{-\frac{1}{2}}([-1,0])} \leq \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}})E(\eta, f)$$

$$(2.2.20) \quad \|\nabla_z v\|_{X^{-\frac{1}{2}}([-1,0])} \leq \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}})(1 + \|\eta\|_{C_*^{\frac{3}{2}+\varepsilon}})E(\eta, f).$$

2. *If $\eta \in H^{s+\frac{1}{2}}$ with $s > \frac{1}{2} + \frac{d}{2}$ then $\nabla_{x,z}v \in C([-1, 0]; H^{-\frac{1}{2}})$ and*

$$(2.2.21) \quad \|\nabla_{x,z}v\|_{X^{-\frac{1}{2}}([-1,0])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})E(\eta, f).$$

Remark. 1. By (2.2.8), we have

$$E(\eta, f) \leq K(1 + \|\eta\|_{W^{1,\infty}})\|f\|_{H^{\frac{1}{2}}}.$$

However, we keep in the estimates (2.2.19)-(2.2.20) the quantity $E(\eta, f)$ instead of $\|f\|_{H^{\frac{1}{2}}}$ because $E(\eta, f)$ is controlled by the Hamiltonian, which is conserved under the flow. Moreover, as we shall derive blow-up criteria involving only Holder norms of the solution, we avoid using $\|f\|_{H^{\frac{1}{2}}}$.

2. The estimates (2.2.19), (2.2.20), (2.2.21) were proved in Proposition 4.3, [102] as *a priori estimates* (see the proof there). It is worth noting that we establish here a real regularity result.

Proof. Denote $I = [-1, 0]$.

1. Observe first that by changing variables,

$$(2.2.22) \quad \|\nabla_{x,z}v\|_{L^2(I;L^2)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}})\|\nabla_{x,y}\phi\|_{L^2(\Omega_h)} = \mathcal{F}(\|\eta\|_{W^{1,\infty}})E(\eta, f).$$

Applying the interpolation Lemma A.0.23, we obtain $\nabla_x v \in X^{-\frac{1}{2}}(I)$ and

$$(2.2.23) \quad \begin{aligned} \|\nabla_x v\|_{X^{-\frac{1}{2}}(I)} &\lesssim \|\nabla_x v\|_{L^2(I;L^2)} + \|\partial_z \nabla_x v\|_{L^2(I;H^{-1})} \\ &\lesssim \|\nabla_{x,z}v\|_{L^2(I;L^2)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}})E(\eta, f). \end{aligned}$$

We are left with (2.2.20). Again, by virtue of Lemma A.0.23 and (2.2.22), it suffices to prove

$$\|\partial_z^2 v\|_{L^2(I;H^{-1})} \leq \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}})(1 + \|\eta\|_{C_*^{\frac{3}{2}+\varepsilon}})E(\eta, f)$$

A natural way is to compute $\partial_z^2 v$ using (2.2.13)

$$\partial_z^2 v = -\alpha \Delta_x v - \beta \cdot \nabla_x \partial_z v + \gamma \partial_z v$$

and then estimate the right-hand side. However, this will lead to a loss of $\frac{1}{2}$ derivative of η . To remedy this, further cancellations coming from the structure of the equation need to be invoked. We have

$$\begin{aligned} (\partial_y \phi)(x, \rho(x, z)) &= \frac{1}{\partial_z \rho} \partial_z v(x, z) =: (\Lambda_1 v)(x, z), \\ (\nabla_x \phi)(x, \rho(x, z)) &= \left(\nabla_x - \frac{\nabla_x \rho}{\partial_z \rho} \partial_z \right) v(x, z) =: (\Lambda_2 v)(x, z). \end{aligned}$$

Set $U := \Lambda_1 v - \nabla_x \rho \Lambda_2 v$, whose trace at $z = 0$ is actually equal to $G(\eta)f$. Then, using the equation $\Delta_{x,y} \phi = 0$, it was proved in [7] (see the formula (3.19) there) that $\partial_z U$ has the divergence form

$$\partial_z U = \nabla_x \cdot (\partial_z \rho \Lambda_2 v).$$

Then, by the interpolation Lemma A.0.23, it is readily seen that $U \in C(I; H^{-\frac{1}{2}})$ and

$$\|U\|_{C(I; H^{-\frac{1}{2}})} \lesssim \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}}) E(\eta, f).$$

Now, from the definition of $\Lambda_{1,2}$ one can compute

$$\partial_z v = \frac{(U + \nabla_x \rho \cdot \nabla_x v) \partial_z \rho}{1 + |\nabla_x \rho|^2} =: Ua + \nabla_x v \cdot b$$

with

$$a := \frac{\partial_z \rho}{1 + |\nabla_x \rho|^2}, \quad b := \frac{\partial_z \rho \nabla_x \rho}{1 + |\nabla_x \rho|^2}.$$

We write $Ua = T_a U + T_U a + R(a, U)$. By Theorem A.0.4 (i),

$$\|T_a U\|_{C(I; H^{-\frac{1}{2}})} \lesssim \|a\|_{C(I; L^\infty)} \|U\|_{C(I; H^{-\frac{1}{2}})} \lesssim \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}}) E(\eta, f).$$

The term $T_U a$ can be estimated by means of Lemma A.0.11 as

$$\|T_U a\|_{C(I; H^{-\frac{1}{2}})} \lesssim \|U\|_{C(I; H^{-\frac{1}{2}})} \|a\|_{C(I; C_*^\varepsilon)} \lesssim \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}}) E(\eta, f).$$

Finally, for the remainder $R(a, U)$ we use (A.0.14), which leads to a loss of $\frac{1}{2}$ derivative for η , to get

$$\|R(U, a)\|_{C(I; H^{-\frac{1}{2}})} \lesssim \|a\|_{C(I; C_*^{\frac{1}{2}+\varepsilon})} \|U\|_{C(I; H^{-\frac{1}{2}})} \lesssim \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}}) (1 + \|\eta\|_{C_*^{\frac{3}{2}+\varepsilon}}) E(\eta, f)$$

where, we have used

$$\|a\|_{C(I; C_*^{\frac{1}{2}+\varepsilon})} \lesssim \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}}) (1 + \|\eta\|_{C_*^{\frac{3}{2}+\varepsilon}}).$$

Finally, the term $b \nabla_x v$ can be treated using the same argument as we have shown that $\nabla_x v \in C(I; H^{-\frac{1}{2}})$. The proof of (2.2.20) is complete.

2. We turn to prove (2.2.21). Observe that by the embedding

$$(2.2.24) \quad \|\eta\|_{C^{1+\varepsilon}} \leq C \|\eta\|_{H^{s+\frac{1}{2}}}$$

with $0 < \varepsilon < s - \frac{1}{2} - \frac{d}{2}$, (2.2.19) implies the estimate of $\nabla_x v$ in (2.2.21). For $\partial_z v$, we follow the above proof of (2.2.20). It suffices to prove $aU \in C(I; H^{-\frac{1}{2}})$ with norm bounded by the right-hand side of (2.2.21). To this end, we write $aU = hU + T_{(a-h)}U + T_U(a-h) + R(a-h, U)$. The proof of (2.2.20), combined with (2.2.24), shows that

$$\|T_U(a-h)\|_{C(I; H^{-\frac{1}{2}})} + \|T_{(a-h)}U\|_{C(I; H^{-\frac{1}{2}})} \lesssim \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) E(\eta, f).$$

Finally, by applying (A.0.13) (notice that $\frac{d}{2} \geq \frac{1}{2}$) and using the estimate

$$\|a\|_{C(I; H^{\frac{d}{2}+\varepsilon})} \lesssim \|a\|_{C(I; H^{s-\frac{1}{2}})} \lesssim \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}),$$

we conclude that

$$\|R(U, a)\|_{C(I; H^{-\frac{1}{2}})} \lesssim \|U\|_{C(I; H^{-\frac{1}{2}})} \|a\|_{C(I; H^{\frac{d}{2}+\varepsilon})} \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) E(\eta, f).$$

□

According to the preceding lemma, the trace $\nabla_{x,z}v|_{z=0}$ is well-defined and belongs to $H^{-\frac{1}{2}}$. Estimates in higher order Sobolev spaces are given in the next proposition.

Proposition 2.2.10 (see [7, Proposition 3.16]). *Let $s > \frac{1}{2} + \frac{d}{2}$, $-\frac{1}{2} \leq \sigma \leq s - \frac{1}{2}$. Assume that $\eta \in H^{s+\frac{1}{2}}$ and $f \in H^{\sigma+1}$ and for some $z_0 \in (-1, 0)$*

$$\|\nabla_{x,z}v\|_{X^{-\frac{1}{2}}([z_0,0])} < +\infty.$$

Then for any $z_1 \in (-1, 0)$, $z_1 > z_0$, we have $\nabla_{x,z}v \in X^\sigma([z_1, 0])$ and

$$\|\nabla_{x,z}v\|_{X^\sigma([z_1,0])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|f\|_{H^{\sigma+1}} + \|\nabla_{x,z}v\|_{X^{-\frac{1}{2}}([z_0,0])} \right\},$$

where \mathcal{F} depends only on σ and z_0, z_1 .

A combination of (2.2.21) and Remark 2.2.2 implies

$$\|\nabla_{x,z}v\|_{X^{-\frac{1}{2}}([-1,0])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^{\frac{1}{2}}}$$

provided $s > \frac{1}{2} + \frac{d}{2}$. With the aid of Proposition 2.2.10, we prove the following identity, which will be used later in the proof of blow-up criteria.

Proposition 2.2.11. *Let $s > \frac{1}{2} + \frac{d}{2}$. Assume that $\eta \in H^{s+\frac{1}{2}}$ and $f \in H^{\frac{3}{2}}$. Then $\phi \in H^2(\Omega_{3h/4})$ and the following identity holds*

$$\int_{\mathbf{R}^d} fG(\eta)f = \|\nabla_{x,y}\phi\|_{L^2(\Omega)}^2.$$

Proof. We first recall from the construction in subsection 2.2.1 that $\phi = u + \underline{f}$, where \underline{f} is defined by (2.2.4) and $u \in H^{1,0}(\Omega)$ is the unique solution of (2.2.6). By the Poincaré inequality of Lemma 2.2.2 and (2.2.6), (2.2.5)

$$\|u\|_{L^2(\Omega_h)} \leq C\|\nabla_{x,y}u\|_{L^2(\Omega)} \leq K(1 + \|\eta\|_{W^{1,\infty}})\|f\|_{H^{\frac{1}{2}}}.$$

Therefore, $\phi \in L^2(\Omega_h)$ and thus, by (2.2.8), $\phi \in H^1(\Omega_h)$. Now, applying Proposition 2.2.10 we have that $v = \phi(x, \rho(x, z))$ satisfies for any $z_1 \in (-1, 0)$

$$\|\nabla_{x,z}v\|_{L^2([z_1,0];H^1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|f\|_{H^{\frac{3}{2}}}.$$

Then using equation (2.2.13) together with the product rules one can prove that

$$\|\partial_z^2 v\|_{L^2([z_1,0];L^2)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|f\|_{H^{\frac{3}{2}}}.$$

By a change of variables we obtain $\nabla_{x,y}\phi \in H^1(\Omega_{3h/4})$ and thus $\phi \in H^2(\Omega_{3h/4})$. Now, taking $\varphi = u \in H^{1,0}(\Omega)$ in the variational equation (2.2.6) gives

$$\int_{\Omega} \nabla_{x,z}\phi \nabla_{x,z}u = 0.$$

Consequently

$$\int_{\Omega} |\nabla_{x,y}\phi|^2 = \int_{\Omega} |\nabla_{x,y}\phi|^2 - \int_{\Omega} \nabla_{x,z}\phi \nabla_{x,z}u = \int_{\Omega} \nabla_{x,y}\phi \nabla_{x,y}\underline{f}.$$

Since $\underline{f} \equiv 0$ in $\Omega \setminus \Omega_{h/2}$, this implies

$$\int_{\Omega} |\nabla_{x,y} \phi|^2 = \int_{\Omega_{3h/4}} \nabla_{x,y} \phi \nabla_{x,y} \underline{f}.$$

We have proved that in $\Omega_{3h/4}$, the harmonic function ϕ is H^2 . Notice in addition that $\phi \equiv 0$ near $\{y = \eta - 3h/4\}$. As $\partial\Omega_{3h/4}$ is Lipschitz ($\eta \in H^{1+\frac{d}{2}+} \subset W^{1,\infty}$), an integration by parts then yields

$$\int_{\Omega} |\nabla_{x,y} \phi|^2 = \int_{\Sigma} \underline{f} \partial_n \phi = \int_{\mathbf{R}^d} \underline{f} G(\eta) \underline{f},$$

which is the desired identity. \square

The next proposition is an improvement of Proposition 2.2.10 in the sense that it gives tame estimates with respect to the highest derivatives of η and f , provided $\nabla_{x,z} v \in L_x^\infty L_x^\infty$.

Proposition 2.2.12 (see [84, Proposition 2.12]). *Let $s > \frac{1}{2} + \frac{d}{2}$, $-\frac{1}{2} \leq \sigma \leq s - \frac{1}{2}$. Assume that $\eta \in H^{s+\frac{1}{2}}$, $f \in H^{\sigma+1}$ and*

$$\nabla_{x,z} v \in L^\infty([z_0, 0]; L^\infty)$$

for some $z_0 \in (-1, 0)$. Then for any $z_1 \in (z_0, 0)$ and $\varepsilon \in (0, s - \frac{1}{2} - \frac{d}{2})$, there exists an increasing function \mathcal{F} depending only on $s, \sigma, z_0, \varepsilon$ such that

$$(2.2.25) \quad \|\nabla_{x,z} v\|_{X^\sigma([z_1, 0])} \leq \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}}) \left\{ \|f\|_{H^{\sigma+1}} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla_{x,z} v\|_{L^\infty([z_0, 0]; L^\infty)} + \|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}([z_0, 0])} \right\}.$$

Besov estimates

Our goal is to establish regularity results for $\nabla_{x,z} v$ in Besov spaces. In particular, we shall need such results in the Zygmund space with negative index $C_*^{-\frac{1}{2}}$, which is one of the new technical issues compared to [11, 3, 7, 8, 102]. To this end, we follow the general strategy in [11] by first parilinearizing equation (2.2.13) and then factorizing this second order elliptic operator into the product of a forward and a backward parabolic operator. The study of $\nabla_{x,z} v$ in $C_*^{-\frac{1}{2}}$ will make use of the maximum principle in Proposition 2.2.5. The proof of the next lemma is straightforward.

Lemma 2.2.13. *Set*

$$(2.2.26) \quad R_1 v = (\alpha - T_\alpha) \Delta_x v + (\beta - T_\beta) \cdot \nabla \partial_z v - (\gamma - T_\gamma) \partial_z v, \quad R_2 v = T_\gamma \partial_z v.$$

Consider two symbols

$$(2.2.27) \quad \begin{aligned} a^{(1)} &= \frac{1}{2} (-i\beta \cdot \xi - \sqrt{4\alpha |\xi|^2 - (\beta \cdot \xi)^2}), \\ A^{(1)} &= \frac{1}{2} (-i\beta \cdot \xi + \sqrt{4\alpha |\xi|^2 - (\beta \cdot \xi)^2}), \end{aligned}$$

which satisfy $a + A = -i\beta \cdot \xi$, $aA = -\alpha |\xi|^2$. Next, set

$$(2.2.28) \quad R_3 = -(T_{a^{(1)}} T_{A^{(1)}} - T_\alpha \Delta) + T_{\partial_z A^{(1)}}.$$

Then we have

$$\mathcal{L} v = (\partial_z - T_{a^{(1)}})(\partial_z - T_{A^{(1)}}) v + R_1 v + R_2 v + R_3 v.$$

The next proposition provides a regularity bootstrap for $\nabla_{x,z}v$ in $B_{\infty,1}^r$ with $r \geq 0$. Its proof is inspired by that of Proposition 4.9 in [102].

Proposition 2.2.14. *Let $\varepsilon_0 > 0$ and $r \in [0, 1 + \varepsilon_0)$. Assume that $\eta \in C_*^{2+\varepsilon_0} \cap L^2$, $f \in H^{\frac{1}{2}}$, $\nabla f \in B_{\infty,1}^r$ and for some $z_0 \in (-1, 0)$*

$$(2.2.29) \quad \nabla_{x,z}v \in \tilde{L}^\infty([z_0, 0]; B_{\infty,1}^{r-\frac{1}{2}}) \cap \tilde{L}^\infty([z_0, 0]; B_{\infty,1}^0)$$

Then, for any $z_1 \in (z_0, 0)$, we have $\nabla_{x,z}v \in C([z_1, 0]; B_{\infty,1}^r)$ and

$$(2.2.30) \quad \|\nabla_{x,z}v\|_{C([z_1, 0]; B_{\infty,1}^r)} \leq K_{\eta, \varepsilon_0} \|\nabla f\|_{B_{\infty,1}^r} + E(\eta, f),$$

where, K_{η, ε_0} is a constant of the form

$$(2.2.31) \quad \mathcal{F}(\|\eta\|_{C_*^{2+\varepsilon_0}} + \|\eta\|_{L^2})$$

with $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ nondecreasing.

Remark. It is important for later applications that our estimate involves only the Besov norm of ∇f and not f itself.

Proposition 2.2.14 is a conditional regularity result. It assumes weaker regularities of $\nabla_{x,z}v$ to derive the regularity in $C([z_1, 0]; B_{\infty,1}^r)$. The later will allow us to estimate the trace $\nabla_{x,z}v|_{z=0}$ in the same space.

Proof. Recall the definitions of R_j $j = 1, 2, 3$ in Lemma 2.2.13. Pick $\varepsilon > 0$ such that $2\varepsilon < \min\{\frac{1}{2}, 1 + \varepsilon_0 - r\}$. We shall frequently use the following fact: for all $s \in \mathbf{R}$ and for all $\delta > 0$, there exists $C > 0$ such that

$$(2.2.32) \quad \frac{1}{C} \|u\|_{C_*^s} \leq \|u\|_{B_{\infty,1}^s} \leq C \|u\|_{C_*^{s+\delta}}.$$

Step 1. In this step, we estimate $R_j v$ in $L^2(J; B_{\infty,1}^{r-\frac{1}{2}})$ for any $J \subset [-1, 0]$. For R_1 we write using the BonyCalculusymboliqueet79 decomposition

$$(\alpha - T_\alpha)\Delta_x v = T_{\Delta_x v} \alpha + R(\Delta_x v, \alpha).$$

Applying (A.0.31) and the assumption (2.2.29) (i) gives

$$\|T_{\Delta_x v} \alpha\|_{\tilde{L}^2 B_{\infty,1}^{r-\frac{1}{2}}} \lesssim \|\alpha\|_{\tilde{L}^2 B_{\infty,1}^{r+\frac{1}{2}+\varepsilon}} \|\Delta_x v\|_{\tilde{L}^\infty C_*^{-1-\varepsilon}} \lesssim K_{\eta, \varepsilon_0} \|\nabla_x v\|_{\tilde{L}^\infty B_{\infty,1}^{-\varepsilon}},$$

where we have used the facts that $r + \frac{1}{2} + 2\varepsilon \leq \frac{3}{2} + \varepsilon_0$ and (by (2.2.17))

$$\|\alpha\|_{\tilde{L}^2 B_{\infty,1}^{r+\frac{1}{2}+\varepsilon}} \lesssim \|\alpha\|_{\tilde{L}^2 C_*^{r+\frac{1}{2}+2\varepsilon}} \lesssim K_{\eta, \varepsilon_0}.$$

Next, noticing that $(\frac{3}{2} + \varepsilon_0) + (-1 - \varepsilon) > 0$ and $\frac{3}{2} + \varepsilon_0 - 1 - \varepsilon \geq r - \frac{1}{2}$, we obtain by using (A.0.32)

$$\|R(\Delta_x v, \alpha)\|_{\tilde{L}^2 B_{\infty,1}^{r-\frac{1}{2}}} \lesssim \|\alpha\|_{\tilde{L}^2 C_*^{\frac{3}{2}+\varepsilon_0}} \|\Delta_x v\|_{\tilde{L}^\infty B_{\infty,1}^{-1-\varepsilon}} \lesssim K_{\eta, \varepsilon_0} \|\nabla_x v\|_{\tilde{L}^\infty B_{\infty,1}^{-\varepsilon}}.$$

The term $(\beta - T_\beta) \cdot \nabla_x \partial_z v$ can be treated in the same way. Lastly, it holds that

$$\|T_{\partial_z v} \gamma\|_{\tilde{L}^2 B_{\infty,1}^{r-\frac{1}{2}}} \lesssim \|\gamma\|_{\tilde{L}^2 B_{\infty,1}^{r-\frac{1}{2}+\varepsilon}} \|\partial_z v\|_{\tilde{L}^\infty C_*^{-\varepsilon}} \lesssim K_{\eta, \varepsilon_0} \|\partial_z v\|_{\tilde{L}^\infty B_{\infty,1}^{-\varepsilon}}$$

and

$$\|R(\partial_z v, \gamma)\|_{\tilde{L}^2 B_{\infty,1}^{r-\frac{1}{2}}} \lesssim \|\gamma\|_{\tilde{L}^2 C_*^{\frac{1}{2}+\varepsilon_0}} \|\partial_z v\|_{\tilde{L}^\infty B_{\infty,1}^{-\varepsilon}} \lesssim K_{\eta,\varepsilon_0} \|\partial_z v\|_{L^\infty B_{\infty,1}^{-\varepsilon}}.$$

Gathering the above estimates leads to

$$\|R_1 v\|_{\tilde{L}^2(J; B_{\infty,1}^{r-\frac{1}{2}})} \lesssim K_{\eta,\varepsilon_0} \|\nabla_{x,z} v\|_{\tilde{L}^\infty(J; B_{\infty,1}^{-\varepsilon})}.$$

On the other hand, $R_2 v$ satisfies (using (A.0.31))

$$\begin{aligned} \|R_2 v\|_{\tilde{L}^2(J; B_{\infty,1}^{r-\frac{1}{2}})} &= \|T_\gamma \partial_z v\|_{\tilde{L}^2(J; B_{\infty,1}^{r-\frac{1}{2}})} \\ &\lesssim \|\gamma\|_{\tilde{L}^2(J; L^\infty)} \|\partial_z v\|_{L^\infty(J; B_{\infty,1}^{r-\frac{1}{2}})} \lesssim K_{\eta,\varepsilon_0} \|\partial_z v\|_{\tilde{L}^\infty(J; B_{\infty,1}^{r-\frac{1}{2}})}, \end{aligned}$$

which is finite due to the assumption (2.2.29).

Next, noticing that (see Notation A.0.7)

$$\mathcal{M}_1^1(a^{(1)}) + \mathcal{M}_1^1(A^{(1)}) + \mathcal{M}_0^1(\partial_z A^{(1)}) \lesssim K_{\eta,\varepsilon_0}$$

we can apply Lemma A.0.16 to deduce that R_3 is of order 1 and

$$\|R_3 v\|_{\tilde{L}^2(J; B_{\infty,1}^{r-\frac{1}{2}})} \leq \|R_3 v\|_{\tilde{L}^\infty(J; B_{\infty,1}^{r-\frac{1}{2}})} \lesssim K_{\eta,\varepsilon_0} \|\nabla_x v\|_{\tilde{L}^\infty(J; B_{\infty,1}^{r-\frac{1}{2}})}.$$

In view of Lemma 2.2.13, we have proved that

$$(\partial_z - T_{a^{(1)}})(\partial_z - T_{A^{(1)}})v = F$$

with

$$\|F\|_{\tilde{L}^2(J; B_{\infty,1}^{r-\frac{1}{2}})} \lesssim_{K_{\eta,\varepsilon_0}} \|\nabla_{x,z} v\|_{\tilde{L}^\infty(J; B_{\infty,1}^{r-\frac{1}{2}} \cap B_{\infty,1}^{-\varepsilon})}.$$

Step 2. Fix $-1 < z_0 < z_1 < 0$ and introduce κ a cut-off function satisfying $\kappa|_{z < z_0} = 0$, $\kappa|_{z > z_1} = 1$. Setting $w = \kappa(z)(\partial_z - T_{A^{(1)}})v$, then

$$(\partial_z - T_{a^{(1)}})w = G := \kappa(z)F + \kappa'(z)(\partial_z - T_{A^{(1)}})v.$$

As $w|_{z=z_0} = 0$, applying Theorem A.0.22 yields for sufficiently large $\delta > 0$ to be chosen, that $w \in C([z_0, 0]; B_{\infty,1}^r)$ and

$$\begin{aligned} \|w\|_{C([z_0, 0]; B_{\infty,1}^r)} &\lesssim \|\kappa(z)F\|_{\tilde{L}^2([z_0, 0]; B_{\infty,1}^{r-\frac{1}{2}})} + \\ &\quad \|\kappa'(z)(\partial_z - T_{A^{(1)}})v\|_{\tilde{L}^2([z_0, 0]; B_{\infty,1}^{r-\frac{1}{2}})} + \|w\|_{L^\infty([z_0, 0]; C_*^{-r_0})}. \end{aligned}$$

Choosing $r_0 > \varepsilon$ and using (2.2.32) we deduce

$$\|w\|_{C([z_0, 0]; B_{\infty,1}^r)} \lesssim_{K_{\eta,\varepsilon_0}} \|\nabla_{x,z} v\|_{\tilde{L}^\infty([z_0, 0]; B_{\infty,1}^{r-\frac{1}{2}} \cap B_{\infty,1}^{-\varepsilon})}.$$

Now, on $[z_1, 0]$, v satisfies

$$(\partial_z - T_{A^{(1)}})\nabla_x v = \nabla w + T_{\nabla_x A^{(1)}} v, \quad \nabla_x v|_{z=0} = \nabla f.$$

After changing $z \mapsto -z$, Theorem A.0.22 gives for sufficiently large $\delta > 0$

$$(2.2.33) \quad \begin{aligned} \|\nabla_x v\|_{C([z_1, 0]; B_{\infty, 1}^r)} &\lesssim_{K_{\eta, \varepsilon_0}} \|\nabla f\|_{B_{\infty, 1}^r} + \|\nabla w\|_{\tilde{L}^\infty([z_1, 0]; B_{\infty, 1}^{r-1})} \\ &\quad + \|T_{\nabla_x A^{(1)}} v\|_{\tilde{L}^\infty([z_1, 0]; B_{\infty, 1}^{r-1})} + \|\nabla_x v\|_{\tilde{L}^\infty([z_1, 0]; C_*^{-\delta})} \\ &\lesssim_{K_{\eta, \varepsilon_0}} \|\nabla f\|_{B_{\infty, 1}^r} + \|\nabla_{x,z} v\|_{\tilde{L}^\infty([z_0, 0]; B_{\infty, 1}^{r-\frac{1}{2}+\varepsilon} \cap B_{\infty, 1}^{-\varepsilon})}. \end{aligned}$$

Then, from the equation $\partial_z v = w + T_{A^{(1)}} v$ we see that $\partial_z v \in C([z_1, 0]; B_{\infty, 1}^r)$ with norm bounded by the right-hand side of (2.2.33). We split

$$\|\nabla_{x,z} v\|_{\tilde{L}^\infty([z_0, 0]; B_{\infty, 1}^{r-\frac{1}{2}} \cap B_{\infty, 1}^{-\varepsilon})}$$

into two norms, one is over $[z_0, z_1]$ and the other is over $[z_1, 0]$. The one over $[z_0, z_1]$ can be bounded by $\|f\|_{H^{\frac{1}{2}}}$ using the estimate (2.2.8). Indeed, the fluid domain corresponding to $[z_0, z_1]$ belongs to the interior of Ω_t , where ϕ is analytic, and thus the result follows from the standard elliptic theory (see for instance the proof of Lemma 2.9, [3]). On the other hand, by choosing a large $\delta > 0$ and interpolating between $B_{\infty, 1}^{-\delta}$ and $B_{\infty, 1}^r$, the term

$$\|\nabla_{x,z} v\|_{\tilde{L}^\infty([z_1, 0]; B_{\infty, 1}^{r-\frac{1}{2}} \cap B_{\infty, 1}^{-\varepsilon})}$$

appearing on the right-hand side of (2.2.33), can be absorbed by $\|\nabla_{x,z} v\|_{\tilde{L}^\infty([z_1, 0]; B_{\infty, 1}^r)}$ on the left-hand side, leaving a term bounded by $\|\nabla_{x,z} v\|_{\tilde{L}^\infty([z_1, 0]; B_{\infty, 1}^{-\delta})}$. Finally, choosing $\delta > \frac{d}{2} + \frac{1}{2}$, we conclude by (2.2.32), Sobolev's embedding and (2.2.19)-(2.2.20) that

$$\|\nabla_{x,z} v\|_{\tilde{L}^\infty([z_1, 0]; B_{\infty, 1}^{-\delta})} \lesssim \|\nabla_{x,z} v\|_{\tilde{L}^\infty([z_1, 0]; H^{-\frac{1}{2}})} \lesssim K_{\eta, \varepsilon_0} E(\eta, f).$$

□

Corollary 2.2.15. *Let $s > \frac{3}{2} + \frac{d}{2}$, $\varepsilon_0 \in (0, s - \frac{3}{2} - \frac{d}{2})$ and $r \in [0, 1 + \varepsilon_0)$. Assume that $\eta \in H^{s+\frac{1}{2}}$ and $f \in H^s$, $\nabla f \in B_{\infty, 1}^r$. Then for any $z \in (-1, 0)$, we have $\nabla_{x,z} v \in C([z, 0]; B_{\infty, 1}^r)$ and*

$$\|\nabla_{x,z} v\|_{C([z, 0]; B_{\infty, 1}^r)} \lesssim_{K_{\eta, \varepsilon_0}} \|\nabla f\|_{B_{\infty, 1}^r} + E(\eta, f).$$

Proof. Under the assumptions on the Sobolev regularity of η and f , we can apply Proposition 2.2.10 in conjunction with (2.2.21) to get for any $z \in (-1, 0)$,

$$\nabla_{x,z} v \in C([z, 0]; H^{s-1}) \hookrightarrow C([z, 0]; C_*^{\frac{1}{2}+\varepsilon_0}) \hookrightarrow C([z, 0]; B_{\infty, 1}^{\frac{1}{2}}).$$

Notice that $\eta \in H^{s+\frac{1}{2}} \hookrightarrow C_*^{2+\varepsilon_0}$ and $\nabla f \in H^{s-1} \hookrightarrow B_{\infty, 1}^{\frac{1}{2}}$. Then the bootstrap provided by Proposition 2.2.14 concludes the proof. □

Considering the case $r = -\frac{1}{2}$, we first establish an a priori estimate.

Proposition 2.2.16. *Assume that $\eta \in C_*^{2+\varepsilon_0} \cap L^2$ for some $\varepsilon_0 > 0$, and $f \in L^\infty$, $\nabla f \in C_*^{-\frac{1}{2}}$. If $\nabla_{x,z} v \in C([z, 0]; C_*^{-\frac{1}{2}})$ for some $z \in (-1, 0)$ then*

$$(2.2.34) \quad \|\nabla_{x,z} v\|_{C([z, 0]; C_*^{-\frac{1}{2}})} \lesssim_{K_{\eta, \varepsilon_0}} \|\nabla f\|_{C_*^{-\frac{1}{2}}} + E(\eta, f).$$

Proof. We follow the proof of Proposition 2.2.15. The first step consists in estimating $R_j v$ in $\tilde{L}^2 C_*^{-1}$. Fix $0 < \varepsilon < \min\{\frac{1}{2}, \varepsilon_0\}$. For $R_1 v$, a typical term can be treated as

$$\begin{aligned} \|(\alpha - T_\alpha)\Delta_x v\|_{\tilde{L}^2(J; C_*^{-1})} &\lesssim \|\alpha\|_{\tilde{L}^2(J; C_*^{\frac{3}{2}+\varepsilon_0})} \|\Delta_x v\|_{\tilde{L}^\infty(J; C_*^{-\frac{3}{2}-\varepsilon})} \\ &\lesssim K_{\eta, \varepsilon_0} \|\nabla_x v\|_{\tilde{L}^\infty(J; C_*^{-\frac{1}{2}-\varepsilon})}. \end{aligned}$$

On the other hand, $R_2 v$ satisfies

$$\|R_2 v\|_{\tilde{L}^2(J; C_*^{-1})} \lesssim \|\gamma\|_{\tilde{L}^2(J; C_*^{\frac{1}{2}+\varepsilon})} \|\partial_z v\|_{\tilde{L}^\infty(J; C_*^{-\frac{1}{2}+\varepsilon})} \lesssim K_{\eta, \varepsilon_0} \|\partial_z v\|_{\tilde{L}^\infty(J; C_*^{-\frac{1}{2}-\varepsilon})}.$$

Since R_3 is of order 1 with norm bounded by K_{η, ε_0} , it holds that

$$\|R_3 v\|_{\tilde{L}^2(J; C_*^{-1})} \lesssim K_{\eta, \varepsilon_0} \|\nabla_x v\|_{\tilde{L}^\infty(J; C_*^{-1})}.$$

Consequently, we obtain

$$(\partial_z - T_{a(1)})(\partial_z - T_{A(1)})v = F$$

with

$$\|F\|_{\tilde{L}^2(J; C_*^{-1})} \lesssim K_{\eta, \varepsilon_0} \|\nabla_{x,z} v\|_{\tilde{L}^\infty(J; C_*^{-\frac{1}{2}-\varepsilon})}.$$

Now, arguing as in the proof of Proposition 2.2.15, one concludes the proof by applying twice Theorem A.0.22, then interpolating the norm $\|\nabla_{x,z} v\|_{\tilde{L}^\infty C_*^{-\frac{1}{2}-\varepsilon}}$ between $\|\nabla_{x,z} v\|_{\tilde{L}^\infty C_*^{-\frac{1}{2}}}$ and $\|\nabla_{x,z} v\|_{\tilde{L}^\infty C_*^{-\delta}}$ with large $\delta > 0$, where the later can be controlled by $E(\eta; f)$ via Sobolev's embedding. \square

Next, we prove a regularity result, assuming $1/2$ more derivative of η .

Proposition 2.2.17. *Assume that $\eta \in C_*^{\frac{5}{2}+\varepsilon_0} \cap L^2$ for some $\varepsilon_0 > 0$, and $f \in L^\infty \cap H^{\frac{1}{2}}$, $\nabla f \in C_*^{-\frac{1}{2}}$. Then, for any $z \in (-1, 0)$ we have $\nabla_{x,z} v \in C([z, 0]; C_*^{-\frac{1}{2}})$ and*

$$(2.2.35) \quad \|\nabla_{x,z} v\|_{C([z, 0]; C_*^{-\frac{1}{2}})} \leq K_{\eta, \varepsilon_0} \left\{ \|\nabla f\|_{C_*^{-\frac{1}{2}}} + (1 + \|\eta\|_{C_*^{\frac{5}{2}+\varepsilon_0}}) \|f\|_{L^\infty} \right\}.$$

Proof. We still follow the proof of Proposition 2.2.15. The first step consists in estimating $R_j v$ in $\tilde{L}^2 C_*^{-1}$. For $R_1 v$, a typical term can be treated as

$$\begin{aligned} \|(\alpha - T_\alpha)\Delta_x v\|_{\tilde{L}^2(J; C_*^{-1})} &\lesssim \|\alpha\|_{\tilde{L}^2(J; C_*^{2+\varepsilon_0})} \|\Delta_x v\|_{\tilde{L}^\infty(J; C_*^{-2})} \\ &\lesssim K_{\eta, \varepsilon_0} (1 + \|\eta\|_{C_*^{\frac{5}{2}+\varepsilon_0}}) \|\nabla_x v\|_{\tilde{L}^\infty(J; C_*^{-1})}. \end{aligned}$$

On the other hand, $R_2 v$ satisfies

$$\|R_2 v\|_{\tilde{L}^2(J; C_*^{-1})} \lesssim \|\gamma\|_{\tilde{L}^2(J; L^\infty)} \|\partial_z v\|_{\tilde{L}^\infty(J; C_*^{-1})} \lesssim K_{\eta, \varepsilon_0} \|\partial_z v\|_{\tilde{L}^\infty(J; C_*^{-1})}.$$

Since R_3 is of order 1 with norm bounded by K_{η, ε_0} , it holds that

$$\|R_3 v\|_{\tilde{L}^2(J; C_*^{-1})} \lesssim K_{\eta, \varepsilon_0} \|\nabla_x v\|_{\tilde{L}^\infty(J; C_*^{-1})}.$$

Consequently, we obtain

$$(\partial_z - T_{a(1)})(\partial_z - T_{A(1)})v = F$$

with

$$\|F\|_{\tilde{L}^2(J; C_*^{-1})} \lesssim K_{\eta, \varepsilon_0} (1 + \|\eta\|_{C_*^{\frac{5}{2} + \varepsilon_0}}) \|\nabla_{x,z} v\|_{\tilde{L}^\infty(J; C_*^{-1})}.$$

Then, arguing as in the proof of Proposition 2.2.15, one concludes the proof by applying twice Theorem A.0.22: once with $q = 2$, $\delta \gg 1$ and once with $q = 1$ and $\delta = 1$ so that Proposition 2.2.5 can be invoked to have

$$\|\nabla_{x,z} v\|_{\tilde{L}^\infty(J; C_*^{-1})} \lesssim_{K_{\eta, \varepsilon_0}} \|f\|_{L^\infty}.$$

□

2.2.3 Estimates for the Dirichlet-Neumann operator

We now apply the elliptic estimates in the previous subsection to study the continuity of the Dirichlet-Neumann operator. Put

$$\zeta_1 = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho}, \quad \zeta_2 = \nabla_x \rho.$$

By the definition (2.2.7), the Dirichlet-Neumann operator is given by

$$(2.2.36) \quad \begin{aligned} G(\eta)f &= \zeta_1 \partial_z v - \zeta_2 \cdot \nabla_x v|_{z=0} \\ &= h^{-1} \partial_z v + (\zeta_1 - h^{-1}) \partial_z v - \zeta_2 \cdot \nabla_x v|_{z=0}, \end{aligned}$$

where v is the solution to (2.2.1).

Proposition 2.2.18. *Let $s > \frac{3}{2} + \frac{d}{2}$, $\eta \in H^{s+\frac{1}{2}}$ and $f \in H^s$. Then we have*

$$(2.2.37) \quad \|G(\eta)f\|_{H^{s-1}} \leq_{K_{\eta, \varepsilon_0}} \|f\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \left\{ \|\nabla f\|_{B_{\infty,1}^0} + E(\eta, f) \right\}.$$

Proof. Notice first that by the Sobolev embedding, $\eta \in C_*^{2+\varepsilon_0}$. Using the formula (2.2.36) and the tame estimate (A.0.24) we obtain

$$\|G(\eta)f\|_{H^{s-1}} \lesssim_{K_{\eta, \varepsilon_0}} \|\nabla_{x,z} v|_{z=0}\|_{H^{s-1}} + \|\eta\|_{H^s} \|\nabla_{x,z} v|_{z=0}\|_{L^\infty}.$$

Under the hypotheses, Corollary 2.2.15 is applicable with $r = 0$. Hence, in view of (2.2.32), it holds that

$$\forall z \in (-1, 0), \quad \|\nabla_{x,z} v\|_{C^0([z,0]; B_{\infty,1}^0)} \lesssim_{K_{\eta, \varepsilon_0}} \|\nabla f\|_{B_{\infty,1}^0} + E(\eta, f).$$

Noticing embedding $B_{\infty,1}^0 \hookrightarrow L^\infty$, we deduce

$$\|G(\eta)f\|_{H^{s-1}} \lesssim_{K_{\eta, \varepsilon_0}} \|f\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \left\{ \|\nabla f\|_{B_{\infty,1}^0} + E(\eta, f) \right\},$$

which is the desired estimate. □

Proposition 2.2.19. *We have the following estimates for the Dirichlet-Neumann operator in Zygmund spaces.*

1. Let $s > \frac{3}{2} + \frac{d}{2}$, $\varepsilon_0 \in (0, s - \frac{3}{2} - \frac{d}{2})$ and $r \in (0, 1 + \varepsilon_0)$. Assume that $\eta \in H^{s+\frac{1}{2}}$ and $f \in H^s$, $\nabla f \in B_{\infty,1}^r$. Then we have

$$(2.2.38) \quad \|G(\eta)f\|_{B_{\infty,1}^r} \lesssim_{K_{\eta, \varepsilon_0}} \|\nabla f\|_{B_{\infty,1}^r} + E(\eta, f),$$

where recall that K_{η, ε_0} is defined by (2.2.31).

2. Let $\varepsilon_0 > 0$. Assume that $\eta \in C_*^{\frac{5}{2}+\varepsilon_0}$, $f \in L^\infty \cap H^{\frac{1}{2}}$ and $\nabla f \in C_*^{-\frac{1}{2}}$, then

$$(2.2.39) \quad \|G(\eta)f\|_{C_*^{-\frac{1}{2}}} \lesssim_{K_{\eta,\varepsilon_0}} \|\nabla f\|_{C_*^{-\frac{1}{2}}} + (1 + \|\eta\|_{C_*^{\frac{5}{2}+\varepsilon_0}}) \|f\|_{L^\infty}.$$

3. Let $\varepsilon_0 > 0$. Assume that $\eta \in C_*^{2+\varepsilon_0} \cap H^{1+\frac{d}{2}+}$ and $f \in H^{\frac{1}{2}+\frac{d}{2}}$, $\nabla f \in C_*^{-\frac{1}{2}}$, then

$$(2.2.40) \quad \|G(\eta)f\|_{C_*^{-\frac{1}{2}}} \lesssim_{K_{\eta,\varepsilon_0}} \|\nabla f\|_{C_*^{-\frac{1}{2}}} + E(\eta, f).$$

Proof. We first notice that $\|\zeta_j|_{z=0}\|_{C_*^{1+\varepsilon_0}} \lesssim K_{\eta,\varepsilon_0}$.

1. Using the BonyCalculusymboliqueet79 decomposition for the right-hand side of (2.2.36), we see that (2.2.38) is a consequence of Corollary 2.2.15, (A.0.28), (A.0.29) and the embedding $B_{\infty,1}^0 \hookrightarrow L^\infty$.

2. For (2.2.39) one applies the product rule (A.0.25) and Proposition 2.2.17.

3. For (2.2.40) we first remark that owing to Proposition 2.2.10, the assumptions $\eta \in H^{1+\frac{d}{2}+}$, $f \in H^{\frac{1}{2}+\frac{d}{2}}$ imply

$$z \in (-1, 0), \quad \nabla_{x,z}v \in C([z, 0]; H^{-\frac{1}{2}+\frac{d}{2}}) \hookrightarrow C([z, 0]; C_*^{-\frac{1}{2}}).$$

Therefore, the a priori estimate of Proposition 2.2.16 yields

$$\|\nabla_{x,z}v\|_{C([z,0]; C_*^{-\frac{1}{2}})} \lesssim_{K_{\eta,\varepsilon_0}} \|f\|_{H^{\frac{1}{2}}} + E(\eta, f),$$

which, combined with (A.0.25), concludes the proof. \square

To conclude this section, let us recall the following result on the *shape derivative* of the Dirichlet-Neumann operator.

Theorem 2.2.20 (see [71, Theorem 3.21]). *Let $s > \frac{1}{2} + \frac{d}{2}$, $d \geq 1$ and $\psi \in H^{\frac{3}{2}}$. Then the map*

$$G(\cdot)\psi : H^{s+\frac{1}{2}} \rightarrow H^{\frac{1}{2}}$$

is differentiable and for any $f \in H^{s+\frac{1}{2}}$,

$$d_\eta G(\eta)\psi \cdot f := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (G(\eta + \varepsilon f)\psi - G(\eta)f) = -G(\eta)(Bf) - \operatorname{div}(Vf)$$

where B and V are functions of (η, ψ) as in (2.1.8).

2.3 Parilinearization and symmetrization of the system

Throughout this section, we assume that (η, ψ) is a solution to (2.1.7) on a time interval $I = [0, T]$ and

$$(2.3.1) \quad \begin{cases} \eta \in L^\infty(I; H^{s+\frac{1}{2}}) \cap L^1(I; C_*^{\frac{5}{2}+\varepsilon_*}), \\ \psi \in L^\infty(I; H^s), \nabla_x \psi \in L^1(I; B_{\infty,1}^1) \\ s > \frac{3}{2} + \frac{d}{2}, \varepsilon_* > 0 \\ \inf_{t \in I} \operatorname{dist}(\eta(t), \Gamma) \geq h > 0. \end{cases}$$

We fix from now on

$$0 < \varepsilon < \min\{\varepsilon_*, \frac{1}{2}\}$$

and define the quantities

$$(2.3.2) \quad \begin{aligned} \mathcal{A} &= \|\eta\|_{C_*^{2+\varepsilon_*}} + \|\eta\|_{L^2} + \|\nabla_x \psi\|_{B_{\infty,1}^0} + E(\eta, \psi), \\ \mathcal{B} &= \|\eta\|_{C_*^{\frac{5}{2}+\varepsilon_*}} + \|\nabla_x \psi\|_{B_{\infty,1}^1} + 1. \end{aligned}$$

Our goal is to derive estimates for (η, ψ) in $L^\infty(I; H^{s+\frac{1}{2}} \times H^s)$ by means of \mathcal{A} and \mathcal{B} and keep them *linear in \mathcal{B}* .

2.3.1 Paralinearization of the Dirichlet-Neumann operator

Our goal is to obtain error estimates for $G(\eta)\psi$ when expanding it in paradifferential operators. More precisely, as in Proposition 3.14, [3], we will need such expansion in terms of the first two symbols defined by

$$(2.3.3) \quad \begin{aligned} \lambda^{(1)} &:= \sqrt{(1 + |\nabla\eta|^2)|\xi|^2 - (\nabla\eta \cdot \xi)^2}, \\ \lambda^{(0)} &:= \frac{1 + |\nabla\eta|^2}{2\lambda^{(1)}} \left[\operatorname{div}(\alpha^{(1)} \nabla\eta) + i\partial_\xi \lambda^{(1)} \cdot \nabla \alpha^{(1)} \right] \end{aligned}$$

with

$$\alpha^{(1)} := \frac{1}{1 + |\nabla\eta|^2} (\lambda^{(1)} + i\nabla\eta \cdot \xi).$$

Set $\lambda := \lambda^{(1)} + \lambda^{(0)}$.

To study $G(\eta)\psi$, we reconsider the elliptic problem (2.2.1), *i.e.*,

$$(2.3.4) \quad \Delta_{x,y} \phi = 0 \text{ in } \Omega, \quad \phi|_\Sigma = \psi, \quad \partial_n \phi|_\Gamma = 0.$$

Let

$$v(x, z) = \phi(x, \rho(x, z)) \quad (x, z) \in S = \mathbf{R}^d \times (-1, 0)$$

as in Section 2.2.2. Then, by (2.2.13), v satisfies $\mathcal{L}v = 0$ in S . Applying Proposition 2.2.12 with $\sigma = s - 1$ and Corollary 2.2.15 with $r = 0$ we obtain for any $z \in (-1, 0)$

$$(2.3.5) \quad \begin{aligned} \|\nabla_{x,z} v\|_{X^{s-1}([z,0])} &\lesssim_A \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \left\{ \|\nabla\psi\|_{B_{\infty,1}^0} + E(\eta, f) \right\}, \\ &\lesssim_{\mathcal{A}} \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

On the other hand, Corollary 2.2.15 with $r = 1$ yields for any $z \in (-1, 0)$

$$(2.3.6) \quad \|\nabla_{x,z} v\|_{C([z,0]; B_{\infty,1}^1)} \lesssim_{\mathcal{A}} \|\nabla\psi\|_{B_{\infty,1}^1} + E(\eta, f) \lesssim_{\mathcal{A}} \mathcal{B}.$$

Lemma 2.3.1. *We have*

$$\partial_z^2 v + T_\alpha \Delta_x v + T_\beta \cdot \nabla_x \partial_z v - T_\gamma \partial_z v - T_{\partial_z v} \gamma = F_1,$$

where, for all $I \in (-1, 0]$, F_1 satisfies

$$\|F_1\|_{Y^{s+\frac{1}{2}}(I)} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \right\}.$$

Proof. From equation (2.2.13) and the BonyCalculusymboliqueet79 decomposition, we see that

$$F_1 = -R_1 v = -(\alpha - T_\alpha)\Delta_x v - (\beta - T_\beta) \cdot \nabla \partial_z v + R(\gamma, \partial_z v).$$

Writing $(\alpha - T_\alpha)\Delta_x v = (\alpha - h^2 - T_{\alpha-h^2})\Delta_x v + (h^2 - T_{h^2})\Delta_x v$, we estimate using (2.3.6)

$$\begin{aligned} \|(\alpha - h^2 - T_{\alpha-h^2})\Delta_x v\|_{L^2 H^s} &\lesssim \|T_{\Delta_x v}(\alpha - h^2)\|_{L^2 H^s} + \|R(T_{\Delta_x v}, \alpha - h^2)\|_{L^2 H^s} \\ &\lesssim \|\Delta_x v\|_{L^2 L^\infty} \|(\alpha - h^2)\|_{L^2 H^s} \\ &\lesssim_{\mathcal{A}} \mathcal{B} \|\eta\|_{H^{s+\frac{1}{2}}}. \end{aligned}$$

Since $(h^2 - T_{h^2})$ is a smoothing operator, there holds by Remark 2.2.2

$$\|(h^2 - T_{h^2})\Delta_x v\|_{L^2 H^s} \lesssim \|\nabla_x v\|_{L^2 L^2} \lesssim (1 + \|\eta\|_{W^{1,\infty}}) \|\psi\|_{H^{\frac{1}{2}}} \lesssim_A \|\psi\|_{H^s}$$

The other terms of F_1 can be treated similarly. \square

The next step consists in studying the paradifferential equation satisfied by the *good-unknown* (see [11] and the reference therein)

$$u := v - T_b \rho \quad \text{with } b := \frac{\partial_z v}{\partial_z \rho}.$$

Notice that $b|_{z=0} = B$. Estimates for b is now provided.

Lemma 2.3.2. *For any $I \in (-1, 0]$, we have*

$$(2.3.7) \quad \|b\|_{L^\infty(I; L^\infty)} \lesssim_{\mathcal{A}} 1,$$

$$(2.3.8) \quad \|\nabla_{x,z} b\|_{L^\infty(I; L^\infty)} \lesssim_{\mathcal{A}} \mathcal{B},$$

$$(2.3.9) \quad \|\nabla_{x,z}^2 b\|_{L^\infty(I; C_*^{-1})} \lesssim_{\mathcal{A}} \mathcal{B}.$$

Proof. We first recall the lower bound (2.2.11)

$$(2.3.10) \quad \partial_z \rho \geq \frac{h}{2}.$$

Observe that with respect to the L^∞ -norm in z , ρ and η have the same Zygmund regularity, hence

$$(2.3.11) \quad \|\nabla_{x,z}^2 \rho\|_{L^\infty(I; C_*^{\varepsilon_*})} + \|\nabla_{x,z}^3 \rho\|_{L^\infty(I; C_*^{-1+\varepsilon_*})} \lesssim_{\mathcal{A}} 1.$$

Next, applying Corollary 2.2.15 with $r = 0$ yields

$$(2.3.12) \quad \|\nabla_{x,z} v\|_{C(I; B_{\infty,1}^0)} \lesssim_{\mathcal{A}} 1.$$

On the other hand, recall from (2.3.6) that

$$(2.3.13) \quad \|\nabla_{x,z} v\|_{C(I; B_{\infty,1}^1)} \lesssim_{\mathcal{A}} \mathcal{B}.$$

Using equation (2.2.13), $\partial_z^2 v$ can be expressed in terms of (α, β, γ) and $(\Delta_x v, \nabla_x \partial_z v, \partial_z v)$. It then follows from (2.3.13), (2.3.11) and Lemma A.0.14 that

$$(2.3.14) \quad \|\partial_z^2 v\|_{C(I; B_{\infty,1}^0)} \lesssim_{\mathcal{A}} \mathcal{B}.$$

Let us now consider

$$\partial_z^3 v = -\alpha \Delta_x \partial_z v - \partial_z \alpha \Delta_x v - \beta \cdot \nabla_x \partial_z^2 v - \partial_z \beta \cdot \nabla_x \partial_z v + \gamma \partial_z^2 v + \partial_z \gamma \partial_z v.$$

We notice the following bounds

$$\|\partial_z \alpha\|_{C(I; C_*^{\frac{1}{2}})} + \|\partial_z \beta\|_{C(I; C_*^{\frac{1}{2}})} + \|\partial_z \gamma\|_{C(I; C_*^{-\frac{1}{2}})} \lesssim_{\mathcal{A}} 1,$$

which can be proved along the same lines as the proof of (2.2.17).

Then using the above estimates and (A.0.25) one can derive

$$(2.3.15) \quad \|\partial_z^3 v\|_{C(I; C_*^{-1})} \lesssim_{\mathcal{A}} \mathcal{B}.$$

The estimates (2.3.7), (2.3.8), (2.3.9) are consequences of the above estimates and the Leibniz rule. \square

Lemma 2.3.3. *We have*

$$Pu := \partial_z^2 u + T_\alpha \Delta_x u + T_\beta \cdot \nabla_x \partial_z u - T_\gamma \partial_z u = F_2,$$

where, for all $I \in (-1, 0]$, F_2 satisfies

$$\|F_2\|_{L^2(I; H^s)} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \right\}.$$

Remark. Compared with the equation satisfied by v in Lemma 2.3.1, the introduction of the good-unknown u helps eliminate the bad term $T_{\partial_z v} \gamma$, which is not controlled in $L^2 H^s$.

Proof. We will write $A \sim B$ if

$$\|A - B\|_{L^2(I; H^s)} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \right\}.$$

From Lemma 2.3.1, we see that

$$(2.3.16) \quad Pu = Pv - PT_b \rho = T_{\partial_z v} \gamma - PT_b \rho + F_1$$

and $F_1 \sim 0$. Therefore, it suffices to prove that $PT_b \rho \sim T_\gamma \partial_z v$.

In the expression of $PT_b \rho$, we observe that owing to Lemma 2.3.2, all the terms containing ρ and $\nabla_{x,z} \rho$ are ~ 0 , hence

$$(2.3.17) \quad PT_b \rho \sim T_b \partial_z^2 \rho + T_\alpha T_b \Delta \rho + T_\beta \cdot T_b \nabla \partial_z \rho.$$

Next, we find an elliptic equation satisfied by ρ . Remark that $w(x, y) := y$ is a harmonic function in Ω . Then, under the change of variables $(x, z) \mapsto (x, \rho(x, z))$, $(x, z) \in S = \mathbf{R}^d \times (-1, 0)$,

$$\tilde{w}(x, z) := w(x, \rho(x, z)) = \rho(x, z)$$

satisfies

$$\mathcal{L}\rho = (\partial_z^2 + \alpha \Delta_x + \beta \cdot \nabla_x \partial_z - \gamma \partial_z) \rho = 0.$$

Then, by parilinearizing as in Lemma 2.3.1 we obtain

$$\partial_z^2 \rho + T_\alpha \Delta_x \rho + T_\beta \cdot \nabla_x \partial_z \rho - T_{\partial_z \rho} \gamma \sim 0,$$

where we have used the fact that $T_\gamma \partial_z \rho \sim 0$. Consequently,

$$T_b \partial_z^2 \rho + T_b T_\alpha \Delta_x \rho + T_b T_\beta \cdot \nabla_x \partial_z \rho - T_b T_{\partial_z \rho} \gamma \sim 0.$$

Comparing with (2.3.17) leads to

$$PT_b \rho \sim [T_\alpha, T_b] \Delta \rho + [T_\beta, T_b] \nabla \partial_z \rho + T_b T_{\partial_z \rho} \gamma.$$

By Lemma 2.3.2, it is easy to check that $[T_\alpha, T_b]$ is of order -1 and

$$\|[T_\alpha, T_b] \Delta \rho\|_{L^2 H^s} \lesssim_{\mathcal{A}} \mathcal{B} \|\Delta_x \rho\|_{L^2 H^{s-1}} \lesssim_{\mathcal{A}} \mathcal{B} \|\eta\|_{H^{s+\frac{1}{2}}}.$$

In other words, $[T_\alpha, T_b] \Delta \rho \sim 0$. By the same argument, we get $[T_\beta, T_b] \nabla \partial_z \rho \sim 0$. Finally, since

$$T_b T_{\partial_z \rho} \gamma \sim T_{b \partial_z \rho} \gamma = T_{\partial_z v} \gamma$$

we conclude that $PT_b \rho \sim T_\gamma \partial_z v$. \square

Next, in the spirit of Lemma 2.2.13, we factorize P into two parabolic operators.

Lemma 2.3.4. *Define*

$$\begin{aligned} a^{(0)} &= \frac{1}{A^{(1)} - a^{(1)}} \left(i \partial_\xi a^{(1)} \partial_x A^{(1)} - \gamma a^{(1)} \right), \\ A^{(0)} &= \frac{1}{a^{(1)} - A^{(1)}} \left(i \partial_\xi a^{(1)} \partial_x A^{(1)} - \gamma A^{(1)} \right) \end{aligned}$$

so that

$$(2.3.18) \quad a^{(1)} + A^{(1)} = -i\beta \cdot \xi, \quad a^{(1)} A^{(1)} = -\alpha |\xi|^2.$$

Set $a = a^{(1)} + a^{(0)}$, $A = A^{(1)} + A^{(0)}$ and $R = T_a T_A - T_\alpha \Delta$. Then we have

$$P = (\partial_z - T_a)(\partial_z - T_A) + R$$

and for any $I \Subset (-1, 0]$,

$$\|Ru\|_{L^2(I; H^s)} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \right\}.$$

Proof. From the definitions of a, A , we can check that

$$(2.3.19) \quad \begin{aligned} a^{(1)} A^{(1)} + \frac{1}{i} \partial_\xi a^{(1)} \cdot \partial_x A^{(1)} + a^{(1)} A^{(0)} + a^{(0)} A^{(1)} &= -\alpha |\xi|^2, \\ a + A &= -i\beta \cdot \xi + \gamma. \end{aligned}$$

A direct computation shows that

$$R = (T_a T_A - T_\alpha \Delta) + ((T_a + T_A) + (T_\beta \cdot \nabla - T_\gamma)) \partial_z = T_a T_A - T_\alpha \Delta$$

by the second equation of (2.3.19). Now, we write

$$T_a T_A = T_{a^{(1)}} T_{A^{(1)}} + T_{a^{(1)}} T_{A^{(0)}} + T_{a^{(0)}} T_{A^{(1)}} + T_{a^{(0)}} T_{A^{(0)}}.$$

We have the following bounds

$$\begin{aligned}\mathcal{M}_{\frac{3}{2}}^1(a^{(1)}) + \mathcal{M}_{\frac{3}{2}}^1(A^{(1)}) &\lesssim \mathcal{F}(\|\eta\|_{C_*^{2+\varepsilon}})(1 + \|\eta\|_{C_*^{\frac{5}{2}}}), \\ \mathcal{M}_{\frac{1}{2}}^1(a^{(1)}) + \mathcal{M}_{\frac{1}{2}}^1(A^{(1)}) &\lesssim \mathcal{F}(\|\eta\|_{C_*^{2+\varepsilon}}), \\ \mathcal{M}_{\frac{1}{2}}^0(a^{(0)}) + \mathcal{M}_{\frac{1}{2}}^0(A^{(0)}) &\lesssim \mathcal{F}(\|\eta\|_{C_*^{2+\varepsilon}})(1 + \|\eta\|_{C_*^{\frac{5}{2}}}), \\ \mathcal{M}_0^0(a^{(0)}) + \mathcal{M}_0^0(A^{(0)}) &\lesssim \mathcal{F}(\|\eta\|_{C_*^{2+\varepsilon}}).\end{aligned}$$

Then, applying Theorem A.0.4 (ii) we obtain

$$(2.3.20) \quad \begin{aligned}\|T_{a^{(0)}}T_{A^{(0)}} - T_{a^{(0)}A^{(0)}}\|_{H^{\mu-\frac{1}{2}} \rightarrow H^\mu} &\lesssim \Xi, \\ \|T_{a^{(0)}}T_{A^{(1)}} - T_{a^{(0)}A^{(1)}}\|_{H^{\mu+\frac{1}{2}} \rightarrow H^\mu} &\lesssim \Xi, \\ \|T_{a^{(1)}}T_{A^{(0)}} - T_{a^{(1)}A^{(0)}}\|_{H^{\mu+\frac{1}{2}} \rightarrow H^\mu} &\lesssim \Xi, \\ \|T_{a^{(1)}}T_{A^{(1)}} - T_{a^{(1)}A^{(1)}} - T_{\frac{1}{i}\partial_\xi a^{(1)} \cdot \partial_x A^{(1)}}\|_{H^{\mu+\frac{1}{2}} \rightarrow H^\mu} &\lesssim \Xi,\end{aligned}$$

where Ξ denotes any constant of the form

$$\mathcal{F}(\|\eta\|_{C_*^{2+\varepsilon}})(1 + \|\eta\|_{C_*^{\frac{5}{2}}}).$$

Therefore, the first equation of (2.3.19) implies

$$\|Ru\|_{L^2H^s} \lesssim_{\mathcal{A}} \mathcal{B} \|\nabla_x u\|_{L^2H^{s-\frac{1}{2}}}$$

where, we have replaced $\|u\|_{L^2H^{s+\frac{1}{2}}}$ by $\|\nabla_x u\|_{L^2H^{s-\frac{1}{2}}}$ according to Remark A.0.1. Finally, writing $\nabla_x u = \nabla_x v - T_{\nabla_x b} \rho - T_b \nabla_x \rho$ we conclude by means of (2.3.5) and (2.3.7) that

$$(2.3.21) \quad \|\nabla_x u\|_{L^2H^{s-\frac{1}{2}}} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \right\}.$$

□

Proposition 2.3.5. *It holds that*

$$G(\eta)\psi = T_\lambda(\psi - T_B\eta) + T_V \cdot \nabla\eta + F$$

with F satisfying

$$\|F\|_{H^{s+\frac{1}{2}}} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \right\}.$$

Proof. A combination of Lemma 2.3.3 and Lemma 2.3.4 yields

$$(\partial_z - T_a)(\partial_z - T_A)u = F_2,$$

where, F_2 satisfies for all $I \in (-1, 0]$,

$$(2.3.22) \quad \|F_2\|_{L^2(I; H^s)} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \right\}.$$

The proof proceeds in two steps.

