Compactifications hétérotiques avec flux

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Table of contents

Part A

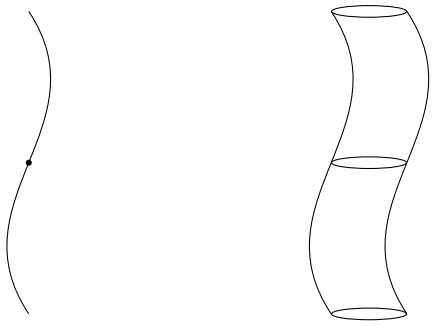
Introduction

Chapter I

Broad Introduction

We start with a very broad overview in order to provide general motivations for the questions adressed later on in the present thesis.

String theory proposes to replace the notion of point particle and the 'worldline' they follow when evolving in time by a one dimensional object, the string, which can be open or closed, and the 'worldsheet' it sweaps out in a d-dimensional spacetime M. One then studies the motion of this string in some spacetime, i.e. the dynamics of maps X^μ describing the embedding of the worldsheet into spacetime. String theory therefore consists in a physical theory for the fields living on the worldsheet, and its properties are closely connected to the spacetime physics through the embedding maps X^μ. In particular, the properties required for the spacetime physics impose severe constraints on the type of theory living on the worldsheet, which will therefore be a particular type of quantum field theory, namely a conformal field theory. Such a



proposal appeared in the late 60's as an attempt to explain the dynamics of hadrons subject to the strong interaction. Various inherent features of the theory, such as the unavoidable presence of a massless spin-two excitation, not present in the hadronic bestiary, appeared however as fatal problems in trying to describe solely

the strong interaction. These problems combined with the development of Quantum Chromodynamics in the beginning of the 70's and its quick successes were enough to discard String Theory as a theory of the strong interaction.

It was understood only later that if the typical string scale is close to the Plank scale, one could understand the massless spin-two excitation as a graviton, particle mediating the gravitational interaction at the quantum level. It was then shown that this massless excitation interacted in a way consistent with the covariance laws of General Relativity, hence one could hope that String Theory may actually describe a consistent quantum theory of gravity.

As mentioned above, String Theory contains closed and open strings, and consists in a two-dimensional sigma model defined on the worldsheet spanned by the string, valued in some target spacetime, and exhibiting conformal invariance for consistency of the theory. In the Ramond-Neveu-Schwarz formalism, this two-dimensional sigma model is required also to admit some amount of supersymmetry in order to contain fermionic excitations in spacetime. Moreover, purely bosonic String Theory contains in its spectrum a tachyon which tends to signal an instability in the theory, and which is not present for the superstring. Requiring consistency of the theory at the quantum level, namely that the conformal invariance of the sigma model is non-anomalous, imposes the dimension of the target spacetime of the superstring to be ten, the purely bosonic string being defined in twenty-six spacetime dimensions.

The properties of the various fundamental particles observed in Nature are understood as the various oscillation modes of strings. In the limit in which the energies are much smaller than the typical string scale, the strings indeed appear as point-like particle, and the lowest energy string excitations can be given a description in terms of a low-energy effective quantum field theory.

The above mentioned graviton appears in the excitation spectrum of the closed string. On the other hand, the open string spectrum naturally contains an abelian gauge field. If one also attaches so-called Chan-Paton factors to the extremities of the open string, then one can incorporate also non-abelian interactions, and hence potentially reproduce the Standard Model or GUT gauge group.

There are five consistent String Theories: the type I, type IIA and IIB, and the $\mathrm{Spin}(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ heterotic strings. The two heterotic string theories contain only closed strings, as well as the two type II string theories in absence of D-branes. The idea behin the construction of the heterotic strings involves exploiting the fact that the right and left-moving sectors of the closed string are basically independent

of one another, allowing to define a hybrid theory in which the right-moving sector is supersymmetric, and the left-moving one is that of the bosonic string. Even though these theories only contain closed strings, they do include non-abelian gauge fields as part of their spectrum. The heterotic string may be considered as the most economical way to implement both supersymmetry and gauge symmetry in spacetime. The various string theories introduced above all lead to a low energy supergravity effective action in 10 dimensions, with minimal supersymmetry for the type I and heterotic theories, extended $\mathcal{N} = 2$ for the type II theories, and are actually all connected to each other by various perturbative and non-perturbative duality transformations, hence forming a web of theories which then appears as various perturbative limits of a single eleven-dimensional theory called M-theory.

In order to make contact with four-dimensional physics, one looks at string backgrounds whose spacetime consists in the cartesian product of, say, four-dimensional Minkowski spacetime and some compact manifold whose characteristic length scale is much smaller than the typical length scales probed in particle physics experiments. The main drawback of such a compactification approach is that the freedom related to the shape and size of the internal manifold leads in the four-dimensional spacetime to the presence a collection of massless scalar fields, whose presence is not relevant from a phenomenological point of view.

One direction in order to tackle this problem is to turn on fluxes, namely to allow for a non-trivial profile along the internal manifold for the fields corresponding to the various massless string excitations. This typically fixes at least part of the compactification moduli.

The goal of the thesis will be to study various aspects of heterotic compactifications with fluxes turned on. Gauge fluxes have already been considered quite extensively in the context of compactifications on a particular type of compact space corresponding to Calabi-Yau manifolds. We will consider in this thesis compactifications which are also characterized by a non-trivial NS-NS flux, which is part of the gravitational excitation sector of the closed string, and whose geometric interpretation is that of a non-trivial torsion on the compactification manifold. One of the main advantages of the heterotic string is that it allows for a purely worldsheet description of fluxes, contrary to type II which should also contains Ramond-Ramond fluxes, which do not admit a known description at the worldsheet level.

After introducing tools common to the various parts of the thesis (part A ch. A.II), we will in a first part (part B) be interested in a large class of heterotic

torsional compactifications known in the litterature as Fu-Yau compactifications. We will define and compute a natural object associated to these compactification, the *dressed elliptic genus*, which, as we will see, captures various properties related to the topology of the compactification manifold and the gauge bundle over it. We will also give a purely mathematical definition independent of any sigma model with target the geometry of interest, and which naturally generalizes the elliptic genus of a holomorphic vector bundle over a Calabi-Yau manifold to holomorphic vector bundles over the non-Kähler total space of a two-torus principal fibration over a Calabi-Yau d-fold.

In a second part (part C) we will use the results of the first part to compute one-loop corrections to various couplings in the low-energy effective action corresponding to these Fu-Yau compactifications. These so-called *threshold corrections* will be expressed in two different forms in order to discuss different physical aspects.

Finally in a last part (part D), we will introduce and describe new non-compact heterotic supergravity solutions generalizating already known torsional solution such as warped Eguchi-Hanson or warped orbifoldized resolved conifold.

This Ph.D. thesis led to three articles:

- *New supersymmetric index of heterotic compactifications with torsion*, with Dan Israël, arXiv:1509.05704, JHEP 1512 (2015) 069,
- *Dressed elliptic genus of heterotic compactifications with torsion and general bundles*, with Dan Israël, arXiv:1606.08982, JHEP 1608 (2016) 176,
- *Threshold corrections in heterotic flux compactifications*, with Carlo Angelantonj and Dan Israël, arXiv:1611.09442,

which mainly correspond to chapter B.II, chapter B.III and chapter C.II respectively, as well as to some yet unpublished work:

- *Heterotic Flux Solutions From Sasaki-Einstein Manifolds*, with Nick Halmagyi, Dan Israël and Eirik Eik Svanes,

corresponding to chapter D.II.

Chapter II

Heterotic supergravity and BPS equations

II.1 Heterotic supergravity action

Let us introduce the supergravity theory obtained in the low-energy limit of the heterotic string. It will allow for a neat derivation of the BPS system of equations.

The theory consists in 10-dimensional $\mathcal{N} = 1$ supergravity theory coupled to $\mathcal{N} = 1$ super Yang-Mills with $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $E_8 \times E_8$ gauge group. The allowed gauge groups are fixed by the requirement of vanishing of the gauge and gravitational anomalies through the Green-Schwarz mechanism.

The bosonic part of the action describes the dynamics of a metric tensor g, a rank-two antisymmetric Kalb-Ramond B-field, a dilaton ϕ and a gauge connection A whose curvature we denote F, and is given in string frame by the following expression:

$$S[g, B, \phi, A] = \int_M \mathrm{vol}_g \, e^{-2\phi} \left\{ \mathcal{R}(g) + 4|\mathrm{d}\phi|^2 - \frac{1}{2}|H|^2 + \frac{\alpha'}{4}\left(|R|^2 - |F|^2\right) \right\}, \tag{A.II.1.1}$$

where $\mathcal{R}(g)$ denotes the scalar curvature of g, and the higher derivative $|R|^2$ term involves the curvature of a connection ∇ on the tangent bundle T_X and is required by quantum consistency of the theory, namely the vanishing of the Lorentz and gauge anomaly forms through the Green-Schwarz mechanism. F and R are taken

to be anti-Hermitian:

$$F = \mathrm{d}A + A \wedge A\,, \tag{A.II.1.2a}$$

$$R = \mathrm{d}\nabla + \nabla \wedge \nabla\,. \tag{A.II.1.2b}$$

The presence of this $|R|^2$ term in the action and the precise choice of spin connection will be explained in more details when we deal with the worldsheet theory, in which the Green-Schwarz mechanism corresponds to a compensation between the non-classically gauge invariant action of the two-dimensional theory and a one-loop anomaly.

Here and in the following, the norm of a p-form ω is defined by:

$$|\omega|^2 = \frac{1}{p!} g^{\mu_1\nu_1} \dots g^{\mu_p\nu_p} \omega_{\mu_1\dots\mu_p} \omega_{\nu_1\dots\nu_p}\,. \tag{A.II.1.3}$$

The norms of the Lie algebra valued two-forms R and F are computed with respect to the Killing form $-\,\mathrm{tr}$ (in the vector representation, see below). The field strength H of the Kalb-Ramond B-field actually also receives an α' correction due to the requirement of anomaly cancellation, and reads locally:

$$H = \mathrm{d}B + \frac{\alpha'}{4}(\mathrm{CS}(\nabla) - \mathrm{CS}(A))\,, \tag{A.II.1.4}$$

where the Chern-Simons 3-forms, defined by:

$$\mathrm{CS}(\nabla) = \mathrm{tr}\left(\nabla \wedge \mathrm{d}\nabla + \frac{2}{3}\nabla \wedge \nabla \wedge \nabla\right)\,, \tag{A.II.1.5a}$$

$$\mathrm{CS}(A) = \mathrm{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right)\,, \tag{A.II.1.5b}$$

are such that:

$$\mathrm{dCS}(\nabla) = \mathrm{tr}R \wedge R \tag{A.II.1.6a}$$

$$\mathrm{dCS}(A) = \mathrm{tr}F \wedge F\,. \tag{A.II.1.6b}$$

The non-standard behaviour under spacetime Lorentz and gauge transformations of the B-field whose field strengh is given by eq. (A.II.1.4) is known as the Green-Schwarz mechanism. We will discuss it again from a worldsheet sigma model point of view in the following, B.I.1, context in which this Green-Schwarz mechanism will play a major role in the rest of the thesis.

Even though the connection B is only defined locally, the expression eq. (A.II.1.4)

leads to the globally defined heterotic Bianchi identity:

$$\mathrm{d}H = \frac{\alpha'}{4}(\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F)\,. \tag{A.II.1.7}$$

The gauge trace naturally appearing in the above expression is actually $\frac{1}{30}\mathrm{Tr}$, where Tr is the trace in the 496-dimensional adjoint representation of the gauge group. In the case of $\mathrm{Spin}(32)/\mathbb{Z}_2$ it is related to the trace in the 32-dimensional representation tr by the relation[1]:

$$\mathrm{Tr}(F \wedge F) = 30\,\mathrm{tr}(F \wedge F)\,. \tag{A.II.1.8}$$

Moreover, the same notation is used in the $E_8 \times E_8$ case even though the latter does not admit a 32-dimensional representation, hence defining tr in that case.

Let us denote for an arbitrary p-form F by $F \vee F$ the following symmetric rank two tensor:

$$(F \vee F)_{M_1 N_1} = g^{M_2 N_2} \ldots g^{M_p N_p} F_{M_1 M_2 \ldots M_p} F_{N_1 N_2 \ldots N_p} \tag{A.II.1.9}$$

Extremization of the above supergravity/Yang-Mills action with respect to the field configurations leads to the following set of equations of motion:

$$\mathrm{Ric}(g) - 2\nabla^{\mathrm{LC}}(\mathrm{d}\phi) - \frac{1}{4}H \vee H - \frac{\alpha'}{4}\,(R \vee R - F \vee F) = 0\,, \tag{A.II.1.10a}$$

$$\mathcal{R}(g) - 4\Delta\phi - 4|\mathrm{d}\phi|^2 - \frac{1}{2}|H|^2 + \frac{\alpha'}{4}\left(|R|^2 - |F|^2\right) = 0\,, \tag{A.II.1.10b}$$

$$\mathrm{d}^\dagger\left(e^{2\phi}H\right) = 0\,, \tag{A.II.1.10c}$$

$$\mathrm{d}_A^\dagger\left(e^{2\phi}F\right) + \frac{1}{2}\,e^{2\phi} \star (F \wedge \star H) = 0\,, \tag{A.II.1.10d}$$

where the dagger denotes the adjoint with respect to the metric, ∇^{LC} the Levi-Civita connection, and d_A the gauge covariant exterior derivative which is locally written $\mathrm{d}_A = \mathrm{d} + A$ and $\star$ the Hodge duality operation defined in terms of the metric g. As we will see later, these equations correspond to the lowest order approximation in an expansion in the Regge slope α'. These equations are therefore corrected by 'stringy' contributions, beyond the point-like approximation, corresponding to including loop diagrams in the worldsheet sigma-model. α' being a dimensionful parameter, carrying dimension $(\mathrm{lenght})^2$, a dimensionless parameter can be defined using a typical length scale in the geometry l as α'/l^2. When the Hull connection ∇^{H} is chosen on the tangent bundle, cf. end of section A.II.2, these equations of motion

[1] More generically $\mathrm{Tr}(F \wedge F) = (n-2)\,\mathrm{tr}(F \wedge F)$ for $SO(n)$.

start receiving corrections at order $\mathcal{O}(\alpha')$. Different choices of connection correspond to field redefinitions in the supergravity theory, or different regularization schemes in the worldsheet theory, and lead to $\mathcal{O}(\alpha')$ corrections to the equations of motion eq. (A.II.1.10). We refer the reader to the next section and to the litterature for more details on the subtelties concerning this choice of connection [1–7].

II.2 Compactification and BPS conditions

In order to make contact with 4-dimensional physics, one has to consider spacetime topologies taking the form of a cartesian product of a 4-dimensional non-compact maximally symmetric spacetime and a 6-dimensional internal manifold X:

$$M = M_4 \times X \,, \tag{A.II.2.1}$$

In the context of heterotic compactifications, the additional data of the gauge bundle should also be dealt with by specifying a vector bundle E over X, whose structure group we embbed into one of E_8 factors, the remaining 4-dimensional spacetime gauge group corresponding to the commutant of the structure group of E inside E_8, times the unbroken E_8 factor.

The question raised by Strominger and Hull [8,9] some time ago is to determine the properties which should be satisfied by the internal (X, E) data so that one preserves at least $\mathcal{N} = 1$ supersymmetry in spacetime. Let us briefly recall the answer to this question.

4-dimensional Poincaré invariance actually allows to consider warped products instead of simply cartesian products, namely settings in which the spacetime metric comes with a warp factor depending on the internal coordinates. In Einstein frame, the metric is therefore written:

$$ds^2 = e^{2\Delta(x)} \left(ds^2(M_4) + ds^2(X) \right) \,, \tag{A.II.2.2}$$

where x denotes the internal coordinates, and $\Delta(x)$ some warp factor. In the following, greek letters will denote external spacetime indices, latin letters will denote internal indices and capital latin letters denote 10-dimensional indices. The requirement that M_4 is a maximally symmetric spacetime imposes that gauge and torsion fluxes and derivative of the dilaton field do not have legs along the external direc-

tions:

$$F_{\mu a} = F_{\mu\nu} = 0\,, \tag{A.II.2.3a}$$

$$H_{\mu ab} = H_{\mu\nu a} = H_{\mu\nu\rho} = 0\,, \tag{A.II.2.3b}$$

$$\partial_\mu \phi = 0\,, \tag{A.II.2.3c}$$

but nothing prevents them from exhibiting a non-trivial profile on the internal manifold X.

A supersymmetric vacuum is left invariant by at least one supercharge. Since in a classical vacuum the fermionic fields have a vanishing vacuum expectation value, one needs only to ensure that the variation of the fermionic fields vanish in the vacuum, since the variation of bosonic fields contain fermionic fields. The spinorial fields entering the heterotic supergravity action are the gravitino, the dilatino and the gaugino. Under a local supersymmetry transformation of Grassmanian parameter ϵ, they transform as:

$$\text{Gravitino variation: } \nabla^{\mathrm{B}}_M \epsilon = 0\,, \tag{A.II.2.4a}$$

$$\text{Dilatino variation: } \left(\not\nabla \phi - \frac{1}{2} H\right) \epsilon = 0\,, \tag{A.II.2.4b}$$

$$\text{Gaugino variation: } F\epsilon = 0\,. \tag{A.II.2.4c}$$

As mentioned already, the choice of connecton ∇ on the tangent bundle is a subtle question. We will discuss this point a bit more later, but let us mention at this point that the above described supergravity theory, in particular the action and the supersymmetry variations written above should be understood as a truncation in an α' expansion, with α' the Regge slope parameter, coupling constant of the worldsheet theory (to be introduce in chapter B.I). It was shown that when ∇ is chosen to be the so-called Hull connection ∇^{H}, the first α' corrections to the action appear at order $\mathcal{O}(\alpha'^3)$, and the first corrections to the supersymmetry variations at order $\mathcal{O}(\alpha'^2)$. If another connection is to be used, all possible connections being related to each other by field redefinitions in the supergravity theory or by different regularization schemes in the worldsheet σ-model, then $\mathcal{O}(\alpha')$ corrections start to appear in the supersymmetry variations.

The first assumption ensuring the existence a supersymmetric vacuum will therefore be the existence of a globally defined spinor ϵ. Among the above system of equations, eq. (A.II.2.4a) is referred to as the Killing Spinor Equations. In eq. (A.II.2.4a),

the connection used is a connection with torsion, the Bismut connection:

$$\nabla^{\mathrm{B}}_M = \nabla_M - \frac{1}{2} H_M \,, \tag{A.II.2.5}$$

where ∇ is the Levi-Civita connection. The Bismut connection is the only Hermitian connection admitting torsion. One in particular sees here that the field strength of the Kalb-Ramond B-field plays the role of a contorsion tensor.

We have introduced above some compact notation for the H flux, which for a generic n-form F reads:

$$F_{M_1 \dots M_p} := \frac{1}{(n-p)!} \Gamma^{M_{p+1} \dots M_n} F_{M_1 \dots M_p, M_{p+1} \dots M_n} \,, \tag{A.II.2.6}$$

with the antisymmetrized product of gamma matrices:

$$\Gamma^{M_1 \dots M_p} := \Gamma^{[M_1} \dots \Gamma^{M_p]} \,. \tag{A.II.2.7}$$

The existence of a supersymmetric vacuum is equivalent to the existence of a spinor field on M satisfying eq. (A.II.2.4).

The 16-dimensional spinorial irreducible representation in 10 dimensions splits under the ansatz eq. (A.II.2.1) as:

$$SO(1,9) \to SO(1,3) \times SO(6) \tag{A.II.2.8}$$

$$16 \to (2,4) \oplus (\bar{2}, \bar{4}) \,, \tag{A.II.2.9}$$

and we decompose the supersymmetry parameter as:

$$\epsilon = \eta_+ \otimes \zeta_+ + \eta_- \otimes \zeta_- \,, \tag{A.II.2.10}$$

with $\eta_+^* = \eta_-$ and $\zeta_+^* = \zeta_-$ Weyl spinors.

The set of equations eq. (A.II.2.4) therefore leads to separate constraints on the external and the internal geometry. Concerning the external geometry, eq. (A.II.2.4b) together with the absence of torsion on M_4 leads to the integrability condition:

$$\Gamma^{\mu\nu} \nabla_\mu \nabla_\nu = 0 \,, \tag{A.II.2.11}$$

with the Levi-Civita connection ∇, which in turn leads to the fact that M_4 has a vanishing scalar curvature, hence is Minkowski spacetime.

Concerning the internal manifold, eq. (A.II.2.4a) leads to:

$$\nabla^{\mathrm{B}}_a \zeta = 0\,, \tag{A.II.2.12}$$

namely to the question of the existence of a covariantly constant spinor ζ with respect to the Bismut connection on X. The existence of such a spinor leads to a restriction on the structure group of the orthonormal frame bundle on X, which is reduced from $SO(6) \simeq SU(4)$ to $SU(3)$, [8–10]. X is then said to admit an $SU(3)$ structure. In the absence of torsion, this property reduces to the fact that X admits $SU(3)$ holonomy.

One sees from eq. (A.II.2.12) that $\zeta_+^\dagger \zeta_+$ is a constant, which can always be set to 1 by properly normalizing the spinor ζ_+. Since ζ_+ is globally defined on X, it can be used to build a globally defined real 2-form and a globally defined complex 3-form as follows. One first defines:

$$I_a{}^b = -i\zeta_-^\dagger \Gamma_a{}^b \zeta_-\,, \tag{A.II.2.13}$$

which can easily be shown to satisfy:

$$I_a{}^b I_b{}^c = -\delta_a^c\,, \tag{A.II.2.14}$$

to which corresponds an almost complex structure (endomorphism of the tangent bundle T_X squaring to minus the identity) showing that X is an almost complex manifold, and which is compatible with the metric. By rotating the metric with this tensor, one obtains the following 2-form:

$$J = \frac{1}{2} J_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b\,, \tag{A.II.2.15}$$

with:

$$J_{ab} = I_a{}^c g_{cb}\,. \tag{A.II.2.16}$$

It is a real $(1,1)$-form with respect to the almost complex structure I. Using the fact that J is covariantly constant with respect to the Bismut connection, one can show that the Nijenhuis tensor vanishes, showing that X is actually a complex manifold, equipped with a hermitian metric. We will denote in the following a generic local patch of complex coordinates by $(z^i, \bar{z}^{\bar{\imath}})$.

One can also define a complex 3-form, which is a $(3,0)$ with respect to the

complex structure:

$$\Omega = e^{-2\phi}\zeta_-^t \Gamma_{ijk}\zeta_- \, \mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}z^k \,, \tag{A.II.2.17}$$

which can be shown to be holomorphic. One therefore sees that the canonical bundle of X admits a globally defined section, hence is trivial:

$$K_X \simeq \mathcal{O}_X \,. \tag{A.II.2.18}$$

J and Ω satisfy the $SU(3)$ structure conditions:

$$J \wedge \Omega = 0 \,, \tag{A.II.2.19a}$$

$$-\frac{i}{8}\,\Omega \wedge \bar{\Omega} = \frac{1}{3!}\,J^3 \,, \tag{A.II.2.19b}$$

therefore the pair (J, Ω) is what is usually referred to as the $SU(3)$ structure on X.

In terms of this data, the dilatino equation can be shown to be equivalent to the *conformally balanced equation*:

$$\mathrm{d}\left(||\Omega||J^2\right) = 0 \,, \tag{A.II.2.20}$$

where $||\cdot||$ is the norm corresponding to the hermitian scalar product $(\cdot,\cdot)_J$ defined with respect to the fundamental form J.

The physical fields, i.e. metric, torsion and dilaton field are then expressed in terms of the geometric data:

$$g = J(\cdot, I\cdot) \,, \tag{A.II.2.21a}$$

$$H = \mathrm{d}^c J \,, \tag{A.II.2.21b}$$

$$\mathrm{d}\phi = -\frac{1}{4}\mathrm{d}\log(||\Omega||) \,, \tag{A.II.2.21c}$$

with $\mathrm{d}^c = i(\bar{\partial} - \partial)$.

One in particular sees that whenever torsion is present, even though the canonical bundle is trivial, the geometry will not be Kähler, hence not Calabi-Yau.

Let us schematically explain why we are actually dealing here with a subclass of $SU(3)$ structure manifolds. One can split the generic Riemannian structure group Lie algebra $so(6)$ as follows:

$$so(6) = su(3) \oplus \underbrace{so(6)/su(3)}_{su(3)^\perp} \,, \tag{A.II.2.22}$$

and accordingly split a generic torsion $T_{ab}{}^{c}$ viewed as an element of $\Omega^1 \otimes \Omega^2 \simeq \Omega^1 \otimes so(6) \simeq \Omega^1 \otimes su(3) \oplus \Omega^1 \otimes su(3)^{\perp}$ and decompose the intrinsic torsion $\tilde{T}$ (the $\Omega^1 \otimes su(3)^{\perp}$ part of the torsion) into irreducible $su(3)$ representations as follows:

$$\Omega^1 \otimes su(3)^{\perp} = (1 \oplus 1) \oplus (8 \oplus 8) \oplus (6 \oplus \bar{6}) \oplus (3 \oplus \bar{3}) \oplus (3 \oplus \bar{3})\,, \tag{A.II.2.23}$$

$$\tilde{T} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5\,, \tag{A.II.2.24}$$

where the five 'torsion classes' correspond to:

$$\mathcal{W}_1 \text{ complex scalar}\,, \tag{A.II.2.25a}$$

$$\mathcal{W}_2 \text{ complex primitive } (1,1)\text{-form}\,, \tag{A.II.2.25b}$$

$$\mathcal{W}_3 \text{ real primitive } (2,1)+(1,2)\text{-form}\,, \tag{A.II.2.25c}$$

$$\mathcal{W}_4 \text{ real } (1,0)+(0,1)\text{-form}\,, \tag{A.II.2.25d}$$

$$\mathcal{W}_5 \text{ complex } (1,0)\text{-form}\,. \tag{A.II.2.25e}$$

They appear as various components in the exterior derivative of J and Ω and therefore characterize the non-closure of these forms:

$$\mathrm{d}J = \frac{3}{2}\mathrm{Im}\left(\bar{\mathcal{W}}_1\Omega\right) + \mathcal{W}_4 \wedge J + \mathcal{W}_3\,, \tag{A.II.2.26a}$$

$$\mathrm{d}\Omega = \mathcal{W}_1 J^2 + \mathcal{W}_2 \wedge J + \bar{\mathcal{W}}_5 \wedge \Omega\,. \tag{A.II.2.26b}$$

In the case of our heterotic BPS constraints, we just saw that X should be a complex manifold, which corresponds to the subclass of $SU(3)$ structure manifolds with $\mathcal{W}_1 = \mathcal{W}_2 = 0$. The vanishing of the three torsion classes $\mathcal{W}_1$, $\mathcal{W}_3$ and $\mathcal{W}_4$ would imply that X is symplectic, but we know that this is not the case since X is non-Kähler. Even though it is not Kähler, we saw it still satisfies the weaker *conformally balanced* condition:

$$\mathrm{d}(||\Omega||J^2) = 0\,. \tag{A.II.2.27}$$

The only remaining constraint is a peculiarity of the Bismut connection, namely that $\mathcal{W}_4 = \mathrm{Re}\,\mathcal{W}_5$.

Remain to be discussed the constraints on the vector bundle E. They are given by the vanishing of the supersymmetry variation of the gaugino eq. (A.II.2.4c), which once splited into components along the internal manifold gives:

$$\left(\Gamma^{ij}F_{ij} + 2\Gamma^{i\bar{j}}F_{i\bar{j}} + \Gamma^{\bar{i}\bar{j}}F_{\bar{i}\bar{j}}\right)\zeta = 0\,, \tag{A.II.2.28}$$

leading to a constraint of holomorphy on the vector bundle:

$$F_{(2,0)} = F_{(0,2)} = 0\,, \tag{A.II.2.29}$$

and also to a constraint of primitivity:

$$F \wedge J^2 = 0\,. \tag{A.II.2.30}$$

Together, these two constraints eqs. (A.II.2.29) and (A.II.2.30) are referred to as the zero-slope Hermite-Yang-Mills equations, as the second one can be seen as a particular case of the more generic equation

$$F \wedge J^2 = -\frac{i\,\mathrm{rk}(E)\lambda}{3}\,J^3\,, \tag{A.II.2.31}$$

with

$$\lambda = 6\pi\frac{\mu(E)}{\int_X J^3}, \tag{A.II.2.32}$$

where one has defined the slope of the holomorphic vector bundle E in terms of its first Chern class and rank as follows:

$$\mu(E) = \frac{\int_X c_1(E)\wedge J^{n-1}}{\mathrm{rk}(E)}\,. \tag{A.II.2.33}$$

A very beautiful theorem by Donaldson, Uhlenbeck and Yau then relates the existence of such a Hermite-Yang-Mills connection to the algebraic geometry of the vector bundle E:

Theorem (DUY): *A holomorphic vector bundle E on a compact Kähler manifold X admits a Hermite-Yang-Mills connections if and only if E is polystable.*

The notion of stability involved in this theorem is that of slope stability[2] of the vector bundle E. A holomorphic vector bundle E is said to be slope stable (resp. slope semi-stable) if for all proper non-trivial $\mathcal{O}_X$-subsheaf $F \subset E$, one has $\mu(F) < \mu(E)$ (resp. $\mu(F) \leqslant \mu(E)$). A holomorphic vector bundle is polystable if it decomposes into a Whitney sum with stable summands of same slope.

This theorem was proven by Donaldson in the case of algebraic surfaces [11], and was then generalized by Uhlenbeck and Yau to any compact Kähler manifolds [12].

[2] Or more precisely slope semi-stability in the zero-slope case.

Let us therefore summarize the BPS constraints on the internal geometry $E \to X$ imposed by the requirement of preserving at least $\mathcal{N} = 1$ supersymmetry in spacetime:

- X is complex and Hermitian,
- X has a trivial canonical bundle: $K_X \simeq \mathcal{O}_X$,
- The Gauduchon metric $\sqrt{||\Omega||}J$ is balanced: $\mathrm{d}\left(||\Omega||J^2\right) = 0$,
- E is a stable holomorphic vector bundle,

to which one should not forget to add the Bianchi identity:

$$\mathrm{d}H = \frac{\alpha'}{4}(\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F)\,, \tag{A.II.2.34}$$

tying together the data of the two bundles T_X and E through the H-flux.

The α' expansion in the case of the heterotic string is a subtle question, as we alredy mentioned in section A.II.1. This is illustrated for instance by the fact that the Bianchi identity eq. (A.II.2.34) above mixes different orders in α'. This implies in particular that one should actually be particularly careful when discussing a large volume supergravity limit, since this mixing of orders typically fixes the size of the internal manifold at string scales.

One can then wonder whether the BPS constaints together with the Bianchi identity imply the equations of motion eq. (A.II.1.10). It was shown that it is indeed the case, provided that the connection ∇ is chosen so that it satisfies also an instanton condition at order α'^0, namely if [7, 13]:

$$R\,\zeta = O(\alpha')\,, \tag{A.II.2.35}$$

translating again into the pair of Hermite-Yang-Mills equations with zero slope for the spin connection:

$$R_{(2,0)} = R_{(0,2)} = O(\alpha')\,, \tag{A.II.2.36a}$$

$$R \wedge J^2 = O(\alpha')\,. \tag{A.II.2.36b}$$

The Hull connection given by:

$$\nabla^{\mathrm{H}}_a = \nabla^{\mathrm{LC}}_a + \frac{1}{2}H_a\,, \tag{A.II.2.37}$$

satisfies this constraint and is the standard choice of spin connection usually made in the Bianchi identity and the action [2, 14]. It seems in general in contradiction with the fact that the left-hand side of the Bianchi identity $\mathrm{d}^c J$ is a $(2,2)$-form, hence the right-hand-side should also be a $(2,2)$-form, which is generically the case if one chooses the Chern connection which locally reads $\nabla^{\mathrm{C}} = \mathrm{d} + 2\partial \log ||\Omega||$. But then one should modify the supersymmetry variation with an $\mathcal{O}(\alpha')$ contribution and replace the Hull connection in the action by ∇^{C}. This modification of the supersymmetry variation leads therefore also to a modification of the BPS system of equations.

In view of the above comments about the subtleties involved in actually making sense of a consistent truncation in the α' expansion, hence of a consistent supergravity large-volume description of heterotic flux compactifications, one can relies on a purely worldsheet approach. Indeed, contrary to type II which contains Ramond-Ramond fluxes whose worldsheet description is not known, the NS-NS torsion flux admits a natural description at the worldsheet level, as we will describe in the next section.

II.3 Solutions to the BPS system

Very few families of solutions are known to the BPS system of equations and heterotic Bianchi identity introduced above. The latter actually constitutes the most complicated equation to solve for a given ansatz, and is also the most difficult constraint to understand from a purely mathematical point of view. Let alone solving the Bianchi identity, finding genuine compact $SU(3)$-structure geometries is a difficult task, see for instance [15] for a construction on toric 3-folds.

A simple way to handle this equation is to impose a pointwise equality at the level of forms:

$$\mathrm{tr}\,(R \wedge R) = \mathrm{tr}\,(F \wedge F)\ . \tag{A.II.3.1}$$

This can be done in particular by imposing that the gauge connection is equal to the spin connection. This type of solution is characterized by a vanishing torsion and a constant dilaton, and is referred to as the standard embedding of the spin connection into the gauge connection. The internal manifold is then Calabi-Yau at one-loop order in α', and is then deformed order by order. The condition of space-time supersymmetry is equivalent to having $(2,2)$ supersymmetry on the worldsheet (cf. below, chapter B.I), imposing that the target space is Kähler. It is moreover possible to find a regularization scheme of the worldsheet sigma model preserving

$(2,2)$ supersymmetry, hence to restrict oneselves to flows in the space of Kähler metrics on X when correcting the metric under the worldsheet RG flow. One actually has a flow preserving the Kähler class of the metric [16], making it possible to correct order by order in α' to maintain superconformal invariance [17].

This setup leads to $E_6 \times E_8$ gauge group in spacetime.

Another class of solutions consists in allowing for a more generic gauge bundle over a Calabi-Yau manifold. Depending on the rank r of the holomorphic vector bundle, this has the advantage of leading to smaller gauge groups in spacetime, $SU(5) \times E_8$ for $r = 3$ and $SO(10) \times E_8$ for $r = 4$, hence more interesting from a phenomenological point of view. We will not extend more the discussion on Calabi-Yau compactifications, and turn now to a class of non-Kähler compactifications, supporting torsion flux.

When dealing with 4-dimensional $\mathcal{N} = 2$ heterotic compactifications, one typically thinks of Calabi-Yau compactifications on $K3 \times T^2$. Actually, this geometry is a particular case of a more generic type of compactifications whose topology consists in a principal two-torus bundle over a warped $K3$ surface:

$$T^2 \hookrightarrow X \xrightarrow{\pi} \mathcal{S}\,. \tag{A.II.3.2}$$

Generic such compactifications lead in spacetime to minimal $\mathcal{N} = 1$ supersymmetry, but a subclass of these correspond to enhanced $\mathcal{N} = 2$ supersymmetry. Moreover these compactifications, known in the litterature as Fu-Yau compactifications, generically support H-flux.

This family of compactifications constitutes the single well-known class of compactifications with torsion. The vector bundle over the total space consists in the pullback of a stable holomorphic vector bundle over K3.

These solutions were first obtained by Dasgupta, Rajesh and Sethi from type IIB orientifolds by S-duality [18], and subsequently studied geometrically by Goldstein and Prokushkin in [19], where their $SU(3)$ structure was made explicit. Fu and Yau managed to solve the Bianchi identity in [20], using the Chern connection for which the Bianchi identity reduces to a complex Monge-Ampère equation for the warp factor (with a sequel [21] discussing more physical aspects), while a different choice of connection was put forward in [22]. The choice of Chern connection however would necessitate to include $\mathcal{O}(\alpha')$ corrections to the supersymmetry variations of the fermions, hence to the BPS system of equations, which would vanish if one decided to use the Hull connection. In the Fu-Yau context, the Hodge type homo-

geneity argument at order α' of the Bianchi identity which can leads to choose the Chern connection is actually also satisfied by the Hull connection, since all the forms entering the identity are actually horizontal, i.e. with legs only along the $K3$ base, hence necessarily of Hodge type $(2,2)$. These compactifications lead to $\mathcal{N}=2$ or $\mathcal{N}=1$ supersymmetry in space-time. The first class of torsion gauged linear sigma model (cf. section B.I.2 below) that was obtained by Adams and collaborators [23] was especially designed to give a worldsheet theory for the former.

Explicitly, taking a two-torus of moduli T and U, see eq. (eq. (B.II.1.11)), the metric on the internal six-dimensional manifold X is chosen to be of the form:

$$\mathrm{d}s^2 = e^{2A(y)}\mathrm{d}s^2(\mathcal{S}) + \frac{U_2}{T_2}\left|\mathrm{d}x^1 + T\mathrm{d}x^2 + \pi^\star\alpha\right|^2 , \tag{A.II.3.3}$$

where $\mathrm{d}s^2(\mathcal{S})$ is a Ricci-flat metric on $\mathcal{S}$ and e^{2A} is a warp factor depending on the K3 coordinates only. The connection one-form α on $\mathcal{S}$ is such that :

$$\iota = \mathrm{d}x^1 + T\mathrm{d}x^2 + \pi^\star\alpha\,, \tag{A.II.3.4}$$

is a globally defined $(1,0)$ form on X. We then define the complex curvature two-form ω on $\mathcal{S}$ through:

$$\tfrac{1}{2\pi}\,\mathrm{d}\iota = \pi^\star\omega\,, \tag{A.II.3.5}$$

that we expand in terms of the T^2 complex structure as

$$\omega = \omega_1 + T\omega_2\,. \tag{A.II.3.6}$$

The metric eq. (A.II.3.3) is globally defined provided that $\omega_\ell \in H^2(\mathcal{S},\mathbb{Z})$.

As was shown by Goldstein and Prokushkin, a solution of the supersymmetry conditions is obtained provided that ω has no component in $\Lambda^{0,2}T^\star\mathcal{S}$ and is primitive, *i.e.* such that

$$\omega\wedge J_{\mathcal{S}} = 0\,. \tag{A.II.3.7}$$

One can then obtain the fundamental $(1,1)$-form J and the complex $(3,0)$-form Ω characterizing the $SU(3)$ structure in terms of the Kähler form and holomorphic two-form on $\mathcal{S}$, $J_{\mathcal{S}}$ and $\Omega_{\mathcal{S}}$, as

$$\Omega = \pi^*\left(\Omega_{\mathcal{S}}\right)\wedge\iota\,, \tag{A.II.3.8a}$$

$$J = \pi^*\left(e^{2\phi}J_{\mathcal{S}}\right) + \frac{iU_2}{2T_2}\iota\wedge\bar{\iota}\,. \tag{A.II.3.8b}$$

Solutions with extended $\mathcal{N}=2$ supersymmetry in four dimensions, *i.e.* with $SU(2)$ structure, are obtained by imposing the extra condition $\omega \in H^{(1,1)}(\mathcal{S})$. This is the relevant case for the torsion gauged linear sigma-models that we consider in this thesis.

One therefore chooses ω_1 and ω_2 in the Picard lattice of $\mathcal{S}$, defined by:

$$\mathrm{Pic}(\mathcal{S}) = H^2(\mathcal{S},\mathbb{Z}) \cap H^{(1,1)}(\mathcal{S})\,, \tag{A.II.3.9}$$

whose rank, the so-called *Picard number* is denoted $\rho(\mathcal{S})$. Let us define a set of complex topological charges $\{M^n, n=1,\ldots,\rho(\mathcal{S})\}$, belonging to the lattice $\mathbb{Z}+T\,\mathbb{Z}$, and choose a basis of $\mathrm{Pic}(\mathcal{S})$, $\{\varpi_n, n=1,\ldots,\rho(\mathcal{S})\}$. One expands the curvature of the two-torus bundle as

$$\omega = \sum_{n=1}^{\rho(\mathcal{S})} M^n \varpi_n\,. \tag{A.II.3.10}$$

The vector bundle over X is obtained as the pullback of a holomorphic gauge bundle E on $\mathcal{S}$ satisfying the zero-slope limit of the Hermite-Yang-Mills equations, see eqns. (eq. (B.II.1.2a),eq. (B.II.1.2b)). On $K3$ it implies anti-self-duality, *i.e.* that the bundle E corresponds to an anti-instanton background. Fu and Yau showed in [20] that one can find a smooth solution to the Bianchi identity for the warp factor, using the Chern connection, provided the following tadpole condition holds,

$$\int_{\mathcal{S}} \mathrm{ch}_2(E) + \frac{U_2}{T_2} d_{mn} M^m \bar{M}^n + 24 = 0\,, \tag{A.II.3.11}$$

written the basis (eq. (A.II.3.10)), where d_{mn} is the intersection matrix on $H^2(\mathcal{S},\mathbb{Z})$.

$K3\times T^2$ compactifications appear in this setting as the very particular case in which the fibration is trivial, namely in which $\omega=0$ and the instanton number of the gauge bundle is the largest:

$$N := -\int_{\mathcal{S}} \mathrm{ch}_2(E) = 24\,. \tag{A.II.3.12}$$

In the presence of fluxes, one typically expects that at least part of the moduli is frozen. This is indeed the case here, as we will discuss in more detail in chapter B.II. What one discovers is that the presence of torsion flux leads to the quantization of the torus fibre moduli. This point will be discussed in more details a bit later.

A more generic holomorphic vector bundle is actually obtained by tensoring the above described non-abelian bundle with line bundles over the total space of the torus fibration. The connection on these lines bundles would reduce to Wilson lines

in the $K3 \times T^2$ case, and we may therefore in the following loosely refer to the torus *Wilson lines* moduli in the following, even when the torus fibration is non-trivial. As we will see, contrary to the complex structure and Kähler moduli of the torus, the Wilson line moduli are not quantized by the torsion flux.

Part B

Fu-Yau dressed elliptic genus

Chapter I

Worldsheet theory

I.1 Non-Linear Sigma Model

As described in the supergravity approach, there are lots of subtleties involved in the large volume expansion in α'/l^2, with l a typical length scale of the geometry of interest. One instance of these subtlelties occurs in the modified Bianchi identity which mixes different orders in α'. When considering a spacetime of the form $M_4 \times X$ with X a six-dimensional compact manifold, the integrated Bianchi identity indeed leads to a tadpole condition which freezes at least part of the internal geometry to string scale. Fortunately, the heterotic string doesn't have RR fluxes, hence all fluxes are amenable to a worldsheet description, which is more fundamental point of view in the sense that it allows to take into account stringy phenomena which are typically ignored in a large volume description. The worldsheet approach however does not capture non-perturbative effects in the string perturbation theory (i.e. in the string coupling constant g_s), i.e. the presence of NS5-branes, magnetic duals of the fundamental string.

The worldsheet theory consists in a conformally invariant theory exhibiting supersymmetry in the right-moving sector only and describing the embedding of the string in a target spacetime. The exact expression of the strongly coupled CFT describing a given string vacua is however often not known apart at some very specific point in the compactification moduli space, for instance at Gepner or orbifold points. Another approach is to consider sigma-models on the worldsheet that we expect to admit a non-trivial infrared fixed-point relevant to describe a heterotic flux vacuum. The sigma-model should be such that the infrared CFT breaks up

into various pieces:

$$(c,\bar{c}) = \underbrace{(4,6)}_{\text{non-compact directions}} + \underbrace{(-26,-26)}_{(b,c)-\text{ghosts system}} + \underbrace{(0,11)}_{(\beta,\gamma)-\text{ghosts system}} + \underbrace{(22,9)}_{\text{internal}} . \tag{B.I.1.1}$$

The first three pieces will be referred to as the external CFT, and the $(22,9)$ part corresponds to the internal CFT, whose NLSM we introduce now. We will in the following further split the internal $(22,9)$ CFT above into two pieces:

$$(22,9) = (6+r,9) + (16-r,0)\,, \tag{B.I.1.2}$$

and also refer to the $(6+r,9)$ piece as the internal CFT. The context will always be clear enough so that one knows what 'internal' refers to.

The BPS conditions described in section A.II.2 ensuring at least $\mathcal{N}=1$ in spacetime were shown by Banks and Dixon [24] to be equivalent from a worldsheet point of view to the requirement of extended $(0,2)$ supersymmetry in the worldsheet sigma-model, together with a quantization of the $U(1)$ R-symmetry charges [25]. In parts B and C, we will be particularly interested in compactifications leading to extended $\mathcal{N}=2$ in spacetime. It was shown that this leads to a further enhancement of the supersymmetry algebra on the worldsheet to $(0,4)\oplus(0,2)$, cf. for instance [26].

It is actually quite convenient to work solely with explicit $(0,1)$ supersymmetry in order to exhibit the involved degrees of freedom. Let us denote by $(\sigma^{\pm},\theta)$ the coordinates on a local $\mathbb{R}^{2|1}$ patch of $(0,1)$-superspace. The NLSM describes the dynamics of matter fields contained in chiral and Fermi superfields whose component expansion is:

$$\text{Chiral: } X = x + \theta\,\psi\,, \tag{B.I.1.3a}$$

$$\text{Fermi: } \Gamma = \gamma + \theta\,G\,, \tag{B.I.1.3b}$$

where x is a complex bosonic field, ψ a right-moving Weyl fermion, γ a left-moving Weyl fermion, and G a complex bosonic auxiliary field. Let us also introduce the supercovariant derivative as follows:

$$D_+ = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial\sigma^+}\,. \tag{B.I.1.4}$$

We can then write a generic action describing the dynamics of a collection of chiral

superfields $\{X^a\}_{a=1\dots6}$ and a collection of Fermi superfields $\{\Gamma^s\}_{s=1\dots32}$ as follows:

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\mathrm{d}\theta\, \left\{E_{ab}(X)\, D_+X^a\partial_-X^b + \delta_{st}\Gamma^s\left(D_+\Gamma^t + D_+X^a A^t{}_{ua}(X)\Gamma^u\right)\right\}\,, \tag{B.I.1.5}$$

where we introduced generic couplings $E_{ab}(X)$ and $A^s{}_{ta}(X)$ depending on the chiral superfields. One can expand this Lagrangian in components and classically integrate out the auxiliary fields to obtain in a local patch of the target:

$$\begin{aligned}\mathcal{S} =& \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\{E_{ab}(x)\,\partial_+x^a\partial_-x^b + ig_{ab}(x)\,\psi^a\left(\partial_-\psi^b + \partial_-x^a L^b_{(-)ac}\psi^c\right) + \\ &+ i\delta_{st}\gamma^s\left(\partial_+\gamma^t + \partial_+x^a A^t{}_{ua}(x)\gamma^u\right) + \frac{1}{2}F_{stab}(x)\psi^a\psi^b\gamma^s\gamma^t\}\,,\end{aligned} \tag{B.I.1.6}$$

where we have introduced the symmetric tensor $g_{ab} = E_{(ab)}$ which has the interpretation of a metric on the target space X described locally by the coordinates $\{x^a\}$. The antisymmetric tensor $B_{ab} = E_{[ab]}$ defines the 2-form $B = B_{ab}\mathrm{d}x^a\wedge\mathrm{d}x^b$ and has the interpretation of a B-field on X. In particular it appears through its exterior derivative $H = \mathrm{d}B$ as a contorsion term in the Lorentz connection:

$$L^a_{(-)bc} = \Gamma^a{}_{bc} - \frac{1}{2}H^a{}_{bc}\,, \tag{B.I.1.7}$$

in accordance with the previous supergravity approach (section A.II.2), and where $\Gamma^a{}_{bc}$ denote the Christoffel symbols. $A^s{}_{ta}$ are interpreted as the components of a non-abelian connection on a principal bundle over X. The 32 fermions $\{\gamma^s\}$ are coupled to the pullback of this connection on the worldsheet. The coefficients $F^s{}_{tab}$ appearing in the Fermi interaction term then denote the components of the curvature of this connection.

The above theory contains chiral fermions, hence develops anomalies under spacetime Lorentz and gauge transformations. Under a spacetime Lorentz transformation of parameter η and gauge transformation of parameter ϵ, the one-loop effective action transforms as [4, 27]:

$$\delta_{\eta,\epsilon}\, S_{\text{eff}} \propto \frac{1}{8\pi\alpha'}\int x^*\left(\mathrm{tr}(\eta\mathrm{d}\nabla_{(+)}) - \mathrm{tr}(\epsilon\mathrm{d}A)\right)\,, \tag{B.I.1.8}$$

where $\nabla_{(+)}$ denote the spin connection:

$$\nabla^\alpha_{(+)\beta c} = e^\alpha_a e^b_\beta\, L^a_{(+)bc}\,, \tag{B.I.1.9}$$

with α, β tangent space indices, and with:

$$L^a_{(+)bc} = \Gamma^a{}_{bc} + \frac{1}{2} H^a{}_{bc}\,, \tag{B.I.1.10}$$

the Lorentz connection with opposite torsion compared to eq. (B.I.1.7). This variation of the fermion measure can be compensated by a redefinition of the local expression of the H-field:

$$\mathrm{d}B \to \mathrm{d}B - \frac{\alpha'}{4}\left(\mathrm{CS}\left(\nabla_{(+)}\right) - \mathrm{CS}(A)\right)\,. \tag{B.I.1.11}$$

This is a worldsheet version of the Green-Schwarz mechanism [28]. Even though the above expression eq. (B.I.1.11) is only local, it leads to the globally defined constraint:

$$\mathrm{d}H = \frac{\alpha'}{4}\left(\mathrm{tr}\left(R_+ \wedge R_+\right) - \mathrm{tr}\left(F \wedge F\right)\right)\,, \tag{B.I.1.12}$$

and one sees that it implies the equality of the second Chern characters of the tangent bundle and the vector bundle (whose structure group is embedded into $E_8 \times E_8$).

The equations of motion eq. (A.II.1.10) are understood from the 2-dimensional worldsheet as the requirement of conformal invariance of the sigma model, which is equivalent to the vanishing of the beta functions, which we give below[1], computed on the sphere up to first order α':

$$\beta_g = \mathrm{Ric}(g) - 2\nabla(\mathrm{d}\phi) - \frac{1}{4} H \vee H - \frac{\alpha'}{4}\left(R \vee R - F \vee F\right) + O(\alpha'^2)\,, \tag{B.I.1.13a}$$

$$\beta_\phi = \mathcal{R}(g) - 4\Delta\phi - 4|\mathrm{d}\phi|^2 - \frac{1}{2}|H|^2 + \frac{\alpha'}{4}\left(|R|^2 - |F|^2\right) + O(\alpha'^2)\,, \tag{B.I.1.13b}$$

$$\beta_B = \mathrm{d}^\dagger\left(e^{2\phi} H\right) + O(\alpha'^2)\,, \tag{B.I.1.13c}$$

$$\beta_A = \mathrm{d}^\dagger_A\left(e^{2\phi} F\right) + \frac{1}{2}\, e^{2\phi} \star \left(F \wedge \star H\right) + O(\alpha'^2)\,, \tag{B.I.1.13d}$$

where we also added the beta function for the dilaton, whose coupling to the worldsheet field we did not consider in the action eq. (B.I.1.5). The coupling of the dilaton to the worldsheet is a bit peculiar since it spoils the classical invariance under conformal transformations of the theory, but which is of course restored at the quantum level once one enforces the vanishing of the β functions. The coupling takes the following form:

$$\int_\Sigma \mathrm{vol}_h\, \phi\, \mathcal{R}(h)\,, \tag{B.I.1.14}$$

[1] In the critical dimension $d = 10$.

with h an auxiliary worldsheet metric. The piece corresponding to the zero-mode part of the dilaton field is then proportional to the Euler number of the worldsheet, hence organises the string loop expansion.

One therefore sees that the supergravity equations of motions correspond to the lowest order of the β functions in the sigma model perturbation theory. The all-order in α' expansion equation of motions are actually not known. The first strategy to adopt would be to suppose that the geometry singled out by the full equations of motion should be quite close to the solution of the low order equations of motions, and that one can correct it order by order. On the other hand, one can consider the problem from a Wilsonian point of view, and consider that the non-linear sigma model described above actually flows in the infrared to the conformal field theory of interest, hence may be a fine description enough to describe part of the four-dimensional physics, the part which is independent of the renormalization group flow, corresponding the topological properties of the target geometry. Pushing the reasoning even further, one can then look for the simplest theory in the same bassin of attraction as the above described non-linear sigma model, if possible linear, since it would capture this topological data as well as the NLSM and may be easier to manipulate. Such linear ultraviolet completions take the form of two-dimensional supersymmetric gauge theories, that we will now discuss.

I.2 Gauged Linear Sigma Model

It will turn out to be quite convenient to keep manifest $(0,2)$ supersymmetry in the following. We will therefore consider gauge theories in $(0,2)$ superspace, whose generic local $\mathbb{R}^{2|2}$ patch of coordinates we will denote $(\sigma^+, \sigma^-, \theta, \bar{\theta})$. Some classes of these supersymmetric gauge theories constitute an ultraviolet completion of the heterotic NLSM.

Such GLSMs were first introduced by Witten [29] in order to give a physical understanding of various connections between non-linear sigma models with Calabi-Yau target, and some Landau-Ginzburg orbifolds, by arguing that the two theories may actually sit on the RG flow of a same theory in the ultraviolet depending on some continuous parameter. Depending on the value of this parameter, the theory belongs either to a geometric phase flowing to the NLSM, or to a Landau-Ginzburg orbifold phase.

Such kind of models are particularly adapted to the description of a large class of Calabi-Yau manifolds built as complete intersections in toric varieties, and monad

bundles over them.

As we will explain later in section B.II.1, these GLSMs were extended in order to also describe the torsional $\mathcal{N} = 2$ compactifications which will be of interest to us [23].

Let us start by describing the field content of such theories. We will work with a $U(1)$ gauge theory for simplicity of the exposure, but one can generalize to higher rank abelian or non-abelian gauge groups.

I.2.1 superfields

We define the superspace derivatives and supercharges as follows:

$$\mathcal{Q}_+ = \partial_\theta + i\bar{\theta}\partial_+ \quad , \qquad \bar{\mathcal{Q}}_+ = -\partial_{\bar{\theta}} - i\theta\partial_+ \,, \tag{B.I.2.1a}$$

$$D_+ = \partial_\theta - i\bar{\theta}\partial_+ \quad , \qquad \bar{D}_+ = -\partial_{\bar{\theta}} + i\theta\partial_+ \,. \tag{B.I.2.1b}$$

The non-trivial anti-commutators are then

$$\{\bar{D}_+, D_+\} = 2i\partial_+ \quad , \qquad \{\bar{\mathcal{Q}}_+, \mathcal{Q}_+\} = -2i\partial_+ \tag{B.I.2.2}$$

Chiral superfields are defined by the constraint that they are annihilated by half of the superspace derivatives. This constraint leads to the following component expansion:

$$\bar{D}_+\Phi = 0 \implies \Phi = \phi + \sqrt{2}\theta\lambda_+ - i\theta\bar{\theta}\partial_+\phi, \tag{B.I.2.3}$$

where ϕ is a complex boson, and λ_+ a right-moving Weyl fermion.

Fermi superfields on the other hand have as a bottom component a left-moving fermion. They satisfy generically the constraint:

$$\bar{D}_+\Gamma = \sqrt{2}E(\Phi_i)\,, \tag{B.I.2.4}$$

where E is a holomorphic function which quantifies the non-chirality of the superfield. We will assume later that this function $E(\Phi_i)$ vanishes for simplicity, but we will keep it for now. Fermi superfields therefore have the following component expansion:

$$\Gamma = \gamma_- + \sqrt{2}\theta G - \sqrt{2}\bar{\theta}E(\Phi_i) - i\theta\bar{\theta}\partial_+\gamma_-\,, \tag{B.I.2.5}$$

where G is an auxiliary bosonic field.

Gauge superfields are actually described by a pair of $(0,2)$ superfields $\mathcal{A}$ and $\mathcal{V}$.

Super-gauge transformations act as

$$\mathcal{A} \to \mathcal{A} + \frac{i}{2}(\bar{\Xi} - \Xi) \quad , \qquad \mathcal{V} \to \mathcal{V} - \frac{1}{2}\partial_-(\Xi + \bar{\Xi}) \tag{B.I.2.6}$$

where Ξ is a chiral superfield. In the so-called Wess-Zumino gauge things get simpler, even though one should be careful when dealing with classically non gauge-invariant actions as it will be the case in chapter B.II. The residual gauge symmetry is

$$\Xi = \rho - i\theta\bar{\theta}\partial_+\rho \tag{B.I.2.7}$$

with real ρ, while the component expansion of $\mathcal{A}$ and $\mathcal{V}$ reads

$$\mathcal{A} = \theta\bar{\theta}^+ A_+ \tag{B.I.2.8a}$$

$$\mathcal{V} = A_- - 2i\theta\bar{\mu}_- - 2i\bar{\theta}^+\mu_- + 2\theta\bar{\theta}^+ D \tag{B.I.2.8b}$$

where D is a real auxiliary field. Accordingly the components $A_\pm = A_0 \pm A_1$ of the gauge field are shifted under the residual gauge transformations as

$$A_\pm \xrightarrow{\rho} A_\pm - \partial_\pm\rho \tag{B.I.2.9}$$

The field strength superfield, which is chiral, is

$$\Upsilon = \bar{D}_+(\partial_-\mathcal{A} + i\mathcal{V}) = -2\left(\mu_- - i\theta(D - iF_{+-}) - i\theta\bar{\theta}^+\partial_+\mu_-\right) \tag{B.I.2.10}$$

with $2F_{+-} = \partial_- A_+ - \partial_+ A_-$. We define the gauge-covariant superderivatives as:

$$\mathfrak{D}_+ = (\partial_\theta - i\bar{\theta}\nabla_+) = D_+ + Q\bar{\theta}A_+ \tag{B.I.2.11a}$$

$$\bar{\mathfrak{D}}_+ = (-\partial_{\bar{\theta}^+} + i\theta\nabla_+) = \bar{D}_+ - Q\theta A_+ \,. \tag{B.I.2.11b}$$

where $\nabla_\pm$ are ordinary covariant derivatives.

Let us make a side comment. If one wishes to discuss theories closer to $(2,2)$ models, one should add to the above gauge superfield an extra Fermi superfield Σ which together with the gauge multiplet would constitute the $(2,2)$ gauge multiplet. One can then notice that the theory is actually invariant under an extra fermionic gauge symmetry [30]:

$$\Sigma \to \Sigma + i\Omega \,, \tag{B.I.2.12a}$$

$$\Gamma \to \Gamma + 2i\Omega E(\Phi) \,, \tag{B.I.2.12b}$$

with Ω a chiral Fermi superfield parameter. Σ is not chiral but rather satisfies $\bar{D}_+\Sigma = \sigma + \sqrt{2}\theta\beta - i\theta\bar{\theta}\partial_+\sigma$, where one has gauged away two out of the four components of Σ thanks to the fermionic gauge symmetry. This Σ superfield will not be necessary in the following.

Charged matter superfields of charge Q need to satisfy the gauge-covariant constraint:

$$\bar{\mathfrak{D}}_+\Phi = 0 \tag{B.I.2.13}$$

which is solved by

$$\Phi = \phi + \sqrt{2}\theta\lambda_+ - i\theta\bar{\theta}\nabla_+\phi\,. \tag{B.I.2.14}$$

In other words, since

$$\bar{\mathfrak{D}}_+ = e^{Q\mathcal{A}}\bar{D}_+e^{-Q\mathcal{A}} \tag{B.I.2.15}$$

We have that

$$\Phi = e^{Q\mathcal{A}}\Phi_0 \tag{B.I.2.16}$$

where Φ_0 is a superfield obeying the standard chirality constraint $\bar{D}_+\Phi_0 = 0$.

Similarly, a *charged Fermi superfield* of charge q can be obtained as $\Gamma = e^{q\mathcal{A}}\Gamma_0$ where Γ_0 satisfies $\bar{D}_+\Gamma_0 = \sqrt{2}E$. Hence the superfield Γ has the component expansion:

$$\Gamma = \gamma_- + \sqrt{2}\theta G - \sqrt{2}\bar{\theta}E(\Phi) - i\theta\bar{\theta}\nabla_+\gamma_-\,, \tag{B.I.2.17}$$

where as before E is a holomorphic function of the chiral superfields.

We will also see later that so-called *shift superfields* have a crucial role to play in describing generic heterotic compactifications. They correspond to chiral superfields charged axially under the gauge symmetry.

In Wess-Zumino gauge supersymmetry transformations should be followed by a supergauge transformation of chiral superfield parameter $\Xi_{wz} = i\bar{\epsilon}\theta\, A_+$ in order to restore the gauge choice. The explicit supersymmetric transformation of the various component fields under the full transformation, defined as $\delta_{\epsilon,\bar{\epsilon}} = \left(\epsilon Q_+ - \bar{\epsilon}\bar{Q}_+ + \delta_{\text{v.m.}}\right)$ is given by:

$$\delta_{\epsilon,\bar\epsilon}\,\phi = -\bar\epsilon\lambda \qquad \text{(B.I.2.18a)} \qquad \delta_{\epsilon,\bar\epsilon}\,\bar\phi = \epsilon\bar\lambda \qquad \text{(B.I.2.18g)}$$

$$\delta_{\epsilon,\bar\epsilon}\,\lambda = i\epsilon\nabla_+\phi \qquad \text{(B.I.2.18b)} \qquad \delta_{\epsilon,\bar\epsilon}\,\bar\lambda = -i\bar\epsilon\nabla_+\bar\phi \qquad \text{(B.I.2.18h)}$$

$$\delta_{\epsilon,\bar\epsilon}\,\gamma = -\bar\epsilon G \qquad \text{(B.I.2.18c)} \qquad \delta_{\epsilon,\bar\epsilon}\,\bar\gamma = \epsilon\bar G \qquad \text{(B.I.2.18i)}$$

$$\delta_{\epsilon,\bar\epsilon}\,G = -i\epsilon\nabla_+\gamma \qquad \text{(B.I.2.18d)} \qquad \delta_{\epsilon,\bar\epsilon}\,\bar G = i\bar\epsilon\nabla_+\bar\gamma \qquad \text{(B.I.2.18j)}$$

$$\delta_{\epsilon,\bar\epsilon}\,\mu = \frac{1}{\sqrt{2}}\bar\epsilon(F_{01}+iD) \qquad \text{(B.I.2.18e)} \qquad \delta_{\epsilon,\bar\epsilon}\,\bar\mu = -\frac{1}{\sqrt{2}}\epsilon(F_{01}-iD) \qquad \text{(B.I.2.18k)}$$

$$\delta_{\epsilon,\bar\epsilon}\,F_{01} = \frac{i}{\sqrt{2}}\partial_+(\epsilon\mu-\bar\epsilon\bar\mu) \qquad \text{(B.I.2.18f)} \qquad \delta_{\epsilon,\bar\epsilon}\,D = \frac{1}{\sqrt{2}}\partial_+(\epsilon\mu+\bar\epsilon\bar\mu)\,, \qquad \text{(B.I.2.18l)}$$

for ϵ and $\bar\epsilon$ constant Grassmann parameters.

