Arc colorings and cycles in digraphs

Yandong Bai

Université Paris-Sud

École Doctorale: Informatique

Laboratoire: Laboratoire de Recherche en Informatique

Spécialité: Informatique

THÈSE DE DOCTORAT

Par
Yandong BAI

Arc Colorings and Cycles in Digraphs

Date de soutenance: 28/11/2014

Composition du jury:

Directeur de thèse:	Hao LI	Directeur de Recherche
Rapporteurs:	Yaojun CHEN	Professeur
	Tomáš KAISER	Maître de Conférences
Examinateurs:	Eric ANGEL	Professeur
	Philippe DAGUE	Professeur
	Cristina BAZGAN	Professeur
	Réza NASERASR	Chargé de Recherche

Contents

Abstract

This thesis studies arc colorings and cycles of digraphs. It focuses on the following topics: vertex-distinguishing proper arc colorings in digraphs, short cycles in digraphs with forbidden subdigraphs, disjoint cycles and cycle factors in bipartite tournaments, universal arcs in tournaments, and directed cuts in a type of Cayley digraph.

The thesis consists of an introductory chapter (Chapter 1), a conclusion chapter (Chapter 7) and five research chapters (Chapters 2-6). Each research chapter is based on an original article, which has been published, or submitted for publication, in an international journal.

In Chapter 1, the basic terminology and notation used in this thesis are introduced, and then a brief introduction to the research contents and main results is given.

In Chapter 2, the (semi-)vertex-distinguishing proper arc coloring of digraphs is introduced. Denote by $\chi'^2_{vd}(D)$ (resp. $\chi'^2_{svd}(D)$) the minimum number of colors required for a vertex-distinguishing (resp. semi-vertex-distinguishing) proper 2-type arc coloring of D. We give tight upper bounds for $\chi'^2_{vd}(D)$ and $\chi'^2_{svd}(D)$ in terms of its order and degrees. Furthermore, the values of $\chi'^2_{vd}(D)$ for some regular digraphs D are given. Besides, we show that the values of $\chi'^2_{vd}(D)$ and $\chi'^2_{svd}(D)$ will not be changed if the coloring, in addition, required to be equitable.

In Chapter 3, we concentrate on the Caccetta-Häggkvist conjecture, which claims that every digraph on n vertices without directed cycles of lengths at most l contains a vertex with outdegree at most $\frac{n-1}{l}$. As a partial support of the conjecture, Razborov [A. Razborov, On the Caccetta-Häggkvist conjecture with forbidden subgraphs, J. Graph Theory (2012) 1-13] verified the case $l = 3$ for a specific family of digraphs.

We generalize Razborov's result by verifying the conjecture for $l \geq 4$ on a similar specific family of digraphs.

In Chapter 4, we consider the vertex-disjoint cycles in bipartite tournaments. Let $t_1, \ldots, t_r \in [4, 2q]$ be any r even integers, where $q \geq 2$ and $r \geq 1$ are two integers. We show that every bipartite tournament with minimum outdegree at least $qr - 1$ contains r vertex-disjoint directed cycles of lengths $t'_1, \ldots, t'_r$ such that $t'_i = t_i$ for $t_i = 0 \ (mod\ 4)$ and $t'_i \in \{t_i, t_i + 2\}$ for $t_i = 2 \ (mod\ 4)$, where $1 \leq i \leq r$. The special case $q = 2$ of the result verifies the bipartite tournament case of a conjecture proposed by Bermond and Thomassen, claiming that every digraph with minimum outdegree at least $2r - 1$ contains at least r vertex-disjoint directed cycles.

In Chapter 5, cycle factors in regular bipartite tournaments are considered. We show that every k-regular bipartite tournament B with $k \geq 3$ has two complementary cycles of lengths 6 and $|V(B)| - 6$, unless B is isomorphic to a special digraph. Also, we show that every k-connected regular bipartite tournament has a cycle factor consisting of k cycles.

In Chapter 6, universal arcs and directed cuts are considered. Let T be a tournament with at least 3 vertices. We show that T has a universal arc if and only if T is strong, and also show that every arc of T is universal if and only if T is 2-connected or T belongs to a special class of 1-connected tournaments. Let $\mathbb{Z}_2^k$ be the set of binary vectors with length k and let $S_k = \{2^{i-1} : i = 1, \ldots, k\}$. We deal with directed cuts in the Cayley digraph $X(\mathbb{Z}_2^k, S_k)$. To be precise, we obtain a lower bound of the maximum number of arcs contained in a directed cut of $X(\mathbb{Z}_2^k, S_k)$ and the minimum number of directed cuts required to cover the arcs of $X(\mathbb{Z}_2^k, S_k)$.

Chapter 7 concludes with a survey of this thesis and some problems for further consideration.

Keywords: vertex-distinguishing proper arc colorings; Caccetta-Häggkvist conjecture; vertex-disjoint cycles; cycle factors; universal arcs; directed cuts

Résumé

Cette thèse étudie la coloration d'arcs et de cycles dans les graphes orientés. Elle se concentre sur les sujets suivants: la coloration propre d'arcs avec des sommet-distingué dans les graphes orientés, les cycles courts dans les graphes orientés avec des sous-graphes interdits, les cycles sommet-disjoints et les cycle-facteurs dans les tournois bipartis, les arcs universels dans les tournois, et des coupes dirigées dans un graphe Cayley orienté.

La thèse se compose d'un chapitre d'introduction (Chapitre 1), un chapitre de conclusion (Chapitre 7) et cinq chapitres de recherche (Chapitres 2-6). La thèse est basée sur cinq articles originaux publiés ou présentés dans des journaux. Les principaux résultats sont les suivants.

Dans le chapitre 1, la terminologie de base et la notation utilisée dans cette thèse sont introduits, puis une brève introduction au contenu de recherche et les principaux résultats est donnée.

Dans le chapitre 2, nous introduisons la coloration propre d'arcs avec des (semi-) sommet-distingué dans les graphes orientés. Nous avons proposé une conjecture sur le nombre arc-chromatique (semi-) sommet-distingué et nous avons aussi donné quelque résultats partiels.

Dans le chapitre 3, nous avons étendu un résultat de Razborov en prouvant que la conjecture de Caccetta-Häggkvist est vraie pour certains graphes orientés avec des sous-graphes interdits.

Dans le chapitre 4, nous avons montré que chaque tournoi biparti avec degré sortant minimum au moins $qr - 1$ contient r cycles de sommets-disjoints de toutes longueurs données $t'_1, \ldots, t'_r$ de telle sorte que $t'_i = t_i$ pour $t_i = 0 \ (mod\ 4)$ et $t'_i \in$

$\{t_i, t_i+2\}$ pour $t_i = 2\ (mod\ 4)$, où $1 \leq i \leq r$ et $t_i \in [4, 2q]$ est arbitraire. Le cas spécial $q = 2$ confirme le cas du tournoi biparti de la conjecture de Bermond-Thomassen.

Dans le chapitre 5, nous avons montré que chaque tournoi biparti k-Regulier avec $k \geq 3$ que l'on notera B a deux cycles complémentaires de longueurs 6 et $|V(B)| - 6$, à moins que B soit isomorphe à un graphe spécifique, tayant ainsi une conjecture sur des 2-cycles-facteurs dans les tournois bipartis. En outre, nous montrons que tous les tournois bipartis réguliers ont un k-cycle-facteur.

Dans le chapitre 6, nous donnons une condition nécessaire et suffisante pour l'existence d'un arc universel dans un tournoi et nous caractérisons tous les tournois où chaque arc est universel. Nous donnons aussi une bonne borne pour la taille d'une coupe max et nous montrons la valeur exacte pour le nombre coupe-abri dans un graphe Cayley orienté.

Chapitre 7 se termine par une enquête de cette thèse et certains problèmes pour un examen plus approfondi.

Mots-clés: coloration propre d'arcs avec des sommet-distingué; conjecture de Caccetta-Häggkvist; cycles sommet-disjoints; cycle-facteurs; arcs universel; coupe orienté

Chapter 1

Introduction

Graph theory studies the properties of various graphs and is a branch of discrete mathematics. The earliest known paper on graph theory was given by Euler in 1736, which discussed the seven bridges of Königsberg. The first book on graph theory is "*Theorie der endlichen und unendlichen Graphen*", which was written by Kőnig and published in 1936. After the appearance of this book, the subject has gone through a remarkable development. In particular, in the recent decades, graph theory has experienced explosive growth concurrent with the growth of computer science. Moreover, since graphs can be used to model many types of relations and processes, the results of graph theory have wide applications in chemistry, physics, biology and computer science.

This thesis focuses on the following topics: vertex-distinguishing proper arc colorings in digraphs, short cycles in digraphs with forbidden subdigraphs, disjoint cycles and cycle factors in bipartite tournaments, universal arcs in tournaments, and directed cuts in a type of Cayley digraph. The results obtained concerning these topics are contained in five distinct research chapters (Chapters 2-6). Each chapter is based on an original article, which has been published, or submitted for publication, in an international journal.

In this chapter, we give a short but relatively complete introduction of this thesis. In Section 1, the basic terminology and notation are given. Sections 2, 3, 4, 5 and 6 are devoted to the main results on vertex-distinguishing proper arc colorings in

digraphs, short cycles with forbidden subdigraphs in digraphs, vertex-disjoint cycles in bipartite tournaments, cycle factors in bipartite tournaments, universal arcs in tournaments and directed cuts of a type of Cayley digraphs, respectively. The final section, Section 7, concludes with some problems deserving further consideration.

1.1 Basic terminology and notation

In this section, we give some basic terminology and notation that will be used in the thesis. For those not defined here, we follow Bang-Jensen and Gutin [18].

Graph and digraph

A *graph* G is a pair $(V(G), E(G))$ consisting of a nonempty set $V(G)$ of vertices and a set $E(G)$, distinct from $V(G)$, of edges. Similarly, a *digraph* or *directed graph* D is a pair $(V(D), E(D))$ consisting of a nonempty set $V(D)$ of vertices and a set $E(D)$, distinct from $V(D)$, of arcs. Alternatively, a digraph can be regarded as a graph such that every edge has a direction. Throughout the thesis, a graph always means an *undirected* graph. Unless otherwise stated, the letter G denotes a graph and the letter D denotes a digraph.

Subgraph and subdigraph

A graph G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. Moreover, if G' is a subgraph of G and G' contains all the edges $uv \in E(G)$ with $u, v \in V(G')$, then G' is an *induced subgraph* of G; and if if G' is a subgraph of G and $V(G') = V(G)$, then G' is a *spanning subgraph* of G. Similarly, we can define a *subdigraph*, an *induced subdigraph* and a spanning subdigraph of a digraph D.

Order and size

The number of vertices of a graph (a digraph) is its *order* and the number of edges (arcs) is its *size*.

Finite and simple

A graph (digraph) is *finite* if both its vertex set and edge set (arc set) are finite. A graph is *simple* if no edge is incident with only one vertex and no two edges incident with the same two vertices. A digraph is *simple* if no arc is incident with only one

vertex and no two arcs have both the same starting vertex and the same ending vertex. Unless otherwise stated, all graphs and digraphs considered in this thesis are finite and simple.

Degree

For an edge uv of a graph G, say u is a *neighbor* of v and vice versa. For a vertex v of G, let $N(v)$ be its *neighborhood*, i.e., the set of neighbors of v, and let $d(v) = |N(v)|$ be its *degree*. Denote by $\delta(G)$ and $\Delta(G)$ the *minimum degree* and *maximum degree* of G, respectively. Let $n_d(G)$ be the number of vertices with degree d in G.

