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Sandro Franceschi

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Présentée par

Sandro Franceschi

**Approche analytique
pour le mouvement brownien réfléchi dans des cônes**

dirigée par **Irina Kourkova** et **Kilian Raschel**

soutenue le vendredi 8 décembre 2017 devant le jury composé de :

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À ma grand-mère,
À ma mère.

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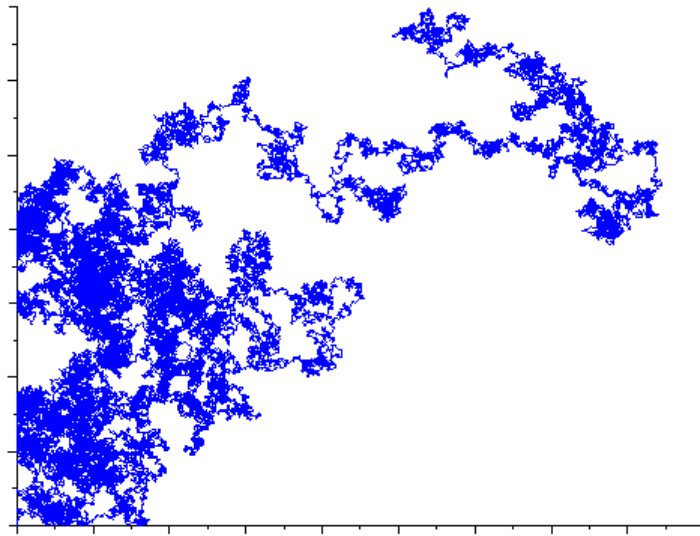
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Un exemple de trajectoire du mouvement Brownien réfléchi dans le quart de plan

An example of path of reflected Brownian motion in the quadrant

Résumé

Le mouvement Brownien réfléchi de manière oblique dans le quadrant, introduit par Harrison, Reiman, Varadhan et Williams dans les années 80, est un objet largement analysé dans la littérature probabiliste. Cette thèse, qui présente l'étude complète de la mesure invariante de ce processus dans tous les cônes du plan, a pour objectif plus global d'étendre au cadre continu une méthode analytique développée initialement pour les marches aléatoires dans le quart de plan par Fayolle, Iasnogorodski et Malyshev dans les années 70. Cette approche est basée sur des équations fonctionnelles, reliant des fonctions génératrices dans le cas discret et des transformées de Laplace dans le cas continu. Ces équations permettent de déterminer et de résoudre des problèmes frontière satisfaits par ces fonctions génératrices. Dans le cas récurrent, cela permet de calculer explicitement la mesure invariante du processus avec rebonds orthogonaux, dans le chapitre 2, et avec rebonds quelconques, dans le chapitre 3. Les transformées de Laplace des mesures invariantes sont prolongées analytiquement sur une surface de Riemann induite par le noyau de l'équation fonctionnelle. L'étude des singularités et l'application de méthodes du point col sur cette surface permettent de déterminer l'asymptotique complète de la mesure invariante selon toutes les directions dans le chapitre 4.

Mots clés. Mouvement Brownien réfléchi dans les cônes; Distribution stationnaire; Transformée de Laplace; Fonctions génératrices; Méthode des invariants de Tutte; Problème de valeur frontière; Transformation conforme; Analyse asymptotique; Méthode du point col; Surface de Riemann.

Abstract

Obliquely reflected Brownian motion in the quadrant, introduced by Harrison, Reiman, Varadhan and Williams in the eighties, has been studied a lot in the probabilistic literature. This thesis, which presents the complete study of the invariant measure of this process in all the cones of the plan, has for overall aim to extend to the continuous framework an analytic method initially developed for random walks in the quarter plane by Fayolle, Iasnogorodski and Malyshev in the seventies. This approach is based on functional equations which link generating functions in the discrete case and Laplace transform in the continuous case. These equations allow to determine and to solve boundary value problems satisfied by these generating functions. In the recurrent case, it permits to compute explicitly the invariant measure of the process with orthogonal reflexions, in the chapter 2, and with any reflexions, in the chapter 3. The Laplace transform of the invariant measure is analytically extended to a Riemann surface induced by the kernel of the functional equation. The study of singularities and the use of saddle point methods on this surface allows to determine the full asymptotics of the invariant measure along every directions in the chapter 4.

Key words. Reflected Brownian motion in cones; Stationary distribution; Laplace transform; Generating function; Tutte's invariant approach; Boundary value problem; Conformal mapping; Asymptotic analysis, Saddle-point method; Riemann surface.

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Introduction

Résumé

Cette thèse est composée de trois chapitres principaux 2, 3 et 4, qui peuvent être lus de manière indépendante et qui sont issus de trois articles publiés ou destinés à l'être. Cette introduction vise à présenter les objets, les outils et les méthodes utilisés qui se retrouvent dans ces diverses parties. Après un bref focus sur les temps d'occupation et la notion de réflexion dans le cas de la dimension un (section 1.1), nous présenterons le mouvement Brownien réfléchi avec dérive dans le quart de plan (section 1.2). Nous nous attarderons notamment sur les équations fonctionnelles qui relient certaines fonctions génératrices et qui sont à la base de notre approche analytique. Nous résumerons ensuite les principes de cette méthode et nous évoquerons la théorie des problèmes frontière qui nous permettra de résoudre ces équations fonctionnelles et d'énoncer nos principaux résultats (section 1.3).

Cette approche nous permettra entre autres de calculer explicitement la mesure invariante de notre processus (chapitres 2 et 3) et de déterminer son asymptotique complète (chapitre 4). C'est en fait l'étude des marches aléatoires dans le quart de plan qui a historiquement permis de développer ce type de méthode et qui a inspiré cette thèse (annexe A).

Un lecteur averti et désireux de rentrer directement au cœur du sujet pourra commencer sa lecture à la section 1.2 qui introduit l'objet principal de notre étude : le mouvement Brownien réfléchi dans le quadrant. La section 1.1 introduit quant à elle des notions classiques mais subtiles de calcul stochastique en lien direct avec notre étude et prend le soin de résoudre en dimension un le problème posé -en dimension deux- dans cette thèse.

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1.1 Temps d'occupation et réflexion

Les mesures d'occupation nous intéressent particulièrement car elles sont intimement liées aux mesures invariantes, aux fonctions de Green et aux processus réfléchis qui sont au cœur de cette thèse. Notre objectif sera tout d'abord dans la section 1.1.1 d'introduire en dimension un des objets comme le temps local et d'analyser ses liens avec les questions de réflexion. Ensuite dans la section 1.1.2 nous ferons quelques rappels sur les mesures invariantes et les fonctions de Green des processus de Markov. Tout cela nous servira par la suite à aborder plus facilement la section 1.2 où l'on étudiera le mouvement Brownien réfléchi en dimension deux. Enfin, dans la section 1.1.3, nous étudierons de manière détaillée, via des équations fonctionnelles, le cas du mouvement Brownien réfléchi avec dérive, et nous calculerons sa mesure invariante et ses fonctions de Green. Cet exemple simple nous permettra d'illustrer en dimension un la méthode analytique, exposée dans la section 1.3, qui sera au centre de notre étude tout au long de cette thèse.

1.1.1 Temps local et réflexion

Soit (B_t) le mouvement brownien réel standard. On cherche à étudier le temps que le processus passe en un point $a \in \mathbb{R}$. Il ne sert à rien d'étudier pour cela $\int_0^t \mathbb{1}_{B_s=a} ds$. En effet cette intégrale est nulle car $\mathbb{E}(\int_0^t \mathbb{1}_{B_s=a} ds) = \int_0^t \mathbb{P}(B_s = a) ds = 0$. La notion intéressante sera celle de densité du temps d'occupation. Pour plus de détails sur de nombreux résultats de cette section, on se référera par exemple aux livres de Revuz et Yor [89] et Yen et Yor [103].

Définition 1 (Mesure d'occupation et temps local). *On définit μ_t la mesure d'occupation par*

$$\mu_t(A) = \int_0^t \mathbb{1}_A(B_s) ds,$$

pour tout ensemble mesurable $A \subset \mathbb{R}$. C'est le temps passé par le processus dans l'ensemble A avant le temps t . Une telle mesure admet une densité L_t^\bullet par rapport à la mesure de Lebesgue telle que $\mu_t(da) = L_t^a da$ (i.e. $\mu_t(A) = \int_A L_t^a da$). Cette densité, appelée temps local, est donc définie pour tout $a \in \mathbb{R}$ par

$$L_t^a = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{\{a \leq B_s < a+\epsilon\}} ds.$$

Proposition 2 (Formule du temps d'occupation). *Presque sûrement la fonction $(t, a) \mapsto L_t^a$ est continue et croissante en t . De plus $t \mapsto L_t^a$ ne croît que lorsque $B_t = a$ i.e. le support de $d_t L_t^a$ coïncide avec l'ensemble $\{t \geq 0 : B_t = a\}$. Pour toute fonction f mesurable positive on a presque sûrement*

$$\int_0^t f(B_s) ds = \int_{\mathbb{R}} f(a) L_t^a da.$$

Cette dernière égalité transforme une intégrale temporelle en intégrale spatiale.

Remarque 3 (Semi-martingales continues et temps local symétrique). Ces résultats et les suivants se généralisent aux semi-martingales continues X_t telle que $X_t = X_0 + M_t + A_t$ où M est une martingale locale (issue de 0) et A un processus à variation finie, en remplaçant

la mesure d'intégration ds par $d\langle X \rangle_s$. Dans ce cas $a \mapsto L_t^a$ n'est plus continue mais càdlàg et on peut définir la notion de temps local symétrique

$$\tilde{L}_t^a := \frac{1}{2}(L_t^a + L_t^{a-}) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{a-\epsilon < X_s < a+\epsilon\}} d\langle X \rangle_s. \quad (1.1)$$

Dans le cas d'une martingale locale continue $L_t^a = \tilde{L}_t^a$.

Comme on va le voir, la notion de réflexion est intimement liée aux temps locaux. La réflexion en dimension un peut être obtenue simplement grâce à la valeur absolue. On va donc maintenant appliquer formellement (et illégalement) la formule d'Itô à la fonction valeur absolue (qui n'est pas \mathcal{C}^2 , il faudrait en fait procéder par approximation) pour comprendre d'où vient la formule de Tanaka. Soit $f = |\cdot - a|$. On a $f' = \text{sgn}(\cdot - a)$ et $f'' = 2\delta_a$ où δ_a est le Dirac en a . Si on pouvait appliquer le théorème d'Itô on aurait donc

$$|B_t - a| = |B_0 - a| + \int_0^t \text{sgn}(B_s - a) dB_s + \frac{1}{2} \int_0^t 2\delta_a(B_s) ds.$$

En remarquant qu'au sens des distributions on a $L_t^a = \int \delta_a(u) L_t^u du = \int_0^t \delta_a(B_s) ds$, on obtient la formule suivante.

Proposition 4 (Formule de Tanaka). *Soit B_t le mouvement Brownien standard et L_t^a son temps local en $a \in \mathbb{R}$, on a alors*

$$|B_t - a| = |B_0 - a| + \int_0^t \text{sgn}(B_s - a) dB_s + L_t^a.$$

Remarque 5 (Temps local et réflexion). Le processus $W_t := \int_0^t \text{sgn}(B_s - a) dB_s$ est un mouvement brownien d'après la caractérisation de Lévy car $\langle \int_0^t \text{sgn}(B_s - a) dB_s \rangle_t = \int_0^t (\text{sgn}(B_s - a))^2 d\langle B \rangle_s = \int_0^t ds = t$. En prenant $a = 0$ on obtient la décomposition

$$|B_t| = W_t + L_t^0, \quad (1.2)$$

qui fera écho à notre manière de définir notre processus réfléchi en dimension deux, voir la définition 28. On peut donc percevoir le temps local en 0 comme une dérive infiniment forte qui s'accroît uniquement lorsque le processus touche 0 et qui l'empêche ainsi de franchir 0, obligeant le processus à se réfléchir sur cette frontière.

On remarquera que si Y_t est une martingale locale continue et si $X_t = |Y_t|$ est sa valeur absolue, on a l'égalité

$$L_t^a(X) = L_t^a(Y) + L_t^{(-a)-}(Y).$$

En particulier, en notant \tilde{L} le temps local symétrique de l'équation (1.1) on a

$$L_t^0(B) = \tilde{L}_t^0(|B|) = \frac{1}{2} L_t^0(|B|). \quad (1.3)$$

Remarque 6 (*Skew Brownian motion*). On reprend la même formule que dans l'équation (1.2) mais en pénalisant le temps local grâce à un coefficient. On considère l'équation stochastique $X_t = W_t + \alpha \tilde{L}_t^0$ où W_t est un mouvement Brownien standard et \tilde{L}_t^0 est le temps local symétrique en 0 du processus X . On se référera à l'article de Harrison et Shepp [54] qui introduit ce processus, nommé *Skew Brownian motion*, et démontre les résultats

suivants. Un tel processus X existe (et est unique) si et seulement si $|\alpha| \leq 1$. Ce processus est en fait une diffusion obtenue avec le mouvement Brownien standard en modifiant de manière indépendante le signe de chacune de ses excursions partant de 0, chaque excursion étant de signe positif avec probabilité $p = \frac{\alpha+1}{2}$ et négative avec probabilité $1-p$. Ainsi en diminuant l'impact du temps local grâce au coefficient α , le processus est en quelque sorte réfléchi en 0 avec probabilité p et traverse 0 avec probabilité $1-p$. En prenant $\alpha = 1$ on a $p = 1$ et on retrouve le mouvement Brownien réfléchi.

Remarque 7 (Formule d'Itô-Tanaka). On peut en fait étendre la formule de Tanaka et ainsi généraliser de manière optimale la formule d'Itô. Si X_t est une semi-martingale et f une différence de deux fonctions convexes on a

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s) dX_s + \frac{1}{2} \int_{-\infty}^{+\infty} L_t^a f''(da),$$

où f'_- est la dérivée à gauche de f et f'' sa dérivée seconde au sens des distributions qui est ici une mesure positive. En particulier, $f(X_t)$ est encore une semi-martingale. Réciproquement, si cette formule est vraie et que X_t et $f(X_t)$ sont des semi-martingales continues alors f est la différence de deux fonctions convexes.

Nous citons ici le Lemme de Skorokhod, qui sert dans la démonstration de plusieurs résultats énoncés ici, mais dont le résultat a aussi une valeur en soi qui nous permet de mieux comprendre le temps local et le phénomène de réflexion.

Lemme 8 (Lemme de Skorokhod). *Soit w une fonction continue définie sur \mathbb{R}_+ telle que $w(0) \geq 0$. Il existe une unique paire de fonctions (x, l) satisfaisant*

- $x = w + l$,
- x est positive,
- l est croissante, s'annule en 0 et dl_s a pour support $\{s : x(s) = 0\}$.

De plus l est donnée par

$$l(t) = \sup_{s \leq t} (-w(s) \vee 0).$$

L'identité suivante caractérise la loi du temps local en 0 d'un mouvement Brownien.

Proposition 9 (Identité de Lévy). *Soit W_t le mouvement Brownien standard, L_t^0 son temps local en 0 et $S_t = \sup_{s \leq t} (-W_s)$ le supremum de son opposé. Alors on a l'égalité en loi suivante*

$$(W_t + S_t, S_t; t \geq 0) \stackrel{(loi)}{=} (|W_t|, L_t^0; t \geq 0).$$

Remarque 10 (Loi jointe). Par un principe de réflexion on peut montrer que $\mathbb{P}(S_t \geq a, B_t \geq b) = \mathbb{P}(B_t \geq 2a + b)$ pour tout $a \geq 0$ et $-b \leq a$. Avec quelques changements de variable on obtient ainsi grâce à l'identité de Lévy la loi jointe de $(|W_t|, L_t^0)$ qui vaut

$$\mathbb{P}(|W_t| \in dx, L_t^0 \in dy) = 2 \frac{x+y}{\sqrt{2\pi t^3}} e^{-\frac{(x+y)^2}{2t}} \mathbb{1}_{x \geq 0, y \geq 0} dx dy.$$

Le mouvement Brownien sans dérive réfléchi en 0 est juste la valeur absolue d'un mouvement Brownien. Dans cette thèse nous nous intéresserons au mouvement Brownien réfléchi avec dérive, qui n'est pas la valeur absolue d'un mouvement Brownien avec dérive, mais la valeur absolue d'un processus nommé bang-bang, voir Shreve [93].

Définition 11 (Processus bang-bang). Soit μ une dérive et W_t un mouvement Brownien standard. Un processus bang-bang Y_t de paramètre μ est un processus tel que

$$dY_t = dW_t + \text{sgn}(Y_t)\mu dt.$$

Proposition 12 (Généralisation de l'identité de Lévy). Soient $W_t + \mu t$ un mouvement Brownien avec dérive μ issu de 0 et $S_t = \sup_{s \leq t} (-W_s - \mu s)$. Soient Y_t un processus bang-bang de paramètre μ et L_t^0 son temps local en 0. On a l'égalité en loi

$$(W_t + \mu t + S_t, S_t; t \geq 0) \stackrel{(loi)}{=} (|Y_t|, L_t^0(Y); t \geq 0).$$

On pourra lire l'article de Cherny et Shiryaev [19] qui explique la propriété précédente et généralise l'identité de Lévy.

Définition 13 (Mouvement Brownien avec dérive réfléchi). On définit X_t , un mouvement Brownien réfléchi de variance $\sigma \geq 0$, de dérive μ et issu de $x_0 \in \mathbb{R}_+$, comme la semi-martingale vérifiant l'égalité suivante

$$X_t = x_0 + \sqrt{\sigma}W_t + \mu t + L_t,$$

où L_t est le temps local symétrique en 0 de X_t et W_t est un mouvement Brownien standard.

De la même manière que dans (1.3) on remarquera que le temps local symétrique L_t vaut la moitié du temps local de X en 0 et est en fait le temps local du processus bang-bang dont X est la valeur absolue. C'est pour cette raison et pour ne pas alourdir les notations par la suite qu'on ne le note pas \tilde{L}_t^0 .

Remarque 14 (Semi-groupe). On cherche à calculer le semi-groupe du processus bang-bang et du mouvement Brownien réfléchi avec dérive. Ce qui suit est un résultat de Zvonkin [104], on pourra se référer au livre de Mallein et Yor [77] pour plus de détails. Pour tout point de départ $x \in \mathbb{R}$ on a

$$\begin{aligned} \mathbb{E}_x(f(Y_t)) &= \mathbb{E} \left(f(x + W_t) \exp \left(\mu \int_0^t \text{sgn}(W_s) dW_s - \frac{\mu^2}{2} t \right) \right) \\ &= \mathbb{E} \left(f(x + W_t) \exp \left(\mu(|W_t| - L_t^0) - \frac{\mu^2}{2} t \right) \right) \\ &= \mathbb{E} \left(\frac{f(x + |W_t|) + f(x - |W_t|)}{2} \exp \left(\mu(|W_t| - L_t^0) - \frac{\mu^2}{2} t \right) \right), \end{aligned}$$

où l'on a utilisé la formule de Girsanov pour la première égalité, la formule de Tanaka pour la seconde égalité et on a remarqué que W_t a une chance sur deux d'être positif pour la troisième égalité. On peut maintenant se servir de la loi jointe de $(|W_t|, L_t^0)$ vue à la remarque 10, et après calcul on peut obtenir $p(t, x, y) = \mathbb{P}_x(Y_t \in dy)/dy$, la densité de transition du semi-groupe du processus bang-bang Y_t . On trouve

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(|y-x|+\mu t)^2}{2t}} + \frac{\mu}{\sqrt{2\pi t}} e^{-2\mu(|x-y|)} \left(1 - F \left(\frac{|x-y| - \mu t}{\sqrt{t}} \right) \right),$$

où F est la fonction de répartition de la Gaussienne centrée réduite. Ainsi en dimension un, on peut obtenir explicitement le semi-groupe du mouvement Brownien réfléchi avec dérive qui vaut donc $p(t, x, y) + p(t, x, -y)$ pour tout x et y positifs. Cette méthode, et notamment l'étape où l'on applique la formule de Girsanov, ne peut pas s'appliquer en dimension deux au processus, que nous introduirons dans la définition 28, qui est réfléchi de manière oblique dans le quart de plan. C'est ce qui rend nécessaire l'exploitation d'autres méthodes pour en savoir plus sur ce type de processus en dimensions supérieures.

1.1.2 Fonctions de Green et mesures invariantes

Nous faisons ici quelques brefs rappels sur les mesures invariantes et les fonctions de Green, qui sont intimement liées aux mesures d'occupation. Nous insistons sur le fait que ces deux notions sont semblables, la première prenant sens dans le cas récurrent et la deuxième dans le cas transient. Ainsi la mesure invariante peut être interprétée comme la *proportion de temps moyenne* que le processus passe dans un ensemble tandis que la mesure de Green sera la *quantité totale de temps moyenne* que le processus passe dans un ensemble.

On notera X_t un processus de Markov homogène à valeur dans \mathbb{R} associé au semi-groupe de transition P tel que $P(t, x_0, A) = \mathbb{P}_{x_0}(X_t \in A)$ ayant une densité p telle que $p(t, x_0, x)dx = P(t, x_0, dx)$. En dimension un on dit qu'un tel processus est :

- *récurrent* si pour tout point $x \in \mathbb{R}$, presque sûrement il existe une suite de temps (aléatoires) t_n qui tend vers l'infini et tel que $X_{t_n} = x$,
- *transient* si presque sûrement le processus tend vers l'infini.

Les notions de fonction de Green et de mesure invariante s'étendent naturellement aux processus en dimensions supérieures. Pour plus de détails sur les résultats qui suivent on pourra se référer au livre de Mörters et Peres [81] et aux articles de Revuz [88] et Hunt [58, 59] en ce qui concerne les fonctions de Green. Pour les notions d'invariance et de transience chez les processus de Markov ainsi que pour les mesures invariantes on se référera au livre de Revuz et Yor [89] et aux articles de Azema *et al.* [3, 4].

1.1.2.1 Transience

Si le processus est transient, pour tout ensemble mesurable *borné* A on a presque sûrement

$$\int_0^\infty \mathbb{1}_A(X_s)ds < \infty$$

et on pose alors la définition suivante.

Définition 15 (Fonctions de Green). *Soit $A \subset \mathbb{R}$ un ensemble mesurable. La mesure de Green de A du processus X partant de x_0 est définie par*

$$G_A^{x_0} = \mathbb{E}_{x_0} \left(\int_0^\infty \mathbb{1}_A(X_s)ds \right) = \int_0^\infty \mathbb{P}_{x_0}(X_s \in A)ds.$$

C'est la moyenne du temps total passé par le processus dans l'ensemble A en partant de x_0 . Lorsqu'elle existe, on appelle fonction de Green et on note $g_x^{x_0}$ la densité de cette mesure par rapport à la mesure de Lebesgue (i.e $G_{dx}^{x_0} = g_x^{x_0}dx$). La fonction de Green vérifie

$$g_x^{x_0} = \int_0^\infty p(t, x_0, x)dt.$$

Remarque 16 (Espérance du temps local et fonction de Green). Supposons que X est une semi-martingale telle que $\langle X \rangle_t = \sigma t$, par exemple un mouvement Brownien de variance σ avec dérive. On notera μ_t la mesure d'occupation de X et L_t^a son temps local en a . On a pour tout ensemble A mesurable $\mu_t(A) = \int_A L_t^a da = \int_0^t \mathbb{1}_A(X_s)\sigma ds$ et donc $\mathbb{E}_{x_0}\mu_t(A) = \int_A (\mathbb{E}_{x_0}L_t^a)da = \int_0^t \mathbb{P}_{x_0}(X_s \in A)\sigma ds = \int_A (\int_0^t p(s, x_0, a)\sigma ds)da$. On a donc

$$\int_0^t p(s, x_0, x)ds = \frac{1}{\sigma} \mathbb{E}_{x_0} L_t^x \xrightarrow[t \rightarrow \infty]{} g_x^{x_0} \text{ et } \frac{1}{\sigma} \mathbb{E}_{x_0} \mu_t(A) \xrightarrow[t \rightarrow \infty]{} G_A^{x_0}. \quad (1.4)$$

La proposition suivante donne une propriété analogue de la formule du temps d'occupation de la proposition 2.

Proposition 17 (Noyau de Green). *Si $f : \mathbb{R} \rightarrow \mathbb{R}_+$ est une fonction mesurable*

$$Gf(x_0) := \int_{\mathbb{R}} f(x) g_x^{x_0} dx = \mathbb{E}_{x_0} \left[\int_0^\infty f(X_t) dt \right].$$

Démonstration. Le théorème de Fubini implique

$$\begin{aligned} \int_{\mathbb{R}} f(x) g_x^{x_0} dx &= \int_{\mathbb{R}} f(x) \int_0^\infty p(t, x_0, x) dt dx = \int_0^\infty \int_{\mathbb{R}} f(x) p(t, x_0, x) dx dt \\ &= \int_0^\infty \mathbb{E}_{x_0}[f(X_t)] dt = \mathbb{E}_{x_0} \left[\int_0^\infty f(X_t) dt \right]. \end{aligned}$$

□

Remarque 18 (Noyau du potentiel). Les fonctions de Green sont aussi appelées noyau de Green. Dans le cas du mouvement Brownien en dimension d supérieure ou égale à trois, ce dernier est transient et ses fonctions de Green sont les noyaux du potentiel. En effet dans ce cas, pour une certaine constante $c(d)$ dépendant de la dimension, on a $g_x^{x_0} = c(d)|x - x_0|^{2-d}$. Ainsi, modulo une constante multiplicative, la fonction de Green $g_x^{x_0}$ est harmonique sur $\mathbb{R}^d \setminus \{x_0\}$ et est solution de l'équation fondamentale du Laplacien $\frac{1}{2}\Delta u = -\delta_{x_0}$. Par convolution Gf est alors solution de l'équation de Poisson $\frac{1}{2}\Delta u = -f$.

1.1.2.2 Récurrence

Si le processus est récurrent, pour tout A de mesure de Lebesgue non nulle on a $\lim_{t \rightarrow \infty} \int_0^t \mathbf{1}_A(X_s) ds = \infty$. On s'intéresse alors plutôt à la proportion de temps limite

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{1}_A(X_s) ds$$

dont on verra qu'elle laisse stable le processus.

Définition 19 (Mesure invariante). *Une mesure invariante Π est une mesure qui laisse stable le semi-groupe, c'est-à-dire qui vérifie pour tout ensemble mesurable A et tout $t \geq 0$,*

$$\Pi(A) = \int_{\mathbb{R}} P(t, x, A) d\Pi(x). \quad (1.5)$$

De manière équivalente, en posant $P^t f(x) := \mathbb{E}_x[f(X_t)]$, on peut remplacer la condition (1.5) par

$$\int_{\mathbb{R}} f(x) \Pi(dx) = \int_{\mathbb{R}} P^t f(x) d\Pi(x),$$

pour toute fonction f dans $L^1(\Pi)$. Lorsqu'elle existe, on notera π la densité de la mesure invariante par rapport à la mesure de Lebesgue. Lorsque Π est une mesure de probabilité on parle de distribution stationnaire.

Remarque 20 (Théorèmes ergodiques). La mesure $\Pi(A)$ représente la proportion moyenne du temps passé en A par le processus. D'après [98, Lemme 18.9] et [89, X.(3.12)] on a

$$P(t, x_0, \cdot) \xrightarrow[t \rightarrow \infty]{} \Pi \quad \text{et} \quad \frac{1}{t} \int_0^t \mathbb{1}_A(X_s) ds \xrightarrow[t \rightarrow \infty]{} \Pi(A) \quad p.s.$$

Il existe une formule analogue à celle de la proposition 17 :

$$\int_{\mathbb{R}} f(x) d\Pi(x) = \mathbb{E}_{\Pi}[f(X_t)] = \mathbb{E}_{\Pi} \left[\frac{1}{t} \int_0^t f(X_s) ds \right]. \quad (1.6)$$

Proposition 21 (Générateur). Soit \mathcal{G} le générateur du processus, i.e. $\mathcal{G}f = \frac{d^+}{dt} P^t f|_{t=0}$ défini sur $D_{\mathcal{G}} := \{f : \mathbb{R} \rightarrow \mathbb{R} : \|t^{-1}(P^t f - f) - \mathcal{G}f\|_{\infty} \xrightarrow[t \rightarrow 0]{} 0\}$. Si Π est une mesure invariante, pour tout $f \in D_{\mathcal{G}}$ on a,

$$\int_{\mathbb{R}} \mathcal{G}f(x) d\Pi(x) = 0. \quad (1.7)$$

La réciproque est vraie sur un espace métrique séparable localement compact si Π est une mesure de probabilité et si le processus est càdlàg, voir Liggett [73, Théorème 3.37].

La formule (1.7) est l'analogie de $\Pi(P - I) = 0$ pour les chaînes de Markov ayant la matrice P pour matrice de transition ou de $\Pi Q = 0$ pour les chaînes de Markov à temps continu ayant la matrice Q pour générateur infinitésimal.

Remarque 22 (Générateur du mouvement Brownien réfléchi). Il est connu qu'un mouvement Brownien B_t a pour générateur $\mathcal{G} = \frac{1}{2}\Delta$ et que $D_{\mathcal{G}} = \mathcal{C}_{uni}^2$ où on note \mathcal{C}_{uni}^2 l'ensemble des fonctions f deux fois dérivables telles que f, f' et f'' sont uniformément continues. Nous allons en déduire que le mouvement Brownien réfléchi $|B_t|$ a pour générateur $\frac{1}{2}\Delta$ et que son domaine de définition est

$$\tilde{D}_{\mathcal{G}} = \{f \in \mathcal{C}_{uni}^2(\mathbb{R}_+) : f'(0) = 0\}.$$

On note P^t et p le semi groupe et la densité de transition de B_t et de même \tilde{P}^t et \tilde{p} le semi-groupe et la densité de transition de $|B_t|$. Pour toute fonction f définie sur \mathbb{R}_+ on définit \tilde{f} telle que $\tilde{f}(x) = f(|x|)$ pour tout $x \in \mathbb{R}$. Le fait que $\tilde{p}(t, x, y) = p(t, x, y) + p(t, x, -y)$ pour tout x et y positifs implique que $\tilde{P}^t f(x) = P^t \tilde{f}(x)$ et donc que B_t et $|B_t|$ ont même générateur (mais des domaines différents). On en déduit que $\tilde{f} \in D_{\mathcal{G}}$ si et seulement si $f \in \tilde{D}_{\mathcal{G}}$. Or $\tilde{f} \in \mathcal{C}_{uni}^2$ si et seulement si $f \in \mathcal{C}_{uni}^2$ et $f'(0) = 0$. On en déduit le domaine $\tilde{D}_{\mathcal{G}}$.

Remarque 23 (Conditions de Neumann). La fonction de transition du mouvement Brownien réfléchi est en fait solution de l'équation de la chaleur avec conditions de Neuman aux bords. En reprenant les notations de la remarque 22 on sait, d'après les équations de Kolmogorov directes (forward), que $\tilde{p}(t, x_0, x)$ est solution fondamentale de l'équation de Fokker-Planck avec condition au bord

$$\partial_t u(t, x) = \frac{1}{2} \partial_x^2 u(t, x), \quad u(0, x) = \delta_{x_0} \quad \text{et} \quad \partial_x u(t, 0) = 0.$$

Par convolution cela implique que $P^t f(x)$ est solution de l'équation aux dérivées partielles avec conditions de Neuman

$$\partial_t u(t, x) = \frac{1}{2} \partial_x^2 u(t, x), \quad u(0, x) = f(x) \quad \text{et} \quad \partial_x u(t, 0) = 0.$$

On retrouve en dimension deux le même phénomène, le mouvement Brownien réfléchi dans le quadrant permet de trouver une solution à l'équation de la chaleur avec conditions de Neuman *obliques* sur les bords, voir l'article de Harrison et Reiman [49].

1.1.3 Mouvement Brownien avec dérive réfléchi en dimension un

Dans cette section nous étudions le mouvement Brownien réfléchi en 0, introduit dans la définition 13,

$$X_t = x_0 + \sqrt{\sigma}W_t + \mu t + L_t.$$

En particulier nous cherchons à calculer explicitement la mesure invariante dans le cas récurrent et les fonctions de Green dans le cas transient. L'objectif principal sera d'illustrer, dans le cas simple de la dimension un, la méthode analytique que nous détaillerons en dimension 2 dans la section 1.3. Pour cela nous allons déterminer, dans le cas transient (resp. récurrent), une équation fonctionnelle impliquant la transformée de Laplace des fonctions de Green (resp. de la mesure invariante). Nous procéderons de manière identique en dimension deux dans la section 1.2.7. Les transformées de Laplace sont en fait l'analogie continu des fonctions génératrices dans le cas discret.

1.1.3.1 Calcul de la mesure invariante

Dans le cas du mouvement Brownien réfléchi avec dérive introduit dans la définition 13, nous allons généraliser la formule (1.7) à des fonctions f qui ne sont pas dans l'ensemble $\tilde{D}_{\mathcal{G}}$ de la remarque 22, c'est-à-dire telles que $f'(0) \neq 0$. On note $\mathcal{C}_b^2(\mathbb{R}_+)$ l'ensemble des fonctions \mathcal{C}^2 de \mathbb{R}_+ bornées ainsi que leurs dérivées premières et secondes. On rappelle que le générateur du mouvement Brownien de variance σ avec dérive μ est

$$\mathcal{G}_{\sigma,\mu}(f) = \frac{1}{2}\sigma f'' + \mu f'.$$

Proposition 24 (*Basic adjoint relationship* en dimension un). *Soient $X_t = X_0 + \sqrt{\sigma}W_t + \mu t + L_t$, et $\mu < 0$. Le processus est récurrent positif et on note Π sa mesure invariante. On a alors pour $f \in \mathcal{C}_b^2(\mathbb{R}_+)$*

$$0 = \int_{\mathbb{R}} \mathcal{G}_{\sigma,\mu}f(x)d\Pi(x) + f'(0)\frac{1}{2}\sigma\pi(0). \quad (1.8)$$

Démonstration. D'après la formule d'Itô si f est une fonction \mathcal{C}^2 on a

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_t + \frac{1}{2} \int_0^t f''(X_s)d\langle X \rangle_s \\ &= f(X_0) + \int_0^t f'(X_s)(\sqrt{\sigma}dW_s + \mu ds + dL_s) + \frac{1}{2} \int_0^t f''(X_s)\sigma ds. \end{aligned}$$

On prend l'espérance en choisissant Π comme distribution initiale, ce qui donne

$$\mathbb{E}_{\Pi}[f(X_t)] = \mathbb{E}_{\Pi}[f(X_0)] + \underbrace{\mathbb{E}_{\Pi} \left[\int_0^t f'(X_s)\sqrt{\sigma}dW_s \right]}_{=0 \text{ car c'est l'espérance d'une martingale comme } f' \text{ est bornée}} + \mathbb{E}_{\Pi} \left[\int_0^t \mathcal{G}_{\sigma,\mu}f(X_s)ds \right] + \mathbb{E}_{\Pi} \left[\int_0^t f'(0)dL_s \right]$$

En prenant $t = 1$ et grâce à la formule (1.6) on a alors $0 = \int_{\mathbb{R}} \mathcal{G}_{\sigma,\mu}f(x)d\Pi(x) + f'(0)\mathbb{E}_{\Pi}L_1$. D'après la remarque 16, $\mathbb{E}_{x_0}(L_t^a) = \sigma \int_0^t p(s, x_0, a)ds$. En intégrant cette égalité contre la mesure Π on trouve $\mathbb{E}_{\Pi}(L_1) = \frac{1}{2}\sigma\pi(0)$ en se souvenant que dans la définition 13, L_t est la moitié du temps local de X en 0. \square

La formule (1.8) généralise ainsi la formule (1.7) et fait écho à la *Basic adjoint relationship* en dimension deux que nous verrons dans la proposition 42. De la même manière qu'en dimension deux, lorsqu'on définit les mesures invariantes sur les bords dans l'équation (1.18), on peut interpréter $\mathbb{E}_\Pi L_1$ comme une mesure invariante sur 0, i.e. le bord où la réflexion a lieu. De même on interprète f' comme le générateur sur le bord 0.

Remarque 25 (Processus ralenti). Pour se conforter dans cette interprétation du générateur sur le bord 0 on peut ralentir le processus. Par un changement de variable temporel du type $\rho \cdot L_s + s = t$ on peut définir un nouveau processus ralenti \tilde{X} tel que $\tilde{X}_t = X_s$. Lorsque $\rho = 0$ on retrouve le mouvement Brownien réfléchi et lorsque $\rho \rightarrow \infty$ le processus devient absorbé aux bords. On peut alors montrer que le générateur de \tilde{X} vaut

$$\tilde{\mathcal{G}}f(x) = \begin{cases} \frac{1}{2}f''(x) & \text{si } x > 0, \\ \frac{1}{\rho}f'(0) & \text{si } x = 0. \end{cases}$$

On se référera au livre de Varadhan [98, Chapitre 16] pour plus de détails.

Proposition 26 (Mesure invariante). *Si $\mu < 0$, pour tout $x \in \mathbb{R}_+$ on a*

$$\pi(x) = -\frac{2\mu}{\sigma}e^{\frac{2\mu}{\sigma}x}. \quad (1.9)$$

Démonstration. Grâce à la proposition 24, en prenant $f(x) = e^{\theta x}$ on obtient

$$0 = (\mu\theta + \frac{1}{2}\sigma\theta^2) \int_{\mathbb{R}} e^{\theta x} d\Pi(x) + \theta \frac{1}{2}\sigma\pi(0).$$

En notant $\varphi(\theta) = \int_{\mathbb{R}} e^{\theta x} d\Pi(x) = \mathbb{E}_\Pi e^{\theta X_t}$, la transformée de Laplace de la mesure invariante, on obtient alors

$$\varphi(\theta) = \frac{-\frac{1}{2}\sigma\pi(0)}{\mu + \frac{1}{2}\sigma\theta}.$$

Comme Π est une mesure de probabilité on a $\varphi(0) = 1$ ce qui donne $\frac{1}{2}\sigma\pi(0) = -\mu$. En inversant la transformée de Laplace on obtient alors la formule (1.9). \square

1.1.3.2 Calcul des fonctions de Green

Nous allons maintenant procéder de la même manière que pour la mesure invariante et déduire les fonctions de Green grâce à une équation fonctionnelle.

Proposition 27 (Fonction de Green). *Si $\mu > 0$, pour tout $x \in \mathbb{R}_+$ la fonction de Green du processus X vaut*

$$g_x^{x_0} = \frac{1}{\mu}e^{\frac{2\mu}{\sigma}(x-x_0)}\mathbf{1}_{\{0 \leq x < x_0\}} + \frac{1}{\mu}\mathbf{1}_{\{x_0 \leq x\}}.$$

Démonstration. D'après la formule d'Itô si f est une fonction \mathcal{C}^2 on a

$$\begin{aligned} f(X_t) &= f(x_0) + \int_0^t f'(X_s) dX_t + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s \\ &= f(x_0) + \int_0^t f'(X_s) (dW_t + \mu dt + dL_t) + \frac{1}{2} \int_0^t f''(X_s) \sigma ds. \end{aligned}$$

On applique cette formule à $f(x) = e^{\theta x}$ pour $\theta < 0$ et on prend l'espérance, ce qui donne

$$\mathbb{E}_{x_0}[e^{\theta X_t}] = e^{\theta x_0} + \underbrace{\theta \mathbb{E}_{x_0} \left[\int_0^t e^{\theta X_s} dW_s \right]}_{=0 \text{ car c'est l'espérance d'une martingale}} + (\mu\theta + \frac{1}{2}\sigma\theta^2)\mathbb{E}_{x_0} \left[\int_0^t e^{\theta X_s} ds \right] + \theta\mathbb{E}_{x_0} \left[\int_0^t e^{\theta X_s} dL_s \right].$$

Comme $\theta < 0$ et que $X_t \xrightarrow[t \rightarrow \infty]{} \infty$ (car $\mu > 0$), on a $\lim_{t \rightarrow \infty} \mathbb{E}[e^{\theta X_t}] = 0$. On fait alors tendre t vers l'infini dans l'équation et on obtient :

$$\begin{aligned} 0 &= e^{\theta x_0} + (\mu\theta + \frac{1}{2}\sigma\theta^2)\mathbb{E}_{x_0} \left[\int_0^\infty e^{\theta X_s} ds \right] + \theta\mathbb{E}_{x_0} \left[\int_0^\infty e^{\theta X_s} dL_s \right] \\ &= e^{\theta x_0} + (\mu\theta + \frac{1}{2}\sigma\theta^2)\psi^{x_0}(\theta) + \theta\mathbb{E}L_\infty \end{aligned}$$

car $e^{\theta X_s} = 1$ sur le support de dL_s qui est l'ensemble $\{s \geq 0 : X_s = 0\}$ et où on a noté

$$\psi^{x_0}(\theta) = \int_0^\infty e^{\theta x} g_x^{x_0} dx = \mathbb{E}_{x_0} \left[\int_0^\infty e^{\theta X_s} ds \right]$$

la transformée de Laplace des fonctions de Green de X_t . D'après la remarque 16 on a $\mathbb{E}L_\infty = \frac{1}{2}\sigma g_0^{x_0}$ en se souvenant que L_t vaut la moitié du temps local de X . En évaluant l'équation fonctionnelle en $\theta^* = -2\mu/\sigma$ on trouve que $\mathbb{E}L_\infty = -\frac{e^{\theta^* x_0}}{\theta^*} = \frac{\sigma}{2\mu} e^{-\frac{2\mu}{\sigma} x_0}$. On obtient donc

$$\psi^{x_0}(\theta) = -\frac{e^{\theta x_0} + \theta \frac{\sigma}{2\mu} e^{-\frac{2\mu}{\sigma} x_0}}{\mu\theta + \frac{1}{2}\sigma\theta^2}.$$

En inversant cette transformée de Laplace on trouve explicitement la formule des fonctions de Green recherchée. \square

1.2 Mouvement brownien réfléchi dans le quart de plan

Nous allons maintenant introduire l'objet principal de notre étude : le mouvement Brownien réfléchi de manière oblique dans le quart de plan. Dans la section 1.2.1 nous faisons un bref panorama de la littérature existante sur le sujet. Nous poursuivons en définissant le processus en tant que semi-martingale dans la section 1.2.2. La section 1.2.3 explique quant à elle pourquoi tous nos résultats se généralisent au cas des cônes dans le quart de plan. Nous faisons ensuite, dans la section 1.2.4, un bref focus sur un problème de sous-martingale qui caractérise le processus. Puis nous énonçons les conditions de récurrence et de transience du processus dans la section 1.2.5 et nous introduisons les mesures invariantes et les fonctions de Green sur les frontières dans la section 1.2.6. Nous finissons par montrer dans la section 1.2.7 les équations fonctionnelles qui sont les formules clés de notre étude analytique présentée dans la section 1.3.

1.2.1 Panorama

Il existe une vaste littérature sur le mouvement Brownien réfléchi dans le quart de plan ou quadrant (mais aussi dans des orthants, une généralisation en dimension supérieure du quadrant). Tout d'abord, il sert d'approximation aux grands réseaux de files d'attente (*large queuing networks*), voir Foddy [42], Baccelli et Fayolle [5], Foschini [43] et Reiman

[86]); c'était la motivation initiale de son étude, voir l'annexe D. Dans le même genre, c'est l'analogie continu des marches aléatoires dans le quart de plan, voir l'annexe A. De manière différente, il est aussi étudié pour ses fonctions de Lyapunov dans Dupuis et Williams [35], ses points cônes du mouvement Brownien dans Le Gall [69], ses relations d'entrelacement et ses probabilités croisées dans Dubédat [34], et pour un aspect qui nous intéresse particulièrement ici, pour sa récurrence/transience dans Hobson et Rogers [57]. Le comportement asymptotique de la distribution stationnaire a été récemment étudié dans divers articles comme Harrison et Hasenbein [52], Dai et Miyazawa [28, 29] et enfin dans le chapitre 4 de cette thèse issu de [44] qui détermine un développement asymptotique complet selon toutes directions. Il existait jusqu'ici peu de résultats donnant des expressions exactes de la distribution stationnaire. On mentionnera tout de même Foddy [42], Foschini [43] (s'occupant du cas particulier où le mouvement Brownien a pour matrice de covariance l'identité dans [42] et avec des conditions de symétrie très fortes sur la dérive et les vecteurs de réflexion dans [43]), Baccelli et Fayolle [5] (sur une diffusion ayant un comportement assez spécial aux bords), Harrison et Williams [56], Dieker et Moriarty [31] (au sujet du cas spécial où la mesure stationnaire est exponentielle). Le chapitre 2 issu de l'article [46] détermine lui une formule explicite dans le cas particulier des réflexions orthogonales. Enfin le chapitre 3 issu de l'article [47] résout totalement le problème dans le cas général.

1.2.2 Définition, existence et unicité

Contrairement à la dimension un où l'on n'a pas le choix dans la direction de la réflexion, en dimension supérieure les rebonds ne sont pas nécessairement orthogonaux sur les bords et peuvent être obliques. On va définir Z_t un mouvement Brownien réfléchi avec dérive dans le quart de plan comme une semi-martingale en s'inspirant de la formule de Tanaka en dimension un, voir l'équation (1.2). Soient

$$\left\{ \begin{array}{l} \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2} \text{ une matrice de covariance définie positive,} \\ \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathbb{R}^2 \text{ un vecteur de dérive non nul,} \\ R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2} \text{ une matrice de « rebond ».} \end{array} \right. \quad (1.10)$$

La matrice R est composée de deux vecteurs de rebond donnant la direction de la réflexion sur les axes, $R_1 = \begin{pmatrix} r_{11} \\ r_{21} \end{pmatrix}$ le long de l'axe des ordonnées et $R_2 = \begin{pmatrix} r_{12} \\ r_{22} \end{pmatrix}$ le long de l'axe des abscisses.

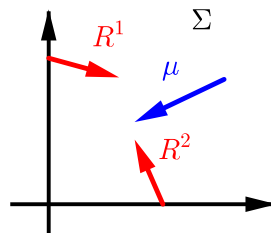


FIGURE 1.1 : Vecteurs dérive et rebond

Définition 28 (SRBM). On définit $(Z)_{t \in \mathbb{R}^+}$ un mouvement brownien avec dérive réfléchi au bord du quart de plan \mathbb{R}_+^2 associé à (Σ, μ, R) comme la semi-martingale telle que pour tout $t \in \mathbb{R}_+$ on a

$$Z_t = Z_0 + W_t + \mu t + RL_t \in \mathbb{R}_+^2,$$

où

- Z_0 est la distribution initiale du processus,
- $(W_t)_{t \in \mathbb{R}^+}$ est un mouvement brownien plan issu de 0 de matrice de covariance Σ ,
- L_t^i est un processus continu et croissant qui s'accroît uniquement lorsque $Z_i(\cdot) = 0$ c'est à dire que le processus touche le bord, i.e. $\int_{\{t: Z_t^i > 0\}} dL_t^i = 0$, pour $i = 1, 2$.

Un tel processus est aussi appelé SRBM pour *semimartingale reflected Brownian motion*.

Remarque 29 (Temps local sur les axes). Le processus $L_t = \begin{pmatrix} L_t^1 \\ L_t^2 \end{pmatrix}$ est, aux constantes r_{11} et r_{22} près, le temps local que le processus Z passe sur les bords. Cela veut dire que $r_{11}L_t^1$ est le temps local sur l'axe des ordonnées et $r_{22}L_t^2$ le temps local sur l'axe des abscisses.

Coordonnée par coordonnée on a les notations suivantes,

$$\begin{cases} Z_t^1 = x + W_t^1 + \mu_1 t + r_{11}L_t^1 + r_{12}L_t^2, \\ Z_t^2 = y + W_t^2 + \mu_2 t + r_{21}L_t^1 + r_{22}L_t^2. \end{cases}$$

Proposition 30 (Existence et unicité). *Le mouvement Brownien avec dérive réfléchi dans le quart de plan associé à (Σ, μ, R) existe pour toute distribution initiale de Z_0 si et seulement si*

$$(r_{11} > 0, \quad r_{22} > 0, \quad \det R > 0) \quad \text{ou} \quad (r_{11} > 0, \quad r_{22} > 0, \quad r_{12} > 0, \quad r_{21} > 0). \quad (1.11)$$

Dans ce cas le processus est unique en distribution pour toute distribution initiale donnée.

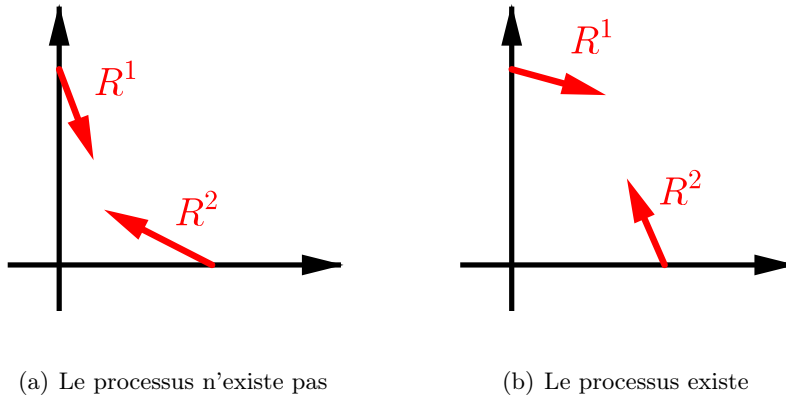


FIGURE 1.2 : Conditions d'existence

Remarque 31 (Interprétation géométrique des conditions d'existence). Les conditions $r_{11} > 0$ et $r_{22} > 0$ servent à ce que les rebonds sur les bords soient dirigés vers l'intérieur du quart de plan. La condition $\det R > 0$ ou ($r_{12} > 0$ et $r_{21} > 0$) est équivalente à ce qu'il existe une combinaison convexe des vecteurs de rebonds qui pointe dans le quart de plan, le cas limite $\det R = 0$ a lieu lorsque les deux vecteurs de rebonds sont opposés. Moralement cette condition sert à ce que le coin ne soit pas trop attractif pour que le processus ne vienne pas s'y coller. On peut montrer que la quantité de temps que le processus passe dans le coin est nulle (au sens de la mesure de Lebesgue), voir Varadhan et Williams [97].

Remarque 32 (Dimensions supérieures). En dimension d quelconque on peut définir de manière similaire une semi-martingale réfléchie dans un l'orthant \mathbb{R}_+^d . Un tel processus existe si et seulement si la matrice R de dimension d est complètement- \mathcal{S} . En dimension deux les conditions (1.11) sont donc équivalentes à ce que la matrice de réflexion R soit complètement- \mathcal{S} . Pour rappel une \mathcal{S} -matrice est une matrice telle qu'il existe un vecteur $w \geq 0$ tel que $Rw \geq 0$. Une matrice est complètement- \mathcal{S} si toutes ses sous-matrices principales sont des \mathcal{S} -matrices.

Démonstration. On pourra consulter les articles de Reiman et Williams [87] et Taylor et Williams [95] pour une démonstration de l'existence et de l'unicité du processus en dimension quelconque. Nous ne donnerons ici que quelques éléments de preuve de l'existence et de l'unicité. On veut définir (L_t^2) pour qu'il soit un processus continu croissant qui ne s'accroît que lorsque $Z_t^2 = 0$ c'est à dire lorsque le processus touche le bord (l'axe des abscisses). Au moins jusqu'au temps $\tau = \inf\{u : Z_u^1 = 0\}$ (lorsque Z_t touche l'axe des ordonnées) on peut définir

$$r_{22}L_t^2 = - \inf_{0 \leq s \leq t} [(y + W_s^2 + \mu_2 s)^-].$$

Avec cette définition Z_t^2 est clairement positif et L_t^2 est croissant. Ce dernier croît uniquement lorsque $\inf_{0 \leq s \leq t} [(y + W_s^2 + \mu_2 s)^-] = [(y + W_t^2 + \mu_2 t)^-]$ c'est à dire quand $Z_t^2 = 0$. L_t^2 agit comme une dérive infiniment forte quand le processus touche le bord pendant un intervalle de temps de mesure de Lebesgue nulle. On peut en déduire assez rapidement la construction du processus (même après le temps τ) du moins tant que le processus n'atteint pas le « coin » $(0, 0)$. Le processus peut en fait être caractérisé par un problème de martingale. Montrons l'unicité de Z_t^2 et donc de L_t^2 (avant τ). Supposons que z_t^a et z_t^b sont deux solutions associées à l_t^a et l_t^b . On a alors

$$d[z_t^a - z_t^b]^2 = 2(z_t^a - z_t^b)r_{22}d(l_t^a - l_t^b) = -2r_{22}z_t^a dl_t^b - 2r_{22}z_t^b dl_t^a \leq 0$$

car dl_t^a est nul quand $z_t^a \neq 0$, $dl_t^a \geq 0$ et $z_t^a \geq 0$ (idem avec b). La positivité de $[z_t^a - z_t^b]^2$ et $z_0^a = z_0^b = y$ permet de conclure que $z_t^a \equiv z_t^b$. \square

1.2.3 Du quart de plan aux cônes convexes

Nous allons expliquer dans cette section pourquoi étudier le mouvement Brownien dans le quart de plan est équivalent à l'étudier dans des cônes et comment il est ainsi possible de généraliser aux cônes tous les résultats qu'on obtient dans le quadrant. En effet, à un isomorphisme près, étudier le mouvement Brownien dans le quart de plan avec une matrice de covariance arbitraire Σ est équivalent à étudier un mouvement Brownien dans un cône d'angle

$$\beta = \arccos - \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}},$$

avec covariance identité. On peut consulter l'annexe du chapitre 3 ou les articles d'Aspandiiarov *et al.* [1, eq (23)] et de Sarantsev [92, Lemma 3.23]. Nous relierons ici les paramètres clés (angles des vecteurs de réflexion et de la dérive) avant et après la transformation linéaire. Définissons la transformation linéaire T , qui vérifie $T\Sigma T^\top = \text{Id}$, telle que

$$T = \begin{pmatrix} \frac{1}{\sin \beta} & \cot \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 \\ 0 & \frac{1}{\sqrt{\sigma_{22}}} \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ 0 & \sqrt{\sigma_{22}} \end{pmatrix} \begin{pmatrix} \sin \beta & -\cos \beta \\ 0 & 1 \end{pmatrix}. \quad (1.12)$$

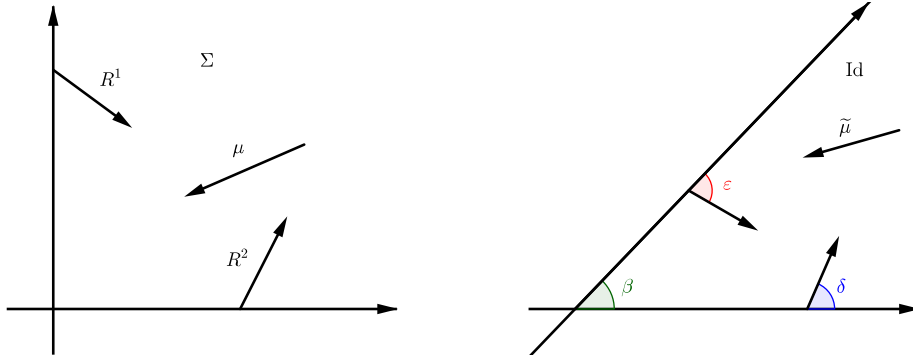


FIGURE 1.3 : La transformation linéaire T dans (1.12) du quart de plan au cône d'angle β

En appliquant l'isomorphisme T , le mouvement Brownien réfléchi associé à (Σ, μ, R) devient un mouvement Brownien (avec covariance identité) dans un cône d'angle β et de paramètres $(\text{Id}, T\mu, TR)$. Les nouveaux angles de réflexion sont δ et ϵ (cf. Figure 1.3) tels que

$$\begin{cases} \tan \delta = \frac{\sin \beta}{a + \cos \beta}, & \cos \delta = \frac{a + \cos \beta}{\sqrt{a^2 + 2a \cos \beta + 1}}, & \sin \delta = \frac{\sin \beta}{\sqrt{a^2 + 2a \cos \beta + 1}}, \\ \tan \epsilon = \frac{\sin \beta}{b + \cos \beta}, & \cos \epsilon = \frac{b + \cos \beta}{\sqrt{b^2 + 2b \cos \beta + 1}}, & \sin \epsilon = \frac{\sin \beta}{\sqrt{b^2 + 2b \cos \beta + 1}}, \end{cases} \quad (1.13)$$

où $a = \frac{r_{12}}{r_{22}} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}}$ et $b = \frac{r_{21}}{r_{11}} \sqrt{\frac{\sigma_{11}}{\sigma_{22}}}$. La nouvelle dérive est $\tilde{\mu} = T\mu$, où

$$\tilde{\mu}_1 = \frac{\mu_1}{\sqrt{\sigma_{11}}} \frac{1}{\sin \beta} + \frac{\mu_2}{\sqrt{\sigma_{22}}} \cot \beta \quad \text{et} \quad \tilde{\mu}_2 = \frac{\mu_2}{\sqrt{\sigma_{22}}}.$$

De la même manière, si on considère \tilde{Z} un mouvement Brownien réfléchi dans un cône convexe de paramètres $(\tilde{\Sigma}, \tilde{\mu}, \tilde{R})$, on se ramène à $Z = T^{-1}\tilde{Z}$ qui est un mouvement Brownien réfléchi dans le quadrant de paramètres (Σ, μ, R) , avec

$$\Sigma = T^{-1}\tilde{\Sigma}(T^{-1})^\top, \quad \mu = T^{-1}\tilde{\mu} \quad \text{et} \quad R = T^{-1}\tilde{R}.$$

Sous réserve d'existence, soient Π (resp. $\tilde{\Pi}$) la mesure invariante de Z (resp. \tilde{Z}) et π (resp. $\tilde{\pi}$) sa densité. On remarque facilement, voir l'annexe 3.6, que

$$\Pi = \tilde{\Pi} \circ T \quad \text{et} \quad \pi = |\det T| \tilde{\pi} \circ T.$$

Ces formules permettent donc de généraliser aux cônes les résultats obtenus dans le quadrant.

Remarque 33 (Paramètre α). Pour π la constante d'Archimède, si on pose

$$\alpha = \frac{\delta + \epsilon - \pi}{\beta}, \quad (1.14)$$

on trouve aisément que la condition d'existence (1.11) du mouvement Brownien réfléchi en tant que semi-martingale comme défini dans la définition 28 équivaut à $\alpha < 1$. Voir la section suivante 1.2.4 pour plus de détails. La valeur de ce rapport α intervient en fait dans de nombreux critères et propriétés satisfaites par le processus. Par exemple $\alpha = 0$ est ce qu'on appellera le cas de symétrie oblique (*Skew-symmetric case*) ou encore le fait que α soit un entier négatif interviendra dans le critère de Dieker et Moriarty [31] pour que la distribution stationnaire soit une somme d'exponentielle. Voir la section 3.5 pour plus de détails.

1.2.4 Problème de sous-martingale

Contrairement à ce que nous avons fait jusqu'ici, on pourrait définir le processus non pas comme une semi-martingale (voir Définition 28) mais à l'aide d'un problème de sous-martingale. On pourra consulter l'article de Varadhan et Williams [97] pour une étude rigoureuse d'un tel processus *sans dérive* défini de cette manière. Ce problème de sous-martingale traduit mathématiquement les propriétés suivantes. Pour un processus de Markov fort à trajectoire continue :

- le processus évolue dans un cône et se comporte comme un mouvement Brownien à l'intérieur de celui ci,
- le processus se réfléchit instantanément sur les frontières du cône, les directions de rebonds étant constantes le long de chaque coté,
- la quantité de temps que le processus passe dans le coin du cône est nulle.

On notera \mathcal{G} le générateur infinitésimal du processus à l'intérieur du cône, voir (1.17) et \mathcal{C}_b^2 l'ensemble des fonctions \mathcal{C}^2 dans le cône bornées ainsi que toutes leurs dérivées partielles jusqu'à l'ordre deux. Une solution de ce problème de sous-martingale partant de $z \in \mathbb{R}_+^2$ est une mesure de probabilité P_z sur l'espace des chemins Z continus dans \mathbb{R}_+^2 telle que

- $P_z(Z_t = z) = 1$,
- $f(Z_t) - \int_0^t \mathcal{G}f(Z_s)ds$ est une P_z sous-martingale lorsque $f \in \mathcal{C}_b^2$ est une fonction constante dans un voisinage de 0 et telle que $\partial_{R_i} f \geq 0$ pour $i = 1, 2$ respectivement sur l'ordonnée et l'abscisse,
- $\mathbb{E}_{P_z} \int_0^\infty \mathbf{1}_0(Z_s)ds = 0$.

Une famille $(P_z, z \in \mathbb{R}_+^2)$, où P_z vérifie ce problème avec point de départ en z , est une solution du problème de sous-martingale. Si une telle famille existe elle est alors unique et vérifie la propriété de Markov forte.

Proposition 34 (Problème de sous-martingale). *Un processus satisfaisant le problème de sous-martingale ci-dessus existe si et seulement si*

$$\alpha = \frac{\epsilon + \delta - \pi}{\beta} < 2.$$

Un tel processus est une semi-martingale si et seulement si

$$\alpha = \frac{\epsilon + \delta - \pi}{\beta} < 1.$$

Lorsque $\alpha \geq 2$ un processus existe mais qui ne vérifie pas la condition de passer un temps nul dans le coin. Le processus est alors absorbé en 0 et est une semi-martingale.

Démonstration. L'article de Varadhan et Williams [97] montre l'existence du processus en tant que solution du problème de sous-martingale. L'article de Williams [99] donne alors des conditions nécessaires et suffisantes pour qu'un tel processus soit une semi-martingale. \square

On retrouve ainsi la remarque 33 de la section précédente. Par ailleurs on peut aussi montrer que (avec ou sans dérive) le processus touche le coin 0 presque sûrement si et seulement si $\alpha > 0$, voir section 1.2.5 ou Varadhan [98, Chapitre 17].

1.2.5 Récurrence et transience

Définition 35 (Récurrence et transience). *Pour tout voisinage $V \subset \mathbb{R}_+^2$ de 0 on note $\tau_V = \inf\{t \geq 0 : Z(t) \in V\}$ le temps de retour dans le voisinage V . Le processus est dit récurrent si*

$$\mathbb{P}_x(\tau_V < \infty) = 1$$

pour tout voisinage de zéro V de mesure de Lebesgue positive et pour tout point de départ $x \in \mathbb{R}_+^2$. Le processus Z est dit récurrent positif si

$$\mathbb{E}_x[\tau_V] < \infty$$

et récurrent nul si $\mathbb{E}_x[\tau_V] = \infty$ pour tout voisinage de zéro V de mesure de Lebesgue positive et pour tout point de départ $x \in \mathbb{R}_+^2$ loin de V . Lorsque le processus n'est pas récurrent on dit qu'il est transient.

Définition 36 (Mesure invariante). *Une mesure Π est dite invariante pour le processus Z si pour toute fonction de \mathbb{R}_+^2 mesurable positive f on a*

$$\mathbb{E}_\Pi[f(Z_t)] = \int_{\mathbb{R}_+^2} f(z) d\Pi(z).$$

Une telle mesure est aussi appelée distribution stationnaire du processus Z .

Dans notre cas, l'existence d'une mesure invariante sera équivalente à la récurrence du processus. On se référera aux articles de Azema *et al.* [3, 4] pour plus de détails sur les notions de récurrence et de transience chez les processus de Markov ainsi que les liens avec les mesures invariantes.

Proposition 37 (Existence et unicité d'une mesure invariante). *L'existence d'un unique processus Z récurrent positif admettant une unique mesure invariante Π est équivalente aux conditions suivantes :*

$$r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0, \tag{1.15}$$

$$r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0. \tag{1.16}$$

Dans ce cas, la mesure Π est absolument continue par rapport à la mesure de Lebesgue et on note $\pi(z) = \pi(z_1, z_2)$ sa densité.

Démonstration. On pourra lire Hobson et Rogers [57], Harrison et Williams [55], Harrison et Hasenbein [52], pour plus de précisions et des démonstrations sur l'existence et l'unicité de la mesure invariante. Pour l'existence d'une densité par rapport à la mesure de Lebesgue on pourra consulter Dai [22], Dai et Harrison [27], Harrison et Williams [55]. L'article Dai et Miyazawa [29] donne des interprétations géométriques des deux conditions, utiles pour les études asymptotiques de la mesure invariante ou des fonctions de Green. \square

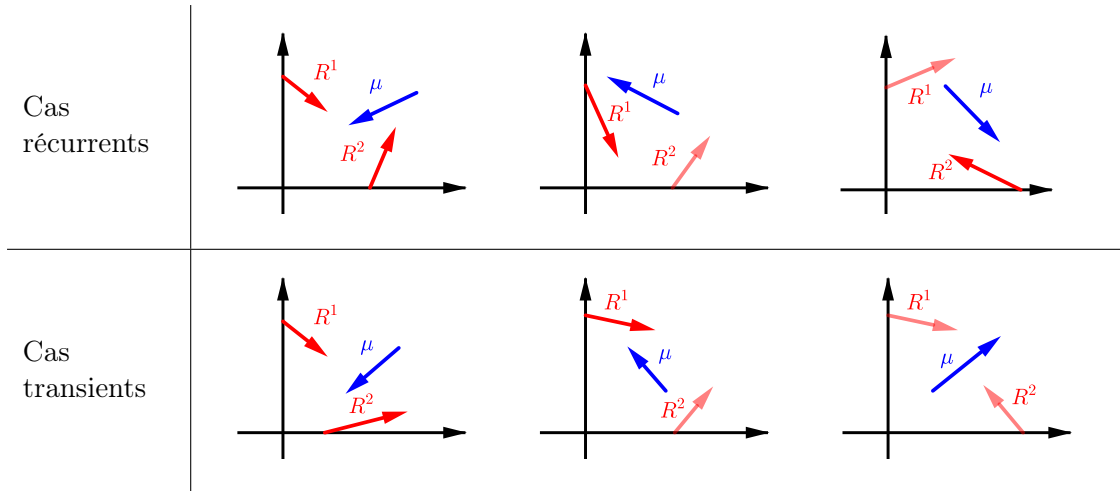


FIGURE 1.4 : Illustration des conditions de récurrence

Remarque 38 (Interprétation des conditions). • La condition (1.15) équivaut au fait que R soit une \mathcal{P} -matrice (tous les mineurs sont positifs). Elle implique que (R^1, R^2) soit une base directe. En particulier R est inversible. Cette condition qui dit que le coin n'est pas « trop » attractif sert à l'existence du processus en tant que semi-martingale.

- La condition (1.16) équivaut à $R^{-1}\mu < 0$ ou encore au fait que (R^2, μ) et (μ, R^1) sont des bases directes. Cette condition sert à ce que, si le processus arrive sur un bord avec la direction de la dérive, il rebondisse « vers l'autre bord » afin de ne pas partir vers l'infini le long du bord qu'il vient de toucher. Il s'agit d'une compétition entre la dérive et les vecteurs de rebond.

Remarque 39 (Cas de la dérive nulle). On remarquera que les conditions de récurrence (1.15) et (1.16) ne font pas intervenir la matrice de co-variance Σ . Cela s'explique car on se situe dans le cas d'une dérive μ non nulle et cette dernière l'emporte sur les temps longs et les grandes distances sur le comportement global de la diffusion. Dans le cas de dérive nulle le processus est récurrent et admet une mesure invariante si et seulement si

$$0 \leq \alpha < 2,$$

voir Williams [100]. Pour rappel α défini dans l'équation (1.14) dépend de la covariance et des vecteurs de rebond. Si on note $p(r, \theta)$ la densité de la mesure invariante en coordonnées polaires dans un cône d'angle β , d'après Williams [100] on a

$$p(r, \theta) = r^{-\alpha} \sin(\delta - \alpha\theta).$$

Remarque 40 (Comparaison avec le cas discret). Les conditions du cas discret et du cas continu se ressemblent. C'est à cause des conditions pour que le coin ne soit pas « trop » attractif qu'elles ne sont pas exactement similaires. Pour que le processus soit récurrent positif on ne peut pas avoir par exemple dans le cas continu $\mu_1 < 0$, $\mu_2 < 0$, $r_{22}\mu_1 - r_{12}\mu_2 < 0$, $r_{11}\mu_2 - r_{21}\mu_1 < 0$, $\frac{r_{11}r_{22} - r_{12}r_{21}}{0} < 0$ ce qui serait possible dans le cas discret, comme dans le cas 2 de la figure A.2 de l'annexe A.

1.2.6 Mesures invariantes et fonctions de Green

On étudiera tout d'abord les mesures invariantes sur les frontières dans le cas récurrent et en particulier la *Basic Adjoint Relationship* (BAR) qui nous servira à démontrer l'équation fonctionnelle qui sera au cœur de notre étude analytique. Ensuite, dans le cas transient, nous tâcherons de définir de manière analogue les fonctions de Green sur les frontières.

1.2.6.1 Cas récurrent

Soit Z un mouvement brownien avec dérive réfléchi au bord du quart de plan \mathbb{R}_+^2 associé à (Σ, μ, R) tel que

$$Z_t = z_0 + W_t + \mu t + RL_t \in \mathbb{R}_+^2.$$

On suppose que les conditions (1.15) et (1.16) sont satisfaites et que le processus est donc récurrent positif et admet une unique distribution stationnaire (ou mesure invariante) Π de densité π . Pour tout $A \subset \mathbb{R}_+^2$, $\Pi(A)$ mesure la *proportion de temps* que le processus passe en A . Soit \mathcal{G} le générateur de $(W_t + \mu t)$ qui est aussi le générateur de Z à l'intérieur de \mathbb{R}_+^2 . On a

$$\mathcal{G}f(z) = \frac{1}{2} \sum_{i,j=1}^2 \sigma_{i,j} \frac{\partial^2 f}{\partial z_1 \partial z_2}(z) + \sum_{i=1}^2 \mu_i \frac{\partial f}{\partial z_i}(z). \quad (1.17)$$

De la même manière qu'en dimension un dans la remarque 22, on définit $D_{\mathcal{G}}$ l'ensemble

$$D_{\mathcal{G}} = \{f \in \mathcal{C}_{uni}^{(2)} : \partial_{R^1} f(0, y) = 0 \text{ et } \partial_{R^2} f(x, 0) = 0\}.$$

Pour tout $f \in D_{\mathcal{G}}$ on peut montrer qu'on a $\int \mathcal{G}f d\Pi = 0$. Comme dans la proposition 24 en dimension un, nous allons montrer la *Basic Adjoint Relationship*, proposition 42, qui généralise cette formule. Pour cela on va d'abord définir ce qu'on peut considérer en un certain sens comme les générateurs du processus sur les frontières. Pour $i = 1, 2$, on pose

$$D_i f := \partial_{R^i} f = \langle R^i | \nabla f \rangle.$$

La BAR explique pourquoi on considère que ce sont les générateurs sur les axes. Cela fait aussi écho à la remarque 25 qui détermine en dimension un le générateur sur le bord du processus réfléchi ralenti. On définit maintenant ce qu'on appellera les mesures invariantes sur les bords.

Définition 41 (Mesures invariantes sur les frontières). *On définit ν_1 et ν_2 les mesures sur les frontières, telles que pour tout $A \subset \mathbb{R}_+^2$ on a pour $i = 1, 2$,*

$$\nu_i(A) := \mathbb{E}_{\Pi} \left[\int_0^1 1_{\{Z(u) \in A\}} dL_u^i \right] = \mathbb{E}_{\Pi} \left[\frac{1}{t} \int_0^t 1_{\{Z(u) \in A\}} dL_u^i \right]. \quad (1.18)$$

Le support de ν_1 est sur l'axe des ordonnées et celui de ν_2 sur l'axe des abscisses. Ces mesures admettent des densités par rapport à la mesure de Lebesgue. Voir [27, 55].

On peut donc dire que $\nu_i(A)$ mesure la *proportion de temps local* que le processus passe sur le bord en A . Voir Dai et Harrison [27], Harrison et Williams [55] pour plus de détails sur ces mesures. Par définition des ν_i , on peut montrer par approximation que

$$\mathbb{E}_\Pi \left[\frac{1}{t} \int_0^t f(Z_u) dL_u^i \right] = \int_{\mathbb{R}_+^2} f(x) \nu_i(dx), \quad \text{pour } f \geq 0. \quad (1.19)$$

On retrouve l'analogie sur les bords de la propriété fondamentale de la mesure invariante de la définition 36. Avant d'énoncer la BAR on définit enfin $\mathcal{C}_b^2(\mathbb{R}_+^2)$ l'ensemble des fonctions \mathcal{C}^2 de \mathbb{R}_+^2 bornées ainsi que toutes leurs dérivées partielles jusqu'à l'ordre deux.

Proposition 42 (*Basic adjoint relationship*). *On a pour tout $f \in \mathcal{C}_b^2(\mathbb{R}_+^2)$,*

$$\int_{\mathbb{R}_+^2} \mathcal{G}f(z) \Pi(dz) + \sum_{i=1,2} \int_{\mathbb{R}_+^2} D_i f(z) \nu_i(dz) = 0. \quad (1.20)$$

Cette relation caractérise la distribution stationnaire : si Π , ν_1 et ν_2 ont des densités positives et intégrables alors ce sont les mesures invariantes du processus.

Démonstration. On montre ici uniquement le fait que (1.20) est une condition nécessaire pour que Π soit une mesure stationnaire. Cette relation a été initialement démontrée par Harrison et Williams [55] et étendue par la suite, voir par exemple Dai et Harrison [27]. On peut se référer à Dai et Kurtz [26], Williams [101] pour la réciproque qui implique que la BAR caractérise la mesure invariante. En prenant l'espérance sur Π de la formule d'Itô appliquée à Z pour tout $f \in \mathcal{C}_b^2(\mathbb{R}_+^2)$, on obtient

$$\mathbb{E}_\Pi[f(Z_t) - f(Z_0)] = \mathbb{E}_\Pi \left[\int_0^t \nabla f(Z_s) dW_s \right] + \int_0^t \mathbb{E}_\Pi[\mathcal{G}f(Z(s))] ds + \sum_{i=1}^2 \mathbb{E}_\Pi \left[\int_0^t D_i f(Z_s) dL_s^i \right].$$

Grâce à l'équation (1.19), en appliquant le théorème de Fubini et en constatant que $\int_0^t \nabla f(Z_s) dW_s$ est une martingale (car ∇f est borné) d'espérance nulle, on obtient alors

$$0 = 0 + t \int_{\mathbb{R}_+^2} \mathcal{G}f(z) \Pi(dz) + \sum_{i=1}^2 t \int_{\mathbb{R}_+^2} D_i f(z) \nu_i(dz).$$

□

1.2.6.2 Cas transient

On suppose maintenant qu'on est dans le cas où Z_t est transient. À notre connaissance et contrairement au cas récurrent, le cas transient n'est presque pas étudié dans la littérature. On peut définir les fonctions (ou mesures) de Green comme suit.

Définition 43 (Fonctions de Green). *La fonction de Green $G_A^{z_0}$ où $z_0 \in \mathbb{R}_+^2$ et $A \subset \mathbb{R}_+^2$ borné est définie dans le cas transient par*

$$G_A^{z_0} = \int_0^\infty P(t, z_0, A) dt = \mathbb{E}_{z_0} \left[\int_0^\infty \mathbf{1}_{\{Z_t \in A\}} dt \right] < \infty$$

où $P(t, z_0, A) = \mathbb{P}[Z(t) \in A | Z(0) = z_0] = \mathbb{E}_{z_0}[\mathbf{1}_{Z(t) \in A}]$ est la fonction de transition de Z . La densité de G^{z_0} est

$$g_z^{z_0} = \int_0^\infty p(t, z_0, z) dt,$$

où $p(t, z_0, z)$ est la densité de transition du processus Z .

La fonction de Green $G_A^{z_0}$ mesure donc la *quantité totale moyenne de temps* passé en A en partant de z_0 . Comme on l'a fait dans le cas récurrent pour les mesures invariantes on va définir h_1 et h_2 les fonctions (ou mesures) de Green sur le bord.

Définition 44 (Fonctions de Green sur les frontières). *On définit h_1 et h_2 les fonctions de Green sur les frontières telles que pour $z_0 \in \mathbb{R}_+^2$ et $A \subset \mathbb{R}_+^2$ borné on a pour $i = 1, 2$,*

$$h_i^{z_0}(A) = \mathbb{E}_{z_0} \left[\int_0^\infty 1_{\{Z(u) \in A\}} dL_u^i \right] < \infty.$$

Le support de h_1 est sur l'axe des ordonnées et celui de h_2 sur l'axe des abscisses.