Let us now describe the dynamics of these superfields.

I.2.2 Lagrangians

We give the Lagrangian describing the dynamics of the above introduced superfields, first in a manifestly supersymmetric expression and then in components.

Let us start with the kinetic term for a chiral field Φ of charge Q, whose components expansion is given by eq. (B.I.2.14). It is given by

$$\mathcal{L}_{\text{c.m.}} = -\frac{i}{2}\int \mathrm{d}^2\theta\,\bar\Phi\mathcal{D}_-\Phi\,, \qquad \text{(B.I.2.19a)}$$

$$= \frac{1}{2}\left(\nabla_+\bar\phi\nabla_-\phi+\nabla_-\bar\phi\nabla_+\phi\right)+i\bar\lambda_+\nabla_-\lambda_+ + i\sqrt{2}Q\left(\lambda_+\mu_-\bar\phi+h.c.\right)+Q|\phi|^2D\,. \qquad \text{(B.I.2.19b)}$$

Let us now move on to the case of a Fermi superfield of charge q. One has the following component expansion:

$$\mathcal{L}_{\text{f.m.}} = -\frac{1}{2}\int \mathrm{d}^2\theta\,\bar\Gamma\Gamma\,, \qquad \text{(B.I.2.20a)}$$

$$= i\bar\gamma_-\nabla_+\gamma_- + |G|^2 - |E(\phi)|^2 - \left(E'(\phi)\bar\gamma_-\lambda_+ + h.c.\right)\,, \qquad \text{(B.I.2.20b)}$$

The gauge kinetic term is written in terms of the field strength superfield, which is a chiral superfield

$$\Upsilon = \bar D_+(\partial_-\mathcal{A}+i\mathcal{V}) = -2\left(\mu_- - i\theta(D-iF_{+-}) - i\theta\bar\theta^+\partial_+\mu_-\right)\,, \qquad \text{(B.I.2.21)}$$

with $2F_{+-} = \partial_- A_+ - \partial_+ A_-$.

$$\mathcal{L}_{\text{v.m.}} = -\frac{1}{8}\int \mathrm{d}^2\theta\, \bar{\Upsilon}\Upsilon\,, \tag{B.I.2.22a}$$

$$= i\bar{\mu}_-\partial_+\mu_- + \frac{1}{2}D^2 + \frac{1}{2}F_{01}^2\,. \tag{B.I.2.22b}$$

Including a possible constant FI term, of parameter $t = ir + \frac{\theta}{2\pi}$, one has

$$\mathcal{L}_{\text{FI}} = \frac{t}{4}\int \mathrm{d}\theta\, \Upsilon + h.c. = -rD + \tfrac{\theta}{2\pi}F_{01}\,, \tag{B.I.2.23}$$

where the imaginary part of the Fayet-Iliopoulos parameter precisely corresponds to this continuous parameter whose value indicates in which 'phase' one is sitting.

The last term in the GLSM is the superpotential term, given by a set of holomorphic functions J of the chiral superfields which play a similar role as the already encountered holomorphic function $E(\Phi)$ in the definition of the Fermi superfields. It has to satisfy the constraint $E^a J_a = 0$ in order to preserve supersymmetry. (where a runs over the Fermi superfields). It reads

$$\mathcal{L}_{\text{J}} = \frac{\alpha}{2}\int \mathrm{d}\theta\, \Gamma J(\Phi) + h.c. \tag{B.I.2.24a}$$

$$= \frac{\alpha}{\sqrt{2}}\left(GJ(\phi) - \gamma_-\lambda_+\partial_\phi J(\phi)\right) + h.c. \tag{B.I.2.24b}$$

After solving for the auxiliary fields of the full theory one gets the scalar potential

$$V(\phi) = \frac{e^2}{8}\left(Q|\phi|^2 - r\right)^2 + \frac{\alpha}{2}|J|^2 + |E|^2\,, \tag{B.I.2.25}$$

which defines the vacua of the theory $Q|\phi|^2 = r$, $J = 0$ and $E = 0$ modulo gauge transformations.

Let us define the following anticommuting supersymmetry transformation, where we make ϵ and $\bar{\epsilon}$ commuting and equal to 1:

$$\mathcal{Q} := (\delta_{\epsilon,\bar{\epsilon}})|_{\epsilon=\bar{\epsilon}=1}\,, \tag{B.I.2.26}$$

which will be particularly useful in chapter B.II when we will define and compute the *dressed elliptic genus* by localization.

Then one can show that the action corresponding to all the Lagrangian intro-

duced above are actually $\mathcal{Q}$-exact:

$$S_{\text{c.m.}} = \frac{1}{g^2}\,\mathcal{Q}\,\nu_{\text{c.m.}}\,, \tag{B.I.2.27a}$$

$$S_{\text{f.m.}} = \frac{1}{f^2}\,\mathcal{Q}\,\nu_{\text{f.m.}}\,, \tag{B.I.2.27b}$$

$$S_{\text{v.m.}} = \frac{1}{e^2}\,\mathcal{Q}\,\nu_{\text{v.m.}}\,, \tag{B.I.2.27c}$$

$$S_{\text{FI}} = t\mathcal{Q}\,\nu_{\text{FI}} + h.c.\,, \tag{B.I.2.27d}$$

$$S_{\text{J}} = \alpha\mathcal{Q}\,\nu_{\text{J}} + h.c.\,, \tag{B.I.2.27e}$$

with the following simple functionals:

$$\nu_{\text{c.m.}} = \int \mathrm{d}^2x\,\left(i\bar{\phi}\nabla_-\lambda - i\sqrt{Q}\bar{\phi}\bar{\mu}\phi\right)\,, \tag{B.I.2.28a}$$

$$\nu_{\text{f.m.}} = -\int \mathrm{d}^2x\,\gamma\bar{G}\,, \tag{B.I.2.28b}$$

$$\nu_{\text{v.m.}} = -\frac{i}{\sqrt{2}}\int \mathrm{d}^2x\,\mu\,(D + iF_{01})\,, \tag{B.I.2.28c}$$

$$\nu_{\text{FI}} = \frac{1}{\sqrt{2}}\int \mathrm{d}^2x\,\mu\,, \tag{B.I.2.28d}$$

$$\nu_{\text{J}} = -\frac{\alpha}{\sqrt{2}}\int \mathrm{d}^2x\,\gamma J(\phi)\,. \tag{B.I.2.28e}$$

We included explicit couplings g, f, e in front the chiral, Fermi and gauge lagrangians respectively.

I.2.3 $U(1)$ charges, anomalies and example

In addition to the $U(1)$ gauge group, the theory is also invariant under a global $U(1)_{\text{R}}$ symmetry, flowing in the IR to the right-moving R-symmetry of the $N = 2$ superconformal algebra. We consider theories which in addition contain a flavor $U(1)_{\text{L}}$ symmetry, which is used to implement the left-moving spectral flow, and also constitutes part of the linearly realized spacetime gauge group. The above described Lagrangians, including the superpotential, should be classically invariant under these various $U(1)$ groups.

The theory containing chiral fermions, the various $U(1)$ symmetries are poten-

tially anomalous, and one should ensure that the *'t Hooft anomaly matrix*:

$$\mathcal{A} := \begin{pmatrix} \mathcal{A}_{U(1)\cdot U(1)} & \mathcal{A}_{U(1)\cdot U(1)_{\mathrm{L}}} & \mathcal{A}_{U(1)\cdot U(1)_{\mathrm{R}}} \\ \mathcal{A}_{U(1)_{\mathrm{L}}\cdot U(1)} & \mathcal{A}_{U(1)_{\mathrm{L}}\cdot U(1)_{\mathrm{L}}} & \mathcal{A}_{U(1)_{\mathrm{L}}\cdot U(1)_{\mathrm{R}}} \\ \mathcal{A}_{U(1)_{\mathrm{R}}\cdot U(1)} & \mathcal{A}_{U(1)_{\mathrm{R}}\cdot U(1)_{\mathrm{L}}} & \mathcal{A}_{U(1)_{\mathrm{R}}\cdot U(1)_{\mathrm{R}}} \end{pmatrix}, \tag{B.I.2.29}$$

with coefficients computed from loops of chiral fermions:

$$\mathcal{A}_{A\cdot B} = \sum_{\text{right-moving Weyl fermions}} Q_A Q_B - \sum_{\text{left-moving Weyl fermions}} q_A q_B \,, \tag{B.I.2.30}$$

takes the following prescribed form:

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & \bar{c}/3 \end{pmatrix}, \tag{B.I.2.31}$$

whose entries are computed in the UV from the OPE of the currents corresponding to the various $U(1)$ symmetries. Notice that one should not forget the contribution from the gaugino, which is a left-moving Weyl fermion charged under the $U(1)_{\mathrm{L}}$. We will not enter into the detail of such OPE computations, and will take the following result for granted [30]: whenever the above equation eq. (B.I.2.31) is satisfied, one ensures that:

- the anomalies for $U(1)$, $U(1)_{\mathrm{L}}$ and $U(1)_{\mathrm{R}}$ vanish,
- the mixed anomalies $U(1)\cdot U(1)_{\mathrm{L}}$ and $U(1)\cdot U(1)_{\mathrm{R}}$ vanish,
- the currents of the left and right global $U(1)_{\mathrm{L}}$ and $U(1)_{\mathrm{R}}$ decouple,
- the central charges of the IR superconformal field theory take the prescribed values $(c,\bar{c}) = (\frac{2}{3}\bar{c}+r,\bar{c})$,
- the holomorphic vector bundle over the target space variety in the geometric phase has prescribed rank r.

The vanishing of the $U(1)$ gauge anomaly can be interpreted from the spacetime point of view as the above mentioned requirement of equality of the second Chern characters $\mathrm{ch}_2(T_X) = \mathrm{ch}_2(E)$.

Complete intersections in weighted projective spaces: Let us introduce a particular class of theories, defined by the following field content: a collection of

chiral superfields Φ_i whose lowest component correspond to homogeneous coordinates in an ambient weighted projective space, a chiral superfield P which somehow plays the role of a Lagrange multiplier and two sets of chiral Fermi superfields $\tilde{\Gamma}_\alpha$ and Γ_a. These Fermi superfields are also chosen to satisfy a chirality constraint for simplicity, namely the E function in eq. (B.I.2.4) vanishes. One also introduces two sets of transverse quasi-homogeneous holomorphic polynomials $G^\alpha(\Phi_i)$ and $J^a(\Phi_i)$.

Let us denote the gauge charges of these fields in the following way:

	Φ_i	P	$\tilde{\Gamma}_\alpha$	Γ_a
$U(1)_{\text{v.m.}}$	Q_i	Q_P	Q_α	Q_a

(B.I.2.32)

with Q_P and Q_α generically negative.

The superpotential eq. (B.I.2.24) is then chosen to be of the form:

$$\mathcal{L}_{\text{pot}} = \mathcal{L}_{\text{T}} + \mathcal{L}_{\text{V}}\,, \tag{B.I.2.33}$$

with:

$$\mathcal{L}_{\text{T}} = -\int \mathrm{d}\theta\, \tilde{\Gamma}_\alpha G^\alpha(\Phi_i) + h.c.\,, \tag{B.I.2.34}$$

and:

$$\mathcal{L}_{\text{V}} = -\int \mathrm{d}\theta\, \Gamma_a P J^a(\Phi_i) + h.c.\,. \tag{B.I.2.35}$$

The quasi-homogeneous polynomials are chosen to be of the appropriate degree to preserve gauge invariance at the classical level and, geometrically, to obtain a hypersurface of vanishing first Chern class.

Let us indeed write down once again the bosonic potential:

$$V(\phi_i, p) = \frac{e^2}{8}\left(\sum_i Q_i|\phi_i|^2 + Q_P|p|^2 - r\right)^2 + \sum_\alpha |G^\alpha|^2 + |p|^2 \sum_a |J^a|^2\,. \tag{B.I.2.36}$$

The supersymmetric vacua of the theory are then defined by the vanishing of this potential, which is written as a sum of three squares. Let us consider the 'phase' $r > 0$. Then since $Q_P < 0$, the vanishing of the first term implies that at least one of the ϕ_i's has a non-vanishing expectation value in the vacuum. The vanishing of the last term then implies that p has a vanishing vacuum expectation value. Finally, modding out by the gauge group, the vanishing of the second terms cuts out the

complete intersection defined by the vanishing of the polynomials G^α:

$$\bigcap_{\alpha=1}^{p} \{\phi_i \mid G^\alpha(\phi_i) = 0\} \ . \tag{B.I.2.37}$$

Second, the holomorphic vector bundle is built, in the simplest examples, by the set of $r+1$ Fermi multiplets Γ_a. Indeed, r of these left-handed Weyl fermions can be shown to transform as sections of a rank r holomorphic vector bundle E, determined by the short exact sequence[2]

$$0 \longrightarrow E \xrightarrow{\iota} \bigoplus_{a=1}^{r+1} \mathcal{O}(Q_a) \xrightarrow{\otimes J^a} \mathcal{O}(-Q_P) \longrightarrow 0\,, \tag{B.I.2.38}$$

one out of the $r+1$ Weyl fermions obtaining a mass through the Yukawa coupling appearing in eq. (B.I.2.35).

Let us add as a side remark that in the presence of the fermionic gauge transformation discussed previously, with this time non-chiral Fermi superfields $\bar{D}_+\Gamma_a = \sqrt{2}E_a(\Phi)$, the holomorphic vector bundle E can be shown to be given by the cohomology of the sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\otimes E_a} \bigoplus_{a=1}^{r+1} \mathcal{O}(Q_a) \xrightarrow{\otimes J^a} \mathcal{O}(-Q_P) \longrightarrow 0\,. \tag{B.I.2.39}$$

We will consider only models with chiral Fermi superfields in the following for simplicity.

The phase $r < 0$ on the other hand is characterised by a non-vanishing VEV for p, hence breaking down the $U(1)$ gauge group to a discrete $\mathbb{Z}_{Q_P}$ gauge group, leading to a Landau-Ginzburg orbifold.

Allowing for higher rank gauge group on the worldsheet leads to more complicated phase diagrams. In the following we will basically restrict ourselves to rank 1 gauge groups for simplicity of the exposure.

Let us make a few extra comments about the gauge charges, and their relation to spacetime properties. In addition to the quadratic constraint corresponding to the vanishing of the anomaly, there are also some linear constraints arising from the requirement of describing a Calabi-Yau complete intersection. One indeed has then

[2]For simplicity of the presentation we do not consider adding the fermionic gauging, cf. comment below eq. (B.I.2.11).

to select the G^α with the appropriate degrees:

$$\sum_i Q_i + \sum_\alpha Q_\alpha = 0 \Leftrightarrow c_1(T_X) = 0\,. \tag{B.I.2.40}$$

One can in addition add the extra bundle requirement:

$$\sum_a Q_a + Q_P = 0 \Leftrightarrow c_1(E) = 0\,. \tag{B.I.2.41}$$

The two conditions eqs. (B.I.2.40) and (B.I.2.41) precisely ensure that the theory is invariant under the global $U(1)_{\rm L} \times U(1)_{\rm R}$ symmetry.

Let us finally say that the Fayet-Iliopoulos parameter generically exhibit a non-trivial RG flow, with a one-loop log divergent diagram, proportional to the sum of the scalar charges of the theory. In order to be able to make sense without any ambiguity of statements like '$r > 0$', one needs to ensure the non-renormalisation of r by adding, if necessary, a chiral/Fermi pair of spectator fields $\left(\tilde\Phi, \tilde\Gamma_{\tilde\Phi}\right)$ to the model together with a mass superpotential [30]:

$$\mathcal{L}_{(\tilde\Phi,\tilde\Gamma_{\tilde\Phi})} = \int \mathrm{d}\theta\, m_{\tilde\Phi}\tilde\Phi\tilde\Gamma_{\tilde\Phi}\,, \tag{B.I.2.42}$$

which therefore do not have an impact on the infrared theory, and with the appropriate gauge charges $\left(Q_{\tilde\Phi}, Q_{\tilde\Gamma_{\tilde\Phi}}\right) = (-\sum_i Q_i - Q_P, \sum_i Q_i + Q_P)$.

I.3 Elliptic genus

I.3.1 Field-theoretic and geometric definitions

Given a $(0,2)$ SCFT, or any ultraviolet completion of it in the form of a non-linear sigma model with target a holomorphic vector bundle E over a Calabi-Yau manifold X, or a gauged linear sigma model describing in the IR of its geometric phase such a NLSM, one defines the elliptic genus as a refined Witten index in which one also keeps track of the grading of the Hilbert space of the theory by a global left $U(1)_{\rm L}$ charge, in addition to the grading by the energy. Alternatively, this elliptic genus can be viewed a partition function in the Ramond-Ramond sector of the theory with boundary conditions twisted by an extra $U(1)_{\rm L}$ fugacity insertion in the trace:

$$Z_{\rm ell}(\tau, z) := \mathrm{tr}_{\rm RR}\left\{(-1)^F y^{J_0} q^{L_0 - \frac{c}{24}} \bar q^{\bar L_0 - \frac{\bar c}{24}}\right\}\,, \tag{B.I.3.1}$$

with $q = \exp(2i\pi\tau)$ and $y = \exp(2i\pi z)$ and the total fermion number $F = F_{\mathrm{L}} + F_{\mathrm{R}}$ (in the SCFT, one has $F = J_0 + \bar{J}_0$).

One can understand this definition in the following way, illuminating the fact that the elliptic genus is an index. Consider the right-moving supercharge Q_{R} which according to the right-moving super-Poincaré algebra squares to the right-moving Hamiltonian

$$Q_{\mathrm{R}}^2 = \bar{L}_0 - \frac{\bar{c}}{24} \,. \tag{B.I.3.2}$$

Its index, namely the difference between the number of bosonic and fermionic states in the $\bar{L}_0 = \bar{c}/24$ eigenspace, is given by a trace involving only the right-moving operators in the definition eq. (B.I.3.1):

$$\mathrm{index}\,(Q_{\mathrm{R}}) = \mathrm{tr}(-1)^{F_{\mathrm{R}}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \,, \tag{B.I.3.3}$$

since states with $\bar{L}_0 \neq \bar{c}/24$ come in pairs of opposite $(-1)^{F_{\mathrm{R}}}$ eigenvalue. This index receives however contributions from the whole tower of left-moving excitations, hence is typically divergent. One therefore regularizes this index by inserting a convergence factor commuting with the right-moving supercharge, and this can be generically done by labelling the left-moving excitation by their L_0 and J_0 eigenvalue, and by signing them by the fermion number $(-1)^{F_{\mathrm{L}}}$, giving the elliptic genus eq. (B.I.3.1). Witten gave it the interpretation of a $U(1)$-equivariant index for a Dirac-like operator on the loop space of X [31].

As an index, the elliptic genus is independent of the RG flow, hence can be computed at any point on the flow starting from any UV completion of the SCFT of interest. This is precisely the reason why the GLSM approach will be of particular use in this thesis, since they are enough to capture the topological properties encoded in the elliptic genus. It was shown recently [32–34] how to compute the elliptic genus directly at the level of the gauged linear sigma-model, using supersymmetric localization [35].

It is very interesting that there actually exists an independent definition of the elliptic genus in terms of the geometric data of the holomorphic vector bundle E over X which precisely coincides with the field-theoretic definition. In this context the elliptic genus is defined as the holomorphic Euler characteristic of a formal series with bundle coefficients:

$$\mathbb{E}_{q,y} = \bigotimes_{n=0}^{\infty} \bigwedge\nolimits_{-yq^n} E^\star \otimes \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{-y^{-1}q^n} E \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_X^\star \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_X \,, \tag{B.I.3.4}$$

where one defined the total antisymmetric and symmetric powers:

$$\bigwedge_t E = \sum_{k=1}^{\infty} t^k \bigwedge^k E \,, \quad S_t T_X = \sum_{k=1}^{\infty} t^k \, S^k \, T_X \,, \tag{B.I.3.5}$$

$\bigwedge^k$ and S^k being respectively the k-th exterior product and the k-th symmetric product.

The elliptic genus is then given in terms of the holomorphic Euler characteristic of $\mathbb{E}_{q,y}$:

$$Z_{\text{ell}}(\tau, z) = q^{\frac{r-d}{12}} y^{-\frac{r}{2}} \chi(\mathbb{E}_{q,y}) \,. \tag{B.I.3.6}$$

We recall that for any holomorphic vector bundle $E \to X$, the holomorphic Euler characteristic is given by:

$$\chi(E) := \sum_{k \geq 0} (-1)^k \dim H^k(X, \mathcal{O}_X(E)) \,. \tag{B.I.3.7}$$

It follows from the Hirzebruch-Riemann-Roch theorem that this purely topological quantity can be computed from local curvature data:

$$\chi(E) = \int_X \text{ch}(E) \text{td}(X) \,, \tag{B.I.3.8}$$

where appear the total Chern charater of E and the total Todd class of T_X.

From a physics point of view, the formal series with bundle coefficients eq. (B.I.3.4) corresponds precisely to the excitation spectrum of the heterotic string. From a mathematical point of view, the elliptic genus is related to the concept of elliptic cohomology [36]. The elliptic genus generalizes the χ_y genus of Hirzebruch, for which we have[3]:

$$\begin{aligned} \chi_y(X) &:= \sum_{p,q} (-1)^q y^p h^{p,q} \\ &= \sum_p y^p \sum_q (-1)^q \dim H^q \left(X, \bigwedge^p T_X^\star \right) \\ &= \sum_p y^p \chi \left(\bigwedge^p T_X^\star \right) \\ &= \int_X \text{ch} \left(\bigwedge_y T_X^\star \right) \text{td}(X) \\ &= \int_X \prod_{i=1}^{d} x_i \frac{1 + y e^{-x_i}}{1 - e^{-x_i}} \,, \end{aligned} \tag{B.I.3.9}$$

[3] We restrict ourselves to $E = T_X$ here, i.e. to $(2,2)$ supersymmetry on the worldsheet from the field theory point of view. The generalization to a generic holomorphic vector bundle E is simple.

where we used the Atiyah-Singer index theorem, and with the Chern roots defined by the splitting principle

$$\mathrm{c}(T_X) := \prod_{i=1}^{d}(1+x_i)\,. \tag{B.I.3.10}$$

Generalizing to the infinite dimensional loop space, one looks at the equivariant χ_y genus $\chi_y(q, \mathcal{L}X)$ and regularizing the divergent integral, one can show that the natural object appearing is indeed the geometric elliptic genus eq. (B.I.3.6) (with $E = T_X$). Let us also mention that in the original definition of Ochanine [37], the elliptic genus is a particular case of multiplicative genus, i.e. a ring homomorphism ϕ from the oriented cobordism ring of X to some ring of automorphic functions, whose formal *logarithm* series

$$\log_\phi(u) := \sum \frac{\phi(\mathbb{P}^{2n})}{2n+1}u^{2n+1} \tag{B.I.3.11}$$

is given by an elliptic integral. We will however not use this definition in the following.

The elliptic genus is well-behaved under modular and elliptic properties, which can be understood from the worldsheet point of view as originating from the modular invariance of the theory and the spectral flow. Together they imply that the elliptic genus is a weak holomorphic Jacobi form of weight 0 and index $r/2$, with r the rank of E. Let us quickly recall what a Jacobi form is. We define the Jacobi group as the semi-direct product $\Gamma^J(1) := SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. For a function $\phi : \mathbb{H}\times\mathbb{C} \to \mathbb{C}$, we define the Petersson slash operation as:

$$(\phi|_{k,m,\chi}\gamma)(\tau,z) = \frac{\chi(g)}{(c\tau+d)^k}e^{2i\pi m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu\right)}\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right), \tag{B.I.3.12}$$

with $\gamma = \left(g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}, (\lambda,\mu)\right) \in \Gamma^J(1)$. A weak holomorphic Jacobi form of weight k, index m and character χ is then a holomorphic function $\phi : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ satisfying $\phi|_{k,m}\gamma = \phi\,, \forall\gamma \in \Gamma^J(1)$, with an extra requirement of holomorphy at the cusp:

$$\phi(\tau,z) = \sum_{4nm-r^2\geq 0} c(n,r)q^n y^r\,. \tag{B.I.3.13}$$

As an example, let us give the elliptic genus of a $(4,4)$ SCFT arising in the IR of the non-linear sigma model with $K3$ target space, equipped with its tangent bundle. It can be computed anywhere in the moduli space of K3 compactifications, for instance

using Landau–Ginzburg orbifolds [38–40], or toroidal orbifolds [41].

$$Z_{K3}(\tau,z)=8\left[\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2+\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\right)^2\right], \tag{B.I.3.14}$$

where the various Jacobi theta functions are defined in appendix E.I. One recovers the Euler number of $K3$: $e(K3)=Z_{K3}(\tau,0)=24$.

I.3.2 Moonshine

a Monstrous moonshine

The first instance of moonshine phenomena [42], namely unexpected connections between modular objects and the representation theory of certain sporadic groups, connects the largest sporadic group, the monster group $\mathbb{M}$, and $j(\tau)$, unique weight zero modular form for $SL_2(\mathbb{Z})$, defined by

$$j(\tau)=J_{SL_2(\mathbb{Z})}(\tau)+744=\frac{\left(\theta_2(\tau|0)^8+\theta_3(\tau|0)^8+\theta_4(\tau|0)^8\right)^3}{8\eta(\tau)^{24}}, \tag{B.I.3.15}$$

where $J_{SL_2(\mathbb{Z})}$ is the hauptmodul for $SL_2(\mathbb{Z})$, namely an isomorphism between the compact genus 0 complex curves:

$$J_{SL_2(\mathbb{Z})}(\tau):SL_2(\mathbb{Z})\backslash\mathbb{H}\cup\{\infty\}\xrightarrow{\sim}\mathbb{P}^1, \tag{B.I.3.16}$$

This hauptmodul admits the following q-expansion:

$$J_{SL_2(\mathbb{Z})}(\tau)=q^{-1}+196884\,q+21493760\,q^2+8642909970\,q^3+\dots \tag{B.I.3.17}$$

The observation of John McKay consists in noticing that:

$$196884=196883+1, \tag{B.I.3.18a}$$

$$21493760=21296876+196883+1, \tag{B.I.3.18b}$$

$$8642909970=842609326+21296876+2\cdot196883+2\cdot1, \tag{B.I.3.18c}$$

where appear on the right hand side the dimension of irreducible representations $R_1,R_{196883},R_{21296876},R_{842609326}$ of the largest sporadic group $\mathbb{M}$. Arise then the

question of the existence of a graded module

$$V = \bigoplus_{n=0}^{\infty} V_n \,, \tag{B.I.3.19}$$

with the first vector spaces given by $V_0 = R_1$, $V_1 = \{0\}$, $V_2 = R_1 \oplus R_{196883}$, $V_3 = R_1 \oplus R_{196883} \oplus R_{21296876}$, and for which the graded dimension is:

$$\dim(V;\tau) := \sum_{n=0}^{\infty} \dim(V_n) q^n = q J_{SL_2(\mathbb{Z})}(\tau) \,. \tag{B.I.3.20}$$

Requiring furthermore that for each element $g \in \mathbb{M}$ the so-called *McKay-Thompson series* defined by

$$J_g(\tau) := q^{-1} \sum_{n=0}^{\infty} \mathrm{tr}_{V_n}(g) q^n \,, \tag{B.I.3.21}$$

constitutes the hauptmodul for a genus 0 subgroup H_g of $SL_2(\mathbb{R})$ leads to the unicity, if existence, of the module V. V was built [43] as the Hilbert space in the Ramond sector of the chiral CFT built out of 24 free bosons compactified on the orbifold $(\mathbb{R}^{24}/\Lambda_{\mathrm{Leech}})/\mathbb{Z}_2$. This theory has holomorphic partition function $\mathrm{tr}_{\mathrm{R}}\, q^{L_0 - c/24} = J_{SL_2(\mathbb{Z})}(\tau)$, and $\mathbb{M}$ as a symmetry group. Modularity of the graded traces eq. (B.I.3.21) can be understoog conveniently from this physics point of view.

b Mathieu moonshine

Consider, as mentioned above, a non-linear sigma model with target space a $K3$ surface equipped with its tangent bundle. One can show that the hyperkähler nature of the $K3$ target space actually implies that the worldsheet theory actually exhibits extended $(4,4)$ supersymmetry. The striking observation made by Eguchi, Ooguri and Tachikawa in [44] consist in noticing that if one expands the $K3$ elliptic genus in terms of $N = 4$ super-Virasoro characters, the coefficients of the expansion are all integers corresponding to the dimension of representation of the sporadic group M_{24}, one of the five sporadic simple Mathieu groups.

Consider indeed a non-linear sigma model with target a K3 surface whose symmetry algebra splits into a chiral and an anti-chiral piece:

$$(N = 4 \text{ super-Virasoro}) \otimes (N = 4 \text{ super-Virasoro}) \,. \tag{B.I.3.22}$$

In particular, the spectrum of the theory organizes itself into representations of the

$N = 4$ superconformal algebras:

$$\mathcal{H} = \sum_{i,\bar{j}} N_{i\bar{j}}\, \mathcal{H}_i \otimes \bar{\mathcal{H}}_{\bar{j}}\,. \tag{B.I.3.23}$$

The moduli space of such $K3$ sigma models takes the form:

$$\mathcal{M} = O(4,20;\mathbb{Z})\backslash O(4,20;\mathbb{R})/O(4,\mathbb{R}) \times O(20,\mathbb{R})\,. \tag{B.I.3.24}$$

The spectrum of states eq. (B.I.3.23) has a fairly complicated structure and is not known exactly at a generic point in the above moduli space.

The elliptic genus eq. (B.I.3.1), namely the following partial index in the Ramond-Ramond sector of the theory

$$Z_{\text{ell}}(\tau, z) := \text{tr}_{\text{RR}} \left\{ (-1)^{F_{\text{L}}} y^{J_0} q^{L_0 - \frac{c}{24}} (-1)^{F_{\text{R}}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right\}\,, \tag{B.I.3.25}$$

is actually a holomorphic quantity, hence receives only contributions from the BPS states defined as those states satisfying $\bar{L}_0 = \bar{c}/24$ (cf. comment below eq. (B.I.3.3)), i.e. the right-moving ground states. Such states then organize themselves in terms of the left-moving $N = 4$ representations as follows:

$$\mathcal{H}_{\text{BPS}} = 20 \cdot \mathcal{H}_{h=\frac{1}{4},j=0} \oplus 2 \cdot \mathcal{H}_{h=\frac{1}{4},j=\frac{1}{2}} \oplus \bigoplus_{n=1}^{\infty} M_n \otimes \mathcal{H}_{h=n+\frac{1}{4},j=\frac{1}{2}}\,, \tag{B.I.3.26}$$

where the M_n's are multiplicity vector spaces, which depend on the point in moduli space.

The expansion of the elliptic genus takes the following form:

$$Z_{K3}(\tau, z) = 24\, \text{ch}_{h=\frac{1}{4},l=0}(\tau, z) + \sum_{n=0}^{\infty} A_n\, \text{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau, z)\,, \tag{B.I.3.27}$$

where $\text{ch}_{h=1/4,l=0}$ is the character of the massless representation of isospin zero and $\text{ch}_{h=k+1/4,l=1/2}$ are characters of massive representations of isospin one-half. The multiplicities precisely correspond to

$$A_n = \text{tr}_{M_n} (-1)^{F_{\text{R}}}\,, \tag{B.I.3.28}$$

and do not depend on the point in moduli space. In the spirit of McKay-Thompson series, The coefficients $\{A_n\}$ of the expansion are surprisingly related to dimensions of M_{24} irreducible representations as $A_0 = -2$, $A_1 = 90 = 45 + \overline{45}$, etc..., i.e.

$A_n = \dim(R_n)$ for some representation of M_{24}.

In the spirit of the McKay-Thompson series in the context of the Monstruous moonshine, and supposing that the spectrum of BPS states do carry an action of M_{24}, one can look at characters:

$$\dim(R_n) \longrightarrow \mathrm{tr}_{R_n}\, g\,, \tag{B.I.3.29}$$

where g corresponds to some element of M_{24}. These characters of course only depend on the conjugacy class of g. From the field theory point of view, this would correspond to compute twining genera of the form:

$$Z_{\mathrm{ell}}(\tau, z) := \mathrm{tr}_{\mathrm{RR}} \left\{ g(-1)^{F_{\mathrm{L}}} y^{J_0} q^{L_0 - \frac{c}{24}} (-1)^{F_{\mathrm{R}}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right\}, \tag{B.I.3.30}$$

which are expected to be modular objects under the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \, c \equiv 0 \mod N \right\}, \tag{B.I.3.31}$$

of the modular group $SL_2(\mathbb{Z})$, with $N = \mathrm{ord}(g)$. Using in particular constraints from modularity, Gaberdiel and collaborators [45] determined such twining genera, and indeed showed that their coefficients in an $N = 4$ character expansion corresponded the characters of representations of M_{24}.

This Mathieu moonshine can be extended to $K3$ compactifications with arbitrary gauge bundles [46], despite the fact that the underlying two-dimensional theory has only $(0, 4)$ supersymmetry (hence no $N = 4$ on the holomorphic side) thanks to the universality properties of the new supersymmetric index dictated by its modular properties [47].

Chapter II

Computation of the dressed elliptic genus

This chapter is built out of the article: *New supersymmetric index of heterotic compactifications with torsion*, with Dan Israël, arXiv:1509.05704, JHEP 1512 (2015) 069.

II.1 Gauged linear sigma-models with torsion

Let us now turn to the main goal of this thesis, namely the computation of what we called the *dressed elliptic genus* associated to Fu-Yau compactifications. As we will see, this dressed elliptic genus naturally generalizes the notion of elliptic genus in the context of the non-Kähler compactifications we consider here. Moreover, from the dressed elliptic genus derives naturally a quantity known in the literature as the *new supersymmetric index* [48], which in particular counts the BPS states in spacetime [49].

In this part we will compute by localization in a torsional GLSM the dressed elliptic genus corresponding to such compactifications. We will restrict in this chapter to the cases where all the 'Wilson line' moduli are turned off. Their inclusion requires to generalize the GLSM description, and will be the topic of the next chapter (chapter B.III). We will then introduce a purely mathematical quantity associated to holomorphic vector bundles over non-Kähler principal torus fibration over Calabi-Yau d-folds, which we will loosely call a modified holomorphic Euler characteristic, and show that this object actually coincides with the Fu-Yau dressed elliptic genus computed in a field-theoretic framework.

The dressed elliptic genus constitutes, among other things, the building block in computing the one-loop corrected threshold corrections to the gauge and gravitational couplings of the low energy effective supergravity action corresponding to Fu-Yau compactifications, which we will discuss later in part C. We will restrict to the case without 'Wilson line' moduli for simplicity of the computation and of the exposure, but no conceptual reasons would prevent us in principle to include such moduli.

As discussed in the introductory section A.II.2, one considers a compactification of the $E_8 \times E_8$ heterotic string down to four dimension on a manifold taking the form of cartesian product $M_4 \times X$ in string frame[1]. One also breaks the $E_8 \times E_8$ gauge bundle by adding to the data a vector bundle E on the internal manifold X.

As discussed, the internal manifold X should be a complex manifold with trivial canonical sheaf $K_X \simeq \mathcal{O}_X$, characterized by a global holomorphic $(3,0)$-form Ω. In addition the internal manifold should satisfy the so-called *conformally balanced* equation:

$$\mathrm{d}\left(||\Omega||J \wedge J\right) = 0\,, \tag{B.II.1.1}$$

and the bundle over it should satisfy the *Hermite-Yang-Mills* equations:

$$F_{(2,0)} = F_{(0,2)} = 0\,, \tag{B.II.1.2a}$$

$$F \wedge J \wedge J = 0\,. \tag{B.II.1.2b}$$

Finally, the data of the manifold X and the holomorphic vector bundle E over it are tied together through the three-form flux by the heterotic Bianchi identity:

$$\mathrm{d}H = 2i\partial\bar{\partial}J = \frac{\alpha'}{4}\left(\mathrm{tr}(R \wedge R) - \mathrm{tr}(F \wedge F)\right)\,, \tag{B.II.1.3}$$

with the flux given by:

$$H = d^c J\,. \tag{B.II.1.4}$$

The constraint that the manifold should be conformally balanced is a weaker condition than Kählerity, and one indeed sees from eq. (B.II.1.4) that whenever torsion is present, the geometry will not be Kähler. eq. (B.II.1.2a),eq. (B.II.1.2b) can be rephrased by demanding that E should be a stable holomorphic vector bundle. Finally, eq. (B.II.1.3) is a consequence of the Green-Schwarz mechanism, and is usually the hardest condition to verify. As mentioned in section A.II.1, the right-hand side

[1] In Einstein frame, there is an extra warp factor.

of eq. (B.II.1.3) is computed using a connection with torsion on the tangent bundle; various possible choices correspond to different regularization schemes in the underlying non-linear sigma-model [3].

The type of solution to the above set of constraints we are interested in correspond to a principal two-torus bundle over a warped $K3$ surface $\mathcal{S}$ together with the pullback of a stable holomorphic vector bundle E over the base to the total space. As we tried to motivate in the introduction, since the quantities we are ultimately interested in are topological in nature, an ultraviolet completion the NLSM with such a target space should be enough to procede to the computation.

The first class of torsion GLSM that was obtained by Adams and collaborators [23] was especially designed to give a worldsheet theory for the flux compactifications under consideration.

The microscopic description of Fu-Yau compactifications as torsion GLSMs provides some evidence for their consistency at the quantum level, beyond the supergravity regime[2] [50, 51]. This approach was also used in [52] to compute their massless spectra using Landau-Ginzburg methods, and in [53] to obtain exact statements about their duality symmetries. A very interesting aspect of the latter work, which will play an important role in the chapter, was that covariance of the theory under perturbative $O(2,2;\mathbb{Z})$ dualities along the two-torus fibre imposes that its moduli are those of a $c = 2$ rational conformal field theory.

We will derive the new supersymmetric index directly from torsion GLSMs corresponding to Fu-Yau compactifications with $\mathcal{N} = 2$ supersymmetry, using supersymmetric localization. Several steps of the derivation are similar to the computation of the elliptic genera for 'ordinary' gauged linear sigma-models [32–34]. There are however important new aspects related to the presence of gauge anomalies cancelled against classically non gauge-invariant interactions. With the choice of supercharge $\mathcal{Q}$ appropriate to the problem, the action of the torsion multiplet, representing the torus fibre, is not $\mathcal{Q}$-exact, and the measure in field space is not $\mathcal{Q}$-invariant; as we will demonstrate, supersymmetric localization makes sense nonetheless for the full theory.

[2] As for ordinary GLSMs the arguments leading to the absence of destabilization by worldsheet instantons in torsion GLSMs should be taken with a grain of salt. In the present case however space-time $\mathcal{N} = 2$ supersymmetry presumably prevents such corrections from contributing to the effective superpotential.

II.1.1 Anomalous gauged linear sigma-model for the base

As the first step of this construction, one considers a standard (0,2) gauged linear sigma-model for the K3 base; generically such model suffers from gauge anomalies, that, in the usual case of Calabi-Yau GLSMs, should be made to vanish by a suitable choice of field content hence of gauge bundle in space-time. In the present case, one cancels instead the anomalous variation of the functional measure against a classically non-gauge-invariant Lagrangian for a *torsion multiplet* modeling the T^2 principal bundle, thereby realizing the Green-Schwarz mechanism on the worldsheet.

For simplicity of the discussion, we restrict ourselves in the following discussion to the case of a $U(1)$ gauge group on the worldsheet; the generalization to higher rank gauge groups is rather straightforward and will be briefly mentioned in section B.II.4. The conventions we use for $(0,2)$ superfields, as well as the components Lagrangian, are given in section B.I.2.

As already discussed in the introductory part of this thesis, a $(0,2)$ gauged linear sigma-model for a complete intersection Calabi-Yau manifold in a weighted projective space [29] contains first a set of n $(0,2)$ chiral multiplets Φ_i, as well as a set of p Fermi multiplets $\tilde{\Gamma}_\alpha$, interacting through the superpotential

$$\mathcal{L}_{\mathrm{T}} = \int \mathrm{d}\theta\, \tilde{\Gamma}_\alpha G^\alpha(\Phi_i) + h.c.\,, \tag{B.II.1.5}$$

where the $G^\alpha(\phi_i)$ are quasi-homogeneous polynomials of the appropriate degree to preserve gauge invariance at the classical level and, geometrically, to obtain a hypersurface of vanishing first Chern class. This Calabi-Yau hypersurface corresponds then to the complete intersection $\bigcap_{\alpha=1}^{p} \{\phi_i \mid G^\alpha(\phi_i) = 0\}$.

Second, the holomorphic vector bundle is described, in the simplest examples, by adding a set of $r+1$ Fermi multiplets Γ_a, a single chiral multiplet P and the superpotential

$$\mathcal{L}_{\mathrm{V}} = \int \mathrm{d}\theta\, P\, \Gamma_a\, J^a(\Phi_i) + h.c.\,, \tag{B.II.1.6}$$

where the J^a are again quasi-homogeneous polynomials. In the geometrical "phase", where the real part of the Fayet-Iliopoulos coupling is taken large and positive, one expects that the model flows to a non-linear sigma-model on the CY hypersurface with left-handed fermionic degrees of freedom transforming as sections of a rank s holomorphic vector bundle E, determined by the short exact sequence:

$$0 \longrightarrow E \xrightarrow{\iota} \bigoplus_{a=1}^{s+1} \mathcal{O}(Q_a) \xrightarrow{\otimes J^a} \mathcal{O}(-Q_P) \longrightarrow 0\,. \tag{B.II.1.7}$$

As the $(0,2)$ multiplets contain chiral fermions there are potentially gauge anomalies on the worldsheet that should be canceled. The model should also contain a non-anomalous global right-moving $U(1)$ symmetry which corresponds in the infrared to the $U(1)_{\mathrm{R}}$ symmetry of the $N=2$ superconformal algebra, and a global left-moving $U(1)_{\mathrm{L}}$ symmetry, used to implement the left-moving GSO projection.

The variation of the effective Lagrangian under a super-gauge transformation of chiral parameter Ξ reads

$$\delta_{\Xi}\mathcal{L}_{\text{eff}} = -\frac{\mathcal{A}}{4}\int \mathrm{d}\theta\, \Xi\Upsilon \;\; + h.c., \tag{B.II.1.8}$$

with Υ the field strength superfield, and the anomaly coefficient

$$\mathcal{A} = \sum_i {Q_i}^2 + {Q_P}^2 - \sum_\alpha {Q_\alpha}^2 - \sum_a {Q_a}^2, \tag{B.II.1.9}$$

which measures the difference between the second Chern character of the tangent bundle of the base manifold and the second Chern character of the vector bundle over the latter, cf. section B.I.2. If one considers a model with $\mathcal{A} \neq 0$, then the theory is at this point ill-defined quantum mechanically.

II.1.2 Two-torus principal bundle and anomaly cancellation

In the original work of Adams and collaborators [23], the two-torus bundle over the K3 base is built up by first constructing a $\mathbb{C}^*\times\mathbb{C}^*$ non-compact bundle, and then changing complex structure in field space, allowing to discard the decoupled non-compact part from the $\mathbb{C}^*\times\mathbb{C}^* = \mathbb{C}\times T^2$ bundle, while preserving $(0,2)$ supersymmetry. We refer the reader to section B.I.2 for our conventions for supersymmetric gauge theories in $(0,2)$ superspace.

To start, one introduces two extra chiral multiplets $\Omega_1 = (\omega_1, \chi_1)$ and $\Omega_2 = (\omega_2, \chi_2)$, whose (imaginary) shift symmetry is gauged as

$$\delta_{\Xi}\,\Omega_\ell = -i\,M_\ell\,\Xi\;, \quad M_\ell \in \mathbb{Z}\;, \quad \ell = 1,2\,. \tag{B.II.1.10}$$

The compact bosonic fields $\mathrm{Im}(\omega_\ell)$ will ultimately parametrize the torus fibre.

A generic two-torus is characterized by a complex structure $T = T_1 + iT_2$ and a complexified Kähler modulus $U = U_1 + iU_2$, such that the metric and Kalb-Ramond

field are given by

$$G = \frac{U_2}{T_2}\begin{pmatrix} 1 & T_1 \\ T_1 & |T|^2 \end{pmatrix}\,, \quad B = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}. \tag{B.II.1.11}$$

The Lagrangian for Ω_1 and Ω_2, corresponding to a complexification of this two-torus, reads [53]:

$$\begin{aligned}
\mathcal{L}_s = &-\frac{iU_2}{4T_2}\int \mathrm{d}^2\theta\ \Big(\Omega_1+\bar\Omega_1+T_1\left(\Omega_2+\bar\Omega_2\right)+2(M_1+T_1M_2)A_+\Big)\times \\
&\qquad\times\Big(\partial_-\left(\Omega_1-\bar\Omega_1+T_1\left(\Omega_2-\bar\Omega_2\right)\right)+2i(M_1+T_1M_2)A_-\Big) \\
&-\frac{iU_2T_2}{4}\int \mathrm{d}^2\theta\ \Big(\Omega_2+\bar\Omega_2+2M_2A_+\Big)\Big(\partial_-\left(\Omega_2-\bar\Omega_2\right)+2iM_2A_-\Big) \\
&+\frac{iU_1}{4}\int \mathrm{d}^2\theta\ \Big\{\Big(\Omega_1+\bar\Omega_1+2M_1A_+\Big)\Big(\partial_-\left(\Omega_2-\bar\Omega_2\right)+2iM_2A_-\Big)- \\
&\qquad-\Big(\Omega_2+\bar\Omega_2+2M_2A_+\Big)\Big(\partial_-\left(\Omega_1-\bar\Omega_1\right)+2iM_1A_-\Big)\Big\} \\
&-\frac{iN^i}{2}\int \mathrm{d}\theta\,\Upsilon\,\Omega_i+h.c.
\end{aligned} \tag{B.II.1.12}$$

The couplings between the chiral superfields Ω_ℓ and the field strength superfield Υ contain field-dependent Fayet-Iliopoulos (FI) terms (last line) that are classically non-invariant under (super)gauge transformations:

$$\delta_\Xi\mathcal{L}_s = -\frac{N^iM_i}{2}\int \mathrm{d}\theta^+\,\Upsilon\,\Xi+h.c.\,. \tag{B.II.1.13}$$

This gauge variation should be such that it compensates the one-loop anomaly eq. (B.II.1.9) of the base GLSM; this can be viewed as a worldsheet incarnation of the Green-Schwarz mechanism. Finally, in order for the action to be single-valued under $\omega_i \sim \omega_i + 2i\pi$ in any instanton sector, the couplings N^i should be integer-valued.

Moduli quantization

In order to restrict the non-compact $\mathbb{C}^*\times\mathbb{C}^*$ fibration described above to a T^2 fibration while maintaining $(0,2)$ worldsheet supersymmetry, one has to define a complex structure in field space that allows for a decoupling of the real part of these multiplets. This is compatible with supersymmetry provided that the couplings between the gaugini and the fermionic components of the superfields Ω_i vanish [23]. It amounts to the following relations between the Fayet-Iliopoulos parameters and

the charges [53]

$$N^1 = -\frac{U_2}{T_2}\mathrm{Re}(M) - U_1 M_2 \ \in \mathbb{Z}\,, \tag{B.II.1.14a}$$

$$N^2 = -\frac{U_2}{T_2}\mathrm{Re}(\bar{T}M) + U_1 M_1 \ \in \mathbb{Z}\,, \tag{B.II.1.14b}$$

with the complex charge M defined as

$$M = M_1 + T M_2\,. \tag{B.II.1.15}$$

Using these relations the gauge-variation of the field-dependent Fayet-Iliopoulos term reads:

$$\delta_\Xi \mathcal{L}_s = \frac{U_2}{2T_2}|M|^2 \int \mathrm{d}\theta\, \Upsilon\Xi + h.c.\,, \tag{B.II.1.16}$$

which should be cancelled against the gauge anomaly from the chiral fermions in order to get a consistent quantum theory. One obtains the condition

$$\sum_i {Q_i}^2 + {Q_P}^2 - \sum_\alpha {Q_\alpha}^2 - \sum_a {Q_a}^2 - \frac{2U_2}{T_2}|M|^2 = 0\,, \tag{B.II.1.17}$$

reproducing the tadpole condition from the integrated Bianchi identity in Fu-Yau compactifications [21], see section A.II.3.

The torus moduli T and U are partially quantized by the pair of supersymmetry conditions eq. (B.II.1.14); in a model with worldsheet gauge group $U(1)^k$ one obtains one such condition for each complex charge M_κ, hence the moduli are generically fully quantized. As was shown in [53], covariance of the theory under T-duality symmetries along the fibre provides another way of understanding quantization of the torus moduli. Under the transformation $U \mapsto -1/U$ in $PSL(2;\mathbb{Z})_U$, each complex charge M is mapped to $-\bar{U}M$. For consistency this charge should belong to the same lattice as the original one, namely $\bar{U}M \in \mathbb{Z} + T\mathbb{Z}$.

Demanding that this property holds for every topological charge in the model is actually a non-trivial statement. Generically, this is true if and only if the elliptic curve $E_T = \mathbb{C}/(\mathbb{Z} + T\mathbb{Z})$ admits a non-trivial endomorphism

$$\begin{aligned} E_T &\to E_T \\ z &\mapsto \bar{U} z\,, \end{aligned} \tag{B.II.1.18}$$

which is known as *complex multiplication*. This property holds if and only if both U and T are valued in the same imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$ with D

the discriminant of a positive definite integral quadratic form:

$$D = b^2 - 4ac < 0 \ , \quad a, b, c \in \mathbb{Z} \ , \quad a > 0 \, . \tag{B.II.1.19}$$

Crucially, $c = 2$ conformal field theories with a two-torus target space are rational iff their T and U moduli satisfy these conditions [54, 55]; this property will play an important role in section B.II.3.

One could also consider incorporating in the torsion GLSM terms corresponding to extra Abelian gauge bundles over the total space X (that would be Wilson lines along the torus in the $K3 \times T^2$ case), which are indeed allowed by the space-time supersymmetry constraints [21]. This generalization is discussed in chapter B.III.

Torsion multiplet

Whenever the supersymmetry conditions eq. (B.II.1.14) are met, the non-compact real parts of $\Omega_{1,2}$ decouple and one can reorganize their imaginary parts into a *torsion multiplet* $\Theta = (\alpha, \chi)$, with

$$\alpha = \mathrm{Im}\,(\omega_1) + T\,\mathrm{Im}\,(\omega_2), \tag{B.II.1.20}$$

$$\chi = \mathrm{Re}\,(\chi_1) + \bar{T}\,\mathrm{Re}(\chi_2), \tag{B.II.1.21}$$

shifted as $\delta_\Xi \Theta = -M\,\Xi$ under supergauge transformations, with the complex charge M defined in eq. (B.II.1.15). The Lagrangian is given (omitting temporarily the topological B-field term for simplicity) by

$$\mathcal{L}_{\text{t.m.}} = -i\frac{U_2}{T_2}\int \mathrm{d}^2\theta \left(\bar{\Theta} + 2i\bar{M}A_+\right)\mathcal{D}_-\Theta - \frac{\bar{M}}{2}\frac{U_2}{T_2}\int \mathrm{d}\theta\,\Theta\Upsilon \ + h.c., \tag{B.II.1.22}$$

where the superspace covariant derivative reads $\mathcal{D}_-\Theta = \partial_-\Theta - \frac{iM}{2}(2\partial_- A_+ + iA_-)$.

As usual going to Wess-Zumino gauge is convenient in order to exhibit the physical degrees of freedom; in the present situation one should not forget nevertheless that the theory is not classically gauge invariant, hence such gauge choice only makes sense in the path integral of the full quantum theory, including the base GLSM, as will be clear below when supersymmetric localization will be put into action.

In this gauge the torsion multiplet contains a compact complex boson coupled chirally to a gauge field and a free right-moving Weyl fermion. After going to

Euclidean signature[3] and some rescaling of the fields, one has

$$\frac{T_2}{U_2}\,\mathcal{L}_{\text{t.m.}} = \nabla_z\bar{\alpha}\nabla_{\bar{z}}\alpha + \nabla_z\alpha\nabla_{\bar{z}}\bar{\alpha} - \frac{1}{2}\left(M\bar{\alpha} + \bar{M}\alpha\right)a_{z\bar{z}} + 2\bar{\chi}\partial\chi, \qquad \text{(B.II.1.23)}$$

with $\nabla_z\alpha = \partial\alpha + Ma_z$ and $\nabla_{\bar{z}}\alpha = \bar{\partial}\alpha + Ma_{\bar{z}}$, and where $a_{z\bar{z}} = 2\left(\partial a_{\bar{z}} - \bar{\partial}a_z\right)$ denotes the field strength of the gauge field. After integrating by parts, one gets the following Lagrangian

$$\frac{T_2}{U_2}\,\mathcal{L}_{\text{t.m.}} = 2\partial\bar{\alpha}\bar{\partial}\alpha + 2Ma_{\bar{z}}\partial\bar{\alpha} + 2\bar{M}a_{\bar{z}}\partial\alpha + 2|M|^2a_za_{\bar{z}} + 2\bar{\chi}\partial\chi + t.d.\,, \quad \text{(B.II.1.24)}$$

where the left-moving $U(1)$ currents $\partial\alpha$ and $\partial\bar{\alpha}$ are coupled to the gauge fields, but not the right-moving ones.

Since we are working in Wess-Zumino gauge, the appropriate supersymmetry transformations are given by

$$\delta_\epsilon = \left(\epsilon Q_+ - \bar{\epsilon}\bar{Q}_+ + \delta_{\text{gauge}}\right), \qquad \text{(B.II.1.25)}$$

where δ_{gauge} refers to the supergauge transformation which is needed to restore Wess-Zumino gauge after the supersymmetry transformation, corresponding to the chiral parameter $\Xi_{wz} = i\bar{\epsilon}\theta^+a_{\bar{z}}$. The transformation properties of the different component fields are listed in section B.I.2, eq. (B.I.2.18); the Lagrangian for the torsion multiplet is not invariant under this transformation, but its variation is precisely such that it compensates the variation of the effective action of the base GLSM under the gauge transformation back to WZ gauge.

To summarize, a consistent torsion gauged linear sigma-model is given by a base K3 GLSM whose gauge anomaly is canceled by a torsion multiplet, provided that the tadpole condition eq. (B.II.1.17) holds. Finally one has to choose the $U(1)_{\text{L}}$ and $U(1)_{\text{R}}$ charges of all the multiplets in order to cancel the global anomalies, and to obtain the correct central charges of the IR superconformal algebra and rank of the vector bundle. The global charges of the torsion multiplet, proportional to their gauge charge M, correspond naturally to charges under a shift symmetry. Consistent choices of global charges will be given in section B.II.4.

[3] One first Wick rotates to Euclidean time $\sigma^2 = -i\sigma^0$. Complex coordinates are then defined by $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$.

II.2 Dressed elliptic genus of $\mathcal{N} = 2$ compactifications

We consider $\mathcal{N} = 2$ compactifications to four dimensions of the $E_8 \times E_8$ heterotic string theory. For any $(0, 2)$ superconformal field theory with $(c, \bar{c}) = (22, 9)$ corresponding to the 'internal' degrees of freedom of such compactification, the new supersymmetric index is defined as the following trace over the Hilbert space in the right Ramond sector:

$$Z_{\text{NEW}}(\tau, \bar{\tau}) = \frac{1}{\eta(\tau)^2} \text{Tr}_{\text{R}} \left[\bar{J}_0 (-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right] , \tag{B.II.2.1}$$

where F_R is the right-moving fermion number and $\bar{J}_0$ the zero-mode of the right-moving R-current, which is part of the (right-moving) superconformal algebra. In general, this index is independent of D-term deformations, while it is sensitive to F-term deformations [48].

It was observed in [56, 57] that the threshold corrections to the gauge and gravitational couplings of $\mathcal{N} = 2$ heterotic string compactifications on $\text{K3} \times T^2$ are easily obtained in terms of the new supersymmetric index eq. (B.II.2.1). Furthermore Harvey and Moore showed in [49] that it counts the four-dimensional BPS states as

$$-\frac{1}{2i\eta^2} Z_{\text{NEW}}(q, \bar{q}) = \sum_{\text{BPS vectors}} q^{\Delta} \bar{q}^{\bar{\Delta}} - \sum_{\text{BPS hypers}} q^{\Delta} \bar{q}^{\bar{\Delta}} . \tag{B.II.2.2}$$

One of the goals of this paper is to extend this analysis to Fu-Yau geometries. Formulaneq. (B.II.2.2) was proven using representation theory of the $\mathcal{N} = 4$ superconformal algebra underlying the $K3$ CFT. As was explained in [26], non-linear sigma-models with a Fu-Yau target space are invariant under the action of the generators of a $(0, 2) \oplus (0, 4)$ superconformal algebra, at the classical level, hence we expect that a similar reasoning holds in the present case.

Let us perform explicitely the left-moving GSO projection using standard orbifold formulæ, see *e.g.* [40], and write the new supersymmetric index in the following form:

$$Z_{\text{NEW}}(\tau, \bar{\tau}) := \frac{\bar{\eta}^2 E_4(\tau)}{\eta^{10}} \frac{1}{2} \sum_{\gamma, \delta = 0}^{1} q^{\gamma^2} \left\{ \left(\frac{\theta_1(\tau | z)}{\eta(\tau)} \right)^{8-r} Z_{\text{FY}}(\tau, \bar{\tau}, z) \right\} \Big|_{z = \frac{\gamma\tau + \delta}{2}} , \tag{B.II.2.3}$$

where we have defined what we will call the *dressed elliptic genus*:

$$Z_{\text{FY}}(\tau, \bar{\tau}, z) = \frac{1}{\bar{\eta}^2} \text{Tr}^{\text{int}}_{\text{RR}} \left[(-1)^F y^{J_0} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \bar{J}_0 \right] , \tag{B.II.2.4}$$

with the superscript 'int' meaning that the trace is restricted to the internal $(22,9)$ CFT. This dressed elliptic genus will constitute the main character in this part of the thesis. As we just saw, this objet is the building block in deriving the new supersymmetric index.

Let us make a few comments at this stage about this dressed elliptic genus we just defined. When dealing with Calabi-Yau compactifications, i.e. $\mathcal{N}=1$ vacua, the typical object one introduces is the elliptic genus, cf. section B.I.3:

$$Z_{\text{ELL}}(\tau,z)=\text{Tr}_{\text{RR}}\left[(-1)^F y^{J_0} q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right], \tag{B.II.2.5}$$

However, as we will see in the following, the elliptic genus vanishes identically in the context of $T^2\to X\to \text{CY}_{n-1}$ compactifications. This can be seen from various points of view. From the field-theoretic point of view, we will see that the culprit if the fermionic zero-mode which is necessarily part of the sigma-model modelization of the torus fibre. One can also see it as a consequence of the purely mathematical definition of Ochanine of an elliptic genus, see [37]. A non-vanishing index can be obtained by inserting an extra $\bar{J}_0$ in the trace to saturate the fermionic zero-mode of the torus, and one precisely recovers the dressed elliptic genus. As we will see in the following, in the case where the torus fibration is trivial, the dressed elliptic genus further factorizes in the elliptic genus of the base and the Narain partition function and oscillators contribution of the torus fibre.

II.2.1 Dressed elliptic genus of K3 $\times$ $\mathbf{T^2}$ compactifications

We first review the computation of the new supersymmetric index in the familiar case of K3 $\times$ T^2 compactifications of the $E_8\times E_8$ heterotic string, without Wilson lines for simplicity. We emphasize the role of the left-moving GSO projection and the formulation of the index as a chiral orbifold in order to facilitate the generalization to Fu-Yau compactifications in the next subsection.

We assume that the gauge bundle lies in the first E_8 only. More specially, we consider a gauge bundle E of structure group G with the embedding $G\subset SO(2r)\subset E_8$. The internal CFT is then the tensor product of a $(0,2)$ theory with $(c,\bar{c})=(14,9)$ and a $(c,\bar{c})=(8,0)$ theory corresponding to the second E_8 factor.

Using the factorization of the $(c,\bar{c})=(14,9)$ CFT in the two-torus and K3 factors, hence the decomposition of the corresponding $(0,2)$ superconformal algebra

into the direct sum $(0,2)\oplus(0,4)$, we split the right-moving R-current as follows:

$$\bar{J} = \bar{J}^{\mathrm{T}^2} + \bar{J}^{\mathrm{K3}}\,. \tag{B.II.2.6}$$

It allows to expand the superconformal index into the sum of two terms. For the second one, we get

$$\mathrm{Tr}_{\mathrm{R}}\left[\bar{J}_0^{\mathrm{K3}}(-1)^{F_R}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right] = 0\,, \tag{B.II.2.7}$$

for two different reasons. First, the fermionic partners of the T^2 have a pair of fermionic zero modes of opposite fermion numbers, hence the trace over the two-torus Hilbert space vanishes. Second the K3 SCFT has $\mathcal{N}=(0,4)$ superconformal symmetry, hence the eigenvalues of $\bar{J}_0^{\mathrm{K3}}$, which are twice the eigenvalues of the Cartan current of the $SU(2)_1$ R-symmetry, come in pairs of opposite sign [49].

In order to trace over the internal Hilbert space of the theory we have to define a left-moving GSO projection corresponding to the first E_8 factor. We assume the existence of a $U(1)_{\mathrm{L}}$ left-moving symmetry, acting on the $(0,4)$ SCFT describing the K3 surface as on the remaining $(8-r)$ free left-moving Weyl fermions of the first E_8.

We consider the dressed elliptic genus as introduced above, i.e. the twining partition function in the RR sector, with a chemical potential y for this $U(1)_{\mathrm{L}}$ symmetry, which we denoted Z_{FY} above and will denote here $Z_{\mathrm{K3}\times T^2}$:

$$Z_{\mathrm{K3}\times T^2}(\tau,\bar{\tau},z) = \frac{1}{\bar{\eta}(\bar{\tau})^2}\mathrm{Tr}_{\mathrm{RR},\mathcal{H}_{\mathrm{K3}\times T^2}}\left[e^{2i\pi z J_0}\bar{J}_0(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right]\,, \tag{B.II.2.8}$$

with J_0 the left-moving $U(1)$ current, $(-1)^F = \exp i\pi(J_0-\bar{J}_0)$ and where the trace is over the Hilbert space of the K3 $\times$ T^2 $(0,2)$ superconformal field theory with $(c,\bar{c}) = (6+r,9)$. Then the new supersymmetric index is obtained as a sum over the sectors of the chiral $\mathbb{Z}_2$ quotient corresponding to the left-moving GSO projection, cf. eq. (B.II.2.3):

$$Z_{\mathrm{NEW}}(\tau,\bar{\tau}) = \frac{\bar{\eta}^2 E_4(\tau)}{2\eta^{10}}\sum_{\gamma,\delta=0}^{1} q^{\gamma^2}\left\{\left(\frac{\theta_1(\tau|z)}{\eta(\tau)}\right)^{8-r} Z_{\mathrm{K3}\times T^2}(\tau,\bar{\tau},z)\right\}\Big|_{z=\frac{\gamma\tau+\delta}{2}}\,, \tag{B.II.2.9}$$

where the modular form $E_4(\tau)$ comes from the contribution of the second E_8 factor, see appendix E.I.