Outdegree and indegree

For an arc uv (or (u, v)) of a digraph D, write $u \to v$ and say v is an *outneighbor* of u and u is an *inneighbor* of v. For a vertex v of D, let $N^+(v)$ and $N^-(v)$ be its *outneighborhood* and *inneighborhood*, i.e., the set of outneighbors and the set of inneighbors of v, respectively. Let $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$ be its *outdegree* and *indegree*, respectively. Denote by $\delta^+(D)$, $\delta^-(D)$, $\Delta^+(D)$ and $\Delta^-(D)$ the minimum outdegree, minimum indegree, maximum outdegree and maximum indegree of D. Define the minimum degree $\delta(D)$ and maximum degree $\Delta(D)$ of D as follows,

$$\delta(D) = \min\{\delta^+(D),\ \delta^-(D)\},\quad \Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}. \tag{1.1}$$

Let $n_{d^+}(D)$ and $n_{d^-}(D)$ be the numbers of vertices with outdegree d^+ and indegree d^- in D, respectively.

Regular digraph

Let k be a nonnegative integer. A digraph D is *k-regular* if every vertex has both outdegree and indegree k.

$\mathbf{D(k)}$ and $\mathbf{D(k, l)}$

Let k and l be two nonnegative integers. Denote by $D(k)$ the family of digraphs in which every vertex has outdegree at most k and $D(k, l)$ the family of digraphs in which every vertex has either outdegree at most k or indegree at most l.

$\pi(\mathbf{G})$ and $\pi(\mathbf{D})$

For a graph G and a digraph D, let

$$\pi(G) = \min\{k \in \mathbf{Z} : \binom{k}{d} \geq n_d(G) \ for \ \delta(G) \leq d \leq \Delta(G)\} \tag{1.2}$$

and

$$\pi(D) = \min \left\{ k \in \mathbf{Z} : \begin{array}{l} \binom{k}{d^+} \geq n_{d^+}(D) \ for \ \delta^+(D) \leq d^+ \leq \Delta^+(D) \\ \binom{k}{d^-} \geq n_{d^-}(D) \ for \ \delta^-(D) \leq d^- \leq \Delta^-(D) \end{array} \right\}. \tag{1.3}$$

Edge coloring

Let k be a positive integer. A *k-edge-coloring* of a graph G is an assignment of k colors to the edges of G. An edge coloring of G is *proper* if no two adjacent edges receive the same color. Denote by $\chi'(G)$ the minimum number of colors required for a proper edge coloring of a graph G.

VDPE coloring and vdec-graph

A *vertex-distinguishing proper edge coloring* (abbreviated *VDPE coloring*) of a graph G is a proper edge coloring where no two vertices are incident with the same set of colors. Note that a graph has a VDPE coloring if and only if it contains no isolated edge and at most one isolated vertex. Such a graph is referred to as a *vertex-distinguishing edge-colorable graph* (abbreviated *vdec-graph*). Denote by $\chi'_{vd}(G)$ the minimum number of colors required for a VDPE coloring of a vdac-graph G.

Arc coloring

Let k be a positive integer. A *k-arc-coloring* of a digraph D is an assignment of k colors to the arcs of D. An arc coloring of D is *1-type proper* if no two consecutive arcs receive the same color, and is *2-type proper* if no two arcs with a common tail or with a common head receive the same color. Denote by $\chi'^1(D)$ and $\chi'^2(D)$ the minimum numbers of colors required for a 1-type and 2-type arc coloring of D, respectively.

VDPA coloring and vdac-digraph

Define the *out-arc set* (*in-arc set*) of v to be the set of arcs starting with v (resp. ending with v). A *vertex-distinguishing proper arc coloring* (*semi-vertex-*

distinguishing proper arc coloring) of D is a proper arc coloring with no two vertices (no three vertices) have the same set of colors either for their out-arc sets or for their in-arc sets. Such a coloring is abbreviated *VDPA coloring* (*semi-VDPA coloring*). A digraph is a *vdac-digraph* (svdac-digrpah) if it has a VDPA coloring (semi-VDPA coloring). Clearly, every vdac-digraph is also a svdac-digraph.

Let $n_S^+ = n_S^+(D)$ ($n_S^- = n_S^-(D)$) be the number of vertices with out-arc set (in-arc set) assigned color set S. Alternatively, a (semi-)VDPA coloring of D can be defined as a proper arc coloring such that $n_S^+ \leq 1$ and $n_S^- \leq 1$ ($n_S^+ \leq 2$ and $n_S^- \leq 2$) for any color set S. Denote by $\chi_{vd}'^1(D)$ and $\chi_{vd}'^2(D)$ the minimum numbers of colors required for a 1-type VDPA coloring and a 2-type VDPA coloring of a vdac-digraph D, respectively.

Equitable edge (arc) coloring

For a proper k-edge-coloring (a proper k-arc-coloring) f of a graph G (a digraph D), let E_α be the set of edges (arcs) colored by α and let $e_\alpha = |E_\alpha|$, f is called *equitable* if $|e_\alpha - e_\beta| \leq 1$ for any two colors $\alpha, \beta \in \{1, \ldots, k\}$. Let $\chi_e'(G)$ ($\chi_e'^2(D)$) be the minimum number of colors required for an equitable proper edge (arc) coloring of G (of D) and let $\chi_{evd}'(G)$ ($\chi_{evd}'^i(D)$) be the minimum number of colors required for an equitable VDPE coloring (equitable i-type VDPA coloring, $i \in \{1, 2\}$) of a vdec-graph G (a vdac-digraph D).

Directed path and cycle

A *directed path* or a *directed k-path* of a digraph D is a sequence of vertices $v_1, v_2, \ldots, v_k$ with $v_1v_2, \ldots, v_{k-1}v_k \in E(D)$, and a *directed cycle* or a *directed k-cycle* of D is a sequence of vertices $v_1, v_2, \ldots, v_k$ with $v_1v_2, \ldots, v_{k-1}v_k, v_kv_1 \in E(D)$. Throughout this thesis, a cycle (path) in a digraph always means a *directed* cycle (path).

Vertex-disjoint and arc-disjoint cycles (paths)

Two cycles (paths) are called *vertex-disjoint* (*arc-disjoint*) if they have no common vertex (arc).

Hamilton cycle and Hamiltonian digraph

A digraph D is *Hamiltonian* if it has a *Hamilton cycle*, i.e., a cycle containing all

vertices of D.

Cycle factor

A *cycle factor* of a digraph D is a spanning subdigraph of D whose components are vertex-disjoint cycles. A *k-cycle-factor* of D is a cycle factor consisting of k cycles. Note that a 1-cycle-factor is a Hamilton cycle, i.e., a cycle containing all the vertices of D, and a 2-cycle-factor consists of two *complementary* cycles. We say that D contains all k-cycle-factors if for any possible cycle-lengths $n_1, \ldots, n_k$ with $|V(D)| = n_1 + \cdots + n_k$ there exists a k-cycle-factor with cycle-lengths $n_1, \ldots, n_k$ respectively in D.

Connectivity

A digraph D is *strong* or *1-connected* if there exists a path from u to v for any two vertices u and v of D. Call D is k-connected if the removal of any set of fewer than k vertices results in a strong digraph. A digraph is *cycle-connected* if every two vertices are in a common cycle.

Tournament, bipartite tournament and multipartite tournament

A *tournament* is an orientation of a complete graph and a *bipartite tournament* (*multipartite tournament*) is an orientation of a complete bipartite (*multipartite* or *c-partite*, $c \geq 3$ is an integer) graph.

Regular bipartite tournament $\mathbf{F_{4\cdot k}}$

Define $F_{4\cdot k}$ to be the k-regular bipartite tournament consisting of four independent sets K, L, M, N each of cardinality k, and all possible arcs from K to L, from L to M, from M to N and from N to K (See Figure 1.1).

Cut vertex

A vertex v of a graph (digraph) is a *cut vertex* if its edge set (arc set) can be partitioned into two nonempty sets E_1 and E_2 such that the subgraphs (subdigraphs) induced by E_1 and E_2 have just the vertex v in common.

Universal arc

An arc uv of a digraph D is *universal* if for any vertex w of D there exists a cycle containing both uv and w.

Max cut

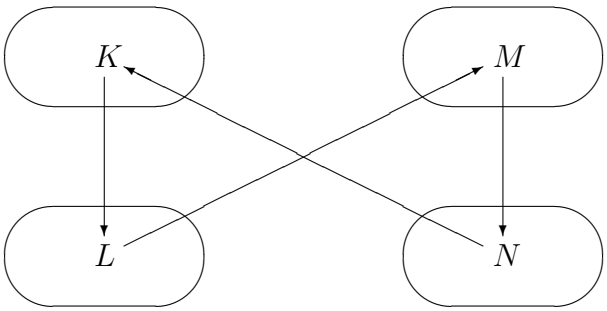

Figure 1.1: Regular bipartite tournament $F_{4\cdot k}$.

For a partition $\{V_1, V_2\}$ of $V(G)$, the set (V_1, V_2) of edges crossing this partition, i.e.,

$$(V_1, V_2) = \{uv \in E(G) : u \in V_1, v \in V_2\}, \tag{1.4}$$

is called a *cut* of G. For a partition $\{V_1, V_2\}$ of $V(D)$, the set (V_1, V_2) of arcs going from V_1 to V_2, i.e.,

$$(V_1, V_2) = \{(u, v) \in E(D) : u \in V_1, v \in V_2\}, \tag{1.5}$$

is called a *directed cut* of D. When no confusion occurs, we use "cut" to denote "directed cut" in a digraph. A *max cut* is a cut of largest size in a graph (digraph).

Let $f(G)$ be the size of a maximum cut of G. Define $f(m)$ to be the minimum of $f(G)$ over graphs of size m. Let $g(D)$ be the size of a maximum directed cut of D. Define $g(m)$ to be the minimum of $g(D)$ over digraphs of size m.

Directed cut cover

A *k-cut-cover* of a digraph D is a family of k directed cuts such that each arc of D belongs to at least one cut. The *cut cover number* $c(D)$ of D is the minimum k for which D has a k-cut-cover.

Cayley graph and Cayley digraph

Let $\mathcal{G}$ be an additive group, and let S be a subset of $\mathcal{G}$ that is closed under taking inverses and does not contain the identity. The *Cayley graph* $X(\mathcal{G}, S)$ is defined with

vertex set $\mathcal{G}$ and edge set

$$\{xy : y - x \in S\}. \tag{1.6}$$

If S is an arbitrary subset of $\mathcal{G}$, then we can define the *Cayley digraph* $X(\mathcal{G}, S)$ with vertex set $\mathcal{G}$ and arc set

$$\{(x, y) : y - x \in S\}. \tag{1.7}$$

Cayley digraph $\mathbf{X}(\mathbb{Z}_2^{\mathbf{k}}, \mathbf{S_k})$

The digraph $\mathbf{X}(\mathbb{Z}_2^{\mathbf{k}}, \mathbf{S_k})$ is a Cayley digraph $X(\mathcal{G}, S)$ with $\mathcal{G} = \mathbb{Z}_2^k$ consists of all binary vectors of length k and

$$S = S_k = \{e_1, \ldots, e_k\}, \tag{1.8}$$

where

$$e_i = \{0 \ldots 1 \ldots 0\} \tag{1.9}$$

in which the ith position is assigned the number "1" and each one of other positions is assigned the number "0". Alternatively, it is a digraph with vertex set $V = \{v_0, v_1, \ldots, v_{2^k-1}\}$ and arc set $E = \{v_i v_j : j - i \equiv 2^t (\text{mod } 2^k)$,where $t \in \{0, 1, ..., k - 1\}\}$.