On peut donc dire que $h_i(A)$ mesure la *quantité totale moyenne de temps local* passé sur un bord en A en partant de z_0 . Par densité des fonctions continues par morceaux et approximation on obtient l'analogie de la formule (1.19), ce qui donne

$$\mathbb{E}^{z_0} \left[\int_0^\infty f(Z_t) dL_t^i \right] = \int_{\mathbb{R}_+} f(x) h_i^{z_0}(x) dx. \quad (1.21)$$

1.2.7 Fonctions génératrices et équation fonctionnelle

Maintenant que nous avons introduit les mesures invariantes sur les bords, nous allons montrer l'équation fonctionnelle qui sera au cœur de l'étude analytique développée dans cette thèse. Cette formule clé relie les transformées de Laplace des mesures invariantes à l'intérieur et sur les bords du quart de plan. Les transformées de Laplace jouent le rôle de fonctions génératrices. De la même manière nous déterminerons ensuite une équation fonctionnelle similaire pour les fonctions de Green.

1.2.7.1 Équation fonctionnelle pour la mesure invariante

Définition 45 (Transformée de Laplace des mesures invariantes). *On définit φ la transformée de Laplace de la mesure invariante par*

$$\varphi(\theta) := \iint_{\mathbb{R}_+^2} e^{\theta \cdot z} \Pi(dz) = \mathbb{E}_\Pi[e^{\theta \cdot Z_t}] \quad (1.22)$$

où on note $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ et $\theta \cdot Z = (\theta_1 Z^1 + \theta_2 Z^2)$. De même on définit φ_1 et φ_2 les transformées de Laplace des mesures invariantes sur les frontières par

$$\varphi_2(\theta_1) := \int_{\mathbb{R}_+^2} e^{\theta_1 z_1} \nu_2(dz) = \mathbb{E}_\Pi \left[\int_0^1 e^{\theta_1 Z_t^1} dL_t^2 \right],$$

et

$$\varphi_1(\theta_2) := \int_{\mathbb{R}_+^2} e^{\theta_2 z_2} \nu_1(dz) = \mathbb{E}_\Pi \left[\int_0^1 e^{\theta_2 Z_t^2} dL_t^1 \right].$$

Les égalités de ces deux dernières définitions sont dues à la formule (1.19).

Proposition 46 (Équation fonctionnelle de la mesure invariante). *Sous les conditions de récurrence (1.15) et (1.16) et lorsque $\varphi(\theta) < \infty$, on a la formule suivante,*

$$-\gamma(\theta)\varphi(\theta) = \gamma_1(\theta)\varphi_2(\theta_1) + \gamma_2(\theta)\varphi_1(\theta_2), \quad (1.23)$$

où on note

$$\begin{cases} \gamma(\theta) = \frac{1}{2}\theta \cdot \Sigma\theta + \theta \cdot \mu = \frac{1}{2}(\sigma_{11}\theta_1^2 + 2\sigma_{12}\theta_1\theta_2 + \sigma_{22}\theta_2^2) + \mu_1\theta_1 + \mu_2\theta_2, \\ \gamma_1(\theta) = R^1 \cdot \theta = r_{11}\theta_1 + r_{21}\theta_2, \\ \gamma_2(\theta) = R^2 \cdot \theta = r_{12}\theta_1 + r_{22}\theta_2. \end{cases} \quad (1.24)$$

Cette formule est une équation d'équilibre entre ce qui se passe à l'intérieur du quart de plan (terme de gauche) et ce qui se passe sur ses bords (terme de droite). On remarquera que γ , qu'on appelle le noyau, est un polynôme du second degré à deux variables dépendant de la matrice de covariance Σ et de la dérive μ . Les fonctions γ_1 et γ_2 sont quant à elles des polynômes de degré un à deux variables dépendant uniquement des vecteurs de rebond R^1 et R^2 .

Démonstration. Nous donnons ici les grandes lignes de la démonstration, on se référera à l'article de Dai et Miyazawa [28, Lemme 4.1.] pour une preuve complète. La BAR, formule (1.20), nous dit que

$$\int_{\mathbb{R}_+^2} Lf(z)\Pi(dz) + \sum_{i=1,2} \int_{\mathbb{R}_+^2} D_i f(z)\nu_i(dz) = 0 \quad \forall f \in \mathcal{C}_b^2(\mathbb{R}_+^2).$$

En évaluant cette égalité en prenant $f(z) = e^{\theta \cdot z} = e^{\theta_1 z_1 + \theta_2 z_2}$ on obtient directement l'équation fonctionnelle :

$$\int_{\mathbb{R}_+^2} \left(\frac{1}{2} \sum_{i,j=1}^2 \sigma_{i,j} \theta_i \theta_j + \sum_{i,j=1}^2 \mu_i \theta_i \right) e^{\theta \cdot z} \Pi(dz) + \sum_{i=1,2} \int_{\mathbb{R}_+^2} R^i \cdot \theta e^{\theta \cdot z} \nu_i(dz) = 0.$$

En fait on ne peut pas appliquer directement la BAR car $f \notin \mathcal{C}_b^2(\mathbb{R}_+^2)$, il faudrait pour être rigoureux procéder par approximation, voir Dai et Miyazawa [28]. \square

1.2.7.2 Équation fonctionnelle pour les fonctions de Green

Définition 47 (Transformée de Laplace des fonctions de Green). *On définit $\psi^{z_0}(\theta)$ la transformée de Laplace des mesures de Green par*

$$\psi^{z_0}(\theta) := \iint_{\mathbb{R}^2} e^{\theta \cdot z} g_z^{z_0} dz = \mathbb{E}_{z_0} \left[\int_0^\infty e^{\theta \cdot Z_t} dt \right].$$

De même on définit $\psi_1^{z_0}$ et $\psi_2^{z_0}$ les transformées de Laplace des fonctions de Green sur les bords

$$\psi_2^{z_0}(\theta_1) := \int_{\mathbb{R}} e^{\theta_1 z_1} h_2^{z_0}(dz) = \mathbb{E}_{z_0} \left[\int_0^\infty e^{\theta_1 Z_t^1} dL_t^2 \right],$$

et

$$\psi_1^{z_0}(\theta_2) := \int_{\mathbb{R}} e^{\theta_2 z_2} h_1^{z_0}(dz) = \mathbb{E}_{z_0} \left[\int_0^\infty e^{\theta_2 Z_t^2} dL_t^1 \right].$$

L'égalité de la première définition ci-dessus s'obtient de la même manière que la proposition 17. Les égalités des deux dernières définitions découlent de la formule (1.21).

Proposition 48 (Équation fonctionnelle des fonctions de Green). *Dans le cas transient, i.e. lorsque le processus tend vers l'infini presque sûrement, et si $\psi^{z_0}(\theta) < \infty$, $\psi_1^{z_0}(\theta_1) < \infty$ et $\psi_2^{z_0}(\theta_2) < \infty$ on a pour tout θ_1 et $\theta_2 < 0$ la formule suivante,*

$$-\gamma(\theta)\psi^{z_0}(\theta) = \gamma_1(\theta)\psi_1^{z_0}(\theta_2) + \gamma_2(\theta)\psi_2^{z_0}(\theta_1) + e^{\theta \cdot z_0}$$

où γ , γ_1 et γ_2 sont définis dans la proposition 46.

De même que pour l'équation fonctionnelle de la mesure invariante, cette formule est une équation de balance reliant ce qui se passe à l'intérieur du quart de plan et sur ses bords. Elle diffère uniquement de la formule (1.23) par le terme constant $e^{\theta \cdot z_0}$, on qualifiera ainsi cette équation de non-homogène. A notre connaissance cette formule n'apparaît pas encore dans la littérature.

Démonstration. La démonstration de cette nouvelle formule clé est une conséquence de la formule d'Itô. Pour $f \in \mathcal{C}_b^2(\mathbb{R}_+^2)$ on a

$$f(Z_t) - f(Z_0) = \int_0^t \nabla f(Z_s) \cdot dW_s + \int_0^t Lf(Z_s) ds + \sum_{i=1}^2 \int_0^t D_i f(Z_s) dL_t^i.$$

Évaluons cette égalité en choisissant $f(z) = e^{\theta \cdot z}$ et prenons-en l'espérance. On sait que $\int_0^t \nabla f(Z_s) \cdot dW_s$ est une martingale car ∇f est borné (car $\theta_i < 0$) et donc son espérance est nulle. En choisissant $f(z) = e^{\theta \cdot z}$ et prenant l'espérance de la précédente égalité on obtient

$$\mathbb{E}_{z_0}[e^{\theta \cdot Z_t}] - e^{\theta \cdot z_0} = 0 + \gamma(\theta)\mathbb{E}_{z_0} \left[\int_0^t e^{\theta \cdot Z_s} ds \right] + \sum_{i=1}^2 \gamma_i(\theta)\mathbb{E}_{z_0} \left[\int_0^t e^{\theta \cdot Z_s} dL_t^i \right].$$

On fait alors tendre t vers l'infini. Comme Z tend presque sûrement vers l'infini et que $\theta_i < 0$ on a $\mathbb{E}_{z_0}[e^{\theta \cdot Z_t}] \xrightarrow{t \rightarrow \infty} 0$. Les espérances étant finies par hypothèse, on obtient alors la formule recherchée,

$$0 - e^{\theta \cdot z_0} = \gamma(\theta)\mathbb{E}_{z_0} \left[\int_0^\infty e^{\theta \cdot Z_s} ds \right] + \sum_{i=1}^2 \gamma_i(\theta)\mathbb{E}_{z_0} \left[\int_0^\infty e^{\theta \cdot Z_s} dL_t^i \right].$$

□

1.3 Approche analytique et résultats

1.3.1 Principes

Comme nous l'avons fait dans l'article [45] écrit pour la conférence AofA 2016, nous détaillons ici les grandes étapes de l'approche analytique qui nous guidera pendant toute cette thèse. Dans la littérature (voir par exemple Malyshev [79], Fayolle et Iasnogorodski [36], Fayolle *et al.* [37], Kourkova et Raschel [64, 65]), l'approche analytique repose sur six étapes :

- (i) Trouver l'équation fonctionnelle reliant les fonctions génératrices ou les transformées de Laplace des fonctions étudiées (fonctions de Green, mesures invariantes, ...);

- (ii) Définir la surface de Riemann associée au noyau, prolonger méromorphiquement les fonctions génératrices, trouver les fonctions de collage conformes ;
- (iii) Introduire le groupe lié au processus ;
- (iv) Dédire de l'équation fonctionnelle un problème frontière (*boundary value problem*) ;
- (v) Résoudre ce problème frontière ;
- (vi) Étudier les singularités des fonctions étudiées (fonctions de Green, mesures invariants, etc.), inverser les transformées de Laplace et appliquer la méthode du point col sur la surface de Riemann.

Avant de commenter ces différentes étapes, précisons qu'elles permettent toutes ensembles d'étudier les trois problèmes principaux suivants :

- (P1) Trouver l'expression explicite des fonctions génératrices (nécessite (i), (ii), (iv) et (v)) ; voir les chapitres 2 et 3 ;
- (P2) Étudier la nature algébrique de ces fonctions (nécessite (ii) et (iii)) ; voir les chapitres 2 et 3 ;
- (P3) Déterminer l'asymptotique des fonctions étudiées selon différents paramètres (nécessite (ii) et (vi)) ; voir le chapitre 4.

Le point (i) traduit les propriétés inhérentes au processus et a déjà été facilement obtenu dans la section 1.2.7. Le point (ii) est le plus technique ; il est cependant crucial car il permet d'obtenir des quantités clés (comme les fonctions de collage conformes qui apparaissent dans la formule exacte de (v)). L'idée d'introduire le groupe lié au processus (iii) a été proposée dans Malyshev [79] et développée dans un contexte combinatoire dans Bousquet-Mélou et Mishna [10]. Le point (iv), montré pour la première fois dans Fayolle et Iasnogorodski [36], est maintenant standard (voir Fayolle *et al.* [37]) et découle de manipulations algébriques de l'équation fonctionnelle obtenue en (i). Le point (v) utilise la littérature spécifique dédiée aux problèmes frontière, nos références principales sont les livres de Litvinchuk [74], Muskhelishvili [82] et Gakhov [48], voir l'annexe C. Finalement le point (vi) permet de déterminer l'asymptotique. Il utilise des théorèmes de transfert reliant les pôles des transformées de Laplace à l'asymptotique des fonctions étudiées, on peut se référer au livre de Doetsch [32], ainsi qu'un double raffinement de la méthode classique du point col : la méthode uniforme de la plus grande pente, nos livres de références sont ceux de Fedoryuk [40, 41].

Dans les sections suivantes nous présentons ces différents points ainsi que les principaux résultats de cette thèse, afin d'avoir une vue d'ensemble des outils utilisés et d'aborder plus facilement la lecture des chapitres suivants. Dans la section 1.3.2 nous présentons succinctement certains aspects techniques des points (ii) et (iii) en introduisant la surface de Riemann et le groupe associé au noyau du processus. Dans la section 1.3.3 nous développons les points (iv) et (v) en énonçant les problèmes frontière (BVP) satisfaits par les transformées de Laplace et en donnant les formules explicites obtenues par la résolution de ces problèmes. On invite le lecteur non initié aux problèmes frontière à lire l'annexe C qui est une brève introduction à cette théorie. Enfin dans la section 1.3.4 le point (vi) sera abordé. Nous y évoquons les théorèmes de transfert et la méthode du point col qui nous permet d'énoncer nos résultats asymptotiques.

1.3.2 Surface de Riemann

Nous avertissons le lecteur que cette section, la plus concise possible, est très technique. Si la première lecture peut sembler aride, le lecteur pourra s'y référer par la suite lorsqu'il en aura besoin.

1.3.2.1 Noyau

Le noyau γ défini dans l'équation (1.24) peut se réécrire des deux manières suivantes,

$$\gamma(\theta_1, \theta_2) = \tilde{a}(\theta_2)\theta_1^2 + \tilde{b}(\theta_2)\theta_1 + \tilde{c}(\theta_2) = a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1), \quad (1.25)$$

où a, b, c (resp. $\tilde{a}, \tilde{b}, \tilde{c}$) sont des polynômes en θ_1 (resp. θ_2). L'équation $\gamma(\theta_1, \theta_2) = 0$ définit une fonction algébrique bivaluée $\Theta_1(\theta_2)$ par $\gamma(\Theta_1(\theta_2), \theta_2) = 0$, et de même $\Theta_2(\theta_1)$ tel que $\gamma(\theta_1, \Theta_2(\theta_1)) = 0$. Les expressions des branches sont données par

$$\Theta_2^\pm(\theta_1) = \frac{-b(\theta_1) \pm \sqrt{d(\theta_1)}}{2a(\theta_1)},$$

où $d(\theta_1) = b^2(\theta_1) - 4a(\theta_1)c(\theta_1)$ est le discriminant. Le polynôme d a deux zéros, réels et de signes opposés ; ils sont notés θ_1^\pm et ils sont des points de branchement de la fonction algébrique Θ_2 . De la même manière on définit Θ_1^\pm et ses points de branchement θ_2^\pm . Finalement, on remarque que d est négatif sur $\mathbb{R} \setminus [\theta_1^-, \theta_1^+]$. Par conséquent, les branches Θ_2^\pm prennent des valeurs complexes conjuguées sur cet ensemble. Par la suite, cela impliquera que l'ensemble \mathcal{R} défini dans l'équation (1.29) est symétrique par rapport à l'axe réel, voir section 1.3.2.5.

1.3.2.2 Surface de Riemann

Introduisons maintenant la surface de Riemann associée aux fonctions algébriques Θ_2 et Θ_1 . On définit la surface de Riemann \mathcal{S} comme l'ensemble des zéros du noyau, on a

$$\mathcal{S} = \{(\theta_1, \theta_2) \in \mathbb{C}^2 : \gamma(\theta_1, \theta_2) = 0\}.$$

Le degré de γ implique que la surface \mathcal{S} est de genre 0 et est une sphère de Riemann, i.e., homéomorphe à $\mathbb{C} \cup \{\infty\}$, voir le chapitre 4. Nous verrons dans l'annexe A que dans le cas discret la surface de Riemann associée à la marche aléatoire est un tore. La surface \mathcal{S} admet une paramétrisation (ou uniformisation) rationnelle très utile donnée par

$$\theta_1(s) = \frac{\theta_1^+ + \theta_1^-}{2} + \frac{\theta_1^+ - \theta_1^-}{4} \left(s + \frac{1}{s} \right), \quad \theta_2(s) = \frac{\theta_2^+ + \theta_2^-}{2} + \frac{\theta_2^+ - \theta_2^-}{4} \left(\frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s} \right), \quad (1.26)$$

où on a encore noté

$$\beta = \arccos -\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}. \quad (1.27)$$

On remarquera qu'il s'agit du même angle rencontré dans la section 1.2.3. L'équation $\gamma(\theta_1(s), \theta_2(s)) = 0$ est vérifiée et $\mathcal{S} = \{(\theta_1(s), \theta_2(s)) : s \in \mathbb{C} \cup \{\infty\}\}$.

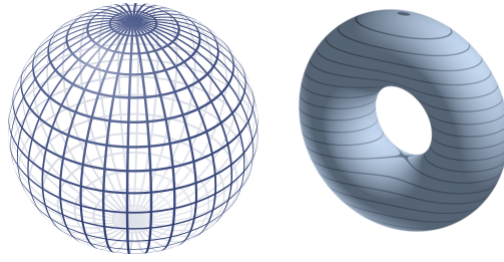


FIGURE 1.5 : Représentation d'une sphère de Riemann et d'un tore

1.3.2.3 Automorphismes de Galois et groupe du processus

Finale­ment nous introduisons la notion de groupe du processus, similaire à la notion de groupe de la marche dans le cadre discret (voir Malyshev [79], Fayolle *et al.* [37], Bousquet-Mélou et Mishna [10]). Ce groupe $\langle \zeta, \eta \rangle$ est généré par ζ et η , données par

$$\zeta(\theta_1, \theta_2) = \left(\theta_1, \frac{c(\theta_1)}{a(\theta_1)} \frac{1}{\theta_2} \right), \quad \eta(\theta_1, \theta_2) = \left(\frac{\tilde{c}(\theta_2)}{\tilde{a}(\theta_2)} \frac{1}{\theta_1}, \theta_2 \right),$$

avec la notation (1.25). Par construction, les générateurs satisfont $\gamma(\zeta(\theta_1, \theta_2)) = \gamma(\eta(\theta_1, \theta_2)) = 0$ dès que $\gamma(\theta_1, \theta_2) = 0$. Dit autrement, ils sont les automorphismes (de recouvrement) de la surface \mathcal{S} . Puisque $\zeta^2 = \eta^2 = 1$, le groupe $\langle \zeta, \eta \rangle$ est un groupe diédral, qui est fini si et seulement si $\zeta \circ \eta$ (ou de manière équivalente $\eta \circ \zeta$) est d'ordre fini.

Si on ré-exprime les automorphismes ζ et η avec la paramétrisation (1.26) on obtient

$$\zeta(s) = \frac{1}{s}, \quad \eta(s) = \frac{e^{2i\beta}}{s}. \quad (1.28)$$

Ces transformations laissent invariant $\theta_1(s)$ et $\theta_2(s)$, respectivement, voir (1.26). Cette formulation servira dans la preuve du lemme 53.

1.3.2.4 Prolongement des transformées de Laplace

Le problème frontière que nous allons établir repose sur une courbe qui réside en dehors du domaine naturel de définition des transformées de Laplace (à savoir, le demi plan de partie réelle négative). De la même manière, l'étude asymptotique utilisera la méthode de la plus grande pente sur des courbes qui n'appartiennent pas au domaine initial de définition. Nous avons donc besoin de prolonger le domaine de définition des transformées de Laplace. On suppose que les conditions de récurrence (1.15) et (1.16) sont vérifiées.

Théorème 49 (Prolongement analytique). *La fonction φ_1 peut être prolongée méromorphiquement sur le plan coupé $\mathbb{C} \setminus [\theta_2^+, \infty)$.*

Démonstration. La première étape est de prolonger méromorphiquement $\varphi_1(\theta_2)$ à l'ensemble ouvert et simplement connexe $\{\theta_2 \in \mathbb{C} : \Re \theta_2 \leq 0 \text{ ou } \Re \Theta_1^-(\theta_2) < 0\}$, en posant d'après (1.23)

$$\varphi_1(\theta_2) = \frac{\gamma_2}{\gamma_1}(\Theta_1^-(\theta_2), \theta_2) \varphi_2(\Theta_1^-(\theta_2)).$$

Le prolongement est alors immédiat (voir le chapitre 4 et l'article [44] pour les détails). Il est alors possible de poursuivre le prolongement à \mathcal{S} tout entier en utilisant les propriétés d'invariance des automorphismes ζ et η satisfaites par les transformées de Laplace sur la surface \mathcal{S} . \square

1.3.2.5 Une hyperbole importante

Pour une utilisation ultérieure, nous avons besoin d'introduire la courbe

$$\mathcal{R} = \{\theta_2 \in \mathbb{C} : \gamma(\theta_1, \theta_2) = 0 \text{ et } \theta_1 \in (-\infty, \theta_1^-)\} = \Theta_2^\pm((-\infty, \theta_1^-)). \quad (1.29)$$

La courbe \mathcal{R} est symétrique par rapport à l'axe réel, voir la Figure 1.6 (cela découle du fait que d est négatif sur $(-\infty, \theta_1^-)$, voir la section 1.3.2.1). De plus, c'est une branche d'une hyperbole, voir Baccelli et Fayolle [5]. On notera $\mathcal{G}_{\mathcal{R}}$ le domaine ouvert de \mathbb{C} borné par \mathcal{R} et contenant 0, voir Figure 1.6.

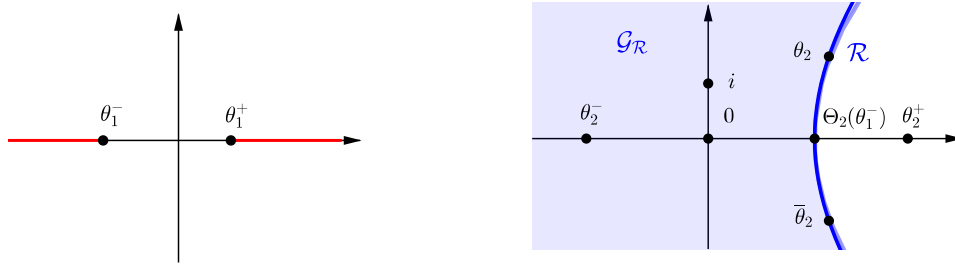


FIGURE 1.6 : Gauche : le discriminant d a deux racines θ_1^- et θ_1^+ de signes opposés. Droite : la courbe \mathcal{R} dans (1.29) est symétrique par rapport à l'axe horizontal et $\mathcal{G}_{\mathcal{R}}$ est le domaine bleu

1.3.2.6 Fonction de collage conforme

Pour construire une solution particulière aux problèmes frontière que nous allons établir dans les sections suivantes, on cherche une fonction w qui satisfait les propriétés suivantes.

- (i) w est analytique sur $\mathcal{G}_{\mathcal{R}}$, continue dans $\overline{\mathcal{G}_{\mathcal{R}}}$ et non bornée à l'infini,
- (ii) w est injective sur $\mathcal{G}_{\mathcal{R}}$,
- (iii) $w(\theta_2) = w(\overline{\theta_2})$ pour tout $\theta_2 \in \mathcal{R}$.

Une telle fonction w est appelée fonction de collage conforme (*conformal gluing function*). La propriété conforme vient de (i) et (ii), et le collage vient de (iii). La fonction w colle ensemble les parties supérieure et inférieure de l'hyperbole \mathcal{R} .

Définition 50 (Fonction de collage conforme). *On peut exprimer w qui vérifie (i), (ii) et (iii) en fonction des polynômes de Tchebychev généralisés*

$$T_a(x) = \cos(a \arccos(x)) = \frac{1}{2} \left\{ (x + \sqrt{x^2 - 1})^a + (x - \sqrt{x^2 - 1})^a \right\}$$

comme suit :

$$w(\theta_2) = T_{\frac{\pi}{\beta}} \left(- \frac{2\theta_2 - (\theta_2^+ + \theta_2^-)}{\theta_2^+ - \theta_2^-} \right), \quad (1.30)$$

où on rappelle que

$$\beta = \arccos - \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}. \quad (1.31)$$

Idée de preuve. Nous verrons dans le chapitre 2 qu'il y a au moins deux manières de trouver un tel w . Tout d'abord, il s'avère qu'il existe dans la littérature des expressions de collages conformes pour les courbes relativement simples comme les hyperboles, voir Baccelli et Fayolle [5, Equation (4.6)]. Ici (comme on le verra dans le chapitre 2), on utilise la sphère de Riemann \mathcal{S} . En effet, de nombreux aspects techniques s'avèrent être plus simples sur cette surface. La fonction de collage w peut donc se définir sur \mathcal{S} et son expression est plus simple en utilisant la paramétrisation de \mathcal{S} introduite en 1.3.2.2. On trouve dans la formule (2.20) du chapitre 2 que

$$w(\theta_2(s)) = -\frac{1}{2}\{(-s)^{\frac{\pi}{\beta}} + (-s)^{-\frac{\pi}{\beta}}\} = -\frac{1}{2}\{e^{\frac{\pi}{\beta}\log(-s)} + e^{-\frac{\pi}{\beta}\log(-s)}\}. \quad (1.32)$$

□

1.3.3 Calcul de la mesure invariante et problème frontière

Cette section énonce le problème frontière qui nous permet d'obtenir une formule explicite pour la mesure invariante. On supposera donc que les conditions de récurrence (1.15) et (1.16) sont vérifiées. Un problème frontière contient en règle générale une condition de régularité et une condition frontière. On rappelle par ailleurs au lecteur qu'une brève introduction à la théorie des problèmes frontière se trouve dans l'annexe C.

1.3.3.1 Problème frontière dans le cas orthogonal

Cette section annonce les résultats principaux du chapitre 2. Dans le cas des réflexions orthogonales (voir la Figure 1.7), R est la matrice identité, et on a $\gamma_1(\theta_1, \theta_2) = \theta_1$ et $\gamma_2(\theta_1, \theta_2) = \theta_2$. On pose

$$\psi_1(\theta_2) = \frac{1}{\theta_2}\varphi_1(\theta_2), \quad \psi_2(\theta_1) = \frac{1}{\theta_1}\varphi_2(\theta_1). \quad (1.33)$$

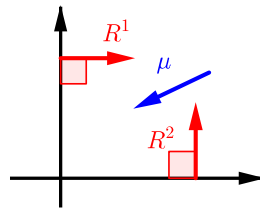


FIGURE 1.7 : Rebonds orthogonaux R^1 et R^2 et dérive μ

Le lemme suivant reprend le résultat de la proposition 64 du chapitre 2.

Lemme 51 (Problème frontière dans le cas orthogonal). *La fonction ψ_1 dans (1.33) satisfait le problème frontière suivant :*

- (i) ψ_1 est méromorphe sur $\mathcal{G}_{\mathcal{R}}$ avec un seul pôle en 0, d'ordre 1 et de résidu $\varphi_1(0)$, et s'annule en l'infini;
- (ii) ψ_1 est continue sur $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{0\}$ et

$$\psi_1(\overline{\theta_2}) = \psi_1(\theta_2), \quad \forall \theta_2 \in \mathcal{R}. \quad (1.34)$$

Schéma de preuve. Les conditions de régularité du point (i) découlent du théorème 49, qui fournit à la fonction un domaine (maximal) de prolongement méromorphe et donc la formule de prolongement permet de déterminer les pôles. Considérons maintenant (ii). En évaluant l'équation fonctionnelle (prolongée) (1.23) en $(\theta_1, \Theta_2^\pm(\theta_1))$, on obtient $\psi_1(\Theta_2^\pm(\theta_1)) + \psi_2(\theta_1) = 0$, qui implique immédiatement que

$$\psi_1(\Theta_2^+(\theta_1)) = \psi_1(\Theta_2^-(\theta_1)). \quad (1.35)$$

En choisissant $\theta_1 \in (-\infty, \theta_1^-)$, les deux quantités $\Theta_2^+(\theta_1)$ et $\Theta_2^-(\theta_1)$ sont complexes conjuguées l'une de l'autre, voir section 1.3.2.1. L'équation (1.35) peut être reformulée comme (1.34), en utilisant la définition (1.29) de la courbe \mathcal{R} . \square

Le problème frontière établi dans le lemme 51 est appelé problème frontière de Carleman homogène avec décalage ou avec *shift* (le *shift* est ici la conjugaison complexe, mais la théorie générale s'applique à des *shift* plus généraux, voir l'annexe C). Il a une forme plus simple que le problème frontière du cas discret du lemme 130 de l'annexe C, car il n'y a pas de terme inhomogène (comme $\pi_{0,0}$) et aussi car dans les coefficients en face des fonctions inconnues il n'y a pas de fonction algébrique (comme Y_0) impliquée.

Grâce à sa forme particulièrement simple, on peut résoudre ce problème de manière explicite en utilisant les deux étapes suivantes :

- Utiliser la fonction de collage conforme w (introduite dans la section 1.3.2.6) pour construire une solution particulière du problème frontière du lemme 51.
- La solution du problème frontière du lemme 51 est unique (grâce à un lemme des invariants, voir Litvinchuk [74, Section 10.2, Lemma 2]). Cela veut dire que la solution construite ci-dessus coïncide avec la fonction ψ_1 .

Dans le chapitre 2 on explique que la méthode ci-dessus peut être vue comme une variation de l'approche par invariant de Tutte, initialement introduite par Tutte pour résoudre une équation fonctionnelle intervenant dans l'énumération de graphes colorés, voir Tutte [96].

Dans le cas des réflexions orthogonales, cette méthode amène au principal résultat du chapitre 2, voir le Théorème 61, que nous exposons ici :

Théorème 52 (Formule explicite dans le cas orthogonal). *On suppose que R est la matrice identité. La transformée de Laplace φ_1 vaut*

$$\varphi_1(\theta_2) = \frac{-\mu_1 w'(0)}{w(\theta_2) - w(0)} \theta_2.$$

1.3.3.2 Nature algébrique de la transformée de Laplace

Dans le chapitre 2 nous montrons de plus que la nature algébrique de la transformée de Laplace est liée à la finitude du groupe du processus introduit dans la section 1.3.2.3. Grâce à la formule (1.28), pour β défini dans (1.27) on a :

Lemme 53 (Finitude du groupe). *Le groupe $\langle \zeta, \eta \rangle$ est fini si et seulement si $\pi/\beta \in \mathbb{Q}$.*

On a alors la proposition suivante :

Théorème 54 (Algébricité). *La solution φ_1 du Théorème 52 et la fonction de collage conforme w définie dans (1.30) sont algébriques si et seulement si le groupe $\langle \zeta, \eta \rangle$ est fini.*

Un tel résultat existe aussi pour les marches aléatoires, voir le théorème 135 de l'annexe A et les articles de Bousquet-Mélou et Mishna [10], Bernardi *et al.* [6].

1.3.3.3 Problème frontière dans le cas général

Nous allons maintenant établir et donner la solution du problème frontière dans le cas général (de réflexions arbitraires pas nécessairement orthogonales) qui est le résultat principal du chapitre 3. Définissons pour $\theta_2 \in \mathcal{R}$

$$G(\theta_2) = \frac{\gamma_1}{\gamma_2}(\Theta_1^-(\theta_2), \theta_2) \frac{\gamma_2}{\gamma_1}(\Theta_1^-(\theta_2), \bar{\theta}_2). \quad (1.36)$$

On obtient le résultat qui suit de la même manière que le lemme 51, voir la proposition 81 du chapitre 3 pour un résultat plus détaillé.

Lemme 55 (Problème frontière dans le cas général). *La fonction φ_1 satisfait le problème frontière suivant :*

- (i) φ_1 est méromorphe sur $\mathcal{G}_{\mathcal{R}}$ avec au plus un pôle p d'ordre 1 et est bornée à l'infini ;
- (ii) φ_1 est continue sur $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{p\}$ et

$$\varphi_1(\bar{\theta}_2) = G(\theta_2)\varphi_1(\theta_2), \quad \forall \theta_2 \in \mathcal{R}. \quad (1.37)$$

A cause de la présence de la fonction $G \neq 1$ dans (1.37), le problème frontière (encore homogène avec *shift*) est plus complexe que celui rencontré dans le lemme 51 et ne peut pas être résolu uniquement grâce à un lemme des invariants. A la place, sa résolution est bien plus technique et fait appel à la théorie générale des problèmes frontière, voir l'annexe C. Les solutions s'expriment en terme d'intégrale de Cauchy et de la fonction de collage w de la section 1.3.2.6. Lorsque la dérive est négative, $\mu_1 < 0$ et $\mu_2 < 0$, le théorème 75 du chapitre 3 donne alors le résultat qui suit.

Théorème 56 (Formule explicite de la transformée de Laplace). *Pour tout $\theta_2 \in \mathcal{G}_{\mathcal{R}}$ la transformée de Laplace φ_1 est égale à*

$$\varphi_1(\theta_2) = \nu_1(\mathbb{R}_+) \left(\frac{w(0) - w(p)}{w(\theta_2) - w(p)} \right)^{-\chi} \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(\theta) \left[\frac{w'(\theta)}{w(\theta) - w(\theta_2)} - \frac{w'(\theta)}{w(\theta) - w(0)} \right] d\theta \right\}, \quad (1.38)$$

où p est l'éventuel pôle de φ_1 , χ égal à 0 ou -1 selon la valeur des paramètres, et \mathcal{R}^- est la moitié de partie imaginaire négative de l'hyperbole \mathcal{R} .

On consultera le théorème 75 pour une version plus détaillée du résultat. De même, il existe une formule analogue pour φ_2 et l'équation fonctionnelle (1.23) permet donc de donner une formule explicite pour la transformée de Laplace bi-variée φ . En inversant ces transformées de Laplace on obtient les mesures invariantes.

Remarque 57 (*Skew symétrie et nature algébrique*). Cette formule explicite permet, dans la section 3.5 du chapitre 3, de redémontrer des résultats classiques, voir Harrison et Williams [56], sur la *skew symétrie* qui caractérise les cas où la mesure invariante a une forme produit. Cela nous permet aussi d'énoncer quelques résultats en lien avec la nature algébrique des transformées de Laplace et avec le résultat de Dieker et Moriarty [31] qui caractérise les cas où la densité stationnaire est une somme d'exponentielles.

Remarque 58 (*Asymptotique*). Dans la section 3.4 du chapitre 3 nous montrons que cette formule explicite permet de retrouver des résultats asymptotiques pour les mesures invariantes sur les bords obtenus par Dai et Miyazawa [29] de la forme $\lim_{x \rightarrow \infty} \frac{\nu_1(x)}{x^\kappa e^{-\tau_2 x}} = b$, où $\kappa \in \{-\frac{3}{2}, -\frac{1}{2}, 0, 1\}$, τ_2 dépend des paramètres et b est une constante positive.

1.3.4 Asymptotique, singularités et méthode du point col

1.3.4.1 Aperçu

On suppose que les conditions (1.15) et (1.16) sont vérifiées et donc que le processus est récurrent et admet une unique distribution stationnaire (ou mesure invariante). Soit Z_∞ un vecteur aléatoire qui a la distribution stationnaire Π du mouvement Brownien réfléchi. Dai et Miyazawa [28] ont obtenu le résultat asymptotique suivant : pour $c \in \mathbb{R}_+^2$,

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}[\langle c | Z_\infty \rangle \geq x]}{f_c(x)} = 1, \quad \text{avec} \quad f_c(x) = bx^{\kappa_c} e^{-\alpha_c x}$$

où les constantes α_c et κ_c peuvent être calculées explicitement en fonction de (Σ, μ, R) , $\kappa_c \in \{-\frac{3}{2}, -\frac{1}{2}, 0, 1\}$ et b est une constante non nulle. Cela représente la limite de la mesure invariante de l'ensemble hachuré sur la figure 1.8 lorsque x tend vers l'infini.

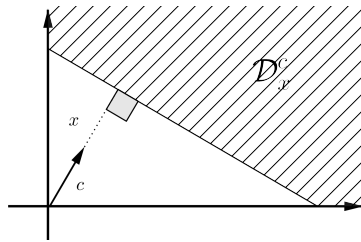


FIGURE 1.8 : Le domaine hachuré \mathcal{D}_x^c vérifie $\Pi(\mathcal{D}_x^c) \underset{x \rightarrow \infty}{\sim} f_c(x)$

Les preuves de ce type de résultats sont la plupart du temps basées sur des prolongements analytiques des transformées de Laplace et sur l'étude de leurs singularités. Cela permet d'appliquer des lemmes de transfert qui déduisent le comportement asymptotique d'une fonction, en fonction du type des singularités de sa transformée de Laplace. La proposition suivante donne une idée du type de lemme utilisé, on se réfère au livre de Doetsch [32].

Lemme 59 (Lemme de transfert). *Soit \tilde{f} la transformée de Laplace d'une fonction f . Soit g un prolongement analytique de \tilde{f} tel que*

$$g(z) - \frac{c_0}{(\alpha_0 - z)^k}$$

est analytique pour $\Re z < \alpha_1$ avec $\alpha_1 > \alpha_0$. Alors, sous des conditions techniques sur g énoncées dans [28, Annexe C], on a :

$$f(x) \underset{x \rightarrow \infty}{\sim} \frac{c_0}{\Gamma(k)} x^{k-1} e^{-\alpha_0 x}.$$

On se référera au livre de Doetsch [32, Théorème 35.1 et 37.1] et à Dai et Miyazawa [28, Lemme C.1 et C.2] pour une formulation détaillée de cette proposition.

Dans le chapitre 4 issu de l'article [44] nous résolvons un problème plus ardu soulevé dans l'article de Dai et Miyazawa [28, §8], à savoir le calcul de l'asymptotique de $\mathbb{P}[Z_\infty \in xc + B]$ lorsque $x \rightarrow \infty$, où $c \in \mathbb{R}_+^2$ est un vecteur directionnel et $B \subset \mathbb{R}_+^2$ un ensemble compact quelconque. De plus, nous sommes en mesure de trouver le développement asymptotique complet de la densité $\pi(x_1, x_2)$ de Π lorsque $x_1, x_2 \rightarrow \infty$ et $x_2/x_1 \rightarrow \tan(\alpha)$, pour tout angle $\alpha \in (0, \pi/2)$ donné. Pour obtenir ce résultat plus fin nous prolongeons les transformées de Laplace sur la surface de Riemann associée au noyau et nous appliquons ensuite des théorèmes de transfert et la méthode du point col.

1.3.4.2 Résultat principal

Nous devons tout d'abord introduire certaines notations. L'équation $\gamma(\theta) = 0$ détermine une ellipse \mathcal{E} sur \mathbb{R}^2 passant par l'origine, voir la Figure 1.9. On se restreint ici au cas de la dérive négative, $\mu_1 < 0$ et $\mu_2 < 0$, bien que notre méthode s'applique aussi aux autres cas sans difficultés supplémentaires.

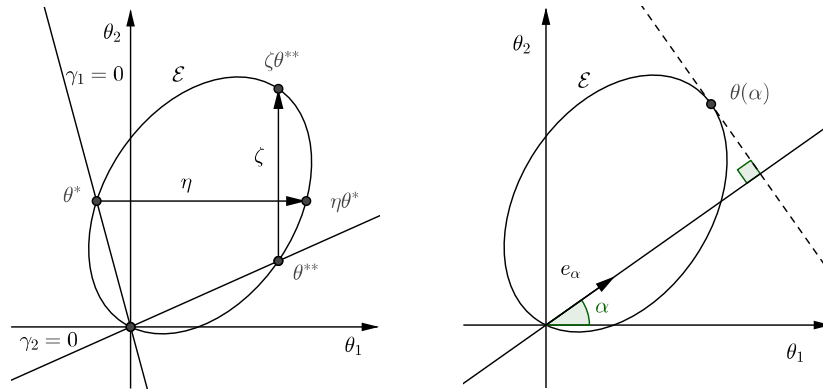


FIGURE 1.9 : Gauche : représentation de l'ellipse \mathcal{E} , des droites $\{\gamma_1(\theta) = 0\}$, $\{\gamma_2(\theta) = 0\}$, et des points θ^* , θ^{**} , $\eta\theta^*$ et $\zeta\theta^{**}$. Droite : interprétation géométrique du point $\theta(\alpha)$ sur \mathcal{E}

Ensuite, considérons les droites $\{\gamma_1(\theta) = 0\}$ et $\{\gamma_2(\theta) = 0\}$, dépendant uniquement de la matrice de réflexion R . Elles coupent l'ellipse \mathcal{E} à l'origine. La droite $\{\gamma_1(\theta) = 0\}$ (resp. $\{\gamma_2(\theta) = 0\}$) coupe l'ellipse à un deuxième point appelé θ^* (resp. θ^{**}). Pour énoncer nos résultats, on a besoin de définir les images sur \mathcal{E} de ces points via les automorphismes ζ et η , introduits dans la section 1.3.2.3. C'est-à-dire que pour le point $\theta^* = (\theta_1^*, \theta_2^*) \in \mathcal{E}$ il existe un unique point $\eta\theta^* = (\eta\theta_1^*, \theta_2^*) \in \mathcal{E}$ possédant la même deuxième coordonnée. De la même manière, il existe un unique point $\zeta\theta^{**} = (\theta_1^{**}, \zeta\theta_2^{**}) \in \mathcal{E}$ possédant la même première coordonnée que $\theta^{**} = (\theta_1^{**}, \theta_2^{**}) \in \mathcal{E}$. Les points θ^* , θ^{**} , $\eta\theta^*$ et $\zeta\theta^{**}$ sont dessinés sur la figure 1.9, les deux derniers s'avèreront être des pôles des transformées de Laplace. Leurs coordonnées peuvent être calculées explicitement.

Enfin, pour un angle donné $\alpha \in [0, \pi/2]$, définissons le point $\theta(\alpha)$ sur l'ellipse \mathcal{E} par

$$\theta(\alpha) = \operatorname{argmax}_{\theta \in \mathcal{E}} \langle \theta | e_\alpha \rangle, \quad \text{où } e_\alpha = (\cos \alpha, \sin \alpha). \quad (1.39)$$

Les coordonnées de $\theta(\alpha)$ peuvent être données explicitement. On peut construire $\theta(\alpha)$ géométriquement comme on le voit sur la figure 1.9. Ce point s'avèrera être le point col que nous utiliserons dans la démonstration du théorème ci-dessous.

On introduit les ensembles de paramètres

$$\mathcal{Q}_{--} = \{((\Sigma, \mu, R), \alpha) : \gamma_1(\eta\theta(\alpha)) < 0 \text{ and } \gamma_2(\zeta\theta(\alpha)) < 0\}$$

et \mathcal{Q}_{+-} , \mathcal{Q}_{-+} et \mathcal{Q}_{++} de manière similaire.

Le résultat suivant est une reformulation du théorème 102 du chapitre 4. Il fournit le terme principal du développement asymptotique de $\pi(r \cos \alpha, r \sin \alpha)$ lorsque $r \rightarrow \infty$ et $\alpha \rightarrow \alpha_0$, voir la figure 1.10. Un résultat identique existe dans le cas discret, voir le théorème 132 de l'annexe A.

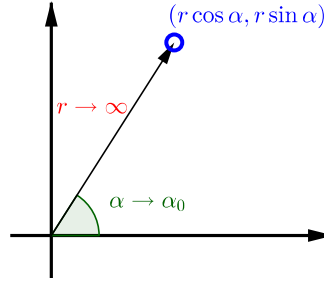


FIGURE 1.10 : Illustration de la direction asymptotique du théorème 60

Théorème 60 (Asymptotique de la mesure invariante). Soit $(x, y) = (r \cos \alpha, r \sin \alpha)$ avec $\alpha_0 \in (0, \pi/2)$. On suppose que $\theta(\alpha) \in \mathbb{R}_+^2$. Alors lorsque $r \rightarrow \infty$ et $\alpha \rightarrow \alpha_0$ on a

$$\pi(re_\alpha) = (1 + o(1)) \cdot \begin{cases} \frac{C_0}{\sqrt{r}} e^{-r\langle e_\alpha | \theta(\alpha) \rangle} & \text{in } \mathcal{Q}_{--}, \\ C_1 e^{-r\langle e_\alpha | \eta\theta^* \rangle} & \text{in } \mathcal{Q}_{+-}, \\ C_2 e^{-r\langle e_\alpha | \zeta\theta^{**} \rangle} & \text{in } \mathcal{Q}_{-+}, \\ C_1 e^{-r\langle e_\alpha | \eta\theta^* \rangle} + C_2 e^{-r\langle e_\alpha | \zeta\theta^{**} \rangle} & \text{in } \mathcal{Q}_{++}, \end{cases} \quad (1.40)$$

où C_0 , C_1 et C_2 sont des constantes qui peuvent s'exprimer en fonction de φ_1 et φ_2 et des paramètres.

Les exposants dans les intégrales sont issus soit des pôles $\eta\theta^*$ et $\zeta\theta^{**}$ soit du point col $\theta(\alpha)$. Dans le chapitre 4 les constantes mentionnées dans le théorème 60 sont déterminées en fonction de φ_1 et φ_2 , obtenues dans les chapitres précédents.

Étapes clés de la preuve. La première étape consiste à prolonger méromorphiquement φ_1 sur $\mathbb{C} \setminus [\theta_2^+, \infty)$ ou sur la surface de Riemann \mathcal{S} , idem pour φ_2 , voir la Section 1.3.2.4. Alors par l'équation fonctionnelle (1.23) et la formule d'inversion des transformées de Laplace (on se réfère à Doetsch [32] et Brychkov *et al.* [13]), la densité $\pi(x_1, x_2)$ peut s'exprimer comme une intégrale double. En utilisant des techniques standards d'analyse complexe comme le théorème des résidus, on se réduit au cas d'une intégrale simple. On obtient :

$$\begin{aligned} \pi(x_1, x_2) &= \frac{-1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} e^{-x_1\theta_1 - x_2\theta_2} \frac{\gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1)}{\gamma(\theta)} d\theta_1 d\theta_2 \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi_2(\theta_1) \gamma_2(\theta_1, \Theta_2^+(\theta_1)) e^{-x_1\theta_1 - x_2\Theta_2^+(\theta_1)} \frac{d\theta_1}{\sqrt{\tilde{d}(\theta_1)}} \\ &\quad + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi_1(\theta_2) \gamma_1(\Theta_1^+(\theta_2), \theta_2) e^{-x_1\Theta_1^+(\theta_2) - x_2\theta_2} \frac{d\theta_2}{\sqrt{\tilde{d}(\theta_2)}}. \end{aligned}$$

Ces intégrales sont typiques pour appliquer la méthode du point col, voir Fedoryuk [41].

Les coordonnées du point col sont les points critiques des fonctions dans les exponentielles,

$$\cos(\alpha)\theta_1 + \sin(\alpha)\Theta_2^+(\theta_1) \quad \text{et} \quad \cos(\alpha)\Theta_1^+(\theta_2) + \sin(\alpha)\theta_2.$$

Le point col est en fait le point $\theta(\alpha)$ défini dans l'équation (1.39). On doit alors déformer le contour d'intégration jusqu'à un nouveau contour qui coïncide avec le contour de plus grande pente au voisinage du point col. Lorsqu'on déforme ce contour on doit prendre

en compte les pôles des intégrandes ci-dessus, qui s'avèrent être les points $\eta\theta^*$ et $\zeta\theta^{**}$, et leurs résidus. Le terme principal de l'asymptotique est alors déterminé par le pôle si le contour croise un pôle lorsqu'il est déformé et par le point col sinon. On peut aussi par cette méthode obtenir tout le développement asymptotique en prenant en compte les pôles et le développement asymptotique complet donné par le point col, voir la section 4.5.1 du chapitre 4. \square

Nous avons ici explicité l'asymptotique pour chaque direction α donnée en fonction des paramètres. Il est possible de présenter ce théorème sous une autre forme, en fixant les paramètres (Σ, μ, R) et en faisant varier l'angle α . La figure 1.11 permet de visualiser cette approche, on constate que sept cas sont possibles. On se référera à la section 4.5.2 du chapitre 4 pour plus de détails.

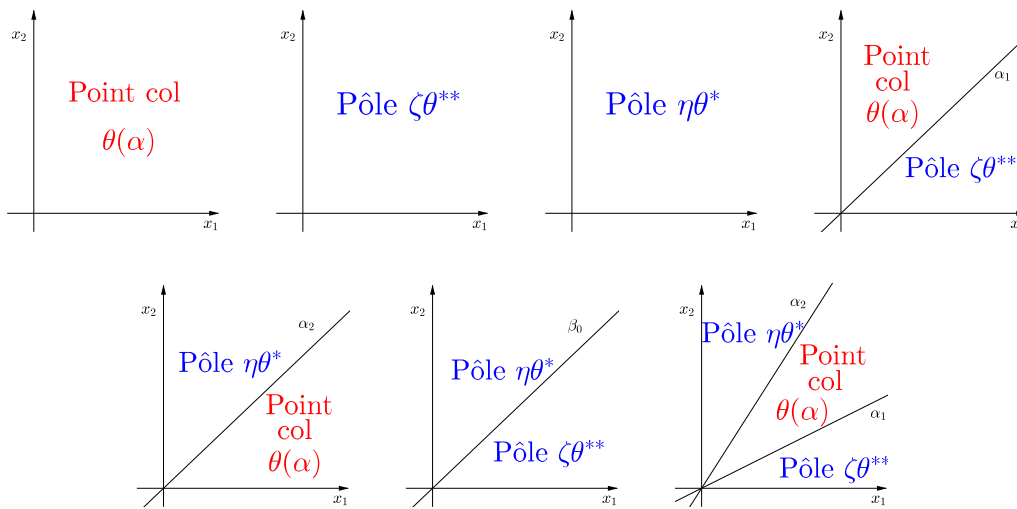


FIGURE 1.11 : Nature de l'asymptotique (pôle ou point col) en fonction de la direction α ; les différents cas dépendent des paramètres

1.3.5 Fonctions de Green et perspectives

Après s'être intéressé au cas récurrent et aux mesures invariantes dans la section précédente, nous passons en revue le cas transiet et les fonctions de Green. Cette question n'a encore jamais été étudiée dans la littérature à notre connaissance. En revanche, dans le cadre discret les fonctions de Green ont été étudiées grâce à la méthode analytique, par exemple dans les articles de Kourkova et Malyshev [66] dans le cas réfléchi et Kourkova et Raschel [64] dans le cas absorbé aux bords.

Nous avons montré dans cette introduction la proposition 48 qui donne une équation fonctionnelle, similaire à celle du cas récurrent, reliant les transformées de Laplace des fonctions de Green. Pour rappel, lorsque $\psi^{z_0}(\theta) < \infty$, $\psi_1^{z_0}(\theta_1) < \infty$ et $\psi_2^{z_0}(\theta_2) < \infty$ on a pour tout θ_1 et $\theta_2 < 0$ l'équation fonctionnelle suivante,

$$-\gamma(\theta)\psi^{z_0}(\theta) = \gamma_1(\theta)\psi_1^{z_0}(\theta_2) + \gamma_2(\theta)\psi_2^{z_0}(\theta_1) + e^{\theta \cdot z_0}$$

où γ , γ_1 et γ_2 sont définis dans la proposition 46 et ψ^{z_0} , $\psi_1^{z_0}$ et $\psi_2^{z_0}$ sont les transformées de Laplace des fonctions de Green dans le quart de plan et sur les frontières, voir la définition 47.

Il s'agit alors de savoir si, tout comme pour la mesure invariante, il est possible d'exploiter cette équation grâce à la méthode analytique que nous avons présentée et d'en déduire l'asymptotique de la fonction de Green G^{z_0} et une formule explicite pour sa transformée de Laplace ψ^{z_0} . Ce travail est actuellement en cours et n'en est qu'à ses débuts. Nous nous contentons ici de faire deux remarques sur les différences et les difficultés supplémentaires que ce problème soulève :

- les fonctions de Green ne sont pas des mesures finies sur \mathbb{R}_+^2 contrairement aux mesures invariantes ; il n'y a donc pas, a priori, de domaine initial naturel de définition des transformées de Laplace ; l'étape de prolongement sur la surface de Riemann s'avère être ainsi plus complexe ;
- l'équation fonctionnelle possède un terme inhomogène (comme dans le cas discret, voir l'annexe A), à savoir $e^{\theta \cdot z_0}$; ce terme conduit à un problème frontière inhomogène qui est plus complexe à résoudre, voir l'annexe C.

Rappelons que G est défini dans l'équation (1.36) et définissons g tel que

$$g(\theta_2) = \frac{e^{(\Theta_1^-(\theta_2), \theta_2) \cdot z_0}}{\gamma_2(\Theta_1^-(\theta_2), \theta_2)} - \frac{e^{(\Theta_1^-(\theta_2), \bar{\theta}_2) \cdot z_0}}{\gamma_2(\Theta_1^-(\theta_2), \bar{\theta}_2)}.$$

Nous faisons alors deux conjectures.

Conjecture 1 (Problème frontière pour les fonctions de Green). La fonction $\psi_1^{z_0}$ satisfait un problème frontière du type :

- $\psi_1^{z_0}$ est méromorphe sur $\mathcal{G}_{\mathcal{R}}$ et continue sur $\overline{\mathcal{G}_{\mathcal{R}}}$;
- $\psi_1^{z_0}$ vérifie la condition frontière

$$\psi_1^{z_0}(\bar{\theta}_2) = G(\theta_2)\psi_1^{z_0}(\theta_2) + g(\theta_2), \quad \forall \theta_2 \in \mathcal{R}.$$

Conjecture 2 (Formule explicite pour les fonctions de Green). La transformée de Laplace $\psi_1^{z_0}$ peut s'écrire de la forme suivante :

$$\psi_1^{z_0}(\theta_2) = \frac{X(\theta_2)}{2i\pi} \int_{\mathcal{R}^-} \frac{w'(t)g(t)}{X^+(t)(w(t) - w(\theta_2))} dt, \quad \forall \theta_2 \in \mathcal{G}_{\mathcal{R}}$$

où X est une solution du problème frontière homogène du type de celle de la formule (1.38) et X^+ est la valeur limite sur \mathcal{R} de la fonction X définie sur $\mathcal{G}_{\mathcal{R}}$.

Après la lecture de cette introduction le lecteur pourra lire l'annexe A. Elle reprend point par point les différentes étapes de la méthode analytique dans le cas des marches aléatoires dans le quadrant. Cette annexe permet ainsi de mettre en parallèle le cadre continu avec le cadre discret qui a inspiré cette thèse.

Le lecteur pourra ensuite lire l'annexe B qui illustre comment cette méthode analytique peut aussi servir à résoudre d'autres problèmes comme déterminer les fonctions harmoniques dans le quart de plan avec condition de Dirichlet.

Tutte's invariant approach for Brownian motion reflected in the quadrant

Ce chapitre est issu de l'article [46] co-écrit avec Kilian Raschel et publié en 2017 dans *ESAIM Probability and Statistics*.

Abstract

We consider a Brownian motion with drift in the quarter plane with orthogonal reflection on the axes. The Laplace transform of its stationary distribution satisfies a functional equation, which is reminiscent from equations arising in the enumeration of (discrete) quadrant walks. We develop a Tutte's invariant approach to this continuous setting, and we obtain an explicit formula for the Laplace transform in terms of generalized Chebyshev polynomials.

Résumé

On considère un mouvement Brownien avec dérive dans le quart de plan avec réflexion orthogonale sur les axes. La transformée de Laplace de sa distribution stationnaire satisfait une équation fonctionnelle, qui rappelle les équations apparaissant dans l'énumération des marches (discrètes) dans le quadrant. On développe une approche par les invariants de Tutte dans ce cadre continu, et on obtient une formule explicite pour la transformée de Laplace en terme de polynômes de Tchebychev généralisés.

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2.1 Introduction and main results

2.1.1 Reflected Brownian motion in the quadrant

The object of study here is the reflected Brownian motion with drift in the quarter plane

$$Z(t) = Z_0 + W(t) + \mu t + RL(t), \quad \forall t \geq 0, \quad (2.1)$$

associated to the triplet (Σ, μ, R) , composed of a non-singular covariance matrix, a drift and a reflection matrix, see Figure 2.1:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}.$$

In Equation (2.1), Z_0 is any initial point in \mathbb{R}_+^2 , the process $(W(t))_{t \geq 0}$ is an unconstrained planar Brownian motion with covariance matrix Σ starting from the origin, and for $i = 1, 2$, $L^i(t)$ is a continuous non-decreasing process, that increases only at time t such that $Z^i(t) = 0$, namely $\int_0^t \mathbf{1}_{\{Z^i(s) \neq 0\}} dL^i(s) = 0$, for all $t \geq 0$. The columns R^1 and R^2 represent the directions in which the Brownian motion is pushed when the axes are reached.

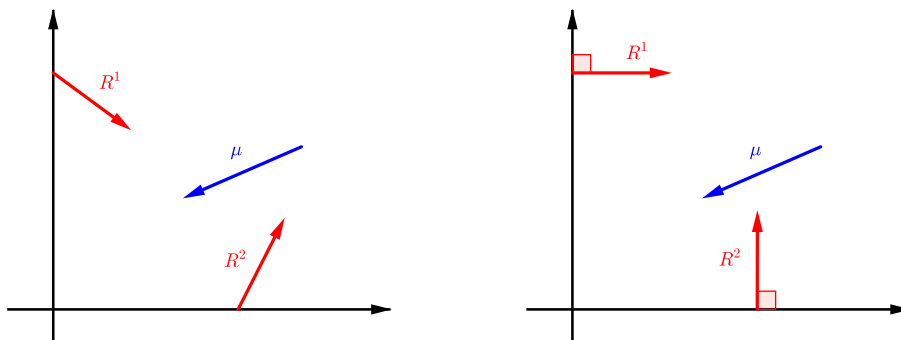


Figure 2.1: Drift μ and reflection vectors R^1 and R^2 in non-orthogonal and orthogonal cases

The reflected Brownian motion $(Z(t))_{t \geq 0}$ associated with (Σ, μ, R) is well defined [56, 101], and is a fundamental stochastic process, from many respects. There is a large literature on reflected Brownian motion in the quadrant (and also in orthants, generalization to higher dimension of the quadrant). First, it serves as an approximation of large queuing networks [42, 5]; this was the initial motivation for its study. In the same vein, it is the continuous counterpart of (random) walks in the quarter plane, which are an important combinatorial and probabilistic object, see [10, 6]. In other directions, it is studied for its Lyapunov functions [35], cone points of Brownian motion [69], intertwining relations and crossing probabilities [34], and of particular interest for us, for its recurrence or transience [57]. Its stationary distribution exists and is unique if and only if the following (geometric) conditions are satisfied:

$$r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0, \quad r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0. \quad (2.2)$$

(For orthogonal reflections, (2.2) is equivalent for the drift μ to have two negative coordinates.) Moreover, the asymptotics of the stationary distribution (when it exists) is now well known, see [52, 28, 44].

There exist, however, very few results giving an exact expression for the stationary distribution, and the main contribution of this paper is precisely to propose a method (based on boundary value problems) for deriving an explicit formula for the (Laplace transform of the) stationary distribution. Our study constitutes one of the first attempts to apply these techniques to reflected Brownian motion, after [43] (under the strong symmetry condition $\mu_1 = \mu_2$, $\sigma_{11} = \sigma_{22}$ and symmetric reflection vectors R^1, R^2 in (2.1)), [42] (with the identity covariance matrix Σ) and [5] (with a diffusion having a quite special behavior on the boundary). We also refer to [14] for the analysis of reflected Brownian motion in bounded domains by complex analysis techniques.

2.1.2 Laplace transform of the stationary distribution

Under assumption (2.2), that we shall do throughout the manuscript, the stationary distribution is absolutely continuous w.r.t. the Lebesgue measure, see [56, 22]. We denote its density by $\pi(x) = \pi(x_1, x_2)$. Let the moment generating function (or Laplace transform) of π be defined by

$$\varphi(\theta) = \mathbb{E}_\pi[e^{\langle \theta | Z \rangle}] = \iint_{\mathbb{R}_+^2} e^{\langle \theta | x \rangle} \pi(x) dx.$$

The above integral converges at least for $\theta = (\theta_1, \theta_2) \in \mathbb{C}^2$ such that $\Re \theta_1 \leq 0$ and $\Re \theta_2 \leq 0$. We further define two finite boundary measures ν_1 and ν_2 with support on the axes, by mean of the formula

$$\nu_i(B) = \mathbb{E}_\pi \left[\int_0^1 \mathbf{1}_{\{Z(t) \in B\}} dL^i(t) \right].$$

The measure ν_i is continuous w.r.t. the Lebesgue measure [56], and may be viewed as the boundary invariant measure. We define the moment generating function of ν_1 and ν_2 by

$$\varphi_2(\theta_1) = \int_{\mathbb{R}_+} e^{\theta_1 x_1} \nu_2(x_1) dx_1, \quad \varphi_1(\theta_2) = \int_{\mathbb{R}_+} e^{\theta_2 x_2} \nu_1(x_2) dx_2.$$

The functions φ_1 and φ_2 exist *a priori* for values of the argument with non-positive real parts. There is a functional equation between the Laplace transforms φ , φ_1 and φ_2 , see (2.5) in Section 2.2, which is reminiscent of the functional equation counting (discrete) quadrant walks [10, 6].

2.1.3 Main result

We derive an explicit expression for φ_1 , and therefore also for φ_2 and φ by the functional equation (2.5), in the particular case where R is the identity matrix, which means that the reflections are orthogonal (Figure 2.1, right). Define the generalized Chebyshev polynomial by (for $a \geq 0$)

$$T_a(x) = \cos(a \arccos(x)) = \frac{1}{2} \left\{ (x + \sqrt{x^2 - 1})^a + (x - \sqrt{x^2 - 1})^a \right\}.$$

It admits an analytic continuation on $\mathbb{C} \setminus (-\infty, -1)$, and even on \mathbb{C} if a is a non-negative integer. We also need to introduce

$$\theta_2^\pm = \frac{(\mu_1 \sigma_{12} - \mu_2 \sigma_{11}) \pm \sqrt{(\mu_1 \sigma_{12} - \mu_2 \sigma_{11})^2 + \mu_1^2 \det \Sigma}}{\det \Sigma} \quad (2.3)$$

(notice that the sign of θ_2^\pm is \pm , see Figure 2.2), as well as the angle (related to the correlation coefficient of the Brownian motion $(W(t))_{t \geq 0}$)

$$\beta = \arccos \left(-\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} \right). \quad (2.4)$$

Theorem 61. *Let R be the identity matrix in (2.1). The Laplace transform φ_1 is equal to*

$$\varphi_1(\theta_2) = \frac{-\mu_1 w'(0)}{w(\theta_2) - w(0)} \theta_2,$$

where the function w can be expressed in terms of the generalized Chebyshev polynomial $T_{\frac{\pi}{\beta}}$ as follows:

$$w(\theta_2) = T_{\frac{\pi}{\beta}} \left(-\frac{2\theta_2 - (\theta_2^+ + \theta_2^-)}{\theta_2^+ - \theta_2^-} \right).$$

Accordingly, φ_1 can be continued meromorphically on the cut plane $\mathbb{C} \setminus (\theta_2^+, \infty)$.

There exists, of course, an analogous expression for $\varphi_2(\theta_1)$, and the functional equation (2.5) finally gives a simple explicit formula for the bivariate Laplace transform φ .

Let us now give some comments around Theorem 61.

- It connects two *a priori* unrelated objects: the stationary distribution of reflected Brownian motion in the quadrant and a particular special function, viz, a generalized Chebyshev polynomial (which is a hypergeometric function). The expression that we obtain is quite tractable: as an example we will recover the well-known case of one-dimensional reflected Brownian motion (Section 2.3.5).
- To prove Theorem 61, we apply a constructive (and combinatorial in nature) variation of the boundary value method of [37], recently introduced in [6] as Tutte's invariant approach [96], see our Section 2.3. This paper is one of the first attempts to apply boundary value techniques to (continuous) diffusions in the quadrant, after [42] (which concerns very particular cases of the covariance matrix, essentially the identity matrix) and [5] (on diffusions with completely different behavior on the boundary).
- The recent paper [44] obtains the exact asymptotic behavior of the stationary distribution along any direction in the quadrant. The constant in that asymptotics involves both functions φ_1 and φ_2 (see [44]), and can thus be made explicit with our Theorem 61.
- Theorem 61 is well suited for asymptotic analysis: we shall derive the asymptotics of the stationary distribution along the axes, by using classical arguments from singularity analysis of Laplace transforms (see the reference book [32]). See our Section 2.4.
- Theorem 61 implies that the Laplace transform is algebraic if and only if a certain group (to be properly introduced later on) is finite, see Section 2.5.3. This result has an analogue in the discrete setting, see [10]. In the same vein, the authors of [31] give necessary and sufficient conditions for the stationary density of two-dimensional reflected Brownian motion with negative drift in a wedge to have the form of a sum of exponentials. The intersection with our results is the polynomial case $\frac{\pi}{\beta} \in \mathbb{Z}$.

- More than the Brownian motion in the quadrant, all results presented here (including Theorem 61) concern the Brownian motion in two-dimensional cones (by a simple linear transformation of the cones). This is a major and interesting difference of the continuous case in comparison with the discrete case, which also illustrates that the analytic approach is very well suited to that context.

Though being self-contained (this is one of the reasons why we focus on the case of orthogonal reflections), this chapter based on [46] is part of a larger project, dealing with any reflection matrix R (as in Figure 2.1, left), see Chapter 3. Let us finally mention that an extended abstract of [46] and of [44] may be found in [45].

2.2 Analytic preliminaries and continuation of the Laplace transforms

In this section we state the key functional equation (a kernel equation, see Section 2.2.1), which is the starting point of our entire analysis. We study the kernel (a second degree polynomial in two variables) in Section 2.2.2. Finally, we continue the Laplace transforms to larger domains (Section 2.2.3), which will be used in Section 2.3 to state a boundary value problem (BVP).

2.2.1 A kernel functional equation

We have the following key functional equation between the Laplace transforms:

$$-\gamma(\theta)\varphi(\theta) = \gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1), \quad (2.5)$$

where

$$\begin{cases} \gamma(\theta) = \frac{1}{2}\langle\theta|\sigma\theta\rangle + \langle\theta|\mu\rangle = \frac{1}{2}(\sigma_{11}\theta_1^2 + \sigma_{22}\theta_2^2 + 2\sigma_{12}\theta_1\theta_2) + \mu_1\theta_1 + \mu_2\theta_2, \\ \gamma_1(\theta) = \langle R^1|\theta\rangle = r_{11}\theta_1 + r_{21}\theta_2, \\ \gamma_2(\theta) = \langle R^2|\theta\rangle = r_{12}\theta_1 + r_{22}\theta_2. \end{cases}$$

By definition of the Laplace transforms, this equation holds at least for any $\theta = (\theta_1, \theta_2)$ with $\Re\theta_1 \leq 0$ and $\Re\theta_2 \leq 0$.

To prove (2.5) the main idea is to use an identity called basic adjoint relationship (first proved in [55] in some particular cases, then extended in [27]), which characterizes the stationary distribution, see [26, 24, 23]. (It is the continuous analogue of the equation $\pi Q = 0$, with π the stationary distribution of a recurrent continuous-time Markov chain having infinitesimal generator matrix Q .) This basic adjoint relationship connects the stationary distribution π and the corresponding boundary measures ν_1 and ν_2 . We refer to [42, 27, 28] for the details.

As stated in [23, Open problem 1], it is an open problem to prove that there is no finite *signed* measure which satisfies the basic adjoint relationship (by signed we mean a measure taking positive and negative values). Going back to our case, we shall see that the functional equation (2.5) admits a unique solution (the stationary distribution); in particular there is no solution of the functional equation with sign changes.

2.2.2 Kernel

By definition, the kernel of Equation (2.5) is the polynomial

$$\gamma(\theta_1, \theta_2) = \frac{1}{2}(\sigma_{11}\theta_1^2 + 2\sigma_{12}\theta_1\theta_2 + \sigma_{22}\theta_2^2) + \mu_1\theta_1 + \mu_2\theta_2.$$

It can be alternatively written as

$$\gamma(\theta_1, \theta_2) = \tilde{a}(\theta_2)\theta_1^2 + \tilde{b}(\theta_2)\theta_1 + \tilde{c}(\theta_2) = a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1),$$

where

$$\begin{cases} \tilde{a}(\theta_2) = \frac{1}{2}\sigma_{11}, & \tilde{b}(\theta_2) = \sigma_{12}\theta_2 + \mu_1, & \tilde{c}(\theta_2) = \frac{1}{2}\sigma_{22}\theta_2^2 + \mu_2\theta_2, \\ a(\theta_1) = \frac{1}{2}\sigma_{22}, & b(\theta_1) = \sigma_{12}\theta_1 + \mu_2, & c(\theta_1) = \frac{1}{2}\sigma_{11}\theta_1^2 + \mu_1\theta_1. \end{cases} \quad (2.6)$$

The equation $\gamma(\theta_1, \theta_2) = 0$ defines a two-valued algebraic function $\Theta_1(\theta_2)$ such that $\gamma(\Theta_1(\theta_2), \theta_2) = 0$, and similarly $\Theta_2(\theta_1)$ such that $\gamma(\theta_1, \Theta_2(\theta_1)) = 0$. Their expressions are given by

$$\Theta_1^\pm(\theta_2) = \frac{-\tilde{b}(\theta_2) \pm \sqrt{\tilde{d}(\theta_2)}}{2\tilde{a}(\theta_2)}, \quad \Theta_2^\pm(\theta_1) = \frac{-b(\theta_1) \pm \sqrt{d(\theta_1)}}{2a(\theta_1)},$$

where \tilde{d} and d are the discriminants of the kernel:

$$\begin{cases} \tilde{d}(\theta_2) = \theta_2^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2\theta_2(\mu_1\sigma_{12} - \mu_2\sigma_{11}) + \mu_1^2, \\ d(\theta_1) = \theta_1^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2\theta_1(\mu_2\sigma_{12} - \mu_1\sigma_{22}) + \mu_2^2. \end{cases}$$

The polynomials \tilde{d} and d have two zeros, real and of opposite signs, they are denoted by θ_2^\pm and θ_1^\pm , respectively: θ_2^\pm is introduced in (2.3) and

$$\theta_1^\pm = \frac{(\mu_2\sigma_{12} - \mu_1\sigma_{22}) \pm \sqrt{(\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \mu_2^2 \det \Sigma}}{\det \Sigma}.$$

Equivalently, θ_1^\pm and θ_2^\pm are the branch points of the algebraic functions Θ_2 and Θ_1 .

Finally, notice that d is positive on (θ_1^-, θ_1^+) and negative on $\mathbb{R} \setminus [\theta_1^-, \theta_1^+]$. Accordingly, the branches Θ_2^\pm take real and complex conjugate values on these sets, respectively. A similar statement holds for Θ_1^\pm .

2.2.3 Continuation of the Laplace transforms

In Section 2.3 we shall state a boundary condition for the functions φ_1 and φ_2 , on curves which lie outside their natural domains of definition (the half-plane with negative real-part). We therefore need to continue these functions, which is done in the result hereafter.

Lemma 62. *We can continue meromorphically $\varphi_1(\theta_2)$ to the open and simply connected set*

$$\{\theta_2 \in \mathbb{C} : \Re \theta_2 \leq 0 \text{ or } \Re \Theta_1^-(\theta_2) < 0\}, \quad (2.7)$$

by setting

$$\varphi_1(\theta_2) = \frac{\gamma_2}{\gamma_1}(\Theta_1^-(\theta_2), \theta_2)\varphi_2(\Theta_1^-(\theta_2)).$$

Lemma 62 is immediate (one can find a refined version of it in Chapter 4). A similar statement holds for the Laplace transform φ_2 . Anticipating slightly, we notice that the domain in (2.7) contains the domain $\mathcal{G}_{\mathcal{R}}$ in Figure 2.2.

2.3 Statement of the boundary value problem

2.3.1 An important hyperbola

For further use, we need to introduce the curve

$$\mathcal{R} = \{\theta_2 \in \mathbb{C} : \gamma(\theta_1, \theta_2) = 0 \text{ and } \theta_1 \in (-\infty, \theta_1^-)\} = \Theta_2^\pm((-\infty, \theta_1^-)). \quad (2.8)$$

By the results of Section 2.2.2, d is negative on the interval $(-\infty, \theta_1^-)$, and thus the curve \mathcal{R} is symmetrical w.r.t. the real axis, see Figure 2.2. Furthermore, it has a simple structure, as shown by the following elementary result, taken from [5, Lemma 9]:

Lemma 63. *The curve \mathcal{R} in (2.8) is a (branch of a) hyperbola, given by the equation*

$$\sigma_{22}(\sigma_{12}^2 - \sigma_{11}\sigma_{22})x^2 + \sigma_{12}^2\sigma_{22}y^2 - 2\sigma_{22}(\sigma_{11}\mu_2 - \sigma_{12}\mu_1)x = \mu_2(\sigma_{11}\mu_2 - 2\sigma_{12}\mu_1). \quad (2.9)$$

Proof. We saw in Section 2.2.2 that whenever $\theta_1 \in (-\infty, \theta_1^-)$, the branches $\Theta_2^+(\theta_1)$ and $\Theta_2^-(\theta_1)$ are complex conjugate. Define x and y by $\Theta_2^+(\theta_1) = x + iy$. Then from the identities

$$\Theta_2^+ + \Theta_2^- = -\frac{b(\theta_1)}{a(\theta_1)} = -2\frac{\sigma_{12}\theta_1 + \mu_2}{\sigma_{22}} = 2x \quad \text{and} \quad \Theta_2^+ \cdot \Theta_2^- = \frac{c(\theta_1)}{a(\theta_1)} = \frac{\sigma_{11}\theta_1^2 + 2\mu_1\theta_1}{\sigma_{22}} = x^2 + y^2,$$

we obtain that x and y are real solutions of (2.9). \square

We shall denote by $\mathcal{G}_{\mathcal{R}}$ the open domain of \mathbb{C} bounded by \mathcal{R} and containing 0, see Figure 2.3. Obviously $\overline{\mathcal{G}_{\mathcal{R}}}$, the closure of $\mathcal{G}_{\mathcal{R}}$, is equal to $\mathcal{G}_{\mathcal{R}} \cup \mathcal{R}$.

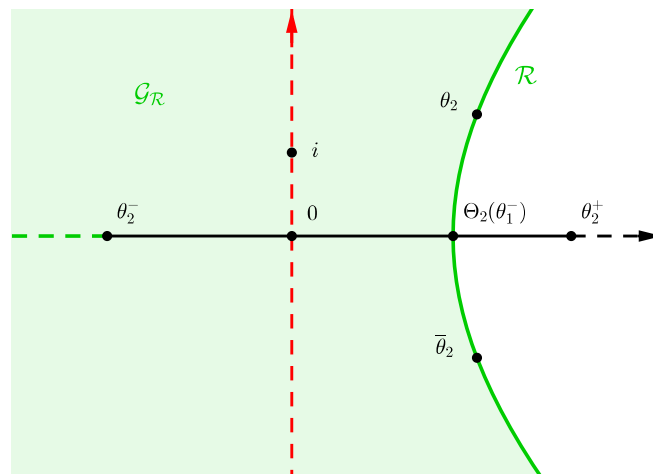


Figure 2.2: The curve \mathcal{R} in (2.8) is symmetric w.r.t. the horizontal axis, and $\mathcal{G}_{\mathcal{R}}$ is the domain in green

2.3.2 BVP for orthogonal reflections

In the case of orthogonal reflections, R is the identity matrix in (2.1), and we have $\gamma_1(\theta_1, \theta_2) = \theta_1$ and $\gamma_2(\theta_1, \theta_2) = \theta_2$. We set

$$\psi_1(\theta_2) = \frac{1}{\theta_2} \varphi_1(\theta_2), \quad \psi_2(\theta_1) = \frac{1}{\theta_1} \varphi_2(\theta_1). \quad (2.10)$$

Proposition 64. *The function ψ_1 satisfies the following BVP:*

- (i) ψ_1 is meromorphic on $\mathcal{G}_{\mathcal{R}}$ with a single pole at 0, of order 1 and residue $\varphi_1(0)$, and vanishes at infinity,
- (ii) ψ_1 is continuous on $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{0\}$ and

$$\psi_1(\overline{\theta_2}) = \psi_1(\theta_2), \quad \forall \theta_2 \in \mathcal{R}. \quad (2.11)$$

Proof. Using the formula (2.10), Point (i) is equivalent to proving that φ_1 is analytic in $\mathcal{G}_{\mathcal{R}}$ and is bounded at infinity. Both properties are obvious in the half-plane $\{\theta_2 \in \mathbb{C} : \Re \theta_2 < 0\}$, thanks to the definition of φ_1 as a Laplace transform. In the domain of $\mathcal{G}_{\mathcal{R}}$ where $\Re \theta_2 \geq 0$, one can use the continuation formula given in Lemma 62, and the conclusion follows.