Step 1. As in the proof of Proposition 2.2.30, we fix $-1 < z_0 < z_1 < 0$ and introduce κ a cut-off function satisfying $\kappa|_{z < z_0} = 0$, $\kappa|_{z > z_1} = 1$. Setting $w = \kappa(z)(\partial_z - T_A)u$, then

$$(\partial_z - T_a)w = G := \kappa(z)F_2u + \kappa'(z)(\partial_z - T_A)u.$$

We now bound G in $L^2([z_0, 0]; H^s)$. First, it follows directly from (2.3.22) that

$$(2.3.23) \quad \|\kappa(z)F_2u\|_{Y^{s+\frac{1}{2}}([z_0,0])} \lesssim \|\kappa(z)Ru\|_{L^2([z_0,0];H^s)} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \right\} =: \Pi.$$

Next, notice that $p := \kappa'(z)(\partial_z - T_A)u$ is non vanishing only for $z \in I := [z_0, z_1]$. In the light of Lemma 2.3.2,

$$\|\nabla_{x,z}u\|_{L^2(I;H^s)} \lesssim_{\mathcal{A}} \mathcal{B} \|\eta\|_{H^{s+\frac{1}{2}}} + \|\nabla_{x,z}v\|_{L^2(I;H^s)}.$$

Hence

$$\|(\partial_z - T_A)u\|_{L^2(I;H^s)} \lesssim_{\mathcal{A}} \|\nabla_{x,z}u\|_{L^2(I;H^s)} \lesssim_{\mathcal{A}} \mathcal{B} \|\eta\|_{H^{s+\frac{1}{2}}} + \|\nabla_{x,z}v\|_{L^2(I;H^s)}.$$

The fluid domain corresponds to $[z_0, z_1]$ is a strip lying in the interior of Ω_h , where the harmonic function ϕ is smooth by the standard elliptic theory. In particular, there holds (see for instance the proof of Lemma 2.9, [3])

$$\|\nabla_{x,z}v\|_{L^2(I;H^s)} \lesssim_A \|\psi\|_{H^{\frac{1}{2}}}.$$

Therefore, we can estimate

$$\begin{aligned} \|p\|_{L^2([z_0,0];H^s)} &\lesssim \|(\partial_z - T_A)u\|_{L^2([z_0,z_1];H^s)} \\ &\lesssim_{\mathcal{A}} \mathcal{B} \|\eta\|_{H^{s+\frac{1}{2}}} + \|\nabla_{x,z}v\|_{L^2([z_0,z_1];H^s)}, \\ &\lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^{\frac{1}{2}}} \right\}. \end{aligned}$$

This, combined with (2.3.23), yields

$$(2.3.24) \quad \|G\|_{Y^{s+\frac{1}{2}}([z_0,0])} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \right\} =: \Pi.$$

Consequently, as $w|_{z=z_0} = 0$, we can apply Theorem A.0.21 to have $\|w\|_{X^{s+\frac{1}{2}}([z_0,0])} \lesssim \Pi$, which implies

$$(2.3.25) \quad \|\partial_z u - T_A u\|_{X^{s+\frac{1}{2}}([z_1,0])} \lesssim \Pi.$$

Step 2. We will write $f_1 \sim f_2$ provided $\|f_1 - f_2\|_{X^{s+\frac{1}{2}}([z_1,0])} \leq \Pi$. By parilinearizing (using the BonyCalculusymboliqueet79 decomposition and Theorem A.0.10) we have

$$\frac{1 + |\nabla\rho|^2}{\partial_z\rho} \partial_z v - \nabla\rho \cdot \nabla v \sim T_{\frac{1+|\nabla\rho|^2}{\partial_z\rho}} \partial_z v + 2T_{b\nabla\rho} \cdot \nabla\rho - T_{b\frac{1+|\nabla\rho|^2}{\partial_z\rho}} \partial_z\rho - T_{\nabla\rho} \cdot \nabla v - T_{\nabla v} \cdot \nabla\rho.$$

Then replacing v with $u + T_b\rho$ we obtain, after some computations, that

$$\frac{1 + |\nabla\rho|^2}{\partial_z\rho} \partial_z v - \nabla\rho \cdot \nabla v \sim T_{\frac{1+|\nabla\rho|^2}{\partial_z\rho}} \partial_z u - T_{\nabla\rho} \cdot \nabla u + T_{b\nabla\rho - \nabla v} \cdot \nabla\rho.$$

Now, using (2.3.25) allows us to replace the normal derivative $\partial_z v$ with the "tangential derivative" $T_A v$, leaving a remainder which is ~ 0 . Therefore,

$$T_{\frac{1+|\nabla\rho|^2}{\partial_z\rho}} \partial_z u - T_{\nabla\rho} \cdot \nabla u \sim T_\Lambda u + T_{b\nabla\rho - \nabla v} \cdot \nabla\rho$$

with

$$\Lambda := \frac{1 + |\nabla\rho|^2}{\partial_z\rho} A - i\nabla\rho \cdot \xi.$$

One can check that $\Lambda|_{z=0} = \lambda = \lambda^{(1)} + \lambda^{(0)}$ as announced. On the other hand, at $z = 0$,

$$b\nabla\rho - \nabla v = B\nabla\eta - \nabla\psi = V, \quad u = \psi - T_B\eta.$$

In conclusion, we have proved that

$$G(\eta)\psi \sim T_\lambda(\psi - T_B\eta) + T_V \cdot \nabla\eta.$$

□

2.3.2 Parilinearization of the full system

Lemma 2.3.6. *There exists a nondecreasing function \mathcal{F} such that*

$$H(\eta) = T_\ell\eta + f,$$

where $\ell = \ell^{(2)} + \ell^{(1)}$ with

$$(2.3.26) \quad \ell^{(2)} = \left(1 + |\nabla\eta|^2\right)^{-\frac{1}{2}} \left(|\xi|^2 - \frac{(\nabla\eta \cdot \xi)^2}{1 + |\nabla\eta|^2}\right), \quad \ell^{(1)} = -\frac{i}{2}(\partial_x \cdot \partial_\xi)\ell^{(2)},$$

and $f \in H^s$ satisfying

$$\|f\|_{H^s} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|\eta\|_{C_*^{\frac{5}{2}}} \|\nabla\eta\|_{H^{s-\frac{1}{2}}}.$$

Proof. We first apply Theorem A.0.10 with $u = \nabla\eta$, $\mu = s - \frac{1}{2}$ and $\rho = \frac{3}{2}$ to have

$$\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}} = T_p\nabla\eta + f_1, \quad p = \frac{1}{(1 + |\nabla\eta|^2)^{\frac{1}{2}}} I - \frac{\nabla\eta \otimes \nabla\eta}{(1 + |\nabla\eta|^2)^{\frac{3}{2}}}$$

with f_1 satisfying

$$\|f_1\|_{H^{s-\frac{1}{2}+\frac{1}{2}}} \leq \mathcal{F}(\|\nabla\eta\|_{L^\infty}) \|\nabla\eta\|_{C_*^{\frac{3}{2}}} \|\nabla\eta\|_{H^{s-\frac{1}{2}}}.$$

Hence,

$$H(\eta) = -\operatorname{div}(T_p\nabla\eta + f_1) = T_{p\xi, \xi-i\operatorname{div} p\xi}\eta - \operatorname{div} f_1.$$

This gives the conclusion with $l^{(2)} = p\xi \cdot \xi$, $l^{(1)} = -i\operatorname{div} p\xi$, $f = -\operatorname{div} f_1$. □

We next parilinearize the other nonlinear terms. Recall the notations

$$B = \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1 + |\nabla\eta|^2}, \quad V = \nabla\psi - B\nabla\eta.$$

For later estimates on B , we write

$$(2.3.27) \quad \begin{aligned} B &= \frac{\nabla\eta}{1 + |\nabla\eta|^2} \cdot \nabla\psi + \frac{1}{1 + |\nabla\eta|^2} G(\eta)\psi \\ &=: K(\nabla\eta) \cdot \nabla\psi + L(\nabla\eta)G(\eta)\psi + G(\eta)\psi, \end{aligned}$$

where K and L are smooth function in $L^\infty(\mathbf{R}^d)$ and satisfy $K(0) = L(0) = 0$. From this expression and the BonyCalculusymboliqueet79 decomposition, one can easily prove the following.

Lemma 2.3.7. *We have*

$$(2.3.28) \quad \|(V, B)\|_{B_{\infty,1}^1} \lesssim_{\mathcal{A}} \mathcal{B},$$

$$(2.3.29) \quad \|(V, B)\|_{L^\infty} \lesssim_{\mathcal{A}} 1.$$

Lemma 2.3.8. *We have*

$$\frac{1}{2} |\nabla\psi|^2 - \frac{1}{2} \frac{(\nabla\eta \cdot \nabla\psi + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} = T_V \cdot \nabla\psi - T_V T_B \cdot \nabla\eta - T_B G(\eta)\psi + f,$$

with $f \in H^s$ and

$$\|f\|_{H^s} \lesssim_{\mathcal{A}} \mathcal{B} \{ \|\eta\|_{H^s} + \|\psi\|_{H^s} \}.$$

Proof. Consider

$$F(a, b, c) = \frac{1}{2} \frac{(ab + c)^2}{1 + |a|^2}, \quad (a, b, c) \in \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}.$$

We compute

$$\partial_a F = \frac{(ab + c)}{1 + |a|^2} \left(b - \frac{(ab + c)}{1 + |a|^2} a \right), \quad \partial_b F = \frac{(ab + c)}{1 + |a|^2} a, \quad \partial_c F = \frac{(ab + c)}{1 + |a|^2}.$$

Taking $a = \nabla\eta$, $b = \nabla\psi$, and $c = G(\eta)\psi$ gives

$$\partial_a F = BV, \quad \partial_b F = B\nabla\eta, \quad \partial_c F = B.$$

The estimate (2.2.38) with $r = 0$ gives

$$\|(a, b, c)\|_{L^\infty} \lesssim_{\mathcal{A}} 1.$$

Next, Proposition 2.2.18 implies

$$\|(a, b, c)\|_{H^{s-1}} \lesssim_{\mathcal{A}} \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s}.$$

On the other hand, the estimate (2.2.38) with $r = 1$ implies

$$\|(a, b, c)\|_{C_*^1} \lesssim_A \mathcal{B}.$$

Using the above estimates, we can apply Theorem A.0.10 with $\rho = 1$ to have

$$\frac{1}{2} \frac{(\nabla\eta \cdot \nabla\psi - G(\eta)\psi)^2}{1 + |\nabla\eta|^2} = T_{VB} \cdot \nabla\eta + T_{B\nabla\eta} \cdot \nabla\psi + T_B G(\eta)\psi + f_1,$$

with

$$\|f_1\|_{H^{s-1+1}} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \right\}.$$

By the same theorem, there holds

$$\frac{1}{2} |\nabla\psi|^2 = T_{\nabla\psi} \cdot \nabla\psi + f_2, \quad \|f_2\|_{H^{s-1+1}} \lesssim_{\mathcal{A}} \mathcal{B} \|\psi\|_{H^s}.$$

At last, we deduce from Theorem A.0.4 (ii) (with $m = m' = 0$, $\rho = \frac{1}{2}$) and the estimates for (B, V) in Lemma 2.3.7 that

$$\|(T_{BV} - T_V T_B) \cdot \nabla\eta\|_{H^{s-\frac{1}{2}+\frac{1}{2}}} \lesssim_{\mathcal{A}} \mathcal{B} \|\nabla\eta\|_{H^{s-\frac{1}{2}}}$$

A combination of the above parilinearizations concludes the proof. \square

Lemma 2.3.9. *We have*

$$\|T_{\partial_t B}\eta\|_{H^s} \lesssim_{\mathcal{A}} \mathcal{B} \|\eta\|_{H^{s+\frac{1}{2}}}.$$

Proof. Applying the paraproduct rule (A.0.19) gives

$$\|T_{\partial_t B}\eta\|_{H^s} \lesssim \|\partial_t B\|_{C_*^{-\frac{1}{2}}} \|\eta\|_{H^{s+\frac{1}{2}}}.$$

The proof thus boils down to showing $\|\partial_t B\|_{C_*^{-\frac{1}{2}}} \lesssim_{\mathcal{A}} \mathcal{B}$. By Theorem 2.2.20 for the shape derivative of the Dirichlet-Neumann, we have

$$\partial_t [G(\eta)\psi] = G(\eta)(\partial_t \psi - B\partial_t \eta) - \operatorname{div}(V\partial_t \eta).$$

From the formulas of V, B and the definition of $G(\eta)\psi$, the water waves system (2.1.7) can be rewritten as

$$(2.3.30) \quad \begin{cases} \partial_t \eta = B - V \cdot \nabla \eta, \\ \partial_t \psi = -V \cdot \nabla \psi - g\eta + \frac{1}{2}V^2 + \frac{1}{2}B^2 + H(\eta). \end{cases}$$

We first estimate using Lemma 2.3.7 and (A.0.24)

$$\|\operatorname{div}(V\partial_t \eta)\|_{C_*^{-\frac{1}{2}}} \lesssim \|VB - V(V \cdot \nabla \eta)\|_{C_*^{\frac{1}{2}}} \lesssim \|VB - V(V \cdot \nabla \eta)\|_{C_*^1} \lesssim_{\mathcal{A}} \mathcal{B}.$$

Similarly, we get

$$\|\partial_t \psi - B\partial_t \eta\|_{L^\infty} \lesssim_{\mathcal{A}}, \quad \|\partial_t \psi - B\partial_t \eta\|_{C_*^1} \lesssim_{\mathcal{A}} \mathcal{B}.$$

Consequently, the estimate (2.2.39) yields

$$\|G(\eta)(\partial_t \psi - B\partial_t \eta)\|_{C_*^{-\frac{1}{2}}} \lesssim_{\mathcal{A}} \mathcal{B},$$

from which we conclude the proof. Remark that the estimate (2.2.40) is not applicable to $G(\eta)(\partial_t \psi - B\partial_t \eta)$ since under the assumption (2.3.1) we only have $\partial_t \psi - B\partial_t \eta \in H^{\frac{d}{2}+}$ (due to the bad term $H(\eta)$) and not $H^{\frac{1}{2}+\frac{d}{2}+}$. \square

We now have all the ingredients needed to parilinearize (2.1.7).

Proposition 2.3.10. *There exists a nondecreasing function \mathcal{F} such that with $U := \psi - T_B \eta$ there holds*

$$(2.3.31) \quad \begin{cases} \partial_t \eta + T_V \cdot \nabla \eta - T_\lambda U = f_1, \\ \partial_t U + T_V \cdot \nabla U + T_\ell \eta = f_2, \end{cases}$$

with (f_1, f_2) satisfying

$$\|(f_1, f_2)\|_{H^{s+\frac{1}{2}} \times H^s} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \right\}.$$

Proof. The first equation is an immediate consequence of the equation $\partial_t \eta = G(\eta)\psi$ and Proposition 2.3.5. For the second one, we use the second equation of (2.1.7) and Lemmas 2.3.6, 2.3.8 to get

$$\partial_t \psi - T_B G(\eta)\psi + T_V(\nabla \psi - T_B \cdot \nabla \eta) + T_\ell \eta = R$$

with

$$\|R\|_{H^s} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\psi\|_{H^s} + \|\eta\|_{H^{s+\frac{1}{2}}} \right\}.$$

Next, differentiating U with respect to t yields

$$\partial_t U = \partial_t \psi - T_B \partial_t \eta - T_{\partial_t B} \eta = \partial_t \psi - T_B G(\eta) \psi - T_{\partial_t B} \eta,$$

where the H^s -norm of $T_{\partial_t B} \eta$ is controlled by means of Lemma 2.3.9.

On the other hand,

$$\nabla \psi - T_B \nabla \eta = \nabla U + T_{\nabla B} \eta$$

and by (2.3.28)

$$\|T_V T_{\nabla B} \eta\|_{H^s} \lesssim_{\mathcal{A}} \|T_{\nabla B} \eta\|_{H^s} \lesssim_{\mathcal{A}} \|\nabla B\|_{L^\infty} \|\eta\|_{H^s} \lesssim_{\mathcal{A}} \mathcal{B} \|\eta\|_{H^s}.$$

The proof is complete. \square

2.3.3 Symmetrization of the system

As in [3] we shall deal with a class of symbols having a special structure that we recall here .

Definition 2.3.11. Given $m \in \mathbf{R}$, Σ^m denotes the class of symbols a of the form $a = a^{(m)} + a^{(m-1)}$ with

$$a^{(m)}(x, \xi) = F(\nabla \eta(x), \xi), \quad a^{(m-1)}(x, \xi) = \sum_{|\alpha|=2} F_\alpha(\nabla \eta(x), \xi) \partial_x^\alpha \eta(x)$$

such that

1. T_a maps real-valued functions to real-valued functions;
2. F is a C^∞ real-valued function of $(\zeta, \xi) \in \mathbf{R}^d \times \mathbf{R}^d \setminus \{0\}$, homogeneous of order m in ξ , and there exists a function $K = K(\zeta) > 0$ such that

$$F(\zeta, \xi) \geq K(\zeta) |\xi|^m, \quad \forall (\zeta, \xi) \in \mathbf{R}^d \times \mathbf{R}^d \setminus \{0\};$$

3. the F_α s are complex-valued functions of $(\zeta, \xi) \in \mathbf{R}^d \times \mathbf{R}^d \setminus \{0\}$, homogeneous of order $m-1$ in ξ .

In what follows, we often need an estimate for u from $T_a u$. For this purpose, we prove the next proposition.

Proposition 2.3.12. *Let $m, \mu, M \in \mathbf{R}$. Then, for all $a \in \Sigma^m$, there exists a nondecreasing function \mathcal{F} such that*

$$(2.3.32) \quad \|u\|_{H^{\mu+m}} \leq \mathcal{F}(\|\eta\|_{C_*^2}) (\|T_a u\|_{H^\mu} + \|u\|_{H^{-M}}),$$

$$(2.3.33) \quad \|u\|_{C_*^{\mu+m}} \leq \mathcal{F}(\|\eta\|_{C_*^2}) \left(\|T_a u\|_{C_*^\mu} + \|u\|_{C_*^{-M}} \right),$$

here \mathcal{F} depends only on m, μ, M and the functions F, F_α given in Definition 2.3.11 of the class Σ^m .

Remark. The same result was proved in Proposition 4.6 of [3] where the constant in the right hand side reads $\mathcal{F}(\|\eta(t)\|_{H^{s-1}})$, $s > 2 + \frac{d}{2}$.

Proof. We give the proof for (2.3.32), the proof for (2.3.33) follows similarly. We write $a = a^{(m)} + a^{(m-1)}$. Set $b = \frac{1}{a^{(m)}}$. Applying Theorem A.0.4 (ii) with $\rho = \varepsilon$ gives $T_b T_a^{(m)} = I + r$ where r is of order $-\varepsilon$ and

$$(2.3.34) \quad \|ru\|_{H^{\mu+\varepsilon}} \leq \mathcal{F}(\|\nabla \eta\|_{C^\varepsilon}) \|u\|_{H^\mu} \leq \mathcal{F}(\|\eta\|_{C^{1+\varepsilon}}) \|u\|_{H^\mu}.$$

Then, setting $R = -r - T_b T_{a^{(m-1)}}$ we have

$$(2.3.35) \quad (I - R)u = T_b T_a u.$$

Let us consider the symbol $a^{(m-1)}$ having the structure given by Definition 2.3.11. Applying (A.0.25) and (A.0.27) yields for $|\alpha| = 2$ and uniformly for $|\xi| = 1$,

$$\|F_\alpha(\nabla\eta, \xi)\partial_x^\alpha \eta\|_{C_*^{-1+\varepsilon}} \leq \|F_\alpha(\nabla\eta, \xi)\|_{C_*^1} \|\partial_x^\alpha \eta\|_{C_*^{-1+\varepsilon}} \leq \mathcal{F}(\|\eta\|_{C_*^2}).$$

Similar estimates also hold when taking ξ -derivatives of $F_\alpha(\nabla\eta, \xi)\partial_x^\alpha \eta$. Consequently, $a^{(m-1)} \in \dot{\Gamma}_{-1+\varepsilon}^{m-1}$ and thus by Proposition A.0.6,

$$\|T_{a^{(m-1)}} u\|_{H^{\mu-m+\varepsilon}} \leq \mathcal{F}(\|\eta\|_{C_*^2}) \|u\|_{H^\mu}.$$

Because $b \in \Gamma_0^{-m}$ with semi-norm bounded by $\mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}})$ we get

$$(2.3.36) \quad \|T_b T_{a^{(m-1)}} u\|_{H^{\mu+\varepsilon}} \leq \mathcal{F}(\|\eta\|_{C_*^2}) \|u\|_{H^\mu}.$$

Combining (2.3.34) with (2.3.36) yields

$$\|Ru\|_{H^{\mu+\varepsilon}} \leq \mathcal{F}(\|\eta\|_{C_*^2}) \|u\|_{H^\mu}.$$

In other words, R is a smoothing operator of order $-\varepsilon$. Now, multiplying both sides of (2.3.35) by $1 + R + \dots + R^N$ leads to

$$u - R^N u = (1 + R + \dots + R^N) T_b T_a u.$$

On the one hand, using the fact that R is of order 0, we get

$$\begin{aligned} \|(1 + R + \dots + R^N) T_b T_a u\|_{H^{\mu+m}} &\leq \mathcal{F}(\|\eta\|_{C_*^2}) \|T_b T_a u\|_{H^{\mu+m}} \\ &\leq \mathcal{F}(\|\eta\|_{C_*^2}) \|T_a u\|_{H^\mu}. \end{aligned}$$

On the other hand, that R is of order $-\varepsilon$ implies

$$\|R^N u\|_{H^{\mu+m}} \leq \mathcal{F}(\|\eta\|_{C_*^2}) \|u\|_{H^{\mu+m-N\varepsilon}}.$$

Therefore, by choosing N sufficiently large we conclude the proof. \square

For the sake of conciseness, we give the following definition.

Definition 2.3.13. Let $m \in \mathbf{R}$ and consider two families of operators of order m ,

$$\{A(t) : t \in [0, T]\}, \quad \{B(t) : t \in [0, T]\}.$$

We write $A \sim B$ (in Σ^m) if $A - B$ is of order $m - \frac{3}{2}$ and the following condition is fulfilled: for all $\mu \in \mathbf{R}$, there exists a nondecreasing function \mathcal{F} such that for a.e. $t \in [0, T]$,

$$\|A(t) - B(t)\|_{H^\mu \rightarrow H^{\mu-m+\frac{3}{2}}} \leq \mathcal{F}(\|\eta(t)\|_{C_*^2}) (1 + \|\eta(t)\|_{C_*^{\frac{5}{2}}}).$$

Proposition 2.3.14. For any $a \in \Sigma^m$ and $b \in \Sigma^{m'}$, it holds that

$$T_a T_b \sim T_c$$

(in $\Sigma^{m+m'}$) with

$$c = a^{(m)} b^{(m')} + a^{(m-1)} b^{(m')} + a^{(m-1)} b^{(m')} + \frac{1}{i} \partial_\xi a^{(m)} \partial_x b^{(m')}.$$

Proof. 1. Since the principal symbol $a^{(m)}(t)$ contains only the first order derivatives of η , applying the nonlinear estimate (A.0.26) we obtain

$$M_{3/2}^m(a^{(m)}(t)) \leq \mathcal{F}(\|\eta(t)\|_{C_*^{1+\varepsilon}})(1 + \|\eta(t)\|_{C_*^{\frac{5}{2}}}).$$

On the other hand,

$$M_{1/2}^m(a^{(m)}(t)) \leq \mathcal{F}(\|\eta(t)\|_{C_*^{\frac{3}{2}}})$$

and

$$M_0^m(a^{(m)}(t)) \leq \mathcal{F}(\|\eta(t)\|_{C_*^{1+\varepsilon}}).$$

2. The subprincipal symbol $a^{(m-1)}(t)$ depends linearly on $\partial^\alpha \eta$, $|\alpha| = 2$ and nonlinearly on $\nabla \eta$. Hence $a^{(m-1)} \in \Gamma_{1/2}^{m-1}$ and by (A.0.24) and (A.0.26) we have uniformly for $|\xi| = 1$,

$$\begin{aligned} & \|F_\alpha(\nabla \eta(t, x), \xi) \partial_x^\alpha \eta(t, x)\|_{C_*^{\frac{1}{2}}} \\ & \leq \| [F_\alpha(\nabla \eta(t, \cdot), \xi) - F_\alpha(0, \xi)] \partial_x^\alpha \eta(t, \cdot) \|_{C_*^{\frac{1}{2}}} + |F_\alpha(0, \xi)| \|\partial_x^\alpha \eta(t, \cdot)\|_{C_*^{\frac{1}{2}}} \\ & \leq \mathcal{F}(\|\eta(t)\|_{C_*^{\frac{3}{2}}}) \|\eta(t)\|_{C_*^{\frac{5}{2}}}. \end{aligned}$$

The same estimates hold when taking ξ -derivatives, consequently

$$M_{1/2}^{m-1}(a^{(m-1)}(t)) \leq \mathcal{F}(\|\eta(t)\|_{C_*^{\frac{3}{2}}}) \|\eta(t)\|_{C_*^{\frac{5}{2}}}.$$

On the other hand,

$$M_0^{m-1}(a^{(m-1)}(t)) \leq \mathcal{F}(\|\eta(t)\|_{C_*^2}).$$

3. We now write

$$T_a T_b = T_{a^{(m)}} T_{b^{(m')}} + T_{a^{(m-1)}} T_{b^{(m')}} + T_{a^{(m)}} T_{b^{(m'-1)}} + T_{a^{(m-1)}} T_{b^{(m'-1)}}.$$

Using 1. and 2., we deduce by virtue of Theorem A.0.4 (ii) with $\rho = 3/2$ that

$$\begin{aligned} & \|T_{a^{(m)}} T_{b^{(m')}} - T_{a^{(m)} b^{(m')}} + \frac{1}{2} \partial_\xi a^{(m)} \partial_x b^{(m')}\|_{H^\mu \rightarrow H^{\mu - (m+m') + \frac{3}{2}}} \\ & \leq \mathcal{F}(\|\eta(t)\|_{C_*^{1+\varepsilon}})(1 + \|\eta(t)\|_{C_*^{\frac{5}{2}}}). \end{aligned}$$

The same theorem, applied with $\rho = 1/2$, yields

$$\begin{aligned} & \|T_{a^{(m-1)}} T_{b^{(m')}} - T_{a^{(m-1)} b^{(m')}}\|_{H^\mu \rightarrow H^{\mu - (m+m') + \frac{3}{2}}} \leq \mathcal{F}(\|\eta(t)\|_{C_*^2})(1 + \|\eta(t)\|_{C_*^{\frac{5}{2}}}), \\ & \|T_{a^{(m)}} T_{b^{(m'-1)}} - T_{a^{(m-1)} b^{(m')}}\|_{H^\mu \rightarrow H^{\mu - (m+m') + \frac{3}{2}}} \leq \mathcal{F}(\|\eta(t)\|_{C_*^2})(1 + \|\eta(t)\|_{C_*^{\frac{5}{2}}}). \end{aligned}$$

Finally, applying Theorem A.0.4 (i) leads to

$$\|T_{a^{(m-1)}} T_{b^{(m'-1)}}\|_{H^\mu \rightarrow H^{\mu - (m+m') + 2}} \leq \mathcal{F}(\|\eta(t)\|_{C_*^2}).$$

Putting the above estimates together we conclude that $T_a T_b \sim T_c$ in $\Sigma^{m+m'}$. \square

Using the preceding Proposition, one can easily verify that Proposition 4.8 in [3] is still valid:

Proposition 2.3.15. *Let $q \in \Sigma^0$, $p \in \Sigma^{\frac{1}{2}}$, $\gamma \in \Sigma^{\frac{3}{2}}$ defined by*

$$\begin{aligned} q &= (1 + |\partial_x \eta|^2)^{-\frac{1}{2}}, \\ p &= (1 + |\partial_x \eta|^2)^{-\frac{5}{4}} \sqrt{\lambda^{(1)}} + p^{(-1/2)}, \\ \gamma &= \sqrt{\ell^{(2)} \lambda^{(1)}} + \sqrt{\frac{\ell^{(2)}}{\lambda^{(1)}}} \frac{\operatorname{Re} \lambda^{(0)}}{2} - \frac{i}{2} (\partial_\xi \cdot \partial_x) \sqrt{\ell^{(2)} \lambda^{(1)}}, \end{aligned}$$

where

$$p^{(-1/2)} = \frac{1}{\gamma^{(3/2)}} \left\{ q^{(0)} \ell^{(1)} - \gamma^{(1/2)} p^{(1/2)} + i \partial_\xi \gamma^{(3/2)} \partial_x p^{(1/2)} \right\}.$$

Then, it holds that

$$T_p T_\lambda \sim T_\gamma T_q, \quad T_q T_\ell \sim T_\gamma T_p, \quad (T_\gamma)^* \sim T_\gamma.$$

We are now in position to perform the symmetrization.

Proposition 2.3.16. *Introduce two new unknowns*

$$\Phi_1 = T_p \eta, \quad \Phi_2 = T_q U.$$

Then $\Phi_1, \Phi_2 \in L^\infty([0, T], H^s)$ and

$$(2.3.37) \quad \begin{cases} \partial_t \Phi_1 + T_V \cdot \nabla \Phi_1 - T_\gamma \Phi_2 = F_1, \\ \partial_t \Phi_2 + T_V \cdot \nabla \Phi_2 + T_\gamma \Phi_1 = F_2, \end{cases}$$

where, there exists a nondecreasing function \mathcal{F} independent of η, ψ such that for a.e. $t \in [0, T]$, there holds

$$(2.3.38) \quad \|(F_1, F_2)\|_{H^s \times H^s} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \right\}.$$

Proof. It follows directly from system (2.3.31) that Φ_1, Φ_2 satisfy

$$(2.3.39) \quad \begin{cases} \partial_t \Phi_1 + T_V \cdot \nabla \Phi_1 - T_\gamma \Phi_2 = T_p f_1 + T_{\partial_t p} \eta + [T_V \cdot \nabla, T_p] \eta + R_1, \\ \partial_t \Phi_2 + T_V \cdot \nabla \Phi_2 + T_\gamma \Phi_1 = T_q f_2 + T_{\partial_t q} U + [T_V \cdot \nabla, T_q] U + R_2, \end{cases}$$

where

$$R_1 = (T_p T_\lambda - T_\gamma T_q) \psi, \quad R_2 = -(T_q T_\ell - T_\gamma T_p) \eta.$$

Let Π denote the right-hand side of (2.3.38). According to Proposition 2.3.15,

$$\|R_1\|_{H^s} + \|R_2\|_{H^s} \lesssim \Pi.$$

On the other hand, Proposition 2.3.10 implies

$$\|T_p f_1\|_{H^s} + \|T_q f_2\|_{H^s} \lesssim \Pi.$$

Owing to Lemma 2.3.7 and the norm estimates for symbols in Proposition 2.3.14, the composition rule of Theorem A.0.4 (ii) (with $\rho = 1$) yields

$$\|[T_V \cdot \nabla, T_p] \eta\|_{H^s} + \|[T_V \cdot \nabla, T_q] U\|_{H^s} \lesssim \Pi.$$

It remains to prove

$$\|T_{\partial_t p}\|_{H^{s+\frac{1}{2}} \rightarrow H^s} + \|T_{\partial_t q}\|_{H^s \rightarrow H^s} \lesssim_{\mathcal{A}} \mathcal{B}.$$

To this end, we first recall from the first equation of (2.3.30) that $\partial_t \eta = B - V \cdot \nabla \eta$. Hence $\|\partial_t \eta\|_{W^{1,\infty}} \lesssim_{\mathcal{A}} \mathcal{B}$ and

$$\mathcal{M}_0^{1/2}(\partial_t p^{(1/2)}) + \mathcal{M}_0^0(\partial_t q) \lesssim_{\mathcal{A}} \mathcal{B},$$

which, combined with Theorem A.0.4 (i), yields

$$\|T_{\partial_t p^{(1/2)}}\|_{H^{s+\frac{1}{2}} \rightarrow H^s} + \|T_{\partial_t q}\|_{H^s \rightarrow H^s} \lesssim_{\mathcal{A}} \mathcal{B}.$$

We are thus left with the estimate of $\|T_{\partial_t p^{(-1/2)}}\|_{H^{s+\frac{1}{2}} \rightarrow H^s}$. According to Proposition A.0.6, it suffices to show

$$(2.3.40) \quad \mathcal{M}_{-1}^{-1/2}(\partial_t p^{(-1/2)}) \lesssim_{\mathcal{A}} \mathcal{B}.$$

Recall that $p^{(-1/2)}$ is of the form

$$p^{(-1/2)} = \sum_{|\alpha|=2} F_\alpha(\nabla \eta, \xi) \partial_x^\alpha \eta,$$

where the F_α are smooth functions in $\xi \neq 0$ and homogeneous of order $-1/2$. Hence,

$$\partial_t p^{(-1/2)} = \sum_{|\alpha|=2} [\partial_t F_\alpha(\nabla \eta, \xi)] \partial_x^\alpha \eta + \sum_{|\alpha|=2} F_\alpha(\nabla \eta, \xi) \partial_t \partial_x^\alpha \eta.$$

It is easy to see that

$$\mathcal{M}_0^{-\frac{1}{2}}([\partial_t F_\alpha(\nabla \eta, \xi)] \partial_x^\alpha \eta) \lesssim_{\mathcal{A}} 1.$$

For the main term $F_\alpha(\nabla \eta, \xi) \partial_t \partial_x^\alpha \eta$ we use the first equation of (2.3.30) to have

$$\partial_t \partial_x^\alpha \eta = \partial_x^\alpha (B - V \nabla_x \eta).$$

Hence

$$\|\partial_t \partial_x^\alpha \eta\|_{C_*^{-1}} \leq \|B - V \nabla_x \eta\|_{C_*^1} \lesssim_{\mathcal{A}} \mathcal{B}.$$

The product rule (A.0.25) then implies

$$\mathcal{M}_{-1}^{-\frac{1}{2}}(F_\alpha(\nabla \eta, \xi) \partial_t \partial_x^\alpha \eta) \lesssim_{\mathcal{A}} \mathcal{B},$$

which concludes the proof of (2.3.40) and hence of the proposition. \square

2.4 A priori estimates and blow-up criteria

2.4.1 A priori estimates

First of all, it follows straightforwardly from Proposition 2.3.16 that the water waves system can be reduced to a single equation of a complex-valued unknown as follows.

Proposition 2.4.1. *Assume that (η, ψ) is a solution to (2.1.7) and satisfies (2.3.1). Let Φ_1, Φ_2 be as in Proposition 2.3.16, then*

$$\Phi := \Phi_1 + i\Phi_2 = T_p \eta + iT_q U$$

satisfies

$$(2.4.1) \quad (\partial_t + T_V \cdot \nabla + iT_\gamma) \Phi = F,$$

$$(2.4.2) \quad \|F(t)\|_{H^s} \lesssim_{\mathcal{A}} \mathcal{B} \left\{ \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \right\}.$$

In order to obtain H^s estimate for Φ , we shall commute equation (2.4.1) with an elliptic operator φ of order s and then perform an L^2 -energy estimate. Since $\gamma^{(3/2)}$ is of order $3/2 > 1$, we need to choose φ as a function of $\gamma^{(3/2)}$ as in [3]:

$$(2.4.3) \quad \varphi := (\gamma^{(3/2)})^{2s/3},$$

and take $\varphi = T_\varphi \Phi$. To obtain energy estimates in terms of the original variables η and ψ , it is necessary to link them with this new unknown φ .

Lemma 2.4.2. *We have*

$$(2.4.4) \quad \|\varphi\|_{L^2} \lesssim_{\mathcal{A}} \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s},$$

$$(2.4.5) \quad \|\eta\|_{H^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \lesssim_{\mathcal{A}} \|\varphi\|_{L^2} + \|\eta\|_{L^2} + \|\psi\|_{L^2}.$$

Proof. Recall that $p \in \Sigma^s$, $q \in \Sigma^0$, and $\varphi \in \Sigma^s$ since $\gamma^{(\frac{3}{2})} \in \Sigma^{\frac{3}{2}}$. The estimate (2.4.4) is then a direct consequence of Theorem A.0.4 (i). To prove (2.4.5) we apply Proposition 2.3.12 twice to get

$$\|\eta\|_{H^{s+\frac{1}{2}}} \lesssim_A \|T_\varphi T_p \eta\|_{L^2} + \|\eta\|_{L^2},$$

$$\|\psi\|_{H^s} \lesssim_{\mathcal{A}} \|T_\varphi T_q \psi\|_{L^2} + \|\psi\|_{L^2}.$$

Clearly, $\|T_\varphi T_p \eta\|_{L^2} \leq \|\varphi\|_{L^2}$, hence

$$\|\eta\|_{H^{s+\frac{1}{2}}} \lesssim_{\mathcal{A}} \|\varphi\|_{L^2} + \|\eta\|_{L^2}.$$

On the other hand,

$$\begin{aligned} \|T_\varphi T_q \psi\|_{L^2} &\leq \|T_\varphi T_q U\|_{L^2} + \|T_\varphi T_q T_B \eta\|_{L^2} \\ &\leq \|\varphi\|_{L^2} + \|T_\varphi T_q T_B \eta\|_{L^2} \\ &\lesssim_{\mathcal{A}} \|\varphi\|_{L^2} + \|\eta\|_{H^{s+\frac{1}{2}}} \\ &\lesssim_{\mathcal{A}} \|\varphi\|_{L^2} + \|\eta\|_{L^2}. \end{aligned}$$

This completes the proof of (2.4.5). \square

Proposition 2.4.3. *There exists a nondecreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on s, ε_*, h such that for any $t \in [0, T]$,*

$$(2.4.6) \quad \frac{d}{dt} \|\varphi\|_{L^2}^2 \leq \mathcal{F}(\mathcal{A}) \mathcal{B}(\|\eta\|_{L^2} + \|\psi\|_{L^2} + \|\varphi\|_{L^2}) \|\varphi\|_{L^2}.$$

Proof. We see from (2.4.1) that φ solves the equation

$$(2.4.7) \quad (\partial_t + T_V \cdot \nabla + iT_\gamma) \varphi = T_\varphi F + G$$

where

$$G = T_{\partial_t \varphi} \Phi + [T_V \cdot \nabla, T_\varphi] \Phi + i[T_\gamma, T_\varphi] \Phi.$$

First, remark that since $\partial_{\xi \varphi} \cdot \partial_x \gamma^{(3/2)} = \partial_{\xi \gamma^{(3/2)}} \cdot \partial_x \varphi$ we can apply Lemma 2.3.14 twice: once with $m = s$, $m' = \frac{3}{2}$, $\rho = \frac{3}{2}$ and once with $m = \frac{3}{2}$, $m' = s$, $\rho = \frac{3}{2}$ to find

$$\|[T_\varphi, T_\gamma]\|_{H^s \rightarrow L^2} \lesssim_{\mathcal{A}} \mathcal{B}.$$

On the other hand, Theorem A.0.4 (ii) applied with $\rho = 1$ gives

$$\|[T_V \cdot \nabla, T_\varphi]\|_{H^s \rightarrow L^2} \lesssim_{\mathcal{A}} \mathcal{B}.$$

Next, we write $\partial_t \varphi = L(\nabla \eta, \xi) \partial_t \nabla \eta$ for some smooth function L homogeneous of order s in ξ , where by the first equation of (2.3.30) $\|\partial_t \nabla \eta\|_{L^\infty} \lesssim_{\mathcal{A}} \mathcal{B}$. Hence

$$\|T_{\partial_t \varphi}\|_{H^s \rightarrow L^2} \lesssim_{\mathcal{A}} \mathcal{B}.$$

Putting the above estimates together leads to

$$\|G\|_{L^2} \lesssim_{\mathcal{A}} \mathcal{B} \|\Phi\|_{H^s}.$$

On the other hand, Proposition 2.3.12 applied to $u = \Phi$, $a = \varphi \in \Sigma^s$ yields

$$\|\Phi\|_{H^s} \lesssim_{\mathcal{A}} \|\varphi\|_{L^2} + \|\eta\|_{L^2} + \|\psi\|_{L^2}.$$

Therefore,

$$\|G\|_{L^2} \lesssim_{\mathcal{A}} \mathcal{B} (\|\varphi\|_{L^2} + \|\eta\|_{L^2} + \|\psi\|_{L^2}).$$

On the other hand, (2.4.2) together with (2.4.5) implies

$$(2.4.8) \quad \|T_\varphi F\|_{L^2} \lesssim_{\mathcal{A}} \mathcal{B} (\|\varphi\|_{L^2} + \|\eta\|_{L^2} + \|\psi\|_{L^2}).$$

Now, using Theorem A.0.4 (iii) and the proof of Lemma 2.3.14 we easily find that

$$(2.4.9) \quad \|(T_V \cdot \nabla) + (T_V \cdot \nabla)^*\|_{L^2 \rightarrow L^2} \lesssim_{\mathcal{A}} \mathcal{B}.$$

On the other hand, according to Proposition 2.3.15, $(T_\gamma)^* \sim T_\gamma$, so

$$(2.4.10) \quad \|(T_\gamma) - (T_\gamma)^*\|_{L^2 \rightarrow L^2} \lesssim_{\mathcal{A}} \mathcal{B}.$$

Therefore, by an L^2 -energy estimate for (2.4.7) we end up with (2.4.6). \square

Proposition 2.4.4. *Set $W = (\eta, \psi)$, $\mathcal{H}^r = H^{r+\frac{1}{2}} \times H^r$. Then, there exists a nondecreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on s, ε_*, h such that for a.e. $t \in [0, T]$,*

$$\|W(t)\|_{\mathcal{H}^s}^2 \leq \mathcal{F}(P^1(t)) \|W(0)\|_{\mathcal{H}^s}^2 + \mathcal{F}(P^1(t)) \int_0^t \mathcal{B}(r) \|W(r)\|_{\mathcal{H}^s}^2 dr$$

with

$$P^1(t) := \sup_{r \in [0, t]} \mathcal{A}(r).$$

Proof. Integrating (2.4.6) over $[0, t]$ and using (2.4.4)-(2.4.5), we obtain

$$(2.4.11) \quad \begin{aligned} \|W(t)\|_{\mathcal{H}^s}^2 &\lesssim_{\mathcal{A}} \|W(t)\|_{L^2 \times L^2}^2 + \|\varphi\|_{L^2}^2 \\ &\lesssim_{\mathcal{A}} \|W(t)\|_{L^2 \times L^2}^2 + \|W(0)\|_{\mathcal{H}^s}^2 + \int_0^t \mathcal{F}(\mathcal{A}(r)) \mathcal{B}(r) \|W(r)\|_{\mathcal{H}^s}^2 dr. \end{aligned}$$

Recall the system (2.3.30) satisfied by W :

$$\begin{cases} \partial_t \eta = B - V \cdot \nabla \eta, \\ \partial_t \psi = -V \cdot \nabla \psi - g\eta + \frac{1}{2}V^2 + \frac{1}{2}B^2 + H(\eta). \end{cases}$$

A standard L^2 estimate for each equation gives

$$\frac{d}{dt} \|W(t)\|_{L^2 \times L^2}^2 \lesssim_{\mathcal{A}} \mathcal{B} \|W(t)\|_{\mathcal{H}^s}^2.$$

Hence

$$\|W(t)\|_{L^2 \times L^2}^2 \leq \|W(0)\|_{L^2 \times L^2}^2 + \int_0^t \mathcal{F}(\mathcal{A}(r)) \mathcal{B}(r) \|W(r)\|_{\mathcal{H}^s}^2 dr.$$

Plugging this into (2.4.11) we conclude the proof. \square

Let us denote the Sobolev norm and the "Strichartz norm" of the solution by

$$(2.4.12) \quad \begin{aligned} M_{\sigma, T} &= \|(\eta, \psi)\|_{L^\infty([0, T]; H^{\sigma+\frac{1}{2}} \times H^\sigma)}, \\ M_{\sigma, 0} &= \|(\eta, \psi)|_{t=0}\|_{H^{\sigma+\frac{1}{2}} \times H^\sigma}, \\ N_{r, T} &= \|(\eta, \nabla \psi)\|_{L^1([0, T]; W^{r+\frac{1}{2}} \times B_{\infty, 1}^1)}. \end{aligned}$$

We next derive from Proposition 2.4.3 an a priori estimate for $tM_{s, T}$ using the control of $N_{r, T}$.

Theorem 2.4.5. *Let $d \geq 1$, $h > 0$ and*

$$s > \frac{3}{2} + \frac{d}{2}, \quad r > 2.$$

Then there exists a nondecreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on (s, r, h, d) such that for all $T \in [0, 1)$ and all (η, ψ) solution to (2.1.7) with

$$\begin{aligned} (\eta, \psi) &\in L^\infty\left([0, T]; H^{s+\frac{1}{2}} \times H^s\right), \\ (\eta, \nabla \psi) &\in L^1\left([0, T]; W^{r+\frac{1}{2}, \infty} \times B_{\infty, 1}^1\right), \\ \inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma) &> h, \end{aligned}$$

there holds

$$(2.4.13) \quad M_{s, T} \leq \mathcal{F}(M_{s, 0} + T\mathcal{F}(M_{s, T}) + N_{r, T}).$$

Proof. Pick

$$0 < \varepsilon < \frac{1}{2} \min\left\{\frac{1}{2}, r-1, s-\frac{3}{2}-\frac{d}{2}\right\}.$$

By Remark 2.2.2, $E(\eta, \psi) \leq \mathcal{F}(\|\eta\|_{C_*^{1+\varepsilon}})\|\psi\|_{H^{\frac{1}{2}}}$. Therefore, by applying Proposition 2.4.4 we obtain

$$M_{s, T} \leq M_{s, 0} K(T) \exp\left(K(T) \int_0^T \mathcal{B}ig(\|(\eta, \nabla \psi)(t)\|_{C_*^{\frac{5}{2}+\varepsilon} \times B_{\infty, 1}^1} + 1 \mathcal{B}ig) dt\right)$$

with

$$K(T) := \mathcal{F} \mathcal{B}ig\left(\sup_{t \in [0, T]} (\|(\eta, \psi)(t)\|_{C_*^{2+\varepsilon} \times C_*^\varepsilon} + \|(\eta, \psi)\|_{L^2 \times H^{\frac{1}{2}}}) \mathcal{B}ig\right).$$

Therefore, it suffices to show for all $t \leq T$

$$\|(\eta, \psi)(t)\|_{C_*^{2+\varepsilon} \times C_*^\varepsilon} + \|(\eta, \psi)(t)\|_{L^2 \times H^{\frac{1}{2}}} \leq \mathcal{F}(M_{s, 0} + TM_{s, T}).$$

By Sobolev's embeddings, this reduces to

$$\|(\eta, \psi)(t)\|_{H^{s+\frac{1}{2}-\varepsilon} \times H^{s-\varepsilon}} \leq \mathcal{F}(M_{s,0} + TM_{s,T}) \quad \forall t \leq T.$$

Using the Sobolev estimate for the Dirichlet-Neumann in Proposition 2.2.18 in conjunction with Remark 2.2.2 we get

$$(2.4.14) \quad \|\eta(t) - \eta(0)\|_{H^{s-1}} \leq \int_0^t \|\partial_t \eta(\tau)\|_{H^{s-1}} d\tau = \int_0^t \|G(\eta(\tau))\psi(\tau)\|_{H^{s-1}} d\tau \leq T\mathcal{F}(M_{s,T}).$$

Consequently, it follows by interpolation that

$$(2.4.15) \quad \begin{aligned} \|\eta(t)\|_{H^{s+\frac{1}{2}-\varepsilon}} &\leq \|\eta(0)\|_{H^{s+\frac{1}{2}-\varepsilon}} + \|\eta(t) - \eta(0)\|_{H^{s+\frac{1}{2}-\varepsilon}} \\ &\leq M_{s,0} + \|\eta(t) - \eta(0)\|_{H^{s-1}}^\theta \|\eta(t) - \eta(0)\|_{H^{s+\frac{1}{2}}}^{1-\theta} \quad \theta \in (0,1) \\ &\leq M_{s,0} + T^\theta \mathcal{F}(M_s(T)). \end{aligned}$$

The estimate for $\|\psi(t)\|_{H^{s-\varepsilon}}$ follows along the same lines using the second equation of (2.1.7) (or (2.3.30)) and interpolation. \square

2.4.2 Blow-up criteria

Taking $\sigma > 2 + \frac{d}{2}$ and

$$(2.4.16) \quad (\eta_0, \psi_0) \in H^{\sigma+\frac{1}{2}} \times H^\sigma, \quad \text{dist}(\eta_0, \Gamma) > h > 0,$$

we know from Theorem 1.1 in [3] that there exists a time $T \in (0, \infty)$ such that the Cauchy problem for system (2.1.7) with initial data (η_0, ψ_0) has a *unique solution*

$$(\eta, \psi) \in C\left([0, T]; H^{\sigma+\frac{1}{2}} \times H^\sigma\right)$$

satisfying

$$\sup_{t \in [0, T]} \text{dist}(\eta(t), \Gamma) > \frac{h}{2}.$$

The *maximal time of existence* $T^* > 0$ then can be defined as

$$(2.4.17) \quad T^* = T^*(\eta_0, \psi_0, \sigma, h) := \sup \left\{ T' > 0 : \begin{array}{l} \text{the Cauchy problem for (2.1.7) with data} \\ (\eta_0, \psi_0) \text{ satisfying (2.4.16) has a solution } (\eta, \psi) \in C([0, T']; H^{\sigma+\frac{1}{2}} \times H^\sigma) \\ \text{satisfying } \inf_{[0, T']} \text{dist}(\eta(t), \Gamma) > 0 \end{array} \right\}.$$

It should be emphasized that T^* depends not only on (η_0, ψ_0) and σ but also on the initial depth h . By the uniqueness statement of Proposition 6.4, [3] (it is because of this Proposition that we require the separation condition in the definition (2.4.17)) the solution (η, ψ) is defined for all $t < T^*$ and

$$(\eta, \psi) \in C\left([0, T^*]; H^{\sigma+\frac{1}{2}} \times H^\sigma\right),$$

which will be called the *maximal solution*.

We recall the following lemma from [103] (see Lemma 9.20 there).

Lemma 2.4.6. *Let $\mu > 1 + \frac{d}{2}$. Then, there exists a constant $C > 0$ such that*

$$\|u\|_{B_{\infty,1}^1} \leq C(1 + \|u\|_{C_*^1}) \ln(e + \|u\|_{H^\mu}^2)$$

provided the right-hand side is finite.

Proof. For the sake of completeness, we present the proof of this lemma, taken from [103]. Given an integer N , we have by the Bernstein inequality

$$\begin{aligned} \|u\|_{B_{\infty,1}^1} &= \sum_{j=0}^N 2^j \|\Delta_j u\|_{L^\infty} + \sum_{j>N} 2^j \|\Delta_j u\|_{L^\infty} \\ &\leq (N+1) \|u\|_{C_*^1} + \sum_{j>N} 2^{j(1+\frac{d}{2}-\mu)} 2^{j\mu} \|\Delta_j u\|_{L^2}. \end{aligned}$$

As $1 + \frac{d}{2} - \mu < 0$, it follows by Hölder's inequality for sequence that there exists $C > 0$ independent of N such that

$$\|u\|_{B_{\infty,1}^1} \leq (N+1) \|u\|_{C_*^1} + C 2^{-N(\mu-1-\frac{d}{2})} (\|u\|_{H^\mu} + e).$$

Choosing $N \sim \ln(e + \|u\|_{H^\mu})$ so that $2^{-N(\mu-1-\frac{d}{2})} (\|u\|_{H^\mu} + e) \sim 1$, we obtain the desired inequality. \square

Proposition 2.4.7. *Let $d \geq 1, h > 0, \sigma > 2 + \frac{d}{2}, T > 0$. Let*

$$(\eta, \psi) \in C([0, T]; H^{\sigma+\frac{1}{2}} \times H^\sigma), \quad \inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma) > h > 0$$

be a solution to (2.1.7). Fix $\varepsilon_ \in (0, \sigma - \frac{3}{2} - \frac{d}{2})$. Then there exists a nondecreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on $(\sigma, \varepsilon_*, h, d)$ such that*

$$M_{\sigma, T}^2 \leq \mathcal{F}(P^2(t)) (M_{\sigma, 0}^2 + 2e) e^{\int_0^t Q(r) dr} - 2e$$

with

$$\begin{aligned} Q(r) &:= 1 + \|\nabla \psi(r)\|_{C_*^1} + \|\eta(r)\|_{C_*^{2+\varepsilon_*}}, \\ P^2(t) &:= \sup_{r \in [0, t]} (\|\eta(r)\|_{C_*^{2+\varepsilon_*}} + \|\nabla \psi(r)\|_{B_{\infty,1}^0} + \mathcal{H}(0)). \end{aligned}$$

Proof. Recall the definition of $\mathcal{A}(t)$:

$$\mathcal{A}(t) = \|\eta\|_{C_*^{2+\varepsilon_*}} + \|\nabla \psi\|_{B_{\infty,1}^0} + \|\eta\|_{L^2} + E(\eta, \psi).$$

Proposition 2.2.11 tells us

$$E(\eta(t), \psi(t)) \leq \int_{\mathbf{R}^d} \psi G(\eta) \psi,$$

hence

$$\|\eta\|_{L^2} + E(\eta, \psi) \leq \mathcal{H}(t) = \mathcal{H}(0),$$

$\mathcal{H}(t)$ being the total energy (2.1.9) at time t . Here, we remark that the conservation of \mathcal{H} follows by proving $\frac{d}{dt} \mathcal{H}(t) = 0$, which can be justified under our the regularity \mathcal{H}^s . Therefore, Proposition 2.4.4 applied with $s = \sigma > \frac{3}{2} + \frac{d}{2}$ yields

$$(2.4.18) \quad \|W(t)\|_{\mathcal{H}^\sigma}^2 \leq \mathcal{F}(P^2(t)) \|W(0)\|_{\mathcal{H}^\sigma}^2 + \mathcal{F}(P^2(t)) \int_0^t \mathcal{B}(r) \|W(r)\|_{\mathcal{H}^\sigma}^2 dr$$

with

$$P^2(t) := \sup_{r \in [0, t]} (\|\eta(r)\|_{C_*^{2+\varepsilon_*}} + \|\nabla\psi(r)\|_{B_{\infty,1}^0} + \mathcal{H}(0)).$$

Next, as $\nabla\psi \in H^{s-1}$ with $s-1 > 1 + \frac{d}{2}$, we can apply Lemma 2.4.6 to have

$$\|\nabla\psi\|_{B_{\infty,1}^1} \leq C(1 + \|\nabla\psi\|_{C_*^1}) \ln(e + \|\psi\|_{H^\sigma}) \leq C(1 + \|\nabla\psi\|_{C_*^1}) \ln(2e + \|\psi\|_{H^\sigma}^2).$$

Consequently,

$$\mathcal{B}(r) \leq C(1 + \|\nabla\psi(r)\|_{C_*^1} + \|\eta(r)\|_{C_*^{2+\varepsilon_*}}) \ln(2e + \|W(r)\|_{\mathcal{H}^\sigma}^2).$$

In view of (2.4.18), this implies

$$\begin{aligned} \|W(t)\|_{\mathcal{H}^\sigma}^2 &\leq \mathcal{F}(P^2(t))\|W(0)\|_{\mathcal{H}^\sigma}^2 + \\ &\quad \mathcal{F}(P^2(t)) \int_0^t Q(r) \ln(2e + \|W(r)\|_{\mathcal{H}^\sigma}^2) \|W(r)\|_{\mathcal{H}^\sigma}^2 dr \end{aligned}$$

with $Q(r) := 1 + \|\nabla\psi(r)\|_{C_*^1} + \|\eta(r)\|_{C_*^{2+\varepsilon_*}}$. Finally, using a Grönwall type argument as in [18] we conclude that

$$\|W(t)\|_{\mathcal{H}^\sigma}^2 \leq \mathcal{F}(P^2(t))(\|W(0)\|_{\mathcal{H}^\sigma}^2 + 2e) \exp(e^{\mathcal{F}(P^2(t)) \int_0^t Q(r) dr}) - 2e.$$

□

Remark. Using (2.4.18) and Grönwall's lemma we obtain the exponential bound

$$\|W(t)\|_{\mathcal{H}^\sigma}^2 \leq \mathcal{F}(P^2(t))\|W(0)\|_{\mathcal{H}^\sigma}^2 \exp(\mathcal{F}(P^2(t)) \int_0^t \mathcal{B}(r) dr)$$

provided $\sigma > \frac{3}{2} + \frac{d}{2}$ only.

Theorem 2.4.8. *Let $d \geq 1, h > 0$, and $\sigma > 2 + \frac{d}{2}$. Let*

$$(\eta_0, \psi_0) \in H^{\sigma+\frac{1}{2}} \times H^\sigma, \quad \text{dist}(\eta_0, \Gamma) > h > 0.$$

Let $T^ = T^*(\eta_0, \psi_0, \sigma, h)$ be the maximal time of existence defined by (2.4.17) and*

$$(2.4.19) \quad (\eta, \psi) \in L^\infty([0, T^*]; H^{\sigma+\frac{1}{2}} \times H^\sigma)$$

be the maximal solution of (2.1.7) with prescribed data (η_0, ψ_0) . If T^ is finite, then for all $\varepsilon > 0$, it holds that*

$$(2.4.20) \quad P_\varepsilon(T^*) + \int_0^{T^*} Q_\varepsilon(t) dt + \frac{1}{h(T^*)} = +\infty,$$

where

$$\begin{aligned} P_\varepsilon(T^*) &= \sup_{t \in [0, T^*)} \|\eta(t)\|_{C_*^{2+\varepsilon}} + \|\nabla\psi(t)\|_{B_{\infty,1}^0}, \\ Q_\varepsilon(t) &= \|\eta(t)\|_{C_*^{\frac{5}{2}+\varepsilon}} + \|\nabla\psi(t)\|_{C_*^1}, \\ h(T^*) &= \inf_{t \in [0, T^*)} \text{dist}(\eta(t), \Gamma). \end{aligned}$$

Consequently, if T^* is finite then for all $\varepsilon > 0$

$$(2.4.21) \quad P_\varepsilon^0(T^*) + \int_0^{T^*} Q_\varepsilon^0(t) dt + \frac{1}{h(T^*)} = +\infty,$$

where

$$\begin{aligned} P_\varepsilon^0(T) &= \sup_{t \in [0, T]} \|\eta(t)\|_{C_*^{2+\varepsilon}} + \|(V, B)(t)\|_{B_{\infty,1}^0}, \\ Q_\varepsilon^0(t) &= \|\eta(t)\|_{C_*^{\frac{5}{2}+\varepsilon}} + \|(V, B)(t)\|_{C_*^1}. \end{aligned}$$

Proof. Suppose that $T^* < +\infty$ and for some $\varepsilon > 0$

$$K := P_\varepsilon(T^*) + \int_0^{T^*} Q_\varepsilon(t) dt + \frac{1}{h(T^*)} < +\infty.$$

Let $T \in [0, T^*)$ be arbitrary then $h(T) \geq h(T^*) \geq 1/K > 0$. As $\sigma > 2 + \frac{d}{2}$, it follows from Proposition 2.4.7 that

$$(2.4.22) \quad M_{\sigma, T} \leq \mathcal{F} \left(M_{\sigma, 0} + \mathcal{H}(0) + P_\varepsilon(T) + \int_0^T Q_\varepsilon(t) dt \right) =: L$$

for some increasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending on $1/K$. On the other hand, from the a priori estimate in Proposition 5.2, [3] we deduce that the existence time for local solutions can be chosen uniformly for data lying in a bounded subset of $H^{\sigma+\frac{1}{2}} \times H^\sigma$ and satisfy uniformly the separation condition (H_0) . In particular, call T_1 be the time of existence for data in the ball $B(0, L)$ of $H^{\sigma+\frac{1}{2}} \times H^\sigma$ whose surface is away from the bottom a distance (at least) $1/K$. Choosing $\eta(T^* - \frac{T_1}{2})$ as such a datum we can prolong the solution up to the time $T^* + \frac{T_1}{2}$. This contradicts the maximality of T^* and thus the blow-up criterion (2.4.20) is proved. Finally, (2.4.21) is a consequence of (2.4.20) and the facts that

$$(2.4.23) \quad \begin{aligned} \|\nabla \psi\|_{B_{\infty,1}^0} &\leq C \|V\|_{B_{\infty,1}^0} + C \|B\|_{B_{\infty,1}^0} \|\nabla \eta\|_{C_*^{\varepsilon*}}, \\ \|\nabla \psi\|_{B_{\infty,1}^1} &\leq C \|V\|_{B_{\infty,1}^1} + C \|B\|_{B_{\infty,1}^1} \|\nabla \eta\|_{C_*^{1+\varepsilon*}}. \end{aligned}$$

□

Now we give the proof of Corollary 2.1.3 which is stated again for the reader's convenience.

Corollary 2.4.9. *Let $T \in (0, +\infty)$ and (η, ψ) be a distributional solution to system (2.1.7) on the time interval $[0, T]$ such that $\inf_{[0, T]} \text{dist}(\eta(t), \Gamma) > 0$. Then the following property holds: if one knows a priori that for some $\varepsilon_0 > 0$*

$$(2.4.24) \quad \sup_{[0, T]} \|(\eta(t), \nabla \psi(t))\|_{C_*^{\frac{5}{2}+\varepsilon_0} \times C_*^1} < +\infty$$

then $(\eta(0), \psi(0)) \in H^\infty(\mathbf{R}^d)^2$ implies that $(\eta, \psi) \in L^\infty([0, T]; H^\infty(\mathbf{R}^d))^2$.

Proof. Take $\sigma > 2 + \frac{d}{2}$ be arbitrary, it suffices to prove that if $(\eta(0), \psi(0)) \in H^{\sigma+\frac{1}{2}} \times H^\sigma$ then $(\eta, \psi) \in L^\infty([0, T]; H^{\sigma+\frac{1}{2}} \times H^\sigma)$. Since $\sigma > 2 + \frac{d}{2}$, according to the Cauchy theory in [3] one has a maximal solution

$$(\eta, \psi) \in L^\infty([0, T_\sigma]; H^{\sigma+\frac{1}{2}} \times H^\sigma).$$

By the uniqueness statement of this Cauchy theory, we only need to show that $T_\sigma > T$. Suppose that $T_\sigma \leq T < +\infty$ we get by applying (2.4.21) that for all $\varepsilon > 0$

$$\sup_{t \in [0, T_\sigma)} \|(\eta, \nabla \psi)(t)\|_{C_*^{\frac{5}{2}+\varepsilon} \times C_*^1} + \frac{1}{h(T_\sigma)} = +\infty.$$

On the other hand, by our assumption, $h(T') \geq h(T_\sigma) \geq h(T) > 0$ (by the assumption) for all $T' < T_\sigma$, hence for all $\varepsilon > 0$,

$$\sup_{t \in [0, T_s)} \|(\eta, \nabla \psi)(t)\|_{C_*^{\frac{5}{2}+\varepsilon} \times C_*^1} = +\infty,$$

which contradicts (2.4.24). \square

2.5 Contraction of the solution map

Our goal in this section is to prove a contraction estimate for two solutions to (2.1.7) in weaker norms. This will be used in the proof of the convergence of the approximate scheme and in establishing uniqueness for the Cauchy theory in our companion paper [87]. On the other hand, the proof will make use of the Strichartz estimate in the same paper. To get started, we have by straightforward computations the following assertion: (η, ψ) is a solution to system (2.1.7) if and only if

$$(\partial_t + T_V \cdot \nabla + \mathcal{L}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} = f(\eta, \psi)$$

with

$$(2.5.1) \quad \mathcal{L} := \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} \begin{pmatrix} 0 & -T_\lambda \\ T_\ell & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix}, \quad f(\eta, \psi) := \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}.$$

where

$$(2.5.2) \quad \begin{aligned} f^1(\eta, \psi) &= G(\eta)\psi - (T_\lambda(\psi - T_B\eta) - T_V \cdot \nabla \eta), \\ f^2(\eta, \psi) &= -\frac{1}{2}|\nabla \psi|^2 + \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} \\ &\quad + T_V \cdot \nabla \psi - T_B T_V \cdot \nabla \eta - T_B G(\eta)\psi - H(\eta) + T_\ell \eta - g\eta. \end{aligned}$$

Assume that (η_1, ψ_1) and (η_2, ψ_2) are two solutions of system (2.1.7) on $[0, T]$ and satisfy

$$(\eta_j, \psi_j) \in L^\infty \left([0, T]; H^{s+\frac{1}{2}} \times H^s \right) \cap L^p \left([0, T]; W^{r+\frac{1}{2}, \infty} \times W^{r, \infty} \right), \quad j = 1, 2$$

with

$$s > \frac{3}{2} + \frac{d}{2}, \quad r > 2.$$

Assume in addition that there exists $h > 0$ such that

$$\sup_{t \in [0, T]} \text{dist}(\eta_j(t), \Gamma) \geq h \quad j = 1, 2.$$

Denote for $j = 1, 2$

$$(2.5.3) \quad \begin{aligned} M_{\sigma, T}^j &= \|(\eta^j, \psi^j)\|_{L^\infty([0, T]; H^{\sigma+\frac{1}{2}} \times H^\sigma)}, \\ M_{\sigma, 0}^j &= \|(\eta^j, \psi^j)|_{t=0}\|_{H^{\sigma+\frac{1}{2}} \times H^\sigma}, \\ Z_{r, T}^j &= \|(\eta^j, \psi^j)\|_{L^p([0, T]; W^{r+\frac{1}{2}, \infty} \times W^{r, \infty})}. \end{aligned}$$

Set

$$\delta\eta = \eta_1 - \eta_2, \quad \delta\psi = \psi_1 - \psi_2, \quad \delta B = B_1 - B_2, \quad \delta V = V_1 - V_2.$$

Define the following quantities

$$(2.5.4) \quad \begin{aligned} P_S(t) &= \|\delta\eta(t)\|_{H^{s-1}} + \|\delta\psi(t)\|_{H^{s-\frac{3}{2}}}, \\ P_H(t) &= \|\delta\eta(t)\|_{C_*^{r-1}} + \|\delta\psi(t)\|_{C_*^{r-\frac{3}{2}}}, \\ P_{S,T} &= \|P_S\|_{L^\infty([0,T])}, \quad P_{H,T} = \|P_H\|_{L^p([0,T])}, \\ P(t) &= P_S(t) + P_H(t), \quad P_T = P_{S,T} + P_{H,T}. \end{aligned}$$

Notation 2.5.1. Throughout this section, we write $A \lesssim B$ if there exists a non-decreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that $A \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2)B$.