1.2 Arc colorings

The edge coloring problem is one of the fundamental problems in graph theory and has been studied extensively by many researchers. Clearly, every graph G satisfies $\chi'(G) \geq \Delta(G)$. For a bipartite graph G, König showed that $\chi'(G) = \Delta(G)$. In 1964, Vizing proved that $\chi'(G) \leq \Delta(G) + 1$ for every simple graph G. In this thesis, we mainly consider VDPA colorings of digraphs. First, we present some background and motivation. To be precise, we summarize the main results on proper arc colorings of digraphs and VDPE colorings of graphs. After this, we introduce VDPA colorings of digraphs.

1.2.1 Proper arc colorings

In 1972, Harner and Entringer [48] first considered the 1-type arc colorings of digraphs. The following results have been obtained.

Theorem 1.1 (Harner and Entringer [48]). *Let D be a digraph and k a positive integer. Then $\lceil \log_2 \chi(D) \rceil \leq \chi'^1(D) \leq \min\{k : \chi(D) \leq \binom{k}{\lfloor k/2 \rfloor}\}$.*

Theorem 1.2 (Harner and Entringer [48]). *Let T_n be the transitive tournament on n vertices. Then $\chi'^1(T_n) = \lceil \log_2 n \rceil$.*

Theorem 1.3 (Harner and Entringer [48]). *There exists a digraph D with underlying graph K_n and $\chi'^1(D) = \min\{k : \chi(D) \leq \binom{k}{\lfloor k/2 \rfloor}\}$.*

Theorem 1.4 (Harner and Entringer [48]). *Let D be an acyclic digraph on n vertices. Then $\chi'^1(D) \leq \lceil \log_2 n \rceil$.*

Theorem 1.2 and Theorem 1.3 imply that the bounds in Theorem 1.1 are tight. A digraph $D = (V(D), E(D))$ is *symmetric* if $uv \in E(D)$ implies that $vu \in E(D)$ for any arc uv. In 1981, Poljak and Rödl [77] got the arc chromatic number of symmetric digraphs with respect to its chromatic number.

Theorem 1.5 (Poljak and Rödl [77]). *Let D be a symmetric digraph and k a positive integer. Then $\chi'^1(D) = \min\{k : \chi(D) \leq \binom{k}{\lfloor k/2 \rfloor}\}$.*

In 2006, Bessy *et al.* [24] improved the lower bound in Theorem1.1 for digraphs with no sink or no source.

Theorem 1.6 (Bessy *et al.* [24]). *Let D be a digraph with chromatic number $\chi(D)$.*
(1) If D has no sink, then $\chi'^1(D) \geq \log_2(\chi(D) + 1)$.
(2) If D has no sink and no source, then $\chi'^1(D) \geq \log_2(\chi(D) + 2)$.

Moreover, arc colorings of digraphs with degree restrictions have attracted special attention. Let

$$\Phi(k) = \max\{\chi'^1(D) : D \in D(k)\}, \quad \Phi(k, l) = \max\{\chi'^1(D) : D \in D(k, l)\}. \tag{1.10}$$

Note that the problem of finding $\chi'^1(D)$ of a digraph D is NP-hard (see Poljak and Rödl [77]), and is equal to the problem of finding the minimum number of directed cuts of D such that each arc of D belongs to at least one directed cut.

For convenience, denote the function $\min\{k : n \leq \binom{k}{\lfloor k/2 \rfloor}\}$ mentioned in Theorem 1.1 by $c(n)$. In 2006, Alon *et al.* [6] and Bessy *et al.* [24] got the following similar results independently.

Theorem 1.7 (Alon *et al.* [6]). *Let k and l be two nonnegative integers. Then*
(1) $\chi'^1(D) \leq c(2k+1)$ *for any digraph* $D \in D(k)$.
(2) $\chi'^1(D) \leq c(2k+2l+2)$ *for any digraph* $D \in D(k,l)$.
(3) $\chi'^1(D) \leq c(k+l+1)$ *for any acyclic digraph* $D \in D(k,l)$.

Theorem 1.8 (Bessy *et al.* [24]). *Let k and l be two positive integers. Then*
(1) *If* $k \geq 2$, *then* $\chi'^1(D) \leq c(2k)$ *for any digraph* $D \in D(k)$.
(2) *If* $k+l \geq 3$, *then* $\chi'^1(D) \leq c(2k+2l)$ *for any digraph* $D \in D(k,l)$.

Theorem 1.9 (Bessy *et al.* [24]). *Let k and l be two positive integers. Then*
(1) $\max\{\log_2(2k+3), c(k+1)\} \leq \Phi(k) \leq c(2k+1)$.
(2) $\max\{\log_2(2k+2l+43), c(k+1), c(l+1)\} \leq \Phi(k,l) \leq c(2k+2l+2)$.

Some special cases have been considered.

Theorem 1.10 (Bessy *et al.* [24]). *Let k and l be two positive integers.*
(1) $\Phi(k,0) = \Phi(k)$.
(2) $\Phi(k,1) = \Phi(k)$ *or* $\Phi(k,k) = \Phi(k)+1$.
(3) $\Phi(k+1) = \Phi(k)+2$.
(4) *If* $\Phi(k) = \Phi(k-1)$ *or* $\Phi(k) = \Phi(k+1)$, *then* $\Phi(k,1) = \Phi(k)$.

Theorem 1.11 (Bessy *et al.* [24]). (1) $\Phi(1,1) = \Phi(1,0) = \Phi(1) = 3$.
(2) $\Phi(2,2) = \Phi(2,1) = \Phi(2,0) = \Phi(2) = 4$.
(3) $\Phi(3,3) = \Phi(3,2) = 5$.

In 2011, Bai *et al.* [11] proved the following result.

Theorem 1.12 (Bai *et al.* [11]). $\Phi(k,k) \leq c(2k+1)+1$ *and* $\Phi(4,4) = \Phi(3,3) = 5$.

In 2013, Xu *et al.* [103] considered $\Phi(5,5)$ and $\Phi(6,6)$.

Theorem 1.13 (Xu *et al.* [103]). $5 \leq \Phi(5,5) \leq \Phi(6,6) \leq 6$.

The 2-type proper arc coloring of digraphs is much simpler than the 1-type one. It was considered in [100] and the exact arc chromatic number was obtained.

Theorem 1.14 ([100]). *Let D be a digraph. Then $\chi'^2(D) = \max\{\Delta^+(D), \Delta^-(D)\}$.*

1.2.2 Vertex-distinguishing proper edge colorings

The VDPE coloring of graphs was introduced and studied independently by Aigner et al. [3], by Burris and Schelp [31] and by Horňák and Soták [57]. In 1997, Burris and Schelp [31] conjectured that $\chi'_{vd}(G) \leq |V(G)|+1$. Bazgan et al. [20] verified this conjecture in 1999.

Theorem 1.15 (Bazgan et al. [20]). *Let G be a vdec-graph. Then $\chi'_{vd}(G) \leq |V(G)|+1$.*

Note that the above result is sharp by considering the complete graphs with even order. Burris and Schelp [31] also proposed the following conjecture.

Conjecture 1.1. Let G be a vdec-graph. Then $\chi'_{vd}(G) \in \{\pi(G), \pi(G)+1\}$.

One can see that Conjecture 1.1 is analogous to the Vizing's Theorem on edge colorings. As for this conjecture we do not even know whether the bound of $\chi'_{vd}(G) \leq \pi(G)+c$ holds for some fixed constant c. However, the conjecture has been verified for some special classes of graphs, including complete graphs, complete bipartite graphs, paths, cycles and some trees by Burris and Schelp [31], union of paths, union of cycles by Balister [13], two families of cubic graphs, ladders and unions of K_4, by Taczuk and Woźniak [91] and any graph G with $\Delta(G) \geq \sqrt{2|V(G)|}+4$ and $\delta(G) \geq 5$ by Balister et al. [15]. In addition, graphs with big maximum degree or small maximum degree are considered. Bazgan et al. [19] showed that $\chi'_{vd}(G) \leq \Delta(G)+5$ if $\delta(G) > |V(G)|/3$. Balister et al. [13] showed that $\pi(G) \leq \chi'_{vd}(G) \leq \pi(G)+5$ if $\Delta(G) = 2$. Also, it is worth noting that Burris and Schelp [31] gave an upper bound of $\chi'_{vd}(G)$ for a vdec-graph G as follows.

Theorem 1.16 (Burris and Schelp [31]). *Let G be a vdec-graph with maximum degree Δ. Then $m_1 \leq \chi'_{vd}(G) \leq (\Delta+1)\lfloor 2m_2+5 \rfloor$, where*

$$m_1 = \max\{k! n_k^{1/k} + \frac{k-1}{2} : 1 \leq k \leq \Delta\} \text{ and } m_2 = \max\{n_k^{1/k} : 1 \leq k \leq \Delta\}. \quad (1.11)$$

Besides, Balister [12] considered the VDPE coloring of random graphs and gave a strong bound.

Theorem 1.17 (Balister [12]). *Let G be a random graph on n vertices with edge probability $p = p(n)$. If $\frac{pn}{log\ n}, \frac{(1-p)n}{log\ n} \to \infty$ as $n \to \infty$, then the probability that $\chi'_{vd}(G) = \Delta$ goes to 1 as $n \to \infty$.*

For more details on VDPE colorings, we refer the readers to see [12, 13, 15, 19, 38, 83, 91].

Besides, in a proper edge coloring of a graph, instead of requiring that any two vertices have different color sets, it can be required that any two adjacent vertices have different color sets. Such a coloring is called *adjacent vertex-distinguishing proper edge coloring* (abbreviated *adjacent VDPE coloring*). It was introduced by Zhang et al. [108] and has also been considered intensively. Let $\chi'_{avd}(G)$ be the smallest number of colors required for a adjacent VDPE coloring of a graph G. In 2002, Zhang et al. [108] proposed the following conjecture.

Conjecture 1.2 (Zhang et al. [108]). Let $G \neq C_5$ be a connected graph with at least 3 vertices. Then $\chi'_{avd}(G) \leq \Delta(G) + 2$.

Note that some special graphs, including paths, cycles, trees, complete bipartite graphs and complete graphs, were verified in [108]. Many results have been obtained concerning the bound of $\chi'_{avd}(G)$. Among them, the following results are of special importance.

Theorem 1.18 (Akbari et al. [4]). *Let $G \neq C_5$ be a connected graph with at least 3 vertices. Then $\chi'_{avd}(G) \leq 3\Delta(G)$.*

Dai and Bu [37] improved the above result by one, i.e., $\chi'_{avd}(G) \leq 3\Delta(G) - 1$. Balister et al. [14] gave a general bound depending on the chromatic number $\chi(G)$ of G.

Theorem 1.19 (Balister et al. [14]). *Let $G \neq C_5$ be a connected graph with at least 3 vertices. Then $\Delta(G) \leq \chi'_{avd}(G) \leq \Delta(G) + O(log\chi(G))$.*

In support of Conjecture 1.2, Hatami [51] showed the following result.