The partition function over the two-torus degrees of freedom is straightforward.

For a torus with complex and Kähler moduli T and U the soliton sum $\Xi_{2,2}(U,T|\tau,\bar\tau)$ is given by

$$\Xi_{2,2}(\tau,\bar\tau|T,U) = \sum_{m_i,n_i\in\mathbb{Z}} \exp\left(-\frac{\pi}{\tau_2}\frac{U_2}{T_2}\left|m_1+n_1\tau+T(m_2+n_2\tau)\right|^2+2i\pi U(m_1n_2-n_2m_1)\right). \tag{B.II.2.10}$$

Then the contribution to the partition function eq. (B.II.2.8) reads:

$$\begin{aligned}\mathrm{Tr}_{\mathrm{RR},\mathcal{H}_{T^2}}&\left[\bar J_0(-1)^{\bar J_0}q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24}\right]\\ &=\frac{1}{2i\pi}\frac{\partial}{\partial\alpha}\Big|_{\alpha=1/2}\mathrm{Tr}_{\mathrm{RR},\mathcal{H}_{T^2}}\left[e^{2i\pi\alpha\bar J_0}q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24}\right]\\ &=\frac{\Xi_{2,2}(T,U)}{\eta^2\bar\eta^2}\,\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}\frac{\theta_1(\bar q,e^{2i\pi\alpha})}{\bar\eta}=\frac{\Xi_{2,2}(T,U)}{i\eta^2}\,.\end{aligned} \tag{B.II.2.11}$$

Finally one needs to compute the trace over the Hilbert space of the $(0,4)$ theory with a K3 target space. Let us consider the case of the standard embedding of the spin connection in the gauge connection, enhancing the supersymmetry of the K3 SCFT to $\mathcal{N}=(4,4)$. Then plugging back the expression eq. (B.II.2.11) into eq. (B.II.2.9), and tracing over the Hilbert space of the 6 free Weyl fermions with twisted boundary conditions, one gets finally the index in terms of the K3 elliptic genus [49]:

$$Z_{\mathrm{NEW}}=\frac{E_4(\tau)\Xi_{2,2}}{2i\eta^{12}\bar\eta^2}\sum_{\gamma,\delta=0}^{1}q^{\gamma^2}\left(\frac{\theta_1(\tau,\frac{\gamma\tau+\delta}{2})}{\eta(\tau)}\right)^6 Z^{\mathrm{ELL}}_{\mathrm{K3}}\left(\tau,\frac{\gamma\tau+\delta}{2}\right), \tag{B.II.2.12}$$

where the $(2,2)$ elliptic genus of K3 is defined by

$$Z^{\mathrm{ELL}}_{\mathrm{K3}}(\tau,z)=\mathrm{Tr}_{\mathrm{RR},\mathcal{H}_{\mathrm{K3}}}\left[e^{2i\pi zJ_0}(-1)^F q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24}\right]. \tag{B.II.2.13}$$

As we shall see in the next section, this localization method which we refer to in section B.I.3 can be generalized to compute the dressed elliptic genus of Fu-Yau compactifications.

The new supersymmetric index of K3$\times T^2$ compactifications is actually universal, *i.e.* independent of the choice of gauge bundle, as was shown in [47], and reviewed recently in [46]. The quantity $\tau_2 Z_{\mathrm{NEW}}$ should be a non-holomorphic modular form of weight -2, with a pole at the infinite cusp (this will remain valid in the case of

Fu-Yau compactifications). Factorizing the index as

$$Z_{\text{NEW}}(\tau,\bar\tau) = -2i\frac{\Xi_{2,2}(\tau,\bar\tau)}{\eta(\tau)^4}\mathcal{G}_{\text{K3}}(\tau)\,, \tag{B.II.2.14}$$

one can show that $\eta^{20}\mathcal{G}_{\text{K3}}(\tau)$ should be a holomorphic modular form of weight 10, hence proportional to E_4E_6. Due to the relation eq. (B.II.2.2) the space-time anomaly cancellation condition $n_H - n_V = 244$ fixes the coefficient to one. Hence the expression eq. (B.II.2.12), obtained from the standard embedding with $(4,4)$ worldsheet supersymmetry, extends to any $(0,4)$ compactification; it means in particular that the "Mathieu moonshine" is a property of the $K3\times T^2$ new supersymmetric index regardless of the choice of gauge bundle [46]. Determining whether this property extends to Fu-Yau compactifications is one of the motivations for the present work.

II.2.2 Dressed elliptic genus of Fu-Yau compactifications

We now turn to the main point of this chapter, the computation of the dressed elliptic genus of Fu-Yau compactifications based on their worldsheet formulation as torsion gauged linear sigma-models.

The starting point of the computation is the same as for $\text{K3}\times T^2$ compactifications. However in the case of torsion GLSMs one cannot split the worldsheet theory as a tensor product of the T^2 and the K3 factors, as none of them makes sense as a quantum theory in isolation. We assume as before that the gauge bundle over the total space, which is the pullback of a Hermite-Yang-Mills gauge bundle E over the K3 base, whose structure group G is embedded as $G\subset SO(2r)\subset E_8$ in the first E_8 factor.

As explained above, the quantity we want to compute is written in operator language as follows:

$$Z_{\text{FY}}\,(\tau,\bar\tau,z) = \frac{1}{\bar\eta(\bar\tau)^2}\text{Tr}_{\text{RR},\mathcal{H}_{\text{FY}}}\left[e^{2i\pi z J_0}\bar J_0(-1)^F q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24}\right]\,, \tag{B.II.2.15}$$

the trace being taken into the Hilbert space of the $(0,2)$ superconformal theory obtained as the infrared fixed point of the torsion GLSM.

A crucial point at this stage is that the right-moving fermions $(\chi,\bar\chi)$ associated with the T^2 factor, that belong to the torsion multiplet, are free in the Wess-Zumino gauge, see the Lagrangian eq. (B.II.1.24), in particular not coupled to the components of gauge multiplet; this is the feature of the theory that eventually leads to

$\mathcal{N}=2$ supersymmetry in space-time. The right-moving R-current of the superconformal algebra, whose zero-mode $\bar{J}^0$ appears in the trace eq. (B.II.2.15), is of the form

$$\bar{J} = \bar{\chi}\chi + \cdots , \tag{B.II.2.16}$$

where the ellipsis stands for (i) a term in $\bar{\partial}\alpha$, as the bottom component of the torsion multiplet can have a shift R-charge, (ii) the contributions of the chiral and Fermi multiplets and (iii) $\mathcal{Q}$-exact terms, where $\mathcal{Q}$ is the localization supercharge (see next section), relating the exact R-current to the Noether one defined in the UV theory. Because there are two right-moving fermionic zero-modes χ_0 and $\bar{\chi}_0$ that need to be saturated in the path integral, and that there are no interactions involving these fermionic fields in the Lagrangian, we do not have to care about these extra terms in any case, as their contribution to the path integral vanishes.

In summary, the new supersymmetric index of Fu-Yau compactifications follows from the twisted partition function eq. (B.II.2.15) that can be formulated as a path integral. Considering the theory on a two-dimensional Euclidean torus of complex structure τ, the quantity to compute can be schematically written as

$$\begin{aligned} Z_{\text{FY}}(\tau,\bar{\tau},z) = \frac{1}{\bar{\eta}(\bar{\tau})^2} \int \mathscr{D}A_-\mathscr{D}A_+\mathscr{D}\mu\mathscr{D}\bar{\mu}\mathscr{D}D\, e^{-\frac{1}{e^2}S_{\text{v.m.}}[A,\mu,D]-t\,S_{\text{FI}}(A,D)} \times \\ \times \int \mathscr{D}\phi_i\mathscr{D}\bar{\phi}_i\mathscr{D}\lambda_i\mathscr{D}\bar{\lambda}_i\, e^{-\frac{1}{g^2}S_{\text{c.m.}}[\phi_i,\lambda_i,A,D,a_{\text{L}}]} \times \\ \times \int \mathscr{D}\gamma_a\mathscr{D}\bar{\gamma}_a\mathscr{D}G_a\mathscr{D}\bar{G}_a\, e^{-\frac{1}{f^2}S_{\text{f.m.}}[\gamma_a,G_a,A,a_{\text{L}}]-S_{\text{J}}[\gamma_a,G_a,\phi_i,\lambda_i]} \times \\ \times \int \mathscr{D}\alpha\mathscr{D}\bar{\alpha}\mathscr{D}\chi\mathscr{D}\bar{\chi}\, e^{-S_{\text{t.m.}}[\alpha,\chi,A,a_{\text{L}}]} \int \frac{\mathrm{d}^2w}{2\tau_2}\,\bar{\chi}\chi\,, \end{aligned} \tag{B.II.2.17}$$

where we have included a background gauge field for the $U(1)_{\text{L}}$ global symmetry

$$a_{\text{L}} = \frac{\pi z}{2i\tau_2}(\mathrm{d}w - \mathrm{d}\bar{w})\,, \tag{B.II.2.18}$$

in order to implement the twisted boundary conditions.[4] The torsion multiplet will be coupled chirally to this flat connection, in the same way as it couples to the dynamical gauge field, see eq. (B.II.1.24).

The left- and right-moving fermions have periodic boundary conditions along both one-cycles of the worldsheet torus. We have also included for latter convenience coupling constants $1/g^2$ and $1/f^2$ in front of the chiral and Fermi multiplets actions,

[4] Strictly speaking, we define the twisted path integral for real y (corresponding to twisted boundary conditions along the space-like cycle) and consider an analytic continuation of the result, see [58].

respectively $S_{\text{c.m.}}$ and $S_{\text{f.m.}}$, besides the usual $1/e^2$ factor in front of the vector multiplet action $S_{\text{v.m.}}$ and t in front of the Fayet-Iliopoulos term S_{FI}. Finally $S_{\text{t.m.}}$ denotes the torsion multiplet action.

To take care of the gauge redundancy one should in principle introduce a gauge-fixing procedure and the corresponding Faddeev-Popov ghosts; however it does not really impact the computation of the path integral through supersymmetric localization that will follow, see [59] for details.

Having set the calculation in functional language will allow us to deal with it using localization techniques. In this formulation one sees that the insertion of the $\bar{J}_0$ operator only contributes through the free right-moving fermion χ which is part of the torsion multiplet, and this insertion appears as a prescription to deal with the fermionic zero modes. This will be important in proving that the supersymmetric localization method is valid in this context, as we shall explain below.

II.3 Dressed elliptic genus through localization

In this section we obtain the dressed elliptic genus of Fu-Yau compactifications, defined by eq. (B.II.2.15), allowing to compute their new supersymmetric index using eq. (B.II.2.3). We consider the case of a $U(1)$ worldsheet gauge group; the main result is given by equation eq. (B.II.3.50). The generalization to higher rank will be provided in the next section.

II.3.1 Review of the localization principle

Let we briefly recall the motivation underlying the supersymmetric localization techniques. Consider the vacuum expectation value of a given operator $O[\phi]$ in quantum field theory admitting a symmetry at the classical and quantum level, generated by a Grassmann-odd operator $\mathcal{Q}$, which is such that $\mathcal{Q}^2 = \delta_{\text{B}}$, with δ_{B} the generator of a bosonic symmetry of the theory (for instance a Lorentz or gauge symmetry). We wish to compute

$$\langle O \rangle = \int \mathcal{D}\phi \, e^{-S[\phi]} O[\phi] \, . \tag{B.II.3.1}$$

The operator is taken to be $\mathcal{Q}$-closed, i.e. $\{\mathcal{Q}, O\} = 0$. The action and measure are also $\mathcal{Q}$-closed, since $\mathcal{Q}$ generates a symmetry at the classical and quantum levels. The choise of supercharge is in general not unique for a given theory.

The idea is that one can deform the theory by adding a positive definite $\mathcal{Q}$-exact term to the action, i.e. look at the quantity

$$\langle O\rangle_t = \int \mathcal{D}\phi\, e^{-S[\phi]-t\mathcal{Q}V} O[\phi]\,, \tag{B.II.3.2}$$

where $V[\phi]$ is a fermionic quantity such that $\delta_{\mathrm{B}} V = 0$. The quantity we are interested in is therefore $\langle O\rangle_{t=0}$. The crucial observation is that $\langle O\rangle_t$ does not actually depend on the value of t. Indeed

$$\begin{aligned}\frac{\partial}{\partial t}\langle O\rangle_t &= -\int \mathcal{D}\phi\, e^{-S[\phi]-t\mathcal{Q}V}\,(\mathcal{Q}V)\, O[\phi] \\ &= -\int \mathcal{Q}\left(\mathcal{D}\phi\, e^{-S[\phi]-t\mathcal{Q}V} V O[\phi]\right) \\ &= 0\,,\end{aligned} \tag{B.II.3.3}$$

after application of Stokes theorem in field space[5]. Since the integral does not depend on the value of t, it can be evaluated in different regimes, such a as $t \to 0$ or $t \to \infty$, the results being equal. This kind of manipulation can lead to highly non-trivial functional identities, where the two side of the identities relate to different regimes of the parameter t.

In the integral above, the limit $t \to \infty$ is particularly interesting because the integral localizes to $\mathcal{Q}$-fixed points, i.e. to a BPS sub-locus of field space on which $\mathcal{Q}V$ vanishes, and which in the most favourable cases turns out to be finite dimensional, hence reducing the problem of computing a complicated infinite dimensional path integral to a finite dimensional integral. In this limit, the saddle point approximation becomes exact. We decompose

$$\phi = \phi_0 + \frac{\delta\phi}{\sqrt{t}}\,, \tag{B.II.3.4}$$

into a piece parametrizing the BPS locus and a fluctuation orthogonal to it. This leads to

$$\langle O\rangle = \int \mathrm{d}\phi_0 \mathcal{D}\delta\phi\, e^{-S\left[\phi_0+\frac{\delta\phi}{\sqrt{t}}\right]-t\left(\mathcal{Q}V(\phi_0)+\frac{\partial^2\mathcal{Q}V}{\partial\phi^2}(\phi_0)\frac{(\delta\phi)^2}{t}+\mathcal{O}\left(t^{-3/2}\right)\right)} O\left[\phi_0+\frac{\delta\phi}{\sqrt{t}}\right] \tag{B.II.3.5}$$

One thus gets, after taking the limit $t \to \infty$ and performing the Gaussian integral

[5] We suppose that no boundary terms appear here.

over the fluctuation:

$$\langle O \rangle \underset{t\to\infty}{=} \int \mathrm{d}\phi_0 \, e^{-S[\phi_0]} O\left[\phi_0\right] \frac{1}{\sqrt{\det\left(\frac{\partial^2 \mathcal{Q}V}{\partial\phi^2}[\phi_0]\right)}} \tag{B.II.3.6}$$

In the 2-dimensional quantum field theories considered in this thesis, the Lagrangian are actually $\mathcal{Q}$-exact, so one does not even need to add a $\mathcal{Q}$-exact deformation term to the action to apply these techniques of supersymmetric localization. In particular, one can normalize the fields such that the kinetic terms comes with a factor $\frac{1}{e^2}$ for the gauge fields and $\frac{1}{g^2}$ or $\frac{1}{f^2}$ for the matter fields, cf. eq. (B.I.2.27), and consider the ultraviolet limit $e \to 0$, $g \to 0$ and $f \to 0$ where the kinetic terms dominate and the theory becomes free.

II.3.2 Justification of the supersymmetric localization

Supersymmetric localization techniques have been successfully applied to compute the elliptic genera of ordinary (0,2) gauged linear sigma-models, see [32–34]. Our goal is to extend these results to compute the dressed elliptic genus of Fu-Yau compactifications, using the torsion gauged linear sigma-models.

One immediate objection to this project is that, as mentioned above, the contribution of the torsion multiplet to the action is not invariant under the supersymmetry transformations eq. (B.II.1.25); furthermore, the operator insertion

$$\int \frac{\mathrm{d}^2 w}{2\tau_2} \, \bar{\chi}\chi \tag{B.II.3.7}$$

in the path integral eq. (B.II.2.17) is obviously not supersymmetric. As we will see below, these two obstacles can be successfully overcome.

The GLSM corresponding to the base contains a $(0,2)$ vector multiplet together with $(0,2)$ chiral and Fermi multiplets (conventions related to $(0,2)$ superspace are gathered in section B.I.2). The first step in applying localization is to choose a fermionic symmetry of the theory, we therefore define the following supercharge, cf. eq. (B.I.2.26):

$$\mathcal{Q} := \left(\delta_{\epsilon,\bar{\epsilon}}\right)\big|_{\epsilon=\bar{\epsilon}=1} \, , \tag{B.II.3.8}$$

One can then show that the $(0,2)$ Lagrangians describing the dynamics of the $U(1)$ vector multiplet, the Fermi multiplet and the chiral multiplets, as well as the superpotential and Fayet-Iliopoulos terms are actually all exact with respect to the above defined supercharge, cf. eqs. (B.I.2.27) and (B.I.2.28). In an ordinary GLSM,

this would imply immediately that the path integral is independent of the coupling constants e, f and g and of the FI parameter t.

To understand what happens in the present situation, let us write the contribution of the base and of the vector multiplet to the TGLSM as

$$S_{\mathrm{K3}} = \frac{1}{e^2}\mathcal{Q}\nu_{\mathrm{v.m.}} + \mathcal{Q}\nu\,, \tag{B.II.3.9}$$

where the first term is the vector multiplet action, written as a $\mathcal{Q}$-exact term, and $\mathcal{Q}\nu$ denotes the ($\mathcal{Q}$-exact as well) contribution of the chiral and Fermi multiplets and of the constant FI term. The functional integral we aim to compute is of the schematic form:

$$Z_{\mathrm{FY}}(\tau,\bar\tau,z) = \frac{1}{\bar\eta(\bar\tau)^2}\int \mathscr{D}\Phi\mathscr{D}\Gamma\mathscr{D}\mathcal{A}\mathscr{D}\mathcal{V}\mathscr{D}\Theta\; e^{-\frac{1}{e^2}\mathcal{Q}\mu_{\mathrm{v.m.}}-\mathcal{Q}\nu}e^{-S_{\mathrm{t.m.}}[\Theta,\mathcal{A},\mathcal{V}]}\int\frac{\mathrm{d}^2w}{2\tau_2}\;\bar\chi\chi \tag{B.II.3.10}$$

One considers then the derivative with respect to $1/e^2$:

$$\begin{aligned}&\frac{\partial Z_{\mathrm{FY}}(\tau,\bar\tau,z)}{\partial(1/e^2)} = \\ &= -\frac{1}{\bar\eta(\bar\tau)^2}\int \mathscr{D}\Phi\mathscr{D}\Gamma\mathscr{D}\mathcal{A}\mathscr{D}\mathcal{V}\mathscr{D}\Theta\;\mathcal{Q}\mu_{\mathrm{v.m.}}\; e^{-\frac{1}{e^2}\mathcal{Q}\mu_{\mathrm{v.m.}}-\mathcal{Q}\nu}e^{-S_{\mathrm{t.m.}}[\Theta,\mathcal{A},\mathcal{V}]}\int\frac{\mathrm{d}^2w}{2\tau_2}\;\bar\chi\chi\,.\end{aligned} \tag{B.II.3.11}$$

As mentioned above, the operator insertion $\int\frac{\mathrm{d}^2w}{2\tau_2}\;\bar\chi\chi$ has the effect of saturating the fermionic zero modes present in the measure $\mathscr{D}\Theta$ over the torsion multiplet. Hence its variation under the action of the supercharge $\mathcal{Q}$, while non-zero, leads to terms which do not saturate the fermionic zero modes anymore, and thus do not contribute to the path integral.

Since the supersymmetry transformation we are considering contains a supergauge transformation of chiral parameter $\Xi_{wz}|_{\bar\epsilon=1}$, see eq. (B.II.1.25), there is a non-trivial transformation of the functional measure over the chiral and Fermi multiplets due to the gauge anomaly. At the same time, the torsion multiplet action is not classically invariant under the action of the supercharge, see eq. (B.II.1.16). Whenever the tadpole condition eq. (B.II.1.17) is satisfied, these two variations cancel each other:

$$\mathcal{Q}\left(\mathscr{D}\Phi\mathscr{D}\Gamma e^{-S_{\mathrm{t.m.}}[\Theta,\mathcal{A},\mathcal{V}]}\right) = 0\,. \tag{B.II.3.12}$$

In conclusion, whenever the quantum anomaly of the base GLSM is canceled against the classical contribution from the torsion multiplet, we get as in more familiar

examples

$$\frac{\partial}{\partial(1/e^2)} Z_{\text{FY}}(\tau,\bar\tau,z) = \\ = -\frac{1}{\bar\eta(\bar\tau)^2}\int \mathcal{Q}\left(\mathscr{D}\Phi\mathscr{D}\Gamma\mathscr{D}\mathcal{A}\mathscr{D}\mathcal{V}\; e^{-\frac{1}{e^2}\mathcal{Q}\mu_{\text{v.m.}}-\mathcal{Q}\nu-W[\mathcal{A},\mathcal{V}]}\,\mu_{\text{v.m.}}\int\frac{\mathrm{d}^2w}{2\tau_2}\;\bar\chi\chi\right) = 0\,, \tag{B.II.3.13}$$

using an analogue of Stokes' theorem in field space. The result of the path integral is then independent of the gauge coupling, allowing to take a free-field limit $e \to 0$. The same reasoning allows to take the limit $g \to 0$ and $f \to 0$ in the chiral and Fermi multiplets actions respectively. By rescaling the superfields $\Phi' = \Phi/g$ and $\Gamma' = \Gamma/f$ one sees that the superpotential couplings do not contribute to the path integral which is localized in the free-field limit of the theory, as far as the base GLSM is concerned.

A similar argument regarding the dependence of the path integral on the torsion multiplet couplings would fail, as the torsion multiplet action is not $\mathcal{Q}$-exact, being not even $\mathcal{Q}$-closed.[6] Nevertheless, this action is Gaussian hence the path integral can be performed exactly. As expected, it implies that the result of the path integral computation does depend on the moduli (T, U) of the principal two-torus bundle in the Fu-Yau geometry.

As in [33] the localization locus is parametrized by the holonomies of the gauge field modulo gauge transformations, corresponding to Wilson lines along the two cycles for the worldsheet torus,

$$a = \frac{\pi}{2i\tau_2}(\bar u\,\mathrm{d}w - u\,\mathrm{d}\bar w)\,, \tag{B.II.3.14}$$

the complex parameter u taking values in the torus

$$u \in \mathcal{M}_{\text{BPS}} = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}\,, \tag{B.II.3.15}$$

to avoid gauge redundancy. One could therefore expect, if everything worked safely and following the idea developped previously in section II.3.1, a result of the form

$$Z_{\text{FY}}(\tau,\bar\tau,z) = \int_{\mathcal{M}_{\text{BPS}}}\frac{\mathrm{d}^2u}{\tau_2}e^{-S[u,\bar u]}Z_{\text{one-loop}}(u,\bar u,z)\,, \tag{B.II.3.16}$$

[6] Even if it were the case (this is what happens if one considers $K3\times T^2$ examples), the $\mathcal{Q}$ variation of $\int\frac{\mathrm{d}^2z}{2\tau_2}\;\bar\chi\chi$ would then contribute to the path integral.

in analogy with eq. (B.II.3.6). This expression is however ill-defined due to the presence of various bosonic and fermionic zero-modes contributing to the one-loop determinant originating from the path integral over quadratic fluctuations around the finite dimensional BPS locus. Indeed, the bosons ϕ_i of gauge charge Q_i and left charge q_i^{L} contained in the chiral multiplets are charged under both the flat $U(1)$-gauge connection and the background flat $U(1)_{\mathrm{L}}$ connection, hence can become massless at some points on the BPS locus defined by

$$Q_i u + q_i^{\mathrm{L}} z = 0 \mod \mathbb{Z}[\tau]\,. \tag{B.II.3.17}$$

We will denote the singular locus defined in this way by the various chiral multiplets of the theory $\mathcal{M}_{\mathrm{sing}}$. The torsion multiplet does not introduce such singularities, cf. section II.3.4 below.

In [33], Benini, Eager, Hori and Tachikawa proposed a prescription to deal with these zero-modes for the computation of the elliptic genus of $(2,2)$ and $(0,2)$ gauge theories, which consists in first working at finite gauge coupling $e \neq 0$, i.e. in particular in the presence of the zero modes of the gaugino $\mu_0, \bar{\mu}_0$ and the auxiliary D-field[7] D_0, the latter acting through the quartic potential it generates as a regulator. The same prescription is followed here in presence of the torsion multiplet, and we will explaine how most of the steps that go into the derivation of the elliptic genus by Benini, Eager, Hori and Tachikawa [33], especially the reduction of the integral over the gauge holonomies into a contour integral of the one-loop determinants, carry over to the present situation without significant modifications.

The prescription therefore consists as we said to work first at a small but non-vanishing value of the gauge coupling e, and then cut out small disks of radius ϵ around each of the singular points in $\mathcal{M}_{\mathrm{sing}}$. Let us denote by $\mathcal{M}_{\mathrm{BPS}}^{\epsilon}$ this BPS locus with disks removed. It so happens, by a careful study of the behaviour of the contribution of the fields becoming massless in the $e \to 0$ limit, that if one first takes the limit $\epsilon \to 0$ and *only then*[8] the limit $e \to 0$, then the contribution of the integral restricted to the disks vanishes.

The above described prescription therefore leads to an expression of the form:

$$Z_{\mathrm{FY}} = \lim_{e,\epsilon \to 0} \int_{\mathbb{R}} \mathrm{d}D_0 \int_{\mathcal{M}_{\mathrm{BPS}}^{\epsilon}} \frac{\mathrm{d}^2 u}{\tau_2} \exp\left(-\frac{1}{2e^2} D_0^2 - i r D_0\right) f_e(u, \bar{u}, D_0)\,, \tag{B.II.3.18}$$

[7] We follow the convention of [33], D is here the Euclidean D-field, rescaled by π/τ_2

[8] One can also take the double limit $e \to 0$ and $\epsilon \to 0$ provided the bound $\epsilon < e^{M+1}$ is satisfied, where M is the number of bosonic fields vanishing at the singular point of interest.

where $f_e(u,\bar{u},D_0)$ corresponds to the result of the path integral over all fields apart the zero modes of the D-field and the gauge field, in the limit $g, f \to 0$, with finite gauge coupling e, including the contribution from the torsion multiplet. In addition to the zero-mode of the auxilary D-field, the vector multiplet also contains the zero-modes of the gaugino, whose contribution was included in $f_e(u,\bar{u},D_0)$. Explicitely, the fermionic measure corresponding to this zero-mode should be saturated, hence pulling down coulings between the gaugini and the bosons in the chiral multiplets:

$$f_e(u,\bar{u},D_0) = \int \mathrm{d}\mu_0 \mathrm{d}\bar{\mu}_0 \left\langle \left| \int \mathrm{d}^2 w \sum_i Q_i \mu \bar{\lambda}_i \phi_i \right|^2 \right\rangle . \tag{B.II.3.19}$$

The crucial point in our torsional theory is that since the torsion multiplet has no coupling to the gaugini, as was explained in section B.II.1, it is not involved in the saturation of their zero-modes.

Furthermore, the torsion multiplet has no coupling to the auxiliary D-field by supersymmetry; as a consequence, the $\bar{u}$-dependence of the one-loop determinant lies entirely in the contribution from the chiral multiplets of the base. One has in the limit:

$$f_e(u,\bar{u},D_0) \underset{e\to 0}{\longrightarrow} h(\tau,z,u,D_0)\, Z_{\text{base}}(\tau,z,u,D_0)\, Z_{\text{torsion}}(\tau,\bar{\tau},u,z)\,, \tag{B.II.3.20}$$

where $h(\tau,z,u,D_0)$ precisely arises from the saturation of the gaugini zero-modes, Z_{base} corresponds to the one-loop determinants arising from the integrals over quadratic fluctuations and includes the multiplets contributing to the $K3$ base. Z_{base} will be given in section II.3.3. Finally, Z_{torsion} corresponds to the path integral over the fields in the torsion multiplet coupled to the flat gauge connection u, and which we will determine soon, in section II.3.4.

Let us now tackle a subtle point which will appear more precisely in sections II.3.3 and II.3.4 where we compute Z_{base} and the contibution Z_{torus} from the torsion multiplet. The definition of the one-loop determinants for the various fields of the base involves infinite products which require to be renormalized, and all together determinants involve a non-holomorphic (in the holonomy u) exponential factor whose argument is proportional to the anomaly of the base GLSM eq. (B.II.1.9). The precise expression of this exponential anomalous factor depends on the prescription one takes for the determinant of the Dirac operator on the torus coupled to a flat $U(1)$-connection, various prescriptions being related by the addition of finite counterterms in the sigma model. On the other hand, the torsion multiplet contribution

will also come with a non-holomorphic exponential factor (all the non-holomorphicity in u is contained in this anomalous factor) which, whenever the adapted prescription is chosen for the determinants of the Dirac operator and the tadpole condition eq. (B.II.1.17) is satisfied, compensates the exponential factor from the base. We will now be sloppy with the notations, and assume that this cancellation of anomalous factors has occured in the product $Z_{\text{base}}(\tau, z, u, D_0)\, Z_{\text{torsion}}(\tau, \bar{\tau}, u, z)$. In particular, all the remaining non-holomorphicity in u comes from $Z_{\text{base}}(\tau, z, u, D_0)$.

Following [33], one has

$$f_e(u, \bar{u}, D_0) \underset{e\to 0}{\longrightarrow} -\frac{1}{\pi D_0}\frac{\partial}{\partial \bar{u}}\left[Z_{\text{base}}(\tau, z, u, \bar{u}, D_0)\, Z_{\text{torsion}}(\tau, \bar{\tau}, u, z)\right], \tag{B.II.3.21}$$

which is a total derivative in the anti-holomorphic holonomy, allowing to reduce the integral over the u-plane to a contour integral:

$$Z_{\text{FY}} = \lim_{e,\epsilon\to 0}\int_{\mathbb{R}} \frac{\mathrm{d}D_0}{2\pi i D_0} e^{-\frac{1}{2e^2}D_0^2 - irD_0} \oint_{\partial\mathcal{M}^{\epsilon}_{\text{BPS}}} \mathrm{d}u\, Z_{\text{base}}(\tau, z, u, \bar{u}, D_0) Z_{\text{torsion}}(\tau, \bar{\tau}, u, z)\,. \tag{B.II.3.22}$$

One should however be careful to avoid the origin in the D_0 complex plane, by translating the D_0 contour integral below the real axis at a distance δ, to get a contour Γ_δ. Let us see what happens for a given component of $\partial\mathcal{M}^{\epsilon}_{\text{BPS}}$, encircling a singular point $u^\star$. We can suppose that the gauge charge of all fields becoming massless at $u^\star$ have the same sign (if this is not the case, it is possible to achieve it by mixing the gauge and $U(1)_{\text{L}}$ charges.). It therefore makes sence to split the singular points as

$$\partial\mathcal{M}^{\epsilon}_{\text{BPS}} = \partial\mathcal{M}^{\epsilon+}_{\text{BPS}} \cup \partial\mathcal{M}^{\epsilon-}_{\text{BPS}}, \tag{B.II.3.23}$$

according to the sign of the fields becoming massless there.

As we will see in section II.3.3, in addition to the singularity at $D_0 = 0$, their are poles coming from Z_{base}, which involves factors with a pole at $D_0 = iQ_i\epsilon^2$ due to the bosons ϕ_i satisfying $Q_i u^\star + q_i^{\text{L}} z = 0$. Therefore, provided $0 < \delta \ll \epsilon^2$, the contour in D_0-plane avoids such poles. Simple application of Cauchy theorem (on the one hand, $\partial\mathcal{M}^{\epsilon+}_{\text{BPS}}$ does not contribute, and on the other hand the integral over the contour Γ'_δ defined in section II.3.2 vanishes) then allows to conclude that:

$$Z_{\text{FY}} = \oint_{\partial\mathcal{M}^{\epsilon-}_{\text{BPS}}} \mathrm{d}u\, Z_{\text{base}}(\tau, z, u, \bar{u}, 0) Z_{\text{torsion}}(\tau, \bar{\tau}, u, z)\,. \tag{B.II.3.24}$$

Had we chosen to deform the contour in D_0 plane above the real axis, we would

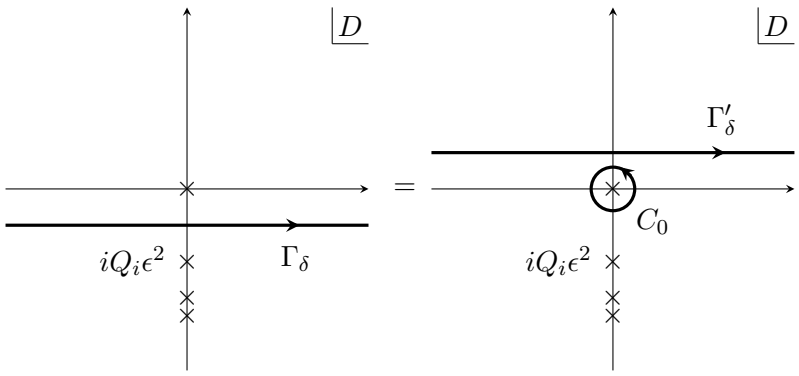


Figure II.1: Poles for $u^\star \in \partial\mathcal{M}^{\epsilon-}_{\text{BPS}}$ and contour integral over Γ'_-, equivalent to the contour over the sum of Γ'_δ and C_0.

have obtained:

$$Z_{\text{FY}} = -\oint_{\partial\mathcal{M}^{\epsilon+}_{\text{BPS}}} \mathrm{d}u\, Z_{\text{base}}(\tau, z, u, \bar{u}, 0) Z_{\text{torsion}}(\tau, \bar{\tau}, u, z)\,. \tag{B.II.3.25}$$

Both expressions agree, since the sum of residues of a meromorphic function on the torus vanishes.

For a rank-one gauge group, the formula for the dressed elliptic genus eq. (B.II.2.15) is therefore of the form:

$$Z_{\text{FY}}(\tau, \bar{\tau}, z) = \pm\frac{1}{2i\pi} \sum_{u^\star \in \mathcal{M}^{\pm}_{\text{sing}}} \oint_{\mathcal{C}(u^\star)} \Sigma_{\text{1-loop}}(\tau, \bar{\tau}, z, u)\,, \tag{B.II.3.26}$$

where

$$\Sigma_{\text{1-loop}}(\tau, \bar{\tau}, z, u) = Z_{\text{base}}(\tau, z, u, \bar{u}, 0) Z_{\text{torsion}}(\tau, \bar{\tau}, u, z)\,, \tag{B.II.3.27}$$

which we split into the contribution from the various fields as follows:

$$\Sigma_{\text{1-loop}}(\tau, \bar{\tau}, z, u) = \frac{1}{\bar{\eta}(\bar{\tau})^2} Z_A \times \left(\prod_{\Phi_i} Z_{\Phi_i}\right) \times \left(\prod_{\Gamma_a} Z_{\Gamma_a}\right) \times Z_\chi \times Z_{\text{torus}}\,, \tag{B.II.3.28}$$

the various factors in the above formula being the one-loop contributions of the various multiplets around the localization locus, and $\mathcal{C}(u^\star)$ denoting a contour around the singularity $u^\star$. $\mathcal{M}^{+}_{\text{sing}}$ and $\mathcal{M}^{-}_{\text{sing}}$ forms a partition of the set of poles of the product of chiral multiplet determinants according to the splitting eq. (B.II.3.23).

In the case where the gauge group G has an arbitrary rank, the formula general-

izes using a notion of residue in higher dimensions, the Jeffrey-Kirwan residue [60]; one obtains the following expression for the twining partition function in terms of the one-loop determinant [34]:

$$Z_{\text{FY}}(\tau,\bar{\tau},z)=\frac{1}{|W|}\sum_{u^\star\in\mathcal{M}_{\text{sing}}}\underset{u=u^\star}{\text{JK-Res}}\left(\text{Q}(u^\star),\eta\right)\Sigma_{\text{1-loop}}\,, \tag{B.II.3.29}$$

with now $\Sigma_{\text{1-loop}}$ a meromorphic rank(G)-form, and $|W|$ the order of the Weyl group. The sum does not depend on the choice of co-vector $\eta\in\mathfrak{h}^*$, in the dual of the Cartan subalgebra, per singular locus $u^\star$. We refer the reader to [34] for more details on the definition of the Jeffrey-Kirwan residue.

Let us now discuss the various contribution to $\Sigma_{\text{1-loop}}$, from the base and the torus fibre.

II.3.3 Contribution of the K3 base

As we have noticed previously, the contributions from the chiral and Fermi multiplets corresponding to the $K3$ base, as well as from the gauge multiplets, are similar to those appearing in the elliptic genus computed in [33]. However, since in the present context issues of gauge invariance are crucial, we need to be a little bit more careful regarding the definition of the chiral fermionic determinants. In the end, taking into account the contribution of the torsion multiplet, the tadpole condition will translate into a cancellation of the prefactors in these expressions, as discussed above.

In order to define the determinant of a chiral Dirac operator $\nabla(u)$ coupled to a (background) flat gauge field, one has to specify a way to split the determinant of the self-adjoint operator $\nabla^\dagger(u)\nabla(u)$ into a 'holomorphic' part and an 'anti-holomorphic' part. According to Quillen's theorem [61], the zeta-regularized determinant of the former is given by (see *e.g.* [62] for a discussion in a similar context):

$$\text{Det}_\zeta\,\nabla(u)^\dagger\nabla(u)=e^{\frac{\pi}{\tau_2}(u-\bar{u})^2}|\theta_1(\tau|u)|^2\,, \tag{B.II.3.30}$$

where u is here a compact notation which takes into account both the $U(1)$ gauge field and the background $U(1)_{\text{L}}$. Splitting $(u-\bar{u})^2=(u^2-u\bar{u})+(\bar{u}^2-u\bar{u})$, one can define the chiral determinant as:

$$\text{Det}\,\nabla(u)=e^{\frac{\pi}{\tau_2}(u^2-u\bar{u})}\theta_1(\tau|u)\,, \tag{B.II.3.31}$$

modulo an overall factor independent of u; other definitions can be interpreted as corresponding to different choices of local counterterms.

With this prescription, as was argued by Witten in [63] in a related context, the gauge functional obtained after the path integral over the fermionic degrees of freedom can be viewed as a holomorphic section of a holomorphic line bundle over the space of gauge connections. The determinant is indeed annihilated by the covariant derivative $\frac{D}{D\bar{u}} = \frac{\partial}{\partial \bar{u}} + \frac{\pi}{\tau_2} u$ (restricted to its zero-mode part in the present situation). It turns out that this choice, besides its nice geometrical interpretation, is naturally compatible with the contribution from the torsion multiplet Lagrangian, see eq. (B.II.3.43) below, leading to an expression without modular anomalies.

Equipped with this result, one can express the contribution of a $(0,2)$ chiral multiplet Φ_i of gauge charge Q_i and $U(1)_{\mathrm{L}}$ charge q_i^{L}, and a Fermi multiplet Γ_a of gauge charge Q_a and $U(1)_{\mathrm{L}}$ charge q_a^{L} as:

$$Z_{\Phi_i}(\tau, u, z) = i e^{-\frac{\pi}{\tau_2}(\upsilon^2 - \upsilon\bar{\upsilon})} \frac{\eta(\tau)}{\theta_1(\tau|\upsilon)} \quad , \ \upsilon = Q_i u + q_i^{\mathrm{L}} z \,, \tag{B.II.3.32a}$$

$$Z_{\Gamma_a}(\tau, u, z) = i e^{\frac{\pi}{\tau_2}(\upsilon^2 - \upsilon\bar{\upsilon})} \frac{\theta_1(\tau|\upsilon)}{\eta(\tau)} \quad , \ \upsilon = Q_a u + q_a^{\mathrm{L}} z \,. \tag{B.II.3.32b}$$

Finally, the contribution from the vector multiplet reads, considering a $U(1)$ gauge group,

$$Z_A(\tau) = -2i\pi\eta(\tau)^2 \mathrm{d}u \,. \tag{B.II.3.33}$$

These contributions combine to give Z_{base}. The final step will be to evaluate the contribution of the torsion multiplet.

II.3.4 Torsion multiplet determinant

In this section we derive the contribution of the torsion multiplet to the partition function $Z_{\mathrm{FY}}(\tau, \bar{\tau}, y)$ at the localization locus. In the functional integral formulation it takes the form

$$\int \mathscr{D}\alpha_1 \mathscr{D}\alpha_2 \mathscr{D}\chi \mathscr{D}\bar{\chi} \; e^{-S_{\mathrm{torsion}}[\alpha_1, \alpha_2, \chi, \bar{\chi}, a, a_{\mathrm{L}}]} \int \frac{\mathrm{d}^2 w}{2\tau_2} \; \bar{\chi}\chi. \tag{B.II.3.34}$$

One is thus dealing with two compact bosons chirally coupled to a flat gauge field, together with a free right-moving Weyl fermion. As was noticed before, the action is not $\mathcal{Q}$-exact, however it is Gaussian hence can be computed explicitly.

Let us evaluate first the contribution of the free fermion, which is completely decoupled from the vector multiplet. Taking into account the insertion $\int \frac{\mathrm{d}^2 z}{2\tau_2} \; \bar{\chi}\chi$

into the functional integral,[9] one gets, using formula eq. (E.I.1.8a):

$$Z_\chi(\bar\tau) = \partial_\mu \int \mathscr{D}\chi \mathscr{D}\bar\chi \; e^{-\int \mathrm{d}^2 w \, 2\bar\chi\partial\chi + \mu \int \frac{\mathrm{d}^2 w}{2\tau_2}\,\bar\chi\chi}\Big|_{\mu=0} = \bar\eta(\bar\tau)^2 \,. \tag{B.II.3.35}$$

Orthogonal torus Now we compute the contribution from the 'axion field' $\alpha = \alpha_1 + T\alpha_2$. It corresponds to a pair of chiral bosons coupled to a gauge field (a_+, a_-). Considering first an orthogonal torus with no B-field, for each of them one has to compute a path integral of the form:

$$\int \mathscr{D}\varphi \;\; \exp\left\{-\frac{R^2}{2\pi}\int \mathrm{d}^2 w \;\left(\partial_+\varphi\partial_-\varphi + 2a_+\partial_-\varphi + a_-a_+\right)\right\} , \tag{B.II.3.36}$$

where $\varphi \sim \varphi + 2\pi$. Here a_z and $a_{\bar z}$ refer to both the dynamical gauge field and to the background $U(1)_{\mathrm{L}}$ gauge field. In the present context, because of localization one focuses on the zero mode of the gauge fields, i.e. the holonomies on the worldsheet two-torus.

At the fermionic radius $R_{\mathrm{f}} = \frac{1}{\sqrt{2}}$, the bosonic action appearing in the path integral eq. (B.II.3.36) is actually nothing but the bosonized form of the chiral determinant eq. (B.II.3.31) (up to an anti-holomorphic determinant independent of the gauge field) that was considered by Witten in [63], hence motivating the choice made above for the latter. In the bosonic formulation this prescription amounts to set the coefficient of the a_-a_+ term, which can be affected by local counterterms, to one, and implies that the classical variation of the Lagrangian under a gauge transformation is independent of φ.

The different instanton sectors of the free compact boson φ on the torus obey the periodicity condition

$$\varphi\left(z + k + \tau l, \bar z + k + \bar\tau l\right) = \varphi\left(z, \bar z\right) + 2\pi\left(km + ln\right) \,, \tag{B.II.3.37}$$

with winding numbers $m, n \in \mathbb{Z}$. The solution for the zero-modes ϕ_0 is then given by:

$$\varphi_0\left(w, \bar w\right) = \frac{i\pi}{\tau_2}\left(w\left(m\bar\tau - n\right) - \bar w\left(m\tau - n\right)\right) \,. \tag{B.II.3.38}$$

Plugging this into the classical action and adding the contribution from the quantum

[9] In our conventions, $\int \mathrm{d}^2 w = 2\tau_2$

fluctuations leads to

$$Z_{S^1}(\tau,\bar\tau,u,z) = \exp\left(-\frac{2\pi}{\tau_2}R^2 u\bar u\right)\frac{R}{\sqrt{\tau_2}|\eta(\tau)|^2}\times \\ \times \sum_{m,n\in\mathbb{Z}} \exp\left(-\frac{\pi R^2}{\tau_2}|m\tau-n|^2 - 2iR^2\left(m\bar\tau - n\right)\left(a_{\bar z}\right)_0\right). \quad \text{(B.II.3.39)}$$

Poisson resummation formula appendix E.I applied to the dummy variable n allows to rewrite this as

$$Z_{S^1}(\tau,\bar\tau,u,z) = \frac{1}{|\eta(\tau)|^2}\exp\left(-\frac{2\pi}{\tau_2}R^2(u\bar u - u^2)\right)\times \\ \times \sum_{m,n\in\mathbb{Z}} \exp\left(\frac{i\pi\tau}{2}\left(\frac{n}{R}+Rm\right)^2 - \frac{i\pi\bar\tau}{2}\left(\frac{n}{R}-Rm\right)^2 - i\pi Ru\left(\frac{n}{R}+Rm\right)\right), \quad \text{(B.II.3.40)}$$

Setting the radius to the free fermion radius $R_{\mathrm{f}} = \frac{1}{\sqrt{2}}$, the above expression can be recast as a finite sum over the different spin structures on the worldsheet torus, namely

$$Z_{S^1}(\tau,\bar\tau,u,z;R_{\mathrm{f}}) = \frac{1}{2}\frac{1}{|\eta(\tau)|^2}e^{-\frac{2\pi}{\tau_2}R_{\mathrm{f}}^2\left[(Mu+m_{\mathrm{L}}z)(M\bar u+m_{\mathrm{L}}z)-(Mu+m_{\mathrm{L}}z)^2\right]}\times \\ \times \sum_{k,l=0}^{1}\theta\begin{bmatrix}k\\l\end{bmatrix}(\tau|Mu+m_{\mathrm{L}}z)\bar\theta\begin{bmatrix}k\\l\end{bmatrix}(\bar\tau|0), \quad \text{(B.II.3.41)}$$

where we have restored the shift charge M and added the coupling to the background $U(1)_{\mathrm{L}}$ gauge field. Taking into account the second S^1 is straightforward, since the two circles factorize.

Comparing the holomorphic part of the partition function eq. (B.II.3.41) to the contribution of a left-moving fermion coming from a charged Fermi multiplet of the base GLSM, one has in the former case an independent sum over the spin structures (k,l) on the worldsheet two-torus, while in the latter case the spin structure is chosen periodic along both one-cycles.[10] This simple observation clarifies some statements about topology-changing T-dualities, mixing the torus and gauge bundles, that were originally proposed by Evslin and Minasian in [64] in the effective theory context, and discussed by one of the authors in the torsion GLSM framework [53] (see also [65]

[10] We consider in this discussion that we are in the left GSO sector given by $z = 0$ for simplicity. Considering a different sector does not change the outcome of the argument; the important point is that the spin structures of all free Fermi multiplets are identical (considering that the gauge group lies in a single E_8).

for related comments). Such duality, that exchanges a line bundle over the base $\mathcal{S}$ and a circle bundle at the fermionic radius, is indeed a symmetry of the twining partition function Z_{FY} built from eq. (B.II.3.41) only in the sector $(k = 0, l = 0)$, in which case the two corresponding left-moving fermions, from the Fermi multiplet and from the left-moving component of the fermionized S^1 fibre, have identical (odd) spin structure. Including the independent sum over the spin structures (k, l) of the latter does not respect this symmetry.

In order to generalize the results obtained at the fermionic radius to compact bosons of arbitrary rational radius squared, it is convenient to rewrite the previous expression in terms of $\widehat{\mathfrak{su}(2)}$ theta functions at level 2 (see appendix E.I):

$$Z_{S^1}(\tau,\bar\tau,u,z;R_{\text{f}}) = \frac{1}{|\eta(\tau)|^2} e^{-\frac{2\pi}{\tau_2}R_{\text{f}}^2\left[(Mu+m_{\text{L}}z)(M\bar u+m_{\text{L}}z)-(Mu+m_{\text{L}}z)^2\right]}\times \\ \times \sum_{s\in\mathbb{Z}_4}\Theta_{s,2}\left(\tau\,|2(Mu+m_{\text{L}}z)\right)\bar\Theta_{s,2}(\bar\tau|0)\,, \quad \text{(B.II.3.42)}$$

although the sum over spin structures is no longer explicit.

Whenever a compact boson is at radius $R = \sqrt{\frac{k}{l}}$ (with k and l coprime integers), the corresponding $c = 1$ conformal field theory becomes rational. One can then reorganize the sum in eq. (B.II.3.40) over infinitely many $\widehat{u(1)}_L \times \widehat{u(1)}_R$ representations into a finite sum over representations of the chiral algebra, much as in the case of the fermionic radius that we have discussed previously. In terms of $\widehat{\mathfrak{su}(2)}$ theta functions level kl, one obtains then

$$Z_{S^1}\left(\tau,\bar\tau,u,z;R=\sqrt{\tfrac{k}{l}}\right) = \frac{1}{|\eta(\tau)|^2} e^{-\frac{2\pi}{\tau_2}R^2\left[(Mu+m_{\text{L}}z)(M\bar u+m_{\text{L}}z)-(Mu+m_{\text{L}}z)^2\right]}\times \\ \times \sum_{s,\bar s\in\mathbb{Z}_{2kl},\,\begin{cases} s+\bar s=0 & [2k] \\ s-\bar s=0 & [2l]\end{cases}} \Theta_{s,kl}\left(\tau\,\middle|\,\tfrac{2}{l}(Mu+m_{\text{L}}z)\right)\bar\Theta_{\bar s,kl}(\bar\tau|0)\,. \quad \text{(B.II.3.43)}$$

Arbitrary rational torus As we have reviewed in section B.II.1, covariance under $O(2,2;\mathbb{Z})$ implies that the moduli of the (spacetime) two-torus should always be those of a rational $c = 2$ conformal field theory, *i.e.* with T and U belonging to the same imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$ with $D < 0$. Reducing the corresponding Narain lattice to a sum over characters of the underlying chiral algebra of the rational theory can be done explicitly, as previously, for any given example. However to write down the result explicitely in a uniform way for all cases requires a more abstract point of view.

A precise relation between rational Narain lattices and the data of rational conformal field theories with $c = 2$ was given in a beautiful article by Hosono, Lian, Oguiso and Yau [66]. Whenever the Narain lattice $\Gamma^{2,2}(T,U)$ is rational, *i.e.* with T and U belonging to the same $\mathbb{Q}(\sqrt{D})$, the even positive definite lattices

$$\Pi_{\rm L} := \Gamma^{2,2}(T,U) \cap \mathbb{R}^{2,0} \ , \quad \Pi_{\rm R} := \Gamma^{2,2}(T,U) \cap \mathbb{R}^{0,2} \tag{B.II.3.44}$$

have rank two. Conversely, a rational CFT with $c = 2$ is given by a *triple* $(\Gamma_{\rm L}, \Gamma_{\rm R}, \phi)$, where $\Gamma_{\rm L,R}$ are even positive definite lattices of rank two, and ϕ an isometry mapping one discriminant group to the other, *i.e.* an application $\phi : \Gamma_{\rm L}^\vee/\Gamma_{\rm L} \to \Gamma_{\rm R}^\vee/\Gamma_{\rm R}$, with $\Gamma_{\rm L}^\vee = {\rm Hom}(\Gamma_{\rm L}, \mathbb{Z})$, preserving the bilinear form; it is known also as the gluing map.

To each even positive definite lattice of rank two one can associate an integral quadratic form $Q = ax^2 + bxy + cy^2$ with, choosing a basis, $2a = (e_1, e_1)$, $b = (e_1, e_2)$ and $2c = (e_2, e_2)$. The $GL_2(\mathbb{Z})$ equivalent classes of quadratic forms, $\bar{\mathcal{C}}$, are isomorphic to the $GL_2(\mathbb{Z})$ equivalent classes of even positive definite lattices of rank two, $[\Gamma]$, characterized by their invariant discriminant (resp. determinant) $4ac - b^2 := -D$. Restricting the former classes to $SL_2(\mathbb{Z})$ equivalence classes $\mathcal{C}$, one obtains for each D an Abelian group of finite rank, equipped with a composition law known as the *Gauss product* $\mathcal{C} \star \mathcal{C}'$.[11]

Likewise, for a given determinant D, the equivalence classes of rational Narain lattices $\Gamma^{2,2}(T,U)$ under the $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ action on T and U correspond to equivalent classes of quadratic forms, $\mathcal{C}_T$ and $\mathcal{C}_U$. The equivalence classes of the left and right lattices defining the rational CFT, $[\Gamma_{\rm L}]$ and $[\Gamma_{\rm R}]$, are then given in terms of the equivalent classes of the moduli by:

$$[\Gamma_{\rm L}] = \left[\Gamma_{\overline{\mathcal{C}_T \star (\mathcal{C}_U)^{-1}}}\right] \ , \quad [\Gamma_{\rm R}] = \left[\Gamma_{\overline{\mathcal{C}_T \star \mathcal{C}_U}}\right] \ . \tag{B.II.3.45}$$

This provides the data of the $c = 2$ rational CFT for any rational $(2,2)$ Narain lattice.

One can now express the result of the path integral in terms of the rational CFT data, in other words in terms of the theta-functions Θ_μ^Γ associated with the left and right lattices $\Gamma_{\rm L}$ and $\Gamma_{\rm R}$, see appendix E.I for details. Let us consider the case of a $U(1)$ (worldsheet) gauge group. Taking an orthonormal basis, the two-dimensional vector of topological charges corresponding to the two-torus bundle,

[11] Whenever the quadratic forms are not primitive, *i.e.* such that $\gcd(a,b,c) > 1$, these statements should be slightly modified, see [66] for details.

see section B.II.1, is of the form

$$p_{\rm M} = \sqrt{\frac{2U_2}{T_2}} \begin{pmatrix} M_1 + T_1 M_2 \\ T_2 M_2 \end{pmatrix} . \tag{B.II.3.46}$$

One can check that this vector belongs actually to the lattice $\Gamma^{2,2}(T,U) \cap \mathbb{R}^{2,0}$, hence to the left lattice of the rational CFT. A convenient assignment of $U(1)_{\rm L}$ shift charges for cancellation of global anomalies is to take them proportional to the gauge charges, with a coefficient of proportionality λ, see section B.II.4.

One obtains then the following one-loop contribution from the bosonic degrees of freedom of the torsion multiplet, for arbitrary T and U in the same imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$:

$$Z_{\rm torus}(\tau,\bar\tau,u,z;T,U) = \exp\left(-\frac{2\pi}{\tau_2}\left[(u+\lambda z)(\bar u+\lambda z) - (u+\lambda z)^2\right]\langle p_{\rm M}, p_{\rm M}\rangle_{\Gamma_{\rm L}}\right) \times \\ \times \sum_{\mu\in\Gamma_{\rm L}^\vee/\Gamma_{\rm L}} \frac{\Theta_\mu^{\Gamma_{\rm L}}(\tau|(u+\lambda z)p_{\rm M})}{\eta(\tau)^2} \frac{\bar\Theta_{\varphi(\mu)}^{\Gamma_R}(\bar\tau|0)}{\bar\eta(\bar\tau)^2} . \tag{B.II.3.47}$$

where $\langle\star,\star\rangle_{\Gamma_{\rm L}}$ is the inner product on $\Gamma_{\rm L}$. Hence the quadratic prefactor of this expression is written in terms of the norm of the vector $p_{\rm M} \in \Gamma_{\rm L}$ giving the topological charges of the torus bundle

$$\langle p_{\rm M}, p_{\rm M}\rangle_{\Gamma_{\rm L}} = \frac{2U_2}{T_2}|M|^2 . \tag{B.II.3.48}$$

$O(2,2;\mathbb{Z})$ T-duality transformations are mapped, under the correspondence between rational Narain lattices and rational CFTs summarized above, to isometries of the triple providing the rational CFT data, hence preserve eq. (B.II.3.48). It provides an elegant explanation of the invariance of this expression, which gives also the contribution of the torus bundle to the integrated Bianchi identity, under the perturbative duality group [64]; as we have shown, this property is intimately related to the rational nature of the Narain lattice.

II.3.5 The result

Having dealt separately with the contribution of each type of multiplet, one can write the full one-loop determinant, in the case of a rank one gauge group, as follows:

$$\begin{aligned}\Sigma_{\text{1-loop}}(\tau,\bar\tau,z) = &-2i\pi\eta(\tau)^2\prod_{\Phi_i}\frac{i\eta(\tau)}{\theta_1(\tau|Q_iu+q_i^{\text{L}}z)}\prod_{\Gamma_a}\frac{i\theta_1(\tau|Q_au+q_a^{\text{L}}z)}{\eta(\tau)}\times\\ &\times\sum_{\mu\in\Gamma_{\text{L}}^\vee/\Gamma_{\text{L}}}\frac{\Theta_\mu^{\Gamma_{\text{L}}}(\tau|(u+\lambda z)p_{\text{M}})}{\eta(\tau)^2}\frac{\bar\Theta^{\Gamma_R}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}\times\\ &\times\exp\Big[\frac{\pi}{\tau_2}\underbrace{\Big(-\sum_i {Q_i}^2+\sum_a {Q_a}^2+\frac{2U_2}{T_2}|M|^2\Big)}_{=0}\Big(u^2-u\bar u\Big)\Big]\,\mathrm{d}u\,.\end{aligned}\tag{B.II.3.49}$$

The first line corresponds to the K3 base and the second line to the two-torus fibre.

In this computation we had chosen a prescription for the determinant of a chiral Dirac operator consistent with the torsion multiplet contribution, such that at the end the various factors present in the last line of eq. (B.II.3.49) cancel each other whenever the gauge charges satisfy the tadpole condition eq. (B.II.1.17), as anounced below eq. (B.II.3.20). Notice that there are also factors linear in y, corresponding to the $U(1)_{\text{L}}$ global anomaly, that we did not include in the equation for sake of clarity; likewise, they cancel among themselves in an anomaly-free model.

Thus the one-loop determinant $\Sigma_{\text{1-loop}}$ is a holomorphic function of the gauge field holonomy. Simplifying the above expression and injecting it in the contour integral, one gets for the twining partition function eq. (B.II.2.15) the result:

$$\begin{aligned}Z_{\text{FY}}(\tau,\bar\tau,z) = \pm\,\eta(\tau)^2\sum_{u^\star\in\mathcal{M}^\pm_{\text{sing}}}\oint_{\mathcal{C}(u^\star)}\mathrm{d}u\,\prod_{\Phi_i}\left(i\frac{\eta(\tau)}{\theta_1(\tau|Q_iu+q_i^{\text{L}}z)}\right)\times\\ \times\prod_{\Gamma_a}\left(i\frac{\theta_1(\tau|Q_au+q_a^{\text{L}}z)}{\eta(\tau)}\right)\sum_{\mu\in\Gamma_{\text{L}}^\vee/\Gamma_{\text{L}}}\frac{\Theta_\mu^{\Gamma_{\text{L}}}(\tau|(u+\lambda z)p_{\text{M}})}{\eta(\tau)^2}\frac{\bar\Theta^{\Gamma_R}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}\,.\end{aligned}\tag{B.II.3.50}$$

This quantity is then plugged into eq. (B.II.2.3) to finally give the new supersymmetric index of Fu-Yau compactifications.

For consistency the new supersymmetric index that we have computed should behave properly under transformations of the modular group $PSL(2,\mathbb{Z})_\tau$. As mentioned above, $\tau_2 Z_{\text{NEW}}(\tau,\bar\tau)$ should be a (non-holomorphic) modular form of weight

-2. Tracking this statement back to the modular behaviour of the non-holomorphic twining partition function Z_{FY}, one should check that whenever the anomaly cancellation condition is satisfied, the latter behaves as a weak Jacobi form of weight 0 and index $\frac{r}{2}$, where r is the rank of the vector bundle, although it is not holomorphic in τ.

This behavior will be checked first at the level of the torsion GLSM in the next section once we have given the charge assignment, and then later on using the geometrical formula that we will define is section B.II.5.

II.4 Generalization to higher rank, global charges

In this section we first generalize the results obtained above to the case of a higher rank gauge group on the worldsheet, provide a consistent assignment of global charges and then consider a concrete example, in which the base manifold is a quartic in $\mathbb{P}^3$.

II.4.1 Higher rank gauge groups on the worldsheet

As mentioned above, the whole construction of the torsion linear sigma-model can be carried on with a larger Abelian[12] gauge group $G = U(1)^k$. For each component $U(1)_\kappa$, one introduces:

- A $(0,2)$ vector multiplet $(A_{+\kappa}, A_{-\kappa})$,
- A chiral multiplet P_κ,
- A set of $r^\kappa + 1$ Fermi multiplets Γ_{a_κ},
- A set of quasi-homogeneous polynomials $J_{a_\kappa}(\Phi_i)$,

and a superpotential

$$\mathcal{L} = \int \mathrm{d}\theta^+ \, \tilde{\Gamma}_\alpha G^\alpha(\Phi_i) + \int \mathrm{d}\theta^+ \, \sum_\kappa P_\kappa \, \Gamma_{a_\kappa} J^{a_\kappa}(\Phi_i) + h.c.\,. \tag{B.II.4.1}$$

The generalization to higher rank gauge group of the twining partition function $Z_{\text{FY}}(\tau, \bar{\tau}, y)$ in terms of Jeffrey-Kirwan residues is immediate, as explained before.

[12] One could consider general non-Abelian gauge groups, with the torsion multiplet charged under their Abelian part.

Using results from reference [34] one gets the twining partition function

$$Z_{\mathrm{FY}}(\tau,\bar\tau,z)=\left(-2i\pi\eta(\tau)^2\right)^{\mathrm{rank}(G)}\times \sum_{u^\star\in\mathcal{M}_{\mathrm{sing}}}\underset{u=u^\star}{\text{JK-Res}}\left(\mathrm{Q}(u^\star),\eta\right)\left\{\prod_{\Phi_i}\frac{i\eta(\tau)}{\theta_1(\tau|Q_i^\kappa u_\kappa+q_i^{\mathrm{L}}z)}\prod_{\Gamma_a}\frac{i\theta_1(\tau|Q_a^\kappa u_\kappa+q_a^{\mathrm{L}}z)}{\eta(\tau)}\right. \left.\sum_{\mu\in\Gamma_{\mathrm{L}}^\vee/\Gamma_{\mathrm{L}}}\frac{\Theta_\mu^{\Gamma_{\mathrm{L}}}(\tau|p_{\mathrm{M}}^\kappa u_\kappa+\hat p_m z)}{\eta(\tau)^2}\frac{\bar\Theta^{\Gamma_R}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}\bigwedge_{\kappa=1}^{\mathrm{rank}(G)}\mathrm{d}u_\kappa\right. \tag{B.II.4.2}$$

where one defines a vector $p_{\mathrm{M}}^\kappa\in\Gamma_{\mathrm{L}}$ as eq. (B.II.3.48) for each κ and an extra vector $\hat p_m$ defining the shift charge for the $U(1)_{\mathrm{L}}$ global symmetry. To simplify the notations, Φ_i denotes all the chiral multiplets in the model, and Γ_a all the Fermi multiplets. Whenever gauge and global anomalies are absent no extra factors appear in the one-loop determinant, as before.