Let us now prove (ii). Evaluating the (continued) functional equation (2.5) at $(\theta_1, \Theta_2^{\pm}(\theta_1))$, we obtain

$$\psi_1(\Theta_2^{\pm}(\theta_1)) + \psi_2(\theta_1) = 0,$$

which immediately implies that

$$\psi_1(\Theta_2^+(\theta_1)) = \psi_1(\Theta_2^-(\theta_1)). \quad (2.12)$$

Choosing $\theta_1 \in (-\infty, \theta_1^-)$, the two quantities $\Theta_2^+(\theta_1)$ and $\Theta_2^-(\theta_1)$ are complex conjugate the one of the other, see Section 2.2. Equation (2.12) can then be reformulated as (2.11), using the definition (2.8) of the curve \mathcal{R} . \square

2.3.3 Conformal gluing function and invariant theorem

The BVP stated in Proposition 64 is called a homogeneous BVP with shift (the shift stands here for the complex conjugation, but the theory applies to more general shifts, see [74]). Due to its particularly simple form, we can solve it in an explicit way, using the two following steps:

- Using a certain conformal mapping (to be introduced below), we can construct a particular solution to the BVP of Proposition 64.
- The solution to the BVP of Proposition 64 is unique, see Lemma 65 below (taken from Lemma 2 in [74, Section 10.2]). In other words, two different solutions must coincide, and the explicit solution constructed above must be the function ψ_1 .

Lemma 65 (Invariant lemma). *The problem of finding functions f such that*

- (i) f is analytic in $\mathcal{G}_{\mathcal{R}}$ and continuous in $\overline{\mathcal{G}_{\mathcal{R}}}$,
- (ii) f satisfies the boundary condition (2.11),

does not have non-trivial solutions in the class of functions f vanishing at infinity.

To construct a particular solution to the BVP of Proposition 64, we shall use the function

$$w(\theta_2) = T_{\frac{\pi}{\beta}} \left(- \frac{2\theta_2 - (\theta_2^+ + \theta_2^-)}{\theta_2^+ - \theta_2^-} \right) \quad (2.13)$$

introduced in Theorem 61. Let us first establish some of its properties.

Lemma 66. *The function w in (2.13) is such that:*

- (i) w is analytic in \mathcal{G} , continuous in $\overline{\mathcal{G}_{\mathcal{R}}}$ and unbounded at infinity,
- (ii) w is injective in \mathcal{G} (onto $\mathbb{C} \setminus (-\infty, -1]$),
- (iii) $w(\theta_2) = w(\overline{\theta_2})$ for all $\theta_2 \in \mathcal{R}$.

The function w is called a conformal gluing function. The conformal property comes from (i) and (ii), and the gluing from (iii): w glues together the upper and lower parts of the hyperbola \mathcal{R} . There are at least two ways for proving Lemma 66. First, it turns out that in the literature there exist expressions for conformal gluing functions for relatively simple curves: circles, ellipses, and also for hyperbolas, see [5, Equation (4.6)]. Following this way, we could obtain the expression (2.13) for w given above and its different properties stated in Lemma 66. Instead, we would like to use the Riemann sphere given by

$$\{(\theta_1, \theta_2) \in \mathbb{C}^2 : \gamma(\theta_1, \theta_2) = 0\}.$$

Indeed, as we shall see in Section 2.5, many (not to say all) technical aspects, in particular finding the conformal mapping, happen to be quite simpler on that surface. The proof of Lemma 66 is thus postponed to Section 2.5.

Under the symmetry condition ($\mu_1 = \mu_2$, $\sigma_{11} = \sigma_{22}$ and symmetric reflection vectors R_1, R_2 in (2.1)), Foschini also obtained an expression for that conformal mapping, see [43, Figure 3].

2.3.4 Proof of Theorem 61

We are now ready to prove our main result. Let us introduce the function

$$f(\theta_2) = \psi_1(\theta_2) - \varphi_1(0) \frac{w'(0)}{w(\theta_2) - w(0)}.$$

The key point is that f satisfies the assumptions of Lemma 65.

The first item (i) of Lemma 65 holds by construction: the only possible pole of f is at 0 (ψ_1 has a unique pole at 0, see Proposition 64, and since w is injective, see Lemma 66, the equation $w(\theta_2) - w(0) = 0$ has only one solution, viz, $\theta_2 = 0$). However, a series expansion shows that the residue of f at 0 is 0, in other words 0 is a removable singularity.

Point (ii) is also clear, since both ψ_1 and w satisfy the boundary condition (2.11). Furthermore, f vanishes at infinity, since on the one hand ψ_1 does, and on the other hand w goes to infinity at infinity. Using Lemma 65, we conclude that $f = 0$. It remains to show that $\varphi_1(0) = -\mu_1$. For this it is enough to evaluate the functional equation (2.5), first at $\theta_2 = 0$, and then at $\theta_1 = 0$.

2.3.5 Diagonal covariance and dimension one

Assuming that the covariance matrix Σ is diagonal, we have $\beta = \frac{\pi}{2}$ in (2.4), and $T_{\frac{\pi}{\beta}} = T_2$ is the second Chebyshev polynomial, given by $T_2(x) = 2x^2 - 1$. The formula (2.13) for w together with the expression (2.3) of θ_2^\pm yields

$$w(\theta_2) - w(0) = \frac{8}{\mu_1^2 \sigma_{11} \sigma_{22} + \mu_2^2 \sigma_{22}^2} \theta_2 \left(\theta_2 + \frac{2\mu_2}{\sigma_{22}} \right).$$

The formula of Theorem 61 then gives

$$\varphi_1(\theta_2) = \frac{-\mu_1 w'(0)}{w(\theta_2) - w(0)} \theta_2 = -\frac{2\mu_1 \mu_2}{\sigma_{22}} \frac{1}{\theta_2 + \frac{2\mu_2}{\sigma_{22}}},$$

and finally, after some elementary computations and the use of functional equation (2.5), we obtain

$$\varphi(\theta) = \frac{2\mu_1/\sigma_{11}}{(\theta_1 + 2\mu_1/\sigma_{11})} \frac{2\mu_2/\sigma_{22}}{(\theta_2 + 2\mu_2/\sigma_{22})}. \quad (2.14)$$

By inversion of the Laplace transform, we reach the conclusion that

$$\nu_1(x_2) = \frac{2\mu_1 \mu_2}{\sigma_{22}} e^{\frac{2\mu_2}{\sigma_{22}} x_2}, \quad \nu_2(x_1) = \frac{2\mu_1 \mu_2}{\sigma_{11}} e^{\frac{2\mu_1}{\sigma_{11}} x_1}, \quad \pi(x_1, x_2) = \frac{4\mu_1 \mu_2}{\sigma_{11} \sigma_{22}} e^{\frac{2\mu_1}{\sigma_{11}} x_1 + \frac{2\mu_2}{\sigma_{22}} x_2}.$$

Let us do two comments around (2.14). Firstly, the product-form expression (2.14) is easily explained by the skew-symmetric condition (see [49, Equation (10.2)])

$$2\Sigma = R \cdot \text{diag}(R)^{-1} \cdot \text{diag}(\Sigma) + \text{diag}(\Sigma) \cdot \text{diag}(R)^{-1} \cdot R^\top,$$

which is obviously satisfied if R is the identity matrix and Σ is diagonal (above, $\text{diag}(R)$ denotes the diagonal matrix with same diagonal entries as R and R^\top is the transpose matrix of R). Secondly, our method works also to study the one-dimensional case. Let indeed

$$X_t = X_0 + W_t + \mu t + L_t^0$$

be a one-dimensional reflected Brownian motion with drift μ , where W is a Brownian motion of variance $\sigma > 0$ and L_t^0 is the local time at 0. Applying Itô formula to $(e^{\theta X_t})$ and taking expected value over the invariant measure π , we obtain, with $\varphi(\theta) = \mathbb{E}_\pi[e^{\theta X_1}]$,

$$\left(\frac{\sigma}{2} \theta + \mu \right) \varphi(\theta) = -\mathbb{E}_\pi[L_1^0].$$

This is the functional equation in dimension one (compare with (2.5)). Evaluating this identity at 0 and remembering that $\varphi(0) = 1$ we find $\mathbb{E}_\pi[L_1^0] = -\mu$, and then

$$\varphi(\theta) = \frac{2\mu/\sigma}{\theta + 2\mu/\sigma}.$$

This coincides with (2.14).

2.3.6 Statement of the BVP in the general case

We would like to close Section 2.3 by stating the BVP in the case of arbitrary reflections (non necessarily orthogonal). Let us define for $\theta_2 \in \mathcal{R}$

$$G(\theta_2) = \frac{\gamma_1}{\gamma_2}(\Theta_1^-(\theta_2), \theta_2) \frac{\gamma_2}{\gamma_1}(\Theta_1^-(\theta_2), \overline{\theta_2}).$$

Using the same line of arguments as in the proof of Proposition 64, we obtain the following result:

Proposition 67. *The function φ_1 satisfies the following BVP:*

- (i) φ_1 is meromorphic on $\mathcal{G}_{\mathcal{R}}$ with at most one pole p of order 1, and is bounded at infinity,
- (ii) φ_1 is continuous on $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{p\}$ and

$$\varphi_1(\overline{\theta_2}) = G(\theta_2)\varphi_1(\theta_2), \quad \forall \theta_2 \in \mathcal{R}. \quad (2.15)$$

Due to the presence of the function $G \neq 1$ in (2.15), this BVP (still homogeneous with shift) is more complicated than the one encountered in Proposition 64, and cannot be solved thanks to an invariant lemma. Instead, the resolution is less combinatorial and far more technical, and the solution should be expressed in terms of Cauchy integrals and the conformal mapping w of Lemma 66. This will be achieved in Chapter 3.

From the viewpoint of the general BVP of Proposition 67, we thus solved in this note a case where the variables θ_1 and θ_2 could be separated in the quantity (the right-hand side of the functional equation (2.5))

$$\gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1).$$

Let us say a few words on another case (the only other case, in fact) where the variables can also be separated, namely, when $\gamma_1 = \gamma_2$. In this case the reflections are parallel, and the function φ_1 (instead of ψ_1) satisfies the BVP of Proposition 64, with no pole in $\mathcal{G}_{\mathcal{R}}$. With Lemma 65, it has to be a constant. In other words, there is no invariant measure, which is in accordance with the fact that with parallel reflections the condition of ergodicity (2.2) is obviously not satisfied.

2.4 Singularity analysis

2.4.1 Statement of the result

In the literature, an important aspect of reflected Brownian motion in the quarter plane (and more generally in orthants) is the asymptotics of its stationary distribution, see indeed [28, 44] for the asymptotics of the interior measure, and [29] for the boundary measures. With our Theorem 61 and a classical singularity analysis of Laplace transforms (our main reference for this is the book [32]), we can easily obtain such asymptotic expansions.

Precisely, we identify three regimes, depending on the sign of $\Theta_1(\theta_2^+)$. The fact that the latter quantity determines the asymptotics can be easily explained: as we shall see in Section 2.5, the pole $\theta_2 \in \mathbb{C} \setminus [\theta_2^+, \infty)$ of φ_1 that is the closest to the origin is (provided it exists) a solution to $\Theta_1^+(\theta_2) = 0$. The relative locations of $\Theta_1^+(\theta_2)$ and 0 will therefore

decide which of that pole or of the algebraic singularity θ_2^+ will be the first singularity of φ_1 .

Define the constants

$$C_1 = \frac{-\varphi_1(\theta_2^+) 2\frac{\pi}{\beta} \sin(\frac{\pi}{\beta}\pi)}{(w(\theta_2^+) - w(0))\sqrt{\theta_2^+ - \theta_2^-}}, \quad C_2 = \frac{-\mu_1 w'(0)\theta_2^+ \sqrt{\theta_2^+ - \theta_2^-}}{2\frac{\pi}{\beta} \sin(\frac{\pi}{\beta}\pi)}. \quad (2.16)$$

Proposition 68. *As $x_2 \rightarrow \infty$, the asymptotics of $\nu_1(x_2)$ is given by*

$$\nu_1(x_2) = \begin{cases} x_2^{-\frac{3}{2}} e^{-\theta_2^+ x_2} \left(\frac{-C_1}{2\sqrt{\pi}} + o(1) \right) & \text{if } \Theta_1(\theta_2^+) < 0, \\ x_2^{-\frac{1}{2}} e^{-\theta_2^+ x_2} \left(\frac{C_2}{\sqrt{\pi}} + o(1) \right) & \text{if } \Theta_1(\theta_2^+) = 0, \\ e^{\frac{2\mu_2}{\sigma_{22}} x_2} \left(\frac{2\mu_1\mu_2}{\sigma_{22}} + o(1) \right) & \text{if } \Theta_1(\theta_2^+) > 0, \end{cases}$$

where the constants C_1 and C_2 are defined in (2.16).

Note that we obtain (for the particular case of orthogonal reflections) the asymptotic result of Dai and Miyazawa in [28, Theorem 6.1], with the additional information of the explicit expression for the constants.

Note that $\beta = \pi/k$ for $k \geq 2$ implies $\Theta_1(\theta_2^+) > 0$ and that k cannot be 1 due to the condition on Σ .

2.4.2 Proof of Proposition 68

Proposition 68 is an easy consequence of the singularity analysis of φ_1 and of classical transfer theorems, as [32, Theorem 37.1]. Due to the expression of φ_1 in Theorem 61, there are two sources of singularities: the singularities of w and the points θ_2 where the denominator $w(\theta_2) - w(0)$ of φ_1 vanishes.

Let us first study the singularities of w . In fact, the function w can not only be analytically continued on $\mathcal{G}_{\mathcal{R}}$ as claimed in Lemma 66, but on the whole of the cut plane $\mathbb{C} \setminus [\theta_2^+, \infty)$. Further, except if $\pi/\beta \in \mathbb{Z}$, in which case w is a polynomial, w has an algebraic-type singularity at θ_2^+ , given by the following result:

Lemma 69. *If $\pi/\beta \notin \mathbb{Z}$, φ_1 has an algebraic-type singularity at θ_2^+ , in the neighborhood of which it admits the expansion:*

$$\varphi_1(\theta_2) = \begin{cases} \varphi_1(\theta_2^+) + C_1 \sqrt{\theta_2^+ - \theta_2} + O(\theta_2 - \theta_2^+) & \text{if } w(\theta_2^+) - w(0) \neq 0, \\ \frac{C_2}{\sqrt{\theta_2^+ - \theta_2}} + O(1) & \text{if } w(\theta_2^+) - w(0) = 0, \end{cases}$$

where C_1 and C_2 are defined in (2.16).

Proof. Lemma 69 follows from doing an expansion of φ_1 at θ_2^+ , using (the elementary) Lemma 70 below. \square

Lemma 70. *If a is an integer, the generalized Chebyshev polynomial T_a is the classical Chebyshev polynomial. If not, T_a is not a polynomial, and admits an analytic extension on $\mathbb{C} \setminus (-\infty, -1]$. The point -1 is an algebraic-type singularity, and there is the expansion*

$$T_a(x) = \cos(a\pi) + a\sqrt{2} \sin(a\pi) \sqrt{x+1} + O(x+1). \quad (2.17)$$

Proof. The considerations on the algebraic nature of T_a are clear, and (2.17) comes from making an expansion of $T_a(x) = \cos(a \arccos(x))$ in the neighborhood of -1 . \square

We now turn to the singularities introduced by the denominator of φ_1 ; these singularities are poles.

Lemma 71. *There are two cases concerning the poles of φ_1 in the cut plane $\mathbb{C} \setminus [\theta_2^+, \infty)$:*

- $\Theta_1(\theta_2^+) \leq 0$, then φ_1 is analytic on $\mathbb{C} \setminus [\theta_2^+, \infty)$,
- $\Theta_1(\theta_2^+) > 0$, then φ_1 is meromorphic on $\mathbb{C} \setminus [\theta_2^+, \infty)$, with poles on the segment $[0, \theta_2^+]$ only. The closest pole to the origin is at $-2\frac{\mu_2}{\sigma_{22}}$ and has order one.

The proof of Lemma 71 is postponed to Section 2.5, since the tools that we shall introduce there will simplify it. Then Proposition 68 is an easy consequence of Lemmas 69 and 71.

2.5 Riemann sphere and related facts

We have many objectives in this section, which all concern the set of zeros of the kernel

$$\mathcal{S} = \{(\theta_1, \theta_2) \in \mathbb{C}^2 : \gamma(\theta_1, \theta_2) = 0\}.$$

We add some complexity here, by using the framework of Riemann surfaces. In return, many technical aspects become more intrinsic and some key quantities admit nice and natural interpretations. We first (Section 2.5.1) study the structure of \mathcal{S} , as a Riemann surface. Then we find a simple formula for the conformal mapping w (Section 2.5.2). We finally introduce the notion of group of the model (Section 2.5.3), similar to the notion of group of the walk in the discrete setting [79, 37, 10], and we prove that the algebraic nature of the solution is related to the finiteness of this group.

2.5.1 Uniformisation

Due to the degree of the kernel γ , the surface \mathcal{S} has genus 0 and is a Riemann sphere $\mathbb{C} \cup \{\infty\}$, see [44]. It thus admits a rational parametrization (or uniformisation) $\{(\theta_1(s), \theta_2(s)) : s \in \mathbb{C} \cup \{\infty\}\}$, given by

$$\begin{cases} \theta_1(s) = \frac{\theta_1^+ + \theta_1^-}{2} + \frac{\theta_1^+ - \theta_1^-}{4} \left(s + \frac{1}{s} \right), \\ \theta_2(s) = \frac{\theta_2^+ + \theta_2^-}{2} + \frac{\theta_2^+ - \theta_2^-}{4} \left(\frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s} \right), \end{cases} \quad (2.18)$$

with β as in (2.4). The equation $\gamma(\theta_1(s), \theta_2(s)) = 0$ is valid for any $s \in \mathcal{S}$. We will often represent a point $s \in \mathcal{S}$ by the pair of coordinates $(\theta_1(s), \theta_2(s))$.

Any point $\theta_1 \in \mathbb{C}$ has two images on \mathcal{S} . More specifically, let $s \in \mathcal{S}$ be defined by $\theta_1(s) = \theta_1$. Then the two points are given by $(\theta_1(s), \theta_2(s))$ and $(\theta_1(s), \theta_2(1/s))$. They are always different, except if θ_1 is a branch point θ_1^\pm . Similarly, any $\theta_2 \in \mathbb{C}$ corresponds to two points on \mathcal{S} , viz, $(\theta_1(s), \theta_2(s))$ and $(\theta_1(e^{2i\beta}/s), \theta_2(s))$.

Any point or domain of \mathbb{C} can be represented on the Riemann sphere, and there is the following correspondance (which is easily proved by using the formulas (2.18)), see Figure 2.3:

- The branch points θ_1^\pm and θ_2^\pm are located at ± 1 and $\pm e^{i\beta}$;
- The point of coordinates $(0, 0)$ is at s_0 , and ∞ is at 0 and ∞ ;
- The real points $(\theta_1, \theta_2) \subset \mathbb{R}^2$ such that $\gamma(\theta_1, \theta_2) = 0$ form the unit circle;
- The curve \mathcal{R} is sent to $(-\infty, 0)$, and if $\theta_2 \in \mathcal{R}$ corresponds to $s \in (-\infty, 0)$, then $\overline{\theta_2} \in \mathcal{R}$ corresponds to $1/s$; the domain $\mathcal{G}_{\mathcal{R}}$ is the cone bounded by $(0, -\infty)$ and $(0, -e^{i\beta}\infty)$.

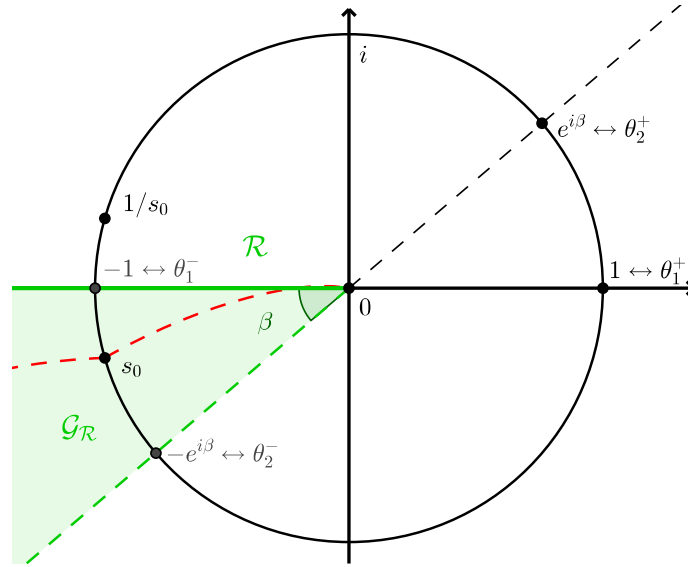


Figure 2.3: The set \mathcal{S} viewed as the Riemann sphere, and important points and curves on it

2.5.2 Conformal mapping

In this section, we first show how to obtain the expression (2.13) of w in terms of generalized Chebyshev polynomials, and we prove Lemma 66. Let us first notice that any function $f(\theta_2)$ can be lifted in a function of $F(s)$, by mean of the formula

$$F(s) = f(\theta_2(s)).$$

In particular, $W(s)$ stands for the lifted conformal mapping $w(\theta_2(s))$. Reciprocally, for a function $F(s)$ to define a univalent function $f(\theta_2)$, the condition $F(s) = F(e^{2i\beta}/s)$ needs to hold, due to the obvious identity $\theta_2(s) = \theta_2(e^{2i\beta}/s)$.

Proof of Lemma 66. We first translate on W the properties of Lemma 66 stated for w . First, W has to be analytic in the open cone $\mathcal{G}_{\mathcal{R}}$ (in \mathcal{S} , this is the cone delimited by $(0, -\infty)$ and $(0, -e^{i\beta}\infty)$) and continuous on the closed cone $\overline{\mathcal{G}_{\mathcal{R}}}$ except at 0 , see (i). Second, W has to be injective on $\mathcal{G}_{\mathcal{R}}$, see (ii). Finally, the boundary condition (iii) has to be replaced by the pair of conditions

$$\begin{cases} W(s) = W(1/s), & \forall s \in \mathcal{R}, \\ W(s) = W(e^{2i\beta}/s), & \forall s \in (0, -e^{i\beta}\infty). \end{cases} \quad (2.19)$$

We easily come up with the (at this point: conjectured) formula

$$W(s) = -\frac{1}{2} \left\{ (-s)^{\frac{\pi}{\beta}} + (-s)^{-\frac{\pi}{\beta}} \right\} = -\frac{1}{2} \left\{ e^{\frac{\pi}{\beta} \log(-s)} + e^{-\frac{\pi}{\beta} \log(-s)} \right\}, \quad (2.20)$$

where we make use of the principal determination of the logarithm. (Since the conditions (2.19) are invariant under multiplication by a constant, we can choose the constant in front of the right-hand side of (2.20). We choose $-1/2$, so as to match the expression $W(s)$ with $w(\theta_2(s))$, w being as in Theorem 61: $w(\Theta_2(\theta_1^-)) = T_{\frac{\pi}{\beta}}(\cos \beta) = \cos \pi = -1 = W(-1)$.)

Let us briefly verify each property. First, the analyticity property (i) is clear from the properties of the logarithm. In order to show (ii), we first remark that $W(s) = W(t)$ if and only if for some $k \in \mathbb{Z}$,

$$s = t^{\pm 1} e^{2ik\beta}. \quad (2.21)$$

Since s and t both belong to the cone $\mathcal{G}_{\mathcal{R}}$, we must have $s = t$, and therefore W is injective. Finally, (iii) is clear from the construction of W .

To obtain the expression of w in terms of the generalized Chebyshev polynomial, we use the fact that $\theta_2(s) = \theta_2$ yields the formula

$$s = \frac{e^{i\beta}}{\theta_2^+ - \theta_2^-} \left(2\theta_2 - (\theta_2^+ + \theta_2^-) \pm 2\sqrt{(\theta_2^+ - \theta_2)(\theta_2^- - \theta_2)} \right).$$

The proof is complete. □

Thanks to the expression (2.20) of W derived above, we can now prove Lemma 71, concerning the (eventual) poles of φ_1 in the cut plane $\mathbb{C} \setminus [\theta_2^+, \infty)$.

Proof of Lemma 71. Using the expression of φ_1 obtained in Theorem 61, we conclude that the poles of φ_1 must be located at points θ_2 where $w(\theta_2) = w(0)$. Using the function W in (2.20), we reformulate the latter identity as $W(s) = W(s_0)$, see Figure 2.3, and thanks to (2.21) we obtain $s = (s_0)^{\pm 1} e^{2ik\beta}$. The closest pole to the origin s_0 is then $1/s_0$, which corresponds to the point

$$\theta_2 = -2 \frac{\mu_2}{\sigma_{22}}$$

such that $\Theta_1^+(\theta_2) = 0$, since $\theta_1(s_0) = \theta_1(1/s_0) = 0$. The sign of $\Theta_1(\theta_2^+)$ then determines if $1/s_0$ is before or after $e^{i\beta}$ (which corresponds to θ_2^+) on the unit circle, see Figure 2.3. The conditions given in terms of sign of $\Theta_1(\theta_2^+)$ follows. Finally, let us notice that $w(\theta_2^+) = w(0)$ if $\Theta_1(\theta_2^+) = 0$ and that $w(\theta_2^+) \neq w(0)$ if $\Theta_1(\theta_2^+) < 0$. □

2.5.3 Group of the model and nature of the solution

The notion of group of the model has been introduced by Malyshev [79] in the context of random walks in the quarter plane. It turns out to be an important characteristic of the model, in particular to decide whether generating functions or Laplace transforms are algebraic or D-finite functions, see [10].

It can be introduced directly on the kernel γ : with the notation (2.6), this is the group $\langle \zeta, \eta \rangle$ generated by ζ and η , given by

$$\zeta(\theta_1, \theta_2) = \left(\theta_1, \frac{c(\theta_1)}{a(\theta_1)} \frac{1}{\theta_2} \right), \quad \eta(\theta_1, \theta_2) = \left(\frac{\tilde{c}(\theta_2)}{\tilde{a}(\theta_2)} \frac{1}{\theta_1}, \theta_2 \right).$$

By construction, the generators satisfy $\gamma(\zeta(\theta_1, \theta_2)) = \gamma(\eta(\theta_1, \theta_2)) = 0$ as soon as $\gamma(\theta_1, \theta_2) = 0$. In other words, there are (covering) automorphisms of the surface \mathcal{S} . Since $\zeta^2 = \eta^2 = 1$, the group $\langle \zeta, \eta \rangle$ is a dihedral group, which is finite if and only if the element $\zeta\eta$ (or $\eta\zeta$) has finite order.

With the above definition, it is not clear how to see if the group is finite, nor to see if this has any implication on the problem. In fact, we have:

Proposition 72. *The group $\langle \zeta, \eta \rangle$ is finite if and only if $\pi/\beta \in \mathbb{Q}$.*

The proof of Proposition 72 is simple, once the elements ζ and η have been reformulated on the sphere \mathcal{S} :

$$\zeta(s) = \frac{1}{s}, \quad \eta(s) = \frac{e^{2i\beta}}{s}.$$

These transformations leave invariant $\theta_1(s)$ and $\theta_2(s)$, respectively, see (2.18). In particular, we have the following result (consequence of the Lemma 74), which connects the nature of the solution of the BVP to the finiteness of the group. Such a result holds for discrete walks, see [10, 6].

Proposition 73. *The solution φ_1 given in Theorem 61 is algebraic if and only if the group $\langle \zeta, \eta \rangle$ is finite.*

The proof of Proposition 73 builds on the following elementary result:

Lemma 74. *Let $a \geq 0$. The generalized Chebyshev polynomial T_a is*

- rational if $a \in \mathbb{Z}$,
- algebraic (and not polynomial) if $a \in \mathbb{Q} \setminus \mathbb{Z}$,
- not algebraic if $a \in \mathbb{R} \setminus \mathbb{Q}$.

However, even in the non-algebraic case, the Chebyshev polynomial T_a always satisfies a linear differential equation with coefficients in \mathbb{R} , since it can be written as the particular hypergeometric function ${}_2F_1([-a, a], [1/2], 1 - x)$, where (with $(A)_k = A(A+1) \cdots (A+(k-1))$)

$${}_2F_1([A, B], [C], t) = \sum_{k=0}^{\infty} \frac{(A)_k (B)_k}{(C)_k} \frac{t^k}{k!}.$$

Explicit expression for the stationary distribution of reflected Brownian motion in a wedge

Ce chapitre est issu de l'article [47] co-écrit avec Kilian Raschel et en cours de révision aux *Annales de l'Institut Henri Poincaré*.

Abstract

For Brownian motion in a (two-dimensional) wedge with negative drift and oblique reflection on the axes, we derive an explicit formula for the Laplace transform of its stationary distribution (when it exists), in terms of Cauchy integrals and generalized Chebyshev polynomials. To that purpose we solve a Carleman-type boundary value problem on a hyperbola, satisfied by the Laplace transforms of the boundary stationary distribution.

Résumé

Pour le mouvement Brownien dans un cône (de dimension deux) avec dérive négative et réflexion oblique sur les axes, on détermine une formule explicite pour la transformée de Laplace de sa distribution stationnaire (quand elle existe), en terme d'intégrale de Cauchy et de polynômes de Tchebychev généralisés. Dans cet objectif on résout un problème frontière du type Carleman sur une hyperbole, satisfait pas la transformée de Laplace de la distribution stationnaire sur la frontière.

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3.1 Introduction and main results

Since its introduction in the eighties by Harrison, Reiman, Varadhan and Williams [49, 50, 97, 100, 99], reflected Brownian motion in the quarter plane has received a lot of attention from the probabilistic community. However, and surprisingly, finding a general explicit expression of the stationary distribution has been left as an open problem. The present chapter solves this problem in a complete and unified way.

Reflected Brownian motion in two-dimensional cones

The reflected Brownian motion with drift in the quarter plane $\mathbb{R}_+^2 := [0, \infty)^2$ (or equivalently in arbitrary convex wedges, by performing a simple linear transformation, cf. Appendix 3.6) can be written as

$$Z(t) = Z_0 + B(t) + \mu \cdot t + R \cdot L(t), \quad \forall t \geq 0, \quad (3.1)$$

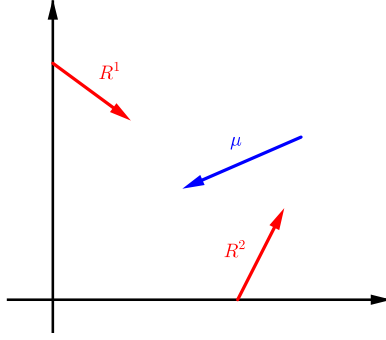
where

- Z_0 is any initial point in the quadrant,
- B is a Brownian motion with covariance $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ starting from the origin,
- $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ denotes the interior drift,
- $R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ is the reflection matrix,
- $L = \begin{pmatrix} L^1 \\ L^2 \end{pmatrix}$ is the local time.

For $i \in \{1, 2\}$, $L^i(t)$ is a continuous non-decreasing process starting from 0 (i.e., $L^i(0) = 0$), increasing only at time t such that $Z^i(t) = 0$, viz., $\int_0^t \mathbf{1}_{\{Z^i(s) \neq 0\}} dL^i(s) = 0$, for all $t \geq 0$. The columns R^1 and R^2 represent the directions in which the Brownian motion is pushed when the axes are reached, see Figure 3.1.

The reflected Brownian motion $(Z(t))_{t \geq 0}$ associated with (Σ, μ, R) is well defined [97, 99], and is a fundamental stochastic process. This process, together with its multi-dimensional version (a semimartingale reflected Brownian motion in the positive orthant \mathbb{R}_+^d , as well as in convex polyhedrons), has been extensively explored. It has been studied in depth, with focuses on its definition and semimartingale properties [97, 100, 26, 101], its recurrence or transience [100, 22, 57, 16, 12, 11, 25], the possible particular (e.g., product) form of its stationary distribution [56, 31], its Lyapunov functions [35], its links with other stochastic processes [69, 34, 70], its use to approximate large queuing networks [42, 5, 55, 63, 62], the asymptotics of its stationary distribution [52, 28, 44, 90], numerical methods to compute the stationary distribution [22, 27], links with complex analysis [42, 5, 14, 46], comparison and coupling techniques [91, 92], etc.

The main contribution of the present paper is to find an exact expression for the stationary distribution (via its Laplace transforms, to be introduced in (3.3) and (3.4)), thanks to the theory of boundary value problems (BVPs), see our Theorem 75. This is one of the first attempts to apply boundary value techniques to diffusions in the quadrant, after

Figure 3.1: Drift μ and reflection vectors R^1 and R^2

[43] (under the symmetry conditions $\mu_1 = \mu_2$, $\sigma_{11} = \sigma_{22}$, and symmetric reflection vectors in (3.1)), [42] (which concerns very specific cases of the covariance matrix, essentially the identity), [5] (on diffusions with a special behavior on the boundary), [46] (orthogonal reflections, solved by Tutte's invariant approach [96, 6]); see also the introduction of [28], where the authors allude to the possibility of such an approach.

Main results

The reflected Brownian motion defined in (3.1) exists if and only if

$$\{r_{11} > 0, \quad r_{22} > 0, \quad \det R > 0\} \quad \text{or} \quad \{r_{11} > 0, \quad r_{22} > 0, \quad r_{12} > 0, \quad r_{21} > 0\}.$$

This condition is equivalent for R to be completely- \mathcal{S} , which is actually a necessary and sufficient condition to the existence of reflected Brownian motion in arbitrary dimension, see [95, 87].

As for the stationary distribution, it exists if and only if

$$r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0, \quad r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0, \quad (3.2)$$

see [52], and in that case it is absolutely continuous w.r.t. the Lebesgue measure [22, 27, 55], with density denoted by $\pi(x) = \pi(x_1, x_2)$. Assumption (3.2) implies in particular that R is invertible and $R^{-1}\mu < 0$, which turns out to be a necessary condition for the existence of the stationary distribution in any dimension, see [55]. From now, we will assume that (3.2) is satisfied. Let the Laplace transform of π be defined by

$$\varphi(\theta) = \mathbb{E}_\pi[\exp(\theta \cdot Z)] = \iint_{\mathbb{R}_+^2} \exp(\theta \cdot x)\pi(x)dx. \quad (3.3)$$

Furthermore we define two finite boundary measures ν_1 and ν_2 such that, for $A \subset \mathbb{R}_+$,

$$\nu_1(A) = \mathbb{E}_\pi \left[\int_0^1 \mathbf{1}_{\{Z(t) \in \{0\} \times A\}} dL^1(t) \right], \quad \nu_2(A) = \mathbb{E}_\pi \left[\int_0^1 \mathbf{1}_{\{Z(t) \in A \times \{0\}\}} dL^2(t) \right].$$

The ν_i have their supports on the axes and may be viewed as boundary invariant measures. They are continuous w.r.t. the Lebesgue measure, see [56]. We define their Laplace transform by

$$\varphi_1(\theta_2) = \int_{\mathbb{R}_+} \exp(\theta_2 x_2)\nu_1(x_2)dx_2, \quad \varphi_2(\theta_1) = \int_{\mathbb{R}_+} \exp(\theta_1 x_1)\nu_2(x_1)dx_1. \quad (3.4)$$

The following functional equation relates the Laplace transforms:

$$-\gamma(\theta)\varphi(\theta) = \gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1), \quad (3.5)$$

where we have noted

$$\begin{cases} \gamma(\theta) = \frac{1}{2}(\theta \cdot \Sigma\theta) + \theta \cdot \mu = \frac{1}{2}(\sigma_{11}\theta_1^2 + 2\sigma_{12}\theta_1\theta_2 + \sigma_{22}\theta_2^2) + \mu_1\theta_1 + \mu_2\theta_2, \\ \gamma_1(\theta) = R^1 \cdot \theta = r_{11}\theta_1 + r_{21}\theta_2, \\ \gamma_2(\theta) = R^2 \cdot \theta = r_{12}\theta_1 + r_{22}\theta_2. \end{cases} \quad (3.6)$$

The integrals (3.3) and (3.4) (resp. Equation (3.5)) exist (resp. holds) at least for values of $\theta = (\theta_1, \theta_2)$ with $\Re\theta_1 \leq 0$ and $\Re\theta_2 \leq 0$. To prove the functional equation (3.5), the main idea is to use an identity called basic adjoint relationship (BAR); see [46, Section 2.1] and [28, 42] for details.

In addition to (3.2), which guarantees existence and uniqueness of the stationary distribution, we shall assume that both coordinates of the drift are negative:

$$\mu_1 < 0, \quad \mu_2 < 0. \quad (3.7)$$

This hypothesis (also done in [43, 42, 44]) is only technical, and allows us to reduce the number of cases to handle. In Section 3.3.7 we comment on the case of a drift μ with one non-negative coordinate (having two non-negative coordinates is obviously incompatible with (3.2)).

Our main result (Theorem 75) can be stated as follows. Although most of the notation of Theorem 75 is to be defined in Sections 3.2 and 3.3, we have chosen to present it in full detail straightaway. In the first reading:

- one may simply focus on the type of the formula (3.8) stated in Theorem 75, namely the product of a function by the exponential of a contour integral (the same type of formula appears in the discrete setting, for the generating functions of stationary distributions of reflected random walks in the quadrant, see [37, Chapter 5] and [67]);
- it is also interesting to dissect (3.8) and to notice that certain quantities in that formula depend only on the behavior of the process in the interior of the quadrant (w and \mathcal{R}^-), some others depend only on the reflection vectors (function G), while the remaining ones mix properties of the interior and boundary of the quarter plane ($\nu_1(\mathbb{R}_+)$, p and χ).

Theorem 75. *Under the assumptions (3.2) and (3.7), the Laplace transform φ_1 in (3.4) is equal to*

$$\varphi_1(\theta_2) = \nu_1(\mathbb{R}_+) \left(\frac{w(0) - w(p)}{w(\theta_2) - w(p)} \right)^{-\chi} \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(\theta) \left[\frac{w'(\theta)}{w(\theta) - w(\theta_2)} - \frac{w'(\theta)}{w(\theta) - w(0)} \right] d\theta \right\}, \quad (3.8)$$

where

- $\nu_1(\mathbb{R}_+) = \frac{r_{12}\mu_2 - r_{22}\mu_1}{\det R}$, see Lemma 77,

- the function w , related to generalized Chebyshev polynomials, is defined in (3.24),
- the pole p is defined in (3.13),
- the index χ equals 0 or -1 , according to the values of the parameters, see (3.32),
- the curve \mathcal{R}^- , a (half) branch of hyperbola oriented from the vertex to infinity, is defined in (3.17), see also Figure 3.4,
- the function G is introduced in (3.18),
- to define the function $\log G(\theta)$, we use the determination of the logarithm taking a value in $(-i\pi, i\pi]$ at the vertex of \mathcal{R}^- and varying continuously over the curve \mathcal{R}^- , see Section 3.3.6.

The function $\varphi_2(\theta_1)$ equals $\varphi_1(\theta_1)$ in (3.8) after the change of parameters

$$\sigma_{11} \leftrightarrow \sigma_{22}, \quad \mu_1 \leftrightarrow \mu_2, \quad r_{11} \leftrightarrow r_{22}, \quad r_{12} \leftrightarrow r_{21}.$$

The functional equation (3.5) finally gives an explicit formula for the bivariate Laplace transform φ .

Let us now give some comments around Theorem 75.

- Theorem 75 completely generalizes the results of [43] (with symmetry conditions), [42] (with the identity covariance matrix Σ) and [46] (with orthogonal reflection on the axes). It offers the first explicit expression of the Laplace transforms, covering all the range of (non-degenerate) parameters (Σ, μ, R) , thereby solving an old open problem.
- It is worth remarking that the expression (3.8) is intrinsically non-continuous in terms of the parameters. Indeed, the index χ can take two different values (namely, 0 and 1). For this reason, (3.8) actually contains two different formulas. In addition, in the subcase $\chi = 0$, the function G may take two different values (-1 and $+1$) at the vertex of \mathcal{R}^- . See Remarks 82 and 92 for further related comments.
- The paper [44] obtains the exact asymptotics of the stationary distribution along any direction in the quarter plane, see [44, Theorems 22–28]. Constants in these asymptotics involve the functions φ_1 and φ_2 in (3.4), and can thus be made explicit with Theorem 75.
- The statement of Theorem 75 (namely, an expression of the Laplace transform as a Cauchy integral), as well as the techniques we shall employ to prove it (viz., reduction to BVPs with shift), are reminiscent of the results and methods used for discrete random walks in the quarter plane, see [37] for a modern reference, and [79, 36] for historical breakthroughs.
- Altogether, Theorem 75 illustrates that the analytic approach consisting in solving quarter plane problems via BVPs is better suited for diffusions than for discrete random walks. We can actually treat any wedge, covariance matrix, drift vector and reflection vectors (see Corollary 76 below), whereas in the discrete case, hypotheses should be done on the boundedness of the jumps (only small steps are considered in [37, 67, 10, 6]) and on the cone (typically, half and quarter planes only).

Theorem 75 further leads to an explicit expression for the Laplace transform of the stationary distribution of reflected Brownian motion in an arbitrary convex wedge, as stated in the following corollary, whose proof is postponed to Appendix 3.6.

Corollary 76. *Let \tilde{Z} be a reflected Brownian motion in a wedge of angle $\beta \in (0, \pi)$, of covariance matrix $\tilde{\Sigma}$, drift $\tilde{\mu}$ and reflection matrix $\tilde{R} = (\tilde{R}^1, \tilde{R}^2)$, corresponding to the angles of reflection δ and ϵ on Figure 3.10. Assume it is recurrent and note $\tilde{\pi}$ the stationary distribution and $\tilde{\varphi}$ its Laplace transform. Let*

$$T_1 = \begin{pmatrix} \frac{1}{\sin \beta} & \cot \beta \\ 0 & 1 \end{pmatrix}, \quad T_1^{-1} = \begin{pmatrix} \sin \beta & -\cos \beta \\ 0 & 1 \end{pmatrix}.$$

Then

$$\tilde{\varphi}(\tilde{\theta}) = \varphi(T_1^\top \tilde{\theta}),$$

where φ is the Laplace transform (3.8) of Theorem 75 associated to (Σ, μ, R) , with

$$\Sigma = T_1 \tilde{\Sigma} T_1^\top, \quad \mu = T_1^{-1} \tilde{\mu} \quad \text{and} \quad R = T_1^{-1} \tilde{R}.$$

Structure of the chapter

- Section 3.2: statement of the kernel functional equation, analytic preliminaries, continuation of the Laplace transforms and definition of an important hyperbola
- Section 3.3: statement and proof that the Laplace transforms satisfy BVP of Carleman-type on branches of hyperbolas, transformation of the Carleman BVP with shift into a (more classical) Riemann BVP, study of the conformal mapping allowing this transformation, resolution of the BVP
- Section 3.4: asymptotics of the stationary distribution, links with Dai and Miyazawa's [29] asymptotic results
- Section 3.5: simplifications of the integral expression of Theorem 75 for models satisfying the skew-symmetric condition, for finite group models, then for orthogonal reflections (leading to a new proof of the results of [46]), links with Dieker and Moriarty's results [31]
- Appendix 3.6: equivalence between Brownian motion in the quarter plane and Brownian motion in convex wedges

3.2 A kernel functional equation and continuation of the Laplace transforms

3.2.1 Kernel

The kernel of (3.5) is γ , a second degree polynomial in both variables θ_1 and θ_2 . The equation $\gamma(\theta_1, \theta_2) = 0$ with $\theta_1, \theta_2 \in \mathbb{C}$ defines (two-valued) algebraic functions $\Theta_1^\pm(\theta_2)$ and $\Theta_2^\pm(\theta_1)$, by

$$\gamma(\Theta_1^\pm(\theta_2), \theta_2) = \gamma(\theta_1, \Theta_2^\pm(\theta_1)) = 0.$$

Solving these equations readily yields

$$\begin{cases} \Theta_1^\pm(\theta_2) = \frac{-(\sigma_{12}\theta_2 + \mu_1) \pm \sqrt{\theta_2^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2\theta_2(\mu_1\sigma_{12} - \mu_2\sigma_{11}) + \mu_1^2}}{\sigma_{11}}, \\ \Theta_2^\pm(\theta_1) = \frac{-(\sigma_{12}\theta_1 + \mu_2) \pm \sqrt{\theta_1^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2\theta_1(\mu_2\sigma_{12} - \mu_1\sigma_{22}) + \mu_2^2}}{\sigma_{22}}. \end{cases} \quad (3.9)$$

The polynomials under the square roots in (3.9) have two zeros (sometimes called branch points), real and of opposite signs. They are denoted by θ_2^\pm and θ_1^\pm , respectively:

$$\begin{cases} \theta_2^\pm = \frac{(\mu_1\sigma_{12} - \mu_2\sigma_{11}) \pm \sqrt{(\mu_1\sigma_{12} - \mu_2\sigma_{11})^2 + \mu_1^2 \det \Sigma}}{\det \Sigma}, \\ \theta_1^\pm = \frac{(\mu_2\sigma_{12} - \mu_1\sigma_{22}) \pm \sqrt{(\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \mu_2^2 \det \Sigma}}{\det \Sigma}. \end{cases} \quad (3.10)$$

Lemma 77. *One has $\begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix} = \begin{pmatrix} \nu_1(\mathbb{R}_+) \\ \nu_2(\mathbb{R}_+) \end{pmatrix} = -R^{-1}\mu.$*

Proof. Equation (3.4) evaluated at 0 gives $\varphi_i(0) = \nu_i(\mathbb{R}_+)$. We now evaluate (3.5) at $\theta_2 = 0$, divide by θ_1 and finally evaluate at $\theta_1 = 0$. This yields $-\mu_1 = r_{11}\varphi_1(0) + r_{12}\varphi_2(0)$. In a similar way, we obtain $-\mu_2 = r_{21}\varphi_1(0) + r_{22}\varphi_2(0)$, thereby concluding the proof. \square

Note that Lemma 77 gives, as announced in Theorem 75,

$$\nu_1(\mathbb{R}_+) = \frac{r_{12}\mu_2 - r_{22}\mu_1}{\det R}, \quad \nu_2(\mathbb{R}_+) = \frac{r_{21}\mu_1 - r_{11}\mu_2}{\det R},$$

which by (3.2) are positive: remind that this positivity condition is necessary for the existence of the stationary distribution [55].

To conclude this paragraph, notice that the set (the real points of the zero set of the kernel)

$$\{(\theta_1, \theta_2) \in \mathbb{R}^2 : \gamma(\theta_1, \theta_2) = 0\}$$

defines an ellipse, see Figure 3.2. Introduced in [28], this ellipse offers the possibility of presenting many analytical quantities (as the branch points) in a clear and compact way.

3.2.2 Meromorphic continuation and poles of the Laplace transforms

In Section 3.3.2 we shall state a boundary condition for φ_1 , on a curve lying outside its natural domain of definition (namely, the half plane with negative real part). The statement hereafter (straightforward consequence of the functional equation (3.5), see Proposition 91 for an extended version) proposes a meromorphic continuation on a domain containing the latter curve.

Lemma 78. *The Laplace transform $\varphi_1(\theta_2)$ can be extended meromorphically to the open and simply connected set*

$$\{\theta_2 \in \mathbb{C} : \Re \theta_2 \leq 0 \text{ or } \Re \Theta_1^-(\theta_2) < 0\}, \quad (3.11)$$

by mean of the formula

$$\varphi_1(\theta_2) = \frac{\gamma_2}{\gamma_1}(\Theta_1^-(\theta_2), \theta_2)\varphi_2(\Theta_1^-(\theta_2)). \quad (3.12)$$

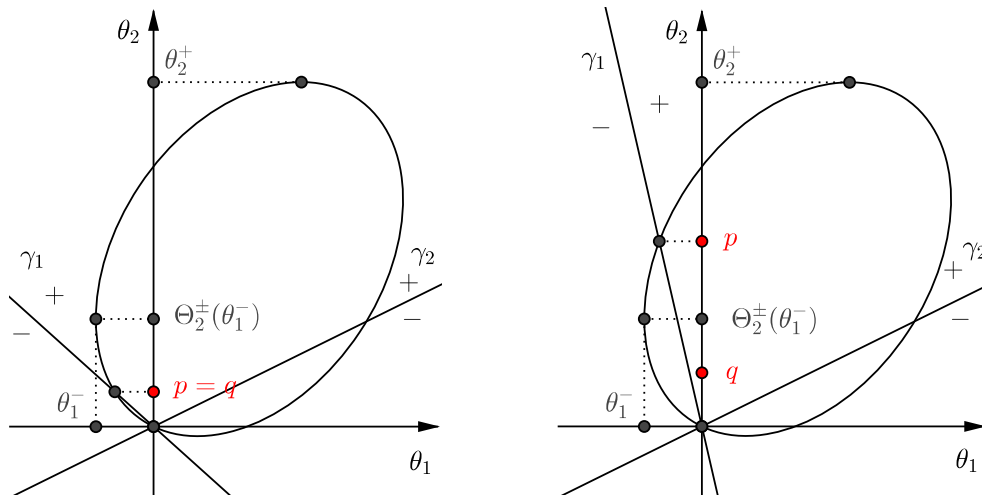


Figure 3.2: The ellipse and the two straight lines are the sets of real points (θ_1, θ_2) which cancel γ , γ_1 and γ_2 , respectively. Location of p (see (3.13)) and q (see (3.23)) according to the sign of $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$

Proof. The domain (3.11) is simply connected by [44, §2.4]. The formula (3.12), see [44, Lemma 6], is a direct consequence of the functional equation (3.5) evaluated at $(\Theta_1^-(\theta_2), \theta_2)$, first on the (non-empty) open domain

$$\{\theta_2 \in \mathbb{C} : \Re \theta_2 < 0 \text{ and } \Re \Theta_1^-(\theta_2) < 0\}. \quad \square$$

Due to the continuation formula (3.12), the only possible pole of φ_1 in the domain (3.11) will come from a cancelation of the denominator γ_1 . More precisely, let p be the (unique, when it exists) non-zero point such that

$$\gamma_1(\Theta_1^-(p), p) = 0. \quad (3.13)$$

It follows from (3.13) that p satisfies a second degree polynomial equation with real coefficients. As one of the roots is 0 the other one must be p , which is then real and equals (when it exists)

$$p = \frac{2r_{11}(\mu_1 r_{21} - \mu_2 r_{11})}{r_{11}^2 \sigma_{22} - 2r_{11} r_{21} \sigma_{12} + r_{21}^2 \sigma_{11}}. \quad (3.14)$$

Formula (3.13) means that p is the ordinate of the intersection point between the ellipse $\gamma = 0$ and the line $\gamma_1 = 0$, see Figure 3.2. The intersection point always exists but sometimes its abscissa is associated to Θ_1^- (when p exists, i.e., $\gamma_1(\Theta_1^-(p), p) \geq 0$) and sometimes to Θ_1^+ (when p doesn't exist, i.e., $\gamma_1(\Theta_1^-(p), p) < 0$), the limit case being $p = \theta_2^+$, see Figure 3.3.

Let us finally remark that the pole that φ_1 may have at p is necessarily simple, due to the expression (3.6) of γ_1 .

Remark 79. The continuation formula (3.12) uses the branch Θ_1^- in (3.9) and not Θ_1^+ . The reason is that Θ_1^- is the small branch of the algebraic function Θ_1^\pm , taking the value 0 at 0. Note, this is peculiar to the case of a drift with negative coordinates (our hypothesis (3.7)). More details can be found in the proof of Theorem 11 in [44].

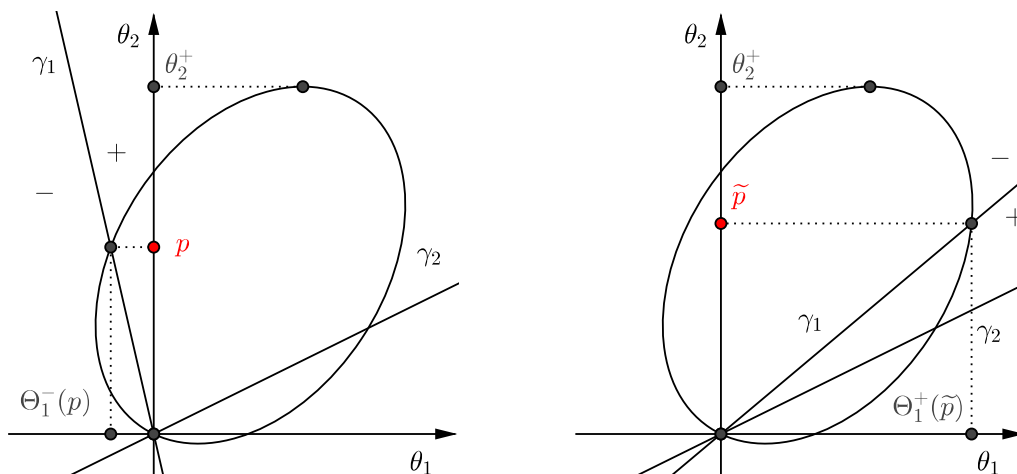


Figure 3.3: The intersection point between the ellipse $\gamma = 0$ and the straight line $\gamma_1 = 0$. On the left side p exists. On the right side p doesn't exist, although \tilde{p} , defined by that $\gamma_1(\Theta_1^+(\tilde{p}), \tilde{p}) = 0$, exists

3.2.3 An important hyperbola

For further use, we need to introduce the curve

$$\mathcal{R} = \{\theta_2 \in \mathbb{C} : \gamma(\theta_1, \theta_2) = 0 \text{ and } \theta_1 \in (-\infty, \theta_1^-)\} = \Theta_2^\pm((-\infty, \theta_1^-)). \quad (3.15)$$

It is symmetrical w.r.t. the real axis, see Figure 3.4. Indeed, the discriminant of Θ_2^\pm (i.e., the polynomial under the square root in (3.9)) is positive on (θ_1^-, θ_1^+) and negative on $\mathbb{R} \setminus [\theta_1^-, \theta_1^+]$. Accordingly, the branches Θ_2^\pm take respectively real and complex conjugate values on the sets above. Furthermore, \mathcal{R} has a simple structure, as shown by the following elementary result:

Lemma 80 (Lemma 9 in [5]). *The curve \mathcal{R} in (3.15) is a branch of hyperbola, given by the equation*

$$\sigma_{22}(\sigma_{12}^2 - \sigma_{11}\sigma_{22})x^2 + \sigma_{12}^2\sigma_{22}y^2 - 2\sigma_{22}(\sigma_{11}\mu_2 - \sigma_{12}\mu_1)x = \mu_2(\sigma_{11}\mu_2 - 2\sigma_{12}\mu_1). \quad (3.16)$$

We denote the part of \mathcal{R} with negative imaginary part by

$$\mathcal{R}^- = \{\theta_2 \in \mathcal{R} : \Im \theta_2 \leq 0\}, \quad (3.17)$$

see Figure 3.4. We further denote by $\mathcal{G}_{\mathcal{R}}$ the open domain of \mathbb{C} containing 0 and bounded by \mathcal{R} , see again Figure 3.4. The closure of $\mathcal{G}_{\mathcal{R}}$ is equal to $\mathcal{G}_{\mathcal{R}} \cup \mathcal{R}$ and will be noted $\overline{\mathcal{G}_{\mathcal{R}}}$.

The domain in (3.11) strictly contains $\overline{\mathcal{G}_{\mathcal{R}}}$, see [44]. Thanks to Lemma 78, this implies that the Laplace transform $\varphi_1(\theta_2)$ can be extended meromorphically to a domain containing $\overline{\mathcal{G}_{\mathcal{R}}}$. To see the above inclusion we refer to [44], where the boundaries of the domain (3.11) (called $\Delta \cup \{s_0\}$ in [44]) is computed, see in particular [44, Figures 10 and 11]. The technique used in [44] is the parametrization of the zero set of the kernel (3.48).

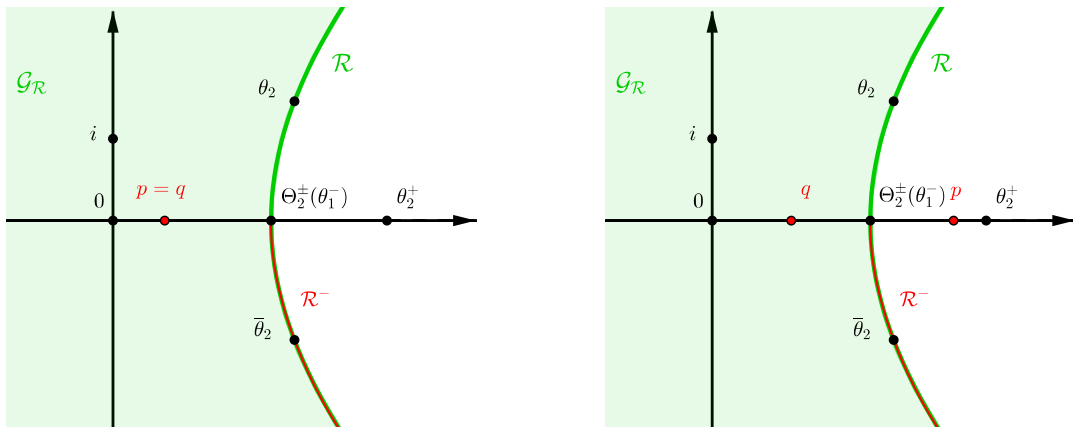


Figure 3.4: The curve \mathcal{R} in (3.15) is symmetric w.r.t. the horizontal axis, and $\mathcal{G}_\mathcal{R}$ is the domain in green. The curve \mathcal{R}^- is the half branch of \mathcal{R} with negative imaginary part. The points p and q are used to define the conformal mapping W , see (3.13) and (3.25). If the pole p is in $\mathcal{G}_\mathcal{R}$ then $q = p$ (figure on the left), otherwise $q = \frac{1}{2}\Theta_2^\pm(\theta_1^-)$ (figure on the right), see (3.23)

3.3 A proof of Theorem 75 via reduction to BVPS

3.3.1 Methodology and positioning of this work regarding [43, 42, 5, 28]

Schematically, our argument for the proof of Theorem 75 is composed of the following steps:

- (1) presentation of the functional equation, analytic preliminaries, meromorphic continuation of the Laplace transforms (Section 3.2);
- (2) statement of a Carleman BVP with shift satisfied by the Laplace transforms (Section 3.3.2);
- (3) introduction of a conformal mapping, allowing to transform the latter BVP into a more classical Riemann BVP (Section 3.3.3), see Figure 3.5;
- (4) statement of the Riemann BVP (Section 3.3.4);
- (5) definition and study of the index (denoted by χ in Theorem 75), which turns out to have a crucial role in solving the Riemann BVP (Section 3.3.5);
- (6) resolution of the BVP (Section 3.3.6).

Except for the study of the index (item (5), which is specific to our problem at hand), the above structuration of the proof dates back to (in chronological order) [36] (for the discrete setting, which later led to the book [37]), [43], [42] and [5].

To begin the discussion, let us remind that the process in [5] is absorbed on the boundary and then relaunched in the quadrant after an exponential time; it is therefore totally different from ours. Let us also recall that the analytic study of reflected Brownian motion in the quadrant was initiated in [43, 42], but both [43] and [42] did quite restrictive

assumptions on the process (symmetry conditions $\mu_1 = \mu_2$, $\sigma_{11} = \sigma_{22}$, and symmetric reflection vectors in [43], identity covariance matrix in [42]).

In this paper we follow the same steps (1)–(6) and use a synthetic approach of [36, 43, 42, 5], to eventually go further than the existing literature and prove Theorem 75. Some technical details of the present work are borrowed from [43, 42, 5] and more particularly from [5]. Although being different, the stochastic process of [5] and ours share the property of satisfying a functional equation with the same left-hand side (as in (3.5)). Accordingly, technical details will be very similar as soon as they concern the kernel; this is the case of Section 3.2.

On the other hand, many points of our analysis profoundly differ from that of [5]: the resolution of the BVP is different as the right-hand side of (3.5) is not comparable to that of [5]. Moreover we make our main result as tractable as possible: see our Section 3.5, where we present many techniques to simplify the analytic expression derived in Theorem 75. We further also propose asymptotic considerations, in close relation to those of [28].

3.3.2 Carleman BVP

For $\theta_2 \in \mathcal{R}$, define

$$G(\theta_2) = \frac{\gamma_1}{\gamma_2}(\Theta_1^-(\theta_2), \theta_2) \frac{\gamma_2}{\gamma_1}(\Theta_1^-(\bar{\theta}_2), \bar{\theta}_2). \quad (3.18)$$

Notice that $G(\theta_2)G(\bar{\theta}_2) = 1$ and that for $\theta_2 \in \mathcal{R}$ one has $\Theta_1^-(\bar{\theta}_2) = \Theta_1^-(\theta_2)$. Let us also recall that p is defined in (3.13).

Proposition 81 (Carleman BVP with shift). *The function φ_1 in (3.4)*

(1) *is meromorphic on $\mathcal{G}_{\mathcal{R}}$,*

- *without pole on $\mathcal{G}_{\mathcal{R}}$ if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) < 0$,*
- *with a single pole on $\mathcal{G}_{\mathcal{R}}$ at p of order one if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) > 0$,*
- *without pole on $\mathcal{G}_{\mathcal{R}}$ and with a single pole of order one on the boundary \mathcal{R} of $\mathcal{G}_{\mathcal{R}}$, at $p = \Theta_2^\pm(\theta_1^-)$, if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$,*

(2) *is continuous on $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{p\}$ and bounded at infinity,*

(3) *satisfies the boundary condition*

$$\varphi_1(\bar{\theta}_2) = G(\theta_2)\varphi_1(\theta_2), \quad \forall \theta_2 \in \mathcal{R}. \quad (3.19)$$

It is worth mentioning that the condition on the sign of $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$ has a clear geometric meaning: indeed, $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$ is negative (resp. positive) if and only if the straight line corresponding to γ_1 crosses the ellipse below (resp. above) the ordinate $\Theta_2^\pm(\theta_1^-)$; see Figure 3.2, left (resp. right).

Remark 82. Item (1) of Proposition 81 shows that according to the values of the parameters, various cases exist regarding the singularities of the Laplace transform in the domain $\mathcal{G}_{\mathcal{R}}$. This is the reason why there isn't a unique expression for the Laplace transform in our main Theorem 75, but two different expressions.

Proof of Proposition 81. First of all, it follows from Lemma 78 that φ_1 is meromorphic in $\mathcal{G}_{\mathcal{R}}$ and may have a pole of order one at p . Indeed, due to the continuation formula (3.12), the only potential pole p of φ_1 in $\mathcal{G}_{\mathcal{R}}$ should be a zero of γ_1 . It is then on the real line and characterized by (3.13). Moreover, p defined by (3.13) is smaller than $\Theta_2^{\pm}(\theta_1^-)$ (i.e., $p \in \mathcal{G}_{\mathcal{R}}$) if and only if the geometric condition $\gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-)) > 0$ is satisfied, see Figure 3.2. This demonstrates the first item of Proposition 81.

The second item (in particular the fact that φ_1 is bounded at infinity) comes from Lemma 78 together with the fact that (3.4) implies that φ_1 (resp. φ_2) is bounded on the set $\{\theta_2 \in \mathbb{C} : \Re \theta_2 \leq 0\}$ (resp. $\{\theta_2 \in \mathbb{C} : \Re \theta_1 \leq 0\}$).

To prove the boundary condition (3.19) (that we announced in [46, Proposition 7]), we consider θ_1 such that $\Re \theta_1 < 0$, and evaluate the functional equation (3.5) at $(\theta_1, \Theta_2^{\pm}(\theta_1))$. This implies

$$\frac{\gamma_1}{\gamma_2}(\theta_1, \Theta_2^{\pm}(\theta_1))\varphi_1(\Theta_2^{\pm}(\theta_1)) + \varphi_2(\theta_1) = 0,$$

which in turn yields

$$\frac{\gamma_1}{\gamma_2}(\theta_1, \Theta_2^+(\theta_1))\varphi_1(\Theta_2^+(\theta_1)) = \frac{\gamma_1}{\gamma_2}(\theta_1, \Theta_2^-(\theta_1))\varphi_1(\Theta_2^-(\theta_1)). \quad (3.20)$$

Restricting (3.20) to values of $\theta_1 \in (-\infty, \theta_1^-)$, for which $\Theta_2^+(\theta_1)$ and $\Theta_2^-(\theta_1)$ are complex conjugate (see Section 3.2), we reach the conclusion that

$$\varphi_1(\overline{\theta_2}) = \frac{\gamma_1}{\gamma_2}(\theta_1, \theta_2) \frac{\gamma_2}{\gamma_1}(\theta_1, \overline{\theta_2}) \varphi_1(\theta_2),$$

which, by definition (3.18) of G , exactly coincides with the boundary condition (3.19). Although we do not exclude a priori the denominators in (3.20) to vanish, note that this does not happen for $\theta_1 \in (-\infty, \theta_1)$ since then the imaginary part of $\Theta_2^{\pm}(\theta_1)$ is non-zero. \square

The BVP established in Proposition 81 belongs to the class of homogeneous Carleman (or Riemann-Carleman) BVPs with shift, see [74], the shift being here the complex conjugation.

In some cases, the function G in (3.19) can be factorized, leading to an interesting particular case, that we comment below. As we shall see, the well-known skew-symmetric condition

$$2\Sigma = R \cdot \text{diag}(R)^{-1} \cdot \text{diag}(\Sigma) + \text{diag}(\Sigma) \cdot \text{diag}(R)^{-1} \cdot R^{\top} \quad (3.21)$$

(equivalent for the stationary distribution $\pi(x_1, x_2)$ to have a product-form) gives a family of examples where such a factorization holds. In (3.21) we have noted $\text{diag}(A)$ the diagonal matrix with the same diagonal coefficients as those of A .

Remark 83. If there exists a rational function F such that

$$G(\theta_2) = \frac{F(\theta_2)}{F(\overline{\theta_2})},$$

one can transform the boundary condition (3.19) for φ_1 with $G \neq 1$ into a boundary condition for $\varphi_1 \cdot F$ with $G = 1$, namely,

$$(\varphi_1 \cdot F)(\overline{\theta_2}) = (\varphi_1 \cdot F)(\theta_2).$$

Then the associated BVP could be solved using Tutte's invariants [6]. This is what has been done in [46], for the particular case of orthogonal reflections (corresponding to $F(\theta_2) = \frac{1}{\theta_2}$).

We show in Section 3.5.2 that such a rational function F always exists in the skew-symmetric case (3.21), and from this we derive a rational expression of the Laplace transform.

The existence of a rational function F factorizing G as above is reminiscent of the notion of decoupling function or telescoper, introduced in [6, 33].

However, a rational factorization term F as in Remark 83 does not exist in general. More precisely, in the case $\frac{\beta}{\pi} \in \mathbb{Q}$, with

$$\beta = \arccos -\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}, \quad (3.22)$$

we prove in Section 3.5.4 that the existence of F is equivalent for the quantity (3.52) (a norm) to take the value 1, and then we compute a possible expression for F , see (3.54).

On the other hand, if $\frac{\beta}{\pi} \notin \mathbb{Q}$, it is still an open problem to characterize the parameters (Σ, μ, R) for which F exists. (Notice that β is the angle of the wedge in which after linear transformation, the covariance matrix of the Brownian motion $B(t)$ in (3.1) is the identity, see Appendix 3.6, in particular Figure 3.10.)

As a consequence, we cannot systematically use Tutte's invariants technique: we are left with transforming the BVP of Proposition 81 into a more classical one, using a certain conformal mapping having a very convenient gluing property.

3.3.3 Conformal gluing

Our main result in this section is to prove that the function W defined by Equation (3.25) below satisfies the properties of Lemma 84, allowing to transform the Carleman BVP with shift on the curve \mathcal{R} of Proposition 81 into a classical BVP on the segment $[0, 1]$.

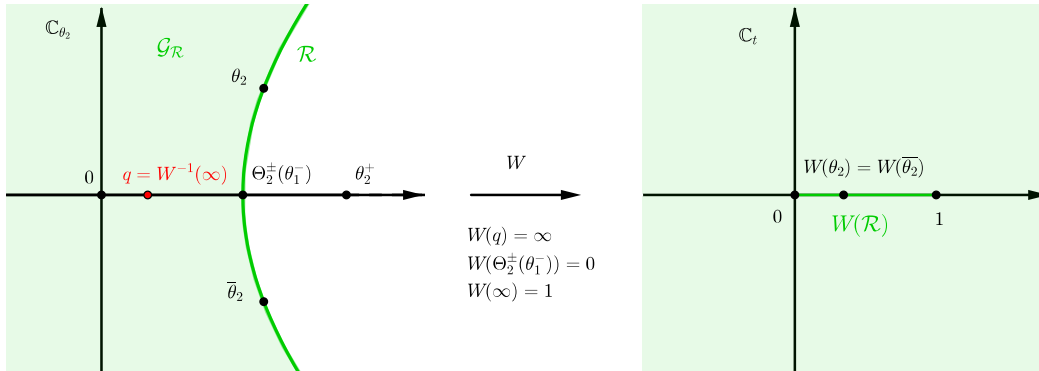


Figure 3.5: Domains, curves and points related to the Carleman BVP with shift on \mathcal{R} (left) and the standard BVP on $[0, 1]$ (right)

First we need to define q by

$$q = \begin{cases} p & \text{if } \gamma_1(\theta_1^-, \Theta_2^{\pm}(\theta_1^-)) > 0, \text{ i.e., if } \varphi_1 \text{ admits } p \in \mathcal{G}_{\mathcal{R}} \text{ as a pole,} \\ \frac{1}{2}\Theta_2^{\pm}(\theta_1^-) & \text{otherwise.} \end{cases} \quad (3.23)$$

Note, the choice $\frac{1}{2}\Theta_2^\pm(\theta_1^-)$ is arbitrary: any point in $\mathcal{G}_\mathcal{R}$ would have been suitable. See Figures 3.2, 3.4 and 3.5.

The function W is built on the function w below (note, w is introduced in [46, Theorem 1]; under the symmetry conditions $\mu_1 = \mu_2$, $\sigma_{11} = \sigma_{22}$, and symmetric reflection vectors in (3.1), Foschini [43] also obtained an expression for the conformal mapping w , see [43, Figure 3]; see finally [5, Equation (4.6)] for a related formulation of w):

$$w(\theta_2) = T_{\frac{\pi}{\beta}} \left(-\frac{2\theta_2 - (\theta_2^+ + \theta_2^-)}{\theta_2^+ - \theta_2^-} \right), \quad (3.24)$$

which itself uses the branch points (3.10), the generalized Chebyshev polynomial ($a \geq 0$)

$$T_a(x) = \cos(a \arccos(x)) = \frac{1}{2} \left\{ (x + \sqrt{x^2 - 1})^a + (x - \sqrt{x^2 - 1})^a \right\},$$

and the angle β in (3.22). Then we define

$$W(\theta_2) = \frac{w(\theta_2) - w(\Theta_2^\pm(\theta_1^-))}{w(\theta_2) - w(q)} = \frac{w(\theta_2) + 1}{w(\theta_2) - w(q)}. \quad (3.25)$$

The last equality is due to the identity

$$\Theta_2^\pm(\theta_1^-) = \frac{\theta_2^+ + \theta_2^-}{2} - \frac{\theta_2^+ - \theta_2^-}{2} \cos \beta.$$

We can see it by direct computation or using the uniformization (3.48), see [46, Section 5] for more details. Then we have $w(\Theta_2^\pm(\theta_1^-)) = T_{\frac{\pi}{\beta}}(\cos \beta) = \cos \pi = -1$.

Lemma 84. *The function W in (3.25)*

- (i) *is analytic in $\mathcal{G}_\mathcal{R} \setminus \{q\}$, continuous in $\overline{\mathcal{G}_\mathcal{R}} \setminus \{q\}$ and bounded at infinity,*
- (ii) *is one-to-one from $\mathcal{G}_\mathcal{R} \setminus \{q\}$ onto $\mathbb{C} \setminus [0, 1]$,*
- (iii) *satisfies $W(\theta_2) = W(\overline{\theta_2})$ for all $\theta_2 \in \mathcal{R}$.*

Proof. It can be found in [46, Lemma 6] that w in (3.24)

- (i') *is analytic in $\mathcal{G}_\mathcal{R}$, continuous in $\overline{\mathcal{G}_\mathcal{R}}$ and unbounded at infinity (T_a admits an analytic continuation on $\mathbb{C} \setminus (\theta_2^+, \infty)$, and even on \mathbb{C} if a is a non-negative integer: in that case T_a is the classical Chebyshev polynomial of the first kind),*
- (ii') *is one-to-one from $\mathcal{G}_\mathcal{R}$ onto $\mathbb{C} \setminus (-\infty, -1]$,*
- (iii') *satisfies $w(\theta_2) = w(\overline{\theta_2})$ for all $\theta_2 \in \mathcal{R}$.*

Here we want to define another conformal gluing function, which glues together the upper part and the lower part of the hyperbola onto the segment $[0, 1]$, and which sends the point q in (3.23) at infinity, see Figure 3.5. For this reason we set W as in (3.25): by construction $W(\Theta_2^\pm(\theta_1^-)) = 0$, $W(\infty) = 1$ and $W(q) = \infty$. The proof of Lemma 84 follows from the above-mentioned properties (i')–(iii') of w together with the definition (3.25) of W . \square

Remark 85. The algebraic nature of the mapping w in (3.24) (or equivalently W in (3.25)) is directly related to the rationality of $\frac{\beta}{\pi}$. Precisely, as shown in [46, Proposition 13], the following behaviors are possible:

- If $\frac{\beta}{\pi} \in \mathbb{Q}$, then w is algebraic;
- If in addition $\frac{\pi}{\beta} \in \mathbb{N}$ (and only in this case), then w is a polynomial;
- If $\frac{\beta}{\pi} \notin \mathbb{Q}$, then w is non-algebraic.

3.3.4 Reduction to a standard BVP

Thanks to the gluing function W in (3.25), we are able to reformulate the Carleman BVP as a standard BVP for an open contour. See Figure 3.5 for a compact view of the two complex planes associated to the Carleman's and Riemann's BVPs. Define ψ_1 by

$$\psi_1(t) = \varphi_1 \circ W^{-1}(t), \quad \forall t \in \mathbb{C} \setminus [0, 1] \quad (3.26)$$

(note, ψ_1 is meromorphic on $\mathbb{C} \setminus [0, 1]$). Equivalently we have $\varphi_1(\theta_2) = \psi_1 \circ W(\theta_2)$ for $\theta_2 \in \mathcal{G}_{\mathcal{R}}$. Obviously W^{-1} is not well defined on $[0, 1]$; however, it does admit upper and lower limits for $t \in [0, 1]$:

$$(W^{-1})^+(t) = \lim_{\substack{u \rightarrow t \\ \Im u > 0}} W^{-1}(u), \quad (W^{-1})^-(t) = \lim_{\substack{u \rightarrow t \\ \Im u < 0}} W^{-1}(u),$$

and similarly for $\psi_1^+(t)$ and $\psi_1^-(t)$. Then for $\theta_2 \in \mathcal{R}$ and $t = W(\theta_2) = W(\overline{\theta_2})$, we have

$$\varphi_1(\theta_2) = \begin{cases} \psi_1^+(t) & \text{if } \Im \theta_2 > 0, \\ \psi_1^-(t) & \text{if } \Im \theta_2 < 0, \end{cases} \quad \varphi_1(\overline{\theta_2}) = \begin{cases} \psi_1^-(t) & \text{if } \Im \theta_2 > 0, \\ \psi_1^+(t) & \text{if } \Im \theta_2 < 0. \end{cases}$$

Define further

$$H(t) = G((W^{-1})^-(t)), \quad \forall t \in [0, 1]. \quad (3.27)$$

Then Proposition 81 becomes:

Proposition 86 (Riemann BVP). *The function ψ_1 in (3.26)*

- (1) *is analytic in $\mathbb{C} \setminus [0, 1]$, bounded at infinity if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \leq 0$ and admitting a simple pole at infinity otherwise,*
- (2) *is continuous on $[0, 1]$ from below (with limits ψ_1^-) and above (with limits ψ_1^+), bounded at 0 and 1 (except if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$: in this case it has a pole of order one at 0),*
- (3) *satisfies, with H defined in (3.27), the boundary condition*

$$\psi_1^+(t) = H(t)\psi_1^-(t), \quad \forall t \in [0, 1]. \quad (3.28)$$

Proof. Items (1) and (2) directly follow from the corresponding items in Proposition 81. With the above definitions, the boundary equation (3.19) becomes

$$\begin{cases} \psi_1^-(t) = G(\theta_2)\psi_1^+(t) & \text{if } \Im \theta_2 > 0, \\ \psi_1^+(t) = G(\theta_2)\psi_1^-(t) & \text{if } \Im \theta_2 < 0. \end{cases}$$

Since $\frac{1}{G(\theta_2)} = G(\overline{\theta_2}) = H(t)$ if $\Im \theta_2 > 0$, and $G(\theta_2) = H(t)$ if $\Im \theta_2 < 0$, the last item follows. \square

3.3.5 Index of the BVP

The resolution of BVPs as in Proposition 86 heavily depends on the *index* χ (see, e.g., [74, Section 5.2]), which is related to the variation of argument of H on $[0, 1]$:

$$\Delta = [\arg H]_0^1, \quad \delta = \arg H(0) \in (-\pi, \pi], \quad \chi = \left\lfloor \frac{\delta + \Delta}{2\pi} \right\rfloor. \quad (3.29)$$

Δ quantifies the variation of argument of H on $[0, 1]$ and $\arg H(1) = \delta + \Delta$. Since $(W^{-1})^{-1}([0, 1]) = \mathcal{R}^-$, Δ in (3.29) can be equivalently written $[\arg G]_{\mathcal{R}^-}$ (from the vertex to infinity).