Theorem 1.20 (Hatami [51]). *Let $G \neq C_5$ be a connected graph with at least 3 vertices and $\Delta(G) > 10^{20}$. Then $\Delta(G) \leq \chi'_{avd}(G) \leq \Delta(G) + 300$.*

For more details on adjacent VDPE colorings, we refer the readers to [4, 14, 37, 46, 51, 56, 98, 99, 108].

Motivated by the conjectures and results on VDPE colorings for undirected graphs mentioned above, we introduce and study the analogous problem for digraphs, i.e., the *vertex-distinguishing proper arc coloring* (abbreviated *VDPA coloring*) of digraphs.

1.2.3 Vertex-distinguishing proper arc colorings

Note that an isolated vertex can be regarded both as a source and as a sink. One can check that the following fact holds.

Fact 1.1. *A digraph D is a vdac-digraph (resp. svdac-digraph) if and only if D contains at most one source (resp. two sources) and at most one sink (resp. two sinks).*

Now we condier $\chi'^1_{vd}(D)$ and $\chi'^2_{vd}(D)$ of D. It is clear that $\chi'^1_{vd}(D) \geq \chi'^1(D)$ and $\chi'^2_{vd}(D) \geq \max\{\chi'^2(D), \pi(D)\}$. Note that $\chi'^2(D) = \Delta(D)$ and $\pi(D) \geq \Delta(D)$. Thus $\chi'^2_{vd}(D) \geq \pi(D)$. In this thesis, we mainly consider the 2-type arc coloring of digraphs. Unless otherwise stated, the proper arc coloring mentioned below always means the 2-type arc coloring. Analogous to Conjecture 1.1 for undirected graphs, we propose the following conjecture for digraphs.

Conjecture 1.3. Let D be a vdac-digraph. Then $\chi'^2_{vd}(D) = \pi(D)$.

Despite Conjecture 1.3 remains unsolved, some good progresses concerning it have been obtained in Chapter 2. To be precise, we give upper bounds for $\chi'^2_{vd}(D)$ and $\chi'^2_{svd}(D)$ respectively. In particular, the value of $\chi'^2_{vd}(D)$ is obtained for some regular digraph D. Moreover, we show that the values of $\chi'^2_{vd}(D)$ and $\chi'^2_{svd}(D)$ will not be changed if the coloring, in addition, required to be equitable.

1.3 Short cycles

The study of cycles is one of the most important and most studied problems in graph theory. There are many papers seeking sufficient conditions for the existence of a Hamilton cycle in a graph (digraph). In this thesis, we focus on short cycles in digraphs and especially consider the Caccetta-Häggkvist conjecture.

1.3.1 Caccetta-Häggkvist conjecture

The famous Caccetta-Häggkvist conjecture (which developed in 1978 and generalised as an earlier conjecture of Behzad et al. [21]) is one of the most famous conjectures in digraph theory. It concerns the length of short cycles and has inspired years of research into sufficient conditions for short cycles in digraphs. There are two equivalent statements of the conjecture.

Conjecture 1.4 (Caccetta and Häggkvist [32])**.** Every digraph on n vertices without cycles of lengths at most l contains a vertex with outdegree at most $\frac{n-1}{l}$.

Conjecture 1.5 (Caccetta and Häggkvist [32])**.** Every digraph on n vertices with minimum outdegree r contains a cycle of length at most $\lceil \frac{n}{r} \rceil$.

For Conjecture 1.4, it is obviously true for $l = 2$. Note that a digraph with a 2-cycle verifies Conjecture 1.5 and thus it suffices to consider oriented graphs. Also, note that lots of work has been done around the Caccetta-Häggkvist conjecture. To be precise, for Conjecture 1.5, it has been proved for $r = 2$ by Caccetta and Häggkvist [32], for $r = 3$ by Hamidoune [50], for $r = 4, 5$ by Hoáng and Reed [55], and for $r < \sqrt{n/2}$ by Shen [86]. Recently, Lichiardopol [69] proved the conjecture for

oriented graphs with independence number two. Besides, some weaker statements were obtained. Chvátal and Szemerédi [34] proved that every oriented graph on n vertices with minimum outdegree r contains a cycle of length at most $\frac{2n}{r+1}$, and they also proved that such an oriented graph contains a cycle of length at most $\frac{n}{r}+2500$. In 1988, Nishimura [75] reduced the constant from 2500 to 304. In 1998, Shen [87] reduced the constant to 73 and this is the best known improvement.

Theorem 1.21 (Shen [85]). *Every digraph on n vertices with minimum outdegree r contains a cycle of length at most $\lceil \frac{n}{r} \rceil + 73$.*

Recall that the minimum degree of D is $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. It is natural to consider the minimum degree that forces an l-cycle. In 2010, Kelly et al. [62] proposed the following conjecture.

Conjecture 1.6 (Kelly et al. [62]). Let $l \geq 4$ be an integer and let $k \geq 3$ be the smallest integer such that k does not divide l. Then there exists an integer $n_0 = n_0(l)$ such that every oriented graph D on $n \geq n_0$ vertices with $\delta(D) \geq \lfloor n/k \rfloor + 1$ contains an l-cycle.

In the same paper, they gave an affirmative answer to the conjecture for l that is not a multiple of 3.

Theorem 1.22 (Kelly et al. [62]). *Let $l \geq 4$ and let D be an oriented graph on $n \geq 10^{10}l$ vertices with $\delta(D) \geq \lfloor n/3 \rfloor + 1$. Then D contains an l-cycle.*

Later, in 2013, Kühn et al. [63] proved Conjecture 1.6 asymptotically for the case when l is large enough compared to k and $k \geq 7$.

Theorem 1.23 (Kühn et al. [63]). *Let $k \geq 7$ and $l \geq 10^7 k^6$. Suppose that $k \geq 3$ is the smallest integer that does not divide l. Then for all $\eta > 0$ there exists an integer $n_1 = n_1(\eta, l)$ such that every oriented graph D on at least $n \geq n_1$ vertices with $\delta(D) \geq (1+\eta)n/k$ contains an l-cycle.*

The special case of $r = \lceil n/3 \rceil$ has attracted most interest, which states that every digraph with minimum outdegree $\lceil n/3 \rceil$ has a 3-cycle, i.e., a (directed) triangle. Caccetta and Häggkvist [32] gave the first weaker result for this case.

Theorem 1.24 (Caccetta and Häggkvist [32]). *For $\alpha \geq (3-\sqrt{5})/2 \approx 0.3819$, every digraph D with $\delta^+(D) \geq \alpha n$ contains a triangle.*

Bondy [26] showed that $\alpha \geq (2\sqrt{6}-3)/5 \approx 0.3797$ suffices, Shen [85] relaxed it to $\alpha \geq 3-\sqrt{7} \approx 0.3542$, Hamburger et al. [49] improved it to 0.3531, and Hladký et al. [54] improved the bound to 0.3465. The best know value of α, till now, is as follows.

Theorem 1.25 (Joannis de Verclos et al. [60]). *Let D be a digraph on n vertices and let $\alpha \geq 0.3386$. If $\delta^+(D) \geq \alpha n$, then D contains a triangle.*

De Graaf et al. [39] firstly considered the minimum degree instead of the minimum outdegree.

Theorem 1.26 (De Graaf et al. [39]). *For $\beta \geq 0.349$, every digraph D with $\delta(D) \geq \beta n$ contains a triangle.*

Shen [85] showed that $\beta \geq 0.348$ suffices. The best known value of β, till now, is as follows.

Theorem 1.27 (Hamburger et al. [49]). *Let D be a digraph on n vertices and let $\beta \geq 0.346$. If $\delta(D) \geq \beta n$, then D contains a triangle.*

1.3.2 Caccetta-Häggkvist conjecture with forbidden subdigraphs

In particular, characterizing some forbidden subdigraphs is another meaningful way to consider this conjecture. In 2012, Razborov [80] verified the case $l = 3$ with three well defined (induced) forbidden subdigraphs.

Theorem 1.28 ([80]). *Let Γ be an oriented graph on n vertices without directed triangles. If*

(1) Γ *contains none of the oriented graphs of Figure 1.2 as an induced subdigraph, or*

(2) Γ *contains none of the oriented graphs of Figure 1.3 as a subdigraph (not necessarily induced),*

then Γ contains a vertex with outdegree at most $\frac{n-1}{3}$.

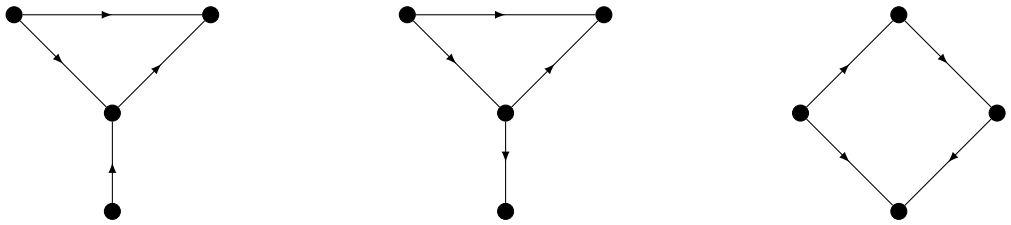

Figure 1.2: Forbidden oriented graphs for Theorem 1.28 (1).

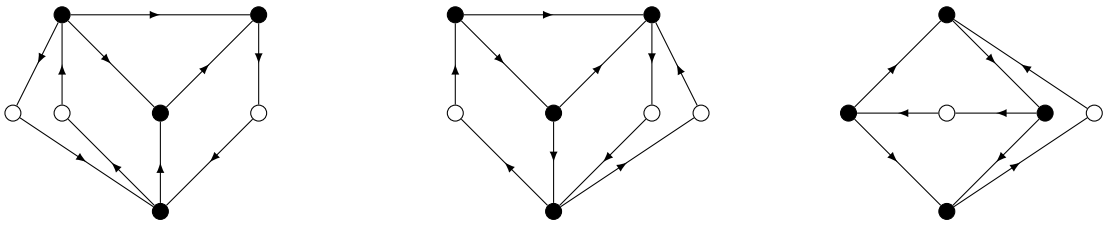

Figure 1.3: Forbidden oriented graphs for Theorem 1.28 (2).

In Chapter 3, we generalize Razborov's result by verifying the conjecture for $l \geq 4$ with $l + 1$ well defined (induced) forbidden subdigraphs.

1.4 Disjoint cycles

There are two types of disjoint cycles in undirected graphs (digraphs), namely, vertex-disjoint cycles and edge-disjoint cycles (resp. arc-disjoint cycles). We will consider disjoint cycles of any lengths and of given length, respectively.

1.4.1 Disjoint cycles of any lengths

The study of vertex-disjoint cycles in undirected graphs has been considered significantly. One of the famous results regarding this is as follows.

Theorem 1.29 (Corrádi and Hajnal [36])**.** *Let G be an undirected graph on at least $3r$ vertices and $\delta(G) \geq 2r$. Then G contains at least r vertex-disjoint cycles.*

The complete 3-partite graph with each partite set having exactly r vertices shows that the result is best possible. By using induction on the minimum degree for Theorem 1.29, we have the following corollary.