One then has to choose a charge assignment for the fields which is compatible with the various anomaly cancellations, and gives the required value for the central charges and the rank of the spacetime gauge bundle. Let us assign the charges in the following way:

	Φ_i	P_κ	$\tilde\Gamma_\alpha$	Γ_{a_κ}	Θ
$U(1)_\iota$	Q_i^ι	$-d_\kappa^\iota$	Q_α^ι	$Q_{a_\kappa}^\iota$	M_ℓ^ι
$U(1)_{\mathrm{L}}$	0	1	0	-1	0

(B.II.4.3)

One sees that whenever the tadpole conditions

$$\sum_i Q_i^\epsilon Q_i^{\epsilon'}+\sum_\kappa d_\kappa^\epsilon d_\kappa^{\epsilon'}-\sum_\alpha Q_\alpha^\epsilon Q_\alpha^{\epsilon'}-\sum_{\kappa,a_\kappa}Q_{a_\kappa}^\epsilon Q_{a_\kappa}^{\epsilon'}-\langle p_{\mathrm{M}}^\epsilon,p_{\mathrm{M}}^{\epsilon'}\rangle=0\quad\forall\epsilon,\epsilon'\in\{1,2,...,k\}\,, \tag{B.II.4.4}$$

are satisfied, then the various local and global anomalies vanish. Furthermore, the rank r of the holomorphic vector bundle is then given by construction by $r=\sum_\kappa r^\kappa$, the central charges take the appropriate values $(c,\bar c)=(6+r,9)$ and the left and right global $U(1)$ current decouple, see [52] for details.[13] Finally this choice of charge is consistent with a space-time gauge bundle having vanishing first Chern class.

This choice of global charges implies that in the geometrical "phase" of the torsion GLSM, which corresponds to taking the residues at the poles of the chiral multiplets

[13] Compared to the work of Adams and Lapan, we have shifted all the $U(1)_L$ charges using the gauge shift $\tilde u_\kappa:=u_\kappa-\sum_\iota(d^{-1})_\kappa^\iota z$. Our choice of charges turns out to be more appropriate in order to discuss the link with the geometrical formula of section B.II.5.

Φ_i (i.e. points $u^\star$ such that $Q_i^\kappa u_\kappa^\star \in \mathbb{Z} + \tau\mathbb{Z}$), the contribution from the torsion multiplet has no y-dependence, in keep with the geometrical formula that we define in section B.II.5. From a geometrical point of view, the meaning of this absence of y-dependence is that the torus fibre should not contribute to the rank of the holomorphic vector bundle. This assertion becomes transparent when we examine the modular behavior of Z_{FY}.

Modular transformations

Let us denote by d the complex dimension of the base, k the rank of the world-sheet gauge group and r the rank of the space-time holomorphic vector bundle. Using the results of appendix E.I, the behavior of $\Sigma_{\text{1-loop}}$ under the $SL(2,\mathbb{Z})$ modular transformations is straightforward. Under a modular T-transformation $\tau \mapsto \tau + 1$, the torsion multiplet contribution is by itself invariant.[14] The remaining contribution comes from the base, and gives:

$$\Sigma_{\text{1-loop}}(\tau+1, \bar\tau+1, z, u_\kappa) = e^{-\frac{i\pi}{6}(d-r)}\Sigma_{\text{1-loop}}(\tau, \bar\tau, z, u_\kappa)\,. \tag{B.II.4.5}$$

Under an S-transformation $\tau \mapsto -1/\tau$, one finds the following transformation rule:

$$\Sigma_{\text{1-loop}}\left(-\frac{1}{\tau}, -\frac{1}{\bar\tau}, \frac{z}{\tau}, \frac{u_\kappa}{\tau}\right) = i^{d-r}\exp\left\{-\frac{i\pi}{\tau}\left(\sum_{\Phi_i} {v_i}^2 + \sum_{P_\kappa} {v_\kappa}^2 - \sum_{\tilde\Gamma_\alpha} {v_\alpha}^2 - \sum_{\Gamma_{a_\kappa}} v_{a_\kappa}^2 - \langle v, v\rangle_{\Gamma_L}\right)\right\}\Sigma_{\text{1-loop}}(\tau, \bar\tau z y, u_\kappa)\,, \tag{B.II.4.6}$$

where

$$v_i = Q_i^\epsilon u_\epsilon\,, \tag{B.II.4.7a}$$
$$v_\kappa = -d_\kappa^\epsilon u_\epsilon + z\,, \tag{B.II.4.7b}$$
$$v_\alpha = Q_\alpha^\epsilon u_\epsilon\,, \tag{B.II.4.7c}$$
$$v_{a_\kappa} = Q_{a_\kappa}^\epsilon u_\epsilon - z\,, \tag{B.II.4.7d}$$
$$v = p_{\text{M}}^\epsilon u_\epsilon\,. \tag{B.II.4.7e}$$

The charge assignement given by eq. (B.II.4.3) was precisely designed such that the gauge and global anomalies vanish provided that the tadpole conditions eq. (B.II.4.4)

[14] The isometry ϕ preserving the bilinear form, $\langle\mu,\mu\rangle_{\Gamma_L} = \langle\phi(\mu),\phi(\mu)\rangle_{\Gamma_R}$.

hold. One gets then:

$$\Sigma_{\text{1-loop}}\left(-\frac{1}{\tau},-\frac{1}{\bar{\tau}},\frac{z}{\tau},\frac{u_\kappa}{\tau}\right)=i^{d-r}\exp\left[\frac{2i\pi}{\tau}\frac{r}{2}z^2\right]\Sigma_{\text{1-loop}}(\tau,\bar{\tau},z,u_\kappa)\,. \tag{B.II.4.8}$$

One concludes that, though non-holomorphic in τ, the twining partition function Z_{FY} transforms as a weak Jacobi form of index $\frac{r}{2}$ and weight zero. This result will be derived again starting from the geometrical formula that we provide in section section B.II.5.

II.4.2 Example of the quartic

We illustrate here the formula giving the twining partition function Z_{FY} of Fu-Yau compactifications in terms of the torsion GLSM data with a simple example, namely a quartic hypersurface in $\mathbb{P}^3$ with a rank four gauge bundle [52]. Following eq. (B.II.4.3), let the charges for the base be:

	$\Phi^{i=1,\dots,4}$	P	$\tilde{\Gamma}$	$\Gamma_{a=1,\dots,5}$	Θ
$U(1)$	1	-5	-4	1	M_ℓ
$U(1)_{\text{L}}$	0	1	0	-1	0

, (B.II.4.9)

with, in addition, the moduli (T,U) and the topological charge M of the torus fibre chosen such that the tadpole condition eq. (B.II.1.17) is satisfied. The full one-loop determinant writes

$$\begin{aligned}\Sigma_{\text{1-loop}}=&\left[-2i\pi\eta(\tau)^2\right]\left[i\frac{\eta(\tau)}{\theta_1(\tau\,|u)}\right]^4\left[i\frac{\eta(\tau)}{\theta_1(\tau\,|-5u+z)}\right]\left[i\frac{\theta_1(\tau\,|-4u)}{\eta(\tau)}\right]\times\\&\times\left[i\frac{\theta_1(\tau\,|u-z)}{\eta(\tau)}\right]^5\left[\sum_{\mu\in\Gamma_{\text{L}}^\vee/\Gamma_{\text{L}}}\frac{\Theta_\mu^{\Gamma_{\text{L}}}(\tau\,|p_{\text{M}}u)}{\eta(\tau)^2}\frac{\bar{\Theta}^{\Gamma_R}_{\varphi(\mu)}(\bar{\tau}|0)}{\bar{\eta}(\bar{\tau})^2}\right]\mathrm{d}u\,.\end{aligned} \tag{B.II.4.10}$$

Landau-Ginzburg phase

One can first provide the result in a form that one would obtain by a direct computation in the Landau-Ginzburg regime of the base GLSM. For this purpose one selects the set of poles $\mathcal{M}^-_{\text{sing}}=\left\{u=-\frac{k+\tau l-z}{5}\;\middle|\;k,l\in[\![0,4]\!]\right\}$. Plugging the

one-loop determinant into the contour integral $\frac{1}{2i\pi}\oint$ leads to:

$$Z_{\text{FY}}(\tau,\bar\tau,z) = -\frac{i}{\eta(\tau)\bar\eta(\bar\tau)^2}\sum_{k,l=0}^{4}\oint_{u=-\frac{k+\tau l-z}{5}} \mathrm{d}u\ \frac{\theta_1\left(\tau\,|u-z\right)^5}{\theta_1\left(\tau\,|u\right)^4}\frac{\theta_1\left(\tau\,|-4u\right)}{\theta_1\left(\tau\,|-5u+z\right)}\times \\ \times \sum_{\mu\in\Gamma_{\text{L}}^\vee/\Gamma_{\text{L}}}\Theta_\mu^{\Gamma_{\text{L}}}\left(\tau\,|p_{\text{M}}u\right)\bar\Theta^{\Gamma_R}_{\varphi(\mu)}(\bar\tau|0)\,. \tag{B.II.4.11}$$

Evaluating the residues, one has

$$Z_{\text{FY}}(\tau,\bar\tau,z) = \frac{1}{5\eta(\tau)^4\bar\eta(\bar\tau)^2}\sum_{k,l=0}^{4}(-1)^{k+l}e^{i\pi l^2\tau}\frac{\theta_1\left(\tau\left|-\frac{k+\tau l}{5}-\frac{4z}{5}\right.\right)^5}{\theta_1\left(\tau\left|-\frac{k+\tau l}{5}+\frac{z}{5}\right.\right)^4}\times \\ \times\,\theta_1\left(\tau\left|\frac{4(k+\tau l)}{5}-\frac{4z}{5}\right.\right)\sum_{\mu\in\Gamma_{\text{L}}^\vee/\Gamma_{\text{L}}}\Theta_\mu^{\Gamma_{\text{L}}}\left(\tau\left|\left(-\frac{k+\tau l}{5}+\frac{z}{5}\right)p_{\text{M}}\right.\right)\bar\Theta^{\Gamma_R}_{\varphi(\mu)}(\bar\tau|0)\,. \tag{B.II.4.12}$$

Geometrical phase

An expression corresponding to a direct computation in the geometrical formulation of the index, see section B.II.5, is obtained by considering the contribution of the pole $u=0$, which is of order 4.[15] Plugging the one loop determinant into the contour integral $-\frac{1}{2i\pi}\oint$ leads to the expression

$$Z_{\text{FY}}(\tau,\bar\tau,z) = \frac{i}{\eta(\tau)\bar\eta(\bar\tau)^2}\oint_{u=0}\mathrm{d}u\ \frac{\theta_1\left(\tau\,|u-z\right)^5}{\theta_1\left(\tau\,|u\right)^4}\frac{\theta_1\left(\tau\,|-4u\right)}{\theta_1\left(\tau\,|-5u+z\right)}\times \\ \times\sum_{\mu\in\Gamma_{\text{L}}^\vee/\Gamma_{\text{L}}}\Theta_\mu^{\Gamma_{\text{L}}}\left(\tau\,|p_{\text{M}}u\right)\bar\Theta^{\Gamma_R}_{\varphi(\mu)}(\bar\tau|0)\,. \tag{B.II.4.13}$$

To conclude this section, let us consider one specific consistent choice of two-torus fibre. To illustrate what happens for a non-orthogonal torus with non-vanishing B-field, one takes the Wess-Zumino-Witten theory $\widehat{\mathfrak{su}(3)}_1$.[16] It corresponds to a $c=2$ toroidal rational CFT with T and U both equal to the cubic root of unity $j=\exp\left(\frac{2i\pi}{3}\right)$, satisfying the quadratic equation $j^2+j+1=0$. Hence T and U belong to the same imaginary quadratic number field $\mathbb{Q}(\sqrt{-3})$.

[15] As was noted earlier, the expressions that one gets by choosing the poles in $\mathcal{M}^-_{\text{sing}}$ (Landau-Ginzburg picture) or in $\mathcal{M}^+_{\text{sing}}$ (geometrical picture) coincide.

[16] This is a special case of the construction discussed in [67].

A consistent choice of topological charge is given by $M_1 = M_2 = 2$, corresponding to the following vector in the root lattice $\mathfrak{su}(3) \simeq A_2$:

$$p_{\rm M} = \sqrt{2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} , \tag{B.II.4.14}$$

written in an orthonormal basis. The root lattice A_2 has discriminant group $A_2^\vee/A_2 \simeq \mathbb{Z}_3$. Hence, in terms of the $SU(3)$ theta functions

$$\Theta_\mu^{A_2}(\tau|\lambda) = \sum_{\gamma \in A_2 + \mu} q^{\frac{1}{2}\langle\gamma,\gamma\rangle} e^{2i\pi\langle\gamma,\lambda\rangle} , \tag{B.II.4.15}$$

one has the following twining partition function

$$\begin{aligned} Z_{\rm FY}(\tau,\bar\tau,z) = {} & \frac{1}{5\eta(\tau)^4\bar\eta(\bar\tau)^2} \sum_{k,l=0}^{4} (-1)^{k+l} e^{i\pi l^2\tau} \frac{\theta_1\left(\tau\left|-\frac{k+\tau l}{5} - \frac{4z}{5}\right.\right)^5}{\theta_1\left(\tau\left|-\frac{k+\tau l}{5} + \frac{z}{5}\right.\right)^4} \times \\ & \times \theta_1\left(\tau\left|\frac{4(k+\tau l)}{5} - \frac{4z}{5}\right.\right) \sum_{\mu\in\mathbb{Z}_3} \Theta_\mu^{A_2}\left(\tau\left|\left(-\frac{k+\tau l}{5} + \frac{z}{5}\right) p_{\rm M}\right.\right) \bar\Theta_{\varphi(\mu)}^{A_2}(\bar\tau|0) , \end{aligned} \tag{B.II.4.16}$$

that we have evaluated in the Landau-Ginzburg phase.

Notice that this model is non-supersymmetric in spacetime, as the primitivity condition (A.II.3.7) is not satisfied, the two-form ω being necessarily proportional to the Kähler form of the base $J_{\rm K3}$ (cf. section A.II.3). Supersymmetric examples are easily obtained with higher rank worldsheet gauge groups; instead of dealing with such examples in detail, we will provide below a formulation of the index which is independent of the choice of GLSM. However, this absence of spacetime supersymmetry is not explicit in the GLSM, and one would need to compute explicitely the massless spectrum of the model in the Landau-Ginzburg phase to see the absence of spacetime supersymmetry.

II.5 A geometrical formula for the genus

The elliptic genus of a complex manifold X of dimension d, of holomorphic tangent bundle T_X, with a holomorphic vector bundle $\mathcal{E}$ of rank r over it, can be defined independently of its realization as the target space of a $(0,2)$ superconformal

field theory. One defines the formal power series:

$$\mathbb{E}_{q,y} = \bigotimes_{n=0}^{\infty} \bigwedge\nolimits_{-yq^n} E^\star \otimes \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{-y^{-1}q^n} E \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_X^\star \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_X \,, \tag{B.II.5.1}$$

where

$$\bigwedge\nolimits_t E = 1 + t\,E + t^2 \bigwedge\nolimits^2 E + \cdots \,, \quad S_t T_X = 1 + t\,T_X + t^2\,S^2\,T_X + \cdots \,, \tag{B.II.5.2}$$

$\bigwedge^k$ and S^k being respectively the k-th exterior product and the k-th symmetric product. The elliptic genus corresponding to this bundle is defined as follows:

$$Z_{\text{ELL}}(X, E|\tau, z) = q^{\frac{r-d}{12}} y^{-\frac{r}{2}} \int_X \text{ch}\,(\mathbb{E}_{q,y})\,\text{td}(T_X)\,, \tag{B.II.5.3}$$

where $\text{ch}\,(\mathbb{E}_{q,y})$ is the total Chern character of the formal power series $\mathbb{E}_{q,y}$ and $\text{td}\,(T_X)$ the total Todd class of the tangent bundle, cf. section B.I.3. Considering that X is a Calabi-Yau manifold, that E has vanishing first Chern class, and that the anomaly condition $\text{ch}_2(T_X) = \text{ch}_2(E)$ is satisfied (that is, we consider an anomaly-free heterotic Calabi-Yau compactification), the elliptic genus is a weak Jacobi form of weight 0 and index $r/2$.

This geometrical formula has been checked against $(0,2)$ Landau-Ginzburg results in [39], and directly compared with the results of supersymmetric localization for $(2,2)$ GLSMs in [33, 34], building on previous works in the physical and mathematical literature [40, 68–71].

In the present context, there is a natural generalization of this geometrical formulation of the Calabi-Yau elliptic genera, defining a non-holomorphic genus for a two-torus bundle over a K3 surface $\mathcal{S}$, $T^2 \hookrightarrow X \overset{\pi}{\to} \mathcal{S}$, endowed with a rank r gauge bundle E. The relevant geometrical data of such non-Kähler manifold is given by

- The holomorphic tangent bundle $T_{\mathcal{S}}$ over the base, with $c_1(T_{\mathcal{S}}) = 0$,
- A rank r holomorphic vector bundle E over $\mathcal{S}$, with $c_1(E) = 0$, whose pullback provides the gauge bundle of the compactification on X,[17]
- A rational Narain lattice $\Gamma(T, U)$ with $T, U \in \mathbb{Q}(\sqrt{D})$, or equivalently a triple $[\Gamma_{\text{L}}, \Gamma_{\text{R}}, \phi]$ defining a $c = 2$ toroidal rational CFT,

[17] Considering that the holomorphic gauge bundle has vanishing first Chern class is not mandatory for getting consistent heterotic compactifications; it is enough that $c_1(E) \in H^2(\mathcal{S}, 2\mathbb{Z})$ (*i.e.* vanishing of the second Stieffel-Whitney class) to ensure that the bundle admits spinors.

- A pair of anti-self-dual two-forms ω_1 and ω_2 in $H^2(\mathcal{S},\mathbb{Z})\cap\Lambda^{1,1}T^\star_{\mathcal{S}}$.

We define then the *modified holomorphic Euler characteristic* associated to the above data as: as:[18]

$$\chi(X,E,\omega|\tau,\bar\tau,z)=q^{\frac{r-2}{12}}w^{-\frac{r}{2}}\int_{\mathcal{S}}\mathrm{ch}\,(\mathbb{E}_{q,y})\,\mathrm{td}\,(T_{\mathcal{S}})\sum_{\mu\in\Gamma_{\mathrm{L}}^\vee/\Gamma_{\mathrm{L}}}\frac{\Theta_\mu^{\Gamma_{\mathrm{L}}}\left(\tau\,\middle|\,\frac{p_\omega}{2i\pi}\right)}{\eta(\tau)^2}\frac{\bar\Theta^{\Gamma_{\mathrm{R}}}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}\,, \tag{B.II.5.4}$$

where the two-component vector p_ω valued in $H^2(\mathcal{S})\times H^2(\mathcal{S})$ reads, taking an orthonormal basis on Γ_{L}:

$$p_\omega=\sqrt{\frac{2U_2}{T_2}}\begin{pmatrix}\omega_1+T_1\,\omega_2\\ T_2\,\omega_2\end{pmatrix}\,, \tag{B.II.5.5}$$

which generalizes eq. (B.II.3.46). This vector belongs to a formal extension of the left momentum lattice Γ_{L}, which is now a module over $H^2(\mathcal{S},\mathbb{Z})$.

It turns out that this object actually coincides with the Fu-Yau dressed elliptic genus we computed from the worldsheet sigma model:

$$\chi(X,E,\omega|\tau,\bar\tau,z)=Z_{\mathrm{FY}}(\tau,\bar\tau,z)\,, \tag{B.II.5.6}$$

hence providing an alternative definition of the dressed elliptic genus purely in terms of the geometrical data associated to the sigma model target space.

A more explicit expression can be obtained using the splitting principle. Let $c(T_{\mathcal{S}})=\prod_{i=1}^2(1+\nu_i)$ and $c(E)=\prod_{a=1}^r(1+\xi_a)$ denote the total Chern classes of the respective bundles. We have then

$$Z_{\mathrm{FY}}(X,E,\omega|\tau,\bar\tau,z)=\int_{\mathcal{S}}G(\tau,\bar\tau,z,\nu,\xi,p_\omega)\,, \tag{B.II.5.7}$$

where

$$\begin{aligned}G(\tau,\bar\tau,z,\nu,\xi,p_\omega)=&\prod_{a=1}^r\frac{i\theta_1(\tau\left|\frac{\xi_a}{2i\pi}-z\right.)}{\eta(\tau)}\prod_{i=1}^2\frac{\eta(\tau)\nu_i}{i\theta_1(\tau\left|\frac{\nu_i}{2i\pi}\right.)}\times\\ &\times\sum_{\mu\in\Gamma_{\mathrm{L}}^\vee/\Gamma_{\mathrm{L}}}\frac{\Theta_\mu^{\Gamma_{\mathrm{L}}}\left(\tau\,\middle|\,\frac{p_\omega}{2i\pi}\right)}{\eta(\tau)^2}\frac{\bar\Theta^{\Gamma_{\mathrm{R}}}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}\,.\end{aligned} \tag{B.II.5.8}$$

[18] Notice that in eq. (B.II.5.4) the tangent bundle of the base $\mathcal{S}$, rather than of the total space X, appears. This makes sense as the Chern classes of T_X are 'horizontal', *i.e.* with no components along the torus fibre.

II.5.1 Modular properties

The behaviour of $G(\tau,\bar\tau,z,\nu,\xi,p_\omega)$ under $PSL(2,\mathbb{Z})_\tau$ is easily derived using the results of appendix appendix E.I. Under the T-transformation $\tau\mapsto\tau+1$, one gets

$$G(\tau+1,\bar\tau+1,z,\nu,\xi,p_\omega)=e^{-\frac{i\pi}{6}(2-r)}G(\tau,\bar\tau,z,\nu,\xi,p_\omega)\,, \tag{B.II.5.9}$$

exactly as in the GLSM computation of section B.II.4.

The contributions of the holomorphic vector bundle E and of the tangent bundle $T_{\mathcal{S}}$ to $G(\tau,\bar\tau,z,\nu,\xi,p_\omega)$ behave under an S-transformation $\tau\mapsto-1/\tau$ as

$$\prod_{a=1}^{r}\frac{\theta_1\left(-\frac{1}{\tau}\left|\frac{\xi_a/2i\pi-z}{\tau}\right.\right)}{\eta\left(-\frac{1}{\tau}\right)}=\prod_{a=1}^{r}\left\{-ie^{\frac{i\pi}{\tau}\left(\frac{\xi_a}{2i\pi}-z\right)^2}\frac{\theta_1(\tau\left|\frac{\xi_a}{2i\pi}-z\right.)}{\eta(\tau)}\right\}\,, \tag{B.II.5.10a}$$

$$\prod_{i=1}^{2}\frac{\eta\left(-\frac{1}{\tau}\right)\frac{\nu_i}{\tau}}{\theta_1\left(-\frac{1}{\tau}\left|\frac{\nu_i/2i\pi}{\tau}\right.\right)}=\frac{1}{\tau^2}\prod_{i=1}^{2}\left\{ie^{-\frac{i\pi}{\tau}\left(\frac{\nu_i}{2i\pi}\right)^2}\frac{\eta(\tau)\nu_i}{\theta_1(\tau\left|\frac{\nu_i}{2i\pi}\right.)}\right\}\,. \tag{B.II.5.10b}$$

One recognizes on the right-hand side of eqs. (B.II.5.10a) and (B.II.5.10b), the second Chern characters of the vector bundle and of the tangent bundle:

$$\mathrm{ch}_2(E)=\frac{1}{2}\,\mathrm{tr}\left(\frac{i}{2\pi}F\right)^2=\frac{1}{2}\sum_{a=1}^{r}{\xi_a}^2\,, \tag{B.II.5.11a}$$

$$\mathrm{ch}_2(T_{\mathcal{S}})=\frac{1}{2}\,\mathrm{tr}\left(\frac{i}{2\pi}R\right)^2=\frac{1}{2}\sum_{i=1}^{2}{\nu_i}^2\,, \tag{B.II.5.11b}$$

Combining these expressions with the contribution from the torus fibre, obtained using the modular transformation of theta-functions given by eq. (E.I.1.15) in appendix E.I, one gets

$$\begin{aligned}G\left(-\frac{1}{\tau},-\frac{1}{\bar\tau},\frac{z}{\tau},\frac{\nu}{\tau},\frac{\xi}{\tau},\frac{p_\omega}{\tau}\right)=\\ -(-i)^r\tau^{-2}e^{2i\pi\frac{r}{2}\frac{z^2}{\tau}+\frac{2i\pi}{\tau}\frac{\mathrm{ch}_2(E)-\mathrm{ch}_2(T_{\mathcal{S}})}{(2i\pi)^2}}e^{\frac{i\pi}{\tau}\frac{\langle p_\omega,p_\omega\rangle}{(2i\pi)^2}}G(\tau,\bar\tau,z,\nu,\xi,p_\omega)\,,\end{aligned} \tag{B.II.5.12}$$

with

$$\langle p_\omega,p_\omega\rangle=\frac{2U_2}{T_2}(\omega_1+T\omega_2)\wedge(\omega_1+\bar T\omega_2)=-\frac{2U_2}{T_2}\omega\wedge\star_{\mathcal{S}}\bar\omega\,, \tag{B.II.5.13}$$

using the anti-self-duality property of the complex two-form ω.

In conclusion we obtain that G, although non-holomorphic in τ, transforms as a

Jacobi form of weight -2 and index $\frac{r}{2}$, whenever the anomaly cancellation condition

$$\mathrm{ch}_2(E) - \frac{U_2}{T_2}\omega \wedge \star_{\mathcal{S}} \bar{\omega} = \mathrm{ch}_2(T_{\mathcal{S}}) \tag{B.II.5.14}$$

is satisfied. This condition corresponds exactly to the Bianchi identity for Fu-Yau compactifications, see section A.II.3. After integrating G over $\mathcal{S}$, through a Taylor expansion to second order in the differential forms, the non-holomorphic genus Z_{FY} transforms then as a Jacobi form of weight zero and index $\frac{r}{2}$.

Let us remark here that the expression of the geometrical formula eq. (B.II.5.4) is pretty much fixed by its modular behavior. In particular, the absence of z-dependence in the torus fibration contribution, hence the absence of shift in the rank of bundle from the latter under modular transformations, is compatible with the result that one obtains when evaluating the twining partition function of the gauged linear sigma-model description in the geometrical phase.

We indeed expect that the geometrical formula eq. (B.II.5.4) and the GLSM formula eq. (B.II.4.2) for the non-holomorphic genus $Z_{\mathrm{FY}}(\tau, \bar{\tau}, z)$ coincide. A general mathematical proof should follow from a natural generalization of the arguments in [69, 70], first to $(0, 2)$ Calabi-Yau examples and second to the Fu-Yau geometries under consideration in the present article. We provide below a proof of this statement in a simple case.

II.6 Proof of the geometrical formula

Independently of physics, the non-holomorphic genus (C.II.1.5) is of valuable mathematical interest. The elliptic genera of holomorphic gauge bundles of vanishing first Chern class over Calabi-Yau manifolds define Jacobi forms only if $\mathrm{ch}_2(\mathcal{T}_M) = \mathrm{ch}_2(\mathcal{V})$. In [72] a modified elliptic genus was defined by Gritsenko, in order to preserve this modular behavior even in particular when $\mathrm{ch}_2(\mathcal{T}_M) \neq \mathrm{ch}_2(\mathcal{V})$. In the present context there is an alternative definition motivated by physics; heterotic compactifications with $\mathrm{ch}_2(\mathcal{T}_M) \neq \mathrm{ch}_2(\mathcal{V})$ can be made anomaly-free if one adds an appropriate two-torus bundle over the Calabi-Yau manifold, leading naturally to the non-holomorphic genus (C.II.1.5) transforming as a Jacobi form.

Let us give a proof of the fact that the modified holomorphic Euler characteristic given by eq. (B.II.5.4) actually coincides with the dressed elliptic genus in the case where the $K3$ base $\mathcal{S}$ is constructed as a subvariety of a projective space $\mathbb{P}^n$ or more

generally of a weighted projective space $V = \mathbb{P}^n(q_0, ..., q_n)$. We restrict to the case where the subvariety does not intersect the singular loci of the ambient space.

One has the following dual of the normal bundle sequence:

$$0 \to N^\star_{\mathcal{S}/V} \to T^\star_V|_{\mathcal{S}} \to T^\star_{\mathcal{S}} \to 0\,, \tag{B.II.6.1}$$

which gives the following long exact sequence in sheaf cohomology:

$$\begin{aligned} \ldots \to H^1(\mathcal{S}, N^\star_{\mathcal{S}/V}) \xrightarrow{\alpha_1} H^1(\mathcal{S}, T^\star_V|_{\mathcal{S}}) \to H^1(\mathcal{S}, T^\star_{\mathcal{S}}) \to \\ \to H^2(\mathcal{S}, N^\star_{\mathcal{S}/V}) \xrightarrow{\alpha_2} H^2(\mathcal{S}, T^\star_V|_{\mathcal{S}}) \to H^2(\mathcal{S}, T^\star_{\mathcal{S}}) \to \ldots\,. \end{aligned} \tag{B.II.6.2}$$

The exactness of this sequence gives in particular:

$$H^{1,1}_{\bar\partial}(\mathcal{S}) \simeq \mathrm{coker}(\alpha_1) \oplus \ker(\alpha_2)\,. \tag{B.II.6.3}$$

For the proof we will restrict to the case where the hypersurface $\mathcal{S}$ is *favourable*, namely that $\ker(\alpha_2)$ is trivial and α_1 is surjective. In this case, all the elements of $H^{1,1}_{\bar\partial}(\mathcal{S})$ can be understood as being inherited from the ambient space. Moreover, since for a weighted projective space one has $\mathrm{Pic}(\mathbb{P}^n(q_0, ..., q_n)) = \mathbb{Z}$ as a finitely generated abelian group, we have that $\mathrm{rk}(\mathrm{Pic}(\mathcal{S})) = 1$[19]. Following the mathematical literature and given two holomorphic vector bundles E and F over $\mathcal{S}$, we define the formal series with bundle coefficients:

$$\mathbb{E}_{q,y}(E,F) := \bigotimes_{n=1}^{\infty} \left(\bigwedge_{-yq^{n-1}} F^\star \otimes \bigwedge_{-y^{-1}q^n} F \otimes S_{q^n} E^\star \otimes S_{q^n} E \right)\,. \tag{B.II.6.4}$$

We also consider a heterotic Narain lattice $\Gamma(T, U, V)$ with a left coupling to p_ω characterizing the torus bundle, see eq. (B.III.3.1), of partition function:

$$\mathcal{Z}(\tau, \bar\tau, p_\omega) = \sum_{\mu \in \Gamma_{\mathrm{L}}^\vee/\Gamma_{\mathrm{L}}} \frac{\Theta^{\Gamma_{\mathrm{L}}}_\mu\left(\tau \,|\, \frac{p_\omega}{2i\pi}\right)}{\eta(\tau)^2} \frac{\bar\Theta^{\Gamma_{\mathrm{R}}}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}\,, \tag{B.II.6.5}$$

Given this data, and for E a holomorphic vector bundle of rank r over $\mathcal{S}$, we intro-

[19] This actually implies that one restricts to examples which are non-supersymmetric in spacetime, since the two-form ω then fails to be primitive with respect to the base. However, this restriction can be straightforwardly overcome, see the end of this appendix. Moreover, neither from the two-dimensional QFT nor from the mathematical viewpoint this seems to play an important role.

duce the following modified holomorphic Euler characteristic:

$$\chi(\mathcal{S},E,\omega) := q^{\frac{r-(n-1)}{12}} y^{-\frac{r}{2}} \int_{\mathcal{S}} \mathrm{ch}(\mathbb{E}_{q,y}(T_{\mathcal{S}},E))\,\mathrm{td}(T_{\mathcal{S}})\,\mathcal{Z}(\tau,\bar{\tau},\omega)\,. \tag{B.II.6.6}$$

Given a holomorphic vector bundles E of formal Chern roots $\{x_a\}$, let us define the following objects:

$$f(E) := \mathrm{ch}\left(\bigotimes_{n=1}^{\infty}\left(\bigwedge\nolimits_{-yq^{n-1}} E^{\star}\otimes\bigwedge\nolimits_{-y^{-1}q^{n}} E\right)\right)\,, \tag{B.II.6.7a}$$

$$g(E) := \mathrm{ch}\left(\bigotimes_{n=1}^{\infty}\left(S_{q^n}E^{\star}\otimes S_{q^n}E\right)\right)\mathrm{td}\,(E)\,. \tag{B.II.6.7b}$$

Using the total Chern characters for the total symmetric and skew-symmetric products:

$$\mathrm{ch}(S_tE) = \prod_a \frac{1}{1-te^{x_a}}\,,\qquad \mathrm{ch}\left(\bigwedge\nolimits_t E\right) = \prod_a (1+te^{x_a})\,, \tag{B.II.6.8}$$

one can show that:

$$f(E) = \prod_a\left\{\prod_{n=1}^{\infty}\left(1-yq^{n-1}e^{-x_a}\right)\left(1-y^{-1}q^{n}e^{x_a}\right)\right\}\,, \tag{B.II.6.9a}$$

$$g(E) = \prod_a\left\{x_a\prod_{n=1}^{\infty}\frac{1}{(1-q^{n-1}e^{-x_a})\,(1-q^{n}e^{x_a})}\right\}\,, \tag{B.II.6.9b}$$

leading to:

$$f(E) = \prod_a\left\{q^{-1/12}y^{1/2}e^{-x_a/2}\,\frac{i\theta_1\left(\tau\,\middle|\,\frac{x_a}{2i\pi}-z\right)}{\eta(\tau)}\right\}\,, \tag{B.II.6.10a}$$

$$g(E) = \prod_a\left\{q^{1/12}e^{x_a/2}\,\frac{\eta(\tau)\,x_a}{i\theta_1\left(\tau\,\middle|\,\frac{x_a}{2i\pi}\right)}\right\}\,. \tag{B.II.6.10b}$$

In our context one has the following defining short exact sequences for the hypersurface $\mathcal{S}$ and the rank r holomorphic vector bundle E over it:

$$0\to T_{\mathcal{S}}\to T_V|_{\mathcal{S}}\to\mathcal{O}_V(k)|_{\mathcal{S}}\to 0\,, \tag{B.II.6.11a}$$

$$0\to E\to\bigoplus_{a=0}^{r}\mathcal{O}(Q_a)|_{\mathcal{S}}\overset{\otimes J^a}{\to}\mathcal{O}(-Q_P)|_{\mathcal{S}}\to 0\,. \tag{B.II.6.11b}$$

Using multiplicative properties of f and g, one obtains:

$$f\left(\bigoplus_{a=0}^{r}\mathcal{O}(Q_a)\right) = f(E)f(\mathcal{O}(-Q_P))\,,\qquad g(T_V|_{\mathcal{S}}) = g(T_{\mathcal{S}})g(\mathcal{O}_V(k)|_{\mathcal{S}})\,. \tag{B.II.6.12}$$

The formal Chern roots are defined through the following total Chern classes:

$$c(\mathcal{O}(m)) = 1 + mH\,, \qquad c(V) = \prod_{i=0}^{n}(1 + q_i H)\,, \tag{B.II.6.13}$$

leading to:

$$\begin{aligned} f(E)\,\left[\frac{i\theta_1\left(\tau\,\middle|-\frac{Q_P H}{2i\pi} - z\right)}{\eta(\tau)}\right]\Bigg|_{\mathcal{S}} = \\ = q^{-r/12} y^{r/2} e^{-\left(\sum Q_a + Q_P\right)\frac{H}{2}} \prod_{a=0}^{r}\left[\frac{i\theta_1\left(\tau\,\middle|\frac{Q_a H}{2i\pi} - z\right)}{\eta(\tau)}\right]\Bigg|_{\mathcal{S}}\,, \end{aligned} \tag{B.II.6.14}$$

Exploiting the Euler exact sequence:

$$0 \to \mathcal{O}_V \to \bigoplus_{i=0}^{n}\mathcal{O}_V(q_i) \to T_V \to 0\,, \tag{B.II.6.15}$$

one also has:

$$g(T_{\mathcal{S}})\,\left[\frac{\eta(\tau)\,kH}{i\theta_1\left(\tau\,\middle|\frac{kH}{2i\pi}\right)}\right]\Bigg|_{\mathcal{S}} = q^{\frac{n-1}{12}}\eta(\tau)^2\prod_{i=0}^{n}\left[\frac{\eta(\tau)\,q_i H}{i\theta_1\left(\tau\,\middle|\frac{q_i H}{2i\pi}\right)}\right]\Bigg|_{\mathcal{S}}\,. \tag{B.II.6.16}$$

The two equations above give the contributions from the tangent bundle and from the holomorphic vector bundle E in terms of the embedding in the ambient space V. Concerning the lattice part, $\mathrm{Pic}(\mathcal{S})$ being of rank one, one necessarily has $\omega_\ell = m_\ell H$.

Turning the integral over the hypersurface to an integral over the ambient space via:

$$\int_{\mathcal{S}}\varphi = \int_V c_1(\mathcal{O}_V(k))\varphi\,, \tag{B.II.6.17}$$

One obtains:

$$\begin{aligned} \chi(\mathcal{S}, E, \omega) = \eta(\tau)^2\prod_{i=0}^{n} q_i \int_V H^{n+1}\prod_{a=0}^{r}\left[\frac{i\theta_1\left(\tau\,\middle|\frac{Q_a H}{2i\pi} - z\right)}{\eta(\tau)}\right]\left[\frac{\eta(\tau)}{i\theta_1\left(\tau\,\middle|-\frac{Q_P H}{2i\pi} - z\right)}\right] \\ \prod_{i=0}^{n}\left[\frac{\eta(\tau)}{i\theta_1\left(\tau\,\middle|\frac{q_i H}{2i\pi}\right)}\right]\left[\frac{i\theta_1\left(\tau\,\middle|\frac{kH}{2i\pi}\right)}{\eta(\tau)}\right]\,\mathcal{Z}(\tau, \bar\tau, \mathfrak{m}H)\,. \end{aligned} \tag{B.II.6.18}$$

Using the fact that the hyperplane class is normalized such that:

$$\prod_i q_i \int_V H^n = 1\,, \tag{B.II.6.19}$$

together with the residue formula, to turn $\int_V H^{n+1}\varphi(H)$ into a contour integral around the origin of the complex plane $\oint_{u=0} \mathrm{d}u\, \varphi(2i\pi u)$, one concludes that this modified holomorphic Euler characteristic coincides with the formula that would be obtained in the geometrical phase of the torsional GLSM, *i.e.* considering the contour integral around $u = 0$ in eq. (B.II.3.50):

$$Z_{\text{FY}}(X, E, \omega) = \chi(\mathcal{S}, E, \omega)\,. \tag{B.II.6.20}$$

This proof can be generalized by allowing for non-favorable hypersurfaces, or by considering complete intersections in more generic toric varieties as ambient spaces. One would then end up with higher dimensional residue operations, leading to Jeffrey-Kirwan residue formulæ type. The generalization to the non-toric case is left for later work.

Let us conclude by saying that non-holomorphic genus is presumably, as the CY elliptic genera, providing a generating functional for the indices of a family of Dirac operators, each transforming in a representation of the bundle specified by a given term in the expansion of (B.II.5.1). A possible interpretation is that, in the present case, one considers a similar problem for Dirac operators related, in the string theory context, to Kaluza-Klein modes with momenta (p_L, p_R) along the two-torus fibre, in their right Ramond ground state and with, roughly speaking, arbitrary left-moving oscillator modes along the tangent bundle of the base $\mathcal{T}_S$ and the gauge bundle $\mathcal{V}$. Because of the non-trivial fibration, one may expect a grading according to the toroidal left momentum p_L, as our explicit formula (C.II.1.5) suggests. Making this correspondence more precise is a very interesting project.

Following what was said concerning the absence of spacetime supersymmetry of the Fermat quartic example. The $(0, 4)$ supersymmetry actually does not seem at all crucial from the GLSM perspective, hence all the construction of the torus fibration can be generalized to a torus fibration over a Calabi-Yau d-fold. It would be extremely interesting to have a criteria at the level of the GLSM to know whether or not the model is supersymmetric in spacetime, without having to compute the massless spectrum in the Landau-Ginzburg phase using $\bar{Q}$-cohomology.

Chapter III

Generic bundle and dressed elliptic genus

This chapter is built out of the article: *Dressed elliptic genus of heterotic compactifications with torsion and general bundles*, with Dan Israël, arXiv:1606.08982, JHEP 1608 (2016) 176.

III.1 Torsional geometry and its GLSM

In the previous chapter we considered only gauge bundles which are pullbacks of stable holomorphic bundles over the $K3$ base. It is known [21] that an additional Abelian gauge bundle over the total space of the principal T^2 bundle, that would reduce to a set of Wilson lines on T^2 for a $K3 \times T^2$ compactification, is allowed by space-time supersymmetry. The main objective of the present chapter is to include them in the torsion GLSM and in the computation of the new supersymmetric index.

As discussing in the previous part, a large class of vector bundles compatible with supersymmetry [21] consists first of the pullback of a stable holomorphic vector bundle over $\mathcal{S}$, satisfying the integrated Bianchi identity:

$$\int_{\mathcal{S}} \mathrm{ch}_2(E) + 24 - \frac{U_2}{T_2} \int_{\mathcal{S}} \omega \wedge \star_{\mathcal{S}} \bar{\omega} = 0 \,. \tag{B.III.1.1}$$

We assume in the following that the structure group of this vector bundle is embedded in the first E_8 factor of the heterotic gauge group.

Second, one can consider also an Abelian bundle over the total space X, whose connection is of the form:

$$A = \mathcal{T}^a \, \mathrm{Re}(\bar{V}^a \, \iota) \,, \tag{B.III.1.2}$$

depending on 8 complex parameters V^a. It reduces to a set of Wilson lines for $K3 \times T^2$ compactifications, which constitute particular cases of this construction. Therefore, we will loosely call them Wilson lines thereafter.

For simplicity we will embed the structure group of this bundle in the second E_8 factor. In eq. (B.III.1.2) $\{\mathcal{T}^a\}$ forms a basis of $\mathfrak{H}_8$, its Cartan subalgebra. The tadpole condition eq. (B.III.1.1) is unchanged hence depends only on the second Chern character of the vector bundle over the base and on the torus moduli T and U. The present chapter extends the results of [73] where only special points in the moduli space where these Abelian gauge bundles over the total space where turned off were considered.

III.1.1 The gauged linear sigma-model with torsion

The construction of the torsion GLSM starts following precisely the same steps as in absence of Wilson lines, namely by introducing a $(0,2)$ GLSM describing the $K3$ base as a complete intersection in a weighted projective space, the model differing from a standard CY GLSM by the requirement of a *non-vanishing* gauge anomaly, which one then compensates by a worldsheet Green-Schwarz mechanism in a way that mimics the principal torus fibration in the spacetime picture. The base GLSM is fairly standard, built out of a $(0,2)$ vector multiplet, as well as chiral and Fermi multiplets. We refer the reader to section B.I.2 for the expression of the various Lagrangians.

The torus fibre on the other hand is modelized by a pair $\{\Omega^\ell = (\omega^\ell, \chi^\ell)\}_{\ell=1,2}$ of chiral superfields charged axially under the worldsheet gauge field. The key point is that, as we saw in chapter B.II, their Lagrangangian contains, in addition to standard kinetic terms and minimal coupling to the gauge superfield, a field-dependent Fayer-Iliopoulos term

$$-\frac{ih_\ell}{4}\int \mathrm{d}\theta\, \Upsilon\Omega^\ell + \text{h.c.}\,, \tag{B.III.1.3}$$

which is obviously non-invariant under the gauge symmetry, but whose gauge variance precisely compensates the anoaly from the base.

At this point, the geometry obtained after integrating out the massive gauge field is that of a $(\mathbb{C}^*)^2$ bundle over $K3$. To decouple the real part of the shift multiplets Ω^ℓ, in order to restrict to a T^2 bundle while preserving $(0,2)$ supersymmetry, one should cancel their couplings to the gaugini. This leads to the quantization conditions eq. (B.II.1.14). With at least a rank-two worldsheet gauge group, U and T are

generically quantized such that the underlying $c = 2$ CFT with a 2-torus target space is rational, see appendix E.II.

Using the relations eq. (B.II.1.14), the anomaly cancellation condition can be written in a simple form:

$$\mathcal{A} - \frac{2U_2}{T_2}|\mathfrak{m}|^2 = 0\,, \tag{B.III.1.4}$$

where $\mathfrak{m} := m^1 + Tm^2$ is the complex topological charge, which is the worldsheet counterpart of the tadpole condition eq. (B.III.1.1).

The extra novelty with respect to the previous chapter is that we previously chose to reorganize the remaining degrees of freedom obtained after decoupling of the real part of Ω^ℓ into a *torsion multiplet* [52], in order to exhibit more explicitly the torus sub-bundle inside the $(\mathbb{C}^*)^2$ bundle, but we will stick here to a formulation in terms of shift multiplets as the Abelian bundle in target-space will be more naturally described in this framework.

III.1.2 Abelian connections over the total space

In order to describe a target-space Abelian gauge bundle over the total space X as eq. (B.III.1.2), one needs to enlarge the torsion GLSM framework. For simplicity, we embed the structure group of E in the first E_8 and the structure group of the Abelian bundle in the second E_8.

From the worldsheet perspective, each line bundle is mapped to a left-moving Weyl fermion λ_- in a Fermi multiplet Λ, transforming as a section of this bundle. In components, a connection of the type eq. (B.III.1.2) corresponds to a kinetic term like $\bar{\lambda}_- \left(\partial_+ \left(\omega - \bar{\omega}\right) + 2mA_+\right) \lambda_-$ in the Lagrangian of the two-dimensional supersymmetric gauge theory. It will be convenient to bosonize these left-moving fermions, as one will be able to consider them and the shift multiplets for the two-torus on the same footing.

As one defines the GLSM in $(0,2)$ superspace, one needs to add enough degrees of freedom to form a multiplet. One first bosonizes λ_- into a chiral and real compact boson, and embeds it in a neutral chiral multiplet B, of components $B = (b, \bar{b}, \xi_+, \bar{\xi}_+)$, as the left-moving, compact imaginary part of b. Of course, such a procedure introduces extra degrees of freedom. For each multiplet B, one has:

- The real part of b which is non-compact,
- The right-moving fermions ξ_+ and $\bar{\xi}_+$,

- The right-moving part of $\mathrm{Im}(b)$.

All of these extra degrees of freedom are an artifact of the bosonization procedure. Naturally the right-moving part of $\mathrm{Im}(b)$ cannot decouple from the Lagrangian of the theory, as it would give Lagrangians for chiral bosons. However as we shall see at the end of the computation, the contribution from those degrees of freedom will appear in the dressed elliptic genus as an overall finite and non-vanishing multiplicative factor.

The dynamics of the chiral multiplets $\{B^n\}_{n=1,\dots,8}$ is described by the following Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{Wilson}} = &-\frac{i\mathcal{E}_{mn}}{8}\int \mathrm{d}^2\theta \left(B^m+\bar{B}^m\right)\partial_-\left(B^n-\bar{B}^n\right)\\ &-\frac{i\beta_{\ell n}}{16}\int \mathrm{d}^2\theta \left(\Omega^\ell+\bar{\Omega}^\ell+2m^\ell\mathcal{A}\right)\partial_-\left(B^n-\bar{B}^n\right)\,, \end{aligned}\tag{B.III.1.5}$$

where $\mathcal{E}_{mn} := \mathcal{G}_{mn}+\mathcal{B}_{mn}$ is such that the corresponding $(8,8)$ toroidal lattice splits into $(E_8)_{\text{L}}\times(E_8)_{\text{R}}$, *i.e.* into two lattices of signatures $(8,0)$ and $(0,8)$ respectively, both isomorphic to the E_8 root lattice, see *e.g.* [74].

The B^n's are chirally coupled to the torus shift multiplets through the off-diagonal terms in the second line of eq. (B.III.1.5), leading to couplings corresponding to the connection eq. (B.III.1.2) in space-time. The parameters $\beta_{\ell n}$ are related to the 'Wilson line' moduli V^a, see eq. (B.III.2.27) in the next section. Unlike the torus moduli (T,U), they are not quantized by the flux.

A discussion about moduli quantization in this context, from the target-space viewpoint, can be found in [75]. In that article examples where the Abelian bundle was not embeded in the commutant of the structure group of E were also considered. They can be incorporated in the present framework without too much effort. One needs to gauge the imaginary shift symmetry of the B^n's, and add an extra axial coupling of the form $\int \mathrm{d}\theta\,\Upsilon\,B^n$, in order to reproduce the gauge anomaly; in other words, at least part of the B^n's become shift multiplets similar to the Ω^ℓ's modeling the two-torus fibre.

The extended fibre Lagrangian: In the following, we will adopt compact notations incorporating both the torus and the Wilson lines by working with a $(10,10)$

lattice whose metric and B-field are:

$$G := \left(\begin{array}{cc|ccc} \frac{U_2}{T_2} & \frac{U_2}{T_2}T_1 & \frac{\beta_{11}}{4} & \frac{\beta_{12}}{4} & \cdots & \frac{\beta_{18}}{4} \\ \frac{U_2}{T_2}T_1 & \frac{U_2}{T_2}|T|^2 & \frac{\beta_{21}}{4} & \frac{\beta_{22}}{4} & \cdots & \frac{\beta_{28}}{4} \\ \hline \frac{\beta_{11}}{4} & \frac{\beta_{21}}{4} & \mathcal{G}_{11} & \mathcal{G}_{12} & \cdots & \mathcal{G}_{18} \\ \frac{\beta_{12}}{4} & \frac{\beta_{22}}{4} & \mathcal{G}_{21} & \mathcal{G}_{22} & \cdots & \mathcal{G}_{28} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{18}}{4} & \frac{\beta_{28}}{4} & \mathcal{G}_{81} & \mathcal{G}_{82} & \cdots & \mathcal{G}_{88} \end{array}\right), \tag{B.III.1.6}$$

$$B := \left(\begin{array}{cc|ccc} 0 & U_1 & \frac{\beta_{11}}{4} & \frac{\beta_{12}}{4} & \cdots & \frac{\beta_{18}}{4} \\ U_1 & 0 & \frac{\beta_{21}}{4} & \frac{\beta_{22}}{4} & \cdots & \frac{\beta_{28}}{4} \\ \hline -\frac{\beta_{11}}{4} & -\frac{\beta_{21}}{4} & 0 & \mathcal{B}_{12} & \cdots & \mathcal{B}_{18} \\ -\frac{\beta_{12}}{4} & -\frac{\beta_{22}}{4} & \mathcal{B}_{21} & 0 & \cdots & \mathcal{B}_{28} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\beta_{18}}{4} & -\frac{\beta_{28}}{4} & \mathcal{B}_{81} & \mathcal{B}_{82} & \cdots & 0 \end{array}\right).$$

We also introduce the following combinations:

$$E = G + B\,, \qquad \bar{E} = G - B. \tag{B.III.1.7}$$

Let us group together the gauge charges and Fayet-Iliopoulos couplings into the following vectors, and denote the various (shift) multiplets by a common letter:

$$v := \left(\begin{array}{c} m^1 \\ m^2 \\ \hline 0 \\ \vdots \\ \vdots \\ 0 \end{array}\right), \qquad h := \left(\begin{array}{c} h_1 \\ h_2 \\ \hline 0 \\ \vdots \\ \vdots \\ 0 \end{array}\right), \qquad \Omega := \left(\begin{array}{c} \Omega^1 \\ \Omega^2 \\ \hline B^1 \\ B^2 \\ \vdots \\ B^8 \end{array}\right). \tag{B.III.1.8}$$

The indices $\{i, j\}$ run over the full set of multiplets $\{\Omega^i\}_{i=1,\ldots,10}$ thereafter.

With these notations, the Lagrangian $\mathcal{L}_{\text{tor}} = \mathcal{L}^0_{\text{tor}} + \mathcal{L}_{\text{Wilson}}$ modelling the two-

torus together with a set of 8 complex Wilson lines reads:

$$\begin{aligned}\mathcal{L}_{\text{tor}} = &-\frac{iE_{ij}}{8}\int \mathrm{d}^2\theta \left(\Omega^i + \bar{\Omega}^i + 2v^i\mathcal{A}\right)\left(\partial_- \left(\Omega^j - \bar{\Omega}^j\right) + 2iv^j\mathcal{V}\right)\\ &-\frac{ih_i}{4}\int \mathrm{d}\theta\, \Upsilon\Omega^i + \text{h.c.}\,.\end{aligned} \tag{B.III.1.9}$$

Upon using the conditions (B.II.1.14), the Lagrangian (B.III.1.9) is given in components, after integrating by parts by:

$$\begin{aligned}\mathcal{L} = \frac{E_{ij}}{8}\Big\{&\partial_+\left(\omega^i + \bar{\omega}^i\right)\partial_-\left(\omega^j + \bar{\omega}^j\right) - \partial_+\left(\omega^i - \bar{\omega}^i\right)\partial_-\left(\omega^j - \bar{\omega}^j\right)\\ &- 2iv^i\partial_-\left(\omega^j - \bar{\omega}^j\right)A_+ - 2iv^j\partial_-\left(\omega^i - \bar{\omega}^i\right)A_+ + 4v^iv^jA_+A_-\\ &+ 2i\chi^i\partial_-\bar{\chi}^j + 2i\bar{\chi}^i\partial_-\chi^j\Big\} + \text{t.d.}\,.\end{aligned} \tag{B.III.1.10}$$

III.2 Dressed elliptic genus of compactifications with torsion

In this section we will extend the computation of the dressed elliptic genus done in section B.II.3.

III.2.1 Dressed elliptic genus and Abelian bundles over the total space

In the formulation of the GLSM used in this work, unlike in the previous chapter, one has also to deal with the spurious degrees of originating from the shift multiplets Ω^ℓ and from the B^n multiplets. All these degrees of freedom are of course artifacts of this formulation and should be decoupled at the end of the computation.

As an intermediate step, one defines a supersymmetric index appropriate for this 'enlarged' $(0,2)$ superconformal field theory as follows:

$$Z_{\text{EXT}}(\tau,\bar{\tau},z) = \frac{1}{\bar{\eta}(\bar{\tau})^{20}}\,\mathrm{Tr}_{\mathcal{H}^{\text{EXT}},\text{RR}}\left\{e^{2i\pi z J_0}\left(\bar{J}_0\right)^{10}(-1)^F q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - \bar{c}/24}\right\}, \tag{B.III.2.1}$$

the trace being taken in the Hilbert space $\mathcal{H}^{\text{EXT}}$ of the SCFT at the infrared fixed point of the torsion GLSM comprising the shift multiplets $\{\Omega^i\}_{i=1,\dots,10}$, in the left and right Ramond sectors.

The extra insertions of the R-current zero mode $\bar{J}_0$ in eq. (B.III.2.1) are needed in order to cancel the extra spurious fermionic zero modes appearing in this formulation. The right-moving R-current of the $(0,2)$ GLSM with the multiplets $\{\Omega^i\}_{i=1,\dots,10}$ is:

$$\bar{J} = G_{ij}\,\bar{\chi}^i\chi^j + \dots\,, \tag{B.III.2.2}$$

where the ellipsis stands for the contribution of the other fields of the theory. From the path integral point of view, this means that each $\bar{J}_0$ insertion has indeed the effect of saturating the fermionic zero modes of a fermion contained in a shift multiplet; hence, having inserted just the right power of this zero-mode, the other terms contributions to the current $\bar{J}_R$ do not play any role in the computation. Additionally to the right-moving fermions χ^i, one gets first a contribution from the non-compact real part of the bosons ω^i, which is completely factorized. Remains finally the contributions from the right-moving part of $\mathrm{Im}(b^n)$, which will be discussed in due time.

From this intermediate partition function Z_{EXT} one can then extract the dressed elliptic genus of interest that we define as,

$$Z^{\mathrm{W}}_{\mathrm{FY}}(\tau,\bar{\tau},z) = \frac{1}{\bar{\eta}(\bar{\tau})^2}\,\mathrm{Tr}_{\mathcal{H}^{\mathrm{W}}_{\mathrm{RR}}}\left\{e^{2i\pi z J_0}\bar{J}_0(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right\}, \qquad \text{(B.III.2.3)}$$

where $\mathcal{H}^{\mathrm{W}}_{\mathrm{RR}}$ is the Hilbert space of the SCFT corresponding to the $(0,2)$ non-linear sigma model of central charges $(c,\bar{c}) = (14+r,9)$ and target space $T^2 \hookrightarrow X \overset{\pi}{\to} \mathcal{S}$, with a rank r gauge bundle E in the first E_8 factor and a generic Abelian gauge bundle in the second E_8 factor; while the trace is restricted to the left Ramond sector for the former, we sum over all spin structures for the latter.

The index eq. (B.III.2.3) is the closest analogue of the elliptic genus in the present context, and consists in a non-holomorphic dressing of the elliptic genus of the $K3$ base, which is anomalous with respect to modular transformations, by a $(10,2)$ lattice encoding the principal two-torus bundle and the line bundles over its total space. The new supersymmetric index is then obtained as

$$Z_{\mathrm{NEW}}(\tau,\bar{\tau}) = \frac{\bar{\eta}^2}{2\eta^2}\sum_{\gamma,\delta=0}^{1} q^{\gamma^2}\left\{\left(\frac{\theta_1(\tau\,|z)}{\eta(\tau)}\right)^{8-r} Z^{\mathrm{W}}_{\mathrm{FY}}(\tau,\bar{\tau},z)\right\}\Bigg|_{z=\frac{\gamma\tau+\delta}{2}}. \qquad \text{(B.III.2.4)}$$

III.2.2 Computation of the dressed elliptic genus through localization

The supersymmetric partition function eq. (B.III.2.1) corresponds in Lagrangian formalism, following the previous chapter, to the following path integral on an Eu-

clidean torus of complex structure τ:

$$\begin{aligned} Z_{\text{EXT}}(\tau,\bar\tau,z) = \frac{1}{\bar\eta(\bar\tau)^{20}} \int \mathscr{D}a_w \mathscr{D}a_{\bar w} \mathscr{D}\lambda \mathscr{D}\bar\lambda \mathscr{D}D \, e^{-\frac{1}{e^2} S_{\text{Gauge}}[a,\lambda,D] - t\, S_{\text{FI}}(a,D)} \times \\ \times \int \prod_I \mathscr{D}\phi_I \mathscr{D}\bar\phi_I \mathscr{D}\psi_I \mathscr{D}\bar\psi_I \, e^{-\frac{1}{g^2} S_{\text{chiral}}[\phi_I,\psi_I,a,D,a_{\text{L}}]} \times \\ \times \int \prod_a \mathscr{D}\gamma_a \mathscr{D}\bar\gamma_a \mathscr{D}G_a \mathscr{D}\bar G_a \, e^{-\frac{1}{f^2} S_{\text{Fermi}}[\gamma_a,G_a,a,a_{\text{L}}] - S_{\text{pot}}[\gamma_a,G_a,\phi_i,\psi_i]} \times \\ \times \int \prod_{i=1}^{s+2} \mathscr{D}\omega^i \mathscr{D}\bar\omega^i \mathscr{D}\chi_i \mathscr{D}\bar\chi_i \, e^{-S_{\text{tor}}[\omega^i,\chi_i,a,a_{\text{L}}]} \left(\int \frac{\mathrm{d}^2 w}{2\tau_2} \, G_{ij} \, \bar\chi_i \chi_j \right)^{10} , \end{aligned} \tag{B.III.2.5}$$

where we have included couplings to a background gauge field for the $U(1)_{\text{L}}$ global symmetry

$$a_{\text{L}} = \frac{\pi z}{i\tau_2} \mathrm{d}w - \frac{\pi z}{i\tau_2} \mathrm{d}\bar w \,, \tag{B.III.2.6}$$

in order to implement the twisted boundary conditions, as well as coupling constants g and f in front of the chiral and Fermi Lagrangians for convenience.

Following what is done in section B.II.3, this path integral localizes to the BPS configurations with respect to the supercharge:

$$\mathcal{Q} = \left(\epsilon \mathcal{Q}_+ - \bar\epsilon \bar{\mathcal{Q}}_+ + \delta_{\text{WZ}} \right) \Big|_{\epsilon=\bar\epsilon=1} , \tag{B.III.2.7}$$

with δ_{WZ} the super-gauge transformation of chiral parameter $\Xi_{\text{WZ}} = i\bar\epsilon\theta a_{\bar w}$ needed to restore Wess-Zumino gauge.

One can wonder whether standard localization arguments apply to the path integral eq. (B.III.2.5). Indeed, as was emphasized above, neither the action, because of the field dependent Fayet-Ilioupoulos couplings, nor the path integral measure, because of the gauge anomaly, are separately gauge-invariant hence supersymmetry-invariant. However, owing to the anomaly cancellation condition eq. (B.III.1.4), one has:

$$\mathcal{Q} \left(\mathscr{D}\Phi \mathscr{D}\Gamma \, e^{-S} \right) = 0 \,. \tag{B.III.2.8}$$

Moreover, the operator $\int \mathrm{d}^2 w \, G_{ij} \, \bar\chi^i \chi^j$ is not annihilated by $\mathcal{Q}$. Thankfully, terms generated by the action of the supercharge do not saturate the fermionic measure hence do not contribute to the path integral. Finally, one can show as in [33] that the whole Lagrangian is actually $\mathcal{Q}$-exact, apart from the torus fibre part.

Gathering these arguments, one can see that the path integral does not depend

on the various couplings of the theory, allowing to compute it in the free-field limit:

$$e, g, f \to 0\,. \tag{B.III.2.9}$$

Notice that even though non-$\mathcal{Q}$-exact, the torus part is Gaussian hence can be computed directly. It implies as expected that the result will depend on the two-torus moduli (T, U) as well as on the Wilson lines moduli V^a.

The localization procedure reduces the path integral to a finite-dimensional integral over the gauge holonomies $(u, \bar{u})$ on the worldsheet torus, the zero-modes of the gauginos and of the auxiliary D-field. We refer to [33] and to the previous chapter for details and for the reduction of the final result to a contour integral in the u complex plane of the one-loop determinant. The contribution from the $K3$ base GLSM fields is the same as in section II.3.3.

Contribution from the extended fibre We compute below the contribution of the 'extended' fibre Lagrangian, introduced in section B.III.1, eq. (B.III.1.9), containing the chiral multiplets $\{\Omega^i\}_{i=1,\dots,10}$.