Remark 87. It is important to notice that $\delta \in (-\pi, \pi]$ in (3.29) corresponds to an arbitrary choice. Any other choice would eventually lead to the same Theorem 75 (though written slightly differently).

First, we compute δ in (3.29).

Lemma 88. *We have*

$$\delta = \begin{cases} 0 & \text{if } \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0, \\ \pi & \text{if } \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0. \end{cases} \quad (3.30)$$

The angle $\delta + \Delta \in (-2\pi, 2\pi)$ and we have

$$\tan \frac{\delta + \Delta}{2} = \frac{\det R \cdot \det \Sigma}{\sigma_{12}(r_{11}r_{22} + r_{12}r_{21}) - \sigma_{22}r_{11}r_{12} - \sigma_{11}r_{22}r_{21}}. \quad (3.31)$$

Note that the denominator of (3.31) can be negative, zero or positive, depending on the parameters.

Proof of Lemma 88. First of all we show the formula (3.30). If first $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0$ then $H(0) = G(\Theta_2^\pm(\theta_1^-)) = 1$, since $\Theta_2^\pm(\theta_1^-) \in \mathbb{R}$ simplifies the quotient (3.18) and then $\delta = \arg H(0) = 0$. In the other case $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$, and we have

$$\lim_{t \rightarrow 0} H(t) = \lim_{\substack{\theta_2 \rightarrow \Theta_2^\pm(\theta_1^-) \\ \theta_2 \in \mathcal{R}^-}} \frac{\theta_2 - \Theta_2^\pm(\theta_1^-)}{\theta_2 - \Theta_2^\pm(\theta_1^-)} = -1.$$

The last equality is due to the fact that the tangent to \mathcal{R} at $\Theta_2^\pm(\theta_1^-)$ is vertical, see Figure 3.5. Indeed if we write $\theta_2 - \Theta_2^\pm(\theta_1^-) = a + ib$ when $\theta_2 \rightarrow \Theta_2^\pm(\theta_1^-)$ with $\theta_2 \in \mathcal{R}^-$, the vertical tangent gives $\frac{a}{b} \rightarrow 0$ and then $\lim_{t \rightarrow 0} H(t) = \lim_{\frac{a}{b} \rightarrow 0} \frac{\frac{a}{b} + i}{\frac{a}{b} - i} = -1$. It implies that $\delta = \pi$.

We are now going to show (3.31). We start by remarking that for $\theta_2 \in \mathcal{R}$, $G(\theta_2) = 1$ if and only if $\theta_2 \in \mathbb{R}$. Accordingly, for $t \in [0, 1]$, $H(t) = 1$ only at $t = 0$. Since $|H| = 1$ on $[0, 1]$, then necessarily $\delta + \Delta \in [-2\pi, 2\pi]$. We now calculate

$$H(1) = \lim_{\substack{\theta_2 \rightarrow \infty \\ \theta_2 \in \mathcal{R}^-}} G(\theta_2).$$

Thanks to Equation (3.9) we easily compute the following limit

$$\lim_{\substack{\theta_2 \rightarrow \infty \\ \theta_2 \in \mathcal{R}^-}} \frac{\Theta_1^-(\theta_2)}{\theta_2} = \frac{-\sigma_{12} - i\sqrt{\det \Sigma}}{\sigma_{11}}.$$

Using this limit together with the definition of G (see (3.18))

$$G(\theta_2) = \frac{\left(r_{11} \frac{\Theta_1^-(\theta_2)}{\theta_2} + r_{21}\right) \left(r_{12} \frac{\Theta_1^-(\theta_2)}{\theta_2} + r_{22}\right)}{\left(r_{12} \frac{\Theta_1^-(\theta_2)}{\theta_2} + r_{22}\right) \left(r_{11} \frac{\Theta_1^-(\theta_2)}{\theta_2} + r_{21}\right)},$$

we obtain that

$$\begin{aligned} H(1) &= \frac{\left(r_{11}(-\sigma_{12} - i\sqrt{\det \Sigma}) + r_{21}\sigma_{11}\right) \left(r_{12}(-\sigma_{12} + i\sqrt{\det \Sigma}) + r_{22}\sigma_{11}\right)}{\left(r_{12}(-\sigma_{12} - i\sqrt{\det \Sigma}) + r_{22}\sigma_{11}\right) \left(r_{11}(-\sigma_{12} + i\sqrt{\det \Sigma}) + r_{21}\sigma_{11}\right)} \\ &= \frac{\sigma_{22}r_{11}r_{12} + \sigma_{11}r_{22}r_{21} - \sigma_{12}(r_{11}r_{22} + r_{12}r_{21}) - i \det R \det \Sigma}{\sigma_{22}r_{11}r_{12} + \sigma_{11}r_{22}r_{21} - \sigma_{12}(r_{11}r_{22} + r_{12}r_{21}) + i \det R \det \Sigma} = \exp(i(\delta + \Delta)). \end{aligned}$$

Remembering that $\arg H(1) = \delta + \Delta \in [-2\pi, 2\pi]$, it gives (3.31) and clearly, $\delta + \Delta$ cannot be equal to $\pm 2\pi$ because $\det R \cdot \det \Sigma \neq 0$. \square

We now prove that

$$\chi = \begin{cases} 0 & \text{if } \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \leq 0, \\ -1 & \text{if } \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) > 0. \end{cases} \quad (3.32)$$

In particular, χ is an intrinsically non-continuous function of the parameters, see also Remark 82.

Recall that the function ψ_1 has been defined in (3.26).

Lemma 89. *The index χ can take only the values 0 and -1 , and we have the dichotomy:*

- $\chi = 0 \iff \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \leq 0 \iff \psi_1$ has no pole at infinity,
- $\chi = -1 \iff \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) > 0 \iff \psi_1$ has a simple pole at infinity.

Note that a simple pole at infinity means that $\psi_1(t) \underset{\infty}{\sim} c \cdot t$ for some non-zero constant c .

Proof. We have already seen in Proposition 81 that the sign of $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$ determines whether φ_1 has a pole in $\mathcal{G}_{\mathcal{R}}$ or not, and thus if ψ_1 has a pole at infinity by the correspondence of Figure 3.5. This shows the two equivalences on the right in the statement of Lemma 89. We are thus left with proving the first two equivalent conditions.

First, if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$, $\delta = \pi$ and we have seen that in this case $H(t) \neq 1$ for all $t \in [0, 1]$. By (3.29), we deduce that $\chi = 0$.

If now $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0$ we notice that

$$\chi = \left\lfloor \frac{\Delta}{2\pi} \right\rfloor = 0 \text{ or } -1.$$

Indeed, we have proved in Lemma 88 that $\Delta \in (-2\pi, 2\pi)$. In particular, the sign of Δ determines χ : if $\text{sgn } \Delta \geq 0$ then $\chi = 0$ and if $\text{sgn } \Delta < 0$, $\chi = -1$. In the rest of the proof, we show that $\text{sgn } \Delta = -\text{sgn } \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$. First, Δ can be computed as

$$\Delta = \arg H(1) = [\arg G]_{\mathcal{R}^-} = \left[\arg \frac{\gamma_1 \bar{\gamma}_2}{\gamma_2 \bar{\gamma}_1}(\Theta_1^-(\theta_2), \theta_2) \right]_{\mathcal{R}^-}.$$

Let $\theta_2 = a - ib \in \mathcal{R}^-$ (we must have $b \geq 0$ and $a \geq \Theta_2^\pm(\theta_1^-) > 0$, see Figure 3.4) and $\theta_1 = \Theta_1^-(\theta_2) \in (-\infty, \theta_1^-]$. Using the expression (3.6) of γ_1 and γ_2 , we obtain

$$\gamma_1 \overline{\gamma_2}(\theta_1, \theta_2) = \gamma_1(\theta_1, a) \gamma_2(\theta_1, a) + r_{21} r_{22} b^2 + ib \theta_1 \det R,$$

from where we deduce that

$$\arg \frac{\gamma_1 \overline{\gamma_2}}{\gamma_2 \overline{\gamma_1}}(\theta_1, \theta_2) = 2 \arctan \frac{b \cdot \theta_1 \cdot \det R}{\gamma_1(\theta_1, a) \gamma_2(\theta_1, a) + r_{21} r_{22} b^2}. \quad (3.33)$$

We now look for the sign of (3.33) when $\theta_2 \rightarrow \Theta_2^\pm(\theta_1^-)$, while remaining in \mathcal{R}^- . This is sufficient to give the sign of Δ , because (3.33) does not change sign on \mathcal{R}^- due to the fact that $G(\theta_2) = 1$ on \mathcal{R}^- if and only if $\theta_2 = \Theta_2^\pm(\theta_1^-)$.

When $\theta_2 \rightarrow \Theta_2^\pm(\theta_1^-)$ we have $b \rightarrow 0$, $a \rightarrow \Theta_2^\pm(\theta_1^-)$ and $\theta_1 \rightarrow \theta_1^-$. We thus have

$$\begin{aligned} \operatorname{sgn} \arg \frac{\gamma_1 \overline{\gamma_2}}{\gamma_2 \overline{\gamma_1}}(\theta_1, \theta_2) &= \operatorname{sgn} b \cdot \operatorname{sgn} \theta_1 \cdot \operatorname{sgn} \det R \cdot \operatorname{sgn} \gamma_2(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \cdot \operatorname{sgn} \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \\ &= (+1)(-1)(+1)(+1) \operatorname{sgn} \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \\ &= -\operatorname{sgn} \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)), \end{aligned}$$

because $b \geq 0$, $\theta_1 < 0$, $\det R > 0$ by (3.2), and $\gamma_2(\theta_1^-, \Theta_2^\pm(\theta_1^-)) > 0$ (because $r_{22} > 0$ and $r_{22}\mu_1 - r_{12}\mu_2 < 0$, see Figure 3.2 to visualize this geometric condition). Then $\operatorname{sgn} \Delta = \operatorname{sgn} \arg \frac{\gamma_1 \overline{\gamma_2}}{\gamma_2 \overline{\gamma_1}}(\theta_1, \theta_2) = -\operatorname{sgn} \gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-))$, concluding the proof. \square

3.3.6 Resolution of the BVP

We are now in position to conclude the proof of Theorem 75. Reformulating Proposition 86, the function ψ_1 in (3.26)

- is sectionally analytic in $\mathbb{C} \setminus [0, 1]$,
- is continuous on $[0, 1]$ from below (with limits ψ_1^-) and above (with limits ψ_1^+),
 - is bounded at the vicinities of $[0, 1]$ if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0$,
 - has a pole of order one at 0 and bounded at 1 if $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$,
- is bounded at infinity if there is no pole before $\Theta_2^\pm(\theta_1^-)$ (then taking the value $\varphi_1(q)$), and with a pole of order one (see Lemma 78) at infinity if not (in short, it has a pole of order $-\chi$ at infinity),
- satisfies $\psi_1^+(t) = H(t)\psi_1^-(t)$ for $t \in [0, 1]$, with index χ given by (3.32), cf. also Lemma 89.

End of proof of Theorem 75. Our main reference for the resolution of the above so-called homogeneous BVP on an open contour is the book [82] of Muskhelishvili, see in particular [82, §79].

First of all, we prove that there exists a non-zero constant c such that

$$\psi_1(t) = c(t-1)^{-\chi} \exp \Gamma(t), \quad (3.34)$$

where Γ is the following function, sectionally analytic on $\mathbb{C} \setminus [0, 1]$:

$$\Gamma(t) = \frac{1}{2i\pi} \int_0^1 \frac{\log H(z)}{z-t} dz. \quad (3.35)$$

To make precise the definition (3.35), we define $\log H(z)$ by the facts that it should vary continuously over $[0, 1]$, and its initial value is such that $\log H(0) = i\delta$ (i.e., 0 if $H(0) = 1$ and $i\pi$ if $H(0) = -1$, see (3.30)).

At the vicinities 0 and 1, we have by [82, §29 and §79] that

$$\exp \Gamma(t) = t^{-\frac{\delta}{2\pi}} \Omega_0(t), \quad \exp \Gamma(t) = (t-1)^{\frac{\delta+\Delta}{2\pi}} \Omega_1(t),$$

for some function Ω_0 (resp. Ω_1) analytic in a neighborhood of 0 (resp. 1) and non-zero at 0 (resp. 1). Then we set

$$X(t) = t^{\frac{\delta}{\pi}} (t-1)^{-\chi} \exp \Gamma(t). \quad (3.36)$$

The function X in (3.36) is sectionally analytic in $\mathbb{C} \setminus [0, 1]$, and by construction of Γ and the Sokhotski-Plemelj formulas, it satisfies the boundary condition (3.28) (see [82, §79] for more details). Furthermore it has a pole of order $-\chi + \frac{\delta}{\pi}$ at infinity and is bounded at 0 and 1: indeed, $\frac{\delta}{\pi}$ and $-\chi$ are both equal to 0 or 1, see Lemmas 88 and 89. Then we consider two cases separately.

• First case: $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) \neq 0$. Then $\delta = 0$, and the function X in (3.36) simplifies into

$$X(t) = (t-1)^{-\chi} \exp \Gamma(t).$$

It satisfies the exact same boundary condition as (3.28). Looking at the ratio $\frac{\psi_1}{X}$, the boundary condition (3.28) gives that on $[0, 1]$,

$$\frac{\psi_1^+}{X^+} = \frac{\psi_1^-}{X^-}.$$

The above ratio is then analytic in the entire plane, even at the vicinities 0 and 1. The point 0 is a regular point and 1 is a removable singularity. Indeed, 1 is an isolated singular point, at which $\frac{\psi_1}{X}$ might be infinite with degree less than unity (namely, $-\chi + \frac{\Delta}{2\pi}$). Moreover, the function $\frac{\psi_1}{X}$ is bounded at infinity, because both X and ψ_1 have a pole of the same order $-\chi$. Thanks to Liouville's theorem, we deduce that $\frac{\psi_1}{X}$ is constant. In conclusion, the formula (3.34) holds in this case.

• Second case: $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$. Then $\delta = \pi$, $\chi = 0$ and $X(t) = t \exp \Gamma(t)$ in (3.36). Firstly, we notice that the function $t\psi_1$ satisfies the boundary condition (3.28), is bounded at 0 and 1, and has a pole of order one at infinity. Moreover, the function X has a pole of order 1 at infinity. Considering then the ratio $\frac{t\psi_1}{X}$, the boundary condition (3.28) implies that on $[0, 1]$,

$$\frac{t\psi_1^+}{X^+} = \frac{t\psi_1^-}{X^-}.$$

The above ratio function is thus analytic in the entire complex plane, including the vicinities 0 and 1. It is indeed bounded at 1, and has a removable singularity at 0: the point 0 is an isolated singular point, at which $\frac{t\psi_1}{X}$ might be infinite with degree less than $\frac{1}{2}$. Using again Liouville's theorem, we deduce that the function $\frac{t\psi_1}{X}$ is a constant. Formula (3.34) therefore also holds.

We now deduce from (3.34) the formula (3.8) stated in Theorem 75. Going from the t -plane back to the θ_2 -plane (see (3.26) and Figure 3.5), one has that for some constant c ,

$$\varphi_1(\theta_2) = \psi_1(W(\theta_2)) = c(W(\theta_2) - 1)^{-\chi} \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(\theta) \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta \right\}. \quad (3.37)$$

Using the equation (3.25) relating W and w , we easily obtain

$$W(\theta_2) - 1 = \frac{w(q) + 1}{w(\theta_2) - w(q)}$$

as well as

$$\frac{W'(\theta)}{W(\theta) - W(\theta_2)} = \frac{w'(\theta)}{w(\theta) - w(\theta_2)} - \frac{w'(\theta)}{w(\theta) - w(q)}.$$

Remembering that in the case $\chi = -1$ one has $q = p$, see (3.23), we finally obtain that for some constant c' ,

$$\varphi_1(\theta_2) = c' \left(\frac{1}{w(\theta_2) - w(p)} \right)^{-\chi} \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(\theta) \frac{w'(\theta)}{w(\theta) - w(\theta_2)} d\theta \right\}.$$

By definition (3.4) of the Laplace transform we have $\varphi_1(0) = \nu_1(\mathbb{R}_+)$. To find the exact constant c' (and thereby our main result (3.8)), we simply evaluate the above formula at $\theta_2 = 0$. \square

Clearly, the integral expression (3.8) of φ_1 is meromorphic in the domain $\mathcal{G}_{\mathcal{R}}$. In Section 3.4 we shall see that it can be meromorphically continued on the much larger domain $\mathbb{C} \setminus [\theta_2^+, \infty)$.

To conclude this part, let us make Remark 87 more precise. In the case $\gamma_1(\theta_1^-, \Theta_2^\pm(\theta_1^-)) = 0$ (i.e., $H(0) = -1$), we have chosen $\delta = \pi$ in (3.29). (Recall that choosing δ is tantamount to fixing a determination of the arg (or log) function.) Remarkably, any other choice of δ would have led to the same explicit expression for φ_1 , though written differently. For instance, if we had taken $\delta = -\pi$ instead, the index χ would have been -1 (instead of 0), and the two determinations of the logarithm would differ by $-2i\pi$. With our notation in the proof of Theorem 75, see (3.35) for the definition of Γ , we would have obtained for some constant c'' ,

$$\psi_1(t) = c'' \frac{t-1}{t} \exp \Gamma(t).$$

This actually corresponds to the formula of Theorem 75 associated with $\chi = -1$.

3.3.7 Generalizations

Our main Theorem 75 is derived under the hypothesis (3.7) that the coordinates (μ_1, μ_2) of the drift are negative. However, the conditions (3.2) (equivalent to the existence of a stationary distribution) allow the drift to have one non-negative coordinate. In the next few lines, we assume that $\mu_1 \geq 0$ or $\mu_2 \geq 0$, and we comment on the slight differences which would have arisen in the analytic treatment of the functional equation (3.5).

In the case of a drift having one non-negative coordinate, the reflected Brownian motion in the quadrant has a pathwise behavior which is quite different (it spends most of its time near the axis towards which the drift is directed). However, from the point of view of our analysis, the differences are only technical. For instance, the continuation formula (3.12) of Lemma 78 may use the other branch Θ_1^+ , as if $\mu_1 > 0$, the small branch (taking value 0 at 0) is Θ_1^+ and not Θ_1^- anymore, see (3.9). As a result, the poles of φ_1 may be located at places different from those described in Proposition 81. Despite these few differences, there still exists a Cauchy-type integral expression for the Laplace transform φ_1 , similar to the one stated in Theorem 75.

Let us also very briefly mention here the case of reflected Brownian motion in higher dimension [49, 56, 12, 11, 25]. Compared to its two-dimensional analogue, much less is known. However, an analogue of the functional equation (3.5) can still be stated (since the BAR exists in any dimension, see [55, 27]). Clearly, our techniques (based on complex analysis) use the dimension 2, and in our opinion, generalizing in higher dimension these BVP techniques is a difficult open problem. In the discrete case too, the case of dimension 3 is less understood. One can mention [20] for some ideas to state a BVP in dimension 3, as well as [8] for more combinatorial techniques.

3.4 Singularity analysis and asymptotics of the boundary distribution

The asymptotics (up to a constant) of the boundary measures has been obtained by Dai and Miyazawa in [29], see also [28, 44] for the interior measure. In this section we show that our expression (3.8) for the Laplace transform φ_1 stated in Theorem 75 is perfectly suited for singularity analysis, and accordingly to study the asymptotics (including the computation of the constant) of the boundary stationary distributions ν_1 and ν_2 . We shall first recall the result of [29] and express it in terms of our notations. Then we will explain how, thanks to Theorem 75, we could obtain a new proof of this result and make the constants explicit.

3.4.1 Asymptotic results

Theorem 90 (Theorem 6.1 of [29]). *The following asymptotics holds:*

$$\lim_{x \rightarrow \infty} \frac{\nu_1(x)}{x^\kappa e^{-\tau_2 x}} = b, \quad (3.38)$$

where $\kappa \in \{-\frac{3}{2}, -\frac{1}{2}, 0, 1\}$ and $\tau_2 \in \{p, p', \theta_2^+\}$ are given in Table 3.1, and b is some positive constant. See Figures 3.6 and 3.7 to visualize geometrically the different cases.

We now propose a series of three remarks on Theorem 90.

- We have already introduced p , see (3.13), and p' is as follows: it is the (unique, when it exists) non-zero point $p' = \Theta_2^+(r)$, with r defined by $\gamma_2(r, \Theta_2^-(r)) = 0$ and $r \leq \Theta_1^+(\theta_2^+)$. It will be convenient to adopt the following notation: we will write $p > \theta_2^+$ (resp. $p' > \theta_2^+$) when p (resp. p') does not exist.

- Theorem 6.1 of [29] deals with the asymptotics of $\nu_1(x, \infty)$, and not with that of $\nu_1(x)$, as stated in Theorem 90 below. However, the two statements are equivalent: when the asymptotics of the density has the form $b x^\kappa e^{-\tau_2 x}$ as in (3.38), the corresponding tail probability is given by the exact same asymptotics (with another constant b), see [28, Lemma D.5].

- Before stating Table 3.1, which gives the values of κ and τ_2 of Theorem 90, we briefly recall Dai and Miyazawa's notations: $\theta^{(2, \max)}$ is the (unique) point of the ellipse $\gamma = 0$ such that $\theta_2^{(2, \max)} = \theta_2^+$, $\tau_2 = \sup\{\theta_2 > 0 : \exists \theta_1 \in \mathbb{R}, \varphi(\theta_1, \theta_2) < \infty\}$ and $\theta^{(2, r)}$ is the intersection point of the straight line $\gamma_1 = 0$ and the ellipse $\gamma = 0$. Notice that the definition of τ_2 does not rely on an analytic continuation of φ .

Cases		Dai and Miyazawa's categories		κ	τ_2
p or $p' \in (0, \theta_2^+)$	1.a	$p < \Theta_2^\pm(\theta_1^-)$	$\tau_2 < \theta_2^{(2,\max)}$	Categories I or II	
	1.b	$\Theta_2^\pm(\theta_1^-) \leq p < p'$		Category I	
	1.c	$\Theta_2^\pm(\theta_1^-) \leq p' < p$		Category III, $\tau_2 \neq \theta_2^{(2,r)}$	
	1.d	$\Theta_2^\pm(\theta_1^-) \leq p = p'$		Category III, $\tau_2 = \theta_2^{(2,r)}$	
p and $p' \geq \theta_2^+$	2.a	p and $p' > \theta_2^+$	$\tau_2 = \theta_2^{(2,\max)}$	Category I, $\theta^{(2,r)} \neq \theta^{(2,\max)}$	
	2.b	$\theta_2^+ = p$		Category I, $\theta^{(2,r)} = \theta^{(2,\max)}$	
	2.c	$\theta_2^+ = p'$		Category II, $\theta^{(2,r)} \neq \theta^{(2,\max)}$	
	2.d	$\theta_2^+ = p = p'$		Category II, $\theta^{(2,r)} = \theta^{(2,\max)}$	

Table 3.1: Dai and Miyazawa's categories expressed with our notations

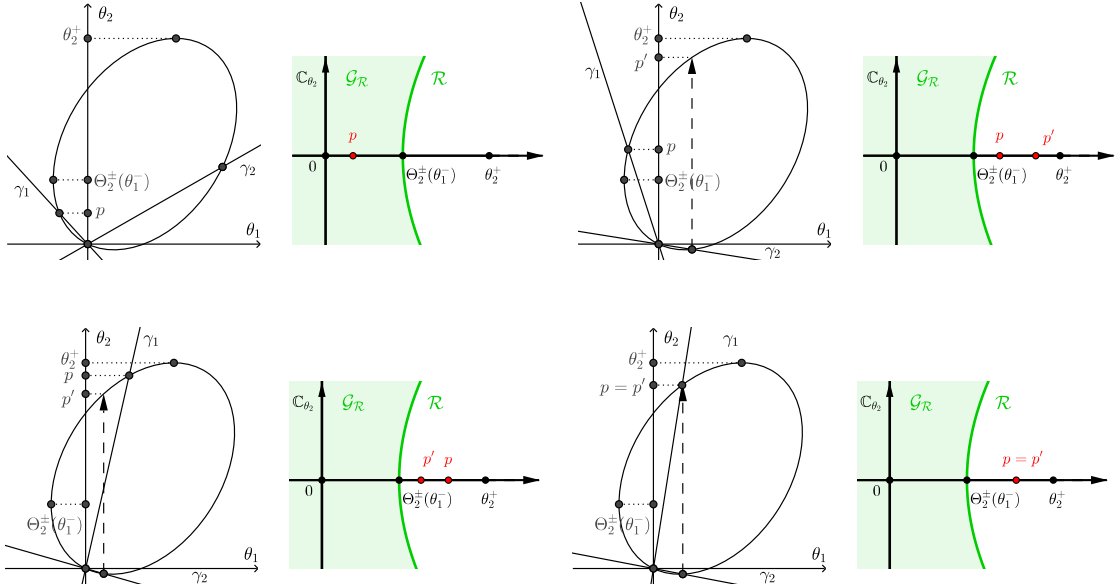


Figure 3.6: Ellipses and curves in the cases 1.a, 1.b, 1.c and 1.d of Table 3.1

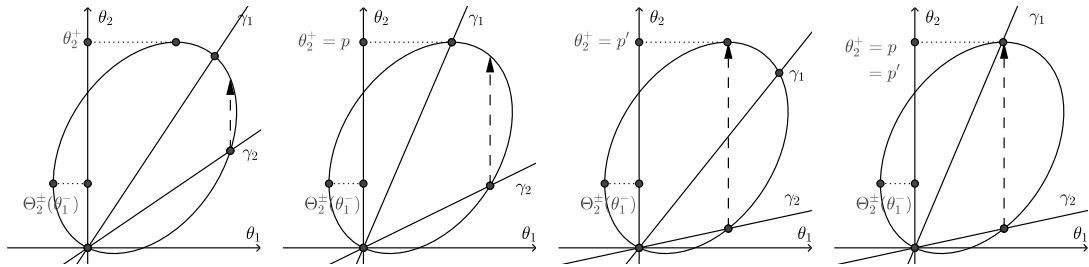


Figure 3.7: Ellipses in the cases 2.a, 2.b, 2.c and 2.d of Table 3.1

3.4.2 Sketch of the proof of Theorem 90

The idea is to study the singularities of φ_1 and to use transfer theorems (such as [32, Theorem 37.1]) relating the asymptotics of a function and the singularities of its Laplace transform. In our case, the singularity closest to 0 will determine the asymptotics. Our aim here is not to propose a complete proof of Theorem 90 (for this we refer to [29]), but rather to illustrate that Theorem 75 indeed implies Theorem 90, with the additional feature of providing an exact expression for the constant b in (3.38). We give details for the case 1.a of Table 3.1, and only sketch the difficulties that would arise in the remaining cases.

Proof of Theorem 90 and computation of b in case 1.a. In that case, the index χ equals -1 and the formula (3.8) gives

$$\varphi_1(\theta_2) = \nu_1(\mathbb{R}_+) \left(\frac{w(0) - w(p)}{w(\theta_2) - w(p)} \right) \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(\theta) \left[\frac{w'(\theta)}{w(\theta) - w(\theta_2)} - \frac{w'(\theta)}{w(\theta) - w(0)} \right] d\theta \right\}.$$

Recall that w is analytic, one-to-one on $\mathcal{G}_{\mathcal{R}}$ and further satisfies $w(\theta_2) \neq w(\theta)$, for all $\theta_2 \in \mathcal{G}_{\mathcal{R}}$ and $\theta \in \mathcal{R}^-$. As a first result, the integral part (thus also its exponential) is analytic in the domain $\mathcal{G}_{\mathcal{R}}$. A second consequence is that $\frac{1}{w(\theta_2) - w(p)}$ has a simple pole at p : $\varphi_1(\theta_2) = \frac{b+o(1)}{\theta_2 - p}$, with

$$b = \nu_1(\mathbb{R}_+) \left(\frac{w(0) - w(p)}{w'(p)} \right) \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(\theta) \left[\frac{w'(\theta)}{w(\theta) - w(p)} - \frac{w'(\theta)}{w(\theta) - w(0)} \right] d\theta \right\}.$$

Theorem 37.1 of [32] gives the announced asymptotics $\nu_1(x) = e^{-px}(b + o(1))$ as $x \rightarrow \infty$. \square

In the other cases, the singularities are not in $\mathcal{G}_{\mathcal{R}}$ and we thus need to extend meromorphically φ_1 in a larger domain:

Proposition 91 (see Theorem 109). *The Laplace transform φ_1 can be meromorphically continued on the domain $\mathbb{C} \setminus [\theta_2^+, \infty)$.*

Proposition 91 has been proved in [44, Theorem 11], see Theorem 109. Note that the formula (3.8) of Theorem 75 provides an alternative, direct, analytic proof, which can be sketched as follows: the equation (3.8) is valid a priori only for θ_2 in $\mathcal{G}_{\mathcal{R}}$. However, considering a Hankel contour similar to that of Figure 3.9, surrounding $[\theta_2^+, \infty)$, we could write φ_1 as an integral over the cut $[\theta_2^+, \infty)$. The study of the so-obtained formula would give the singularities p, p' and θ_2^+ as in Table 3.1, and would lead to the precise asymptotics (we could even obtain the full asymptotic development) and the computation of b .

Remark 92. We have already commented on the fact that within a single formula, Theorem 75 actually captures two different expressions, depending on the value of χ in (3.32). From an asymptotic viewpoint, Section 3.4 shows that different cases exist as well, depending on various parameters. It should be noted that these cases are all different, i.e., δ and χ do not govern the asymptotic behavior of the boundary measures.

3.5 Algebraic nature and simplification of the Laplace transforms

Motivations

In this section we are interested in the following question: in which extent is it possible to simplify the expressions of the Laplace transforms given in Theorem 75? For instance, is this possible that these functions be algebraic or even rational?

This is of paramount importance: first, simplified expressions would lead to an easier analysis, in particular for asymptotic analysis or for taking inverse Laplace transforms; second, understanding the parameters (Σ, μ, R) for which the Laplace transforms are rational should reveal intrinsic structure of the model. In the particular case of the identity covariance matrix Σ , some attempts of simplifications may be found in [42, Chapter 4].

In the literature, this question has received much interest in the discrete setting. One can first think at the famous Jackson's networks [61] and their product form solutions. In a closer context, Latouche and Miyazawa [68], Chen, Boucherie and Goseling [17], obtain geometric necessary and sufficient conditions for the stationary distribution of random walks in the quarter plane to be sums of geometric terms. Such criteria can be applied, e.g., to derive an approximation scheme to error bounds for performance measures of random walks in the quarter plane [18].

In our context of reflected diffusions in the quadrant, analogues of these results are obtained by Dieker and Moriarty [31]: a simple condition (involving the single angle (3.39)) for the stationary density to be a sum of exponential terms is derived. We will discuss the links with our results in Section 3.5.1. In the simplest case the sum of exponential terms consist of one single term: this is the skew-symmetric condition (3.21) of [56]. In this case, a factorization such as in Remark 83 exists and applying an invariant method we are able to solve the skew-symmetric case, see our Section 3.5.2, yielding new proofs to already known results.

Another main reason which can lead to simplified expressions comes from the group of the model, which is a group of symmetries naturally associated with the model (cf. Section 3.5.4). Before being more precise, let us mention that this reason is deeply different than the first one: the group only depends on the covariance matrix and drift vector, and is therefore independent of the reflection matrix, while Dieker and Moriarty's condition also depends on the reflection angles, see (3.39). In the discrete setting, this group was shown to have a decisive influence on the D-finiteness (a function is D-finite if it satisfies a linear differential equation with polynomial coefficients on \mathbb{Q}) of the generating functions, see [37, 10, 33]. For reflected Brownian motion in the quadrant with orthogonal reflections, it is shown in [46] that the Laplace transform is algebraic (note, any algebraic function is D-finite) if and only if the group is finite. We present a structural result in Section 3.5.3.

In Section 3.5.4 we prove a criterion for the algebraicity of the Laplace transform, peculiar to the finite group case. Namely, we show that φ_1 is algebraic if and only if a certain norm is equal to 1, which in turn can be reformulated as a condition on a single angle, this way obtaining a result close to Dieker and Moriarty's condition.

Finally, we focus in Section 3.5.5 on the case of orthogonal reflections, and derive a new proof of the main result of [46], as a consequence of Theorem 75.

3.5.1 Dieker and Moriarty's criterion

For the sake of completeness, let us mention the following result:

Theorem 93 (Theorem 1 in [31]). *The stationary density is a sum of exponentials if and only if*

$$\frac{\epsilon + \delta - \pi}{\beta} \in -\mathbb{N} = -\{0, 1, 2, \dots\}, \quad (3.39)$$

with ϵ and δ in $(0, \pi)$ and

$$\tan \epsilon = \frac{\sin \beta}{\frac{r_{21}}{r_{11}} \sqrt{\frac{\sigma_{11}}{\sigma_{22}}} + \cos \beta}, \quad \tan \delta = \frac{\sin \beta}{\frac{r_{12}}{r_{22}} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos \beta}. \quad (3.40)$$

Notice that this is not the exact statement of [31, Theorem 1], as in Dieker and Moriarty's paper, the Brownian motion is assumed to have an identity covariance matrix and evolves in a wedge with an arbitrary opening angle, whereas we consider Brownian motion with arbitrary covariance matrix but in the quarter plane. A simple linear transform, which is made explicit in Appendix 3.6, makes both statements equivalent. The expression (3.40) of the angles ϵ and δ follows from this transformation.

3.5.2 Skew-symmetric case

The skew-symmetric case holds when the matrix condition (3.21) is satisfied [56]. In dimension two, (3.21) can be reduced to the single equation

$$2\sigma_{12} = \frac{r_{21}}{r_{11}}\sigma_{11} + \frac{r_{12}}{r_{22}}\sigma_{22}. \quad (3.41)$$

Using the identities in (3.62) we easily show that the above condition is equivalent to

$$\epsilon + \delta = \pi,$$

which is the case where the quantity of Dieker and Moriarty's criterion is equal to 0, see (3.39).

It is known, see [56], that condition (3.21) is satisfied if and only if the stationary distribution has a product form, i.e., $\pi(x_1, x_2) = \pi_2(x_1)\pi_1(x_2)$, where the π_i 's are the marginal densities of π . Furthermore it implies that the stationary distribution is exponential, meaning that

$$\pi(x_1, x_2) = \alpha_1 \alpha_2 e^{-\alpha_1 x_1 - \alpha_2 x_2}, \quad \text{with } \alpha = -2 \cdot \text{diag}(\Sigma)^{-1} \cdot \text{diag}(R) \cdot R^{-1} \cdot \mu. \quad (3.42)$$

See for example [49, §10], [56] or [29] for more details on these results.

In fact, in the skew-symmetric case, a rational factorization such as in the Remark 83 exists. It is then possible to find again, in another way, some already known results. Indeed if the skew-symmetric condition holds, we shall prove below that for $\theta_2 \in \mathcal{R}$,

$$G(\theta_2) = \frac{F(\theta_2)}{F(\overline{\theta_2})}, \quad \text{with } F(\theta_2) = \alpha_2 - \theta_2. \quad (3.43)$$

where by (3.42) α_2 takes the value

$$\alpha_2 = \frac{2r_{22}(r_{21}\mu_1 - r_{11}\mu_2)}{\sigma_{22} \det R}. \quad (3.44)$$

Using (3.41) and (3.14) it is easy to remark that $\alpha_2 = p$, which is the only possible pole of φ_1 in $\mathcal{G}_{\mathcal{R}}$ as we observe after Lemma 78. It follows from Proposition 81 that the function $\varphi_1(\theta_2)(\alpha_2 - \theta_2)$

- is bounded on $\mathcal{G}_{\mathcal{R}}$ and converges at infinity to $\nu_1(0)$ (thanks to the initial value theorem and Lemma 78),
- is continuous on $\overline{\mathcal{G}_{\mathcal{R}}}$,
- satisfies the boundary condition $\varphi_1(\theta_2)(\alpha_2 - \theta_2) = \varphi_1(\overline{\theta_2})(\alpha_2 - \overline{\theta_2})$ for all $\theta_2 \in \mathcal{R}$.

Then, using an invariant lemma (see Lemma 2 in [74, Section 10.2]), we conclude that for some constant C ,

$$\varphi_1(\theta_2) = \frac{C}{\alpha_2 - \theta_2}.$$

Evaluating the above equation at 0 and using Lemma 77 gives $C = \frac{\sigma_{11}}{2r_{11}}\alpha_1\alpha_2$. Inverting the Laplace transform implies that the stationary distribution is exponential. Then using the functional equation (3.5), we find that $\frac{\sigma_{11}\sigma_{22}}{4r_{11}r_{22}}\alpha_1\alpha_2\varphi(\theta_1, \theta_2) = \varphi_1(\theta_2)\varphi_2(\theta_1)$, which means that π has a product form.

Proof of Equation (3.43). Let $\theta_2 \in \mathcal{R}$ and note $\theta_1 = \Theta_1^-(\theta_2)$. Elementary computations give

$$\gamma_1(\theta_1, \theta_2)\gamma_2(\theta_1, \overline{\theta_2}) = \theta_1 \left(\frac{2r_{22}}{\sigma_{22}}(r_{21}\mu_1 - r_{11}\mu_2) - \theta_2 \det R \right).$$

To find this, we just have to use the skew-symmetric condition (3.41) and to remark that for $\theta_2 \in \mathcal{R}$ Vieta's formulas give $\theta_2\overline{\theta_2} = \frac{1}{\sigma_{22}}(\sigma_{11}\theta_1^2 + 2\mu_1\theta_1)$ and $\overline{\theta_2} = -\theta_2 - \frac{1}{\sigma_{22}}(2\sigma_{12}\theta_1 + 2\mu_2)$. Then, using the expression (3.44) of α_2 we find $G(\theta_2) = \frac{\alpha_2 - \theta_2}{\alpha_2 - \overline{\theta_2}}$ for $\theta_2 \in \mathcal{R}$. \square

3.5.3 Structural form of the Laplace transforms and finite group

By finite group models, we mean that $\frac{\beta}{\pi} \in \mathbb{Q}$, where β is defined in (3.22), see Section 3.5.4 for more details on the group. In this case, and in this case only, the function W in (3.25) is algebraic (as the generalized Chebyshev polynomial (3.24) is, see Remark 85), yielding the following structural result:

Proposition 94. *If $\frac{\beta}{\pi} \in \mathbb{Q}$, the Laplace transform φ_1 of Theorem 75 is the product of an algebraic function by the exponential of a D-finite function.*

Proof. This easily follows from the fact that the Cauchy integral of a D-finite function is D-finite, see, e.g., [94]. \square

However, it is not true in general that the exponential of a D-finite function is still D-finite.

3.5.4 An algebraicity criterion for finite group models

This section is inspired by [37, Chapter 4], and is independent of the rest of the paper. We present purely algebraic manipulations, thanks to which we can study rational and algebraic solutions of the functional equation (3.5). This question is directly related to Dieker and Moriarty's condition (3.39), as the Laplace transform of sums of exponential terms is obviously rational.

Theorem 95. *Assume that $\frac{\beta}{\pi} \in \mathbb{Q}$. If the functional equation (3.5) admits algebraic solutions (and in particular rational solutions), then necessarily*

$$\frac{\epsilon + \delta - \pi}{\beta} \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}. \quad (3.45)$$

Before proving Theorem 95, let us do some comments, in particular a comparison with [31]. Our criterion involves the same angle as [31, Theorem 1], namely $\frac{\epsilon + \delta - \pi}{\beta}$. However our condition is less restrictive ($-\mathbb{N}$ is included in $\mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$), as we allow not only rational but also algebraic solutions. On the other hand, notice that we are not able to characterize the parameters (Σ, μ, R) for which our algebraic solutions are rational, nor to say whether the Laplace transform φ_1 is one of these algebraic solutions. Finally, our proof does not reveal if Theorem 95 still holds in the case $\frac{\beta}{\pi} \notin \mathbb{Q}$ (although our condition (3.45) does not need $\frac{\beta}{\pi}$ to be rational). Theorem 95 should rather be considered as a structural result on the solutions of the functional equation.

To prove Theorem 95, we shall combine two points of view on the functional equation (3.5). Firstly, following [37, Chapter 4], we shall perform computations in the quotient ring (below, γ is the kernel (3.6))

$$\mathbb{C}(\theta_1, \theta_2) / \gamma(\theta_1, \theta_2), \quad (3.46)$$

which contains all functions of the form $f(\theta_1) + g(\theta_1)\theta_2$, with rational f and g . Secondly, we will also work on a uniformization (or parametrization) of the kernel:

$$\mathcal{S} = \{(\theta_1, \theta_2) \in (\mathbb{C} \cup \{\infty\})^2 : \gamma(\theta_1, \theta_2) = 0\} = \{(\theta_1(s), \theta_2(s)) : s \in \mathbb{C} \cup \{\infty\}\}, \quad (3.47)$$

where

$$\begin{cases} \theta_1(s) = \frac{\theta_1^+ + \theta_1^-}{2} + \frac{\theta_1^+ - \theta_1^-}{4} \left(s + \frac{1}{s} \right), \\ \theta_2(s) = \frac{\theta_2^+ + \theta_2^-}{2} + \frac{\theta_2^+ - \theta_2^-}{4} \left(\frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s} \right). \end{cases} \quad (3.48)$$

Equations (3.47) and (3.48) are easily verified, by evaluating the kernel at $(\theta_1(s), \theta_2(s))$ and using the values of the branch points (3.10) and β in (3.22); see also [46, Section 5].

In this context, the group of the model is the dihedral group $\langle \xi, \eta \rangle$ acting on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, where

$$\xi(s) = \frac{1}{s}, \quad \eta(s) = \frac{e^{2i\beta}}{s}, \quad \delta(s) = \eta\xi(s) = e^{2i\beta}s. \quad (3.49)$$

It is finite if and only if δ in (3.49) has finite order, i.e., if and only if $\frac{\beta}{\pi} \in \mathbb{Q}$.

Finally, we define s_0 such that $(\theta_1(s_0), \theta_2(s_0)) = (0, 0)$; the equations in (3.48) give

$$s_0 = -\frac{\theta_1^+ + \theta_1^- + 2i\sqrt{-\theta_1^+\theta_1^-}}{\theta_1^+ - \theta_1^-} = -\frac{\theta_2^+ + \theta_2^- - 2i\sqrt{-\theta_2^+\theta_2^-}}{\theta_2^+ - \theta_2^-} e^{i\beta}. \quad (3.50)$$

Proof of Theorem 95. We shall mainly work on the Riemann sphere \mathcal{S} introduced in (3.47), and we use the following notation: any function f of the complex variable θ_1 can be lifted on \mathcal{S} , by setting

$$\tilde{f}(s) = f(\theta_1(s)).$$

Most of the time we will lighten the notation, writing $f(s)$ instead.

The starting point is to translate the boundary condition (3.19) to \mathcal{S} :

$$\varphi_1(\delta s) = G(s)\varphi_1(s), \quad (3.51)$$

where δ is as in (3.49), and by definition (3.18) of G one has

$$G(s) = \frac{\gamma_1(s)/\gamma_2(s)}{\gamma_1(\eta s)/\gamma_2(\eta s)}.$$

Assuming that the group has order $2n$ ($\delta^n = 1$, with $\delta = \eta\xi$) and iterating n times (3.51), we get

$$\varphi_1(s) = \varphi_1(\delta^n s) = G(s) \cdot G(\delta s) \cdots G(\delta^{n-1}s) \cdot \varphi_1(s).$$

As an obvious consequence, for a solution in the quotient ring (3.46) to exist, it is necessary for the following quantity to equal 1:

$$N(G) := G(s) \cdot G(\delta s) \cdots G(\delta^{n-1}s) = \frac{\gamma_1(s) \cdot \gamma_1(\delta s) \cdots \gamma_1(\delta^{n-1}s) \cdot \gamma_2(\eta s) \cdot \gamma_2(\delta \eta s) \cdots \gamma_2(\delta^{n-1}\eta s)}{\gamma_2(s) \cdot \gamma_2(\delta s) \cdots \gamma_2(\delta^{n-1}s) \cdot \gamma_1(\eta s) \cdot \gamma_1(\delta \eta s) \cdots \gamma_1(\delta^{n-1}\eta s)}. \quad (3.52)$$

(Our notation $N(G)$ comes from the fact that (3.52) is interpreted as a norm, see [37, Chapter 4].) The key point is that by Lemma 96 (whose proof is technical, and can be found below), $N(G) = 1$ if and only if the condition on the angles (3.45) holds. Condition (3.45) is therefore necessary for the existence of solutions in the quotient ring (3.46), and a fortiori for rational solutions of (3.5).

Conversely, let us assume that $N(G) = 1$. We shall prove the existence of algebraic solutions of the functional equation (3.5). Unfortunately, this does not exclude a priori the existence of non-algebraic solutions, and thus this does not characterize the algebraic nature of φ_1 (clearly there is not uniqueness of the solutions to the fundamental functional equation). Introduce

$$F = 1 + G + G \cdot G(\delta) + \cdots + G \cdot G(\delta) \cdots G(\delta^{n-2}). \quad (3.53)$$

It is easily seen that G can be factorized as

$$G = \frac{F}{F(\delta)}. \quad (3.54)$$

In particular, (3.54) offers a solution to the factorization problem of Remark 83, solution which is peculiar to the case $N(G) = 1$.

Equation (3.54) enables us to reformulate (3.51) as $(\varphi_1 \cdot F)(\delta) = \varphi_1 \cdot F$. Such a function invariant by δ must be a function of s^n , and we deduce that the solutions of (3.51) in the quotient ring have the form $H(s^n)/F(s)$, for an arbitrary function H and F as in (3.53). \square

Lemma 96. *The norm $N(G)$ in (3.52) equals 1 if and only if the geometric condition (3.45) holds.*

Proof. We start from the expression (3.52) of the norm. With (3.6) one has $\gamma_1(\theta) = r_{11}\theta_1 + r_{21}\theta_2$ and $\gamma_2(\theta) = r_{12}\theta_1 + r_{22}\theta_2$. By convenience and also by homogeneity of $N(G)$ we shall equivalently write, with obvious notations,

$$\gamma_1(\theta) = \theta_1 + \rho_1\theta_2, \quad \gamma_2(\theta) = \theta_1 + \rho_2\theta_2.$$

For some fixed value of $\rho \in \mathbb{R}$ (not necessarily positive), consider now the equation

$$\theta_1(s) + \rho \cdot \theta_2(s) = 0, \quad (3.55)$$

where θ_1 and θ_2 are the coordinates of the parametrization (3.48). Due to the particular form of θ_1 and θ_2 , Equation (3.55) has two solutions. One of them is s_0 (and in particular is independent of ρ), where by definition s_0 is the unique value $s \in \mathbb{C}$ such that $\theta_1(s) = \theta_2(s) = 0$, see (3.50). We call the other root $s(\rho)$. Notice that $|s_0| = |s(\rho)| = 1$. Going back to the variable θ , $s(\rho_1)$ (resp. $s(\rho_2)$) corresponds on Figure 3.8 to the point θ^* (resp. θ^{**}).

The norm $N(G)$ in (3.52) can thus be rewritten as

$$\begin{aligned} N(G) &= \frac{(s - s(\rho_1)) \cdot (s - \delta^{-1}s(\rho_1)) \cdots (s - \delta^{-(n-1)}s(\rho_1)) \cdot (s - \eta s(\rho_2)) \cdots (s - \eta \delta^{-(n-1)}s(\rho_2))}{(s - s(\rho_2)) \cdot (s - \delta^{-1}s(\rho_2)) \cdots (s - \delta^{-(n-1)}s(\rho_2)) \cdot (s - \eta s(\rho_1)) \cdots (s - \eta \delta^{-(n-1)}s(\rho_1))} \\ &= \frac{(s - s(\rho_1)) \cdot (s - \delta s(\rho_1)) \cdots (s - \delta^{n-1}s(\rho_1)) \cdot (s - \eta s(\rho_2)) \cdot (s - \delta \eta s(\rho_2)) \cdots (s - \delta^{n-1} \eta s(\rho_2))}{(s - s(\rho_2)) \cdot (s - \delta s(\rho_2)) \cdots (s - \delta^{n-1}s(\rho_2)) \cdot (s - \eta s(\rho_1)) \cdot (s - \delta \eta s(\rho_1)) \cdots (s - \delta^{n-1} \eta s(\rho_1))}. \end{aligned} \quad (3.56)$$

Using the root-coefficient relationships in (3.55), we obtain

$$s(\rho) = \frac{1}{s_0} \frac{(\theta_1^+ - \theta_1^-) + (\theta_2^+ - \theta_2^-) \rho e^{i\beta}}{(\theta_1^+ - \theta_1^-) + (\theta_2^+ - \theta_2^-) \rho e^{-i\beta}} = e^{i(2\omega(\rho) - \arg s_0)},$$

with β as in (3.22) and

$$\tan \omega(\rho) = \frac{(\theta_2^+ - \theta_2^-) \rho \sin \beta}{(\theta_1^+ - \theta_1^-) + (\theta_2^+ - \theta_2^-) \rho \cos \beta}.$$

Note that it follows from (3.10) that

$$\frac{\theta_1^+ - \theta_1^-}{\theta_2^+ - \theta_2^-} = \sqrt{\frac{\sigma_{22}}{\sigma_{11}}}.$$

In particular, $\tan \omega(\rho)$ can be rewritten as

$$\tan \omega(\rho) = \frac{\sin \beta}{\frac{1}{\rho} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos \beta} \quad (3.57)$$

and we easily obtain, with δ and ϵ as in (3.62),

$$\omega(\rho_2) = \delta, \quad \omega(\rho_1) = \beta - \epsilon. \quad (3.58)$$

In the norm $N(G)$ in (3.56), the arguments of the zeros of the numerator (resp. denominator) are at $\mathcal{F}(\rho_1) \cup \mathcal{G}(\rho_2)$ (resp. $\mathcal{F}(\rho_2) \cup \mathcal{G}(\rho_1)$), where

$$\begin{aligned} \mathcal{F}(\rho) &= \{2\omega(\rho) - \arg s_0, 2\omega(\rho) - \arg s_0 + 2\beta, \dots, 2\omega(\rho) - \arg s_0 + 2(n-1)\beta\} \\ &= 2\omega(\rho) - \arg s_0 + 2\beta \times \{0, \dots, n-1\} \\ &= 2\omega(\rho) - \arg s_0 + 2\beta\mathbb{Z} \end{aligned}$$

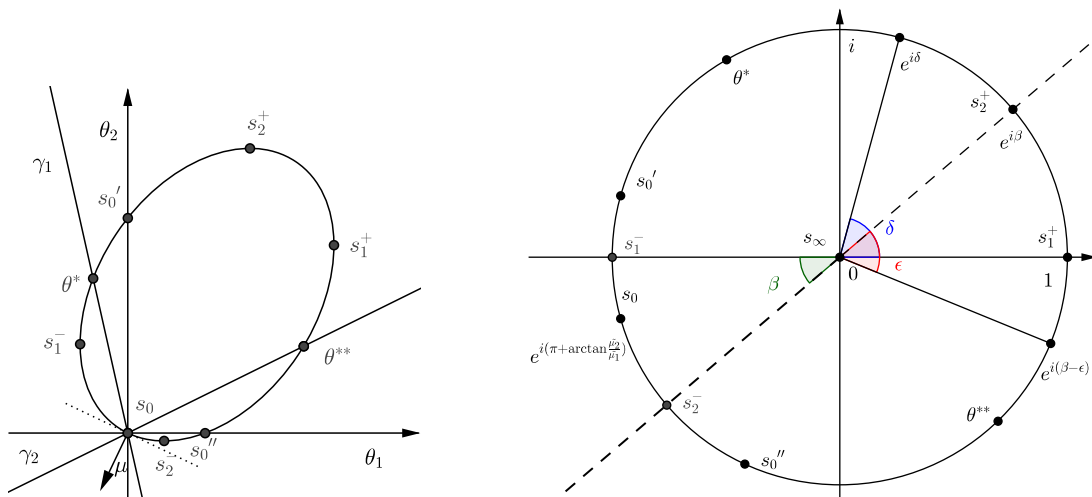


Figure 3.8: The ellipse $\{(\theta_1, \theta_2) \in \mathbb{R}^2 : \gamma(\theta_1, \theta_2) = 0\}$ (left) becomes a circle (right) on \mathcal{S} after uniformization (3.48)

and

$$\begin{aligned} \mathcal{G}(\rho) &= \{-(2\omega(\rho) - \arg s_0) + 2\beta, -(2\omega(\rho) - \arg s_0) + 4\beta, \dots, -(2\omega(\rho) - \arg s_0) + 2n\beta\} \\ &= -(2\omega(\rho) - \arg s_0) + 2\beta \times \{1, \dots, n\} \\ &= -(2\omega(\rho) - \arg s_0) + 2\beta\mathbb{Z}. \end{aligned}$$

(We can replace $\{0, \dots, n-1\}$ and $\{1, \dots, n\}$ by \mathbb{Z} , because the group is finite and of order $2n$ if and only if $\beta = k\pi/n$, for some integer k relatively prime with n .)

The zeros of $\mathcal{F}(\rho_1) \cup \mathcal{G}(\rho_2)$ compensate with those of $\mathcal{F}(\rho_2) \cup \mathcal{G}(\rho_1)$ if and only if the family $\mathcal{F}(\rho_1)$ is canceled by $\mathcal{F}(\rho_2)$, and $\mathcal{G}(\rho_1)$ by $\mathcal{G}(\rho_2)$. Observing that $e^{i\mathcal{F}(\rho_1)} = e^{i\mathcal{F}(\rho_2)}$ (resp. $e^{i\mathcal{G}(\rho_1)} = e^{i\mathcal{G}(\rho_2)}$) if and only if $2\omega(\rho_1) - 2\omega(\rho_2) \in \frac{2\pi}{n}\mathbb{Z}$, we conclude by (3.58), writing that $\frac{\pi}{n} \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$. \square

Remark 97. It is not possible to have $\omega(\rho_1) = \omega(\rho_2)$, since with (3.57) this would imply $\rho_1 = \rho_2$ and thus the two reflection vectors would be parallel, which is excluded by our assumptions (3.2).

3.5.5 Orthogonal reflections

Here we consider the case of orthogonal reflections, which is equivalent for the reflection matrix R in (3.1) to be the identity matrix; see also Figure 3.1. By developing the theory of Tutte's invariants (introduced by Tutte in [96] for the enumeration of properly colored triangulations, and used in [6] for the enumeration of quadrant walks) for the Brownian motion, we proved in [46] the following result:

Theorem 98 (Theorem 1 in [46]). *Let R be the identity matrix in (3.1). The Laplace transform φ_1 is equal to*

$$\varphi_1(\theta_2) = \frac{-\mu_1 w'(0)}{w(\theta_2) - w(0)} \theta_2. \quad (3.59)$$

In this section we derive a new proof of this result, as a consequence of Theorem 75. More generally, the proof below would work for any parameters (Σ, μ, R) such that

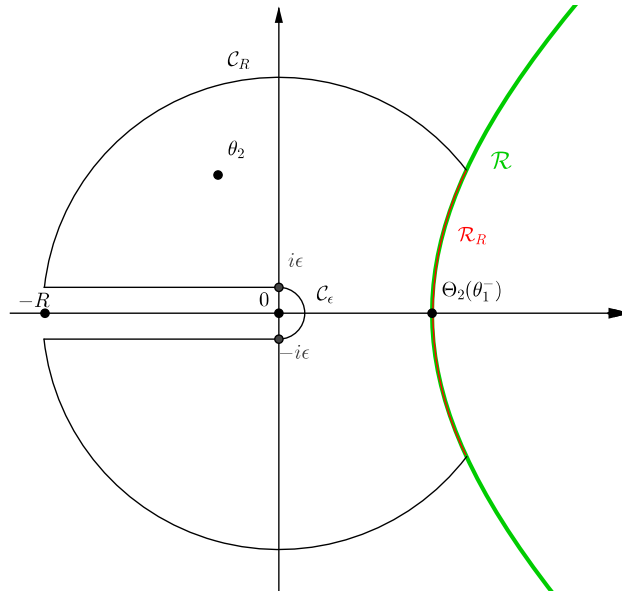


Figure 3.9: Integration contour used in the proof of Theorem 98

$G(\theta_2) = \frac{F(\theta_2)}{F(\bar{\theta}_2)}$ (cf. Remark 83), yielding a rational expression of $\varphi_1(\theta_2)$ in terms of $w(\theta_2)$ and θ_2 .

Proof. Let us first notice that the index $\chi = 0$. Indeed, since $G(\theta_2) = \frac{\bar{\theta}_2}{\theta_2}$, we have $\arg G(\theta_2) = -2 \arg \theta_2 > 0$ for $\theta_2 \in \mathcal{R}$, and thus $\Delta > 0$, see the proof of Lemma 89.

Starting from the formula (3.37), we have for some constant C

$$\varphi_1(\theta_2) = C \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} (\log \bar{\theta} - \log \theta) \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta \right\} = C \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{R}} \log \theta \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta \right\}.$$

To compute the above integral, we first integrate on the contour represented on Figure 3.9. The residue theorem gives

$$\frac{1}{2i\pi} \left\{ \int_{\mathcal{R}_R} + \int_{C_R} + \int_{C_\epsilon} + \int_{-R+i\epsilon}^{i\epsilon} + \int_{-i\epsilon}^{-R-i\epsilon} \right\} \log \theta \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta = \log \theta_2 - \log q. \quad (3.60)$$

It is easy to see that

$$\lim_{R \rightarrow \infty} \frac{1}{2i\pi} \int_{\mathcal{R}_R} \log \theta \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta = \frac{1}{2i\pi} \int_{\mathcal{R}} \log \theta \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta$$

and that in the limits when $\epsilon \rightarrow 0$ and $R \rightarrow \infty$, the contributions on C_ϵ and C_R both converge to 0, because W is analytic at 0 and ∞ , respectively.

Furthermore,

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \lim_{R \rightarrow \infty} \frac{1}{2i\pi} \left\{ \int_{-R+i\epsilon}^{i\epsilon} + \int_{-i\epsilon}^{-R-i\epsilon} \right\} \log \theta \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{2i\pi} \int_{-\infty}^0 (\log(t+i\epsilon) - \log(t-i\epsilon)) \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta \\
&= \frac{1}{2i\pi} \int_{-\infty}^0 2i\pi \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta \\
&= \log \frac{W(0) - W(\theta_2)}{W(\infty) - W(\theta_2)} \\
&= \log \frac{w(\theta_2) - w(0)}{w(q) - w(0)}.
\end{aligned}$$

Above we have used the dominated convergence and the fact that the principal determination of the logarithm gives us $\lim_{\epsilon \rightarrow 0} (\log(t+i\epsilon) - \log(t-i\epsilon)) = 2i\pi$. Letting $R \rightarrow \infty$ and then $\epsilon \rightarrow 0$ in (3.60), we have, for some constants C and C' ,

$$\varphi_1(\theta_2) = C \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{R}} \log \theta \frac{W'(\theta)}{W(\theta) - W(\theta_2)} d\theta \right\} = C' \frac{\theta_2}{w(\theta_2) - w(0)}.$$

Since by Lemma 77 one has $\varphi_1(0) = -\mu_1$, this eventually gives the right constant in (3.59). \square

3.6 Appendix: Equivalence between Brownian motion in wedges and Brownian motion in the quarter plane

We use the notation of Section 3.1. Up to an isomorphism, studying Brownian motion in the quarter plane with arbitrary covariance matrix Σ is equivalent to studying Brownian motion in a cone of angle $\beta = \arccos -\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$, with covariance identity. See for example [1, Equation (23)] and [92, Lemma 3.23]. In this short section we relate the key parameters (angles of the reflection vectors and drift) before and after the linear transformation.

Let us define the linear transforms

$$T = \begin{pmatrix} \frac{1}{\sin \beta} & \cot \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 \\ 0 & \frac{1}{\sqrt{\sigma_{22}}} \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ 0 & \sqrt{\sigma_{22}} \end{pmatrix} \begin{pmatrix} \sin \beta & -\cos \beta \\ 0 & 1 \end{pmatrix}. \quad (3.61)$$

Obviously the reflected Brownian motion associated to (Σ, μ, R) becomes a Brownian motion (with covariance identity) in a wedge of angle β and with parameters $(\text{Id}, T\mu, TR)$. The new angles of reflection are δ and ϵ (cf. Figure 3.10), such that

$$\begin{cases} \tan \delta = \frac{\sin \beta}{a + \cos \beta}, & \cos \delta = \frac{a + \cos \beta}{\sqrt{a^2 + 2a \cos \beta + 1}}, & \sin \delta = \frac{\sin \beta}{\sqrt{a^2 + 2a \cos \beta + 1}}, \\ \tan \epsilon = \frac{\sin \beta}{b + \cos \beta}, & \cos \epsilon = \frac{b + \cos \beta}{\sqrt{b^2 + 2b \cos \beta + 1}}, & \sin \epsilon = \frac{\sin \beta}{\sqrt{b^2 + 2b \cos \beta + 1}}, \end{cases} \quad (3.62)$$

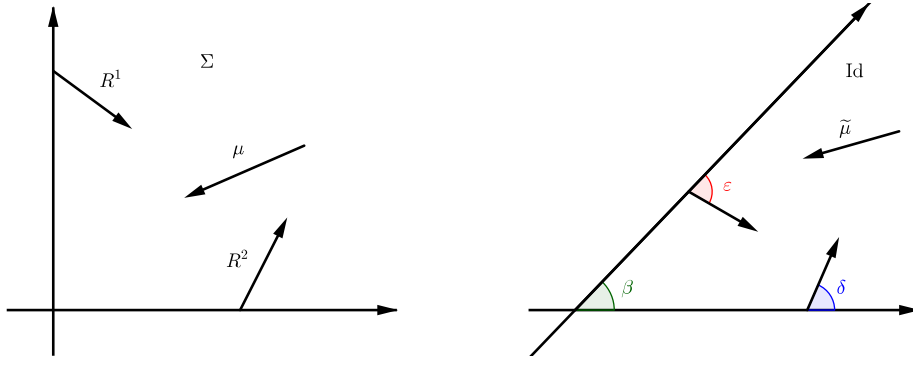


Figure 3.10: The linear transformation T in (3.61) from the quadrant to the wedge of opening angle β

where $a = \frac{r_{12}}{r_{22}} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}}$ and $b = \frac{r_{21}}{r_{11}} \sqrt{\frac{\sigma_{11}}{\sigma_{22}}}$. The new drift is $\tilde{\mu} = T\mu$, where

$$\tilde{\mu}_1 = \frac{\mu_1}{\sqrt{\sigma_{11}}} \frac{1}{\sin \beta} + \frac{\mu_2}{\sqrt{\sigma_{22}}} \cot \beta \quad \text{and} \quad \tilde{\mu}_2 = \frac{\mu_2}{\sqrt{\sigma_{22}}}.$$

Thanks to (3.10) and (3.50), we have $\tan \arg s_0 = \frac{\tilde{\mu}_2}{\tilde{\mu}_1}$, see Figure 3.8.

To conclude the appendix, we prove Corollary 76, which gives an explicit formula for the Laplace transform of the stationary distribution of a reflected Brownian motion in a wedge.

Proof of Corollary 76. Let us arbitrarily choose $\sigma_{11} = \sigma_{22} = 1$ and $\sigma_{12} = -\cos \beta$. It implies that the linear transform T in (3.61) is the same as T_1 in the statement of Corollary 76. We consider the process $Z = T^{-1}\tilde{Z}$, which is a reflected Brownian motion in the quadrant of parameters (Σ, μ, R) , with

$$\Sigma = T^{-1}\tilde{\Sigma}(T^{-1})^\top, \quad \mu = T^{-1}\tilde{\mu} \quad \text{and} \quad R = T^{-1}\tilde{R}.$$

Let Π (resp. $\tilde{\Pi}$) be the invariant measure of Z (resp. \tilde{Z}) and π (resp. $\tilde{\pi}$) its density. It is easy to notice that

$$\Pi = \tilde{\Pi} \circ T \quad \text{and} \quad \pi = |\det T| \tilde{\pi} \circ T.$$

Indeed, by a fundamental property of the invariant measure, for all $x \in \mathbb{R}_+^2$ and all measurable set A in \mathbb{R}_+^2 we have the following limits

$$\mathbb{P}_x[Z_t \in A] \xrightarrow[t \rightarrow \infty]{} \Pi(A), \quad \mathbb{P}_x[Z_t \in A] = \mathbb{P}_x[TZ_t \in TA] = \mathbb{P}_{Tx}[\tilde{Z}_t \in TA] \xrightarrow[t \rightarrow \infty]{} \tilde{\Pi}(TA).$$

It yields $\Pi = \tilde{\Pi} \circ T$. Furthermore by a simple change of variable we have

$$\tilde{\Pi} \circ T(A) = \int_{TA} \tilde{\pi}(\tilde{x}) d\tilde{x} = |\det T| \int_A \tilde{\pi}(Tx) dx,$$

therefore $|\det T| \tilde{\pi} \circ T$ is the density of $\Pi \circ T$ and is then equal to π . Theorem 75 gives the value of φ , the Laplace transform of π . Lastly, a simple change of variable $\tilde{x} = Tx$ in

Equation (3.3) leads to

$$\begin{aligned}\tilde{\varphi}(\tilde{\theta}) &= \iint_{\mathcal{C}_\beta} \exp(\tilde{\theta} \cdot \tilde{x}) \tilde{\pi}(\tilde{x}) d\tilde{x} = \iint_{T^{-1}\mathcal{C}_\beta} \exp(\tilde{\theta} \cdot Tx) \tilde{\pi}(Tx) |\det T| dx \\ &= \iint_{\mathbb{R}_+^2} \exp(T^\top \tilde{\theta} \cdot x) \pi(x) dx = \varphi(T^\top \tilde{\theta}),\end{aligned}$$

where \mathcal{C}_β is the wedge of angle β where the process evolves. □

Asymptotic expansion of stationary distribution for reflected Brownian motion in the quarter plane via analytic approach

Ce chapitre est issu de l'article [44] co-écrit avec Irina Kourkova et publié en 2017 dans le journal *Stochastic Systems*.

Abstract

Brownian motion in \mathbf{R}_+^2 with covariance matrix Σ and drift μ in the interior and reflection matrix R from the axes is considered. The asymptotic expansion of the stationary distribution density along all paths in \mathbf{R}_+^2 is found and its main term is identified depending on parameters (Σ, μ, R) . For this purpose the analytic approach of Fayolle, Iasnogorodski and Malyshev in [37] and [80], restricted essentially up to now to discrete random walks in \mathbf{Z}_+^2 with jumps to the nearest-neighbors in the interior is developed in this article for diffusion processes on \mathbf{R}_+^2 with reflections on the axes.

Résumé

On considère le mouvement Brownien dans \mathbf{R}_+^2 avec covariance Σ et dérive μ à l'intérieur et matrice de réflexion R sur les axes. Le développement asymptotique de la densité de la distribution stationnaire le long de toute trajectoires dans \mathbf{R}_+^2 est trouvé et son terme principal est déterminé en fonction des paramètres (Σ, μ, R) . Dans cet objectif l'approche analytique de Fayolle, Iasnogorodski et Malyshev dans [37] et [80], jusqu'ici restreinte au cas discret des marches aléatoires dans \mathbf{Z}_+^2 avec sauts aux plus proches voisins, est développée dans cet article pour un processus de diffusion sur \mathbf{R}_+^2 réfléchi sur les axes.

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4.1 Introduction and main results

4.1.1 Context

Two-dimensional semimartingale reflecting Brownian motion (SRBM) in the quarter plane received a lot of attention from the mathematical community. Problems such as SRBM existence [87, 95], stationary distribution conditions [51, 55], explicit forms of stationary distribution in special cases [29, 31, 51, 56, 68], large deviations [2, 29, 75, 76] construction of Lyapunov functions [35], and queueing networks approximations [51, 53, 71, 72, 102] have been intensively studied in the literature. References cited above are non-exhaustive, see also [101] for a survey of some of these topics. Many results on two-dimensional SRBM have been fully or partially generalized to higher dimensions.

In this chapter we consider stationary SRBMs in the quarter plane and focus on the asymptotics of their stationary distribution along any path in \mathbf{R}_+^2 . Let $Z(\infty) = (Z_1(\infty), Z_2(\infty))$ be a random vector that has the stationary distribution of the SRBM. In [28], Dai et Myazawa obtain the following asymptotic result: for a given directional vector $c \in \mathbf{R}_+^2$ they find the function $f_c(x)$ such that

$$\lim_{x \rightarrow \infty} \frac{\mathbf{P}(\langle c | Z(\infty) \rangle \geq x)}{f_c(x)} = 1$$

where $\langle \cdot | \cdot \rangle$ is the inner product. In [29] they compute the exact asymptotics of two boundary stationary measures on the axes associated with $Z(\infty)$. In this chapter we solve a harder problem arisen in [28, §8 p.196], the one to compute the asymptotics of

$$\mathbf{P}(Z(\infty) \in xc + B), \text{ as } x \rightarrow \infty,$$

where $c \in \mathbf{R}_+^2$ is any directional vector and $B \subset \mathbf{R}_+^2$ is a compact subset. Furthermore, our objective is to find *the full asymptotic expansion of the density $\pi(x_1, x_2)$ of $Z(\infty)$ as $x_1, x_2 \rightarrow \infty$ and $x_2/x_1 \rightarrow \tan(\alpha)$ for any given angle $\alpha \in]0, \pi/2[$.*

Our main tool is the analytic method developed by V. Malyshev in [80] to compute the asymptotics of stationary probabilities for discrete random walks in \mathbf{Z}_+^2 with jumps to the nearest-neighbors in the interior and reflections on the axes. This method proved to be fruitful for the analysis of Green functions and Martin boundary [64, 66], and also useful for studying some joining the shortest queue models [67]. The article [80] has been a part of Malyshev's global analytic approach to study discrete-time random walks in \mathbf{Z}_+^2 with four domains of spatial homogeneity (the interior of \mathbf{Z}_+^2 , the axes and the origin). Namely, in the book [78] he made explicit their stationary probability generating functions as solutions of boundary problems on the universal covering of the associated Riemann surface and studied the nature of these functions depending on parameters. G. Fayolle and R. Iasnogorodski [36] determined these generating functions as solutions of boundary problems of Riemann-Hilbert-Carleman type on the complex plane. Fayolle, Iasnogorodski and Malyshev merged together and deepened their methods in the book [37]. The latter is entirely devoted to the explicit form of stationary probabilities generating functions for discrete random walks in \mathbf{Z}_+^2 with nearest-neighbor jumps in the interior. The analytic approach of this book has been further applied to the analysis of random walks absorbed on the axes in [64]. It has been also especially efficient in combinatorics, where it allowed to study all models of walks in \mathbf{Z}_+^2 with small steps by making explicit the generating functions of the numbers of paths and clarifying their nature, see [84] and [65].

However, the methods of [37] and [80] seem to be essentially restricted to discrete-time models of walks in the quarter plane with jumps in the interior only to the nearest-neighbors. They can hardly be extended to discrete models with bigger jumps, even at distance 2, nevertheless some attempts in this direction have been done in [39]. In fact, while for jumps at distance 1 the Riemann surface associated with the random walk is the torus, bigger jumps lead to Riemann surfaces of higher genus, where the analytic procedures of [37] seem much more difficult to carry out. Up to now, as far as we know, neither the analytic approach of [37], nor the asymptotic results [80] have been translated to the continuous analogs of random walks in \mathbf{Z}_+^2 , such as SRBMs in \mathbf{R}_+^2 , except for some special cases in [5] and in [42]. This article is the first one in this direction. Namely, the asymptotic expansion of the stationary distribution density for SRBMs is obtained by methods strongly inspired by [80]. The aim of this work goes beyond the solution of this particular problem. It provides the basis for the development of the analytic approach of [37] for diffusion processes in cones of \mathbf{R}_+^2 which is continued in the next articles [46] and [47]. In [47] the authors make explicit Laplace transform of the invariant measure for SRBMs in the quarter plane with general parameters of the drift, covariance and reflection matrices. Following [37], they express it in an integral form as a solution of a boundary value problem and then discuss possible simplifications of this integral formula for some particular sets of parameters. The special case of orthogonal reflections from the axes is the subject of [46]. Let us note that the analytic approach for SRBMs in \mathbf{R}_+^2 which is developed in the present paper and continued by the next ones [46] and [47], looks more transparent than the one for discrete models and deprived of many second order details. Last but not the least, contrary to random walks in \mathbf{Z}_+^2 with jumps at distance 1, it can be easily extended to diffusions in any cones of \mathbf{R}^2 via linear transformations, as we observe in the concluding remarks, see Section 4.5.3.

4.1.2 Reflected Brownian motion in the quarter plane

We now define properly the two-dimensional SRBM and present our results. Let

$$\left\{ \begin{array}{l} \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \in \mathbf{R}^{2 \times 2} \text{ be a non-singular covariance matrix,} \\ \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathbf{R}^2 \text{ be a drift,} \\ R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathbf{R}^{2 \times 2} \text{ be a reflection matrix.} \end{array} \right.$$

Definition 99. *The stochastic process $Z(t) = (Z^1(t), Z^2(t))$ is said to be a reflected Brownian motion with drift in the quarter plane \mathbf{R}_+^2 associated with data (Σ, μ, R) if*

$$Z(t) = Z_0 + W(t) + \mu t + RL(t) \in \mathbf{R}_+^2,$$

where

- (i) $(W(t))_{t \in \mathbb{R}^+}$ is an unconstrained planar Brownian motion with covariance matrix Σ , starting from 0;
- (ii) $L(t) = (L^1(t), L^2(t))$; for $i = 1, 2$, $L^i(t)$ is a continuous and non-decreasing process that increases only at time t such as $Z^i(t) = 0$, namely $\int_0^t 1_{\{Z^i(s) \neq 0\}} dL^i(s) = 0$ $\forall t \geq 0$;

(iii) $Z(t) \in \mathbf{R}_+^2 \forall t \geq 0$.

Process $Z(t)$ exists if and only if $r_{11} > 0$, $r_{22} > 0$ and either $r_{12}, r_{21} > 0$ or $r_{11}r_{22} - r_{12}r_{21} > 0$ (see [95] and [87] which obtain an existence criterion in any dimension). In this case the process is unique in distribution for each given initial distribution of Z_0 .

Columns R^1 and R^2 represent the directions where the Brownian motion is pushed when it reaches the axes, see Figure 4.1.

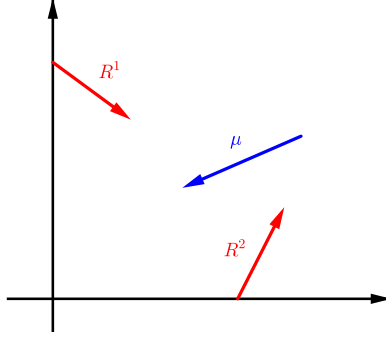


Figure 4.1: Drift μ and reflection vectors R^1 and R^2

Proposition 100. *The reflected Brownian motion $Z(t)$ associated with (Σ, μ, R) is well defined, and its stationary distribution Π exists and is unique if and only if the data satisfy the following conditions:*

$$r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0, \quad (4.1)$$

$$r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0. \quad (4.2)$$

The proof and some more detailed statements can be found in [57, 100, 52]. From now on we assume that conditions (4.1) and (4.2) are satisfied. The stationary distribution Π is absolutely continuous with respect to Lebesgue measure as it is shown in [55] and [22]. We denote its density by $\pi(x_1, x_2)$.