2.5.1 Contraction estimate for f^2

Recall that we consider B, V as functions of (η, ψ) defined by (2.1.8).

Lemma 2.5.2. *We have for a.e. $t \in [0, T]$*

$$\|\delta B(t)\|_{C_*^{-\frac{1}{2}}} + \|\delta V(t)\|_{C_*^{-\frac{1}{2}}} \lesssim P(t).$$

Proof. Assume the estimate for δB . We have

$$\delta V = \nabla \delta \psi - \delta B \nabla \eta_1 - B_2 \nabla \delta \eta.$$

Obviously,

$$\|\nabla \delta \psi(t)\|_{C_*^{-\frac{1}{2}}} \leq \|\delta \psi(t)\|_{C_*^{\frac{1}{2}}} \leq \|\delta \psi(t)\|_{C_*^{r-\frac{3}{2}}} \leq P_H(t).$$

On the other hand,

$$\|B_2 \nabla \delta \eta(t)\|_{C_*^{-\frac{1}{2}}} \leq \|B_2 \nabla \delta \eta(t)\|_{L^\infty} \lesssim \|\delta \eta(t)\|_{W^{1,\infty}} \lesssim P_H(t)$$

From the product rule (A.0.25) for negative Hölder indices, we deduce

$$\|\delta B \nabla \eta_1(t)\|_{C_*^{-\frac{1}{2}}} \lesssim \|\delta B(t)\|_{C_*^{-\frac{1}{2}}} \|\nabla \eta_1(t)\|_{C_*^{\frac{1}{2}+\varepsilon}} \lesssim P(t)$$

with $\varepsilon > 0$ sufficiently small so that $\|\nabla \eta_1(t)\|_{C_*^{\frac{1}{2}+\varepsilon}} \leq C \|\nabla \eta_1(t)\|_{H^{s-\frac{1}{2}}}$. Therefore, we are left with the estimate for δB , for which we use again the formula (2.3.27)

$$B = K(\nabla \eta) \cdot \nabla \psi + L(\nabla \eta) G(\eta) \psi + G(\eta) \psi$$

where K and L are smooth functions, vanishing at 0. Observe that $G(\eta)$ has order 1, hence these three terms have the same regularity structure. We give the proof for the second one since it is a product with the Dirichlet-Neumann operator

$$\begin{aligned} L(\nabla \eta_1) G(\eta_1) \psi_1 - L(\nabla \eta_2) G(\eta_2) \psi_2 &= [L(\nabla \eta_1) - L(\nabla \eta_2)] G(\eta_1) \psi_1 \\ &\quad + L(\nabla \eta_2) [G(\eta_1) \psi_1 - G(\eta_2) \psi_2]. \end{aligned}$$

Let us consider the more difficult term $L(\nabla \eta_2) [G(\eta_1) \psi_1 - G(\eta_2) \psi_2]$. By means of the product rule (A.0.25) it suffices to estimate the $C_*^{-\frac{1}{2}}$ norm of

$$G(\eta_1) \psi_1 - G(\eta_2) \psi_2 = G(\eta_1) \delta \psi - [G(\eta_1) - G(\eta_2)] \psi_2.$$

The Hölder estimate (2.2.40) together with Remark 2.2.2 implies that

$$\|G(\eta_1)\delta\psi\|_{C_*^{-\frac{1}{2}}} \lesssim \|\delta\psi\|_{C_*^{\frac{1}{2}}} + \|\delta\psi\|_{H^{\frac{1}{2}}} \lesssim \|\delta\psi\|_{C_*^{\frac{1}{2}}} + \|\delta\psi\|_{H^{s-\frac{3}{2}}},$$

where we have used the fact that $s > 2$.

For the second term on the right-hand side, we apply Proposition 2.2.20 to have

$$(2.5.5) \quad [G(\eta_1) - G(\eta_2)]\psi_2 = \int_0^1 \left\{ G(\tilde{\eta}(m))(\tilde{B}\delta\eta(t)) + \operatorname{div}(\tilde{V}(m)\delta\eta(t)) \right\} dm$$

where $\tilde{\eta}(m) = \eta_1 + m\delta\eta$, $\tilde{B}(m) = B(\tilde{\eta}(m), \psi_2)$, $\tilde{V}(m) = V(\tilde{\eta}(m), \psi_2)$. Theorem 2.2.6 applied with $\sigma = s - 2$ then yields

$$(2.5.6) \quad \|[G(\eta_1) - G(\eta_2)]\psi_2\|_{H^{s-2}} \lesssim \|\delta\eta\|_{H^{s-1}}.$$

The embedding $H^{s-2} \hookrightarrow C_*^{-\frac{1}{2}}$ then concludes the proof. \square

We introduce the following notation.

Notation 2.5.3. Let $f : \mathbf{R}^d \rightarrow \mathbf{C}^d$ be a function of u , we set

$$d_u f(u)\dot{u} = \lim_{\varepsilon \rightarrow 0} \{f(u + \varepsilon\dot{u}) - f(u)\}.$$

Proposition 2.5.4. *With f^2 defined in (2.5.2), it holds for a.e. $t \in [0, T]$ that*

$$\|f^2(\eta_1, \psi_1)(t) - f^2(\eta_2, \psi_2)(t)\|_{H^{s-\frac{3}{2}}} \lesssim P(t).$$

Proof. It suffices to prove that

$$(2.5.7) \quad \left\| d_\eta f^2(\eta, \psi)\dot{\eta} + d_\psi f^2(\eta, \psi)\dot{\psi} \right\|_{H^{s-\frac{3}{2}}} \lesssim \|\dot{\eta}\|_{H^{s-1}} + \|\dot{\eta}\|_{C_*^{r-1}} + \|\dot{\psi}\|_{H^{s-\frac{3}{2}}} + \|\dot{\psi}\|_{C_*^{r-\frac{3}{2}}}.$$

We have $f^2(\eta, \psi) = I_1 + I_2 + I_3$ with

$$I_1 := H(\eta) + T_l\eta,$$

$$I_2 := -\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2} \frac{(\nabla\eta \cdot \nabla\psi + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} + T_V \cdot \nabla\psi - T_B T_V \cdot \nabla\eta - T_B G(\eta)\psi,$$

$$I_3 := -g\eta.$$

Observe that $d_\psi I_1 = d_\psi I_3 = 0$. The estimate for $d_\eta I_3 \dot{\eta} = -g\dot{\eta}$ is obvious. Observe that I_1 and I_2 are the remainder of the parilinearization of nonlinear functions in Lemmas 2.3.6 and 2.3.8, respectively. Putting $f(x) = x(1 + |x|^2)^{-1/2}$, $x \in \mathbf{R}^d$, we have $-H(\eta) = \operatorname{div} f(\nabla\eta)$. Since

$$-d_\eta f(\nabla\eta)\dot{\eta} = f'(\nabla\eta)\nabla\dot{\eta},$$

it follows that

$$-d_\eta H(\eta)\dot{\eta} = \operatorname{div}(f'(\nabla\eta)\nabla)\dot{\eta} + f'(\nabla\eta)\nabla \cdot \nabla\dot{\eta}.$$

Using the Bony-Calculus symbolique decomposition we get

$$-d_\eta H(\eta)\dot{\eta} = T_{i \operatorname{div}(f'(\nabla\eta)\xi)}\dot{\eta} + T_{-f'(\nabla\eta)\xi \cdot \xi}\dot{\eta} + R = T_{-\ell}\dot{\eta} + R$$

with $\|R\|_{H^{s-3/2}} \lesssim \|\dot{\eta}\|_{H^{s-1}} + \|\dot{\eta}\|_{C_*^{r-1}}$. The Leibnitz rule then implies

$$d_\eta I_1(\eta)\dot{\eta} = T_{\dot{\ell}}\eta + R, \quad \dot{\ell} := d_\eta \ell \dot{\eta},$$

so we only need to show that $\|T_{\dot{\ell}}\eta\|_{H^{s-3/2}} \lesssim \|\dot{\eta}\|_{H^{s-1}} + \|\dot{\eta}\|_{C_*^{r-1}}$. Indeed, observe that $\dot{\ell}$ is of the form

$$\dot{\ell} = F_1(\nabla\eta, \xi)\nabla\dot{\eta} + F_2(\nabla\eta, \xi)\nabla^2\dot{\eta} + F_3(\nabla\eta, \xi)\nabla\dot{\eta}\nabla^2\eta =: \sum_{j=1}^3 G_j(x, \xi),$$

where F_j , $j = 1, 2, 3$ are smooth in $\mathbf{R}^d \times \mathbf{R}^d \setminus \{0\}$, F_1 is homogeneous of order 2 in ξ and F_2, F_3 are homogeneous of order 1 in ξ . By virtue of Theorem A.0.4 (i) and Proposition A.0.6 we see that to obtain the desired bound for $\|T_{\dot{\ell}}\eta\|_{H^{s-3/2}}$ it suffices to prove for $j = 1, 2, 3$

$$\sup_{|\xi|=1} \|\partial_\xi^\alpha G_1(\cdot, \xi)\|_{L^\infty} + \sup_{|\xi|=1} \|\partial_\xi^\alpha G_j(\cdot, \xi)\|_{C_*^{-1}} \lesssim C_\alpha \|\dot{\eta}\|_{C_*^{r-1}} \quad \forall \alpha \in \mathbf{N}^d.$$

This is true because (assuming without loss of generality that $F_j(0, \xi) = 0$, for all ξ) uniformly in $|\xi| = 1$,

$$\begin{aligned} \|F_1(\nabla\eta)\nabla\dot{\eta}\|_{L^\infty} &\lesssim \|\dot{\eta}\|_{W^{1,\infty}} \lesssim \|\dot{\eta}\|_{C_*^{r-1}}, \\ \|F_2(\nabla\eta)\nabla^2\dot{\eta}\|_{C_*^{-1}} &\lesssim \|F_2(\nabla\eta)\|_{C_*^{1+\varepsilon}} \|\nabla^2\dot{\eta}\|_{C_*^{-1}} \lesssim \|\dot{\eta}\|_{C_*^{r-1}} \quad \varepsilon \in (0, s - \frac{3}{2} - \frac{d}{2}), \\ \|F_3(\nabla\eta)\nabla\dot{\eta}\nabla^2\eta\|_{C_*^{-1}} &\lesssim \|F_3(\nabla\eta)\nabla\dot{\eta}\nabla^2\eta\|_{L^\infty} \lesssim \|\dot{\eta}\|_{W^{1,\infty}} \lesssim \|\dot{\eta}\|_{C_*^{r-1}}. \end{aligned}$$

We have shown the desired estimate for I_1 . By inspecting the proof of Lemma 2.3.8, the estimate for the $H^{s-3/2}$ -norm of $d_\eta I_2\dot{\eta} + d_\psi I_2\dot{\psi}$ can be obtained by the same method. \square

2.5.2 Contraction estimate for f^1

Our goal in this subsection is to derive the following estimate.

Proposition 2.5.5. *With f^1 defined as in (2.5.2), it holds for a.e. $t \in [0, T]$ that*

$$\|f^1(\eta_1, \psi_1)(t) - f^1(\eta_2, \psi_2)(t)\|_{H^{s-1}} \lesssim P_H(t) + P_S(t)Q(t)$$

with

$$(2.5.8) \quad Q(t) := 1 + \sum_{j=1}^2 \|\eta_j(t)\|_{C_*^{r+\frac{1}{2}}} + \sum_{j=1}^2 \|\psi_j(t)\|_{C_*^r}.$$

The key point is that the preceding estimate is tame with respect to the highest Hölder norms. Proposition 2.5.5 will be a consequence of

$$(2.5.9) \quad \|d_\eta f^1(\eta, \psi)\dot{\eta}\|_{H^{s-1}} \lesssim \|\dot{\eta}\|_{H^{s-1}} (1 + \|\eta\|_{C_*^{r+\frac{1}{2}}} + \|\psi\|_{C_*^r}) + \|\dot{\eta}\|_{C_*^{r-1}}$$

for all $\dot{\eta} \in H^{s+\frac{1}{2}} \cap C_*^{r+\frac{1}{2}}$, and

$$(2.5.10) \quad \|d_\psi f^1(\eta, \psi)\dot{\psi}\|_{H^{s-1}} \lesssim \|\dot{\psi}\|_{H^{s-\frac{3}{2}}} (1 + \|\eta\|_{C_*^{r+\frac{1}{2}}}) + \|\dot{\psi}\|_{C_*^{r-\frac{3}{2}}}$$

for all $\dot{\psi} \in H^s \cap C_*^r$.

Lemma 2.5.6. *The estimate (2.5.9) holds.*

Proof. From the definition of f^1 and Proposition 2.2.20 we have

$$\begin{aligned} d_\eta f^1(\eta, \psi)\dot{\eta} &= -G(B\dot{\eta}) - \operatorname{div}(V\dot{\eta}) \\ &\quad - \{T_{\dot{\lambda}}(\psi - T_B\eta) - T_{\dot{\lambda}}T_{\dot{B}}\eta - T_{\dot{\lambda}}T_B\dot{\eta} - T_{\dot{V}}\nabla\eta - T_V\nabla\dot{\eta}\} \\ &= \sum_{j=1}^5 I_j, \end{aligned}$$

where $\dot{B} := d_\eta B(\eta, \psi)\dot{\eta}$ (similarly for \dot{V} , $\dot{\lambda}$) and

$$\begin{aligned} I_1 &:= T_{\dot{V}}\nabla\eta, & I_2 &:= -V\nabla\dot{\eta} + T_V\nabla\dot{\eta}, & I_3 &:= -T_{\dot{\lambda}}(\psi - T_B\eta), \\ I_4 &:= T_{\dot{\lambda}}T_{\dot{B}}\eta, & I_5 &:= -G(B\dot{\eta}) - (\operatorname{div} V)\dot{\eta} + T_{\dot{\lambda}}T_B\dot{\eta}. \end{aligned}$$

1. For I_2 we write $I_2 = -T_{\nabla\dot{\eta}}V - R(\nabla\dot{\eta}, V)$ and use (A.0.13), (A.0.21) to estimate

$$\|I_2\|_{H^{s-1}} \lesssim \|V\|_{H^{s-1}} \|\nabla\dot{\eta}\|_{L^\infty} \lesssim \|\dot{\eta}\|_{C_x^{r-1}}.$$

2. Let us study \dot{B} and \dot{V} . For the former, the only nontrivial point is

$$(2.5.11) \quad d_\eta[G(\eta)\psi]\dot{\eta} = -G(\eta)(B\dot{\eta}) - \operatorname{div}(V\dot{\eta}).$$

It holds that

$$\|d_\eta[G(\eta)\psi]\dot{\eta}\|_{H^{s-2}} \lesssim \|\dot{\eta}\|_{H^{s-1}} + \|V\dot{\eta}\|_{H^{s-1}} \lesssim \|\dot{\eta}\|_{H^{s-1}}.$$

Therefore, $\|\dot{B}\|_{H^{s-2}} \lesssim \|\dot{\eta}\|_{H^{s-1}}$. This, together with the relation $V = \nabla\psi - B\nabla\eta$, implies

$$\|\dot{B}\|_{H^{s-2}} + \|\dot{V}\|_{H^{s-2}} \lesssim \|\dot{\eta}\|_{H^{s-1}}.$$

As a consequence, the paraproduct rule (A.0.16) gives (keep in mind that $s > \frac{3}{2} + \frac{d}{2}$)

$$\|I_1\|_{H^{s-1}} \lesssim \|\dot{V}\|_{H^{s-2}} \|\nabla\eta\|_{H^{s-\frac{1}{2}}} \lesssim \|\dot{\eta}\|_{H^{s-1}}.$$

Similarly,

$$\|I_4\|_{H^{s-1}} \lesssim \|T_{\dot{B}}\eta\|_{H^s} \lesssim \|\dot{B}\|_{H^{s-2}} \|\eta\|_{H^{s+\frac{1}{2}}} \lesssim \|\dot{\eta}\|_{H^{s-1}}.$$

3. For I_3 one estimates $\dot{\lambda}$ exactly as for \dot{l} in the proof of Proposition 2.5.4.

4. For I_5 we follow [3] using the following cancellation in Lemma 2.12, [3] whose proof applies equally at our regularity level:

$$G(\eta)B = -\operatorname{div} V + R, \quad \|R\|_{H^{s-1}} \lesssim 1.$$

On the other hand, applying Proposition 3.13 in [7] with $\varepsilon = \frac{1}{2}$ $\sigma = s - \frac{1}{2}$ we obtain the following parilinearizations

$$G(\eta)(B\dot{\eta}) = T_{\lambda^{(1)}}B\dot{\eta} + F(\eta, B\dot{\eta}), \quad G(\eta)(B) = T_{\lambda^{(1)}}B + F(\eta, B)$$

with

$$\|F(\eta, B\dot{\eta})\|_{H^{s-1}} \lesssim \|\dot{\eta}\|_{H^{s-1}}, \quad \|F(\eta, B)\|_{H^{s-1}} \lesssim 1.$$

Then plugging these parilinearizations into the expression of I_5 gives (see [3] pages 482 – 483 for details) $I_5 = J_1 + J_2$ with

$$\begin{aligned} J_1 &= -T_{\lambda^{(1)}}(B\dot{\eta} - T_B\dot{\eta} - T_{\dot{\eta}}B), \\ J_2 &= T_{\lambda^{(0)}}T_B\dot{\eta} + [T_{\dot{\eta}}, T_{\lambda^{(1)}}]B + T_{\dot{\eta}}F(\eta, B) + (\dot{\eta} - T_{\dot{\eta}})\operatorname{div} V - F(\eta, B\dot{\eta}) - T_{\dot{\eta}}R. \end{aligned}$$

Using (A.0.14) we estimate

$$\|J_1\|_{H^{s-1}} \lesssim \|R(B, \dot{\eta})\|_{H^s} \lesssim \|\dot{\eta}\|_{H^{s-1}} \|B\|_{C_*^1} \lesssim \|\dot{\eta}\|_{H^{s-1}} \mathcal{B}ig(1 + \|\eta\|_{C_*^{r+\frac{1}{2}}} + \|\psi\|_{C_*^r} \mathcal{B}ig).$$

For J_2 we only need to take care of the commutator $[T_{\dot{\eta}}, T_{\lambda^{(1)}}]B$. Since $\|B\|_{H^{s-1}} \lesssim 1$ it suffices to prove that $[T_{\dot{\eta}}, T_{\lambda^{(1)}}]$ has order 0 and map $H^{s-1} \rightarrow H^{s-1}$ with norm bounded by the right hand side of (2.5.9). This is in turn a consequence of Theorem A.0.4 (ii) and the fact that $r-1 > 1$. \square

Finally, we prove

Lemma 2.5.7. *The estimate (2.5.10) holds.*

Write $B = B(\eta, \psi)$, $V = V(\eta, \psi)$. Since $G(\eta)\psi$ is linear with respect to ψ we get

$$d_\psi f^1(\eta, \psi)\dot{\psi} = G(\eta)\dot{\psi} - T_\lambda(\dot{\psi} - T_{B(\eta, \psi)}\eta) - T_{V(\eta, \psi)} \cdot \nabla \eta =: R(\eta, \dot{\psi}).$$

Estimate (2.5.10) means that R has order $-1/2$ with respect to $\dot{\psi}$ and map $H^{s-3/2}$ to H^{s-1} . In fact, Proposition 2.3.5 shows that R maps H^s to $H^{s+1/2}$. Here, we will follow the proof of Proposition 2.3.5 except that the good unknown will not be invoked. Lemma 2.5.7 is a consequence of the following.

Lemma 2.5.8. *Let $d \geq 1$, $h > 0$ and*

$$s > \frac{3}{2} + \frac{d}{2}, \quad r > 2.$$

Then there exists a nondecreasing function $\mathcal{F} : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that for any $\eta \in H^{s+\frac{1}{2}}$ satisfying $\operatorname{dist}(\eta, \Gamma) \geq h > 0$ and $\psi \in H^s \cap C^r$, there holds

$$(2.5.12) \quad \begin{aligned} &\|G(\eta)\psi - T_\lambda(\psi - T_B\eta) - T_V \cdot \nabla \eta\|_{H^{s-1}} \\ &\leq \mathcal{F}(\|(\eta, \psi)\|_{H^{s+\frac{1}{2}} \times H^s}) \left\{ \|\psi\|_{H^{s-\frac{3}{2}}} (1 + \|\eta\|_{C_*^{r+\frac{1}{2}}} + \|\psi\|_{C_*^r}) + \|\psi\|_{C_*^{r-\frac{3}{2}}} \right\}. \end{aligned}$$

Proof. We first apply Theorem A.0.4 (i) to have

$$\|T_\lambda T_B \eta\|_{H^{s-1}} + \|T_V \cdot \nabla \eta\|_{H^{s-1}} \leq \mathcal{F}(\|(\eta, \psi)\|_{H^{s+\frac{1}{2}} \times H^s}) \left\{ \|B\|_{C_*^{-\frac{1}{2}}} + \|V\|_{C_*^{-\frac{1}{2}}} \right\}.$$

On the other hand, as in Lemma 2.5.2 it holds that

$$\|B\|_{C_*^{-\frac{1}{2}}} + \|V\|_{C_*^{-\frac{1}{2}}} \leq \mathcal{F}(\|(\eta, \psi)\|_{H^{s+\frac{1}{2}} \times H^s}) \left\{ \|\psi\|_{H^{s-\frac{3}{2}}} + \|\psi\|_{C_*^{r-\frac{3}{2}}} \right\}.$$

Therefore, the proof of (2.5.12) reduces to showing

$$(2.5.13) \quad \begin{aligned} &\|G(\eta)\psi - T_\lambda \psi\|_{H^{s-1}} \\ &\leq \mathcal{F}(\|(\eta, \psi)\|_{H^{s+\frac{1}{2}} \times H^s}) \left\{ \|\psi\|_{H^{s-\frac{3}{2}}} (1 + \|\eta\|_{C_*^{r+\frac{1}{2}}} + \|\psi\|_{C_*^r}) + \|\psi\|_{C_*^{r-\frac{3}{2}}} \right\}. \end{aligned}$$

To this end, let ϕ be the solution to (2.3.4). Let v be as in (2.2.12), which satisfies equation (2.2.13). Let $z_0 \in (-1, 0)$ and denote $J = [z_0, 0]$. Let Π denote the right-hand side of (2.5.13). Again, to alleviate notations, we will write $A \lesssim B$ provided

$$A \leq \mathcal{F}(\|(\eta, \psi)\|_{H^{s+\frac{1}{2}} \times H^s})B.$$

According to Proposition 2.2.16 and Remark 2.2.2 (notice that $s - \frac{3}{2} > \frac{1}{2}$), there holds

$$(2.5.14) \quad \|\nabla_{x,z} v\|_{C(J; C_*^{-\frac{1}{2}})} \lesssim \|\psi\|_{H^{s-\frac{3}{2}}} + \|\psi\|_{C_*^{-\frac{3}{2}}}.$$

On the other hand, applying Proposition 2.2.10 with $\sigma = s - \frac{5}{2} \geq -\frac{1}{2}$ gives

$$(2.5.15) \quad \|\nabla_{x,z} v\|_{X^{s-\frac{5}{2}}(I)} \lesssim \|\psi\|_{H^{s-\frac{3}{2}}}.$$

We will write $g_1 \sim_E g_2$ if the E -norm of $g_1 - g_2$ is bounded by Π . As in Proposition 2.3.3, we set

$$P := \partial_z^2 + T_\alpha \Delta_x + T_\beta \cdot \nabla_x \partial_z - T_\gamma \partial_z.$$

In view of equation (2.2.13), we have

$$0 = (\partial_z^2 + \alpha \Delta_x + \beta \cdot \nabla_x \partial_z - \gamma \partial_z)v = Pv + Qv$$

with

$$Qv := [T_{\Delta v}(\alpha - h^2) + R(\Delta v, \alpha - h^2)] + [(h^2 - T_{h^2})\Delta_x v] + [T_{\nabla \partial_z v} \beta + R(\nabla \partial_z v, \beta)] - [T_{\partial_z v} \gamma + R(\partial_z v, \gamma)].$$

For the first bracket, we use (A.0.19), (A.0.14) and (2.5.14) to have

$$\|T_{\Delta v}(\alpha - h^2)\|_{L^2 H^{s-\frac{3}{2}}} + \|R(\Delta v, \alpha - h^2)\|_{L^2 H^{s-\frac{3}{2}}} \lesssim \|\Delta v\|_{L^\infty C_*^{-\frac{3}{2}}} \|\alpha - h^2\|_{L^2 H^s} \lesssim \Pi.$$

The other terms can be estimated along the same lines. Consequently,

$$Pu \sim_{L^2(J; H^{s-\frac{3}{2}})} 0.$$

Next, with two symbols a, A defined in Lemma 2.3.4, the proof of Lemma 2.3.4 shows that

$$Pv = (\partial_z - T_a)(\partial_z - T_A)v + (T_a T_A - T_\alpha \Delta_x)v.$$

According to the symbolic estimate (2.3.20) and Theorem A.0.4 (ii), $(T_a T_A - T_\alpha \Delta_x)$ has order $\frac{1}{2}$ and

$$\|(T_a T_A - T_\alpha \Delta_x)v\|_{L^2 H^{s-\frac{3}{2}}} \lesssim (1 + \|\eta\|_{C_*^{\frac{5}{2}}}) \|\nabla v\|_{L^2 H^{s-2}} \lesssim (1 + \|\eta\|_{C_*^{\frac{5}{2}}}) \|\psi\|_{L^2 H^{s-\frac{3}{2}}},$$

owing to (2.5.15). We have proved that

$$(\partial_z - T_a)(\partial_z - T_A)v \sim_{L^2(J; H^{s-\frac{3}{2}})} 0,$$

which implies

$$(\partial_z - T_a)(\partial_z - T_A)v \sim_{Y^{s-1}(J)} 0.$$

With this result, one can follow exactly Step 1. of the proof of Proposition 2.3.5 and obtain for some $I = [z_1, 0)$, $z_1 \in (z_0, 0)$ that

$$(2.5.16) \quad \|\partial_z v - T_A v\|_{X^{s-1}(I)} \lesssim \Pi.$$

This allow us to replace the normal derivative $\partial_z v$ with the "tangential derivative" $T_A v$, leaving a term $\sim_{X^{s-1}(I)} 0$. Therefore, we deduce by using the BonyCalculusymboliqueet79 decomposition and the estimates (2.5.14)-(2.5.15) that

$$\begin{aligned} \frac{1 + |\nabla \rho|^2}{\partial_z \rho} \partial_z v - \nabla \rho \cdot \nabla v &\sim_{X^{s-1}(I)} T_{\frac{1+|\nabla \rho|^2}{\partial_z \rho}} \partial_z v - T_{\nabla \rho} \nabla v \\ &\sim_{X^{s-1}(I)} T_{\frac{1+|\nabla \rho|^2}{\partial_z \rho}} T_A v - T_{\nabla \rho} \nabla v \\ &\sim_{X^{s-1}(I)} \sim T_{\frac{1+|\nabla \rho|^2}{\partial_z \rho}} A v - T_{\nabla \rho} \nabla v \\ &\sim_{X^{s-1}(I)} T_\Lambda v \end{aligned}$$

with $\Lambda = \frac{1+|\nabla \rho|^2}{\partial_z \rho} A - i \nabla \eta \cdot \xi$ satisfying $\Lambda|_{z=0} = \lambda$. The proof of (2.5.13) is complete. \square

2.5.3 Contraction estimate for the solution map

In views of the notations (2.5.2), (2.5.4) and (2.5.8) , we have proved in subsections 5.1, 5.2 the following result for a.e. $t \in [0, T]$,

$$\|f(\eta_1, \psi_1)(t) - f(\eta_2, \psi_2)(t)\|_{H^{s-1} \times H^{s-\frac{3}{2}}} \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) (P_H(t) + P_S(t)Q(t)).$$

Consequently, this together with Lemma 2.5.2 implies that the difference of two solutions satisfies

$$(2.5.17) \quad (\partial_t + T_{V_1} \cdot \nabla + \mathcal{L}_1) \begin{pmatrix} \delta \eta \\ \delta \psi \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$(2.5.18) \quad \|(g_1(t), g_2(t))\|_{H^{s-1} \times H^{s-\frac{3}{2}}} \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) \mathcal{B}ig(P_H(t) + P_S(t)Q(t)\mathcal{B}ig).$$

Symmetrization

Now, as in Section 2.3.3 we symmetrize (2.5.17) using the symmetrizer

$$S_1 = \begin{pmatrix} T_{p_1} & 0 \\ 0 & T_{q_1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_{B_1} & I \end{pmatrix}.$$

The dispersive part \mathcal{L} . Recall the Definition 2.3.13 on the equivalence of two families of operators $A(t)$ and $B(t)$, $t \in [0, T]$: $A \sim B$ if

$$\|A(t) - B(t)\|_{H^\mu \rightarrow H^{\mu-m+\frac{3}{2}}} \leq \mathcal{F}(\|\eta(t)\|_{H^{s+\frac{1}{2}}}) \left(1 + \|\eta(t)\|_{C^{r+\frac{1}{2}}}\right).$$

By virtue of Proposition 2.3.15 we obtain

$$\begin{aligned} &\begin{pmatrix} T_{p_1} & 0 \\ 0 & T_{q_1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_{B_1} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ T_{B_1} & I \end{pmatrix} \begin{pmatrix} 0 & -T_{\lambda_1} \\ T_{\ell_1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} T_{p_1} & 0 \\ 0 & T_{q_1} \end{pmatrix} \begin{pmatrix} 0 & -T_{\lambda_1} \\ T_{\ell_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -T_{p_1} T_{\lambda_1} \\ T_{q_1} T_{\ell_1} & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 0 & -T_{\gamma_1} T_{q_1} \\ T_{\gamma} T_{p_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -T_{\gamma_1} \\ T_{\gamma_1} & 0 \end{pmatrix} \begin{pmatrix} T_{p_1} & 0 \\ 0 & T_{q_1} \end{pmatrix}. \end{aligned}$$

Consequently (see (2.5.1) for the definition of \mathcal{L})

$$S_1 \mathcal{L}_1 \sim \begin{pmatrix} 0 & -T_{\gamma_1} \\ T_{\gamma_1} & 0 \end{pmatrix} \begin{pmatrix} T_{p_1} & 0 \\ 0 & T_{q_1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_{B_1} & I \end{pmatrix}.$$

Therefore, if we set

$$\Phi_1 := T_{p_1} \delta \eta, \quad \Phi_2 := T_{q_1} (\delta \psi - T_{B_1} \delta \eta),$$

then Φ_1, Φ_2 satisfy

$$S_1 \mathcal{L}_1 \begin{pmatrix} \delta \eta \\ \delta \psi \end{pmatrix} \sim \begin{pmatrix} -T_{\gamma_1} \Phi_2 \\ T_{\gamma_1} \Phi_1 \end{pmatrix},$$

meaning that

$$\left\| S_1 \mathcal{L}_1 \begin{pmatrix} \delta \eta \\ \delta \psi \end{pmatrix} - \begin{pmatrix} -T_{\gamma_1} \Phi_2 \\ T_{\gamma_1} \Phi_1 \end{pmatrix} \right\|_{H^{s-\frac{3}{2}}} (t) \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) \left(1 + \|\eta_1(t)\|_{C_*^{r+\frac{1}{2}}} \right) P_S(t).$$

The convection part $\partial_t + T_{V_1} \nabla$: one proceeds as in the proof Proposition 2.3.16 and obtain

$$S_1 (\partial_t + T_{V_1} \cdot \nabla) \begin{pmatrix} \delta \eta \\ \delta \psi \end{pmatrix} = (\partial_t + T_{V_1} \cdot \nabla) S_1 \begin{pmatrix} \delta \eta \\ \delta \psi \end{pmatrix} + R = (\partial_t + T_{V_1} \cdot \nabla) \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} + R$$

where the remainder R verifies

$$\|R(t)\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{3}{2}}} \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) \left(1 + \|\eta_1\|_{C^{r+\frac{1}{2}}} + \|\psi_1\|_{C^r} \right) P_S(t).$$

A combination of two parts yields

$$(2.5.19) \quad \begin{cases} \partial_t \Phi_1 + T_{V_1} \cdot \nabla \Phi_1 - T_{\gamma_1} \Phi_2 = F_1 + G_1, \\ \partial_t \Phi_2 + T_{V_1} \cdot \nabla \Phi_2 + T_{\gamma_1} \Phi_2 = F_2 + G_2 \end{cases}$$

where for a.e. $t \in [0, T]$,

$$(2.5.20) \quad \begin{aligned} \|(F_1, F_2)\|_{H^{s-\frac{3}{2}} \times H^{s-\frac{3}{2}}} &\leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) \mathcal{B}ig(1 + \|\eta_1\|_{C^{r+\frac{1}{2}}} + \|\psi_1\|_{C^r} \mathcal{B}ig) P_S(t) \\ &\leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) Q(t) P_S(t), \end{aligned}$$

and from (2.5.17)

$$\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} T_{p_1} g_1 \\ T_{q_1} (g_2 - T_{B_1} g_1) \end{pmatrix}.$$

It follows from (2.5.18) that (G_1, G_2) also satisfy

$$(2.5.21) \quad \|(G_1, G_2)\|_{H^{s-\frac{3}{2}} \times H^{s-\frac{3}{2}}} \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) \mathcal{B}ig(P_H(t) + P_S(t) Q(t) \mathcal{B}ig).$$

Contraction estimates

Put $\Phi := \Phi_1 + i\Phi_2$, then

$$(2.5.22) \quad \partial_t \Phi + T_{V_1} \cdot \nabla \Phi + iT_{\gamma_1} \Phi = F + G := (F_1 + iF_2) + (G_1 + iG_2).$$

We are now back to the situation of Proposition 2.4.1: we shall conjugate (2.5.22) with an operator of order $s - 3/2$ and then perform an L^2 -energy estimate. As in (2.4.3), we choose

$$\wp_1 = (\gamma_1^{(3/2)})^{2(s-\frac{3}{2})/3}, \quad \varphi = T_{\wp_1} \Phi.$$

After conjugating with T_{φ_1} , one obtains

$$(2.5.23) \quad (\partial_t + T_{V_1} \cdot \nabla + iT_{\gamma_1}) \varphi = T_{\varphi_1} (F + G) + H$$

with

$$H := T_{\partial_t \varphi_1} \Phi + [T_{V_1} \cdot \nabla, T_{\varphi_1}] \Phi + i[T_{\gamma_1}, T_{\varphi_1}] \Phi.$$

It is easy to see as in the proof of Proposition 2.4.3 that

$$(2.5.24) \quad \begin{aligned} \|H(t)\|_{H^{s-\frac{3}{2}}} &\leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) Q(t) \|\Phi(t)\|_{H^{s-\frac{3}{2}}} \\ &\leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) Q(t) (\|\varphi(t)\|_{L^2} + \|\Phi(t)\|_{L^2}), \end{aligned}$$

where we have applied Lemma 2.3.12 in the second line.

On the other hand, from the estimates (2.5.20), (2.5.21) for F, G we get

$$(2.5.25) \quad \|T_{\varphi_1} (F + G)\|_{L^2} \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) \mathcal{B}ig(P_H(t) + P_S(t)Q(t)\mathcal{B}ig).$$

Now, multiplying both sides of (2.5.23) by φ and using (2.5.24), (2.5.25), (2.4.9), (2.4.10) lead to

$$\begin{aligned} \frac{d}{dt} \|\varphi(t)\|_{L^2}^2 &\leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) \times \\ &\left\{ \mathcal{B}ig[P_H(t) + Q(t)P_S(t) + Q(t)\|\Phi(t)\|_{L^2} \mathcal{B}ig] \|\varphi(t)\|_{L^2} + Q(t) \|\varphi(t)\|_{L^2}^2 \right\}. \end{aligned}$$

Notice that

$$\|\Phi(t)\|_{L^2} \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2) P_S(t), \quad \int_0^T Q(t) dt \leq T + T^{\frac{1}{p'}} (Z_{r,T}^1 + Z_{r,T}^2),$$

with $\frac{1}{p'} = 1 - \frac{1}{p} > 0$ (recall the notation $Z_{r,T}^j$ in (2.5.3)). Grönwall's lemma then implies (see Notation 2.5.4) for all $t \leq T \leq 1$

$$(2.5.26) \quad \begin{aligned} \|\varphi(t)\|_{L^2} &\leq \mathcal{F}(\dots) \left(\|\varphi(0)\|_{L^2} + \int_0^t [Q(m)P_S(m) + P_H(m)] dm \right) \\ &\leq \mathcal{F}(\dots) \left(\|\varphi(0)\|_{L^2} + T^{\frac{1}{p'}} [(1 + Z_{r,T}^1 + Z_{r,T}^2)P_{S,T} + P_{H,T}] \right) \\ &\leq \mathcal{F}(\dots) \left(P_S(0) + T^{\frac{1}{p'}} P_T \right) \end{aligned}$$

where

$$\mathcal{F}(\dots) = \mathcal{F}(M_{s,T}^1, M_{s,T}^2, Z_{r,T}^1, Z_{r,T}^2).$$

We want to show that $\|(\delta\eta, \delta\psi)\|_{H^{s-1} \times H^{s-\frac{3}{2}}}$ is also controlled by the right-hand side of (2.5.26). To this end, one uses again Proposition 2.3.12 to have

$$\begin{aligned} \|\delta\eta\|_{H^{s-1}} &\lesssim \|T_{\varphi} T_p \delta\eta\|_{L^2} + \|\delta\eta\|_{H^{-\frac{1}{2}}}, \\ \|\delta\psi\|_{H^{s-\frac{3}{2}}} &\lesssim \|T_{\varphi} T_q \delta\psi\|_{L^2} + \|\delta\psi\|_{H^{-\frac{1}{2}}}. \end{aligned}$$

Then, in view of (2.5.26) it remains to estimate $\|\delta\eta\|_{H^{-\frac{1}{2}}}$ and $\|\delta\psi\|_{H^{-\frac{1}{2}}}$. Indeed, we write

$$\begin{aligned} \|\delta\eta(t)\|_{H^{-\frac{1}{2}}} &\leq \|\delta\eta(0)\|_{H^{-\frac{1}{2}}} + \|\delta\eta(t) - \delta\eta(0)\|_{H^{-\frac{1}{2}}} \\ &\leq \|\delta\eta(0)\|_{H^{-\frac{1}{2}}} + \left\| \int_0^t \frac{d}{dt} \delta\eta(m) dm \right\|_{H^{-\frac{1}{2}}} \\ &\leq \|\delta\eta(0)\|_{H^{-\frac{1}{2}}} + T \sup_{t \in [0, T]} \left\| \frac{d}{dt} \delta\eta(t) \right\|_{H^{-\frac{1}{2}}}. \end{aligned}$$

The last term can be written as

$$\frac{d}{dt} \delta\eta(t) = G(\eta_1(t))\psi_1(t) - G(\eta_2(t))\psi_2(t) = G(\eta_1)\delta\psi + [G(\eta_1(t)) - G(\eta_2(t))]\psi_2(t).$$

The Sobolev estimate for the Dirichlet-Neumann operator in Theorem 2.2.6 applied with $\sigma = \frac{1}{2}$ gives

$$\|G(\eta_1)\delta\psi\|_{H^{-\frac{1}{2}}} \lesssim \|\delta\psi\|_{H^{\frac{1}{2}}} \lesssim \|\delta\psi\|_{H^{s-\frac{3}{2}}}.$$

On the other hand, according to (2.5.6)

$$\|[G(\eta_1) - G(\eta_2)]\psi_2\|_{L^2} \leq \|[G(\eta_1) - G(\eta_2)]\psi_2\|_{H^{s-2}} \lesssim \|\delta\eta\|_{H^{s-1}}.$$

Therefore,

$$\|\delta\eta\|_{H^{-\frac{1}{2}}} \lesssim \|\delta\eta(0)\|_{H^{-\frac{1}{2}}} + TP_{S,T}.$$

Using the second equation of (2.3.30) and arguing as above, we find that

$$\|\delta\psi\|_{H^{-\frac{1}{2}}} \lesssim \|\delta\psi(0)\|_{H^{-\frac{1}{2}}} + TP_{S,T}.$$

Putting the above estimates together, we end up with

$$\|(\delta\eta(t), \delta\psi(t))\|_{H^{s-1} \times H^{s-\frac{3}{2}}} \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2, Z_{r,T}^1, Z_{r,T}^2) \left(P_S(0) + T^{\frac{1}{p'}} P_T \right),$$

which implies (recall that we are assuming $s > \frac{3}{2} + \frac{d}{2}$, $r > 2$)

$$(2.5.27) \quad P_{S,T} \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2, Z_{r,T}^1, Z_{r,T}^2) \left(P_S(0) + T^{\frac{1}{p'}} P_T \right).$$

Observe that (2.5.27) is an a priori estimate for the Sobolev norm of the difference of two solutions. To close this estimate, we seek a similar estimate for the Hölder norm, *i.e.*, for $P_{H,T}$. This is achieved by applying the Strichartz estimates in our companion paper [87] to the dispersive equation (2.5.22). According to Theorem 1.1 of [87], if u is a solution to the problem

$$(\partial_t + T_{V_1} \cdot \nabla + iT_{\gamma_1})u = f$$

with $f \in L^\infty([0, T]; H^\sigma)$, $\sigma \in \mathbf{R}$, then it holds that

$$(2.5.28) \quad \|u\|_{L^p W^{\sigma-\frac{d}{2}+\mu}} \leq \mathcal{F}(Z_{r,T}^1) (\|f\|_{L^p H^\sigma} + \|u\|_{L^\infty H^\sigma})$$

where,

$$(2.5.29) \quad \begin{cases} \mu = \frac{3}{20}, p = 4 & \text{when } d = 1, \\ \mu = \frac{3}{10}, p = 2 & \text{when } d \geq 2. \end{cases}$$

Applying this result to $u = \Phi$ with $\sigma = s - \frac{3}{2}$ leads to

$$\|\Phi\|_{L^p W^{s-\frac{3}{2}-\frac{d}{2}+\mu}} \leq \mathcal{F}(Z_{r,T}^1) \left(\|F + G\|_{L^p H^{s-\frac{3}{2}}} + \|\Phi\|_{L^\infty H^{s-\frac{3}{2}}} \right),$$

This, combined with (2.5.20) and (2.5.21), implies for any $2 < r < r' < s - \frac{d}{2} + \mu$

$$\begin{aligned} \|\Phi\|_{L^p W^{r'-\frac{3}{2}}} &\leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2, Z_{r,T}^1, Z_{r,T}^2) \left(P_T + \|\Phi\|_{L^\infty H^{s-\frac{3}{2}}} \right) \\ &\leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2, Z_{r,T}^1, Z_{r,T}^2) P_T. \end{aligned}$$

By interpolating between r' and some lower index, we gain a multiplication factor of the form T^δ , $\delta > 0$ on the right-hand side. Then using the symbolic calculus in Theorem A.0.4 to go back from Φ to $\delta\eta$, $\delta\psi$ we obtain

$$(2.5.30) \quad P_{H,T} \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2, Z_{r,T}^1, Z_{r,T}^2) T^\delta P_T.$$

Combining (2.5.27) and (2.5.30) we end up with a closed a priori estimate for the difference of two solutions of (2.1.7) in terms of Sobolev norm and Strichartz norm: for any $T \leq 1$, there holds

$$P_T \leq \mathcal{F}(M_{s,T}^1, M_{s,T}^2, Z_{r,T}^1, Z_{r,T}^2) (P_S(0) + T^\delta P_T).$$

This implies $P_{T_1} \leq \mathcal{F}(\dots)P_S(0)$ for some $T_1 > 0$ sufficiently small, depending only on $\mathcal{F}(\dots)$. Then iterating this estimate between $[T_1, 2T_1], \dots, [T - T_1, T]$ we obtain the following result.

Theorem 2.5.9. *Let (η_j, ψ_j) , $j = 1, 2$ be two solutions to (2.1.7) on $I = [0, T]$, $0 < T \leq 1$ such that*

$$(\eta_j, \psi_j) \in L^\infty(I; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)) \cap L^p(I; W^{r+\frac{1}{2}}(\mathbf{R}^d) \times W^{r,\infty}(\mathbf{R}^d))$$

with

$$(2.5.31) \quad s > \frac{3}{2} + \frac{d}{2}, \quad 2 < r < s - \frac{d}{2} + \mu;$$

where μ, p are given by (2.5.29) and such that $\inf_{t \in [0, T]} \text{dist}(\eta_j(t), \Gamma) > h > 0$. Set

$$M_{s,T}^j := \|(\eta_j, \psi_j)\|_{L^\infty([0, T]; H^{s+\frac{1}{2}} \times H^s)}, \quad Z_{r,T}^j := \|(\eta_j, \psi_j)\|_{L^p([0, T]; W^{r+\frac{1}{2}} \times W^{r,\infty})}.$$

Consider the differences $\delta\eta := \eta_1 - \eta_2$, $\delta\psi := \psi_1 - \psi_2$ and their norms in Sobolev space and Hölder space:

$$P_T := \|(\delta\eta, \delta\psi)\|_{L^\infty(I; H^{s-1} \times H^{s-\frac{3}{2}})} + \|(\delta\eta, \delta\psi)\|_{L^p(I; W^{r-1, \infty} \times W^{r-\frac{3}{2}, \infty})}.$$

Then there exists a non-decreasing function $\mathcal{F} : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on d, r, s, p, μ , and h such that

$$P_T \leq \mathcal{F}_h(M_{s,T}^1, M_{s,T}^2, Z_{r,T}^1, Z_{r,T}^2) \|(\delta\eta, \delta\psi)|_{t=0}\|_{H^{s-1} \times H^{s-\frac{3}{2}}}.$$

Remark. If the Strichartz estimate (2.5.28) had been proved with a gain of μ' derivative, $\mu' \in (0, \frac{1}{2}]$, then Theorem 2.5.9 would have held with $\mu = \mu'$ in (2.5.31).

Inégalités de STRICHARTZ et problème de CAUCHY

Cet article est une collaboration avec Quang Huy NGUYEN, et fait suite au précédent. Il a été publié dans le Journal of Differential Equations ; voir [87].

3.1 Introduction

3.1.1 Equations

The water waves problem is the study of the motion of an incompressible inviscid fluid, lying above a fixed bottom and below an atmosphere, from which it is separated by a free surface. At equilibrium, this surface is flat. As soon as one perturbs this equilibrium, the surface will be put in motion by the combined action of gravity and surface tension.

The velocity of such a fluid will obey the classical Euler equations of fluid dynamics, with the added difficulty of the moving surface. As such, the domain occupied by the fluid will depend on the time at which it is observed. We thus consider the time-dependent domain

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : (x, y) \in \Omega_t\}$$

where each Ω_t is a domain located underneath a free surface

$$\Sigma_t = \{(x, y) \times \mathbf{R}^d \times \mathbf{R} : y = \eta(t, x)\}$$

and above a fixed bottom $\Gamma = \partial\Omega_t \setminus \Sigma_t$. The physical dimensions are $d = 1, 2$. We make the following important assumption on the domain: *Assumption (H_t)*
 Ω_t is the intersection of the half space

$$\Omega_{1,t} = \{(x, y) \times \mathbf{R}^d \times \mathbf{R} : y < \eta(t, x)\}$$

and an open connected set Ω_2 containing a fixed strip around Σ_t , i.e., there exists $h > 0$ such that

$$\{(x, y) \times \mathbf{R}^d \times \mathbf{R} : \eta(t, x) - h \leq y \leq \eta(t, x)\} \subset \Omega_2.$$

This important hypothesis prevents the bottom from emerging, or even from coming arbitrarily close to the free surface. The study of water waves without it is an open problem.

It is customary in mathematics to simplify the problem further by supposing the motion of the fluid to be irrotational. This covers a large class of physical applications. Now under this additional hypothesis, and if the domain is simply connected, the velocity field v admits a potential $\phi : \Omega \rightarrow \mathbf{R}$, i.e, $v = \nabla\phi$. An important observation by Zakharov [111] is that the motion is then completely determined by the value of the elevation $\eta(t, x)$ and of the trace $\psi(t, x) = \phi(t, x, \eta(t, x))$ of the potential at the surface. We can then find ϕ as the unique variational solution of

$$(3.1.1) \quad \Delta\phi = 0 \text{ in } \Omega_t, \quad \phi(t, x, \eta(t, x)) = \psi(t, x), \quad \partial_n\phi|_{\Gamma} = 0$$

Now following Craig and Sulem [35] to write a compact version of the equations, we introduce the Dirichlet-Neumann operator

$$\begin{aligned} G(\eta)\psi &= \sqrt{1 + |\nabla_x\eta|^2} \left(\frac{\partial\phi}{\partial n} \Big|_{\Sigma} \right) \\ &= (\partial_y\phi)(t, x, \eta(t, x)) - \nabla_x\eta(t, x) \cdot (\nabla_x\phi)(t, x, \eta(t, x)). \end{aligned}$$

The water wave system can now be rewritten as the following so-called Zakharov-Craig-Sulem system on (η, ψ) :

$$(3.1.2) \quad \begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + g\eta - H(\eta) + \frac{1}{2}|\nabla_x\psi|^2 - \frac{1}{2} \frac{(\nabla_x\eta \cdot \nabla_x\psi + G(\eta)\psi)^2}{1 + |\nabla_x\eta|^2} = 0, \end{cases}$$

where $H(\eta)$ is the mean curvature of the free surface:

$$H(\eta) = \operatorname{div} \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}} \right).$$

The vertical and horizontal components of the velocity will play an important role in the analysis of system (3.1.2). These quantities can be expressed in terms of η and ψ as

$$(3.1.3) \quad B = (v_y)|_{\Sigma} = \frac{\nabla_x\eta \cdot \nabla_x\psi + G(\eta)\psi}{1 + |\nabla_x\eta|^2}, \quad V = (v_x)|_{\Sigma} = \nabla_x\psi - B\nabla_x\eta.$$

3.1.2 The problem

In the present paper and its companion [86], we aim to prove local existence for rough data below the energy threshold, using the dispersive properties of this system. The local existence of solutions for the water waves system has been extensively studied by many authors, among them Nalimov [83], Yosihara [109], Coutand-Shkoller [32], Craig [34], Wu [104, 105], Christodoulou-Lindblad [31], Lindblad[75], Lannes [70], Ming-Zhang [82] and for the case with surface tension, in Beyer-Günther [19], Ambrose-Masmoudi [12, 13], Shatah-Zeng [90, 91, 90]. For the full system with gravity and surface tension, in terms of regularity of data the result of Alazard, Burq and Zuily [3] reaches an important level:

$$(3.1.4) \quad (\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d), \quad s > 2 + \frac{d}{2}.$$

Observe that by the formulas (2.1.8), this is the optimal Sobolev index to ensure that the initial velocity field is Lipschitz up to the free surface, which is a quite natural criterion for the flow of fluid particles to be well-defined, in terms of the Cauchy-Lipschitz theorem.

Now, let us look at the linearized around the rest state $(\eta = 0, \psi = 0)$ of (3.1.2), with $g = 0$. It reads

$$\partial_t \Phi + i |D|^{\frac{3}{2}} \Phi = 0,$$

where $\Phi = |D|^{\frac{1}{2}} \eta + i\psi$. This linear equation is dispersive (see paragraph 3.1.3 below), and we expect the full system to exhibit dispersive properties as well. The consequences of this dispersion for long time dynamics have been extensively studied in recent years, starting from the works of Wu [106, 107], by Germain-Masmoudi-Shatah [45, 46], Alazard-Delort [10], Ionescu-Pusateri [58, 57], Hunter-Ifrim-Tataru [52], and Ifrim-Tataru [56, 54].

In this paper, we are interested in the consequences for short time and rough data, the so-called Strichartz estimates. They are a family of local in time estimates improving the Sobolev inequalities for a solution of the system, which can then be used to improve the energy estimates and thus lead to well-posedness with less regularity for initial data. The method to obtain such results for quasi-linear wave equations was developed by Bahouri and Chemin [15] and by Tataru, notably in [97].

However until recently, little was known about Strichartz estimates for water waves systems. In [30], Christianson-Hur-Staffilani proved Strichartz estimate for 2D gravity-capillary waves under another formulation. Then, Alazard-Burq-Zuily obtained in [4] such a result for solutions to (3.1.2) at regularity (3.1.4). We want to improve this in two ways, by proving Strichartz estimates :

- (1) valid for 3D waves,
- (2) that can be used to improve the threshold (3.1.4), for both 2D and 3D waves.

In fact, the method used in [4] relies on a reduction specific to the dimension $d = 1$, so for (1) we need another method. On the other hand, for (2) one need to derive the Strichartz estimates assuming that the solution is less regular than (3.1.4) and consequently, the coefficients appearing in the equation are rougher. Such a program has been carried out by Alazard, Burq and Zuily in [8] for the pure gravity case. In fact, we shall follow here a similar approach, that is, proving dispersive estimates using semiclassical analysis. The main novelty is that here, the equation has infinite speed of propagation, so that we need to construct a parametrix in semiclassical time. Also, we use at the fullest the regularity of the coefficients to expand the lifespan of this parametrix.

The first step in this program is to reduce system (3.1.2) to a single equation to which the method for quasilinear equations can be applied. This uses paradifferential calculus, whose notations and main features are recalled in Appendix A. Specifically, we proved in the companion paper [86] that assuming (η, ψ) to be a solution of (3.1.2) satisfying condition (H_t) for all times $t \in I = [0, T]$, such that

$$(\eta, \psi) \in L^\infty(I; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)) \cap L^p(I; W^{r+\frac{1}{2}, \infty}(\mathbf{R}^d) \times W^{r, \infty}(\mathbf{R}^d)),$$

where

$$(3.1.5) \quad s > \frac{3}{2} + \frac{d}{2}, \quad 2 < r < s - \frac{d}{2} + \frac{1}{2},$$

the system (3.1.2) can be rewritten as

$$(3.1.6) \quad \partial_t u + T_V \cdot \nabla u + iT_{\gamma+\omega} u = f,$$

where the principal symbol γ is of order $3/2$, real-valued; the sub-principal symbol ω is of order $1/2$, complex-valued; the transport field V is the horizontal part of the velocity field at the free surface: $V = (v_x)|_\Sigma$ and the remainder term f satisfies the following *tame estimate* for a.e. $t \in I$

$$\|f(t)\|_{H^s} \leq \tilde{F} \left(\|\eta(t)\|_{H^{s+\frac{1}{2}}}, \|\psi(t)\|_{H^s} \right) \left[1 + \|\eta(t)\|_{W^{r+\frac{1}{2}, \infty}} + \|\psi(t)\|_{W^{r, \infty}} \right].$$

In this article, we shall study equation (3.1.6) independently from its origin in the water waves system, proving a priori Strichartz estimates for its solution. This will imply a priori Strichartz estimates for the gravity-capillary waves system (3.1.2). We will then combine them with the energy and contraction estimates and with a blow-up criterion, all proved in [86], to solve the Cauchy problem at low regularity such that the initial velocity field may fail to be Lipschitz (up to the surface).

3.1.3 Main results

Remark that the linearized system of (3.1.2) around the rest state $(\eta = 0, \psi = 0)$ when $g = 0$ reads

$$\begin{cases} \partial_t \eta - |D_x| \psi = 0, \\ \partial_t \psi - \Delta \eta = 0, \end{cases}$$

which can be written

$$\partial_t \Phi + i|D_x|^{\frac{3}{2}} \Phi = 0, \quad \text{with } \Phi = |D_x|^{\frac{1}{2}} \eta + i\psi.$$

It classically follows from the explicit formula for the solution, Littlewood-Paley decomposition, stationary phase and a TT* argument that

$$\|\Phi\|_{L^p W^{s-\frac{d}{2}-\mu_{opt},\infty}} \lesssim \|\Phi|_{t=0}\|_{H^s}$$

with

$$(3.1.7) \quad \begin{cases} \mu_{opt} = \frac{3}{8}, p = 4 & \text{if } d = 1, \\ \mu_{opt} = \frac{3}{4} - \varepsilon, p = 2 & \text{if } d \geq 2. \end{cases}$$

Our first result states that the fully nonlinear gravity-capillary waves system (3.1.2) satisfies a similar estimate to (3.1.7) with

$$\mu = \frac{3}{20} - \varepsilon \text{ if } d = 1, \quad \mu = \frac{3}{10} - \varepsilon \text{ if } d \geq 2.$$

More precisely, we prove

Theorem 3.1.1. *Let $I = [0, T]$, $d \geq 1$ and $p \geq 1$. Let $V(t, x) : I \times \mathbf{R}^d \rightarrow \mathbf{R}^d$ be a vector field and γ, ω, ω_1 be the symbols defined by (3.2.1). Consider*

$$(3.1.8) \quad \mu \in \left(0, \frac{3}{20}\right), \quad p = 4 \text{ if } d = 1; \quad \mu \in \left(0, \frac{3}{10}\right), \quad p = 2 \text{ if } d \geq 2.$$

Then for any $\sigma \in \mathbf{R}$ there exist $k = k(d) \in \mathbf{N}$ and $\tilde{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non-decreasing such that the following property holds:

if $f \in L^p(I; H^{\sigma-\frac{9}{10}}(\mathbf{R}^d))$ and $u \in L^\infty(I; H^\sigma(\mathbf{R}^d))$ satisfy

$$(\partial_t + T_V \cdot \nabla + iT_{\gamma+\omega})u = f,$$

then we have

$$(3.1.9) \quad \|u\|_{L^p(I; W^{\sigma-\frac{d}{2}+\mu}(\mathbf{R}^d))} \leq \tilde{F}(\Upsilon) \left[\|f\|_{L^p(I; H^{\sigma-\frac{9}{10}}(\mathbf{R}^d))} + \|u\|_{L^\infty(I; H^\sigma(\mathbf{R}^d))} \right],$$

where Υ is the sum of semi-norms of the coefficients, defined by (3.2.23) and (3.4.6):

$$\Upsilon = \|V\|_{L^p(I, W^{1,\infty}(\mathbf{R}^d))} + \mathcal{M}_k(\gamma)(I) + \mathcal{N}_k(\gamma)(I) + \mathcal{N}_k(\omega)(I).$$

As a corollary, this will imply the corresponding Strichartz estimate for the water waves equation. To be concise in the following statements let us define the quantities that control the system:

$$\text{Sobolev norms : } M_{\sigma,T} = \|(\eta, \psi)\|_{L^\infty([0,T]; H^{\sigma+\frac{1}{2}} \times H^\sigma), \quad M_{\sigma,0} = \|(\eta_0, \psi_0)\|_{H^{\sigma+\frac{1}{2}} \times H^\sigma},$$

$$\text{Blow-up norm : } N_{\sigma,T} = \|(\eta, \psi)\|_{L^1([0,T]; W^{\sigma+\frac{1}{2},\infty} \times W^{\sigma,\infty}),$$

$$\text{Strichartz norm : } Z_{\sigma,T} = \|(\eta, \psi)\|_{L^p([0,T]; W^{\sigma+\frac{1}{2},\infty} \times W^{\sigma,\infty}).$$

Corollary 3.1.2. *Let $d \geq 1$, $h > 0$ and $(s, r) \in \mathbf{R}^2$ such that*

$$s > 2 + \frac{d}{2} - \mu, \quad 2 < r < s - \frac{d}{2} + \mu,$$

with μ and p as in (3.1.8). Then there exists a non-decreasing $\tilde{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that for all $T \in (0, 1]$, for all (η, ψ) smooth solution of (3.1.2) on $[0, T]$ satisfying $\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma) > h$, there holds

$$Z_r(T) \leq \tilde{F} \left(T \tilde{F} (M_s(T) + Z_r(T)) \right).$$

In [86], we have established the following energy estimate, blow up criterion and contraction estimate.

Proposition 3.1.3 ([86, Theorem 1.1]). *Let $d \geq 1$, $h > 0$, $p > 1$ and $(r, s) \in \mathbf{R}^2$ such that*

$$(3.1.10) \quad s > \frac{3}{2} + \frac{d}{2}, \quad 2 < r < s + \frac{1}{2} - \frac{d}{2}.$$

Then there exists a non-negative, non-decreasing function \tilde{F} such that: for all $T \in (0, 1]$ and all (η, ψ) smooth solution to (3.1.2) on $[0, T]$ satisfying $\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma) > h$, there holds

$$M_{s,T} \leq \tilde{F} \left(M_{s,0} + T \tilde{F} (M_{s,T} + Z_{r,T}) \right).$$

Proposition 3.1.4 ([86, Theorem 1.2]). *Let $d \geq 1$, $h > 0$ and indices*

$$\frac{3}{2} + \frac{d}{2} < s_0 < s - \frac{1}{2}, \quad 2 < r < s_0 + \frac{1}{2} - \frac{d}{2}.$$

Let $T^ = T^*(\eta_0, \psi_0, h)$ be the maximal time of existence and*

$$(\eta, \psi) \in L^\infty \left([0, T^*]; H^{s+\frac{1}{2}} \times H^s \right)$$

be the maximal solution of (3.1.2) with prescribed data (η_0, ψ_0) satisfying $\text{dist}(\eta_0, \Gamma) > h$. Then if T^ is finite, we have*

$$\limsup_{T \rightarrow T^*} (M_{s_0}(T) + N_r(T)) = +\infty.$$

Proposition 3.1.5 ([86, Theorem 5.9]). *Let p and μ as in (3.1.8). Let (η_j, ψ_j) , $j = 1, 2$ be two solutions to (3.1.2) on $I = [0, T]$, $0 < T \leq 1$ such that*

$$(\eta_j, \psi_j) \in L^\infty(I; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)) \cap L^p(I; W^{r+\frac{1}{2}}(\mathbf{R}^d) \times W^{r,\infty}(\mathbf{R}^d))$$

with

$$s > \frac{3}{2} + \frac{d}{2}, \quad 2 < r < s - \frac{d}{2} + \mu;$$

such that $\inf_{t \in [0, T]} \text{dist}(\eta_j(t), \Gamma) > h > 0$. Set

$$M_{s, T}^j := \|(\eta_j, \psi_j)\|_{L^\infty([0, T]; H^{s+\frac{1}{2}} \times H^s)}, \quad Z_{r, T}^j := \|(\eta_j, \psi_j)\|_{L^p([0, T]; W^{r+\frac{1}{2}, \infty} \times W^{r, \infty})}.$$

Consider the differences $\delta\eta := \eta_1 - \eta_2$, $\delta\psi := \psi_1 - \psi_2$ and their norms in Sobolev space and Hölder space:

$$P_T := \|(\delta\eta, \delta\psi)\|_{L^\infty(I; H^{s-1} \times H^{s-\frac{3}{2}})} + \|(\delta\eta, \delta\psi)\|_{L^p(I; W^{r-1, \infty} \times W^{r-\frac{3}{2}, \infty})}.$$

Then there exists a non-decreasing function $\tilde{F} : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on d, r, s, h such that

$$P_T \leq \tilde{F}(M_{s, T}^1, M_{s, T}^2, Z_{r, T}^1, Z_{r, T}^2) \|(\delta\eta, \delta\psi)|_{t=0}\|_{H^{s-1} \times H^{s-\frac{3}{2}}}.$$

With the above ingredients we can prove our main theorem about the Cauchy problem.

Theorem 3.1.6. *Let $d \geq 1$ and two real numbers r, s satisfying*

$$2 < r < s - \frac{d}{2} + \mu, \quad \mu = \begin{cases} \frac{3}{20} & \text{if } d = 1, \\ \frac{3}{10} & \text{if } d \geq 2. \end{cases}$$

Let $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}} \times H^s$ be such that $\text{dist}(\eta_0, \Gamma) > h > 0$. Then there exists a time $T > 0$ such that the Cauchy problem for (3.1.2) has a unique solution

$$(\eta, \psi) \in L^\infty([0, T]; H^{s+\frac{1}{2}} \times H^s) \cap L^p([0, T]; W^{r+\frac{1}{2}, \infty} \times W^{r, \infty})$$

where $p = 4$ when $d = 1$ and $p = 2$ when $d \geq 2$. Moreover, we have

$$(\eta, \psi) \in C^0([0, T]; H^{s'+\frac{1}{2}} \times H^{s'}), \quad \forall s' < s$$

and

$$\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma) > h/2.$$

Remark. In view of the formulas (2.1.8), the initial velocity field in the Cauchy theory 3.1.6 may fail to be Lipschitz up to the free surface but it becomes Lipschitz at almost all later time. This result is parallel to the result in [8] for pure gravity water waves.