Corollary 1.1 (Corrádi and Hajnal [36]). *Let G be an undirected graph on at least $3r$ vertices and $\delta(G) \geq 2r$. Then G contains at least $\binom{r+1}{2}$ edge-disjoint cycles.*

Note that a trivial fact holds with only the minimum degree condition (one can verify this by deleting the edges of a cycle and using the induction on the minimum degree recursively).

Fact 1.2. *Let G be an undirected graph with minimum degree at least $2r$. Then G contains at least r edge-disjoint cycles.*

Justesen [61] improved Theorem 1.29 by showing the following.

Theorem 1.30 (Justesen [61]). *Let G be a graph on at least $3r$ vertices. If $d(x) + d(y) \geq 4r$ for any two non-adjacent vertices x and y of G, then G contains at least r vertex-disjoint cycles.*

Wang [96] strengthened Justesen's result by showing that the result holds if $d(x)+ d(y) \geq 4r - 1$. Motivated by Theorem 1.29, Bermond and Thomassen [23] proposed an analogous conjecture on vertex-disjoint cycles in 1981. This is regarded as one of the most famous conjectures in digraph theory.

Conjecture 1.7 (Bermond and Thomassen [23]). Let D be a digraph with minimum outdegree at least $2r - 1$. Then D contains at least r vertex-disjoint cycles.

The complete digraph on $2r - 1$ vertices implies that if the conjecture is true then it would be best possible. Note that the conjecture is trivially true for $r = 1$. Thomassen [93] and Lichiardopol et al. [71] proved it for $r = 2$ and $r = 3$, respectively. Bessy et al. [25] verified it for regular tournaments in 2010. In 2014, Bang-Jensen et al. [17] verified it for tournaments and proposed a stronger conjecture.

Conjecture 1.8 (Bang-Jensen et al. [17]). Let D be a digraph with girth at least $g \geq 2$ and with minimum outdegree at least $\frac{g}{g-1}r$. Then D contains at least r vertex-disjoint cycles.

There are also other results concerning Conjecture 1.7. Among them, it is worth mentioning the following one, which, firstly, shows that the minimum outdegree can be bounded by a linear function of r.

Theorem 1.31 (Alon [5]). *Let D be a digraph with minimum outdegree at least $64r$. Then D contains at least r vertex-disjoint cycles.*

For more details on Conjecture 1.7, we refer the readers to see [5,17,23,25,71,93]. Note that the analogous statement (a linear bound of minimum degree int terms of r guaranteeing r vertex-disjoint cycles) for undirected graphs are obvious. One can show the following fact by deleting a shortest cycle and then use induction.

Fact 1.3. *Let G be an undirected graph with minimum degree at least $3r-1$. Then G contains at least r vertex-disjoint cycles.*

As a corollary of Theorem 1.31, we have the following result on the number of arc-disjoint cycles.

Corollary 1.2 (Alon [5]). *Let D be a digraph with minimum outdegree at least r. Then D contains at least $\frac{1}{128}r^2$ arc-disjoint cycles.*

By Conjecture 1.7, we can conjecture the following and this can be regarded as an analogous statement of Fact 1.2.

Conjecture 1.9. Let D be a digraph with minimum outdegree at least $2r-1$. Then D contains at least r^2 arc-disjoint cycles.

The maximum number of arc-disjoint cycles in digraphs was considered by Alon et al. [8] and the following conjecture was proposed.

Conjecture 1.10 (Alon et al. [8]). Let D be a r-regular digraph. Then D contains at least $\binom{r+1}{2}$ arc-disjoint cycles.

Three weaker results have been obtained in the same paper.

Theorem 1.32 (Alon et al. [8]). *Let D be a r-regular digraph. Then D contains at least $5r/2-2$ arc-disjoint cycles.*

Theorem 1.33 (Alon et al. [8]). *Let D be a r-regular digraph. Then D contains at least ϵr^2 arc-disjoint cycles, where $\epsilon = 3/2^{19}$.*

Theorem 1.34 (Alon et al. [8]). *Let D be a r-regular digraph. Then D contains at least $r^2/8$ Inr arc-disjoint cycles.*

Very recently, Lichiardopol [70] obtained some new bounds on the maximum number of arc-disjoint cycles in a digraph. To be precise, it was proved that for $r \leq 4$ the result in Theorem 1.32 is valid for all digraphs with minimum outdegree at least r. Also, it was shown that for $r \geq 4$ every digraph with minimum outdegree at least r contains at least $3r - 4$ arc-disjoint cycles.

1.4.2 Disjoint cycles of given lengths

In 2000, Wang considered vertex-disjoint triangles (3-cycles) in digraphs and showed the following. Here let $\delta^*(D) = \min\{d^+(v) + d^-(v) : v \in V(D)\}$.

Theorem 1.35 (Wang [97]). *Let D be a digraph on n vertices with $\delta^*(D) \geq \lfloor (3n - 3)/2 \rfloor$. Then D contains $\lfloor n/3 \rfloor$ vertex-disjoint triangles.*

Later, in 2005, Zhang and Wang [104] considered vertex-disjoint 4-cycles in digraphs.

Theorem 1.36 (Zhang and Wang [104]). *Let D be a digraph on $4r$ vertices with $\delta^*(D) \geq 6r - 2$. Then D contains r vertex-disjoint 4-cycles unless D is isomorphic to a special digraph.*

In 2010, Lichiardopol [68] considered vertex-disjoint cycles of given length in tournaments. The following result has been obtained.

Theorem 1.37 (Lichiardopol [68]). *Let T be a tournament with $\min\{\delta^+(T), \delta^-(T)\} \geq (q - 1)r - 1$. Then T contains r vertex-disjoint q-cycles.*

Lichiardopol [68] conjectured in the same paper that T contains r vertex-disjoint q-cycles if $\delta^+(T) \geq (q - 1)r - 1$. Motivated by the result and the conjecture above, we consider the analogous problem for bipartite tournaments in Chapter 4. As a corollary, we verify Conjecture 1.7 for bipartite tournaments.

1.5 Cycle factors

1.5.1 Cycle factors in graphs

Note that a 1-cycle-factor of a graph is a Hamilton cycle. The following two results are two fundamental results on Hamilton cycles in graphs.

Theorem 1.38 (Dirac [40]). *Let G be a 2-connected graph on n vertices with minimum degree at least $n/2$. Then G has a Hamilton cycle.*

Theorem 1.39 (Ore [76]). *Let G be a graph on n vertices. If $d(x) + d(y) \geq n$ for any two non-adjacent vertices x and y of G, then G has a Hamilton cycle.*

In Theorem 1.29, if the graph has exactly $3k$ vertices then it has a k-cycle-factor such that each cycle is a triangle. This implies a result on k-cycle-factors of graphs. In 1984, El-Zahar [42] considered the 2-cycle-factors of given cycle-lengths in graphs.

Theorem 1.40 (El-Zahar [42]). *Let G be a graph of order $n = n_1+n_2$ with $n_1, n_2 \geq 3$ and minimum degree at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil$. Then G has a 2-cycle-factor of cycle-lengths n_1 and n_2.*

1.5.2 Cycle factors in digraphs

Note that a 1-cycle-factor of a digraph is a directed Hamilton cycle. Define the minimum semidegree of D to be the minimum of its minimum outdegree and minimum indegree. For an analogue of Theorem 1.38 in digrahs, Ghouila-Houri [45] proved the following result.

Theorem 1.41 (Ghouila-Houri [45]). *Let D be a strong digraph on n vertices with $\delta^+(D)+\delta^-(D) \geq n$. Then D has a Hamilton cycle. In particular, if D has minimum semidegree at least $n/2$ then D has a Hamilton cycle.*

For an analogue of Theorem 1.39, Woodall [101] proved the following result.

Theorem 1.42 (Woodall [101]). *Let D be a strong digraph on n vertices with $d^+(x)+d^-(y) \geq n$ for every pair $x \neq y$ of vertices such that $xy \notin E(D)$. Then D has a Hamilton cycle.*

Woodall's theorem is generalized by Meyniel's theorem as follows.

Theorem 1.43 (Meyniel [74]). *Let D be a strong digraph on n vertices with $d(x) + d(y) \geq 2n - 1$ for any two non-adjacent vertices x and y of D. Then D has a Hamilton cycle.*

For 2-cycle-factors in digraphs, Little and Wang [72] got the following theorem.

Theorem 1.44 (Little and Wang [72]). *Let D be a digraph on n vertices. If $d(x) \geq 3(n-1)/2$, then D has a 2-cycle-factor of any given cycle-lengths, i.e., D is strongly 2-cycle-factorable.*

Amar and Raspaud [10] considered the k-cycle-factors of given lengths in digraphs.

Theorem 1.45 (Amar and Raspaud [10]). *Let D be a digraph on n vertices and at least $(n-1)(n-2)+3$ arcs, and let $n_1, \ldots, n_k$ be k integers with $n = n_1 + \ldots + n_k$ and $n_i \geq 3$ for $i = 1, \ldots, k$. Then D has a k-cycle-factor of cycle-lengths $n_1, \ldots, n_k$ except in two cases:*
(1) *$n = 6$, $n_1 = n_2 = 3$ and D contains an independent set with 3 vertices;*
(2) *$n = 9$, $n_1 = n_2 = n_3 = 3$ and D contains an independent set with 4 vertices.*

One can see that the conditions in these theorems almost guarantee that the digraph is "more" than a tournament. It is natural to consider the cycle factors in tournaments and bipartite tournaments.

1.5.3 Cycle factors in bipartite tournaments

Note that every strong tournament is Hamiltonian and is thus has a 1-cycle-factor. The problem of 2-cycle-factors in 2-connected tournaments was completely solved by Reid [81] and Song [89].

Theorem 1.46 (Reid [81] and Song [89]). *Let T be a 2-connected tournament with $|V(T)| \geq 6$. Then T has a 2-cycle-factor of lengths t and $|V(T)| - t$ for all $3 \leq t \leq |V(T)| - 3$, unless T is isomorphic to T_7.*

The problem of 2-cycle-factors in regular bipartite tournaments was investigated by Song [88], Zhang and Song [106], Zhang et al. [105], and Zhang and Wang [107]. They showed that every k-regular bipartite tournament with $k \geq 2$ has a 2-cycle-factor. Volkmann [94] proved that every regular c-partite tournaments with $c \geq 3$ on at least 8 vertices have a 2-cycle-factor. In [105], it was conjectured that every k-regular bipartite tournament not isomorphic to $F_{4\cdot k}$ has a 2-cycle-factor of all possible cycle-lengths.

Conjecture 1.11 (Zhang et al. [105])**.** Let B be a k-regular bipartite tournament not isomorphic to $F_{4\cdot k}$. Then B has a 2-cycle-factor of even cycle-lengths t and $|V(B)|-t$ for all $4 \leq t \leq |V(B)|-4$.

Conjecture 1.11 is true for $t = 4$. We show that Conjecture 1.11 is true for $t = 6$ in Chapter 5.

1.6 Universal arcs and directed cuts

1.6.1 Universal arcs in digraphs

The concept *universal arc* is very new and was first appeared in [2] in 1999. In the same paper, Ádám introduced the following problem.

Problem 1.1. *Does every cycle-connected digraph contain a universal arc?*

Hetyei [52] conjectured in 2001 that the answer would be yes. Hubenko [58], and Volkmann and Winzen [95] verified this for bipartite tournaments in 2008 and multipartite tournaments in 2009, respectively.