4.1.3 Functional equation for the stationary distribution

Let A be the generator of $(W_t + \mu t)_{t \geq 0}$. For each $f \in \mathcal{C}_b^2(\mathbf{R}_+^2)$ (the set of twice continuously differentiable functions f on \mathbf{R}_+^2 such that f and its first and second order derivatives are bounded) one has

$$Af(z) = \frac{1}{2} \sum_{i,j=1}^2 \sigma_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j}(z) + \sum_{i=1}^2 \mu_i \frac{\partial f}{\partial z_i}(z).$$

Let us define for $i = 1, 2$,

$$D_i f(x) = \langle R^i | \nabla f \rangle$$

that may be interpreted as generators on the axes. We define now ν_1 and ν_2 two finite boundary measures with their support on the axes: for any Borel set $B \subset \mathbf{R}_+^2$,

$$\nu_i(B) = \mathbb{E}_\Pi \left[\int_0^1 1_{\{Z(u) \in B\}} dL^i(u) \right].$$

By definition of stationary distribution, for all $t \geq 0$, $\mathbb{E}_\Pi[f(Z(t))] = \int_{\mathbf{R}_+^2} f(z)\Pi(dz)$. A similar formula holds true for ν_i : $\mathbb{E}_\Pi[\int_0^t f(Z(u))dL^i(u)] = t \int_{\mathbf{R}_+^2} f(x)\nu_i(dx)$. Therefore ν_1 and ν_2 may be viewed as a kind of boundary invariant measures. The basic adjoint relationship takes the following form: for each $f \in \mathcal{C}_b^2(\mathbf{R}_+^2)$,

$$\int_{\mathbf{R}_+^2} Af(z)\Pi(dz) + \sum_{i=1,2} \int_{\mathbf{R}_+^2} D_i f(z)\nu_i(dz) = 0. \quad (4.3)$$

The proof can be found in [55] in some particular cases and then has been extended to a general case, for example in [27]. We now define $\varphi(\theta)$ the two-dimensional Laplace transform of Π also called its moment generating function. Let

$$\varphi(\theta) = \mathbb{E}_\Pi[\exp(\langle \theta | Z \rangle)] = \iint_{\mathbf{R}_+^2} \exp(\langle \theta | z \rangle)\Pi(dz)$$

for all $\theta = (\theta_1, \theta_2) \in \mathbb{C}^2$ such that the integral converges. It does of course for any θ with $\Re \theta_1 \leq 0, \Re \theta_2 \leq 0$. We have set $\langle \theta | Z \rangle = \theta_1 Z^1 + \theta_2 Z^2$. Likewise we define the moment generating functions for $\nu_1(\theta_2)$ and $\nu_2(\theta_1)$ on \mathbf{C} :

$$\varphi_2(\theta_1) = \mathbb{E}_\Pi[\int_0^1 e^{\theta_1 Z_1^i} dL^2(t)] = \int_{\mathbf{R}_+^2} e^{\theta_1 z} \nu_2(dz), \quad \varphi_1(\theta_2) = \mathbb{E}_\Pi[\int_0^1 e^{\theta_2 Z_2^i} dL^1(t)] = \int_{\mathbf{R}_+^2} e^{\theta_2 z} \nu_1(dz).$$

Function $\varphi_2(\theta_1)$ exists a priori for any θ_1 with $\Re \theta_1 \leq 0$. It is proved in [28] that it also does for θ_1 with $\Re \theta_1 \in [0, \epsilon_1]$, up to its first singularity $\epsilon_1 > 0$, the same is true for $\varphi_1(\theta_2)$. The following key functional equation (proven in [28]) results from the basic adjoint relationship (4.3).

Theorem 101. *For any $\theta \in \mathbf{R}_+^2$ such $\varphi(\theta) < \infty$, $\varphi_2(\theta_1) < \infty$ and $\varphi_1(\theta_2) < \infty$ we have the following fundamental functional equation:*

$$\gamma(\theta)\varphi(\theta) = \gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1), \quad (4.4)$$

where

$$\begin{cases} \gamma(\theta) = -\frac{1}{2}\langle \theta | \Sigma \theta \rangle - \langle \theta | \mu \rangle = -\frac{1}{2}(\sigma_{11}\theta_1^2 + \sigma_{22}\theta_2^2 + 2\sigma_{12}\theta_1\theta_2) - (\mu_1\theta_1 + \mu_2\theta_2), \\ \gamma_1(\theta) = \langle R^1 | \theta \rangle = r_{11}\theta_1 + r_{21}\theta_2, \\ \gamma_2(\theta) = \langle R^2 | \theta \rangle = r_{12}\theta_1 + r_{22}\theta_2. \end{cases} \quad (4.5)$$

This equation holds true a priori for any $\theta = (\theta_1, \theta_2)$ with $\Re \theta_1 \leq 0, \Re \theta_2 \leq 0$. It plays a crucial role in the analysis of the stationary distribution.

4.1.4 Results

Our aim is to obtain the asymptotic expansion of the stationary distribution density $\pi(x) = \pi(x_1, x_2)$ as $x_1, x_2 \rightarrow \infty$ and $x_2/x_1 \rightarrow \tan(\alpha_0)$ for any given angle $\alpha_0 \in [0, \pi/2]$.

Notation. We write the asymptotic expansion $f(x) \sim \sum_{k=0}^n g_k(x)$ as $x \rightarrow x_0$ if $g_k(x) = o(g_{k-1}(x))$ as $x \rightarrow x_0$ for all $k = 1, \dots, n$ and $f(x) - \sum_{k=0}^n g_k(x) = o(g_n(x))$ as $x \rightarrow x_0$.

It will be more convenient to expand $\pi(r \cos \alpha, r \sin \alpha)$ as $r \rightarrow \infty$ and $\alpha \rightarrow \alpha_0$. We give our final results in Section 4.5, Theorems 120–123: we find the expansion of

$\pi(r \cos \alpha, r \sin \alpha)$ as $r \rightarrow \infty$ and prove it uniform for α fixed in a small neighborhood $\mathcal{O}(\alpha_0) \subset]0, \pi/2[$ of $\alpha_0 \in]0, \pi/2[$.

In this section, Theorem 102 below announces the main term of the expansion depending on parameters (μ, Σ, R) and a given direction α_0 . Next, in Section 4.1.5 we sketch our analytic approach following the main lines of this paper in order to get the full asymptotic expansion of π . We present at the same time the organization of the article.

Now we need to introduce some notations. The quadratic form $\gamma(\theta)$ is defined in (4.4) via the covariance matrix Σ and the drift μ of the process in the interior of \mathbf{R}_+^2 . Let us restrict ourselves on $\theta \in \mathbf{R}^2$. The equation $\gamma(\theta) = 0$ determines an ellipse \mathcal{E} on \mathbf{R}^2 passing through the origin, its tangent in it is orthogonal to vector μ , see Figure 4.2. Stability conditions (4.1) and (4.2) imply the negativity of at least one of coordinates of μ , see [28, Lemma 2.1]. In this article, in order to shorten the number of pictures and cases of parameters to consider, we restrict ourselves to the case

$$\mu_1 < 0 \text{ and } \mu_2 < 0, \quad (4.6)$$

although our methods can be applied without any difficulty to other cases, we briefly sketch some different details at the end of Section 2.4.

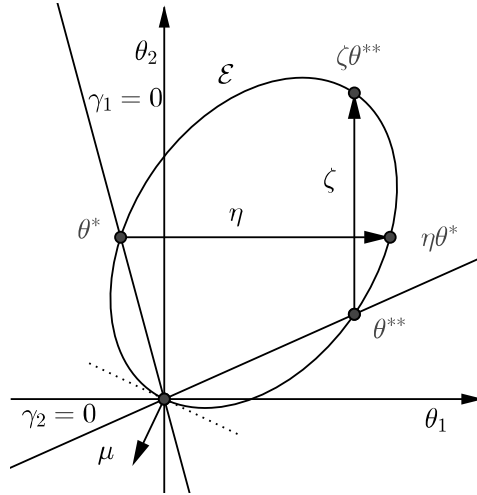
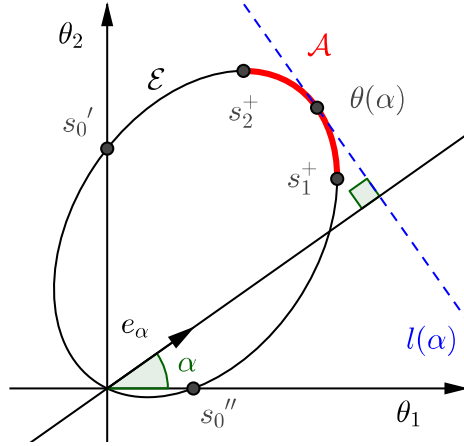


Figure 4.2: Ellipse \mathcal{E} , straight lines $\gamma_1(\theta) = 0$, $\gamma_2(\theta) = 0$, points $\theta^*, \theta^{**}, \eta\theta^*, \zeta\theta^{**}$

Let us call $s_1^+ = (\theta_1(s_1^+), \theta_2(s_1^+)) \in \mathcal{E}$ the point of the ellipse with the maximal first coordinate: $\theta_1(s_1^+) = \sup\{\theta_1 : \gamma(\theta_1, \theta_2) = 0\}$. Let us call s_2^+ the point of the ellipse with the maximal second coordinate. Let \mathcal{A} be the arc of the ellipse with endpoints s_1^+, s_2^+ not passing through the origin, see Figure 4.3. For a given angle $\alpha \in [0, \pi/2]$ let us define the point $\theta(\alpha)$ on the arc \mathcal{A} as

$$\theta(\alpha) = \operatorname{argmax}_{\theta \in \mathcal{A}} \langle \theta | e_\alpha \rangle \quad \text{where } e_\alpha = (\cos \alpha, \sin \alpha). \quad (4.7)$$

Note that $\theta(0) = s_1^+$, $\theta(\pi/2) = s_2^+$, and $\theta(\alpha)$ is an isomorphism between $[0, \pi/2]$ and \mathcal{A} . Coordinates of $\theta(\alpha)$ are given explicitly in (4.50). One can also construct $\theta(\alpha)$ geometrically: first draw a ray $r(\alpha)$ on \mathbf{R}_+^2 that forms the angle α with θ_1 -axis, and then the straight line $l(\alpha)$ orthogonal to this ray and tangent to the ellipse. Then $\theta(\alpha)$ is the point where $l(\alpha)$ is tangent to the ellipse, see Figure 4.3.

Figure 4.3: Arc \mathcal{A} and point $\theta(\alpha)$ on \mathcal{E}

Secondly, consider the straight lines $\gamma_1(\theta) = 0$, $\gamma_2(\theta) = 0$ defined in (4.4) via the reflection matrix R . They cross the ellipse \mathcal{E} in the origin. Furthermore, due to stability conditions (4.1) and (4.2) the line $\gamma_1(\theta) = 0$ [resp. $\gamma_2(\theta) = 0$] intersects the ellipse at the second point $\theta^* = (\theta_1^*, \theta_2^*)$ (resp. $\theta^{**} = (\theta_1^{**}, \theta_2^{**})$) where $\theta_2^* > 0$ (resp. $\theta_1^{**} > 0$). Stability conditions also imply that the ray $\gamma_1(\theta) = 0$ is always "above" the ray $\gamma_2(\theta) = 0$, see [28, Lemma 2.2]. To present our results, we need to define the images of these points via the so-called Galois automorphisms ζ and η of \mathcal{E} . Namely, for point $\theta^* = (\theta_1^*, \theta_2^*) \in \mathcal{E}$ there exists a unique point $\eta\theta^* = (\eta\theta_1^*, \theta_2^*) \in \mathcal{E}$ that has the same second coordinate. Clearly, θ_1^* and $\eta\theta_1^*$ are two roots of the second degree equation $\gamma(\cdot, \theta_2^*) = 0$. In the same way for point $\theta^{**} = (\theta_1^{**}, \theta_2^{**}) \in \mathcal{E}$ there exists a unique point $\zeta\theta^{**} = (\theta_1^{**}, \zeta\theta_2^{**}) \in \mathcal{E}$ with the same first coordinate. Points θ_2^{**} and $\zeta\theta_2^{**}$ are two roots of the second degree equation $\gamma(\theta_1^{**}, \cdot) = 0$. Points θ^* , θ^{**} , $\eta\theta^*$ and $\zeta\theta^{**}$ are pictured on Figure 4.2. Their coordinates are made explicit in (4.33) and (4.34).

Finally let $s_0' = (0, -2\frac{\mu_{22}}{\sigma_{22}})$ be the point of intersection of the ellipse \mathcal{E} with θ_2 -axis and let $s_0'' = (-2\frac{\mu_{11}}{\sigma_{11}}, 0)$ be the point of intersection of the ellipse with θ_1 -axis, see Figure 4.3. The following theorem provides the main asymptotic term of $\pi(r \cos \alpha, r \sin \alpha)$.

Theorem 102. *Let $\alpha_0 \in]0, \pi/2[$. Let $\theta(\alpha)$ be defined in (4.7). Let $\{\theta(\alpha_0), s_0'\}$ (resp. $\{s_0'', \theta(\alpha_0)\}$) be the arc of the ellipse \mathcal{E} with end points s_0' and $\theta(\alpha_0)$ (resp. s_0'' and $\theta(\alpha_0)$) not passing through the origin. We have the following results.*

- (1) *If $\zeta\theta^{**} \notin \{\theta(\alpha_0), s_0'\}$ and $\eta\theta^* \notin \{s_0'', \theta(\alpha_0)\}$, then there exists a constant $c(\alpha_0)$ such that*

$$\pi(r \cos \alpha, r \sin \alpha) \sim \frac{c(\alpha_0)}{\sqrt{r}} \exp\left(-r \langle e_\alpha | \theta(\alpha) \rangle\right) \quad r \rightarrow \infty, \alpha \rightarrow \alpha_0. \quad (4.8)$$

The function $c(\alpha)$ varies continuously on $[0, \pi/2]$, $\lim_{\alpha \rightarrow 0} c(\alpha) = \lim_{\alpha \rightarrow \pi/2} c(\alpha) = 0$.

- (2) *If $\zeta\theta^{**} \in \{\theta(\alpha_0), s_0'\}$ and $\eta\theta^* \notin \{s_0'', \theta(\alpha_0)\}$, then with some constant $c_1 > 0$*

$$\pi(r \cos \alpha, r \sin \alpha) \sim c_1 \exp\left(-r \langle e_\alpha | \zeta\theta^{**} \rangle\right) \quad r \rightarrow \infty, \alpha \rightarrow \alpha_0. \quad (4.9)$$

(3) If $\zeta\theta^{**} \notin \{\theta(\alpha_0), s'_0\}$ and $\eta\theta^* \in \{s''_0, \theta(\alpha_0)\}$, then with some constant $c_2 > 0$

$$\pi(r \cos \alpha, r \sin \alpha) \sim c_2 \exp\left(-r\langle e_\alpha \mid \eta\theta^* \rangle\right) \quad r \rightarrow \infty, \alpha \rightarrow \alpha_0. \quad (4.10)$$

(4) Let $\zeta\theta^{**} \in \{\theta(\alpha_0), s'_0\}$ and $\eta\theta^* \in \{s''_0, \theta(\alpha_0)\}$. If $\langle \zeta\theta^{**} \mid e_{\alpha_0} \rangle < \langle \eta\theta^* \mid e_{\alpha_0} \rangle$, then the asymptotics (4.9) is valid with some constant $c_1 > 0$. If $\langle \zeta\theta^{**} \mid e_{\alpha_0} \rangle > \langle \eta\theta^* \mid e_{\alpha_0} \rangle$, then the asymptotics (4.10) is valid with some constant $c_2 > 0$. If $\langle \zeta\theta^{**} \mid e_{\alpha_0} \rangle = \langle \eta\theta^* \mid e_{\alpha_0} \rangle$, then with some constants $c_1 > 0$ and $c_2 > 0$

$$\pi(r \cos \alpha, r \sin \alpha) \sim c_1 \exp\left(-r\langle e_\alpha \mid \zeta\theta^{**} \rangle\right) + c_2 \exp\left(-r\langle e_\alpha \mid \eta\theta^* \rangle\right) \quad r \rightarrow \infty, \alpha \rightarrow \alpha_0. \quad (4.11)$$

See Figure 4.4 for the different cases. (The arcs $\}a, b\}$ or $\{a, b\}$ of \mathcal{E} are those not passing through the origin where the left or the right end respectively is excluded).

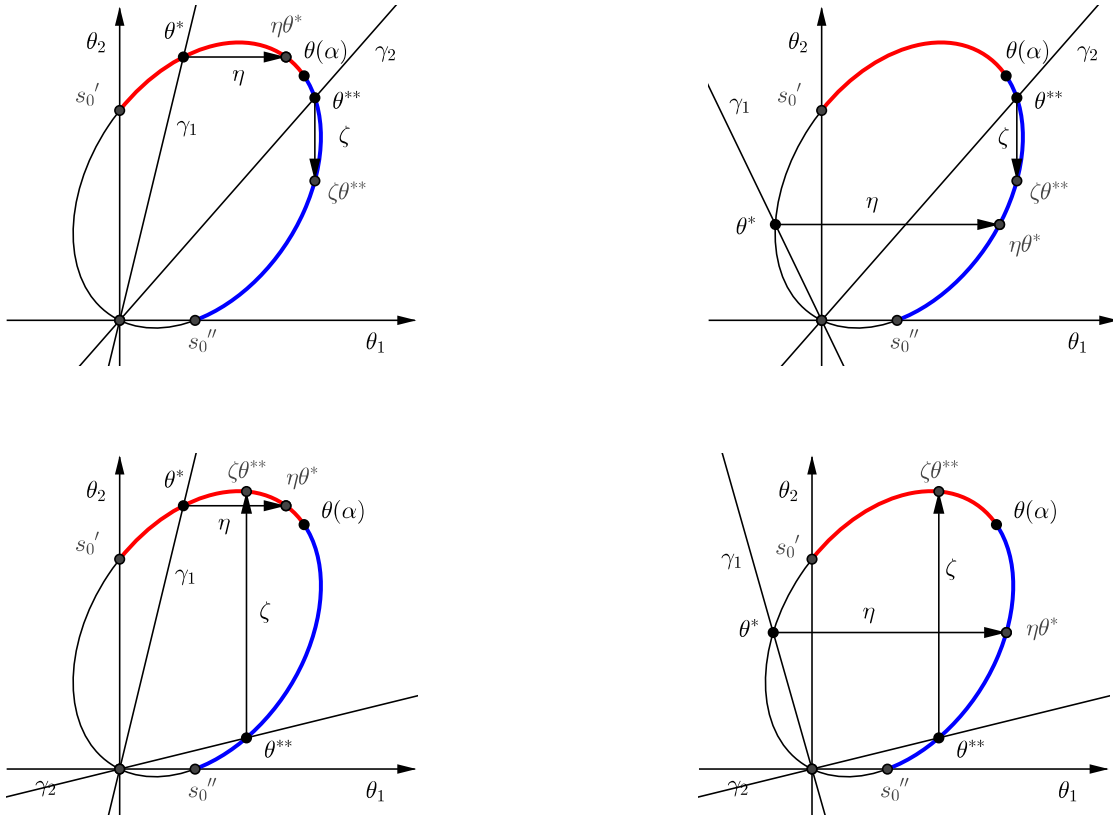


Figure 4.4: Cases (1),(2),(3),(4)

Let us note that the exponents in Theorem 102 are the same as in the large deviation rate function found in [29, Thm 3.2]. The same phenomenon is observed for discrete random walks, cf. [80] and [60].

4.1.5 Sketch of the analytic approach. Organization of the paper

The starting point of our approach is the main functional equation (4.4) valid for any $\theta = (\theta_1, \theta_2) \in \mathbf{C}^2$ with $\Re \theta_1 \leq 0, \Re \theta_2 \leq 0$. The function $\gamma(\theta_1, \theta_2)$ in the left-hand side

is a polynomial of the second order of θ_1 and θ_2 . The algebraic function $\Theta_1(\theta_2)$ defined by $\gamma(\Theta_1(\theta_2), \theta_2) \equiv 0$ is 2-valued and its Riemann surface \mathbf{S}_{θ_2} is of genus 0. The same is true about the 2-valued algebraic function $\Theta_2(\theta_1)$ defined by $\gamma(\theta_1, \Theta_2(\theta_1)) = 0$ and its Riemann surface \mathbf{S}_{θ_1} . The surfaces \mathbf{S}_{θ_1} and \mathbf{S}_{θ_2} being equivalent, we will consider just one surface \mathbf{S} defined by the equation $\gamma(\theta_1, \theta_2) = 0$ with two different coverings. Each point $s \in \mathcal{S}$ has two "coordinates" $(\theta_1(s), \theta_2(s))$, both of them are complex or infinite and satisfy $\gamma(\theta_1(s), \theta_2(s)) = 0$. For any point $s = (\theta_1, \theta_2) \in \mathcal{S}$, there exists a unique point $s' = (\theta_1, \theta_2') \in \mathcal{S}$ with the same first coordinate and there exists a unique point $s'' = (\theta_1', \theta_2) \in \mathcal{S}$ with the same second coordinate. We say that $s' = \zeta s$, i.e. s' and s are related by Galois automorphism ζ of \mathbf{S} that leaves untouched the first coordinate, and that $s'' = \eta s$, i.e. s'' and s are related by Galois automorphism η of \mathbf{S} that leaves untouched the second coordinate. Clearly $\zeta^2 = Id$, $\eta^2 = Id$ and the branch points of $\Theta_1(\theta_2)$ and of $\Theta_2(\theta_1)$ are fixed points of ζ and η respectively. The ellipse \mathcal{E} is the set of points of \mathbf{S} where both "coordinates" are real. The construction of \mathbf{S} and definition of Galois automorphisms are carried out in Section 4.2.

Next, unknown functions $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$ are lifted in the domains of \mathcal{S} where $\{s \in \mathcal{S} : \Re \theta_2(s) \leq 0\}$ and $\{s \in \mathcal{S} : \Re \theta_1(s) \leq 0\}$ respectively. The intersection of these domains on \mathbf{S} is non-empty, both φ_2 and φ_1 are well defined in it. Since for any $s = (\theta_1(s), \theta_2(s)) \in \mathcal{S}$ we have $\gamma(\theta_1(s), \theta_2(s)) = 0$, the main functional equation (4.4) implies:

$$\gamma_1(\theta_1(s), \theta_2(s))\varphi_1(\theta_2(s)) + \gamma_2(\theta_1(s), \theta_2(s))\varphi_2(\theta_1(s)) = 0 \quad \forall s \in \mathcal{S}, \Re \theta_1(s) \leq 0, \Re \theta_2(s) \leq 0.$$

Using this relation, Galois automorphisms and the facts that φ_1 and φ_2 depend just on one "coordinate" (φ_1 depends on θ_2 and φ_2 on θ_1 only), we continue φ_1 and φ_2 explicitly as meromorphic on the whole of \mathcal{S} . This meromorphic continuation procedure is the crucial step of our approach, it is the subject of Section 4.3.1. It allows to recover φ_1 and φ_2 on the complex plane as multivalued functions and determines all poles of all its branches. Namely, it shows that poles of φ_1 and φ_2 may be only at images of zeros of γ_1 and γ_2 by automorphisms η and ζ applied several times. We are in particular interested in the poles of their first (main) branch, and more precisely in the most "important" pole (from the asymptotic point of view, to be explained below), that turns out to be at one of points $\zeta\theta^{**}$ or $\eta\theta^*$ defined above. The detailed analysis of the "main" poles of φ_1 and φ_2 is furnished in Section 4.3.2.

Let us now turn to the asymptotic expansion of the density $\pi(x_1, x_2)$. Its Laplace transform comes from the right-hand side of the main equation (4.4) divided by the kernel $\gamma(\theta_1, \theta_2)$. By inversion formula the density $\pi(x_1, x_2)$ is then represented as a double integral on $\{\theta : \Re \theta_1 = \Re \theta_2 = 0\}$. In Section 4.4.1, using the residues of the function $\frac{1}{\gamma(\theta_1, \cdot)}$ or $\frac{1}{\gamma(\cdot, \theta_2)}$ we transform this double integral into a sum of two single integrals along two cycles on \mathcal{S} , those where $\Re \theta_1(s) = 0$ or $\Re \theta_2(s) = 0$. Putting $(x_1, x_2) = re_\alpha$ we get the representation of the density as a sum of two single integrals along some contours on \mathbf{S} :

$$\pi(re_\alpha) = \frac{1}{2\pi\sqrt{\det \Sigma}} \left(\int_{\mathcal{I}_{\theta_1}^+} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r\langle \theta(s) | e_\alpha \rangle} ds + \int_{\mathcal{I}_{\theta_2}^+} \frac{\varphi_1(s)\gamma_1(\theta(s))}{s} e^{-r\langle \theta(s) | e_\alpha \rangle} ds \right). \quad (4.12)$$

We would like to compute their asymptotic expansion as $r \rightarrow \infty$ and prove it to be uniform for α fixed in a small neighborhood $\mathcal{O}(\alpha_0)$, $\alpha_0 \in]0, \pi/2[$.

These two integrals are typical to apply the saddle-point method, see [41, 83]. The point $\theta(\alpha) \in \mathcal{E}$ defined above is the saddle-point for both of them, this is the subject of Section 4.4.2. The integration contours on \mathbf{S} are then shifted to new ones $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$ which are constructed in such a way that they pass through the saddle-point $\theta(\alpha)$, follow the steepest-descent curve in its neighborhood $\mathcal{O}(\theta(\alpha))$ and are “higher” than the saddle-point w.r.t. the level curves of the function $\langle \theta(s) | e_\alpha \rangle$ outside $\mathcal{O}(\theta(\alpha))$, see Section 4.4.3. Applying Cauchy Theorem, the density is finally represented as a sum of integrals along these new contours and the sum of residues at poles of the integrands we encounter deforming the initial ones:

$$\begin{aligned} \pi(re_\alpha) &= \sum_{p \in \mathcal{P}'_\alpha} \operatorname{res}_p \varphi_2(\theta_1(s)) \frac{\gamma_2(p)}{\sqrt{d(\theta_1(p))}} e^{-r\langle \theta(p) | e_\alpha \rangle} + \sum_{p \in \mathcal{P}''_\alpha} \operatorname{res}_p \varphi_1(\theta_2(s)) \frac{\gamma_1(p)}{\sqrt{\tilde{d}(\theta_2(p))}} e^{-r\langle \theta(p) | e_\alpha \rangle} \\ &= \frac{1}{2\pi\sqrt{\det \Sigma}} \left(\int_{\Gamma_{\theta_1, \alpha}} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r\langle \theta(s) | e_\alpha \rangle} ds + \int_{\Gamma_{\theta_2, \alpha}} \frac{\varphi_1(s)\gamma_1(\theta(s))}{s} e^{-r\langle \theta(s) | e_\alpha \rangle} ds \right). \end{aligned} \quad (4.13)$$

Here \mathcal{P}'_α (resp. \mathcal{P}''_α) is the set of poles of the first order of φ_1 (resp. φ_2) that are found when shifting the initial contour $\mathcal{I}_{\theta_1}^+$ to the new one $\Gamma_{\theta_1, \alpha}$ (resp. $\mathcal{I}_{\theta_2}^+$ to $\Gamma_{\theta_2, \alpha}$), all of them are on the arc $\{s'_0, \theta(\alpha)\}$ (resp. $\{\theta(\alpha), s''_0\}$) of ellipse \mathcal{E} .

The asymptotic expansion of integrals along $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$ is made explicit by the standard saddle-point method in Section 4.4.4. The set of poles $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ is analyzed in Section 4.4.5. In Case (1) of Theorem 102 this set is empty, thus the asymptotic expansion of the density is determined by the saddle-point, its first term is given in Theorem 102. In Cases (2), (3) and (4) this set is not empty. The residues at poles over $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ in (4.13) bring all more important contribution to the asymptotic expansion of $\pi(re_\alpha)$ than the integrals along $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$. Taking into account the monotonicity of function $\langle \theta | e_\alpha \rangle$ on the arcs $\{s'_0, \theta(\alpha)\}$ and on $\{\theta(\alpha), s'_0\}$, they can be ranked in order of their importance: clearly, the term associated with a pole p' is more important than the one with p'' if $\langle p' | e_\alpha \rangle < \langle p'' | e_\alpha \rangle$. In Case (2) (resp. (3)) the most important pole is $\zeta\theta^{**}$ (resp. $\eta\theta^*$), as announced in Theorem 102. In Case (4) the most important of them is among $\zeta\theta^{**}$ and $\eta\theta^*$, as stated in Theorem 102 as well. The expansion of integrals in (4.13) along $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$ via the saddle-point method provides all smaller asymptotic terms than those coming from the poles. Section 4.5 is devoted to the results: they are stated from two points of view in Sections 4.5.1 and 4.5.2 respectively. First, given an angle α_0 , we find the uniform asymptotic expansion of the density $\pi(r \cos(\alpha), r \sin(\alpha))$ as $r \rightarrow \infty$ and $\alpha \in \mathcal{O}(\alpha_0)$ depending on parameters (Σ, μ, R) : Theorems 120–123 of Section 4.5.1 state it in all cases of parameters (1)–(4). Second, in Section 4.5.2, given a set parameters (Σ, μ, R) , we compute the asymptotics of the density for all angles $\alpha_0 \in]0, \pi/2[$, see Theorems 124–126.

Remark. The constants mentioned in Theorem 102 and all those in asymptotic expansions of Theorems 120–126 are specified in terms of functions φ_1 and φ_2 . In the present paper we leave unknown these functions in their initial domains of definition although we carry out explicitly their meromorphic continuation procedure and find all their poles. In [47] the first author and K. Raschel make explicit these functions solving some boundary value problems. This determines the constants in asymptotic expansions in Theorems 102, 120–126.

Future works. The case of parameters such that $\zeta\theta^{**} = \theta(\alpha)$ and $\eta\theta^* \notin \{s'_0, \theta(\alpha)\}$ or

the case such that $\eta\theta^* = \theta(\alpha)$ and $\zeta\theta^{**} \notin \{s_0'', \theta(\alpha)\}$ are not treated in Theorem 102. Theorem 123 gives a partial result but not at all as satisfactory as in all other cases. In fact, in these cases the saddle-point $\theta(\alpha)$ coincides with the “main” pole of φ_1 or φ_2 . The analysis is then reduced to a technical problem of computing the asymptotics of an integral whenever the saddle-point coincides with a pole of the integrand or approaches to it. We leave it for the future work.

In the cases $\alpha = 0$ and $\alpha = \pi/2$, the tail asymptotics of the boundary measures ν_1 and ν_2 has been found in [29] and the constants have been specified in [47]. It would be also possible to find the asymptotics of $\pi(r \cos \alpha, r \sin \alpha)$ where $r \rightarrow \infty$ and $\alpha \rightarrow 0$ or $\alpha \rightarrow \pi/2$. This problem is reduced to obtaining the asymptotics of an integral when the saddle-point $\theta(0)$ or $\theta(\pi/2)$ coincides with a branch point of the integrand φ_1 or φ_2 . It can be solved by the same methods as in [64] for discrete random walks.

4.2 Riemann surface S

4.2.1 Kernel $\gamma(\theta_1, \theta_2)$

The kernel of the main functional equation

$$\gamma(\theta_1, \theta_2) = \frac{1}{2}(\sigma_{11}\theta_1^2 + \sigma_{22}\theta_2^2 + 2\sigma_{12}\theta_1\theta_2) + \mu_1\theta_1 + \mu_2\theta_2$$

can be written as

$$\gamma(\theta_1, \theta_2) = \tilde{a}(\theta_2)\theta_1^2 + \tilde{b}(\theta_2)\theta_1 + \tilde{c}(\theta_2) = a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$$

where

$$\begin{aligned} \tilde{a}(\theta_2) &= \frac{1}{2}\sigma_{11}, & \tilde{b}(\theta_2) &= \sigma_{12}\theta_2 + \mu_1, & \tilde{c}(\theta_2) &= \frac{1}{2}\sigma_{22}\theta_2^2 + \mu_2\theta_2, \\ a(\theta_1) &= \frac{1}{2}\sigma_{22}, & b(\theta_1) &= \sigma_{12}\theta_1 + \mu_2, & c(\theta_1) &= \frac{1}{2}\sigma_{11}\theta_1^2 + \mu_1\theta_1. \end{aligned}$$

The equation $\gamma(\theta_1, \theta_2) \equiv 0$ defines a two-valued algebraic function $\Theta_1(\theta_2)$ such that $\gamma(\Theta_1(\theta_2), \theta_2) \equiv 0$ and a two-valued algebraic function $\Theta_2(\theta_1)$ such that $\gamma(\theta_1, \Theta_2(\theta_1)) \equiv 0$. These functions have two branches:

$$\Theta_1^+(\theta_2) = \frac{-\tilde{b}(\theta_2) + \sqrt{\tilde{d}(\theta_2)}}{2\tilde{a}(\theta_2)}, \quad \Theta_1^-(\theta_2) = \frac{-\tilde{b}(\theta_2) - \sqrt{\tilde{d}(\theta_2)}}{2\tilde{a}(\theta_2)},$$

and

$$\Theta_2^+(\theta_1) = \frac{-b(\theta_1) + \sqrt{d(\theta_1)}}{2a(\theta_1)}, \quad \Theta_2^-(\theta_1) = \frac{-b(\theta_1) - \sqrt{d(\theta_1)}}{2a(\theta_1)}.$$

where

$$\begin{aligned} \tilde{d}(\theta_2) &= \theta_2^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2\theta_2(\mu_1\sigma_{12} - \mu_2\sigma_{11}) + \mu_1^2, \\ d(\theta_1) &= \theta_1^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2\theta_1(\mu_2\sigma_{12} - \mu_1\sigma_{22}) + \mu_2^2. \end{aligned}$$

The discriminant $d(\theta_1)$ (resp. $\tilde{d}(\theta_2)$) has two zeros θ_1^+, θ_1^- (resp. θ_2^+ and θ_2^-) that are both real and of opposite signs:

$$\begin{aligned} \theta_1^- &= \frac{(\mu_2\sigma_{12} - \mu_1\sigma_{22}) - \sqrt{D_1}}{\det \Sigma} < 0, & \theta_1^+ &= \frac{(\mu_2\sigma_{12} - \mu_1\sigma_{22}) + \sqrt{D_1}}{\det \Sigma} > 0, \\ \theta_2^- &= \frac{(\mu_1\sigma_{12} - \mu_2\sigma_{11}) - \sqrt{D_2}}{\det \Sigma} < 0, & \theta_2^+ &= \frac{(\mu_1\sigma_{12} - \mu_2\sigma_{11}) + \sqrt{D_2}}{\det \Sigma} > 0, \end{aligned}$$

with notations $D_1 = (\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \mu_2^2 \det \Sigma$ and $D_2 = (\mu_1\sigma_{12} - \mu_2\sigma_{11})^2 + \mu_1^2 \det \Sigma$. Then $\Theta_2(\theta_1)$ (resp. $\Theta_1(\theta_2)$) has two branch points: θ_1^- and θ_1^+ (resp. θ_2^- and θ_2^+). We can compute:

$$\Theta_2^\pm(\theta_1^-) = \frac{\mu_1\sigma_{12} - \mu_2\sigma_{11} + \frac{\sigma_{12}}{\sigma_{22}}\sqrt{D_1}}{\det \Sigma}, \quad \Theta_2^\pm(\theta_1^+) = \frac{\mu_1\sigma_{12} - \mu_2\sigma_{11} - \frac{\sigma_{12}}{\sigma_{22}}\sqrt{D_1}}{\det \Sigma},$$

$$\Theta_1^\pm(\theta_2^-) = \frac{\mu_2\sigma_{12} - \mu_1\sigma_{22} + \frac{\sigma_{12}}{\sigma_{11}}\sqrt{D_2}}{\det \Sigma}, \quad \Theta_1^\pm(\theta_2^+) = \frac{\mu_2\sigma_{12} - \mu_1\sigma_{22} - \frac{\sigma_{12}}{\sigma_{11}}\sqrt{D_2}}{\det \Sigma}.$$

Furthermore, $d(\theta_1)$ (resp. $\tilde{d}(\theta_2)$) being positive on $]\theta_1^-, \theta_1^+[$ (resp. $]\theta_2^-, \theta_2^+[$) and negative on $\mathbf{R} \setminus [\theta_1^-, \theta_1^+]$ (resp. $\mathbf{R} \setminus [\theta_2^-, \theta_2^+]$), both branches $\Theta_2^\pm(\theta_1)$ (resp. $\Theta_1^\pm(\theta_2)$) take real values on $[\theta_1^-, \theta_1^+]$ (resp. $[\theta_2^-, \theta_2^+]$) and complex values on $\mathbf{R} \setminus [\theta_1^-, \theta_1^+]$ (resp. $\mathbf{R} \setminus [\theta_2^-, \theta_2^+]$).

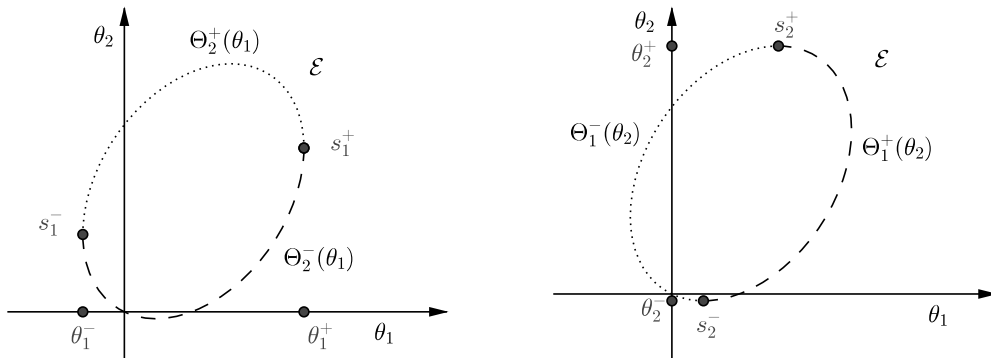


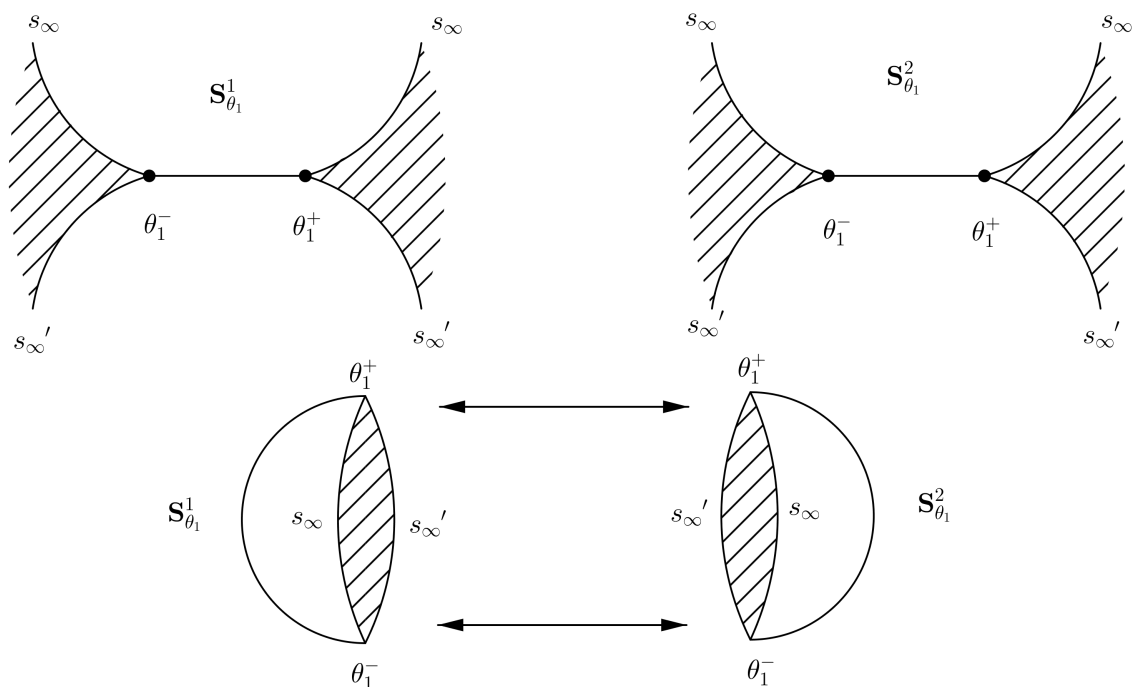
Figure 4.5: Functions $\Theta_2^\pm(\theta_1)$ and $\Theta_1^\pm(\theta_2)$ on $[\theta_1^-, \theta_1^+]$ and $[\theta_2^-, \theta_2^+]$

4.2.2 Construction of the Riemann surface \mathbf{S}

As in [37], we construct the Riemann surface \mathbf{S} of the algebraic function $\Theta_2(\theta_1)$. For this purpose we take two Riemann spheres $\mathbb{C}_{\theta_1}^1 \cup \{\infty\}$ and $\mathbb{C}_{\theta_1}^2 \cup \{\infty\}$, say $\mathbf{S}_{\theta_1}^1$ and $\mathbf{S}_{\theta_1}^2$, cut along $([-\infty^{(l)}, \theta_1^-] \cup [\theta_1^+, \infty^{(l)}])$, and we glue them together along the borders of these cuts, joining the lower border of the cut on $\mathbf{S}_{\theta_1}^1$ to the upper border of the same cut on $\mathbf{S}_{\theta_1}^2$ and vice versa. This procedure can be viewed as gluing together two half-spheres, see Figure 4.6. The resulting surface \mathbf{S} is homeomorphic to a sphere (i.e., a compact Riemann surface of genus 0) and is projected on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by a canonical covering map $h_{\theta_1} : \mathbf{S} \rightarrow \mathbb{C} \cup \{\infty\}$. In a standard way, we can lift the function $\Theta_2(\theta_1)$ to \mathbf{S} , by setting $\Theta_2(s) = \Theta_2^+(h_{\theta_1}(s))$ if $s \in \mathbf{S}_{\theta_1}^1 \subset \mathbf{S}$ and $\Theta_2(s) = \Theta_2^-(h_{\theta_1}(s))$ if $s \in \mathbf{S}_{\theta_1}^2 \subset \mathbf{S}$.

In a similar way one constructs the Riemann surface of the function $\Theta_1(\theta_2)$, by gluing together two copies $\mathbf{S}_{\theta_2}^1$ and $\mathbf{S}_{\theta_2}^2$ of the Riemann sphere \mathbf{S} cut along $([-\infty^{(l)}, \theta_2^-] \cup [\theta_2^+, \infty^{(l)}])$. We obtain again a surface homeomorphic to a sphere where we lift function $\Theta_1(\theta_2)$.

Since the Riemann surfaces of $\Theta_1(\theta_2)$ and $\Theta_2(\theta_1)$ are equivalent, we can and will work on a *single* Riemann surface \mathbf{S} , with two different covering maps $h_{\theta_1}, h_{\theta_2} : \mathbf{S} \rightarrow \mathbb{C} \cup \{\infty\}$. Then, for $s \in \mathbf{S}$, we set $\theta_1(s) = h_{\theta_1}(s)$ and $\theta_2(s) = h_{\theta_2}(s)$. We will often represent a point $s \in \mathbf{S}$ by the pair of its *coordinates* $(\theta_1(s), \theta_2(s))$. These coordinates are of course not independent, because the equation $\gamma(\theta_1(s), \theta_2(s)) = 0$ is valid for any $s \in \mathbf{S}$. One can see \mathbf{S} with points $s_1^\pm = (\theta_1^\pm, \frac{\mu_1\sigma_{12} - \mu_2\sigma_{11} \mp \frac{\sigma_{12}}{\sigma_{22}}\sqrt{D_1}}{\det \Sigma})$, $s_2^\pm = (\frac{\mu_2\sigma_{12} - \mu_1\sigma_{22} \mp \frac{\sigma_{12}}{\sigma_{11}}\sqrt{D_2}}{\det \Sigma}, \theta_2^\pm)$, $s_\infty = (\infty, \infty)$,

Figure 4.6: Construction of the Riemann surface \mathcal{S}

$s_{\infty'} = (\infty', \infty')$ on Figure 4.7. It is the union of $\mathbf{S}_{\theta_1}^1$ and $\mathbf{S}_{\theta_1}^2$ glued along the contour $\mathcal{R}_{\theta_1} = \{s : \theta_1(s) \in \mathbf{R} \setminus]\theta_1^-, \theta_1^+[\}$ that goes from s_{∞} to $s_{\infty'}$ via s_1^- and back to s_{∞} via s_1^+ . It is also the union of $\mathbf{S}_{\theta_2}^1$ and $\mathbf{S}_{\theta_2}^2$ glued along the contour $\mathcal{R}_{\theta_2} = \{s : \theta_2(s) \in \mathbf{R} \setminus]\theta_2^-, \theta_2^+[\}$. This contour goes from s_{∞} to $s_{\infty'}$ and back as well, but via s_2^- and s_2^+ . Let \mathcal{E} be the set of points of \mathbf{S} where both coordinates $\theta_1(s)$ and $\theta_2(s)$ are real. Then

$$\mathcal{E} = \{s \in \mathbf{S} : \theta_1(s) \in [\theta_1^-, \theta_1^+]\} = \{s \in \mathbf{S} : \theta_2(s) \in [\theta_2^-, \theta_2^+]\}.$$

One can see \mathcal{E} on Figures 4.5 and in black on 4.7, it contains all branch points s_1^{\pm} and s_2^{\pm} .

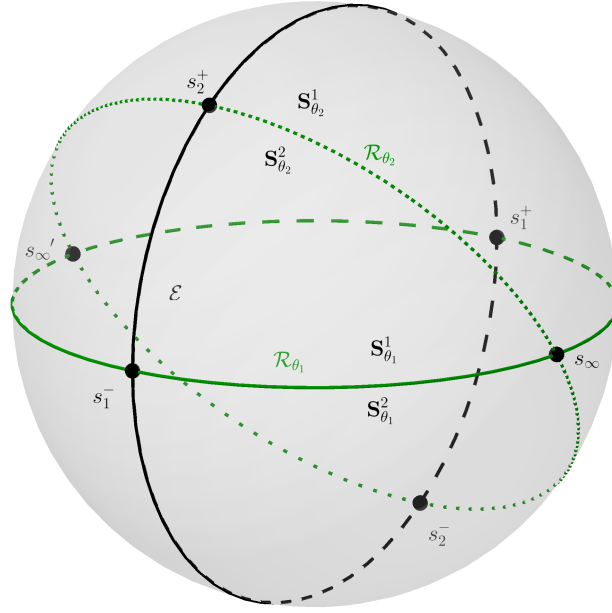
4.2.3 Galois automorphisms η and ζ

Now we need to introduce Galois automorphisms on \mathbf{S} . For any $s \in \mathbf{S} \setminus s_1^{\pm}$ there is a unique $s' \neq s \in \mathbf{S} \setminus s_1^{\pm}$ such that $\theta_1(s) = \theta_1(s')$. Furthermore, if $s \in \mathbf{S}_{\theta_1}^1$ then $s' \in \mathbf{S}_{\theta_1}^2$ and vice versa. On the other hand, whenever $s = s_1^-$ or $s = s_1^+$ (i.e. $\theta_1(s) = \theta_1^{\pm}$ is one of branch points of $\Theta_2(\theta_1)$) we have $s = s'$. Also, since $\gamma(\theta_1(s), \theta_2(s)) = 0$, $\theta_2(s)$ and $\theta_2(s')$ represent both values of function $\Theta_2(\theta_1)$ at $\theta_1 = \theta_1(s) = \theta_1(s')$. By Vieta's theorem we obtain $\theta_2(s)\theta_2(s') = \frac{c(\theta_1(s))}{a(\theta_1(s))}$.

Similarly, for any $s \in \mathbf{S} \setminus s_2^{\pm}$, there exists a unique $s'' \neq s \in \mathbf{S} \setminus s_2^{\pm}$ such that $\theta_2(s) = \theta_2(s'')$. If $s \in \mathbf{S}_{\theta_2}^1$ then $s'' \in \mathbf{S}_{\theta_2}^2$ and vice versa. On the other hand, if $s = s_2^-$ or $s = s_2^+$ (i.e. $\theta_2(s) = \theta_2^{\pm}$ is one of branch points of $\Theta_1(\theta_2)$) we have $s = s''$. Moreover, since $\gamma(\theta_1(s), \theta_2(s)) = 0$, $\theta_1(s)$ and $\theta_1(s'')$ give both values of function $\Theta_1(\theta_2)$ at $\theta_2 = \theta_2(s) = \theta_2(s'')$. Again, by Vieta's theorem $\theta_1(s)\theta_1(s'') = \frac{\tilde{c}(\theta_2(s))}{\tilde{a}(\theta_2(s))}$.

With the previous notations we now define the mappings $\zeta : \mathbf{S} \rightarrow \mathbf{S}$ and $\eta : \mathbf{S} \rightarrow \mathbf{S}$ by

$$\begin{cases} \zeta s = s' & \text{if } \theta_1(s) = \theta_1(s'), \\ \eta s = s'' & \text{if } \theta_2(s) = \theta_2(s'') \end{cases}$$

Figure 4.7: Points of \mathbf{S} with $\theta_1(s)$ or $\theta_2(s)$ real

Following [78] we call them *Galois automorphisms* of \mathbf{S} . Then $\zeta^2 = \eta^2 = \text{Id}$, and

$$\theta_2(\zeta s) = \frac{c(\theta_1(s))}{a(\theta_1(s))} \frac{1}{\theta_2(s)}, \quad \theta_1(\eta s) = \frac{\bar{c}(\theta_2(s))}{\bar{a}(\theta_2(s))} \frac{1}{\theta_1(s)}.$$

Points s_1^- and s_1^+ (resp. s_2^- and s_2^+) are fixed points for ζ (resp. η).

It is known that conformal automorphisms of a sphere (that can be identified to $\mathbb{C} \cup \infty$) are transformations of type $z \mapsto \frac{az+b}{cz+d}$ where a, b, c, d are any complex numbers satisfying $ad - bc \neq 0$. The automorphisms ζ and η , which are conformal automorphisms of \mathbf{S} , have each two fixed points and are involutions (because $\zeta^2 = \eta^2 = \text{Id}$). We can deduce from it that ζ (resp. η) is a symmetry w.r.t. the axis A_1 (resp. A_2) that passes through fixed points s_1^- and s_1^+ (resp. s_2^- and s_2^+). In other words ζ (resp. η) is a rotation of angle π , around A_1 (resp. A_2), see Figure 4.8. Let us draw the axis A orthogonal to the plane generated by the axes A_1 and A_2 and passing through the intersection point of A_1 and A_2 . We denote by β the angle between the axes A_1 and A_2 . Automorphisms $\eta\zeta$ and $\zeta\eta$ are then *rotations of angle 2β and -2β around the axis A* . This axis goes through points s_∞ and s_∞' , which are fixed points for $\eta\zeta$ and $\zeta\eta$, see Figure 4.8.

In the particular case $\Sigma = \text{Id}$, we have $\eta\zeta = \zeta\eta$, the axes A_1 and A_2 are orthogonal. We deduce that $\beta = \frac{\pi}{2}$ and that $\eta\zeta$ and $\zeta\eta$ are symmetries w.r.t. the axis A .

4.2.4 Domains of initial definition of φ_1 and φ_2 on \mathbf{S}

We would like to lift functions $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$ on \mathbf{S} naturally as $\varphi_1(s) = \varphi_1(\theta_2(s))$ and $\varphi_2(s) = \varphi_2(\theta_1(s))$. But it can not be done for all $s \in \mathbf{S}$, $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$ being not defined on the whole of \mathbf{C} . Nevertheless, we are able to do it for points s where $\theta_2(s)$

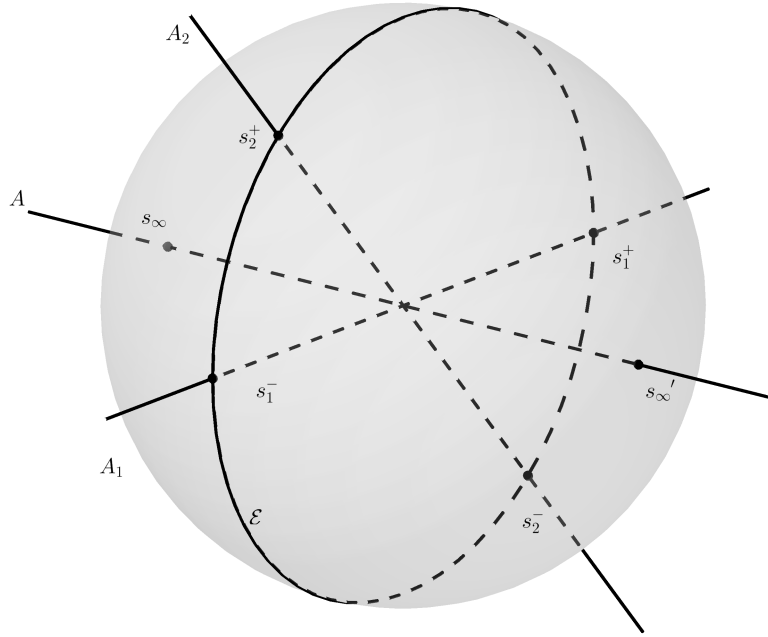


Figure 4.8: Axes A_1 , A_2 and A of Galois automorphisms ζ , η and $\zeta\eta$ respectively

or $\theta_1(s)$ respectively have non-positive real parts. Therefore, in this section we study the domains on \mathbf{S} where it holds true.

For any $\theta_1 \in \mathbf{C}$ with $\mathcal{R}(\theta_1) = 0$, $\Theta_2(\theta_1)$ takes two values $\Theta_2^\pm(\theta_1)$. Let us observe that under assumption that the second coordinate of the interior drift is negative, i.e. $\mu_2 < 0$ we have $\mathcal{R}\Theta_2^-(\theta_1) \leq 0$ and $\mathcal{R}\Theta_2^+(\theta_1) > 0$. Furthermore $\mathcal{R}\Theta_2^-(\theta_1) = 0$ only at $\theta_1 = 0$, and then $\Theta_2^-(\theta_1) = 0$. The domain

$$\Delta_1 = \{s \in \mathcal{S} : \mathcal{R}\theta_1(s) < 0\}$$

is simply connected and bounded by the contour $\mathcal{I}_{\theta_1} = \{s : \mathcal{R}\theta_1(s) = 0\}$.

The contour \mathcal{I}_{θ_1} can be represented as the union of $\mathcal{I}_{\theta_1}^- \cup \mathcal{I}_{\theta_1}^+$, where $\mathcal{I}_{\theta_1}^- = \{s : \mathcal{R}\theta_1(s) = 0, \mathcal{R}\theta_2(s) \leq 0\}$, $\mathcal{I}_{\theta_1}^+ = \{s : \mathcal{R}\theta_1(s) = 0, \mathcal{R}\theta_2(s) > 0\}$, see Figure 4.9.

The contour $\mathcal{I}_{\theta_1}^-$ goes from s_∞ to $s_{\infty'}$ crossing the set of real points \mathcal{E} at $s_0 = (0, 0)$, while $\mathcal{I}_{\theta_1}^+$ goes from s_∞ to $s_{\infty'}$ crossing \mathcal{E} at $s'_0 = (0, -2\frac{\mu_2}{\sigma_{22}})$ where the second coordinate is positive.

In the same way, under assumption that the first coordinate of the interior drift is negative, i.e. $\mu_1 < 0$, for any $\theta_2 \in \mathbf{C}$ with $\mathcal{R}(\theta_2) = 0$, $\Theta_1(\theta_2)$ takes two values $\Theta_1^\pm(\theta_2)$, where $\mathcal{R}\Theta_1^-(\theta_2) \leq 0$ and $\mathcal{R}\Theta_1^+(\theta_2) > 0$, moreover $\mathcal{R}\Theta_1^-(\theta_2) = 0$ only at $\theta_2 = 0$, and then $\Theta_1^-(\theta_2) = 0$. The domain

$$\Delta_2 = \{s \in \mathcal{S} : \mathcal{R}\theta_2(s) < 0\}$$

is simply connected and bounded by the contour $\mathcal{I}_{\theta_2} = \{s : \mathcal{R}\theta_2(s) = 0\}$. The contour \mathcal{I}_{θ_2} can be represented as the union of $\mathcal{I}_{\theta_2}^- \cup \mathcal{I}_{\theta_2}^+$, where $\mathcal{I}_{\theta_2}^- = \{s : \mathcal{R}\theta_2(s) = 0, \mathcal{R}\theta_1(s) \leq 0\}$, $\mathcal{I}_{\theta_2}^+ = \{s : \mathcal{R}\theta_2(s) = 0, \mathcal{R}\theta_1(s) > 0\}$. The contour $\mathcal{I}_{\theta_2}^-$ goes from s_∞ to $s_{\infty'}$, crossing the set

of real points \mathcal{E} at $s_0 = (0, 0)$, while $\mathcal{I}_{\theta_2}^+$ goes from s_∞ to $s_{\infty'}$ crossing \mathcal{E} at $s_0'' = (-2\frac{\mu_1}{\sigma_{11}}, 0)$, see Figure 4.9.

Assume now that the interior drift has both coordinates negative, i.e. (4.6). From what said above, $\mathcal{I}_{\theta_1}^- \setminus s_0 \subset \Delta_2$ and $\mathcal{I}_{\theta_2}^- \setminus s_0 \subset \Delta_1$. The intersection $\Delta_1 \cap \Delta_2$ consists of two connected components, both bounded by $\mathcal{I}_{\theta_1}^-$ and $\mathcal{I}_{\theta_2}^-$. The union $\Delta_1 \cup \Delta_2$ is a connected domain, but not simply connected because of the point s_0 . The domain $\Delta_1 \cup \Delta_2 \cup s_0$ is open, simply connected and bounded by $\mathcal{I}_{\theta_1}^+$ and $\mathcal{I}_{\theta_2}^+$, see Figure 4.9. We set $\Delta = \Delta_1 \cup \Delta_2$.

Note that in the cases of stationary SRBM with drift μ having one of coordinates non-negative, the location of contours $\mathcal{I}_{\theta_1}^+$, $\mathcal{I}_{\theta_1}^-$, $\mathcal{I}_{\theta_2}^+$, $\mathcal{I}_{\theta_2}^-$ on \mathbf{S} is different. For example, assume that $\mu_2 > 0$. Then $\mathcal{R}\Theta_2^-(\theta_1) < 0$ and $\mathcal{R}\Theta_2^+(\theta_1) \geq 0$, the contour $\mathcal{I}_{\theta_1}^-$ goes from s_∞ to $s_{\infty'}$ crossing the set of real points \mathcal{E} at $s_0' = (0, -2\frac{\mu_2}{\sigma_{22}})$ where the second coordinate is negative, while $\mathcal{I}_{\theta_1}^+$ goes from s_∞ to $s_{\infty'}$ crossing \mathcal{E} at $s_0 = (0, 0)$. In order to shorten the number of cases and pictures, we restrict ourselves in this paper to the case (4.6) of both coordinates of μ negative, although all our methods work in these other cases as well.

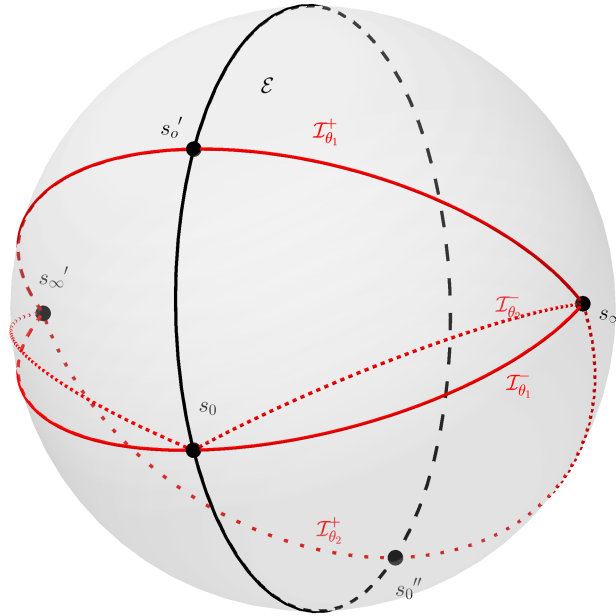


Figure 4.9: Pure imaginary points of \mathcal{S}

4.2.5 Parametrization of \mathbf{S}

It is difficult to visualize on three-dimensional sphere different points, contours, automorphisms and domains introduced above that will be used in future steps. For this reason we propose here an explicit and practical parametrisation of \mathcal{S} . Namely we identify \mathcal{S} to $\mathbb{C} \cup \{\infty\}$ and in the next proposition we explicitly define h_{θ_1} and h_{θ_2} two recoveries introduced in Section 4.2.2. Such a parametrisation allows to visualize better in two dimensions the sphere $\mathcal{S} \equiv \mathbb{C} \cup \{\infty\}$ and all sets we are interested in, as we can see in Figure 4.10.

Proposition 103. *We set the following covering maps*

$$\begin{aligned} h_{\theta_1} : \mathbb{C} \cup \{\infty\} \equiv \mathcal{S} &\longrightarrow \mathbb{C} \cup \{\infty\} \\ s &\longmapsto h_{\theta_1}(s) = \theta_1(s) := \frac{\theta_1^+ + \theta_1^-}{2} + \frac{\theta_1^+ - \theta_1^-}{4} \left(s + \frac{1}{s}\right) \end{aligned}$$

and

$$\begin{aligned} h_{\theta_2} : \mathbb{C} \cup \{\infty\} \equiv \mathcal{S} &\longrightarrow \mathbb{C} \cup \{\infty\} \\ s &\longmapsto h_{\theta_2}(s) = \theta_2(s) := \frac{\theta_2^+ + \theta_2^-}{2} + \frac{\theta_2^+ - \theta_2^-}{4} \left(\frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s}\right), \end{aligned}$$

where

$$\beta = \arccos -\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}.$$

The equation $\gamma(\theta_1(s), \theta_2(s)) = 0$ is valid for any $s \in \mathcal{S}$. Galois automorphisms can be written

$$\zeta(s) = \frac{1}{s}, \quad \eta(s) = \frac{e^{2i\beta}}{s},$$

and $\eta\zeta$ (resp. $\zeta\eta$) is a rotation around $s_\infty \equiv 0$ of angle 2β (resp. -2β) according to counterclockwise direction.

Proof. We set $h_{\theta_1}(s) = \theta_1(s) := \frac{\theta_1^+ + \theta_1^-}{2} + \frac{\theta_1^+ - \theta_1^-}{4} \left(s + \frac{1}{s}\right)$. One can notice that $h_{\theta_1}(1) = \theta_1^+$, $h_{\theta_1}(-1) = \theta_1^-$, $h'_{\theta_1}(1) = 0$, $h'_{\theta_1}(-1) = 0$. This parametrization is practical because it leads to a similar rational recovery h_{θ_2} . In order to make the equation $\gamma(\theta_1(s), \theta_2(s)) = 0$ valid for any $s \in \mathcal{S}$ we naturally set

$$\theta_2(s) = \Theta_2^+(\theta_1(s)) := \frac{-b(\theta_1(s)) + \sqrt{d(\theta_1(s))}}{2a(\theta_1(s))}$$

and we are going to show that $\theta_2(s) = \frac{\theta_2^+ + \theta_2^-}{2} + \frac{\theta_2^+ - \theta_2^-}{4} \left(\frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s}\right)$ where $\beta = \arccos -\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$. We note that $d(\theta_1(s))$ is the opposite of the square of a rational fraction

$$\begin{aligned} d(\theta_1(s)) &= -\det \Sigma(\theta_1(s) - \theta_1^+)(\theta_1(s) - \theta_1^-) = -\det \Sigma\left(\frac{\theta_1^+ - \theta_1^-}{4}\right)^2 \left(-2 + \left(s + \frac{1}{s}\right)\right) \left(2 + \left(s + \frac{1}{s}\right)\right) \\ &= -\det \Sigma\left(\frac{\theta_1^+ - \theta_1^-}{4}\right)^2 \left(s - \frac{1}{s}\right)^2 \leq 0. \end{aligned}$$

Then we have

$$\theta_2(s) = \Theta_2^+(\theta_1(s)) := \frac{-\sigma_{12} \frac{\theta_1^+ + \theta_1^-}{2} + \frac{\theta_1^+ - \theta_1^-}{4} \left(s + \frac{1}{s}\right) - \mu_2 + i\sqrt{\det \Sigma} \left(\frac{\theta_1^+ - \theta_1^-}{4}\right) \left(s - \frac{1}{s}\right)}{\sigma_{22}}. \quad (4.14)$$

Furthermore this parametrization leads to simple expressions for Galois automorphisms η and ζ . We derive immediately that $\theta_1(s) = \theta_1\left(\frac{1}{s}\right)$ and $\theta_2\left(\frac{1}{s}\right) = \Theta_2^-(\theta_1(s))$. Then we have

$$\zeta(s) = \frac{1}{s}.$$

Next we search η as an automorphism of the form $\eta s = \frac{K}{s}$. Since $\theta_2(s)$ is of the form $\theta_2(s) = us + \frac{v}{s} + w$ with constants u, v, w defined by (4.14), then $\theta_2(s) = \theta_2\left(\frac{K}{s}\right)$ with $K = \frac{v}{u}$. This leads to

$$\eta(s) = \frac{K}{s} \text{ with } K = \frac{-\sigma_{12} - i\sqrt{\det \Sigma}}{-\sigma_{12} + i\sqrt{\det \Sigma}}.$$

After setting

$$K = e^{2i\beta} \text{ with } \beta = \arccos -\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$$

we have

$$\zeta(s) = \frac{1}{s}, \quad \eta(s) = \frac{e^{2i\beta}}{s}$$

and then

$$\eta\zeta(s) = e^{2i\beta}s, \quad \zeta\eta(s) = e^{-2i\beta}s.$$

It follows that $\eta\zeta$ and $\zeta\eta$ are just rotations for angles 2β et -2β respectively. By symmetry considerations we can now rewrite

$$\theta_2(s) = \sqrt{uv}\left(\frac{s}{\sqrt{K}} + \frac{\sqrt{K}}{s}\right) + w = \frac{\theta_1^+ - \theta_1^-}{4} \sqrt{\frac{\sigma_{11}}{\sigma_{22}}}\left(\frac{s}{\sqrt{K}} + \frac{\sqrt{K}}{s}\right) + \frac{-\sigma_{12}\left(\frac{\theta_1^+ + \theta_1^-}{2}\right) - \mu_2}{\sigma_{22}}.$$

For $i = 1, 2$ we have $\theta_i^+ - \theta_i^- = 2\frac{\sqrt{D_i}}{\det \Sigma}$ and $\sigma_{11}D_1 = \sigma_{22}D_2$. Then we obtain $\frac{\theta_1^+ - \theta_1^-}{4} \sqrt{\frac{\sigma_{11}}{\sigma_{22}}} = \frac{\theta_2^+ - \theta_2^-}{4}$. Moreover $\frac{-\sigma_{12}\left(\frac{\theta_1^+ + \theta_1^-}{2}\right) - \mu_2}{\sigma_{22}} = \frac{\Theta_2^+(\theta_1^+) + \Theta_2^-(\theta_1^-)}{2} = \frac{\theta_2^+ + \theta_2^-}{2}$ (the last equality follows from elementary geometric properties of an ellipse). It implies

$$h_{\theta_2}(s) = \theta_2(s) = \frac{\theta_2^+ + \theta_2^-}{2} + \frac{\theta_2^+ - \theta_2^-}{4}\left(\frac{s}{\sqrt{K}} + \frac{\sqrt{K}}{s}\right)$$

concluding the proof. \square

Figure 4.10 shows different sets we are interested in according to the parametrization we have just introduced. We have $\theta_1(1) = \theta_1^+$, $\theta_1(-1) = \theta_1^-$, $\theta_2(e^{i\beta}) = \theta_2^+$ et $\theta_2(e^{i(\pi+\beta)}) = \theta_2^-$, $\theta_1(0) = \theta_2(0) = \infty$, $\theta_1(\infty) = \theta_2(\infty) = \infty$. Then we write $s_1^+ = 1$, $s_1^- = -1$, $s_2^+ = e^{i\beta}$, $s_2^- = e^{i(\pi+\beta)}$, $s_\infty = 0$, $s_{\infty'} = \infty$. It is easy to see that

$$\mathcal{E} = \{s \in \mathbb{C} \mid |s| = 1\},$$

and

$$\mathcal{R}_{\theta_1} = \mathbf{R}, \quad \mathcal{R}_{\theta_2} = e^{i\beta}\mathbf{R}.$$

We can determine the equation of the analytic curves of pure imaginary points of θ_i . We have $\mathcal{I}_{\theta_1} = \{s \in \mathcal{S} \mid \theta_1(s) \in i\mathbf{R}\}$. If we write $s = e^{i\omega}$ with $\omega = a + ib \in \mathbb{C}$ we find that $\Re(\theta_1(s)) = \frac{\theta_1^+ + \theta_1^-}{2} + \frac{\theta_1^+ - \theta_1^-}{2} \cos(a) \cosh(b)$. It follows that

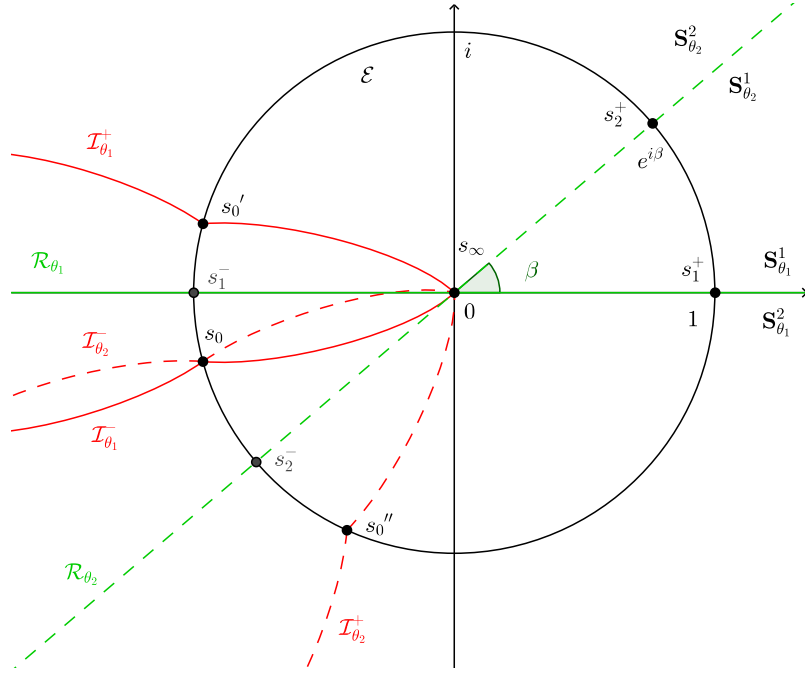
$$\mathcal{I}_{\theta_1} = \{s = e^{i\omega} \in \mathcal{S} \mid \omega = a + ib, a \in \mathbf{R}, b \in \mathbf{R}, \cos(a) \cosh(b) = \frac{\theta_1^+ + \theta_1^-}{\theta_1^- - \theta_1^+}\}.$$

Similarly we have

$$\mathcal{I}_{\theta_2} = \{s = e^{i(\omega+\beta)} \in \mathcal{S} \mid \omega = a + ib, a \in \mathbf{R}, b \in \mathbf{R}, \cos(a) \cosh(b) = \frac{\theta_2^+ + \theta_2^-}{\theta_2^- - \theta_2^+}\}.$$

We can easily notice that

$$\zeta\mathcal{I}_{\theta_1}^- = \mathcal{I}_{\theta_1}^+, \quad \zeta\mathcal{I}_{\theta_1}^+ = \mathcal{I}_{\theta_1}^-, \quad \text{and} \quad \eta\mathcal{I}_{\theta_2}^- = \mathcal{I}_{\theta_2}^+, \quad \eta\mathcal{I}_{\theta_2}^+ = \mathcal{I}_{\theta_2}^-.$$

Figure 4.10: Parametrization of \mathcal{S}

4.3 Meromorphic continuation of φ_1 and φ_2 on \mathbf{S}

4.3.1 Lifting of φ_1 and φ_2 on \mathbf{S} and their meromorphic continuation

Lifting of φ_1 and φ_2 on \mathcal{S} . Since the function $\theta_1 \rightarrow \varphi_2(\theta_1)$ is holomorphic on the set $\{\theta_1 \in \mathbb{C} : \Re\theta_1 < 0\}$ and continuous up to its boundary, we can lift it to $\bar{\Delta}_1 = \{s \in \mathcal{S} : \Re\theta_1(s) \leq 0\}$ as

$$\varphi_2(s) = \varphi_2(\theta_1(s)), \quad \forall s \in \bar{\Delta}_1.$$

In the same way we can lift φ_1 to $\bar{\Delta}_2$ as

$$\varphi_1(s) = \varphi_1(\theta_2(s)), \quad \forall s \in \bar{\Delta}_2.$$

Moreover, by definition of Galois automorphisms, functions φ_1 and φ_2 are invariant w.r.t. η and ζ respectively:

$$\begin{aligned} \varphi_2(\zeta s) &= \varphi_2(\theta_1(\zeta s)) = \varphi_2(\theta_1(s)) = \varphi_2(s), \quad \forall s \in \bar{\Delta}_1, \\ \varphi_1(\eta s) &= \varphi_1(\theta_2(\eta s)) = \varphi_1(\theta_2(s)) = \varphi_1(s), \quad \forall s \in \bar{\Delta}_2. \end{aligned} \quad (4.15)$$

Functions γ_1 and γ_2 can be lifted naturally on the whole of \mathbf{S} as

$$\gamma_1(s) = \gamma_1(\theta_1(s), \theta_2(s)), \quad \gamma_2(s) = \gamma_2(\theta_1(s), \theta_2(s)) \quad \forall s \in \mathbf{S}.$$

Since $\gamma(\theta_1(s), \theta_2(s)) = 0$, then the right-hand side in the main functional equation (4.4) equals zero for any $\theta = (\theta_1(s), \theta_2(s))$ such that $s \in \bar{\Delta}_1 \cap \bar{\Delta}_2$. Thus we have

$$\gamma_1(s)\varphi(s) + \gamma_2(s)\varphi_2(s) = 0, \quad \forall s \in \bar{\Delta}_1 \cap \bar{\Delta}_2. \quad (4.16)$$

Continuation of φ_1 and φ_2 on Δ .

Lemma 104. *Functions φ_1 and φ_2 (defined on $\bar{\Delta}_2$ and $\bar{\Delta}_1$ respectively) can be meromorphically continued on $\Delta \cup \{s_0\}$ by setting*

$$\varphi_1(s) = -\frac{\gamma_2(s)}{\gamma_1(s)}\varphi_2(s) \quad \text{if } s \in \Delta_1,$$

and

$$\varphi_2(s) = -\frac{\gamma_1(s)}{\gamma_2(s)}\varphi_1(s) \quad \text{if } s \in \Delta_2.$$

Furthermore,

$$\gamma_1(s)\varphi_1(s) + \gamma_2(s)\varphi_2(s) = 0 \quad \forall s \in \Delta \cup \{s_0\}, \quad (4.17)$$

$$\varphi_1(s) = \varphi(\eta s), \quad \varphi_2(s) = \varphi(\zeta s) \quad \forall s \in \Delta \cup \{s_0\}. \quad (4.18)$$

Proof. The open set $\Delta_1 \cap \Delta_2$ is non-empty and bounded by the curve $\mathcal{I}_{\theta_1}^- \cup \mathcal{I}_{\theta_2}^-$. Functional equation (4.16) is valid for $s \in \Delta_1 \cap \Delta_2$. It allows us to continue functions φ_1 and φ_2 as meromorphic on Δ as stated in this lemma. The functional equation (4.16) is then valid on the whole of Δ , as well as the invariance formulas (4.18). \square

The function $\varphi_1(s)$ is defined in a neighborhood $\mathcal{O}(s_0)$ of s_0 as $\varphi_1(\theta_2(s))$ for any $s \in \Delta_2 \cap \mathcal{O}(s_0)$ and $-\frac{\gamma_2(s)}{\gamma_1(s)}\varphi_2(\theta_1(s))$ for any $s \in \Delta_1 \cap \mathcal{O}(s_0)$. Furthermore,

$$\lim_{s \rightarrow s_0, s \in \Delta_2} \varphi_1(s) = \mathbb{E}_\pi \left(\int_0^1 dL_t^1 \right)$$

by definition of the function φ_1 . It is easy to see that function $\frac{\gamma_2(s)}{\gamma_1(s)}$ has a removable singularity at s_0 and to compute $\lim_{s \rightarrow s_0} \frac{\gamma_2(s)}{\gamma_1(s)} = \frac{r_{12}\mu_2 - r_{22}\mu_1}{r_{11}\mu_2 - r_{21}\mu_1}$. Hence

$$\lim_{s \rightarrow s_0, s \in \Delta_1} \varphi_1(s) = \lim_{s \rightarrow s_0, s \in \Delta_1} -\frac{\gamma_2(s)}{\gamma_1(s)}\varphi_2(\theta_1(s)) = \frac{r_{22}\mu_1 - r_{12}\mu_2}{r_{11}\mu_2 - r_{21}\mu_1} \mathbb{E}_\pi \left(\int_0^1 dL_t^2 \right).$$

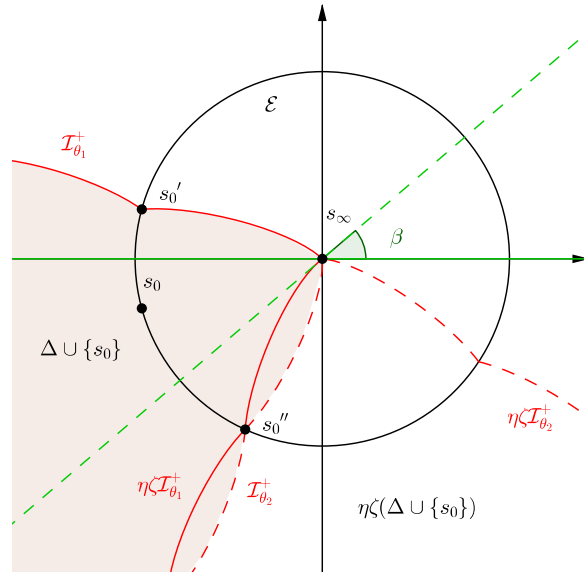
For any $s \in \Delta_1 \cap \Delta_2 \cap \mathcal{O}(s_0)$, by (4.16) $\varphi_1(s) = -\frac{\gamma_2(s)}{\gamma_1(s)}\varphi_2(s)$, from where $\lim_{s \rightarrow s_0, s \in \Delta_2} \varphi_1(s) = \lim_{s \rightarrow s_0, s \in \Delta_1} \varphi_1(s)$. Hence, function $\varphi_1(s)$ has a removable singularity at s_0 , and so is $\varphi_2(s)$ by the same arguments.