The plan of the paper is as follows. First, we prove in Section 3.2 some reduction of the problem, reducing it to a semiclassical equation. In Section 3.3, we construct a microlocal parametrix for this equation. Then in Section 3.4, we use it to prove the Strichartz estimates. The last part, Section 3.5, is devoted to the local existence of solutions. Some needed results about paradifferential calculus are recalled in Appendix A.

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3.2 Reductions of the system

3.2.1 Paradifferential reduction

First of all, we recall precisely the paradifferential reduction of the gravity-capillary system (3.1.2) that we performed in [86], which requires the following symbols:

— Symbols of the Dirichlet-Neumann operator

$$\begin{aligned}\lambda^{(1)} &= \sqrt{(1 + |\nabla\eta|^2)|\xi|^2 - (\nabla\eta \cdot \xi)^2}, \\ \lambda^{(0)} &= \frac{1 + |\nabla\eta|^2}{2\lambda^{(1)}} \left\{ \operatorname{div} \left(\alpha^{(1)} \nabla\eta \right) + i\partial_\xi \lambda^{(1)} \cdot \nabla \alpha^{(1)} \right\}\end{aligned}$$

with

$$\alpha^{(1)} = \frac{\lambda^{(1)} + i\nabla\eta \cdot \xi}{1 + |\nabla\eta|^2}.$$

— Symbols of the mean-curvature operator:

$$\ell^{(2)} := (1 + |\nabla\eta|^2)^{-\frac{1}{2}} \left(|\xi|^2 - \frac{(\nabla\eta \cdot \xi)^2}{1 + |\nabla\eta|^2} \right), \quad \ell^{(1)} := -\frac{i}{2} (\partial_x \cdot \partial_\xi) \ell^{(2)}.$$

— Symbols used for symmetrization

$$q = (1 + (\nabla_x \eta)^2)^{-\frac{1}{2}}, \quad p = (1 + (\nabla_x \eta)^2)^{-\frac{5}{4}} |\xi|^{\frac{1}{2}} + p^{(-\frac{1}{2})},$$

where $p^{(-\frac{1}{2})} = F(\nabla_x \eta, \xi) \partial_x^\alpha \eta$, with $|\alpha| = 2$ and $F \in C^\infty(\mathbf{R} \times \mathbf{R} \setminus \{0\}; \mathbf{C})$ is homogeneous of order $-1/2$ in ξ .

— Symbols in the symmetrized equation:

$$(3.2.1) \quad \begin{aligned}\gamma &:= \sqrt{l^{(2)} \lambda^{(1)}} = \left(|\xi|^2 - \frac{(\nabla\eta \cdot \xi)^2}{1 + |\nabla\eta|^2} \right)^{\frac{3}{4}}, \\ \omega &:= -\frac{i}{2} (\partial_\xi \cdot \partial_x) \sqrt{l^{(2)} \lambda^{(1)}} + \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}} \frac{\operatorname{Re} \lambda^{(0)}}{2}.\end{aligned}$$

Theorem 3.2.1. *Let $s > \frac{3}{2} + \frac{d}{2}$, $2 < r < s - \frac{d}{2} + \frac{1}{2}$ and $p \in [1, \infty]$. Suppose that (η, ψ) a solution of (3.1.2) satisfying condition (H_t) for all times $t \in I$ and*

$$(3.2.2) \quad (\eta, \psi) \in L^\infty(I; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)) \cap L^p(I; W^{r+\frac{1}{2}, \infty}(\mathbf{R}^d) \times W^{r, \infty}(\mathbf{R}^d)),$$

The complex-valued unknown $u := T_p \eta + iT_q(\psi - T_B \eta)$ then satisfies

$$(3.2.3) \quad \partial_t u + T_V \cdot \nabla u + iT\gamma u = f - iT_\omega u,$$

where for a.e. $t \in I$,

$$(3.2.4) \quad \|f(t)\|_{H^s} \leq \tilde{F} \left(\|\eta(t)\|_{H^{s+\frac{1}{2}}}, \|\psi(t)\|_{H^s} \right) \left[1 + \|\eta(t)\|_{W^{r+\frac{1}{2}, \infty}} + \|\psi(t)\|_{W^{r, \infty}} \right].$$

As mentioned in the introductory section, we shall from now on consider (3.2.3) as an independent equation with coefficients V , γ , ω , ω_1 at the following regularity level

$$(3.2.5) \quad \begin{aligned}V &\in L^\infty \left(I; W^{\frac{1}{2}, \infty}(\mathbf{R}^d)^d \right) \cap L^p \left(I; W^{1, \infty}(\mathbf{R}^d)^d \right), \\ \eta &\in L^\infty \left(I; W^{2, \infty}(\mathbf{R}^d) \right) \cap L^p \left(I; W^{\frac{5}{2}, \infty}(\mathbf{R}^d) \right),\end{aligned}$$

which is sufficient for the semi-norms appearing in Theorem 3.1.1.

We give here some preliminary informations on the principal symbol γ . Define

$$\mathbf{C}' = \left\{ \xi \in \mathbf{R}^d : 1/4 \leq |\xi| \leq 4 \right\}.$$

Lemma 3.2.2. *1. The symbol γ is in $L^\infty(I; W^{1,\infty} S^{\frac{3}{2}}) \cap L^p(I; C^{\frac{3}{2}} S^{\frac{3}{2}})$ and for all $\beta \in \mathbf{N}^d$, there exists $\tilde{F}_\beta : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that for all $t \in I$, $\xi \in \mathbf{C}'$,*

$$(3.2.6) \quad \left\| D_\xi^\beta \gamma(t, \xi) \right\|_{W^{1,\infty}} \leq \tilde{F}_\beta (\|\nabla \eta(t)\|_{W^{1,\infty}}),$$

$$(3.2.7) \quad \left\| D_\xi^\beta \gamma(t, \xi) \right\|_{C^{\frac{3}{2}}} \leq \tilde{F}_\beta (\|\nabla \eta(t)\|_{W^{1,\infty}}) \left[1 + \|\nabla \eta(t)\|_{W^{\frac{3}{2},\infty}} \right].$$

2. There exists an absolute constant $C_d > 0$ such that with $c_0 = C_d(1 + \|\nabla \eta\|_{L^\infty(I \times \mathbf{R}^d)})$ we have for all $t \in I$, $x \in \mathbf{R}^d$, and $\xi \in \mathbf{C}'$,

$$(3.2.8) \quad |\det \text{Hess}_\xi(\gamma)(t, x, \xi)| \geq c_0.$$

Proof. The proof of part 1. is straightforward using product rules and Sobolev embedding. For a proof of part 2., we refer to Corollary 4.7 in [8]. \square

3.2.2 Localization in frequency

To prove our estimates, we will follow standard procedure: decomposing the solution using Littlewood-Paley theory and using a parametrix and a TT* argument to derive Strichartz estimates for those dyadic pieces. We will then bring the pieces back together to derive a Strichartz estimate for the original solution. Standard definitions and notations for the Littlewood-Paley decomposition are recalled in appendix A. For $j \geq 0$, the dyadic piece $\Delta_j u$ verifies the equation

$$(3.2.9) \quad (\partial_t + iT_\gamma + T_V \cdot \nabla) \Delta_j u = F_j,$$

where

$$(3.2.10) \quad F_j := \Delta_j F - i\Delta_j(T_\omega u) + i[T_\gamma, \Delta_j] + [T_V, \Delta_j] \cdot \nabla u.$$

Recall that $\Delta_j \Delta_k = 0$ if $|k - j| \geq 2$. In the sequel, we shall always consider $\Delta_j u$ for j large enough, in particular $j \geq 1$ so the spectrum of $\Delta_j u$ is always contained in the annulus

$$\mathcal{C}_j := \left\{ \xi \in \mathbf{R}^d : 2^{j-1} < |\xi| \leq 2^{j+1} \right\}.$$

Thanks to the spectral localization of $\Delta_j u$ we can replace the paradifferential operators with pseudodifferential operators. Such a replacement for the transport term is harmless due to the following lemma.

Lemma 3.2.3 ([8, Lemma 4.9]). *We have*

$$T_V \cdot \nabla \Delta_j u = S_{j-3}(V) \cdot \nabla \Delta_j u + R_j u,$$

where $R_j u$ has its spectrum contained in an annulus $\mathcal{C}(c_1 2^{j-1}, c_2 2^{j+1})$ and satisfies the following estimate

$$\|R_j u\|_{H^s(\mathbf{R}^d)} \leq C \|V\|_{W^{1,\infty}(\mathbf{R}^d)} \|u\|_{H^s(\mathbf{R}^d)}$$

where the constant $C > 0$ is independent of u, V, j .

The preceding lemma was proved in [8] thanks to the special form of the symbol $V(x)\xi$. Here, for the highest order term, let us prove the following more general fact for any paradifferential operator. Let $a \in \Gamma_r^m$, $r > 0$ and define

$$(3.2.11) \quad \forall j \in \mathbf{Z}, \quad S_j(a)(x, \xi) = \psi(2^{-j}D_x)a(x, \xi)$$

the spatial regularization of the symbol a , where ψ is given in the Littlewood-Paley decomposition A.0.1.

Proposition 3.2.4. *For every $j \in \mathbf{N}^*$, define*

$$T_a \Delta_j u = S_{j-3}(a)(x, D_x) \Delta_j u + R'_j u.$$

Then the spectrum of $R'_j u$ is contained in an annulus $\mathcal{C}(c_1 2^{j-1}, c_2 2^{j+1})$ and for every $\mu \in \mathbf{R}$ we have the following norm estimate

$$\|R'_j u\|_{H^{\mu-m+r}(\mathbf{R}^d)} \leq C M_r^m(a) \|u\|_{H^\mu(\mathbf{R}^d)}$$

where the constant $C > 0$ is independent of a, u, j .

Remark. If a is homogeneous in ξ then $S_{j-3}a$ is still homogeneous in ξ . This remark is important in the next part when we multiply both side of our equation by $h^{\frac{3}{2}}$ to derive a semi-classical equation.

Proof. Since $\varrho = 1$ on the support of φ_j for any $j \geq 1$, we see that

$$R'_j u = T_a \Delta_j u - S_{j-3}(a)(x, D_x) \varrho(D_x) \Delta_j u.$$

In the following proof, we shall use the presentation of $\tilde{\text{M}\ddot{\text{A}}\text{t}\text{iv}\text{i}\text{e}\text{r}$ [80] on pseudodifferential and paradifferential operators. To be compatible with [80] we also abuse in notations: by Γ_r^m we denote the class of symbols a satisfying (A.0.1) for any $\xi \in \mathbf{R}^d$ and by M_0^m the semi-norm (A.0.2) where the supremum is taken over $\xi \in \mathbf{R}^d$.

1. By definition of the paradifferential operator T_a we have $T_a v = \text{Op}(\sigma_a \varrho) v$ where $\text{Op}(\sigma_a \varrho)$ denotes the classical pseudodifferential operator with symbol

$$\sigma_a(x, \xi) \varrho(\xi) = \chi(D_x, \xi) a(x, \xi) \varrho(\xi).$$

Hence $R'_j u = \text{Op}(a_j) u$ with

$$a_j(x, \xi) = \sigma_a(x, \xi) \varrho(\xi) \varphi_j(\xi) - S_{j-3}(a)(x, \xi) \varrho(\xi) \varphi_j(\xi).$$

Now, we write

$$a_j = (\sigma_a \varrho \varphi_j - a \varrho \varphi_j) + (a \varrho \varphi_j - S_{j-3}(a) \varrho \varphi_j) = a_j^1 + a_j^2.$$

Applying Proposition 5.8(ii) in [80] gives $a_j^1 \in \Gamma_0^{m-r}$ and (remark that $(\varphi_j)_j$ is bounded in Γ_r^0)

$$M_0^{m-r}(a_j^1) \leq C M_r^m(a \varrho \varphi_j) \leq C M_r^m(a \varrho).$$

On the other hand, if we denote $b = a \varrho \varphi$ then $a_j^2(x, \xi) = b(x, \xi) - \psi_{j-3}(D_x, \xi) b(x, \xi)$. Taking into account the fact that $\text{supp } \varphi_j \subset B(0, C2^j)$ we may estimate

$$\begin{aligned} |a_j^2(x, \xi)| &\leq \sum_{k \geq j-2} |\Delta_k b(x, \xi)| \leq \sum_{k \geq j-2} 2^{-kr} \|b(\cdot, \xi)\|_{W^{r, \infty}} \\ &\leq C 2^{-jr} \|b(\cdot, \xi)\|_{W^{r, \infty}} = C 2^{-jr} |\varphi_j(\xi)| \|a(\cdot, \xi) \varrho(\xi)\|_{W^{r, \infty}} \\ &\leq C(1 + |\xi|)^{m-r} M_r^m(a \varrho), \quad \forall \xi \in \mathbf{R}^d. \end{aligned}$$

By the same method for estimating $|\partial_\xi^\alpha a_j^2|$ we obtain that $a_j^2 \in \Gamma_0^{m-r}$ and hence $a_j \in \Gamma_0^{m-r}$; moreover

$$M_0^{m-r}(a_j) \leq CM_r^m(a\varrho).$$

2. Property (A.0.5) implies in particular that

$$\mathfrak{F}_x(\sigma_a)(\eta, \xi) = 0 \text{ for } |\eta| \geq \varepsilon_2(1 + |\xi|),$$

here we denote \mathfrak{F}_x the Fourier transform with respect the the patial variable x . On the other hand, by definition of the smoothing operator

$$\mathfrak{F}_x S_{j-3}(a)(x, \xi) \varrho(\xi) \varphi_j(\xi) = \psi(2^{-(j-3)}\eta) \mathfrak{F}_x a(\eta, \xi) \varrho(\xi) \varphi(2^{-j}\xi)$$

which vanish if $|\eta| \geq \frac{1}{2}(1 + |\xi|)$. Indeed, if either $|\xi| > 2^{j+1}$ or $|\xi| \leq 2^{j-1}$ then $\varphi(2^{-j}\xi) = 0$. Considering $2^{j-1} < |\xi| \leq 2^{j+1}$ then $|\eta| \geq \frac{1}{2}(1 + |\xi|) > 2^{j-2}$ and thus $\psi(2^{-(j-3)}\eta) = 0$. We have proved the existence of $0 < \varepsilon < 1$ such that

$$(3.2.12) \quad \mathfrak{F}_x a_j(\eta, \xi) = 0 \text{ for } |\eta| \geq \varepsilon(1 + |\xi|).$$

3. By the spectral property (3.2.12) one can use the Bernstein inequalities (see Corollary 4.1.7, [80]) to prove that a_j is a pseudodifferential symbol in the class $S_{1,1}^{m-r}$. Then, applying Theorem 4.3.5 in [80] we conclude that

$$\|R'_j u\|_{H^{\mu-m+r}(\mathbf{R}^d)} = \|\text{Op}(a_j)u\|_{H^{\mu-m+r}(\mathbf{R}^d)} \leq CM_0^{m-r}(a_j) \|u\|_{H^\mu(\mathbf{R}^d)}.$$

Finally, the Fourier transform of $R'_j u$ reads

$$\mathfrak{F}(R'_j u)(\xi) = \int_{\mathbf{R}^d} \mathfrak{F}_x(a_j)(\xi - \eta, \eta) \hat{u}(\eta) d\eta.$$

Using the spectral localization property (3.2.12) and the fact that $\mathfrak{F}_x(a_j)(\xi - \eta, \eta)$ contains the factor $\varphi_j(\eta)$ we conclude that the spectrum of $R'_j u$ is contained in an annulus of size 2^j as claimed. \square

Now we can use the preceding results to rewrite the equation as

$$(3.2.13) \quad (\partial_t + iS_{j-3}(\gamma)(x, D_x) + S_{j-3}(V) \cdot \nabla) \Delta_j u = F'_j,$$

$$(3.2.14) \quad F'_j = F_j + R_j + iR'_j.$$

3.2.3 Regularization of symbols

Now, following the classical method for quasilinear equations pioneered by Bahouri and Chemin in [14] and [15], we further regularize the equation, using a parameter $\delta \in (0, 1)$. By doing so, we aim to construct a parametrix with a regular enough phase to apply the stationary phase argument. This results in a slightly worse remainder term, which will in turn result in slightly worse Strichartz estimates. Eventually, we optimize in δ .

Define for all $(t, x, \xi) \in I \times \mathbf{R}^d \times \mathbf{R}^d$ and $j \geq 0$,

$$S_{(j-3)\delta}(\gamma)(t, x, \xi) = \psi(2^{-(j-3)\delta} D_x) \gamma(t, x, \xi)$$

and similarly $S_{(j-3)\delta}(V)(t, x)$. Let $\varphi_1 \in C^\infty(\mathbf{R}^d)$, with

$$\text{supp } \varphi_1 \subset \mathbf{C}' = \left\{ \xi : \frac{1}{4} \leq |\xi| \leq 4 \right\}, \quad \varphi_1 = 1 \text{ on } \mathbf{C}'' := \left\{ \xi : \frac{1}{3} \leq |\xi| \leq 3 \right\},$$

so that it is 1 on the support of the Littlewood-Paley function φ . Then, equation (3.2.13) can be rewritten as

$$(3.2.15) \quad L_j \Delta u_j(t, x) := (\partial_t + S_{(j-3)\delta}(V) \cdot \nabla + iS_{(j-3)\delta}(\gamma)(x, D_x) \varphi_1(2^{-j} D_x)) \Delta_j u = F_{j\delta},$$

with

$$(3.2.16) \quad F_{j\delta} = i(S_{(j-3)\delta}\gamma(x, D_x) - S_{j-3}\gamma(x, D_x)) \Delta_j u + F'_j + (S_{(j-3)\delta}(V) - S_{j-3}(V)) \cdot \nabla \Delta_j u.$$

The function φ_1 has been inserted to keep into the operator the information about the localization of its solution $\Delta_j u$.

Next, Lemma 3.2.2 shows that the Hessian in ξ of γ is non-degenerate and since $S_{j\delta}(\gamma)$ is a small perturbation of γ when j large enough, we also have

Proposition 3.2.5. *There exists $c_0 > 0$, $j_0 \in \mathbf{N}$, such that*

$$|\det \text{Hess}_\xi (S_{j\delta}(\gamma))(t, x, \xi)| \geq c_0,$$

for all $t \in I$, $x \in \mathbf{R}^d$, $\xi \in \mathbf{C}'$, $j \geq j_0$.

3.2.4 Semi-classical formulation

We now want to prove Strichartz estimates for the homogeneous version of equation (3.2.15):

$$(3.2.17) \quad L_j u_j(t, x) = 0.$$

To this end, we recast the problem in the semi-classical formalism with $h = 2^{-j}$. One need to write the pseudodifferential operators as functions of hD_x . Since the highest order operator is of order $\frac{3}{2}$, we will multiply the equation by $h^{\frac{3}{2}}$. Let us consider the following model operator

$$\partial_t + i |D_x|^{\frac{3}{2}},$$

which becomes

$$h^{\frac{3}{2}} \partial_t + i |hD_x|^{\frac{3}{2}}.$$

To give it the canonical form of a semi-classical equation, we need to put the equation in the semi-classical time $\sigma := h^{-\frac{1}{2}} t$. In our model, the operator would become

$$h \partial_\sigma + i |hD_x|^{\frac{3}{2}}.$$

Here, we have

$$h^{\frac{3}{2}} S_{(j-3)\delta}(\gamma)(t, x, D_x) \varphi_1(2^{-j} D_x) = S_{j\delta}(\gamma)(h^{\frac{1}{2}} \sigma, x, hD_x) \varphi_1(hD_x)$$

because of the homogeneity of the original symbol γ , which is conserved by its spatial regularization. Next, for the change of temporal variable $t = h^{\frac{1}{2}} \sigma$ we set

$$(3.2.18) \quad w_h(\sigma, x) := u_j(h^{\frac{1}{2}} \sigma, x), \quad V_h(\sigma, x) := S_{(j-3)\delta}(V)(h^{\frac{1}{2}} \sigma, x),$$

$$(3.2.19) \quad \Gamma_h(\sigma, x, \xi) := S_{(j-3)\delta}(\gamma)(h^{\frac{1}{2}} \sigma, x, \xi) \varphi_1(\xi),$$

$$(3.2.20) \quad \mathcal{L}_h(\sigma, x) := h \partial_\sigma + i \Gamma_h(x, hD_x) + h^{\frac{1}{2}} V_h \cdot (h \nabla_x),$$

so that

$$(3.2.21) \quad h^{\frac{3}{2}} (L_j u_j)(h^{\frac{1}{2}} \sigma, x) = \mathcal{L}_h w_h(\sigma, x),$$

and we want to establish Strichartz estimates for the semi-classical PDE

$$(3.2.22) \quad \mathcal{L}_h w_h(\sigma, x) = 0.$$

Symbolic calculus. To express the regularity of the symbols involved, we define for $k \in \mathbf{N}$ and J a time interval, the quantities

$$(3.2.23) \quad \begin{aligned} \mathcal{N}_k(\gamma)(J) &:= \sum_{|\beta| \leq k} \sup_{t \in J} \sup_{\xi \in \mathbf{C}'} \left\| D_\xi^\beta \gamma \right\|_{W_x^{1,\infty}(\mathbf{R}^d)}, \\ \mathcal{N}_k(\omega)(J) &:= \sum_{|\beta| \leq k} \sup_{t \in J} \sup_{\xi \in \mathbf{C}'} \left\| D_\xi^\beta \omega \right\|_{L_x^\infty(\mathbf{R}^d)}. \end{aligned}$$

The regularity of V is tracked under the norm

$$(3.2.24) \quad E(J) := L^p(J; W^{1,\infty}(\mathbf{R}^d))^d.$$

To simplify notations, let us set

$$\Xi_k(J) = \|V\|_{E(J)} + \mathcal{N}_k(\gamma)(J).$$

Let us define now our symbol classes.

Definition 3.2.6. Let $m \in \mathbf{R}$, $\mu_0 \in \mathbf{R}^+$, and $a(\sigma, x, \xi, h)$ a smooth function defined on $h^{-\frac{1}{2}}J \times \mathbf{R}^d \times \mathbf{C}' \times (0, h_0]$, with $h_0 > 0$ and smooth in the second and the third argument. We say that $a \in S_{\mu_0}^m(h^{-\frac{1}{2}}J)$ (resp. $a \in \dot{S}_{\mu_0}^m(h^{-\frac{1}{2}}J)$) if there is a function $\tilde{F}_k : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ for every $k \in \mathbf{N}$ (resp. $k \in \mathbf{N}^*$), such that for all $(\sigma, x, \xi, h) \in h^{-\frac{1}{2}}J \times \mathbf{R}^d \times \mathbf{C}' \times (0, h_0]$,

$$(3.2.25) \quad \left| D_x^\alpha D_\xi^\beta a(\sigma, x, \xi, h) \right| \leq \tilde{F}_k(\Xi_{k+1}(J)) h^{m-|\alpha|\mu_0}, \quad |\alpha| + |\beta| \leq k.$$

We need a result on composition of such symbols, whose proof is indeed the same as that of Proposition 4.20, [8].

Proposition 3.2.7. If f is a symbol in $S_\delta^m(h^{-\frac{1}{2}}J)$ (respectively $\dot{S}_\delta^m(h^{-\frac{1}{2}}J)$), with $m \in \mathbf{R}$, and we are given two symbols $U \in \dot{S}_\delta^\delta(h^{-\frac{1}{2}}J)$ and $V \in \dot{S}_\delta^0(h^{-\frac{1}{2}}J)$, then

$$F(\sigma, y, \zeta) := f(\sigma, U(\sigma, y, \zeta), V(\sigma, y, \zeta))$$

is in $S_\delta^m(h^{-\frac{1}{2}}J)$ (respectively $\dot{S}_\delta^m(h^{-\frac{1}{2}}J)$).

Since $\eta \in L^\infty(I, H^{s+\frac{1}{2}}(\mathbf{R}^d))$ and $V \in L^\infty(I, H^s(\mathbf{R}^d))$ with $s > \frac{3}{2} + \frac{d}{2}$ we obtain by using Bernstein inequalities that

Lemma 3.2.8. We have $\Gamma_h, \nabla_x \Gamma_h, V_h \in S_\delta^0(h^{-\frac{1}{2}}I)$.

Remark. Recall that $\mathbf{C}'' = \left\{ \xi \in \mathbf{R}^d : \frac{1}{3} \leq |\xi| \leq 3 \right\}$. In the semi-classical scale, Proposition 3.2.5 translates as

$$(3.2.26) \quad |\det \text{Hess}_\xi(\Gamma_h)(\sigma, x, \xi)| \geq c_0,$$

for $(\sigma, x, \xi, h) \in h^{-\frac{1}{2}}I \times \mathbf{R}^d \times \mathbf{C}'' \times (0, h_0]$, for h_0 small enough, because $\varphi_1 = 1$ on \mathbf{C}'' .

3.2.5 Straightening the transport term

The semi-classical equation (3.2.22) is not perfectly adapted to the construction of a parametrix, the reason being the term of order $h^{\frac{1}{2}}$, which has to be taken into account while constructing the phase. An easy way around this problem is to remark that this is only a transport term, and can be straightened by going to the associated lagrangian coordinates. Consider the solution $X_h(\sigma; y) \in \mathbf{R}^d$ of the differential equation

$$(3.2.27) \quad \begin{cases} \dot{X}_h(\sigma; y) = h^{\frac{1}{2}} V_h(\sigma, X_h(\sigma; y)), \\ X_h(0; y) = y, \end{cases}$$

where $y \in \mathbf{R}^d$. The vector field V_h is in $L^\infty(h^{-\frac{1}{2}}I; H^\infty(\mathbf{R}^d))^d$, and

$$\left| h^{\frac{1}{2}} V_h(\sigma, x) \right| \leq C \|V\|_{L^\infty(I \times \mathbf{R}^d)}, \quad \forall (\sigma, x) \in h^{-\frac{1}{2}}I \times \mathbf{R}^d.$$

Then (3.2.27) has a unique solution on $h^{-\frac{1}{2}}I$. Moreover, we have the following estimates on the flow (for the proof, see Proposition 4.10 in [8]).

Proposition 3.2.9. *At fixed $\sigma \in h^{-\frac{1}{2}}I$, the map $y \mapsto X_h(\sigma; y)$ is in $C^\infty(\mathbf{R}^d; \mathbf{R}^d)$, and there exists functions $\tilde{F}, \tilde{F}_\alpha : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that*

$$(i) \quad \left\| \frac{\partial X_h}{\partial y}(\sigma; \cdot) - I \right\|_{L^\infty} \leq \tilde{F} \left(\|V\|_{E(I)} \right) \left| h^{\frac{1}{2}} \sigma \right|^{\frac{1}{2}},$$

$$(ii) \quad \left\| (D_y^\alpha X_h)(\sigma; \cdot) \right\|_{L^\infty} \leq \tilde{F}_\alpha \left(\|V\|_{E(I)} \right) h^{-\delta(|\alpha|-1)} \left| h^{\frac{1}{2}} \sigma \right|^{\frac{1}{2}}, \quad |\alpha| \geq 2.$$

Corollary 3.2.10. *If T satisfies*

$$(3.2.28) \quad T \tilde{F}(\|V\|_{E(I)}) \ll 1$$

then for any $\sigma \in h^{-\frac{1}{2}}I$, the map $X_h(\sigma)$ is a diffeomorphism from \mathbf{R}^d to itself.

Proof. Proposition 3.2.9 shows that for T small enough as in (3.2.28), the matrix $\frac{\partial X_h}{\partial y}(\sigma; y)$ is invertible. Also, we have

$$|X_h(\sigma; y) - y| \leq h^{\frac{1}{2}} \int_0^{h^{-\frac{1}{2}}T} \left\| V(h^{\frac{1}{2}}s) \right\|_{L^\infty(\mathbf{R}^d)} ds \leq T^{1/p'} \|V\|_{L^p([0, T], L^\infty(\mathbf{R}^d))},$$

with $1/p' + 1/p = 1$. Thus, the map $X_h(\sigma)$ is proper. This enables us to conclude using the Hadamard theorem. □

We will always assume in what follows that the chosen T satisfies (3.2.28). The Strichartz estimates for the original solution can be recovered by summing the ones for the short time, the number of pieces depending only on the $L_t^p L_x^\infty$ -norms of V appearing in the final constant.

Now we have to compute how our semi-classical equation (3.2.22) gets affected by this change of variables. The new unknown will be $v_h(\sigma, y) := w_h(\sigma, X_h(\sigma; y))$. The important quantity is $A := (\Gamma_h(hD_x)w_h)(\sigma, X_h(\sigma; y))$. Taking σ, h, δ as parameters, we have

$$A = (2\pi h)^{-d} \int \int e^{ih^{-1}(X(y)-x') \cdot \eta} \Gamma(X(y), \eta) w(x') dx' d\eta.$$

Now we need to set

$$H_h(\sigma; y, y') := \int_0^1 \frac{\partial X_h}{\partial y}(\sigma; \lambda y + (1-\lambda)y') d\lambda, \quad M_h(\sigma; y, y') := (H_h^T(\sigma; y, y'))^{-1},$$

$$M_h^0(\sigma; y) := \left(\left(\frac{\partial X_h}{\partial y} \right)^T(\sigma; y) \right)^{-1}, \quad J_h(\sigma; y, y') := \left| \det \left(\frac{\partial X_h}{\partial y} \right)(\sigma; y') \right| |\det M_h(\sigma; y, y')|.$$

Proposition 3.2.9 shows that M and M^0 are well defined. Remark that $M^0(y) = M(y, y)$ and that $J(y, y) = 1$. We now change variables in the expression of A , putting $x' := X(y')$. We will then use $X(y) - X(y') = H(y, y')(y - y')$ and set $\eta := M(y, y')\zeta$ to get

$$A = (2\pi h)^{-d} \iint e^{ih^{-1}(y-y')\cdot\zeta} \Gamma(X(y), M(y, y')\zeta) J(y, y') v(y') dy' d\zeta.$$

We have proved that

$$(3.2.29) \quad A = (\Gamma_h(hD_x)w_h)(\sigma, X_h(\sigma; y)) = P_h v_h(\sigma, y),$$

where P_h is a semi-classical pseudodifferential operator of amplitude

$$(3.2.30) \quad \tilde{p}_h(\sigma, y, y', \zeta) := \Gamma_h(\sigma, X_h(\sigma; y), M_h(\sigma; y, y')\zeta) J_h(\sigma; y, y').$$

We define the symbol

$$(3.2.31) \quad p_h(\sigma, y, \zeta) := \tilde{p}_h(\sigma, y, y, \zeta) = \Gamma_h(\sigma, X_h(\sigma; y), M_h^0(\sigma; y)\zeta).$$

We also set

$$(3.2.32) \quad p_h(\sigma, y, \zeta) := \tilde{p}_h(\sigma, y, y, \zeta) = \Gamma_h(\sigma, X_h(\sigma; y), M_h^0(\sigma; y)\zeta).$$

Let us write $I_h := [0, h^{\frac{1}{2}+\delta}]$ and impose a constrain on δ :

$$(3.2.33) \quad 0 < \delta \leq \frac{1}{2}$$

so that for all $\sigma \in h^{-\frac{1}{2}}I_h$ one has

$$(3.2.34) \quad \left| h^{\frac{1}{2}}\sigma \right|^{\frac{1}{2}} \leq h^\delta.$$

Proposition 3.2.11. *For every $k \in \mathbf{N}$, there exists $\tilde{F}_k : \mathbf{R}^+ \rightarrow \mathbf{R}^+$, such that*

$$(3.2.35) \quad \left| D_y^\alpha D_\zeta^\beta p_h(\sigma, y, \zeta) \right| \leq \tilde{F}_k \left(\mathcal{N}_k(\gamma)(I) + \|V\|_{E(I)} \right) \left[1 + h^{-(|\alpha|-1)\delta} \right],$$

where $|\alpha| + |\beta| \leq k$, and $(\sigma, y, \zeta, h) \in h^{-\frac{1}{2}}I_h \times \mathbf{R}^d \times \mathcal{C}' \times (0, h_0]$.

Consequently, we have

$$(3.2.36) \quad p_h \in S_\delta^0(h^{-\frac{1}{2}}I_h) \quad \text{and} \quad \nabla_y p_h \in S_\delta^0(h^{-\frac{1}{2}}I_h).$$

Remark. Remark that Proposition 3.2.7 implies only the first assertion in (3.2.36). In the construction of the phase of our parametrix below, to control the flow (see (3.3.6), (3.3.7)) we need to differentiate p twice in x and thus the first assertion in (3.2.36) implies only $\partial_x^2 p \in S_\delta^{-2\delta}(h^{\frac{1}{2}}I_h)$ while with the second one, we have $\partial_x^2 p \in S_\delta^{-\delta}(h^{\frac{1}{2}}I_h)$. Consequently, the restriction $\sigma \leq h^\delta$ is sufficient instead of requiring $\sigma \leq h^{2\delta}$. This means that the parametrix is constructed in a time of double length by virtue of the second one.

Proof. We will consider $\sigma \in h^{-\frac{1}{2}}I_h$ and $h \in (0, h_0)$ as parameters. Denote $A_k = \mathcal{N}_k(\gamma)(I) + \|V\|_{E(I)}$. First, remark that we can use the identity

$$\left(\frac{\partial X}{\partial y}\right)(y) \cdot M^0(y) = I_d$$

and Proposition 3.2.9 to get

$$(3.2.37) \quad \begin{aligned} \|M^0(y) - I\|_{L^\infty(\mathbf{R}^d)} &\leq \tilde{F}(\|V\|_E) h^\delta, \\ \|D_y^\alpha M^0(y)\|_{L^\infty(\mathbf{R}^d)} &\leq \tilde{F}_\alpha(\|V\|_E) h^{-\delta|\alpha|} \left|h^{\frac{1}{2}}\sigma\right|^{\frac{1}{2}} \leq \tilde{F}_\alpha(\|V\|_E) h^{\delta-\delta|\alpha|}, \quad \text{for } |\alpha| \geq 1. \end{aligned}$$

Now $D_\zeta^\beta p$ is a finite linear combination of terms of the form

$$(D_\xi^{\beta'} \Gamma)(X(y), M^0(y)\zeta) \cdot P_{|\beta|}(M^0(y)) := A \cdot B,$$

where $|\beta'| = |\beta|$, and where $P_{|\beta|}(M^0(y))$ is a homogeneous polynomial of order $|\beta|$ in the coefficients of $M^0(y)$. Hence $D_y^\alpha D_\zeta^\beta p$ is a finite linear combination of terms of the form

$$D_y^{\alpha_1} A \cdot D_y^{\alpha_2} B, \quad \alpha_1 + \alpha_2 = \alpha.$$

Concerning B we use (3.2.37) to find

$$(3.2.38) \quad \begin{cases} |B| \leq \tilde{F}_k(\|V\|_E), \\ |D_y^{\alpha_2} B| \leq \tilde{F}_k(\|V\|_E) h^{\delta-|\alpha_2|\delta} \quad \text{if } |\alpha_2| \geq 1. \end{cases}$$

By the fact that $\Gamma_h \in S_\delta^0(h^{-\frac{1}{2}}I)$ we see that if $\alpha = 0$ then $|AB| \leq \tilde{F}_k(\|V\|_E)$. Considering now $|\alpha| \geq 1$. If $\alpha_1 = 0$ then $\alpha_2 = \alpha \neq 0$ and thus by (3.2.38)

$$|A \cdot \partial_y^\alpha B| \leq \tilde{F}_k(\|V\|_E) h^{\delta-|\alpha|\delta}.$$

From now on, we assume $|\alpha_1| \geq 1$. By virtue of the Faà di Bruno formula, we see that $D_y^{\alpha_1} A$ is a finite linear combination of terms of the form

$$C = \left(D_x^a D_\zeta^{\beta'+b} \Gamma\right)(X(y), M^0(y)\zeta) \prod_{j=1}^r (D_y^{l_j} X(y))^{p_j} (D_y^{l_j} M^0(y)\zeta)^{q_j},$$

where $1 \leq |a| + |b| \leq |\alpha_1|$, $|l_j| \geq 1$, $\sum_{j=1}^r (|p_j| + |q_j|) l_j = \alpha_1$, $\sum_{j=1}^r p_j = a$, $\sum_{j=1}^r q_j = b$. We distinguish 2 cases corresponding to $a = 0$ or $a \neq 0$.

Case 1: $|a| = 0$. Then every p_j is 0, and $\sum_{j=1}^r |q_j| |l_j| = |\alpha_1| \geq 1$, so that at least one of the $|q_j| |l_j|$ is non null. Then using the boundedness of $D_\zeta^{\beta'+b} \Gamma$ (since $\Gamma_h \in S_\delta^0(h^{-\frac{1}{2}}I)$), estimates (3.2.37), and the fact that ζ is bounded on the support of $\varphi_1(M^0(y)\zeta)$ and thus of $(D_\zeta^{\beta'+b})\Gamma(X(y), M^0(y)\zeta)$, we obtain

$$|C| \leq \tilde{F}_k(A_k) h^{\delta-\delta \sum_{j=1}^r |l_j| |q_j|} = \tilde{F}_k(A_k) h^{\delta-\delta|\alpha_1|}.$$

On the other hand, (3.2.38) implies that $|\partial_y^{\alpha_2} B| \leq \tilde{F}_k(\|V\|_E) h^{-|\alpha_2|\delta}$, $\forall \alpha_2 \in \mathbf{N}$. Therefore, we conclude in this case that

$$|C \cdot \partial_y^{\alpha_2} B| \leq \tilde{F}_k(A_k) h^{\delta-|\alpha|\delta}.$$

Case 2: $|a| \geq 1$. We use in this case $\nabla_x \Gamma \in S_\delta^0$, estimate (3.2.37), and Proposition 3.2.9 with the remark above on the boundedness of ζ to obtain

$$\begin{aligned} \left| \left(D_x^a D_\zeta^{\beta'+b} \Gamma \right) (X(y), M^0(y)\zeta) \right| &\leq \tilde{F}_k(A_k) h^{-(|a|-1)\delta}, \\ \left| \left(D_y^{l_j} X(y) \right)^{p_j} \right| &\leq \tilde{F}_k(A_k) h^{-(|l_j|-1)|p_j|\delta}, \\ \left| \left(D_y^{l_j} M^0(y)\zeta \right)^{q_j} \right| &\leq \tilde{F}_k(A_k) h^{-|l_j||q_j|\delta}. \end{aligned}$$

These estimates imply $|C| \leq \tilde{F}_k(A_k) h^M$ with

$$M = -(|a| - 1)\delta - \sum_{j=1}^r (|l_j| - 1)|p_j|\delta - \sum_{j=1}^r |l_j||q_j|\delta = -(|\alpha_1| - 1)\delta$$

and hence $|C| \leq \tilde{F}_k(A_k) h^{-(|\alpha_1|-1)\delta}$. The second inequality in (3.2.38) then yields

$$|C \cdot \partial_y^{\alpha_2} B| \leq \tilde{F}_k(A_k) h^{-(|\alpha|-1)\delta}.$$

Summing up, we obtain in any case the desired estimate and complete the proof. \square

We also have the following result, whose proof follows that of the preceding and is in fact simpler.

Proposition 3.2.12. *For every $k \in \mathbf{N}$, there exists $\tilde{F}_k : \mathbf{R}^+ \rightarrow \mathbf{R}^+$, such that*

$$(3.2.39) \quad \left| D_y^{\alpha_1} D_{y'}^{\alpha_2} D_\zeta^\beta \tilde{p}_h(\sigma, y, y', \zeta) \right| \leq \tilde{F}_k \left(\mathcal{N}_k(\gamma)(I) + \|V\|_{E(I)} \right) h^{-(|\alpha_1|+|\alpha_2|)\delta},$$

where $|\alpha_1| + |\alpha_2| + |\beta| \leq k$, and $(\sigma, y, \zeta, h) \in h^{-\frac{1}{2}}I_h \times \mathbf{R}^d \times \mathcal{C}' \times (0, h_0]$.

Concerning the Hessian of the principal symbol, we derive the following result.

Proposition 3.2.13. *There exist $h_0 > 0$ and $c_0 > 0$ such that*

$$|\det \text{Hess}_\zeta(p_h)(\sigma, y, \zeta)| \geq c_0,$$

for $(\sigma, y, \zeta, h) \in h^{-\frac{1}{2}}I \times \mathbf{R}^d \times \mathbf{C} \times (0, h_0]$. Here, recall that $\mathbf{C} = \left\{ \zeta \in \mathbf{R}^d : \frac{1}{2} \leq |\zeta| \leq 2 \right\}$.

Proof. The Hessians of p_h and Γ_h are conjugated by

$$\text{Hess}_\zeta(p_h)(y, \zeta) = \left(M^0(y) \right)^T \text{Hess}_\zeta \Gamma_h(X(y), M^0(y)\zeta) M^0(y),$$

so the result follows from (3.2.37) and (3.2.26) for h_0 small enough. \square

At last, the transport term disappears, since

$$(3.2.40) \quad \left(h\partial_\sigma w_h + h^{\frac{1}{2}} V_h \cdot (h\nabla_x) w_h \right) (\sigma, X_h(\sigma; y)) = h\partial_\sigma v_h(\sigma, y).$$

Now, using (3.2.29) and (3.2.40), the semi-classical equation (3.2.22) becomes

$$(3.2.41) \quad (\mathcal{L}_h w_h)(\sigma, X_h(\sigma, y)) = (h\partial_\sigma + iP_h) v_h(\sigma, y) = 0$$

via the change of spatial variable $v_h(\sigma, y) := w_h(\sigma, X_h(\sigma; y))$.

3.3 Construction of the parametrix

We want to construct a parametrix for the operator $h\partial_\sigma + iP_h$ (recall that the space-time variables are (σ, y)). To compensate for the loss in powers of h incurred while differentiating our symbols, we will need to restrict ourselves to a small time interval depending on the frequency and the number of derivative used to regularize: $\sigma \in h^{-\frac{1}{2}}I_h = [0, h^\delta]$.

We will look for a parametrix with the following Fourier integral operator form

$$(3.3.1) \quad \mathcal{K}v(\sigma, y) = (2\pi h)^{-d} \iint e^{ih^{-1}(\phi_h(\sigma, y, \eta) - y' \cdot \eta)} \tilde{b}_h(\sigma, y, y', \eta) \chi_1(\eta) v(y') \, dy' \, d\eta.$$

We will take ϕ_h to be a real valued phase such that

$$\phi_h(0, y, \eta) = y \cdot \eta$$

and \tilde{b}_h an amplitude of the form

$$(3.3.2) \quad \tilde{b}_h(\sigma, y, y', \eta) = b_h(\sigma, y, \eta) \psi \left(\frac{\partial \phi_h}{\partial \eta}(\sigma, y, \eta) - y' \right),$$

where $b|_{\sigma=0} = \chi(\eta)$, $\chi \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$ and $\psi \in C_c^\infty(\mathbf{R}^d)$ is such that $\psi(z) = 1$ if $|z| \leq 1$. At last, $\chi_1 \in C_c^\infty(\mathbf{R}^d)$ is 1 on the support of χ .

3.3.1 Construction of the phase

As usual, the phase will be the solution of the eikonal equation associated with the principal symbol of the operator,

$$(3.3.3) \quad \frac{\partial \phi_h}{\partial \sigma} + p_h \left(\sigma, y, \frac{\partial \phi_h}{\partial y} \right) = 0, \quad \phi_h(0, y, \eta) = y \cdot \eta.$$

We will solve this equation with the method of characteristics. Those are the solution of the system

$$(3.3.4) \quad \begin{cases} \dot{y}_h(\sigma; y_0, \eta) = \frac{\partial p_h}{\partial \zeta}(\sigma, y_h(\sigma; y_0, \eta), \zeta_h(\sigma; y_0, \eta)), & y_h(0; y_0, \eta) = y_0, \\ \dot{\zeta}_h(\sigma; y_0, \eta) = -\frac{\partial p_h}{\partial y}(\sigma, y_h(\sigma; y_0, \eta), \zeta_h(\sigma; y_0, \eta)), & \zeta_h(0; y_0, \eta) = \eta. \end{cases}$$

This system has a unique solution on $h^{-\frac{1}{2}}I_h$. Now let us show that for fixed h, η , and σ this flow is a global diffeomorphism from \mathbf{R}^d to itself.

We start by showing that the differential of this map is invertible. Taking h as a parameter, denote by

$$m(\sigma) := (\sigma, y(\sigma; y_0, \eta), \zeta(\sigma, y_0, \eta))$$

the flow-out of $(0, y_0, \eta)$. Differentiate (3.3.4) with respect to y_0 . Then at the point (y_0, η) , there holds

$$(3.3.5) \quad \begin{cases} \frac{\partial y}{\partial y_0}(\sigma) = \frac{\partial^2 p}{\partial y \partial \zeta}(m(\sigma)) \frac{\partial y}{\partial y_0}(\sigma) + \frac{\partial^2 p}{\partial \zeta^2}(m(\sigma)) \frac{\partial \zeta}{\partial y_0}(\sigma), & \frac{\partial y}{\partial y_0}(0) = I_d, \\ \frac{\partial \zeta}{\partial y_0}(\sigma) = -\frac{\partial^2 p}{\partial y^2}(m(\sigma)) \frac{\partial y}{\partial y_0}(\sigma) - \frac{\partial^2 p}{\partial \zeta \partial y}(m(\sigma)) \frac{\partial \zeta}{\partial y_0}(\sigma), & \frac{\partial \zeta}{\partial y_0}(0) = 0. \end{cases}$$

This system is linear, of the form $\dot{U}(\sigma) = M(\sigma)U(\sigma)$. Then, Proposition 3.2.11 and the remark that follows it give

$$(3.3.6) \quad \|M(\sigma)\| \leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}}I_h) + \|V\|_{E(h^{-\frac{1}{2}}I_h)} \right) h^{-\delta}.$$

When we integrate in time over $h^{-\frac{1}{2}}I_h = [0, h^\delta]$, we get

$$(3.3.7) \quad \int_0^\sigma \|M(s)\| \, ds \leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}}I_h) + \|V\|_{E(h^{-\frac{1}{2}}I_h)} \right).$$

The Grönwall inequality then shows that $\|U(\sigma)\|$ is uniformly bounded on $h^{-\frac{1}{2}}I_h$. Now using (3.3.5) and noticing that the coefficients of the first equation involve only derivatives of order 0 and 1 in y of p , we obtain by virtue of Proposition 3.2.11

$$(3.3.8) \quad \left| \frac{\partial y_h}{\partial y_0}(\sigma; y_0, \eta) - Id \right| \leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}}I_h) + \|V\|_{E(h^{-\frac{1}{2}}I_h)} \right) h^\delta.$$

Similarly, since the second equation in (3.3.5) has coefficients containing derivatives of p in y up to order 2, we have

$$(3.3.9) \quad \left| \frac{\partial \zeta_h}{\partial y_0}(\sigma; y_0, \eta) \right| \leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}}I_h) + \|V\|_{E(h^{-\frac{1}{2}}I_h)} \right).$$

Now taking h small enough, (3.3.8) gives the invertibility of the matrix $\frac{\partial y_h}{\partial y_0}(\sigma; y_0, \eta)$, and since

$$|y_h(\sigma; y_0, \eta) - y_0| \leq \int_0^\sigma |y_h'(s, y_0, \eta)| \, ds \leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}}I_h) + \|V\|_{E(h^{-\frac{1}{2}}I_h)} \right) h^\delta$$

for $\sigma \in h^{-\frac{1}{2}}I_h$, the map $y_0 \mapsto y_h(\sigma, y_0, \eta)$ is proper. Therefore it is as announced a global diffeomorphism, and we denote by κ_h its inverse:

$$y_h(\sigma; y_0, \eta) = y \Leftrightarrow y_0 = \kappa_h(\sigma; y, \eta).$$

Then we can define for h_0 small enough, for $(h, \sigma, y, \eta) \in (0, h_0] \times (h^{-\frac{1}{2}}I_h) \times \mathbf{R}^d \times \mathbf{R}^d$ the real-valued phase

$$(3.3.10) \quad \phi_h(\sigma, y, \eta) = y \cdot \eta - \int_0^\sigma p_h(s, y, \zeta_h(s; \kappa_h(s; y, \eta), \eta)) \, ds.$$

Proposition 3.3.1. *The function ϕ defined in (3.3.10) solves the eikonal equation (3.3.3).*

The proof of this proposition is standard (see for example [113], 10.2.2).

The map ϕ is C^1 in σ and C^∞ in (y, η) . We can study the Hessian of this phase in the η variable, using our study of the symbol p .

Proposition 3.3.2. *For h_0 small enough, there exists a constant $M_0 > 0$ such that*

$$|\det \text{Hess}_\eta(\phi_h)(\sigma, y, \eta)| \geq M_0 \sigma^d,$$

for $\sigma \in (h^{-\frac{1}{2}}I_h)$, $y \in \mathbf{R}^d$, $\eta \in \mathbf{C}$ and $h \in (0, h_0]$.

Proof. By differentiating the eikonal equation (3.3.3) twice with respect to η , we find that

$$\begin{aligned} \partial_\sigma(\partial_{\eta_i}\partial_{\eta_j}\phi) &= - \sum_{k,l=1}^d (\partial_{\zeta_k}\partial_{\zeta_l}p) \left(\sigma, y, \frac{\partial\phi}{\partial y} \right) (\partial_{y_k}\partial_{\eta_i}\phi)(\partial_{y_l}\partial_{\eta_j}\phi) \\ &\quad - \sum_{k=1}^d (\partial_{\zeta_k}p) \left(\sigma, y, \frac{\partial\phi}{\partial y} \right) (\partial_{y_k}\partial_{\eta_i}\partial_{\eta_j}\phi). \end{aligned}$$

From the initial conditions of the eikonal equation, we obtain the values of the terms at $\sigma = 0$, so that

$$\partial_\sigma(\partial_{\eta_i}\partial_{\eta_j}\phi)|_{\sigma=0} = -(\partial_{\zeta_i}\partial_{\zeta_j}p)(0, y, \eta),$$

so that

$$\partial_{\eta_i}\partial_{\eta_j}\phi(\sigma, y, \eta) = -(\partial_{\zeta_i}\partial_{\zeta_j}p)(0, y, \eta)\sigma + o(\sigma).$$

Then using Proposition 3.2.13 and taking h_0 small enough, which means σ small, we can conclude the proposition. \square

Now we want estimates of higher orders for the phase and various related quantities. We start by estimating the derivatives of the flow.

Proposition 3.3.3. *There exists $\tilde{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that*

$$(3.3.11) \quad \begin{aligned} \left| \frac{\partial y_h}{\partial y_0}(\sigma; y_0, \eta) - I_d \right| &\leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}}I_h) + \|V\|_{E(h^{-\frac{1}{2}}I_h)} \right) h^\delta, \\ \left| \frac{\partial \zeta_h}{\partial y_0}(\sigma; y_0, \eta) \right| &\leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}}I_h) + \|V\|_{E(h^{-\frac{1}{2}}I_h)} \right), \\ \left| \frac{\partial y_h}{\partial \eta}(\sigma; y_0, \eta) \right| &\leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}}I_h) + \|V\|_{E(h^{-\frac{1}{2}}I_h)} \right) h^\delta, \\ \left| \frac{\partial \zeta_h}{\partial \eta}(\sigma; y_0, \eta) - I_d \right| &\leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}}I_h) + \|V\|_{E(h^{-\frac{1}{2}}I_h)} \right) h^\delta, \end{aligned}$$

for $(h, s, y_0, \eta) \in (0, h_0] \times h^{-\frac{1}{2}}I_h \times \mathbf{R}^d \times \mathcal{C}'$.

Moreover, for every $k \geq 1$ there exists $\tilde{F}_k : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that for $(\alpha, \beta) \in \mathbf{N}^d \times \mathbf{N}^d$

$$\begin{cases} |D_y^\alpha D_\eta^\beta y_h(\sigma)| \leq \mathcal{F}_k(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)) h^{\delta-|\alpha|}, \\ |D_y^\alpha D_\eta^\beta \zeta_h(s)| \leq \mathcal{F}_k(\|V\|_{E_0} + \mathcal{N}_{k+1}(\gamma)) h^{-\delta|\alpha|}. \end{cases}$$

Consequently,

$$y_h \in \dot{S}_\delta^\delta(h^{-\frac{1}{2}}I_h), \quad \zeta_h \in \dot{S}_\delta^0(h^{-\frac{1}{2}}I_h).$$

Proof. The first two estimates of (3.3.11) have already been proven in (3.3.8) and (3.3.9). Similarly, if we differentiate the characteristic system (3.3.4) with respect to η we obtain the last two estimates of (3.3.11).

To prove estimates on higher order derivatives, we proceed by induction and Grönwall inequality (see Proposition 4.21, [8]). \square

We now deduce estimates on some quantities associated to the phase. Define

$$(3.3.12) \quad \theta_h(\sigma, y, y', \eta) := \int_0^1 \frac{\partial \phi_h}{\partial y}(\sigma, \lambda y + (1-\lambda)y', \eta) d\lambda.$$

Corollary 3.3.4. *For every $k \geq 1$ there exists $\tilde{F}_k : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that for $(\alpha, \beta) \in \mathbf{N}^d \times \mathbf{N}^d$ with $|\alpha| + |\beta| = k$, $(\alpha_1, \alpha_2, \beta) \in \mathbf{N}^d \times \mathbf{N}^d \times \mathbf{N}^d$ with $|\alpha_1| + |\alpha_2| + |\beta| = k$,*

$$(3.3.13) \quad |D_y^\alpha D_\eta^\beta \kappa_h(\sigma; y, \eta)| \leq \tilde{F}_k \left(\mathcal{N}_{k+1}(\gamma)(h^{-\frac{1}{2}} I_h) + \|V\|_{E(h^{-\frac{1}{2}} I_h)} \right) h^{\delta - |\alpha|\delta},$$

$$(3.3.14) \quad \left| D_y^\alpha D_\eta^\beta \left(\frac{\partial \phi_h}{\partial y} \right) (\sigma, y, \eta) \right| \leq \tilde{F}_k \left(\mathcal{N}_{k+1}(\gamma)(h^{-\frac{1}{2}} I_h) + \|V\|_{E(h^{-\frac{1}{2}} I_h)} \right) h^{-|\alpha|\delta},$$

$$(3.3.15) \quad \left| D_y^\alpha D_\eta^\beta \left(\frac{\partial \phi_h}{\partial \eta} \right) (\sigma, y, \eta) \right| \leq \tilde{F}_k \left(\mathcal{N}_{k+1}(\gamma)(h^{-\frac{1}{2}} I_h) + \|V\|_{E(h^{-\frac{1}{2}} I_h)} \right) h^{\delta - |\alpha|\delta},$$

$$(3.3.16) \quad |D_y^{\alpha_1} D_{y'}^{\alpha_2} D_\eta^\beta \theta_h(\sigma, y, y', \eta)| \leq \tilde{F}_k \left(\mathcal{N}_{k+1}(\gamma)(h^{-\frac{1}{2}} I_h) + \|V\|_{E(h^{-\frac{1}{2}} I_h)} \right) h^{-(|\alpha_1| + |\alpha_2|)\delta},$$

for all $(h, \sigma, y, y', \eta) \in (0, h_0] \times (h^{-\frac{1}{2}} I_h) \times \mathbf{R}^d \times \mathbf{R}^d \times \mathcal{C}'$.

This means that $\kappa_h \in \dot{S}_\delta^\delta$, $\frac{\partial \phi_h}{\partial y} \in \dot{S}_\delta^0$, $\frac{\partial \phi_h}{\partial \eta} \in \dot{S}_\delta^\delta$.

Proof. The first estimate comes from the relation $y(\sigma; \kappa(\sigma; y, \eta), \eta) = y$ which by differentiation gives

$$\frac{\partial y}{\partial y} \cdot \frac{\partial \kappa}{\partial y} = I_d, \quad \frac{\partial y}{\partial y} \cdot \frac{\partial \kappa}{\partial \eta} = -\frac{\partial y}{\partial \eta}.$$

Now the case $k = 1$ follows from (3.3.11), and by differentiating $k + 1$ times and using an induction we get (3.3.13).

From the definition of ϕ as a solution of the Hamilton-Jacobi equation associated to p , we see that ϕ is a generating function for the Lagrangian surface

$$\left\{ (\sigma, p(\sigma); y, \zeta(\sigma; \kappa(\sigma; y, \eta), \eta); \kappa(\sigma; y, \eta), \eta) \mid \sigma \in (h^{-\frac{1}{2}} I_h), (y, \eta) \in \mathbf{R}^{2d} \right\},$$

so that

$$(3.3.17) \quad \frac{\partial \phi}{\partial y}(\sigma, y, \eta) = \zeta(\sigma; \kappa(\sigma; y, \eta), \eta), \quad \frac{\partial \phi}{\partial \eta}(\sigma, y, \eta) = \kappa(\sigma; y, \eta).$$

This immediately implies (3.3.15), and since $\zeta \in \dot{S}_\delta^0$, $\kappa \in \dot{S}_\delta^\delta$ and obviously $\eta \in \dot{S}_\delta^0$, we can use Proposition 3.2.7 to get (3.3.14).

At last (3.3.16) follows directly from the definition of θ and from (3.3.14). \square

3.3.2 Construction of the amplitude

The first step in constructing the amplitude is to compute the expression

$$J_h(\sigma, y, y', \eta) := e^{-ih^{-1}\phi_h(\sigma, y, \eta)} P_h \left(e^{ih^{-1}\phi_h(\sigma, y, \eta)} \tilde{b}_h(\sigma, y, y', \eta) \right).$$

This is a classical computation, identical to the one performed in [8], section 4.7.1. This yields, taking h, σ, y', η as parameters, for all $N \in \mathbf{N}^*$,

$$\begin{aligned} J(y) = & p \left(y, \frac{\partial \phi}{\partial y}(y) \right) b(y) \Psi(y) + \sum_{1 \leq |\alpha| \leq N-1} \frac{h^{|\alpha|}}{\alpha!} D_z^\alpha \left[(\partial_\eta^\alpha \tilde{p})(y, z, \theta(y, z)) b(z) \right] \Big|_{z=y} \Psi(y) \\ & + U_N(y) + R_N(y) + S_N(y), \end{aligned}$$

where $\Psi(y) = \psi \left(\frac{\partial \phi_h}{\partial \eta}(\sigma, y, \eta) - y' \right)$ which has been defined in (3.3.2), and θ has been defined in (3.3.12). The first remainder contain all the terms where Ψ is differentiated at least once,

$$(3.3.18) \quad U_N(y) := \sum_{1 \leq |\alpha| \leq N-1} \frac{h^{|\alpha|}}{\alpha!} \sum_{0 \leq |\beta| \leq |\alpha|-1} \binom{\alpha}{\beta} D_z^\beta [(\partial_\eta^\alpha \tilde{p})(y, z, \theta(y, z)) b(z)]|_{z=y} D_y^{\alpha-\beta} \Psi(y),$$

the second is the Taylor remainder due to the change of phase,

$$(3.3.19) \quad R_N(y) := \frac{1}{(2\pi h)^d} \iint e^{ih^{-1}(y-z) \cdot \mu} \kappa(\mu) r_N(y, z, \mu) \tilde{b}(y) dy d\mu$$

with

$$r_N(y, z, \mu) = \sum_{|\alpha|=N} \frac{N}{\alpha!} \int_0^1 (1-\lambda)^{N-1} (\partial_\eta^\alpha \tilde{p})(y, z, \mu + \lambda\theta(y, z)) \mu^\alpha d\lambda,$$

and where $\kappa \in C_c^\infty$ is 1 on the support of $\tilde{p}(y, z, \mu + \theta(z, z'))$, which is compact locally in η because the phase is locally bounded in η . The last one comes from this κ term, and it is

$$(3.3.20) \quad S_N(y) := \frac{1}{(2\pi)^d} \sum_{|\alpha| \leq N-1} \sum_{|\beta|=N} N \frac{h^{|\alpha|+|\beta|}}{\alpha! \beta!} \iint_0^1 (1-\lambda)^{N-1} \mu^\beta \hat{\kappa}(\mu) f_{\alpha, \beta}(y, y + \lambda h \mu) d\lambda d\mu,$$

where

$$f_{\alpha, \beta}(y, z) = \partial_z^\beta D_z^\alpha [(\partial_\eta^\alpha \tilde{p})(y, z, \theta(y, z)) \tilde{b}(z)].$$

Now ϕ satisfies the eikonal equation (3.3.3), so that we have

$$(3.3.21) \quad e^{-ih^{-1}\phi(y)} (h\partial_\sigma + iP) \left(e^{ih^{-1}\phi(y)} \tilde{b}(y) \right) \\ = h \left(\partial_\sigma b + i \sum_{1 \leq |\alpha| \leq N-1} \frac{h^{|\alpha|-1}}{\alpha!} D_z^\alpha [(\partial_\eta^\alpha \tilde{p})(y, z, \theta(y, z)) b(z)]|_{z=y} \right) \Psi(y) \\ + (hb\partial_\sigma \Psi + iU_N) + iR_N + iS_N.$$

We want this to be $\mathcal{O}(h^{N+1})$. Let

$$(3.3.22) \quad T_N := \partial_\sigma b + i \sum_{1 \leq |\alpha| \leq N-1} \frac{h^{|\alpha|-1}}{\alpha!} D_z^\alpha [(\partial_\eta^\alpha \tilde{p})(y, z, \theta(y, z)) b(z)]|_{z=y},$$

so that we want $T_N = \mathcal{O}(h^N)$.

Writing

$$\mathcal{L} := \partial_\sigma + \sum_{i=1}^d a_i(y) \partial_{y_i} + c(y),$$

where

$$(3.3.23) \quad \begin{cases} a_i(y) := (\partial_{\eta_i} p) \left(y, \frac{\partial \phi}{\partial y}(y) \right), \\ c(y) := \sum_{i=1}^d (\partial_{\eta_i} \partial_{y_i} \tilde{p}) \left(y, y, \frac{\partial \phi}{\partial y}(y) \right) + \sum_{i,j=1}^d (\partial_{\eta_i} \partial_{\eta_j} p) \left(y, \frac{\partial \phi}{\partial y}(y) \right) (\partial_{y_i} \partial_{y_j} \phi)(y), \end{cases}$$

we can rewrite (3.3.22) as

$$(3.3.24) \quad T_N = \mathcal{L}b(y) + i \sum_{2 \leq |\alpha| \leq N-1} \frac{h^{|\alpha|-1}}{\alpha!} D_z^\alpha [(\partial_\eta^\alpha \tilde{p})(y, z, \theta(y, z)) b(z)] \Big|_{z=y}.$$

For a fixed ν satisfying

$$(3.3.25) \quad 0 < \nu \leq 1 - \delta$$

we look for b under the form

$$(3.3.26) \quad b = \sum_{k=0}^N h^{k\nu} b_k.$$

Inserting this ansatz into (3.3.24) gives, after a change of indices,

$$(3.3.27) \quad \begin{aligned} T_N &= \sum_{k=0}^N h^{k\nu} \mathcal{L}b_k(y) \\ &+ i \sum_{k=1}^{N+1} h^{k\nu} \sum_{2 \leq |\alpha| \leq N-1} \frac{h^{|\alpha|-1-\nu}}{\alpha!} D_z^\alpha [(\partial_\eta^\alpha \tilde{p})(y, z, \theta(y, z)) b_{k-1}(z)] \Big|_{z=y}. \end{aligned}$$

We take b_0 as a solution of

$$(3.3.28) \quad \begin{cases} \mathcal{L}b_0 = 0, \\ b_0(\eta)|_{\sigma=0} = \chi(\eta), \end{cases}$$

where $\chi \in C_c^\infty(\mathbf{R}^d \setminus \{0\})$ is the Cauchy data needed for b .

Then we will recursively construct b_k , $1 \leq k \leq N$ as a solution of

$$(3.3.29) \quad \begin{cases} \mathcal{L}b_k = F_{j-1} := -i \sum_{2 \leq |\alpha| \leq N-1} \frac{h^{|\alpha|-1-\nu}}{\alpha!} D_z^\alpha [(\partial_\eta^\alpha \tilde{p})(y, z, \theta(y, z)) b_{k-1}(z)] \Big|_{z=y} \\ b_k|_{\sigma=0} = 0. \end{cases}$$

Again, (3.3.28) and (3.3.29) are solved by the method of characteristics. First we study the highest-order coefficients.