We consider first the bosonic terms in the Lagrangian eq. (B.III.1.10). Let us define the compact bosons:

$$\alpha^i := \mathrm{Im}(\omega^i)\,, \tag{B.III.2.10}$$

and proceed to a Wick rotation. α^1 and α^2 describe the coordinates on the two-torus of moduli T an U, while the other α^i correspond to the lattice $(E_8)_{\mathrm{L}} \times (E_8)_{\mathrm{R}}$. Setting aside the decoupled real part of ω^i, the bosonic part of the Lagrangian is then:

$$\mathcal{L}_{\mathrm{bos}} = \frac{E_{ij}}{2}\Big\{\bar{\partial}\alpha^i\partial\alpha^j + v^i\partial\alpha^j A_{\bar{w}} + v^j\partial\alpha^i A_{\bar{w}} + v^i v^j A_w A_{\bar{w}}\Big\}\,. \tag{B.III.2.11}$$

The fields α^i satisfy the periodicity conditions:

$$\alpha^i(z + k + \tau l, \bar{z} + k + \bar{\tau} l) = \alpha^i(z, \bar{z}) + 2\pi(k w_i + l n_i)\,. \tag{B.III.2.12}$$

The zero mode part of the compact bosons is then:

$$\alpha_0^i(z, \bar{z}) = \frac{i\pi}{\tau_2}\left\{z(w_i\bar{\tau} - n_i) - \bar{z}(w_i\tau - n_i)\right\}\,, \tag{B.III.2.13}$$

where n_i and w_i represent respectively the momentum and winding numbers. At the localization locus the gauge fields are reduced to their holonomies on the worldsheet

two-torus:

$$A^0 = \frac{\pi\bar{u}}{i\tau_2}\,\mathrm{d}w - \frac{\pi u}{i\tau_2}\,\mathrm{d}\bar{w}\,. \tag{B.III.2.14}$$

Plugging these expressions into (B.III.2.11) leads to the zero modes part of the action:

$$\mathcal{S}^0_{\rm bos} = \frac{\pi E_{ij}}{\tau_2}\Big\{(w_i\tau - n_i)(w_j\bar{\tau} - n_j) - v^i(w_j\bar{\tau} - n_j)u - v^j(w_i\bar{\tau} - n_i)u + v^iv^ju\bar{u}\Big\}\,. \tag{B.III.2.15}$$

The partition function is given as a sum over the momenta and windings:

$$\mathcal{Z}^0_{\rm bos} = \sum_{(w_i,n_i)\in\mathbb{Z}^{20}} \exp\left(-\mathcal{S}^0_{\rm bos}\right)\,. \tag{B.III.2.16}$$

Adopting obvious matrix notations, we can write the action as:

$$\mathcal{S}^0_{\rm bos} = \frac{\pi}{\tau_2}\Big\{n\cdot Gn + F\cdot n + |\tau|^2 w\cdot Gw - 2\bar{\tau}u\,Gv\cdot w + u\bar{u}\,v\cdot Gv\Big\}\,, \tag{B.III.2.17}$$

where we have defined:

$$F := -(\bar{\tau}E + \tau\bar{E})w + 2u\,Gv\,. \tag{B.III.2.18}$$

After performing a Poisson resummation on each variable n_i, one gets:

$$\begin{aligned}\mathcal{Z}^0_{\rm bos} = \frac{\sqrt{\tau_2}^{10}}{\sqrt{\det G}}\,e^{-\frac{\pi}{\tau_2}v\cdot Gv\,u\bar{u}} \sum_{(w,n)\in\mathbb{Z}^{20}} &\exp\Big\{-\pi\tau_2\left(n - \frac{F}{2i\tau_2}\right)\cdot G^{-1}\left(n - \frac{F}{2i\tau_2}\right)\Big\} \\ &\times\exp\Big\{-\frac{\pi}{\tau_2}\left(|\tau|^2 w\cdot Gw - 2\bar{\tau}u\,Gv\cdot w\right)\Big\}\,.\end{aligned} \tag{B.III.2.19}$$

Let us introduce the left and right momenta:

$$P_{\rm L} = \frac{1}{\sqrt{2}}\,G^{-1}\big(n - (B-G)\,w\big)\ ,\quad P_{\rm R} = \frac{1}{\sqrt{2}}\,G^{-1}\big(n - (B+G)\,w\big)\,. \tag{B.III.2.20}$$

One then has, after adding the contribution from the quantum fluctuations[1]:

$$\mathcal{Z}_{\rm bos} = \frac{1}{|\eta(\tau)|^{20}}\,e^{-\frac{\pi}{\tau_2}v\cdot Gv\,(u\bar{u}-u^2)} \sum_{(w,n)\in\mathbb{Z}^{20}} q^{\frac{1}{2}P^2_{\rm L}}\bar{q}^{\frac{1}{2}P^2_{\rm R}}e^{-2i\pi\sqrt{2}\,u\,v\cdot GP_{\rm L}}, \tag{B.III.2.21}$$

with $P^2_{\rm L} = P_{\rm L}\cdot GP_{\rm L}$ and $P^2_{\rm R} = P_{\rm R}\cdot GP_{\rm R}$. Let us introduce the following (20) × (20)

[1] We set $q := \exp(2i\pi\tau)$.

matrices:

$$\mathcal{M} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \quad \mathbb{I} = \begin{pmatrix} 0 & \mathbb{I}_{10} \\ \mathbb{I}_{10} & 0 \end{pmatrix}. \tag{B.III.2.22}$$

In terms of these matrices, one has:

$$\frac{1}{2}P_{\rm L}^2 = \frac{1}{4}\begin{pmatrix} n & w \end{pmatrix}(\mathcal{M}+\mathbb{I})\begin{pmatrix} n \\ w \end{pmatrix}, \quad \frac{1}{2}P_{\rm R}^2 = \frac{1}{4}\begin{pmatrix} n & w \end{pmatrix}(\mathcal{M}-\mathbb{I})\begin{pmatrix} n \\ w \end{pmatrix}. \tag{B.III.2.23}$$

The spurious contributions to the path integral, a consequence of the formulation of the GLSM in terms of shift multiplets, are dealt with as follows. First, the real part of each complex boson ω^i gives a $V/(\sqrt{\tau_2}|\eta|^2)$ contribution, proportional to the infinite volume V of their target space, which factorizes completely from the result. Second, the anti-holomorphic contribution of the right-moving part of $\mathrm{Im}(b_n)$ is also completely factorized, given that its zero-modes contribution span an E_8 root lattice of signature $(0,8)$. Indeed by construction the Wilson lines deformation do not involve this sub-lattice of the $(10,10)$ lattice corresponding to the 'extended' fibre, see eq. (B.III.1.6).

It leads eventually to a expression similar to standard heterotic lattices with Wilson lines, in terms of 8 complex moduli V^a, together with an extra left coupling of the torus fibre to the worldsheet gauge holonomy:

$$\begin{aligned}\mathcal{Z}_{\rm bos} =& \frac{1}{\eta(\tau)^{18}\bar\eta(\bar\tau)^2}\,\exp\left\{-\frac{\pi}{\tau_2}\frac{U_2}{T_2}|\mathfrak{m}|^2\,(u\bar u - u^2)\right\}\times \\ &\times \sum_{\substack{(n_1,n_2,w_1,w_2)\in\mathbb{Z}^4,\\ N\in\Gamma_{8,0}}} q^{\frac{1}{4}|p_{\rm L}|^2}\bar q^{\frac{1}{4}|p_{\rm R}|^2}\exp\left(-2i\pi u\,\mathrm{Re}\left(\mathfrak{m}\,\overline{p_{\rm L}^0}\right)\right),\end{aligned} \tag{B.III.2.24}$$

with the following standard complex expression for the momenta:

$$|p_{\rm R}|^2 = \frac{1}{(T_2U_2 - \sum_a (V_2^a)^2)}\left|-n_1T + n_2 + w_1U + w_2\left(TU - \sum_a (V^a)^2\right) + N_aV^a\right|^2, \tag{B.III.2.25a}$$

$$|p_{\rm L}|^2 = |p_{\rm R}|^2 + 4(n_1w_1 + n_2w_2) + N_aN^a\,, \tag{B.III.2.25b}$$

and where $p_{\rm L}^0$ is the left-moving momentum along the two-torus, in the absence of Abelian bundle, written in complex notation:

$$p_{\rm L}^0 := p_{\rm L}|_{V^a=0} = \frac{1}{\sqrt{U_2T_2}}(-n_1T + n_2 + U(w_1 + Tw_2))\,. \tag{B.III.2.26}$$

The relation between the complex Wilson line moduli $V^a = V_1^a + TV_2^a$ and the couplings $\beta_{\ell n}$ is then given by:

$$4iV_1^a = \left(1 + \frac{T_1^2}{T_2}\right)\beta_{1a} - \frac{T_1}{T_2}\beta_{2a}\,, \tag{B.III.2.27a}$$

$$4iV_2^a = \frac{T_1}{T_2}\beta_{1a} - \frac{1}{T_2}\beta_{2a}\,. \tag{B.III.2.27b}$$

Let us finally consider the contribution from the free fermions $\chi^i, \bar{\chi}^i$. After Wick rotation of eq. (B.III.1.10), one has:

$$\mathcal{L}_{\text{fer}} = \frac{G_{ij}}{2}\,\bar{\chi}^i\,\partial\chi^j\,. \tag{B.III.2.28}$$

On the other hand, the right-moving current is of the form:

$$\bar{J} = G_{ij}\,\bar{\chi}^i\chi^j + \dots\,, \tag{B.III.2.29}$$

where the ellipsis stand for the contribution of all the other fields and possible $\mathcal{Q}$-exact terms. As discussed previously, a $(\bar{J}_0)^{10}$ allows to handle all the fermionic zero-modes originating from the torus fibre and Wilson lines fermions, see eq. (B.III.2.3), and one obtains a $\bar{\eta}(\bar{\tau})^2$ contribution for each of the 10 free fermions which is canceled by the $1/\bar{\eta}^{20}$ in the definition of the intermediate supersymmetric index defined in eq. (B.III.2.1).

The result Assembling all pieces together, namely the contributions of the chiral and Fermi multiplets from the $K3$ base, of the $U(1)$ vector multiplets, and those from the torus fibre and Wilson lines, one obtains:

$$\begin{aligned} Z_{\text{FY}}^{\text{W}}(\tau,\bar{\tau},z) = \pm\left(-2i\pi\eta(\tau)^2\right)\times \\ \sum_{u^\star\in\mathcal{M}_{\text{sing}}^{\pm}}\oint_{u=u^\star}\mathrm{d}u\left\{\prod_{\Phi_i}\frac{i\eta(\tau)}{\theta_1(\tau|Q_iu+q_i^{\text{L}}z)}\prod_{\Gamma_a}\frac{i\theta_1(\tau|Q_au+q_a^{\text{L}}z)}{\eta(\tau)}\right. \\ \left.\sum_{(p_{\text{L}},p_{\text{R}})\in\Gamma_{10,2}}\frac{q^{\frac{1}{4}|p_{\text{L}}|^2}}{\eta(\tau)^{18}}\frac{\bar{q}^{\frac{1}{4}|p_{\text{R}}|^2}}{\bar{\eta}(\bar{\tau})^2}\exp\left(-2i\pi u\,\text{Re}\left(\mathfrak{m}\,\overline{p_{\text{L}}^0}\right)\right)\right\}, \end{aligned} \tag{B.III.2.30}$$

Thanks to the tadpole condition eq. (B.III.1.1), the global factor from the determinants

$$\exp\left\{-\frac{\pi\,(u^2-u\bar{u})}{\tau_2}\left(\sum_{\text{chiral}}Q^2 - \sum_{\text{fermi}}Q^2 - \frac{2U_2}{T_2}|\mathfrak{m}|^2\right)\right\}\,,$$

which is non holomorphic in the gauge field holonomy (hence potentially forbidding the reduction to a contour integral) vanishes. There are also similar non-holomorphic terms involving the $U(1)_{\mathrm{L}}$ global charges which vanish owing to the cancellation of the corresponding (mixed) anomalies.

A consistent choice of global charges, as we have already discussed in the previous chapter, is to assign $U(1)_{\mathrm{L}}$ charge $+1$ to the chiral multiplet P, charges -1 to the Fermi multiplets Γ_a, both appearing in the superpotential term eq. (B.II.1.6), and vanishing $U(1)_{\mathrm{L}}$ charge to all other multiplets.

In the formula eq. (B.III.2.30), $\mathcal{M}^{\pm}_{\mathrm{sing}}$ corresponds to one of two sets of singularities in the u plane for the determinants of chiral multiplets (from the $K3$ base), of positive or negative gauge charge respectively [34]. Both choices are equivalent since the sum of residues of a meromorphic function on the torus vanishes, however the natural interpretation of the formula is different in both cases. In general, the expression obtained from $\mathcal{M}^{+}_{\mathrm{sing}}$ would correspond to a Landau–Ginzburg type of computation.

Picking up $\mathcal{M}^{-}_{\mathrm{sing}}$, giving typically a contour integral around a pole at the origin, and provides the result that one would obtain by a direct computation in the geometrical 'phase', flowing in the IR to a (large volume) non-linear sigma-model. In the next section, we will provide a corresponding geometrical formula for the index, while the equivalence between both expressions is proven in section B.II.6, when the $K3$ surface is a subvariety of a weighted projective space.

In the computation of the index that we have presented in this section, we considered a $U(1)$ worldsheet gauge group for clarity. The result can be generalized for higher rank gauge groups in terms of a sum of Jeffrey-Kirwan residues using the results from [34], as we have done in the previous chapter. Instead of going along this route we will instead move to the geometrical formulation of the supersymmetric index, which is expected to be equally valid for any formulation, or UV completion, of the worldsheet theory underlying the $\mathcal{N} = 2$ compactifications with torsion.

III.3 Geometrical formulation of the dressed elliptic genus

In exactly a similar fashion as the in case without Abelian bundle over the total space, one proposes a purely geometrical formula in terms of a modified holomorphic Euler characteristic.

The relevant data of such non-Kähler heterotic compactification involves a heterotic Narain lattice $\Gamma(T, U, V)$ of signature $(10, 2)$, with T and U belonging to the

same imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$. One notices that the main difference with respect to the case in absence of Wilson lines basically consists in the replacement of the rational CFT data $(\Gamma_{\rm L}, \Gamma_{\rm R}, \phi)$, cf. section II.3.4 by the data of this Narain lattice $\Gamma(T, U, V)$.

The data also involves a pair of anti-self-dual $(1,1)$-forms ω_1 and ω_2 which we gather into the following vector:

$$p_\omega := \sqrt{\frac{U_2}{T_2}}\,(\omega_1 + T\omega_2)\,. \tag{B.III.3.1}$$

This vector as it is belongs to a formal extension, over $H^2(\mathcal{S},\mathbb{Z})$, of the winding sub-lattice of the $\Gamma^{2,2}(T,U)$ toroidal lattice. As explained in appendix E.II, the compatibility conditions eq. (E.II.1.12) between $\omega_{1,2}$ and the lattice ensure that it actually belongs to (a formal extension of) the left lattice $\Gamma_{\rm L}$ of the two-torus. Notice that p_ω involves the moduli U and T of the torus with the Wilson lines turned off, and not those corresponding to the physical Kaluza-Klein metric on T^2.

In appendix E.II, we discuss in more detail the quantization of the torus moduli, and the compatibility between the choice of rational Narain lattice and of the pair of two-forms (ω_1,ω_2). These anti-self-dual two-forms generate a rank two[2] even negative-definite lattice Γ_ω, which is a sub-lattice of $H^2(\mathcal{S},\mathbb{Z})$.

Notice that p_ω involves the moduli U and T of the torus with the Wilson lines turned off, and not those corresponding to the physical Kaluza-Klein metric on T^2.

III.3.1 Modified Euler characteristic

Using the notations introduced above and following sections B.I.3 and B.II.5, we define the dressed elliptic genus of a holomorphic vector bundle E over a $K3$ surface $\mathcal{S}$, with a given $(10,2)$ lattice comprising the T^2 fibre of the principal bundle $T^2 \hookrightarrow X \xrightarrow{\pi} \mathcal{S}$, and the set of Abelian connections over X, as the following modified Euler characteristic:

$$\begin{aligned} Z^{\rm W}_{\rm FY}\left(X,E,\omega\middle|\,\tau,\bar\tau,z\right) = q^{\frac{r-2}{12}}\,y^{\frac{r}{2}}\int_{\mathcal{S}}\mathrm{ch}\left(\mathbb{E}_{q,y}\right)\mathrm{td}\left(T_{\mathcal{S}}\right)\times \\ \times\sum_{(p_{\rm L},p_{\rm R})\in\Gamma_{10,2}}\frac{q^{\frac14|p_{\rm L}|^2}}{\eta(\tau)^{18}}\frac{\bar q^{\frac14|p_{\rm R}|^2}}{\bar\eta(\bar\tau)^2}e^{-\mathrm{Re}\left(p_\omega\,\overline{p^0_{\rm L}}\right)}\,, \end{aligned} \tag{B.III.3.2}$$

[2] We don't consider the degenerate case where ω_1 and ω_2 are colinear.

where $\overline{p^0_{\rm L}}$ is defined in eq. (B.III.2.26).

The proof that this formula actually coincides with the GLSM result, eq. (B.III.2.30), is given in the previous chapter for the case without Wilson lines, when $\mathcal{S}$ is constructed as a subvariety of a weighted projective space $V = \mathbb{P}^n(q_0, ..., q_n)$. The proof in the present context follows exactly the same lines as in absence of Wilson lines, simply replacing the partition function of the rational toroidal $c = 2$ CFT by the partition function of the Narain lattice $\Gamma_{10,2}$, with no extra modifications.

Let $c(\mathcal{T}_\mathcal{S}) = \prod_{i=1}^2(1+\nu_i)$ and $c(E) = \prod_{a=1}^r(1+\xi_a)$ denote the total Chern classes of the respective bundles, making use of the splitting principle. One can write

$$Z^{\rm W}_{\rm FY}(X, E, \omega|\tau, \bar\tau, z) = \int_\mathcal{S} G(\tau, \bar\tau, z, \nu, \xi, p_\omega)\,, \tag{B.III.3.3}$$

with integrand

$$\begin{aligned} G(\tau, \bar\tau, z, \nu, \xi, p_\omega) = \prod_{a=1}^r \frac{i\theta_1\left(\tau\left|\frac{\xi_a}{2i\pi} - z\right.\right)}{\eta(\tau)} \prod_{i=1}^2 \frac{\eta(\tau)\nu_i}{i\theta_1(\tau\left|\frac{\nu_i}{2i\pi}\right.)} \times \\ \times \sum_{(p_{\rm L}, p_{\rm R})\in\Gamma_{10,2}} \frac{q^{\frac14|p_{\rm L}|^2}}{\eta(\tau)^{18}} \frac{\bar q^{\frac14|p_{\rm R}|^2}}{\bar\eta(\bar\tau)^2} e^{-\,{\rm Re}\left(p_\omega\,\overline{p^0_{\rm L}}\right)}\,. \end{aligned} \tag{B.III.3.4}$$

III.3.2 Modular properties

The ordinary elliptic genus of a rank r holomorphic vector bundle of vanishing first Chern class over a K3 surface $\mathcal{S}$, satisfying the condition $c_2(E) = c_2(T_\mathcal{S})$, is a weak Jacobi form of weight 0 and index $r/2$ with the same character, or multiplier system, as $(\theta_1/\eta)^{r-2}$.

The dressed elliptic genus $Z^{\rm W}_{\rm FY}$ that we have defined for non-Kähler $T^2 \hookrightarrow X \xrightarrow{\pi} \mathcal{S}$ principal bundles, though non-holomorphic in τ by construction, is holomorphic in the z variable and transforms also as a weak Jacobi form[3] of weight 0 and index $\frac{r}{2}$, with the same character as $(\theta_1/\eta)^{r-2}E_4/\eta^8$, as can be seen by a trivial generalization of the computation that we have presented in the previous chapter.

[3] If the fibre of the principal bundle is one-dimensional (*i.e.* an S^1 rather than a T^2), in the absence of Wilson lines, the non-trivial part of the index $\int_\mathcal{S}\prod_{a=1}^r \frac{i\theta_1\left(\tau\left|\frac{\xi_a}{2i\pi}-z\right.\right)}{\eta(\tau)} \prod_{i=1}^2 \frac{\eta(\tau)\nu_i}{i\theta_1(\tau\left|\frac{\nu_i}{2i\pi}\right.)} \sum_{\mu\in\mathbb{Z}_{2k}} \Theta_{\mu,k}\left(\tau\left|\frac{m^1\omega_1}{2i\pi}\right.\right)\Theta_{\varphi(\mu),k}(-\bar\tau|0)$ has some similarity with skew-holomorphic Jacobi forms, defined by Skoruppa [76], but fails to satisfy a heat equation.

III.3.3 Decomposition into weak Jacobi forms

An explicit expression of the dressed elliptic genus can then be obtained, with minimal knowledge of the underlying geometrical data. The following formula holds [72]:

$$\theta_1(\tau|z+\xi) = \exp\left\{-\frac{\pi^2}{6}E_2(\tau)\xi^2 + \frac{\theta_1'(\tau|z)}{\theta_1(\tau|z)}\xi - \sum_{n\geqslant 2}\wp^{(n-2)}(\tau,z)\frac{\xi^n}{n!}\right\}\theta_1(\tau|z)\,, \tag{B.III.3.5}$$

where $\wp$ is the Weierstrass elliptic function and $\wp^{(n)} := \frac{\partial^n}{\partial z^n}\wp$. Expanding the integrand (B.II.5.8) and keeping only the top degree form terms, one obtains

$$\begin{aligned} Z^{\rm W}_{\rm FY} &= (-i)^r \frac{\theta_1(\tau|z)^r}{\eta(\tau)^{r+4}} \sum_{(p_{\rm L},p_{\rm R})\in\Gamma_{10,2}} \frac{q^{\frac{1}{4}|p_{\rm L}|^2}}{\eta(\tau)^{18}} \frac{\bar q^{\frac{1}{4}|p_{\rm R}|^2}}{\bar\eta(\bar\tau)^2} \times \\ &\times \int_{\mathcal S}\left\{-\frac{E_2(\tau)}{24}\sum_i \nu_i^2 + \left(\frac{E_2(\tau)}{24} - \frac{\wp(\tau,z)}{2(2i\pi)^2}\right)\sum_a \xi_a^2 + \frac{1}{2}{\rm Re}\left(p_\omega\,\overline{p^0_{\rm L}}\right)^2\right\}\,, \end{aligned} \tag{B.III.3.6}$$

where we have used the fact that the first Chern class of the holomorphic vector bundle $c_1(\mathcal V)$ vanishes. Using the definition of the instanton number, the fact that $\int_{\mathcal S} {\rm ch}_2(\mathcal T_{\mathcal S}) = -24$, and that the ordinary elliptic genus of a $(4,4)$ non-linear sigma-model on $K3$, namely

$$Z^{K3}_{\rm ell}(\tau,z) = 8\left\{\left(\frac{\theta_2(\tau|z)}{\theta_2(\tau|0)}\right)^2 + \left(\frac{\theta_3(\tau|z)}{\theta_3(\tau|0)}\right)^2 + \left(\frac{\theta_4(\tau|z)}{\theta_4(\tau|0)}\right)^2\right\}\,, \tag{B.III.3.7}$$

is related to the Weierstrass $\wp$-function by the following formula:

$$Z^{K3}_{\rm ell}(\tau,z) = \frac{6}{\pi^2}\frac{\wp(\tau,z)\theta_1(\tau|z)^2}{\eta(\tau)^6}\,, \tag{B.III.3.8}$$

one can write the index as sum of three terms in the following way:

$$\begin{aligned} Z^{\rm W}_{\rm FY} &= -(-i)^r \sum_{(p_{\rm L},p_{\rm R})\in\Gamma_{10,2}} \frac{q^{\frac{1}{4}|p_{\rm L}|^2}}{\eta(\tau)^{18}} \frac{\bar q^{\frac{1}{4}|p_{\rm R}|^2}}{\bar\eta(\bar\tau)^2} \times \\ &\left(\frac{\vartheta_1(\tau|z)}{\eta(\tau)}\right)^{r-2}\left\{\frac{N}{24}Z^{K3}_{\rm ell}(\tau,z) + \frac{N-24}{12}\frac{\theta_1(\tau|z)^2}{\eta(\tau)^6}E_2(\tau) - \frac{\theta_1(\tau|z)^2}{2\,\eta(\tau)^6}\int_{\mathcal S}{\rm Re}\left(p_\omega\,\overline{p^0_{\rm L}}\right)^2\right\}\,. \end{aligned} \tag{B.III.3.9}$$

This expression can be rewritten as:

$$Z_{\rm FY}^{\rm W} = -(-i)^r \sum_{(p_{\rm L},p_{\rm R})\in\Gamma_{10,2}} \frac{q^{\frac{1}{4}|p_{\rm L}|^2}}{\eta(\tau)^{18}} \frac{\bar{q}^{\frac{1}{4}|p_{\rm R}|^2}}{\bar{\eta}(\bar{\tau})^2} \left(\frac{\vartheta_1(\tau|z)}{\eta(\tau)}\right)^{r-2} \mathcal{Z}_N(\tau,z,p_{\rm L})\,, \qquad \text{(B.III.3.10)}$$

where:

$$\mathcal{Z}_N(\tau,z,p_{\rm L}) = \frac{N}{12}\,\phi_{0,1}(\tau,z) + \left(-\frac{N-24}{12}\,E_2(\tau) + m(p_{\rm L},\omega)\right)\phi_{-2,1}(\tau,z) \qquad \text{(B.III.3.11)}$$

with the standard weak Jacobi forms of index 1:

$$\begin{aligned}\phi_{0,1}(\tau,z) &= 4\left(\frac{\theta_2(\tau|z)^2}{\theta_2(\tau|0)^2} + \frac{\theta_3(\tau|z)^2}{\theta_3(\tau|0)^2} + \frac{\theta_4(\tau|z)^2}{\theta_4(\tau|0)^2}\right)\,,\\ \phi_{-2,1}(\tau,z) &= -\frac{\theta_1(\tau|z)^2}{\eta(\tau)^6}\,,\end{aligned} \qquad \text{(B.III.3.12)}$$

and where we have defined:[4]

$$m(p_{\rm L},\omega) := \frac{1}{2}\int_{\mathcal{S}} {\rm Re}\left(p_\omega\,\overline{p_{\rm L}^0}\right)^2\,. \qquad \text{(B.III.3.13)}$$

The expression (B.III.3.11) that we have obtained shows that the index depends on the vector bundle $\mathcal{V}$ only through its instanton number N. The data characterizing the principal two-torus bundle is encoded in (B.III.3.13), that intertwines Γ_ω with $\Gamma_{2,2}(T,U)$. Equation (B.III.3.11) is a good starting point to compute the threshold corrections to the gauge and gravitational couplings, that will be given in the next part.

III.4 Moonshine properties of the index?

As discussed quickly in the introductory part when we introduced the elliptic genus, non-linear sigma-models on $K3$ with $(4,4)$ supersymmetry have a rather mysterious relationship with the Mathieu group $\mathbb{M}_{24}$, that was first observed in [44] (and explored later on by many authors), by expanding the elliptic genus of the former into characters of the $\mathcal{N}=4$ superconformal algebra:

$$Z_{\rm ell}^{K3}(\tau,z) = 24\,{\rm ch}_{h=1/4,l=0}(\tau,z) + \sum_{n=0}^{\infty} A_n\,{\rm ch}_{h=k+1/4,l=1/2}(\tau,z)\,, \qquad \text{(B.III.4.1)}$$

[4] $m(p_{\rm L},\omega)$ is an integer since the Picard lattice ${\rm Pic}(\mathcal{S}) = H^2(\mathcal{S},\mathbb{Z}) \cap H^{1,1}_{\bar\partial}(\mathcal{S})$ is even.

where $\mathrm{ch}_{h=1/4,l=0}$ is the character of the massless representation of isospin zero and $\mathrm{ch}_{h=k+1/4,l=1/2}$ are characters of massive representations of isospin one-half. The coefficients $\{A_n\}$ of the expansion are indeed related to dimensions of $\mathbb{M}_{24}$ irreducible representations as $A_0 = -2$, $A_1 = 90 = 45 + \overline{45}$, etc...

As we discussed in section b, this Mathieu moonshine can be extended to $K3$ compactifications with arbitrary gauge bundles [46]. Since Fu-Yau compactifications encompass and largely extend such $K3 \times T^2$ compactifications, it is therefore legitimate to investigate possible moonshine phenomena for the torsional compactifications investigated here. A first step is naturally to look for possible hints of relations with the group $\mathbb{M}_{24}$.

One can actually expand $\mathcal{Z}_N(\tau, z, p_{\mathrm{L}})$, the summand appearing in the dressed elliptic genus (B.III.3.6), in terms of $\mathcal{N} = 4$ characters as follows:

$$\mathcal{Z}_N(\tau, z, p_{\mathrm{L}}) = N\,\mathrm{ch}_{h=1/4,l=0}(\tau, z) + \sum_{n=0}^{\infty} \tilde{A}_n\,\mathrm{ch}_{h=n+1/4,l=1/2}(\tau, z)\,, \tag{B.III.4.2}$$

with

$$\tilde{A}_n(N, p_{\mathrm{L}}, \omega) = \frac{N}{24} A_n + \frac{N-24}{12}\,B_n - m(p_{\mathrm{L}}, \omega)\,C_n\,, \tag{B.III.4.3}$$

where the integer coefficients B_n and C_n are defined by the expansions:

$$\frac{q^{1/8} E_2}{\eta^3} = \sum_{k=0}^{\infty} B_k\,q^k\,, \tag{B.III.4.4a}$$

$$\frac{q^{1/8}}{\eta^3} = \sum_{k=0}^{\infty} C_k\,q^k\,. \tag{B.III.4.4b}$$

In particular the $\{C_n\}$ give the number of partition into three kinds of integers. They can be expressed by the following recursive relations:

$$\begin{aligned} C_0 &= 1\,, \\ C_n &= \frac{3}{k} \sum_{l=0}^{n-1} C_\ell\,\sigma_1(n-l),\ \forall n \in \mathbb{N}^*, \end{aligned} \tag{B.III.4.5}$$

where $\sigma_1(n) = \sum_{d|n} d$. Moreover, one can expand the Eisenstein series E_2 as

$$E_2(\tau) = 1 - 24 \sum_{k=1}^{\infty} \sigma_1(k)\,q^k\,. \tag{B.III.4.6}$$

One thus has the following relation between these two sequences of coefficients:

$$\begin{aligned} \forall n \in \mathbb{N},\ B_n &= C_n - 24 \sum_{l=0}^{n-1} C_\ell\, \sigma_1(n-l) \\ &= (1-8n)\, C_n\,. \end{aligned} \tag{B.III.4.7}$$

Using these relations, one obtains the final expression for the coefficients $\{\tilde{A}_n\}$ of the expansion into $N=4$ characters, eq. (B.III.4.2), as

$$\tilde{A}_n = 2(8n-1)C_n + N\,\frac{A_n - 2(8n-1)C_n}{24} - m(p_{\rm L},\omega)C_n\,. \tag{B.III.4.8}$$

The first coefficients of this expansion are explicitly:

n	$\tilde{A}_n$
0	$-2 - m(p_{\rm L})$
1	$42 + 2N - 3\,m(p_{\rm L})$
2	$270 + 8N - 9\,m(p_{\rm L})$
3	$1012 + 22N - 22\,m(p_{\rm L})$
4	$3162 + 58N - 51\,m(p_{\rm L})$
5	$8424 + 132N - 108\,m(p_{\rm L})$
6	$20774 + 294N - 221\,m(p_{\rm L})$
7	$47190 + 604N - 429\,m(p_{\rm L})$
8	$102060 + 1210N - 810\,m(p_{\rm L})$
9	$210018 + 2318N - 1479\,m(p_{\rm L})$
10	$417120 + 4334N - 2640\,m(p_{\rm L})$

(B.III.4.9)

All the coefficients $\{\tilde{A}_n\}$ are integer numbers, which is not obvious from their expression (B.III.4.8). This follows from a quite intriguing property, which may shed some light on the ordinary Mathieu moonshine, namely that:

$$\forall n \in \mathbb{N}\,, \quad A_n - 2(8n-1)C_n \equiv 0 \mod 24\,, \tag{B.III.4.10}$$

where the $\{A_n\}$ are the coefficients of the expansion of the $K3$ elliptic genus into $\mathcal{N}=4$ representations – hence encode the information about $\mathbb{M}_{24}$ representations – and where C_n are defined by (B.III.4.4a). In fact one has a stronger result:

$$\forall n \in \mathbb{N}\,, \quad A_n - 2(8n-1)C_n \equiv 0 \mod 48\,. \tag{B.III.4.11}$$

To see it, let us define:

$$A(q) := \sum_{n=0}^{\infty} A_n q^n \,, \qquad C(q) := \frac{q^{1/8}}{\eta(q)^3} \,, \tag{B.III.4.12}$$

and consider the function:

$$\nu(q) := A(q) - 2\left(8q\frac{\partial}{\partial q} - 1\right) C(q) \,. \tag{B.III.4.13}$$

This function then has the following q-expansion:

$$\nu(q) = \sum_{n=0}^{\infty} \Big(A_n - 2(8n-1)C_n\Big) q^n \,. \tag{B.III.4.14}$$

Using the fact that $q\partial_q = (2i\pi)^{-1}\partial_\tau$ and that:

$$E_2(\tau) = \frac{12}{i\pi}\frac{\partial}{\partial\tau}\eta(\tau) \,, \tag{B.III.4.15}$$

one computes

$$\nu(q) = A(q) + 2\,\frac{q^{1/8}}{\eta(q)^3}\,E_2(q) \,. \tag{B.III.4.16}$$

One can then use eq. (7.16) of [77] to obtain:

$$\nu(q) = 48\,\frac{q^{1/8}}{\eta(q)^3}\,F^{(2)}(q) \,, \tag{B.III.4.17}$$

with:

$$F^{(2)}(q) = \sum_{\substack{0<m<n \\ n\not\equiv m \bmod 2}} (-1)^n\, m\, q^{\frac{mn}{2}} \,, \tag{B.III.4.18}$$

the relevant point being that 48 divides $\nu(q)$. It may be possible to generate other identities of the type eq. (B.III.4.11) by considering twists by insertion of $\mathbb{M}_{24}$ elements.

Coming back to the possible moonshine behavior of the index (B.III.4.2), given that the coefficients $m(p_{\rm L})$ can be arbitrary large (negative) integers, depending on the left-moving momentum along the two-torus fibre, the decomposition of the coefficients $\{\tilde{A}_n\}$ into dimensions of irreducible representations of $\mathbb{M}_{24}$, or any other sporadic group, is far from obvious. If these coefficients were corresponding each to the dimension of a given representation, the term in $m(p_{\rm L},\omega)$ could indicate the number of times such representation appears in the module for this $p_{\rm L}$.

By considering the problem from another point of view, one could notice that the quantity $\mathcal{Z}_N(\tau, z, p_{\rm L})$ appearing in eq. (B.III.3.11) is actually similar to a twining partition function in the context of the standard Mathieu moonshine [45, 78], if one sets aside the contribution in $m(p_{\rm L}, \omega)$ which involves the torus fibre. Further investigations are under way [79].

Conclusion

Let us conclude this chapter by saying that, as mentioned in the beginning of chapter B.II, the dressed elliptic genus leads naturally to the new supersymmetric index, which is the natural starting point for computing the gauge and gravitational one-loop threshold corrections appearing in the low-energy four-dimensional effective action of $\mathcal{N} = 2$ heterotic compactifications. Given that the torsional compactifications considered here represent a large fraction of those, including as a subset the familiar $K3\times T^2$ compactifications, computing these thresholds is a rather important task. Starting from eq. (B.III.3.9) in the present chapter, one can reach an expression in terms of standard weak almost holomorphic modular forms for the threshold corrections, and exploit the whole machinery developped in [80, 81] for performing modular integrals by unfolding the integration domain against Niebur-Poincaré series. These results will be presented in chapter C.II.

The computation of the heterotic threshold corrections could also shed light on $\mathcal{N} = 2$ type IIA/heterotic dualities. Potential duals of torsional heterotic compactifications were proposed in [75], as Calabi-Yau three-folds admitting a K3 fibration without compatible elliptic fibration with section. Given that the Abelian bundle moduli are not quantized by H-flux, unlike the T and U moduli of the two-torus fibre, one can in principle compute the associated prepotential governing the complex structure moduli from the heterotic threshold corrections, and compare with the type II expectations, as was done for $K3 \times T^2$ in [46]. We plan to perform such quantitative checks, that would extend significantly our current knowledge of type IIA/heterotic $\mathcal{N} = 2$ dualities.

Part C

Threshold corrections in $\mathcal{N} = 2$ heterotic compactifications

Chapter I

Effective $\mathcal{N} = 2$ supergravity and threshold corrections

In this part, we continue focusing on compactifications of the heterotic string leading to extended $\mathcal{N} = 2$ in four-dimensional effective theory. We will introduce the kind of topological quantities related to such compactifications one is typically interested in, namely the loop corrections to the couplings appearing in the low energy effective action, especially those receiving correction only from the BPS states of the theory, and which are deeply connected to the *dressed elliptic genus* introduced and computed in the context of Fu-Yau compactifications in chapter B.II and further discussed in chapter B.III.

As mentioned above, we will be interested in a particular type of couplings entering the four-dimensional supergravity effective action corresponding to the above heterotic compactifications. In order to introduce these couplings, let us give some generic comments about the effective action for a generic $\mathcal{N} = 2$ heterotic compactification.

The field content of the four-dimensional low energy effective theory consists in a gravitational multiplet consisting of the metric tensor, two gravitini and a graviphoton. It also contains vector multiplets, consisting of the gauge bosons as well as their associated pair of gaugini, and one complex scalar. Finally, the theory contains a collection of hypermultiplets consisting of a single fermionic field and four scalar fields.

The moduli space of the theory is then written locally as a cartesian product corresponding to the contribution of the scalars contained in the vector and hyper

multiplets:

$$\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H \,, \tag{C.I.0.1}$$

$\mathcal{N}=2$ local supersymmetry imposing constraints on the structure carried by these two factors. $\mathcal{M}_V$ plays the major role for us, since its structure and the various quantum corrections it receives directly contribute to the couplings of interest to us. $\mathcal{M}_V$ is a special Kähler manifold, meaning that in addition to being Kähler, the whole data is basically fixed by the knowledge of a unique function, the prepotential $f(z^i)$, where $\{z^i\}_{i=0,1,\ldots,N_V}$ denotes a local patch of coordinates on $\mathcal{M}_V$. Denoting by:

$$f_{i_1,\ldots,i_k}(z) = \frac{\partial^k}{\partial z^{i_1}\ldots\partial z^{i_k}} f(z) \,, \tag{C.I.0.2}$$

the Kähler potential is given by:

$$K = -\log\Big(-2\mathrm{Im}(z^i f_i)\Big) \,, \tag{C.I.0.3}$$

from which the Kähler metric obtained:

$$G_{i\bar{\jmath}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{\jmath}}} \,. \tag{C.I.0.4}$$

We will be interested in the following to the one-loop contribution to the threshold corrections of the gauge coupling. The kinetic term for the gauge fields is given by:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\,\Xi_{ij} F^i \wedge \star F^j \,. \tag{C.I.0.5}$$

This term is related by supersymmetry to a parity-violating contribution of the form:

$$\mathcal{L}_{\text{odd}} = -\frac{1}{2}\,\Theta_{ij} F^i \wedge F^j \,. \tag{C.I.0.6}$$

The coupling functions Ξ_{ij} and Θ_{ij} are also completely fixed in terms of the prepotential:

$$\Xi_{ij} = -\frac{1}{2}\,\mathrm{Im}N_{ij} \,, \tag{C.I.0.7a}$$

$$\Theta_{ij} = \frac{1}{2}\,\mathrm{Re}N_{ij} \,, \tag{C.I.0.7b}$$

with:

$$N_{ij} = \bar{f}_{ij} + 2i\,\frac{z^k z^l\,\mathrm{Im}f_{ik}\,\mathrm{Im}f_{jl}}{z^m z^n\,\mathrm{Im}f_{mn}} \,. \tag{C.I.0.8}$$

Let us focus on the contibution from a given simple factor G in the spacetime gauge

group, and normalize its coupling as follows:

$$\int \frac{2}{g_G^2} \operatorname{tr}\left(F^a \wedge \star F^a\right) . \tag{C.I.0.9}$$

As mentionned above the gauge coupling actually depends on the vector moduli, the goal being precisely to make explicit this dependence at one-loop.

The running of the above gauge coupling takes the following form:

$$\frac{16\pi^2}{g_G^2(\mu)} = \left(\frac{16\pi^2}{g_s^2} + \Delta_{\text{univ}}\right) + \beta_G \log\left(\frac{M_s^2}{\mu^2}\right) + \Delta_G \,, \tag{C.I.0.10}$$

with the tree level contribution being given by the string coupling g_s. The universal component Δ_{univ} does not depend on the gauge group under consideration, and typically originates from taking into account the backreaction on the geometry due to the presence of a non-trivial gauge configuration [82]. One usually absorbs this piece into a redefinition of the string coupling:

$$\frac{16\pi^2}{\tilde{g}_s^2} := \frac{16\pi^2}{g_s^2} + \Delta_{\text{univ}} \,. \tag{C.I.0.11}$$

We will follow this convention in the following, and actually drop the tilde on the string coupling.

The second term corresponds to the contribution from the massless states of the theory, hence to the result on would obtain in a purely field-theoretic computation. This contribution is of course plagued by IR divergences, and the energy scale μ correspond to some IR regularization scheme, which we will briefly discuss later. Finally, the last term corresponds to the contribution from the whole tower of stringy massive states running into the loop, and depends on the compactification details.

In a similar fashion, one will also be interested in the one-loop threshold corrections to the gravitational coupling:

$$\frac{16\pi^2}{g_{\text{grav}}^2(\mu)} = \frac{16\pi^2}{g_s^2} + \beta_{\text{grav}} \log\left(\frac{M_s^2}{\mu^2}\right) + \Delta_G \,, \tag{C.I.0.12}$$

The Lagrangian also contains the equivalent parity odd counterpart of the gravitational coupling corresponding to a term of the form $R \wedge R$ in the effective Lagrangian.

As we discussed in section A.II.1 and section B.I.1, these higher derivative terms are intrinsically related to the Green-Schwarz mechanism of anomaly cancellations, which requires to include such one-loop (in the sigma model perturbation theory)

contributions to the supergravity action. Let us describe schematically how the expression of the one-loop threshold corrections is related to descendants of an object called the new supersymmetric index, which has already been introduced in section B.II.2.

Let us focus on a simple factor G_a of the spacetime gauge group for concreteness, the same king of reasoning applying in the same way to the R^2 gravitational term in the effective action. As explained above, one expects the gauge coupling g_a to have the following one-loop expression:

$$\left.\frac{16\pi^2}{g_a^2(\mu)}\right|_{\text{one-loop}} = \beta_G \log\left(\frac{M_s^2}{\mu^2}\right) + \Delta_G\,, \tag{C.I.0.13}$$

which splits into a logarithmic contribution from the massless fields proportional to the field-theoretic β-function, and the contribution Δ_a from the whole tower of massive fields running into the loop. As is usual in field theory, massless fields come with infrared divergences, whose regularization requires the introduction of an energy scale denoted μ in the above expression. Let us simply mention that for instance in the case of $K3\times T^2$ compactifications, a very quick way to derive the expression of the threshold as the intgral over the fundamental domain of $SL_2(\mathbb{Z})$ of almost holomorphic modular form consists in first looking at second derivatives with respect to, say, the moduli of the torus, which do not contain any IR divergent pieces. However, we will ultimately be interested in computing the threshold corrections corresponding to $\mathcal{N}=2$ Fu-Yau compactifications for which we know that such moduli are actually quantized, both belonging to the same imaginary quadratic field:

$$T, U \in \mathbb{Q}(\sqrt{D})\,, \tag{C.I.0.14}$$

with D the discriminant of some positive definite integer quadratic form.

There are therefore to possible approaches. Either one can formally analytically continue the expression of the thresholds to generic complex values of the torus fibre moduli, hence allowing to follow the above mentioned approach, or follow the more rigorous *background field method* [83–85], which consists in replacing the four-dimensional CFT descibing the flat Minkowski spacetime by some other CFT characterized by a non-trivial profile of the gravitational sector fields, hence generating a mass gap in the theory, providing a natural IR cutoff μ^2. Among other good properties, this approach preserves modular invariance, and the results can be matched unambiguously with the field-theoretic results. We will not enter to

much into the details of this computation below, and refer the reader to the existing litterature.

Schematically, one wants to compute a correlation function in the worldsheet CFT quadratic in the field strength of the gauge field corresponding to simple gauge factor G_a under consideration. One way to do this is to try and turn on a background gauge field in the theory by deforming it with an exactly marginal operator, and collect the quadratic terms in the modified partition function. However, as described for instance in the original paper of Kaplunovsky [82], turning on a non-trivial gauge field contributes to the energy-momentum tensor, hence adds a source to the Einstein equations, imposing to also turn on a background gravitational field for consistency. In the zero ghost number picture, the gauge vertex operator is written:

$$V_{\text{gauge}}(z,\bar{z}) = F^a_{\mu\nu} J^a(z) \left(X^\mu(z,\bar{z}) \bar{\partial} X^\nu(z,\bar{z}) + \psi^\mu(\bar{z}) \psi^\nu(\bar{z}) \right) , \tag{C.I.0.15}$$

but the right-moving part of this vertex operator is not a well-defined $(0,1)$ operator. This problem is solved if one replaces the CFT corresponding to the four-dimensional spacetime by a curved background with the same central charge and such that modular invariance is preserved. One such theory is $\mathbb{R}_{1/\sqrt{k+2}} \times SU(2)_k$, and provides a mass gap $\mu^2 = 1/(k+2)$ in the spectrum of the theory. The vertex operator in the regulated theory is:

$$V_{\text{gauge}}(z,\bar{z}) = \vec{F}_a \cdot \vec{J}_a (\bar{J}^3 + \psi^1 \psi^2) \, , \tag{C.I.0.16}$$

with still a labelling a simple factor of the gauge group. J^i_a is in the Cartan of the gauge group, in the adjoint representation. The gravitational marginal deformation is:

$$V_{\text{grav}}(z,\bar{z}) = \mathcal{R} J^3 (\bar{J}^3 + \psi^1 \psi^2) \, . \tag{C.I.0.17}$$

This IR regulator is removed in the flat limit $k \to \infty$. The partition function of the regulated theory is then written as:

$$Z(\mu) = \frac{\Gamma_0(\mu)}{V(\mu)} Z(0) \, , \tag{C.I.0.18}$$

where $V(\mu) = 1/8\pi\mu^3$ is the volume of the three-sphere, and the partition function in the unregulated theory splits into the flat spacetime contribution and the

contribution from the internal CFT:

$$Z(0)(\tau,\bar{\tau}) = \frac{1}{\tau_2 \left|\eta\right|^4} \sum_{k,l=0}^{1} \frac{\theta\left[\begin{smallmatrix}k\\l\end{smallmatrix}\right](\tau,0)}{\eta} Z_{\text{int}}\left[\begin{smallmatrix}k\\l\end{smallmatrix}\right](\tau,\bar{\tau})\,. \tag{C.I.0.19}$$

Finally $\Gamma_0(\mu)$ is proportional to the $SO(3)_{k/2}$ partition function and can be conveniently written as:

$$\Gamma_0(\mu) = -\frac{\partial_{\mu^2} X(\mu)}{2\pi}\,, \tag{C.I.0.20}$$

with

$$X(\mu) = P(\mu) - P(2\mu)\,, \tag{C.I.0.21}$$

and

$$P(\mu) = \sqrt{\tau_2} \sum_{(m,n)\in\mathbb{Z}^2} q^{\frac{1}{4}(m\mu+n/\mu)^2} \bar{q}^{\frac{1}{4}(m\mu-n/\mu)^2}\,. \tag{C.I.0.22}$$

The deformation of the theory by the marginal operators eqs. (C.I.0.16) and (C.I.0.17) is then understood as a boost of the fermionic and $SU(2)$ charge lattice leading to:

$$\begin{aligned}\delta L_0 = &\left(\bar{\mathcal{P}} + \bar{I}\right)(\mathcal{R} I + F_a \mathcal{Q}_a) + \\ &+ \frac{-1+\sqrt{1+(k+2)(k_a F_a^2 + k\mathcal{R})}}{2} \left(\frac{\left(\bar{\mathcal{P}} + \bar{I}\right)^2}{k+2} + \frac{(F_a \mathcal{Q}_a + \mathcal{R} I)^2}{k_a F_a^2 + k\mathcal{R}^2} \right)\,,\end{aligned} \tag{C.I.0.23}$$

where $\bar{\mathcal{P}}$ is the zero mode of the antiholomorphic helicity current $\psi^1\psi^2$, Q_a the zero mode of the holomorphic current J_a and $I, \bar{I}$ the zero modes of the $SU(2)$ currents $J^3, \bar{J}^3$. One can then expand the deformed partition function $Z_{F,\mathcal{R}}(\mu)$ in powers of F_a and $\mathcal{R}$ and collect the quadractic terms to obtain the following correlation functions in a supersymmetric vacuum:

$$\langle F_a^2 \rangle = 8\pi^2 \tau_2^2 \langle \bar{\mathcal{P}}^2 \rangle \left(\langle Q_a^2 \rangle - \frac{k_a}{8\pi\tau_2} \right)\,, \tag{C.I.0.24a}$$

$$\langle \mathcal{R}^2 \rangle = 8\pi^2 \tau_2^2 \langle \bar{\mathcal{P}}^2 \rangle \left(\langle I^{\,2} \rangle - \frac{k}{8\pi\tau_2} \right)\,, \tag{C.I.0.24b}$$

whose integral over the fundamental domain of $SL_2(\mathbb{Z})$ corresponds precisely to the one-loop gauge and gravitational couplings. One computes for instance the following

expression:

$$\left.\frac{16\pi^2}{g_a^2(\mu)}\right|_{\text{one-loop}} = \frac{i}{\pi^2 V(\mu)}\int_{\mathcal{F}} \mathrm{d}\nu\, \frac{\tau_2}{|\eta|^4}\partial_{\mu^2}X(\mu)\sum_{k,l=0}^{1}\frac{\partial_\tau\theta\left[\begin{smallmatrix}k\\l\end{smallmatrix}\right](\tau,0)}{\eta}\mathrm{Tr}^{\text{int}}_{k,l}\left(\langle Q_a^2\rangle - \frac{k_a}{8\pi\tau_2}\right). \tag{C.I.0.25}$$

Sending the regulator to zero by going to the flat $k\to\infty$ limit, one obtains:

$$\left.\frac{16\pi^2}{g_a^2(\mu)}\right|_{\text{one-loop}} = \beta_a \log\frac{M_s^2}{\mu^2} + \Delta_a = \int_{\mathcal{F}} \mathrm{d}\nu\, Z_a(\tau,\bar\tau)\,, \tag{C.I.0.26}$$

where $Z_a(\tau,\bar\tau)$ corresponds to the new supersymmetric index Z_{NEW} with an extra insertion of $\left(Q_a^2 - \frac{1}{8\pi\tau_2}\right)$ in the trace:

$$Z_a(\tau,\bar\tau) = \frac{\tau_2}{\eta(\tau)^2}\,\mathrm{Tr}_{\text{R}}\left\{\left(Q_a^2 - \frac{1}{8\pi\tau_2}\right)\bar J_0(-1)^{F_{\text{R}}}q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24}\right\}. \tag{C.I.0.27}$$

Similarly, one can show that in the flat limit, the one-loop gravitational coupling is given by:

$$\left.\frac{16\pi^2}{g_{\text{grav}}^2(\mu)}\right|_{\text{one-loop}} = \beta_{\text{grav}}\log\frac{M_s^2}{\mu^2} + \Delta_{\text{grav}} = \frac{1}{24}\int_{\mathcal{F}}\mathrm{d}\nu\,\left\{\tau_2\hat E_2(\tau)\,Z_{\text{NEW}}(\tau,\bar\tau)\right\}, \tag{C.I.0.28}$$

where $\hat E_2 = E_2 - 3/\pi\tau_2$ is the weight-2 non-holomorphic modular Eisenstein series defined in appendix E.I. In the next chapter, we will compute these threshold corrections for the $\mathcal{N}=2$ Fu-Yau compactifications.

Let us simply add as a comment that we chose in this introductory chapter a certain prescription among others, the background field method, in order to derive the generic expression of the thresholds, and then removed the IR cutoff by going to the flat limit, giving the divergent expressions eqs. (C.I.0.26) and (C.I.0.28). Any other regularization scheme consistent with the Fu-Yau $\mathcal{N}=2$ vacua would also be acceptable. For instance, in [80, 81, 86], whose results concerning modular integrals are to be used later on, used another prescription. The common advantage that both prescriptions share is that they preserve modular invariance.

Chapter II

One-loop corrections to BPS saturated couplings

This chapter is built out of the article: *Threshold corrections in heterotic flux compactifications*, with Carlo Angelantonj and Dan Israël, arXiv:1611.09442.

II.1 $\mathcal{N} = 2$ thresholds and new supersymmetric index

The goal of this chapter is to exhibit a physical application of the dressed elliptic genus defined and computed in chapters B.II and B.III. We will explicitely compute the one-loop threshold corrections to the gauge and gravitational couplings introduced in chapter C.I.

The one-loop running of the coupling constant associated with a simple factor G of the space-time gauge group is, as we discuss in the introduction, expressed through the relation:

$$\frac{16\pi^2}{g_G^2(\mu)} = \frac{16\pi^2}{g_s^2} + \beta_G \log \frac{M_s^2}{\mu^2} + \Delta_G \,. \tag{C.II.1.1}$$

The second term in the right-hand-side of eq. (C.II.1.1) corresponds to the contribution from the massless multiplets, hence to the running one would compute in a field theoretic setting. It is proportional to the gauge-theory beta-function β_G. The last term Δ_G incorporates the contribution from the whole tower of massive fields, hence describes the stringy part of the one-loop correction to gauge coupling.

A similar expression holds for the one-loop threshold correction to the gravita-

tional coupling as well:

$$\frac{16\pi^2}{g^2_{\rm grav}(\mu)} = \frac{16\pi^2}{g_s^2} + \beta_{\rm grav} \log \frac{M_s^2}{\mu^2} + \Delta_{\rm grav} \,. \tag{C.II.1.2}$$

These threshold corrections have been studied in great details for $K3\times T^2$ compactifications, see the introduction for a partial list of relevant references. Extended $\mathcal{N}=2$ supersymmetry in spacetime highly constrains these corrections; in particular, they only receive contribution from BPS states. As discussed in the introduction, it turns out that they all can be expressed as the integral over the fundamental domain of the worldsheet torus modular group of descendants of a quantity known as the new supersymmetric index [48]. This objet is independent of the moduli of the K3 surface, but depends on the torus and Wilson line moduli of the compactification.

This type of integral can be computed using the standard orbit method that was developped for $K3 \times T^2$ compactifications, which consists in unfolding the integration domain against the Narain lattice partition function [82]. This approach is convenient for studying the D-instanton corrections in the type I S-duals (see *e.g.* [87, 88]), however it hides the explicit covariance under the perturbative duality group $O(2,2;\mathbb{Z})$ of the two-torus, that occurs also in the $\mathcal{N}=2$ compactifications with torsion under study [53].

Another approach, developed recently in [80, 81, 86] , suggests to maintain the explicit covariance under T-duality by instead keeping the Narain partition function intact, expanding the remaining weak almost holomorphic modular form in terms of (absolutely convergent) Niebur-Poincaré series, and finally unfolding the integration domain against the latter. This approach not only has the advantage of keeping T-duality manifest and the analytic structure of the amplitude transparent, but rather it is the best (if not the only) way to extract physical couplings for values of the moduli close to the string scale, where the conventional expansion might fail to converge. This is especially useful for the present class of models, given that the volume of the two-torus fiber is generically frozen by the fluxes to a small value in string units.

Following this approach, we obtain in this section compact and T-duality covariant expressions for the threshold corrections, written in a chamber-independent form, *i.e.* valid for any values of the moduli of the torus fiber. The results depend explicitely on the topology of the principal two-torus bundle, *i.e.* on the choice of a pair of anti-self-dual $(1,1)$ forms on the $K3$ base.

We will consider thereafter an alternative representation of the threshold corrections in terms of a Fourier series expansion in the Kähler modulus T of the torus fiber [81], enlightening the origin of the various contributions, especially those corresponding to the worldsheet instantons wrapping the T^2. These corrections, that would be, for $\mathrm{Spin}(32)/\mathbb{Z}_2$ compactifications, S-dual to D1-instanton corrections in type I compactifications with Ramond-Ramond fluxes, are particularly interesting. Indeed, topologically, the two-torus is not a proper two-cycle of the total space of the bundle, but only a torsion two-cycle. Nevertheless as we will find the instanton corrections take the form of a infinite sum over the wrapping number.

We computed this dressed elliptic genus for Fu-Yau compactifications, from which derives naturally the new supersymmetric index, buidling block of these threshold corrections. We did the computation first in absence of Abelian bundle over the total space, cf. chapter B.II, and then including such 'Wilson line' moduli, cf. chapter B.III.

As we already discussed, a main difference with the conventional $K3\times T^2$ compactifications is that both the complex structure and complexified Kähler moduli of the two-torus fiber are now generically quantized. However Abelian bundles over the total space, that would reduce to Wilson lines in the $K3\times T^2$ case, have moduli which are not quantized by the three-form flux. In the following, we will turn off such exta moduli for simplicity of the computations and of the exposure.

These compactifications are also characterized by the pullback of a holomorphic vector bundle E over the $K3$ base. For definiteness, we will embed its structure group in the first E_8 factor of the $E_8\times E_8$ heterotic gauge group.

Then the new supersymmetric index $Z_{\rm NEW}$, which we computed above is expressed in terms of a non-holomorphic dressed elliptic genus $Z_{\rm FY}(\tau,\bar\tau,z)$ through

$$Z_{\rm NEW}(\tau,\bar\tau)=\frac{\bar\eta^2 E_4(\tau)}{2\eta^{10}}\sum_{\gamma,\delta=0}^{1}q^{\gamma^2}\left\{\left(\frac{\theta(\tau,z)}{\eta(\tau)}\right)^{8-r}Z_{\rm FY}(\tau,\bar\tau,z)\right\}\Big|_{z=\frac{\gamma\tau+\delta}{2}}\,, \qquad \text{(C.II.1.3)}$$

where we have defined the non-holomorphic dressed elliptic genus as follows:

$$Z_{\rm FY}(\tau,\bar\tau,z)=\frac{1}{\bar\eta(\bar\tau)^2}\,\mathrm{Tr}_{\rm RR,\mathcal{H}_{\rm FY}}\left(e^{2i\pi z J_0}\bar J_0(-1)^F q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24}\right)\,, \qquad \text{(C.II.1.4)}$$

the trace being taken into the Hilbert space of the $(0,2)$ superconformal theory corresponding to the compactification. This dressed elliptic genus, which is holomorphic in z but not in τ, transforms as a weak Jacobi form of weight 0 and index $r/2$, where r is the rank of the holomorphic vector bundle E. This object somehow appeared as

the natural ireeducible index to look at. The same non-holomorphic dressed elliptic genus can be defined for $K3\times T^2$, which corresponds to the particular case in which the torus fibration is trivial, hence the dressed elliptic genus factorizes into the usual elliptic genus of $K3$ and the partition function of the signature $(2,2)$ Narain lattice of the two-torus.

The dressed elliptic genus eq. (C.II.1.4) was computed in the previous parts, and is given by :

$$Z_{\rm FY}(X,E,\omega|\tau,\bar\tau,z)=q^{\frac{r-2}{12}}w^{-\frac{r}{2}}\int_{\mathcal{S}}\mathrm{ch}\left(\mathbb{E}_{q,y}\right)\mathrm{td}\left(T_{\mathcal{S}}\right)\sum_{\mu\in\Gamma_{\rm L}^\vee/\Gamma_{\rm L}}\frac{\Theta_\mu^{\Gamma_{\rm L}}\left(\tau\,\middle|\,\frac{p_\omega}{2i\pi}\right)}{\eta(\tau)^2}\frac{\bar\Theta^{\Gamma_{\rm R}}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}\,, \tag{C.II.1.5}$$

where the two-component vector p_ω valued in $H^2(\mathcal{S})\times H^2(\mathcal{S})$ reads, taking an orthonormal basis on $\Gamma_{\rm L}$:

$$p_\omega=\sqrt{\frac{2U_2}{T_2}}\begin{pmatrix}\omega_1+T_1\,\omega_2\\ T_2\,\omega_2\end{pmatrix}\,, \tag{C.II.1.6}$$

This vector belongs to a formal extension of the left momentum lattice $\Gamma_{\rm L}$, which is a module over $H^2(\mathcal{S},\mathbb{Z})$.

A more explicit expression was obtained using the splitting principle. With $c(T_{\mathcal{S}})=\prod_{i=1}^2(1+\nu_i)$ and $c(E)=\prod_{a=1}^r(1+\xi_a)$ denote the total Chern classes of the respective bundles, we have:

$$Z_{\rm FY}(X,E,\omega|\tau,\bar\tau,z)=\int_{\mathcal{S}}G(\tau,\bar\tau,z,\nu,\xi,p_\omega)\,, \tag{C.II.1.7}$$

where

$$\begin{aligned}G(\tau,\bar\tau,z,\nu,\xi,p_\omega)=&\prod_{a=1}^r\frac{i\theta_1(\tau\,\left|\,\frac{\xi_a}{2i\pi}-z\right.)}{\eta(\tau)}\prod_{i=1}^2\frac{\eta(\tau)\nu_i}{i\theta_1(\tau\,\left|\,\frac{\nu_i}{2i\pi}\right.)}\times\\&\times\sum_{\mu\in\Gamma_{\rm L}^\vee/\Gamma_{\rm L}}\frac{\Theta_\mu^{\Gamma_{\rm L}}\left(\tau\,\middle|\,\frac{p_\omega}{2i\pi}\right)}{\eta(\tau)^2}\frac{\bar\Theta^{\Gamma_{\rm R}}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}\,.\end{aligned} \tag{C.II.1.8}$$

Let us try and massage a bit the above expression of the non-holomorphic genus in order to obtain a form more adapted to the computation of the threshold corrections.