Functions φ_1 and φ_2 can be then of course continued to $\bar{\Delta}$. Moreover we have the following lemma.

Lemma 105. *The domains $\bar{\Delta} \cup \eta\zeta\bar{\Delta}$ and $\bar{\Delta} \cup \zeta\eta\bar{\Delta}$ are simply connected.*

Proof. Since $\eta\zeta$ and $\zeta\eta$ are just rotations for a certain angle 2β or -2β , it suffices to check that $\eta\zeta\mathcal{I}_{\theta_1}^+ \subset \bar{\Delta}$ and that $\zeta\eta\mathcal{I}_{\theta_2}^+ \in \bar{\Delta}$. In fact, $\zeta\mathcal{I}_{\theta_1}^+ = \mathcal{I}_{\theta_1}^- \subset \bar{\Delta}_2$. Since $\eta\bar{\Delta}_2 = \bar{\Delta}_2$, it follows that $\eta\mathcal{I}_{\theta_1}^- \subset \bar{\Delta}_2 \subset \bar{\Delta}$. By the same arguments $\zeta\eta\mathcal{I}_{\theta_2}^+ \in \bar{\Delta}$. One can refer to Figure 4.11. \square

Now we would like to continue function φ_1 (resp. φ_2) on $\eta\zeta\bar{\Delta}$ (resp. $\zeta\eta\bar{\Delta}$) as $\varphi_1(s) = G(s)\varphi_1(\zeta\eta s)$ for all $s \in \eta\zeta\bar{\Delta}$, where $G(s)$ is a known function and $\varphi_1(\zeta\eta s)$ is well defined since $\zeta\eta s \in \bar{\Delta}$. We could then continue this procedure for $(\eta\zeta)^2\bar{\Delta}$, $(\eta\zeta)^3\bar{\Delta}$, (resp. $(\zeta\eta)^2\bar{\Delta}$, $(\zeta\eta)^3\bar{\Delta}$) etc and hence to define φ_1 (resp. φ_2) on the whole of \mathbf{S} . Unfortunately, the domain $\bar{\Delta}$ is closed, from where it will be difficult to establish that the function is meromorphic. From the other hand, neither $\Delta \cup \eta\zeta\bar{\Delta}$ nor $(\Delta \cup s_0) \cup \eta\zeta(\Delta \cup s_0)$ are simply connected,

Figure 4.11: Δ and $\eta\zeta\Delta$

there is a “gap” at s_0'' . See figure 4.11. To avoid this technical complication, we will first continue φ_1 and φ_2 on a slightly bigger open domain Δ^ϵ defined as follows. Let

$$\Delta_1^\epsilon = \{s : \Re\theta_1(s) < \epsilon\}, \quad \Delta_2^\epsilon = \{s : \Re\theta_2(s) < \epsilon\} \quad (4.19)$$

and

$$\Delta^\epsilon = \Delta_1^\epsilon \cup \Delta_2^\epsilon \quad (4.20)$$

Let us fix any $\epsilon > 0$ small enough. For any $\theta_1 \in \mathbf{C}$ with $\Re\theta_1 = \epsilon$, the function $\Theta_2(\theta_1)$ takes two values $\Theta_2^\pm(\theta_1)$ where $\Re(\Theta_2^-(\theta_1)) < 0$ and $\Re(\Theta_2^+(\theta_1)) > 0$. The domain Δ_1^ϵ is bounded by the contour $\mathcal{I}_{\theta_1}^\epsilon = \mathcal{I}_{\theta_1}^{\epsilon,-} \cup \mathcal{I}_{\theta_1}^{\epsilon,+}$ where $\mathcal{I}_{\theta_1}^{\epsilon,-}$, $\mathcal{I}_{\theta_1}^{\epsilon,+}$ both go from s_∞ to $s_{\infty'}$, $\mathcal{I}_{\theta_1}^{\epsilon,-} \subset \Delta_2$ and $\mathcal{I}_{\theta_1}^{\epsilon,+} \cap \Delta = \emptyset$. See Figure 4.12. The same is true about the contour $\mathcal{I}_{\theta_2}^\epsilon = \mathcal{I}_{\theta_2}^{\epsilon,-} \cup \mathcal{I}_{\theta_2}^{\epsilon,+}$ limiting Δ_2^ϵ , namely $\mathcal{I}_{\theta_2}^{\epsilon,-} \subset \Delta_1$ and $\mathcal{I}_{\theta_2}^{\epsilon,+} \cap \Delta = \emptyset$.

Lemma 106. *Functions $\varphi_1(s)$ and $\varphi_2(s)$ can be continued as meromorphic functions on Δ^ϵ . Moreover equation (4.17) and the invariance formulas (4.18) remain valid.*

Proof. For any $s \in \Delta_1^\epsilon \setminus \Delta$, we have $\zeta s \in \Delta_2 \subset \Delta$, except for $s = s_0'$, for which $\zeta s_0' = s_0$. Anyway, function $\varphi_2(s)$ can be continued as meromorphic function on Δ_1^ϵ/Δ as :

$$\varphi_2(s) = \varphi_2(\zeta s), \quad \forall s \in \Delta_1^\epsilon \setminus \Delta.$$

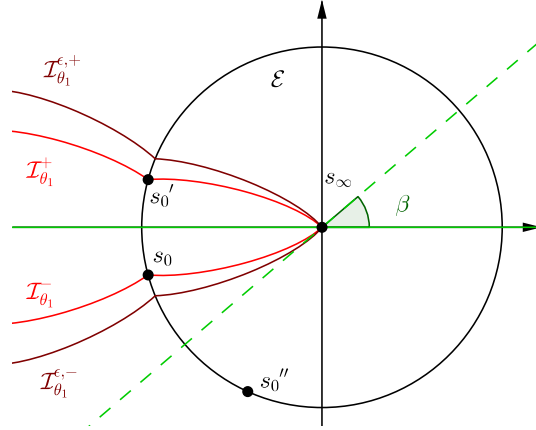
Then $\varphi_1(s)$ can be continued on the same domain by (4.17):

$$\varphi_1(s) = -\frac{\gamma_2(s)}{\gamma_1(s)}\varphi_2(s) \quad \forall s \in \Delta_1^\epsilon \setminus \Delta.$$

Similarly, the formulas

$$\varphi_1(s) = \varphi_1(\eta s), \quad \varphi_2(s) = -\frac{\gamma_1(s)}{\gamma_2(s)}\varphi_1(s) \quad \forall s \in \Delta_2^\epsilon \setminus \Delta$$

determine the meromorphic continuation of $\varphi_1(s)$ and $\varphi_2(s)$ on $\Delta_2^\epsilon \setminus \Delta$. \square

Figure 4.12: $\mathcal{I}_{\theta_1}^{\epsilon,-}$ and $\mathcal{I}_{\theta_1}^{\epsilon,+}$

Lemma 107. *The domains $\Delta^\epsilon \cap \eta\zeta\Delta^\epsilon$ and $\Delta^\epsilon \cap \zeta\eta\Delta^\epsilon$ are open simply connected domains. Function $\varphi_1(s)$ can be continued as meromorphic on $\Delta^\epsilon \cup \eta\zeta\Delta^\epsilon$ by the formula :*

$$\varphi_1(s) = \frac{\gamma_1(\zeta\eta s)\gamma_2(\eta s)}{\gamma_2(\zeta\eta s)\gamma_1(\eta s)}\varphi_1(\zeta\eta s), \quad \forall s \in \eta\zeta\Delta^\epsilon \text{ continuation by rotation of } 2\beta. \quad (4.21)$$

Function $\varphi_2(s)$ can be continued as meromorphic on $\Delta^\epsilon \cup \zeta\eta\Delta^\epsilon$ by the formula :

$$\varphi_2(s) = \frac{\gamma_2(\eta\zeta s)\gamma_1(\zeta s)}{\gamma_1(\eta\zeta s)\gamma_2(\zeta s)}\varphi_2(\eta\zeta s), \quad \forall s \in \zeta\eta\Delta^\epsilon \text{ continuation by rotation of } -2\beta. \quad (4.22)$$

Proof. We have shown in the proof of Lemma 105 that $\eta\zeta\mathcal{I}_{\theta_1}^+ \subset \bar{\Delta} \subset \Delta^\epsilon$, and that $\zeta\eta\mathcal{I}_{\theta_2}^+ \subset \bar{\Delta} \subset \Delta^\epsilon$. Since $\zeta\eta$ and $\eta\zeta$ are just rotations, this implies that $\Delta^\epsilon \cap \eta\zeta\Delta^\epsilon$ and $\Delta^\epsilon \cap \zeta\eta\Delta^\epsilon$ are non-empty open simply connected domains, and that $\Delta^\epsilon \cup \eta\zeta\Delta^\epsilon$ and $\Delta^\epsilon \cup \zeta\eta\Delta^\epsilon$ are simply connected.

Let us take $s \in \Delta^\epsilon \cap \eta\zeta\Delta^\epsilon$. Then $\zeta\eta s \in \Delta^\epsilon \cap \zeta\eta\Delta^\epsilon$ and we can write by (4.17)

$$\gamma_1(\zeta\eta s)\varphi_1(\zeta\eta s) + \gamma_2(\zeta\eta s)\varphi_2(\zeta\eta s) = 0. \quad (4.23)$$

Furthermore, we have shown in the proof of Lemma 105 that $\zeta\eta\mathcal{I}_{\theta_2}^+ \in \bar{\Delta}_1$. It follows that for all ϵ small enough $\zeta\eta\mathcal{I}_{\theta_2}^{\epsilon,+} \in \Delta_1$, and hence $\Delta^\epsilon \cap \zeta\eta\Delta^\epsilon \subset \Delta_1^\epsilon$. Since $\zeta\Delta_1^\epsilon = \Delta_1^\epsilon$, then $\zeta(\Delta^\epsilon \cap \zeta\eta\Delta^\epsilon) \subset \Delta_1^\epsilon \subset \Delta_\epsilon$. Then $\zeta(\zeta\eta s) = \eta s \in \Delta^\epsilon$ and we can write (4.17) and (4.18) at this point as well:

$$\gamma_1(\eta s)\varphi_1(\eta s) + \gamma_2(\eta s)\varphi_2(\eta s) = 0. \quad (4.24)$$

$$\varphi_1(\eta s) = \varphi_1(s) \quad (4.25)$$

$$\varphi_2(\eta s) = \varphi_2(\zeta\eta s). \quad (4.26)$$

Combining (4.25) and (4.24) we get $\varphi_1(s) = -\gamma_2(\eta s)\varphi_2(\eta s)/\gamma_1(\eta s)$ from where by (4.26)

$$\varphi_1(s) = -\frac{\gamma_2(\eta s)}{\gamma_1(\eta s)}\varphi_2(\zeta\eta s). \quad (4.27)$$

Due to (4.23)

$$\varphi_2(\zeta\eta s) = -\frac{\gamma_1(\zeta\eta s)}{\gamma_2(\zeta\eta s)}\varphi_1(\zeta\eta s). \quad (4.28)$$

Substituting (4.28) into (4.27), we obtain the formula (4.21) valid for any $s \in \Delta^\epsilon \cap \eta\zeta\Delta^\epsilon$. By principle of analytic continuation this allows to continue φ_1 on $\eta\zeta\Delta^\epsilon$ as meromorphic function. The proof is completely analogous for φ_2 . \square

We may now in the same way, using formulas (4.21) and (4.22), continue function $\varphi_1(s)$ (resp. $\varphi_2(s)$) as meromorphic on $(\eta\zeta)^2\Delta^\epsilon$, $(\eta\zeta)^3\Delta^\epsilon$ (resp. $(\zeta\eta)^2\Delta^\epsilon$, $(\zeta\eta)^3\Delta^\epsilon$) etc proceeding each time by rotation for the angle 2β [resp. -2β]. Namely we have the following lemma.

Lemma 108. *For any $n \geq 1$ the domains $\Delta^\epsilon \cup \eta\zeta\Delta^\epsilon \cup \dots \cup (\eta\zeta)^n\Delta^\epsilon$ and $\Delta^\epsilon \cup \zeta\eta\Delta^\epsilon \cup \dots \cup (\zeta\eta)^n\Delta^\epsilon$ are open simply connected domains. Function $\varphi_1(s)$ can be continued as meromorphic subsequently on $\eta\zeta\Delta^\epsilon, (\eta\zeta)^2\Delta^\epsilon, \dots, (\eta\zeta)^n\Delta^\epsilon$ by the formulas :*

$$\varphi_1(s) = \frac{\gamma_1(\zeta\eta s)\gamma_2(\eta s)}{\gamma_2(\zeta\eta s)\gamma_1(\eta s)}\varphi_1(\zeta\eta s), \quad \forall s \in (\eta\zeta)^k\Delta^\epsilon, k = 1, 2, \dots, n, \quad \text{continuation by rotation of } 2\beta. \quad (4.29)$$

Function $\varphi_2(s)$ can be continued as meromorphic on $\zeta\eta\Delta^\epsilon, (\zeta\eta)^2\Delta^\epsilon, \dots, (\zeta\eta)^n\Delta^\epsilon$ by the formulas :

$$\varphi_2(s) = \frac{\gamma_2(\eta\zeta s)\gamma_1(\zeta s)}{\gamma_1(\eta\zeta s)\gamma_2(\zeta s)}\varphi_2(\eta\zeta s), \quad \forall s \in (\zeta\eta)^k\Delta^\epsilon, k = 1, 2, \dots, n, \quad \text{continuation by rotation of } -2\beta. \quad (4.30)$$

Proof. We proceed by induction on $k = 1, 2, \dots, n$. For $k = 1$, this is the subject of the previous lemma. For any $k = 2, \dots, n$, assume the formula (4.29) for any $s \in (\eta\zeta)^{k-1}\Delta^\epsilon$. The domain $(\eta\zeta)^{k-1}\Delta^\epsilon \cap (\eta\zeta)^k\Delta^\epsilon = (\eta\zeta)^{k-1}(\Delta^\epsilon \cap \eta\zeta\Delta^\epsilon)$ is a non empty open domain by Lemma 107, $(\eta\zeta)^{k-1}$ being just the rotation for the angle $2(k-1)\beta$. The formula (4.29) is valid for any $s \in (\eta\zeta)^{k-1}\Delta^\epsilon \cap (\eta\zeta)^k\Delta^\epsilon$ by induction assumption. Hence, by the principle of meromorphic continuation it is valid for any $s \in (\eta\zeta)^k\Delta^\epsilon$. The same is true for the formula (4.30). \square

Proceeding as in Lemma 108 by rotations, we will continue φ_1 soon on the first half of \mathbf{S} , that is $\mathbf{S}_{\theta_2}^1$, then the whole of \mathbf{S} and go further, turning around \mathbf{S} for the second time, for the third, etc up to infinity. In fact, each time we complete this procedure on one of two halves of \mathbf{S} , we recover a new branch of the function φ_1 as function of $\theta_2 \in \mathbf{C}$. So, going back to the complex plane, we continue this function as multivalued and determine all its branches. The same is true for φ_2 if we proceed by rotations in the opposite direction. This procedure could be presented better on the universal covering of \mathbf{S} , but for the purpose of the present paper it is enough to complete it only on one-half of \mathbf{S} , that is to study just the first (main) branch of φ_1 and φ_2 . We summarize this result in the following theorem. We recall that $\mathbf{S} = \mathbf{S}_{\theta_1}^1 \cup \mathbf{S}_{\theta_1}^2$ and we denote by $\mathbf{S}_{\theta_1}^1$ the half that contains s'_0 (and not s_0 , as $\zeta s_0 = s'_0$). In the same way $\mathbf{S} = \mathbf{S}_{\theta_2}^1 \cup \mathbf{S}_{\theta_2}^2$ and we denote by $\mathbf{S}_{\theta_2}^1$ the half that contains s''_0 (and not s_0 , as $\eta s_0 = s''_0$), see Figure 4.10.

Theorem 109. *For any $s \in \mathbf{S}_{\theta_2}^1$ there exists $n \geq 0$ such that $(\zeta\eta)^n s \in \bar{\Delta}$. Let us define*

$$\varphi_1(s) = \frac{\gamma_1((\zeta\eta)^n s) \dots \gamma_1(\zeta\eta s) \gamma_2(\eta(\zeta\eta)^{n-1} s) \dots \gamma_2(\eta s)}{\gamma_2((\zeta\eta)^n s) \dots \gamma_2(\zeta\eta s) \gamma_1(\eta(\zeta\eta)^{n-1} s) \dots \gamma_1(\eta s)}\varphi_1((\zeta\eta)^n s) \quad (4.31)$$

Then the function $\varphi_1(s)$ is meromorphic on $\mathbf{S}_{\theta_2}^1$. For any $s \in \mathbf{S}_{\theta_1}^1$, there exists $n \geq 0$ such that $(\eta\zeta)^n s \in \bar{\Delta}$. Let us define

$$\varphi_2(s) = \frac{\gamma_2((\eta\zeta)^n s) \dots \gamma_2(\eta\zeta s) \gamma_1(\zeta(\eta\zeta)^{n-1} s) \dots \gamma_1(\eta s)}{\gamma_1((\eta\zeta)^n s) \dots \gamma_1(\eta\zeta s) \gamma_2(\zeta(\eta\zeta)^{n-1} s) \dots \gamma_2(\zeta s)} \varphi_1((\eta\zeta)^n s) \quad (4.32)$$

Then the function $\varphi_2(s)$ is meromorphic on $\mathbf{S}_{\theta_1}^1$.

Proof. It is a direct corollary of Lemma 105 and Lemma 108. \square

4.3.2 Poles of functions φ_1 and φ_2 on \mathbf{S}

It follows from meromorphic continuation procedure that all poles of $\varphi_1(s)$ and $\varphi_2(s)$ on \mathbf{S} are located on the ellipse \mathcal{E} , they are images of zeros of γ_1 and γ_2 by automorphisms η and ζ applied several times. Then all poles of all branches of $\varphi_2(s)$ (resp. $\varphi_1(s)$) on \mathbf{C}_{θ_1} (resp. \mathbf{C}_{θ_2}) are on the real segment $[\theta_1^-, \theta_1^+]$ (resp. $[\theta_2^-, \theta_2^+]$).

Notations of arcs on \mathcal{E} . Let us remind that we denote by $\{s_1, s_2\}$ an arc of the ellipse \mathcal{E} with ends at s_1 and s_2 not passing through the origin, see Theorem 102. From now on, we will denote in square brackets $]s_1, s_2[$ or $[s_1, s_2]$ an arc of \mathcal{E} going in the anticlockwise direction from s_1 to s_2 .

In order to compute the asymptotic expansion of stationary distribution density, we are interested in poles of φ_1 on the arc $]s_0'', s_2^+[$ and in those of φ_2 on the arc $]s_1^+, s_0'[,$ To determine the main asymptotic term, we are particularly interested in the pole of $\varphi_1(\theta_2(s))$ on $]s_0'', s_2^+[$ closest to s_0'' and in the one of $\varphi_2(\theta_1(s))$ on $]s_1^+, s_0'[,$ closest to s_0' . We identify them in this section.

We remind that θ^* is a zero of $\gamma_1(s)$ on \mathcal{E} different from s_0 and that θ^{**} is a zero of $\gamma_2(s)$ on \mathcal{E} different from s_0 . Their coordinates are

$$\begin{aligned} \theta^* &= 2 \frac{r_{21}\mu_1 - r_{11}\mu_2}{r_{21}^2\sigma_{11} - 2r_{11}r_{21}\sigma_{12} + r_{11}^2\sigma_{22}} \left(-r_{21}, r_{11} \right), \\ \theta^{**} &= 2 \frac{r_{12}\mu_2 - r_{22}\mu_1}{r_{22}^2\sigma_{11} - 2r_{22}r_{12}\sigma_{12} + r_{12}^2\sigma_{22}} \left(r_{22}, -r_{12} \right) \end{aligned} \quad (4.33)$$

Their images by automorphisms η and ζ have the following coordinates:

$$\begin{aligned} \eta\theta^* &= \left(-\frac{r_{11}}{\sigma_{11}r_{21}}(\sigma_{22}\theta_2^* + 2\mu_2), \theta_2^* \right), \\ \zeta\theta^{**} &= \left(\theta_1^{**}, -\frac{r_{22}}{\sigma_{22}r_{12}}(\sigma_{11}\theta_1^{**} + 2\mu_1) \right). \end{aligned} \quad (4.34)$$

Lemma 110. (1) If $\theta^{**} \in]s_0, s_1^+[$, then $\zeta\theta^{**}$ is a pole of $\varphi_2(\theta_1(s))$ on $]s_1^+, s_0'[,$

(2) If $\theta^* \in]s_2^+, s_0'[,$ then $\eta\theta^*$ is a pole of $\varphi_1(\theta_2(s))$ on $]s_0'', s_2^+[$.

Proof. By meromorphic continuation procedure

$$\varphi_2(\zeta\theta^{**}) = \frac{\gamma_2(\eta\theta^{**})\gamma_1(\theta^{**})\varphi_2(\eta\theta^{**})}{\gamma_2(\theta^{**})\gamma_1(\eta\theta^{**})}. \quad (4.35)$$

Let us check that the numerator in (4.35) is non zero, this will prove the statement (1) the lemma.

It is clear that $\gamma_1(\theta^{**}) \neq 0$ due to stability conditions (4.1) and (4.2).

Suppose that $\gamma_2(\eta\theta^{**}) = 0$. This could be only if $\eta\theta^{**} = \theta^{**} \in]s_0, s_1^+[$, thus $\theta^{**} = s_2^-$ where $\theta_2(s_2^-) < 0$ and consequently $\varphi_1(\eta\theta^{**}) < \infty$. But by meromorphic continuation of $\varphi_2(\theta_1(s))$ to the arc $\{s \in \mathcal{E} : \theta_2(s) < 0\}$ we have: $\varphi_2(\eta\theta_1^{**}) = -\frac{\gamma_1(\eta\theta^{**})\varphi_1(\eta\theta_2^{**})}{\gamma_2(\eta\theta^{**})}$, from where by (4.35)

$$\varphi_2(\zeta\theta^{**}) = -\frac{\gamma_1(\theta^{**})}{\gamma_2(\theta^{**})}\varphi_1(\eta\theta_2^{**}).$$

Then $\zeta\theta^{**}$ is clearly a pole of φ_2 , this finishes the proof of statement (1) of the lemma in this particular case. Otherwise $\gamma_2(\eta\theta^{**}) \neq 0$.

Let us finally check that $\varphi_2(\eta\theta^{**}) \neq 0$. Let us first observe that $\varphi_2(\theta_1(s)) \neq 0$ for any $s \in \mathcal{E}$ with one of two coordinates non-positive. In fact, if the first coordinate $\theta_1(s)$ of s is non-positive, then $\varphi_2(\theta_1(s)) \neq 0$ by its definition. If s has the second coordinate $\theta_2(s)$ non-positive, then $\varphi_2(\theta_1(s)) = -\frac{\gamma_1(s)}{\gamma_2(s)}\varphi_1(\theta_2(s))$ where $\gamma_1(s)$ can not have zeros with the second coordinate non-positive by stability conditions and neither $\varphi_1(\theta_2(s))$ by its definition. Hence, $\varphi_2(\theta_1(s)) \neq 0$ on the arc $\{s \in \mathcal{E} : \theta_1(s) \leq 0 \text{ or } \theta_2(s) \leq 0\}$.

It remains to consider the case where both coordinates of $\eta\theta^{**}$ are positive, i.e. $\theta^{**} \in]\eta s'_0, s_1^+[$ where the parameters are such that $s_2^{1,+} > \theta_2(\eta s'_0) > 0$ and to show that $\varphi_2(\eta\theta^{**}) \neq 0$. Suppose the contrary, that $\varphi_2(\eta\theta^{**}) = 0$. Then there are zeros of φ_2 on $] \eta s_1^+, s'_0[$ and among these zeros there exists θ^0 the closest one to s'_0 . By meromorphic continuation

$$\varphi_2(\theta_1^0) = \frac{\gamma_2(\eta\zeta\theta^0)\gamma_1(\zeta\theta^0)\varphi_2(\eta\zeta\theta^0)}{\gamma_2(\zeta\theta_1^0)\gamma_1(\eta\zeta\theta^0)}, \quad (4.36)$$

where $\eta\zeta\theta^0 \in]\theta_0, s''_0[$. First of all, we note that $\varphi_2(\eta\zeta\theta^0) \neq 0$ if $\eta\zeta\theta^0 \in [\theta_0, s'_0[$, since θ^0 is the closest zero to s'_0 , and $\varphi_2(\eta\zeta\theta^0) \neq 0$ if $\eta\zeta\theta^0 \in [s'_0, s''_0[$, because one of coordinates of $\eta\zeta\theta^0$ is non-positive within this segment. Hence, $\varphi_2(\eta\zeta\theta^0) \neq 0$ for any point $\eta\zeta\theta^0 \in]\theta_0, s''_0[$.

Furthermore, since $\eta\zeta\theta^0 \in]\theta_0, s''_0[$, then $\eta\zeta\theta^0 \neq \theta^{**}$ and thus $\gamma_2(\eta\zeta\theta^0) \neq 0$ except for $\eta\zeta\theta^0 = s_0$. As for this particular case $\eta\zeta\theta^0 = s_0$, we would have $\varphi_2(\theta^0) = -\gamma_1(s''_0)\varphi_1(s_0)\gamma_2^{-1}(s''_0) \neq 0$, so that $\theta^0 = \zeta\eta s_0$ can not be a zero of φ_2 .

The point $\zeta\theta^0 \in \zeta[\eta\theta^{**}, s'_0] = \zeta[\eta s'_0, \theta^{**}]$ that is the segment $[\eta s'_0, \theta^{**}]$ rotated for the angle -2β . Hence $\zeta\theta^0$ is located on \mathcal{E} below θ^{**} . Then $\gamma_1(\zeta\theta^0) = 0$ combined with $\gamma_2(\theta^{**}) = 0$ is impossible by stability conditions (4.1) and (4.2). Thus $\gamma_1(\zeta\theta^0) \neq 0$. It follows from (4.36) that $\varphi_2(\theta_1^0) \neq 0$. Thus there exist no zeros of φ_2 on $] \eta s_1^+, s'_0[$ and finally $\varphi_2(\eta\theta^{**}) \neq 0$. Therefore the numerator in (4.35) is non zero, hence $\zeta\theta^{**}$ is a pole of φ_2 .

The reasoning for θ^* is the same. □

Lemma 111. (i) Assume that $\theta^p \in]s_1^+, s'_0[$ is a pole of $\varphi_2(\theta_1)$ and it is the closest pole to s'_0 .

If the parameters (Σ, μ) are such that $\theta_2(s_1^+) \leq 0$, or the parameters (Σ, μ, R) are such that $\theta_2(s_1^+) > 0$ but $\eta\zeta\theta^p \notin]\eta s_1^+, s_0[$, then $\gamma_2(\zeta\theta^p) = 0$ where $\zeta\theta^p \in]s_1^+, s_0[$ and θ^p is a pole of the first order.

If the parameters (Σ, μ, R) are such that $\theta_2(s_1^+) > 0$ and $\eta\zeta\theta^p \in]\eta s_1^+, s_0[$, then either $\gamma_2(\zeta\theta^p) = 0$ where $\zeta\theta^p \in]s_0, s_1^+[$ or $\gamma_1(\eta\zeta\theta^p) = 0$. Furthermore, in this case, if $\gamma_2(\zeta\theta^p)$ and $\gamma_1(\eta\zeta\theta^p)$ do not equal zero simultaneously, then θ^p is a pole of the first order.

(ii) Assume that $\theta^p \in]s''_0, s_2^+[$ is a pole of $\varphi_1(\theta_2)$ and it is the closest pole to s''_0 .

If the parameters (Σ, μ) are such that $\theta_1(s_2^+) \leq 0$, or the parameters (Σ, μ, R) are such that $\theta_1(s_2^+) > 0$ but $\zeta\eta\theta^p \notin]s_0, \zeta s_2^+[$, then $\gamma_1(\eta\theta^p) = 0$ where $\eta\theta^p \in]s_0, s_2^+[$ and θ^p is a pole of the first order.

If the parameters (Σ, μ, R) are such that $\theta_1(s_2^+) > 0$ and $\zeta\eta\theta^p \in]s_0, \eta s_1^+[$, then either $\gamma_1(\eta\theta^p) = 0$ where $\eta\theta^p \in]s_2^+, s_0[$ or $\gamma_2(\zeta\eta\theta^p) = 0$. Furthermore, in this case, if $\gamma_1(\eta\theta^p)$ and $\gamma_2(\zeta\eta\theta^p)$ do not equal zero simultaneously, then θ^p is a pole of the first order.

Proof. Due to meromorphic continuation procedure we have

$$\varphi_2(\theta_1^p) = \frac{\gamma_2(\eta\zeta\theta^p)\gamma_1(\zeta\theta^p)\varphi_2(\eta\zeta\theta_1^p)}{\gamma_2(\zeta\theta^p)\gamma_1(\eta\zeta\theta^p)} \quad (4.37)$$

where $\eta\zeta\theta^p \in]\theta^p, s_0''[$.

Assume that $\eta\zeta\theta^p \in]\theta^p, s_0[$. In this case point $\eta\zeta\theta^p$ has the second coordinate positive and so does $\zeta\theta^p \in]s_0, s_1^+[$. It follows that $\theta_2(s_1^+) > 0$ and $\eta\zeta\theta^p \in \eta]s_0, s_1^+[\cap\{\theta : \theta_2 > 0\} =]\eta s_1^+, s_0[$.

Thus, if the parameters (Σ, μ) are such that $\theta_2(s_1^+) \leq 0$, or the parameters (Σ, μ, R) are such that $\theta_2(s_1^+) > 0$ but $\eta\zeta\theta^p \notin]\eta s_1^+, s_0[$, then the second coordinate of $\eta\zeta\theta^p$ is non-positive, i.e. $\eta\zeta\theta^p \in [s_0, s_0''[$. In this case

$$\varphi_2(\eta\zeta\theta_1^p) = -\frac{\gamma_1(\eta\zeta\theta^p)}{\gamma_2(\eta\zeta\theta^p)}\varphi_1(\eta\zeta\theta^p) \quad (4.38)$$

from where by (4.37)

$$\varphi_2(\theta_1^p) = -\frac{\gamma_1(\zeta\theta^p)\varphi_1(\eta\zeta\theta_2^p)}{\gamma_2(\zeta\theta^p)}. \quad (4.39)$$

Since $\varphi_1(\eta\zeta\theta_2^p)$ is finite for for any $\eta\zeta\theta^p \in [s_0, s_0''[$ by its initial definition, the formula (4.39) implies that $\gamma_2(\zeta\theta^p) = 0$ and the pole θ^p is of the first order.

If parameters (Σ, μ, R) are such that $\theta_2(s_1^+) > 0$ and $\eta\zeta\theta^p \in]\eta s_1^+, s_0[$, then either $\eta\zeta\theta^p \in [s_0'', s_0]$ or $\eta\zeta\theta^p \in]s_0, \theta^p[$. In the first case we have (4.39) as previously from where $\gamma_2(\zeta\theta^p) = 0$ and the pole $\zeta\theta^p$ is of the first order. Let us turn to the second case $\eta\zeta\theta^p \in]\theta^p, s_0[$ for which we will use the formula (4.37). The pole θ^p being the closest to s_0' , then $\eta\zeta\theta^p$ can not be a pole of φ_2 on $] \theta^p, s_0'[$. It can neither be a pole of φ_2 on $[s_0', s_0]$, since this function is initially well defined on this segment. Hence in formula (4.37) $\varphi_2(\eta\zeta\theta_1^p) \neq \infty$ for $\eta\zeta\theta^p \in]\theta^p, s_0'[$. It follows from (4.37) that either $\gamma_2(\zeta\theta^p) = 0$ or $\gamma_1(\eta\zeta\theta^p) = 0$ and if these two equalities do not hold simultaneously, then pole θ^p must be of the first order.

The proof in the case (ii) is symmetric. □

Figure 4.13 gives two illustrations of Lemmas 110 and 111.

On the left figure the parameters are such that $\theta_1(s_2^+) > 0$ and $\theta_2(s_1^+) > 0$. Let us look at zeros θ^* of γ_1 and θ^{**} of γ_2 different from s_0 . We see $\theta^* \in]s_2^+, s_0[$, then $\eta\theta^*$ is the first candidate for the closest pole of φ_1 to s_0'' on $]s_0'', s_2^+[$. We also see $\theta^{**} \notin [s_0, \zeta s_2^+]$, then there are no other candidates. Hence the closest pole of φ_1 to s_0'' on $]s_0'', s_2^+[$ is $\eta\theta^*$. Since $\theta^{**} \in]s_0, s_1^+[$, then $\zeta\theta^{**}$ is the first candidate for the closest pole of φ_2 to s_0' on $]s_1^+, s_0'[$. Furthermore, $\theta^* \in]\eta s_1^+, s_0[$, so that $\zeta\eta\theta^*$ is the second candidate to be the closest pole of φ_2 to s_0' on $]s_1^+, s_0'[$. We see at the picture that $\zeta\eta\theta^*$ is closer to s_0' than $\zeta\eta^{**}$.

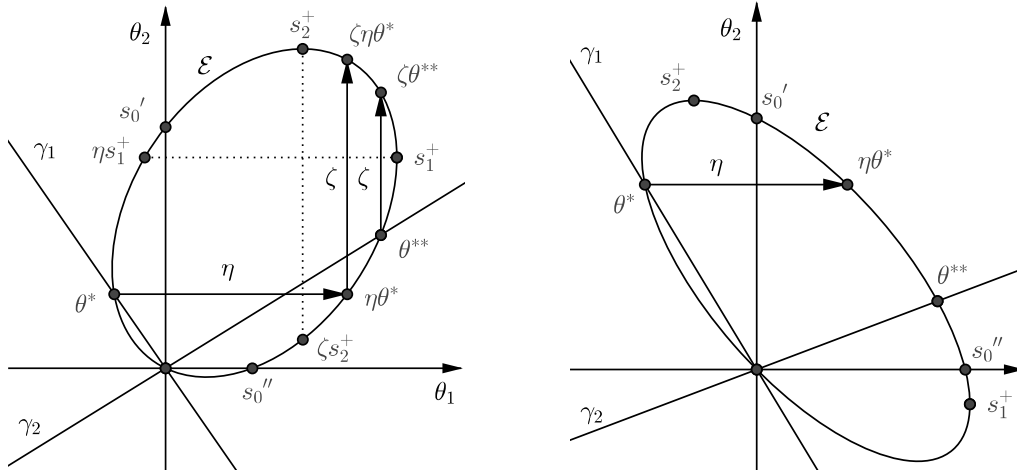


Figure 4.13: On the left figure : $\eta\theta^*$ is the closest pole of φ_1 to s_0'' on $]s_0'', s_2^+[$, $\zeta\eta\theta^*$ is the closest pole of φ_2 to s_0' on $]s_1^+, s_0[$. On the right figure: $\eta\theta^*$ is the closest pole of φ_1 to s_0'' on $]s_0'', s_2^+[$, there are no poles of φ_2 on $]s_1^+, s_0[$

On the right figure the parameters are such that $\theta_1(s_2^+) < 0$ and $\theta_2(s_1^+) < 0$. We see $\theta^* \in]s_2^+, s_0[$, then $\eta\theta^*$ is immediately the closest pole of φ_1 to s_0'' on $]s_0'', s_2^+[$. Since $\theta^{**} \notin]s_0, s_1^+[$, then there are no poles of φ_2 on $]s_1^+, s_0[$.

We will also need the following two lemmas.

Lemma 112. (1) Assume that $\theta_2(s_1^+) > 0$. Then for any $s \in]\eta s_1^+, s_0[$ we have $\theta_2(\zeta\eta s) > \theta_2(\eta s)$.

(2) Assume that $\theta_1(s_2^+) > 0$. Then for any $s \in]s_0, \zeta s_2^+[$ we have $\theta_1(\eta\zeta s) > \theta_1(\zeta s)$.

Proof. Since $\theta_2(s_1^+) > 0$, then $\theta_2(\zeta\eta s_0) - \theta_2(\eta s_0) > 0$. Consider the function $f(s) = \theta_2(\zeta s) - \theta_2(s)$ for $s \in [s_0'', s_1^+]$. It depends continuously on s on this arc. We note that $f(s_0'') = \theta_2(\zeta\eta s_0) - \theta_2(\eta s_0) > 0$, $f(s_1^+) = 0$. Furthermore, since $s_1^- \notin [s_0'', s_1^+]$, then $f(s) \neq 0$ for all $s \in]s_0'', s_1^+[$. Hence $f(s) > 0$ for all $s \in]s_0'', s_1^+[$, from where $\theta_2(\zeta\eta s) - \theta_2(\eta s) = f(\eta s) > 0$ for any $s \in \eta]s_0'', s_1^+[=]\eta s_1^+, s_0[$. The proof in the other case is symmetric. \square

Lemma 113. Assume that $\gamma_2(s)$ has a zero $\theta^{**} \in]s_0, s_1^+[$ and $\gamma_1(s)$ has a zero $\theta^* \in]s_2^+, s_0[$. Then one of the following three assertions holds true :

- (i) The closest pole of $\varphi_2(\theta_1(s))$ to s_0' on $]s_1^+, s_0'[$ is $\zeta\theta^{**}$, the closest pole of $\varphi_1(\theta_2(s))$ to s_0'' on $]s_0'', s_2^+[$ is $\eta\theta^*$, both of them are of the first order.
- (ii) The closest pole of $\varphi_2(\theta_1(s))$ to s_0' on $]s_1^+, s_0'[$ is $\zeta\theta^{**}$, it is of the first order. The closest pole of $\varphi_1(\theta_2(s))$ to s_0'' on $]s_0'', s_2^+[$ is $\eta\zeta\theta^{**}$ where $\theta_1(\eta\zeta\theta^{**}) > \theta_1(\zeta\theta^{**})$.
- (iii) The closest pole of $\varphi_2(\theta_1(s))$ to s_0' on $]s_1^+, s_0'[$ is $\zeta\eta\theta^*$ where $\theta_2(\zeta\eta\theta^*) > \theta_2(\eta\theta^*)$. The closest pole of $\varphi_1(\theta_2(s))$ to s_0'' on $]s_0'', s_2^+[$ is $\eta\theta^*$, it is of the first order.

The case (ii) is illustrated on Figure 4.13.

Proof. By Lemma 110 there exist poles of the function $\varphi_1(\theta_2(s))$ on $]s_0'', s_2^+[$. By Lemma 111 under parameters such that $\theta_1(s_2^+) \leq 0$ or $\theta_1(s_2^+) > 0$ and $\theta^{**} \notin]s_0, \zeta s_2^+[$, $\eta\theta^*$ is the closest pole to s_0'' and it is of the first order. By the same lemma under parameters such that $\theta_1(s_2^+) > 0$ and $\theta^{**} \in]s_0, \zeta s_2^+[$, either $\eta\theta^*$ or $\eta\zeta\theta^{**}$ is the closest pole to s_0' . By Lemma 111, if $\gamma_1(\zeta\theta^*) \neq 0$, pole $\eta\theta^*$ is of the first order. Condition $\gamma_1(\zeta\theta^*) \neq 0$ is equivalent to $\zeta\theta^* \neq \theta^{**}$. This means just that pole $\eta\theta^*$ is different from $\eta\zeta\theta^{**}$ which is another candidate for the closest pole to s_0'' . By Lemma 112 $\theta_1(\eta\zeta\theta^{**}) > \theta_1(\zeta\theta^{**})$. To summarize, one of two following statements holds true:

- (a1) Point $\eta\theta^*$ is the closest pole of $\varphi_1(\theta_2(s))$ to s_0'' on $]s_0'', s_2^+[$ and it is of the first order;
- (b1) The parameters are such that $\theta_1(s_2^+) > 0$ and $\theta^{**} \in]s_0, \zeta s_2^+[$. Point $\eta\zeta\theta^{**}$ is the closest pole of $\varphi_1(\theta_2(s))$ to s_0'' on $]s_0'', s_2^+[$ and $\theta_1(\eta\zeta\theta^{**}) > \theta_1(\zeta\theta^{**})$.

By Lemmas 110, 111, 112 and the same considerations, one of the following statements about $\varphi_2(\theta_1)$ holds true:

- (a2) Pole $\zeta\theta^{**}$ is the closest pole of $\varphi_2(\theta_1(s))$ to s_0' on $]s_1^+, s_0'[$ and it is of the first order.
- (b2) The parameters are such that $\theta_2(s_1^+) > 0$, $\theta^* \in]\eta s_1^+, s_0[$, point $\zeta\eta\theta^*$ is the closest pole of $\varphi_2(\theta_1(s))$ to s_0' on $]s_1^+, s_0'[$ and $\theta_2(\zeta\eta\theta^*) > \theta_2(\eta\theta^*)$.

Let us finally prove that (b1) and (b2) can not hold true simultaneously. Assume that $\theta_2(s_1^+) > 0$, $\theta^* \in]\eta s_1^+, s_0[$, $\theta_1(s_2^+) > 0$, $\theta^{**} \in]s_0, \zeta s_2^+[$ and e.g. (b2), that is $\zeta\eta\theta^*$ is the closest pole to s_0' . Note that in this case $\zeta\theta^{**} \in]s_2^+, s_0'[$. Then $\zeta\eta\theta^*$ is closer to s_0' than the pole $\zeta\theta^{**}$ on this segment or coincides with it. Hence $\theta_1(\zeta\eta\theta^*) \leq \theta_1(\zeta\theta^{**})$ and $\theta_2(\zeta\eta\theta^*) \leq \theta_2(\zeta\theta^{**})$. By Lemma 112 $\theta_1(\eta\theta^*) = \theta_1(\zeta\eta\theta^*)$ and $\theta_1(\zeta\theta^{**}) < \theta_1(\eta\zeta\theta^{**})$, $\theta_2(\eta\theta^*) < \theta_2(\zeta\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) = \theta_2(\eta\zeta\theta^{**})$. Then $\theta_1(\eta\theta^*) < \theta_1(\eta\zeta\theta^{**})$, $\theta_2(\eta\theta^*) < \theta_2(\eta\zeta\theta^{**})$. This means that that $\eta\theta^*$ is the closest pole of $\varphi_1(\theta_2(s))$ to s_0'' , $\eta\theta^* \neq \eta\zeta\theta^{**}$, so that (b1) is impossible for $\varphi_1(\theta_2(s))$, then we have (a1).

In the same way assumption (b1) leads to (a2). Thus (b1), (b2) can not hold true simultaneously, the lemma is proved. \square

4.4 Contribution of the saddle-point and of the poles to the asymptotic expansion

4.4.1 Stationary distribution density as a sum of integrals on \mathbb{S}

By the functional equation (4.4) and the inversion formula of Laplace transform (we refer to [32] and [13]), the density $\pi(x_1, x_2)$ can be represented as a double integral

$$\begin{aligned} \pi(x_1, x_2) &= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} e^{-x_1\theta_1 - x_2\theta_2} \varphi(\theta_1, \theta_2) d\theta_1 d\theta_2 \\ &= \frac{-1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} e^{-x_1\theta_1 - x_2\theta_2} \frac{\gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1)}{\gamma(\theta)} d\theta_1 d\theta_2. \end{aligned} \quad (4.40)$$

We now reduce it to a sum of single integrals.

Lemma 114. For any $(x_1, x_2) \in \mathbf{R}_+^2$

$$\pi(x_1, x_2) = I_1(x_1, x_2) + I_2(x_1, x_2)$$

where

$$I_1(x_1, x_2) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi_2(\theta_1) \gamma_2(\theta_1, \Theta_2^+(\theta_1)) e^{-x_1\theta_1 - x_2\Theta_2^+(\theta_1)} \frac{d\theta_1}{\sqrt{d(\theta_1)}} \quad (4.41)$$

and

$$I_2(x_1, x_2) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi_1(\theta_2) \gamma_1(\Theta_1^+(\theta_2), \theta_2) e^{-x_1\Theta_1^+(\theta_2) - x_2\theta_2} \frac{d\theta_2}{\sqrt{\tilde{d}(\theta_2)}}. \quad (4.42)$$

Proof. By inversion formula (4.40)

$$\begin{aligned} \pi(x_1, x_2) &= \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi_2(\theta_1) e^{-x_1\theta_1} \left(\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\gamma_2(\theta)}{\gamma(\theta)} e^{-x_2\theta} d\theta \right) d\theta_1 \\ &\quad + \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi_1(\theta_2) e^{-x_2\theta_2} \left(\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\gamma_1(\theta)}{\gamma(\theta)} e^{-x_1\theta} d\theta \right) d\theta_2. \end{aligned}$$

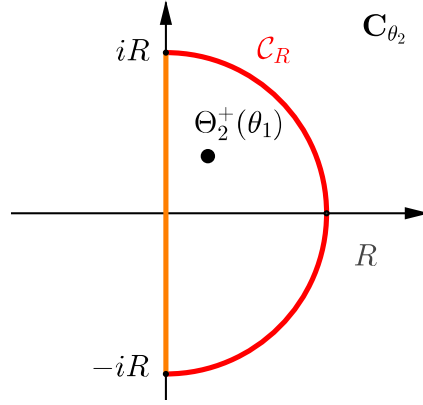
Now it suffices to show the following formulas

$$\frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\gamma_2(\theta) e^{-x_2\theta}}{\gamma(\theta)} d\theta = \frac{\gamma_2(\theta_1, \Theta_2^+(\theta_1))}{\sqrt{d(\theta_1)}} e^{-x_2\Theta_2^+(\theta_1)}, \quad (4.43)$$

$$\frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\gamma_1(\theta) e^{-x_1\theta}}{\gamma(\theta)} d\theta = \frac{\gamma_1(\Theta_1^+(\theta_2), \theta_2)}{\sqrt{\tilde{d}(\theta_2)}} e^{-x_1\Theta_1^+(\theta_2)}. \quad (4.44)$$

Let us prove (4.43). For any $\theta_1 \in i\mathbf{R} \setminus \{0\}$, the function $\gamma(\theta) = \frac{\sigma_{22}}{2}(\theta_2 - \Theta_2^+(\theta_1))(\theta_2 - \Theta_2^-(\theta_1))$ has two zeros $\Theta_2^+(\theta_1)$ and $\Theta_2^-(\theta_1)$. Their real parts are of opposite signs: $\Re(\Theta_2^-(\theta_1)) < 0$ and $\Re(\Theta_2^+(\theta_1)) > 0$. Thus for any fixed $\theta_1 \in i\mathbf{R} \setminus \{0\}$, function $\frac{\gamma_2(\theta) e^{-x_2\theta}}{\gamma(\theta)}$ of the argument θ_2 has two poles on the complex plane \mathbf{C}_{θ_2} , one at $\Theta_2^-(\theta_1)$ with negative real part and another one at $\Theta_2^+(\theta_1)$ with positive real part. Let us construct a contour $\mathcal{C}_R = [-iR, iR] \cup \{Re^{it} \mid t \in]-\pi/2, \pi/2[\}$ composed of the purely imaginary segment $[-iR, iR]$ and the half of the circle with radius R and center 0 on \mathbf{C}_{θ_2} , see Figure 4.14. For R large enough $\Theta_2^+(\theta_1)$ is inside the contour. The integral of $\frac{\gamma_2(\theta) e^{-x_2\theta}}{\gamma(\theta)}$ over this contour taken in the counter-clockwise direction equals the residue at the unique pole of the integrand:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{C}_R} \frac{\gamma_2(\theta) e^{-x_2\theta}}{\gamma(\theta)} d\theta &= \text{res}_{\theta_2=\Theta_2^+(\theta_1)} \frac{\gamma_2(\theta) e^{-x_2\theta}}{\gamma(\theta)} = \frac{\gamma_2(\theta_1, \Theta_2^+(\theta_1))}{(\sigma_{22}/2)(\Theta_2^+(\theta_1) - \Theta_2^-(\theta_1))} e^{-x_2\Theta_2^+(\theta_1)} \\ &= \frac{\gamma_2(\theta_1, \Theta_2^+(\theta_1))}{\sqrt{d(\theta_1)}} e^{-x_2\Theta_2^+(\theta_1)} \quad \text{for all large enough } R \geq 0. \end{aligned} \quad (4.45)$$


 Figure 4.14: Contour \mathcal{C}_R on \mathbf{C}_{θ_2} .

Let us take the limit of this integral as $R \rightarrow \infty$:

$$\lim_{R \rightarrow \infty} \int_{\mathcal{C}_R} \frac{\gamma_2(\theta) e^{-x_2 \theta_2}}{\gamma(\theta)} d\theta_2 = - \lim_{R \rightarrow \infty} \int_{-iR}^{iR} \frac{\gamma_2(\theta) e^{-x_2 \theta_2}}{\gamma(\theta)} d\theta_2 + \lim_{R \rightarrow \infty} \int_{\{Re^{it} | t \in]-\frac{\pi}{2}, \frac{\pi}{2}[\}} \frac{\gamma_2(\theta) e^{-x_2 \theta_2}}{\gamma(\theta)} d\theta_2. \quad (4.46)$$

The last term equals

$$\lim_{R \rightarrow \infty} \int_{\{Re^{it} | t \in]-\frac{\pi}{2}, \frac{\pi}{2}[\}} \frac{\gamma_2(\theta) e^{-x_2 \theta_2}}{\gamma(\theta)} d\theta_2 = \lim_{R \rightarrow \infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\gamma_2(\theta_1, Re^{it})}{\gamma(\theta_1, Re^{it})} e^{-x_2 Re^{it}} i Re^{it} dt. \quad (4.47)$$

We note that $\sup_{R>0} \sup_{t \in]-\frac{\pi}{2}, \frac{\pi}{2}[} |i Re^{it} \frac{\gamma_2(\theta_1, Re^{it})}{\gamma(\theta_1, Re^{it})}| < \infty$ and $\sup_{R>0} \sup_{t \in]-\frac{\pi}{2}, \frac{\pi}{2}[} |e^{-x_2 Re^{it}}| \leq 1$. Furthermore $|e^{-x_2 Re^{it}}| = e^{-x_2 R \cos t} \rightarrow 0$ as $R \rightarrow \infty$ for all $t \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Then by dominated convergence theorem the limit (4.47) equals 0 as $R \rightarrow \infty$. Hence, due to (4.45) and (4.46)

$$\frac{\gamma_2(\theta_1, \Theta_2^+(\theta_1))}{(\sigma_{22}/2)(\Theta_2^+(\theta_1) - \Theta_2^-(\theta_1))} e^{-x_2 \Theta_2^+(\theta_1)} = \lim_{R \rightarrow \infty} \int_{\mathcal{C}_R} \frac{\gamma_2(\theta) e^{-x_2 \theta_2}}{\gamma(\theta)} d\theta_2 = \int_{-i\infty}^{i\infty} \frac{\gamma_2(\theta) e^{-x_2 \theta_2}}{\gamma(\theta)} d\theta_2,$$

that proves (4.43) for any $\theta_1 \in i\mathbf{R} \setminus \{0\}$. The proof of (4.44) is analogous.

Note also that the integral

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi_2(\theta_1) \gamma_2(\theta_1, \Theta_2^+(\theta_1)) e^{-x_1 \theta_1 - x_2 \Theta_2^+(\theta_1)} \frac{d\theta_1}{\sqrt{d(\theta_1)}}$$

is absolutely convergent. In fact $\sup_{\theta_1 \in i\mathbf{R}} |\varphi_2(\theta_1)| \leq \nu_2(\mathbf{R}_+^2)$ by definition of φ_2 . It is elementary to see that $\sup_{\theta_1 \in i\mathbf{R}} |\gamma_2(\theta_1, \Theta_2^+(\theta_1)) d^{-1/2}(\theta_1)| < \infty$. Furthermore, for any $\theta_1 \in i\mathbf{R}$, $\Re \Theta_2^+(\theta_1) = \sigma_{22}^{-1}(-\mu_2 + \Re \sqrt{d(\theta_1)})$, thus for some constant $c > 0$ we have $\Re \Theta_2^+(\theta_1) > c |\Im \theta_1|$. Then the integral is absolutely convergent. This concludes the proof of formula (4.41). The proof of (4.42) is completely analogous. \square

Remark These integrals are equal to those on the Riemann surface \mathbf{S} along properly oriented contours $\mathcal{I}_{\theta_1}^+$ and $\mathcal{I}_{\theta_2}^+$ respectively. Thanks to the parametrization of Section 4.2.5 we have

$$\frac{d\theta_1}{\sqrt{d(\theta_1)}} = \frac{d\theta_2}{\sqrt{\tilde{d}(\theta_2)}} = \frac{id s}{s \sqrt{\det \Sigma}}. \quad (4.48)$$

Then we can write for $x = (x_1, x_2) \in \mathbf{R}_+^2$ the density $\pi(x_1, x_2)$ as a sum of two integrals on \mathbf{S} :

$$I_1 + I_2 = \frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\mathcal{I}_{\theta_1}^+} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-(\theta(s)|x)} ds + \frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\mathcal{I}_{\theta_2}^+} \frac{\varphi_1(s)\gamma_1(\theta(s))}{s} e^{-(\theta(s)|x)} ds.$$

4.4.2 Saddle-point

Let us fix $\alpha \in]0, \pi/2[$ and put $(x_1, x_2) = r e_\alpha = r(\cos(\alpha), \sin(\alpha))$ where $\alpha \in]0, \pi/2[$. Our aim now is to find the asymptotic expansion of $\pi(r \cos(\alpha), r \sin(\alpha))$, that is the one of the sum

$$I_1 + I_2 = \frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\mathcal{I}_{\theta_1}^+} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds + \frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\mathcal{I}_{\theta_2}^+} \frac{\varphi_1(s)\gamma_1(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds \quad (4.49)$$

as $r \rightarrow \infty$ and to prove that for any $\alpha_0 \in]0, \pi/2[$ this asymptotic expansion is uniform in a small neighborhood $\mathcal{O}(\alpha_0) \in]0, \pi/2[$.

These integrals are typical to apply the saddle-point method, see [41] or [83]. Let us study the function $\langle \theta(s) | e_\alpha \rangle$ on \mathbf{S} and its critical points.

Lemma 115. (i) For any $\alpha \in]0, \pi/2[$ function $\langle \theta(s) | e_\alpha \rangle$ has two critical points on \mathbf{S} denoted by $\theta^+(\alpha)$ and $\theta^-(\alpha)$. Both of them are on ellipse \mathcal{E} , $\theta^+(\alpha) \in]s_1^+, s_2^+[$, $\theta^-(\alpha) \in]s_1^-, s_2^-]$. Both of them are non-degenerate.

(ii) The coordinates of $\theta^+(\alpha) = (\theta_1^+(\alpha), \theta_2^+(\alpha))$ are given by formulas :

$$\begin{aligned} \theta_1^\pm(\alpha) &= \frac{\mu_2\sigma_{12} - \mu_1\sigma_{22}}{\det \Sigma} \pm \frac{1}{\det \Sigma} \sqrt{\frac{D_1}{1 + \frac{\tan(\alpha)^2}{(\sigma_{22} - \tan(\alpha)\sigma_{12})^2} \det \Sigma}} \\ \theta_2^\pm(\alpha) &= \frac{\mu_1\sigma_{12} - \mu_2\sigma_{11}}{\det \Sigma} \pm \frac{1}{\det \Sigma} \sqrt{\frac{D_2}{1 + \frac{\tan(\alpha)^2}{(\sigma_{11} - \tan(\alpha)\sigma_{12})^2} \det \Sigma}} \end{aligned} \quad (4.50)$$

where notations $D_1 = (\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \mu_2^2 \det \Sigma$ and $D_2 = (\mu_1\sigma_{12} - \mu_2\sigma_{11})^2 + \mu_1^2 \det \Sigma$ are used. With the parametrization of Section 4.2.5 the corresponding points on \mathbf{S} are such that:

$$s_\pm(\alpha)^2 = \frac{\cos \alpha (\theta_1^+ - \theta_1^-) + \sin \alpha (\theta_2^+ - \theta_2^-) e^{i\beta}}{\cos \alpha (\theta_1^+ - \theta_1^-) + \sin \alpha (\theta_2^+ - \theta_2^-) e^{-i\beta}}.$$

(iii) Function $\theta^+(\alpha)$ is an isomorphism between $[0, \pi/2]$ and $\mathcal{A} = [s^{1,+}, s^{2,+}]$, $\lim_{\alpha \rightarrow 0} \theta^+(\alpha) = s_1^+$, $\lim_{\alpha \rightarrow \pi/2} \theta^+(\alpha) = s_2^+$. Function $\theta^-(\alpha)$ is an isomorphism between $[0, \pi/2]$ and $[s^{1,-}, s^{2,-}]$, $\lim_{\alpha \rightarrow 0} \theta^-(\alpha) = s_1^-$, $\lim_{\alpha \rightarrow \pi/2} \theta^-(\alpha) = s_2^-$.

(iv) Function $\langle \theta(s) | e_\alpha \rangle$ is strictly increasing on the arc $[\theta^-(\alpha), \theta^+(\alpha)]$ of \mathcal{E} and strictly decreasing on the arc $[\theta^+(\alpha), \theta^-(\alpha)]$. Namely, $\theta^+(\alpha)$ is its maximum on \mathcal{E} and $\theta^-(\alpha)$ is its minimum:

$$\theta^+(\alpha) = \operatorname{argmax}_{s \in \mathcal{E}} \langle \theta(s) | e_\alpha \rangle \quad \theta^-(\alpha) = \operatorname{argmin}_{s \in \mathcal{E}} \langle \theta(s) | e_\alpha \rangle.$$

Proof. Let us look for critical points with coordinates (θ_1, θ_2) of $\langle \theta(s) | e_\alpha \rangle$ on \mathbf{S} . Equation $(\theta_1 \cos(\alpha) + \theta_2 \sin(\alpha))'_{\theta_1} = 0$ implies $\tan(\alpha) \frac{d\theta_2}{d\theta_1} = -1$. Substituting it into equation

$\gamma(\theta_1, \theta_2(\theta_1))'_{\theta_1} \equiv 0$ and writing also $\gamma(\theta_1, \theta_2) \equiv 0$ we get the system of two equations

$$\begin{cases} -\sigma_{11}\theta_1 \tan(\alpha) + \sigma_{22}\theta_2 + \sigma_{12}\theta_1 - \sigma_{12}\theta_2 \tan(\alpha) - \mu_1 \tan(\alpha) + \mu_2 = 0 \\ \sigma_{11}\theta_1^2 + \sigma_{22}\theta_2^2 + 2\sigma_{12}\theta_1\theta_2 + \mu_1\theta_1 + \mu_2\theta_2 = 0 \end{cases}$$

from where we compute $\theta^-(\alpha) = (\theta_1^-(\alpha), \theta_2^-(\alpha))$ and $\theta^+(\alpha) = (\theta_1^+(\alpha), \theta_2^+(\alpha))$ explicitly as announced in (4.50). We check directly that $\frac{d^2\theta_2}{d\theta_1} \neq 0$ at these points, so they are non-degenerate critical points. It is also easy to see from (4.50) that $\theta_1^-(\alpha)$ is strictly increasing from branch point θ_1^- to $\theta_1(\theta_2^-)$ and that $\theta_1^+(\alpha)$ is strictly decreasing from branch point θ_1^+ to $\theta_1(\theta_2^+)$ when α runs the segment $[0, \pi/2]$. In the same way $\theta_2^-(\alpha)$ is strictly decreasing from $\theta_2(\theta_1^-)$ to θ_2^- and $\theta_2^+(\alpha)$ is strictly increasing from $\theta_2(\theta_1^+)$ to θ_2^+ when α runs the segment $[0, \pi/2]$. This proves assertions (i)–(iii).

Finally, since there are no critical points on \mathcal{E} except for $\theta^+(\alpha)$ and $\theta^-(\alpha)$, function $\langle \theta(s) | e_\alpha \rangle$ is monotonous on the arcs $[\theta^-(\alpha), \theta^+(\alpha)]$ and $[\theta^+(\alpha), \theta^-(\alpha)]$. In view of the inequality $\langle \theta^+(\alpha) | e_\alpha \rangle > \langle \theta^-(\alpha) | e_\alpha \rangle$, assertion (iv) follows. \square

Notation of the saddle-point. From now on we are interested in point $\theta^+(\alpha)$ that we denote by $\theta(\alpha)$ for shortness.

The steepest-descent contour γ_α . The level curves $\{s : \Re\langle \theta(s) | e_\alpha \rangle = \langle \theta(\alpha) | e_\alpha \rangle\}$ are orthogonal at $\theta(\alpha)$ and subdivide its neighborhood into four sections. The curves of steepest descent $\{s : \Im\langle \theta(s) | e_\alpha \rangle = 0\}$ on \mathbf{S} are orthogonal at $\theta(\alpha)$ as well, see Lemma 1.3, Chapter IV in [40]. One of them coincides with \mathcal{E} . We denote the other one by γ_α . The real part $\Re\langle \theta(s) | \alpha \rangle$ is strictly increasing on γ_α as s goes far away from $\theta(\alpha)$, see [41, Section 4.2]. The level curves of functions $\Re\langle \theta(s) | e_\alpha \rangle$ and $\Im\langle \theta(s) | e_\alpha \rangle$ are pictured in Figure 4.15.

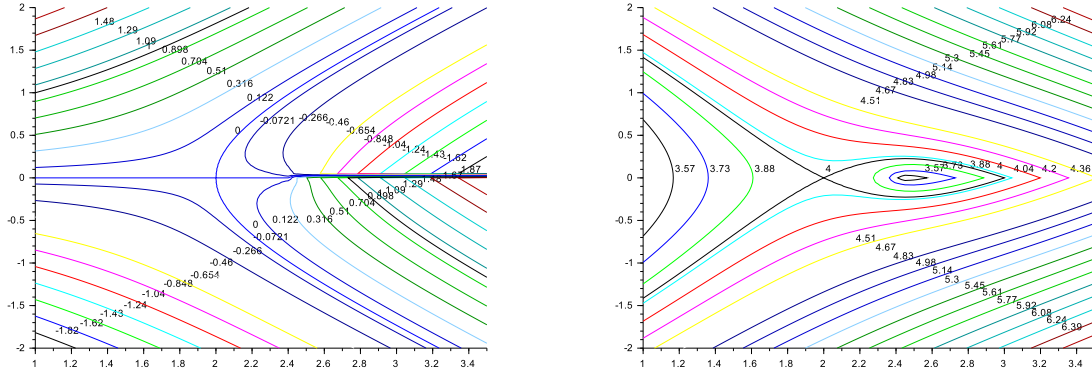


Figure 4.15: Level sets of $\Im\langle \theta(s) | e_\alpha \rangle$ and $\Re\langle \theta(s) | e_\alpha \rangle$

Let $z_{\alpha,+} = (\theta_1(z_{\alpha,+}), \theta_2(z_{\alpha,+}))$ and $z_{\alpha,-} = (\theta_1(z_{\alpha,-}), \theta_2(z_{\alpha,-}))$ be the end points of γ_α where $\Im\theta_1(z_{\alpha,-}) > 0$ and $\Im\theta_1(z_{\alpha,-}) < 0$. We can fix end points $z_{\alpha,-}$ and $z_{\alpha,+}$ in such a way that $\forall \alpha \in \mathcal{O}(\alpha_0)$ and some small $\epsilon > 0$

$$\Re\langle z_{\alpha,\pm} | e_\alpha \rangle = \langle \theta(\alpha) | e_\alpha \rangle + \epsilon.$$

For technical reasons we choose ϵ small enough such that $\Re\theta_1(z_{\alpha,\pm}) \in]\theta_1^-, \theta_1^+[$ and $\Re\theta_2(z_{\alpha,\pm}) \in]\theta_2^-, \theta_2^+[$.

4.4.3 Shifting the integration contours

Our aim now is to shift the integration contours $\mathcal{I}_{\theta_1}^+$ and $\mathcal{I}_{\theta_2}^+$ in (4.49) up to new contours $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$ respectively which coincide with γ_α in a neighborhood of $\theta(\alpha)$ on \mathbf{S} and are “higher” than $\theta(\alpha)$ in the sense of level curves of the function $\Re\langle\theta(s) | e_\alpha\rangle$, that is $\Re\langle\theta(s) | e_\alpha\rangle > \Re\langle\theta(\alpha) | e_\alpha\rangle + \epsilon$ for any $s \in \Gamma_{\theta_i, \alpha} \setminus \gamma_\alpha$ with $i = 1, 2$. When shifting the contours we should of course take into account the poles of the integrands and the residues at them.

Let us construct $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$. We set

$$\Gamma_{\theta_1, \alpha}^{1,+} = \{s : \Re\theta_1(s) = \Re\theta_1(z_{\alpha,+}), \Im\theta_1(z_{\alpha,+}) \leq \Im\theta_1(s) \leq V(\alpha)\}$$

where $V(\alpha) > 0$ will be defined later. Then the end points of $\Gamma_{\theta_1, \alpha}^{1,+}$ are $z_{\alpha,+}$ and $Z_{\alpha,+}$ where $\Re\theta_1(z_{\alpha,+}) = \Re\theta_1(Z_{\alpha,+})$, $\Im\theta_1(Z_{\alpha,+}) = V(\alpha)$. Next

$$\Gamma_{\theta_1, \alpha}^{2,+} = \{s : \Im\theta_1(s) = V(\alpha), 0 \leq \Re\theta_1(s) \leq \Re\theta_1(z_{\alpha,+})\}$$

if $\Re\theta_1(z_{\alpha,+}) > 0$ and

$$\Gamma_{\theta_1, \alpha}^{2,+} = \{s : \Im\theta_1(s) = V(\alpha), 0 \geq \Re\theta_1(s) \geq \Re\theta_1(z_{\alpha,+})\}$$

if $\Re\theta_1(z_{\alpha,+}) < 0$. This contour goes from $Z_{\alpha,+}$ up to $Z_{\alpha,+}^0$ on \mathcal{I}_{θ_1} with $\Re(\theta_1(s)) = 0$, $\Im(\theta_1(s)) = V(\alpha)$. Finally $\Gamma_{\theta_1, \alpha}^{3,+}$ coincides with $\mathcal{I}_{\theta_1}^+$ from $Z_{\alpha,+}^0$ up to infinity :

$$\Gamma_{\theta_1, \alpha}^{3,+} = \{s : \Re\theta_1(s) = 0, \Im\theta_1(s) \geq V(\alpha)\}.$$

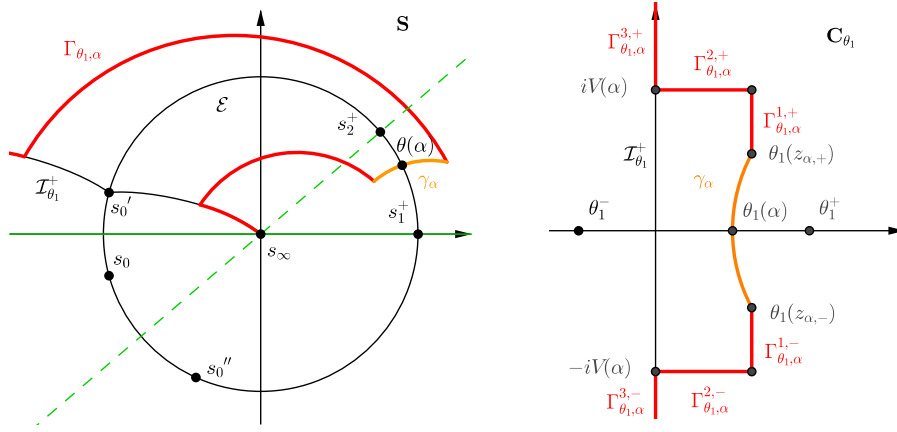
We define in the same way $\Gamma_{\theta_1, \alpha}^{1,-} = \{s : \Re\theta_1(s) = \Re\theta_1(z_{\alpha,-}), -V(\alpha) \leq \Im s \leq \Im\theta_1(z_{\alpha,-})\}$. The end points of $\Gamma_{\theta_1, \alpha}^{1,-}$ are $z_{\alpha,-}$ and $Z_{\alpha,-}$ where $\Re\theta_1(z_{\alpha,-}) = \Re\theta_1(Z_{\alpha,-})$, $\Im\theta_1(Z_{\alpha,-}) = -V(\alpha)$. Next $\Gamma_{\theta_2, \alpha}^{2,-} = \{s : \Im\theta_1(s) = -V(\alpha), 0 \leq \Re\theta_1(s) \leq \Re\theta_1(z_{\alpha,-})\}$ or $\Gamma_{\theta_2, \alpha}^{2,-} = \{s : \Im\theta_1(s) = -V(\alpha), 0 \geq \Re\theta_1(s) \geq \Re\theta_1(z_{\alpha,-})\}$ according to the sign of $\Re\theta_1(z_{\alpha,-})$. It goes from $Z_{\alpha,-}$ to $Z_{\alpha,-}^0$ on \mathbf{S}_{θ_1} with $\Re(\theta_1(Z_{\alpha,-}^0)) = 0$, $\Im(\theta_1(Z_{\alpha,-}^0)) = -V(\alpha)$. Finally $\Gamma_{\theta_1, \alpha}^{3,+}$ coincides with $\mathcal{I}_{\theta_1}^+$ from $Z_{\alpha,-}^0$ up to infinity. Then contour $\Gamma_{\theta_1, \alpha} = \Gamma_{\theta_1, \alpha}^{3,-} \cup \Gamma_{\theta_1, \alpha}^{2,-} \cup \Gamma_{\theta_1, \alpha}^{1,-} \cup \gamma_\alpha \cup \Gamma_{\theta_1, \alpha}^{1,+} \cup \Gamma_{\theta_1, \alpha}^{2,+} \cup \Gamma_{\theta_1, \alpha}^{3,+} \subset \mathbf{S}_{\theta_1}^1$. One can visualize this contour on Figure 4.16 : in the left picture it is drawn on parametrized \mathbf{S} , in the right picture it is projected on the complex plane \mathbf{C}_{θ_1} .

The contour $\Gamma_{\theta_2, \alpha}$ is constructed analogously with respect to θ_2 -coordinate, $\Gamma_{\theta_2, \alpha} = \Gamma_{\theta_2, \alpha}^{3,-} \cup \Gamma_{\theta_2, \alpha}^{2,-} \cup \Gamma_{\theta_2, \alpha}^{1,-} \cup \gamma_\alpha \cup \Gamma_{\theta_2, \alpha}^{1,+} \cup \Gamma_{\theta_2, \alpha}^{2,+} \cup \Gamma_{\theta_2, \alpha}^{3,+} \subset \mathbf{S}_{\theta_2}^1$. The curve of steepest descent γ_α is common for $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$.

Let us recall that poles of $\varphi_1(s)$ and $\varphi_2(s)$ on \mathbf{S} may occur only at \mathcal{E} . Let us also recall the convention that an arc $\}a, b\{$ on \mathcal{E} is the one with ends a and b which does not include $s_0 = (0, 0)$.

Notation of the sets of poles \mathcal{P}'_α and \mathcal{P}''_α . Let \mathcal{P}'_α be the set of poles of the first order of the function $\varphi_2(\theta_1(s))$ on the arc $\}\theta(\alpha), s'_0\{$. Let \mathcal{P}''_α be the set of poles of the first order of the function $\varphi_1(\theta_2(s))$ on the arc $\}\theta(\alpha), s''_0\{$.

Then the following lemma holds true.


 Figure 4.16: Contour $\Gamma_{\theta_1, \alpha}$ on parametrized \mathbf{S} and projected on \mathbf{C}_{θ_1} .

Lemma 116. *Let $\alpha_0 \in]0, \pi/2[$ be such that $\theta(\alpha_0)$ is not a pole of $\varphi_1(\theta_2(s))$ neither of $\varphi_2(\theta_1(s))$. If $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ is not empty, then for any $\alpha \in \mathcal{O}(\alpha_0)$*

$$\begin{aligned} \pi(re_\alpha) &= \sum_{p \in \mathcal{P}'_\alpha} \text{res}_p \varphi_2(\theta_1(s)) \frac{\gamma_2(p)}{\sqrt{d(\theta_1(p))}} e^{-r(\theta(p)|e_\alpha)} + \sum_{p \in \mathcal{P}''_\alpha} \text{res}_p \varphi_1(\theta_2(s)) \frac{\gamma_1(p)}{\sqrt{\tilde{d}(\theta_2(p))}} e^{-r(\theta(p)|e_\alpha)} \\ &+ \frac{1}{2\pi\sqrt{\det \Sigma}} \left(\int_{\Gamma_{\theta_1, \alpha}} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds + \int_{\Gamma_{\theta_2, \alpha}} \frac{\varphi_1(s)\gamma_1(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds \right). \end{aligned} \quad (4.51)$$

If $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ is empty, representation (4.51) stays valid where the corresponding sums over $p \in \mathcal{P}'_\alpha$ and $p \in \mathcal{P}''_\alpha$ are omitted.

Proof. It follows from the assumption of the lemma that $\theta(\alpha)$ is not a pole of $\varphi_1(\theta_2(s))$ neither of $\varphi_2(\theta_1(s))$ for any α in a small enough neighborhood $\mathcal{O}(\alpha_0)$. Then we use the representation of the density (4.49) and apply Cauchy theorem shifting the contours to $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$. \square

In order to find the asymptotic expansion of the density $\pi(r \cos(\alpha), r \sin(\alpha))$, we have to evaluate now the contribution of the residues at poles in (4.51) and the one of integrals along shifted contours $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$. This is the subject of the next two sections.

4.4.4 Asymptotics of integrals along shifted contours $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$

To finish the construction of $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$, it remains to specify $V(\alpha)$. For that purpose we consider closer the function

$$f_\alpha(s) = \langle \theta(s) | e_\alpha \rangle = \theta_1(s) \cos \alpha + \theta_2(s) \sin \alpha.$$

Let us define the projection of this function on \mathbf{C}_{θ_1} :

$$f_\alpha(\theta_1) = \theta_1 \cos \alpha + \Theta_2^+(\theta_1) \sin \alpha, \quad \theta_1 \in \mathbf{C}_{\theta_1}.$$

Clearly $f_\alpha(s) = f_\alpha(\theta_1(s)) = \langle \theta(s) | e_\alpha \rangle$ on $\mathbf{S}_{\theta_1}^1$.

Lemma 117. (i) For any fixed $u \in [\theta_1^-, \theta_1^+]$ the function $v \rightarrow \Re(f_\alpha(u+iv))$ is increasing on $[0, \infty[$ and decreasing on $] -\infty, 0]$.