Lemma 3.3.5. *For $1 \leq i \leq d$, $a_i \in S_\delta^0(h^{-\frac{1}{2}}I_h)$.*

Proof. We know from Proposition 3.2.11 and the remark that follows it that $\partial_{\eta_i} p \in S_\delta^0$. From Corollary 3.3.4 we have that $\frac{\partial \phi}{\partial y} \in \dot{S}_\delta^0$. Then, using Proposition 3.2.7 gives the lemma. \square

Now consider the differential equation

$$\begin{cases} \dot{Y}(\sigma) = a(\sigma, Y(\sigma), \eta), \\ Y(0) = y. \end{cases}$$

From Lemma 3.3.5, a is bounded, so the system has a unique solution on $h^{-\frac{1}{2}}I_h$. Now we remark that

$$\frac{\partial a}{\partial y} = \frac{\partial^2 p}{\partial y \partial \eta} + \frac{\partial^2 p}{\partial \eta^2} \cdot \frac{\partial^2 \phi}{\partial y^2}.$$

Thus, using Proposition 3.2.11 and noticing as in the proof of Corollary 3.3.4 that since $\frac{\partial \phi}{\partial y} = \zeta(\kappa(y))$, we can differentiate to get $\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \zeta}{\partial y} \cdot \frac{\partial \kappa}{\partial y}$, which is bounded by (3.3.11) and (3.3.13), we find that $\frac{\partial a}{\partial y}$ is bounded.

Proceeding as in the proof of Proposition 3.2.9, we differentiate the equation in y and use Grönwall lemma to deduce that

$$\left| \frac{\partial Y}{\partial y} - I_d \right| \leq \tilde{F} \left(\mathcal{N}_2(\gamma)(h^{-\frac{1}{2}} I_h) + \|V\|_{E(h^{-\frac{1}{2}} I_h)} \right) h^\delta,$$

so that the map $y \mapsto Y(\sigma; y, \eta)$ is a global diffeomorphism with inverse $\mu(\sigma; Y, \eta)$. Now again differentiating the equation and using the Faa-di-Bruno formula, we can prove by induction the following result.

Lemma 3.3.6. *The functions Y and μ both belong to $\dot{S}_\delta^\delta(h^{-\frac{1}{2}} I_h)$.*

Now we see that

$$\frac{d}{d\sigma} [b_j(\sigma, Y(\sigma))] = \left(\frac{\partial b_j}{\partial \sigma} + a \cdot \nabla b_j \right) (\sigma, Y(\sigma)) = -(cb_j)(\sigma, Y(\sigma)) + F_{j-1}(\sigma, Y(\sigma)),$$

so that the unique solution of (3.3.28) and (3.3.29) is

$$(3.3.30) \quad \begin{cases} b_0(\sigma, y, \eta) = \chi(\eta) \exp \left(\int_0^\sigma c(s, Y(s; \mu(\sigma, y, \eta), \eta), \eta) ds \right), \\ b_j(\sigma, y, \eta) = \int_0^\sigma e^{\int_\sigma^s c(s', Y(s'; \mu(\sigma, y, \eta), \eta), \eta) ds'} F_{j-1}(s, Y(s; \mu(\sigma, y, \eta), \eta), \eta) ds. \end{cases}$$

The main result in the construction of the amplitude is the following proposition on the regularity of the b_j s.

Proposition 3.3.7. *The symbols b_j are in $S_\delta^0(h^{-\frac{1}{2}} I_h)$.*

Proof. Step 1. We start by showing that

$$(3.3.31) \quad e^{\int_\sigma^s c(s', Y(s'; \mu(\sigma, y, \eta), \eta), \eta) ds'} \in S_\delta^0(h^{-\frac{1}{2}} I_h).$$

This will be a consequence of the simple remark that if $a \in S_\delta^m$ with $m \geq 0$, then $e^a \in S_\delta^0$. So we need to show that

$$(3.3.32) \quad \int_\sigma^s c(s', Y(s'; \mu(\sigma, y, \eta), \eta), \eta) ds' \in S_\delta^0(h^{-\frac{1}{2}} I_h).$$

First, from Lemma 3.3.6 we now that Y and μ both belong to \dot{S}_δ^δ , and then Proposition 3.2.7 implies that $Y(s'; \mu(\sigma, y, \eta), \eta) \in \dot{S}_\delta^\delta$. Therefore, once again, by Proposition 3.2.7 we only need to prove

$$(3.3.33) \quad \int_\sigma^s c(s', y, \eta) ds' \in S_\delta^0(h^{-\frac{1}{2}} I_h).$$

(Recall (3.3.23) for the definition of c .)

From Corollary 3.3.4 we know that $\frac{\partial \phi}{\partial y} \in \dot{S}_\delta^0$ and $\frac{\partial^2 \phi}{\partial y^2} \in S_\delta^{-\delta}$. On the other hand, one can deduce from Proposition 3.2.11 that

$$\partial_\eta^2 p(y, \eta) \in S_\delta^0, \quad (\partial_{y'} \partial_\eta \tilde{p})(y, y, \eta) \in S_\delta^0.$$

Then, by Proposition 3.2.7 we get

$$(\partial_\eta^2 p)(y, \partial_y \phi(y)) \in S_\delta^0, \quad (\partial_{y'} \partial_\eta \tilde{p})(y, y, \partial_y \phi(y)) \in S_\delta^0$$

and hence

$$(\partial_\eta^2 p)(y, \partial_y \phi(y)) \partial_y^2 \phi(y) \in S_\delta^{-\delta}.$$

Consequently,

$$\int_\sigma^s (\partial_\eta^2 p)(s', y, \partial_y \phi(s', y)) \partial_y^2 \phi(s', y) ds' \in S_\delta^0, \quad \int_\sigma^s (\partial_{y'} \partial_\eta \tilde{p})(s', y, \partial_y \phi(s', y), \eta) ds' \in S_\delta^\delta.$$

With this, we get (3.3.32) and thus (3.3.31).

Step 2. We now need to prove that

$$(3.3.34) \quad \int_0^\sigma F_{j-1}(s, Y(s; \mu(\sigma, y, \eta), \eta)) ds \in S_\delta^0(h^{-\frac{1}{2}} I_h).$$

We write for any $2 \leq |\iota| \leq N-1$,

$$G_{j-1}(\sigma, y, \eta) := h^{|\iota|-1-\nu} D_z^\iota [(\partial_\eta^\iota \tilde{p})(\sigma, y, z, \theta(\sigma, y, z), \eta) b_{k-1}(z)] \Big|_{z=y}.$$

Since $Y \in \dot{S}_\delta^\delta$, by Proposition 3.2.7 we see that to obtain (3.3.34) it suffices to prove that for any $\Lambda = (\alpha, \beta)$ with $|\alpha| + |\beta| = k \geq 0$ there holds

$$(3.3.35) \quad |D^\Lambda G_{j-1}(s, y, \eta)| \leq \tilde{F}_k \left(\mathcal{N}_{k+1}(\gamma)(h^{-\frac{1}{2}} I_h) + \|V\|_{E(h^{-\frac{1}{2}} I_h)} \right) h^{-\delta-|\alpha|\delta}$$

where $D^\Lambda = (D_y^\alpha, D_\eta^\beta)$. Again, the Faa-di-Bruno formula implies that $D^\Lambda G_{j-1}$ is a finite linear combination of terms of the form $K_1 \cdot K_2$ with

$$\begin{aligned} K_1 &= h^{|\iota|-1-\nu} D^{\Lambda_1} \left\{ D_z^{\iota_1} [(\partial_\eta^{\iota_1} \tilde{p})(\sigma, y, z, \theta(y, z), \eta)] \Big|_{z=y} \right\}, \\ K_2 &= D^{\Lambda_2} D_z^{\iota_2} b_{j-1}(\sigma, y, \eta), \end{aligned}$$

where $\Lambda_i = (\alpha_i, \beta_i)$, $|\Lambda_1| + |\Lambda_2| = |\Lambda|$, $|\iota_1| + |\iota_2| = |\iota|$. By induction, there holds

$$|K_2| \leq \tilde{F}_k(\dots) h^{-(|\alpha_2|+|\iota_2|)\delta}.$$

On the other hand, thanks to Proposition 3.2.12, we can deduce without any difficulty that

$$|K_1| \leq \tilde{F}_k(\dots) h^{|\iota|-1-\nu} h^{-(|\alpha_1|+|\iota_1|)\delta}.$$

Consequently, $|K_1 \cdot K_2| \leq \tilde{F}_k(\dots) h^M$ where, since $|\iota| \geq 2$,

$$(3.3.36) \quad \begin{aligned} M &= |\iota| - 1 - \nu - (|\alpha_1| + |\iota_1|)\delta - (|\alpha_2| + |\iota_2|)\delta = |\iota|(1-\delta) - 1 - \nu - |\alpha|\delta \\ &\geq 2(1-\delta) - 1 - \nu - |\alpha|\delta = 1 - \nu - 2\delta - |\alpha|\delta \geq -\delta - |\alpha|\delta. \end{aligned}$$

Therefore, we obtain (3.3.35). \square

Remark. If instead of (3.3.26), one takes b of the usual form $b = \sum h^k b_k$ then a similar computation shows that step 2 of the above proof does not work.

In summary, we have proved that

Proposition 3.3.8. *Let ϕ_h be the solution to the eikonal equation (3.3.3) and $b_j \in S_\delta^0(h^{-\frac{1}{2}}I_h)$ given by the formula (3.3.30). We have*

$$(3.3.37) \quad e^{-ih^{-1}\phi} (h\partial_\sigma + iP) \left(e^{ih^{-1}\phi} \tilde{b} \right) = hT_N\Psi + hb\partial_\sigma\Psi + iU_N + iR_N + iS_N.$$

with

$$(3.3.38) \quad T_N = ih^{(N+1)\nu} \sum_{2 \leq |\alpha| \leq N-1} \frac{h^{|\alpha|-1-\nu}}{\alpha!} D_z^\alpha \left[(\partial_\eta^\alpha \tilde{p})(y, z, \theta(y, z)) b_N(z) \right] \Big|_{z=y},$$

and U_N, R_N, S_N given by (3.3.18), (3.3.19), (3.3.20) respectively.

Define the "error" of the parametrix to the exact solution as

$$(3.3.39) \quad R_h(\sigma, y) := (h\partial_\sigma + iP)(\mathcal{K}v)(\sigma, y).$$

Then using the preceding proposition and our study of the phase ϕ and the amplitude before, we can prove using the stationary phase method as in Proposition 4.31, [8] that $\mathcal{K}v$ defined in (3.3.1) is a good parametrix in the following sense

Proposition 3.3.9. *Take M_0 an integer. Then for any $N \in \mathbf{N}$, there exists a function $\tilde{F}_N : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that*

$$\sup_{0 < \sigma \leq h^\delta} \|R_h(\sigma, \cdot)\|_{H^{M_0}(\mathbf{R}^d)} \leq \tilde{F}_N(\mathcal{N}_k(\gamma) + \|V\|_E) h^N \|v\|_{L^2}.$$

Remark. In Proposition 4.31, [8], $\|v\|_{L^2}$ on the right hand side of the preceding estimate is replaced by $\|v\|_{L^1}$. Let us remark how to get $\|v\|_{L^2}$ as above. According to Lemma 4.33, [8] R_h can be written as

$$R_h(\sigma, y) = \int H_h(\sigma, y, y') v(y') dy'$$

where the kernel H_h satisfies the following property: let $k_0 > d$ be an integer then we have for some $\rho > 0$

$$(3.3.40) \quad \sup_{\sigma \in [0, h^\delta], y \in \mathbf{R}^d, y' \in \mathbf{R}^d, |\eta| \leq C} \langle m(\sigma, y, y', \eta) \rangle^{k_0} |D_y^\beta H_h(\sigma, y, y')| \leq C_{\beta, N} h^{\rho N},$$

with

$$m(\sigma, y, y', \eta) = \partial_\eta \phi(\sigma, y, \eta) - y'.$$

We only need to bound R_h in L^2 , the bounds for $\partial_y^\beta R_h$ in L^2 follow similarly. Now by the Schur test it suffices to prove that

$$\sup_{\sigma \in [0, h^\delta], y' \in \mathbf{R}} \int |H_h(y, y')| dy \leq C_N h^{\rho N}, \quad \sup_{\sigma \in [0, h^\delta], y \in \mathbf{R}} \int |H_h(y, y')| dy' \leq C_N h^{\rho N}$$

In view of (3.3.40) this reduces to

$$\begin{aligned} \sup_{\sigma \in [0, h^\delta], y' \in \mathbf{R}, |\eta| \leq C} \int \langle m(\sigma, y, y', \eta) \rangle^{-k_0} dy &\leq C_N h^{\rho N}, \\ \sup_{\sigma \in [0, h^\delta], y \in \mathbf{R}, |\eta| \leq C} \int \langle m(\sigma, y, y', \eta) \rangle^{-k_0} dy' &\leq C_N h^{\rho N}. \end{aligned}$$

The first inequality was proved in Lemma 4.32, [8]. For the second one, the obvious change of variables $y' \mapsto \tilde{y} := \partial_\xi \phi(\sigma, y, \eta) - y'$ gives the conclusion.

3.4 Strichartz estimates

We first derive Strichartz estimates for the semi-classical equation (3.2.41). If v_h^0 is the initial datum for this equation, recall that the parametrix $\mathcal{K}v_h^0$ is defined by (3.3.1), where ϕ and b were constructed in the preceding section. The kernel of \mathcal{K} is

$$K_h(\sigma, y, y') = (2\pi h)^{-d} \int e^{ih^{-1}(\phi_h(\sigma, y, \eta) - y' \cdot \eta)} \tilde{b}_h(\sigma, y, y', \eta) \chi_1(\eta) d\eta,$$

so that

$$\mathcal{K}v_h^0 = \int K_h(\sigma, y, y') v_h^0(y') dy'.$$

The parametrix \mathcal{K} at time 0 is a good approximation of the initial value, as proved below.

Lemma 3.4.1. *For any integer M_0 greater than $d/2$, we have*

$$(3.4.1) \quad \mathcal{K}v_h^0(0, y) = v_h^0(y) + r_h(y),$$

$$(3.4.2) \quad \|r_h\|_{H^{M_0}(\mathbf{R}^d)} \leq \tilde{F}_N(\dots) h^N \|v_h^0\|_{L^2(\mathbf{R}^d)}, \quad \forall N \in \mathbf{N}.$$

Proof. Keeping in mind the initial conditions imposed on ϕ and b , and the fact that v_h^0 is localized in frequency, we have equation (3.4.1) with

$$r_h(y) = (2\pi h)^{-d} \iint e^{ih^{-1}(y-y') \cdot \eta} \chi(\eta) (1 - \Psi(y - y')) v_h^0(y') dy' d\eta.$$

Now for $|\beta| \leq M_0$, $D_y^\beta r_h(y)$ is a finite linear combination of terms of the form

$$h^{-d-|\beta_1|} \iint e^{ih^{-1}(y-y') \cdot \eta} \eta^{\beta_1} \chi(\eta) \Psi_{\beta_1}(y - y') v_h^0(y') dy' d\eta, \quad |\beta_1| \leq |\beta|,$$

where $|y - y'| \geq 1$ on the support of Ψ_{β_1} . This is a convolution of $v_h^0(y')$ with

$$w_h(Y) := h^{-d-|\beta_1|} \int e^{ih^{-1}Y \cdot \eta} \eta^{\beta_1} \chi(\eta) \Psi_{\beta_1}(Y) d\eta$$

with $|Y| \geq 1$ on the support of Ψ_{β_1} . This is an oscillating integral, and integrating by parts with the vector field

$$L = \frac{1}{|Y|^2} \sum Y_j \partial_{\eta_j}$$

yields

$$w_h(Y) = h^{M-d-|\beta_1|} \int e^{ih^{-1}Y \cdot \eta} (-L)^M (\eta^{\beta_1} \chi(\eta)) \Psi_{\beta_1}(Y) d\eta.$$

Hence, the L^1 norm of w_h is bounded by $\tilde{F}_M(\dots) h^{M-d-|\beta_1|}$ for all $M \in \mathbf{N}$, so that for $|\beta| \leq M_0$,

$$\|D_y^\beta r_h\|_{L^2} \leq \tilde{F}_M(\dots) h^{M-d-|\beta_1|} \|v_h^0\|_{L^2}.$$

This concludes the proof of (3.4.2). □

Now define \mathcal{T}_h the propagator of our (homogeneous) semi-classical equation, i.e.,

$$(3.4.3) \quad \begin{cases} (h\partial_\sigma + iP_h)(\mathcal{T}_h(\sigma, \sigma_0)v_h^0)(y) = 0, \\ (\mathcal{T}_h(\sigma_0, \sigma_0)v_h^0)(y) = v_h^0(y), \end{cases}$$

where $h \in (0, h_0]$ with h_0 small enough, $0 < |\sigma - \sigma_0| \leq h^\delta$, $y \in \mathbf{R}^d$ and $\widehat{v_h^0}$ supported in \mathbf{C} . Then using the Duhamel formula and (3.3.39), (3.4.1) we can write

$$(3.4.4) \quad \mathcal{T}_h(\sigma, \sigma_0)v_h^0 = \mathcal{K}v_h^0(\sigma - \sigma_0) - \mathcal{T}_h(\sigma, \sigma_0)r_h - \int_{\sigma_0}^{\sigma} \mathcal{T}_h(\sigma, s)R_h(s) ds.$$

By classical energy estimates, we have that \mathcal{T}_h is bounded on Sobolev spaces (and notably on L^2), uniformly in time. This, combined with Proposition 3.3.9, (3.4.2) and (3.4.4), gives

$$\sup_{0 \leq \sigma \leq h^\delta} \|\mathcal{K}v_h^0(\sigma)\|_{L^2} \leq \tilde{F}(\dots) \|v_h^0\|_{L^2}.$$

Thus to use the classical TT* argument and prove Strichartz estimates, we only need to prove the following lemma.

Lemma 3.4.2. *There holds for any $0 < \sigma \leq h^\delta$,*

$$\|\mathcal{K}(\sigma)\mathcal{K}^*(\sigma')v_h^0\|_{L^\infty(\mathbf{R}^d)} \leq \tilde{F}(\Xi_k) h^{-\frac{d}{2}} |\sigma - \sigma'|^{-\frac{d}{2}} \|v_h^0\|_{L^1(\mathbf{R}^d)},$$

where \mathcal{K}^* denotes the adjoint of \mathcal{K} .

Proof. Here we follow the proof of Theorem 10.8, [113]. The bound of the Lemma will be implied by an L^∞ bound on the kernel of $\mathcal{K}(\sigma)\mathcal{K}^*(\sigma')$, which is

$$W(\sigma, \sigma', x, z) := \frac{1}{(2\pi h)^{2d}} \iiint e^{\frac{i}{h}(\phi_h(\sigma, x, \eta) - \phi_h(\sigma', z, \zeta) - y \cdot (\eta - \zeta))} B dy d\zeta d\eta$$

where $B \in S_\delta^0$.

In the (y, ζ) variables, ϕ_h is non-degenerate and it is stationary at $\zeta = \eta$, $y = \partial_\zeta \phi_h(\sigma', z, \zeta)$. Thus using stationary phase we get

$$W(\sigma, \sigma', x, z) = \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}(\phi_h(\sigma, x, \eta) - \phi_h(\sigma', z, \eta))} B' dy d\zeta d\eta$$

where again $B' \in S_\delta^0$.

Now the phase of this oscillating integral is

$$\begin{aligned} \tilde{\phi} &:= \phi_h(\sigma, x, \eta) - \phi_h(\sigma', z, \eta) \\ &= (\sigma - \sigma')(p_h(0, x, \eta) + \mathcal{O}(|s| + |s'|)) + \langle x - z, \eta + \sigma' F(\sigma', x, z, \eta) \rangle \end{aligned}$$

where F is in S_δ^0 with seminorms controled by $\tilde{F}(\Xi_k)$, and the constant of the \mathcal{O} is also of this form. The phase is stationary when

$$\partial_\eta \tilde{\phi} = (I + \sigma' \partial_\eta F)(x - z) + (t - s)(\partial_\eta p_h + \mathcal{O}(|\sigma| + |\sigma'|)) = 0,$$

and since for h small, σ' is small and thus $(I + \sigma' \partial_\eta F)$ is invertible, the phase can only be stationary when $x - z = \mathcal{O}(\sigma - \sigma')$. The Hessian is then

$$\partial_\eta^2 \tilde{\phi} = (\sigma - \sigma')(\partial_\eta^2 p_h(0, x, \eta) + \mathcal{O}(|\sigma| + |\sigma'|)).$$

Since $\partial_\eta^2 p_h(0, x, \eta)$ is non-singular, stationary phase gives for $|\sigma - \sigma'| > Ch$ that

$$|W(\sigma, \sigma', x, z)| \leq \tilde{F}(\Xi_k) h^{-\frac{d}{2}} |\sigma - \sigma'|^{-\frac{d}{2}},$$

and for $|\sigma - \sigma'| < Ch$, it is easily seen that

$$|W(\sigma, \sigma', x, z)| \leq \tilde{F}(\Xi_k) h^{-d} \leq \tilde{F}(\Xi_k) h^{-\frac{d}{2}} |\sigma - \sigma'|^{-\frac{d}{2}}$$

also. This concludes the proof of the Lemma. \square

Now the TT* argument (see Theorem 10.7, [113]) can be invoked to prove the Strichartz estimates for the parametrix :

Proposition 3.4.3. *For any $2 \leq p \leq \infty$, $1 \leq q \leq \infty$ such that*

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

there is a nonnegative nondecreasing function \tilde{F} such that for $0 < h \leq h_0$ small enough, there holds

$$\|\mathcal{K}v_h^0\|_{L^p((0, h^\delta); L^q(\mathbf{R}^d))} \leq \tilde{F}(\Xi_k) h^{-\frac{1}{p}} \|v_h^0\|_{L^2(\mathbf{R}^d)}.$$

This, Proposition 3.3.9, (3.4.2) and (3.4.4), the boundedness of \mathcal{T}_h on Sobolev spaces and Sobolev embeddings give the same Strichartz estimates for (3.4.3).

Corollary 3.4.4. *For any $2 < p \leq \infty$, $1 \leq q \leq \infty$ such that*

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

there is a nonnegative nondecreasing function \tilde{F} such that for $0 < h \leq h_0$ small enough, for any $\sigma_0 \in h^{-\frac{1}{2}}I$, there holds

$$\|\mathcal{T}_h v_h^0\|_{L^p((\sigma_0, \sigma_0 + h^\delta); L^q(\mathbf{R}^d))} \leq \tilde{F}(\Xi) h^{-\frac{1}{p}} \|v_h^0\|_{L^2(\mathbf{R}^d)}.$$

Now, recall from (3.2.41) that with $v_h(\sigma, y) = w_h(\sigma, X_h(\sigma, y))$ there holds

$$(\mathcal{L}_h w_h)(\sigma, X_h(\sigma, y)) = (h\partial_\sigma + iP_h) v_h(\sigma, y) = 0.$$

Denoting by $\mathcal{S}_h(\sigma, \sigma_0)$ the flow map of $\mathcal{L}_h w_h(\sigma, x) = 0$ we deduce immediately from Corollary 3.4.4 the following estimates.

Corollary 3.4.5. *Let $\chi \in C_c^\infty(\mathbf{R}^d)$ be supported in $\mathbf{C} = \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. Take $2 < p \leq \infty$, $1 \leq q \leq \infty$ such that*

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

There is a nonnegative nondecreasing function \tilde{F} such that for $0 < h \leq h_0$ small enough, for any $\sigma_0 \in h^{-\frac{1}{2}}I$, there holds with $w_h^0 := \chi(hD_y)w_0$, for any L^2 function w_0 , that

$$\|\mathcal{S}_h w_h^0\|_{L^p((\sigma_0, \sigma_0 + h^\delta); L^q(\mathbf{R}^d))} \leq \tilde{F}(\Xi_k) h^{-\frac{1}{p}} \|w_h^0\|_{L^2(\mathbf{R}^d)}.$$

We are now in position to derive Strichartz estimates for the operator L_j , whose flow map is denoted by \mathcal{S}_j , using the relation (3.2.21):

$$h^{\frac{3}{2}}(L_j u_j)(h^{\frac{1}{2}}\sigma, x) = \mathcal{L}_h(\sigma, x)w_h(\sigma, x), \quad w_h(\sigma, x) = u_j(h^{\frac{1}{2}}\sigma, x) \quad h = 2^{-j}.$$

Theorem 3.4.6. *Let $I_j = [t_0, t_0 + 2^{-j(\delta + \frac{1}{2})}]$. There exist $k \in \mathbf{N}$, and $j_0 \in \mathbf{N}$ such that for any $s \in \mathbf{R}$ and $\varepsilon > 0$ there exist $\tilde{F}, \tilde{F}_\varepsilon : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that if we have*

$$\begin{cases} L_j u_j = 0, \\ u_j(t_0) = u_j^0, \end{cases}$$

where u_j , u_j^0 and $F_{j\delta}$ are supported in the annulus $\mathbf{C}_j = \{\xi : \frac{1}{C}2^j \leq |\xi| \leq C2^j\}$, then there exist $k = k(d)$ and $j_0 \in \mathbf{N}$ such that for $j \geq j_0$, we have

$$\begin{aligned} \|u_j\|_{L^4(I_j, W^{s-\frac{1}{8}, \infty}(\mathbf{R}))} &\leq \tilde{F}(N) \|u_j^0\|_{H^s(\mathbf{R})}, \quad \text{if } d = 1, \\ \|u_j\|_{L^{2+\varepsilon}(I_j, W^{s-\frac{d}{2}+\frac{3}{4}-\varepsilon, \infty}(\mathbf{R}^d))} &\leq \tilde{F}_\varepsilon(N) \|u_j^0\|_{H^s(\mathbf{R}^d)}, \quad \text{if } d = 2. \end{aligned}$$

We now glue together the Strichartz estimates in the preceding proposition to get Strichartz estimates in the full time interval.

Corollary 3.4.7. *Recall that $I = [0, T]$. Put $\varsigma = \frac{1}{2} + \delta$. There exist $k \in \mathbf{N}$, and $j_0 \in \mathbf{N}$ such that for any $s \in \mathbf{R}$ and $\varepsilon > 0$, there exist $\tilde{F}, \tilde{F}_\varepsilon : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that if we have*

$$(3.4.5) \quad \begin{cases} L_j u_j(t, x) = F_{j\delta}(t, x), \\ u_j(t_0, x) = u_j^0(x), \end{cases}$$

where u_j , u_j^0 and $F_{j\delta}$ are supported in the annulus $\mathbf{C}_j = \{\xi : \frac{1}{C}2^j \leq |\xi| \leq C2^j\}$, then there exist $k = k(d)$ and $j_0 \in \mathbf{N}$ such that for $j \geq j_0$, there holds

$$\begin{aligned} \|u_j\|_{L^4(I, W^{s-\frac{1}{2}+\frac{3}{8}-\frac{\varsigma}{4}, \infty}(\mathbf{R}))} &\leq \tilde{F}(\Xi_k) \left(\|F_{j\delta}\|_{L^4(I, H^{s-\varsigma}(\mathbf{R}))} + \|u_j\|_{L^\infty(I, H^s(\mathbf{R}))} \right), \quad \text{if } d = 1, \\ \|u_j\|_{L^2(I, W^{s-\frac{d}{2}+\frac{3}{4}-\frac{\varsigma}{2}-\varepsilon, \infty}(\mathbf{R}^d))} &\leq \tilde{F}_\varepsilon(\Xi_k) \left(\|F_{j\delta}\|_{L^2(I, H^{s-\varsigma}(\mathbf{R}^d))} + \|u_j\|_{L^\infty(I, H^s(\mathbf{R}^d))} \right), \quad d \geq 2. \end{aligned}$$

Proof. Take a cut-off $\chi \in C_c^\infty(0, 2)$ equal to one on $[\frac{1}{2}, \frac{3}{2}]$. Define for $0 \leq m \leq [2^{\varsigma j} T] - 2$ the interval $I_{j,m} := [m2^{\varsigma j}, (m+2)2^{\varsigma j}]$, and the associated cut-off $\chi_{j,m}(t) := \chi\left(\frac{t-m2^{\varsigma j}}{2^{\varsigma j}}\right)$. We have

$$L_j(\chi_{j,m} u_j) = \chi_{j,m} F_{j\delta} + 2^{\varsigma j} \chi' \left(\frac{tm2^{\varsigma j}}{2^{\varsigma j}} \right) u_j,$$

with $\chi_{j,m} u_j(k2^{\varsigma j}) = 0$. Then applying Theorem 3.4.6 to $\chi_{j,m} u_j$ with the help of the Duhamel formula, noticing that the flow maps $\mathcal{S}(t, \tau)$ are bounded on Sobolev spaces and $\chi_{j,m} = 1$ on $((m+\frac{1}{2})2^{\varsigma j}, (m+\frac{3}{2})2^{\varsigma j})$, we find for $d \geq 2$

$$\begin{aligned} \|u_j\|_{L^2(((m+\frac{1}{2})2^{\varsigma j}, (m+\frac{3}{2})2^{\varsigma j}), W^{s-\frac{d}{2}+\frac{3}{4}-\varepsilon, \infty})} &\leq \tilde{F}(\Xi_k) \left(\|F_{j\delta}\|_{L^1((m2^{\varsigma j}, (m+2)2^{\varsigma j}), H^s)} + 2^{\varsigma j} \left\| \chi' \left(\frac{t-m2^{\varsigma j}}{2^{\varsigma j}} \right) u_j \right\|_{L^1(I, H^s)} \right) \\ &\leq \tilde{F}(\Xi_k) \left(2^{-\varsigma j/2} \|F_{j\delta}\|_{L^2((m2^{\varsigma j}, (m+2)2^{\varsigma j}), H^s)} + \|u_j\|_{L^\infty(I, H^s)} \right). \end{aligned}$$

Then we multiply both sides by $2^{-\varsigma j/2}$ and use the fact that u_j and $F_{j\delta}$ are supported in annulus to find

$$\begin{aligned} \|u_j\|_{L^2(((m+\frac{1}{2})2^{\varsigma j}, (m+\frac{3}{2})2^{\varsigma j}), W^{s-\frac{d}{2}+\frac{3}{4}-\varepsilon, \infty})} &\leq \tilde{F}(\Xi_k) \left(\|F_{j\delta}\|_{L^2((m2^{\varsigma j}, (m+2)2^{\varsigma j}), H^{s-\varsigma})} + 2^{\varsigma j/2} \|u_j\|_{L^\infty(I, H^s)} \right). \end{aligned}$$

At last, elevating at the power 2 and summing back the pieces, and adding the control of the first and last pieces using Theorem 3.4.6, we find the result as claimed.

The case $d = 1$ follows along the same lines. \square

The next step is to derive Strichartz estimates for the non-regularized equation. For these estimates, one need the following higher order semi-norm of γ

$$(3.4.6) \quad \mathcal{M}_k(\gamma)(J) := \sum_{|\beta| \leq k} \sup_{\xi \in \mathbf{C}'} \left\| D_\xi^\beta \gamma \right\|_{L^p(J; W_x^{\frac{3}{2}, \infty}(\mathbf{R}^d))},$$

and we put

$$\tilde{\Xi} = \mathcal{M}_k(\gamma)(J) + \mathcal{N}_k(\gamma)(J) + \mathcal{N}_k(\omega)(J) + \|V\|_{E(J)}.$$

Corollary 3.4.8. *There exists $k \in \mathbf{N}$, and $j_0 \in \mathbf{N}$ such that for any $s \in \mathbf{R}$ and $\varepsilon > 0$ there exist $\tilde{F}, \tilde{F}_\varepsilon : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that if we have*

$$(3.4.7) \quad \begin{cases} (\partial_t + iT_\gamma + T_V \cdot \nabla) u_j = F_j, \\ u_j(0) = u_j^0, \end{cases}$$

where u_j, u_j^0 and F_j are supported in the annulus $\mathbf{C}_j = \{\xi : \frac{1}{C}2^j \leq |\xi| \leq C2^j\}$, then there exist $k = k(d)$ and $j_0 \in \mathbf{N}$ such that for $j \geq j_0$, we have

— if $d = 1$,

$$\|u_j\|_{L^4(I, W^{s-\frac{1}{2}+\frac{3}{20}, \infty}(\mathbf{R}))} \leq \tilde{F}(\tilde{\Xi}_k) \left(\|F_j\|_{L^4(I, H^{s-\frac{9}{10}}(\mathbf{R}))} + \|u_j\|_{L^\infty(I, H^s(\mathbf{R}))} \right),$$

— if $d \geq 2$,

$$\|u_j\|_{L^2(I, W^{s-\frac{d}{2}+\frac{3}{10}-\varepsilon, \infty}(\mathbf{R}^d))} \leq \tilde{F}_\varepsilon(\tilde{\Xi}_k) \left(\|F_j\|_{L^2(I, H^{s-\frac{9}{10}}(\mathbf{R}^d))} + \|u_j\|_{L^\infty(I, H^s(\mathbf{R}^d))} \right),$$

for $j \geq j_0$.

Proof. By (3.2.14), (3.2.15) and (3.2.16) we have that if u_j is a solution of (3.4.7) then u_j is also a solution of (3.4.5) with

$$\begin{aligned} F_{j\delta} &= F_j + R_j + iR'_j + (S_{(j-3)\delta}\gamma - S_{j-3}\gamma) \Delta_j u + (S_{(j-3)\delta}(V) - S_{j-3}(V)) \cdot \nabla \Delta_j u \\ &=: F_j + \tilde{F}_j. \end{aligned}$$

From Lemma 3.2.3 we have that R_j and R'_j are of order 0. On the other hand, with $p = 4$ if $d = 1$ and $p = 2$ if $d \geq 2$, there holds

$$\begin{aligned} \|(S_{(j-3)\delta}(V) - S_{j-3}(V)) \cdot \nabla u_j\|_{L^p(I, H^{s-(1-\delta)}(\mathbf{R}^d))} &\leq \|V\|_{E(J)} \|u_j\|_{L^\infty(I, H^s(\mathbf{R}^d))}, \\ \|(S_{(j-3)\delta}\gamma - S_{j-3}\gamma) u_j\|_{L^p(I, H^{s-\frac{3}{2}(1-\delta)}(\mathbf{R}^d))} &\leq \mathcal{M}_k(\gamma)(J) \|u_j\|_{L^\infty(I, H^s(\mathbf{R}^d))}. \end{aligned}$$

Those are classical regularization results (See e.g. [100], Section 1.3).

We deduce that

$$\|\tilde{F}_j\|_{L^p(I, H^{s-\frac{3}{2}(1-\delta)}(\mathbf{R}^d))} \leq \tilde{F}(\tilde{\Xi}_k) \|u_j\|_{L^\infty(I, H^s(\mathbf{R}^d))}.$$

Now, choosing $\delta = \frac{2}{5}$ gives $\varsigma = \frac{3}{2}(1-\delta)$ and then Corollary 3.4.7 concludes the proof. \square

Finally, we prove our main theorem.

Proof of Theorem 3.1.1

Let u be as in the statement of the theorem, by (3.2.3) u is a solution of

$$\partial_t u + iT_\gamma u + T_V \cdot \nabla u = f - iT_\omega u.$$

Then, by (3.2.9) $\Delta_j u$ solves

$$(\partial_t + iT_\gamma + T_V \cdot \nabla) \Delta_j u = F_j,$$

with

$$F_j := \Delta_j f - i\Delta_j(T_\omega u) + i[T_\gamma, \Delta_j]u + [T_V, \Delta_j] \cdot \nabla u.$$

Notice that F_j has spectrum in \mathbf{C}_j . Applying the symbolic calculus Theorem A.0.4 we deduce that

$$\begin{aligned} \|\Delta_j(T_{\omega_1} u)\|_{L^p(I; H^{s-\frac{1}{2}})} &\leq C\mathcal{N}_k(\omega_1) \|u\|_{L^\infty(I; H^s)} \\ \|[T_\gamma, \Delta_j]u\|_{L^p(I; H^{s-\frac{1}{2}})} &\leq C\mathcal{N}_k(\gamma) \|u\|_{L^\infty(I; H^s)}, \\ \|[T_V, \Delta_j] \cdot \nabla u\|_{L^p(I; H^s)} &\leq C\|V\|_E \|u\|_{L^\infty(I; H^s)}. \end{aligned}$$

Then we can use Corollary 3.4.8 on $\Delta_j u$ to prove

$$\|\Delta_j u\|_{L^p(I, W^{s-\frac{d}{2}+\mu, \infty}(\mathbf{R}^d))} \leq \tilde{F}(\mathcal{N}_k(\gamma) + \|V\|_E) \left(\|f\|_{L^p(I, H^{s-\frac{9}{10}}(\mathbf{R}^d))} + \|u\|_{L^\infty(I, H^s(\mathbf{R}^d))} \right)$$

for $j \geq j_0$, and using the bound

$$\|\Delta_j u\|_{L^p(I, W^{s-\frac{d}{2}+\mu, \infty}(\mathbf{R}^d))} \leq C2^{j\mu} \|u\|_{L^\infty(I, H^s(\mathbf{R}^d))} \leq C\|u\|_{L^\infty(I, H^s(\mathbf{R}^d))}$$

for $j < j_0$, we finally obtain

$$\|u\|_{L^p(I, W^{s-\frac{d}{2}+\mu-\varepsilon, \infty}(\mathbf{R}^d))} \leq \sum_j 2^{-j\varepsilon} \|\Delta_j u\|_{L^p(I, W^{s-\frac{d}{2}+\mu, \infty}(\mathbf{R}^d))}$$

which is bounded by the desired quantity.

3.5 Cauchy problem

We are now in position to derive the Cauchy theory announced in Theorem 3.1.6. Let

$$(\eta_0, \psi_0) \in H^{s+\frac{1}{2}} \times H^s, \quad s > 2 - \frac{d}{2} + \mu$$

be the initial data such that $\text{dist}(\eta_0, \Gamma) > h > 0$. We regularize (η_0, ψ_0) to a sequence $(\eta_0^\varepsilon, \psi_0^\varepsilon) \in H^\infty \times H^\infty$ converging to (η_0, ψ_0) in $H^{s+\frac{1}{2}} \times H^s$. Then we can choose a uniform $h_0 > 0$ such that $\text{dist}(\eta_0^\varepsilon, \Gamma) > h_0$. For each initial condition $(\eta_0^\varepsilon, \psi_0^\varepsilon)$ we know from the local well-posedness theory in [3] that there exists a smooth solution $(\eta^\varepsilon, \psi^\varepsilon)$ to (3.1.2) with the maximal life time interval $[0, T_\varepsilon^*)$. Applying our a priori estimate of Proposition 3.1.3 and the Strichartz estimate of Corollary 3.1.2 give for each $\varepsilon > 0$

$$M_{s,T}^\varepsilon + Z_{r,T}^\varepsilon \leq \tilde{F}_{h_0} \left(M_{s,0}^\varepsilon + T^\delta \tilde{F} (M_{s,T}^\varepsilon + Z_{r,T}^\varepsilon) \right), \quad \forall T \in [0, T_\varepsilon^*), \quad \forall \varepsilon > 0.$$

with obvious notations. Combining this estimate with the blow-up criterion in Proposition 2.4.8 one deduces by standard argument that there exists a time $T > 0$ uniformly in $\varepsilon > 0$ such that $T_\varepsilon^* > T$. Set $I = [0, T]$. By virtue of Proposition 2.5.9, the sequence $(\eta^\varepsilon, \psi^\varepsilon)$ is Cauchy in

$$X^{s-\frac{3}{2}, r-\frac{3}{2}} := C^0(I; H^{s-1}(\mathbf{R}^d) \times H^{s-\frac{3}{2}}(\mathbf{R}^d)) \cap L^p(I; W^{r-\frac{1}{2}}(\mathbf{R}^d) \times W^{r-1, \infty}(\mathbf{R}^d))$$

and therefore converges strongly to some (η, ψ) in $X^{s,r}$. On the other hand, this sequence is bounded in

$$Y^{r,s} := (\eta, \psi) \in L^\infty(I; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)) \cap L^p(I; W^{r+\frac{1}{2}}(\mathbf{R}^d) \times W^{r,\infty}(\mathbf{R}^d)).$$

Therefore, it converges strongly to (η, ψ) in $X^{s',r'}$ for any $s' < s$, $r' < r$ and weakly to (η, ψ) in $Y^{r,s}$. Consequently, one can pass to the limit in the system (3.1.2) as $\varepsilon \rightarrow 0$ to have that (η, ψ) is a distributional solution to (3.1.2). Here, we remark that the only nontrivial point is to pass to the limit in the Dirichlet-Neumann operator $G(\eta_\varepsilon)\psi_\varepsilon$, this is done for example in [8], Corollary 5.16. Finally, by interpolation it holds that

$$(\eta, \psi) \in C^0(I; H^{s'+\frac{1}{2}}(\mathbf{R}^d) \times H^{s'}(\mathbf{R}^d)), \quad \forall s' < s,$$

which completes the proof of Theorem 3.1.6.

Troisième partie
Fond émergent

Capillarité dans un canal

Cette note est inédite.

4.1 Introduction

The water waves problem studies the motion of a fluid in a container, separated from the atmosphere by a free-moving interface, under the action of gravity and surface tension. Here we consider the cases of an infinite canal and of a rectangular basin, with vertical walls.

More precisely, the fluid occupies a time-dependent region Ω_t situated below a moving surface Σ_t . Thus the vertical — up-pointing — direction e_y is distinguished from the horizontal directions e_{x_1}, e_{x_2} . To simplify matters we assume that Σ_t is the graph of a function $\eta(t, x_1, x_2)$ — this means that we forbid configurations where the fluid overhangs itself, like rolls or breakers. Thus for $t \in [0, T]$,

$$\Omega_t := \{(x_1, x_2, y) \in M \times \mathbf{R}; b(x) < y < \eta(t, x), x := (x_1, x_2)\},$$

where $M = (0, l) \times \mathbf{R}$ in the case of an infinite canal or $M = (0, l) \times (0, L)$ in the case of a rectangular basin, and where b is a real continuous function on M , describing the topography of the bottom. The function η is real, continuous, and defined on $[0, T] \times M$. We define by Γ_t the portion of the fixed boundary that is underwater,

$$\Gamma_t := \partial\Omega_t \setminus \Sigma_t.$$

An important assumption on the domain is that the surface only meets the container along the vertical walls. Mathematically this means that for $t \in [0, T]$,

$$(B) \quad \exists h_t > 0, b(x) < \eta(t, x) - h_t, \forall x \in M.$$

The fluid is assumed to be perfect, incompressible, non-viscous, and of constant density and temperature. Thus its velocity field $u(t, x_1, x_2, y) \in \mathbf{R}^3$, defined for $t \in [0, T], (x_1, x_2, y) \in \overline{\Omega}_t$, follows the incompressible Euler equation

$$\begin{cases} \partial_t u + u \cdot \nabla_{x,y} u + \nabla_{x,y} P = -ge_y, \\ \nabla_{x,y} \cdot u = 0, \end{cases}$$

where $x := (x_1, x_2)$ stands for the horizontal variables, $P(t, x, y) \in \mathbf{R}$ is the pressure, and $g \geq 0$ is the acceleration of gravity, supposed uniform and constant. To simplify the study and focus on the dynamics of the surface, it is customary to impose in addition the condition

$$\operatorname{curl}_{x,y} v = 0,$$

which is conserved by the flow.

The free surface is assumed to move with the fluid velocity, thus

$$\partial_t \eta(t, x) = \nu(t, x) \cdot u(t, x, \eta(t, x)) = u_y(t, x, \eta(t, x)) - \nabla \eta(t, x) \cdot u_x(t, x, \eta(t, x))$$

where $\nu = (-\nabla \eta, 1)$ is the exterior normal to Σ_t , and $u := (u_x, u_y)$, with $u_x := (u_{x_1}, u_{x_2})$. The velocity field value at the boundary Γ_t of the container needs to satisfy the no-penetration conditions

$$u \cdot n = 0$$

where n is the normal to Γ_t . This and the equations prescribe the normal derivative of the pressure at Γ_t , thus we need only to prescribe the pressure at the free surface Σ_t . The presence of surface tension corresponds to a jump of the pressure across the interface, proportional to its mean curvature

$$H(\eta) := \operatorname{div} \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right).$$

Assuming the atmospheric pressure to be a constant — which can be normalized to 0 since the equation depends only on its gradient — this means that

$$P(t, x, \eta(t, x)) = -\kappa H(\eta)(t, x),$$

where $\kappa \geq 0$ is the surface tension coefficient.

Under the incompressibility and irrotationality conditions, the velocity of the fluid is the gradient of a harmonic function $\phi : \Omega_t \rightarrow \mathbf{R}$. Thus

$$v = \nabla_{x,y} \phi,$$

with

$$(4.1.1) \quad \Delta_{x,y} \phi = 0 \quad \text{in } \Omega_t.$$

The boundary conditions on ϕ translate as

$$(4.1.2) \quad \partial_n \phi = 0 \quad \text{on } \Gamma_t.$$

Thus if we know the value of ϕ at the free surface Σ_t , we can solve the Laplace problem (4.1.1) in Ω_t with this Dirichlet datum at Σ_t and Neumann condition at the rest of the boundary $\partial\Omega_t$. Thus we only need to know the evolution of the function

$$\psi(t, x) := \phi(t, x, \eta(t, x)).$$

From the Euler equation and the Pressure value at the surface, we see that

$$\partial_t \phi = -g\eta + \kappa H(\eta) - \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} (\partial_y \phi)^2 \quad \text{on } \Sigma_t,$$

while the dynamic boundary condition becomes

$$\partial_t \eta = \partial_y \phi - \nabla \eta \cdot \nabla \phi \quad \text{on } \Sigma_t.$$

Thus, by introducing the (rescaled) Dirichlet-Neumann map

$$(G(\eta)\psi)(t, x) = \partial_y \phi(t, x, \eta(t, x)) - \nabla \eta(t, x) \cdot \nabla \phi(t, x, \eta(t, x)) = \sqrt{1 + |\nabla \eta|^2} \partial_n \phi|_{y=\eta(t, x)},$$

we obtain the system

$$(4.1.3) \quad \begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta - \kappa H(\eta) + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = 0, \end{cases}$$

which is due to Zakharov ([111]) and Craig-Sulem ([35]). Here the advantage is that η and ψ are both real-valued functions of $(t, x) \in [0, T] \times M$.

In this case, the surface Σ_t and the container Γ intersect, and the exact nature of the boundary conditions to be imposed at this intersection are unknown. The Cauchy problem has been extensively studied in the case where Σ_t and Γ are clearly separated, corresponding to a laterally infinite ocean with a fixed separation between the bottom and the free surface — see for instance Nalimov ([83]), Shinbrot ([93]), Yosihara ([109, 110]), Craig ([34]), Beyer and Günther ([19]), and Wu ([104, 105]), and the works of Alazard, Burq, and Zuily ([3, 7, 8, 4]). However, if this separation condition is removed, almost no such results exist. One exception, when surface tension is absent, is in the case described above of an infinite canal or a rectangular basin with vertical walls. In this setting, local existence has been proved in [9] using the observation that the angle between the free surface and the walls is necessarily a right angle, and simple reflection and periodisation tricks to deduce local existence from results valid on the Torus.

In this article, we prove a similar result in presence of surface tension, thus $\kappa \neq 0$, and we normalize it to 1. Here, there is no physical obligation for the surface to meet the boundary at a right angle, however we will see that if such is the case initially, it remains true for a short time. To encode this information, we will introduce for $s > 1 + \frac{d}{2}$ the space

$$H_p^s(M) := \{u \in H^s(M); \partial_\nu u = 0 \text{ on } \partial M\},$$

where ν is the normal to ∂M . In the case of the rectangular basin, the normal is not defined at the corners, but this means that both $\partial_{x_1} u$ and $\partial_{x_2} u$ are 0 there. Since $s > 1 + \frac{d}{2}$, $H^s(M) \subset C^1(\overline{M})$ so that the definition makes sense. This corresponds to Sobolev functions that meet the walls of the container at right angles. Our theorem is as follows.

Theorem 4.1.1. *Take $s \in (3 - \frac{3}{10}, 3)$, $2 < r < s + \frac{3}{10} - 1$, and $M = (0, l) \times \mathbf{R}$ or $M = (0, l) \times (0, L)$. Consider initial data*

$$(\eta_0, \psi_0) \in H_p^{s+\frac{1}{2}}(M) \times H_p^s(M),$$

satisfying Assumption (1.1) at initial time $t = 0$. Then there exists a time $T > 0$ depending only on the norms of those initial data and the h_0 in Assumption (1.1) such that the Cauchy Problem of (4.1.3) admits a unique solution

$$(\eta, \psi) \in C\left([0, T]; H_p^{s+\frac{1}{2}}(M) \times H_p^s(M)\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(\overline{M}) \times C^r(\overline{M})\right),$$

satisfying Assumption (1.1) for each time $t \in [0, T]$.

- Remark.*
1. The condition $s < 3$ is a necessary limitation to the method. Even starting from smooth initial data, the symmetrization trick will transform η to an $H^{s+\frac{1}{2}}$ function of the Torus, with $s < 3$, so that the solution we construct will only have this regularity. See Section 4.2 for details.
 2. To solve the Cauchy problem at this low regularity, the energy estimates are insufficient, and one need to use Strichartz estimates for this equation on the Torus. Nguyen and the author have proved such Strichartz estimates and used them to solve the Cauchy Problem, in the case of the whole space, in [86, 87]. We explain in Section 4.3 how this adapt to the Torus.
 3. The use of such Strichartz estimates explains why the uniqueness is only known to holds in the space

$$C\left([0, T]; H_p^{s+\frac{1}{2}}(M) \times H_p^s(M)\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(\overline{M}) \times C^r(\overline{M})\right).$$

4. In [9], Alazard, Burq and Zuily proved the corresponding result in the absence of surface tension, within the class of uniformly local Sobolev spaces. In the case of the infinite canal, this means that no decay of the data at infinity is required. However in the presence of surface tension, the propagation speed is infinite, and such a result is unlikely to hold.

4.2 Reduction to the Torus

The symmetrization procedure we follow is due to Boussinesq (see [24], p.37) and has been used by Alazard, Burq and Zuily in [9] for the pure gravity case. This procedure can be describe as follows.

Take a smooth function u in \overline{M} , where we take $M = (0, l) \times \mathbf{R}$ to simplify. We can symmetrize this function with respect to the line $x_1 = 0$, giving us a function on $(-l, l) \times \mathbf{R}$, with the same values at $x_1 = -l$ and $x_1 = l$. Thus this function can be thought as periodic in x_1 , and thus as an even function on $\mathbf{T}_{2l}^1 \times \mathbf{R}$, where $\mathbf{T}_{2l}^1 := \mathbf{R}/2l\mathbf{Z}$ is the flat torus of period $2l$. For the case of $M = (0, l) \times (0, L)$, one performs successive symmetrizations along the line $x_1 = 0$ and $x_2 = 0$ (the order of these operations does not matter, obviously.) Then one can lift the function to the flat inhomogeneous Torus $\mathbf{T}_{2l, 2L}^2 := (\mathbf{R}/2l\mathbf{Z}) \times (\mathbf{R}/2L\mathbf{Z})$.

Now the problem with this symmetrization procedure is that even if the original function u is smooth, its reflexion with respect to $x_1 = 0$ is not guaranteed to be smooth. In fact, taking the example of a linear, non-constant function shows that a Lipschitz singularity appears in general. However, in the case of functions satisfying $\partial_{x_1} u = 0$, the singularity is of higher order. The exact mapping properties of this reflexion, in terms of Sobolev spaces, are summarized in the following one-dimensional Proposition from [9].

For a smooth compactly supported function v on $[0, +\infty)$, define its extension v^{ev} to \mathbf{R} by

$$(4.2.1) \quad v^{\text{ev}}(y) := \begin{cases} v(y) & \text{if } y \geq 0 \\ v(-y) & \text{if } y < 0. \end{cases}$$

This map obviously lift to distributions, and thus to Sobolev spaces.

Proposition 4.2.1. (*Proposition 6.5 of [9].*)

1. Assume that $0 \leq s < \frac{3}{2}$. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $H^s(0, +\infty)$ to $H^s(\mathbf{R})$.
2. Assume that $\frac{3}{2} \leq s < \frac{7}{2}$. Then the map $v \mapsto v^{\text{ev}}$ is continuous from the space $\{v \in H^s(0, +\infty), v'(0) = 0\}$ to $H^s(\mathbf{R})$.

Then we can define the symmetrized-periodized extension of a function $u \in C_0^\infty(\overline{M})$ as follows. For $M = (0, l) \times \mathbf{R}$,

$$(4.2.2) \quad u^\sharp(x_1, x_2) := \begin{cases} u(x_1, x_2) & \text{if } 0 \leq x_1 \leq l \\ u(-x_1, x_2) & \text{if } -l \leq x_1 < 0 \\ u(x_1 - 2kl, x_2) & \text{if } -l + 2kl \leq x_1 \leq l + 2kl, k \in \mathbf{Z}. \end{cases}$$

This again lift to Sobolev spaces, and since the periodization procedure preserves regularity we have the following Corollary.

Corollary 4.2.2. *Let $M = (0, l) \times \mathbf{R}$.*

1. *Assume that $0 \leq s < \frac{3}{2}$. Then the map $u \mapsto u^\sharp$ is continuous from $H^s(M)$ to $H^s(\mathbf{T}_{2l}^1 \times \mathbf{R})$.*
2. *Assume that $\frac{3}{2} \leq s < \frac{7}{2}$. Then the map $u \mapsto u^\sharp$ is continuous from $H_p^s(M)$ to $H^s(\mathbf{T}_{2l}^1 \times \mathbf{R})$.*

Thus the regularity $s \in (3 - 3/10, 3)$ of Theorem 4.1.1 is sufficient for both $\eta_0 \in H_c^{s+\frac{1}{2}}(M)$ and $\psi_0 \in H_c^s(M)$ to keep their regularity by this procedure. This explains the higher limit $s < 3$.

In the case $M = (0, l) \times (0, L)$ we pose

$$(4.2.3) \quad u^\sharp(x_1, x_2) := \begin{cases} u(x_1, x_2) & \text{if } 0 \leq x_1 \leq l, 0 \leq x_2 \leq L \\ u(-x_1, x_2) & \text{if } -l \leq x_1 < 0, 0 \leq x_2 \leq L \\ u(x_1, -x_2) & \text{if } 0 \leq x_1 \leq l, -L \leq x_2 < 0 \\ u(-x_1, -x_2) & \text{if } -l \leq x_1 < 0, -L \leq x_2 < 0 \\ u(x_1 - 2kl, x_2 - 2KL) & \text{if } -l + 2kl \leq x_1 \leq l + 2kl, \\ & -L + 2KL \leq x_2 \leq L + 2KL, k, K \in \mathbf{Z}. \end{cases}$$

We again have the following Corollary.

Corollary 4.2.3. *Let $M = (0, l) \times (0, L)$.*

1. *Assume that $0 \leq s < \frac{3}{2}$. Then the map $u \mapsto u^\sharp$ is continuous from $H^s(M)$ to $H^s(\mathbf{T}_{2l, 2L}^2)$.*
2. *Assume that $\frac{3}{2} \leq s < \frac{7}{2}$. Then the map $u \mapsto u^\sharp$ is continuous from $H_p^s(M)$ to $H^s(\mathbf{T}_{2l, 2L}^2)$.*

Now, we have reduced our initial data to Sobolev functions on the flat Torus (or the flat cylinder). In Section 4.3, we will show how to solve the Cauchy problem for those initial data. Suppose we have done so, obtaining functions

$$(\tilde{\eta}, \tilde{\psi}) \in C\left([0, T]; H^{s+\frac{1}{2}}(N) \times H^s(N)\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(N) \times C^r(N)\right)$$

Here and in what follows,

$$(4.2.4) \quad \begin{aligned} N &:= \mathbf{T}_{2l}^1 \times \mathbf{R} & \text{if } M = (0, l) \times \mathbf{R}, \\ N &:= \mathbf{T}_{2l, 2L}^2 & \text{if } M = (0, l) \times (0, L). \end{aligned}$$

We now have to show that the restriction of those functions to the original base M solves the original Cauchy Problem. We start by showing that it is in the good functional space.

Lemma 4.2.4. *Take (η, ψ) as above, the restriction to M of the unique solution to the Cauchy problem with initial data $(\eta_0^\sharp, \psi_0^\sharp)$. Then*

$$(\eta, \psi) \in C\left([0, T]; H_p^{s+\frac{1}{2}}(M) \times H^s(M)_p\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(\overline{M}) \times C^r(\overline{M})\right),$$

and $(\tilde{\eta}(t), \tilde{\psi}(t)) = (\eta^\sharp(t), \psi^\sharp(t))$ for all $t \in [0, T]$.

Proof. The fact that

$$(\eta, \psi) \in C\left([0, T]; H^{s+\frac{1}{2}}(M) \times H^s(M)\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(\overline{M}) \times C^r(\overline{M})\right)$$

is evident, but we still need to prove the right angle conditions.

Take the case $M = (0, l) \times \mathbf{R}$. The water waves equation (4.1.3) is left invariant by the translation

$$x_1 \mapsto x_1 + 2kl, \quad l \in \mathbf{Z},$$

and by the reflection

$$x_1 \mapsto -x_1.$$

Thus, since the initial data $(\eta_0^\sharp, \psi_0^\sharp)$ is also invariant by those transformations, the solution keeps this invariance property, as a consequence of uniqueness. Now this obviously entails this right angle result.

The case $M = (0, l) \times (0, L)$ follows along the same lines. \square

Now the above argument obviously applies also to the harmonic potential ϕ in the periodized fluid domain, so that $\partial_{x_1}\phi(0, x_2, y) = 0$ and the same on the other boundaries, and $\tilde{\phi} = \sharp\phi$. As a consequence, at a point $(t, x) \in [0, T] \times M$,

$$G(\eta)\psi(t, x) = G(\tilde{\eta})\tilde{\psi}(t, x).$$

Thus the functions (η, ψ) solve the original Cauchy problem, uniqueness being insured by the requirement that $(\eta^\sharp, \psi^\sharp) = (\tilde{\eta}, \tilde{\psi})$, for which uniqueness is already known. Thus the proof of Theorem 4.1.1 is reduced to solving the Cauchy Problem in the periodized space N .

4.3 The Cauchy problem in the periodized space

In the new periodized setting, the bottom is parametrized by the function b^\sharp which, since it does not satisfy any right angle condition, has only limited regularity. It however stays continuous, and this will be all that we need, taking into account (1.1), which becomes

$$(B') \quad \exists h_t > 0, b(x) < \eta(t, x) - h_t, \forall x \in N.$$

The following Proposition allows one to solve the water waves in N .

Proposition 4.3.1. *Take $N = \mathbf{T}_{2l}^1 \times \mathbf{R}$ or $N = \mathbf{T}_{2l, 2L}^2$. Suppose $s > 3 - \frac{3}{10}$ and $2 < r < s + \frac{3}{10} - 1$. Then if*

$$\left(\tilde{\eta}_0, \tilde{\psi}_0\right) \in H^{s+\frac{1}{2}}(N) \times H^s(N)$$

satisfy condition B' , there exists a time $T > 0$ depending only on the norms of those initial data and the h_0 in Assumption (B') such that the Cauchy Problem of (4.1.3) admits a unique solution

$$(\tilde{\eta}, \tilde{\psi}) \in C\left([0, T]; H^{s+\frac{1}{2}}(N) \times H^s(N)\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(N) \times C^r(N)\right),$$

satisfying Assumption (B') for each time $t \in [0, T]$.

The corresponding result was proved in [86] and [87] by Nguyen and the author, in the case of the whole space in arbitrary dimension. Most of the proof is a straightforward adaptation of those results, as we explain below. The only apparent difficulty is that the compactness properties of N would seem to precludes the use of dispersion, and thus of Strichartz estimates for such

an equation with infinite speed of propagation. However, as explained below, the fact that our parametrix is constructed on semiclassical times is sufficient to recover the estimates, as in the strategy of Burq, Gerard and Tzvetkov for Shrödinger equation on compact manifolds ([25]).

The main tool used in the proof is Bony's paradifferential calculus, which is well-known to work indistinctly in euclidean space using Littlewood-Paley decomposition, or in the Torus using Fourier series. We thus can easily extend it to work on N in both cases. The first step of the proof is a paradifferential reduction of the equations, which rest on an analysis of elliptic regularity of the Dirichlet-Neumann operator, which stays identical in the periodic setting, and on paraproduct estimates. Thus, dropping the tildes in the variables for convenience, the analysis in [86] shows that the equation is equivalent to the paradifferential equation

$$(4.3.1) \quad \partial_t u + T_V \cdot \nabla u + iT_\gamma u = f$$

for the complex-valued unknown $u := T_p \eta + iT_q(\psi - T_B \eta)$. Here,

$$B := \frac{\nabla \eta \cdot \nabla \psi + G(\eta)\psi}{1 + |\nabla \eta|^2}, \quad V := \nabla \psi - B \nabla \eta$$

are the vertical and horizontal trace of the velocity at the surface and are thus controlled by the unknowns η and ψ , the symbols p and q are elliptic, respectively of order $1/2$ and 0 , and smooth functions of $\nabla \eta$. The symbol γ is again elliptic, of order $3/2$, and a smooth function of $\nabla \eta$. The remainder f satisfies

$$\|f\|_{H^s} \leq \mathcal{F} \left(\|\eta\|_{H^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \left[1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|\psi\|_{W^{r,\infty}} \right].$$

Now denoting by $M_s(T)$ the supremum in time of the Sobolev norms of the solution, and by $Z_r(T)$ the L^2 norm in time of their Hölder norms, we can use classical energy estimates to prove that

$$M_s(T) \leq \mathcal{F} \left(M_s(0) + T^{\frac{1}{2}} \mathcal{F} (M_s(T) + Z_r(T)) \right).$$

Then, if we can prove that $Z_r(T)$ is bounded by the same quantity, proving contraction estimates in a lower order norm and using a quasilinear scheme of convergence as in [86] will complete the proof.

We thus only have to prove that

$$Z_r(T) \leq \mathcal{F} \left(M_s(0) + T^{\frac{1}{2}} \mathcal{F} (M_s(T) + Z_r(T)) \right),$$

which are the so-called Strichartz estimates (with loss).

4.4 Strichartz estimates

The main Proposition in this section is the existence of a-priori Strichartz estimates (with loss of derivatives) in the periodic setting.

Proposition 4.4.1. *Take $N = \mathcal{T}_{2l}^1 \times \mathbf{R}$ or $N = \mathbf{T}_{2l,2L}^2$. Suppose $s > 3 - 3/10$ and $2 < r < s + 3/10 - 1$. Then there is a positive non-decreasing function \mathcal{F} such that if*

$$(\tilde{\eta}, \tilde{\psi}) \in C^0 \left([0, T]; H^{s+\frac{1}{2}}(N) \times H^s(N) \right)$$

is a solution to (4.1.3) satisfying condition (B'), then

$$(\tilde{\eta}, \tilde{\psi}) \in L^2 \left([0, T]; C^{r+\frac{1}{2}}(N) \times C^r(N) \right)$$

and there holds

$$Z_r(T) \leq \mathcal{F} \left(M_s(0) + T^{\frac{1}{2}} \mathcal{F}(M_s(T) + Z_r(T)) \right).$$

The strategy used in [87] to prove the corresponding estimates in the Euclidian setting starts with the symmetrized paradifferential form of the equation, where the remainder is controlled in terms of the Strichartz and Sobolev norms due to the classical parilinearization process and several cancellations in the equations. This process applies verbatim to the periodic setting. It reduces the problem to proving the estimates for an equation of the form

$$\partial_t u + iT_\gamma u + T_V \cdot \nabla u = 0,$$

where the T_a s are paradifferential operators whose norms are controlled through the energy norm of the solution.

Then, to simplify the form of the equation, we change variables to follow the velocity flow, which cancel out the transport term. This again is identical.

The next step is the crucial one. The operator T_γ is of order 3/2, and thus the equation has infinite speed of propagation. But since we are only interested in local results in time, and thus to high frequencies (low frequencies being controlled through Sobolev inequalities), we can put the equation in semiclassical form, where the small parameter h is the inverse of the frequency $\langle \xi \rangle^{-1}$. Then, rescaling the time by putting $\sigma := h^{-\frac{1}{2}}t$, we obtain a semiclassical equation of the form

$$(h\partial_\sigma + iP)v = 0,$$

Then the method of Bahouri-Chemin [15] is used : we regularize the coefficient to lie in the $S_{1,\delta}$ class, construct a WKB parametrix on a small time interval of length $[0, h^\delta]$, and use a TT* argument to deduce Strichartz estimates. and re-sum the time intervals and the frequency pieces, before optimizing on δ to minimize the loss of derivatives.