Theorem 1.47 (Hubenko [58])**.** *Let B be a cycle-connected bipartite tournament. Then every maximal cycle of B has a universal arc.*

Theorem 1.48 (Volkmann and Winzen [95])**.** *Let D be a multipartite tournament. If D is cycle-connected, then D contains a universal arc. If D is 1-connected with $\delta(D) \geq 2$, then every longest cycle of D contains a universal arc.*

1.6.2 Universal arcs in tournaments

Let $\mathbf{T}_\mathbf{s}^*$ be the set of 1-connected tournaments with one cut vertex v such that the subtournaments induced by $N^+(v)$ and $N^-(v)$ are 1-connected.

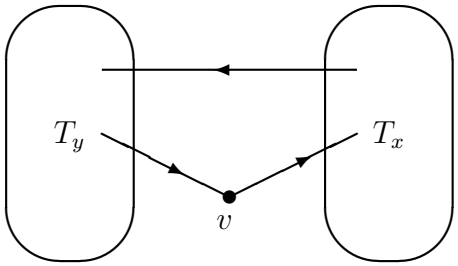

Figure 1.4: A tournament in $\mathbf{T}_\mathbf{s}^*$.

Note that an arc in a Hamilton cycle is obviously a universal arc and a tournament not 1-connected has no universal arc. Recall that a tournament is 1-connected if and only if it has a Hamilton cycle (see [33]). So a 1-connectivity can guarantee the existence of a universal arc in a tournament. In Chapter 6, we show that the converse statement is also true. Moreover, we show that every arc of a tournament is universal if and only if it is 2-connected or belong to $\mathbf{T}_\mathbf{S}^*$.

1.6.3 Max cuts in graphs and digraphs

The well-known Max Cut problem asks for a largest cut in a graph (digraph). It is a NP-hard problem and has been the focus of extensive study, both from the algorithmic aspect in computer science and the extremal aspect in combinatorics. The algorithmic problem asks for good algorithms that determine $f(G)$ and $g(G)$. The extremal problem asks for the value of $f(m)$ and $g(m)$. Here we focus on the extremal problem and especially consider the digraphs. We first give some progress of the problem.

Note that a random bipartition of an undirected graph G gives a cut with expected size $|E(G)|/2$. Thus $f(m) \geq m/2$. In 1973, Edwards showed the following theorem.

Theorem 1.49 (Edwards [41])**.**

$$f(m) \geq \lceil \frac{m}{2} + \frac{-1+\sqrt{1+8m}}{8} \rceil.$$

The bound is tight by considering the complete graphs. In 1998, Alon and Halperin [7] gave the following lower bound for $f(m)$.

Theorem 1.50 (Alon and Halperin [7])**.** *For $m = \binom{n}{2} + k$ with $0 \leq k < n$, we have*

$$f(m) \geq \lfloor \frac{n^2}{4} \rfloor + \min\{\lceil \frac{n}{2} \rceil, f(k)\}.$$

Furthermore, they conjectured that the "=" in the above theorem always holds.

Conjecture 1.12 (Alon and Halperin [7])**.** For $m = \binom{n}{2} + k$ with $0 \leq k < n$, we have

$$f(m) = \lfloor \frac{n^2}{4} \rfloor + \min\{\lceil \frac{n}{2} \rceil, f(k)\}.$$

For more results on cuts in graphs, we refer to [65, 78]. It is easy to see that $g(m) \geq f(m)/2$. So $g(m) \geq m/4$ and furthermore by Theorem 1.49 we have

Theorem 1.51.

$$g(m) \geq \lceil \frac{m}{4} + \frac{-1+\sqrt{1+8m}}{16} \rceil.$$

Alon [6] and Lehel [66] considered the maximum directed cuts in digraphs with degree restrictions and obtained the following results respectively.

Theorem 1.52 (Alon [6])**.** *If D is a digraph with m edges and contains no vertex with outdegree larger than k, then $g(D) \geq (\frac{1}{4} + \frac{1}{8k+4})m$.*

Theorem 1.53 (Lehel [66])**.** *If $D \in D(k,k)$ is acyclic (i.e., contains no cycle) and has m arcs, then $g(D) \geq (\frac{1}{4} + \frac{1}{8k+4})m$.*

As mentioned in Section 1.2.1, the minimum cut cover problem is equivalent to the 1-type arc coloring of digraphs. For more introduction on cut cover of digraphs, one can see in Section 1.2.1.

1.6.4 Directed cuts in a type of Cayley digraph

Note that $|\mathbb{Z}_2^k| = 2^k$ and $|S_k| = k$. So $d^-(v) = d^+(v) = k$ for every vertex v of $X(\mathbb{Z}_2^k, S_k)$, and $|E(X(\mathbb{Z}_2^k, S_k))| = k2^k$. Alon *et al.* [6] showed that for every digraph D with m arcs and maximum outdegree at most d,

$$g(D) \geq (\frac{1}{4} + \frac{1}{8d+4})m. \tag{1.12}$$

Thus

$$g(X(\mathbb{Z}_2^k, S_k)) \geq (\frac{1}{4} + \frac{1}{8k+4})k2^k. \tag{1.13}$$

Let

$$h(n) = \min\{p : \binom{p}{\lfloor p/2 \rfloor} \geq n\}, \tag{1.14}$$

where n and p are positive integers. Alon *et al.* [6] proved that the arcs of digraphs in which every vertex has either outdegree at most k or indegree at most k can be covered by $h(4k+2)$ cuts. Bai *et al.* [11] showed that $h(2k+1)+1$ cuts suffice. It follows that

$$c(X(\mathbb{Z}_2^k, S_k)) \leq h(2k+1) + 1. \tag{1.15}$$

However, it is not best possible. In Chaper 6, we give better bound for $g(X(\mathbb{Z}_2^k, S_k))$ and exact value of $c(X(\mathbb{Z}_2^k, S_k))$.

Chapter 2

Vertex-Distinguishing Proper Arc Colorings of Digraphs

2.1 Introduction

For a digraph D, we define $B^D = (X, Y; E)$ to be a (unique) corresponding balanced bipartite graph satisfying that $X = Y = V(D)$ and $E(B^D) = \{xy : x \in X, y \in Y, \overrightarrow{xy} \in A(D)\}$. Let $\delta^X(B^D)$ and $\Delta^X(B^D)$ (resp. $\delta^Y(B^D)$ and $\Delta^Y(B^D)$) be the minimum and the maximum of degrees of vertices in X (resp. in Y) , respectively. Put $\delta(B^D) = \min\{\Delta^X(B^D), \Delta^Y(B^D)\}$ and $\Delta(B^D) = \max\{\delta^X(B^D), \delta^Y(B^D)\}$. One can check that the following facts hold by definitions.

$$\begin{aligned} \Delta^+(D) = \Delta^X(B^D),\ \Delta^-(D) = \Delta^Y(B^D), \\ \delta^+(D) = \delta^X(B^D),\ \delta^-(D) = \delta^Y(B^D), \\ \Delta(B^D) = \Delta(D),\ \delta(B^D) = \delta(D). \end{aligned} \tag{2.1}$$

The proper arc coloring of D is now equivalent to the proper edge coloring of B^D. Note that $\chi'(B) = \Delta(B)$ for any bipartite graph B. The following fact (see also in [100]) holds directly.

Fact 2.1. *Let D be a digraph. Then $\chi'^2(D) = \chi'(B^D) = \Delta(B^D) = \Delta(D)$.*

A proper edge coloring of a bipartite graph $B = (X, Y; E)$ is called *partial-vertex-*

distinguishing if no two vertices in X and no two vertices in Y are incident with the same set of colors. A bipartite graph is *partial-vertex-distinguishing edge-colorable* or a *pvdec-bipartite graph* if it has a partial-vertex-distinguishing proper edge coloring (abbreviated *PVDPE coloring*). Note that B is a pvdec-bipartite graph if and only if both X and Y contain at most one isolated vertex.

One can check that the VDPA coloring of a vdac-digraph D is equivalent to the PVDPE coloring of B^D. Denote by $\chi'_{pvd}(B)$ the minimum number of colors required for a partial-vertex-distinguishing proper edge coloring of B. Let $\delta(B) = \min\{\delta_X(B), \delta_Y(B)\}$ and $\Delta(B) = \max\{\Delta_X(B), \Delta_Y(B)\}$.

Fact 2.2. *Let D be a vdac-digraph. Then $\chi'^2_{vd}(D) = \chi'_{pvd}(B^D)$.*

Fact 2.3. *There is a one-to-one correspondence between digraphs (not necessarily simple and loops are allowed) with order n and balanced bipartite graphs with order $2n$.*

Let $B = (X, Y; E)$ be a bipartite graph. Let $n^X_{d_X}$ (resp. $n^Y_{d_Y}$) be the number of vertices of degree d_X in X (resp. degree d_Y in Y) in B. Define

$$\pi'(B) = \min\left\{ k \in \mathbf{Z} : \begin{array}{l} \binom{k}{d_X} \geq n^X_{d_X} \ for\ \delta_X \leq d_X \leq \Delta_X \\ \binom{k}{d_Y} \geq n^Y_{d_Y} \ for\ \delta_Y \leq d_Y \leq \Delta_Y \end{array} \right\}. \tag{2.2}$$

It is clear that $\chi'_{pvd}(B) \geq \pi'(B)$. Analogous to Conjecture 1.3 we conjecture that $\chi'_{pvd}(B) = \pi'(B)$.

Let α and β be two colors of a proper edge coloring of an undirected graph G. An (α, β)-Kempe-path is a maximal path in G consisting of edges colored by α and β. Note that the colors α and β appear alternatively in an (α, β)-Kempe-path.

In the rest of this chapter, the proofs for the results of digraphs will be transferred to the proofs the corresponding results of the balanced bipartite graphs and the Kemp-path will play an important role in the proofs.

2.2 Vertex-distinguishing proper arc colorings of digraphs

Note that an isolated vertex can be regarded both as a source and as a sink. One can check that the following fact holds.

Fact 2.4. *A digraph D is a vdac-digraph (resp. svdac-digraph) if and only if D contains at most one source (resp. two sources) and at most one sink (resp. two sinks).*

Despite Conjecture 1.3 remains unsolved, some good progresses concerning it have been obtained. In particular, we get the following result.

Theorem 2.1. *Let D be a vdac-digraph on n vertices and $t \geq 1$ an integer. If $\delta(D) \geq \frac{n-1}{t}$, then $\chi'^2_{vd}(D) \leq \min\{n, \Delta(D) + t\}$.*

Corollary 2.1. *Let D be a vdac-digraph on n vertices. Then $\chi'^2_{vd}(D) \leq n$.*

We will show that $\chi'^2_{vd}(\overrightarrow{K}_n) = n$, see in Theorem 2.3 as follows, where $\overrightarrow{K}_n$ is a complete symmetric digraph on n vertices. This implies that Theorem 2.1 and Corollary 2.1 cannot be improved in general.

Proof of Theorem 2.1. For a proper edge coloring f of a bipartite graph B, let $F(u)$ be the set of colors incident with u. It suffices to show the following result.

Lemma 2.1. *Let B be a pvdec-balanced bipartite graph on $2n$ vertices and with $\delta(B) \geq \frac{n-1}{t}$, where $t \geq 1$ is an integer. If $\Delta(B) \leq n-1$, then $\chi'_{pvd}(B) \leq \min\{n, \Delta(B) + t\}$. If $\Delta(B) = n$, then $\chi'_{pvd}(B) \in \{n, n+1\}$.*

Proof. We first prove the following claim, on which the above lemma is heavily based.