The following formula holds [72]:

$$\theta_1(\tau|z+\xi)=\exp\left\{-\frac{\pi^2}{6}E_2(\tau)\xi^2+\frac{\theta_1'(\tau|z)}{\theta_1(\tau|z)}\xi-\sum_{n\geqslant 2}\wp^{(n-2)}(\tau,z)\frac{\xi^n}{n!}\right\}\theta_1(\tau|z)\,, \tag{C.II.1.9}$$

where $\wp$ is the Weierstrass elliptic function and $\wp^{(n)}:=\frac{\partial^n}{\partial z^n}\wp$. Expanding the integrand eq. (B.II.5.8) and keeping only the top degree form terms, one obtains

$$\begin{aligned}Z^{\rm W}_{\rm FY}=&(-i)^r\frac{\theta_1(\tau|z)^r}{\eta(\tau)^{r+4}}\sum_{(p_{\rm L},p_{\rm R})\in\Gamma_{10,2}}\frac{q^{\frac{1}{4}|p_{\rm L}|^2}}{\eta(\tau)^{18}}\frac{\bar q^{\frac{1}{4}|p_{\rm R}|^2}}{\bar\eta(\bar\tau)^2}\times\\
&\times\int_{\mathcal{S}}\left\{-\frac{E_2(\tau)}{24}\sum_i\nu_i^2+\left(\frac{E_2(\tau)}{24}-\frac{\wp(\tau,z)}{2(2i\pi)^2}\right)\sum_a\xi_a^2+\frac{1}{2}{\rm Re}\left(p_\omega\,\overline{p^0_{\rm L}}\right)^2\right\}\,,\end{aligned} \tag{C.II.1.10}$$

where we have used the fact that the first Chern class of the holomorphic vector bundle $c_1(\mathcal{V})$ vanishes. Using the definition of the instanton number, the fact that $\int_{\mathcal{S}}{\rm ch}_2(\mathcal{T}_{\mathcal{S}})=-24$, and that the ordinary elliptic genus of a $(4,4)$ non-linear sigma-model on $K3$, namely

$$Z^{K3}_{\rm ell}(\tau,z)=8\left\{\left(\frac{\theta_2(\tau|z)}{\theta_2(\tau|0)}\right)^2+\left(\frac{\theta_3(\tau|z)}{\theta_3(\tau|0)}\right)^2+\left(\frac{\theta_4(\tau|z)}{\theta_4(\tau|0)}\right)^2\right\}\,, \tag{C.II.1.11}$$

is related to the Weierstrass $\wp$-function by the following formula:

$$Z^{K3}_{\rm ell}(\tau,z)=\frac{6}{\pi^2}\frac{\wp(\tau,z)\theta_1(\tau|z)^2}{\eta(\tau)^6}\,, \tag{C.II.1.12}$$

one can write the index as sum of three terms in the following way:

$$\begin{aligned}Z^{\rm W}_{\rm FY}=&-(-i)^r\sum_{(p_{\rm L},p_{\rm R})\in\Gamma_{10,2}}\frac{q^{\frac{1}{4}|p_{\rm L}|^2}}{\eta(\tau)^{18}}\frac{\bar q^{\frac{1}{4}|p_{\rm R}|^2}}{\bar\eta(\bar\tau)^2}\times\\
&\left(\frac{\vartheta_1(\tau|z)}{\eta(\tau)}\right)^{r-2}\left\{\frac{N}{24}Z^{K3}_{\rm ell}(\tau,z)+\frac{N-24}{12}\frac{\theta_1(\tau|z)^2}{\eta(\tau)^6}E_2(\tau)-\frac{\theta_1(\tau|z)^2}{2\,\eta(\tau)^6}\int_{\mathcal{S}}{\rm Re}\left(p_\omega\,\overline{p^0_{\rm L}}\right)^2\right\}\,.\end{aligned} \tag{C.II.1.13}$$

One therefore obtain the following very simple expression as a sum of three terms:

$$Z_{\rm FY} = \frac{1}{\eta(\tau)^2\bar\eta(\bar\tau)^2}\sum_{\mu\in\Gamma_{\rm L}^\star/\Gamma_{\rm L}}\sum_{\substack{p_{\rm L}\in\Gamma_{\rm L}+\mu\\ p_{\rm R}\in\Gamma_{\rm R}+\varphi(\mu)}} q^{\frac12\langle p_{\rm L},p_{\rm L}\rangle_{\Gamma_{\rm L}}}\bar q^{\frac12\langle p_{\rm R},p_{\rm R}\rangle_{\Gamma_{\rm R}}}\times$$
$$\times\left\{\frac{n}{24}\left(\frac{\theta(\tau,z)}{\eta(\tau)}\right)^{r-2}Z_{\rm ell}^{K3}(\tau,z)+\frac{\theta(\tau,z)^r}{12\,\eta(\tau)^{r+4}}\,(n-24)\hat E_2(\tau)\right.$$
$$\left.-\frac{\theta(\tau,z)^r}{2\,\eta(\tau)^{r+4}}\left(\int_{\mathcal S}\langle p_{\rm L},p_\omega\rangle_{\Gamma_{\rm L}}^2-\frac{n-24}{2\pi\tau_2}\right)\right\}. \qquad \text{(C.II.1.14)}$$

The definition of the various functions entering the above expression are summarized in appendix E.I. The left and right momenta $p_{\rm L}$ and $p_{\rm R}$ belong to the even lattices $\Gamma_{\rm L}$ and $\Gamma_{\rm R}$[1] shifted by μ and $\varphi(\mu)$ respectively, where μ is an element of the discriminant group $\Gamma_{\rm L}^\star/\Gamma_{\rm L}$ and $\varphi:\Gamma_{\rm L}^\star/\Gamma_{\rm L}\to\Gamma_{\rm R}^\star/\Gamma_{\rm R}$ is an isometry [55]. In the above expression, $\langle\cdot,\cdot\rangle_\Gamma$ denotes the scalar product on the even lattice Γ. We define then

$$f(p_{\rm L},\omega):=\int_{\mathcal S}\langle p_{\rm L},p_\omega\rangle_{\Gamma_{\rm L}}^2-\frac{n-24}{2\pi\tau_2}=\int_{\mathcal S}\left(\langle p_{\rm L},p_\omega\rangle_{\Gamma_{\rm L}}^2-\frac{1}{4\pi\tau_2}\langle p_\omega,p_\omega\rangle_{\Gamma_{\rm L}}\right). \qquad \text{(C.II.1.15)}$$

where we have used the tadpole condition (B.II.1.17).

Taking into account the remaining free fermions and performing the left GSO projection, one obtains then for the new supersymmetric index:

$$Z_{\rm NEW}(\tau,\bar\tau)=\frac{E_4(\tau)}{\eta(\tau)^{12}}\sum_{\mu\in\Gamma_{\rm L}^\star/\Gamma_{\rm L}}\sum_{\substack{p_{\rm L}\in\Gamma_{\rm L}+\mu\\ p_{\rm R}\in\Gamma_{\rm R}+\varphi(\mu)}} q^{\frac12\langle p_{\rm L},p_{\rm L}\rangle_{\Gamma_{\rm L}}}\bar q^{\frac12\langle p_{\rm R},p_{\rm R}\rangle_{\Gamma_{\rm R}}}\times$$
$$\times\frac12\sum_{\gamma,\delta=0}^{1}q^{\gamma^2}\left\{\frac{n}{24}\left(\frac{\theta(\tau,z)}{\eta(\tau)}\right)^6Z_{\rm ell}^{K3}(\tau,z)+\right. \qquad \text{(C.II.1.16)}$$
$$\left.+\frac{\theta(\tau,z)^8}{12\,\eta(\tau)^{12}}\,(n-24)\hat E_2(\tau)-\frac{\theta(\tau,z)^8}{2\,\eta(\tau)^{12}}f(p_{\rm L},\omega)\right\}\Bigg|_{z=\frac{\gamma\tau+\delta}{2}}. \qquad \text{(C.II.1.17)}$$

Notice that the modular behaviour of the third term with a momentum insertion is ensured, since by construction the sum of the three terms is well-behaved and the first two terms are also by themselves weak almost holomorphic modular forms of weight -2.

Finally, in terms of standard weak almost holomorphic modular forms, the result

[1] These lattices are defined as $\Gamma_{\rm L}=\Gamma_{2,2}(T,U)\cap\mathbb{R}^{2,0}$ and $\Gamma_{\rm L}=\Gamma_{2,2}(T,U)\cap\mathbb{R}^{0,2}$ and are both of rank two because the corresponding $c=2$ CFT is rational.

takes a relatively simple form:

$$Z_{\text{NEW}}(\tau,\bar\tau) = \sum_{\mu\in\Gamma_{\text{L}}^\star/\Gamma_{\text{L}}} \sum_{\substack{p_{\text{L}}\in\Gamma_{\text{L}}+\mu\\ p_{\text{R}}\in\Gamma_{\text{R}}+\varphi(\mu)}} q^{\frac{1}{2}\langle p_{\text{L}},p_{\text{L}}\rangle_{\Gamma_{\text{L}}}}\bar q^{\frac{1}{2}\langle p_{\text{R}},p_{\text{R}}\rangle_{\Gamma_{\text{R}}}} \times \\ \times \left(-\frac{n}{12}\frac{E_4E_6}{\Delta}+\frac{n-24}{12}\frac{E_4^2\hat E_2}{\Delta}-\frac{f(p_{\text{L}},\omega)}{2}\frac{E_4^2}{\Delta}\right), \tag{C.II.1.18}$$

which will allow us to use the techniques developed in [80, 81, 86], and reviewed briefly in the next section, to perform the integration over the fundamental domain of the worldsheet torus modular group leading to the various threshold corrections.

The formula eq. (C.II.1.10) that we used as a starting point was derived in the previous part from a geometrical definition of the dressed elliptic genus, that coincides with the result obtained directly from a gauged linear sigma model using supersymmetric localization as we have proven there. We expect that this formula holds in full generality for all $\mathcal{N}=2$ compactifications with torsion of the class discussed in this work, even for those without an obvious GLSM realization.

This result contains as a special case the standard $K3\times T^2$ compactifications, corresponding to the limiting case where the gauge instanton number n equals 24 and where the momentum insertion $f(p_{\text{L}},\omega)$ vanishes.

II.2 Niebur-Poincaré Series

Integrals of the type

$$\int_{\mathcal{F}} \mathrm{d}\nu\, \Phi(\tau)\, \Lambda_{2,2}(T,U;\tau) \tag{C.II.2.1}$$

are quite common in string theory, since they compute the one-loop correction to couplings in the low-energy effective action. Here $d\nu = \mathrm{d}\tau_1\mathrm{d}\tau_2\,\tau_2^{-2}$ is the $SL(2,\mathbb{Z})$ invariant measure, while $\mathcal{F}=\mathbb{H}/SL(2,\mathbb{Z})$ is the fundamental domain of the modular group, $\mathbb{H}$ being the Poincaré upper complex plane. $\Lambda_{2,2}(T,U;\tau)$ is the partition function associated to the $(2,2)$ dimensional Narain lattice, depending on the Kähler and complex structure moduli of the compactification torus as well as on the Teichmüller parameter τ of the worldsheet torus, while $\Phi(\tau)$ is a, a priori, generic function invariant under the action of the modular group, whose explicit expression depends on the kind of coupling we are interested in. For those of interest in this paper, the automorphic function is weak quasi holomorphic modular function, in the sense that it has zero weight, it is holomorphic in the τ variable, aside from possible explicit τ_2 dependence via the Eisenstein series $\hat E_2$, and has a simple pole

at the cusp $\tau = i\infty$. Holomorphy is a consequence of the fact that the couplings we are interested in receive contributions only from BPS states.

While the traditional way of computing the integral (C.II.2.1) relies on the $SL(2,\mathbb{Z})$ orbit decomposition of the Narain partition function, in [80, 81, 86] a new method has been proposed whereby the fundamental domain is unfolded against the automorphic function Φ itself. This way of proceeding has the clear advantage of keeping manifest the perturbative T-duality symmetries at all steps, and expresses the final result as a sum over BPS states. Moreover, singularities associated to states becoming massless at special points in the Narain moduli space are easily revealed in this representation.

In order to implement this strategy, it is essential that Φ be represented as an absolutely convergent Poincaré series, so that the unfolding of the fundamental domain is justified. This is actually the case, since any weak quasi-holomorphic modular form can be uniquely decomposed in terms of so-called Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$, where w is the modular weight, κ determines the growth of the function at the cusp, while s is a generic complex parameter. The Poincaré series representation of $\mathcal{F}(s,\kappa,w)$ is

$$\mathcal{F}(s,\kappa,w) = \frac{1}{2}\sum_{(c,d)=1} (c\tau+d)^{-w}\,\mathcal{M}_{s,w}\left(-\frac{\kappa\tau_2}{|c\tau+d|^2}\right)\times \\ \times\exp\left\{-2i\pi\kappa\left(\frac{a}{c}-\frac{c\tau_1+d}{c|c\tau+d|^2}\right)\right\}, \tag{C.II.2.2}$$

in terms of the Whittaker M-function, $\mathcal{M}_{s,w}(y) = |4\pi y|^{-w/2}\,M_{\frac{w}{2}\mathrm{sgn}(y),s-\frac{1}{2}}(4\pi|y|)$.

We refer the interested reader to [80, 81, 86] for a general discussion of Niebur-Poincaré series. In the following we shall only remind that for negative weight, the choice $s = 1-\frac{w}{2}+n$ is rather special, since the Niebur-Poincaré series are quasi holomorphic and absolutely convergent. As a result,

$$\Phi(\tau) = \sum_{n,\ell} d_\ell(n)\,\mathcal{F}(1-\frac{w}{2}+n,\ell,w)\,, \tag{C.II.2.3}$$

where the coefficients $d_\ell(n)$ are uniquely determined by matching the principal parts of the q-Laurent expansion of the two sides of the equation. In eqs. (C.II.3.3), (C.II.3.15) and (C.II.3.25) we list the decomposition of interest for us, while we refer to [81, 86] for more general cases.

Since any weak quasi holomorphic modular form can be decomposed in terms of Niebur-Poincaré series, for the purpose of computing modular integrals it suffices to

consider the basic integral

$$I(s) = \text{R.N.} \int_{\mathcal{F}} \mathrm{d}\nu \, \mathcal{F}(s,1,0) \, \Lambda_{2,2}(T,U;\tau) \, . \tag{C.II.2.4}$$

Here we have selected $\kappa = 1$, the only case of interest in string theory. The symbol R.N. (that we shall omit in the following, assuming that all integrals are properly renormalised) implies that the integral has been properly renormalised in order to cope with the infrared (logarithmic) divergences ascribed to massless states running in the loop. Our modular invariant prescription amounts at cutting-off the fundamental domain at large $\tau_2 > \mathcal{T}$, thus removing the singular behaviour of light states in the $\mathcal{T} \to \infty$ limit [80,81,86]. This prescription to handle the infrared divergences coming from the massless excitations is different form the one we chose in chapter C.I. As mentioned already there, these two IR regulators share the nice property of preserving modular invariance, ensuring the absence of spacetime Lorentz and gauge anomalies [89]. The infrared divergence regularization problem will not play any further role in the following.

Upon unfolding the fundamental domain against $\mathcal{F}(s,1,0)$ one gets [81]

$$I(s) = \sum_{\text{BPS}} \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2} \, \mathcal{M}_{s,0}(-\tau_2) \, e^{-\pi\tau_2(p_{\text{L}}^2 + p_{\text{R}}^2)/2} \tag{C.II.2.5}$$

where the sum is restricted only to the BPS states satisfying $p_{\text{L}}^2 - p_{\text{R}}^2 = 2$. The integral can be straightforwardly evaluated to yield [81]

$$\begin{aligned} I_\alpha(s,w) &= \int_{\mathcal{F}} \mathrm{d}\nu \; \tau_2^\alpha \sum_{p_{\text{L}},p_{\text{R}}} q^{\frac{1}{2}p_{\text{L}}^2} \, \bar{q}^{\frac{1}{2}p_{\text{R}}^2} \, \mathcal{F}(s,\kappa,w) \\ &= \sum_{p_{\text{L}},p_{\text{R}}} \delta\left(p_{\text{L}}^2 - p_{\text{R}}^2 - 2\kappa\right) (4\pi\kappa)^{1-\alpha} \left(\frac{p_{\text{L}}^2}{2\kappa}\right)^{-\frac{|w|}{2}-\alpha-s+1} \Gamma\left(\alpha + \frac{|w|}{2} + s - 1\right) \\ &\quad \times {}_2F_1\left(\alpha + \frac{|w|}{2} + s - 1, s - \frac{|w|}{2}; 2s; \frac{2\kappa}{p_{\text{L}}^2}\right) , \end{aligned} \tag{C.II.2.6}$$

with $p_{\text{L}}^2 := \langle p_{\text{L}}, p_{\text{L}} \rangle_{\Gamma_{\text{L}}}$ and $p_{\text{R}}^2 := \langle p_{\text{R}}, p_{\text{R}} \rangle_{\Gamma_{\text{R}}}$, and where we have allowed for a non-trivial weight of the Niebur-Poincaré series to compensate for Wilson lines and/or for momentum insertions in the Narain partition function [81]. As we shall see in the next section, this representation of the modular integral clearly spells out possible IR divergences ascribed to new states becoming massless at points of symmetry enhancement.

The integral (C.II.2.5) can actually be given an alternative representation whenever the BPS constraint is solved before the τ_2 integral is evaluated. The resulting representation defines a Fourier series expansion in the U_1 variable, which is only valid in special regions of moduli space (corresponding to large volume) [86]. For the case of momentum insertions we need to slightly generalise the construction of [86], and we shall present the new results in section section C.II.4.

II.3 Threshold corrections

We are now ready to compute the one-loop threshold corrections to the gauge and gravitational coupling for $\mathcal{N}=2$ heterotic compactifications with torsion, starting from eq. (C.II.1.18) and using the techniques that were summarized in section C.II.2. Note that the actual models we are considering only exist at special points of the Narain moduli space compatible with the three-form flux. Nevertheless, we shall try to keep the moduli arbitrary and treat them as continuous variables, so that the expressions can be conveniently adapted to any special realization.[2]

II.3.1 Gravitational threshold corrections

In order to compute the threshold correction to the gravitational coupling, one has to compute the following modular integral:

$$\Lambda_{\rm grav} = \beta_{\rm grav} \log \frac{M_s^2}{\mu^2} + \Delta_{\rm grav} = \frac{1}{24} \int_{\mathcal{F}} {\rm d}\nu \, \left\{ \tau_2 \hat{E}_2(\tau) \, Z_{\rm NEW}(\tau,\bar\tau) \right\} . \qquad \text{(C.II.3.1)}$$

Using eq. (C.II.1.18), one thus has to compute:

$$\begin{aligned} 24\Lambda_{\rm grav} = & \sum_{\mu,p_{\rm L},p_{\rm R}} \int_{\mathcal{F}} {\rm d}\nu \; \tau_2 \, q^{\frac{1}{2}\langle p_{\rm L},p_{\rm L}\rangle_{\Gamma_{\rm L}}} \bar q^{\frac{1}{2}\langle p_{\rm R},p_{\rm R}\rangle_{\Gamma_{\rm R}}} \times \\ & \times \left\{ -\frac{n}{12} \frac{\hat{E}_2 E_4 E_6}{\Delta} + \frac{n-24}{12} \frac{\hat{E}_2^2 E_4^2}{\Delta} - \frac{f(p_{\rm L},\omega)}{2} \frac{\hat{E}_2 E_4^2}{\Delta} \right\} . \end{aligned} \qquad \text{(C.II.3.2)}$$

where here and in the following, the momentum sum $\sum_{\mu,p_{\rm L},p_{\rm R}}$ is a compact notation for $\sum_{\mu\in\Gamma_{\rm L}^\star/\Gamma_{\rm L}} \sum_{p_{\rm L}\in\Gamma_{\rm L}+\mu} \sum_{p_{\rm R}\in\Gamma_{\rm R}+\varphi(\mu)}$.

Following [81] we rewrite the weak almost holomorphic modular forms entering in the integrands above in terms of Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$. One has the

[2] In particular, when ω_1 and ω_2 in eq. (C.II.1.6) are proportional to each other, only one complex torus modulus is stabilized by the flux and the other one remains.

following decompositions:

$$\begin{aligned}\frac{\hat{E}_2E_4E_6}{\Delta} &= \mathcal{F}(2,1,0) - 5\mathcal{F}(1,1,0) - 144\,,\\ \frac{\hat{E}_2^2E_4^2}{\Delta} &= \tfrac{1}{5}\mathcal{F}(3,1,0) - 4\mathcal{F}(2,1,0) + 13\mathcal{F}(1,1,0) + 144\,,\\ \frac{\hat{E}_2E_4^2}{\Delta} &= \tfrac{1}{40}\mathcal{F}(3,1,-2) - \tfrac{1}{3}\mathcal{F}(2,1,-2)\,.\end{aligned} \tag{C.II.3.3}$$

Regularizing the IR divergence and performing the modular integral by unfolding the integration domain against the Niebur-Poincaré series, one obtains

$$\begin{aligned}\Lambda_{\rm grav} = \sum_{\rm BPS}\Bigg\{ &-\frac{m(p_{\rm L})}{48}\left(\frac{3\ {}_2F_1\left(2,4,6,t^{-1}\right)}{20t^4} - \frac{2\ {}_2F_1\left(1,3,4,t^{-1}\right)}{3t^3}\right)\\ &-\frac{n}{12\times 24}\left(\frac{{}_2F_1\left(2,2,4,t^{-1}\right)}{t^2} - \frac{5\ {}_2F_1\left(1,1,2,t^{-1}\right)}{t}\right)\\ &+\frac{n-24}{24}\left(\frac{{}_2F_1\left(2,3,6,t^{-1}\right)}{20t^3} - \frac{{}_2F_1\left(1,2,4,t^{-1}\right)}{3t^2}\right)\\ &+\frac{n-24}{12\times 24}\left(\frac{2\ {}_2F_1\left(3,3,6,t^{-1}\right)}{5t^3} - \frac{4\ {}_2F_1\left(2,2,4,t^{-1}\right)}{t^2} + \frac{13\ {}_2F_1\left(1,1,2,t^{-1}\right)}{t}\right)\Bigg\}\\ &+(n-12)\mathcal{I}_{\rm DKL}\,,\end{aligned} \tag{C.II.3.4}$$

where $t := p_{\rm L}^2/2$, $m(p_{\rm L},\omega) := \int_{\mathcal{S}}\langle p_{\rm L}, p_\omega\rangle^2_{\Gamma_{\rm L}}$ and

$$\mathcal{I}_{\rm DKL} := \int_{\mathcal{F}} {\rm d}\nu\ \tau_2 \sum_{\mu,p_{\rm L},p_{\rm R}} q^{\frac{1}{2}p_{\rm L}^2}\,\bar{q}^{\frac{1}{2}p_{\rm R}^2} = -\log\left(4\pi e^{-\gamma}T_2U_2\,|\eta(T)\eta(U)|^4\right) \tag{C.II.3.5}$$

is the Dixon-Kaplunovsky-Louis integral, where γ is the Euler-Mascheroni constant. As already explained, in the above expression, $\sum_{\rm BPS}$ is a shorthand for $\sum_{p_{\rm L},p_{\rm R}}\delta(p_{\rm L}^2 - p_{\rm R}^2 - 2)$, in other words the sum over perturbative half-BPS states.

Fortunately, this complicated expression simplifies considerably in the cases of interest here, and one ends up with standard polynomial and logarithmic functions of the last argument, cf. appendix E.I. One ends up with the following simple

expression for the gravitational threshold corrections:

$$\begin{aligned}\Lambda_{\text{grav}} = \sum_{\text{BPS}} \Bigg\{ & 1 + \frac{n-24}{24}\,\frac{3}{2t} + \left(t - \frac{11}{12}\right)\log\left(\frac{t-1}{t}\right) + \\ & + \frac{m(p_{\text{L}},\omega)}{24}\left[6 - \frac{3}{4t^2} - \frac{5}{2t} + 6\left(t - \frac{11}{12}\right)\log\left(\frac{t-1}{t}\right)\right]\Bigg\} \\ & + (n-12)\,\mathcal{I}_{\text{DKL}}\,. \end{aligned} \tag{C.II.3.6}$$

This expression is clearly independent of the choice of chamber in the Narain moduli space.

Setting $n = 24$ and $m(p_{\text{L}}, \omega) = 0$ to make the torus fibration trivial, one obtains the result for $K3 \times T^2$ compactifications:

$$\Lambda_{\text{grav}} = \sum_{\text{BPS}} \left\{1 + \left(t - \frac{11}{12}\right)\log\left(\frac{t-1}{t}\right)\right\} + 12\,\mathcal{I}_{\text{DKL}}\,. \tag{C.II.3.7}$$

Note that these expressions are potentially divergent if $t = 1$, i.e. at point of symmetry enhancement where $p_{\text{L}}^2 = 2$. The presence or not of these divergences clearly depends of the actual values of the Kähler and complex structure moduli.

Finally from eq. (C.II.3.6) we can extract the value of gravitational β-function, which is the coefficient of the trace anomaly [90]:

$$\beta_{\text{grav}} = n - 12\,. \tag{C.II.3.8}$$

This coefficient is related to the relative number of hypermultiplets and vector multiplets. Comparing eq. (C.II.3.7) with known results from $K3 \times T^2$ [91], we get the normalisation:

$$\beta_{\text{grav}} = \frac{24 + n_{\text{H}} - n_{\text{V}}}{22}\,. \tag{C.II.3.9}$$

Hence $n_{\text{H}} - n_{\text{V}}$, *i.e.* the difference between the number of massless hypermultiplets and vector multiplets (including S, containing the dilaton, and the graviphoton), depends on the instanton number n of the vector bundle E, hence indirectly on the data of the principal two-torus bundle through the integrated Bianchi identity eq. (B.II.1.17).

II.3.2 Gauge threshold corrections

The expression eq. (C.II.1.18) for the new supersymmetric index is independent of the rank of the gauge bundle. In order to compute explicitely the correction

to the gauge couplings one has to choose a particular sub-class of bundles; we will consider below the case of a bundle of structure group $SU(2)$, embedded into one of the two E_8 factors of the gauge group, with arbitary instanton number $0 \leqslant n \leqslant 24$. It will allow to compare easily with classical results for $K3 \times T^2$ with the standard embedding of the spin connection into the gauge connection, and vanishing Wilson lines, *i.e.* models with a rank one bundle and $n = 24$.

Corrections to the E_8 coupling Let us start with the one-loop correction to the gauge coupling corresponding to the unbroken E_8 factor of the spacetime gauge group. The threshold correction is given by:

$$\Lambda_{E_8} = \beta_{E_8} \log \frac{M_s^2}{\mu^2} + \Delta_{E_8} = \int_{\mathcal{F}} \mathrm{d}\nu \, Z_{E_8}(\tau, \bar{\tau}) \,, \tag{C.II.3.10}$$

where $Z_{E_8}(\tau, \bar{\tau})$ corresponds to the new supersymmetric index with an extra insertion of $\left(Q_{E_8}^2 - \frac{1}{8\pi\tau_2}\right)$ in the trace:

$$Z_{E_8}(\tau, \bar{\tau}) = \frac{\tau_2}{\eta(\tau)^2} \, \mathrm{Tr}_{\mathrm{R}} \left\{ \left(Q_{E_8}^2 - \frac{1}{8\pi\tau_2} \right) \bar{J}_0 (-1)^{F_{\mathrm{R}}} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right\} \tag{C.II.3.11}$$

Let us define $\tilde{D}_w := (-4w)^{-1} D_w$, where D_w is the modular covariant derivative as defined in appendix E.I. The insertion $\left(Q_{E_8}^2 - \frac{1}{8\pi\tau_2}\right)$ corresponds then to acting in Z_{NEW} on the character of the affine E_8 algebra, namely $E_4(\tau)$ with the operator $\tilde{D}_4$. Using the fact that:

$$D_4 E_4 = \frac{2}{3} \left(E_6 - \hat{E}_2 E_4 \right) \,, \tag{C.II.3.12}$$

One obtains:

$$Z_{E_8} = \frac{\hat{E}_2 E_4 - E_6}{24\Delta} \, \tau_2 \sum_{\mu, p_{\mathrm{L}}, p_{\mathrm{R}}} q^{\frac{1}{2} p_{\mathrm{L}}^2} \, \bar{q}^{\frac{1}{2} p_{\mathrm{R}}^2} \left\{ -\frac{n}{12} E_6 + \frac{n - 24}{12} \hat{E}_2 E_4 - \frac{f(p_{\mathrm{L}}, \omega)}{2} E_4 \right\} \,, \tag{C.II.3.13}$$

i.e.:

$$\begin{aligned} Z_{E_8} = \frac{1}{24\Delta} \, \tau_2 \sum_{\mu, p_{\mathrm{L}}, p_{\mathrm{R}}} q^{\frac{1}{2} p_{\mathrm{L}}^2} \, \bar{q}^{\frac{1}{2} p_{\mathrm{R}}^2} \times \\ \left\{ -\frac{n - 12}{6} \hat{E}_2 E_4 E_6 + \frac{n}{12} E_6^2 + \frac{n - 24}{12} \hat{E}_2^2 E_4^2 - \frac{f(p_{\mathrm{L}}, \omega)}{2} (\hat{E}_2 E_4^2 - E_4 E_6) \right\} \,. \end{aligned} \tag{C.II.3.14}$$

In addition to eq. (C.II.3.3), one has the following decompositions into Niebur-Poincaré series:

$$\begin{aligned}\frac{E_6^2}{\Delta} &= \mathcal{F}(1,1,0) - 1008\,,\\ \frac{E_4E_6}{\Delta} &= \tfrac{1}{6}\mathcal{F}(2,1,-2)\,.\end{aligned} \tag{C.II.3.15}$$

One then performs the modular integral to get:

$$\begin{aligned}\Lambda_{E_8} = \sum_{\text{BPS}} \Bigg\{ &- \frac{m(p_{\text{L}})}{48}\left(\frac{3\ {}_2F_1\left(2,4,6,t^{-1}\right)}{20t^4} - \frac{{}_2F_1\left(1,3,4,t^{-1}\right)}{t^3}\right)\\ &- \frac{n-12}{6\times 24}\left(\frac{{}_2F_1\left(2,2,4,t^{-1}\right)}{t^2} - \frac{5\ {}_2F_1\left(1,1,2,t^{-1}\right)}{t}\right)\\ &+ \frac{n-24}{24}\left(\frac{{}_2F_1\left(2,3,6,t^{-1}\right)}{20t^3} - \frac{{}_2F_1\left(1,2,4,t^{-1}\right)}{2t^2}\right) + \frac{n\ {}_2F_1\left(1,1,2,t^{-1}\right)}{12t}\\ &+ \frac{n-24}{12\times 24}\left(\frac{2\ {}_2F_1\left(3,3,6,t^{-1}\right)}{5t^3} - \frac{4\ {}_2F_1\left(2,2,4,t^{-1}\right)}{t^2} + \frac{13\ {}_2F_1\left(1,1,2,t^{-1}\right)}{t}\right)\Bigg\}\\ &- 2(n+12)\,\mathcal{I}_{\text{DKL}}\,.\end{aligned} \tag{C.II.3.16}$$

Once again, for such integer values of the arguments, the hypergeometric functions simplify dramatically, cf. eq. (E.I.2.3), and one ends up with the following simple expression:

$$\begin{aligned}\Lambda_{E_8} = \sum_{\text{BPS}} \Bigg\{ &1 + \frac{n-24}{12t} + (t-1)\log\left(\frac{t-1}{t}\right) +\\ &+ \frac{m(p_{\text{L}},\omega)}{4}\left[1 - \frac{1}{6t^2} - \frac{1}{2t} + (t-1)\log\left(\frac{t-1}{t}\right)\right]\Bigg\}\\ &- 2(n+12)\,\mathcal{I}_{\text{DKL}}\,.\end{aligned} \tag{C.II.3.17}$$

From this expression we can read off immediately the expression of the β-function:

$$\beta_{E_8} = -2(n+12)\,. \tag{C.II.3.18}$$

Setting $n = 24$ and $m(p_{\text{L}},\omega) = 0$, one obtains:

$$\Lambda_{E_8} = \sum_{\text{BPS}}\left\{1 + (t-1)\log\left(\frac{t-1}{t}\right)\right\} - 72\,\mathcal{I}_{\text{DKL}}\,, \tag{C.II.3.19}$$

which coincides with the already known result for $K3\times T^2$ [81].

Corrections to the E_7 coupling For definiteness, and as stated in the introduction of this section, we focus on the case in which the vector bundle over the compact manifold has an $SU(2)$ structure group, such that the unbroken gauge group in spacetime contains a E_7 factor.

As before, computing the threshold correction corresponds to performing the modular integral of a descendant of the new supersymmetric index, i.e. with a $\left(Q_{E_7}^2 - \frac{1}{8\pi\tau_2}\right)$ insertion in the trace:

$$\Lambda_{E_7} = \beta_{E_7} \log \frac{M_s^2}{\mu^2} + \Delta_{E_7} = \int_{\mathcal{F}} \mathrm{d}\nu \, Z_{E_7}(\tau, \bar{\tau}) \,, \tag{C.II.3.20}$$

with:

$$Z_{E_7}(\tau, \bar{\tau}) = \frac{\tau_2}{\eta(\tau)^2} \, \mathrm{Tr}_{\mathrm{R}} \left\{ \left(Q_{E_7}^2 - \frac{1}{8\pi\tau_2} \right) \bar{J}_0 (-1)^{F_{\mathrm{R}}} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right\} \tag{C.II.3.21}$$

In functional picture, the extra operator insertion acts as $\tilde{D}_w$ on every $E_4(\tau)$ and $E_6(\tau)$ factor in the new supersymmetric index but not on the $E_4(\tau)$ corresponding to the unbroken E_8 factor of the gauge group, which was treated in the previous section. One has the following identities, due to Ramanujan:

$$D_4 E_4 = \frac{2}{3} \left(E_6 - \hat{E}_2 E_4 \right) \,, \tag{C.II.3.22a}$$

$$D_6 E_6 = E_4^2 - \hat{E}_2 E_6 \,. \tag{C.II.3.22b}$$

One thus obtains:

$$Z_{E_7} = \frac{\tau_2}{24\Delta} \sum_{\mu, p_{\mathrm{L}}, p_{\mathrm{R}}} q^{\frac{1}{2}p_{\mathrm{L}}^2} \bar{q}^{\frac{1}{2}p_{\mathrm{R}}^2} \times \tag{C.II.3.23}$$

$$\left\{ -\frac{n-12}{6} \hat{E}_2 E_4 E_6 + \frac{n}{12} E_4^3 + \frac{n-24}{12} \hat{E}_2^2 E_4^2 - \frac{f(p_{\mathrm{L}}, \omega)}{2} (\hat{E}_2 E_4^2 - E_4 E_6) \right\} \,. \tag{C.II.3.24}$$

In addition to eqs. (C.II.3.3) and (C.II.3.15) one has the following decomposition into Niebur-Poincaré series:

$$\frac{E_4^3}{\Delta} = \mathcal{F}(1,1,0) + 720 \,. \tag{C.II.3.25}$$

It gives:

$$\begin{aligned}\Lambda_{E_7} = \sum_{\text{BPS}} \Bigg\{ & -\frac{m(p_{\text{L}})}{48}\left(\frac{3\ {}_2F_1\left(2,4,6,t^{-1}\right)}{20t^4} - \frac{{}_2F_1\left(1,3,4,t^{-1}\right)}{t^3}\right) \\ & + \frac{n}{12\times 24}\frac{{}_2F_1\left(1,1,2,t^{-1}\right)}{t} + \frac{n-24}{24}\left(\frac{{}_2F_1\left(2,3,6,t^{-1}\right)}{20t^3} - \frac{{}_2F_1\left(1,2,4,t^{-1}\right)}{2t^2}\right) \\ & + \frac{n-24}{12\times 24}\left(\frac{2\ {}_2F_1\left(3,3,6,t^{-1}\right)}{5t^3} - \frac{4\ {}_2F_1\left(2,2,4,t^{-1}\right)}{t^2} + \frac{13\ {}_2F_1\left(1,1,2,t^{-1}\right)}{t}\right)\Bigg\} \\ & + 4(n-6)\mathcal{I}_{\text{DKL}}\,. \end{aligned} \tag{C.II.3.26}$$

Once again, for such integer values of the arguments, the hypergeometric functions simplify dramatically, cf. eq. (E.I.2.3), and one ends up with the following simple expression:

$$\begin{aligned}\Lambda_{E_7} = \sum_{\text{BPS}} \Bigg\{ & 1 + \frac{n-24}{12t} + (t-1)\log\left(\frac{t-1}{t}\right) \\ & + \frac{m(p_{\text{L}},\omega)}{4}\left[1 - \frac{1}{6t^2} - \frac{1}{2t} + (t-1)\log\left(\frac{t-1}{t}\right)\right]\Bigg\} \\ & + 4(n-6)\,\mathcal{I}_{\text{DKL}}\,. \end{aligned} \tag{C.II.3.27}$$

We can once again read directly the β-function:

$$\beta_{E_7} = 4(n-6)\,. \tag{C.II.3.28}$$

Setting $n = 24$ and $m(p_{\text{L}},\omega) = 0$, one obtains:

$$\Lambda_{E_7} = \sum_{\text{BPS}} \left\{1 + (t-1)\log\left(\frac{t-1}{t}\right)\right\} + 72\,\mathcal{I}_{\text{DKL}}\,, \tag{C.II.3.29}$$

corresponding indeed to the already known result for $K3\times T^2$.

Universality property of the gauge threshold corrections The presence of $\mathcal{N}=2$ supersymmetry in spacetime hints towards some universality properties of the thresholds, as in the $K3\times T^2$ case. The difference of the two gauge thresholds indeed turns out to be universal. Using the fact that:

$$E_4^3 - E_6^2 = 1728\Delta\,, \tag{C.II.3.30}$$

one obtains for the difference of the two integrands:

$$Z_{E_8} - Z_{E_7} = -6n\,\tau_2 \sum_{\mu,p_{\rm L},p_{\rm R}} q^{\frac{1}{2}p_{\rm L}^2}\,\bar{q}^{\frac{1}{2}p_{\rm R}^2} \tag{C.II.3.31}$$

leading to an integer multiple of the Dixon-Kaplunovsky-Louis integral for the thresholds:

$$\Lambda_{E_8} - \Lambda_{E_7} = (\beta_{E_8} - \beta_{E_7})\,\mathcal{I}_{\rm DKL} = -6n\,\mathcal{I}_{\rm DKL}\,. \tag{C.II.3.32}$$

Setting $n = 24$, one recovers the well-known result:

$$\Lambda_{E_8} - \Lambda_{E_7} = -144\,\mathcal{I}_{\rm DKL}\,. \tag{C.II.3.33}$$

II.4 Fourier series and worldsheet instanton corrections

The results obtained in the previous section encapsulate in a compact and $O(2,2;\mathbb{Z})$ invariant way the threshold corrections to the gauge and gravitational couplings. It is useful to present the result in a different way, which allows one to isolate the contributions from worldsheet instantons, using a Fourier series expansion [86].

The role of worldsheet instantons is particularly interesting to investigate in these $\mathcal{N} = 2$ torsional compactifications, whose topology corresponds to the total space of the principal bundle $T^2 \hookrightarrow X \stackrel{\pi}{\to} \mathcal{S}$. The relevant instantons in this context are holomorphic maps from the worldsheet two-torus to the target-space T^2.

In the present context neither the $K3$ base nor the T^2 fiber are cycles of the total space $\mathcal{M}$ of the principal bundle; in particular the two-torus is only a torsion two-cycle. One may wonder therefore whether an infinite tower of instanton corrections appears in the result; as we will see below, it turns out to be the case.

Starting from Spin(32)/$\mathbb{Z}_2$ ten-dimensional heterotic strings, our results lead to interesting insights on non-perturbative corrections to Type I compactifications with Ramond-Ramond flux. Under heterotic/type I S-duality, the one-loop heterotic computations capture both perturbative and non-pertubative corrections on the type I side, in particular the contribution of Euclidean D1-brane worldsheets wrapping the two-torus [88, 92]. This is a quite interesting result, as D-instantons corrections in the presence of RR fluxes have not been investigated in detail to our knowledge.

II.4.1 The Fourier series expansion

Let us now focus on an alternative representation in terms of a Fourier series expansion of the integral:

$$\mathcal{I}_g(s) := \int_{\mathcal{F}} \mathrm{d}\nu\, \tau_2 \sum_{\mu, p_{\mathrm{L}}, p_{\mathrm{R}}} g(p_{\mathrm{L}}\sqrt{\tau_2})\, q^{\frac{1}{2}p_{\mathrm{L}}^2} \bar{q}^{\frac{1}{2}p_{\mathrm{R}}^2}\, \mathcal{F}(s,1,w)\,, \tag{C.II.4.1}$$

with some momentum insertion $g(p_{\mathrm{L}}\sqrt{\tau_2})$, which in our case will correspond to $f(p_{\mathrm{L}},\omega)$.

In order to obtain this alternative Fourier series representation, one first performs the τ_1 integral which imposes the BPS constaint on the momenta, then solves the constraint and performs a suitable Poisson resummation.

Explicitly, one expands the Niebur-Poincaré series in terms of the Whittaker M-function, which is then itself expressed in terms of the confluent hypergeometric function ${}_1F_1$,

$$\begin{aligned} \mathcal{M}_{s,w}(-t) &= (4\pi t)^{-w/2} M_{-w/2,s-1/2}(4\pi t) \\ &= (4\pi t)^{s-w/2}\, e^{-2\pi t}\, {}_1F_1(s+w/2; 2s; 4\pi t)\,. \end{aligned} \tag{C.II.4.2}$$

The hypergeometric function ${}_1F_1$ satisfies:

$${}_1F_1(a;2a+n;y) = \Gamma(a-\tfrac{1}{2}) \left(\frac{y}{4}\right)^{\frac{1}{2}-a} e^{y/2} \sum_{\ell=0}^{n} \frac{(-n)_\ell\, (2a-1)_\ell}{(2a+n)_\ell\, \ell!} (a+\ell-\tfrac{1}{2})\, I_{a+\ell-\frac{1}{2}}(y/2)\,. \tag{C.II.4.3}$$

In these expressions $(x)_\ell = \Gamma(x+\ell)/\Gamma(x) = x(x+1)\ldots(x+l-1)$ is the Pochhammer symbol or rising factorial. It satisfies, $(-x)_\ell = (-1)^\ell (x-\ell+1)_\ell$.

This strategy can be applied first to compute the Fourier series expansion in absence of momentum insertion:

$$\mathcal{I}(s) := \int_{\mathcal{F}} \mathrm{d}\nu\, \mathcal{F}(s,1,0)\, \Lambda_{2,2}(T,U)\,, \tag{C.II.4.4}$$

where:

$$\Lambda_{2,2}(T,U) := \tau_2 \sum_{\mu, p_{\mathrm{L}}, p_{\mathrm{R}}} q^{\frac{1}{2}p_{\mathrm{L}}^2} \bar{q}^{\frac{1}{2}p_{\mathrm{R}}^2} \tag{C.II.4.5}$$

is the modular invariant partition function of the signature $(2,2)$ Narain lattice. It is evaluated at some particular points in moduli space specified by the quantization condition $T, U \in \mathbb{Q}[\sqrt{D}]$, although the computation below, by itself, could be done for any T and U as nowhere we make use of these conditions.

The Fourier expansion of eq. (C.II.4.4) was computed in [86]. The result splits into zero, negative and positive frequency parts:

$$\mathcal{I}(s) = \mathcal{I}^{(-)}(s) + \mathcal{I}^{(0)}(s) + \mathcal{I}^{(+)}(s)\,, \tag{C.II.4.6}$$

with:

$$\begin{aligned}
\mathcal{I}^{(0)}(s) &= 2^{4s-3}\sqrt{4\pi}\,\Gamma\left(s-\frac{1}{2}\right)\sum_{(n^1,n^2)=1}(U_2\tilde{T}_2)^s\left(U_2+\tilde{T}_2+|U_2-\tilde{T}_2|\right)^{1-2s}\,,\\
\mathcal{I}^{(+)}(s) &= \frac{1}{2}\sum_{M>0}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_T}\frac{e^{2i\pi M(U_1-\tilde{T}_1)}}{M}\mathcal{M}_{s,0}\left(\frac{M}{2}\left(U_2+\tilde{T}_2-|U_2-\tilde{T}_2|\right)\right)\times\\
&\quad\times\mathcal{W}_{s,0}\left(\frac{M}{2}\left(U_2+\tilde{T}_2+|U_2-\tilde{T}_2|\right)\right)\,,
\end{aligned} \tag{C.II.4.7}$$

the negative frequency part being obtained by complex conjugation.

Using the relations between the Whittaker functions $M_{k,m}$, $W_{k,m}$ and the modified Bessel functions of the first and second kind [81]:

$$\begin{aligned}
\mathcal{M}_{s,0}(\pm y) &= 2^{2s-1}\Gamma\left(s+\frac{1}{2}\right)\sqrt{4\pi|y|}\,I_{s-\frac{1}{2}}(2\pi|y|)\,,\\
\mathcal{W}_{s,0}(\pm y) &= 2\sqrt{|y|}\,K_{s-\frac{1}{2}}(2\pi|y|)\,,
\end{aligned} \tag{C.II.4.8}$$

and focusing on the fundamental chamber $U_2 > \tilde{T}_2$ for definiteness, one can rewrite the positive frequency part in the following way:

$$\begin{aligned}
\mathcal{I}^{(+)}(s) &= 2^{2s+1}\sqrt{\pi}\,\Gamma\left(s+\frac{1}{2}\right)\times\\
&\quad\times\sum_{M>0}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sqrt{U_2\tilde{T}_2}\,e^{2i\pi M(U_1-\tilde{T}_1)}I_{s-\frac{1}{2}}(2\pi M\tilde{T}_2)K_{s-\frac{1}{2}}(2\pi MU_2)\,,
\end{aligned} \tag{C.II.4.9}$$

where one recognizes the sum over comprime integers n^1, n^2 as a sum over cosets in the quotient of the modular group $\Gamma = SL_2(\mathbb{Z})_T$ by the stabilizer of the cusp at infinity Γ_∞. Notice the presence of a factor of 2 since the pairs (n^1, n^2) and $(-n^1, -n^2)$ correspond to the same coset γ.

We now want to compute the Fourier series expansion of an integral of the same

type but with the extra $f(p_{\rm L},\omega)$ weight 2 momentum insertion, namely:

$$\mathcal{I}_f(s) := \int_{\mathcal{F}} \mathrm{d}\nu\, \mathcal{F}(s,1,-2)\, \tau_2 \sum_{\mu,p_{\rm L},p_{\rm R}} q^{\frac{1}{2}p_{\rm L}^2} \bar{q}^{\frac{1}{2}p_{\rm R}^2} f(p_{\rm L},\omega)\,, \tag{C.II.4.10}$$

with:

$$f(p_{\rm L},\omega) = \int_{\mathcal{S}} \langle p_{\rm L}, p_\omega\rangle^2_{\Gamma_{\rm L}} - \frac{n-24}{2\pi\tau_2}\,. \tag{C.II.4.11}$$

Upon unfolding the fundamental domain $\mathcal{F}$ against the Niebur-Poincaré series $\mathcal{F}(s,1,-2)$ one gets:

$$\begin{aligned}\mathcal{I}_f(s) &= \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, \mathcal{M}_{s,-2}(\tau_2) e^{-2\pi i \tau_1} \sum_{\mu,p_{\rm L},p_{\rm R}} q^{\frac{1}{2}p_{\rm L}^2} \bar{q}^{\frac{1}{2}p_{\rm R}^2} f(p_{\rm L},\omega) \\ &= \sum_{\rm BPS} \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2}\, \mathcal{M}_{s,-2}(\tau_2) f(p_{\rm L},\omega)\, e^{-\pi\tau_2(|p_{\rm L}|^2+|p_{\rm R}|^2)/2}\,. \end{aligned} \tag{C.II.4.12}$$

The τ_1 integration variable acts as a Lagrange multiplier to restrict the lattice sum to the contributions $m_1 n^1 + m_2 n^2 = 1$, where we have expanded the momenta in a complex basis:

$$p_L = \frac{1}{\sqrt{T_2 U_2}} \Big(m_2 - T m_1 + \bar{U}(n^1 + T n^2) \Big) \tag{C.II.4.13a}$$

$$p_R = \frac{1}{\sqrt{T_2 U_2}} \Big(m_2 - T m_1 + U(n^1 + T n^2) \Big) \tag{C.II.4.13b}$$

As explained above, first one has to solve the BPS constraint $m_1 n^1 + m_2 n^2 = 1$. In general, for any pair of co-prime integers (n^1, n^2), Bézout's lemma ensures that one can find another pair of co-primes $(\tilde{m}_1, \tilde{m}_2)$ such that $\tilde{m}_1 n^1 + \tilde{m}_2 n^2 = 1$. The solutions of the BPS constraints are then of the form:

$$\begin{aligned} m_1 &= \tilde{m}_1 + \tilde{M} n^2\,, \\ m_2 &= \tilde{m}_2 - \tilde{M} n^1\,, \end{aligned} \tag{C.II.4.14}$$

with $\tilde{M} \in \mathbb{Z}$. Upon inserting this expression into the integrand one notices that the complex structure T and the charges defining p_ω always appear in the combination $\tilde{T} = \gamma \cdot T$ so that the sum over (n^1, n^2) reduces to a sum over images with respect to $SL(2;\mathbb{Z})_T$. At this point one has to Poisson resum over the variable $\tilde{M}$ to obtain the desired Fourier series expansion in U. Notice that the momenta are at most linear in $\tilde{M}$ which imply that both the argument in the exponential and the $f(p_{\rm L},\omega)$ insertion in eq. (C.II.4.12) are polynomials of second degree in $\tilde{M}$. One gets the

following schematic expression for this integral eq. (C.II.4.10):

$$\mathcal{I}_f(s) = 2\sum_{M\in\mathbb{Z}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sqrt{U_2\tilde{T}_2}e^{2i\pi M(U_1-\tilde{T}_1)}\times \\ \times\int_0^\infty\frac{\mathrm{d}\tau_2}{\tau_2^{3/2}}\,\mathcal{M}_{s,-2}(-\tau_2)\left(\frac{\alpha}{\tau_2^2}+\frac{\beta}{\tau_2}+\delta\right)\exp\left(-\frac{A}{\tau_2}-B\tau_2\right). \tag{C.II.4.15}$$

The precise expression of the various coefficients in the above schematic expression is determined in appendix E.III. We recall that

$$\begin{aligned}\mathcal{M}_{s,-2}(y) &= 4\pi y\, M_{1,s-1/2}(4\pi y)\\ &= (4\pi y)^{s+1}\,e^{-2\pi y}\,{}_1F_1(s-1;2(s-1)+2;4\pi y)\,,\end{aligned} \tag{C.II.4.16}$$

that, together with eq. (C.II.4.3) yields:

$$\mathcal{M}_{s,-2}(y) = 2^{2s-3}\,\Gamma(s-\tfrac{3}{2})\,(4\pi y)^{5/2}\times \\ \times\sum_{\ell=0}^{2}(-1)^\ell\frac{(3-\ell)_\ell\,(2s-3)_\ell}{(2s)_\ell\,\ell!}\,(s+\ell-\tfrac{3}{2})\,I_{s+\ell-\frac{3}{2}}(2\pi y)\,. \tag{C.II.4.17}$$

Plugging this expression into the integral in eq. (C.II.4.15) yields:

$$4^{2s}\,\pi^{5/2}\,\Gamma(s-\tfrac{3}{2})\sum_{\ell=0}^{2}(-1)^\ell\frac{(3-\ell)_\ell\,(2s-3)_\ell}{(2s)_\ell\,\ell!}\,(s+\ell-\tfrac{3}{2})\times \\ \times\int_0^\infty\mathrm{d}t\,\left(\frac{\alpha}{t}+\beta+\delta t\right)\,I_{s+\ell-\frac{3}{2}}(2\pi t)\,e^{-A/t-Bt}\,. \tag{C.II.4.18}$$

The relevant values of the coefficients A, B, α, β and δ computed in appendix E.III are the following:

$$A = \pi M^2 U_2\tilde{T}_2\,, \tag{C.II.4.19a}$$

$$B = \pi\left(\frac{U_2}{\tilde{T}_2}+\frac{\tilde{T}_2}{U_2}\right)\,, \tag{C.II.4.19b}$$

$$\alpha = -U_2^2M^2\tilde{N}^i_{(1)}d_{ij}\tilde{N}^j_{(1)}\,, \tag{C.II.4.19c}$$

$$\beta = 2iU_2M\,\frac{U_2+\tilde{T}_2}{\tilde{T}_2}\,\tilde{N}^i_{(1)}d_{ij}\tilde{N}^j_{(2)} - \frac{U_2}{2\pi\tilde{T}_2}\,\tilde{N}^i_{(2)}d_{ij}\tilde{N}^j_{(2)}\,, \tag{C.II.4.19d}$$

$$\delta = \left(\frac{U_2+\tilde{T}_2}{\tilde{T}_2}\right)^2\,\tilde{N}^i_{(2)}d_{ij}\tilde{N}^j_{(2)}\,, \tag{C.II.4.19e}$$

where $\tilde{N}_{(1)} := \mathrm{Re}(\tilde{N})$ and $\tilde{N}_{(2)} := \mathrm{Im}\,(\tilde{N})$.

One then plugs eq. (C.II.4.18) into eq. (C.II.4.15), and splits the later into its zero, positive and negative frequency parts:

$$\mathcal{I}_f(s) = \mathcal{I}_f^{(-)}(s) + \mathcal{I}_f^{(0)}(s) + \mathcal{I}_f^{(+)}(s)\,. \tag{C.II.4.20}$$

Zero-frequency mode One has explicitely for the zero mode part of the Fourier expansion:

$$\mathcal{I}_f^{(0)}(s) = 2\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sqrt{U_2\tilde{T}_2}\int_0^\infty \frac{\mathrm{d}t}{t^{3/2}}\,\mathcal{M}_{s,-2}(-t)\left(\frac{\beta^{(0)}}{t}+\delta\right)\exp\left(-Bt\right)\,, \tag{C.II.4.21}$$

which we can rewrite, using the results above, as:

$$\begin{aligned}\mathcal{I}_f^{(0)}(s) = 2^{4s+1}\,\pi^{5/2}\,\Gamma(s-\tfrac{3}{2})\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sqrt{U_2\tilde{T}_2}\sum_{\ell=0}^{2}(-1)^\ell\frac{(3-\ell)_\ell\,(2s-3)_\ell}{(2s)_\ell\,\ell!}\,(s+\ell-\tfrac{3}{2})\times\\ \times\left(-\beta^{(0)}\frac{\partial}{\partial B}+\delta\frac{\partial^2}{\partial B^2}\right)\int_0^\infty\frac{\mathrm{d}t}{t}\;I_{s+\ell-\frac{3}{2}}(2\pi t)\,e^{-Bt}\,.\end{aligned} \tag{C.II.4.22}$$

One can obtain a very explicit expression in the form:

$$\begin{aligned}\mathcal{I}_f^{(0)}(s) = 2^{4s+1}\,\pi^{5/2}\,\Gamma(s-\tfrac{3}{2})\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sqrt{U_2\tilde{T}_2}\sum_{\ell=0}^{2}(-1)^\ell\frac{(3-\ell)_\ell\,(2s-3)_\ell}{(2s)_\ell\,\ell!}\,(s+\ell-\tfrac{3}{2})\times\\ \times\left(\beta^{(0)}F^{(0)}_{1,s+l-\frac{3}{2}}(B,2\pi)+\delta F^{(0)}_{2,s+l-\frac{3}{2}}(B,2\pi)\right)\,,\end{aligned} \tag{C.II.4.23}$$

where the functions $F^{(0)}_{n,\nu}$ are defined and computed in appendix E.I.

Positive frequency modes Let us now consider the positive frequency part, the negative part being obtained from it by complex conjugation. The contribution of positive modes reads:

$$\begin{aligned}\mathcal{I}_f^{(+)}(s) = 2\sum_{M>0}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sqrt{U_2\tilde{T}_2}e^{2i\pi M(T_1-\tilde{U}_1)}\times\\ \times\int_0^\infty\frac{\mathrm{d}\tau_2}{\tau_2^{3/2}}\,\mathcal{M}_{s,-2}(-\tau_2)\left(\frac{\alpha}{\tau_2^2}+\frac{\beta}{\tau_2}+\delta\right)\exp\left(-\frac{A}{\tau_2}-B\tau_2\right)\,.\end{aligned} \tag{C.II.4.24}$$

One can again rewrite it as:

$$\mathcal{I}_f^{(+)}(s) = 2^{4s+1}\,\pi^{5/2}\,\Gamma(s-\tfrac{3}{2})\sum_{M>0}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sqrt{U_2\tilde{T}_2}\sum_{\ell=0}^{2}(-1)^\ell\frac{(3-\ell)_\ell\,(2s-3)_\ell}{(2s)_\ell\,\ell!}\times$$
$$\times\,(s+\ell-\tfrac{3}{2})\left(\alpha-\beta\frac{\partial}{\partial B}+\delta\,\frac{\partial^2}{\partial B^2}\right)\int_0^\infty\frac{\mathrm{d}t}{t}\;I_{s+\ell-\frac{3}{2}}(2\pi t)\,e^{-A/t-Bt}\,, \tag{C.II.4.25}$$

leading to the expression:

$$\mathcal{I}_f^{(+)}(s) = \;2^{4s+1}\,\pi^{5/2}\,\Gamma(s-\tfrac{3}{2})\sum_{M>0}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sqrt{U_2\tilde{T}_2}\sum_{\ell=0}^{2}(-1)^\ell\frac{(3-\ell)_\ell\,(2s-3)_\ell}{(2s)_\ell\,\ell!}\times$$
$$\times\,(s+\ell-\tfrac{3}{2})\left(\alpha F_{0,s+l-\frac{3}{2}}(A,B,2\pi)+\beta F_{1,s+l-\frac{3}{2}}(A,B,2\pi)+\delta F_{2,s+l-\frac{3}{2}}(A,B,2\pi)\right)\,, \tag{C.II.4.26}$$

where the functions $F_n(A,B,C)$ are defined in appendix E.I, and depend on the coeffients A,B and C mainly through $u_\pm=\sqrt{A}(\sqrt{B+C}\pm\sqrt{B-C})$.

Putting all pieces together, one has the following compact expressions for the Fourier expansion of the three threshold corrections:

Gravitational threshold corrections:

$$\begin{aligned}24\Lambda_{\rm grav} =& -\frac{n}{12}\,(\mathcal{I}(2)-5\mathcal{I}(1)-144\,\mathcal{I}_{\rm DKL})\\ &+\frac{n-24}{12}\left(\frac{1}{5}\mathcal{I}(3)-4\mathcal{I}(2)+13\mathcal{I}(1)+144\,\mathcal{I}_{\rm DKL}\right)\\ &-\frac{1}{2}\left(\frac{1}{40}\mathcal{I}_f(3)-\frac{1}{3}\mathcal{I}_f(2)\right)\,.\end{aligned} \tag{C.II.4.27}$$

E_8 **threshold corrections:**

$$\begin{aligned}24\Lambda_{E_8} =& -\frac{n-12}{6}\,(\mathcal{I}(2)-5\mathcal{I}(1)-144\,\mathcal{I}_{\rm DKL})\\ &+\frac{n-24}{12}\left(\frac{1}{5}\mathcal{I}(3)-4\mathcal{I}(2)+13\mathcal{I}(1)+144\,\mathcal{I}_{\rm DKL}\right)\\ &+\frac{n}{12}\,(\mathcal{I}(1)-1008\,\mathcal{I}_{\rm DKL})-\frac{1}{2}\left(\frac{1}{40}\mathcal{I}_f(3)-\frac{1}{2}\mathcal{I}_f(2)\right)\,.\end{aligned} \tag{C.II.4.28}$$

E_7 threshold corrections:

$$\begin{aligned}24\Lambda_{E_7} =& -\frac{n-12}{6}\left(\mathcal{I}(2)-5\mathcal{I}(1)-144\,\mathcal{I}_{\rm DKL}\right)\\ &+\frac{n-24}{12}\left(\frac{1}{5}\mathcal{I}(3)-4\mathcal{I}(2)+13\mathcal{I}(1)+144\,\mathcal{I}_{\rm DKL}\right)\\ &+\frac{n}{12}\left(\mathcal{I}(1)+720\,\mathcal{I}_{\rm DKL}\right)-\frac{1}{2}\left(\frac{1}{40}\mathcal{I}_f(3)-\frac{1}{2}\mathcal{I}_f(2)\right)\,.\end{aligned} \tag{C.II.4.29}$$

II.4.2 A simple subclass of models

The Fourier series expansion that we have obtained above is not easy to analyse, in particular because the two-torus metric and the intersection form on the base $d_{ij} = \int_{\mathcal{S}} \omega_i \wedge \omega_j$ are intertwined in a non trivial way in the momentum insertion $\int \langle p_{\rm L}, p_\omega\rangle^2$. In order to unveil the role of the worldsheet instantons, we consider below a subclass of models that, although not really special from the physical point of view, allow to present the results in a much simpler way.

Noticing that the interpretation in terms of worldsheet instantons does not depend on the precise moduli of the torus fiber, let us consider for convenience examples in which the momentum insertion $\int \langle p_{\rm L}, p_\omega\rangle^2$ is proportional to $\langle p_{\rm L}, p_{\rm L}\rangle := p_{\rm L}^2$, namely the case where:

$$f(p_{\rm L},\omega) = (n-24)\left(p_{\rm L}^2 - \frac{1}{2\pi\tau_2}\right)\,, \tag{C.II.4.30}$$

where the proportionality constant in front of the $p_{\rm L}^2$ term is fixed by modularity, and where one made use of the tadpole condition eq. (B.II.1.17). It amounts to a particular relation between the torus metric and the intersection form d_{ij}, see section II.4.3.