To implement this strategy in the periodic setting, it has been remarked by Burq-Gérard-Tzvetkov [25] that one can use a partition of unity to reduce to initial data compactly supported in a coordinate patch, construct the parametrix as in the Euclidian setting, and observe that for sufficiently small initial support and sufficiently short time (depending only on the energy norm of the solution), the finite speed of propagation implies that the paramatrix is still supported in a coordinate patch, so that the TT* argument can be applied verbatim, before summing up the spatial pieces. Then coming back to the Bahouri-Chemin method, one re-sum the time intervals and the frequency pieces, before optimizing on δ to minimize the loss of derivatives, which yields the result.

Estimations a priori pour un fond émergent

Cet article est disponible sur ArXiv ; voir [85].

5.1 Introduction

Suppose we are given a fixed smooth simply connected domain \mathcal{O} of \mathbf{R}^n , with $n \geq 2$. We call \mathcal{M} its boundary, which we assume to be connected. An incompressible fluid fills a time-dependent domain $\Omega_t \subset \mathcal{O}$, delimited by \mathcal{M} and a time dependent hypersurface \mathcal{S}_t . We suppose this surface to be at all times connected, and such that Ω_t is always compact and simply connected. The part of \mathcal{M} that bounds Ω_t , called the bottom, is thus time-dependent. We denote it by \mathcal{B}_t .

Our last hypotheses on the domain is that for all times the intersection between \mathcal{M} and \mathcal{S}_t is along a time-dependent compact codimension 2 submanifold, the water line \mathcal{L}_t . This intersection is supposed to be transverse, so that notably the contact angle along \mathcal{L}_t is bounded away from 0 on each compact time interval. The situation is drawn in figure 5.1.

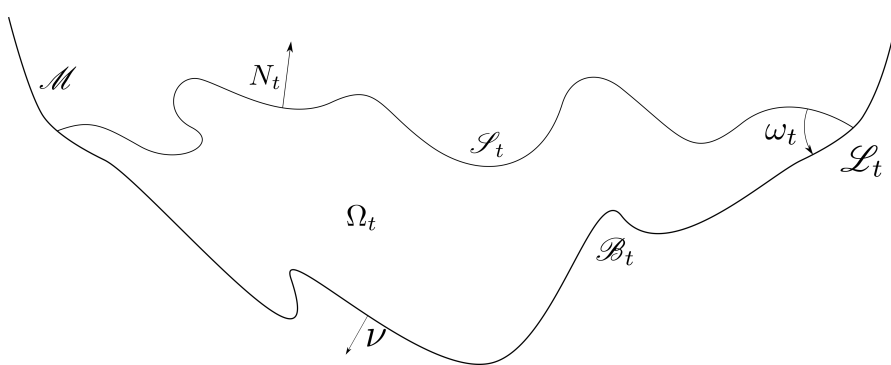


Figure 5.1 – Geometry of the problem

The fluid is described by its velocity $v \in \mathbf{R}^n$ defined for each t on the domain Ω_t , which

satisfies the incompressible Euler equations in a constant gravity field

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p - g e_n, \\ \nabla \cdot v = 0, \end{cases}$$

where for each time t , the function $p : \Omega_t \rightarrow \mathbf{R}$ is the pressure of the fluid. The constant $g \geq 0$ measures gravity, and e_n is a fixed unitary length vector which we think of as the upward direction. The fluid domain moves with the velocity field, and pressure at the boundary is 0, so that

$$(BC) \quad \begin{cases} \mathbf{D}_t := \partial_t + v \cdot \nabla \text{ is tangent to } \cup_t \Omega_t \subset \mathbf{R}^{n+1}, \\ p(t, x) = 0, \quad x \in \mathcal{S}_t. \end{cases}$$

Here \mathbf{D}_t is the material derivative, and the first condition equivalently says that the velocity of \mathcal{S}_t is given by $v \cdot N$ with N the unit outward normal to \mathcal{S}_t , and that $v \cdot \nu = 0$ with ν the unit outward normal on \mathcal{M} .

At a time t and a point $x \in \mathcal{L}_t$ of the waterline, the angle between \mathcal{B}_t and \mathcal{S}_t measured in the plane of ν and N , will be denoted $\omega(x)$.

Our objective is to give a local well-posedness result for the associated Cauchy problem. In this paper, we concern ourselves with establishing a priori estimates. The study of the water-waves problem has a long story, starting with Cauchy in [28]. The rigorous derivation of local existence in Sobolev spaces, however, was only established in 1997, by Wu ([104, 105]). Then a number of other proofs, improving on the regularity, the shape of the bottom, or using different approaches, appeared in the last 20 years. We only quote Beyer and Gunther in [19], Christodoulou and Lindblad in [31] Lannes in [70], Coutand and Shkoller in [32], Alazard, Burq and Zuily in [3, 7], Hunter, Ifrim and Tataru in [52], and with vorticity, Castro and Lannes in [27], and Shatah and Zeng in [90, 91, 92].

However, all of those papers assume a laterally infinite ocean, where there is no contact line. One trick to study such a configuration, when the walls are vertical, is to periodize and symmetrize. This was done by Alazard, Burq and Zuily in [9], for right angles, and later developed in the case of general angles by Kinsey and Wu ([62]) and then by Wu ([108]). The case of a more general angle has only been tackled, to the best of our knowledge, by Ming and Wang in [81]. In this paper, they study the Dirichlet to Neumann operator associated with such a configuration, in 2D, and give a complete description of its singularities at the corner.

The model of the Euler equation, with the boundary condition described above, is only an approximation of the real physical phenomenon. In Practice, viscosity and surface tension are fundamental to a precise description of the motion close to the corner. Steps in this direction have been done by Guo and Tice for stability of the equilibrium in [50] and by Tice and Zheng in [112] for well-posedness, both for the Stoke flow.

Our theorem, stated informally, is as follows. The notation H^s is for the Sobolev spaces based on L^2 .

Theorem 5.1.1. *Suppose \mathcal{S}_t , a C^2 in time family of H^s hypersurfaces, and $v \in C^2(H^s(\Omega_t))$, are solution of the equations.*

Here $s > 1 + \frac{n}{2}$, and $s < \frac{1}{2} + \frac{\pi}{2\bar{\omega}}$, where $\bar{\omega} > 0$ is a number, such that $\forall t, \forall x \in \mathcal{L}_t, \omega(x) \leq \bar{\omega}$. Notice this implies $\omega < \frac{\pi}{n+1}$.

Assume also that there is a number $a_0 > 0$ such that the Taylor coefficient $a := -\nabla_N p \geq a_0 > 0$ for all t , and a number $\underline{\omega} > 0$ such that $\omega \geq \underline{\omega}$ for all t .

Then, for some energy $E(t) = E(\Omega_t, v(t, \cdot))$, to be defined below, and giving a control of \mathcal{S}_t in H^s and $v \in H^s(\Omega_t)$, there exists a time $T > 0$, depending only on the norms of the initial data,

such that $\forall t \in [0, T]$,

$$E(t) \leq E(0) + \int_0^t \mathcal{F}(E(t')) dt',$$

where \mathcal{F} is an increasing function depending only on $\underline{\omega}$, s , a_0 , and a neighborhood of the initial data in the rougher topology $H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}}(\Omega_t)$.

To make this Theorem precise, we need to precise the topology on H^s hypersurfaces, which will be done in 5.3, and to define the Energy. Then Proposition 5.5.1 gives the control of the unknowns from E and Proposition 5.5.2 gives the time T and the estimation on the evolution of E .

A few remarks are in order. First, in the classical case of a well-separated bottom and free surface, we would have the same Theorem, without the restrictions on the angle. The level of regularity, which corresponds to $v \in C^1$ by Sobolev embedding, is the best we can find without using dispersive properties (see for example [7]). Notice we do not assume the field v to be irrotational.

Second, the condition on the angle arises because of the presence of an edge in the domain. In such a domain, the elliptic regularity theory degenerates. This elliptic regularity is needed to make sense of the equations, since p solves an elliptic problem. It also comes into play often in the analysis. The allowed range for s is the one where elliptic regularity works as in smooth domains, as will be seen in Section 5.3.

At last, we expect to be able to prove local well-posedness for the same problem, under the same regularity and with the conditions on the Taylor coefficients and the angle being satisfied initially, for a time depending only on the norms of the initial data a_0 , and $\underline{\omega}$.

In Section 5.2, we study heuristically the infinite-dimensional geometry of the problem, derive the linearized equation, and explain its consequences on our strategy. In Section 5.3, we develop all the analytical tools needed to study moving hypersurfaces with boundaries and moving domains with edges the elliptic regularity theory. Since the problem is fully non-linear, a classical strategy is to find a way to differentiate the equation to reduce it to a quasilinear form, which we hope to be equivalent to the original one. Usually, one would differentiate in space. However, this only work for translation-invariant equations, which is not the case of this problem. Instead, we take advantage of the time-translation invariance, and differentiate in time. This is accomplished in Section 5.4. At last, the Energy is defined and studied in Section 5.5, where the two main Propositions are stated.

In our analysis, we are heavily indebted to two works, in which we draw heavily. The first is the book by Dauge, [39], from which we take the analysis of the elliptic problem. Our main contribution in this domain is to precise the dependence of the constants in the geometry. To the best of our knowledge, the div-curl lemma is new. The second work is the series of three articles by Shatah and Zeng, [90, 91, 92], who developed a coordinate-free framework for the analysis of the water waves problem. Although the analysis ends up being quite different, due to the failure of the mean curvature to quasi-linearize the equations, the coordinate-free framework, most of the notations, and a few computations come directly from there.

5.2 Geometry of the problem

In this section we heuristically derive the linearized equation from the energy. In particular, we do not discuss the smoothness of the objects involved. Most of this section is from the work [90] by Shatah and Zeng, where they study the case of a droplet. We show that this heuristic analysis stays valid in our case, and explain its consequences for our strategy.

5.2.1 Lagrangian formulation

Under the conditions (BC), the Euler equation (E) is easily seen to admit a conserved energy

$$E_0 = \int_{\Omega_t} \frac{|v|^2}{2} dx + g \int_{\Omega_t} x^n dx,$$

where x_n is the coordinate of x along e^n . We want to express (E) as a minimizer of an associated Lagrangian, under the constraints (BC).

For this, we introduce the Lagrangian coordinates by solving the ODE

$$\frac{dx}{dt} = v(t, x), \quad x(0) = y,$$

which gives the spatial path of a fluid particle initially at position $y \in \Omega_0$. Then we introduce for each t the diffeomorphism $u(t, \cdot) : \Omega_0 \rightarrow \Omega_t$ as the flow of this ODE. The divergence free condition on v induces that u preserves the volume, and now $v = u_t \circ u^{-1}$. For any vector field w on Ω_t , we write $\bar{w} = w \circ u$ defined on Ω_0 , and the chain rule implies

$$(5.2.1) \quad \mathbf{D}_t w = \partial_t w + \nabla_v w = \bar{w}_t \circ u^{-1}.$$

Here and in all the following, $\nabla_v w := \langle v, \nabla w \rangle$ where $\langle \cdot, \cdot \rangle$ is the scalar product.

A solution of the Euler equation is thus a particular path, starting from identity, in the infinite dimensional manifold

$$\Gamma := \{ \Phi : \Omega_0 \rightarrow \mathbf{R}^n \mid \Phi \text{ is volume preserving and } \Phi(\mathcal{B}_0) \subset \mathcal{M} \}.$$

Its tangent space at the point Φ is

$$T_\Phi \Gamma := \{ \bar{w} : \Omega_0 \rightarrow \mathbf{R}^n \mid \nabla \cdot w = 0 \text{ on } \Phi(\Omega_0) \text{ and } w \cdot \nu = 0 \text{ on } \Phi(\mathcal{B}_0), \text{ for } w = \bar{w} \circ \Phi^{-1} \}.$$

The energy now takes the form

$$E_0 = \frac{1}{2} |u_t|_{L^2(\Omega_0)}^2 + g\mathbb{G}(u) := \int_{\Omega_0} \frac{|u_t|^2}{2} dy + g \int_{\Omega_0} u^n dy.$$

This suggest that the associated Lagrangian action is

$$\int \mathbb{L}(u) dt = \iint_{\Omega_0} \frac{|u_t|^2}{2} dy dt - g \int \mathbb{G}(u) dt.$$

It then follows from classical variational principles that a minimizer of this action is a path u in Γ whose velocity $v(t)$ should satisfy the equation

$$(5.2.2) \quad \bar{\mathcal{D}}_t u_t + g\mathbb{G}'(u) = 0.$$

Here $\bar{\mathcal{D}}$ is the covariant derivative on Γ for the L^2 metric. We have that Γ is a submanifold of the space of diffeomorphisms from Ω_0 , equipped with L^2 metric, whose tangent space is simply the space of vector fields \bar{w} on Ω_0 . Its covariant derivative along a path u is simply \bar{w}_t , so that we have for an element $\bar{w} \in T\Gamma$ defined above the path $u(t)$ the formula

$$(5.2.3) \quad \bar{\mathcal{D}}_t \bar{w} = \bar{w}_t - \Pi_u(u_t, \bar{w}).$$

Here Π is the second fundamental form of Γ as a submanifold of this space of diffeomorphisms.

Hodge decomposition To compute $\Pi(u_t, \bar{w})$ we observe that any vector field X in Ω can be decomposed uniquely as

$$X = w + \nabla\phi,$$

where ϕ is defined as the solution of

$$(5.2.4) \quad \begin{cases} \Delta\phi = \nabla \cdot X \text{ in } \Omega, \\ \phi|_{\mathcal{S}} = 0, \\ \nabla_\nu\phi|_{\mathcal{B}} = \langle X, \nu \rangle. \end{cases}$$

Thus w verifies

$$\begin{cases} \nabla \cdot w = 0 \text{ in } \Omega, \\ w|_{\mathcal{S}} = X|_{\mathcal{S}}, \\ \langle w, \nu \rangle|_{\mathcal{B}} = 0. \end{cases}$$

This decomposition is easily seen to be L^2 orthogonal. Thus we can identify

$$(5.2.5) \quad (T_\Phi\Gamma)^\perp = \{-(\nabla\phi) \circ \Phi \mid \phi|_{\mathcal{S}} = 0\}.$$

However since in (5.2.4) we define ϕ by inverting the Laplace operator in a domain with corner, the parts w and $\nabla\phi$ of the decomposition are not necessarily smooth, even if X is. Thus (5.2.5) is to be understood only in a heuristic sense.

Covariant derivative Now coming back to (5.2.3), we see that for a path $u(t)$ in Γ with velocity field $u_t = \bar{v}$, and a vector field \bar{w} defined on it, there holds

$$\bar{\mathcal{D}}_t\bar{w} = \bar{w}_t + (\nabla p_{v,w}) \circ u,$$

where

$$\begin{cases} \Delta p_{v,w} = -\text{tr}(DvDw) \text{ in } \Omega, \\ p_{v,w}|_{\mathcal{S}} = 0, \\ \nabla_\nu p_{v,w}|_{\mathcal{B}} = -\Pi_{\mathcal{B}}(v, w), \end{cases}$$

with $\Pi_{\mathcal{B}}$ the second fundamental form of the bottom. This can be inferred from (5.2.4) by taking $X = \bar{w}_t \circ u^{-1} = \mathbf{D}_t w$.

Now this is in Lagrangian coordinates, and it can be rewritten in Eulerian coordinates, using (5.2.1). This gives

$$\bar{\mathcal{D}}_t\bar{w} = (\mathbf{D}_t w + \nabla p_{v,w}) \circ u = (\partial_t w + \nabla_v w + \nabla p_{v,w}) \circ u.$$

Gravity force We then compute $\mathbb{G}'(u)$. For any $\bar{w} \in T_u\Gamma$, take a path in Γ indexed by ε and starting from u at $\varepsilon = 0$, with tangent vector at $\varepsilon = 0$ equal to \bar{w} . Then

$$\begin{aligned} \langle \mathbb{G}'(u), \bar{w} \rangle_{L^2(\Omega_0)} &= \frac{d}{d\varepsilon} \int_{u(\Omega_0)} x^n dx \\ &= \int_{u(\Omega_0)} \mathbf{D}_\varepsilon x^n dx \\ &= \int_{u(\Omega_0)} \langle w, \nabla x^n \rangle dx \\ &= \int_{u(\mathcal{S}_0)} w^\perp x^n dS \\ &= \int_{u(\Omega_0)} \langle w, \nabla \mathcal{H}(x^n|_{u(\mathcal{S}_0)}) \rangle dx. \end{aligned}$$

Here we have used two times the Green formula, and the terms on $u(\mathcal{B})$ vanish since $\langle w, \nu \rangle = 0$ there. We have replaced ∇x^n with $\nabla \mathcal{H}(x^n|_{u(\mathcal{S}_0)})$, where \mathcal{H} is the harmonic extension with homogeneous Neumann condition on the bottom, so that now $\nabla \mathcal{H}(x^n|_{u(\mathcal{S}_0)}) \in T_u \Gamma$ and we can identify it with $\mathbb{G}'(u)$.

Then the Euler-Lagrange equations (5.2.2) for our action become in Eulerian coordinates

$$(5.2.6) \quad \partial_t v + v \cdot \nabla v = -\nabla p_{v,v} - g \nabla \mathcal{H}(x^n|_{\mathcal{S}_t}) = -\nabla p - g e^n,$$

with $p = p_{v,v} + g(e^n - \mathcal{H}(x^n|_{\mathcal{S}_t}))$ the physical pressure. Combined with the constraint that $v \circ u \in T_u \Gamma$ is the velocity vector of the domain, this gives the Euler equations (E) with boundary conditions (BC).

5.2.2 The linearized equation

To help us study the Euler equations, we want to find a way to linearize them around a given solution, i.e. a path $u(t)$ in Γ , such that its velocity $\bar{v} = u_t \in T_u \Gamma$ satisfies the Euler equations. Since we see the Euler equation as a geodesic flow with potential, the natural linearization is through the Jacobi equation. It is the equation that a time-dependent vector field $\bar{w} \in T_u(t)\Gamma$ defined on the path u has to satisfy if, by moving the curve u by the flow of \bar{w} , we want it to stay a solution of the Euler equations. Classically, this is

$$(5.2.7) \quad \bar{\mathcal{D}}_t^2 \bar{w} + \bar{\mathcal{R}}(u_t, \bar{w})u_t + g \bar{\mathcal{D}}^2 \mathbb{G}(u) \bar{w} = 0,$$

where $\bar{\mathcal{R}}$ is the Riemann curvature tensor of Γ at the point $u(t)$, and $\bar{\mathcal{D}}^2 \mathbb{G}(u)$ is the Hessian of \mathbb{G} . We need to compute $\bar{\mathcal{R}}(u_t, \bar{w})u_t$ and $\bar{\mathcal{D}}^2 \mathbb{G}(u) \bar{w}$, or at least their principal parts, from their bilinear forms. To do this, we consider for a given $\bar{w} \in T_u(t)\Gamma$ a family of curves $u(t, \varepsilon) \in \Gamma$ such that $u(t, 0) = u(t)$, and $\partial_\varepsilon u(t, 0) = \bar{w}$. Then we extend \bar{w} to be the tangent vector in ε .

The Riemann curvature We use the classical formula

$$\begin{aligned} \langle \bar{\mathcal{R}}(\bar{v}, \bar{w})\bar{v}, \bar{w} \rangle_{L^2(\Omega_0)} &= \langle \Pi_u(\bar{v}, \bar{v}), \Pi_u(\bar{w}, \bar{w}) \rangle_{L^2(\Omega_0)} - |\Pi_u(\bar{v}, \bar{w})|_{L^2(\Omega_0)}^2 \\ &= \int_{\Omega_t} \langle \nabla p_{v,v}, \nabla p_{w,w} \rangle dx - \int_{\Omega_t} |\nabla p_{v,w}|^2 dx. \end{aligned}$$

Then assuming enough regularity on v and w , we can repetitively use the Green formula and the definition of $p_{v,v}$ to find

$$\begin{aligned} \int_{\Omega_t} \langle \nabla p_{v,v}, \nabla p_{w,w} \rangle dx &= \int_{\Omega_t} p_{v,v} \operatorname{tr}(DwDw) dx + \int_{\mathcal{B}_t} p_{v,v} \nabla_\nu p_{w,w} dS \\ &= - \int_{\Omega_t} \langle \nabla p_{v,v}, \nabla_w w \rangle dx + \int_{\mathcal{B}_t} p_{v,v} (\langle \nabla_w \nu, w \rangle + \langle \nabla_w w, \nu \rangle) dS \\ &= \int_{\Omega_t} D^2 p_{v,v}(w, w) dx - \int_{\mathcal{S}_t} w^\perp \nabla_w p_{v,v} dS. \end{aligned}$$

Here we have also used the identity

$$\langle \nabla_w \nu, w \rangle + \langle \nabla_w w, \nu \rangle = w \langle w, \nu \rangle = 0,$$

where w is taken as a derivation, because $\langle w, \nu \rangle = 0$ on \mathcal{B} , and $\nabla \cdot w = 0$.

Then $p_{v,v} = 0$ on \mathcal{S}_t , and thus $\nabla_w p_{v,v} = w^\perp \nabla_N p_{v,v}$. Then a last application of the Green formula gives

$$\int_{\Omega_t} \langle \nabla p_{v,v}, \nabla p_{w,w} \rangle dx = \int_{\Omega_t} \langle w, \nabla \mathcal{H}(-\nabla_N p_{v,v} w^\perp) \rangle dx + \int_{\Omega_t} D^2 p_{v,v}(w, w) dx.$$

Now the second term is expected to be more regular, so that

$$\bar{\mathcal{R}}(\bar{v}, \bar{w})\bar{v} \approx (\mathcal{R}_0(v)w) \circ u$$

where $\mathcal{R}_0(v)w = \nabla \mathcal{H}(-\nabla_N p_{v,v} w^\perp)$.

The gravity term To compute $\mathbb{G}''(u)$, we use the formulas (5.4.30) and (5.5.11) for the evolutions of the normal and the surface element of a surface moving with divergence-free velocity w :

$$\begin{aligned} \langle \mathbb{G}''(u)\bar{w}, \bar{w} \rangle_{L^2(\Omega_0)} &= \partial_\varepsilon \langle \mathbb{G}'(u), \bar{w} \rangle_{L^2(\Omega_0)} - \langle \mathbb{G}'(u), \bar{\mathcal{D}}_\varepsilon \bar{w} \rangle_{L^2(\Omega_0)} \\ &= \partial_\varepsilon \int_{\mathcal{S}} x^n w^\perp dS - \int_{\mathcal{S}} x^n N \cdot (\mathbf{D}_\varepsilon w + \nabla p_{w,w}) dS \\ &= \int_{\mathcal{S}} w^n w^\perp - x^n w \cdot ((Dw)_*(N))^\top + x^n N \cdot \mathbf{D}_\varepsilon w \\ &\quad - x^n w^\perp \nabla_N w \cdot N - x^n N \cdot \mathbf{D}_\varepsilon w - x^n N \cdot \nabla p_{w,w} dS \\ &= \int_{\mathcal{S}} (w^n w^\perp - x^n \nabla_w w \cdot N - x^n N \cdot \nabla p_{w,w}) dS. \end{aligned}$$

Here \top and \perp are the projection to vector fields respectively tangent and normal to \mathcal{S} . Notice that this operation has limited regularity when \mathcal{S} is not smooth. But using repeated Green formulas give

$$\begin{aligned} - \int_{\mathcal{S}} x^n N \cdot \nabla p_{w,w} dS &= - \int_{\Omega} \nabla \mathcal{H}(x^n|_{\mathcal{S}}) \cdot \nabla p_{w,w} + \mathcal{H}(x^n|_{\mathcal{S}}) \text{tr}((Dw)^2) dx \\ &\quad + \int_{\mathcal{B}} \mathcal{H}(x^n|_{\mathcal{S}}) \nu \cdot \nabla p_{w,w} dS \\ &= - \int_{\Omega} \nabla \mathcal{H}(x^n|_{\mathcal{S}}) \cdot \nabla_w w dx + \int_{\mathcal{S}} x^n \nabla_w w \cdot N dS \\ &\quad + \int_{\mathcal{B}} \mathcal{H}(x^n|_{\mathcal{S}}) (\nabla_w w \cdot \nu + \nabla_w \nu \cdot w) dS \\ &= \int_{\Omega} D^2 \mathcal{H}(x^n|_{\mathcal{S}})(w, w) dx + \int_{\mathcal{S}} x^n \nabla_w w \cdot N - w^\perp \nabla_w \mathcal{H}(x^n|_{\mathcal{S}}) dS. \end{aligned}$$

Then by noticing that

$$\nabla_w \mathcal{H}(x^n|_{\mathcal{S}}) = \nabla_w \top x^n + w^\perp \mathcal{N}(x^n|_{\mathcal{S}}) = w^n - w^\perp N^n + w^\perp \mathcal{N}(x^n|_{\mathcal{S}}),$$

we find

$$\begin{aligned} \langle \mathbb{G}''(u)\bar{w}, \bar{w} \rangle_{L^2(\Omega_0)} &= \int_{\mathcal{S}} (w^\perp)^2 (N^n - \mathcal{N}(x^n|_{\mathcal{S}})) dS + \int_{\Omega} D^2 \mathcal{H}(x^n|_{\mathcal{S}})(w, w) dx \\ &= \int_{\Omega} w \cdot \nabla \mathcal{H}[(N^n - \mathcal{N}(x^n|_{\mathcal{S}})) w^\perp] dx + \int_{\Omega} D^2 \mathcal{H}(x^n|_{\mathcal{S}})(w, w) dx. \end{aligned}$$

Thus again the second term is more regular, so that

$$\mathbb{G}''(u)\bar{w} \simeq (\mathcal{G}w) \circ u$$

where $\mathcal{G}w = \nabla \mathcal{H} [(N^n - \mathcal{N}(x^n|_{\mathcal{S}})) w^\perp]$.

Thus the linearized equation becomes in Eulerian coordinates

$$(5.2.8) \quad \mathcal{D}_t^2 w + \mathcal{R}_0(v)w + g\mathcal{G}w = \text{bounded terms.}$$

we observe that both \mathcal{R}_0 and \mathcal{G} are of order 1, and that their form is similar. In fact, we can write

$$(5.2.9) \quad (\mathcal{R}_0 + g\mathcal{G})w = \nabla \mathcal{H}(aw^\perp) =: \mathcal{R}_g w,$$

where a is the Rayleigh-Taylor coefficient

$$(5.2.10) \quad a := -\nabla_N(p_{v,v} + \mathcal{H}(x^n|_{\mathcal{S}}) - x^n) = -\nabla_N p,$$

where p is again the physical pressure.

The Rayleigh-Taylor coefficient It can be seen on this equation that there is stability (meaning exponential decay of the solution) only if the Rayleigh-Taylor condition

$$(5.2.11) \quad a(t, x) \geq c > 0, \forall x \in \Omega_t$$

holds for all times, with c an arbitrary positive constant.

Assuming enough regularity, we can compute a at the triple line. There holds

$$\nabla_\nu p_{v,v} = \langle \nu, \nabla p_{v,v} \rangle = \langle \nu^\top, \nabla^\top p_{v,v} \rangle + \langle \nu, N \rangle \nabla_N p_{v,v} = \langle \nu, N \rangle \nabla_N p_{v,v},$$

because $p_{v,v} = 0$ on \mathcal{S} . Here, A^\top refer to the tangent part at \mathcal{S} . On the other hand,

$$\nabla_\nu p_{v,v} = \langle v, \nabla_\nu v \rangle = -\langle \nu, \nabla_\nu v \rangle,$$

because $\langle v, \nu \rangle = 0$ on \mathcal{B} . Therefore, assuming $\langle \nu, N \rangle \neq 0$, we find

$$(5.2.12) \quad -\nabla_N p_{v,v} = \frac{\nu \cdot \nabla_\nu v}{\nu \cdot N}.$$

A similar computation can be performed on the gravity part :

$$0 = \nabla_\nu \mathcal{H}(x^n|_{\mathcal{S}}) = \langle \nu^\top, \nabla^\top \mathcal{H}(x^n|_{\mathcal{S}}) \rangle + \langle \nu, N \rangle \nabla_N \mathcal{H}(x^n|_{\mathcal{S}}) \\ = \langle e^n, \nu - \langle \nu, N \rangle N \rangle + \langle \nu, N \rangle \nabla_N \mathcal{H}(x^n|_{\mathcal{S}}),$$

and since $\nabla_N x^n = N^n$, we find

$$(5.2.13) \quad -g \nabla_N (\mathcal{H}(x^n|_{\mathcal{S}}) - x^n) = g \frac{\nu^n}{\langle \nu, N \rangle},$$

again assuming $\langle \nu, N \rangle \neq 0$.

Thus putting together (5.2.12) and (5.2.13) gives for $\langle \nu, N \rangle \neq 0$ that

$$(5.2.14) \quad a = \frac{g\nu^n + \langle \nu, \nabla_\nu v \rangle}{\langle \nu, N \rangle}.$$

To see what this means, we specialize to 2D situations, with zero velocity field. Then $\langle \nu, N \rangle = -\cos(\omega)$, with ω the angle between the bottom and the free surface, so that the condition $a > 0$ gives the situation under study as stable.

Of course, when the velocity field is non zero, it can counterbalance the effect of gravity and change this situation.

5.3 Analysis on moving domains

In this section, we develop the norms and estimates we need for our analysis. The main objective is to derive estimates whose constants are independent of the domain.

5.3.1 Surface coordinates

Our first objective is to give a coordinates description of hypersurfaces in a given H^{s_0} neighborhood. Take $s_0 > \frac{n+1}{2}$. Using local coordinates, one can easily define what it means to be an H^r function on a given H^{s_0} hypersurface with boundary \mathcal{S} . For $s_0 > r > -s_0$, those are simply functions whose coordinates representatives are locally in $H^r(\mathbf{R}^{n-1})$ for interior coordinates and $H^r(\mathbf{R}_+^{n-1})$ for boundary coordinates. Here \mathbf{R}_+^{n-1} is the upper half-plane, and H^r functions are simply restrictions of functions that are H^r in the whole plane.

It is easy to see that this produce a Banach space, and a norm can be chosen by taking a covering of \mathcal{S} by a finite number of coordinates patch, and an adapted partition of unity. However such a norm is dependent on those choices of coordinates, and therefore we will not use it. After that, one can define a topology on the space of H^{s_0} surfaces with boundary contained in our fixed bottom hypersurface \mathcal{M} by saying that two are close if a diffeomorphism from one to the other is close to identity in H^{s_0} norm. It is quite easy to see that the subspace of such surfaces whose intersection with \mathcal{M} is transverse is an open set, and therefore we can consider a neighborhood of a given smooth hypersurface \mathcal{S}_* consisting entirely of H^{s_0} surfaces intersecting \mathcal{M} transversally. By density, any hypersurface is included in one such neighborhood.

Now we will construct such a neighborhood. Take a compact, smooth, reference hypersurface \mathcal{S}_* , whose intersection with \mathcal{M} is transverse, and whose boundary is this intersection \mathcal{L}_* . We want to represent close enough surfaces as graphs over \mathcal{S}_* , and for this we need a good collar neighborhood of \mathcal{S}_* . We cannot use normal coordinates because since we want to represent surfaces with boundary contained in \mathcal{M} , we need to straighten \mathcal{M} in some way. We accomplish this through the following lemma. Recall that \mathcal{O} is the domain whose boundary is \mathcal{M} .

Lemma 5.3.1. *There exists a smooth unit vector field X , defined on a neighborhood of \mathcal{S}_* in \mathcal{O} , that is always transverse to \mathcal{S}_* and always tangent to \mathcal{M} . There exists $\delta > 0$ such that the flow of X*

$$\phi : \mathcal{S}_* \times [-\delta, \delta] \rightarrow \mathbf{R}^n$$

is a smooth diffeomorphism from its domain to a neighborhood of \mathcal{S}_ in \mathcal{O} .*

Proof. One start by constructing X_1 always normal to \mathcal{S}_* away from its boundary. For this, we consider \mathcal{S}_* only as an hypersurface with boundary of \mathbf{R}^n . We take an open submanifold of it, which is an hypersurface of \mathbf{R}^n . Now we take the normal to this hypersurface, and we extend it to a neighborhood in \mathbf{R}^n .

Then in a neighborhood of \mathcal{L}_* in \mathcal{M} , we can construct a smooth vector field X_2 tangent to \mathcal{M} and transverse to \mathcal{L}_* by extending the normal to \mathcal{L}_* in \mathcal{M} . Then we can extend it in a neighborhood of \mathcal{L}_* in $\bar{\mathcal{O}}$ to a smooth vector field tangent to \mathcal{M} and transverse to \mathcal{S}_* , because their intersection is transverse.

To finish, we can cover a small enough neighborhood of \mathcal{S}_* in $\bar{\mathcal{O}}$ with this neighborhood where X_2 is defined, and an open set whose closure is in the interior of \mathcal{O} , and where X_1 is well defined. At last, we can use a partition of unity to glue them smoothly to the vector field X .

The existence of ϕ , its regularity, and the fact that it is a diffeomorphism for small enough δ is a simple consequence of the theory of ODEs. \square

If we fix an H^{s_0} norm on \mathcal{S}_* , we can express a neighborhood of it in the space of H^{s_0} surfaces with boundary in \mathcal{M} by the condition that there exists a diffeomorphism F between the two satisfying

$$\|F - \text{Id}\|_{H^{s_0}(\mathcal{S}_*; \mathbf{R}^n)} < \delta_1.$$

For δ_1 small enough, this implies transversality of all the surfaces in the neighborhood. Taking again δ_1 small enough, those surfaces are all contained in the collar neighborhood we just defined. Then in those collar coordinates, again for δ_1 small, those are necessarily graphs above \mathcal{S}_* . Therefore, we can represent such a neighborhood by functions $\eta_{\mathcal{S}}$ defined on \mathcal{S}_* with small enough H^{s_0} norms, and those give diffeomorphisms

$$\Phi_{\mathcal{S}}(p) := \phi(p, \eta_{\mathcal{S}}(p))$$

in $H^s(\mathcal{S}_*; \mathbf{R}^n)$.

All those notions can be restricted to \mathcal{L}_* , so that \mathcal{L} is the graph of a function $\eta_{\mathcal{L}}$ which is the trace of $\eta_{\mathcal{S}}$ on \mathcal{L}_* , giving a diffeomorphism $\Phi_{\mathcal{L}}$ which is also the trace of $\Phi_{\mathcal{S}}$. If $n = 2$, \mathcal{L} is just two points, and these terms are well-defined because $\eta_{\mathcal{S}}$ is H^{s_0} , and therefore continuous. For $n \geq 3$, those are traces in the Sobolev sense, and those traces are well-defined in $H^{s_0 - \frac{1}{2}}(\mathcal{L}_*)$ since $s_0 > \frac{n+1}{2}$.

Definition 5.3.2. For $\delta > 0$ and $s_0 > \frac{n+1}{2}$, we define $\Lambda(\mathcal{S}_*, s_0, \delta)$ as the neighborhood of H^{s_0} hypersurfaces \mathcal{S} such that their associated $\eta_{\mathcal{S}}$ satisfies $\|\eta_{\mathcal{S}}\|_{H^{s_0}(\mathcal{S}_*)} < \delta$.

For surfaces \mathcal{S} in $\Lambda(\mathcal{S}_*, s_0, \delta)$, we can define the Sobolev norms $H^r(\mathcal{S})$, for $-s_0 \leq r \leq s_0$, from the reference norm on \mathcal{S}_* . In the analysis of the evolution problem, we will work with surfaces in $\Lambda_* := \Lambda(\mathcal{S}_*, s - \frac{1}{2}, \delta)$, with $s > 1 + n/2$, and where $\delta > 0$ is small enough that all the above properties hold. However, our surfaces will really be of H^s class. The reason for this is that we do not want any smallness condition in the condition in the norm where the dynamics take place. The set Λ_* takes the role of a control neighborhood, and by taking \mathcal{S}_* close to the initial surface \mathcal{S}_0 , we can treat any case.

Since being in Λ_* is sufficient to have a well-defined $\Phi_{\mathcal{S}}$, we can use its H^s norm to measure the regularity of \mathcal{S} . More precisely, for $\mathcal{S} \in \Lambda_*$, if both \mathcal{S}_* and \mathcal{S} are in H^s , we write

$$(5.3.1) \quad |\mathcal{S}|_s := \|\Phi_{\mathcal{S}}\|_{H^s(\mathcal{S}_*)}.$$

Of course, any other choice of reference surface \mathcal{S}_* provides an equivalent quantity, as soon as both are defined.

We also write

$$(5.3.2) \quad |\mathcal{L}|_{s-\frac{1}{2}} := \|\Phi_{\mathcal{L}}\|_{H^{s-\frac{1}{2}}(\mathcal{L}_*)}$$

in dimension $n \geq 3$. In dimension $n = 2$, \mathcal{L} consist only of two point, whose position on \mathcal{M} is controlled by the condition $\mathcal{S} \in \Lambda_*$, so that we do not need to control its regularity.

The procedure to prove estimates with constants uniform in Λ_* is to prove them on \mathcal{S}_* and then pushing them forward to \mathcal{S} . If we do not study norms greater than $H^{s-\frac{1}{2}}(\mathcal{S})$, this will only involve the $H^{s-\frac{1}{2}}(\mathcal{S}_*)$ norms of $\Phi_{\mathcal{S}}$ and $\Phi_{\mathcal{S}}^{-1}$, which are uniformly bounded for $\mathcal{S} \in \Lambda_*$. For example, it is easy to prove the following product estimates, which will be used freely in the paper.

Proposition 5.3.3. *If $s > 1 + \frac{n}{2}$, \mathcal{S}_* is a reference hypersurface, δ small enough and $\mathcal{S} \in \Lambda_*$, then for any functions $f \in H^{s_1}(\mathcal{S})$ and $g \in H^{s_2}(\mathcal{S})$, with $s_1 \leq s_2 \leq s - \frac{1}{2}$, there holds*

$$\begin{aligned} \|fg\|_{H^{s_1+s_2-\frac{n-1}{2}}(\mathcal{S})} &\leq C \|f\|_{H^{s_1}(\mathcal{S})} \|g\|_{H^{s_2}(\mathcal{S})} \quad \text{if } s_2 < \frac{n-1}{2} \text{ and } 0 < s_1 + s_2, \\ \|fg\|_{H^{s_1}(\mathcal{S})} &\leq C \|f\|_{H^{s_1}(\mathcal{S})} \|g\|_{H^{s_2}(\mathcal{S})} \quad \text{if } s_2 > \frac{n-1}{2} \text{ and } 0 \leq s_1 + s_2. \end{aligned}$$

Here C depends only on Λ_* .

Similar inequalities hold on \mathcal{L} in dimension $n \geq 3$.

5.3.2 From the curvature to the surface

Recall that the mean curvature κ of \mathcal{S} is defined as the trace of the second fundamental form Π , whose definition is in turn

$$\Pi(v, w) = -\langle \nabla_v N, w \rangle$$

for $v, w \in T\mathcal{S}$.

The regularity of the hypersurface \mathcal{S} can be measured by its curvature κ and the curvature κ_l of its boundary \mathcal{L} taken as a hypersurface of \mathcal{M} (this is only needed in dimension greater than 3). More precisely, we have the following lemmas, distinguishing between dimension 2 and dimension greater than 3.

Lemma 5.3.4. *For $n = 2$, take $s > 2$, a reference hypersurface \mathcal{S}_* , and $\delta > 0$ small enough. Assume the hypersurface \mathcal{S} is in Λ_* , and $\kappa \in H^{s-2}(\mathcal{S})$. Then the surface \mathcal{S} is actually H^s , and we have the following estimates on its geometry:*

$$|\mathcal{S}|_s + \|\Pi\|_{H^{s-2}(\mathcal{S})} + \|\mathbf{N}\|_{H^{s-1}(\mathcal{S})} \leq C \left(1 + \|\kappa\|_{H^{s-2}(\mathcal{S})}\right).$$

Lemma 5.3.5. *For $n \geq 3$, take $s > 1 + \frac{n}{2}$, a reference hypersurface \mathcal{S}_* , and $\delta > 0$ small enough. Assume the hypersurface \mathcal{S} is in Λ_* , and $\kappa \in H^{s-2}(\mathcal{S})$, $\kappa_l \in H^{s-\frac{5}{2}}(\mathcal{L})$. Then the surface \mathcal{S} is actually H^s , and we have the following estimates on its geometry:*

$$|\mathcal{S}|_s + \|\Pi\|_{H^{s-2}(\mathcal{S})} + \|\mathbf{N}\|_{H^{s-1}(\mathcal{S})} \leq C \left(1 + \|\kappa\|_{H^{s-2}(\mathcal{S})} + \|\kappa_l\|_{H^{s-\frac{5}{2}}(\mathcal{L})}\right).$$

We also have estimates on the geometry of \mathcal{L} :

$$|\mathcal{L}|_{s-\frac{1}{2}} + \|\Pi_l\|_{H^{s-\frac{5}{2}}(\mathcal{L})} + \|n_l\|_{H^{s-\frac{3}{2}}(\mathcal{L})} \leq C \left(1 + \|\kappa\|_{H^{s-2}(\mathcal{S})} + \|\kappa_l\|_{H^{s-\frac{5}{2}}(\mathcal{L})}\right).$$

Proof. The proof is standard, and we only give a sketch of it. It rests on the identity

$$(5.3.3) \quad -\Delta_{\mathcal{S}} \Pi = -\mathcal{D}^2 \kappa + |\Pi|^2 \Pi - \kappa \Pi^2,$$

which is proved in [90] and stays valid for a hypersurface with boundary. For the case $n \geq 3$, the same identity holds for on \mathcal{L} in \mathcal{M} . Then one can use $\Phi_{\mathcal{L}}$ to transfer it to an elliptic equation on some derivatives of η_l , and use elliptic regularity to find the regularity of \mathcal{L} and the above estimates. Using again $\Phi_{\mathcal{S}}$ and elliptic estimates, this time in domains with boundary, keeping in mind the regularity of \mathcal{L} as boundary data, we find the regularity of \mathcal{S} and the estimates. In dimension $n = 2$, we only need to use the identity on \mathcal{S} , since as remarked above, the boundary data consists only of two point whose range are bounded by the condition $\mathcal{S} \in \Lambda_*$. \square

The advantage of this lemma is that now to control the regularity of \mathcal{S} , we only need to control κ and κ_l , which are invariantly defined.

5.3.3 Internal coordinates

We can easily define Sobolev norms on Ω by considering Sobolev functions as restrictions of functions Sobolev on \mathbf{R}^n . Then

$$\|u\|_{H^r(\Omega)} = \inf \left\{ \|U\|_{H^r(\mathbf{R}^n)}; u = U|_{\Omega} \right\}.$$

This way, the constants of Sobolev injections theorems are independent of the domain Ω . Also, if Ω_t is a continuous one-parameter family of such domains, we can use that to define the classes $C^k(H^r(\Omega_t))$ of functions k -differentiable in t with values in $H^r(\Omega_t)$, simply by requiring that an extension of the function to \mathbf{R}^n be C^k in time with value in $H^r(\mathbf{R}^n)$. It is easy to see that any other reasonable definition of $C^k(H^r(\Omega_t))$ coincides with this one, which is in particular independent of the chosen (continuous) extension operator.

Our objective in this section is to construct a diffeomorphism from Ω to Ω_* with maximal regularity. As can be seen from the boundaryless case, any construction based on an affine change of variable would be only of H^s regularity, while we want it to be $H^{s+\frac{1}{2}}$. As we will see, the existence of this diffeomorphism is a consequence of Sobolev extension theorems in domains with edges. All of those are based on the following theorem in the model case of the quarter-space.

Lemma 5.3.6. *For $m \in \mathbf{N}_*$, the mapping $u \mapsto \{(f_k)_{0 \leq k \leq m-1}, (g_l)_{0 \leq l \leq m-1}\}$ defined by*

$$f_k = \partial_z^k u|_{x=0}, \quad g_l = \partial_x^l u|_{z=0}$$

for $u \in C^\infty(\overline{\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{n-2}})$ has a unique continuous extension from $H^m(\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^{n-2})$ onto the subspace of

$$\prod_{k=0}^{m-1} H^{m-k-\frac{1}{2}}(\mathbf{R}^+ \times \mathbf{R}^{n-2}) \times \prod_{l=0}^{m-1} H^{m-l-\frac{1}{2}}(\mathbf{R}^+ \times \mathbf{R}^{n-2})$$

defined by

$$\begin{aligned} & - \partial_x^l f_k(0) = \partial_z^k g_l(0), \quad l+k < m-1 \text{ and} \\ & - \int_0^1 \|\partial_x^l f_k(t) - \partial_z^k g_l(t)\|_{L^2(\mathbf{R}^{n-2})}^2 \frac{dt}{t} < +\infty, \quad l+k = m-1. \end{aligned}$$

It has a continuous right inverse, the extension operator.

This is a trivial extension of theorem 1.5.2.4 in [48].

With smooth local charts for the manifold with corner Ω_* , we can transfer results on the quarter-space to results close to \mathcal{L}_* . One such example is the Sobolev extension theorem, used in the following Proposition on global coordinates.

Proposition 5.3.7. *For $\delta > 0$ small enough, and $s > 1 + \frac{n}{2}$, for any $\mathcal{S} \in \Lambda_*$, there exists a global diffeomorphism Φ_Ω from Ω_* to Ω , restricting to $\Phi_{\mathcal{S}}$ on \mathcal{S}_* , and satisfying*

$$\|\Phi_\Omega\|_{H^s(\Omega_*)} + \|\Phi_\Omega^{-1}\|_{H^s(\Omega)} \leq C,$$

with C uniform in Λ_* .

Furthermore, if \mathcal{S}_* and \mathcal{S} are both in H^s , then

$$\|\Phi_\Omega\|_{H^{s+\frac{1}{2}}(\Omega_*)} + \|\Phi_\Omega^{-1}\|_{H^{s+\frac{1}{2}}(\Omega)} \leq C [1 + |\mathcal{S}|_s].$$

Proof. As stated, we want the boundary value for Φ_Ω to be $\Phi_{\mathcal{S}}$ on \mathcal{S}_* . On \mathcal{B}_* , which is a compact hypersurface with boundary \mathcal{L}_* , we need it to restrict to an H^s diffeomorphism to \mathcal{B} ,

with value $\Phi_{\mathcal{L}}|_{\mathcal{L}_*}$ on \mathcal{L}_* . Since this $\Phi_{\mathcal{L}}|_{\mathcal{L}_*}$ is only $H^{s-\frac{1}{2}}$, we need a diffeomorphism of maximal regularity. To define such a diffeomorphism, we use the following construction. Recall that we have constructed in Lemma 5.3.1 a smooth vector field X whose restriction to \mathcal{M} is a tangent vector field in a neighborhood of \mathcal{L}_* , and such that for $p \in \mathcal{L}_*$, $\Phi_{\mathcal{L}}(p) := \phi(p, \eta_{\mathcal{L}}(p))$ with ϕ the flow of X , and $\eta_{\mathcal{L}}$ an $H^{s-\frac{1}{2}}$ function on \mathcal{L}_* . We can extend X to a smooth tangent vector field on the whole of \mathcal{M} by gluing it to the null vector field using a partition of unity. Then we can extend $\eta_{\mathcal{L}}$ to an H^s function $\eta_{\mathcal{B}}$ on \mathcal{B}_* , using for example a harmonic extension. Then defining $\Phi_{\mathcal{B}}(p) := \phi(p, \eta_{\mathcal{B}}(p))$, with ϕ still the flow of our extended vector field, we get the promised diffeomorphism. Since $\Phi_{\mathcal{L}}$ is close to identity and we have extended X by the null vector field, this diffeomorphism is close to identity.

Then we want to construct Φ_{Ω} as

$$\Phi_{\Omega} := \text{Id} + E(\Phi_{\mathcal{L}} - \text{Id}, \Phi_{\mathcal{B}} - \text{Id}),$$

where $E(a, b)$ is a Sobolev extension of (a, b) in Ω . Using $H^{s+\frac{1}{2}}$ local coordinates and a partition of unity, we only have to construct such an extension in the model case of the half-plane, which is trivial, and of the quarter-space, where we want to use Lemma 5.3.6.

The construction of this E is immediate for the case of integers $m \geq 2$, because we only need to prove that we can find the f_k, g_l , with $f_0 = \Phi_{\mathcal{L}} - \text{Id}$, $g_0 = \Phi_{\mathcal{B}} - \text{Id}$, and with the compatibility conditions. The only condition needed between $\Phi_{\mathcal{L}} - \text{Id}$ and $\Phi_{\mathcal{B}} - \text{Id}$ is their equality at \mathcal{L}_* . Then finding the other f_k, g_l is a simple exercise.

Then in the general case, since $s > 2$, and since the operator E is defined on $H^m \cap H^{m+1}$, we can use interpolation between consecutive integers to find our E .

Continuity of E and smallness of the boundary values give us that Φ_{Ω} is a diffeomorphism satisfying the conclusions of the Proposition. \square

We can deduce from the existence those coordinates and from Lemma 5.3.6 the extension theorem we will use in the analysis :

Proposition 5.3.8. *For $\delta > 0$ small enough, and $s > 1 + \frac{n}{2}$, for any $\mathcal{L} \in \Lambda_*$, and for $1 \leq \sigma < s$, the restriction operator taking a smooth function in Ω to its trace on \mathcal{L} extends to an operator from $H^{\sigma}(\Omega)$ onto $H^{\sigma-\frac{1}{2}}(\mathcal{L})$. This operator and its right inverse are uniformly continuous in $\mathcal{L} \in \Lambda_*$.*

The same statement holds for restrictions to \mathcal{B} .

Proof. Again we interpolate between integers $m \geq 1$. We only need to show that any $f \in H^{\sigma-\frac{1}{2}}(\mathcal{L})$ can be associated with a $g \in H^{\sigma-\frac{1}{2}}(\mathcal{B})$ so that f and g satisfy the compatibility conditions. For $m \geq 2$, this is only continuity at the edge, which is easily achieved. For $m = 1$, we need to find g such that $\int_0^1 \|f(t) - g(t)\|_{L^2(\mathbf{R}^{n-2})}^2 \frac{dt}{t} < +\infty$. This is obtained by taking $g = f$. \square

We also remark that since functions in Sobolev spaces $H^{\sigma}(\Omega)$, $\sigma > \frac{1}{2}$ are extendible to functions in $H^{\sigma}(\mathbf{R}^n)$, and since we can take traces on \mathcal{L} (or on \mathcal{B}) for those functions, the trace part of the previous Proposition is valid up to $\sigma > \frac{1}{2}$.

We will also need product estimates, which have exactly the same form as the one on \mathcal{L} . Again they will be used liberally.

5.3.4 Elliptic regularity

Our next point of order is to study two operators that appear frequently in the analysis. The first is the harmonic extension operator \mathcal{H} , which takes a function $f \in H^{\sigma+\frac{1}{2}}$, $0 \leq \sigma \leq s - \frac{1}{2}$,

on \mathcal{S} to the solution of

$$\begin{cases} \Delta \mathcal{H}(f) = 0 \text{ in } \Omega, \\ \mathcal{H}(f)|_{\mathcal{S}} = f, \\ \nabla_{\nu} \mathcal{H}(f)|_{\mathcal{B}} = 0. \end{cases}$$

The second one takes an $H^{\sigma-1}$ function g on Ω and an $H^{\sigma-\frac{1}{2}}$ function h on \mathcal{B} to the solution $q =: \Delta^{-1}(g, h)$ of

$$\begin{cases} \Delta q = g \text{ in } \Omega, \\ q|_{\mathcal{S}} = 0, \\ \nabla_{\nu} q|_{\mathcal{B}} = h. \end{cases}$$

We would like to prove that those mapping are continuous with value in $H^{\sigma+1}(\Omega)$. These are both particular cases of the more general problem of finding the regularity of the solution q of the problem

$$(5.3.4) \quad \begin{cases} \Delta u = g \text{ in } \Omega, \\ u|_{\mathcal{S}} = f, \\ \nabla_{\nu} u|_{\mathcal{B}} = h, \end{cases}$$

with $(f, g, h) \in H^{\sigma+\frac{1}{2}}(\mathcal{S}) \times H^{\sigma-1}(\Omega) \times H^{\sigma-\frac{1}{2}}(\mathcal{B})$, where $0 \leq \sigma \leq s - \frac{1}{2}$, with $h = 0$ if $\sigma < \frac{1}{2}$ because it would not be defined in a strong sense.

For future reference, we give the full existence and regularity theory for those problems, and not only the a priori estimates.

In this endeavor, we are faced with two challenges. The first is technical: in order to use our estimates in the evolution problem, our constants have to be of the form $C(1 + |\mathcal{S}|_s)$, with C uniform in Λ_* . To solve this, we use the global coordinates Φ_{Ω} defined in Proposition 5.3.7 to pull back the problem to Ω_* , which gives us a family of problems with coefficients bounded by a constant of the form we want. Then we prove a regularity theory for those problems, using freely the information that the surface is in Λ_* , but using the information that it is in H^s only once. This will give us the regularity for our problems, with constant as above.

The second challenge is deeper. The domain Ω_* has an edge, and it is well known that elliptic problems in domain with corners or edges give solution which have in general a limited regularity at the corner, whatever the smoothness of the data. We will prove below, using variational methods, that an H^1 solution always exists. If $\sigma > 0$, one would expect from the case of a regular boundary that the solution should be $H^{\sigma+1}$. However, in general for domains with corner, the solution is not necessarily $H^{\sigma+1}$ at the edge. To be more specific, it can be decomposed between a regular $H^{\sigma+1}$ part and an explicit sums of singularities of the form r^{λ} or $r^{\lambda} \ln r$ where r is the distance to the edge, and the λ are a discrete set of real numbers, here of the form $(k + 1/2)\pi/\omega$ with ω the contact angle. Therefore, the first singularity to appear, for $\lambda = \frac{\pi}{2\omega}$, limits the regularity of the solution to $H^{1+\frac{\pi}{2\omega}-}$. To avoid the presence of those singularities in the evolution problem, we restrict our attention to the case where $\omega < \pi/(n+1)$, so that we can take the regularity of the surface to be H^s with $1 + \frac{n}{2} < s < \frac{\pi}{2\omega} - \frac{1}{2}$ and have at the same time solutions of (5.3.4) with the expected regularity, and enough regularity of the surface to find solutions to the Cauchy problem. Our analysis follows closely the method in [39].

Because the meaning of the problem changes from variational to classical as σ increases, we recast it as follows. First, we define for each $\mathcal{S} \in \Lambda_*$ the bilinear form

$$a_{\Omega}(u, v) := \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx = \int_{\Omega_*} \nabla_* u \cdot \overline{\nabla_* v} \, dx_*,$$

where ∇_* and dx_* are the pullback by Φ of the gradient and the Lebesgue measure to Ω_* . They both derive from the pullback of the Euclidean metric to Ω , giving a bounded family of $H^{s-\frac{1}{2}}$ metrics on Ω_* . We also identify u and v with their pullback to keep notations simple. Those forms are well-defined on $H^1(\Omega_*)$.

Define the variational space

$$V := \{v \in H^1(\Omega_*), v = 0 \text{ on } \mathcal{S}_*\}.$$

The family of diffeomorphisms Φ_Ω induce an uniformly bounded family of isomorphism between V and the space $V_\Omega := \{v \in H^1(\Omega), v = 0 \text{ on } \mathcal{S}\}$. Therefore if a is strongly coercive on V_Ω with a constant independent of Ω , then it will be strongly coercive on V uniformly in Λ_* . This means that we have to prove that for any $v \in V_\Omega$,

$$\|v\|_{H^1(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)},$$

with C depending only on Λ_* .

To prove this, we first remark that since Ω is Lipschitz, the space V is the adherence for the H^1 norm of $C_c^\infty(\bar{\Omega} \setminus \mathcal{S})$. Also, because $s - \frac{1}{2} > \frac{n+1}{2}$, the set Λ_* is bounded in the L^∞ topology, and therefore all the domains Ω are contained in a band delimited by two parallel hyperplanes. The function in $C_c^\infty(\bar{\Omega} \setminus \mathcal{S})$ are simply the restriction of smooth functions that are zero near the ‘‘upper’’ hyperplane. Because we have defined the H^s norms in Ω as the quotient norm from \mathbf{R}^n , our inequality is a consequence of the fact that for those smooth function,

$$\|v\|_{H^1} \leq C \|\nabla v\|_{L^2},$$

with C depending only on the distance between those hyperplanes, which is simply the Poincaré inequality. In summary, we have proved the following lemma.

Lemma 5.3.9. *There exists a constant C depending only on Λ_* such that for any \mathcal{S} in Λ_* , the form a_Ω satisfies for any $u \in V$*

$$\|u\|_{H^1(\Omega)}^2 \leq Ca(u, u).$$

Therefore by Lax-Milgram, the family a generates a bounded family of isomorphisms A_Ω^* between V and its dual V' , defined by $(A_\Omega^* u)(v) = a_\Omega(u, v)$.

To simplify the notations, we now omit the subscript Ω from our operators, keeping in mind that we are really dealing with a family of problems on which our estimates have to be uniform.

The meaning of our problem (5.3.4) changes, as σ increases, from variational to classical. To be precise, we introduce the following family A of operators $A^{(\sigma)}$.

— For $0 \leq \sigma \leq \frac{1}{2}$, we take

$$A^{(\sigma)} : H^{\sigma+1}(\Omega_*) \rightarrow H^{\sigma+\frac{1}{2}}(\mathcal{S}_*) \times (V^{1-\sigma})',$$

where $V^{1-\sigma} := \{v \in H^{1-\sigma}(\Omega_*); v|_{\mathcal{S}_*} = 0\}$. It is defined by

$$A^{(\sigma)} u = (u|_{\mathcal{S}_*}, g),$$

where for any $v \in V^{1-\sigma}$,

$$g(v) = a(u, v).$$

— For $\frac{1}{2} < \sigma < 1$, we take

$$A^{(\sigma)} : H^{\sigma+1}(\Omega_*) \rightarrow H^{\sigma+\frac{1}{2}}(\mathcal{S}_*) \times (H^{\sigma-1}(\Omega_*)) \times H^{\sigma-\frac{1}{2}}(\mathcal{B}_*),$$

defined by

$$A^{(\sigma)}u = (u|_{\mathcal{S}_*}, g, (\nabla_\nu)_*u|_{\mathcal{B}_*}),$$

where for any $v \in H^{1-\sigma}(\Omega_*)$,

$$g(v) = a(u, v) - \int_{\mathcal{B}_*} (\nabla_\nu)_*u \bar{v},$$

with $(\nabla_\nu)_*u$ the pullback to \mathcal{B}^* of $\nabla_\nu u$ on \mathcal{B} . Here we have identified $H^{\sigma-1}$ as the dual to $H^{1-\sigma}$ since $0 < 1 - \sigma < \frac{1}{2}$.

— For $1 \leq \sigma \leq s - \frac{1}{2}$, we take

$$A^{(\sigma)} : H^{\sigma+1}(\Omega_*) \rightarrow H^{\sigma+\frac{1}{2}}(\mathcal{S}_*) \times H^{\sigma-1}(\Omega_*) \times H^{\sigma-\frac{1}{2}}(\mathcal{B}_*),$$

defined by

$$A^{(\sigma)}u = (u|_{\mathcal{S}_*}, -\Delta_*u, (\nabla_\nu)_*u|_{\mathcal{B}_*}),$$

where Δ_* is the Laplace operator for the pulled-back metric.

We remark that this family of operators correspond more properly to $-\Delta$, which of course does not change anything.

Because of Green's identity on the domain Ω , there holds

$$a(u, v) = \int_{\Omega_*} (-\Delta_*)u \bar{v} + \int_{\mathcal{S}_*} (\nabla_\nu)_*u \bar{v} + \int_{\mathcal{B}_*} (\nabla_\nu)_*u \bar{v}$$

for regular functions.

The expression

$$\int_{\Omega_*} \nabla_*u \cdot \overline{\nabla_*v} dx_*$$

makes sense when the integral is interpreted as a duality product in $H^\sigma \times H^{-\sigma}$, for $0 \leq \sigma \leq \frac{1}{2}$. Therefore if $0 \leq \sigma \leq \frac{1}{2}$, for $u \in H^{1+\sigma}$, $a(u, \cdot)$ can be thought of as a linear form on $V^{1-\sigma}$. The definition is to be interpreted in this sense.

If v is zero on \mathcal{S} and the functions are regular enough, there holds

$$a(u, v) - \int_{\mathcal{B}_*} (\nabla_\nu)_*u \bar{v} = \int_{\Omega_*} (-\Delta_*)u \bar{v},$$

and again if we take the integral as a duality product, for $\frac{1}{2} < \sigma \leq 1$ the right-hand side makes sense for $u \in H^{1+\sigma}$ as a linear form on $v \in H^{1-\sigma}$. It is again in this sense that the definition is to be interpreted.

For $\sigma = 1$, the Green formula tells us the classical and variational formulations coincide.

Proposition 5.3.10. *The operator $A^{(0)}$ is an isomorphism between $H^1(\Omega_*)$ and $H^{\frac{1}{2}}(\mathcal{S}_*) \times (V)'$, whose inverse has bounded norm as \mathcal{S} varies in Λ_* .*

Proof. Being given $(f, g) \in H^{\frac{1}{2}}(\mathcal{S}_*) \times (V)'$, We want to find $u \in H^{\sigma+1}(\Omega_*)$ such that $f = u|_{\mathcal{S}_*}$ and $g(v) = a(u, v)$ for $v \in V$. We consider a Sobolev extension $f \mapsto \tilde{f}$ from $H^{\frac{1}{2}}(\mathcal{S}_*)$ to $H^1(\Omega_*)$. It exists by a construction similar to the one in the proof of Proposition 5.3.7. Then we use the strong coercivity of a to find $\tilde{u} \in V$ such that

$$a(\tilde{u}, v) = g(v) - a(\tilde{f}, v),$$

and we set $u = \tilde{u} + \tilde{f}$. The fact that the constant associated to this construction is uniform in Λ_* comes from the uniformity of the constant in the coercivity of a . \square

We link those different formulations using the following injections $I_{\sigma,\sigma'}$ of the target of $A^{(\sigma')}$ into the target of $A^{(\sigma)}$, for $\sigma < \sigma'$.

- If $0 \leq \sigma < \sigma' < \frac{1}{2}$, they are the canonical embeddings of the space $H^{\sigma+\frac{1}{2}}(\mathcal{S}_*) \times (V^{1-\sigma})'$ in $H^{\sigma'+\frac{1}{2}}(\mathcal{S}_*) \times (V^{1-\sigma'})'$.
- If $0 \leq \sigma < \frac{1}{2} < \sigma'$, we pose $I_{\sigma,\sigma'}(f, g, h) = (f, g')$ with

$$g'(v) = g(v) + \int_{\mathcal{B}_*} h \bar{v}$$

which define an embedding from $H^{\sigma+\frac{1}{2}}(\mathcal{S}_*) \times (H^{1-\sigma}(\Omega_*))' \times H^{\sigma-\frac{1}{2}}(\mathcal{B}_*)$ to $H^{\sigma+\frac{1}{2}}(\mathcal{S}_*) \times (V^{1-\sigma})'$.

- If $\frac{1}{2} < \sigma < \sigma'$, again we take the trivial embedding.

We remark that the definition does not depend on a , and is therefore the same for all $\mathcal{S} \in \Lambda_*$. The following is immediate.

Lemma 5.3.11. 1. For $0 \leq \sigma < \sigma' < \sigma''$,

$$I_{\sigma,\sigma'} \circ I_{\sigma',\sigma''} = I_{\sigma,\sigma''}.$$

2. For $0 \leq \sigma < \sigma'$,

$$A^{(\sigma)}|_{H^{1+\sigma'}} = I_{\sigma,\sigma'} \circ A^{(\sigma')}.$$

3. For $0 \leq \sigma < \sigma'$, $I_{\sigma,\sigma'}$ is compact.

As a consequence of this and of Proposition 5.3.10, being given $(f, g) \in H^{\sigma+\frac{1}{2}}(\mathcal{S}_*) \times (V^{1-\sigma})'$ for $0 \leq \sigma < \frac{1}{2}$, or $(f, g, h) \in H^{\sigma+\frac{1}{2}}(\mathcal{S}_*) \times H^{\sigma-1}(\Omega_*) \times H^{\sigma-\frac{1}{2}}(\mathcal{B}_*)$ for $\frac{1}{2} < \sigma \leq s - \frac{1}{2}$, we can always define an H^1 variational solution u by

$$u = (A^{(0)})^{-1} I_{0,\sigma}(f, g, h).$$

Because we have included transversality in our definition of Λ_* , for any \mathcal{S} in it and any point on its waterline \mathcal{L} , we can define a contact angle as the angle between the inward normal vector to \mathcal{L} in \mathcal{B} and the inward normal vector to \mathcal{L} in \mathcal{S} . This defines a continuous function on \mathcal{L} because $s - \frac{1}{2} > \frac{n+1}{2}$ and $n \geq 2$. Therefore by taking δ small enough, which shrinks Λ_* , we can assume that all the angles of all the surfaces in Λ_* lie in an interval $[\underline{\omega}, \bar{\omega}]$ with $0 < \underline{\omega}$ and $\bar{\omega} < 2\pi$. Our aim is then to prove the following theorem.