(ii) There exist constants $d_1 \leq 0$, $d_2 > 0$ and $V > 0$ such that:

$$\inf_{u \in [\theta_1^-, \theta_1^+]} \Re(f_\alpha(u+iv)) \geq d_1 + d_2 \sin(\alpha)|v| \quad \forall v \geq V \text{ and } \forall v \leq -V, \quad \forall \alpha \in]0, \pi/2[. \quad (4.52)$$

Proof. We compute :

$$\Re(f_\alpha(u+iv)) = \cos(\alpha)u + \frac{\sin(\alpha)}{\sigma_{22}}(-\sigma_{12}u - \mu_2 + \Re\sqrt{d(u+iv)})$$

with the discriminant $d(u+iv) = (\det \Sigma)(u+iv - \theta_1^-)(\theta_1^+ - u - iv)$. Then

$$\Re\sqrt{d(u+iv)} = \sqrt{\det \Sigma} \sqrt{|(u+iv - \theta_1^-)(\theta_1^+ - u - iv)|} \cos\left(\frac{\omega_-(u+iv) + \omega_+(u+iv)}{2}\right)$$

where $\omega_-(u+iv)$ et $\omega_+(u+iv)$ are defined as $\omega_-(u+iv) = \arg(\theta_1^+ - u - iv)$ and $\omega_+(u+iv) = \arg(u+iv - \theta_1^-)$, see Figure 4.17. We have

$$\begin{aligned} \cos\left(\frac{\omega_-(u+iv) + \omega_+(u+iv)}{2}\right) &= \sqrt{\frac{1}{2} \cos(\omega_-(u+iv) + \omega_+(u+iv)) + \frac{1}{2}} \\ &= \sqrt{\frac{1}{2} \cos(\omega_-(u+iv)) \cos(\omega_+(u+iv)) - \frac{1}{2} \sin(\omega_-(u+iv)) \sin(\omega_+(u+iv)) + \frac{1}{2}} \\ &= \sqrt{\frac{(u - \theta_1^-)(\theta_1^+ - u) - v(-v)}{2|(u+iv - \theta_1^-)(\theta_1^+ - u - iv)|} + \frac{1}{2}}. \end{aligned}$$

Thus

$$\begin{aligned} \Re(f_\alpha(u+iv)) &= \cos(\alpha)u \\ &+ \frac{\sin(\alpha)}{\sigma_{22}} \left(-\sigma_{12}u - \mu_2 + \sqrt{\frac{1}{2} \sqrt{(u - \theta_1^-)(\theta_1^+ - u) + v^2 + |(u+iv - \theta_1^-)(\theta_1^+ - u - iv)|}} \right) \end{aligned}$$

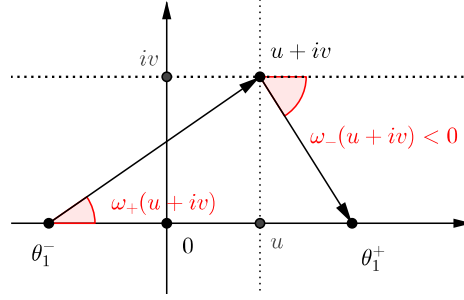
Both statements of the lemma follow directly from this representation. \square

We may now choose $V(\alpha)$ and such that

$$V(\alpha) = \max \left(V, \frac{\langle \theta(\alpha) | e_\alpha \rangle + \epsilon - d_1}{d_2 \sin(\alpha)} \right) \quad (4.53)$$

in accordance with notations of Lemma 117. This concludes the construction of $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$.

The asymptotic expansion of integrals along these contours is given in the following lemma. The main contribution comes from the integrals along γ_α , while all other parts of integrals are proved to be exponentially negligible by construction.


 Figure 4.17: $\omega_-(u + iv)$ et $\omega_+(u + iv)$

Lemma 118. *Let $\alpha_0 \in]0, \pi/2[$ and $\mathcal{O}(\alpha_0)$ a small enough neighborhood of α_0 . Then when $r \rightarrow \infty$ uniformly for $\alpha \in \mathcal{O}(\alpha_0)$ we have*

$$\frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\Gamma_{\theta_1, \alpha}} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds \sim \sum_{l=0}^k \frac{c_{\theta_1}^l(\alpha)}{r^l \sqrt{r}} e^{-r(\theta(\alpha)|e_\alpha)}, \quad (4.54)$$

$$\frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\Gamma_{\theta_2, \alpha}} \frac{\varphi_1(s)\gamma_1(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds \sim \sum_{l=0}^k \frac{c_{\theta_2}^l(\alpha)}{r^l \sqrt{r}} e^{-r(\theta(\alpha)|e_\alpha)}. \quad (4.55)$$

The constants $c_{\theta_1}^l(\alpha)$, $c_{\theta_2}^l(\alpha)$, $l = 0, 1, 2, \dots$ depend continuously of α and can be made explicit in terms of functions φ_1 and φ_2 and their derivatives at $\theta(\alpha)$. Namely

$$c_{\theta_1}^0(\alpha) = \frac{1}{\sqrt{2\pi \det \Sigma}} \frac{\varphi_2(s(\alpha))\gamma_2(\theta(\alpha))}{s(\alpha)\sqrt{f_\alpha''(s(\alpha))}},$$

$$c_{\theta_2}^0(\alpha) = \frac{1}{\sqrt{2\pi \det \Sigma}} \frac{\varphi_1(s(\alpha))\gamma_1(\theta(\alpha))}{s(\alpha)\sqrt{f_\alpha''(s(\alpha))}}.$$

Proof. By Lemma 117 (i) and by (4.48) for any $r > 0$.

$$\left| \int_{\Gamma_{\theta_1, \alpha}^{\pm}} \frac{\varphi_2(\theta_1(s))\gamma_2(s)}{s\sqrt{\det \Sigma}} \exp^{-r(\theta(s)|e_\alpha)} ds \right| \leq 2V(\alpha) \sup_{s \in \Gamma_{\theta_1, \alpha}^{\pm}} \left| \frac{\varphi_2(\theta_1(s))\gamma_2(s)}{\sqrt{d(\theta_1(s))}} \right| e^{-r(\theta(\alpha)|e_\alpha) - r\epsilon}. \quad (4.56)$$

The length of $\Gamma_{\theta_2, \alpha}^{\pm}$ being smaller than $(\theta_1^+ - \theta_1^-)$, by Lemma 117 (ii) and by (4.48) for any $r > 0$

$$\left| \int_{\Gamma_{\theta_1, \alpha}^{\pm}} \frac{\varphi_2(\theta_1(s))\gamma_2(s)}{s\sqrt{\det \Sigma}} \exp^{-r(\theta(s)|e_\alpha)} ds \right| \leq (\theta_1^+ - \theta_1^-) \sup_{s \in \Gamma_{\theta_2, \alpha}^{\pm}} \left| \frac{\varphi_2(\theta_1(s))\gamma_2(s)}{\sqrt{d(\theta_1(s))}} \right| e^{-r(d_1 + d_2 \sin(\alpha))V(\alpha)} \quad (4.57)$$

where due to the choice (4.53) of $V(\alpha)$

$$e^{-r(d_1 + d_2 \sin(\alpha))V(\alpha)} \leq e^{-r(\theta(\alpha)|e_\alpha) - r\epsilon}. \quad (4.58)$$

Finally note that for any $s \in \Gamma_{\theta_1, \alpha}^{3, \pm}$

$$\frac{\gamma_2(s)}{\sqrt{d(\theta_1(s))}} = r_{1,2} \frac{\theta_1(s)}{\sqrt{d(\theta_1(s))}} + r_{22} \frac{-b(\theta_1(s)) + \sqrt{d(\theta_1(s))}}{2a(\theta_1(s))\sqrt{d(\theta_1(s))}}$$

where $\Re\theta_1(s) = 0$, $\Im\theta_1(s) \geq V$. Then there exists a constant $D > 0$ such that $|\gamma_2(s)d^{-1/2}(\theta_1(s))| \leq D$ for any $s \in \Gamma_{\theta_1, \alpha}^{3, \pm}$ and any $\alpha \in]0, \pi/2[$. Moreover $|\varphi_2(\theta_1(s))| \leq \nu_1(\mathbf{R}_+)$ for any $s \in \mathcal{I}_{\theta_1}$. Thus by Lemma 117 (ii) and by (4.48)

$$\begin{aligned} & \left| \int_{\Gamma_{\theta_1, \alpha}^{3, \pm}} \frac{\varphi_2(\theta_1(s))\gamma_2(s)}{s\sqrt{\det \Sigma}} \exp^{-r\langle \theta(s) | e_\alpha \rangle} ds \right| \leq 2D\nu_1(\mathbf{R}_+) \int_{V(\alpha)}^{\infty} e^{-r(d_1+d_2 \sin(\alpha)v)} dv \\ & \leq 2D\nu_1(\mathbf{R}_+) \frac{1}{c \sin(\alpha)V(\alpha)} e^{-r(d_1+d_2 \sin(\alpha)V(\alpha))} \leq 2D\nu_1(\mathbf{R}_+) \frac{1}{c \sin(\alpha)V(\alpha)} e^{-r\langle \theta(\alpha) | e_\alpha \rangle - r\epsilon}. \end{aligned} \quad (4.59)$$

The contours $\Gamma_{\theta_1, \alpha}^{i, \pm}$ for $i = 1, 2$ being far away from poles of φ_2 and zeros of $d(\theta_1(s))$ for all $\alpha \in \mathcal{O}(\alpha_0)$, $\sup_{\alpha \in \mathcal{O}(\alpha_0)} \sup_{s \in \Gamma_{\theta_1, \alpha}^{i, \pm}} \left| \frac{\varphi_2(\theta_1(s))\gamma_2(s)}{\sqrt{d(\theta_1(s))}} \right| < \infty$ for $i = 1, 2$, and of course $\sup_{\alpha \in \mathcal{O}(\alpha_0)} (\sin(\alpha)V(\alpha))^{-1}$ and $\sup_{\alpha \in \mathcal{O}(\alpha_0)} V(\alpha)$ are finite as well. It follows that for some constant $C > 0$, any $r > 0$ and any $\alpha \in \mathcal{O}(\alpha_0)$

$$\left| \int_{\Gamma_{\theta_1, \alpha}^{1, \pm} \cup \Gamma_{\theta_1, \alpha}^{2, \pm} \cup \Gamma_{\theta_1, \alpha}^{3, \pm}} \frac{\varphi_2(\theta_1(s))\gamma_2(s)}{s\sqrt{\det \Sigma}} e^{-r\langle \theta(s) | e_\alpha \rangle} ds \right| \leq C e^{-r\langle \theta(\alpha) | e_\alpha \rangle - r\epsilon}. \quad (4.60)$$

As for the contour γ_α of the steepest descent of the function $\langle \theta(s) | e_\alpha \rangle$, we apply the standard saddle-point method, see e.g. Theorem 1.7, Chapter IV in [40]: for any $k > 0$ when $r \rightarrow \infty$, uniformly $\forall \alpha \in \mathcal{O}(\alpha_0)$,

$$\frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\gamma_\alpha} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r\langle \theta(s) | e_\alpha \rangle} ds \sim \sum_{l=0}^k \frac{c_{\theta_1}^l(\alpha)}{r^l \sqrt{r}} e^{-r\langle \theta(\alpha) | e_\alpha \rangle}, \quad (4.61)$$

where $c_{\theta_1}^0(\alpha)$ is given explicitly in the statement of the lemma and all other constants $c_{\theta_1}^l(\alpha)$ can be written in terms of the same functions and their derivatives at $\theta(\alpha)$. Thus (4.54) is proved and the proof of (4.55) for the integral over $\Gamma_{\theta_2, \alpha}$ is absolutely analogous. \square

4.4.5 Contribution of poles into the asymptotics of $\pi(r \cos(\alpha), r \sin(\alpha))$

Once Lemma 118 established the asymptotics of integrals along shifted contours $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$, let us come back to Lemma 116 and evaluate the contribution to the density of residues at poles over $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$. There are two possibilities:

- (i) $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ is empty, then the asymptotics of the density is determined by the saddle-point via Lemma 118.
- (ii) $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ is not empty. Then due to monotonicity of the function $\langle \theta(s) | e_\alpha \rangle$ on \mathcal{E} , see Lemma 115 (iv), for any $p \in \mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ we have $\langle \theta(p) | e_\alpha \rangle < \langle \theta(\alpha) | e_\alpha \rangle$. Hence all residues at poles $p \in \mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ bring more important contribution to the asymptotic expansion as $r \rightarrow \infty$ than integrals over $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$.

First of all, we would like to distinguish the set of parameters (Σ, μ, R) under which (i) or (ii) hold true. Secondly, under (ii), we would like to find the most important pole from the asymptotic point of view. Let us look closer at the arc $\{s'_0, \theta(\alpha)\}$. Under parameters such that $\theta_1(s_2^+) < 0$ we have $s'_0 \in]s_1^+, s_2^+[$, see Figure 4.18, the left picture. Then for some $\alpha' \in]0, \pi/2[$ $\theta(\alpha') = s'_0$. This arc written in square brackets in the anticlockwise direction is $]s'_0, \theta(\alpha)[$ for any $\alpha \in]\alpha', \pi/2[$ and the function $\langle \theta(s) | e_\alpha \rangle$ is increasing when s runs from s'_0 to $\theta(\alpha)$. For any $\alpha \in [0, \alpha'[$ this arc is written $]\theta(\alpha), s'_0[$ and the function $\langle \theta(s) | e_\alpha \rangle$ is decreasing when s runs from $\theta(\alpha)$ so s'_0 . Under parameters such that $\theta_1(s_2^+) \geq 0$, we have $s'_0 \notin]s_1^+, s_2^+[$, see Figure 4.18 the right picture, from where this arc is written $]\theta(\alpha), s'_0[$ for any $\alpha \in]0, \pi/2[$. The function $\langle \theta(s) | e_\alpha \rangle$ is decreasing when s runs from $\theta(\alpha)$ to s'_0 .

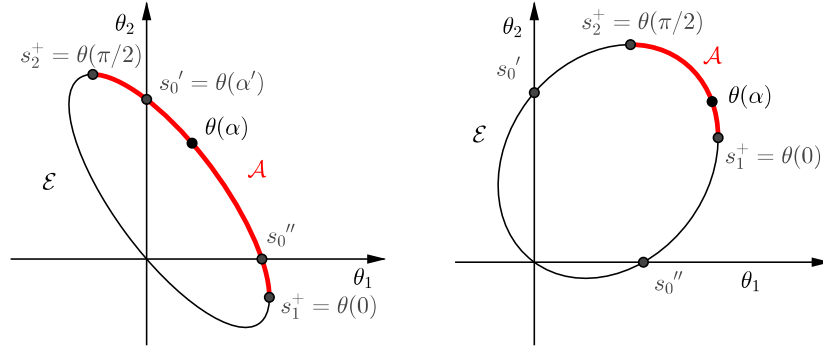


Figure 4.18: The arc $\mathcal{A} = [s_1^+, s_2^+]$ if $\theta_1(s_2^+) < 0$, $\theta_2(s_1^+) < 0$ on the left picture, if $\theta_1(s_2^+) > 0$, $\theta_2(s_1^+) > 0$ on the right picture

The important conclusion is that in all cases, the pole p of φ_2 on the arc $\{s'_0, \theta(\alpha)\}$ with the smallest $\langle \theta(p) | e_\alpha \rangle$ is the closest to s'_0 . In the same way we can consider the arc $\{s''_0, \theta(\alpha)\}$ and find out, due to monotonicity of the function $\langle \theta(s) | e_\alpha \rangle$, that the pole of φ_1 with the smallest $\langle \theta(p) | e_\alpha \rangle$ is the closest to s''_0 . We know from Lemmas 110–113 the way that these poles are related to zeros of γ_1 and γ_2 . Now we summarize this information in the following theorem.

Theorem 119. (a) Let $\zeta\theta^{**} \notin \{\theta(\alpha), s'_0\}$, $\eta\theta^* \notin \{\theta(\alpha), s''_0\}$. Then \mathcal{P}'_α and \mathcal{P}''_α are both empty, $\theta(\alpha)$ is not a pole of φ_1 and neither of φ_2 .

(b) Let $\zeta\theta^{**} \in]\theta(\alpha), s'_0\}$ and $\eta\theta^* \notin \{\theta(\alpha), s''_0\}$. Then

$$\min_{p \in \mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle = \langle \zeta\theta^{**} | e_\alpha \rangle \quad (4.62)$$

and this minimum over $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ is achieved at the unique element $p = \zeta\theta^{**}$ which is a pole of the first order of φ_2 .

(c) Let $\zeta\theta^{**} \notin \{\theta(\alpha), s'_0\}$ and $\eta\theta^* \in \{s''_0, \theta(\alpha)\}$. Then

$$\min_{p \in \mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle = \langle \eta\theta^* | e_\alpha \rangle \quad (4.63)$$

and this minimum over $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ is achieved at the unique element $p = \eta\theta^*$ which is a pole of the first order of φ_1 .

(d) Let $\zeta\theta^{**} \in]\theta(\alpha), s'_0\}$ and $\eta\theta^* \in \{s''_0, \theta(\alpha)\}$.

If $\langle \zeta\theta^{**} | e_\alpha \rangle < \langle \eta\theta^* | e_\alpha \rangle$, then (4.62) is valid. If $\langle \zeta\theta^* | e_\alpha \rangle > \langle \eta\theta^{**} | e_\alpha \rangle$, then (4.63) is valid. In both cases the minimum over $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ is achieved at the unique element which is the pole of the first order $p = \zeta\theta^{**}$ of φ_2 or the pole of the first order $p = \eta\theta^*$ of φ_1 respectively.

If $\langle \zeta\theta^{**} | e_\alpha \rangle = \langle \eta\theta^* | e_\alpha \rangle$, then

$$\min_{p \in \mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle = \langle \zeta\theta^{**} | e_\alpha \rangle = \langle \eta\theta^* | e_\alpha \rangle. \quad (4.64)$$

This minimum over $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$ is achieved at exactly two elements $p = \zeta\theta^{**}$ and $p = \eta\theta^*$ which are poles of the first order of φ_1 and φ_2 respectively.

Proof. (a) Let $\theta_1(s_2^+) < 0$ and let $\alpha > \alpha'$ defined above. Then $\theta_1(\alpha) < 0$ and all points of the arc $\{\theta(\alpha), s'_0\}$ have the first coordinate negative, so that function $\varphi_2(\theta_1(s))$ is initially well defined at them and holomorphic. Let now $\theta_1(s_2^+) < 0$ and $\alpha \in]0, \alpha'[$ or $\theta_1(s_2^+) \geq 0$. Then $\theta_1(\alpha) > 0$ and the arc $\{\theta(\alpha), s'_0\}$ written in the anticlockwise direction is $[\theta(\alpha), s'_0]$. Assume that $\varphi_2(\theta_1(s))$ has poles on $[\theta(\alpha), s'_0[$ and θ^p is the closest to s'_0 . Then by Lemma 111 either $\gamma_2(\zeta\theta^p) = 0$ or parameters are such that $\theta_2(s_1^+) > 0$, $\eta\zeta\theta^p \in]\eta s_1^+, s_0[$ and $\gamma_1(\eta\zeta\theta^p) = 0$. In the first case $\zeta\theta^p = \theta^{**}$ is a zero of γ_2 different from s_0 . This implies $\theta^p = \zeta\theta^{**} \in [\theta(\alpha), s'_0[$ which is impossible by assumptions. In the second case $\eta\zeta\theta^p = \theta^*$ is a zero of γ_1 different from s_0 . This implies $\zeta\theta^p = \eta\theta^* \in \eta] \eta s_1^+, s_0[=]s''_0, s_1^+[\subset]s''_0, \theta(\alpha)[=]\theta(\alpha), s''_0\}$ that contradicts the assumptions as well. Hence $\varphi_2(\theta_1(s))$ has no poles on the open arc $]\theta(\alpha), s'_0\}$ and neither at $\theta(\alpha)$, \mathcal{P}'_α is empty, The reasoning for \mathcal{P}''_α is the same.

(b) By stability conditions (4.1) and (4.2) $\theta_1^{**} > 0$, then $\zeta\theta_1^{**} > 0$. Thus $\theta_1(\alpha) > 0$, in the case $\theta_1(s_2^+) < 0$ the angle α must be smaller than α' and the arc $]\theta(\alpha), s'_0\}$ should be written $]\theta(\alpha), s'_0[$. By Lemma 110 there exist poles of function $\varphi_2(\theta_1(s))$ on this arc and $\zeta\theta^{**}$ is one among them. By Lemma 111 $\zeta\theta^{**}$ can not be the closest pole to s'_0 only if the parameters are such that $\theta_2(s_1^+) > 0$ and for some $\theta^p \in]\theta(\alpha), s'_0[$ such that $\eta\zeta\theta^p \in]\eta s_1^+, s_0[$ $\gamma_1(\eta\zeta\theta^p) = 0$. But then $\eta\zeta\theta^p = \theta^*$ is a zero of γ_1 different from s_0 . It follows $\zeta\theta^p = \eta\theta^* \in \eta] \eta s_1^+, s_0[=]s''_0, s_1^+[\subset]s''_0, \theta(\alpha)[=]\theta(\alpha), s''_0\}$ that is impossible by assumptions. Hence by Lemma 111 $\zeta\theta^{**}$ is the closest pole to s'_0 of $\varphi_2(\theta_1(s))$ and it is of the first order. The function $\langle \theta(s) | e_\alpha \rangle$ being decreasing on $]\theta(\alpha), s'_0[$ when s runs the arc in the anticlockwise direction, thus

$$\min_{p \in \mathcal{P}'_\alpha} \langle \theta(p) | e_\alpha \rangle = \langle \zeta\theta^{**} | e_\alpha \rangle, \quad (4.65)$$

and the minimum is achieved on the unique element $\zeta\theta^{**}$.

If \mathcal{P}''_α is empty then the statement (b) is proved.

Assume that \mathcal{P}''_α is not empty. Then there exist poles of $\varphi_1(\theta_2(s))$ on the arc $]\theta(\alpha), s''_0\}$. Since function $\varphi_1(\theta_2(s))$ is initially well defined and holomorphic at all points with the second coordinate negative, then $\theta_2(\alpha) > 0$ and the arc is $]s''_0, \theta(\alpha)[$ when written in the anticlockwise direction. Let θ^p be a pole of $\varphi_1(\theta_2(s))$ which is the closest to s''_0 . Then by Lemma 111 either $\gamma_1(\eta\theta^p) = 0$ or parameters are such that $\theta_1(s_2^+) > 0$, $\zeta\eta\theta^p \in]s_0, \zeta s_2^+[$ and $\gamma_2(\zeta\eta\theta^p) = 0$. In the first case $\eta\theta^p = \theta^*$ is a zero of γ_1 different from s_0 . This implies $\theta^p = \eta\theta^* \in]s''_0, \theta(\alpha)[$ which is impossible by assumptions. In the second case $\zeta\eta\theta^p = \theta^{**}$ where $\eta\theta^p = \zeta\theta^{**} \in]s_0, \zeta s_2^+[=]s_2^+, s'_0[\subset]\theta(\alpha), s'_0[$. Thus $\theta^p = \eta\zeta\theta^{**}$ is the closest pole to s''_0 . Hence, the closest pole of the first order coincides with it or is further away from s''_0 .

Since the function $\langle \theta(s) | e_\alpha \rangle$ is increasing on $]s_0'', \theta(\alpha)[$ when s is running from s_0'' to $\theta(\alpha)$, we derive

$$\min_{p \in \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle \geq \langle \eta \zeta \theta^{**} | e_\alpha \rangle.$$

But by Lemma 112

$$\theta_1(\eta \zeta \theta^{**}) > \theta_1(\zeta \theta^{**}), \quad \theta_2(\eta \zeta \theta^{**}) = \theta_2(\zeta \theta^{**})$$

from where

$$\langle \eta \zeta \theta^{**} | e_\alpha \rangle > \langle \zeta \theta^{**} | e_\alpha \rangle.$$

Thus, whenever \mathcal{P}''_α is non empty,

$$\min_{p \in \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle > \langle \zeta \theta^{**} | e_\alpha \rangle.$$

This inequality combined with (4.65) finishes the proof of (b).

The proof of (c) is symmetric.

(d) Since $\theta_2^* = \eta \theta_2^* > 0$ and $\theta_1^{**} = \zeta \theta_1^{**} > 0$ by stability conditions (4.1) and (4.2), then $\theta(\alpha)$ has both coordinates positive. The corresponding arcs written in the anticlockwise direction are $]\theta(\alpha), s_0'[\subset]s_1^+, s_0'[\$ and $]s_0'', \theta(\alpha)[\subset]s_0'', s_2^+[\$. By Lemma 111 $\zeta \theta^{**}$ is a pole of $\varphi_2(\theta_1(s))$ on the first of these arcs while $\eta \theta^*$ is a pole of $\varphi_1(\theta_2(s))$ on the second one. Then one of the statements of Lemma 113 (i), (ii) or (iii) holds true.

Under the statement (i), taking into account the monotonicity of the function $\langle \theta(s) | e_\alpha \rangle$ on the arcs, we derive immediately that $\min_{p \in \mathcal{P}'_\alpha} \langle \theta(p) | e_\alpha \rangle = \langle \zeta \theta^{**} | e_\alpha \rangle$, and this minimum is achieved on the unique element $p = \zeta \theta^{**}$. We derive also that $\min_{p \in \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle = \langle \eta \theta^* | e_\alpha \rangle$ and this minimum is achieved on the unique element $p = \eta \theta^*$. Thus, under the statement (i) of Lemma 113, the theorem is immediate.

Assume now (ii) of Lemma 113. Again by monotonicity of $\langle \theta(s) | e_\alpha \rangle$ we deduce $\min_{p \in \mathcal{P}'_\alpha} \langle \theta(p) | e_\alpha \rangle = \langle \zeta \theta^{**} | e_\alpha \rangle$ where the minimum is achieved at the unique element $\zeta \theta^{**}$. Under (ii) all poles of $\varphi_1(\theta_2(s))$ on $]s_0'', \theta(\alpha)[$ are not closer to s_0'' than $\eta \zeta \theta^{**}$, so that either \mathcal{P}''_α is empty or

$$\min_{p \in \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle \geq \langle \eta \zeta \theta^{**} | e_\alpha \rangle.$$

By Lemma 112 $\theta_1(\eta \zeta \theta^{**}) > \theta_1(\zeta \theta^{**})$, $\theta_2(\eta \zeta \theta^{**}) = \theta_2(\zeta \theta^{**})$ from where $\langle \eta \zeta \theta^{**} | e_\alpha \rangle > \langle \zeta \theta^{**} | e_\alpha \rangle$. Hence

$$\min_{p \in \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle > \langle \zeta \theta^{**} | e_\alpha \rangle,$$

and finally

$$\min_{p \in \mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle = \langle \zeta \theta^{**} | e_\alpha \rangle \quad (4.66)$$

where the minimum is achieved on the unique element $\zeta \theta^{**}$. From the other hand, the pole $\eta \theta^* \in]s_0'', \theta(\alpha)[$ of $\varphi_1(\theta_2(s))$ in this case is not closer to s_0'' than $\eta \zeta \theta^{**}$. Then the inequality

$$\langle \eta \theta^* | e_\alpha \rangle \geq \langle \eta \zeta \theta^{**} | e_\alpha \rangle > \langle \zeta \theta^{**} | e_\alpha \rangle \quad (4.67)$$

is valid.

Under the statement (iii) of Lemma 113, by symmetric arguments, $\min_{p \in \mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha} \langle \theta(p) | e_\alpha \rangle = \langle \eta \theta^* | e_\alpha \rangle$ where the minimum is achieved on the unique element $\eta \theta^*$, while $\langle \eta \theta^* | e_\alpha \rangle < \langle \zeta \theta^{**} | e_\alpha \rangle$. This concludes the proof of the lemma. \square

4.5 Asymptotic expansion of the density $\pi(r \cos(\alpha), r \sin(\alpha))$, $r \rightarrow \infty$, $\alpha \in \mathcal{O}(\alpha_0)$

4.5.1 Given angle α_0 , asymptotic expansion of the density as a function of parameters (Σ, μ, R)

We are now ready to formulate and prove the results. In this section we fix an angle $\alpha_0 \in]0, \pi/2[$ and give the asymptotic expansion of the density of stationary distribution depending on parameters (Σ, μ, R) , and more precisely on the position of zeros of γ_1 and γ_2 on ellipse \mathcal{E} .

In the first theorem parameters (Σ, μ, R) are such that the asymptotic expansion is determined by the saddle-point.

Theorem 120. *Let $\alpha_0 \in]0, \pi/2[$, $\mathcal{O}(\alpha_0)$ is a small enough neighborhood of α_0 . Assume that $\zeta^{\theta^{**}} \notin \{\theta(\alpha_0), s'_0\}$, $\eta^{\theta^*} \notin \{\theta(\alpha_0), s''_0\}$. Then there exist constants $c^l(\alpha)$, $l = 0, 1, 2, \dots$, such that for any $k > 0$:*

$$\pi(r \cos(\alpha), r \sin(\alpha)) \sim \sum_{l=0}^k \frac{c^l(\alpha)}{r^l \sqrt{r}} e^{-r(\theta(\alpha)|e_\alpha)}, \quad \text{as } r \rightarrow \infty, \quad \text{uniformly for } \alpha \in \mathcal{O}(\alpha_0). \quad (4.68)$$

Constants $c^l(\alpha)$ $l = 0, 1, 2, \dots$ depend continuously on α and can be expressed in terms of functions φ_1 and φ_2 and their derivatives at $\theta(\alpha)$. Namely

$$c^0(\alpha) = c_{\theta_1}^0(\alpha) + c_{\theta_2}^0(\alpha) \quad (4.69)$$

where $c_{\theta_1}^0(\alpha)$ and $c_{\theta_2}^0(\alpha)$ are defined in Lemma 118.

Proof. By Lemma 115 (iii) $\theta(\alpha)$ depends continuously on α , then $\zeta^{\theta^{**}} \notin \{\theta(\alpha), s'_0\}$, $\eta^{\theta^*} \notin \{\theta(\alpha), s''_0\}$ for all $\alpha \in \mathcal{O}(\alpha_0)$. By Theorem 119 (a) the sets \mathcal{P}'_α and \mathcal{P}''_α are both empty, furthermore, $\theta(\alpha)$ is not a pole of φ_1 and neither of φ_2 . Then by Lemma 116 the density equals the sum of integrals along shifted contours $\Gamma_{\theta_1, \alpha}$ and $\Gamma_{\theta_2, \alpha}$ the asymptotics of which is found in Lemma 118, $c^l(\alpha) = c_{\theta_1}^l(\alpha) + c_{\theta_2}^l(\alpha)$, $l = 0, 1, 2, \dots$ \square

In the second theorem parameters (Σ, μ, R) are such that the most important terms of the asymptotic expansion come from the poles of φ_1 or φ_2 and the smaller ones come from the saddle-point.

Theorem 121. *Let $\alpha_0 \in]0, \pi/2[$, $\mathcal{O}(\alpha_0)$ is a small enough neighborhood of α_0 . Assume that $\zeta^{\theta^{**}} \in \{\theta(\alpha_0), s'_0\}$ or $\eta^{\theta^*} \in \{\theta(\alpha_0), s''_0\}$. Assume also that $\theta(\alpha_0)$ is not a pole of $\varphi_1(\theta_2(s))$ neither of $\varphi_2(\theta_1(s))$. Then for any $k > 0$ when $r \rightarrow \infty$, uniformly for $\alpha \in \mathcal{O}(\alpha_0)$ we have*

$$\begin{aligned} \pi(r \cos(\alpha), r \sin(\alpha)) \sim & \sum_{p \in \mathcal{P}'_{\alpha_0}} \text{res}_p \varphi_2(\theta_1(s)) \frac{\gamma_2(p)}{\sqrt{d(\theta_1(p))}} e^{-r(\theta(p)|e_\alpha)} \\ & + \sum_{p \in \mathcal{P}''_{\alpha_0}} \text{res}_p \varphi_1(\theta_2(s)) \frac{\gamma_1(p)}{\sqrt{\tilde{d}(\theta_2(p))}} e^{-r(\theta(p)|e_\alpha)} \\ & + \sum_{l=0}^k \frac{c^l(\alpha)}{r^l \sqrt{r}} e^{-r(\theta(\alpha)|e_\alpha)}. \end{aligned} \quad (4.70)$$

Constants $c^l(\alpha)$ $l = 0, 1, 2, \dots$ are the same as in Theorem 120. Furthermore

- (i) If $\zeta\theta^{**} \in \theta(\alpha_0), s'_0\}$ and $\eta\theta^* \notin \theta(\alpha_0), s''_0\}$, then the main term in the expansion (4.70) is at $p = \zeta\theta^{**}$.
- (ii) If $\zeta\theta^{**} \notin \theta(\alpha_0), s'_0\}$ and $\eta\theta^* \in \theta(\alpha_0), s''_0\}$, then the main term in (4.70) is at $p = \eta\theta^*$.
- (iii) Let $\zeta\theta^{**} \in \theta(\alpha_0), s'_0\}$ and $\eta\theta^* \in \theta(\alpha_0), s''_0\}$. If $\langle \zeta\theta^{**} | e_{\alpha_0} \rangle < \langle \eta\theta^* | e_{\alpha_0} \rangle$, then the main term in (4.70) is at $p = \zeta\theta^{**}$.
 If $\langle \zeta\theta^{**} | e_{\alpha_0} \rangle > \langle \eta\theta^* | e_{\alpha_0} \rangle$, then main term in (4.70) is at $p = \eta\theta^*$.
 If $\langle \zeta\theta^{**} | e_{\alpha_0} \rangle = \langle \eta\theta^* | e_{\alpha_0} \rangle$, then two the most important terms in the expansion (4.70) are at $p = \zeta\theta^{**}$ and at $p = \eta\theta^*$.

Proof. Point $\theta(\alpha_0)$ being not a pole of φ_1 neither of φ_2 , one can choose $\mathcal{O}(\alpha_0)$ small enough such that $\theta(\alpha)$ is not a pole of no one of these functions and $\mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha = \mathcal{P}'_{\alpha_0} \cup \mathcal{P}''_{\alpha_0}$ for all $\alpha \in \mathcal{O}(\alpha_0)$. By assumptions $\zeta\theta^{**} \in \theta(\alpha_0), s'_0\}$ or $\eta\theta^* \in \theta(\alpha_0), s''_0\}$, then by Theorem 119 (b), (c) or (d) $\mathcal{P}'_{\alpha_0} \cup \mathcal{P}''_{\alpha_0}$ is not empty. Finally by virtue of Lemma 116 and Lemma 118 the representation (4.70) holds true.

Let us study the main asymptotic term. Statements (i), (ii) and (iii) for $\alpha = \alpha_0$ follow directly from Theorem 119 (b), (c) and (d). They remain valid for any $\alpha \in \mathcal{O}(\alpha_0)$ due to the continuity of the functions $\alpha \rightarrow \langle \theta(p) | e_\alpha \rangle$ for any $p \in \mathcal{P}'_{\alpha_0} \cup \mathcal{P}''_{\alpha_0}$. \square

Remark. Under parameters such that $\zeta\theta^{**} \in \theta(\alpha_0), s'_0\}$, $\eta\theta^* \in \theta(\alpha_0), s''_0\}$ and $\langle \zeta\theta^{**} | e_{\alpha_0} \rangle = \langle \eta\theta^* | e_{\alpha_0} \rangle$ (case (iii)), for any fixed angle $\alpha < \alpha_0$, the main asymptotic term is at $\eta\theta^*$ and the second one is at $\zeta\theta^{**}$; for any fixed angle $\alpha > \alpha_0$ the pole $\zeta\theta^{**}$ provides the main asymptotic term and $\eta\theta^*$ gives the second one. If $r \rightarrow \infty$ and $\alpha \rightarrow \alpha_0$, both of these terms should be taken into account.

In Theorem 121 $\theta(\alpha_0)$ is assumed not to be a pole of φ_1 and neither of φ_2 , that is why Lemma 116 applies. Nevertheless, it may happen (for a very few angles and under some sets of parameters) that $\theta(\alpha_0)$ is a pole of one of these functions. In this case the following theorem holds true.

Theorem 122. Let $\alpha_0 \in]0, \pi/2[$. Assume that $\zeta\theta^{**} \in \theta(\alpha_0), s'_0\}$ or $\eta\theta^* \in \theta(\alpha_0), s''_0\}$. Assume also that $\theta(\alpha_0)$ is a pole of $\varphi_1(\theta_2(s))$ or of $\varphi_2(\theta_1(s))$. Then for any $\delta > 0$ there exists a small enough neighborhood $\mathcal{O}(\alpha_0)$ such that

$$\begin{aligned} \pi(r \cos(\alpha), r \sin(\alpha)) &\sim \sum_{p \in \mathcal{P}'_{\alpha_0}} \text{res}_p \varphi_2(\theta_1(s)) \frac{\gamma_2(p)}{\sqrt{d(\theta_1(p))}} e^{-r \langle \theta(p) | e_\alpha \rangle} \\ &+ \sum_{p \in \mathcal{P}''_{\alpha_0}} \text{res}_p \varphi_1(\theta_2(s)) \frac{\gamma_1(p)}{\sqrt{\tilde{d}(\theta_2(p))}} e^{-r \langle \theta(p) | e_\alpha \rangle} \\ &+ o(e^{-r(\langle \theta(\alpha) | e_\alpha \rangle - \delta)}) \quad r \rightarrow \infty, \text{ uniformly } \forall \alpha \in \mathcal{O}(\alpha_0) \end{aligned} \quad (4.71)$$

Furthermore, the main term in this expansion is the same as in Theorem 121, cases (i), (ii) and (iii).

Proof. For any $\delta > 0$ one can choose $\tau' \in s'_0, \theta(\alpha_0)\}$ and $\tau'' \in s''_0, \theta(\alpha_0)\}$ close enough to $\theta(\alpha_0)$ so that $\mathcal{P}'_{\alpha_0} \subset s'_0, \tau'\}$ and $\mathcal{P}''_{\alpha_0} \subset s''_0, \tau''\}$. Furthermore τ' and τ'' can be chosen close enough to α_0 so that $\langle \theta(\alpha_0) | e_{\alpha_0} \rangle - \langle \tau' | e_{\alpha_0} \rangle < \delta/4$ and $\langle \theta(\alpha_0) | e_{\alpha_0} \rangle - \langle \tau'' | e_{\alpha_0} \rangle < \delta/4$.

Then by continuity of the functions $\alpha \rightarrow \langle \theta(\alpha) | e_\alpha \rangle$, $\alpha \rightarrow \langle \tau' | e_\alpha \rangle$, $\alpha \rightarrow \langle \tau'' | e_\alpha \rangle$ one can fix a small enough neighborhood $\mathcal{O}(\alpha_0)$ such that

$$\langle \theta(\alpha) | e_\alpha \rangle - \langle \tau' | e_\alpha \rangle < \delta/2, \quad \langle \theta(\alpha) | e_\alpha \rangle - \langle \tau'' | e_\alpha \rangle < \delta/2, \quad \forall \alpha \in \mathcal{O}(\alpha_0). \quad (4.72)$$

Next, we shift the integration contours in (4.49) $\mathcal{I}_{\theta_1}^+$ and $\mathcal{I}_{\theta_2}^+$ to the new ones $\Gamma'_{\theta_1, \alpha}$ and $\Gamma''_{\theta_2, \alpha}$ going through τ' and τ'' respectively that we construct as follows: $\Gamma'_{\theta_1, \alpha} = \Gamma_{\theta_1, \alpha}^{\prime 1} \cup \Gamma_{\theta_1, \alpha}^{\prime 2, \pm} \cup \Gamma_{\theta_1, \alpha}^{\prime 3, \pm}$ where $\Gamma_{\theta_1, \alpha}^{\prime 1} = \{s : \Re \theta_1(s) = \Re \theta_1(\tau'), -V(\alpha) \leq \Im \theta_1(s) \leq V(\alpha)\}$, $\Gamma_{\theta_1, \alpha}^{\prime 2, \pm} = \{s : \Im \theta_1(s) = \pm V(\alpha), 0 \leq \Re \theta_1(s) \leq \Re \theta_1(\tau')\}$ if $\Re \theta_1(\tau') > 0$ and $\Gamma_{\theta_1, \alpha}^{\prime 2, \pm} = \{s : \Im \theta_1(s) = \pm V(\alpha), 0 \geq \Re \theta_1(s) \geq \Re \theta_1(\tau')\}$ if $\Re \theta_1(\tau') < 0$, finally $\Gamma_{\theta_1, \alpha}^{\prime 3, +} = \{s : \Re \theta_1(s) = 0, \Im \theta_1(s) \geq V(\alpha)\}$, $\Gamma_{\theta_1, \alpha}^{\prime 3, -} = \{s : \Re \theta_1(s) = 0, \Im \theta_1(s) \leq -V(\alpha)\}$. The construction of $\Gamma''_{\theta_2, \alpha}$ is analogous. The value $V(\alpha)$ is fixed as:

$$V(\alpha) = \max \left(V, \frac{\langle \tau' | e_\alpha \rangle - d_1}{d_2 \sin(\alpha)}, \frac{\langle \tau'' | e_\alpha \rangle - d_1}{d_2 \sin(\alpha)} \right) \quad (4.73)$$

with notations from Lemma 117. Thanks to the representation (4.49) and Cauchy theorem

$$\begin{aligned} \pi(re_\alpha) &= \sum_{p \in \mathcal{P}'_{\alpha_0}} \text{res}_p \varphi_2(\theta_1(s)) \frac{\gamma_2(p)}{\sqrt{d(\theta_1(p))}} e^{-r(\theta(p)|e_\alpha)} + \sum_{p \in \mathcal{P}''_{\alpha_0}} \text{res}_p \varphi_1(\theta_2(s)) \frac{\gamma_1(p)}{\sqrt{\tilde{d}(\theta_2(p))}} e^{-r(\theta(p)|e_\alpha)} \\ &+ \frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\Gamma'_{\theta_1, \alpha}} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds + \frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\Gamma''_{\theta_2, \alpha}} \frac{\varphi_1(s)\gamma_1(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds. \end{aligned} \quad (4.74)$$

Applying Lemma 117 (i) for the estimation of integrals along $\Gamma'_{\theta_1, \alpha}$ and $\Gamma''_{\theta_1, \alpha}$, and the same lemma (ii) for the estimation of those along $\Gamma_{\theta_1, \alpha}^{\prime \pm 2}$, $\Gamma_{\theta_2, \alpha}^{\prime \pm 2}$, $\Gamma_{\theta_1, \alpha}^{\prime \pm 3}$ and $\Gamma_{\theta_2, \alpha}^{\prime \pm 3}$ exactly as in Lemma 118 and in view of (4.73) we can show that with some constant $C > 0$

$$\begin{aligned} \left| \int_{\Gamma'_{\theta_1, \alpha}} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds \right| &\leq C e^{-r\langle \tau' | e_\alpha \rangle}, \\ \left| \int_{\Gamma''_{\theta_1, \alpha}} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds \right| &\leq C e^{-r\langle \tau'' | e_\alpha \rangle} \quad \forall r > 0, \forall \alpha \in \mathcal{O}(\alpha_0). \end{aligned}$$

Hence, by (4.72)

$$\int_{\Gamma'_{\theta_1, \alpha}} \frac{\varphi_2(s)\gamma_2(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds + \int_{\Gamma''_{\theta_2, \alpha}} \frac{\varphi_1(s)\gamma_1(\theta(s))}{s} e^{-r(\theta(s)|e_\alpha)} ds = o(e^{-r(\langle \theta(\alpha) | e_\alpha \rangle - \delta)}) \quad (4.75)$$

as $r \rightarrow \infty$ uniformly $\forall \alpha \in \mathcal{O}(\alpha_0)$. This finishes the proof of the representation (4.71). The analysis of the main term is the same as in Theorem 121. \square

It remains to study the cases of parameters such that

- (O1) $\zeta\theta^{**} = \theta(\alpha_0)$ and $\eta\theta^* \notin \{s''_0, \theta(\alpha_0)\}$
- (O2) $\eta\theta^* = \theta(\alpha_0)$ and $\zeta\theta^{**} \notin \{s''_0, \theta(\alpha_0)\}$.

By Lemma 110 this means that $\theta(\alpha_0)$ is a pole of one of functions φ_1 or φ_2 . Since in both cases $\eta\theta^* \notin \{s''_0, \theta(\alpha)\}$, $\zeta\theta^{**} \notin \{s''_0, \theta(\alpha)\}$, we derive by the same reasoning as in Theorem 119 (a) that $\mathcal{P}'_{\alpha_0} \cup \mathcal{P}''_{\alpha_0}$ is empty. The following theorem is valid.

Theorem 123. *Assume that α_0 is such that the assumptions on parameters (O1) or (O2) are valid. Then for any $\delta > 0$ there exists a small enough neighborhood $\mathcal{O}(\alpha_0)$ such that*

$$\pi(r \cos(\alpha), r \sin(\alpha)) = o(e^{-r((\theta(\alpha)|e^\alpha) - \delta)}) \quad r \rightarrow \infty, \quad \text{uniformly } \forall \alpha \in \mathcal{O}(\alpha_0). \quad (4.76)$$

Proof. We choose τ' and τ'' according to (4.72) and proceed exactly as in the proof of Theorem 122. \square

Remark. In Theorems 122 and 123 $\theta(\alpha_0)$ is a pole of one of the functions φ_1 or φ_2 , hence at least one of the integrals (4.49) can not be shifted to $\Gamma_{\theta_1, \alpha_0}$ or $\Gamma_{\theta_2, \alpha_0}$ going through $\theta(\alpha_0)$. Furthermore, although for any $\alpha \in \mathcal{O}(\alpha_0)$, $\alpha \neq \alpha_0$, this shift is possible, the uniform asymptotic expansion by the saddle-point method as in Lemma 118 does not stay valid, that is why we are not able to specify small asymptotic terms in Theorem 122 neither to obtain a more precise result in Theorem 123. This should be possible if we consider the double asymptotics $r \rightarrow \infty$ and $\alpha \rightarrow \alpha_0$ and apply the (more advanced) saddle-point method in the special case when the saddle-point is approaching a pole of the integrand. We do not do it in the present paper.

Remark. Assumptions of theorems 120 — 123 are expressed in terms of positions on ellipse \mathcal{E} of points $\zeta\theta^{**}$ and $\eta\theta^*$ that are images of zeros of γ_1 and γ_2 on \mathcal{E} by Galois automorphisms. They can be also expressed in terms of the following simple inequalities.

Under parameters such that $\theta_1(\alpha_0) > 0$, we have $\zeta\theta^{**} \neq \{s'_0, \theta(\alpha_0)\}$ iff $\theta^{**} \neq \{s_0, \zeta\theta(\alpha_0)\}$ that is equivalent to $\gamma_2(\zeta\theta(\alpha)) < 0$. Under parameters such that $\theta_1(\alpha_0) \leq 0$, we have always $\zeta\theta^{**} \neq \{s'_0, \theta(\alpha_0)\}$ because $\theta_1(\zeta\theta^{**}) > 0$ by stability conditions, in this case we have also $\gamma_2(\zeta\theta(\alpha_0)) \geq 0$. We come to the following conclusions.

(i) Assumption $\zeta\theta^{**} \neq \{s'_0, \theta(\alpha_0)\}$ is equivalent to the one that $\gamma_2(\zeta\theta(\alpha_0)) < 0$ or $\theta_1(\alpha_0) \leq 0$.

Assumption $\zeta\theta^{**} \in \{s'_0, \theta(\alpha_0)\}$ is equivalent to the one that $\gamma_2(\zeta\theta(\alpha_0)) > 0$ and $\theta_1(\alpha_0) > 0$.

(ii) Assumption $\eta\theta^* \neq \{s''_0, \theta(\alpha_0)\}$ is equivalent to the one that $\gamma_1(\eta\theta(\alpha_0)) < 0$ or $\theta_2(\alpha_0) \leq 0$.

Assumption $\eta\theta^* \in \{s''_0, \theta(\alpha_0)\}$ is equivalent to the one that $\gamma_1(\eta\theta(\alpha_0)) > 0$ and $\theta_2(\alpha_0) > 0$.

4.5.2 Given parameters (Σ, μ, R) , density asymptotics for all angles $\alpha_0 \in]0, \pi/2[$

In this section we state the asymptotics of the density for all angles $\alpha_0 \in]0, \pi/2[$ once parameters (Σ, μ, R) are fixed. Theorems 124 – 126 below are direct corollaries of Theorems 120 – 123 and elementary geometric properties of ellipse \mathcal{E} and straight lines $\gamma_1(\theta) = 0$ and $\gamma_2(\theta) = 0$, therefore we do not give their proofs. To shorten the presentation, we restrict ourselves to the main term in the formulations of the results, although of course further terms of the expansions could be written. The different cases of Theorem 124 are illustrated by Figures 4.19–4.25.

Theorem 124. *Let $\theta_1(s_2^+) > 0$, $\theta_2(s_1^+) > 0$.*

(i) Let $\gamma_2(s_1^+) \leq 0$ and $\gamma_1(s_2^+) \leq 0$. Then for any $\alpha_0 \in]0, \pi/2[$ we have :

$$\pi(r \cos \alpha, r \sin \alpha) \sim \frac{c(\alpha_0)}{\sqrt{r}} \exp(-r \langle \theta(\alpha) \mid e_\alpha \rangle), \quad r \rightarrow \infty, \alpha \rightarrow \alpha_0, \quad (4.77)$$

where the constant $c(\alpha_0)$ depends continuously on $\alpha_0 \in]0, \pi/2[$ and $\lim_{\alpha_0 \rightarrow 0} c(\alpha_0) = \lim_{\alpha_0 \rightarrow \pi/2} c(\alpha_0) = 0$.

(ii) Let $\gamma_2(s_1^+) > 0$ and $\gamma_1(s_2^+) \leq 0$.

(iia) Let $\gamma_2(\zeta s_2^+) \geq 0$ or equivalently $\left. \frac{d\Theta_2^+(\theta_1)}{d\theta_1} \right|_{\theta_1^{**}} \geq 0$. Then for any $\alpha_0 \in]0, \pi/2[$ we have

$$\pi(r \cos \alpha, r \sin \alpha) \sim d_1 \exp(-r \langle \zeta \theta^{**} \mid e_\alpha \rangle), \quad r \rightarrow \infty, \alpha \rightarrow \alpha_0, \quad (4.78)$$

with some constant $d_1 > 0$.

(iib) Let $\gamma_2(\zeta s_2^+) < 0$ or equivalently $A^{**} \equiv \left. \frac{d\Theta_2^+(\theta_1)}{d\theta_1} \right|_{\theta_1^{**}} < 0$. Define $\alpha_1 = \arctan(-1/A^{**}) \in]0, \pi/2[$. Then for any $\alpha_0 \in]0, \alpha_1[$ we have (4.78) and for any $\alpha \in]\alpha_1, \pi/2[$ we have (4.77).

(iii) Let $\gamma_2(s_1^+) < 0$ and $\gamma_1(s_2^+) \geq 0$.

(iiia) Let $\gamma_1(\eta s_1^+) \geq 0$ or equivalently $\left. \frac{d\Theta_1^+(\theta_2)}{d\theta_2} \right|_{\theta_2^*} \geq 0$. Then for any $\alpha_0 \in]0, \pi/2[$ we have

$$\pi(r \cos \alpha, r \sin \alpha) \sim d_2 \exp(-r \langle \eta \theta^* \mid e_\alpha \rangle), \quad r \rightarrow \infty, \alpha \rightarrow \alpha_0, \quad (4.79)$$

with some constant $d_2 > 0$.

(iiib) Let $\gamma_1(\eta s_1^+) < 0$ or equivalently $A^* \equiv \left. \frac{d\Theta_1^+(\theta_2)}{d\theta_2} \right|_{\theta_2^*} < 0$. Define $\alpha_2 = \arctan(-A^*) \in]0, \pi/2[$. Then for any $\alpha_0 \in]0, \alpha_2[$ we have (4.77) and for any $\alpha \in]\alpha_2, \pi/2[$ we have (4.79).

(iv) Let $\gamma_2(s_1^+) > 0$ and $\gamma_1(s_2^+) > 0$.

(iva) Let $\theta_1(\zeta \theta^{**}) \leq \theta_1(\eta \theta^*)$ and $\theta_2(\zeta \theta^{**}) \leq \theta_2(\eta \theta^*)$ where at least one of inequalities is strict. Then for any $\alpha_0 \in]0, \pi/2[$ we have (4.78).

(ivb) Let $\theta_1(\zeta \theta^{**}) \geq \theta_1(\eta \theta^*)$ and $\theta_2(\zeta \theta^{**}) \geq \theta_2(\eta \theta^*)$ where at least one of inequalities is strict. Then for any $\alpha \in]0, \pi/2[$ we have (4.79).

(ivc) Let $\theta_1(\zeta \theta^{**}) \leq \theta_1(\eta \theta^*)$ and $\theta_2(\zeta \theta^{**}) \geq \theta_2(\eta \theta^*)$ where at least one of the inequalities is strict. Let us define $\beta_0 = \arctan \frac{\theta_1(\zeta \theta^{**}) - \theta_1(\eta \theta^*)}{\theta_2(\eta \theta^*) - \theta_2(\zeta \theta^{**})}$. Then for any $\alpha_0 \in]0, \beta_0[$ we have (4.78), for any $\alpha_0 \in]\beta_0, \pi/2[$ we have (4.79) and for $\alpha_0 = \beta_0$ we have

$$\pi(r \cos \alpha, r \sin \alpha) \sim d_1 \exp(-r \langle \zeta \theta^{**} \mid e_\alpha \rangle) + d_2 \exp(-r \langle \eta \theta^* \mid e_\alpha \rangle), \quad r \rightarrow \infty, \alpha \rightarrow \alpha_0. \quad (4.80)$$

(ivd) Let $\theta_1(\zeta \theta^{**}) \geq \theta_1(\eta \theta^*)$ and $\theta_2(\zeta \theta^{**}) \leq \theta_2(\eta \theta^*)$. Let us define angles α_1 and α_2 as in (ii) and (iii). Then $0 < \alpha_1 \leq \alpha_2 < \pi/2$; for any $\alpha_0 \in]0, \alpha_1[$ we have (4.78), for any $\alpha_0 \in]\alpha_1, \alpha_2[$ we have (4.77) and for any $\alpha_0 \in]\alpha_2, \pi/2[$ we have (4.79).

Theorem 125. Let $\theta_1(s_2^+) \leq 0$, $\theta_2(s_1^+) \leq 0$.

- (i) Let $\gamma_2(s_1^+) \leq 0$ and $\gamma_1(s_2^+) \leq 0$. Then for any $\alpha_0 \in]0, \pi/2[$ the asymptotics (4.77) is valid.
- (ii) Let $\gamma_2(s_1^+) > 0$ and $\gamma_1(s_2^+) \leq 0$. Let $A^{**} \equiv \left. \frac{d\Theta_2^+(\theta_1)}{d\theta_1} \right|_{\theta_1^{**}}$. Then $\alpha_1 = \arctan(-1/A^{**}) \in]0, \pi/2[$. For any $\alpha_0 \in]0, \alpha_1[$ the asymptotics (4.78) is valid and for any $\alpha \in]\alpha_1, \pi/2[$ the asymptotics (4.77) holds true.
- (iii) Let $\gamma_2(s_1^+) < 0$ and $\gamma_1(s_2^+) \geq 0$. Let $A^* \equiv \left. \frac{d\Theta_1^+(\theta_2)}{d\theta_2} \right|_{\theta_2^*}$. Then $\alpha_2 = \arctan(-A^*) \in]0, \pi/2[$. For any $\alpha_0 \in]0, \alpha_2[$ the asymptotics (4.77) is valid and for any $\alpha \in]\alpha_2, \pi/2[$ the asymptotics (4.79) holds true.
- (iv) Let $\gamma_2(s_1^+) > 0$ and $\gamma_1(s_2^+) > 0$. Then either $\theta_1(\zeta\theta^{**}) < \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) > \theta_2(\eta\theta^*)$, or $\theta_1(\zeta\theta^{**}) > \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) < \theta_2(\eta\theta^*)$, or finally $\theta_1(\zeta\theta^{**}) = \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) = \theta_2(\eta\theta^*)$.
- (iva) Let $\theta_1(\zeta\theta^{**}) < \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) > \theta_2(\eta\theta^*)$. Let us define $\beta_0 = \arctan \frac{\theta_1(\zeta\theta^{**}) - \theta_1(\eta\theta^*)}{\theta_2(\eta\theta^*) - \theta_2(\zeta\theta^{**})}$. Then for any $\alpha_0 \in]0, \beta_0[$ we have (4.78), for any $\alpha_0 \in]\beta_0, \pi/2[$ we have (4.79) and for $\alpha_0 = \beta_0$ we have (4.80).
- (ivb) Let $\theta_1(\zeta\theta^{**}) > \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) < \theta_2(\eta\theta^*)$ or $\theta_1(\zeta\theta^{**}) = \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) = \theta_2(\eta\theta^*)$. Let us define angles α_1 and α_2 as in (ii) and (iii). Then $0 < \alpha_1 \leq \alpha_2 < \pi/2$; for any $\alpha_0 \in]0, \alpha_1[$ we have (4.78), for any $\alpha_0 \in]\alpha_1, \alpha_2[$ we have (4.77) and for any $\alpha_0 \in]\alpha_2, \pi/2[$ we have (4.79).

Theorem 126. Let $\theta_1(s_2^+) > 0$, $\theta_2(s_1^+) \leq 0$.

- (i) Let $\gamma_2(s_1^+) \leq 0$ and $\gamma_1(s_2^+) \leq 0$. Then for any $\alpha_0 \in]0, \pi/2[$ the asymptotics (4.77) is valid.
- (ii) Let $\gamma_2(s_1^+) > 0$ and $\gamma_1(s_2^+) \leq 0$.
 - (iia) Let $\gamma_2(\zeta s_2^+) \geq 0$ or equivalently $\left. \frac{d\Theta_2^+(\theta_1)}{d\theta_1} \right|_{\theta_1^{**}} \geq 0$. Then for any $\alpha_0 \in]0, \pi/2[$ the asymptotics (4.78) is valid.
 - (iib) Let $\gamma_2(\zeta s_2^+) < 0$ or equivalently $A^{**} \equiv \left. \frac{d\Theta_2^+(\theta_1)}{d\theta_1} \right|_{\theta_1^{**}} < 0$. Define $\alpha_1 = \arctan(-1/A^{**}) \in]0, \pi/2[$. Then for any $\alpha_0 \in]0, \alpha_1[$ we have (4.78) and for any $\alpha \in]\alpha_1, \pi/2[$ we have (4.77).
- (iii) Let $\gamma_2(s_1^+) < 0$ and $\gamma_1(s_2^+) \geq 0$. Let $A^* \equiv \left. \frac{d\Theta_1^+(\theta_2)}{d\theta_2} \right|_{\theta_2^*}$. Then $\alpha_2 = \arctan(-A^*) \in]0, \pi/2[$. For any $\alpha_0 \in]0, \alpha_2[$ the asymptotics (4.77) is valid and for any $\alpha \in]\alpha_2, \pi/2[$ the asymptotics (4.79) holds true.
- (iv) Let $\gamma_2(s_1^+) > 0$ and $\gamma_1(s_2^+) > 0$. Then either $\theta_1(\zeta\theta^{**}) \leq \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) \leq \theta_2(\eta\theta^*)$, or $\theta_1(\zeta\theta^{**}) < \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) > \theta_2(\eta\theta^*)$, or $\theta_1(\zeta\theta^{**}) > \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) < \theta_2(\eta\theta^*)$, or finally $\theta_1(\zeta\theta^{**}) = \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) = \theta_2(\eta\theta^*)$.
- (iva) Let $\theta_1(\zeta\theta^{**}) \leq \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) \leq \theta_2(\eta\theta^*)$ where at least one of inequalities is strict. Then for any $\alpha_0 \in]0, \pi/2[$ we have (4.78).

- (ivb) Let $\theta_1(\zeta\theta^{**}) < \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) > \theta_2(\eta\theta^*)$. Let us define $\beta_0 = \arctan \frac{\theta_1(\zeta\theta^{**}) - \theta_1(\eta\theta^*)}{\theta_2(\eta\theta^*) - \theta_2(\zeta\theta^{**})}$. Then for any $\alpha_0 \in]0, \beta_0[$ we have (4.78), for any $\alpha_0 \in]\beta_0, \pi/2[$ we have (4.79) and for $\alpha_0 = \beta_0$ we have (4.80).
- (ivc) Let $\theta_1(\zeta\theta^{**}) > \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) < \theta_2(\eta\theta^*)$ or $\theta_1(\zeta\theta^{**}) = \theta_1(\eta\theta^*)$ and $\theta_2(\zeta\theta^{**}) = \theta_2(\eta\theta^*)$. Let us define angles α_1 and α_2 as in (ii) and (iii). Then $0 < \alpha_1 \leq \alpha_2 < \pi/2$; for any $\alpha_0 \in]0, \alpha_1[$ we have (4.78), for any $\alpha_0 \in]\alpha_1, \alpha_2[$ we have (4.77) and for any $\alpha_0 \in]\alpha_2, \pi/2[$ we have (4.79).

The symmetric theorem for the case $\theta_1(s_2^+) \leq 0$, $\theta_2(s_1^+) > 0$ holds.

4.5.3 Concluding remarks

Let us remark that the approach of this article applies to the SRBM in any cone of \mathbf{R}^2 . Thanks to a linear transformation $T \in \mathbf{R}^{2 \times 2}$, it is easy to transform $Z(t)$, a reflected Brownian motion of parameters (Σ, μ, R) in a cone into $TZ(t)$ a reflected Brownian motion of parameters $(T\Sigma T^t, T\mu, TR)$ in the quarter plane. For example if the initial cone is the set $\{(x, y) | x \geq 0 \text{ and } y \leq ax\}$ for some $a > 0$, we may just take $T = \begin{pmatrix} 1 & -\frac{1}{a} \\ 0 & 1 \end{pmatrix}$. The process $TZ(t)$ lives in a quarter plane. Then the approach of this article applies and its results can be converted to the initial cone by the inverse linear transformation. The analytic approach for discrete random walks is essentially restricted to those with jumps to the nearest neighbors in the interior of the quarter plane. Since a linear transformation can not generally keep the length of jumps, this procedure does not work in the discrete case. That is why the analytic approach in \mathbf{R}^2 has a more general scope of applications.

To conclude this article, we sketch the way of recovering the asymptotic results of Dai and Miyazawa [28] via the approach of this article. Given a directional vector $c = (c_1, c_2) \in \mathbf{R}_+^2$, thanks to the representation of Lemma 114 we obtain

$$\begin{aligned}
\mathbf{P}(\langle c | Z(\infty) \rangle \geq R) &= \int_{\substack{x_1 \geq 0, x_2 \geq 0 \\ c_1 x_1 + c_2 x_2 \geq R}} \pi(x_1, x_2) dx_1 dx_2 \\
&= \int_{\substack{x_1 \geq 0, x_2 \geq 0 \\ c_1 x_1 + c_2 x_2 \geq R}} I_1(x_1, x_2) dx_1 dx_2 + \int_{\substack{x_1 \geq 0, x_2 \geq 0 \\ c_1 x_1 + c_2 x_2 \geq R}} I_2(x_1, x_2) dx_1 dx_2 \\
&= \int_{\mathcal{I}_{\theta_1}^{\epsilon, +}} g_1(\theta_1) \frac{1}{\theta_1} \frac{e^{-\frac{R}{c_1} \theta_1}}{\Theta_2^+(\theta_1) - \theta_1 \frac{c_2}{c_1}} d\theta_1 + \int_{\mathcal{I}_{\theta_1}^{\epsilon, +}} g_1(\theta_1) \frac{-c_2/c_1}{\Theta_2^+(\theta_1)(\Theta_2^+(\theta_1) - \theta_1 \frac{c_2}{c_1})} e^{-\frac{R}{c_2} \Theta_2^+(\theta_1)} d\theta_1 \\
&\quad + \int_{\mathcal{I}_{\theta_2}^{\epsilon, +}} g_2(\theta_2) \frac{1}{\theta_2} \frac{e^{-\frac{R}{c_2} \theta_2}}{\Theta_1^+(\theta_2) - \theta_2 \frac{c_1}{c_2}} d\theta_2 + \int_{\mathcal{I}_{\theta_2}^{\epsilon, +}} g_2(\theta_2) \frac{-c_1/c_2}{\Theta_1^+(\theta_2)(\Theta_1^+(\theta_2) - \theta_2 \frac{c_1}{c_2})} e^{-\frac{R}{c_1} \Theta_1^+(\theta_2)} d\theta_2,
\end{aligned} \tag{4.82}$$

where

$$g_1(\theta_1) = \frac{\varphi_2(\theta_1) \gamma_2(\theta_1, \Theta_2^+(\theta_1))}{\sqrt{d(\theta_1)}}, \quad g_2(\theta_2) = \frac{\varphi_1(\theta_2) \gamma_1(\Theta_1^+(\theta_2), \theta_2)}{\sqrt{d(\theta_2)}}.$$

The first term in (4.81) is just the Laplace transform of the function $h_1(\theta_1) = g_1(\theta_1) \frac{1}{\theta_1} \frac{1}{\Theta_2^+(\theta_1) - \theta_1 \frac{c_2}{c_1}}$, its asymptotics is determined by the smallest real singularity of $h_1(\theta_1)$, see e.g. [32]. This

may be either the branch point θ_1^+ of $\varphi_2(\theta_1)$, or the smallest pole of $h_1(\theta_1)$ on $]0, \theta_1^+[$ whenever it exists, the natural candidates are $\zeta\theta^{**}$, $\zeta\eta\theta^*$ due to Lemmas 110–113 or a point $\theta^c = (\theta_1^c, \theta_2^c)$ such that $\theta_2^c = \Theta_2^+(\theta_1^c) = \theta_1^c \frac{c_2}{c_1}$. To determine the asymptotics of the second integral in (4.81), we shift the integration contour to the new one passing through the saddle-point $\Theta_1(\theta_2^+)$ and take into account the poles of the integrand we encounter, the most important of these poles are those listed above. The asymptotics of two terms in (4.82) is determined in the same way. Combining all these results together we derive the main asymptotic term depending on the parameters that can be either $e^{-\frac{R}{c_1}\theta_1^+}$, $e^{-\frac{R}{c_2}\theta_2^+}$ preceding by $R^{-1/2}$ or $R^{-3/2}$ with some constant, or $e^{-\frac{R}{c_1}\theta_1^c} = e^{\frac{R}{c_2}\theta_2^c}$, $e^{-\frac{R}{c_i}(\zeta\theta^{**})_i}$, $e^{-\frac{R}{c_i}(\eta\theta^*)_i}$, $i = 1, 2$ preceding by some constant and the factor R in some critical cases. This analysis leads to the results of [28].

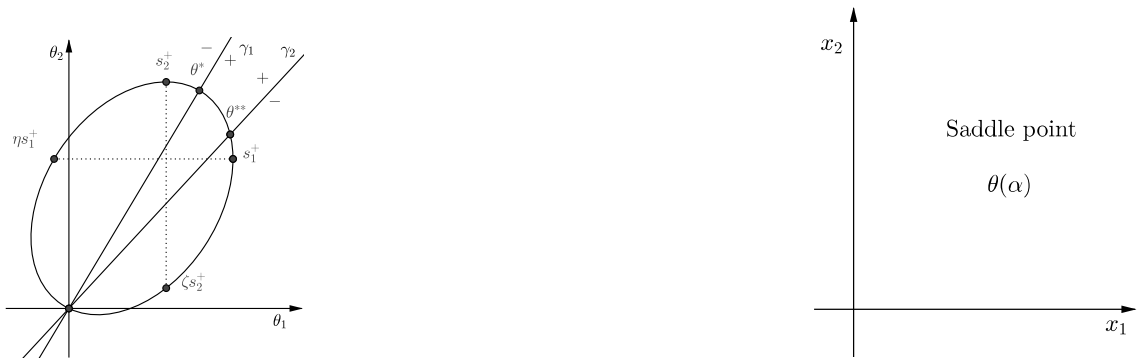


Figure 4.19: Theorem 124 case (i)

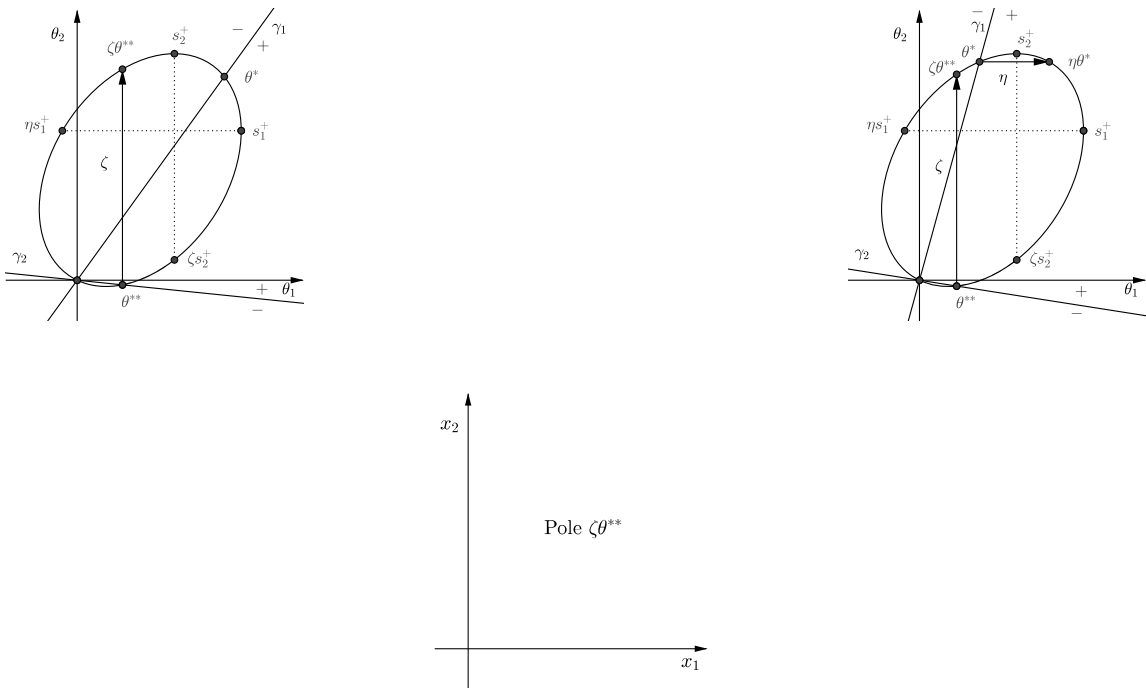


Figure 4.20: Theorem 124 case (iia) and (iva)

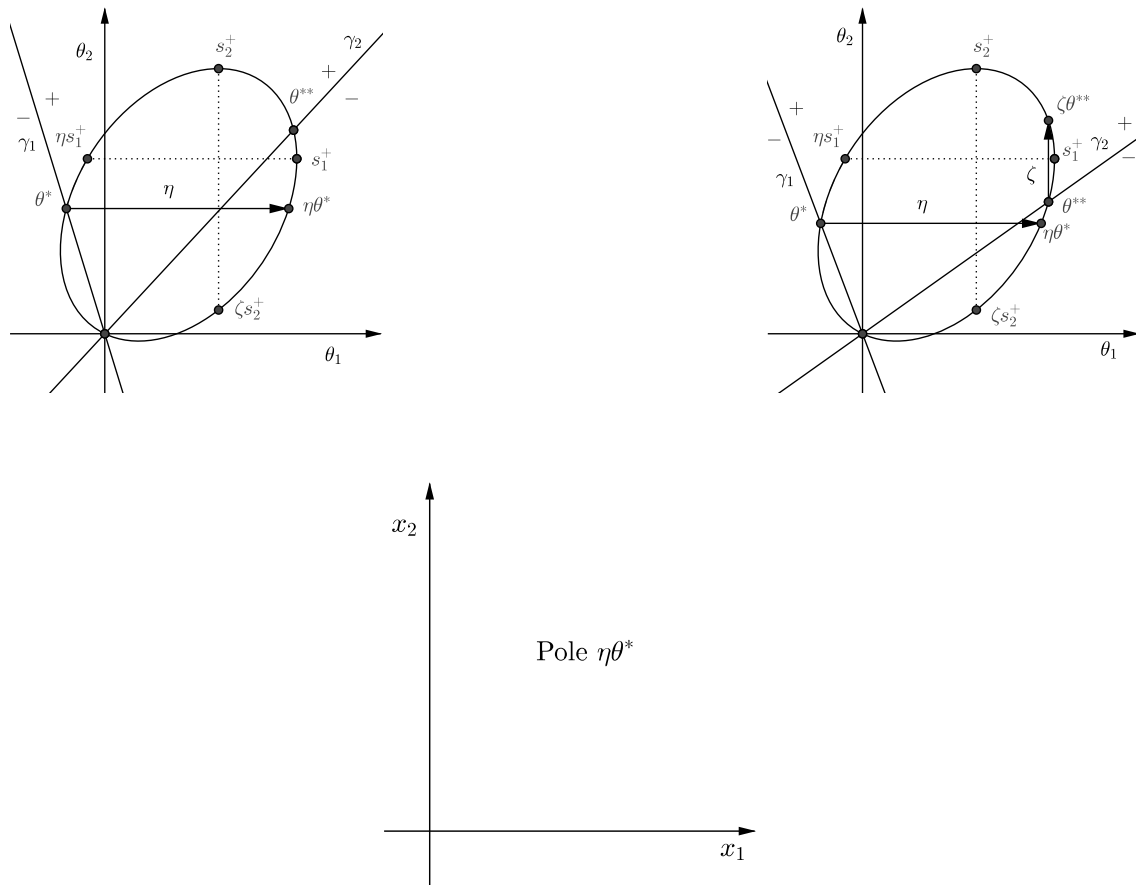


Figure 4.21: Theorem 124 case (iiia) and (ivb)

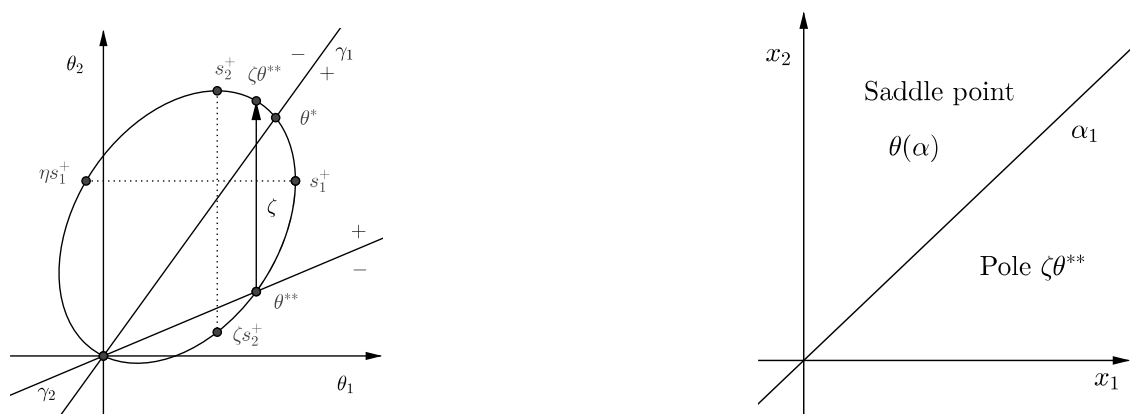


Figure 4.22: Theorem 124 case (iib)

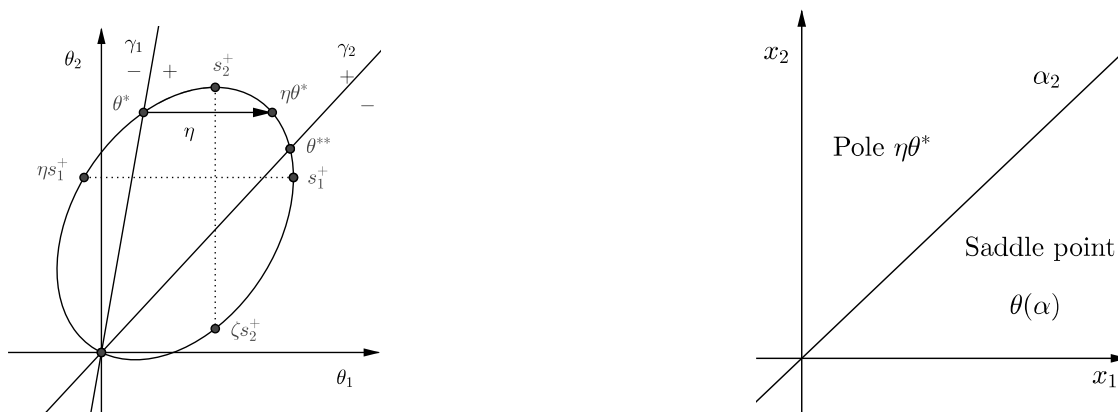


Figure 4.23: Theorem 124 case (iiib)

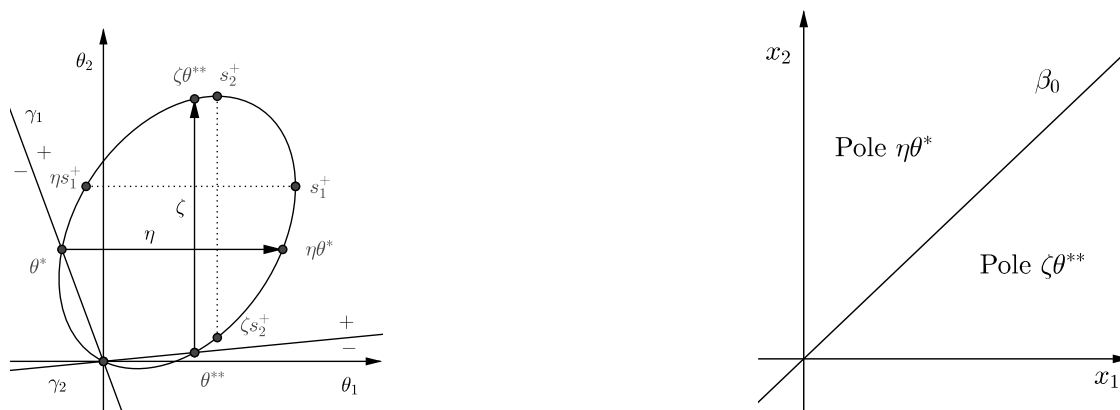


Figure 4.24: Theorem 124 case (ivc)

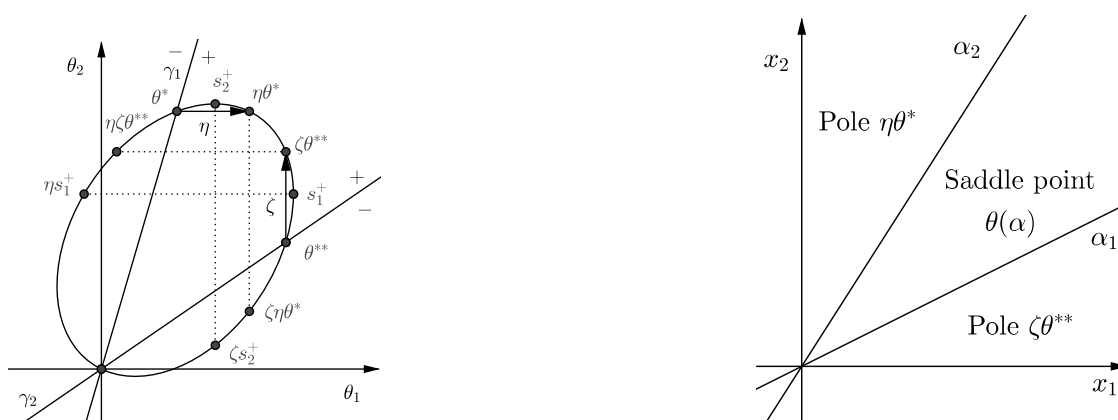


Figure 4.25: Theorem 124 case (ivd)

Annexes

Random walks in the quadrant

Cette annexe est en partie issue de [45] écrit pour la conférence AofA 2016 et paru dans *Proceedings of the 27th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms, Kraków, Poland, 4-8 July*.