Claim 2.1. *Let $B = (X, Y; E)$ be a pvdec-bipartite graph and $t \geq 1$ an integer. Then there exists a proper k-edge-coloring of B with $n_S^X \leq t$ and $n_S^Y \leq t$ for any $S \subseteq \{1, \ldots, k\}$, where k is the minimum integer such that $d_X(k - d_X) \geq n_{d_X}^X - t$ for $\delta_X \leq d_X \leq \Delta_X$ and $d_Y(k - d_Y) \geq n_{d_Y}^Y - t$ for $\delta_Y \leq d_Y \leq \Delta_Y$.*

Proof. With respect to an edge coloring f of B, a vertex $x \in X$ (resp. $y \in Y$) is called *bad* if $n^X_{F(x)} \geq 2$ (resp. $n^Y_{F(y)} \geq 2$) and *good* otherwise. Note that $k \geq \Delta(B) = \chi'(B)$. There exists a proper k-edge coloring of B. Let f_0 be one coloring with minimal number of bad vertices among all the proper k-edge colorings of B.

If $\max\{n^X_S, n^Y_S\} \leq t$ for any $S \subseteq \{1, \ldots, k\}$, then we are done. Assume without loss of generality that $n^X_{F_0(u)} \geq t+1$ for some $u \in X$. Since $t \geq 1$, we have that u is a bad vertex. Note that $d_X(k - d_X)$ can be regarded as the maximum number of distinct color sets that can be obtained from $F_0(u)$ by changing the color of one edge incident with u. Note also that $n^X_{d_X} - t - 1$ can be regarded as the possible maximum number of distinct color sets different from $F_0(u)$. Since $d_X(k - d_X) \geq n^X_{d_X} - t$ for all $\delta_X \leq d_X \leq \Delta_X$, there exist two colors α and β with $\alpha \in F_o(u)$, $\beta \notin F(u)$ and $n^X_{F_0(u)-\{\alpha\}+\{\beta\}} = 0$.

Let $P_1 = u_1 \ldots v_1$ be an (α, β)-Kempe-path with $u_1 = u$. Let f_1 be a new edge coloring of B obtained from f_0 by exchanging the colors α and β on the path P_1. Note that f_1 is proper and the color set of any vertex distinct from u_1 and v_1 remains the same. Since $n^X_{F_0(u)-\{\alpha\}+\{\beta\}} = 0$ and $F_1(u_1) \neq F_1(v_1)$ if $v_1 \in X$, we have $n^X_{F_1(u_1)} = 1$ and u_1 is good with respect to f_1. Recall that u_1 is bad with respect to f_0 and f_0 has minimum number of bad vertices, we have that v_1 is good with respect to f_0 and bad with respect to f_1. Assume without loss of generality that $v_1 \in X$ and let $u_2 \in X$ with $F_1(v_1) = F_1(u_2)$. Note that $u_2 \neq u_1$. Consider the (α, β)-Kempe-path $P_2 = u_2 \ldots v_2$. We interchange the colors α and β on P_2 and denote the new edge coloring by f_2. Note that now u_2 is good and v_2 is bad with respect to f_2.

More generally, for the edge coloring f_{i-1} together with an (α, β)-Kempe-path $P_i = u_i \ldots v_i$, we can get a new proper coloring f_i by exchanging the colors α and β on the path P_{i-1} such that u_i is bad with respect to f_{i-1} and good with respect to f_i, and v_i is good with respect to f_{i-1} and bad with respect to f_i. Moreover, we can continue as above to get another (α, β)-Kempe-path $P_{i+1} = u_{i+1} \ldots v_{i+1}$ such that $F_i(u_{i+1}) = F_i(v_i)$ and $u_{i+1} \notin P_1 \cup \ldots \cup P_i$. Note that we can get an (α, β)-Kempe-path P_{i+1} for every (α, β)-Kempe-path P_i and these (α, β)-Kempe-paths $P_1, \ldots, P_i, P_{i+1}, \ldots$ are pairwise vertex disjoint. It contradicts the fact that

both X and Y are finite sets. □

Consider the case $t = 1$ in the above claim, we have the following claim.

Claim 2.2. *Let $B = (X, Y; E)$ be a pvdec-bipartite graph with bipartition (X, Y). Let k be the minimum integer such that $d_X(k - d_X) \geq n^X_{d_X} - 1$ for all $\delta_X \leq d_X \leq \Delta_X$ and $d_Y(k - d_Y) \geq n^Y_{d_Y} - 1$ for all $\delta_Y \leq d_Y \leq \Delta_Y$. Then $\chi'_{pvd}(B) \leq k$.*

Let $\delta(B) \geq \frac{n-1}{t}$. Since $d_X(\Delta(B) + t - d_X) \geq \frac{n-1}{t} \cdot t = n - 1 \geq n^X_{d_X} - 1$ for all $\delta_X \leq d_X \leq \Delta_X$ and similarly $d_Y(\Delta(B) + t - d_Y) \geq n^Y_{d_Y} - 1$ for all $\delta_Y \leq d_Y \leq \Delta_Y$, then by Claim 2.2 we have $\chi'_{pvd}(B) \leq \Delta(B) + t$.

If $\Delta(B) \leq n - 1$, then $n(d - 1) \geq (d + 1)(d - 1)$ for any $\delta(B) \leq d \leq \Delta(B)$. It follows that $d(n - d) \geq n - 1$. So $d_X(n - d_X) \geq n_X - 1$ for all $\delta_X \leq d_X \leq \Delta_X$ and $d_Y(n - d_Y) \geq n_Y - 1$ for all $\delta_Y \leq d_Y \leq \Delta_Y$. By Claim 2.2, we have $\chi'_{pvd}(B) \leq n$ and thus $\chi'_{pvd}(B) \leq \{n, \Delta(B) + t\}$.

If $\Delta(B) = n$, then $\chi'_{pvd}(B) \geq n$ and similarly as above we have $\chi'_{pvd}(B) \leq n + 1$. So $\chi'_{pvd}(B) \in \{n, n + 1\}$. □

The proof of Theorem 2.1 is complete. □

Corresponding to Claim 2.1 and Claim 2.2, we have two analogous results for digraphs.

Lemma 2.2. *Let D be a vdac-digraph and $t \geq 1$ an integer. Let k be the minimum integer such that $d^+(k - d^+) \geq n_{d^+} - t$ for all $\delta^+(D) \leq d^+ \leq \Delta^+(D)$ and $d^-(k - d^-) \geq n_{d^-} - t$ for all $\delta^-(D) \leq d^- \leq \Delta^-(D)$. Then D has a proper k-arc-coloring with $n^+_S \leq t$ and $n^-_S \leq t$ for any $S \subseteq \{1, \ldots, k\}$.*

Lemma 2.3. *Let D be a vdac-digraph. Let k be the minimum integer such that $d^+(k - d^+) \geq n_{d^+} - 1$ for all $\delta^+(D) \leq d^+(D) \leq \Delta^+(D)$ and $d^-(k - d^-) \geq n_{d^-} - 1$ for all $\delta^-(D) \leq d^-(D) \leq \Delta^-(D)$. Then $\chi'^2_{vd}(D) \leq k$.*

2.3 Semi-vertex-distinguishing proper arc colorings of digraphs

For any vdac-digraph D, till now, we cannot show that $\pi(D)$ colors can guarantee a VDPA coloring but we can show that $\pi(D)$ colors can guarantee a semi-VDPA coloring.

Theorem 2.2. *Let D be a vdac-digraph. Then $\chi'^2_{svd}(D) \leq \pi(D) \leq \chi'^2_{vd}(D)$.*

Proof. Clearly, $\chi'^2_{vd}(D) \geq \pi(D)$. It suffices to show that $\chi'^2_{svd}(D) \leq \pi(D)$, i.e., D has a proper $\pi(D)$-arc-coloring with $n_S^+ \leq 2$ and $n_S^- \leq 2$ for any $S \subseteq \{1, \ldots, \pi(D)\}$. We will get this by showing that D has a a proper $\pi(D)$-arc-coloring with $|n_S^+ - n_{S'}^+| \leq 2$ and $|n_S^- - n_{S'}^-| \leq 2$ for all $S, S' \subseteq \{1, \ldots, k\}$ with $|S| = |S'|$.

Note that a VDPA coloring satisfies that $|n_S^+ - n_{S'}^+| \leq 1$ and $|n_S^- - n_{S'}^-| \leq 1$ for any two color sets S and S'. Define a *balanced* proper arc coloring of a digraph to be a proper arc coloring with $|n_S^+ - n_{S'}^+| \leq 2$ and $|n_S^- - n_{S'}^-| \leq 2$ for any two color sets S and S'. Define an *optimal* k-coloring of D to be a proper arc coloring of D with k colors and with minimal value of

$$\sum_S ((n_S^+)^2 + (n_S^-)^2). \tag{2.3}$$

By definition, on can check that an optimal k-coloring of D exists if and only if $k \geq \chi'^2(D) = \Delta(D)$. We first show that an optimal coloring is also balanced.

Lemma 2.4. *In an optimal k-arc-coloring of D, we have $|n_S^+ - n_{S'}^+| \leq 2$ and $|n_S^- - n_{S'}^-| \leq 2$ for all $S, S' \subseteq \{1, \ldots, k\}$ with $|S| = |S'|$.*

We transfer this problem to bipartite graphs. A proper coloring of a bipartite graph with k colors is called *partial-balanced* if $|n_S^X - n_{S'}^X| \leq 2$ and $|n_S^Y - n_{S'}^Y| \leq 2$ for any two color sets $S, S' \subseteq \{1, \ldots, k\}$ with $|S| = |S'|$, and is called *partial-semi-vertex-distinguishing* if $n_S^X \leq 2$ and $n_S^Y \leq 2$ for any $S \subseteq \{1, \ldots, k\}$.

Define an *optimal* k-edge-coloring of B to be a proper edge coloring of B with k

colors and with minimal value of

$$\sum_{S}((n_S^X)^2 + (n_S^Y)^2). \tag{2.4}$$

Note that an optimal k-edge-coloring exists for every B with $k \geq \chi'(B) = \Delta(B)$. To prove Lemma 2.4, it suffices to show that an optimal edge coloring of a balanced bipartite graph is also partial-balanced.

Lemma 2.5. *For an optimal k-edge-coloring of $B = (X, Y; E)$, we have $|n_S^X - n_{S'}^X| \leq 2$ and $|n_S^Y - n_{S'}^Y| \leq 2$ for all $S, S' \subseteq \{1, \ldots, k\}$ with $|S| = |S'|$.*

Proof. Let f be an optimal edge coloring of B with color set $\{1, \ldots, k\}$. Denote by $[\alpha \diamond \beta]$ the family of subsets of $\{1, \ldots, k\}$ in which each subset contains precisely one of α and β. Let $V_{[\alpha\diamond\beta]}$ be the set of vertices with color sets in $[\alpha \diamond \beta]$. Define an involution $i_{\alpha\beta}$ on subsets in $[\alpha \diamond \beta]$ by interchanging the colors α and β. We first show the following claim.