Theorem 5.3.12. For any \mathcal{S} in Λ_* , for any $0 \leq \sigma < \min(s - 1, \frac{\pi}{2\bar{\omega}})$, the operator $A_{\Omega}^{(\sigma)}$ is an isomorphism. The norm of the inverse is uniformly bounded in Λ_* .

If \mathcal{S} is also in H^s , and if $\sigma < \min(s - \frac{1}{2}, \frac{\pi}{2\bar{\omega}})$, then the operator $A_{\Omega}^{(\sigma)}$ is still an isomorphism, and the norm of the inverse is bounded by $C(1 + |\mathcal{S}|_s)$, with C uniform in Λ_* .

Because we already now that a variational solution u exists, and because $A^{(0)}|_{H^{1+\sigma'}} = I_{0,\sigma} \circ A^{(\sigma)}$, the statements are really on the regularity of this variational solution.

Since $\bar{\Omega}_*$ is compact, the regularity of the solution is equivalent to its regularity in a neighborhood of each point. More precisely, if we show that for any point of $\bar{\Omega}_*$, a solution of the problem supported in a small enough neighborhood of this point (the neighborhood being uniform in Λ_*) satisfies the elliptic regularity estimates, then we can cover $\bar{\Omega}$ by finitely many open sets of this form, take an associated smooth partition of unity, and decompose any solution of the elliptic problem into pieces supported in those open sets, satisfying the elliptic problem with right-hand side the sum of the localized right-hand-side of the original problem, and a more

regular localization term. Those pieces will therefore satisfy the regularity theory, which we can then transfer to the original problem.

Regularity near interior points and near regular points of the boundary is classical, because Δ is an elliptic operator so that Δ_* also is. In fact, the maximal angle does not limit the regularity there. The dependence on the constant comes from the bootstrap nature of the estimates: we prove the regularity at level H^σ by assuming it at level $H^{\sigma-\frac{1}{2}}$, so that we only need to use the information that \mathcal{L} is H^s once.

We concentrate on the case of the neighborhood of a point $x \in \mathcal{L}$. The methods rest on the analysis of a constant-coefficients model operator, which we think of as “frozen” at the point x , the regularity of which will imply the regularity of the original problem.

Model operator The model operator is constructed as follows. We start with the form a on Ω_* . It can be written

$$a(u, v) = \sum_{i,j=1}^n \int_{\Omega_*} \alpha_{i,j} \partial_i u \overline{\partial_j v},$$

where the $\alpha_{i,j}$ are $H^{s-\frac{1}{2}}$ functions in Ω_* . Because $s - \frac{1}{2} > \frac{n+1}{2}$, they are continuous functions and therefore make sense at x . We can then consider the constant coefficient form p on $\mathbf{R}^{n-2} \times \Gamma_*$, where Γ_* is the conical sector of the plane of summit 0 and angle ω_* , the contact angle at x , defined by the formula

$$p(u, v) = \sum_{i,j=1}^n \int_{\mathbf{R}^{n-2} \times \Gamma_*} \alpha_{i,j}(x) \partial_i u \overline{\partial_j v}.$$

Then we further reduce the problem to the form l defined on Γ_* by

$$l(u, v) = \sum_{i,j=1}^2 \int_{\Gamma_*} \alpha_{i,j}(x) \partial_i u \overline{\partial_j v}$$

where we have only kept the derivatives corresponding to the variables in Γ_* . If we had performed this freezing for the euclidean Laplace operator on Ω , we would have found

$$\int_{\Gamma} \nabla u \cdot \nabla v$$

on the $2D$ angular sector of opening ω . Our form l is simply the pull-back to Γ_* by $d\Phi_{\Omega}(x)|_{\Gamma_*}$. It also satisfies a Green formula, the pull-back of the one on Γ , and we can attach it to a family of problems $L^{(\sigma)}$ in the exact same way we have used for A . The constant-coefficients differential operator L it is attached to is simply obtained by freezing the principal part of Δ_* at x and replacing all derivatives in the part tangential to \mathcal{L}_* by 0, and the Neumann condition is for the normal vector with constant coefficients frozen at x and projected onto Γ_* . This is again the pull-back of the Dirichlet-Neumann Laplace operator on Γ .

To further simplify the study of those problems, we remark that this definition is really an invariant notion. The reason is that if we have a local H^s diffeomorphism near x to another neighborhood of a point in the transversal intersection of two hypersurfaces, its differential at x makes sense again because $s - 1 > \frac{n}{2}$, and can be interpreted as a linear diffeomorphism between $\mathbf{R}^{n-2} \times \Gamma_*$ and the corresponding angular sector $\mathbf{R}^{n-2} \times \Gamma'$, sending Γ_* isomorphically onto Γ' . It is then easy to see that the restriction of this linear isomorphism to Γ_* sends l to the constant-coefficients form obtained by sending a to the new domain, and freezing the coefficients

as above. Applying this to the diffeomorphism Φ between Ω_* and Ω , we see that the study of the $L^{(\sigma)}$ is equivalent to the study of the problems derived from the form

$$\int_{\Gamma} \nabla u \cdot \overline{\nabla v} dz$$

in the sector Γ whose angle ω is the original contact angle in Ω . It corresponds to the constant coefficients Laplace operator, with Dirichlet condition on one edge and Neumann condition on the other. Because we want our constants to be uniform in Λ_* , we will study those operators in the fixed domain Γ_* . We will however use the version in Γ to find a particular algebraic condition. Remark that the coefficients of the form l are a fixed number of constants inhabiting a compact set as \mathcal{S} varies in Λ_* .

We denote by B_* the bottom edge and by S_* the surface edge of Γ_* . Our aim is to prove the following Proposition.

Proposition 5.3.13. *For any \mathcal{S} in Λ_* , for any $0 \leq \sigma < \frac{\pi}{2\omega}$, if there exists $u \in H^1(\Gamma_*)$ compactly supported and*

1. $(f, g) \in H^{\sigma+\frac{1}{2}}(S_*) \times (V^{1-\sigma})'$ for $0 \leq \sigma < \frac{1}{2}$,
2. $(f, g, h) \in H^{\sigma+\frac{1}{2}}(S_*) \times (H^{1-\sigma}(\Gamma_*))' \times H^{\sigma-\frac{1}{2}}(B_*)$ for $\frac{1}{2} < \sigma < 1$,
3. $(f, g, h) \in H^{\sigma+\frac{1}{2}} \times H^{\sigma-1}(\Gamma_*) \times H^{\sigma-\frac{1}{2}}(B_*)$ for $1 \leq \sigma$,

such that

$$L^{(0)}u = I_{0,\sigma}(f, g) \quad (\text{resp. } L^{(0)}u = I_{0,\sigma}(f, g, h)),$$

then $u \in H^{\sigma+1}(\Gamma_*)$, with constants of the good form, depending on the support of u .

Here we have abused notation to name $V^{1-\sigma} = \{v \in H^{1-\sigma}(\Gamma_*), v|_{S_*} = 0\}$ and the $I_{\sigma,\sigma'}$ have the obvious definitions. Their properties are the same, except that since Γ_* is not compact, they are only compact on the subspaces of functions supported in a compact K .

The reason of the index $\frac{\pi}{2\omega}$ is algebraic, and comes from the following. We let (r, θ) be the polar coordinates in Γ . For any λ in \mathbf{C} , and any $\sigma \in R$, we define

$$(5.3.5) \quad S^{\lambda,\sigma} := \left\{ v = \sum_{q=0}^Q r^\lambda \log^q r v_q(\theta); v_q \in H^{1+\sigma}([0, \omega_*]) \right\}.$$

We say that such a function is a polynomial if it is polynomial in the Cartesian coordinates z_1, z_2 of the angular sector Γ_* . This is only possible for $\lambda \in \mathbf{N}^*$, or if $v = 0$. We can define L on $S^{\lambda,\sigma}$ by testing against compactly supported function for the duality of $V^{1-\sigma}$ or $H^{1-\sigma}$, as is easily seen. Then we say that for $v \in S^{\lambda,\sigma}$, Lv is polynomial if there exists polynomials (f, g) (resp. (f, g, h)) such that $L^{(0)}v = I_{0,\sigma}(f, g)$ (resp. $L^{(0)}v = I_{0,\sigma}(f, g, h)$). If λ is not in \mathbf{N}^* , then (f, g, h) is zero. At last, L is said to be injective modulo polynomials if $v \in S^{\lambda,\sigma}$ is polynomial as soon as (f, g, h) are polynomials. Again if λ is not an integer, this is just injectivity.

Lemma 5.3.14. *Let $\sigma > 0$. Then L is injective modulo polynomials exactly for $\lambda \neq (k + \frac{1}{2})\frac{\pi}{\omega}$, with $k \in \mathbf{Z}$.*

Proof. It is easily seen that this notion is invariant by linear diffeomorphism, and therefore it is sufficient to check it on the euclidean Laplace operator in Γ . In polar coordinates, this is

$$\Delta = \frac{1}{r^2} [(r\partial_r)^2 + (\partial_\theta)^2],$$

with Dirichlet condition at $\theta = \omega$, and Neumann condition at $\theta = 0$. There, the inward normal is ∂_θ . Because of the classical ellipticity of Δ , this notion is independent of the regularity σ . We can therefore check it for smooth v_q .

Then injectivity modulo polynomials can be computed to be a cascade of ODEs on the v_q of the form

$$\partial_\theta^2 v_q + \lambda^2 v_q + 2\lambda(q+1)v_{q+1} + (q+2)(q+1)v_{q+2} = 0$$

for $q \geq 1$, with $v_{Q+1} = v_{Q+2} = 0$. Then starting from $q = Q$, we show that if $\lambda \neq (k + \frac{1}{2})\frac{\pi}{\omega}$, all the v_q are 0 except v_0 , which is such that v is polynomial. If $\lambda = (k + \frac{1}{2})\frac{\pi}{\omega}$, the fact that $\partial_\theta^2 + \lambda^2$ is not injective yield immediately a counter-example. \square

The rest of the proof is based on the use of the Mellin transform, for which the properties in Appendix A to [39] will suffice. We only recall its definition,

$$\mathcal{M}[u](\lambda) = \int_{\mathbf{R}} r^{-\lambda} u \frac{dr}{r},$$

and the following proposition :

Proposition 5.3.15. — *If $u \in H^\beta(\Gamma_*)$, with $\beta < 1$, and u is compactly supported, then the quantity $\mathcal{M}[u]$ is defined up to $\text{Re}\lambda = \beta - 1$ and analytic in $\text{Re}\lambda < \beta - 1$ with values in $H^\beta([0, \omega_*])$.*

— *If $u \in H^\beta(\Gamma_*)$, with $\beta \geq 1$, and u is compactly supported, then $\mathcal{M}[u]$ is holomorphic in $\text{Re}\lambda < 0$ with values in $H^\beta([0, \omega_*])$. It can be meromorphically extended up to $\text{Re}\lambda < \beta - 1$. Let U denote that extension. For $\text{Re}\lambda \in]k, k+1[$, $k \in \mathbf{N}$, U coincides with $\mathcal{M}[u - P_k u]$, where $P_k u$ denotes the Taylor series of order k of u at 0. Finally, the poles of U are simple and lie in $k \in \mathbf{N}$ with $k < \beta - 1$, and we have*

$$\text{Res}_{k=\lambda} r^\lambda U(\lambda) = P_{k-1} u - P_k u.$$

There is an inversion formula,

$$u(z) = \frac{1}{2\pi} \int_{\mathbf{R}} r^{-\eta+i\zeta} \mathcal{M}[u](-\eta+i\zeta) d\zeta,$$

valid on any vertical line in the domain of holomorphy of $\mathcal{M}[u]$. At last, one can measure the $H^\beta(\Gamma_*)$ regularity of u from the L^2 norm in ζ of the $H^\beta([0, \omega_*], |\zeta|+1)$ norm in η of $\mathcal{M}[u](-\eta+i\zeta)$. Here and in the following, $H^\beta([0, \omega_*], \rho)$ is the H^β space endowed with the norm

$$\|f\|_{\beta, \rho}^2 = \rho^{2\beta} \|f\|_{L^2}^2 + \|D^{[\beta]} f\|_{H^{\beta-[\beta]}}^2.$$

The interest of the Mellin transform is that it changes $r\partial_r$ into λ , so that the problem

$$\begin{cases} \Delta u = g \text{ in } \Gamma, \\ u|_{\theta=0} = f, \\ \nabla_\nu u|_{\theta=\omega} = h, \end{cases}$$

would become, through multiplication by the appropriate power of r and the Mellin transform,

$$\begin{cases} (\partial_\theta^2 + \lambda^2)U = \mathcal{M}[r^2 g] \text{ in } \Omega, \\ U|_{\theta=0} = \mathcal{M}[f], \\ \partial_\theta U|_{\theta=\omega} = \mathcal{M}[r h]. \end{cases}$$

Of course, we need to work in Ω_* and with the associated form l .

To simplify notations, we concentrate on the case $s - 1 \geq \sigma > \frac{1}{2}$, the proof of the other cases being similar.

So suppose we are in the hypotheses of Proposition 5.3.13. Then in Mellin, we give the following definitions.

- The Mellin transform of u is $U(\lambda)$, which is holomorphic in $\operatorname{Re}\lambda < 0$ and defined up to $\operatorname{Re}\lambda = 0$, with values in $H^1([0, \omega_*])$.
- The Mellin transform of (f, r^2g, rh) is $(F, G, H)(\lambda)$, which is meromorphic on $\operatorname{Re}\lambda < \sigma$, defined up to $\operatorname{Re}\lambda = \sigma$, with values in $\mathbf{R} \times (H^{1-\sigma}([0, \omega_*]))' \times \mathbf{R}$ (or $\mathbf{R} \times H^{\sigma-1}([0, \omega_*]) \times \mathbf{R}$). Its poles are concentrated at the non-negative integer values of λ .
- The operator $L^{(0)}$ becomes a holomorphic family of operators $\mathcal{L}^{(0)}(\lambda) : H^1([0, \omega_*]) \rightarrow \mathbf{R} \times (V([0, \omega_*]))'$ defined by

$$\left(\mathcal{L}^{(0)}(\lambda)v \right) (w) := (v(\omega_*), l(\lambda)(v, w)),$$

with $l(\lambda)$ a holomorphic family of forms on $V([0, \omega_*])$ with constant coefficients, bounded as \mathcal{L} varies in Λ_* .

- The $I_{0,\sigma}$ are transformed in a holomorphic family of compact operators $\mathcal{I}_{0,\sigma}(\lambda)$.

Now the equation on u becomes

$$\mathcal{L}^{(0)}(\lambda)U(\lambda) = \mathcal{I}_{0,\sigma}(\lambda)(F(\lambda), G(\lambda), H(\lambda)).$$

It is valid on the common domain of holomorphy of those functions, which is $\operatorname{Re}\lambda < 0$.

Lemma 5.3.16. *For any real numbers $\alpha < \beta$, there is a constant $C_{\alpha,\beta}$ uniform in Λ_* such that for any $\alpha < \operatorname{Re}\lambda < \beta$ with $|\operatorname{Im}\lambda| \gg 1$, again uniformly in Λ_* , $\mathcal{L}^{(0)}(\lambda)$ is invertible and satisfies*

$$\left\| \mathcal{L}^{(0)}(\lambda)^{-1}(F, G) \right\|_{H^1} \leq C_{\alpha,\beta} [\|F\| + \|G\|_{V'}].$$

The family $\mathcal{L}^{(0)}(\lambda)^{-1}$ is meromorphic in \mathbf{C} .

Proof. The first step is to deduce the coercivity of l from the one of a . We can find an $H^{s+\frac{1}{2}}$ diffeomorphism sending a neighborhood of our point x to a neighborhood of 0 in $\mathbf{R}^{n-2} \times \Gamma_*$ such that the frozen forms it produce are the p and l already defined. It suffices to compose our local coordinates map with the inverse of its linearized at x .

Then the coefficients of this form a are regular enough that

$$|p(u, u) - a(u, u)| \leq C \left[\rho \|u\|_{H^1}^2 + \|u\|_{L^2}^2 \right]$$

for any $u \in V$ with support in $B(0, \rho)$, $\rho \leq 1$. Combined with the coercivity of a , this yields

$$\|u\|_{H^1}^2 \leq C \left\{ p(u, u) + \|u\|_{L^2}^2 \right\}$$

if ρ is small enough. Applying this to $v = u(\frac{1}{\rho} \cdot)$ and letting ρ go to 0 gives

$$\|v\|_{H^1}^2 \leq C \left\{ p(v, v) + \|v\|_{L^2}^2 \right\}$$

for v compactly supported, and thus by density for any v . Then applying this to $v(y, z) = \psi(y)w(z)$ with ψ compactly supported, and equal to one on the unit ball of \mathbf{R}^{n-2} , we get

$$\|w\|_{H^1}^2 \leq C \left\{ l(w, w) + \|w\|_{H^1} \|w\|_{L^2} \right\}.$$

Applying this result to $w(z) = \chi(r)r^\lambda u(\theta)$, for χ compactly supported, equal to 1 near $r = 1$ and to 0 near $r = 0$, and for $\alpha < \operatorname{Re}\lambda < \beta$, this yields

$$\|u\|_{H^1(|\lambda|)}^2 \leq C_{\alpha,\beta} \left\{ \operatorname{Re}(\lambda)(u, u) + \|u\|_{H^1(|\lambda|)} \|u\|_{L^2} \right\}.$$

Remarking that $\|u\|_{L^2} \leq C\lambda^{-1} \|u\|_{H^1(|\lambda|)}$, we get for λ big enough

$$\|u\|_{H^1(|\lambda|)}^2 \leq C_{\alpha,\beta} \{ \operatorname{Re}(\lambda)(u, u) \}.$$

This holds for $u \in V([0, \omega_*])$ and is the required coercivity. This is enough to prove the invertibility of $\mathcal{L}(\lambda)$ and the accompanying estimates.

For the meromorphy, we start by observing that for $\lambda, \lambda' \in \mathbf{C}$, the operator $\mathcal{L}^{(0)}(\lambda) - \mathcal{L}^{(0)}(\lambda')$ is compact. Then if λ_0 is such that $\mathcal{L}^{(0)}(\lambda_0)$ is invertible, we can write

$$\mathcal{L}^{(0)}(\lambda) = \mathcal{L}^{(0)}(\lambda_0) \circ (I + \mathcal{Z}(\lambda))$$

with

$$\mathcal{Z}(\lambda) = \mathcal{L}^{(0)}(\lambda_0)^{-1} \circ (\mathcal{L}^{(0)}(\lambda) - \mathcal{L}^{(0)}(\lambda_0)).$$

Thus $\mathcal{Z}(\lambda)$ is a holomorphic family of compact operators, with $I + \mathcal{Z}(\lambda_0) = I$ invertible. This implies the meromorphy of the family of index 0 Fredholm operators $\mathcal{L}^{(0)}(\lambda)$, and therefore the meromorphy of $\mathcal{L}^{(0)}(\lambda)^{-1}$. \square

Then we need to prove the regularity of the family \mathcal{L} .

Lemma 5.3.17. *For any $\operatorname{Re}\lambda \in [0, \sigma]$, if $U \in H^1([0, \omega_*])$ is such that*

$$L^{(0)}(\lambda)U = \mathcal{I}_{0,\sigma}(\lambda)(F, G, H),$$

then $U \in H^{\sigma+1}([0, \omega_])$, and*

$$\|U\|_{H^{\sigma+1}} \leq C \left[|F| + \|G\|_{(H^{1-\sigma})} + |H| \right]$$

with C uniform in λ and in Λ_ .*

Proof. Because the L are constant coefficients, their coefficients when \mathcal{S} varies in Λ inhabit a compact set. Therefore it is sufficient to prove this for a fixed $\mathcal{S} \in \Lambda_*$, with a constant that is an increasing function of the supremum of the coefficients. The operator family L is elliptic, and with constant coefficients. The classical elliptic regularity theory can be applied far from 0, near any interior and boundary point. The constant depends on the coefficients continuously. Then applying this regularity to $w(z) = \chi(r)r^\lambda u(\theta)$ as above, this function being compactly supported away from 0, we immediately find the announced regularity. \square

Then we want to define $U(\lambda)$, which only made sense for $\operatorname{Re}\lambda \leq 0$ initially, in the half-plane $\operatorname{Re}\lambda \leq \sigma$. The formula

$$(5.3.6) \quad U(\lambda) = \mathcal{L}^{(\sigma)}(\lambda)^{-1}(F(\lambda), G(\lambda), H(\lambda))$$

agrees with the definition of $U(\lambda)$ in the left half-plane, because

$$\mathcal{L}^{(0)}|_{H^{1+\sigma}} = I_{0,\sigma} \circ \mathcal{L}^{(\sigma)}.$$

Also we have already seen that $\mathcal{L}^{(0)}(\lambda)^{-1}$ is meromorphic. Its poles are the places where $\mathcal{L}^{(0)}(\lambda)$ fails to be injective. It is immediate that the injectivity of $\mathcal{L}^{(0)}$ on $H^1([0, \omega_*])$ is equivalent

to the injectivity of L on $S^{\lambda,0}$, the space defined in (5.3.5). Because of Lemma 5.3.14, we know that for $\sigma < \frac{\pi}{2\omega}$, this injectivity is true for any λ except the integers, where only injectivity modulo polynomials holds. Therefore the only possible poles of $\mathcal{L}^{(0)}(\lambda)^{-1}$ when σ is in the range of Proposition 5.3.13 are the integers between 0 and σ .

Because of the preceding Lemma, $\mathcal{L}^{(\sigma)}(\lambda)^{-1}$ is also meromorphic with the same poles. Then because F , G and H where also meromorphic with integer poles, (5.3.6) define a meromorphic extension of U to $\operatorname{Re}\lambda < \sigma$, with integer poles.

We want to take the inverse Mellin transform of $U(\lambda)$ along $\operatorname{Re}\lambda = \sigma$. However, because of the possible presence of a pole at σ when $\sigma \in \mathbf{N}$, we need to be careful.

First, take $\sigma' \leq \sigma$ that is not an integer. If already σ is not an integer, we can take $\sigma' = \sigma$. Then $U(\lambda)$ is meromorphic on $\operatorname{Re}\lambda \leq \sigma'$.

Taking the inverse Mellin transform along $\operatorname{Re}\lambda = \sigma'$, we find a function $u_0 \in H_0^{\sigma'+1}(\Gamma_*)$, with norm bounded by

$$\int_{\operatorname{Re}\lambda=\sigma'} \|U(\lambda)\|_{\sigma'+1,|\lambda|}^2 d\lambda.$$

However

$$\|U(\lambda)\|_{\sigma'+1,|\lambda|}^2 \leq C \left[|F(\lambda)| + \|G(\lambda)\|_{\sigma'-1,|\lambda|} + |H(\lambda)| \right],$$

the value for large λ coming from Lemma 5.3.16 and the elliptic regularity, while for the others values of λ we use the qualitative information that $\mathcal{L}^{(s)}$ is invertible to find the equality with constants C_λ and then we take the supremum of those constants because λ is in a compact set. This constant is also uniform in Λ_* because the coefficients of the problems are in a compact set. This means that

$$\|u_0\|_{\sigma'+1} \leq C \left[\|f\|_{\sigma+\frac{1}{2}} + \|g\|_{\sigma-1} + \|h\|_{\sigma-\frac{1}{2}} \right].$$

Now because of the Cauchy formula between $\operatorname{Re}\lambda = 0$ and $\operatorname{Re}\lambda = \sigma'$, we find

$$u - u_0 = 2i\pi \sum_{\mu=0}^{\lfloor \sigma' \rfloor} \operatorname{Res}_{\lambda=\mu} r^\lambda \mathcal{L}^{(\sigma')}(\lambda)^{-1}(F(\lambda), G(\lambda), H(\lambda)).$$

Then writing u_μ for the μ th residue, we see that for χ a cutoff at 0, for $\mu > 0$, $\chi u_\mu \in V$. Since $u \in V$, we deduce immediately that $\chi u_0 \in V$ and therefore u_0 is a polynomial. We want to prove by recurrence that all the u_k are polynomials. By subtracting the ones already known to be polynomials, it is sufficient to assume that

$$\forall \mu \leq k-1, \operatorname{Res}_{\mu=\lambda} r^\lambda U(\lambda) = 0.$$

Then comparing $l(u - u_k, v)$ and $l(u, v)$ for $v \in V$ shows that $a(u_k, v)$ is a polynomial. Then injectivity modulo polynomials shows that u_k is a polynomial. The uniformity of the constants in Λ_* is immediate. This concludes the proof of Proposition 5.3.13 in the case where $\sigma \notin \mathbf{N}$.

If $\sigma \in \mathbf{N}$, we just proved that the regularity holds for $\sigma' < \sigma$. Keeping separate the regular part $u_0 \in H^{\sigma'+1}$ and the residues (which are polynomials), we see that we only need to prove that $\chi u_0 \in H^{\sigma+1}$ with χ a cut-off at 0. Thus we only need to show that

$$\forall \alpha, |\alpha| = \sigma + 1, D^\alpha u_0 \in L^2(\Gamma_*).$$

For this we need to extend

$$w(\lambda) = (\mathcal{D}^\alpha(\lambda)U(\lambda))|_{|\alpha|=\sigma+1}$$

up to $\operatorname{Re}\lambda = \sigma$, with the punctual convergence

$$w(\sigma' + i \cdot) \rightarrow w(\sigma + i \cdot) \text{ in } L^2(\mathbf{R} \times [0, \omega_*]) \cap L^{2,\sigma}(\mathbf{R}; L^2([0, \omega_*])),$$

where $\mathcal{D}^\alpha(\lambda)$ is the Mellin transform of D^α and

$$L^{2,\sigma}(\mathbf{R}; H) := \{u; (1 + |\cdot|)^\sigma \|u(\cdot)\|_H \in L^2(\mathbf{R})\}.$$

Then passing to the limit $\operatorname{Re}\sigma' \rightarrow \operatorname{Re}\sigma$ in the Cauchy formula would give us the result. But the usual limit case for the Mellin transform (again see [39]) gives the desired result. This concludes the proof of Proposition 5.3.13.

Full operator. The objective is now to come prove the regularity of the full problem starting from this model case. Recall that from the full form a we have first constructed a constant coefficients form p on $\mathbf{R}^{n-2} \times \Gamma_*$ by freezing the coefficients at 0. We can associate to this form p a series of problems P as we have done for a and l . The first objective is to derive the following regularity theory for P .

Proposition 5.3.18. *For any \mathcal{S} in Λ_* , for any $0 \leq \sigma < \frac{\pi}{2\omega}$, if there exists $u \in H^1(\mathbf{R}^{n-2} \times \Gamma_*)$ and*

1. $(f, g) \in H^{\sigma+\frac{1}{2}}(\mathbf{R}^{n-2} \times S_*) \times (V^{1-\sigma})'$ for $0 \leq \sigma < \frac{1}{2}$,
2. $(f, g, h) \in H^{\sigma+\frac{1}{2}}(\mathbf{R}^{n-2} \times S_*) \times (H^{1-\sigma}(\mathbf{R}^{n-2} \times \Gamma_*))' \times H^{\sigma-\frac{1}{2}}(\mathbf{R}^{n-2} \times B_*)$ for $\frac{1}{2} < \sigma < 1$,
3. $(f, g, h) \in H^{\sigma+\frac{1}{2}}(\mathbf{R}^{n-2} \times S_*) \times H^{\sigma-1}(\mathbf{R}^{n-2} \times \Gamma_*) \times H^{\sigma-\frac{1}{2}}(\mathbf{R}^{n-2} \times B_*)$ for $1 \leq r\sigma$,

such that

$$P^{(0)}u = I_{0,\sigma}(f, g) \quad (\text{resp. } P^{(0)}u = I_{0,\sigma}(f, g, h)),$$

then $u \in H^{\sigma+1}(\mathbf{R}^{n-2} \times \Gamma_*)$ near 0, with constant of the good form.

We have again used the obvious definition for V and the $I_{0,\sigma}$. This statement is different from the one for L only if $n \geq 3$, so that their is actually a transverse direction $y \in \mathbf{R}^{n-2}$.

Proof. The spirit of the proof is to first go to Fourier in the unbounded variable $y \in \mathbf{R}^{n-2}$, and then use homogeneity to reduce to the dual variable η in the sphere S^{n-3} . Then for such an η , the regularity far from 0 will be a simple consequence of the classical regularity theory, while the regularity near 0 will come from the one of L .

Accordingly, for $\eta \in S^{n-3}$, let $p(\eta)$ be the form deduced from p by going to Fourier in y , or equivalently replacing the ∂_y^α derivatives with multiplication by $i\eta^\alpha$. Let $P(\eta)$ be the associated family of problems. Assume we have $u \in V(\Gamma_*)$ and $(f, g, h) \in H^{\sigma+\frac{1}{2}}(S_*) \times (H^{1-\sigma}(\Gamma_*))' \times H^{\sigma-\frac{1}{2}}(B_*)$ such that

$$P^{(0)}(\eta)u = I_{0,\sigma}(f, g, h).$$

We want to prove that $u \in H^{\sigma+1}(\Gamma_*)$, with the associated constant uniform in Λ_* . Here u need not have compact support, which is important for what follows. We can decompose the regularity between points far from 0 and 0. The regularity at 0 is an immediate consequence of the regularity of L and the easily established compactness of $P(\eta) - L$ on compactly supported functions, which comes from the fact that the associated form $a - p(\eta)$ involves no first-order derivatives.

For the regularity far from 0, we start by remarking that the regularity of P near any point $(0, z) \in \mathbf{R}^{n-2} \times \Gamma_*$ with $|z| = 1$ comes immediately from the constancy of its coefficients and the classical interior and boundary estimates. The constant can be taken uniformly for those points because they form a compact set. Then applying the associated regularity theory and estimates to a cut-off near this set times $e^{i2^\gamma \langle \eta, y \rangle} u(2^\gamma z)$ for $\gamma \in \mathbf{N}^*$ gives the regularity of u in dyadic crowns exhausting $B(0, 1)^c$, uniformly in 2^γ . Then summing the pieces gives the regularity far from 0. The constant are of course smoothly dependent on the coefficients of p

and on $\eta \in S^{n-2}$, which inhabits compact set. Therefore it can be taken uniform in Λ_* and in $\eta \in S^{n-2}$.

Then we derive that for any $\eta \in \mathbf{R}^{n-2}$, if u has compact support in $B(0, 1)$, then $P(\eta)^{(0)}u = I_{0,\sigma}(f, g, h)$ means that $u \in H^{\sigma+1}$ with estimations uniform in the norms $H^r(\Gamma_*, |\eta|)$. This is immediate for $\eta \leq 1$ because those norms are equivalent to the classical Sobolev norm, and because $P(\eta) - P(\eta/|\eta|)$ is compact.

For $\eta > 1$, we simply apply the regularity of $P(\eta/|\eta|)$ to $u(z/|\eta|)$. The fact that the support of this function goes to infinity is the reason we needed estimates far from 0 in the preceding step.

Then by using Fourier, those estimates immediately imply Proposition 5.3.18. \square

Then the remainder of the proof of Theorem 5.3.12 is simple. We first deduce from the regularity of P near 0 the regularity of A near $x \in \mathcal{L}$, by mapping a neighborhood of x to a neighborhood of 0 in $\mathbf{R}^{n-2} \times \Gamma_*$ so that p is its associated frozen coefficients form, then using the $H^{s-\frac{1}{2}}$ (with $s > 1 + \frac{n}{2}$) regularity of the coefficients to write that the form a is close to the form p for functions with support small enough, close to x the size of this support depends only on the Lipschitz norm of those coefficients, so that it is bounded from below for $\mathcal{S} \in \Lambda_*$. Those steps are where the regularity of the solution gets limited to $s + \frac{1}{2}$.

Then combining this with the classic regularity near other points, and using the compactness of Ω_* , we can finish the proof. For the last step of regularity, assuming \mathcal{S} is H^s , one needs simply to use the $H^s(\Omega)$ regularity of the solution, with constants uniform in Λ_* , and then prove the regularity up to $H^{s+\frac{1}{2}}(\Omega)$ by repeating the proof above, taking care that the dependence in the $H^{s-\frac{1}{2}}(\Omega)$ norm of the coefficients, and therefore in $|\mathcal{S}|_s$, is linear.

It can be remarked in the proof above that the regularity is limited by the angle only at \mathcal{L} . Therefore, if $s \geq \frac{\pi}{n+1}$, the regularity of the solution in the full domain will only be $H^{1+\frac{\pi}{2\omega_{\max}}}$, but we still have the following.

Lemma 5.3.19. *Take $s > 1 + \frac{n}{2}$, an H^s surface \mathcal{S}_* as above, and $\delta > 0$ small enough. Assume $\mathcal{S} \in \Lambda_*$ is also in H^s . Take u a variational solution as above, with data $(f, g, h) \in H^{r+\frac{1}{2}} \times H^r \times H^{r-\frac{1}{2}}$, with $0 \leq r \leq s - \frac{1}{2}$. Then u is locally in H^{r+1} near any point of $\Omega \setminus \mathcal{L}$.*

Other elliptic problems Another elliptic problem in Ω we will need to solve is

$$\begin{cases} \Delta u = g \text{ in } \Omega, \\ \nabla_N u|_{\mathcal{S}} = f, \\ \nabla_\nu u|_{\mathcal{B}} = h, \end{cases}$$

subject to the natural condition

$$\int_{\Omega} g \, dx = \int_{\mathcal{S}} f \, dS + \int_{\mathcal{B}} h \, dS.$$

The resolution is completely analogous to the previous one, although it has to be performed modulo constants since they are always solution of the homogeneous problem. The only real change is the model operator, which will of course be the Laplace operator on a sector of angle ω , but with Neumann condition at both sides. It is readily checked that the singularities appear for $\lambda = \frac{k\pi}{\omega}$, $k \in \mathbf{N}^*$, and therefore the solution is (modulo constants) in H^{r+1} with $0 < r \leq \min\left(s - \frac{1}{2}, \frac{\pi}{\omega_{\max}}\right)$.

In the same vein, one obtain the same regularity result for the Dirichlet-Dirichlet problem, assuming the two pieces of data can be pasted together at the angle to form a smooth enough function. The singularities are at the same place as in the Neumann-Neumann case.

5.3.5 The Dirichlet to Neumann operator

The fundamental operator for the dynamic is the Dirichlet to Neumann operator \mathcal{N} , defined on functions on \mathcal{S} by

$$\mathcal{N}f = \nabla_N \mathcal{H}(f),$$

where $\mathcal{H}(f)$ is the harmonic extension of f in Ω , satisfying

$$\begin{cases} \Delta \mathcal{H}(f) = 0 \text{ in } \Omega, \\ \mathcal{H}(f)|_{\mathcal{S}} = f, \\ \nabla_\nu \mathcal{H}(f)|_{\mathcal{B}} = 0. \end{cases}$$

As a consequence of the preceding analysis, this operator is continuous, elliptic, and self-adjoint on $L^2(\mathcal{S})$, for \mathcal{S} in Λ_* .

Proposition 5.3.20. *Let $s > 1 + \frac{n}{2}$. Let \mathcal{S}_* be an $H^{s-\frac{1}{2}}$ reference hypersurface, and δ small enough, so that in the corresponding Λ_* , the maximal angle satisfies $s < \frac{\pi}{2\bar{\omega}}$, and all other geometric conditions imposed above apply. Let $1 \leq \sigma \leq s - \frac{1}{2}$.*

1. *Continuity: there is a constant C , depending only on Λ_* and σ , such that if \mathcal{S} is an hypersurface in Λ_* , and $f \in H^\sigma(\mathcal{S})$, then $\mathcal{N}f \in H^{\sigma-1}(\mathcal{S})$, and*

$$\|\mathcal{N}f\|_{H^{\sigma-1}(\mathcal{S})} \leq C \|f\|_{H^\sigma(\mathcal{S})}.$$

2. *Ellipticity: there is a constant C , depending only on Λ_* and σ , such that if \mathcal{S} is an hypersurface in Λ_* , and $f \in H^{\frac{1}{2}}(\mathcal{S})$ is such that $\mathcal{N}f \in H^{\sigma-1}(\mathcal{S})$, then $f \in H^\sigma(\mathcal{S})$, and*

$$\|f\|_{H^\sigma(\mathcal{S})} \leq C \left[\|\mathcal{N}f\|_{H^{\sigma-1}(\mathcal{S})} + \|f\|_{L^2(\mathcal{S})} \right].$$

3. *Self-adjointness on $L^2(\mathcal{S})$: If f and g are in $H^1(\mathcal{S})$, then*

$$\int_{\mathcal{S}} f \mathcal{N}g \, dS = \int_{\mathcal{S}} g \mathcal{N}f \, dS.$$

Proof. The operator can be defined on $H^{\frac{1}{2}}(\mathcal{S})$ through the following weak formulation : for any (f, g) smooth enough, the Stokes formula is valid since the domain is Lipschitz, and gives

$$\int_{\mathcal{S}} f \mathcal{N}g \, dS = \int_{\Omega} \nabla \mathcal{H}f \cdot \nabla \mathcal{H}g \, dx.$$

Since the right-hand side is well-defined for f and g in $H^{\frac{1}{2}}(\mathcal{S})$, $\mathcal{N}g$ is well defined with values in $H^{-\frac{1}{2}}(\mathcal{S})$ (or more precisely in the dual of $H^{\frac{1}{2}}(\mathcal{S})$, which is contained in $H^{-\frac{1}{2}}(\mathcal{S})$). From elliptic regularity for the Dirichlet-Neumann problem, and since the angle condition is verified, it is well defined and continuous in the range $1 \leq \sigma \leq s - \frac{1}{2}$.

To prove the ellipticity, we simply remark that under the hypotheses, the harmonic extension $\mathcal{H}f$ is well-defined in $H^1(\Omega)$ and satisfies, in the weak sense, the Neumann-Neumann problem with value $\mathcal{N}f$ on \mathcal{S} . Since the angle condition for this problem is also respected, the preceding analysis gives us the required control with an $H^{\frac{1}{2}}(\mathcal{S})$ norm of f in place of the $L^2(\mathcal{S})$ norm. But then a simple interpolation argument enables us to conclude.

To check that \mathcal{N} is symmetric, we observe from the Stokes formula that for f and g in $H^1(\mathcal{S})$,

$$\int_{\mathcal{S}} f \mathcal{N}g \, dS = \int_{\Omega} \nabla \mathcal{H}f \cdot \nabla \mathcal{H}g \, dx = \int_{\mathcal{S}} g \mathcal{N}f \, dS.$$

To check self-adjointness, one should consider the closure $\overline{\mathcal{N}}$ of the Dirichlet-Neumann operator and its adjoint $\overline{\mathcal{N}}^*$, as unbounded operators on $\dot{H}^2(\mathcal{S})$. Then we want to prove that all of those have domain $H^1(\mathcal{S})$. As in the proof of Lannes ([71, Proposition A.14]), we show

$$H^1(\mathcal{S}) \subset \mathcal{D}(\overline{\mathcal{N}}) \subset \mathcal{D}(\overline{\mathcal{N}}^*) \subset H^1(\mathcal{S}).$$

The first inclusion is obvious, the second comes from the symmetry of \mathcal{N} , and the last one from the previous part on ellipticity, with $\sigma = 1$. \square

We can then use Borel's functional calculus (see for example [88] to define complex powers \mathcal{N}^r of \mathcal{N} . Those will form an analytic family of operators, which coincides at integer powers $r = k$ with the operator \mathcal{N}^k defined by applying \mathcal{N} k times. By the properties of \mathcal{N} and complex interpolation, we will find the following Proposition.

Proposition 5.3.21. *Let $s > 1 + \frac{n}{2}$. Let \mathcal{S}_* be an $H^{s-\frac{1}{2}}$ reference hypersurface, and δ small enough, so that in the corresponding Λ_* , the maximal angle satisfies $s < \frac{\pi}{2\delta}$, and all other geometric conditions imposed above apply. Let $0 \leq r \leq s - \frac{1}{2}$.*

Then, uniformly in Λ_ , there exists a self-adjoint unbounded operator \mathcal{N}^r on $L^2(\mathcal{S})$, with domain $H^r(\mathcal{S})$, which is continuous and elliptic from $H^\sigma(\mathcal{S})$ to $H^{\sigma-r}(\mathcal{S})$, for any $r \leq \sigma \leq s - \frac{1}{2}$.*

They form an analytic family of operators, coinciding with classical powers of \mathcal{N} for r -integers, and with the Identity for $r = 0$.

5.3.6 Div-curl Problem

In order to recover the velocity in the following analysis, we will need an elliptic regularity statement for the following problem on a vector field $v : \Omega \rightarrow \mathbf{R}^d$.

$$(5.3.7) \quad \begin{cases} \nabla \cdot v = g \text{ in } \Omega, \\ \nabla \times v = \mu \text{ in } \Omega, \\ (\langle \nabla v, N \rangle^\sharp)^\top = f \text{ on } \mathcal{S} \\ \langle v, \nu \rangle = h \text{ on } \mathcal{B}. \end{cases}$$

Here $\nabla \times v$ is a shorthand for the vorticity form $\mu(X) \cdot Y = \langle \nabla_X v, Y \rangle - \langle \nabla_Y v, X \rangle$ which, as is well-known, can be seen as a function in dimension $n = 2$ and a vector field in dimension $n = 3$. The musical isomorphisms \sharp and \flat respectively raise and lower the indices through the metric. Thus \sharp transforms a linear form into a vector. The metric used being the Euclidean metric in Ω , or the metric induced on \mathcal{S} , \mathcal{M} or \mathcal{L} when the fields are tangent to either \mathcal{S} , \mathcal{B} , or \mathcal{S} . Notice that in the cases of \mathcal{S} and \mathcal{L} , this operation has limited regularity.

In our problem, $g = 0$ and we could assume $\mu = 0$, however this does not simplify the proof, and therefore we may as well study the general case.

The strange boundary condition at \mathcal{S} replaces the condition on $\langle v, N \rangle$, whose regularity would be limited by N and not by v . Of course, since ν is smooth, no such difficulty arises at the bottom.

In this paper, we will not need the existence of a variational solution for such a problem, so we concentrate on regularity theorems. We will have two main difficulties.

The first one is already present in the usual case without corners. The surface \mathcal{S} is H^s , and we will want the velocity field $v \in H^s(\Omega)$. However, if we see v as a vector field on the $H^{s+\frac{1}{2}}$ -manifold with corner Ω , its maximal regularity would be $H^{s-\frac{1}{2}}(\Omega)$, since change of coordinates on vector fields involve multiplication by $D\phi_\Omega$. The way out of this is to transport v by $v \circ \Phi_\Omega$, which

means considering v as an array of functions, whose regularity are only limited at $H^{s+\frac{1}{2}}(\Omega)$, and satisfying some additional relations. Since this is not the canonical way to transport vector fields, it does not preserve being a gradient, and does not transport the divergence and curl correctly. Therefore any technique invariant by change of coordinates on vector fields, like the Hodge decomposition, is useless in this setting. Remark however that there is no problem of definition for the data, which are all less regular functions.

The second difficulty is again due to the presence of the corner. Since we do not want to perform an analysis of the singularities as above, our aim is to reduce the problem to a scalar one, at least locally near the corner, and use the preceding results.

The reason why the boundary data on \mathcal{S} is not under the more classical form $\langle v, N \rangle$ is that N would limit the regularity of this expression, being only in $H^{s-\frac{3}{2}}(\mathcal{S})$ for $\mathcal{S} \in \Lambda_*$. Since ν has maximal regularity on \mathcal{B} , this is not the case on \mathcal{B} .

For the following Proposition, remark that since we only want $v \in H^s(\mathcal{S})$ at the maximum, we do not need to now that \mathcal{S} is H^s . The information that it is in $H^{s-\frac{1}{2}}$, which is included in the hypothesis $\mathcal{S} \in \Lambda_*$, is sufficient. Thus, the constants will not depend on $|\mathcal{S}|_s$.

Proposition 5.3.22. *Take $s > 1 + n/2$, an- H^s hypersurface \mathcal{S}_* and $\delta > 0$ small enough, so that in particular as above, $s < \frac{\pi}{2\omega_{\max}} + \frac{1}{2}$. Then if $\mathcal{S} \in \Lambda_*$, if $(g, \mu, f, h) \in H^{\sigma-1}(\Omega) \times H^{\sigma-1}(\Omega) \times H^{\sigma-\frac{3}{2}}(\mathcal{S}) \times H^{\sigma-\frac{1}{2}}(\mathcal{B})$, with $s-1 \leq \sigma \leq s$, if v is a solution of (5.3.7) in $H^\sigma(\Omega)$, then*

$$\|v\|_{H^\sigma(\Omega)} \leq C \left[\|g\|_{H^{\sigma-1}(\Omega)} + \|\mu\|_{H^{\sigma-1}(\Omega)} + \|f\|_{H^{\sigma-\frac{3}{2}}(\mathcal{S})} + \|h\|_{H^{\sigma-\frac{1}{2}}(\mathcal{B})} + \|v\|_{L^2(\Omega)} \right],$$

where C is uniform in Λ_* .

The dependence on $\|v\|_{L^2}$ is good, since as a part of the Hamiltonian, it is a bounded quantity. Also the condition on the angle correspond to what would be expected for $v = \nabla\psi$ and a smooth geometry, if we wanted ψ to be in H^{s+1} .

Proof. First, remark that by interpolation it is enough to prove the inequality with the quantity $\|v\|_{H^{s-\epsilon}(\Omega)}$ for some $\epsilon > 0$ in place of the L^2 norm. The proof is based on the observation that (5.3.7) implies that the euclidean coordinates of v satisfy

$$(5.3.8) \quad \Delta v^i = \partial_j \mu_j^i + \partial_i f \text{ in } \Omega,$$

so that one can study v^i as a function satisfying an elliptic equation.

The regularity will again be proved locally near any point of $\bar{\Omega}$. Near interior points, equation (5.3.8) is enough.

Near a boundary point x_0 not in the edge \mathcal{L} , the analysis is more involved. Since it is no different in both components, we concentrate on \mathcal{S} . First, we freeze coefficients. More precisely, we use a local coordinate map ψ , of class H^s , and such that $D\psi$ is the identity at x_0 , and a cutoff to transfer the functions v^i close to x to compactly supported functions with value $v^i(\psi)$ close to x_0 . Thus, by pulling back the v^i 's as functions, we avoid the loss of regularity. Then those functions, close to x_0 , satisfy a certain div-curl problem with non-constant coefficients, depending smoothly on $D\psi^{-1}$, which, if frozen at x_0 , give the euclidean divergence and curl operators, and the boundary conditions with the good normals. For example on the divergence part, if χ denotes the inverse of ψ and $v^i = w^i(\chi)$, we have

$$\partial_i \chi^j \partial_j w^i = g(\psi),$$

with $D\chi(0) = I$, and therefore writing

$$\partial_i w^i = \sum_i \partial_i w^i (1 - \partial_i \chi^i) - \sum_{i \neq j} \partial_j \chi^i \partial_i w^j + g$$

gives the control

$$\|\nabla \cdot w\|_{H^{s-1}} \leq \|g\|_{H^{s-1}} + \|Dw\|_{H^{s-1}} \|D\chi - Id\|_{L^\infty} + C(\chi) \|Dw\|_{L^\infty}.$$

Then since $D\chi$ is Lipschitz we find that on a ball of radius ρ ,

$$\|\nabla \cdot w\|_{H^{s-1}} \leq C [\|g\|_{H^{s-1}} + \rho \|w\|_{H^s} + \|w\|_{H^{s-\epsilon}}]$$

for $\epsilon > 0$ small enough, with C uniform in Λ_* , independent of ρ . The same can be done for the curl and for the boundary data. Then one only needs to prove the regularity near x_0 for the frozen constant coefficients, and then take ρ small enough to absorb the term in the left-hand side. Remark that even if the right-hand side of the original problem was 0, we would still need to study the inhomogeneous version since the freezing process produces a right-hand side.

Then one needs only to prove the regularity for a solution of (5.3.7) in a half-space for some function compactly supported near 0. We again transform the problem into (5.3.8), and study each coordinates separately. If e_n is the normal coordinate in our half-space, the coordinates v^i , for $i \neq n$ satisfy a Laplace problem, and in terms of boundary data, we find

$$\nabla_n v^i = \nabla_n \langle v, e_i \rangle = \langle \nabla_n v, e_i \rangle = \langle \nabla_{e_i} v, n \rangle + \langle (\nabla \times v)(n), e_i \rangle,$$

and both of those terms are part of the data. Therefore we have a Neumann boundary condition for v_i , and the regularity is classical. On the other hand, v^n also satisfies a Laplace problem, and

$$\nabla_{e_i} v^n = \langle \nabla_{e_i} v, n \rangle$$

which is part of the data for all $i \neq n$, so that we control the full gradient of v^n on the boundary, and thus using $\|v\|_{H^{s-\epsilon}}$ we can control the value of v^n on the boundary. We therefore have a Dirichlet problem for v^n , and we can easily conclude.

The last step is the control near a point of \mathcal{L} . As above, we reduce it to the same problem in the angular sector $\mathbf{R}^{n-2} \times \Gamma$. The components of v in the unbounded direction, being tangential to all parts of the boundary, can be treated as above. We are left with two components, and we would like to reduce the problem to a 2D system in the angular sector Γ . As in the proof of the scalar problem, we apply the Fourier transform in the unbounded variables. Treating all terms in the tangential variables and all lower order terms as a right-hand side, we have a problem with parameter $\xi \in \mathbf{R}^{n-2}$, of the form

$$(5.3.9) \quad \begin{cases} \nabla \cdot \tilde{v}(\xi) = \tilde{g}(\xi) \text{ in } \Gamma, \\ \nabla \times \tilde{v} = \tilde{\omega}(\xi) \text{ in } \Gamma, \\ (\langle \nabla \tilde{v}(\xi), N \rangle^\#)^\top = \tilde{f}(\xi) \text{ on } S \\ (\langle \nabla \tilde{v}(\xi), \nu \rangle^\#)^\top_b = \tilde{h}(\xi) \text{ on } B. \end{cases}$$

Here $\tilde{v}(\xi)$ is for each ξ a vector field on Γ . Also, \top_b is the projection to vector fields tangent to \mathcal{M} . We need to prove weighted in ξ estimates for this problem. Again, as in the scalar case, it is sufficient to prove them for fixed ξ on the sphere, but with arbitrary large support, and constants independent of the support.

Now since we are on a straight domain, there is no problem anymore to use vectorial methods. One can use Hodge decomposition in the angular sector, which exists since Γ is piecewise regular and the function has compact support, writing $\tilde{v} = \nabla \phi + \nabla^\perp \tilde{\phi}$, where ∇^\perp here is the perpendicular gradient $(\partial_2, -\partial_1)$, and such that $\nabla_N \phi = \tilde{v} \cdot N$ on S and the same on B . Differentiating those boundary conditions along the tangential direction τ , we find that we control

$$\nabla_\tau \nabla_N \phi = \langle \nabla_\tau \tilde{v}, N \rangle = \tilde{f}$$

and since we also control v in $H^{s-\epsilon}$, we control the Neumann data $\partial_N \phi$. The same can be done on the bottom B , so that we can use $\Delta \phi = \operatorname{div} \tilde{v} = \tilde{g}$ and our preceding regularity result for functions satisfying some elliptic problem with Neumann data on both sides to control $\nabla \phi$ as expected.

We at last find $\Delta \tilde{\phi} = \nabla \times \tilde{v} = \tilde{g}$, and since $\langle \nabla^\perp \tilde{\phi}, N \rangle = \nabla_\tau \tilde{\phi} = 0$, we find $\tilde{\phi}$ to be constant on the boundary, and we can again use elliptic regularity for functions with Dirichlet data at both sides to conclude. \square

We notice that for the case $\sigma = s - 1$, the boundary data can be given under the form $\langle v, N \rangle$ and $\langle v, \nu \rangle$, because $N \in H^{s-\frac{3}{2}}(\mathcal{S})$.

5.4 Quasi-linearization

In this section, we want to find a quantity satisfying the linearized equation. As explained in the introduction, we want to differentiate the equation in time. Since

$$\mathbf{D}_t v = -\nabla p - g e_n,$$

we look for an equation on $\nabla p = \nabla p_{v,v} - g \nabla \mathcal{H}(x^n|_{\mathcal{S}}) + g \nabla x^n$ (the term $g e_n$ does not depend on time.) As we will see later, the regularity of ∇p is equivalent to the regularity of its normal part $a = -\nabla_N p|_{\mathcal{S}}$, the Taylor coefficient, so we will in fact prove that a satisfies the linearized equation.

Proposition 5.4.1. *Let $s > 1 + \frac{n}{2}$. Let \mathcal{S}_* be an H^s reference hypersurface, and δ small enough, so that in the corresponding Λ_* , the maximal angle satisfies $s < \frac{1}{2} + \frac{\pi}{2\omega}$, and all other geometric conditions imposed above apply.*

Then if \mathcal{S}_t is a continuous family of H^s hypersurface belonging to Λ_ , if $v \in C(H^s(\Omega_t))$, and if they satisfy the water waves equations, then the Taylor coefficient $a \in C(H^{s-1}(\mathcal{S}_t))$ follows the equation*

$$(5.4.1) \quad \mathbf{D}_t^2 a + a_{\mathcal{N}} a = R.$$

Here, the remainder R is in $C(H^{s-\frac{3}{2}}(\mathcal{S}_t))$, with at each time t

$$(5.4.2) \quad \|R(t)\|_{H^{s-\frac{3}{2}}(\mathcal{S}_t)} \leq Q\left(|\mathcal{S}_t|_s, \|v\|_{H^s(\Omega_t)}\right),$$

where Q is a time-independent polynomial in its variables, whose coefficients depend only on Λ_ and g .*

Proof. This is only a long computation. To start with, we recall that p is the solution of the elliptic equation

$$(5.4.3) \quad \begin{cases} \Delta p = -\operatorname{tr}((Dv)^2) & \text{in } \Omega_t, \\ p|_{\mathcal{S}_t} = 0, \\ \nabla_\nu p|_{\mathcal{B}_t} = -\Pi_{\mathcal{M}}(v, v) + g\nu^n \end{cases}$$

where $\Pi_{\mathcal{M}}(v, w) = -\langle \nabla_\nu v, w \rangle$ is the second fundamental form of \mathcal{M} and ν^n is the component of ν along e_n . As a consequence of Theorem 5.3.12, recalling that $\Pi_{\mathcal{M}}$ is smooth, we find that $p \in C(H^{s+\frac{1}{2}}(\Omega_t))$ with for each time

$$(5.4.4) \quad \|p\|_{H^{s+\frac{1}{2}}(\Omega_t)} \leq Q\left(|\mathcal{S}_t|_s, \|v\|_{H^{s-\frac{1}{2}}(\Omega_t)}\right)$$

where Q is as in the Proposition. The fact that we can use only the $H^{s-\frac{1}{2}}$ norm of v means that the regularity of p is limited by the domain, and not its data.

Then

$$(5.4.5) \quad \mathbf{D}_t \nabla p = -\langle \nabla v, \nabla p \rangle^\sharp + \nabla \mathbf{D}_t p.$$

We want to find the elliptic problem satisfied by $\mathbf{D}_t p$.

Elliptic problem for $\mathbf{D}_t p$ First, using (5.4.3),

$$(5.4.6) \quad \mathbf{D}_t p|_{\mathcal{S}_t} = \mathbf{D}_t (p|_{\mathcal{S}_t}) = 0.$$

In Ω_t ,

$$\Delta \mathbf{D}_t p = \mathbf{D}_t \Delta p + 2\text{tr}(D^2 p \cdot Dv) + \langle \Delta v, \nabla p \rangle,$$

where \cdot represents the matrix product. But, using (5.4.3), we find

$$\mathbf{D}_t \Delta p = -\mathbf{D}_t \text{tr}((Dv)^2) = -2\text{tr}(D\mathbf{D}_t v \cdot Dv) + 2\text{tr}((Dv)^3),$$

and the Euler equations give

$$\mathbf{D}_t \Delta p = 2\text{tr}(D^2 p \cdot Dv) + 2\text{tr}((Dv)^3).$$

Thus we have proved that in Ω_t ,

$$(5.4.7) \quad \Delta \mathbf{D}_t p = 4\text{tr}(D^2 p \cdot Dv) + 2\text{tr}((Dv)^3) + \langle \Delta v, \nabla p \rangle.$$

Since both ν and $\Pi_{\mathcal{M}}$ are smooth and independent of t , we compute easily that on points of \mathcal{B}_t ,

$$(5.4.8) \quad \mathbf{D}_t \nu|_{\mathcal{B}} = \nabla_\nu \nu$$

and for w and w' tangent to \mathcal{B} ,

$$(5.4.9) \quad \mathbf{D}_t (\Pi_{\mathcal{M}}(w, w')) = \Pi_{\mathcal{M}}(\mathbf{D}_t^\top w, w') + \Pi_{\mathcal{M}}(w, \mathbf{D}_t^\top w') - \langle D^2 \nu(v, w), w' \rangle,$$

where \top represents the orthogonal projection to the tangent plane of \mathcal{M} .

On the other hand,

$$\nabla_\nu (\mathbf{D}_t p)|_{\mathcal{B}_t} = \mathbf{D}_t (\nabla_\nu p|_{\mathcal{B}_t}) + \langle \nabla_\nu v, \nabla p \rangle|_{\mathcal{B}_t} + \Pi_{\mathcal{M}}(v, \nabla^\top p).$$

Using the elliptic equation (5.4.3) on p and the preceding computations (5.4.8) and (5.4.9), we find

$$\mathbf{D}_t (\nabla_\nu p|_{\mathcal{B}_t}) = -2\Pi_{\mathcal{M}}(\mathbf{D}_t^\top v, v) + \langle D^2 \nu(v, v), v \rangle|_{\mathcal{B}_t} + g(\nabla_\nu \nu)^n|_{\mathcal{B}_t}.$$

Euler equations let us conclude

$$\mathbf{D}_t (\nabla_\nu p|_{\mathcal{B}_t}) = 2\Pi_{\mathcal{M}}(\nabla^\top p, v) + \langle D^2 \nu(v, v), v \rangle|_{\mathcal{B}_t} - g(\nabla_\nu \nu)^n|_{\mathcal{B}_t}.$$

Therefore,

$$(5.4.10) \quad \nabla_\nu (\mathbf{D}_t p)|_{\mathcal{B}_t} = 3\Pi_{\mathcal{M}}(\nabla^\top p, v) + \langle D^2 \nu(v, v), v \rangle|_{\mathcal{B}_t} - g(\nabla_\nu \nu)^n|_{\mathcal{B}_t} + \langle \nabla_\nu v, \nabla p \rangle|_{\mathcal{B}_t}.$$

By grouping together terms of same regularity, we therefore find for $\mathbf{D}_t p$ the expression (5.4.11)

$$\begin{aligned} \mathbf{D}_t p &= \Delta^{-1} (\langle \Delta v, \nabla p \rangle, \langle \nabla_\nu v, \nabla p \rangle) \\ &\quad + \Delta^{-1} (4\text{tr} (D^2 p \cdot Dv) + 2\text{tr} ((Dv)^3), 3\Pi_{\mathcal{M}}(\nabla^\top p, v) + \langle D^2 \nu(v, v), v \rangle - g(\nabla_\nu \nu)^n), \end{aligned}$$

where $u = \Delta^{-1}(g, h)$ is the solution of

$$\begin{cases} \Delta u = g \text{ in } \Omega_t, \\ u|_{\mathcal{S}_t} = 0, \\ \nabla_\nu u|_{\mathcal{B}_t} = h. \end{cases}$$

The elliptic regularity Theorem 5.3.12, combined with product estimates gives us that the first term of (5.4.11) is in $H^s(\Omega_t)$, while the second is in $H^{s+\frac{1}{2}}$.

If we plug this into (5.4.5), we find

$$\begin{aligned} (5.4.12) \quad \mathbf{D}_t \nabla p &= -\langle \nabla v, \nabla p \rangle^\sharp + \nabla \Delta^{-1} (\langle \Delta v, \nabla p \rangle, \langle \nabla_\nu v, \nabla p \rangle) \\ &\quad + \nabla \Delta^{-1} (4\text{tr} (D^2 p \cdot Dv) + 2\text{tr} ((Dv)^3), 3\Pi_{\mathcal{M}}(\nabla^\top p, v) + \langle D^2 \nu(v, v), v \rangle - g(\nabla_\nu \nu)^n). \end{aligned}$$

The first line of the right-hand side is in $H^{s-1}(\Omega_t)$, the second in $H^{s-\frac{1}{2}}(\Omega_t)$. Let us call

$$\alpha := \Delta^{-1} (\langle \Delta v, \nabla p \rangle, \langle \nabla_\nu v, \nabla p \rangle),$$

and

$$\beta := \Delta^{-1} (4\text{tr} (D^2 p \cdot Dv) + 2\text{tr} ((Dv)^3), 3\Pi_{\mathcal{M}}(\nabla^\top p, v) + \langle D^2 \nu(v, v), v \rangle - g(\nabla_\nu \nu)^n),$$

so that in short,

$$(5.4.13) \quad \mathbf{D}_t \nabla p = -\langle \nabla v, \nabla p \rangle^\sharp + \nabla \alpha + \nabla \beta.$$

Now we need to compute the second derivative in time. For this, we compute the derivative of each of the three terms of (5.4.13).

First term We first compute

$$-\mathbf{D}_t \langle \nabla v, \nabla p \rangle^\sharp = -\langle \nabla \mathbf{D}_t v, \nabla p \rangle^\sharp + \langle \nabla_{\nabla v} v, \nabla p \rangle^\sharp - \langle \nabla v, \mathbf{D}_t \nabla p \rangle^\sharp.$$

From Euler equations, the first term is

$$-\langle \nabla \mathbf{D}_t v, \nabla p \rangle^\sharp = \langle D^2 p, \nabla p \rangle^\sharp.$$

Using the evolution of ∇p from (5.4.13) to express the third term, we find

$$-\mathbf{D}_t \langle \nabla v, \nabla p \rangle^\sharp = \langle D^2 p, \nabla p \rangle^\sharp + 2 \langle \nabla_{\nabla v} v, \nabla p \rangle^\sharp - \langle \nabla v, \nabla \alpha \rangle^\sharp - \langle \nabla v, \nabla \beta \rangle^\sharp.$$

Using the product estimates we can sum this up as

$$(5.4.14) \quad -\mathbf{D}_t \langle \nabla v, \nabla p \rangle^\sharp = \langle D^2 p, \nabla p \rangle^\sharp + R$$

with R satisfying

$$(5.4.15) \quad \|R\|_{H^{s-1}(\Omega_t)} \leq Q \left(|\mathcal{S}_t|_s, \|v\|_{H^s(\Omega_t)} \right)$$

with Q as in the Proposition.

Second term We now compute

$$(5.4.16) \quad \mathbf{D}_t \nabla \alpha = \nabla \mathbf{D}_t \alpha - \langle \nabla v, \nabla \alpha \rangle^\sharp.$$

To find expression for $\mathbf{D}_t \alpha$, we use the same method as for p . We look for an elliptic problem it satisfies.