For definiteness let us consider the gravitational threshold corrections corresponding to such a setting. As discussed previously, it is written:

$$\begin{aligned}24\Lambda_{\rm grav} =& \int_{\mathcal{F}} {\rm d}\nu\,\tau_2 \sum_{\mu,p_{\rm L},p_{\rm R}} q^{\frac{1}{2}p_{\rm L}^2}\bar{q}^{\frac{1}{2}p_{\rm R}^2}\times\\ &\times\left\{-\frac{n}{12}\frac{\hat{E}_2E_4E_6}{\Delta}+\frac{n-24}{12}\frac{\hat{E}_2^2E_4^2}{\Delta}-\frac{f(p_{\rm L},\omega)}{2}\frac{\hat{E}_2E_4^2}{\Delta}\right\}\,,\end{aligned} \tag{C.II.4.31}$$

with ${\rm d}\nu = {\rm d}\tau_1{\rm d}\tau_2/\tau_2^2$ the modular invariant measure. Let us focus on the last term and exploit eq. (C.II.4.30). Once again, we denote by $\Lambda_{2,2}$ the partition function associated with the Narain lattice $\Gamma_{2,2}(T,U)$, including a factor τ_2 making it modular

invariant by itself. As a preliminary step, notice that:

$$(n-24)(D_0 \Lambda_{2,2}) = -\tau_2 \sum_{\mu, p_{\rm L}, p_{\rm R}} f(p_{\rm L}, \omega)\, q^{\frac{1}{2}p_{\rm L}^2} \bar{q}^{\frac{1}{2}p_{\rm R}^2}\,, \tag{C.II.4.32}$$

with D_0 the modular covariant derivative as defined in eq. (E.I.1.28). This allows to reexpress the last term in eq. (C.II.4.31) simply as:

$$\frac{n-24}{2} \int \mathrm{d}\nu\, (D_0 \Lambda_{2,2}) \frac{\hat{E}_2 E_4^2}{\Delta}\,. \tag{C.II.4.33}$$

Using eq. (E.I.1.29), an integration by part then leads to:

$$-\frac{n-24}{2} \int \mathrm{d}\nu\, \Lambda_{2,2} D_{-2} \left(\frac{\hat{E}_2 E_4^2}{\Delta} \right)\,. \tag{C.II.4.34}$$

Exploiting again eq. (E.I.1.29), one computes:

$$D_{-2} \left(\frac{\hat{E}_2 E_4^2}{\Delta} \right) = \frac{1}{6} \frac{E_4^3}{\Delta} + \frac{4}{3} \frac{\hat{E}_2 E_4 E_6}{\Delta} + \frac{1}{2} \frac{\hat{E}_2^2 E_4^2}{\Delta}\,. \tag{C.II.4.35}$$

Plugging this result into eq. (C.II.4.31), one obtains finally:

$$24\Lambda_{\rm grav} = \int_{\mathcal{F}} \mathrm{d}\nu\, \Lambda_{2,2} \left(-\frac{3n-64}{4} \frac{\hat{E}_2 E_4 E_6}{\Delta} - \frac{n-24}{6} \frac{\hat{E}_2^2 E_4^2}{\Delta} - \frac{n-24}{12} \frac{E_4^3}{\Delta} \right)\,. \tag{C.II.4.36}$$

Using the decompositions in terms of Niebur-Poincaré series, one finally obtains:

$$\begin{aligned} 24\Lambda_{\rm grav} = \int_{\mathcal{F}} \mathrm{d}\nu\, \Lambda_{2,2} \Big(& -\frac{n-24}{30} \mathcal{F}(3,1,0) - \frac{n}{12} \mathcal{F}(2,1,0) + \\ & + \frac{3n-52}{2} \mathcal{F}(1,1,0) + 24(n-12) \Big)\,, \end{aligned} \tag{C.II.4.37}$$

which can be written in terms of eq. (C.II.4.4) as:

$$24\Lambda_{\rm grav} = -\frac{n-24}{30} \mathcal{I}(3) - \frac{n}{12} \mathcal{I}(2) + \frac{3n-52}{2} \mathcal{I}(1) + 24(n-12)\, \mathcal{I}_{\rm DKL}\,. \tag{C.II.4.38}$$

Let us split the result into positive, negative and zero-frequency parts:

$$\Lambda_{\rm grav} = \Lambda_{\rm grav}^{(-)} + \Lambda_{\rm grav}^{(0)} + \Lambda_{\rm grav}^{(+)}\,, \tag{C.II.4.39}$$

that will be given separately below.

Zero-frequency mode It turns out that one can have a very explicit expression for the zero mode part of the above expression in terms of real analytic Eisenstein series, defined by:

$$E(z,\rho) := \frac{1}{2}\sum_{(m,n)=1}\frac{\mathrm{Im}(z)^\rho}{|m+zn|^{2\rho}}\,. \tag{C.II.4.40}$$

The zero-frequency mode of the gravitational threshold correction in the above example is then given by:

$$\begin{aligned}\Lambda^{(0)}_{\rm grav} =& \frac{\pi}{90U_2^2}\Big\{15(3n-52)U_2^2\,E(T,1) - 5nU_2\,E(T,2) - 12(n-24)\,E(T,3)\Big\}\\ &+ (n-12)\,\mathcal{I}_{\rm DKL}\,.\end{aligned} \tag{C.II.4.41}$$

In the $K3\times T^2$ case, it reduces to:

$$\Lambda^{(0)}_{\rm grav} = \frac{2\pi}{3}\Big\{5\,E(T,1) - 2T_2^{-1}\,E(T,2)\Big\} + 12\,\mathcal{I}_{\rm DKL}\,. \tag{C.II.4.42}$$

Positive frequency part The positive frequency part can also be written explicitely in terms of the Niebur-Poincaré series themselves, cf. eq. (C.II.4.7):

$$\begin{aligned}\Lambda^{(+)}_{\rm grav} =& \frac{1}{30\times 24}\sum_{M>0}\frac{e^{2i\pi MU_1}}{M}\Big\{30(3n-52)\,\mathcal{W}_{1,0}(MU_2)\mathcal{F}(1,M,0;T)\\ &- 5n\,\mathcal{W}_{2,0}(MU_2)\mathcal{F}(2,M,0;T) - 2(n-24)\,\mathcal{W}_{3,0}(MU_2)\mathcal{F}(3,M,0;T)\Big\}\,,\end{aligned} \tag{C.II.4.43}$$

which reduces for $K3\times T^2$ to:

$$\Lambda^{(+)}_{\rm grav} = \frac{1}{6}\sum_{M>0}\frac{e^{2i\pi MU_1}}{M}\Big\{5\,\mathcal{W}_{1,0}(MU_2)\mathcal{F}(1,M,0;T) - \mathcal{W}_{2,0}(MU_2)\mathcal{F}(2,M,0;T)\Big\}\,. \tag{C.II.4.44}$$

Given that $\mathcal{W}_{1+\ell,0}(MU_2)\sim (MU_2)^{-\ell}e^{-2\pi MU_2}\times$ (polynomial in MU_2), one has in both cases a sum over $M\in\mathbb{Z}_{>0}$ which represents the sum over the wrapping number of a worldsheet instanton around the two-torus fiber of the principal bundle $T^2\hookrightarrow X\xrightarrow{\pi}\mathcal{S}$, which is of volume T_2.

Even though for $n<24$ the torus fiber is only a torsional two-cycle, it appears that worldsheet instantons, corresponding to holomorphic maps from the heterotic worldsheet to X wrapping the fiber, do contribute to the threshold corrections, for any wrapping number.

Would we have decided to work with the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string, this discussion should also be considered in the context of type I flux compactifications via S-duality [18]. Then, the heterotic thresholds encompass both the spacetime perturbative and non-perturbative effects on the type I side, the latter corresponding to Euclidean D1-branes wrapping the torus fiber.

II.4.3 Generic momentum insertion

In section II.4.2, we discussed a simple class of models for which the momentum insertion takes a particularly simple form. Here, we want to understand in more detail the constraint eq. (C.II.4.30).

The data of the compactification involves an even integral lattice Γ_{L} naturally associated to the rational Narain lattice $\Gamma_{2,2}$. In the following we denote this lattice Γ_{L} simply by Γ. One associates to this lattice the theta function $\Theta^{\Gamma} : \mathbb{H} \times (\Gamma \otimes \mathbb{C}) \to \mathbb{C}$:

$$\Theta^{\Gamma}(\tau, z) = \sum_{v \in \Gamma} e^{i\pi(\langle v,v\rangle + 2\langle v,z\rangle)} \,, \tag{C.II.4.45}$$

whose second argument lives in the complexification of the lattice Γ. More precisely, the rational Narain lattice partition function involves such a theta function with an extra characteristic μ, namely the summation vector runs over the shifted lattice $\Gamma + \mu$, where μ belongs to the discriminant group $\Gamma^\star/\Gamma$. In our situation, we actually have two lattices involved in the compactification data, Γ and $\mathrm{Pic}(\mathcal{S})$. The inner product on Γ is denoted $\langle \cdot, \cdot \rangle$, and the one on $\mathrm{Pic}(\mathcal{S})$ is defined via the composition:

$$(\cdot,\cdot) : \mathrm{Pic}(\mathcal{S}) \times \mathrm{Pic}(\mathcal{S}) \to \wedge^2 \mathrm{Pic}(\mathcal{S}) \xrightarrow{\int_{\mathcal{S}}} \mathbb{Z} \,, \tag{C.II.4.46}$$

In our situation, the second argument of the theta function actually lives in a further extension of the lattice Γ:

$$\Theta^{\Gamma}\left(\tau, \frac{p_\omega}{2i\pi}\right) = \sum_{v \in \Gamma} e^{i\pi\langle v,v\rangle + \langle v,p_\omega\rangle} \,, \tag{C.II.4.47}$$

with $p_\omega \in \Gamma \otimes \mathrm{Pic}(\mathcal{S}) \otimes \mathbb{C}$. Hence, denoting by $\{e_i\}$ and $\{\epsilon_a\}$ a basis of Γ and $\mathrm{Pic}(\mathcal{S})$ respectively, we have:

$$p_\omega = \omega^{ia}\, e_i \otimes \epsilon_a \,, \tag{C.II.4.48a}$$

$$v = v^i\, e_i \,, \tag{C.II.4.48b}$$

with $\omega^{ia} \in \mathbb{C}$. The matrix (ω^{ia}) specifies the data of the torus fibration, and is fixed once and for all for a given model. (ω_{ia}) should be viewed as connecting the two *a priori* independent integral even lattices Γ and $\mathrm{Pic}(\mathcal{S})$. In the above function, $\langle v, p_\omega \rangle$ should be understood as the map $\Gamma \times (\Gamma \otimes \mathrm{Pic}(\mathcal{S})) \to \mathrm{Pic}(\mathcal{S})$ naturally induced by the pairing on Γ, also loosely denoted $\langle \cdot, \cdot \rangle$:

$$\langle v, p_\omega \rangle := v^i g_{ij} \omega^{ja} \, \epsilon_a \,, \tag{C.II.4.49}$$

with g the metric on the lattice Γ, namely $g_{ij} := \langle e_i, e_j \rangle$, not to be confused with the metric on the Narain lattice $\Gamma_{2,2}$. We also define the metric d on the lattice $\mathrm{Pic}(\mathcal{S})$ by:

$$d_{ab} := (\epsilon_a, \epsilon_b) = \int_{\mathcal{S}} \epsilon_a \wedge \epsilon_b \,. \tag{C.II.4.50}$$

Let us also define the pull-back metric:

$$\tilde{d}^{ij} := \omega^{ia} \omega^{jb} \, d_{ab} \,. \tag{C.II.4.51}$$

Notice that we can define a natural inner product on $\Gamma \otimes \mathrm{Pic}(\mathcal{S})$, which we denote by a dot, in the following way: given two elements $\alpha = \alpha^{ia} \, e_i \otimes \epsilon_a$ and $\beta = \beta^{ia} \, e_i \otimes \epsilon_a$, we define:

$$\alpha \cdot \beta := g_{ij} d_{ab} \alpha^{ia} \beta^{jb} \,. \tag{C.II.4.52}$$

Let us look at the simplified case for which $\int_{\mathcal{S}} \langle v, p_\omega \rangle^2 \propto (n-24) \langle v, v \rangle$, the $(n-24)$ coefficient originating from the tadpole cancellation condition $p_\omega \cdot p_\omega = 2(n-24)$. Let us simply express the momentum insertion in terms of the lattice data in the following way:

$$\int_{\mathcal{S}} \langle v, p_\omega \rangle^2 = \; v^i v^k \, g_{ij} g_{kl} \tilde{d}^{jl} = \tilde{d}_{ik} v^i v^k \,. \tag{C.II.4.53}$$

Therefore, we see that the insertion is proportional to $(n-24)\langle v, v \rangle$ if and only if $\tilde{d} \propto g$.

Let us give a final comment about the above defined object. One has actually defined a pairing through the following chain of maps:

$$* : \big(\Gamma \times \Gamma \otimes \mathrm{Pic}(\mathcal{S})\big) \times \big(\Gamma \times \Gamma \otimes \mathrm{Pic}(\mathcal{S})\big) \xrightarrow{\langle , \rangle \wedge \langle , \rangle} \mathrm{Pic}(\mathcal{S}) \times \mathrm{Pic}(\mathcal{S}) \xrightarrow{(,)} \mathbb{Z} \,, \tag{C.II.4.54}$$

hence endowing $\Gamma \times \Gamma \otimes \mathrm{Pic}(\mathcal{S})$ with a natural structure of integral lattice. The momentum insertion eq. (C.II.4.53) corresponds simply to $(v, p_\omega) * (v, p_\omega)$. The lattice $(\Gamma \times \Gamma \otimes \mathrm{Pic}(\mathcal{S}), *)$ is in this sence the natural lattice keeping track of the

principal torus fibration structure at the level of the Fu-Yau genus, enlightening the interplay between the two even integral lattices Γ and $\mathrm{Pic}(\mathcal{S})$.

Some comments: To conclude this part, let us say that as mentioned above, by S-duality our results apply to D-instanton corrections in orientifold compactifications with Ramond-Ramond backgrounds fluxes. A better understanding of the physics behind these instanton corrections would involve then studying D1-instanton probes in these flux backgrounds of type I supergravity. In [93] heterotic five-branes wrapping the torus fiber have been studied. However the physics is not the same because the coupling to the NS-NS flux is different. We plan to come back to this problem in the future.

A generalization of our results to models with Abelian gauge bundle over the total space is also worthwile considering, given that the dressed elliptic genus has also been computed in those cases in chapter B.III. These examples are especially important from the four-dimensional effective field-theory perspective, as the threshold corrections will then be functions of the bundle moduli, while the torus moduli are frozen to discrete values for a generic choice of torus principal bundle.

Part D

New non-compact heterotic supergravity solutions

Chapter I

Introduction

As we already discussed in the previous parts, finding compact solutions to the BPS system of equations ensuring minimal supersymmetry in spacetime is a very difficult task. The equation originating from the variation of the dilatino field indeed tells us the the internal manifold X should be a balanced manifold, or more precisely a conformally balanced manifold (due to the presence of a non-constant dilaton field), which is a weaker condition than Kählerity. In addition, the gravitino equation tells us that the manifold X should have a restricted holonomy with respect to a connection with torsion, the Bismut connection, where the NS$\otimes$NS field strength plays the role of torsion. In each dimension Berger's classification of the possible holonomy groups on a Riemannian manifold then restricts the possible type of geometry.

On another hand, we saw that the gaugino equation constraints the connection on the holomorphic vector bundle $E \to X$ to be an instanton configuration. Comes therefore the existence question of such connections on X, let alone their explicit construction. The Donaldson-Uhlenbeck-Yau theorem then allows to rephrase this question into the existence problem of stable holomorphic vector bundles over X.

Finally comes the Bianchi identity, which ties together the data of the NS$\otimes$NS flux and the connection on E. As described in the previous parts, this identity is non-linear in the torsion flux, hence particularly difficult to deal with, and constitutes the main obstacle in exhibiting solutions to the BPS system. We saw that apart from the Fu-Yau solution leading to a four-dimensional flux background, new compact torsional solutions are rare. Moreover, this identity mixes different orders in the α' expansion, and leads to tadpole conditions which typically fix the compact manifold to stringy size, hence making the large volume limit approximation a bit dubious.

A strategy often adopted consists in considering non compact solutions to the BPS system of equations, which are local approximation of compact models. In contrast to the compact spaces they appromimate, these geometries often admit isometries, hence one can often write the metric explicitely. In addition to providing a richer collection of solutions, they also often allow to define a 'large charge' limit in which one can consistently neglect the non-linear $\mathrm{tr}\, R \wedge R$ term in the Bianchi identity.

As an illustration, consider the singular point in the $K3$ moduli space correponding to the standard $T^4/\mathbb{Z}_2$ orbifold. One obtains then a smooth surface by blowing up each or the 16 singularities, i.e. replacing each of them by an Eguchi-Hanson space, whose boundary is $\mathbb{P}^3_{\mathbb{R}} = S^3/\mathbb{Z}_2$. The Kähler metric on Eguchi-Hanson space is written in terms of the $SU(2)$ left-invariant one forms:

$$\sigma_1 = \cos\frac{\psi}{2}\,\mathrm{d}\theta - \sin\frac{\psi}{2}\sin\theta\,\mathrm{d}\phi\,, \tag{D.I.0.1a}$$

$$\sigma_2 = \sin\frac{\psi}{2}\,\mathrm{d}\theta - \cos\frac{\psi}{2}\sin\theta\,\mathrm{d}\phi\,, \tag{D.I.0.1b}$$

$$\sigma_3 = \frac{1}{2}\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\,, \tag{D.I.0.1c}$$

as:

$$\mathrm{d}s^2_{\text{EH}} = \Delta^{-1}\mathrm{d}r^2 + \frac{r^2}{4}\left(\sigma_1^2 + \sigma_2^2 + \Delta\,\sigma_3^2\right)\,, \tag{D.I.0.2}$$

with $\Delta = 1 - \left(\frac{a}{r}\right)^4$, $r \in [a, +\infty]$, $\theta \in [0, \pi]$ and $\phi, \psi \in [0, 2\pi]$. The Eguchi-Hanson space has the topology of the canonical bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$.

This geometry corresponds therefore to the resolution of an A_1 singularity, with resolution parameter a.

In the spirit of the non-Kähler geometries studied in the previous parts, and applying thechniques analog to those of [94], it was shown by Fu, Tseng and Yau in [95] that two-torus principal bundles over Eguchi-Hanson space $T^2 \to X \to \mathcal{O}_{\mathbb{P}^1}(-2)$ with the following metric:

$$\mathrm{d}s^2 = \mathrm{d}s^2_{1,3} + e^{\Phi - \Phi_\infty}\mathrm{d}s^2_{\text{EH}} + \frac{\alpha' U_2}{T_2}\left|\mathrm{d}x^1 + T\mathrm{d}x^2 + \left(m^1 + Tm^2\right)\varpi\right|^2\,, \tag{D.I.0.3}$$

with the connection one-form on the base:

$$\varpi = \frac{a^2}{2r^2}\sigma_3\,, \tag{D.I.0.4}$$

and equipped with a direct sum of line bundles constitute a solution of the BPS sys-

tem and Bianchi identity, by solving the Bianchi identity with the Chern connection for the dilaton in a 'large resolution' limit. It was then shown in [96] instead that in a large NS5-brane charge regime in which one can neglect the spin curvature term in the Bianchi identity, one could define a 'double scaling limit' by:

$$g_s \to 0\,, \qquad \mu = \frac{g_s\sqrt{\alpha'}}{a} \text{ fixed}\,, \tag{D.I.0.5}$$

allowing to decouple the near-horizon physics from the bulk, while keeping fixed the tension of NS5-branes wrapping the resolution cycle fixed, preventing the appearance of new massless degrees of freedom.

The exceptional feature of this decoupled theory is that it can be shown to admit an exact worldsheet CFT description, ensuring its status of exact solution to the full string theory. In [96], Luca Carlevaro, Dan Israël and Marios Petropoulos built this CFT starting from the blown down limit of the decoupled theory where the abelian gauge instanton becomes pointlike, correponding to the so-called Callan-Strominger-Harvey background[1] further compactified on the two torus. In the blowdoan limit, the worldsheet SCFT is therefore:

$$\mathbb{R}^{1,3} \times T^2 \times \mathbb{R}_{\sqrt{2/k}} \times SU(2)_k/\mathbb{Z}_2 \times SO(32)_1|_{\text{L}}\,, \tag{D.I.0.6}$$

in the case of the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic string for definiteness. The performing on it a so-called dynamical deformation, namely a deformation by a marginal operator with field-dependent parameters allows to consistently blow-up the resolution cycle from the worldsheet point of view. We refer the reader to the paper [96] for details.

Later on, Luca Carlevaro and Dan Israël [97] built a genuine $SU(3)$-structure non-compact heterotic solution based on the conifold equipped with a line bundle, which was further studied in [98][2]. Let us briefly recall the ansatz of [97].

In the singular geometry, the asymptotic locally Ricci-flat geometry consist in the conifold, metric cone over $T^{1,1}$. The latter corresponds to the quotient $SU(2) \times SU(2)/U(1)$, the $U(1)$ being embedded symmetrically in the two $SU(2)$ factors. The $SU(2) \times SU(2) \times U(1)$-invariant Ricci-flat metric is written in terms of a pair of $SU(2)$ left-invariant one-forms with common third Euler angle ψ, $(\sigma_i, \tilde{\sigma}_i)$ as:

$$\mathrm{d}s^2_{\text{conifold}} = \mathrm{d}r^2 + r^2\left[\frac{1}{6}\left(\sigma_1^2 + \sigma_2^2 + \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2\right) + \frac{1}{9}\varpi^2\right]\,, \tag{D.I.0.7}$$

[1] More precisely a $\mathbb{Z}_2$ orbifold of it.

[2] In this article, the ansatz was generalized allowing for a breaking of the $\mathbb{Z}_2$ symmetry between the two $\mathbb{P}^1$'s, cf. below.

with

$$\varpi = \sigma_3 + \tilde{\sigma}_3 = \mathrm{d}\psi + \cos\theta_1 \mathrm{d}\phi_1 + \cos\theta_2 \mathrm{d}\phi_2 \tag{D.I.0.8}$$

a connection one-form. The geometry is usually desingularized in the IR by blowing up a two or three sphere, but considering a $\mathbb{Z}_2$ orbifold of the conifold for which the range of ψ is reduced to $[0, 2\pi[$ instead of $[0, 4\pi[$ allows to blow up a four-cycle $\mathbb{P}^1 \times \mathbb{P}^1$, leading to a geometry homeomorphic to the total space of the canonical line bundle $\mathcal{O}(-K) \to \mathbb{P}^1 \times \mathbb{P}^1$. The ansatz for the torsional background geometry is conformal to such a regularized conifold:

$$\mathrm{d}s^2_{1,9} = \mathrm{d}s^2_{1,3} + \frac{3H(r)}{2}\left\{\frac{\mathrm{d}r^2}{f(r)^2} + \frac{r^2}{6}\left(\sigma_1^2 + \sigma_2^2 + \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2\right) + \frac{r^2}{9} f(r)^2 \varpi^2\right\}, \tag{D.I.0.9a}$$

$$H_3 = \frac{\alpha' k}{6}\, g_1(r)^2 (\Omega_1 + \Omega_2) \wedge \varpi\,, \tag{D.I.0.9b}$$

$$A_1 = \frac{1}{4}\left[(\Omega_1 - \Omega_2)\mathbf{p} + g_2 \varpi \mathbf{q}\right] \cdot I\,, \tag{D.I.0.9c}$$

with the two volume forms $\Omega_i = \sin\theta_i \mathrm{d}\theta_i \wedge \mathrm{d}\phi_i$ and $\mathcal{I}$ in the Cartan of $\mathfrak{so}(32)$ or $\mathfrak{e}_8 \times \mathfrak{e}_8$. The dilaton also exhibit a non-constant profile. It was shown numerically that this ansatz indeed solves the BPS equations, as well as the Bianchi identity at leading order in the large charge limit in which $\mathbf{p}^2 = \mathbf{q}^2$ is large in string units compared to the blow-up mode a characterizing the bolt, defined by $f(a) = 0$. One can then decouple the physics near the bolt from the bulk by defining a limit similar in spirit to eq. (D.I.0.5):

$$g_s \to 0\,, \qquad \mu = \frac{g_s \alpha'}{a^2} \text{ fixed}\,, \tag{D.I.0.10}$$

where g_s is the asymptotic string coupling. This leads to the following analytic solution (we set $\rho := r/a$):

$$\mathrm{d}s^2_{1,9} = \mathrm{d}s^2_{1,3} + \frac{4\alpha' Q_5}{\rho^2}\left\{\frac{\mathrm{d}\rho^2}{1 - \frac{1}{\rho^8}} + \frac{\rho^2}{8}\left(\sigma_1^2 + \sigma_2^2 + \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2\right) + \frac{\rho^2}{16}\left(1 - \frac{1}{\rho^8}\right)\varpi^2\right\}, \tag{D.I.0.11a}$$

$$H_3 = \frac{\alpha' Q_5}{4}\left(1 - \frac{1}{\rho^8}\right)(\Omega_1 + \Omega_2) \wedge \varpi\,, \tag{D.I.0.11b}$$

$$F = \frac{1}{4}\left[(\Omega_1 - \Omega_2)\mathbf{p} + \frac{1}{\rho^4}\varpi \mathbf{q}\right] \cdot \mathcal{I}\,, \tag{D.I.0.11c}$$

with fivebrane charge

$$Q_5 := \frac{1}{2\pi^2\alpha'}\int_{\mathbb{P}^3_{\mathbb{R}}} H_3 = \frac{k}{2}\,. \tag{D.I.0.12}$$

The dilaton behaves as:

$$e^{\Phi} = \frac{2\mu}{H_\infty}\left(\frac{Q_5}{\rho^2}\right)\,. \tag{D.I.0.13}$$

It was then shown that the above solution eq. (D.I.0.11) admits an algebraic description in terms of a coset CFT:

$$\frac{SL_2(\mathbb{R})_{k/2}\times\frac{SU(2)_k\times SU(2)_k}{U(1)_{\mathrm{L}}}}{U(1)_{\mathrm{L}}\times U(1)_{\mathrm{R}}}\,, \tag{D.I.0.14}$$

together with the flat CFT $\mathbb{R}^{1,3}$, the left-moving heterotic affine algebra, and a ghost-superghost system. Such a description in principle allows to include all the loop corrections order by order, turning the theory into a full heterotic solution, even away from the large-charge limit.

Chapter II

Heterotic flux solutions from Sasaki-Einstein manifolds

II.1 Expression of the ansatz

This chapter is built out of the article to appear: *Heterotic Flux Solutions From Sasaki-Einstein Manifolds*, with Nick Halmagyi, Dan Israël and Eirik Eik Svanes.

Let us try and generalize the above constructions. We introduce a Sasaki-Einstein manifold M of real dimension $2n-1$, build from an S^1 fibration over a Kähler-Einstein $(n-1)$-fold base B_{n-1}. The metric on the Sasaki-Einstein manifold reads:

$$g_M = g_B + \varpi \otimes \varpi \,, \tag{D.II.1.1}$$

with

$$\varpi := \mathrm{d}\psi - A \tag{D.II.1.2}$$

a connection one-form, A being the Kähler potential one-form on the base B (The Kähler form reds $J_B = 2\mathrm{d}A$). It is known that the metric cone $X := C(M)$ is then a non-compact Calabi-Yau n-fold, with standard metric:

$$g_X = \mathrm{d}r \otimes \mathrm{d}r + r^2 g_M \,. \tag{D.II.1.3}$$

Drawing lessons from the above described heterotic solutions, we propose the fol-

lowing ansatz for a heterotic flux background:

$$\mathrm{d}s^2_{1,9} = \mathrm{d}s^2_{1,9-2n} + H(r)\left\{\frac{\mathrm{d}r^2}{f(r)^2} + r^2\left(\mathrm{d}s^2_B + \frac{f(r)^2}{n^2}\varpi^2\right)\right\}, \tag{D.II.1.4a}$$

$$F = \hat{F} + \mathrm{d}\left[g(r)\varpi\right]\mathbf{q}\cdot\mathcal{I} \tag{D.II.1.4b}$$

with $\hat{F}$ an instantonic gauge field on a holomorphic vector bundle $\hat{E}$ living purely on the Kähler-Einstein base, and g a function to be solved for by the Hermite-Yang-Mills equations. The metric ansatz involves two functions, f and H, which we will refer to as the *squashing* function and the *warping* function respectively. The internal geometry is therefore conformal to a regularized metric cone over the Sasaki-Einstein manifold M.

Let us define the following holomorphic vielbein:

$$E^1 = \frac{\sqrt{H(r)}}{f(r)}\mathrm{d}r + i\frac{rf(r)}{n}\sqrt{H(r)}\varpi\,, \tag{D.II.1.5a}$$

$$E^a = r\sqrt{H(r)}\hat{E}^a\,, \tag{D.II.1.5b}$$

with $\{E^a\}_{a=2,\ldots,n}$ holomorphic vielbein on the Kähler-Einstein base B.

Let us recall the BPS constaints in this context, completely similar to those derived in the introduction of this thesis, in section A.II.2. The internal manifold data should satisfy:

$$\mathrm{d}\left(e^{-2\Phi}J^{n-1}\right) = 0\,, \tag{D.II.1.6a}$$

$$\mathrm{d}\left(e^{-2\Phi}\Omega\right) = 0\,. \tag{D.II.1.6b}$$

The three-form flux is given by:

$$H_3 = \star e^{2\Phi}\mathrm{d}\left(e^{-2\Phi}J^{n-2}\right), \tag{D.II.1.7}$$

and finally, the vector bundle should satisfy the zero-slope Hermite-Yang-Mills equations:

$$F\wedge J^{n-1} = 0\,, \tag{D.II.1.8a}$$

$$F^{(2,0)} = F^{(0,2)} = 0\,. \tag{D.II.1.8b}$$

As usual, one should add to this system the heterotic Bianchi identity:

$$\mathrm{d}H_3 = \frac{\alpha'}{4}\left(\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F \wedge F\right) . \tag{D.II.1.9}$$

We will also require the existence of a 'large-charge' limit, allowing to neglect consistently the term involving the curvature of the spin connection. In the case where connection $\hat{F}$ on the Kähler-Einstein base is abelian, such a large charge limit can be defined easily, similarly to the Eguchi-Hanson or conifold cases. However, it seems more difficult to define such a limit for non-abelian bundles, as we will discuss below.

In terms of our complex vielbein basis eq. (D.II.1.5), the Kähler form and holomorphic n-form read:

$$J = \frac{1}{2i}\sum_{i=1}^{n} E^i \wedge \bar{E}^{\bar{\imath}} , \tag{D.II.1.10a}$$

$$\Omega = E^1 \wedge \bigwedge_{a=2}^{n} E^a , \tag{D.II.1.10b}$$

which when injected into the BPS equations together with the gauge connection yield the following expressions for the three-form flux and dilaton in terms of the ansatz functions:

$$H_3 = -\frac{r^3 f^2 H'}{n}\varpi \wedge J_B , \tag{D.II.1.11a}$$

$$e^{2(\Phi-\Phi_0)} = H^{n-1} , \tag{D.II.1.11b}$$

as well as the following pair of non-linear first order ODEs:

$$\left(\log H^{2-n} r^{2n} f^2\right)' = \frac{2n}{r f^2} , \tag{D.II.1.12a}$$

$$\frac{r^3 f^2 H'}{n} = -\frac{\alpha'}{2}\left(\tau^2 - \mathbf{q}^2 g^2\right) . \tag{D.II.1.12b}$$

The Hermite-Yang-Mills equations allow to obtain exactly the function g:

$$g(r) = \left(\frac{a}{r}\right)^{2(n-1)} \tag{D.II.1.13}$$

completely fixed by the Hermite-Yang-Mills eqs. (D.II.1.8a) and (D.II.1.8b). We have actually made a very strong assumption concerning the bundle $\hat{E}$ over the Kähler-Einstein base, namely that in addition to being a holomorphic stable vector

bundle, its Hermite-Yang-Mills connection G also satisfies:

$$\operatorname{tr} \hat{F} \wedge \hat{F} = -\tau^2 J_B \wedge J_B \,, \tag{D.II.1.14}$$

with τ a *constant* which can be made arbitrarily large. This requirement of course originates from imposing the Bianchi identity in a large charge limit, leading to eq. (D.II.1.12b). We will suppose that this assumption is valid, leaving the proof of existence for a later work.

Case n=2:

Let us consider the particular case where $n = 2$. Let us recall that a $(2n-1)$-dimensional Sasakian manifold (M, g_M) can only satisfy the Einstein equation $\mathrm{Ric}_{g_M} = \mu g_M$ for $\mu = 2(n-1)$. In particular it has positive Ricci curvature. In the case $n = 2$, one recovers the conformally Eguchi-Hanson solution. In this case, the condition eq. (D.II.1.14) is empty since $J_B \wedge J_B$ vanishes identically on the base. One can solve the above system exactly for the warping and squashing functions. The solution for f is:

$$f(r)^2 = 1 - \left(\frac{a}{r}\right)^4 \tag{D.II.1.15}$$

allowing to reduce the equation for H to a Laplace equation on Eguchi-Hanson:

$$\Delta_{\mathrm{EH}} H(r) = -\frac{2\alpha' a^4 \mathbf{q}^2}{r^8} \,, \tag{D.II.1.16}$$

which is generically solved by the following function preserving the $SU(2) \times U(1)$ isometry:

$$H(r) = 1 + \frac{\alpha' \mathbf{q}^2}{4r^2} + \frac{\alpha' \left(\mathbf{q}^2 + C\right)}{8a^2} \log\left(\frac{(r/a)^2 - 1}{(r/a)^2 + 1}\right) \,, \tag{D.II.1.17}$$

with C an integration constant. Defining the following radial variable

$$\frac{r}{a} =: 1 + \epsilon \,, \tag{D.II.1.18}$$

one can probe the local behaviour of the solution near the horizon:

$$H\left(a(1+\epsilon)\right) \underset{\epsilon \to 0^+}{\simeq} \frac{\alpha' \left(\mathbf{q}^2 + C\right)}{8a^2} \log \epsilon \,. \tag{D.II.1.19}$$

Such a logarithmic divergence at the bolt can prevented by choosing $C = -\mathbf{q}^2$. Otherwise, the logarithmic divergence signals the presence of fivebranes smeared on

the resolution two-sphere. In the limit where the blowup parameter is sent to zero, this solution reduces to a $\mathbb{Z}_2$ orbifold of the CHS background [99].

Case n=3:

In the case of a Kähler-Einstein 2-fold B_2, one can choose for instance the following homogeneous spaces:

$$\mathbb{P}^1 \times \mathbb{P}^1 = \frac{SU(2)}{U(1)} \times \frac{SU(2)}{U(1)}\,, \tag{D.II.1.20a}$$

$$\mathbb{P}^2 = \frac{SU(3)}{U(2)}\,, \tag{D.II.1.20b}$$

the first one leading to nothing else but the cone over $T^{1,1}/\mathbb{Z}_2$. In that case, the question of the existence of a stable holomorphic vector bundle satisfying the constaint:

$$\mathrm{tr}\,\hat{F} \wedge \hat{F} = -\tau^2 J_B \wedge J_B\,, \tag{D.II.1.21}$$

is solved, and precisely corresponds to the line bundles discussed for instance in [97, 98]. The singular Ricci-flat conifold indeed has two harmonic two-forms which are inherited by the resolved geomtry by a four-cycle $\mathcal{O}(-K) \to \mathbb{P}^1 \times \mathbb{P}^1$, and which read locally:

$$\varpi_1 = \frac{1}{4}\mathrm{d}\left(\cos\theta_1 \mathrm{d}\phi_1 - \cos\theta_2 \mathrm{d}\phi_2\right)\,, \tag{D.II.1.22a}$$

$$\varpi_2 = \frac{a^2}{4\pi}\,\mathrm{d}\left(\frac{\varpi}{r^4}\right)\,. \tag{D.II.1.22b}$$

The gauge field strength in this context therefore reads:

$$F = \pi\left(\varpi_1 \mathbf{p} + \varpi_2 \mathbf{q}\right) \cdot \mathcal{I}\,. \tag{D.II.1.23}$$

Let us come back to the generic n case. It is very interesting to notice that our ansatz may actually allow for a *non-zero slope* of the stable holomorphic vector bundle $\hat{E}$ on the base B. Indeed, let us suppose that the curvature $\hat{F}$ of the Hermite-Yang-Mills connection on $\hat{E}$ satisfies:

$$\hat{F} \wedge J_B^{n-2} = \frac{c}{n-1}\, J_B^{n-1} \mathrm{id}_{\hat{E}}\,, \tag{D.II.1.24}$$

where the purely imaginary number c is proportional to the slope:

$$c = -2i\pi n \frac{\mu\left(\hat{E}\right)}{\int_B J^{n-1}} \,. \tag{D.II.1.25}$$

Let us at this stage that the slope a holomorphic vector bundle E over a complex n-fold X is defined in terms of its first Chern class by:

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} = \frac{\int_X \mathrm{c}_1(E) \wedge J^{n-1}}{\mathrm{rk}(E)} \,. \tag{D.II.1.26}$$

Starting from the ansatz eq. (D.II.1.4b), let us try and solve for g. Recall that on the cone

$$J = -\frac{1}{2}\mathrm{d}(r^2\varpi) = -r\mathrm{d}r \wedge \varpi + r^2 J_B \,, \tag{D.II.1.27}$$

implying that (only two terms survive in the Newton binomial formula):

$$J^{n-1} = r^{2n-2}\left(J_B^{n-1} - (n-1)\mathrm{d}(\log r) \wedge \varpi \wedge J_B^{n-2}\right) \,. \tag{D.II.1.28}$$

One therefore has:

$$\begin{aligned} F \wedge J^{n-1} = 0 \Leftrightarrow & \left(J_B^{n-1} - (n-1)\mathrm{d}(\log r) \wedge \varpi \wedge J_B^{n-2}\right) \wedge \\ & \wedge \left(\hat{F} + (g'\mathrm{d}r \wedge \varpi - 2gJ_B)\mathbf{q}\cdot\mathcal{I}\right) = 0 \\ \Leftrightarrow & \left[(rg' + 2(n-1)g)\mathbf{q}\cdot\mathcal{I} - c\,\mathrm{id}_{\hat{E}}\right]\mathrm{d}(\log r) \wedge \varpi \wedge J_B^{n-1} = 0 \\ \Leftrightarrow & (rg' + 2(n-1)g)\mathbf{q}\cdot\mathcal{I} - c\,\mathrm{id}_{\hat{E}} = 0 \,. \end{aligned} \tag{D.II.1.29}$$

Therefore, provided that $\mathbf{q}\cdot\mathcal{I} = \mathfrak{q}\,\mathrm{id}_{\hat{E}}$ for some charge $\mathfrak{q}$, we can solve for the function g:

$$g(r) = \left(\frac{a}{r}\right)^{2(n-1)} + \frac{c}{2(n-1)\mathfrak{q}} \,, \tag{D.II.1.30}$$

naturally generalizing eq. (D.II.1.13).

II.2 Numerical solution for vanishing slope

It is quite convenient to work with the following radial coordinate:

$$\rho := \frac{r}{a} \,. \tag{D.II.2.1}$$

We set:

$$k := \tau^2 = \mathbf{q}^2 \,. \tag{D.II.2.2}$$

Let us also define the following *large charge* expansion parameter:

$$\lambda := \frac{a^2}{\alpha' k} \,. \tag{D.II.2.3}$$

Massaging a bit eqs. (D.II.1.12a) and (D.II.1.12b), we obtain:

$$f^2 H' + \frac{n}{2\lambda}\frac{1}{\rho^3}\left(1 - \frac{1}{\rho^{4(n-1)}}\right) = 0 \,, \tag{D.II.2.4a}$$

$$n\rho^2 H\left(f^2 - 1\right) + \rho^3 f f' H + \frac{n(n-2)}{4\lambda}\left(1 - \frac{1}{\rho^{4(n-1)}}\right) = 0 \,. \tag{D.II.2.4b}$$

This system cannot be solve analytically for $n > 2$, and needs to be integrated numerically. We impose that the warping function is asymptotically constant, as one may expect from a brane solution in supergravity. One can find in sections D.II.2 and D.II.2 the plots of the numerical solutions for this system in the case $n = 3$.

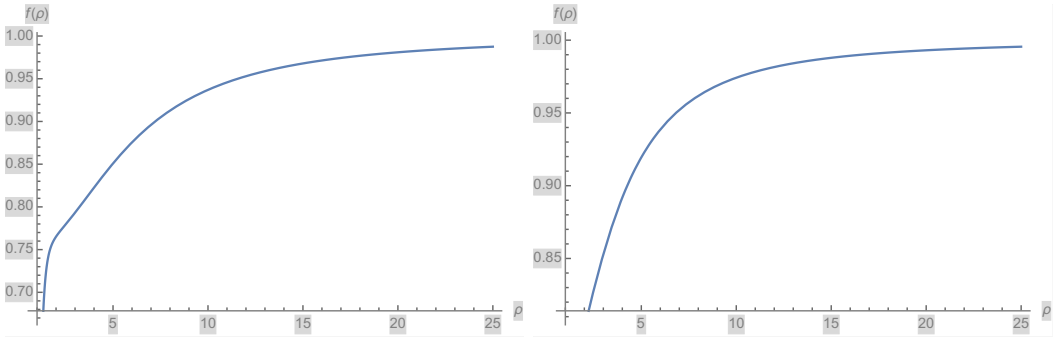


Figure II.1: Plot of the numerical solution for the squashing function $f(\rho)^2$ with parameter $\lambda = 0.0001$ on the left and $\lambda = 0.01$ on the right.

The squashing function vanishes at the bolt $\rho = 1$ and tends to the asymptotic value 1. Our solutions therefore interpolate between squashed resolved cone over Sasaki-Einstein manifolds, and a standard Ricci-flat background asymptotically, where one recovers a the Einstein orbifold $M/\mathbb{Z}_2$, the modding out by $\mathbb{Z}_2$ being required for regularity of the solution at the bolt, cf. section D.II.3 below. At finite radial coordinate, the geometry is squashed by the non-trivial profile of f, hence in non-Kähler, which as we saw corresponds to the presence of torsion.

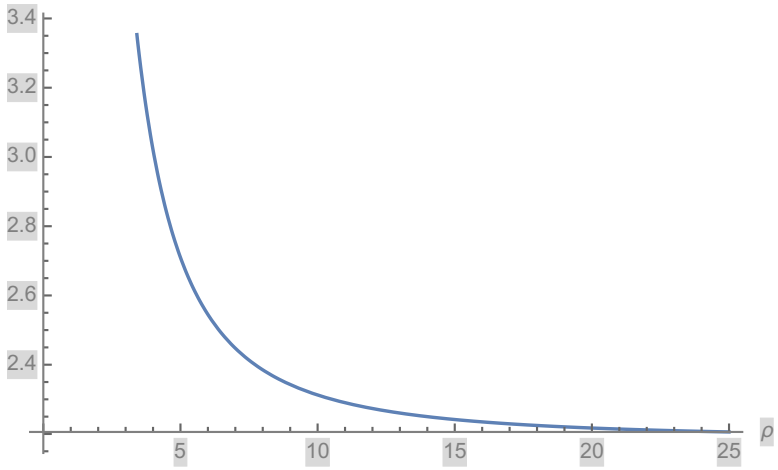


Figure II.2: Plot of the numerical solution for the warping function $H(\rho)$ with parameter $\lambda = 0.01$.

The warp factor behaves like

$$H(\rho) \underset{\rho\sim 1}{\sim} 1 + \frac{n}{\lambda\rho^2}\,, \tag{D.II.2.5}$$

near the bolt. In the blowdown limit, the conformal factor therefore diverges in the IR, the conical singularity therefore corresponds to a strong coupling singularity. This signals the presence of NS5-branes wrapping the vanishing cycle, and corresponding to the limit where the gauge bundle characterized by the charge vector $\mathbf{q}$ becomes point-like.

II.3 Regularity of the solution for non-vanishing slope

Let us check that this one-parameter generalization of the ansatz does not spoil its regularity at the bolt. Recall that we obtained the following coupled system of first order differential equations:

$$\left(\log H^{2-n} r^{2n} f^2\right)' = \frac{2n}{r f^2}\,, \tag{D.II.3.1a}$$

$$r^3 f^2 H' \propto -\tau^2 + \mathbf{q}^2 g^2\,, \tag{D.II.3.1b}$$

with τ^2 defined by:

$$\mathrm{tr}(\hat{F}\wedge\hat{F}) = -\tau^2\, J_B \wedge J_B\,. \tag{D.II.3.2}$$

The charges $\mathbf{q}$ and $\mathfrak{q}$ are related by $2\mathbf{q}^2 = \mathrm{rk}(\hat{E})\mathfrak{q}^2$. The radius at the bolt is defined by the requirement that $\lim_{r\to r_b} f(r) = 0$. Supposing that $H(r_b)H'(r_b) \neq 0$, this leads to:

$$\mathbf{q}^2 g^2(r_b) = \tau^2 \,. \tag{D.II.3.3}$$

The radius at the bolt is therefore defined as the real positive root of the following equation:

$$\mathbf{q}^2 \left(\left(\frac{a}{r_b}\right)^{2(n-1)} + \sqrt{\frac{\mathrm{rk}(\hat{E})}{2}} \frac{c}{2(n-1)||q||^2} \right)^2 = \tau^2 \,. \tag{D.II.3.4}$$

One obtains the following expression for the radius at the bolt:

$$r_b = a \left(\frac{\tau}{||q||} - \sqrt{\frac{\mathrm{rk}(\hat{E})}{8}} \frac{c}{(n-1)||q||^2} \right)^{-\frac{1}{2(n-1)}} , \tag{D.II.3.5}$$

which is real provided

$$c^2 \leq 8 \frac{\tau^2 (n-1)^2}{\mathrm{rk}(\hat{E})} \,. \tag{D.II.3.6}$$

The Bogomolov-Lübke inequality ensures that this is indeed always the case[1].

Now, rewriting eq. (D.II.3.1a) as:

$$(f^2)' = \frac{2n}{r} - \frac{2n}{r} f^2 + (n-2)\frac{H'}{H} f^2 \,, \tag{D.II.3.7}$$

gives the following limit:

$$\lim_{r\to r_b} (f^2)' = \frac{2n}{r_b} \,, \tag{D.II.3.8}$$

showing that f^2 behaves in the vicinity of the bolt as:

$$f^2(r) = 2n \, \frac{r - r_b}{r_b} + \mathcal{O}\left(\left(\frac{r - r_b}{r_b} \right)^2 \right) . \tag{D.II.3.9}$$

We see that the resolved cone metric contains a piece proportional to

$$f^{-2} \mathrm{d}r^2 + \frac{r^2 f^2}{n^2} \mathrm{d}\psi^2 \,. \tag{D.II.3.10}$$

Let us make the change of variable $\mathrm{d}\rho = f^{-1}\mathrm{d}r$. In the vicinity of the bolt, this

[1] The Bogomolov-Lübke inequality gives actually the even stronger condition $c^2 \leq \frac{\tau^2 (n-1)^2}{\mathrm{rk}(\hat{E})}$.

corresponds to setting:

$$\rho := \sqrt{\frac{2r_b(r-r_b)}{n}}\,, \tag{D.II.3.11}$$

leading again in the vicinity of the bolt to:

$$\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\psi^2\,, \tag{D.II.3.12}$$

proving the absence of conical singularity at the bolt provided the range of the angle ψ is restricted to $[0, 2\pi)$.

II.4 Near-horizon solution

One can obtain an analytic solution in a certain *double scaling limit*:

$$g_s \to 0\,, \qquad \mu_n := \frac{g_s \alpha'^{\frac{n-1}{2}}}{a^{n-1}}\,, \tag{D.II.4.1}$$

with the asymptotic string coupling defined by:

$$g_s := e^{\Phi_0} H_\infty^{\frac{n-1}{2}}\,. \tag{D.II.4.2}$$

In this limit, one isolates the physics near the horizon, by decoupling the physics in the IR from the asymptotic region and fixing the heterotic fivebrane tension μ_n. One obtains:

$$H(\rho) = \frac{n}{\lambda \rho^2}\,, \tag{D.II.4.3a}$$

$$f(\rho) = \frac{1}{2}\sqrt{1 - \frac{1}{\rho^{4(n-1)}}}\,, \tag{D.II.4.3b}$$

leading to the following line element:

$$\mathrm{d}s^2_{1,9} = \mathrm{d}s^2_{1,9-2n} + \frac{n\alpha' k}{\rho^2}\left\{\frac{\mathrm{d}\rho^2}{\frac{1}{4}\left(1 - \frac{1}{\rho^{4(n-1)}}\right)} + \rho^2\left[\mathrm{d}s^2_B + \frac{1}{4n^2}\left(1 - \frac{1}{\rho^{4(n-1)}}\right)\varpi^2\right]\right\}. \tag{D.II.4.4}$$

The dilaton reads:

$$e^{\Phi} = \frac{\mu_n}{H_\infty^{\frac{n-1}{2}}}\frac{k^{\frac{n-1}{2}}}{\rho^{n-1}}\,, \tag{D.II.4.5}$$

and the three-form flux:

$$H_{(3)} = -\frac{\alpha' k}{4}\left(1 - \frac{1}{\rho^{4(n-1)}}\right)\varpi \wedge J_B\,. \tag{D.II.4.6}$$

This near-horizon solution (in the case $n = 3$) coincides nicely for values of ρ close to 1 with the asymptotically flat supergravity solution found numerically in the previous section, provided the large charge expansion parameter λ is small compared to 1.

We observe that the blow-up mode a was completely absorbed in the double-scaling parameter μ_n, similarly to what happens for Eguchi-Hanson [96] and the conifold [97, 98]. The double-scaling parameter therefore defines the effective coupling in the double-scaling limit.

Holography:

Let us make a quick comment which would require more work to be developped further. We see that both in the asymptotically Ricci-flat case and in the double-scaling limit solution, the dilaton field is asymptotically linear, suggesting the existence of a holographic description of these supergravity solutions, as argued in [100], and generalizing the standard heterotic fivebrane setup [101]. The dual theory should be a $\mathcal{N} = 1$ *little string theory* [102] living on the extended dimensions of the worldvolume of the fivebranes wrapped on the resolution cycle. These fivebranes can in some sense be considered in analogy with fractional $D3$-branes in the type II solutions.

II.5 Towards solving the Bianchi identity

Let us now discuss some ideas towards solving the existence question of stable holomorphic vector bundles over the Kähler-Einstein B which we now take to be of complex dimension n. In the following we drop the tildes above the bundle data on B, we also drop the subscript B on the fundamental form J_B.

In order to tackle the question of whether a holomorphic vector bundle E (with stucture group G) over a Kähler-Einstein projective base B, equipped with a Hermite-Yang-Mills connection A, further satisfying the very restrictive property that its second Chern character is proportional to $J \wedge J$, where J is the fundamental form on B. We denote by d_A the covariant exterior derivative (which locally reads $\mathrm{d}_A = \mathrm{d} + A$)

and $F = \mathrm{d}_A A$ the curvature of the connection. In the following we will write $\Omega^{p,q} := \bigwedge^{p,q} T^\star_X$ for the bundle of (p,q)-forms on X.

Let us introduce the Lefschetz operator:

$$\begin{aligned} L : \Omega^{p,q}(X) &\to \Omega^{p+1,q+1}(X) \\ \alpha &\to J \wedge \alpha \,. \end{aligned} \tag{D.II.5.1}$$

Let us also introduce the dual Lefschetz operator $\Lambda = *^{-1} L *$. We recall that a (p,q)-form ω is said to be primitive if it is orthogonal to the Kähler form, namely $\Lambda(\omega) = 0$, i.e. it is not of the form $L(\rho)$ for some $(p-1,q-1)$-form ρ.

As stated above, we equip the holomorphic vector bundle E with a Hermite-Einstein structure. The curvature F therefore satisfies:

$$J \lrcorner F = c \, \mathrm{id}_E \,, \tag{D.II.5.2}$$

with c a purely imaginary number proportional to the slope of E:

$$c = -2i\pi \, n \, \frac{\mu(E)}{\int_B J^n} \,, \tag{D.II.5.3}$$

where the slope is defined by eq. (D.II.1.26):

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} = \frac{\int_B \mathrm{c}_1(E) \wedge J^{n-1}}{\mathrm{rk}(E)} \,. \tag{D.II.5.4}$$

The slope therefore quantifies the non-primitivity of the curvature.

Let us split F into a primitive part $\tilde{F}$ and a non-primitive part:

$$F = \tilde{F} + \alpha J \, \mathrm{id}_E \,. \tag{D.II.5.5}$$

Applying Λ on both sides and using eq. (D.II.5.2) then fixes $\alpha = c/n$. One therefore has:

$$\begin{aligned} F \wedge F &= \left(\tilde{F} + \alpha \, J \, \mathrm{id}_E\right) \wedge \left(\tilde{F} + \alpha \, J \, \mathrm{id}_E\right) \\ &= \tilde{F} \wedge \tilde{F} + 2\alpha \, J \wedge \tilde{F} + \alpha^2 \, J \wedge J \, \mathrm{id}_E \,, \end{aligned} \tag{D.II.5.6}$$

leading for the trace to the following quadratic polynomial in the slope:

$$-8\pi^2 \mathrm{ch}_2(E) = \mathrm{tr}\left(\tilde{F} \wedge \tilde{F}\right) + 2\alpha \, J \wedge \mathrm{tr}\left(\tilde{F}\right) + \mathrm{rk}(E) \, \alpha^2 \, J \wedge J \,. \tag{D.II.5.7}$$

The second term in eq. (D.II.5.7) belongs to $\Gamma\left(B, \mathrm{P}^{1,1}(B) \otimes \mathrm{End}(E)\right)$, where $\mathrm{P}^{p,q}(B)$ denotes the bundle of primitive (p,q)-forms. The first term belongs to $\Gamma\left(B, \mathrm{diag}\left(\mathrm{P}^{1,1}\right) \otimes \mathrm{End}(E)\right)$, where the sections of $\mathrm{diag}\left(\mathrm{P}^{q,q}(B)\right)$ are of the type $\omega \wedge \omega$, with ω a section of $\mathrm{P}^{q,q}(B)$. Therefore, no accidental cancellation of these two terms may occur.

Let us mention that we haven't used the fact that B is Kähler-Einstein, namely that the curvature of the Levi-Civita connection on the holomorphic tangent bundle itself satisfies:

$$J \lrcorner R = \lambda \, \mathrm{id}_{T_B} \,, \tag{D.II.5.8}$$

for some purely imaginary constant λ, proportional to the slope of the holomorphic tangent bundle T_B.

Let us now introduce the degree operator H:

$$H = \sum_{k=0}^{2n} (k-n) p_k \,, \tag{D.II.5.9}$$

where $p_k : \bigwedge^\bullet T_B^\star \to \bigwedge^k T_B^\star$ is the projection on the k^{th} pure component. The triplet (L, Λ, H) then satisfy the $\mathfrak{sl}_2(\mathbb{C})$ commutation relations. Let us therefore decompose the first term of eq. (D.II.5.7) into its irreducible components:

$$\mathrm{tr}\left(\tilde{F} \wedge \tilde{F}\right) := \beta_4 + J \wedge \beta_2 + \beta_0 J \wedge J \,, \tag{D.II.5.10}$$

where β_4,β_2 and β_0 are respectively a primitive $(2,2)$-form, $(1,1)$-form and function. Translating the familiar representation theory of $\mathfrak{sl}_2(\mathbb{C})$ to the present reducible representation on the exterior algebra $\bigwedge^\bullet T_B^\star$, we can see that irreducible representations, labelled by j, are given by:

$$\begin{aligned} 0 \leq j \leq \frac{n}{2} \,, \\ -2j + n \leq k \leq 2j + n \,, \end{aligned} \tag{D.II.5.11}$$

with j jumping by half-integer units, and k jumping by even integer units (representing the degree of the form). Therefore, $\mathfrak{sl}_2(\mathbb{C})$ representation theory gives us the following decomposition into irreducible representations in various dimensions :

$$n = 2 : \;\; \Omega^{2,2} = L^2\left(P^{0,0}\right) \,, \tag{D.II.5.12a}$$

$$n = 3 : \;\; \Omega^{2,2} = L\left(P^{1,1}\right) \oplus L^2\left(P^{0,0}\right) \,. \tag{D.II.5.12b}$$

Indeed, let us see what happens for $n = 3$. In this case, one has the following irreducible representations:

$$\begin{aligned} j = 0 &: 3 \leq k \leq 3\,, \\ j = 1/2 &: 2 \leq k \leq 4\,, \\ j = 1 &: 1 \leq k \leq 5\,, \\ j = 3/2 &: 0 \leq k \leq 6\,. \end{aligned} \tag{D.II.5.13}$$

Therefore collecting the various $k = 4$ components, we get:

$$\Omega^4 = L\left(P^2\right) \oplus L^2\left(P^0\right)\,. \tag{D.II.5.14}$$

Moreover, since the decomposition respects the Dolbeault grading, we indeed get eq. (D.II.5.12a). Combining eq. (D.II.5.10) and eq. (D.II.5.7), one therefore obtains that:

$$n = 2: \quad -8\pi^2 \mathrm{ch}_2(E) = \left(\beta_0 + \mathrm{rk}(E)\alpha^2\right) J \wedge J\,, \tag{D.II.5.15a}$$

$$n = 3: \quad -8\pi^2 \mathrm{ch}_2(E) = \left(\beta_0 + \mathrm{rk}(E)\alpha^2\right) J \wedge J + \left(\beta_2 + 2\alpha \mathrm{tr}\tilde{F}\right) \wedge J\,, \tag{D.II.5.15b}$$

where β_0 and β_2 are "Clebsch-Gordan coefficients" obtained from $\tilde{F}$.
Therefore, we see that in the case of a Calabi-Yau 3-fold cone X, we have the second Chern character of the vector bundle $\mathrm{ch}_2(E)$ is proportional to $J \wedge J$, and can be made arbitrarily large provided that we can construct holomorphic stable vector bundles of arbitrarily large degree on a Kähler-Einstein surface.

If we therefore decide to focus on complex surfaces, we then see with a bit of $\mathfrak{sl}_2(\mathbb{C})$ representation theory that

$$-8\pi^2 \mathrm{ch}_2(E) = \mathrm{tr}(F \wedge F) \propto J \wedge J\,, \tag{D.II.5.16}$$

regardless of the fact that the slope is null or not. Provided we restrict ourselves to surfaces which are algebraic, we can then invoke the main result of Li and Qin in [103], which ensures the nonemptiness of the moduli space of slope-stable rank-r vector bundle with first Chern class c_1 and large enough second Chern class c_2 (and with determinant given by a line bundle L used to make the construction), hence ensuring the existence of some sort of "large charge limit", even in the non-abelian case. Once again, this is independent of the value of the slope. Slope considerations may enter into the game for higher dimensional base, for which the decomposition

of forms into irreducible $\mathfrak{sl}_2(\mathbb{C})$ components gives less sharp results. One direction could be to exploit the polynomial structure in the slope of eq. (D.II.5.7), and restrict to large-slope bundle in order to suppress the terms which are not proportional to $J \wedge J$. Remains however open the question of finding such bundles further satisfying the property that the second Chern character is proportional to $J \wedge J$ with *constant* proportionality factor, i.e. not a function of the base coordinates.

An approach to study this question could be to restrict ourselves to homogeneous spaces for the Kähler-Einstein base and homogeneous vector bundles over it [104–106].

More generically, this question can be rephrased in terms of the obtruction for a Chern class form to be harmonic. There is a large mathematical litterature where this kind of questions arise, and the good tools to look at seem to be the so-called Futaki invariants and K-energy functionals [107,108]. This subtle question is left for later work.

II.6 Towards a GLSM description

This section corresponds to a very preliminary work on the worldsheet aspects of the above describe heterotic solutions.

It would be extremely interesting to extend the analysis of [96, 97] in order to provide an exact worldsheet CFT description of some new heterotic backgrounds described above. Such techniques should be extendable for instance in the case of a Kähler-Einstein base

$$\frac{SU(3)}{U(1)^2}\,. \tag{D.II.6.1}$$

We will try to explore such worldsheet CFT description in the near future.

Independently of knowing or not the exact worldsheet CFT underlying the heterotic flux backgrounds of interest, and drawing lessons from the previous parts of the thesis, it would be extremely interesting to exhibit a gauged linear sigma model description, since it could capture at least part of the physics.

In particular, these GLSMs could allow to compute topological quantities such as the elliptic genus, even without knowing the exact CFT.

II.6.1 Toric realization of Eguchi-Hanson $\mathcal{O}_{\mathbb{P}^1}(-2)$

We consider a 2-dimensional a fairly standard $U(1)$ gauge theory in $(0,2)$ superspace with chiral superfields X_1, X_2, X_0, P, Φ and Fermi superfields $\Lambda, \Gamma, \Gamma_\Phi$. The

pair (Φ, Γ_Φ) is a couple of spectator fields, becoming massive in the IR, and whose role is described in section B.I.2. Apart from the kinetic terms and minimal coupling of the matter fields to the gauge multiplet, the Lagrangian contains the following superpotential:

$$\mathcal{L}_{\text{J}} = \int \mathrm{d}\theta \left(\frac{1}{\sqrt{2}} \Gamma P X_0 + m_\Phi \Phi \Gamma_\Phi \right) , \tag{D.II.6.2}$$

the idea being that when the field X_0 develops a vev, the fields P and Γ become massive, with a mass large compared to the gauge coupling e. Once integrated out [109], the effective action at one-loop contains a term of the form

$$\frac{i\,(Q_P - Q_\gamma)}{16\pi} \int \mathrm{d}^2\sigma \mathrm{d}\theta\, \Upsilon \log(X_0)\,, \tag{D.II.6.3}$$

which is precisely the type of field-dependent Fayet-Iliopoulos coupling leading to a non-trivial torsion flux.

We impose the following charge assignement to the fields:

	X_1	X_2	X_0	Λ	P	Γ	Φ	Γ_Φ
$U(1)$	1	1	-2	q	Q	$2-Q$	$-Q$	Q
$U(1)_{\text{L}}$	0	0	0	1	$q/2$	$-q/2$	0	0
$U(1)_{\text{R}}$	0	0	0	0	1	0	1	0

(D.II.6.4)

where

$$Q = \frac{q^2}{4} - \frac{1}{2}\,. \tag{D.II.6.5}$$

We recall that $U(1)_{\text{R}}$ flow in the IR to the R-symmetry of the right-moving $N = 2$ superconformal algebra. The complete freedom on the choice of charge q corresponds from the worldsheet point of view to the existence of a large charge limit. This charge assignements solves the whole set of anomaly cancellation conditions, is such that the superpotential is classically invariant under the various $U(1)$ symmetries and gives $\bar{c} = 6$ for the right moving central charge, and $r = 1$ for the rank. The left-moving central charge is given by $c = \frac{2}{3}\bar{c} + r = 5$. We again refer the reader to section B.I.2 for the generic discussion on $(0, 2)$ GLSMs.

By studying the vacuum manifold of this theory defined by the zero locus of the D-term constraint modded out by the action of gauge group, one sees that this GLSM describes in the IR of the large FI parameter phase a NLSM with target a rank 1 vector bundle over $T^\star_{\mathbb{P}^1}$, as required.

II.6.2 ALE elliptic genus

Let us proceed to the computation of the elliptic genus, for which we can rely on [33]. The full one loop determinant reads:

$$\Sigma_{1-\text{loop}} = -\eta^4 \frac{\theta_1\left(qu+z\right)\theta_1\left((2-Q)u-\frac{q}{2}z\right)}{\theta_1\left(u\right)^2\theta_1\left(2u-z\right)\theta_1\left(-Qu-\frac{q}{2}z\right)}\,, \tag{D.II.6.6}$$

where we dropped the modular argument of the odd Jacobi theta functions. We compute the elliptic genus in the phase defined by the fields P and X_0, i.e. we choose in $\mathcal{M}^-_{\text{sing}}$ (as in eq. (B.II.3.26)), and we explicitely perform the contour integral in the u-plane. The poles are located at:

$$u = \frac{k+\tau l+z}{2}\,, \qquad k,l\in\{0,1\}\,, \tag{D.II.6.7a}$$

$$u = \frac{k+\tau l+\frac{q}{2}z}{|Q|}\,, \qquad k,l\in\{0,\dots,|Q|-1\}\,. \tag{D.II.6.7b}$$

Recall that that we have the following residue:

$$\oint_{u=k+\tau l}\mathrm{d}u\frac{1}{\theta_1(u)} = \frac{i(-1)^{k+l}e^{i\pi l^2\tau}}{\eta^3}\,, \tag{D.II.6.8}$$

from which we finally obtain the following holomorphic expression:

$$\begin{aligned} Z_{\text{ell}} = i\eta(\tau)\Bigg\{&\frac{1}{2}\sum_{k,l=0}^{1}(-1)^{k+l+1}e^{i\pi l^2\tau}\,\frac{\theta_1\left(\frac{q}{2}(k+\tau l+z)+z\right)\theta_1\left(\frac{2-Q}{2}(k+\tau l+z)-\frac{q}{2}z\right)}{\theta_1\left(\frac{k+\tau l+z}{2}\right)^2\theta_1\left(\frac{Q}{2}(k+\tau l+z)-\frac{q}{2}z\right)}+\\ &+\frac{1}{|Q|}\sum_{k,l=0}^{|Q|-1}(-1)^{k+l+1}e^{i\pi l^2\tau}\,\frac{\theta_1\left(\frac{q}{|Q|}(k+\tau l+\frac{q}{2}z)+z\right)\theta_1\left(\frac{2-Q}{|Q|}(k+\tau l+\frac{q}{2}z)-\frac{q}{2}z\right)}{\theta_1\left(\frac{k+\tau l+\frac{q}{2}z}{|Q|}\right)^2\theta_1\left(\frac{2}{|Q|}(k+\tau l+\frac{q}{2}z)-z\right)}\Bigg\}\,. \end{aligned} \tag{D.II.6.9}$$

Alternatively, it would be extremely interesting to write a GLSM describing the double-scaling limit of the warper Eguchi-Hanson solution. It would allow to make the connection with the existing litterature discussing the non-compact elliptic genera of the corresponding CFT and their interesting relations to Mock modular forms, see [58, 59, 110, 111] for instance.