Abstract

Random walks in the quarter plane are an important object both of combinatorics and probability theory. Of particular interest for their study is the analytic approach initiated by Fayolle, Iasnogorodski and Malyshev and further developed by Kourkova and Raschel. In the same way as we have presented it for Brownian motion in the quadrant, the outcomes of this method are explicit expressions for the generating functions of interest, asymptotic analysis of their coefficients, etc. The analytic approach developed in this thesis in the continuous setting have been strongly inspired by the discrete case. The aim of this appendix is twofold: it is first an abstract of existing results obtained thanks to this method for reflected random walks in the quadrant; we further compare various aspects of the discrete and continuous analytic approaches.

Résumé

Les marches aléatoires dans le quart de plan sont un objet important à la fois en combinatoire et en théorie des probabilités. L'approche analytique initiée par Fayolle, Iasnogorodski et Malyshev et par la suite développée par Kourkova et Raschel est d'un intérêt particulier pour l'étude de ces marches. De la même manière que nous l'avons présenté pour le mouvement Brownien réfléchi dans le quadrant, les résultats de cette méthode sont les expressions explicites des fonctions génératrices étudiées, l'analyse asymptotique de leurs coefficients, etc. L'approche analytique développée dans cette thèse dans le cadre continu a été fortement inspirée par le cas discret. L'objectif de cette annexe est double: tout d'abord c'est un résumé des résultats existants obtenus par cette méthode pour les marches aléatoires réfléchies dans le quadrant; on compare de plus les divers aspects de l'approche analytique dans le cadre discret et continu.

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Since the seventies and the pioneered papers [79, 36], random walks in the quarter plane are extensively studied. They are indeed an important object of probability theory and have been studied for their recurrence/transience, for their links with queueing systems [36], representation theory [7], potential theory. The asymptotics for the exit time has been studied in the quadrant [64] and also for multidimensional random walk in a general cone [30]. Moreover, the state space \mathbb{N}^2 offers a natural framework for studying any two-dimensional population [15]; accordingly, quadrant walks appear as models in biology and in finance [21]. Another interest of random walks in the quarter plane is that in the large class of random processes in cones, they form a family for which remarkable exact formulas exist. Moreover, quadrant walks are popular in combinatorics, see [10, 9, 65]. Indeed, many models of walks are in bijection with other combinatorial objects: maps, permutations, trees, Young tableaux, etc. In combinatorics again, famous models have emerged from quadrant walks, as Kreweras' or Gessel's ones, see [10, 9]. Finally, walks in the quadrant are interesting for the numerous tools used for their analysis: combinatorial [10], from complex analysis [79, 36, 37, 64, 65, 6], computer algebra [9], for instance.

This appendix is an abstract of the application of the analytic approach in the discrete case. This thesis is the result of the export of this analytic method for reflected Brownian motion in the quarter plane, after [42, 5]. We further propose a comparative study of the discrete/continuous setting.

This appendix is organized as follows: we first define the random walks in the quadrant and establish the ergodic (or recurrent) conditions (Section A.1); we state the key functional equation (a kernel equation) for the invariant measure, which is the starting point of our entire analysis, and we study the kernel (Section A.2); we state and solve the BVP satisfied by the generating functions (Section A.3); we move to asymptotic results (Section A.4); we introduce the Riemann surface of the model and some important related facts (Section A.5); we briefly discuss some similar results on Green functions in the transient case (Section A.6); then we propose a comparative table of the discrete and continuous setting (Section A.7).

A.1 Recurrence conditions

One considers Z a piecewise homogeneous random walk with sample paths in \mathbb{N}^2 . There are four domains of spatial homogeneity (the interior of \mathbb{N}^2 , the horizontal and vertical axes, the origin), inside of which the transition probabilities (of unit size) are denoted by $p_{i,j}$, $p'_{i,j}$, $p''_{i,j}$ and $p_{i,j}^0$, respectively. See Figure A.1. We then have

$$\begin{cases} \mathbb{P}[Z(n+1) = (x+i, y+j) | Z(n) = (x, y)] = p_{i,j} & \text{if } (x, y) \in \mathbb{Z}_+^{*2} \\ \mathbb{P}[Z(n+1) = (x+i, y+j) | Z(n) = (x, y)] = p'_{i,j} & \text{if } (x, y) \in \mathbb{Z}_+^* \times \{0\} \\ \mathbb{P}[Z(n+1) = (x+i, y+j) | Z(n) = (x, y)] = p''_{i,j} & \text{if } (x, y) \in \{0\} \times \mathbb{Z}_+^* \\ \mathbb{P}[Z(n+1) = (x+i, y+j) | Z(n) = (x, y)] = p_{i,j}^0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Transition probabilities p_{ij} , p'_{ij} , p''_{ij} , p_{ij}^0 are zero when $|i| > 1$ or $|j| > 1$.

The recurrent conditions can be given in terms of the mean jump vectors (E, E', E'') such that

$$\begin{cases} E = (E_x, E_y) = (\sum_{i,j} ip_{i,j}, \sum_{i,j} jp_{i,j}), \\ E' = (E'_x, E'_y) = (\sum_{i,j} ip'_{i,j}, \sum_{i,j} jp'_{i,j}), \\ E'' = (E''_x, E''_y) = (\sum_{i,j} ip''_{i,j}, \sum_{i,j} jp''_{i,j}). \end{cases} \quad (\text{A.1})$$

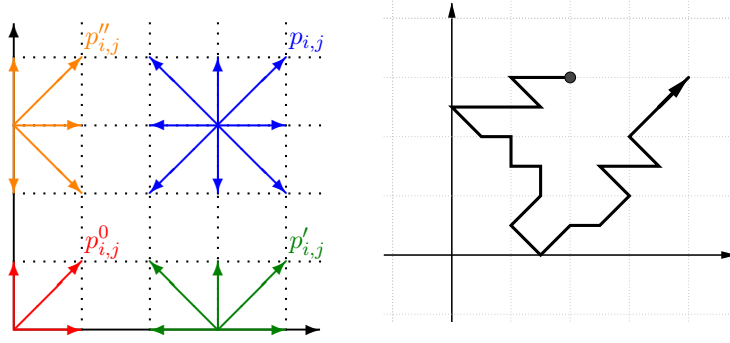


Figure A.1: Transition probabilities of the reflected random walk in the quarter plane with four domains of spatial homogeneity and example of path

They are the equivalent of the drift μ and the reflexion vectors R^1 and R^2 introduced in the continuous setting in equation (1.10), see Figure A.2.

Proposition 127 (Ergodic conditions). *The reflected random walk in the quadrant is positive recurrent (or ergodic) if and only if one of these conditions are satisfied:*

1. $E_x < 0, E_y < 0, E_x E'_y - E_y E'_x < 0, E_y E''_x - E_x E''_y < 0,$
2. $E_x > 0, E_y < 0, E_x E'_y - E_y E'_x < 0,$
3. $E_x < 0, E_y > 0, E_y E''_x - E_x E''_y < 0.$

We refer to [37, Theorem 1.2.1], the proof of this result can be found in the book [38].

Remark 128 (Interpretation of the conditions). • The two last parts of condition 1 are equivalent to the fact that (E', E) and (E, E'') are direct bases. Last part of condition 2 is equivalent to the fact that (E', E) is a direct base and last part of condition 3 to the fact that (E'', E) is a direct base.

- These ergodic conditions express a competition between the mean jump vectors, the drift E on one side and the reflexion vectors E', E'' on the other side, see Figure A.2. Remark 40 compares these conditions with the continuous case.

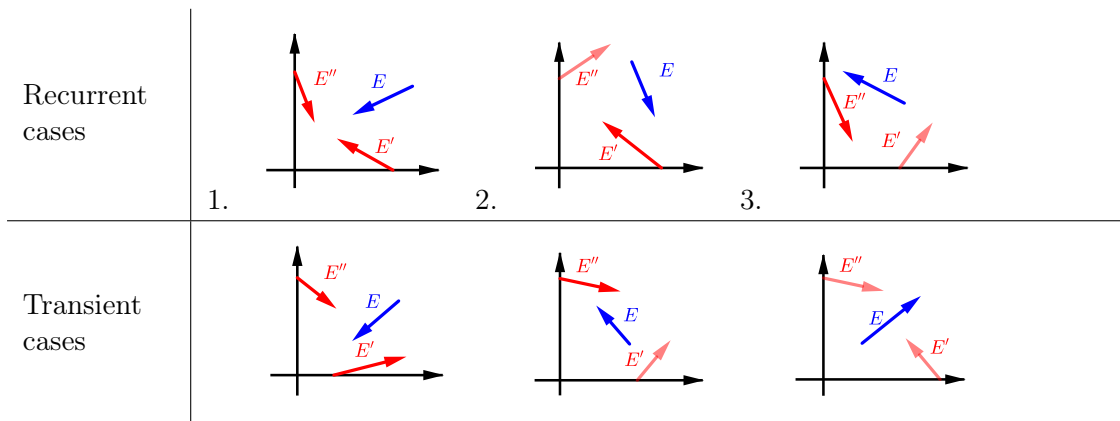


Figure A.2: Ergodic conditions and mean jump vectors

A.2 Functional equation for the invariant measure

This section and the following devoted to the study of invariant measure are based mainly on [37]. The inventory polynomial of the inner domain is called the kernel and equals

$$K(x, y) = xy\{\sum_{-1 \leq i, j \leq 1} p_{i,j} x^i y^j - 1\}. \quad (\text{A.2})$$

The inventory polynomials associated to the other homogeneity domains are

$$k(x, y) = x\{\sum p'_{i,j} x^i y^j - 1\}, \quad \tilde{k}(x, y) = y\{\sum p''_{i,j} x^i y^j - 1\}, \quad k_0(x, y) = \{\sum p_{i,j}^0 x^i y^j - 1\}.$$

Assuming the random walk ergodic, we denote the invariant measure by $\{\pi_{i,j}\}_{i,j \geq 0}$ and introduce the generating functions

$$\pi(x, y) = \sum_{i,j \geq 1} \pi_{i,j} x^{i-1} y^{j-1}, \quad \pi(x) = \sum_{i \geq 1} \pi_{i,0} x^{i-1}, \quad \tilde{\pi}(y) = \sum_{j \geq 1} \pi_{0,j} y^{j-1}.$$

Writing the balance equations at the generating function level, we have (see [37, Equation (1.3.6)] for the original statement):

Lemma 129 (Functional equation for invariant measure). *The fundamental functional equation holds*

$$-K(x, y)\pi(x, y) = k(x, y)\pi(x) + \tilde{k}(x, y)\tilde{\pi}(y) + k_0(x, y)\pi_{0,0}. \quad (\text{A.3})$$

Equation (A.3) holds a priori in the region $\{(x, y) \in \mathbb{C}^2 : |x| \leq 1, |y| \leq 1\}$. Indeed, the $\pi_{i,j}$ sum up to 1, so that the generating functions $\pi(x, y)$, $\pi(x)$ and $\tilde{\pi}(y)$ are well defined on the (bi)disc. The identity (A.3) is a kernel equation, and a crucial role will be played by the kernel (A.2). This polynomial K is of second order in both x and y ; its roots $X(y)$ and $Y(x)$ defined by

$$K(X(y), y) = K(x, Y(x)) = 0 \quad (\text{A.4})$$

are thus algebraic of degree 2. Writing the kernel as $K(x, y) = a(y)x^2 + b(y)x + c(y)$ and defining its discriminant $d(y) = b(y)^2 - 4a(y)c(y)$, one has obviously

$$X(y) = \frac{-b(y) \pm \sqrt{d(y)}}{2a(y)}.$$

The polynomial d has three or four roots, and exactly two of them are located in the unit disc, see [37, Lemma 2.3.8]. They are named y_1, y_2 , cf. Figure A.3. On (y_1, y_2) one has $d(y) < 0$, so that the two values (or branches) of $X(y)$ (that we shall call $X_0(y)$ and $X_1(y)$) are complex conjugate of one another. Similarly, we define Y_0 and Y_1 . In particular, the set

$$\mathcal{M} = X([y_1, y_2]) = \{x \in \mathbb{C} : K(x, y) = 0 \text{ and } y \in [y_1, y_2]\}$$

is symmetrical w.r.t. the real axis (Figure A.3). This curve will be used to set a boundary condition for the unknown function π (Lemma 130). It is the analogue of the curve \mathcal{R} in Section 1.3.2.5 in the continuous setting.

A.3 Statement and resolution of the BVP

The analytic approach of [79, 36, 37] proposes a way for solving the functional equation (A.3), by reduction to a BVP. Generally speaking, a BVP consists of a regularity condition and a boundary condition.

Lemma 130 (BVP for invariant measure). *The function π satisfies the following BVP:*

- π is meromorphic in the bounded domain delimited by \mathcal{M} and has there identified poles;
- for any $x \in \mathcal{M}$,

$$\frac{k(x, Y_0(x))}{\tilde{k}(x, Y_0(x))} \pi(x) - \frac{k(\bar{x}, Y_0(\bar{x}))}{\tilde{k}(\bar{x}, Y_0(\bar{x}))} \pi(\bar{x}) = \frac{k_0(\bar{x}, Y_0(\bar{x}))}{\tilde{k}(\bar{x}, Y_0(\bar{x}))} \pi_{0,0} - \frac{k_0(x, Y_0(x))}{\tilde{k}(x, Y_0(x))} \pi_{0,0}.$$

Proof. The regularity condition follows from Theorem 133, which provides a (maximal) meromorphic continuation of the function π . We now turn to the boundary condition. For $i \in \{0, 1\}$, we evaluate the functional equation (A.3) at $(X_i(y), y)$ and divide by $\tilde{k}(X_i(y), y)$. We then make the difference of the identities corresponding to $i = 0$ and $i = 1$. Finally, we substitute $X_0(y) = x$ and $X_1(y) = \bar{x}$, noting that when $y \in [y_1, y_2]$, $x \in \mathcal{M}$ by construction. Notice that we have chosen the segment $[y_1, y_2]$ connecting the points inside of the unit disc (Figure A.3), in which we know that the generating function $\tilde{\pi}$ is well defined. \square

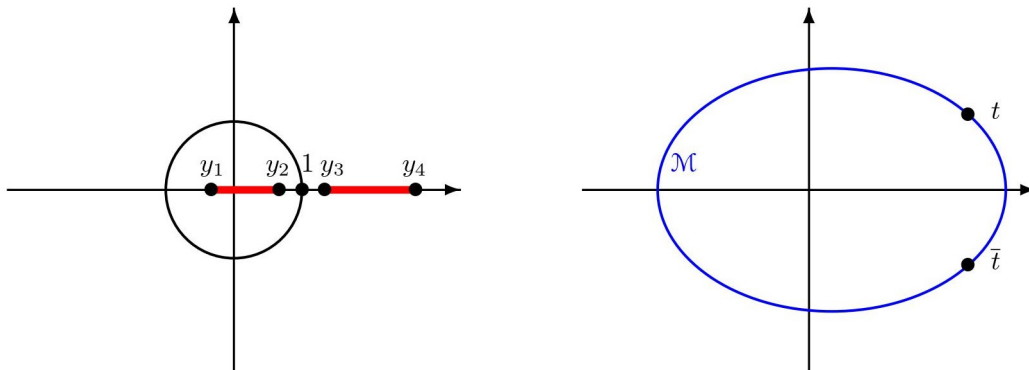


Figure A.3: Left: the polynomial d has three or four roots, denoted by y_1, y_2, y_3, y_4 ; exactly two of them are inside the unit disc. Right: the curve $\mathcal{M} = X([y_1, y_2])$ is symmetrical w.r.t. the horizontal axis

Although Lemma 130 could be written more precisely (by giving the number and the location of the poles of π), we shall prefer the above version, since we focus in this appendix on the main ideas of the analytic approach.

It has a harder form than the BVP in Proposition 64 for the continuous case, because there is an inhomogeneous term (as $\pi_{0,0}$) and also because in the coefficients in front of the unknown functions there is an algebraic function (as Y_0) involved.

Lemma 130 happens to characterize the generating functions, as it eventually leads to an explicit expression for π , see Theorem 131. Before stating this central result (borrowed from [37, Theorem 5.4.3]), we need to introduce a function w called a conformal gluing

function. By definition it satisfies $w(x) = w(\bar{x})$ for $x \in \mathcal{M}$ and is one-to-one inside of \mathcal{M} . This function will be constructed in Theorem 134 of Section A.5 and is reminiscent of the function constructed in the continuous setting in Section 1.3.2.6.

Theorem 131 (Explicit expression for invariant measure). *There exist two functions f and g , constructed from k , \tilde{k} , k_0 and w , such that the following integral formulation for π holds:*

$$\pi(x) = f(x) \int_{\mathcal{M}} g(u) \frac{w'(u)}{w(u) - w(x)} du.$$

A similar contour integral representation exists for $\tilde{\pi}$, and eventually the functional equation (A.3) provides us with an explicit expression for the bivariate function $\pi(x, y)$.

A.4 Asymptotics of the stationary probabilities

The asymptotics of coefficients $\pi_{i,j}$ of unknown generating functions satisfying the functional equation (A.3) has been obtained in [80] via analytic arguments. The author computed the asymptotics of the stationary probabilities $\pi_{i,j}$ as $i, j \rightarrow \infty$ and $j/i = \tan \alpha$, for any given $\alpha \in (0, \pi/2)$. Let us briefly present these results. It is assumed in [80] that the random walk is simple, meaning that

$$p_{-1,1} = p_{1,1} = p_{-1,-1} = p_{1,-1} = 0. \quad (\text{A.5})$$

It is also assumed that both coordinates of the interior drift vector are negative (as in Figure 3.1). For $\alpha \in (0, \pi/2)$, we define the point $(x(\alpha), y(\alpha))$ as follows. Introducing as in [64] the function $P(u, v) = \sum_{i,j} p_{i,j} e^{iu} e^{jv}$ on \mathbb{R}^2 , the mapping

$$(u, v) \mapsto \frac{\nabla P(u, v)}{|P(u, v)|}$$

is a homeomorphism between $\{(u, v) \in \mathbb{R}^2 : P(u, v) = 1\}$ and the unit circle. The point $(u(\alpha), v(\alpha))$ is the unique solution to $\frac{\nabla P(u, v)}{|P(u, v)|} = (\cos \alpha, \sin \alpha)$. Finally, $(x(\alpha), y(\alpha)) = (e^{u(\alpha)}, e^{v(\alpha)})$.

Similarly to the continuous case, see Theorem 60, and following [80], we introduce the sets of parameters

$$\begin{aligned} \mathcal{P}_{--} &= \{(\{p_{i,j}\}, \alpha) : k(\psi(x(\alpha), y(\alpha))) \leq 0 \text{ and } \tilde{k}(\varphi(x(\alpha), y(\alpha))) \leq 0\}, \\ \mathcal{P}_{+-} &= \{(\{p_{i,j}\}, \alpha) : k(\psi(x(\alpha), y(\alpha))) > 0 \text{ and } \tilde{k}(\varphi(x(\alpha), y(\alpha))) \leq 0\}, \end{aligned}$$

and \mathcal{P}_{-+} and \mathcal{P}_{++} accordingly. The automorphisms ζ and η are defined in Section A.5 by (A.9). The following theorem is proven in [80].

Theorem 132 (Asymptotics for invariant measure). *Let $(i, j) = (r \cos \alpha, r \sin \alpha)$ with $\alpha \in (0, \pi/2)$. Then as $r \rightarrow \infty$ we have*

$$\pi_{i,j} = (1 + o(1)) \cdot \begin{cases} \frac{C_0(\alpha)}{\sqrt{r}} x^{-i}(\alpha) y^{-j}(\alpha) & \text{in } \mathcal{P}_{--}, \\ C_1 p_1^{-i} q_1^{-j} & \text{in } \mathcal{P}_{-+}, \\ C_2 p_2^{-i} q_2^{-j} & \text{in } \mathcal{P}_{+-}, \\ C_1 p_1^{-i} q_1^{-j} + C_2 p_2^{-i} q_2^{-j} & \text{in } \mathcal{P}_{++}, \end{cases} \quad (\text{A.6})$$

where C_0, C_1 and C_2 are constants that can be expressed in terms of the functions π and $\tilde{\pi}$. The point (p_1, q_1) is a solution of the system $\{K(x, y) = 0, k(\psi(x, y)) = 0\}$ and similarly, (p_2, q_2) is a solution of $\{K(x, y) = 0, \tilde{k}(\varphi(x, y)) = 0\}$.

Proof. The stationary probabilities $\pi_{i,j}$ are first written as two-dimensional Cauchy integrals, then reduced via the residue theorem to one-dimensional integrals along some contours. The asymptotics of these integrals is characterized either by the saddle point $(x(\alpha), y(\alpha))$ in the case of the set of parameters \mathcal{P}_{--} or by a pole (p_1, q_1) or (p_2, q_2) that is encountered when moving the integration contour to the saddle point; this happens for the sets of parameters $\mathcal{P}_{-+}, \mathcal{P}_{+-}$ and \mathcal{P}_{++} . \square

As illustrated in [64, 67], the assumption (A.5) is not crucial for the applicability of the method. The limiting cases $\alpha = 0$ and $\alpha = \pi/2$ can also be treated via this approach, with some additional technical details (the saddle point then coincides with a branch point of the integrand), it is done in [64].

A.5 Riemann surface and related facts

In Section A.2 the set

$$\mathcal{K} = \{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\} = \{(x, y) \in \mathbb{C}^2 : \sum p_{i,j} x^i y^j = 1\}$$

has appeared very naturally, since in order to state the BVP (our Lemma 130), we introduced the functions $X(y)$ and $Y(x)$, which by construction cancel the kernel, see (A.4).

In this section the central idea is to consider the (global) complex structure of \mathcal{K} . The set \mathcal{K} turns out to be a Riemann surface of genus 1, i.e., a torus. This simply comes from the reformulation of the identity $K(x, y) = 0$ as

$$\{2a(y)x + b(y)\}^2 = d(y).$$

Moreover, the Riemann surface of the square root of a polynomial of degree 3 or 4 is classically a torus (with this terminology, the roots of the discriminant are branch points).

This new point of view on \mathcal{K} brings powerful tools. Of particular interest is a parametrization of \mathcal{K} in terms of Weierstrass elliptic functions:

$$\mathcal{K} = \{(x(\omega), y(\omega)) : \omega \in \mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})\}. \tag{A.7}$$

This parametrization is totally explicit: $x(\omega)$ and $y(\omega)$ are rational functions in the \wp -Weierstrass function and its derivative \wp' (see [37, Lemma 3.3.1]); the periods ω_1 and ω_2 admit expressions as elliptic integrals in terms of $\{p_{i,j}\}$ (cf. [37, Lemma 3.3.2]), etc. Moreover, as any functions of x and/or y , the functions $\pi(x)$ and $\tilde{\pi}(y)$ can be lifted on \mathcal{K} by setting

$$\Pi(\omega) = \pi(x(\omega)), \quad \tilde{\Pi}(\omega) = \tilde{\pi}(y(\omega)). \tag{A.8}$$

Group of the walk

Introduced in [79] in a probabilistic context and further used in [37, 10], the group of the walk is a dihedral group generated by

$$\zeta(x, y) = \left(x, \frac{\sum_i p_{i,-1} x^i}{\sum_i p_{i,+1} x^i} \frac{1}{y} \right), \quad \eta(x, y) = \left(\frac{\sum_j p_{-1,j} y^j}{\sum_j p_{+1,j} y^j} \frac{1}{x}, y \right). \tag{A.9}$$

(One easily verifies that these generators are idempotent: $\zeta^2 = \eta^2 = 1$.) The group $\langle \zeta, \eta \rangle$ can be finite or infinite, according to the order of the element $\zeta \circ \eta$. The generator ζ (resp. η) exchanges the roots in y (resp. in x) of $K(x, y) = 0$. Viewed as a group of birational transformations in [10], we shall rather see it as a group of automorphisms of the Riemann surface \mathcal{K} .

This group has many applications. First, it allows for a continuation of the functions $\pi(x)$ and $\tilde{\pi}(y)$ (Theorem 133 below). It further connects the algebraic nature of the generating functions to the (in)finiteness of the group (Theorem 135). Finally, in the finite group case, elementary algebraic manipulations of the functional equations can be performed (typically, via the computation of certain orbit-sums) so as to eventually obtain D-finite expressions for the unknowns, see [37, Chapter 4] and [10].

Using the structure of the automorphisms of a torus, the lifted versions of ζ and η admit simple expressions [37, Section 3.1.2]:

$$\zeta(\omega) = -\omega + \omega_1 + \omega_2, \quad \eta(\omega) = -\omega + \omega_1 + \omega_2 + \omega_3, \quad (\text{A.10})$$

where, as the periods ω_1 and ω_2 , $\omega_3 \in (0, \omega_2)$ is an elliptic integral [37, Lemma 3.3.3]. Accordingly, the group is finite if and only if $\omega_2/\omega_3 \in \mathbb{Q}$, which provides a nice criterion in terms of elliptic integrals.

Continuation

While the generating function $\pi(x)$ is defined through its power series in the unit disc, it is a priori unclear how to continue it to a larger domain. This is however crucial, since the curve \mathcal{M} on which it satisfies a BVP (Lemma 130) is not included in the unit disc in general.

Theorem 133 (Meromorphic continuation). *The function π can be continued as a meromorphic function to $\mathbb{C} \setminus [x_3, x_4]$.*

We notice that \mathcal{M} does not intersect $[x_3, x_4]$ by [37, Theorem 5.3.3], so that Theorem 133 indeed provides a continuation of the generating function in the domain delimited by \mathcal{M} .

Proof. This result, stated as Theorem 3.2.3 in [37], is a consequence of a continuation of the lifted generating functions (A.8) on the Riemann surface (or, better, on its universal covering — but we shall not go into these details here). The continuation on \mathcal{K} uses the (lifted) functional equation (A.3) and the group of the walk $\langle \zeta, \eta \rangle$. \square

Conformal mapping

In the integral expression of Theorem 131, the conformal gluing function w is all-present, as it appears in the integrand and in f and g as well. The introduction of the Riemann surface \mathcal{K} allows to derive an expression for this function (this is another major interest of introducing \mathcal{K}). Let us recall that ω_1 and ω_2 are the periods of the elliptic functions of the parametrization (A.7), while ω_3 comes out in the lifted expression (A.10) of the automorphisms.

Theorem 134 (Conformal gluing function). *The conformal gluing function w admits the expression:*

$$w(x) = \wp(\wp^{-1}(x; \omega_1, \omega_2); \omega_1, \omega_3),$$

where for $i \in \{2, 3\}$, $\wp(\cdot; \omega_1, \omega_i)$ is the \wp -Weierstrass elliptic function with periods ω_1 and ω_i .

Proof. While on the complex plane, \mathcal{M} is a quartic curve, it becomes on \mathcal{H} much simpler (typically, a segment). This remark (which again illustrates all the benefit of having introduced the Riemann surface) is used in [37, Section 5.5.2] so as to obtain the above expression for w . \square

Algebraic nature of the generating functions

Recall that a function of one variable is D-finite if it satisfies a linear differential equation with polynomial coefficients.

Theorem 135 (Nature of generating functions). *If the group is finite, the generating functions $\pi(x)$ and $\tilde{\pi}(y)$ are D-finite.*

As we have seen, such a result holds for the continuous case, see Proposition 73.

Proof. This follows from manipulations on the Riemann surface, see [37, Chapter 4]. The D-finiteness is proved on \mathbb{R} ; refined results (in the combinatorial context of the enumeration of paths) can be found in [10], where the D-finiteness is proved on \mathbb{Q} . \square

The converse of Theorem 135 is not shown in full generality. It is true in combinatorics, see [65].

A.6 Green functions

This analytic approach has been applied for the analysis of the join the shortest queue problem in [67], and for computing the asymptotics of Green functions for transient random walks in the quarter plane reflected at the axes (see [66]) or killed at the axes (cf. [64]). In this section we just define the Green functions, their generating functions and we state their functional equation in the case of a transient reflecting random walk Z .

Definition 136 (Green functions). *When z and $z_0 = (x_0, y_0) \in \mathbb{Z}_+^2$, Green functions $g_z^{z_0}$ are defined by*

$$g_z^{z_0} = \sum_{n=0}^{\infty} p_{z_0, z}^n = \mathbb{E}_{z_0} \sum_{n=0}^{\infty} \mathbf{1}_{\{Z_n=z\}}$$

where $p_{z_0, z}^n = \mathbb{P}[Z_n = z | Z_0 = z_0] = \mathbb{E}_{z_0}[\mathbf{1}_{Z_n=z}]$ is the transition function of Z .

Then, the Green function $g_z^{z_0}$ is the mean of the number of visits in z starting from z_0 . Let $G(x, y)$ the generating function inside the quadrant and $G^{z_0}(x)$ and $\widetilde{G}^{z_0}(y)$ the generating functions on the boundaries such that

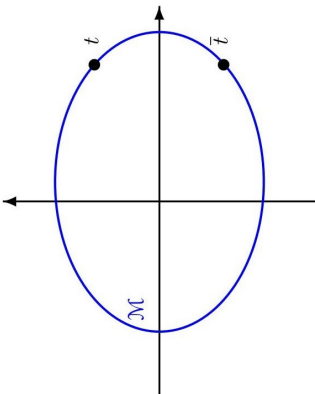
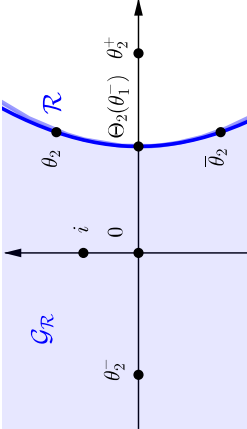
$$G^{z_0}(x, y) = \sum_{i, j \geq 1} g_{(i, j)}^{z_0} x^{i-1} y^{j-1}, \quad G^{z_0}(x) = \sum_{i \geq 1} g_{(i, 0)}^{z_0} x^{i-1}, \quad \widetilde{G}^{z_0}(y) = \sum_{j \geq 1} g_{(0, j)}^{z_0} y^{j-1}.$$

Lemma 137 (Functional equation for Green functions). *The following functional equation holds*

$$-K(x, y)G^{z_0}(x, y) = k(x, y)G^{z_0}(x) + \widetilde{k}(x, y)\widetilde{G}^{z_0}(y) + k_0(x, y)g_0^{z_0} + x^{x_0}y^{y_0}.$$

Thanks to this equation it is then possible to develop the analytic method in order to obtain explicit expression for Green functions and study their asymptotics.

A.7 Discrete setting versus continuous setting

Recurrent case	Discrete setting	Continuous setting
Invariant measure	$\pi_{i,j}$	$\pi(x, y)$
Generating functions	$\pi(x, y) = \sum_{i,j \geq 1} \pi_{i,j} x^{i-1} y^{j-1}$	$\varphi(\theta_1, \theta_2) = \iint_{\mathbb{R}^2_+} e^{\theta_1 z} \Pi(dz)$
Functional equation	$-K(x, y)\pi(x, y) = k(x, y)\pi(x) + \tilde{k}(x, y)\tilde{\pi}(y) + k_0(x, y)\pi_{0,0}$	$-\gamma(\theta)\varphi(\theta) = \gamma_1(\theta)\varphi_2(\theta_1) + \gamma_2(\theta)\varphi_1(\theta_2)$
Kernel	$K(x, y) = xy \{ \sum_{-1 \leq i,j \leq 1} p_{i,j} x^i y^j - 1 \}$	$\gamma(\theta) = \frac{1}{2}(\sigma_{11}\theta_1^2 + 2\sigma_{12}\theta_1\theta_2 + \sigma_{22}\theta_2^2) + \mu_1\theta_1 + \mu_2\theta_2$
Riemann surface	Torus \mathcal{K}	Sphere \mathcal{S}
Galois automorphism	$\zeta(x, y) = \left(x, \frac{\sum_i p_{i,-1} x^i \frac{1}{y}}{\sum_i p_{i,+1} x^i y} \right), \quad \eta(x, y) = \left(\frac{\sum_j p_{-1,j} y^j \frac{1}{x}}{\sum_j p_{+1,j} y^j x}, y \right)$	$\zeta(\theta_1, \theta_2) = \left(\theta_1, \frac{c(\theta_1) \frac{1}{\theta_2}}{a(\theta_1) \theta_2} \right), \quad \eta(\theta_1, \theta_2) = \left(\frac{\tilde{c}(\theta_2) \frac{1}{\theta_1}}{\tilde{a}(\theta_2) \theta_1}, \theta_2 \right)$
Gluing function	$w(x) = \wp(\wp^{-1}(x; \omega_1, \omega_2); \omega_1, \omega_3)$	$w(\theta_2) = T_{\frac{\pi}{\beta}} \left(-\frac{2\theta_2 - (\theta_2^+ + \theta_2^-)}{\theta_2^+ - \theta_2^-} \right)$
Boundary condition	$\frac{k(x, Y_0(x))}{k(x, Y_0(x))} \pi(x) - \frac{k(\bar{x}, Y_0(\bar{x}))}{k(\bar{x}, Y_0(\bar{x}))} \pi(\bar{x}) = \frac{k_0(\bar{x}, Y_0(\bar{x}))}{k(\bar{x}, Y_0(\bar{x}))} \pi_{0,0} - \frac{k_0(x, Y_0(x))}{k(x, Y_0(x))} \pi_{0,0}$	$\varphi_1(\bar{\theta}_2) = G(\theta_2)\varphi_1(\theta_2), \quad \forall \theta_2 \in \mathcal{R}$
Boundary		
Transient case	Discrete setting	Continuous setting
Green functions	$g_z^{z_0} = \sum_{n=0}^{\infty} p_{z_0, z}^n$	$g_z^{z_0} = \int_0^{\infty} p(t, z_0, z) dt$
Generating functions	$G^{z_0}(x, y) = \sum_{i,j \geq 1} g_{(i,j)}^{z_0} x^{i-1} y^{j-1}$	$\psi^{z_0}(\theta) = \iint_{\mathbb{R}^2_+} e^{\theta z} g_z^{z_0} dz$
Functional equation	$-K(x, y)G^{z_0}(x, y) = k(x, y)G^{z_0}(x) + \tilde{k}(x, y)\tilde{G}^{z_0}(y) + k_0(x, y)g_0^{z_0} + x^{x_0} y^{y_0}$	$-\gamma(\theta)\psi^{z_0}(\theta) = \gamma_1(\theta)\psi_1^{z_0}(\theta_2) + \gamma_2(\theta)\psi_2^{z_0}(\theta_1) + e^{\theta \cdot z_0}$

Harmonic functions in the quarter plane for Brownian motion via functional equations

Cette courte annexe est issue de l'annexe de l'article [85] publié en 2014 dans *Stochastic Processes and their Applications*.

Abstract

The goal of this appendix is to illustrate that the analytic method that we have presented in the introduction also allows to solve other types of problems than the study of invariant measures and Green functions. The article [85] finds the discrete harmonic functions in the quarter plane with Dirichlet conditions thanks to functional equations satisfied by the generating functions of the values taken by the harmonic functions. In this appendix this approach is generalized to the continuous case. This analytic method allows to find the harmonic functions within the quarter plane which vanish on the boundary axes.

Résumé

L'objectif de cette annexe est d'illustrer que la méthode analytique que nous avons présentée dans l'introduction permet aussi de résoudre d'autres types de problèmes que l'étude des mesures invariantes et des fonctions de Green. L'article [85] détermine les fonctions harmoniques discrètes dans le quart de plan avec conditions de Dirichlet grâce à des équations fonctionnelles satisfaites par les fonctions génératrices des valeurs prises par les fonctions harmoniques. Dans cette annexe cette approche est généralisée au cas continu. Cette méthode analytique permet ainsi de retrouver les fonctions analytiques dans le quart de plan et nulles aux bords.

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We are interested in finding harmonic functions associated to the Brownian motion in the quarter plane \mathbf{R}_+^2 without drift and with covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}.$$

The latter are the functions which are positive within the quarter plane, which vanish on the boundary axes, and which cancel the generator

$$\mathcal{G}f = \frac{1}{2} \left(\sigma_{11} \frac{\partial^2 f}{\partial x^2} + 2\sigma_{12} \frac{\partial^2 f}{\partial x \partial y} + \sigma_{22} \frac{\partial^2 f}{\partial y^2} \right).$$

B.1 Functional equation

Our first task is to find a functional equation. The Laplace transforms will play the role of the generating functions. For $\theta = (\theta_1, \theta_2)$ and f a function with a non-exponential growth as well as its two first derivatives, we shall note

$$L(f)(\theta) = \int_0^\infty \int_0^\infty f(x, y) e^{-(\theta_1 x + \theta_2 y)} dx dy, \quad \forall \Re \theta_i > 0.$$

Proposition 138 (Functional equation). *The functional equation takes the form*

$$\gamma(\theta) L(f)(\theta) = \frac{1}{2} \left[\sigma_{11} L \left(\frac{\partial f}{\partial x}(0, \cdot) \right) (\theta_2) + \sigma_{22} L \left(\frac{\partial f}{\partial y}(\cdot, 0) \right) (\theta_1) \right], \quad (\text{B.1})$$

where we have noted

$$\gamma(\theta) = \frac{1}{2} \sum_{i,j=1}^2 \sigma_{ij} \theta_i^2 = \frac{1}{2} (\sigma_{11} \theta_1^2 + 2\sigma_{12} \theta_1 \theta_2 + \sigma_{22} \theta_2^2). \quad (\text{B.2})$$

The above quantity is called the kernel of the functional equation.

Proof. The proof is based on a simple integral calculus on the Laplace transform. If f is not growing too quickly, we obtain the three following equalities by doing an integration by part and remembering that $f = 0$ on the boundary of the quarter plane:

$$L(f) = \frac{1}{\theta_1^2} \left(\int_0^\infty \frac{\partial f}{\partial x}(0, y) e^{-\theta_2 y} dy + \int_0^\infty \int_0^\infty \frac{\partial^2 f}{\partial x^2}(x, y) e^{-(\theta_1 x + \theta_2 y)} dx dy \right) \quad (\text{B.3})$$

$$= \frac{1}{\theta_2^2} \left(\int_0^\infty \frac{\partial f}{\partial y}(x, 0) e^{-\theta_1 x} dx + \int_0^\infty \int_0^\infty \frac{\partial^2 f}{\partial y^2}(x, y) e^{-(\theta_1 x + \theta_2 y)} dx dy \right) \quad (\text{B.4})$$

$$= \frac{1}{\theta_1 \theta_2} \int_0^\infty \int_0^\infty \frac{\partial^2 f}{\partial x \partial y}(x, y) e^{-(\theta_1 x + \theta_2 y)} dx dy. \quad (\text{B.5})$$

It is now sufficient to make the linear combination of the three above identities

$$\frac{1}{2} [\sigma_{11} \theta_1^2 (\text{B.3}) + \sigma_{22} \theta_2^2 (\text{B.4}) + 2\sigma_{12} \theta_1 \theta_2 (\text{B.5})],$$

in such a way that $\gamma(\theta)$ appears on one side and $\mathcal{G}(f)$ on the other side. We obtain

$$\begin{aligned} \gamma(\theta) L(f)(\theta) &= \int_0^\infty \int_0^\infty \mathcal{G}(f) e^{\theta_1 x + \theta_2 y} dx dy + \\ &\quad \frac{1}{2} \left[\sigma_{11} \int_0^\infty \frac{\partial f}{\partial x}(0, y) e^{-\theta_2 y} dy + \sigma_{22} \int_0^\infty \frac{\partial f}{\partial y}(x, 0) e^{-\theta_1 x} dx \right]. \end{aligned}$$

We conclude by using the equality $\mathcal{G}(f) = 0$. \square

B.2 Kernel

Our second task is to analyze the basic properties of the kernel (B.2). By studying this polynomial equation of degree two and noticing that $\det \Sigma \geq 0$ we easily find $\Theta_1(\theta_2)$ and $\Theta_2(\theta_1)$, the linear functions defined by $\gamma(\Theta_1(\theta_2), \theta_2) = \gamma(\theta_1, \Theta_2(\theta_1)) = 0$:

$$\Theta_1(\theta_2) = \frac{-\sigma_{12} \pm i\sqrt{\det \Sigma}}{\sigma_{11}}\theta_2, \quad \Theta_2(\theta_1) = \frac{-\sigma_{12} \pm i\sqrt{\det \Sigma}}{\sigma_{22}}\theta_1.$$

We note $k = (-\sigma_{12} + i\sqrt{\det \Sigma})/\sigma_{11} = \sqrt{\frac{\sigma_{22}}{\sigma_{11}}}e^{i\beta}$ and $l = (-\sigma_{12} + i\sqrt{\det \Sigma})/\sigma_{22} = \sqrt{\frac{\sigma_{11}}{\sigma_{22}}}e^{i\beta}$, where β is introduced in equation (1.27). The angle β is such that

$$\tan(\beta) = -\frac{\sqrt{\det \Sigma}}{\sigma_{12}}, \quad \cos(\beta) = -\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}, \quad \sin(\beta) = \sqrt{1 - \frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}}}.$$

With these notations one can rewrite the kernel as follows:

$$\gamma(\theta) = (\theta_1 - k\theta_2)(\theta_1 - \bar{k}\theta_2) = (\theta_2 - l\theta_1)(\theta_2 - \bar{l}\theta_1), \quad \forall \theta_1, \theta_2 \in \mathbb{C}.$$

B.3 Boundary value problem

The third point consists in finding, starting from the functional equation (B.1), explicit expressions for $L(\frac{\partial f}{\partial x}(0, \cdot))(\theta_2)$ and $L(\frac{\partial f}{\partial y}(\cdot, 0))(\theta_1)$, that we will note respectively $L_2(\theta_2)$ and $L_1(\theta_1)$. Then using the functional equation and the inverse Laplace transform we will find f . The scheme of the demonstration happens to be the same as presented in section 1.3: we are going to show that $L_1(\theta_1)$ satisfies a boundary value problem on a certain domain \mathcal{G}_{Θ_1} (which is the analogue of the domain $\mathcal{G}_{\mathcal{R}}$ in section 1.3.2.5), which eventually will lead to the following result.

Proposition 139 (Explicit expression for the Laplace transform). *The following function defines the Laplace transform of an harmonic function:*

$$L_1(\theta_1) = \frac{\mu_1}{\theta_1^{\pi/\beta}}, \tag{B.6}$$

where μ_1 is some non-zero constant.

Proof. Let us first introduce \mathcal{G}_{Θ_1} as the open set bounded by the curve $\Theta_1([0, \infty])$, see Figure B.1 for a graphical representation. Thanks to an analytic continuation, we shall extend the domain of definition of L_1 from $\{\theta_1 \in \mathbb{C} : \Re\theta_1 > 0\}$ to \mathcal{G}_{Θ_1} . On the set

$$\{(\theta_1, \theta_2) \in \mathbb{C}^2 : \Re\theta_1 > 0, \Re\theta_2 > 0, \gamma(\theta) = 0\}$$

we have $\sigma_{11}L_2(\theta_2) + \sigma_{22}L_1(\theta_1) = 0$. This means that $\sigma_{11}L_2(\theta_2) + \sigma_{22}L_1(k\theta_2) = 0$ and $\sigma_{11}L_2(\theta_2) + \sigma_{22}L_1(\bar{k}\theta_2) = 0$ for all θ_2 such as $\Re\theta_2 > 0$. Thanks to these relations,

- L_1 is now well defined and continuous on $\overline{\mathcal{G}_{\Theta_1}} \setminus \{0\}$ and analytic on \mathcal{G}_{Θ_1} (this is the analogue of Lemma 3 conditions (i) and (ii) in [85, p. 3155];
- $L_1(kt) = L_1(\bar{k}t)$ for all $t \in (0, \infty)$, which means that $L_1(\theta_1) = L_1(\bar{\theta}_1)$ for all θ_1 in the boundary of \mathcal{G}_{Θ_1} except at 0 (it is the analogue of Lemma 3 (iii) in [85, p. 3155]).

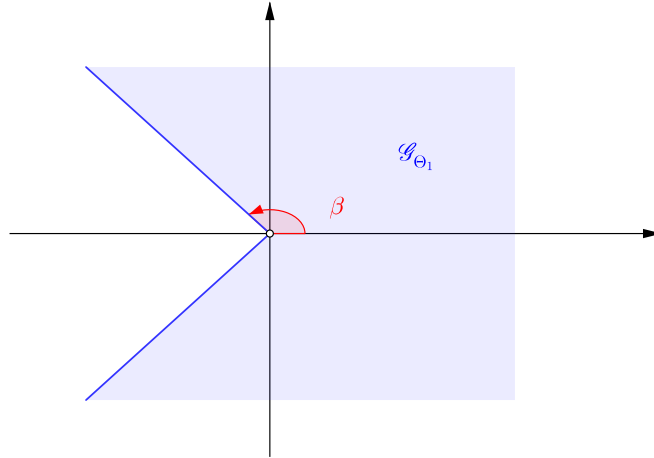


Figure B.1: The domain \mathcal{G}_{θ_1} is a cone of opening angle 2β

Therefore, we have shown that $L_1(\theta_1)$ satisfies the same boundary value problem as the one stated in Lemma 3 of [85].

To solve this boundary value problem we introduce the gluing function

$$w(\theta_1) = 1/\theta_1^{\pi/\beta}.$$

Obviously $w(\theta_1)$ is a conformal mapping from \mathcal{G}_{θ_1} to $\mathbf{C} \setminus \mathbf{R}_-$, which besides satisfies $w(kt) = w(\bar{k}t)$ for all $t \in (0, \infty)$. To avoid any problem at 0, rather than $L_1(\theta_1)$ we consider \mathcal{K} defined by

$$\mathcal{K}(\theta_1) = \frac{1}{L_1(\theta_1) - z_0}$$

for a given non-real z_0 . The function \mathcal{K} has a finite number q of poles on \mathcal{G}_{θ_1} noted x_ℓ , that we can cancel thanks to w and some polynomials Q_ℓ , according to the same procedure as in [85, Section 3.1]. It is possible to construct

$$D(\theta_1) = \sum_{\ell=1}^q Q_\ell \left(\frac{1}{w(\theta_1) - w(x_\ell)} \right)$$

such that $\mathcal{K}(\theta_1) - D(\theta_1)$ is well defined and continuous on $\overline{\mathcal{G}_{\theta_1}}$ (condition Lemma 4 (*ii*) in [85]) and is analytic on \mathcal{G}_{θ_1} (condition Lemma 3 (*i*)). Furthermore for all θ_1 on the boundary of \mathcal{G}_{θ_1} , $\mathcal{K}(\theta_1) - D(\theta_1) = \mathcal{K}(\bar{\theta}_1) - D(\bar{\theta}_1)$, i.e., $\mathcal{K}(kt) - D(kt) = \mathcal{K}(\bar{k}t) - D(\bar{k}t)$ for all $t \in [0, \infty)$ (condition Lemma 3 (*iii*)). With the key Lemma 4 in [85, p. 3158], we deduce that $\mathcal{K}(\theta_1) - D(\theta_1)$ is a constant function which is 0.

We readily deduce that $L_1(\theta_1) = z_0 + 1/S(w(\theta_1))$, where

$$S(X) = \sum_{\ell=1}^q Q_\ell \left(\frac{1}{X - w(x_\ell)} \right).$$

As in the article [85] it is possible to show that $1/S(X)$ must be a polynomial. We conjecture that the choice $\frac{1}{S} = X - z_0$, which leads to the formula (B.6), is the only one which leads to a positive harmonic function. \square

Similarly, one has the expression

$$L_2(\theta_2) = \frac{\mu_2}{\theta_2^{\pi/\beta}}. \tag{B.7}$$

Further, the constants μ_1 and μ_2 in (B.6) and (B.7) are related together: evaluating the functional equation at any point such that $\gamma(\theta) = 0$, we obtain

$$\mu_2 = \mu_1 \left(\frac{\sigma_{22}}{\sigma_{11}} \right)^{1-\pi/(2\beta)}.$$

For example, if we take Σ equal to the identity matrix and thus $\beta = \pi/2$, we find with the functional equation

$$L(f)(\theta) = \frac{1}{\theta_1^2 + \theta_2^2} \left(\frac{1}{\theta_1^2} + \frac{1}{\theta_2^2} \right) = \frac{1}{\theta_1^2 \theta_2^2},$$

which is the Laplace transform of the function $f(x, y) = xy$.

In the general case we find

$$L(f)(\theta) = \frac{\sigma_{11}\mu_1/\theta_1^{\pi/\beta} + \sigma_{22}\mu_2/\theta_2^{\pi/\beta}}{\sigma_{11}\theta_1^2 + 2\sigma_{12}\theta_1\theta_2 + \sigma_{22}\theta_2^2}.$$

Taking the inverse Laplace transform, the solution of the problem has the following form: let $\varphi(x, y) = (x', y')$, where the transformation φ is defined in [85, Eq 5.1]. We further take the notations $x' = \rho' \cos(\theta')$ and $y = \rho' \sin(\theta')$. Then we have

$$f(x, y) = \rho'^{\frac{\pi}{\beta}} \sin \left(\frac{\theta' \pi}{\beta} \right),$$

which indeed is the unique continuous harmonic function (see [85, Eq 5.2]), up to multiplicative constants.

Introduction aux problèmes frontière

Résumé

Ce chapitre est une brève introduction à la théorie des problèmes aux valeurs frontière et vise à donner une idée des méthodes et techniques employées pour les résoudre. Cette présentation suit dans l'ensemble les livres de Litvinchuk [74], Muskhelishvili [82] et Gakhov [48] qui font référence sur le sujet.

Abstract

This chapter is a brief introduction to the theory of boundary value problems and intends to give an idea of methods and techniques used for their resolution. This overall presentation follows the reference books on this subject of Litvinchuk [74], Muskhelishvili [82] and Gakhov [48].

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C.1 Formules de Sokhotski-Plemelj

Les formules de Sokhotski-Plemelj sont à la base de la résolution des problèmes frontière de Riemann. Soit \mathcal{L} un contour (ouvert ou fermé) lisse et orienté et $f \in \mathbb{H}_\mu(\mathcal{L})$ l'ensemble des fonctions μ -höldériennes sur \mathcal{L} pour $0 < \mu \leq 1$. Une fonction est dite sectionnellement holomorphe si elle est holomorphe sur tout le plan complexe privé de \mathcal{L} et admet une limite à droite et une limite à gauche sur \mathcal{L} (privé de ses éventuelles extrémités).

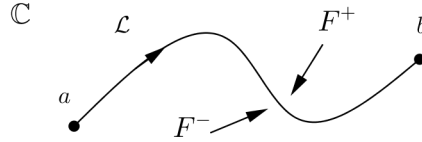


FIGURE C.1 : Courbe lisse ouverte orientée \mathcal{L} d'extrémités a et b et valeurs limites de F à droite et à gauche

Proposition 140 (Formules de Sokhotski-Plemelj). *La fonction*

$$F(z) := \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{f(t)}{t-z} dt, \quad z \notin \mathcal{L}$$

est sectionnellement holomorphe. Les fonction F^+ et F^- de \mathcal{L} prenant les valeurs limites de F respectivement à gauche et à droite vérifient les formules :

$$F^+(t) = \frac{1}{2}f(t) + \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{f(s)}{s-t} ds \quad \text{et} \quad F^-(t) = -\frac{1}{2}f(t) + \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{f(s)}{s-t} ds.$$

Ces formules sont équivalentes aux équations

$$F^+(t) - F^-(t) = f(t) \quad \text{et} \quad F^+(t) + F^-(t) = \frac{1}{i\pi} \int_{\mathcal{L}} \frac{f(s)}{s-t} ds.$$

Les intégrales ci-dessus sont prises au sens de la valeur principale.

Remarque 141 (Fonctions sectionnellement holomorphes pour une discontinuité donnée). Le théorème de Liouville montre que la fonction F définie ci-dessus est l'unique fonction sectionnellement holomorphe satisfaisant l'équation

$$\Phi^+(t) - \Phi^-(t) = f(t) \quad \forall t \in \mathcal{L}$$

et s'annulant en l'infini. Les solutions de cette équation de degré fini à l'infini sont donc de la forme

$$\Phi = F + P$$

où P est un polynôme.

Remarque 142 (Comportement aux extrémités). On peut de plus montrer que si \mathcal{L} est un contour ouvert orienté de l'extrémité a à l'extrémité b , alors au voisinage de l'extrémité c il existe $F_c(z)$, une fonction holomorphe au voisinage de c , telle que

$$F(z) = \frac{\epsilon_c}{2i\pi} f(c) \log(z-c) + F_c(z) \quad \text{où} \quad \epsilon_c = \begin{cases} -1 & \text{si } c = a, \\ 1 & \text{si } c = b. \end{cases} \quad (\text{C.1})$$

C.2 Problème frontière de Riemann

De manière standard un problème frontière est composé d'une condition de régularité sur un domaine et d'une condition frontière sur les bords de ce domaine.

Définition 143 (Problème frontière de Riemann). *On dit que Φ satisfait le problème frontière de Riemann sur \mathcal{L} si :*

- Φ est sectionnellement holomorphe sur $\mathbb{C} \setminus \mathcal{L}$ et admet Φ^+ pour limite à gauche et Φ^- pour limite à droite ; Φ est de degré fini à l'infini ;
- Φ satisfait la condition frontière

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \mathcal{L}$$

où G et g sont des fonctions définies sur \mathcal{L} .

On supposera ici que G et $g \in \mathbb{H}_\mu(\mathcal{L})$ et que G ne s'annule pas sur \mathcal{L} . Lorsque $g = 0$ on parle de problème de Riemann homogène.

C.2.1 Contour fermé

On suppose que le contour \mathcal{L} est fermé et on note \mathcal{L}^+ l'ouvert borné de frontière \mathcal{L} et \mathcal{L}^- le complémentaire de $\mathcal{L}^+ \cup \mathcal{L}$.

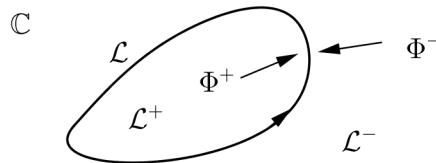


FIGURE C.2 : Courbe lisse fermée orientée \mathcal{L} , domaines \mathcal{L}^+ et \mathcal{L}^- et valeurs limites de Φ à droite et à gauche

Pour résoudre le problème frontière de Riemann nous devons introduire l'*indice*

$$\chi := \frac{1}{2i\pi} [\log G]_{\mathcal{L}} = \frac{1}{2\pi} [\arg G]_{\mathcal{L}}$$

qui quantifie la variation de l'argument de G sur le contour \mathcal{L} dans le sens positif. On suppose sans perdre en généralité que 0 est dans \mathcal{L}^+ . On peut ainsi définir sans problème la fonction mono-valuée

$$\log(t^{-\chi}G(t)), \quad t \in \mathcal{L}$$

qui satisfait la condition de Hölder.

Proposition 144 (Solution du problème de Riemann homogène sur un contour fermé). *Soit*

$$\Gamma(z) := \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\log(t^{-\chi}G(t))}{t-z} dt, \quad z \notin \mathcal{L}.$$

On définit la solution fondamentale du problème frontière de Riemann homogène $X^+(t) = G(t)X^-(t)$ pour $t \in \mathcal{L}$ par

$$X(z) = \begin{cases} \exp \Gamma(z), & z \in \mathcal{L}^+, \\ z^{-\chi} \exp \Gamma(z), & z \in \mathcal{L}^-. \end{cases}$$

La fonction X est de degré $-\chi$ à l'infini. Si Φ est une solution quelconque du problème de Riemann homogène alors $\Phi(z) = X(z)P(z)$ où P est un polynôme.

Si on note k le degré de P , la solution Φ est de degré $k - \chi$ à l'infini. La solution fondamentale X de degré χ est donc la solution homogène non nulle de plus petit degré à l'infini.

Démonstration. Les formules de Sokhotski-Plemelj appliquées à Γ montrent que, en notant pour $t \in \mathcal{L}$, $\tilde{\Gamma}(t) = \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\log(s^{-\chi}G(s))}{s-t} ds$ (au sens de la valeur principale), pour $t \in \mathcal{L}$ on a

$$X^+(t) = e^{\tilde{\Gamma}(t)} \sqrt{t^{-\chi}G(t)} \quad \text{et} \quad X^-(t) = t^{-\chi} e^{\tilde{\Gamma}(t)} \frac{1}{\sqrt{t^{-\chi}G(t)}}$$

et donc que X est solution du problème homogène. Si Φ est une solution quelconque alors comme $X^\pm(z) \neq 0$ pour $z \in \mathcal{L}$, on obtient

$$\frac{\Phi^+}{X^+}(z) = \frac{\Phi^-}{X^-}(z), \quad z \in \mathcal{L}.$$

Par prolongement analytique la fonction $\frac{\Phi}{X}$ est donc holomorphe dans le plan complexe tout entier, de degré fini à l'infini et est donc un polynôme par le théorème de Liouville. \square

Proposition 145 (Solution du problème de Riemann sur un contour fermé). *On définit*

$$\Psi(z) := \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{g(t)}{X^+(t)(t-z)} dt, \quad z \notin \mathcal{L}.$$

Les solutions du problème frontière de Riemann sont de la forme

$$\Phi(z) = X(z)\Psi(z) + X(z)P(z)$$

où P est un polynôme.

Démonstration. La solution fondamentale X^\pm ne s'annule pas sur \mathcal{L} et on a la factorisation $G = \frac{X^+}{X^-}$. Si Φ est solution du problème frontière on a alors

$$\frac{\Phi^+}{X^+}(t) = \frac{\Phi^-}{X^-}(t) + \frac{g}{X^+}(t), \quad t \in \mathcal{L}.$$

La fonction $\frac{\Phi}{X}$ étant de degré fini à l'infini, la remarque 141 donne alors $\frac{\Phi}{X} = \Psi + P$. \square

C.2.2 Contour ouvert

On suppose que la fonction Φ recherchée satisfait le problème frontière de Riemann sur un contour ouvert orienté de l'extrémité a vers l'extrémité b et que Φ est bornée au voisinage de a et b . De manière plus générale on pourrait chercher les solutions admettant des singularités intégrables aux extrémités. On note δ , Δ , ρ_a et ρ_b tels que

$$G(a) = \rho_a e^{i\delta}, \quad \Delta = [\arg G]_{\mathcal{L}} \quad \text{et} \quad G(b) = \rho_b e^{i(\delta+\Delta)}$$

en choisissant $-2\pi < \delta \leq 0$ et la détermination correspondante du logarithme $\log G$. On définit l'indice

$$\chi := \left\lfloor \frac{\delta + \Delta}{2\pi} \right\rfloor.$$

Proposition 146 (Solution du problème de Riemann sur un contour ouvert). *Soit*

$$\Gamma(z) := \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\log(G(t))}{t-z} dt, \quad z \notin \mathcal{L}.$$

La fonction

$$X(z) := (z-b)^{-\chi} e^{\Gamma(z)}$$

est solution du problème de Riemann homogène et est bornée aux extrémités. Cette solution est d'ordre $-\chi$ à l'infini. Si Φ est une solution quelconque du problème homogène elle s'écrit $\Phi(z) = X(z)P(z)$ où P est un polynôme. On définit

$$\Psi(z) := \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{g(t)}{X^+(t)(t-z)} dt, \quad z \notin \mathcal{L}.$$

Les solutions du problème frontière de Riemann non-homogène bornées aux extrémités sont de la forme

$$\Phi(z) = X(z)\Psi(z) + X(z)P(z)$$

où P est un polynôme.

Démonstration. D'après la remarque 142 au voisinage d'une extrémité c on a

$$e^{\Gamma(z)} = (z-c)^{\lambda_c} e^{\Gamma_c(z)}$$

pour Γ_c une fonction holomorphe au voisinage de c et

$$\lambda_a = -\frac{\delta}{2\pi} + i\frac{\log \rho_a}{2\pi} \quad \text{et} \quad \lambda_b = \frac{\delta + \Delta}{2\pi} - i\frac{\log \rho_b}{2\pi}.$$

Comme $\delta < 0$ la fonction $e^{\Gamma(z)}$ est bornée en a . De plus on remarque que la fonction $X(z) = (z-b)^{-\chi} e^{\Gamma(z)}$ est bornée en b (et en a). Le reste de la démonstration est similaire au cas du contour fermé. \square

C.3 Problème frontière de Carleman

Le problème frontière de Carleman est un problème frontière avec un décalage (*shift*) noté $\alpha(t)$ qui est un homéomorphisme du contour \mathcal{L} sur lui-même tel que sa dérivée ne s'annule pas et qui satisfait la condition de Hölder. La plupart du temps la condition $\alpha(\alpha(t)) = t$ est vérifiée et on dit que α est un automorphisme de Carleman de \mathcal{L} . Dans cette thèse la fonction de décalage est la conjugaison complexe.

Définition 147 (Problème frontière de Carleman). *On dit que Φ satisfait le problème frontière de Carleman sur le contour \mathcal{L} fermé (ou ayant ses deux extrémités à l'infini, comme dans cette thèse) si :*

- Φ est holomorphe dans tout le domaine \mathcal{L}^+ borné par \mathcal{L} et continu sur \mathcal{L} ;
- Φ satisfait la condition frontière

$$\Phi(\alpha(t)) = G(t)\Phi(t) + g(t), \quad t \in \mathcal{L}$$

où G et g sont des fonctions définies sur \mathcal{L} .

On supposera ici que G et $g \in \mathbb{H}_\mu(\mathcal{L})$ et que G ne s'annule pas sur \mathcal{L} . Lorsque $g = 0$ on parle de problème de Riemann homogène.

Pour résoudre le problème de Carleman l'idée est d'introduire une fonction de collage conforme. Le théorème suivant établit l'existence de telles fonctions.

Proposition 148 (Fonction de collage conforme). *Soit α un automorphisme de Carleman de la courbe \mathcal{L} . Il existe w une fonction*

- holomorphe sur \mathcal{L}^+ privé d'un point où w a un pôle simple ;
- vérifiant la condition de collage

$$w(\alpha(t)) = w(t), \quad t \in \mathcal{L}.$$

Un tel w établit une transformation conforme (bijection holomorphe) de \mathcal{L}^+ vers le plan complexe privé d'un arc lisse ouvert \mathcal{M} . Cette fonction de collage conforme admet deux points fixes A et B d'image a et b qui sont les extrémités de \mathcal{M} .

Si l'on trouve une telle fonction de collage conforme, on peut ramener le problème frontière de Carleman à un problème de Riemann. On oriente \mathcal{M} de a vers b en les choisissant de manière à ce que l'orientation de \mathcal{L} soit conservée par w . On note alors w^{-1} l'inverse de w et $(w^{-1})^+$ sa limite à gauche et $(w^{-1})^-$ sa limite à droite sur \mathcal{M} . Voir la figure C.3. Pour t sur l'arc \mathcal{L} orienté positivement de B vers A , ces fonctions vérifient

$$(w^{-1})^+(t) = \alpha(t) \quad \text{et} \quad (w^{-1})^-(t) = t.$$

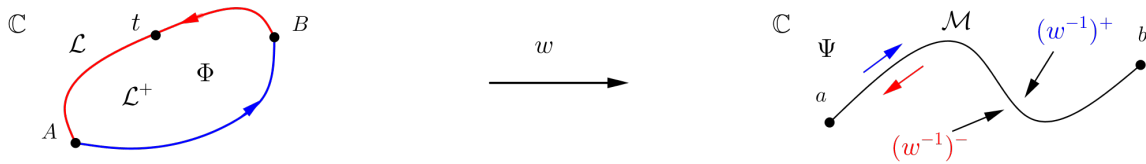


FIGURE C.3 : Fonction de collage conforme de \mathcal{L}^+ vers $\mathbb{C} \setminus \mathcal{M}$

Soit Φ une solution du problème de Carleman, on définit la fonction Ψ telle que

$$\Psi(w(z)) := \Phi(z), \quad z \in \mathcal{L}^+.$$

On a alors

$$\Psi(z) = \Phi(w^{-1}(z)), \quad z \in \mathbb{C} \setminus \mathcal{M}$$

et les limites à gauche et à droite de Ψ sur \mathcal{M} sont

$$\Psi^+(t) = \Phi((w^{-1})^+(t)) \quad \text{et} \quad \Psi^-(t) = \Phi((w^{-1})^-(t)), \quad t \in \mathcal{M}.$$

Proposition 149. *La fonction Ψ satisfait le problème de Riemann associé à l'arc \mathcal{M} et aux fonctions $G((w^{-1})^-(t))$ et $g((w^{-1})^-(t))$:*

- Ψ est sectionnellement holomorphe sur $\mathbb{C} \setminus \mathcal{M}$;
- Ψ satisfait la condition frontière

$$\Psi^+(t) = G((w^{-1})^-(t))\Phi^-(t) + g((w^{-1})^-(t)), \quad t \in \mathcal{M}.$$

Démonstration. La preuve découle directement de la définition 147, de la proposition 148 et des notations ci-dessus. \square

Comme $\Phi = \Psi \circ w$, pour résoudre le problème de Carleman il suffit donc de déterminer la fonction de collage conforme w et de trouver Ψ en résolvant grâce à la section C.2 le problème de Riemann de la proposition 149.

Réseaux de files d'attente

Résumé

Historiquement, la motivation initiale pour étudier le mouvement Brownien réfléchi dans des orthants est due au fait qu'il sert d'approximation aux grands réseaux de files d'attente. Cette annexe présente brièvement un théorème limite en trafic dense qui établit que lorsque l'intensité du trafic tend vers 1, la longueur de la file d'attente, bien renormalisée, converge vers le mouvement Brownien réfléchi dans le quadrant. On se réfère à l'article de Reiman [86].

Abstract

Historically, the initial motivation for the study of reflected Brownian motion in the orthant is because it serves of approximation to large queueing networks. This appendix briefly presents some heavy traffic limit theorem that states that when the network traffic intensity converges to 1, the queue length, properly normalized, converges to reflected Brownian motion in the quadrant. We refer to the article of Reiman [86].

Pour illustrer le lien entre le mouvement Brownien réfléchi et les réseaux de files d'attente on considère deux guichets inter-connectés représentés sur la figure D.1. Les clients arrivent au guichet 1 (resp. 2) avec un processus de renouvellement ayant des temps d'inter-arrivée i.i.d. d'espérance $\frac{1}{\lambda_1}$ (resp. $\frac{1}{\lambda_2}$) et de variance v_1 (resp. v_2). Les temps de service au guichet 1 (resp. 2) sont d'espérance $\frac{1}{\nu_1}$ (resp. $\frac{1}{\nu_2}$) et de variance s_1 (resp. s_2) et sont i.i.d.. Une fois servi, un client du guichet 1 (resp. 2) passe dans la file d'attente du guichet 2 (resp. 1) avec probabilité q_{12} (resp. q_{21}) et quitte le système avec probabilité $1 - q_{12}$ (resp. $1 - q_{21}$).

On suppose que $q_{12}q_{21} < 1$, ce qui implique que tous les clients finissent par quitter le système. On note $q_{11} = q_{22} = 0$, on définit la matrice $Q = (q_{ij})_{i,j=1,2}$ et on pose

$$(\gamma_1, \gamma_2) = (\lambda_1, \lambda_2)(I - Q)^{-1}.$$

On dit que γ_1 (resp. γ_2) est le taux d'entrée net dans la file du guichet 1 (resp. 2). On définit aussi l'intensité du trafic $\rho_i = \frac{\gamma_i}{\nu_i}$ pour $i = 1, 2$. Le système est stable si $\rho_1 < 1$ et $\rho_2 < 1$. Le trafic est dit dense lorsque son intensité ρ_i est proche de 1. On définit enfin N_1 (resp. N_2) le nombre de clients présents dans la file du guichet 1 (resp. 2).

Afin d'énoncer un théorème limite on considère maintenant une suite, indexée par n , de réseaux de files d'attente introduits ci-dessus ayant pour paramètres $\lambda_i^{(n)}, \nu_i^{(n)}, v_i^{(n)}$ et

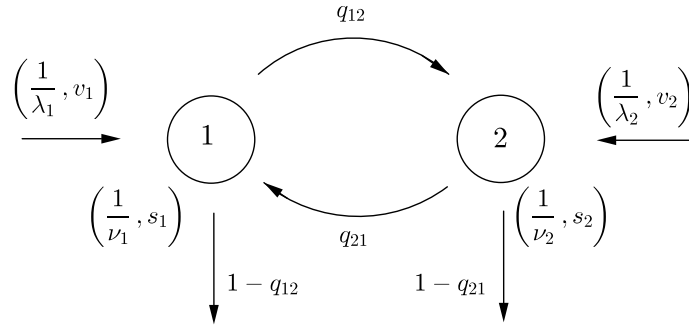


FIGURE D.1 : Un réseau de deux files d'attente

$s_i^{(n)}$. On suppose que Q est indépendant de n . On note $N^{(n)}$ le vecteur du nombre de clients dans les files.

On suppose que le supremum des moments d'ordre 3 de l'amplitude des temps d'inter-arrivée et des temps de service sont finis, voir les conditions (25) et (26) de l'article de Reiman [86].

Proposition 150 (Théorème limite d'approximation du mouvement Brownien réfléchi).
Pour $i = 1, 2$, on suppose que lorsque n tend vers l'infini

$$\lambda_i^{(n)} \rightarrow \lambda_i, \quad \nu_i^{(n)} \rightarrow \nu_i, \quad \nu_i^{(n)} \rightarrow \nu_i, \quad s_i^{(n)} \rightarrow s_i$$

et que

$$\sqrt{n}(\lambda_i^{(n)} - \nu_i^{(n)}) \rightarrow \mu_i,$$

toutes ces limites étant des constantes finies. On a le résultat limite suivant :

$$\frac{1}{\sqrt{n}}N^{(n)}(nt) \xrightarrow[n \rightarrow \infty]{} Z_t, \quad t \geq 0$$

où Z_t est un mouvement Brownien réfléchi dans le quadrant de dérive μ , de matrice de réflexion $R = I - Q$ et de covariance $\Sigma = (\sigma_{ij})$ avec

$$\sigma_{11} = \lambda_1^3 \nu_1 + \nu_1^3 s_1 + \nu_2 q_{21} (1 - q_{21} + q_{21} \mu_2^2 s_2), \quad \sigma_{22} = \lambda_2^3 \nu_2 + \nu_2^3 s_2 + \nu_1 q_{12} (1 - q_{12} + q_{12} \mu_1^2 s_1)$$

et

$$\sigma_{12} = \sigma_{21} = -(\nu_1^3 s_1 q_{12} + \nu_2^3 s_2 q_{21}).$$

On se réfèrera à l'article de Reiman [86] pour un résultat complet et plus détaillé.

Bibliographie

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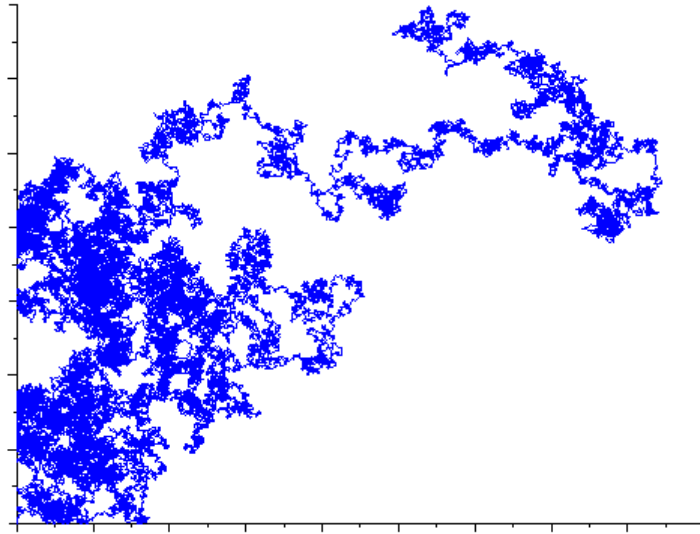
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Un exemple de trajectoire du mouvement Brownien réfléchi dans le quart de plan

An example of path of reflected Brownian motion in the quadrant

Résumé

Le mouvement Brownien réfléchi de manière oblique dans le quadrant, introduit par Harrison, Reiman, Varadhan et Williams dans les années 80, est un objet largement analysé dans la littérature probabiliste. Cette thèse, qui présente l'étude complète de la mesure invariante de ce processus dans tous les cônes du plan, a pour objectif plus global d'étendre au cadre continu une méthode analytique développée initialement pour les marches aléatoires dans le quart de plan par Fayolle, Iasnogorodski et Malyshev dans les années 70. Cette approche est basée sur des équations fonctionnelles, reliant des fonctions génératrices dans le cas discret et des transformées de Laplace dans le cas continu. Ces équations permettent de déterminer et de résoudre des problèmes frontière satisfaits par ces fonctions génératrices. Dans le cas récurrent, cela permet de calculer explicitement la mesure invariante du processus avec rebonds orthogonaux, dans le chapitre 2, et avec rebonds quelconques, dans le chapitre 3. Les transformées de Laplace des mesures invariantes sont prolongées analytiquement sur une surface de Riemann induite par le noyau de l'équation fonctionnelle. L'étude des singularités et l'application de méthodes du point col sur cette surface permettent de déterminer l'asymptotique complète de la mesure invariante selon toutes les directions dans le chapitre 4.

Mots clés. Mouvement Brownien réfléchi dans les cônes ; Distribution stationnaire ; Transformée de Laplace ; Fonctions génératrices ; Méthode des invariants de Tutte ; Problème de valeur frontière ; Transformation conforme ; Analyse asymptotique ; Méthode du point col ; Surface de Riemann.

Abstract

Obliquely reflected Brownian motion in the quadrant, introduced by Harrison, Reiman, Varadhan and Williams in the eighties, has been studied a lot in the probabilistic literature. This thesis, which presents the complete study of the invariant measure of this process in all the cones of the plan, has for overall aim to extend to the continuous framework an analytic method initially developed for random walks in the quarter plane by Fayolle, Iasnogorodski and Malyshev in the seventies. This approach is based on functional equations which link generating functions in the discrete case and Laplace transform in the continuous case. These equations allow to determine and to solve boundary value problems satisfied by these generating functions. In the recurrent case, it permits to compute explicitly the invariant measure of the process with orthogonal reflexions, in the chapter 2, and with any reflexions, in the chapter 3. The Laplace transform of the invariant measure is analytically extended to a Riemann surface induced by the kernel of the functional equation. The study of singularities and the use of saddle point methods on this surface allows to determine the full asymptotics of the invariant measure along every directions in the chapter 4.

Key words. Reflected Brownian motion in cones ; Stationary distribution ; Laplace transform ; Generating function ; Tutte's invariant approach ; Boundary value problem ; Conformal mapping ; Asymptotic analysis, Saddle-point method ; Riemann surface.