It is immediate that

$$(5.4.17) \quad \mathbf{D}_t \alpha|_{\mathcal{S}_t} = 0.$$

In the domain Ω_t ,

$$(5.4.18) \quad \Delta \mathbf{D}_t \alpha = \mathbf{D}_t \Delta \alpha + 2 \operatorname{tr} (D^2 \alpha \cdot Dv) + \langle \Delta v, \nabla \alpha \rangle.$$

Let us concentrate on the first term, using that

$$\Delta \alpha = \langle \Delta v, \nabla p \rangle.$$

We find

$$\mathbf{D}_t \Delta \alpha = \langle \Delta \mathbf{D}_t v, \nabla p \rangle - 2 \operatorname{tr} (\langle D^2 v, \nabla p \rangle \cdot Dv) - \langle \nabla_{\Delta v} v, \nabla p \rangle + \langle \Delta v, \mathbf{D}_t \nabla p \rangle.$$

Thanks once again to Euler's equations,

$$\langle \Delta \mathbf{D}_t v, \nabla p \rangle = - \langle \Delta \nabla p, \nabla p \rangle = \langle \nabla \operatorname{tr} ((Dv)^2), \nabla p \rangle,$$

so that again, using the evolution of ∇p from (5.4.13),

$$\mathbf{D}_t \Delta \alpha = \langle \nabla \operatorname{tr} ((Dv)^2), \nabla p \rangle - 2 \operatorname{tr} (\langle D^2 v, \nabla p \rangle \cdot Dv) - 2 \langle \nabla_{\Delta v} v, \nabla p \rangle + \langle \Delta v, \nabla \alpha \rangle + \langle \Delta v, \nabla \beta \rangle.$$

Combined with (5.4.18), this gives

$$(5.4.19) \quad \mathbf{D}_t \Delta \alpha = \langle \nabla \operatorname{tr} ((Dv)^2), \nabla p \rangle - 2 \operatorname{tr} (\langle D^2 v, \nabla p \rangle \cdot Dv) - 2 \langle \nabla_{\Delta v} v, \nabla p \rangle + 2 \langle \Delta v, \nabla \alpha \rangle + \langle \Delta v, \nabla \beta \rangle + 2 \operatorname{tr} (D^2 \alpha \cdot Dv).$$

At last on \mathcal{B}_t ,

$$\nabla_\nu \mathbf{D}_t \alpha = \mathbf{D}_t (\nabla_\nu \alpha) + \langle \nabla_\nu v, \nabla \alpha \rangle - \langle \nabla_\nu \nu, \nabla \alpha \rangle.$$

The first term can be computed from

$$\nabla_\nu \alpha|_{\mathcal{B}_t} = \langle \nabla_\nu v, \nabla p \rangle,$$

and is

$$\mathbf{D}_t (\nabla_\nu \alpha) = \langle \nabla_\nu \mathbf{D}_t v, \nabla p \rangle + \langle \nabla_{\nabla_\nu v} v, \nabla p \rangle - \langle \nabla_{\nabla_\nu v} v, \nabla p \rangle + \langle \nabla_\nu v, \mathbf{D}_t \nabla p \rangle.$$

Using Euler's equations and (5.4.13) once again,

$$(5.4.20) \quad \nabla_\nu \mathbf{D}_t \alpha = -D^2 p(\nu, \nabla p) + \langle \nabla_{\nabla_\nu v} v, \nabla p \rangle - 2 \langle \nabla_{\nabla_\nu v} v, \nabla p \rangle + 2 \langle \nabla_\nu v, \nabla \alpha \rangle + \langle \nabla_\nu v, \nabla \beta \rangle + \Pi_{\mathcal{M}}(v, \nabla^\top \alpha).$$

Combining (5.4.17), (5.4.19) and (5.4.20) with elliptic and product estimates, we find

$$(5.4.21) \quad \mathbf{D}_t \nabla \alpha = -\nabla \Delta^{-1} (0, D^2 p(\nu, \nabla p)) + R$$

with

$$(5.4.22) \quad \|R\|_{H^{s-1}(\Omega_t)} \leq Q \left(|\mathcal{S}_t|_s, \|v\|_{H^s(\Omega_t)} \right).$$

Third term We finish by computing

$$(5.4.23) \quad \mathbf{D}_t \nabla \beta = \nabla \mathbf{D}_t \beta - \langle \nabla v, \nabla \beta \rangle^\sharp.$$

Again we look for an elliptic problem on $\mathbf{D}_t \beta$.

As above,

$$(5.4.24) \quad \mathbf{D}_t \beta|_{\mathcal{S}_t} = 0.$$

In Ω_t ,

$$\Delta \mathbf{D}_t \beta = \mathbf{D}_t \Delta \beta + 2 \operatorname{tr} (D^2 \beta \cdot Dv) + \langle \Delta v, \beta \alpha \rangle,$$

and

$$\begin{aligned} \mathbf{D}_t \Delta \beta &= 4 \operatorname{tr} (D \mathbf{D}_t \nabla p \cdot Dv) + 4 \operatorname{tr} (D^2 p \cdot D \mathbf{D}_t v) - 8 \operatorname{tr} (D^2 p \cdot Dv \cdot Dv) \\ &\quad + 6 \operatorname{tr} (D \mathbf{D}_t v \cdot Dv \cdot Dv) - 6 \operatorname{tr} ((Dv)^4). \end{aligned}$$

At the end end, each of those terms lies in $H^{s-2}(\Omega_t)$, so that

$$(5.4.25) \quad \|\Delta \mathbf{D}_t \beta\|_{H^{s-2}(\Omega_t)} \leq Q \left(|\mathcal{S}_t|_s, \|v\|_{H^s(\Omega_t)} \right).$$

On the bottom,

$$\nabla_\nu \mathbf{D}_t \beta|_{\mathcal{B}_t} = \mathbf{D}_t (\nabla_\nu \beta|_{\mathcal{B}_t}) + \langle \nabla_\nu v, \nabla \beta \rangle - \langle \nabla_\nu \nu, \nabla \beta \rangle,$$

Since

$$\nabla_\nu \beta|_{\mathcal{B}_t} = 3 \Pi_{\mathcal{M}} (\nabla^\top p, v) + \langle D^2 \nu(v, v), v \rangle - g (\nabla_\nu \nu)^n,$$

we only need to compute the evolution of each of those three terms. The first gives

$$3 \mathbf{D}_t \Pi_{\mathcal{M}} (\nabla^\top p, v) = 3 \Pi_{\mathcal{M}} (\mathbf{D}_t^\top \nabla^\top p, v) + 3 \Pi_{\mathcal{M}} (\nabla^\top p, \mathbf{D}_t^\top v) - 3 \langle D^2 \nu(\nabla^\top p, v), v \rangle,$$

which is easily seen to lie in $H^{s-\frac{3}{2}}(\mathcal{B}_t)$. For the second term,

$$\mathbf{D}_t \langle D^2 \nu(v, v), v \rangle = \langle D^4 \nu(v, v, v), v \rangle + 2 \langle D^2 \nu(\mathbf{D}_t v, v), v \rangle + \langle D^2 \nu(v, v), \mathbf{D}_t v \rangle,$$

which is again in $H^{s-\frac{3}{2}}(\mathcal{B}_t)$. The last term is

$$-g \mathbf{D}_t (\nabla_\nu \nu)^n = -g (\nabla_{\mathbf{D}_t v} \nu)^n + g (D^2 \nu(v, v))^n,$$

still in $H^{s-\frac{3}{2}}(\mathcal{B}_t)$. Putting all of this together, we see that

$$(5.4.26) \quad \|\nabla_\nu \mathbf{D}_t \beta\|_{H^{s-\frac{3}{2}}(\mathcal{B}_t)} \leq Q \left(|\mathcal{S}_t|_s, \|v\|_{H^s(\Omega_t)} \right).$$

Thus in the end, using elliptic regularity,

$$(5.4.27) \quad \|\mathbf{D}_t \beta\|_{H^s(\Omega_t)} \leq Q \left(|\mathcal{S}_t|_s, \|v\|_{H^s(\Omega_t)} \right).$$

Equation on ∇p We have therefore proved that

$$\mathbf{D}_t^2 \nabla p = \langle D^2 p, \nabla p \rangle^\# - \nabla \Delta^{-1} (0, D^2 p(\nu, \nabla p)) + R$$

where

$$\|R\|_{H^{s-1}(\Omega_t)} \leq Q \left(|\mathcal{S}_t|_s, \|v\|_{H^s(\Omega_t)} \right).$$

To finish, we remark that the first two terms of the right-hand side can be rewritten as

$$\frac{1}{2} \nabla (|\nabla p|^2) - \frac{1}{2} \nabla \Delta^{-1} (0, \nabla_\nu |\nabla p|^2) = \frac{1}{2} \nabla \mathcal{H} (|\nabla p|^2|_{\mathcal{S}_t}) - \nabla f,$$

where f is solution of

$$\begin{cases} \Delta f = 2 \langle \nabla p, \nabla \text{tr} ((Dv)^2) \rangle + 2 \text{tr} ((D^2 p)^2) & \text{in } \Omega_t, \\ f|_{\mathcal{S}_t} = 0, \\ \nabla_\nu f|_{\mathcal{S}_t} = 0, \end{cases}$$

and is therefore in $H^s(\Omega_t)$. Remarking that on \mathcal{S}_t , because $p|_{\mathcal{S}_t} = 0$, $\nabla p = \nabla_N p N$, we find

$$\mathbf{D}_t^2 \nabla p = \frac{1}{2} \nabla \mathcal{H} (a^2) + R$$

with R as above. To finish, we notice that

$$\mathcal{H} (a^2) = (\mathcal{H} (a))^2 + H^{s+\frac{1}{2}}(\Omega_t)$$

because of the elliptic problem satisfied by the difference, so that in the end,

$$(5.4.28) \quad \mathbf{D}_t^2 \nabla p = \mathcal{H} (a) \nabla \mathcal{H} (a) + R$$

with

$$(5.4.29) \quad \|R\|_{H^{s-1}(\Omega_t)} \leq Q \left(|\mathcal{S}_t|_s, \|v\|_{H^s(\Omega_t)} \right).$$

Equation on a To transform this into an equation on a , we need to compute the evolution of the normal to the surface, $\mathbf{D}_t N$. We redo the computations of Shatah and Zheng in [90]. First, because $|N| = 1$,

$$\mathbf{D}_t N \perp N.$$

Then we can choose τ_0 tangent to \mathcal{S}_{t_0} at the point $x_0 \in \mathcal{S}_{t_0}$, and transport it in time as a solution of

$$\mathbf{D}_t \tau = \nabla_\tau v, \quad \tau(t_0) = \tau_0.$$

Immediatly,

$$\langle \mathbf{D}_t N, \tau_0 \rangle = - \langle N, \mathbf{D}_t \tau \rangle = - \langle \nabla_{\tau_0} v, N \rangle.$$

Thus

$$(5.4.30) \quad \mathbf{D}_t N = - \left(\langle \nabla v, N \rangle^\# \right)^\top \in H^{s-\frac{3}{2}}(\mathcal{S}_t).$$

This also gives

$$(5.4.31) \quad \langle \mathbf{D}_t^2 N, N \rangle = - |\mathbf{D}_t N|^2 \in H^{s-\frac{3}{2}}(\mathcal{S}_t).$$

Now since $a = -\langle \nabla p, N \rangle|_{\mathcal{S}_t}$, we find

$$\mathbf{D}_t^2 a = -\langle \mathbf{D}_t^2 \nabla p, N \rangle - 2\langle \mathbf{D}_t \nabla p, \mathbf{D}_t N \rangle + a\langle \mathbf{D}_t^2 N, N \rangle,$$

and the last two terms are in $H^{s-\frac{3}{2}}(\mathcal{S}_t)$. Thus by taking the scalar product with N of the trace of (5.4.28),

$$(5.4.32) \quad \mathbf{D}_t^2 a + a_{\mathcal{N}} a = H^{s-\frac{3}{2}}(\mathcal{S}_t),$$

with estimates on the remainder as in the Proposition. \square

5.5 The energy

Using this quasi-linear form (5.4.32) for the equations, an obvious choice for the energy is

$$(5.5.1) \quad E := \int_{\mathcal{S}_t} \left| \mathcal{N}^{s-\frac{3}{2}} \mathbf{D}_t a \right|^2 dS + \int_{\mathcal{S}_t} a \left| \mathcal{N}^{s-1} a \right|^2 dS + \|\mu\|_{H^{s-1}(\Omega_t)}^2.$$

Here, we use the powers of the Dirichlet to Neumann map \mathcal{N} , defined in Subsection 5.3.5, and we integrate according to the surface element dS of \mathcal{S}_t . The first two terms correspond to the energy for the linearized equation satisfied by a . We will see that those terms do not control the vorticity part of v , and therefore we added the $H^{s-1}(\Omega_t)$ norm of $\mu := Dv - Dv^*$, which will be well controlled because the vorticity is a conservation law for Euler equations. We call $E_0 := E(t=0)$ the initial energy.

We first need to show that one can recover the original unknown from this energy.

Proposition 5.5.1. *Let $s > 1 + \frac{n}{2}$. Let \mathcal{S}_* be an H^s reference hypersurface, and δ small enough, so that in the corresponding Λ_* , the maximal angle satisfies $s < \frac{1}{2} + \frac{\pi}{2\bar{\omega}}$, and all other geometric conditions imposed above apply.*

Then if \mathcal{S} is an H^s hypersurface belonging to Λ_ and $v \in H^s(\Omega_t)$ are solution of the equations, and if the Taylor condition $a \geq a_0 > 0$ is satisfied, then E is well-defined, finite, and we have*

$$|\mathcal{S}|_s^2 + \|v\|_{H^s(\Omega)}^2 \leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)}, E \right)$$

where \mathcal{F} is a non-decreasing continuous function of its arguments, depending only on Λ_* , a_0 and $\bar{\omega}$.

Proof. If \mathcal{S} is in H^s and v is in $H^s(\Omega)$, our elliptic regularity theory gives ∇p in $H^{s-\frac{1}{2}}$ and therefore $a \in H^{s-1}(\mathcal{S})$, while the formula for $\mathbf{D}_t a$ tells us it lies in $H^{s-\frac{3}{2}}(\mathcal{S})$. Thus the quantities composing E are all well-defined.

Now to prove our result, we start with the basic preliminary controls

$$\|\nabla p\|_{H^{s-1}(\Omega)} \leq C \|v\|_{H^{s-\frac{1}{2}}(\Omega)}^2$$

because of the elliptic equation (5.4.3) on the pressure, and

$$\|\mathbf{D}_t \nabla p\|_{H^{s-\frac{3}{2}}(\Omega)} \leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)} \right),$$

because of (5.4.13) and the elliptic equations on α and β . We notice that

$$\mathbf{D}_t a = -\langle \mathbf{D}_t \nabla p, N \rangle - \langle \nabla p, \mathbf{D}_t N \rangle = -\langle \mathbf{D}_t \nabla p, N \rangle$$

because $\mathbf{D}_t N$ is tangent to \mathcal{S} and ∇p is normal. Thus

$$(5.5.2) \quad \|a\|_{H^{s-\frac{3}{2}}(\mathcal{S})} + \|\mathbf{D}_t a\|_{H^{s-2}(\mathcal{S})} \leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)} \right).$$

Here the traces on \mathcal{S} are well-defined because $s > 2$.

The next step is to use the ellipticity of \mathcal{N} to find

$$\begin{aligned} \|a\|_{H^{s-1}(\mathcal{S})} &\leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)} \right) \left[\|\mathcal{N}^{s-1} a\|_{L^2(\mathcal{S})} + 1 \right] \\ &\leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)} \right) \left[\left\| \frac{1}{\sqrt{a}} \right\|_{L^\infty(\mathcal{S})} \|\sqrt{a} \mathcal{N}^{s-1} a\|_{L^2(\mathcal{S})} + 1 \right], \end{aligned}$$

so that

$$\|a\|_{H^{s-1}(\mathcal{S})}^2 \leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)} \right) [1 + E].$$

Similarly,

$$\|\mathbf{D}_t a\|_{H^{s-\frac{3}{2}}(\mathcal{S})}^2 \leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)} \right) [1 + E].$$

Next, we want to control ∇p and $\mathbf{D}_t \nabla p$ with those quantities. To avoid the apparition of $|\mathcal{S}|_s$ in the right-hand side, we use the div-curl problems

$$\begin{cases} \nabla \cdot \nabla p = -\text{tr}((Dv)^2) & \text{in } \Omega, \\ \nabla \times \nabla p = 0 & \text{in } \Omega, \\ \langle \nabla \nabla p, N \rangle^\sharp^\top = -(\nabla a)^\top & \text{on } \mathcal{S} \\ \langle \nabla p, \nu \rangle = -\Pi_{\mathcal{M}}(v, v) - g\nu^n & \text{on } \mathcal{B}; \end{cases}$$

and

$$\begin{cases} \nabla \cdot \mathbf{D}_t \nabla p = 3\text{tr}(D^2 p \cdot Dv) + 2\text{tr}((Dv)^3) & \text{in } \Omega, \\ \nabla \times \mathbf{D}_t \nabla p = D^2 p \cdot Dv - (Dv)^* \cdot D^2 p & \text{in } \Omega, \\ \langle \mathbf{D}_t \nabla p, N \rangle = -\mathbf{D}_t a & \text{on } \mathcal{S} \\ \langle \mathbf{D}_t \nabla p, \nu \rangle = 3\Pi_{\mathcal{M}}(\nabla^\top p, v) + \langle D^2 \nu(v, v), v \rangle + g\Pi_{\mathcal{M}}(v, e_n) & \text{on } \mathcal{B}. \end{cases}$$

If s was greater than $3/2 + n/2$, we could use the information $v \in H^{s-\frac{1}{2}}(\Omega)$ directly to conclude. However, here with $s > 1 + n/2$, this information only gives $\text{tr}((Dv)^3) \in H^{s-\frac{5}{2}+}(\Omega)$ for example, and this is not enough to conclude. We therefore need to implement a bootstrap procedure. We see from the above elliptic problems that if $s - \frac{1}{2} < \sigma \leq s$, and if we choose $\epsilon \leq \sigma - s + \frac{1}{2}$,

$$(5.5.3) \quad \|\nabla p\|_{H^{\sigma-\frac{1}{2}}(\Omega)} + \|\mathbf{D}_t \nabla p\|_{H^{\sigma-1}(\Omega)} \leq \mathcal{F} \left(\|v\|_{H^{\sigma-\epsilon}(\Omega)}, E \right).$$

Here we have first proved the estimate on ∇p , then used it to prove the one on $\mathbf{D}_t \nabla p$.

Then we use the problem satisfied by v ,

$$\begin{cases} \nabla \cdot v = 0 & \text{in } \Omega, \\ \nabla \times v = \mu & \text{in } \Omega, \\ \langle \nabla v, N \rangle^\sharp^\top = \frac{1}{a} (\mathbf{D}_t \nabla p)^\top & \text{on } \mathcal{S} \\ \langle v, \nu \rangle = 0 & \text{on } \mathcal{B}, \end{cases}$$

to deduce

$$\|v\|_{H^\sigma(\Omega)} \leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)} \right) \left[1 + E^{\frac{1}{2}} + \|\mathbf{D}_t \nabla p\|_{H^{\sigma-\frac{1}{2}}(\Omega)} \right].$$

Then a simple bootstrap procedure can close the estimates : we find

$$(5.5.4) \quad \|\nabla p\|_{H^{s-\frac{1}{2}}(\Omega)} + \|\mathbf{D}_t \nabla p\|_{H^{s-1}(\Omega)} + \|v\|_{H^s(\Omega)} \leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)}, E \right).$$

To control the regularity of \mathcal{S} , we use the formula

$$\Delta p = \Delta_{\mathcal{S}} p - \kappa \nabla_N p + D^2 p(N, N),$$

so that, since $p|_{\mathcal{S}} = 0$,

$$\kappa = \frac{1}{a} (\Delta p - D^2 p(N, N)).$$

Thus we can conclude that

$$(5.5.5) \quad \|\kappa\|_{H^{s-2}(\mathcal{S})} \leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)}, E \right).$$

If $n = 2$, this is enough. For $n \geq 3$, we need also to control κ_l . The same formula as above, seeing \mathcal{L} as the boundary of \mathcal{B} with exterior normal n gives

$$\kappa_l = -\frac{1}{\nabla_n p} (\Delta_{\mathcal{B}} p - D^2 p(n, n)) = \frac{1}{\langle n, N \rangle a} (\Delta_{\mathcal{B}} p - D^2 p(n, n))$$

where we have used that $p = 0$ on \mathcal{S} to write $\nabla_n p = \langle N, n \rangle \nabla_N p$. Observing that $\langle n, N \rangle$ is bounded from below because $\frac{\pi}{2} > \omega \geq \underline{\omega} > 0$, we conclude

$$(5.5.6) \quad \|\kappa_l\|_{H^{s-\frac{5}{2}}(\mathcal{L})} \leq \mathcal{F} \left(\|v\|_{H^{s-\frac{1}{2}}(\Omega)}, E \right).$$

Here the traces make sense because $s > \frac{5}{2}$ when $n \geq 3$. This concludes the proof. \square

The last proposition is the control on the energy. We need to use a control neighborhood both in \mathcal{S} , which is the role of Λ_* , and in $v \in H^{s-\frac{1}{2}}(\Omega_t)$.

Proposition 5.5.2. *Let $s > 1 + \frac{n}{2}$. Let \mathcal{S}_* be an H^s reference hypersurface, and δ small enough, so that in the corresponding Λ_* , the maximal angle satisfies $s < \frac{1}{2} + \frac{\pi}{2\underline{\omega}}$, and all other geometric conditions imposed above apply. Let $A > 0$.*

Take \mathcal{S}_t a C^2 family of H^s hypersurfaces so that $\mathcal{S}_0 \in \Lambda_$, and $v \in C^2(H^s(\Omega_t))$, satisfying $\|v(0)\|_{H^{s-\frac{1}{2}}(\Omega_0)} < A$. Assume that (\mathcal{S}_t, v) are solutions of the equations, and that the Taylor condition $a(t, \cdot) \geq a_0 > 0$ is satisfied.*

Then there exists a time $T > 0$, depending only on Λ_ , A , a_0 , $\underline{\omega}$, $|\mathcal{S}_0|_s$ and $\|v(0, \cdot)\|_{H^s(\Omega_0)}$, so that for all times $t \in [0, T]$, $\mathcal{S}_t \in \Lambda_*$, $\|v(t)\|_{H^{s-\frac{1}{2}}(\Omega_t)} < A$, and the energy E satisfies*

$$E(t) \leq E(0) + \int_0^t \mathcal{F}(E(t')) dt',$$

where \mathcal{F} is an increasing function of its argument, and depends only on Λ_* , A , a_0 , and $\underline{\omega}$.

Remark that in fact, Λ_* and A can be chosen depending only on the data. Also because the L^∞ evolution of a is controlled by the evolution of $\mathcal{S} \in \Lambda_*$ and v in $H^{s-\frac{1}{2}}$, it is easily seen that a_0 can be chosen depending only on the initial data. Thus at the end the time of validity T of the Proposition only depends on the norms of the initial data. We do not write the Proposition in this way, since the point is that the function \mathcal{F} in the control of the energy is uniform in a neighborhood of the the initial data in a rougher topology.

Proof. The equation being quasilinear, one needs to use control neighborhoods in rougher topologies.

Control neighborhoods First we prove that \mathcal{S} stays in Λ_* and $\|v\|_{H^{s-\frac{1}{2}}(\Omega_t)}$ stays less than A for a short time. This rests on an estimate of the Lagrangian map $u(t, \cdot) : \Omega_0 \rightarrow \Omega_t$, which is the solution of

$$\partial_t u(t, y) = v(t, u(t, y))$$

with initial data $u(0) = I$. It is immediate that for any $0 \leq \sigma \leq s$ and $f \in H^\sigma(\Omega_t)$,

$$\|f \circ u(t, \cdot)\|_{H^\sigma(\Omega_0)} \leq C \|f\|_{H^\sigma(\Omega_t)} \|u(t, \cdot)\|_{H^s(\Omega_0)}^\sigma.$$

Duality then gives

$$\|f \circ u(t, \cdot)\|_{H^{-\sigma}(\Omega_0)} \leq C \|f\|_{H^{-\sigma}(\Omega_t)} \|u(t, \cdot)^{-1}\|_{H^s(\Omega_0)}^\sigma.$$

Then

$$\|u(t, \cdot) - I\|_{H^s(\Omega_0)} \leq C \int_0^t \|v(t', \cdot)\|_{H^s(\Omega_{t'})} \|u(t', \cdot)\|_{H^s(\Omega_0)}^s dt',$$

by writing the ODE in integral form.

Take a large M to be chosen later, and

$$t_0 = \sup \left\{ t; \|v(t', \cdot)\|_{H^s(\Omega_{t'})} + |\mathcal{S}_{t'}|_s < M, \forall t' \in [0, t] \right\}.$$

This time is positive because of the continuity of the solution. Then

$$\|u(t, \cdot) - I\|_{H^s(\Omega_0)} \leq M \int_0^t \|u(t', \cdot)\|_{H^s(\Omega_0)}^s dt'.$$

Thus by ODE estimates, we find a time and a constant C depending only on μ , such that

$$\|u(t, \cdot) - I\|_{H^s(\Omega_0)} \leq Ct.$$

This implies that in local coordinates, the function η that parametrized \mathcal{S} above \mathcal{S}_* grows linearly, so that there is a time t_1 depending only on μ , Λ_* and the norms of the initial data, so that \mathcal{S} stays in Λ_* for $0 \leq t \leq \min\{t_0, t_1\}$.

The same construction, using $u_t = v(t, u(t))$ and $u_{tt} = (\nabla p - ge_n) \circ u$, gives that for a time t_2 depending only on μ , Λ_* , A and the norms of the initial data, if $0 \leq t \leq \min\{t_0, t_1, t_2\}$ then $\|v(t)\|_{H^{s-\frac{1}{2}}(\Omega)} < A$.

Evolution of the Curl The evolution of $\mu = Dv - (Dv)^*$ can easily be computed to be

$$(5.5.7) \quad \mathbf{D}_t \mu = -(Dv)^* \mu - \mu Dv.$$

Then it is easy to obtain energy estimates,

$$(5.5.8) \quad \frac{d}{dt} \|\mu\|_{H^{s-1}(\Omega_t)} \leq C \|v\|_{H^s(\Omega_t)} \|\mu\|_{H^{s-1}(\Omega_t)}^2.$$

Commutators We need to compute the commutator between \mathbf{D}_t and powers of \mathcal{N} . The one we need is

$$(5.5.9) \quad \|[\mathbf{D}_t, \mathcal{N}^\sigma]\|_{L(H^\sigma(\mathcal{S}_t); L^2(\mathcal{S}_t))} \leq C \|v\|_{H^s(\Omega_t)},$$

for $\frac{1}{2} \leq \sigma \leq s-1$, with C depending only on Λ_* . This will be a consequence of

$$\|[\mathbf{D}_t, \mathcal{N}]\|_{L(H^r(\mathcal{S}_t); H^{r-1}(\mathcal{S}_t))} \leq C \|v\|_{H^s(\Omega_t)},$$

for $r \in [1/2, s-1/2]$, which can be proven for $r > 1$ by writing the commutator formula,

$$\begin{aligned} [\mathbf{D}_t, \mathcal{N}]f &= \nabla_N \Delta^{-1} \left(2Dv \cdot D^2 f_{\mathcal{H}} + \nabla_{\Delta v} f_{\mathcal{H}}, \nabla_{(\nabla v)^\top b} f_{\mathcal{H}} - \nabla_{\Pi_b(v)} f_{\mathcal{H}} \right) \\ &\quad - \nabla_{\nabla_N v} f_{\mathcal{H}} - \langle \nabla_{\nabla^\top f} v \cdot N \rangle, \end{aligned}$$

and for $r = 1/2$ by writing the weak formulation of \mathcal{N} , defined by duality from $H^{\frac{1}{2}}(\mathcal{S})$ to its dual by

$$\int_{\mathcal{S}} \mathcal{N} f g \, dS = \int_{\Omega} \nabla f_{\mathcal{H}} \cdot \nabla g_{\mathcal{H}},$$

extending g by $\mathbf{D}_t g = 0$, and computing the time derivative.

Then one can use the formula

$$[\mathbf{D}_t, \mathcal{N}^{k+1}] = [\mathbf{D}_t, \mathcal{N}] \mathcal{N}^k + \mathcal{N} [\mathbf{D}_t, \mathcal{N}^k]$$

and interpolation to conclude.

We will also need the commutator between \mathcal{N} and a ,

$$(5.5.10) \quad \|[a, \mathcal{N}^{s-2}]\|_{L(H^{s-2}(\mathcal{S}_t), H^{\frac{1}{2}}(\mathcal{S}_t))} \leq C \|a\|_{H^{s-1}(\mathcal{S}_t)},$$

which can again be proven by interpolation between integer powers, those one being computed explicitly.

Evolution of the energy Now one can tackle the evolution of the other two terms in the energy. We write

$$E_1 = \int_{\mathcal{S}_t} \left| \mathcal{N}^{s-\frac{3}{2}} \mathbf{D}_t a \right|^2 \, dS$$

and

$$E_2 = \int_{\mathcal{S}_t} a \left| \mathcal{N}^{s-1} a \right|^2 \, dS.$$

We recall from [90] that for a function f defined on \mathcal{S}_t ,

$$(5.5.11) \quad \frac{d}{dt} \int_{\mathcal{S}_t} f \, dS = \int_{\mathcal{S}_t} (\mathbf{D}_t f + f(\mathcal{D} \cdot v^\top - \kappa v^\perp)) \, dS,$$

and because v is divergence-free,

$$\mathcal{D} \cdot v^\top - \kappa v^\perp = \langle \nabla_N v, N \rangle \in H^{s-\frac{3}{2}}(\mathcal{S}_t) \subset L^\infty(\mathcal{S}_t).$$

Thus this second term is harmless in the estimates.

First, we prove that

$$(5.5.12) \quad \left| \frac{d}{dt} E_1 - \left\langle \mathcal{N}^{s-\frac{3}{2}}(\mathbf{D}_t a), \mathcal{N}^{s-\frac{3}{2}}(a_c \mathcal{N} a) \right\rangle_{L^2(\mathcal{S}_t)} \right| \leq \mathcal{F} \left(\|v\|_{H^s(\Omega_t)}, |\mathcal{S}_t|_s \right).$$

Since $\mathbf{D}_t a \in H^{s-\frac{3}{2}}(\mathcal{S}_t) \subset L^\infty(\mathcal{S}_t)$, we have

$$\left| \frac{d}{dt} E_1 - \left\langle \mathbf{D}_t \mathcal{N}^{s-1}(a), a_c \mathcal{N}^{s-1}(a) \right\rangle_{L^2(\mathcal{S}_t)} \right| \leq \mathcal{F} \left(\|v\|_{H^s(\Omega_t)}, |\mathcal{S}_t|_s \right),$$

and (5.5.12) is a consequence of the commutators estimates (5.5.9) and (5.5.10), and of the self-adjointness of powers of \mathcal{N} .

Along the same lines, the commutator estimate (5.5.9) proves

$$(5.5.13) \quad \left| \frac{d}{dt} E_2 - \left\langle \mathcal{N}^{s-\frac{3}{2}}(\mathbf{D}_t a), \mathcal{N}^{s-\frac{3}{2}}(\mathbf{D}_t^2 a) \right\rangle_{L^2(\mathcal{S}_t)} \right| \leq \mathcal{F} \left(\|v\|_{H^s(\Omega_t)}, |\mathcal{S}_t|_s \right).$$

Thus, using Proposition 5.4.1 on the equation satisfied by a , we conclude

$$\left| \frac{d}{dt} E \right| \mathcal{F} \left(\|v\|_{H^s(\Omega_t)}, |\mathcal{S}_t|_s \right).$$

Using Proposition 5.5.1 to control the right-hand side by a function of E , we conclude the inequality of our Proposition on the interval of time $[0, \min \{t_0, t_1, t_2\}]$. Then if we choose M big enough depending only on the initial data and the control neighborhoods, the control of the energy implies that t_0 is bounded from below by a time t_* depending only on the initial data. Also since we have fixed M , t_1 and t_2 only depend on the initial data. Therefore the control is valid up to a time T as in the Proposition. \square

Calcul paradifférentiel

A.0.1 Paradifferential operators

Definition A.0.1. 1. (Littlewood-Paley decomposition) Let $\kappa \in C_0^\infty(\mathbf{R}^d)$ be such that

$$\kappa(\theta) = 1 \quad \text{for } |\theta| \leq 1.1, \quad \kappa(\theta) = 0 \quad \text{for } |\theta| \geq 1.9.$$

Define

$$\kappa_k(\theta) = \kappa(2^{-k}\theta) \quad \text{for } k \in \mathbf{Z}, \quad \varphi_0 = \kappa_0, \quad \text{and} \quad \varphi_k = \kappa_k - \kappa_{k-1} \quad \text{for } k \geq 1.$$

Given a temperate distribution u , we introduce

$$\begin{aligned} S_k u &= \kappa_k(D_x)u \quad \text{for } k \in \mathbf{Z}, \\ \Delta_0 u &= S_0 u, \quad \Delta_k u = S_k u - S_{k-1} u \quad \text{for } k \geq 1. \end{aligned}$$

Then we have the formal dyadic partition of unity

$$u = \sum_{k=0}^{\infty} \Delta_k u.$$

2. (Zygmund spaces) Let $s \in \mathbf{R}$ and $p, q \in [1, \infty]$. The Besov space $B_{p,q}^s(\mathbf{R}^d)$ is defined as the space of all the tempered distributions u satisfying

$$\|u\|_{B_{p,q}^s} := \left\| \left(2^{js} \|\Delta_j u\|_{L^p(\mathbf{R}^d)} \right)_{j=0}^{\infty} \right\|_{\ell^q} < +\infty.$$

When $p = q = \infty$, $B_{p,q}^s$ becomes the Zygmund space denoted by C_*^s .

3. (Hölder spaces) For $k \in \mathbf{N}$, we denote by $W^{k,\infty}(\mathbf{R}^d)$ the usual Sobolev spaces. For $\rho = k + \sigma$ with $k \in \mathbf{N}$ and $\sigma \in (0, 1)$, $W^{\rho,\infty}(\mathbf{R}^d)$ denotes the space of all function $u \in W^{k,\infty}(\mathbf{R}^d)$ such that all the k^{th} derivatives of u are σ -Hölder continuous on \mathbf{R}^d .

Let us review notations and results about BonyCalculusymboliqueet79's paradifferential calculus (see [21, 51, 80]). Here we follow the presentation of Métivier in [80] (se also [7], [8]).

Definition A.0.2. 1. (Symbols) Given $\rho \in [0, \infty)$ and $m \in \mathbf{R}$, $\Gamma_\rho^m(\mathbf{R}^d)$ denotes the space of locally bounded functions $a(x, \xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$

and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x, \xi)$ belongs to $W^{\rho, \infty}(\mathbf{R}^d)$ and there exists a constant C_α such that,

$$(A.0.1) \quad \forall |\xi| \geq \frac{1}{2}, \quad \|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbf{R}^d)} \leq C_\alpha (1 + |\xi|)^{m - |\alpha|}.$$

Let $a \in \Gamma_\rho^m(\mathbf{R}^d)$, we define the semi-norm

$$(A.0.2) \quad M_\rho^m(a) = \sup_{|\alpha| \leq d/2 + 1 + \rho} \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(\cdot, \xi) \right\|_{W^{\rho, \infty}(\mathbf{R}^d)}.$$

2. (Paradifferential operators) Given a symbol a , we define the paradifferential operator T_a by

$$(A.0.3) \quad \widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \psi(\eta) \widehat{u}(\eta) d\eta,$$

where $\widehat{a}(\theta, \xi) = \int e^{-ix \cdot \theta} a(x, \xi) dx$ is the Fourier transform of a with respect to the first variable; χ and ψ are two fixed C^∞ functions such that:

$$(A.0.4) \quad \psi(\eta) = 0 \quad \text{for } |\eta| \leq \frac{1}{5}, \quad \psi(\eta) = 1 \quad \text{for } |\eta| \geq \frac{1}{4},$$

and $\chi(\theta, \eta)$ is defined by $\chi(\theta, \eta) = \sum_{k=0}^{+\infty} \kappa_{k-3}(\theta) \varphi_k(\eta)$.

Remark. We make the following remarks on the preceding definition.

1. The cut-off χ satisfies the following localization property (see [80], page 73) for some $0 < \varepsilon_1 < \varepsilon_2 < 1$

$$(A.0.5) \quad \begin{cases} \chi(\theta, \eta) = 1 & \text{for } |\theta| \leq \varepsilon_1(1 + |\eta|) \\ \chi(\theta, \eta) = 0 & \text{for } |\theta| \geq \varepsilon_2(1 + |\eta|). \end{cases}$$

Therefore, in the definition of $T_a u$, on the Fourier side, $T_a u$ keeps only the regime where u has higher frequency than a . In particular, when $a = 1$, we have $T_1 u = \psi(D_x)u$, hence

$$(T_1 - 1) : H^{-\infty} \rightarrow H^\infty, \quad C_*^{-\infty} \rightarrow C_*^\infty.$$

2. As usual, the paraproduct $\widetilde{T}_a u$ is defined by

$$\widetilde{T}_a u = \sum_{k=0}^{+\infty} S_{k-3} a \Delta_k u.$$

On the Fourier side, $\widetilde{T}_a u$ is thus given by the formula (A.0.3) with $\psi \equiv 1$. Consequently

$$(T_a - \widetilde{T}_a)u = \widetilde{T}_a((1 - \psi(D_x))u)$$

and thus using the fact that for any $m > 0$ (see Theorem 2.82, [17]),

$$\|\widetilde{T}_a v\|_{H^s} \leq C \|a\|_{C_*^{-m}} \|v\|_{H^{s+m}}$$

we obtain

$$(T_a - \widetilde{T}_a) : H^{-\infty} \rightarrow H^\infty, \quad C_*^{-\infty} \rightarrow C_*^\infty$$

provided $a \in C_*^{-\infty}$. For this reason, we do not distinguish $T_a u$ and $\widetilde{T}_a u$ in this paper.

Definition A.0.3. Let $m \in \mathbf{R}$. An operator T is said to be of order m if, for all $\mu \in \mathbf{R}$, it is bounded from H^μ to $H^{\mu-m}$.

Symbolic calculus for paradifferential operators is summarized in the following theorem.

Theorem A.0.4. (Symbolic calculus) Let $m \in \mathbf{R}$ and $\rho \in [0, \infty)$.

(i) If $a \in \Gamma_0^m(\mathbf{R}^d)$, then T_a is of order m . Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$(A.0.6) \quad \|T_a\|_{H^\mu \rightarrow H^{\mu-m}} \leq KM_0^m(a).$$

(ii) If $a \in \Gamma_\rho^m(\mathbf{R}^d)$, $b \in \Gamma_\rho^{m'}(\mathbf{R}^d)$ with $\rho > 0$. Then $T_a T_b - T_{a\sharp b}$ is of order $m + m' - \rho$ where

$$a\sharp b := \sum_{|\alpha| < \rho} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi).$$

Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$(A.0.7) \quad \|T_a T_b - T_{a\sharp b}\|_{H^\mu \rightarrow H^{\mu-m-m'+\rho}} \leq KM_\rho^m(a)M_0^{m'}(b) + KM_0^m(a)M_\rho^{m'}(b).$$

(iii) Let $a \in \Gamma_\rho^m(\mathbf{R}^d)$ with $\rho > 0$. Denote by $(T_a)^*$ the adjoint operator of T_a and by \bar{a} the complex conjugate of a . Then $(T_a)^* - T_{a^*}$ is of order $m - \rho$ where

$$a^* = \sum_{|\alpha| < \rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \bar{a}.$$

Moreover, for all μ there exists a constant K such that

$$(A.0.8) \quad \|(T_a)^* - T_{a^*}\|_{H^\mu \rightarrow H^{\mu-m+\rho}} \leq KM_\rho^m(a).$$

We also need the following definition for symbols with negative regularity.

Definition A.0.5. For $m \in \mathbf{R}$ and $\rho \in (-\infty, 0)$, $\Gamma_\rho^m(\mathbf{R}^d)$ denotes the space of distributions $a(x, \xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x, \xi)$ belongs to $C_*^\rho(\mathbf{R}^d)$ and there exists a constant C_α such that,

$$(A.0.9) \quad \forall |\xi| \geq \frac{1}{2}, \quad \|\partial_\xi^\alpha a(\cdot, \xi)\|_{C_*^\rho} \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}.$$

For $a \in \Gamma_\rho^m$, we define

$$(A.0.10) \quad M_\rho^m(a) = \sup_{|\alpha| \leq 2(d+2)+|\rho|} \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot, \xi) \right\|_{C_*^\rho(\mathbf{R}^d)}.$$

Proposition A.0.6 (see [7, Proposition 2.12]). Let $\rho < 0$, $m \in \mathbf{R}$ and $a \in \dot{\Gamma}_\rho^m$. Then the operator T_a is of order $m - \rho$:

$$(A.0.11) \quad \|T_a\|_{H^s \rightarrow H^{s-(m-\rho)}} \leq CM_\rho^m(a), \quad \|T_a\|_{C_*^s \rightarrow C_*^{s-(m-\rho)}} \leq CM_\rho^m(a).$$

Remark. In the definition (A.0.3) of paradifferential operators, the cut-off ψ removes the low frequency part of u . Therefore, estimates pertaining to $T_a u$ can be relaxed, for example, when $a \in \Gamma_0^m$ and $u \in \mathcal{S}'$ such that $\nabla u \in H^{\sigma+m-1}$ we have

$$\|T_a u\|_{H^\sigma} \leq CM_0^m(a) \|\nabla u\|_{H^{\sigma+m-1}}.$$

Notation A.0.7. Let $I \subset \mathbf{R}$ and $a(z, x, \xi) : I \times \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{C}$ be a family of symbols parametrized by $z \in I$. We denote

$$\mathcal{M}(a) = \sup_{z \in I} M^m(\rho)(a(z, \cdot, \cdot)).$$

The set of such a with $\mathcal{M}(a) < \infty$ is denoted by $\Gamma_\rho^m(\mathbf{R}^d \times I)$.

A.0.2 Paraproducts

Given two functions a, b defined on \mathbf{R}^d we define the remainder

$$(A.0.12) \quad R(a, u) = au - T_a u - T_u a.$$

We shall use frequently various estimates about paraproducts (see chapter 2 in [17], and [7]) which are recalled here.

Theorem A.0.8. 1. Let $\alpha, \beta \in \mathbf{R}$. If $\alpha + \beta > 0$ then

$$(A.0.13) \quad \|R(a, u)\|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbf{R}^d)} \leq K \|a\|_{H^\alpha(\mathbf{R}^d)} \|u\|_{H^\beta(\mathbf{R}^d)},$$

$$(A.0.14) \quad \|R(a, u)\|_{H^{\alpha+\beta}(\mathbf{R}^d)} \leq K \|a\|_{C_*^\alpha(\mathbf{R}^d)} \|u\|_{H^\beta(\mathbf{R}^d)},$$

$$(A.0.15) \quad \|R(a, u)\|_{C_*^{\alpha+\beta}(\mathbf{R}^d)} \leq K \|a\|_{C_*^\alpha(\mathbf{R}^d)} \|u\|_{C_*^\beta(\mathbf{R}^d)}.$$

2. Let s_0, s_1, s_2 be such that $s_0 \leq s_2$ and $s_0 < s_1 + s_2 - \frac{d}{2}$, then

$$(A.0.16) \quad \|T_a u\|_{H^{s_0}} \leq K \|a\|_{H^{s_1}} \|u\|_{H^{s_2}}.$$

If in addition to the conditions above, $s_1 + s_2 > 0$ then

$$(A.0.17) \quad \|au - T_a u\|_{H^{s_0}} \leq K \|a\|_{H^{s_1}} \|u\|_{H^{s_2}}.$$

3. Let $m > 0$ and $s \in \mathbf{R}$. Then

$$(A.0.18) \quad \|T_a u\|_{H^s} \leq K \|a\|_{L^\infty} \|u\|_{H^s},$$

$$(A.0.19) \quad \|T_a u\|_{H^{s-m}} \leq K \|a\|_{C_*^{-m}} \|u\|_{H^s},$$

$$(A.0.20) \quad \|T_a u\|_{C_*^{s-m}} \leq K \|a\|_{C_*^{-m}} \|u\|_{C_*^s},$$

$$(A.0.21) \quad \|T_a u\|_{C_*^s} \leq K \|a\|_{L^\infty} \|u\|_{C_*^s}.$$

Proposition A.0.9. 1. If $s_0, s_1, s_2 \in \mathbf{R}$ satisfying $s_1 + s_2 > 0$, $s_0 \leq s_1$, $s_0 \leq s_2$ and $s_0 < s_1 + s_2 - \frac{d}{2}$, then

$$(A.0.22) \quad \|u_1 u_2\|_{H^{s_0}} \leq K \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}}.$$

2. If $s \geq 0$ then

$$(A.0.23) \quad \|u_1 u_2\|_{H^s} \leq K (\|u_1\|_{H^s} \|u_2\|_{L^\infty} + \|u_2\|_{H^s} \|u_1\|_{L^\infty}).$$

3. If $s \geq 0$ then

$$(A.0.24) \quad \|u_1 u_2\|_{C_*^s} \leq K (\|u_1\|_{C_*^s} \|u_2\|_{L^\infty} + \|u_2\|_{C_*^s} \|u_1\|_{L^\infty}).$$

4. Let $\beta > \alpha > 0$. Then

$$(A.0.25) \quad \|u_1 u_2\|_{C_*^{-\alpha}} \leq K \|u_1\|_{C_*^\beta} \|u_2\|_{C_*^{-\alpha}}.$$

5. Let $s \geq 0$ and $F \in C^\infty(\mathbf{C}^N)$ satisfying $F(0) = 0$. Then there exists a nondecreasing function $\mathcal{F}: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ such that, for all $U \in H^s(\mathbf{R}^d)^N \cap L^\infty(\mathbf{R}^d)^N$,

$$(A.0.26) \quad \|F(U)\|_{H^s} \leq \mathcal{F}(\|U\|_{L^\infty}) \|U\|_{H^s}.$$

6. Let $F \in C^\infty(\mathbf{C}^N)$ satisfying $F(0) = 0$, $s > 0$, and $p, r \in [1, \infty]$. Then there exists a nondecreasing function $\mathcal{F}: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ such that for all $u \in B_{p,r}^s(\mathbf{R}^d)^N \cap L^\infty(\mathbf{R}^d)^N$,

$$(A.0.27) \quad \|F \circ u\|_{B_{p,r}^s} \leq \mathcal{F}(\|u\|_{L^\infty}) \|u\|_{B_{p,r}^s}.$$

Theorem A.0.10 (see [17, Theorem 2.92]). (*Paralinearization*) Let r, ρ be positive real numbers and F be a C^∞ function on \mathbf{R} such that $F(0) = 0$. Assume that ρ is not an integer. For any $u \in H^\mu(\mathbf{R}^d) \cap C_*^\rho(\mathbf{R}^d)$ we have

$$\|F(u) - T_{F'(u)}u\|_{H^{\mu+\rho}(\mathbf{R}^d)} \leq C(\|u\|_{L^\infty(\mathbf{R}^d)}) \|u\|_{C_*^\rho(\mathbf{R}^d)} \|u\|_{H^\mu(\mathbf{R}^d)}.$$

Remark. In Theorem 2.92, [17], there is a restriction that ρ is not an integer. In fact, by following the proof of the same result (but qualitative) in Theorem 5.2.4, [80] one can check that this restriction can be dropped.

Lemma A.0.11. Let $s, r, \alpha \in \mathbf{R}$ satisfy

$$\text{either } r \leq 0, s < \alpha + r \text{ or } r > 0, s < \alpha.$$

Then there exists $C > 0$ such that

$$\|T_a u\|_{H^s} \leq C \|a\|_{L^r} \|u\|_{C_*^\alpha}.$$

Proof. We have by the definition of paraproducts (see Definition A.0.1 and Remark A.0.1),

$$\|T_a u\|_{H^s}^2 \lesssim \sum_{k \geq 0} 2^{2sk} \|S_{k-3} a \Delta_k u\|_{L^2}^2 \lesssim \sum_{k \geq 0} 2^{2sk} \|S_{k-3} a\|_{L^2}^2 \|\Delta_k u\|_{L^\infty}^2.$$

For small k , we have the easy estimate

$$\sum_{k=0}^3 2^{2sk} \|S_{k-3} a\|_{L^2}^2 \|\Delta_k u\|_{L^\infty}^2 \lesssim \|a\|_{L^2}^2 \|u\|_{C_*^\alpha}^2.$$

Consider the case $r \leq 0$ and $s < \alpha + r$. Pick $\varepsilon \in (0, \alpha + r - s)$. For $k \geq 4$, using $S_{k-3} a = \sum_{j=0}^{k-3} \Delta_j a$ we can apply the Hölder inequality to estimate (notice that $r \leq 0$)

$$\begin{aligned} \sum_{k \geq 4} 2^{2sk} \|S_{k-3} a\|_{L^2}^2 \|\Delta_k u\|_{L^\infty}^2 &\lesssim \|u\|_{C_*^\alpha}^2 \sum_{k \geq 4} 2^{2(s-\alpha)k} \mathcal{B}ig\left(\sum_{j=0}^{k-3} \|\Delta_j a\|_{L^2} \mathcal{B}ig\right)^2 \\ &\lesssim \|u\|_{C_*^\alpha}^2 \sum_{k \geq 4} 2^{2(s-\alpha)k} \sum_{j=0}^{k-3} 2^{2rj} \|\Delta_j a\|_{L^2}^2 \sum_{j=0}^{k-3} 2^{-2rj} \\ &\lesssim \|u\|_{C_*^\alpha}^2 \sum_{k \geq 4} 2^{2(s-\alpha-r+\varepsilon)k} \sum_{j=0}^{k-3} 2^{2rj} \|\Delta_j a\|_{L^2}^2 \\ &\lesssim \|u\|_{C_*^\alpha}^2 \|a\|_{H^r}^2. \end{aligned}$$

Now, if $r > 0$ and $s < \alpha$, in the second line of the preceding estimate we observe that the series

$$\sum_{j \geq 0} 2^{-2rj}, \quad \sum_{k \geq 4} 2^{2(s-\alpha)k}$$

converge. This concludes the proof in the second case. \square

We will also need the following commutator estimate between a paradifferential operator and a convective derivative

Lemma A.0.12 (Lemma 2.16 of [7]). *Let $V \in C^0([0, T]; \mathbf{C}_*^{1+\varepsilon}(\mathbf{R}^d))$ and let $p = p(t, x, \xi)$ be a symbol homogeneous of order $m \in \mathbf{R}$ in ξ . Then there exists $C > 0$ independent of p and V , such that for any $t \in [0, T]$ and any $u \in C^0([0, T]; H^m(\mathbf{R}^d))$,*

$$\begin{aligned} & \| [T_p, \partial_t + T_V \cdot \nabla] u(t) \|_{L^2(\mathbf{R}^d)} \\ & \leq C \left\{ \mathcal{M}_0^m(p) \|V(t)\|_{C_*^{1+\varepsilon}} + \mathcal{M}_0^m(\partial_t p + V \cdot \nabla p) \right\} \|u(t)\|_{H^m(\mathbf{R}^d)}. \end{aligned}$$

A.0.3 Paradifferential calculus in Besov spaces

Concerning the symbolic calculus in Besov spaces, we have the following results.

Lemma A.0.13 (see [102, Lemma 2.6]). *Let $s, m, m' \in \mathbf{R}$, $q \in [1, \infty]$ and $\rho \in [0, 1]$.*

(i) *If $a \in \Gamma_0^m(\mathbf{R}^d)$ then*

$$\|T_a\|_{B_{\infty, q}^s \rightarrow B_{q, \infty}^{s-m}} \leq CM_0^m(a).$$

(ii) *If $a \in \Gamma_0^m(\mathbf{R}^d)$, $b \in \Gamma_0^{m'}(\mathbf{R}^d)$ then*

$$\|T_a T_b - T_{ab}\|_{B_{\infty, q}^s \rightarrow B_{q, \infty}^{s-(m+m')+\rho}} \leq CM_\rho^m(a) M_0^{m'}(b) + CM_0^m(a) M_\rho^{m'}(b).$$

Lemma A.0.14 (see [103, Lemma 2.10]). *1. Let $s \in \mathbf{R}$ and $p, q \in [1, \infty]$. Then for any $\sigma > 0$ we have*

$$(A.0.28) \quad \|T_a u\|_{B_{p, q}^s} \leq K \min(\|a\|_{L^\infty} \|u\|_{B_{p, q}^s}, \|a\|_{C_*^{-\sigma}} \|u\|_{B_{p, q}^{s+\sigma}}).$$

2. Let $s > 0$ and $p, q \in [1, \infty]$. Then for any $\sigma \in \mathbf{R}$, we have

$$(A.0.29) \quad \|R(a, u)\|_{B_{p, q}^s} \leq K \|a\|_{C_*^\sigma} \|u\|_{B_{p, q}^{s-\sigma}}.$$

To deal with time-dependent distributions, we use the Chemin-Lerner spaces defined as follows (see Chapter 2, [17]).

Definition A.0.15. For $T > 0$, $s \in \mathbf{R}$, and $p, q, r \in [1, \infty]$, we set

$$(A.0.30) \quad \|u\|_{\tilde{L}^q([0, T]; B_{p, r}^s)} := \left\| \left(2^{js} \|\Delta_j u\|_{L^q([0, T]; L^p(\mathbf{R}^d))} \right)_{j=0}^\infty \right\|_{\ell^r}.$$

Again, when $p = r = \infty$, we denote $\tilde{L}^q([0, T]; B_{p, r}^s) = \tilde{L}^q([0, T]; C_*^s)$. Notice that $\|u\|_{\tilde{L}^\infty(I; C_*^s)} = \|u\|_{L^\infty(I; C_*^s)}$.

The next lemma then follows easily from the proof Lemma A.0.13.

Lemma A.0.16 (see [102, Lemma 2.6]). *Let $s, m, m' \in \mathbf{R}$, $p, q \in [1, \infty]$, $\rho \in [0, 1]$ and $I = [0, T]$.*

(i) *If $a \in \Gamma_0^m(I \times \mathbf{R}^d)$ then*

$$\|T_a\|_{\tilde{L}^p(I; B_{\infty, q}^s) \rightarrow \tilde{L}^p(I; B_{\infty, q}^{s-m})} \leq CM_0^m(a).$$

(ii) *If $a \in \Gamma_0^m(I \times \mathbf{R}^d)$, $b \in \Gamma_0^{m'}(I \times \mathbf{R}^d)$ then*

$$\|T_a T_b - T_{ab}\|_{\tilde{L}^p(I; B_{\infty, q}^s) \rightarrow \tilde{L}^p(I; B_{\infty, q}^{s-(m+m')+\rho})} \leq CM_\rho^m(a) \mathcal{M}_0^{m'}(b) + CM_0^m(a) \mathcal{M}_\rho^{m'}(b).$$

Finally, the following lemma is a direct consequence of Lemma A.0.28.

Lemma A.0.17 (see [103, Lemmas 2.17, 218]). *Let $I = [0, T]$.*

1. *Let $s \in \mathbf{R}$ and $q, q_1, q_2, r \in [1, \infty]$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then for any $\sigma > 0$ we have*

$$(A.0.31) \quad \|T_a u\|_{\tilde{L}^q(I; B_{\infty, r}^s)} \leq K \min \left(\|a\|_{\tilde{L}^{q_1}(I; L^\infty)} \|u\|_{\tilde{L}^{q_2}(I; B_{\infty, r}^s)}, \|a\|_{\tilde{L}^{q_1}(I; C_*^{-\sigma})} \|u\|_{\tilde{L}^{q_2}(I; B_{\infty, r}^{s+\sigma})} \right).$$

2. *Let $s > 0$ and $q, q_1, q_2, r \in [1, \infty]$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then for any $\sigma \in \mathbf{R}$ we have*

$$(A.0.32) \quad \|R(a, u)\|_{\tilde{L}^q(I; B_{\infty, r}^s)} \leq K \|a\|_{\tilde{L}^{q_1}(I; C_*^{-\sigma})} \|u\|_{\tilde{L}^{q_2}(I; B_{\infty, r}^{s+\sigma})}.$$

A.0.4 Parabolic regularity

Given $I \subset \mathbf{R}$, let $\Gamma_\rho^m(I \times \mathbf{R}^d)$ be the set of symbols $a(z, x, \xi)$ satisfying

$$\mathcal{M}_\rho^m(a) = \sup_{z \in I} \sup_{|\alpha| \leq \frac{3d}{2} + \rho + 1} \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(z; \cdot, \xi) \right\|_{W^{\rho, \infty}(\mathbf{R}^d)} < +\infty.$$

We need to study the parabolic evolution equation

$$\partial_z w + T_p w = f, \quad w|_{z=z_0} = w_0,$$

with an elliptic symbol $p \in \Gamma_\rho^m(I \times \mathbf{R}^d)$, which means that

$$p(z; x, \xi) \geq c |\xi|,$$

for some positive constant c .

Define for $\mu \in \mathbf{R}$ the spaces

$$\begin{aligned} X^\mu(I) &= L_z^\infty(I; H^\mu(\mathbf{R}^d)) \cap L_z^2(I; H^{\mu+\frac{1}{2}}(\mathbf{R}^d)), \\ Y^\mu(I) &= L_z^1(I; H^\mu(\mathbf{R}^d)) + L_z^2(I; H^{\mu-\frac{1}{2}}(\mathbf{R}^d)). \end{aligned}$$

We will use the following proposition from [7].

Proposition A.0.18 (Proposition 2.18 of [7]). *Let $\rho \in (0, 1)$, $J = [z_0, z_1] \subset \mathbf{R}$, $p \in \Gamma_\rho^1(\mathbf{R}^d \times J)$ with the assumption that*

$$\text{Rep}(z; x, \xi) \geq c |\xi|,$$

for some positive constant c . Assume that w solves

$$\partial_z w + T_p w = F, \quad w|_{z=z_0} = w_0.$$

Then we have

$$\|w\|_{X^r} \leq K \{ \|w_0\|_{H^r} + \|F\|_{Y^r} \},$$

for some positive constant K depending only on r, ρ, c and $\mathcal{M}_\rho^1(p)$.

In the following study, we will need the following Chemin-Lerner type of time-dependent spaces. See for example [17].

Definition A.0.19. If s is any real number, and if $1 \leq p, q, l \leq \infty$, we define the space $\tilde{L}_z^p(I; B_{q, l}^s(\mathbf{R}^d))$ as the space of tempered distributions u such that

$$\|u\|_{\tilde{L}_z^p(I; B_{q, l}^s(\mathbf{R}^d))} := \left(\sum_{k \in \mathbf{N}} 2^{lks} \|\Delta_k u\|_{L_z^p(I; L^q)}^l \right)^{\frac{1}{l}} < +\infty.$$

For $q = l = 2$, $\tilde{L}_z^p(I; B_{2,2}^s(\mathbf{R}^d)) = \tilde{L}_z^p(I; H^s(\mathbf{R}^d))$. Also for $q = l = \infty$, $\tilde{L}_z^p(I; B_{\infty,\infty}^s(\mathbf{R}^d)) = \tilde{L}_z^p(I; C_*^s(\mathbf{R}^d))$. As can be seen for example in [17], we have

$$\|u\|_{\tilde{L}_z^p(I; B_{q,l}^s)} \leq \|u\|_{L_z^p(I; B_{q,l}^s)} \quad \text{if } l \geq p, \quad \|u\|_{\tilde{L}_z^p(I; B_{q,l}^s)} \geq \|u\|_{L_z^p(I; B_{q,l}^s)} \quad \text{if } l \leq p.$$

The paraproduct properties can be passed to the time dependent spaces as long as the exponents respect the conditions for Hölder inequality.

In the following, we will need the following Bernstein inequalities

Lemma A.0.20. *Let $1 \leq p \leq q \leq \infty, \alpha \in \mathbf{N}^d$. Then it holds that*

$$\begin{aligned} \|\partial^\alpha S_k u\|_{L^q} &\leq C 2^{kd(\frac{1}{p} - \frac{1}{q} + |\alpha|)} \|S_k u\|_{L^p} \quad \text{for } k \in \mathbf{N}, \\ \|\Delta_k u\|_{L^q} &\leq C 2^{kd(\frac{1}{p} - \frac{1}{q} + |\alpha|)} \sup_{|\beta|=|\alpha|} \|\partial^\beta \Delta_k u\|_{L^p} \quad \text{for } k \geq 1. \end{aligned}$$

Theorem A.0.21 (see [7, Proposition 2.18]). *Let $\rho \in (0, 1)$, $J = [z_0, z_1] \subset \mathbf{R}$, $p \in \Gamma_\rho^1(\mathbf{R}^d \times J)$, $q \in \Gamma_0^0(\mathbf{R}^d \times J)$ with the assumption that*

$$\text{Rep}(z; x, \xi) \geq c|\xi|,$$

for some constant $c > 0$. Assume that w solves

$$\partial_z w + T_p w = T_q w + f, \quad w|_{z=z_0} = w_0.$$

Then for any $r \in \mathbf{R}$, if $f \in Y^r(J)$ and $w_0 \in H^r$, we have $w \in X^r(J)$ and

$$\|w\|_{X^r(J)} \leq K \left\{ \|w_0\|_{H^r} + \|f\|_{Y^r(J)} \right\}.$$

for some constant $K = K(\mathcal{M}_\rho^1(\rho), \mathcal{M}_0^0(q), c^{-1})$ nondecreasing in each argument.

Theorem A.0.22 (see [102, Proposition 3.1]). *Let $r \in \mathbf{R}, \ell \in [1, \infty]$ and $1 \leq q \leq p \leq \infty$. Let $\rho \in (0, 1)$, $J = [z_0, z_1] \subset \mathbf{R}$, $a \in \Gamma_\rho^1(\mathbf{R}^d \times J)$ with the assumption that*

$$\text{Rea}(z; x, \xi) \geq c|\xi|,$$

for some constant $c > 0$. Assume that w solves

$$\partial_z w + T_a w = F, \quad w|_{z=z_0} = w_0.$$

If

$$w_0 \in B_{\infty,\ell}^r, \quad F \in \tilde{L}^q(J, B_{\infty,\ell}^{r-1+\frac{1}{q}})$$

and there exists $\delta > 0$ such that $w \in \tilde{L}^p(J, C_*^{-\delta})$, then we have $w \in \tilde{L}^p(J, B_{\infty,\ell}^{r+\frac{1}{p}})$ and

$$\|w\|_{\tilde{L}^p(J, B_{\infty,\ell}^{r+\frac{1}{p}})} \leq K \text{Big} \{ \|w_0\|_{B_{\infty,\ell}^r} + \|F\|_{\tilde{L}^q(J, B_{\infty,\ell}^{r-1+\frac{1}{q}})} + \|w\|_{\tilde{L}^p(J, C_*^{-\delta})} \text{Big} \}.$$

for some constant $K = K(\mathcal{M}_\rho^1(a), c^{-1})$ nondecreasing in each argument. When $p = \infty$, the left-hand side can be replaced by $\|w\|_{C(J, B_{\infty,\ell}^r)}$.

Finally, we recall a classical interpolation lemma.

Lemma A.0.23 (see [76, th 3.1]). *Let $I = (-1, 0)$ and $s \in \mathbf{R}$. Let $u \in L_z^2(I, H^{s+\frac{1}{2}}(\mathbf{R}^d))$ such that $\partial_z u \in L_z^2(I, H^{s-\frac{1}{2}}(\mathbf{R}^d))$. Then $u \in C^0([-1, 0], H^s(\mathbf{R}^d))$ and there exists an absolute constant $C > 0$ such that*

$$\sup_{z \in [-1, 0]} \|u(z, \cdot)\|_{H^s(\mathbf{R}^d)} \leq C \left(\|u\|_{L^2(I, H^{s+\frac{1}{2}}(\mathbf{R}^d))} + \|\partial_z u\|_{L^2(I, H^{s-\frac{1}{2}}(\mathbf{R}^d))} \right).$$

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Résumé

Cette thèse a pour objet l'étude de certains aspects du problème de CAUCHY pour l'équation des vagues. Dans la première partie, on utilise une formulation paradifférentielle pour prouver un critère d'explosion pour les vagues de gravité. On montre ensuite des estimations de STRICHARTZ pour les vagues de capillarité, avant de les utiliser pour résoudre le problème de CAUCHY à faible régularité. Dans la deuxième partie, on prouve des estimations a priori pour les vagues de gravité avec fond émergent.

Mots Clés

Equations aux dérivées partielles, Mécanique des fluides, Dispersion

Abstract

This thesis studies some aspects of the Cauchy problem for the water waves equation. In the first part, we use a paradifferential formulation to prove a blow-up criterion for gravity waves. We then show some Strichartz estimates for capillary waves, and use them to solve the Cauchy problem at low regularity. In the second part, we prove a priori estimates for gravity waves with an emerging bottom.

Keywords

Partial Differential Equations, Fluid Mechanics, Dispersion