II.6.3 Toric realisation of the resolved conifold $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-K)$

We consider a 2-dimensional $U(1)^2$ gauge theory in $(0,2)$ superspace with chiral superfields $X_1, X_2, X_3, X_4, X_0, P, \Phi$ and Fermi superfields $\Lambda_a, \Gamma, \Gamma_\Phi$. The pair (Φ, Γ_Φ) is a couple of spectator fields, becoming massive in the IR. As in the case of warped Eguchi-Hanson, the Lagrangian contains the following superpotential:

$$\mathcal{L}_{\text{J}} = \int \mathrm{d}\theta \left(\frac{1}{\sqrt{2}} \Gamma P X_0 + m_\Phi \Phi \Gamma_\Phi \right) . \tag{D.II.6.10}$$

Mimicking the Eguchi-Hanson case, we consider the following charge assignement:

	X_1	X_2	X_3	X_4	X_0	Λ_a	P	Γ
$U(1)$	1	1	-1	-1	0	q_a	0	0
$\widetilde{U(1)}$	1	1	0	0	-2	$\tilde{q}_a$	$\tilde{Q}$	$2-\tilde{Q}$
$U(1)_{\text{L}}$	0	0	0	0	0	Q_a	$\frac{1}{2}\sum_a \tilde{q}_a Q_a$	$-\frac{1}{2}\sum_a \tilde{q}_a Q_a$
$U(1)_{\text{R}}$	0	0	0	0	0	0	1	0

(D.II.6.11)

where:

$$\tilde{Q} = \frac{1}{4}\sum_a \tilde{q}_a^2 - \frac{1}{2} . \tag{D.II.6.12}$$

The charges of the Λ_a multiplets should satisfy the following Diophantine equations:

$$\sum_a q_a^2 = 4 , \quad \sum_a q_a \tilde{q}_a = 2 , \quad \sum_a q_a Q_a = 0 , \quad \sum_a Q_a^2 = 2 , \tag{D.II.6.13}$$

imposing the minimal number of Fermi multiplets Λ_a to be 3. One example of charge assignement satisfying the above conditions is:

$$\begin{aligned} (q_1, q_2, q_3) &= (0, 0, 2) , \\ (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3) &= (\tilde{q}_1, \tilde{q}_2, 1) , \\ (Q_1, Q_2, Q_3) &= (1, 1, 0) . \end{aligned} \tag{D.II.6.14}$$

One can check that the above charge assignement leads to the right central charges, i.e. $(c, \bar{c}) = (6 + r, 9)$, with a rank $r = 2$ gauge bundle, and is such that the various anomalies and mixed anomalies vanish.

II.6.4 Asymptotically Ricci-flat elliptic genus

We compute the elliptic genus of the model described above using the result obtained in [34] extending the computation of the elliptic genus for a higher rank worldsheet gauge group and giving the index as a sum of Jeffrey-Kirwan residues. The refer the reader to the later article for details, especially concerning the definition of the Jeffrey-Kirwan residue.

One introduces the following charge vectors:

$$\mathsf{Q}_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathsf{Q}_{3,4} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \mathsf{Q}_0 = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad \mathsf{Q}_P = \begin{pmatrix} 0 \\ \tilde{Q} \end{pmatrix}, \tag{D.II.6.15}$$

all belonging to the dual $\mathfrak{h}^*$ of the Cartan algebra $\mathfrak{h} \simeq \mathbb{R}^2$, and defining the various 'phases' of the GLSM, cf. fig. II.3.

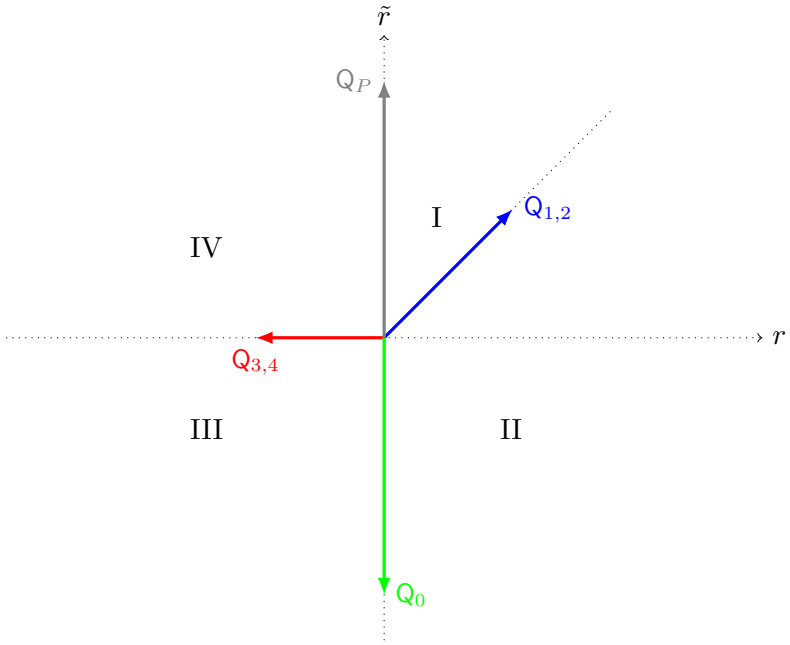


Figure II.3: 'Phases' of the GLSM

The constraints obtained from D-terms are:

$$|X_1|^2 + |X_2|^2 - |X_3|^2 - |X_4|^2 = r \tag{D.II.6.16}$$

$$|X_1|^2 + |X_2|^2 - 2|X_0|^2 + \tilde{Q}|P|^2 = \tilde{r}\,, \tag{D.II.6.17}$$

or taking the difference of the two equations:

$$|X_1|^2 + |X_2|^2 - 2|X_0|^2 + \tilde{Q}|P|^2 = \tilde{r}\,, \tag{D.II.6.18}$$

$$|X_3|^2 + |X_4|^2 - 2|X_0|^2 + \tilde{Q}|P|^2 = \tilde{r} - r\,, \tag{D.II.6.19}$$

from which we see that the phases II and III have the interpretation of geometric phases. The one-loop determinant associated to a $(0,2)$ chiral multiplet X_i and Fermi multiplet Λ_a are respectively:

$$Z_{X_i}(\tau,u,\tilde{u},z) = i\,e^{-\frac{\pi}{\tau_2}(\upsilon^2-\upsilon\bar{\upsilon})}\,\frac{\eta(\tau)}{\theta_1(\tau|\upsilon)}, \qquad \upsilon = q_i u + \tilde{q}_i\tilde{u} + Q_i z\,, \tag{D.II.6.20}$$

$$Z_{\Lambda_a}(\tau,u,\tilde{u},z) = i\,e^{\frac{\pi}{\tau_2}(\upsilon^2-\upsilon\bar{\upsilon})}\,\frac{\theta_1(\tau|\upsilon)}{\eta(\tau)}, \qquad \upsilon = q_a u + \tilde{q}_a\tilde{u} + Q_a z\,. \tag{D.II.6.21}$$

In the case of an abelian worldsheet gauge group of rank n, the contribution from the $(0,2)$ gauge multiplet $(\mathcal{V},\mathcal{A})$ is:

$$Z_{\mathcal{V},\mathcal{A}}(\tau,u,\tilde{u},z) = \Big(-2i\pi\eta(\tau)^2\Big)^n \bigwedge_{k=1}^{n} \mathrm{d}u_k\,. \tag{D.II.6.22}$$

Putting all this together, one obtains for the full one-loop determinant:

$$\Sigma_{\text{1-loop}} = -i\,(2i\pi)^2\,\eta^5\,\frac{\theta_1\Big((2-\tilde{Q})\tilde{u}\Big)\prod_a \theta_1\left(q_a u + \tilde{q}_a\tilde{u} + Q_a z\right)}{\theta_1\left(u+\tilde{u}\right)^2\theta_1\left(-u\right)^2\theta_1\left(-2\tilde{u}\right)\theta_1\Big(\tilde{Q}\tilde{u} + \frac{1}{2}\sum_a \tilde{q}_a Q_a\, z\Big)}\,\mathrm{d}u\wedge\mathrm{d}\tilde{u}\,, \tag{D.II.6.23}$$

where we dropped everywhere the first argument of the odd Jacobi theta functions for clarity. The different chiral superfields define the following singular hyperplanes:

$$H_{1,2} = \left\{\,(u,\tilde{u})\in\mathfrak{h}_{\mathbb{C}} \mid u+\tilde{u}=0\right\}, \tag{D.II.6.24}$$

$$H_{3,4} = \left\{\,(u,\tilde{u})\in\mathfrak{h}_{\mathbb{C}} \mid -u=0\right\}, \tag{D.II.6.25}$$

$$H_{0} = \left\{\,(u,\tilde{u})\in\mathfrak{h}_{\mathbb{C}} \mid -2\tilde{u}=0\right\}, \tag{D.II.6.26}$$

$$H_{P} = \left\{\,(u,\tilde{u})\in\mathfrak{h}_{\mathbb{C}} \mid \tilde{Q}\tilde{u} + \frac{1}{2}\sum_a \tilde{q}_a Q_a\, z = 0\right\}. \tag{D.II.6.27}$$

A real slice of this hyperplane arrangement is represented in fig. II.4 for the case where $\tilde{Q}=2$.

The elliptic genus reads:

$$Z_{\text{ell}}(\tau,z) = \sum_{u^\star\in\mathcal{M}_{\text{sing}}} \underset{u=u^\star}{\text{JK-Res}}\left(\mathsf{Q}(u^\star),\eta\right)\,\Sigma_{\text{1-loop}}\,, \tag{D.II.6.28}$$

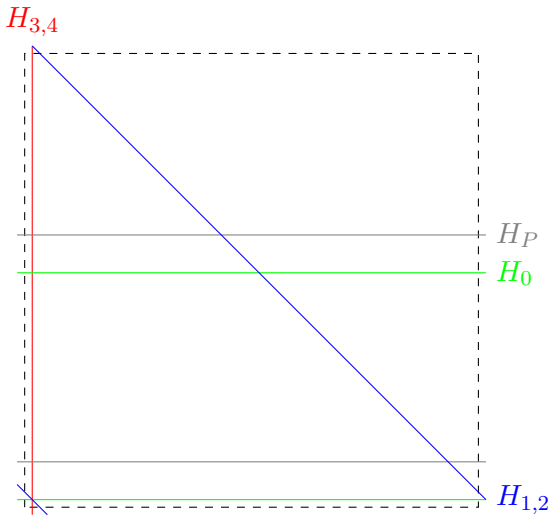


Figure II.4: Real slice of the hyperplane arrangement. Identifications by shift of elements of the worldsheet torus lattice are understood.

where $\mathsf{Q}(u^\star)$ is the set of charges giving the pole u^*, and η is a vector in $\mathfrak{h}^*$ indicated in which phase the index is computed. The result should not depend on the cone in which η belongs, or subchamber in a given cone.

One can for instance consider the phase IV, defined y the hyperplanes H_P and $H_{3,4}$ (cf. fig. II.3) for definiteness. This phase has the advantage to be non degenerate, in the sense that two and only two hyperplanes meet at each pole. One obtains the final expression:

$$Z_{\text{ell}} = \eta^2 \sum_{k,l=0}^{\tilde{Q}-1} (-1)^{k+l} e^{i\pi l^2 \tau} \times \\ \times \left\{ \frac{\theta_1\left((2-\tilde{Q})\tilde{u}\right)}{\theta_1(-2\tilde{u})} \oint \mathrm{d}u \, \frac{\prod_a \theta_1(q_a u + \tilde{q}_a \tilde{u} + Q_a z)}{\theta_1(u+\tilde{u})^2 \, \theta_1(-u)^2} \right\} \Bigg|_{\tilde{u}=\frac{1}{\tilde{Q}}\left(k+\tau l - \frac{1}{2}\sum_a \tilde{q}_a Q_a z\right)} . \tag{D.II.6.29}$$

The residue at the second order poles $u = -\frac{1}{\tilde{Q}}\left(k + \tau l - \frac{1}{2}\sum_a \tilde{q}_a Q_a z\right)$ can then be computed exactly. In the same spirit as the derivation by localization of section B.II.3, the result computed in the various phases actually gives different representations of the same object.

Once again, it would be very interesting to exhibit a GLSM descibing the double-scaling limit solution.

Conclusion and outlook

We were interested in this thesis to various aspects of heterotic compactifications with torsion flux. In part B we defined and computed the dressed elliptic genus for $\mathcal{N} = 2$ Fu-Yau compactifications. It would be extremely interesting to study in more details the mathematical aspects related to the dressed elliptic genus, in particular its nature as the generated function for the indices of some Dirac-like operator graded by the left momentum p_{L}. It would also be interesting to understand better its nature as a modular object, since even though it's shape is can be somehow reminiscent of what is known in the literature as skew-holomorphic Jacobi forms, as defined by Skoruppa [76], it fails to be one of these by not satisfying a heat equation.

A direction towards such a better mathematical understanding would be to study deeper the interplay between the two self-dual lattices Γ_{L} and $\text{Pis}(\mathcal{S})$. We initiated this study in section II.4.3. We also believe that such a better understanding may be of some interest in what concerns the threshold corrections to the gauge and gravitational couplings in the low energy 4-dimensional supergravity action that we computed in chapter C.II, maybe by allowing to simplify their computation is the spirit of what we did in section II.4.2, but this time for a generic model.

It would be interesting to better determine the moonshine properties possibly hidden in the Fu-Yau dressed elliptic genus. In particular, it would be interesting to understand better if the latter could be understood as some sort of twining partition function, beyond its similarity with the results of Gaberdiel [45]. A non negligible obstacle to this program is the explicit momentum depence of all the integer coefficients. One could therefore inspect the expression of the coefficients for a specific model, and a given point in Γ_{L}.

As we already mentioned, genuine compact $SU(3)$-structure vacua of the heterotic string a quite rare. It would be very interesting to be able to discover new solutions to the BPS system of equations and Bianchi identity, for instance by trying to fiber higher-genus Riemann surfaces over positive curvature compact algebraic

surfaces. The absence of continuous isometries of higher genus Riemann surfaces seems to forbid a GLSM construction *à la* Adams and collaborators, but constuctions as in [109, 112] involving logarithmic Fayet-Iliopoulos couplings may provide a different construction.

We exposed in part D work in progress concerning new non-compact heterotic flux solutions. As a step towards truly giving them the status of genuine heterotic flux backgrounds, one should completely solve the existence problem of stable holomorphic vector bundles over the Kähler-Einstein base. As we already stated, we expect that the generic answer will involve some Futaki invariants-type reasoning. This topic being quite involved, it will require more work. Therefore, at least exhibiting one bundle satisfying thegood requirement would already be satisfactory. Extension of line bundles may allow to build non-abelian bundles on the $T^{1,1}$ cone for instance.

An obvious direction would be to try and generalize the worldsheet CFT constructions in the double-scaling limit of the warped Eguchi-Hanson and cone over $T^{1,1}$ [96–98], allowing to elevate the solution to a genuine heterotic string solution, even outside the large charge regime. As we tried to sketch out in section D.II.6, it would be very interesting to write out $(0,2)$ GLSMs flowing in the IR to the geometries of part D, in particular GLSMs describing the geometry in the doubled scaling limit, and to compute by localization the corresponding non-holomorphic elliptic genus, which we expect to be Mock modular. We do not expect however these GLSMs to be related directly to those introduced in section D.II.6, since no marginal deformation allows to deform their respective IR superconformal fixed points into one another. A first step would therefore be to understand better the warped Eguchi-Hanson case in the doule-scaling limit, before trying to generalize. We expect that a construction *à la* Hori and Kapustin [113] using compensator fields and extending the GLSM for the *cigar* $SL_2(\mathbb{R})/U(1)$ may be a good direction.

Another work in progress involves another class of heterotic flux solutions on non-compact hyper-Kähler manifolds, for instance on generic ALE and ALF Gibbons-Hawking spaces, as well as solutions with a $\mathbb{R}^{1,1}$ factor, typically on a product of hyper-Kähler 4-folds, and allowing for the inclusion of fundamental strings along $\mathbb{R}^{1,1}$ sourcing electrically the three-form flux.

In the completely different perspective of studying more abstractly the moduli of heterotic compactifications, different approaches have been adopted. Among them, some extension of the Atiyah algebroid [114, 115] was constructed in order to account

for the presence of the Bianchi identity. Another approach, which coincides with the former provided some topological restrictions are met rather has a generalized geometry flavor [116]. On another hand, recent techniques have been developed in order to broadly generalize the type of sigma models one can construct [117], by allowing for theories with target algebroid-type geometrical structures. Very hypothetically and somehow marrying the above developments on sigma models and heterotic moduli could shed some light on the latter by probing the geometrical structures underlying the Hull-Strominger system with an adapted sigma model.

Part E

Appendices

Appendix I

Modular forms and hypergeometric functions

I.1 Poisson resummation, Theta functions and modular forms

For a general n-dimensional lattice Γ, with A a symmetric positive definite $n \times n$ matrix defining its bilinear form, one has the Poisson resummation formula

$$\sum_{p\in\Gamma} e^{-\pi(p+x)\cdot A(p+x)+2i\pi y\cdot(p+x)} = \frac{1}{\mathrm{vol}(\Gamma)\sqrt{\det A}} \sum_{p\in\Gamma^\vee} e^{-2i\pi p\cdot x-\pi(y+p)\cdot A^{-1}(y+p)}\,, \tag{E.I.1.1}$$

which reduces in the case $\Gamma = \mathbb{Z}$ to

$$\sum_{n=-\infty}^{\infty} e^{-\pi a n^2+2i\pi bn} = \frac{1}{\sqrt{a}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{a}(n-b)^2}\,. \tag{E.I.1.2}$$

The Dedekind eta function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)\,, \tag{E.I.1.3}$$

with $q = e^{2i\pi\tau}$. Its modular properties are

$$\eta(\tau+1) = e^{i\frac{\pi}{12}}\eta(\tau)\,, \tag{E.I.1.4a}$$

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau)\,. \tag{E.I.1.4b}$$

The Jacobi theta functions with characteristics are defined by

$$\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau|u)=\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}(n+\frac{a}{2})^2}e^{2i\pi(n+\frac{a}{2})(u+\frac{b}{2})}\,. \tag{E.I.1.5}$$

One defines

$$\theta_1=-\theta\begin{bmatrix}1\\1\end{bmatrix}\quad\theta_2=\theta\begin{bmatrix}1\\0\end{bmatrix}\quad\theta_3=\theta\begin{bmatrix}0\\0\end{bmatrix}\quad\theta_4=\theta\begin{bmatrix}0\\1\end{bmatrix}\,. \tag{E.I.1.6}$$

One can rewrite the Jacobi theta functions in terms of an infinite product. In particular, θ_1 writes:

$$\theta_1(\tau|u)=-iq^{\frac{1}{8}}w^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)\,(1-wq^n)\left(1-w^{-1}q^{n-1}\right)\,, \tag{E.I.1.7}$$

with $w:=\exp(2i\pi u)$. One has the following properties

$$\frac{\partial}{\partial u}\theta_1(\tau|u)|_{u=0}=2\pi\eta(\tau)^3\,, \tag{E.I.1.8a}$$

$$\theta_1(\tau|-u)=-\theta_1(\tau|u)\,, \tag{E.I.1.8b}$$

$$\oint_{u=k+\tau l}\frac{\mathrm{d}u}{2i\pi}\,\frac{1}{\theta_1(\tau|u)}=i\frac{(-1)^{k+l}e^{i\pi l^2\tau}}{\eta(\tau)^3}\,. \tag{E.I.1.8c}$$

Under modular transformations, the Jacobi theta functions transform as

$$\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau+1|u)=e^{-\frac{i\pi}{4}a(a-2)}\theta\left[\begin{smallmatrix}a\\a+b-1\end{smallmatrix}\right](\tau|u)\,, \tag{E.I.1.9a}$$

$$\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\left(-\frac{1}{\tau}\middle|\,\frac{u}{\tau}\right)=-\sqrt{-i\tau}e^{\frac{i\pi}{2}ab+\frac{i\pi u^2}{\tau}}\theta\left[\begin{smallmatrix}b\\-a\end{smallmatrix}\right](\tau|u)\,. \tag{E.I.1.9b}$$

For $m,n\in\mathbb{Z}$, they satisfy the quasi-periodicity property

$$\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau|u+m+\tau n)=\exp\left(i\pi ma-i\pi\tau n^2-2i\pi n\left(u+\frac{b}{2}\right)\right)\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau|u)\,. \tag{E.I.1.10}$$

The $\widehat{\mathfrak{su}(2)}_k$ theta functions are defined by

$$\Theta_{s,k}(\tau|z)=\sum_{n\in\mathbb{Z}+\frac{s}{2k}}q^{kn^2}e^{2i\pi zkn}\,, \tag{E.I.1.11}$$

with $s\in\mathbb{Z}_{2k}$. Under modular transformations, one has

$$\Theta_{s,k}(\tau+1|z)=e^{i\pi\frac{s^2}{k}}\Theta_{s,k}(\tau|z)\,, \tag{E.I.1.12a}$$

$$\Theta_{s,k}\left(-\frac{1}{\tau}\middle|\,\frac{z}{\tau}\right)=(-i\tau)^{1/2}\frac{1}{\sqrt{2k}}e^{\frac{i\pi}{\tau}\frac{kz^2}{2}}\sum_{s'\in\mathbb{Z}_{2k}}e^{-\frac{i\pi}{k}ss'}\Theta_{s',k}(\tau|z)\,. \tag{E.I.1.12b}$$

They also satisfy a quasi-periodicity property

$$\Theta_{s,k}(\tau|z+m+\tau n) = (-1)^{k(m+n)} e^{-i\pi k\left(\frac{n^2}{2}\tau + nz\right)} \Theta_{s,k}(\tau|z)\,. \tag{E.I.1.13}$$

Following [118], one defines the theta function related to a lattice Γ by

$$\Theta^{\Gamma}_{\mu}(\tau|\lambda) = \sum_{\gamma\in\Gamma+\mu} q^{\frac{1}{2}\langle\gamma,\gamma\rangle} e^{2i\pi\langle\gamma,\lambda\rangle}\,. \tag{E.I.1.14}$$

Under modular transformations, one has

$$\Theta^{\Gamma}_{\mu}(\tau+1|\lambda) = e^{i\pi\langle\mu,\mu\rangle}\Theta^{\Gamma}_{\mu}(\tau|\lambda)\,, \tag{E.I.1.15a}$$

$$\Theta^{\Gamma}_{\mu}\left(-\frac{1}{\tau}\middle|\frac{\lambda}{\tau}\right) = \frac{(-i\tau)^{\frac{\mathrm{rank}(\Gamma)}{2}}}{|\Gamma^{\vee}/\Gamma|^{\frac{1}{2}}} e^{\frac{i\pi}{\tau}\langle\lambda,\lambda\rangle} \sum_{\mu'\in\Gamma^{\vee}/\Gamma} e^{-2i\pi\langle\mu,\mu'\rangle}\Theta^{\Gamma}_{\mu'}(\tau|\lambda)\,. \tag{E.I.1.15b}$$

Let us define the Kronecker delta on the lattice Γ by:

$$\delta_{b,b'} = \frac{1}{|\Gamma^{\vee}/\Gamma|}\sum_{a\in\Gamma^{\vee}/\Gamma} e^{2i\pi\langle a,b-b'\rangle}\,. \tag{E.I.1.16}$$

Given a triplet $(\Gamma_{\mathrm{L}},\Gamma_{\mathrm{R}},\varphi)$, with φ being an isometry between the discriminant group of the two lattices Γ_{L} and Γ_{R}, let us determine the modular behaviour under a S-transformation of the quantity

$$\sum_{\mu\in\Gamma_{\mathrm{L}}^{\vee}/\Gamma_{\mathrm{L}}} \Theta^{\Gamma_{\mathrm{L}}}_{\mu}\left(\tau|\,\lambda\right)\bar{\Theta}^{\Gamma_{\mathrm{R}}}_{\varphi(\mu)}\left(\bar{\tau}|\,0\right)\,. \tag{E.I.1.17}$$

One has

$$\sum_{\mu\in\Gamma_{\mathrm{L}}^{\vee}/\Gamma_{\mathrm{L}}} \Theta^{\Gamma_{\mathrm{L}}}_{\mu}\left(-\frac{1}{\tau}\middle|\frac{\lambda}{\tau}\right)\bar{\Theta}^{\Gamma_{\mathrm{R}}}_{\varphi(\mu)}\left(-\frac{1}{\bar{\tau}}\middle|0\right) = \frac{|\tau|^{\mathrm{rank}(\Gamma)}}{Mathieu|\Gamma_{\mathrm{L}}^{\vee}/\Gamma_{\mathrm{L}}|} e^{\frac{i\pi}{\tau}\langle\lambda,\lambda\rangle}\sum_{\mu}\sum_{\rho,\bar{\rho}} e^{-2i\pi(\langle\mu,\rho\rangle-\langle\varphi(\mu),\bar{\rho}\rangle)}\Theta^{\Gamma_{\mathrm{L}}}_{\rho}\left(\tau|\,\lambda\right)\bar{\Theta}^{\Gamma_{\mathrm{R}}}_{\bar{\rho}}\left(\bar{\tau}|\,0\right) \tag{E.I.1.18}$$

Since φ is an isometry, one has $\langle\varphi(\mu),\bar{\rho}\rangle = \langle\mu,\varphi^{-1}(\bar{\rho})\rangle$. Permuting the sums, using eq. (E.I.1.16), and the fact that $\delta_{\rho,\varphi^{-1}(\bar{\rho})} = \delta_{\bar{\rho},\varphi(\rho)}$, one obtains finally

$$\sum_{\mu\in\Gamma_{\rm L}^\vee/\Gamma_{\rm L}} \Theta_\mu^{\Gamma_{\rm L}}\left(-\frac{1}{\tau}\middle|\,\frac{\lambda}{\tau}\right)\bar{\Theta}^{\Gamma_{\rm R}}_{\varphi(\mu)}\left(-\frac{1}{\bar{\tau}}\middle|\,0\right) = |\tau|^{\mathrm{rank}(\Gamma)} e^{\frac{i\pi}{\tau}\langle\lambda,\lambda\rangle}\times \\ \times \sum_{\mu\in\Gamma_{\rm L}^\vee/\Gamma_{\rm L}} \Theta_\mu^{\Gamma_{\rm L}}(\tau|\,\lambda)\,\bar{\Theta}^{\Gamma_{\rm R}}_{\varphi(\mu)}(\bar{\tau}|\,0)\ . \tag{E.I.1.19}$$

The weight $2k$ ($k>1$) Eisenstein series are holomorphic modular forms given by

$$E_{2k}(\tau) = -\frac{(2k)!}{(2i\pi)^{2k}B_{2k}}\sum_{(m,n)\neq(0,0)}\frac{1}{(m\tau+n)^{2k}}\,, \tag{E.I.1.20}$$

with B_{2k} the Bernoulli numbers. In terms of Jacobi theta functions, one has

$$E_4(\tau) = \frac{1}{2}\left(\theta_2(\tau)^8+\theta_3(\tau)^8+\theta_4(\tau)^8\right)\,. \tag{E.I.1.21}$$

In particular, E_4/η^8 is modular invariant. Finally, a weak Jacobi form of weight k and index t with character χ is a holomorphic function ϕ on $\mathbb{H}\times\mathbb{C}$ which satisfies for $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$:

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{y}{c\tau+d}\right) = \chi\,(g)\,(c\tau+d)^k e^{2i\pi tc\frac{y^2}{c\tau+d}}\,\phi(\tau,y)\,, \tag{E.I.1.22a}$$

$$\phi(\tau,y+\lambda\tau+\mu) = (-1)^{2t(\lambda+\mu)}e^{-2i\pi t(\lambda^2t+2\lambda y)}\,\phi(\tau,y)\quad,\qquad \lambda,\mu\in\mathbb{Z}\,, \tag{E.I.1.22b}$$

and with a Fourier expansion containing only positive powers of q.

We define the odd Jacobi theta function and the Dedekind eta function by the following infinite products:

$$\theta(\tau,z) := -i\,q^{\frac{1}{8}}y^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)\,(1-yq^n)\left(1-y^{-1}q^{n-1}\right)\,. \tag{E.I.1.23a}$$

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,, \tag{E.I.1.23b}$$

with $q := \exp(2i\pi\tau)$ and $y := \exp(2i\pi z)$. The discriminant modular form is given in terms of the Dedekind eta function by:

$$\Delta(\tau) := \eta(\tau)^{24}\,. \tag{E.I.1.24}$$

Given an even integral lattice Γ, whose pairing we denote:

$$\langle \cdot, \cdot \rangle : \Gamma \times \Gamma \to \mathbb{Z}\,, \tag{E.I.1.25}$$

and an element $\mu \in \Gamma^\star/\Gamma$ in its discriminant group, we define its associated theta-function with characteristic μ as a refined generated function:

$$\begin{aligned} \Theta^\Gamma_\mu : \mathbb{H} \times (\Gamma \otimes \mathbb{C}) &\to \mathbb{C} \\ (\tau, z) &\mapsto \sum_{v \in \Gamma + \mu} e^{i\pi(\langle v,v\rangle \tau + 2\langle v,z\rangle)}\,. \end{aligned} \tag{E.I.1.26}$$

Let us recall the definition of the $SL_2(\mathbb{Z})$ normalized Eisenstein series of weight $2w$:

$$E_{2w}(\tau) := \frac{1}{2\zeta(2w)} \sum_{(m,n)\in(\mathbb{Z}^*)^2} \frac{1}{|m+\tau n|^{2w}}\,. \tag{E.I.1.27}$$

We define the following weight-2 modular covariant derivative acting on the space of weight w modular forms:

$$\begin{aligned} D_w :\ \mathcal{M}_w &\to \mathcal{M}_{w+2} \\ f &\mapsto \left(\frac{i}{\pi}\frac{\partial}{\partial\tau} + \frac{w}{2\pi\tau_2}\right) f\,. \end{aligned} \tag{E.I.1.28}$$

Notice that this modular covariant derivative satisfies the Leibniz rule:

$$D_{w+r}(\psi_w \phi_r) = (D_w \psi_w)\phi_r + \psi_w D_r(\phi_r)\,. \tag{E.I.1.29}$$

We give two identities due to Ramanujan involving the Eisenstein series:

$$D_4 E_4 = \frac{2}{3}\left(E_6 - \hat{E}_2 E_4\right)\,, \tag{E.I.1.30a}$$

$$D_6 E_6 = E_4^2 - \hat{E}_2 E_6 \tag{E.I.1.30b}$$

I.2 Hypergeometric and modified Bessel function

The confluent hypergeometric function ${}_1F_1\left(a;c;z\right)$ is defined by:

$${}_1F_1\left(a;c;z\right) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!}\,, \tag{E.I.2.1}$$

with $(q)_n$ the Pochhammer symbol, or rising factorial. The hypergeometric function ${}_2F_1\left(a,b;c;z\right)$ is defined by:

$$ {}_2F_1\left(a,b;c;z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \,. \tag{E.I.2.2} $$

We give the expression of the hypergeometric function ${}_2F_1$ for some specific values of its arguments:

$$ {}_2F_1(2,4,6,t^{-1}) = -\frac{10}{3}t^2 \left(24t^2 + 6(4t-3)t^2 \log\left(\frac{t-1}{t}\right) - 6t - 1\right) , \tag{E.I.2.3a} $$

$$ {}_2F_1(1,3,4,t^{-1}) = -\frac{3}{2}t \left(2t^2 \log\left(\frac{t-1}{t}\right) + 2t + 1\right) , \tag{E.I.2.3b} $$

$$ {}_2F_1(2,2,4,t^{-1}) = -6t^2 \left((2t-1) \log\left(\frac{t-1}{t}\right) + 2\right) , \tag{E.I.2.3c} $$

$$ {}_2F_1(1,1,2,t^{-1}) = -t \log\left(\frac{t-1}{t}\right) , \tag{E.I.2.3d} $$

$$ {}_2F_1(2,3,6,t^{-1}) = 10t^2 \left(12t^2 + 6\left(2t^2 - 3t + 1\right) t \log\left(\frac{t-1}{t}\right) - 12t + 1\right) , \tag{E.I.2.3e} $$

$$ {}_2F_1(1,2,4,t^{-1}) = 3t \left(2t + 2(t-1)t \log\left(\frac{t-1}{t}\right) - 1\right) , \tag{E.I.2.3f} $$

$$ {}_2F_1(3,3,6,t^{-1}) = -30t^3 \left(\left(6t^2 - 6t + 1\right) \log\left(\frac{t-1}{t}\right) + 6t - 3\right) . \tag{E.I.2.3g} $$

In this appendix, we define various functions defined by an integral involving modified Bessel functions, and relevant for the computation of the Fourier series representation of the various threshold corrections in section C.II.4. To obtain the expressions below, one extensively makes use of the following Bessel functions identity:

$$ 2\frac{d}{dx} C_\alpha(x) = C_{\alpha-1}(x) + C_{\alpha+1}(x) \,, \tag{E.I.2.4} $$

where C_α denotes I_α or $e^{i\pi\alpha} K_\alpha$.

Zero-frequency mode: Let us first define $\forall (B,C,\nu,n) \in \mathbb{C}\times\mathbb{R}\times\mathbb{C}\times\mathbb{N}$ such that $\mathrm{Re}(B) \geqslant C$:

$$ F^{(0)}_{n,\nu}(B,C) := (-1)^n \frac{\partial^n}{\partial B^n} \int_0^\infty \frac{\mathrm{d}t}{t} \, I_\nu(Ct)\, e^{-Bt} \,, \tag{E.I.2.5} $$

relevant for the computation of the zero mode component of the Fourier series expansion. Following Erdelyi, one can compute $F^{(0)}_{n,\nu}$ explicitely:

$$F^{(0)}_{0,\nu}(B,C) = \frac{C^\nu \left(B+\sqrt{B^2-C^2}\right)^{-\nu}}{\nu}\,, \tag{E.I.2.6a}$$

$$F^{(0)}_{1,\nu}(B,C) = \frac{C^\nu \left(B+\sqrt{B^2-C^2}\right)^{-\nu}}{\sqrt{B^2-C^2}}\,, \tag{E.I.2.6b}$$

$$F^{(0)}_{2,\nu}(B,C) = \frac{C^\nu \left(B+\sqrt{B^2-C^2}\right)^{-\nu}\left(B+\nu\sqrt{B^2-C^2}\right)}{(B^2-C^2)^{3/2}}\,. \tag{E.I.2.6c}$$

Positive frequency modes: We also define $\forall (A,B,C,\nu,n) \in \mathbb{C}^2\times\mathbb{R}\times\mathbb{C}\times\mathbb{N}$:

$$F_{n,\nu}(A,B,C) := (-1)^n \frac{\partial^n}{\partial B^n}\int_0^\infty \frac{\mathrm{d}t}{t}\, I_\nu(Ct)\, e^{-Bt-A/t}\,, \tag{E.I.2.7}$$

relevant for the computation of the positive frequency modes of the Fourier series expansion. One then computes the following expressions:

$$F_{0,\nu}(A,B,C) = 2I_\nu(u_-)K_\nu(u_+)\,, \tag{E.I.2.8a}$$

$$F_{1,\nu}(A,B,C) = \frac{4A}{u_-^2-u_+^2}\Big(u_- I_{\nu-1}(u_-)K_\nu(u_+) + u_+ I_\nu(u_-)K_{\nu-1}(u_+)\Big)\,, \tag{E.I.2.8b}$$

$$\begin{aligned} F_{2,\nu}(A,B,C) = {} & \frac{8A^2}{u_-\left(u_-^2-u_+^2\right)^3}\Big(2u_-^2u_+\left(u_-^2-u_+^2\right)I_{\nu-1}(u_-)K_{\nu-1}(u_+) \\ & + u_- I_{\nu-2}(u_-)\left(u_-^4K_\nu(u_+) - 2(\nu+1)u_-^2u_+K_{\nu-1}(u_+) - u_+^4K_{\nu-2}(u_+)\right) \\ & - 2u_+^2 I_{\nu-1}(u_-)\left((\nu+1)u_-^2-(\nu-1)u_+^2\right)K_{\nu-2}(u_+)\Big)\,, \end{aligned} \tag{E.I.2.8c}$$

where we have introduced the following convenient combinations:

$$u_\pm := \sqrt{A}\left(\sqrt{B+C}\pm\sqrt{B-C}\right)\,. \tag{E.I.2.9}$$

Appendix II

Rational Narain lattices

II.1 Generic torus

We discuss in some detail quantization of the two-torus moduli and compatibility of the latter with the two-forms $\omega_{1,2}$ characterizing the principal torus bundle. We consider first that there is no Abelian bundle, *i.e.* that 'Wilson lines' are turned off.

Quantization of the torus moduli follows from single-valuelessness of $\exp(iS)$ in any instanton sector [52, 53], or from H-flux quantization in supergravity [75]. It was shown in [53] to derive from covariance of the model under T-duality along the torus fiber. Moreover it was noticed there that these quantization conditions imply that the underlying $c = 2$ CFT with a two-torus target space is rational.

The Narain Lattice $\Gamma_{\rm N} \subset \mathbb{R}^{2,2}$ corresponding to the two-torus of metric and B-field:

$$g = \frac{U_2}{T_2}\begin{pmatrix} 1 & T_1 \\ T_1 & |T|^2 \end{pmatrix}\,, \quad b = \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}. \tag{E.II.1.1}$$

is spanned by

$$\frac{1}{\sqrt{2U_2T_2}}\left\{ \begin{pmatrix} T_2 \\ -T_1 \\ T_2 \\ -T_1 \end{pmatrix} n_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} n_2 + \begin{pmatrix} U_2 \\ -U_1 \\ -U_2 \\ -U_1 \end{pmatrix} w^1 + \begin{pmatrix} T_1U_2 + T_2U_1 \\ -U_1T_1 + U_2T_2 \\ -T_1U_2 + T_2U_1 \\ -U_1T_1 - U_2T_2 \end{pmatrix} w^2 \right\}$$

$$= e^1 n_1 + e^2 n_2 + \tilde{e}_1 w^1 + \tilde{e}_2 w^2\,. \tag{E.II.1.2}$$

where n_ℓ (resp. w_ℓ) are the integer-valued momenta (resp. winding numbers). The inverse of the two-torus metric eq. (E.II.1.1) is $g^{ij} = 2\langle e^i, e^j\rangle\Big|_{\mathbb{R}^{2,0}}$.

The underlying conformal field theory is a rational CFT (*i.e.* with an extended chiral algebra), if $\Gamma_{\rm L} := \Gamma_{\rm N} \cap \mathbb{R}^{2,0}$ and $\Gamma_{\rm R} := \Gamma_{\rm N} \cap \mathbb{R}^{0,2}$ are rank two (even and

positive-definite) lattices [55]. It is equivalent to require that the space of solutions over the integers of the equations

$$T_2 n_1 - U_2 w_1 + (U_1 T_2 - T_1 U_2) w_2 = 0\,, \tag{E.II.1.3a}$$

$$-T_1 n_1 + n_2 + U_1 w_1 - (U_1 T_1 + U_2 T_2) w_2 = 0\,. \tag{E.II.1.3b}$$

has maximal rank (*i.e.* rank two). This is satisfied if and only if $U, T \in \mathbb{Q}(\sqrt{D})$, where D is a discriminant of a positive-definite even quadratic form, in other words

$$D = b^2 - 4ac < 0\ , \quad a, b, c \in \mathbb{Z}\ ,\ a > 0\,. \tag{E.II.1.4}$$

One can eliminate n_ℓ in eq. (E.II.1.2) using eq. (E.II.1.3) and express any element of the lattice $\Gamma_{\rm L}$ as an element of the 'winding lattice' (the sublattice of $\Gamma_{\rm N}$ defined by $n_1 = n_2 = 0$):

$$p_L = \sqrt{\frac{2U_2}{T_2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} w_1 + \sqrt{\frac{2U_2}{T_2}} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} w_2\,, \tag{E.II.1.5}$$

where, to ensure that it is actually an element of $\Gamma_{\rm L}$, w_ℓ have to satisfy the 'quantization conditions'

$$\frac{U_2}{T_2}(w_1 + T_1 w_2) - U_1 w_2 \in \mathbb{Z}\,, \tag{E.II.1.6a}$$

$$\frac{U_2}{T_2}(T_1 w_1 + |T|^2 w_2) + U_1 w_1 \in \mathbb{Z}\,, \tag{E.II.1.6b}$$

The data required to specify a RCFT consists of a triple $(\Gamma_{\rm L}, \Gamma_{\rm R}, \varphi)$, with φ being an isometry between the discriminant group of the two lattices $\Gamma_{\rm L}$ and $\Gamma_{\rm R}$. The corresponding modular-invariant partition function reads:

$$Z = \frac{1}{\eta^2(\tau)\eta^2(-\bar\tau)} \sum_{\mu \in \Gamma_{\rm L}^\star/\Gamma_{\rm L}} \Theta_\mu^{\Gamma_{\rm L}}(\tau|0)\, \Theta_{\varphi(\mu)}^{\Gamma_{\rm R}}(-\bar\tau|0)\ . \tag{E.II.1.7}$$

where

$$\Theta_\mu^\Gamma(\tau|\lambda) = \sum_{\gamma \in \Gamma + \mu} q^{\frac{1}{2}\langle\gamma,\gamma\rangle} e^{2i\pi\langle\gamma,\lambda\rangle} \tag{E.II.1.8}$$

is the theta function with characteristics associated with the lattice Γ. μ is an element of the discriminant group $\Gamma^\vee/\Gamma$ of the lattice and φ is an isometry between the discriminant groups of $\Gamma_{\rm L}$ and $\Gamma_{\rm R}$.

In order to specify the principal two-torus bundle over the $\mathcal{S}$, one should further

choose two anti-self-dual $(1,1)$-forms ω_1 and ω_2 on the $K3$ base $\mathcal{S}$ which define two different integer cohomology classes $[\omega_1], [\omega_2] \in H^2(\mathcal{S}, \mathbb{Z})$. In other words, one should specify a rank-two sublattice Γ_ω of the Picard lattice $\mathrm{Pic}(\mathcal{S}) = H^2(\mathcal{S}, \mathbb{Z}) \cap H^{1,1}_{\bar\partial}(\mathcal{S})$. The metric on this lattice is given by their intersection form:

$$\int_{\mathcal{S}} \omega_i \wedge \omega_j = d_{ij} \,. \tag{E.II.1.9}$$

We remind that the intersection matrix on the lattice of anti-self dual two-forms on $K3$ can be brought to the form

$$(-E_8) \oplus (-E_8) \oplus -2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{E.II.1.10}$$

One is thus endowed with two quadratic even lattices Γ_{L} and Γ_ω. We define first the following element of a formal extension of the winding lattice, valued in $H^2(\mathcal{S}, \mathbb{Z}) \times H^2(\mathcal{S}, \mathbb{Z})$, as:

$$p_\omega = \sqrt{\frac{2U_2}{T_2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \omega_1 + \sqrt{\frac{2U_2}{T_2}} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \omega_2 \,. \tag{E.II.1.11}$$

One should impose that p_ω belongs actually to a formal extension $\Gamma_{\text{L}} \otimes \mathrm{Pic}(\mathcal{S})$ of the left lattice, *i.e.* one should impose 'compatibility conditions', of the same form as eq. (E.II.1.6):

$$\frac{U_2}{T_2}(\omega_1 + T_1\omega_2) - U_1\omega_2 \in H^2(\mathcal{S}, \mathbb{Z}) \,, \tag{E.II.1.12a}$$

$$\frac{U_2}{T_2}(T_1\omega_1 + |T|^2\omega_2) + U_1\omega_1 \in H^2(\mathcal{S}, \mathbb{Z}) \,. \tag{E.II.1.12b}$$

Importantly, these conditions depends on Γ_{N} and not of Γ_{L} only.

To summarize, for a given pair of anti-self dual two-forms and (ω_1, ω_2), defining a rank two (generically) lattice $\Gamma_\omega \in H^2(\mathcal{S}, \mathbb{Z})$, one needs to choose the metric and B-field of the two torus such that two conditions are satisfied:

1. The compatibility condition eq. (E.II.1.12),
2. The tadpole condition $N - 24 = \int_{\mathcal{S}} \langle p_\omega, p_\omega \rangle$, where $N = -\int_{\mathcal{S}} \mathrm{ch}_2(E)$ is the instanton number.

In order to include Abelian bundles, or 'Wilson lines', one first consider an embedding of the toroidal $(2,2)$ lattice into a $(10,2)$ lattice which includes also the

contribution from the E_8 weight lattice. The moduli V^a in eq. (B.III.1.2) corresponds then to off-diagonal deformations of this lattice. Quantization of (T, U), as well as the compatibility conditions eq. (E.II.1.12), are not affected. The 'physical' two-torus metric is not given anymore by eq. (B.II.1.11), but is of the form $g_{IJ} + \frac{\alpha'}{4}\mathrm{Tr}(A_I A_J)$, as for ordinary Wilson lines, however the tadpole and compatibility conditions remains unchanged.

II.2 Orthogonal torus

For illustration let us consider an orthogonal torus with radii $R_1 = \sqrt{p_1/q_1}$ and $R_2 = \sqrt{p_2/q_2}$ hence

$$U = i\sqrt{\frac{p_1 p_2}{q_1 q_2}}\,, \quad T = i\sqrt{\frac{p_2 q_1}{q_2 p_1}}\,. \tag{E.II.2.1}$$

The quantization conditions eq. (E.II.1.6) read then

$$p_1 w_1 \equiv 0 \mod q_1\,, \quad p_2 w_2 \equiv 0 \mod q_2\,. \tag{E.II.2.2}$$

Assuming that $\gcd(q_\ell, p_\ell) = 1$ this is solved by choosing $w_1 = q_1 W_1$ and $w_2 = q_2 W_2$. Elements of the lattice Γ_{L} are then of the form

$$p_L = \sqrt{2p_1 q_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} W_1 + \sqrt{2p_2 q_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} W_2\,, \tag{E.II.2.3}$$

with $W_\ell \in \mathbb{Z}$. A basis of Γ_{L} is then provided by

$$e_1^{\mathrm{L}} = \sqrt{2p_1 q_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \quad e_2^{\mathrm{L}} = \sqrt{2p_2 q_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{E.II.2.4}$$

The modular-invariant partition function associated with this rational Narain lattice reads:

$$Z = \frac{1}{\eta^2(\tau)\eta^2(-\bar\tau)} \prod_{\ell=1}^{2} \sum_{r_\ell \in \mathbb{Z}_{2p_\ell}} \sum_{s_\ell \in \mathbb{Z}_{2q_\ell}} \Theta_{q_\ell r_\ell + p_\ell s_\ell, 2p_\ell q_\ell}(\tau|\,0)\,\Theta_{q_\ell r_\ell - p_\ell s_\ell, 2p_\ell q_\ell}(-\bar\tau|\,0)\,. \tag{E.II.2.5}$$

The isometry $\varphi : \Gamma_{\mathrm{L}}^\star/\Gamma_{\mathrm{L}} \to \Gamma_{\mathrm{R}}^\star/\Gamma_{\mathrm{R}}$ can be determined explicitly by mapping $q_\ell r_\ell \pm p_\ell s_\ell$ into the 'fundamental domain' $\{0, \ldots, 2p_\ell q_\ell - 1\}$.

The compatibility condition between Γ_ω and the orthogonal lattice of moduli

eq. (E.II.2.1) amounts to

$$\omega_{\rm L}^{\ell} := \frac{1}{q_\ell}\,\omega^\ell \in H^2(\mathcal{S},\mathbb{Z})\,,\ \ \ell=1,2\,. \tag{E.II.2.6}$$

Notice that the partition function eq. (E.II.2.5) differs from the partition function for an orthogonal torus with radii $R_\ell = \sqrt{2p_\ell q_\ell}$ precisely in the choice of the isometry φ. In the latter case one has indeed simply

$$Z = \frac{1}{\eta^2(\tau)\eta^2(-\bar\tau)} \prod_{\ell=1}^{2} \sum_{m_\ell\in\mathbb{Z}_{2p_\ell q_\ell}} \Theta_{m_\ell,2p_\ell q_\ell}\,(\,\tau|\,0)\,\Theta_{m_\ell,2p_\ell q_\ell}\,(\,-\bar\tau|\,0)\ . \tag{E.II.2.7}$$

Moreover in this case the chiral lattices $\Gamma_{\rm L}$ and $\Gamma_{\rm R}$ coincide such that the compatiblity condition is trivial.

One can also consider examples obtained from the above by T-duality along the two circles, each of these cases corresponding to a different choice of isometry φ. Satisfying the compatibility condition eq. (E.II.2.1) is then equivalent to considering in each case a different sublattice of $\mathrm{Pic}(\mathcal{S})$. This is one of the dualities studied in [53], which induces a duality action on $\mathrm{Pic}(\mathcal{S})$, leaving the tadpole condition invariant by construction.

Appendix III

Coefficients for the Fourier representation

In this appendix, we will determine the exact coefficients entering in the computation of the Fourier expansion representation of the integral eq. (C.II.4.10):

$$\mathcal{I}_f(s) := \int_{\mathcal{F}} \mathrm{d}\nu\, \mathcal{F}(s,1,-2)\, \tau_2 \sum_{\mu, p_{\mathrm{L}}, p_{\mathrm{R}}} q^{\frac{1}{2}p_{\mathrm{L}}^2} \bar{q}^{\frac{1}{2}p_{\mathrm{R}}^2} f(p_{\mathrm{L}}, \omega)\,, \tag{E.III.0.1}$$

with quadratic momentum insertion:

$$f(p_{\mathrm{L}}, \omega) = \tilde{d}_{ij} p_{\mathrm{L}}^i p_{\mathrm{L}}^j - \frac{n-24}{2\pi\tau_2}\,, \tag{E.III.0.2}$$

with the metric $\tilde{d}$ defined in section II.4.3.

As mentioned in section II.4.1, the first step in deriving the Fourier representation is to first perform the integral over the Lagrange multiplier τ_1 to impose the constraint on the lattice momenta, solve explicitely the constaint, and perform a suitable Poisson resummation.

Let us now introduce some notation for the lattice. First, the $\Gamma_{2,2}(T,U)$ Narain lattice elements can be written in a complex basis as

$$p_L = \frac{1}{\sqrt{T_2 U_2}} \Big(m_2 - T m_1 + \bar{U}(n^1 + T n^2) \Big)$$
$$p_R = \frac{1}{\sqrt{T_2 U_2}} \Big(m_2 - T m_1 + U(n^1 + T n^2) \Big)$$

In complex notation the scalar product becomes

$$\langle p_L, p'_L\rangle = \frac{1}{2}\left(|p_L+p'_L|^2-|p_L|^2-|p_R|^2\right) = \mathrm{Re}\,(p_L\bar{p}'_L)\,. \tag{E.III.0.3}$$

The BPS constraint $\frac{1}{4}(|p_L|^2-|p_R|^2) = m_1 n^1 + m_2 n^2 = 1$ is solved, for coprime (n^1,n^2), as

$$m_1 = m_1^\star + \tilde{M} n^2\,, \quad m_2 = m_2^\star - \tilde{M} n^1\,, \tag{E.III.0.4}$$

where $m_1^\star$ is a modular inverse of n^1 modulo n^2, and $m_2^\star$ a modular inverse of n^2 modulo n^1.

As mentioned in section II.4.1, after solving the constraint on momenta as above, one ends up with an expression of the following form:

$$\sum_{\tilde{M}\in\mathbb{Z}} e^{-\pi a\tilde{M}^2+2i\pi b\tilde{M}}\left(c\tilde{M}^2+d\tilde{M}+e\right)\,, \tag{E.III.0.5}$$

to be Poisson resummed over the variable $\tilde{M}$.

The Poisson resummation formula on $\mathbb{Z}$:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \tilde{f}(k)\,, \tag{E.III.0.6}$$

gives the following general formulae:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} e^{-\lambda n^2} &= \sqrt{\frac{\pi}{\lambda}}\sum_{k\in\mathbb{Z}} e^{-\pi^2k^2/\lambda}\,,\\
\sum_{n\in\mathbb{Z}} n\, e^{-\lambda n^2} &= -i\sqrt{\frac{\pi}{\lambda}}\sum_{k\in\mathbb{Z}} \frac{\pi k}{\lambda}\, e^{-\pi^2k^2/\lambda}\,,\\
\sum_{n\in\mathbb{Z}} n^2\, e^{-\lambda n^2} &= \sqrt{\frac{\pi}{\lambda}}\sum_{k\in\mathbb{Z}} \left(\frac{1}{2\lambda}-\frac{\pi^2k^2}{\lambda^2}\right)\, e^{-\pi^2k^2/\lambda}\,.
\end{aligned} \tag{E.III.0.7}$$

Using these results, one obtains the Poisson resummed expression:

$$\begin{aligned}
&\sum_{\tilde{M}\in\mathbb{Z}} e^{-\pi a\tilde{M}^2+2i\pi b\tilde{M}}\left(c\tilde{M}^2+d\tilde{M}+e\right)\\
&= \frac{1}{\sqrt{a}}\sum_{M\in\mathbb{Z}} e^{-\frac{\pi}{a}(M-b)^2}\left\{\left(\frac{1}{2\pi a}-\left(\frac{b-M}{a}\right)^2\right)c+\frac{i(b-M)}{a}d+e\right\}\\
&= \frac{1}{\sqrt{a}}\sum_{M\in\mathbb{Z}} e^{-\frac{\pi}{a}(M-b)^2}\left\{\left(\frac{1}{2\pi a}c-\frac{b^2}{a^2}c+\frac{ib}{a}d+e\right)+\frac{1}{a}\left(\frac{2b}{a}c-id\right)M-\frac{c}{a^2}M^2\right\}\,.
\end{aligned} \tag{E.III.0.8}$$

In order to determine the various coefficients (a, b, c, d, e), let us expand the left momentum as

$$p_L = \frac{1}{\sqrt{T_2 U_2}} \left(\underbrace{m_2^\star - T m_1^\star + \bar{U}(n^1 + T n^2)}_{P_L^\star} - \tilde{M} \underbrace{(n^1 + T n^2)}_{\hat{P}} \right) \tag{E.III.0.9}$$

In the following one will consider the $SL(2;\mathbb{Z})$ transformation related to the solution of the BPS constraint:

$$\tilde{T} = \frac{m_1^\star T - m_2^\star}{n^1 + n^2 T}\,, \tag{E.III.0.10}$$

implying in particular that:

$$\tilde{T}_2 = \frac{T_2}{|n^1 + n^2 T|^2} = \frac{T_2}{|\hat{P}|^2}\,. \tag{E.III.0.11}$$

We remark also that:

$$\frac{P_L^\star}{\hat{P}} = \bar{U} - \tilde{T} \tag{E.III.0.12}$$

Now we consider the vector of two-forms that appears in the insertion. Considering a basis $\{\varpi_\ell\}$ of $\mathrm{Pic}(\mathcal{S})$, we expand, in complex notation

$$p_w = \sqrt{\frac{U_2}{T_2}} \underbrace{\left(N_1^\ell + T N_2^\ell\right)}_{N^\ell} \varpi_\ell\,, \tag{E.III.0.13}$$

and introduce the intersection form $d_{\ell k} = \int \varpi_\ell \wedge \varpi_k$. Notice that (N_1^ℓ, N_2^ℓ) transforms as a doublet under $SL(2;\mathbb{Z})_T$. From this one can compute the scalar products that appear in the insertion. One obtains for the quadratic momentum insertion:

$$\begin{aligned} f(p_{\rm L}, \omega) &= \int_{\mathcal{S}} \langle p_\omega, p_{\rm L}\rangle^2 - \frac{1}{4\pi\tau_2}\langle p_\omega, p_\omega\rangle \\ &= \frac{1}{\tilde{T}_2^2}\Big\{ d_{\ell k}\,\mathrm{Re}\left(\tilde{N}^\ell\left(U - \bar{\tilde{T}} - \tilde{M}\right)\right)\,\mathrm{Re}\left(\tilde{N}^k\left(U - \bar{\tilde{T}} - \tilde{M}\right)\right) \\ &\quad - \frac{U_2 \tilde{T}_2}{4\pi\tau_2} d_{\ell k}\,\mathrm{Re}\left(\tilde{N}^\ell \overline{\tilde{N}^k}\right)\Big\} \end{aligned} \tag{E.III.0.14}$$

Out of this expression one can first collect the term in $\tilde{M}^2$, namely:

$$c = \frac{d_{\ell k}}{\tilde{T}_2^2}\mathrm{Re}\left(\tilde{N}^\ell\right)\mathrm{Re}\left(\tilde{N}^k\right)\,, \tag{E.III.0.15}$$

then term linear in $\tilde{M}$, which reads (using the symmetry of the intersection form):

$$d = -\frac{2d_{\ell k}}{\tilde{T}_2^2}\left[(U_1-\tilde{T}_1)\,\mathrm{Re}\left(\tilde{N}^\ell\right)-(U_2+\tilde{T}_2)\,\mathrm{Im}\left(\tilde{N}^\ell\right)\right]\mathrm{Re}\left(\tilde{N}^k\right), \qquad \text{(E.III.0.16)}$$

and finally the constant term given by:

$$\begin{aligned} e = {} & \frac{d_{\ell k}}{\tilde{T}_2^2}\left[(U_1-\tilde{T}_1)\,\mathrm{Re}\left(\tilde{N}^\ell\right)-(U_2+\tilde{T}_2)\,\mathrm{Im}\left(\tilde{N}^\ell\right)\right]\times \\ & \times\left[(U_1-\tilde{T}_1)\,\mathrm{Re}\left(\tilde{N}^k\right)-(U_2+\tilde{T}_2)\,\mathrm{Im}\left(\tilde{N}^k\right)\right] \\ & -\frac{U_2}{4\pi\tau_2\tilde{T}_2}d_{\ell k}\,\mathrm{Re}\left(\tilde{N}^\ell\overline{\tilde{N}^k}\right). \end{aligned} \qquad \text{(E.III.0.17)}$$

We are now ready to consider the Poisson resummation of the result, organised in powers of the dual variable M. In the exponential, we have:

$$\begin{aligned} \exp\left(-\frac{\pi\tau_2}{2}(|p_L|^2+|p_R|^2)\right) &= \exp\left(-\frac{\pi\tau_2}{T_2U_2}|P_L^\star-\tilde{M}\hat{P}|^2+2\pi\tau_2\right) \\ &= \exp\left(-\frac{\pi\tau_2}{\tilde{T}_2U_2}|\bar{U}-\tilde{T}-\tilde{M}|^2+2\pi\tau_2\right) \\ &= \exp\left(2\pi\tau_2-\frac{\pi\tau_2}{\tilde{T}_2U_2}|\bar{U}-\tilde{T}|^2\right)\times \\ &\quad\times\exp\left(-\frac{\pi\tau_2}{\tilde{T}_2U_2}\tilde{M}^2+2i\pi\tilde{M}\frac{i\tau_2}{\tilde{T}_2U_2}(\tilde{T}_1-U_1)\right), \end{aligned} \qquad \text{(E.III.0.18)}$$

from which one can once again extract the coefficient of the quadratic and linear terms in $\tilde{M}$ to obtain a and b. Adding up the terms quadratic, linear and constant in the dual dummy variable M, one can finally read up the summand of the Poisson resummed momentum insertion:

$$\begin{aligned} &\sqrt{\frac{\tilde{T}_2U_2}{\tau_2}}\sum_{M\in\mathbb{Z}}\left\{-\frac{U_2^2}{\tau_2^2}\,d_{\ell k}\left[\mathrm{Re}\left(\tilde{N}^\ell\right)M-i\frac{\tau_2}{\tilde{T}_2U_2}(U_2+\tilde{T}_2)\,\mathrm{Im}\left(\tilde{N}^\ell\right)\right]\times\right. \\ &\times\left[\mathrm{Re}\left(\tilde{N}^k\right)M-i\frac{\tau_2}{\tilde{T}_2U_2}(U_2+\tilde{T}_2)\,\mathrm{Im}\left(\tilde{N}^k\right)\right]-\frac{U_2}{2\pi\tau_2\tilde{T}_2}d_{\ell k}\,\mathrm{Im}\left(\tilde{N}^\ell\right)\mathrm{Im}\left(\tilde{N}^k\right)\Bigg\}\times \\ &\times\exp\left\{-\frac{\pi\tilde{T}_2U_2}{\tau_2}\left(M-\frac{i\tau_2}{\tilde{T}_2U_2}(\tilde{T}_1-U_1)\right)^2+2\pi\tau_2-\frac{\pi\tau_2}{\tilde{T}_2U_2}|\bar{U}-\tilde{T}|^2\right\}, \end{aligned} \qquad \text{(E.III.0.19)}$$

to be used in section C.II.4.

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Matthieu SARKIS **16/06/2017**

Sujet : Compactifications hétérotiques avec flux

Résumé : Nous étudions différents aspects liés aux compactifications hétérotiques avec torsion. Nous définissons et calculons le *genre elliptique vêtu* associé aux compactifications Fu-Yau, et exploitons ce résultat pour calculer les corrections de seuil à une boucle de différents couplages BPS-saturés dans l'action effective de supergravité à quatre dimensions. Enfin nous nous intéressons à des solutions supersymétriques non-compactes qui généralisent, entre autres, les solutions hétérotiques connues sur le conifold.

Mots clés : Corde hétérotique, torsion, GLSM, Genre elliptique vêtu, solutions non-compactes

Subject : Heterotic compactifications with flux

Abstract : We study various aspects of heterotic compactifications with torsion. We define and compute the *dressed elliptic genus* associated to Fu-Yau compactifications, and use this result to compute one-loop threshold corrections to various BPS-saturated couplings in the four-dimensional effective supergravity action. Finally, we study non-compact supersymmetric solutions which generalize, among others, the known heterotic solutions on the conifold.

Keywords : Heterotic string, torsion, GLSM, dressed elliptic genus, non-compact solutions