Claim 2.3. *Assume we have an optimal k-edge-coloring f of B and $S \in [\alpha\diamond\beta]$. Then we can change the coloring by interchanging α and β on some edges such that we get a new optimal coloring f' in which both the pair of values n_S^X and $n_{i_{\alpha\beta}S}^X$ (differing by one) and one other pair of values $n_{S'}$ and $n_{i_{\alpha\beta}S'}$ (both in X or both in Y, differing by one) are interchanged, and all other n_{S*}^X and n_{S*}^Y remain the same. Moreover, we have $|n_S^X - n_{i_{\alpha\beta}S}^X| \leq 1$ and $|n_S^Y - n_{i_{\alpha\beta}S}^Y| \leq 1$ for any $S \in [\alpha \diamond \beta]$.*

Proof. Let M_S^X (resp. M_S^Y) be a matching between the vertices in X_S (resp. Y_S) and the vertices in $X_{i_{\alpha\beta}S}$ (resp. $Y_{i_{\alpha\beta}S}$) with maximal number of edges. Let M be the union of $M_S^X \cup M_S^Y$, where $S \in [\alpha \diamond \beta]$. Note that a vertex $v \in X$ (resp. $v \in Y$) is unmatched by M implies that $n_{S(v)}^X > n_{i_{\alpha\beta}S(v)}^X$ (resp. $n_{S(v)}^Y > n_{i_{\alpha\beta}S(v)}^Y$). Define K a matching with $E(K) = \{uv : u$ and v are the endvertices of an (α, β)-Kempe-path of $B\}$. Note that $V(K) = V_{[\alpha\diamond\beta]}$. Let $H_{\alpha,\beta}$ be a graph with $V(H_{\alpha,\beta}) = V_{[\alpha\diamond\beta]}$ and $E(H_{\alpha,\beta}) = E(M) \cup E(K)$. One can check that $\Delta(H_{\alpha,\beta}) \leq 2$ and $H_{\alpha,\beta}$ consists of paths and cycles.

Let v_0 be an arbitrary vertex unmatched by M. One can check that the path of H starting with v_0 will also end with a vertex v_{2l+1} unmatched by M. Denote this path by $P = v_0 \dots v_{2l+1}$. Interchange the colors α and β for the color sets of the vertices of P, i.e., interchange the colors α and β for the color sets of Kempe-paths with endvertices v_{2i} and v_{2i+1} for every $0 \leq i \leq l$. Note that $S(v_{2i+1}) = i_{\alpha\beta}S(v_{2i+2})$ and $S(v_{2i+2}) = i_{\alpha\beta}S(v_{2i+1})$. Assume without loss of generality that $v_0 \in X$.

Assume first that $v_{2l+1} \in Y$. Note that only

$$n^X_{S(v_0)},\ n^X_{i_{\alpha\beta}S(v_0)},\ n^Y_{S(v_{2l+1})},\ n^Y_{i_{\alpha\beta}S(v_{2l+1})}$$

will be changed. Also, we have

$$n^X_{S(v_0)} > n^X_{i_{\alpha\beta}S(v_0)},\ n^Y_{S(v_{2l+1})} > n^Y_{i_{\alpha\beta}S(v_{2l+1})}.$$

Furthermore, $n^X_{S(v_0)}$ and $n^Y_{S(v_{2l+1})}$ decrease by one, and $n_{i^X_{\alpha\beta}S(v_0)}$ and $n^Y_{i_{\alpha\beta}S(v_{2l+1})}$ increase by one. Recall that f is an optimal edge coloring. If $n^X_{S(v_0)} \geq n^X_{i_{\alpha\beta}S(v_0)} + 2$ or $n^Y_{S(v_{2l+1})} \geq n^Y_{i_{\alpha\beta}S(v_{2l+1})} + 2$, then the value of (2.4) will decrease which contradicts the optimality of f. Thus $n^X_{S(v_0)} = n^X_{i_{\alpha\beta}S(v_0)} + 1$ and $n^Y_{S(v_{2l+1})} = n^Y_{i_{\alpha\beta}S(v_{2l+1})} + 1$. Note that the new coloring is also optimal since the value of (2.4) remains the same.

Now assume that $v_{2l+1} \in X$. One can check that $S(v_0) \neq S(v_{2l+1})$. In fact, by assuming without loss of generality that $S(v_0) \cap \{\alpha, \beta\} = \alpha$, we have that $S(v_{2i+1}) \cap \{\alpha, \beta\} = \beta$ if $v_{2i+1} \in X$, and $S(v_{2i+1}) \cap \{\alpha, \beta\} = \alpha$ if $v_{2i+1} \in Y$. Similar to the above analysis, we have that $n^X_{S(v_0)}$ and $n^X_{S(v_{2l+1})}$ will decrease one, and $n^X_{i_{\alpha\beta}S(v_0)}$ and $n^X_{i_{\alpha\beta}S(v_{2l+1})}$ will increase one. By the fact that $n^X_{S(v_0)} > n^X_{i_{\alpha\beta}S(v_0)},\ n^X_{S(v_{2l+1})} > n^X_{i_{\alpha\beta}S(v_{2l+1})}$ and f is optimal. We have $n^X_{S(v_0)} = n^X_{i_{\alpha\beta}S(v_0)} + 1$ and $n^X_{S(v_{2l+1})} = n^X_{i_{\alpha\beta}S(v_{2l+1})} + 1$. The resulting edge coloring is also optimal. □

Denote by $S \vartriangle S'$ the symmetric difference of S and S'. Define $d(S, S') = \frac{1}{2}|S \vartriangle S'|$ the distance of S and S'. Assume the opposite that there exist S_1, S_2 with $|S_1| = |S_2|$ and without loss of generality $n^X_{S_1} \geq n^X_{S_2} + 3$. Among all optimal k-colorings, choose one coloring and sets S_1, S_2 with $d(S_1, S_2)$ is as small as possible.

Let $d = |S_1| = |S_2|$, $S_+ = S_1 \cup S_2$, $S_- = S_1 \cap S_2$ and $[S_-, S_+]_d = \{S : |S| = d \text{ and } S_- \subseteq S \subseteq S_+\}$. Note that $S_1, S_2 \in [S_-, S_+]_d$. Note also that for any $S \neq S_1, S_2$ we have $d(S_1, S_2) > \max\{d(S, S_1), d(S, S_2)\}$. So by the assumption at the beginning of this paragraph we have $n^X_{S_1} > n^X_S > n^X_{S_2}$.

By Claim 2.3 we have that $n^X_{S_1}$ and n^X_S can be interchanged if $d(S_1, S) = 1$. Let α be a color with $\alpha \in S_1$ and $\alpha \notin S_2$. Let $S \in [S_- \cup \{\alpha\}, S_+]_d$. Then $d(S_1, S) < d(S_1, S_2)$. By the minimality of $d(S_1, S_2)$ we have that $n^X_{S_2}$ will interchange with S' such that $d(S_2, S') = 1$ simultaneously. So $|n^X_{S_1} - n^X_S| = 1$. In fact, for any $S \in [S_- \cup \{\alpha\}, S_+]_d$ with $S \neq S_1$, we can interchange $n^X_{S_1}$ with n^X_S by a sequence of steps. It follows that $|n^X_{S_1} - n^X_S| = 1$ for any $S \neq S_1, S_2$. Similarly, we can get that $|n^X_{S_2} - n^X_S| = 1$ for any $S \neq S_1, S_2$. Thus $|n^X_{S_1} - n^X_{S_2}| \leq 2$, which contradicts the assumption that $n^X_{S_1} \geq n^X_{S_2} + 3$. □

If there exists some S with $n_S \geq 3$, assume that $|S| = d$, then by Lemma 2.5 we have $n_S \geq 1$ for every S' with $S' \subseteq \{1, \ldots, k\}$ and $|S'| = |S|$. It follows that $n_d \geq \binom{k}{d} - 1 + 3 = \binom{k}{d} + 2$, contradicting to the definition of k.

The proof of Theorem 2.2 is complete. □

2.4 Vertex-distinguishing proper arc colorings of regular digraphs

Let D^n_d be a d-regular digraph on n vertices. In particular, D^n_n is a complete symmetric digraph on n vertices and D^n_n consists of a complete symmetric digraph on n vertices and n loops. Note that D^n_d is a vdac-digraph and D^n_n is not a simple digraph. For convenience, let

$$k_{d,n} = \min\{k : \binom{k}{d} \geq n\}. \tag{2.5}$$

Note that $\pi(D^n_d) = k_{d,n}$. By Conjecture 1.3 we have the following conjecture.

Conjecture 2.1. $\chi'^2_{vd}(D^n_d) = \pi(D^n_d) = k_{d,n}$.

Although we have not proved it completely, many cases have been verified.

Theorem 2.3. *Let D_d^n be a d-regular digraph on n vertices. Then Conjecture 2.1 holds both for $n \leq 7$ and for $d \in \{1,2\} \cup [k_{2,2n}+4, n]$; moreover, it nearly holds for $d \in [k_{2,n}-2, k_{2,2n}+3]$, where $k_{2,n} = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$ and $k_{2,2n} = \left\lceil \frac{1+\sqrt{1+16n}}{2} \right\rceil$. To be precise, we have*

(1) $\chi'^2_{vd}(D_d^n) = k_{d,n}$ *if* $n \leq 7$;

(2) $\chi'^2_{vd}(D_1^n) = k_{1,n} = n$;

(3) $\chi'^2_{vd}(D_2^n) = k_{2,n} = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$;

(4) $\chi'^2_{vd}(D_d^n) \leq d + k_{2,n} - 2 \leq 2k_{2,n} - 5 = \lceil \sqrt{1+8n} \rceil - 4$ *if* $3 \leq d \leq k_{2,n} - 3$;

(5) $\chi'^2_{vd}(D_d^n) \in \{k_{d,n}, k_{d,n}+1\}$ *and* $k_{d,n} = d+2$ *if* $k_{2,n} - 2 \leq d \leq k_{2,2n} + 3$;

(6) $\chi'^2_{vd}(D_d^n) = k_{d,n} = d+2$ *if* $k_{2,2n} + 4 \leq d \leq n-2$;

(7) $\chi'^2_{vd}(D_d^n) = k_{d,n} = d+1$ *if* $n-1 \leq d \leq n$.

Proof. Recall that a regular vdac-digraph corresponds to a regular balanced pvdec-bipartite-graph. Let B_d^n be a d-regular balanced bipartite graph on $2n$ vertices. Note that B_d^n is a pvdec-bipartite graph. Since the proof of Theorem 2.3 (1) will use the result of Theorem 2.3 (2) and (3), the proofs will be given in this order: (2), (3), (1), (4), (5), (6) and (7). We first show the following claim.

Claim 2.4. *If $k_{2,n} - 2 \leq d \leq n-2$, then $k_{d,n} = d+2$.*

Proof. Clearly, $k_{d,n} \geq d+1$. Since $\binom{d+1}{d} = d+1 < n$ for $d \leq n-2$ and

$$\binom{d+2}{d} = \binom{d+2}{2} = \frac{(d+2)(d+1)}{2} \geq \frac{k_{2,n}(k_{2,n}-1)}{2} = \binom{k_{2,n}}{2} \geq n, \qquad (2.6)$$

we have $k_{d,n} = d+2$. □

(2) Note that $k_1^n = n$. The result follows directly from Corollary 2.1.

(3) It suffices to show the following lemma.

Lemma 2.6. $\chi'_{pvd}(B_2^n) = k_{2,n} = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$.

Proof. B is a hamiltonian cycle or a union of vertex disjoint cycles. Put the $\binom{k}{2}$ copies of the 2-color sets in the order as indicated in Figure 2.1. We will give an

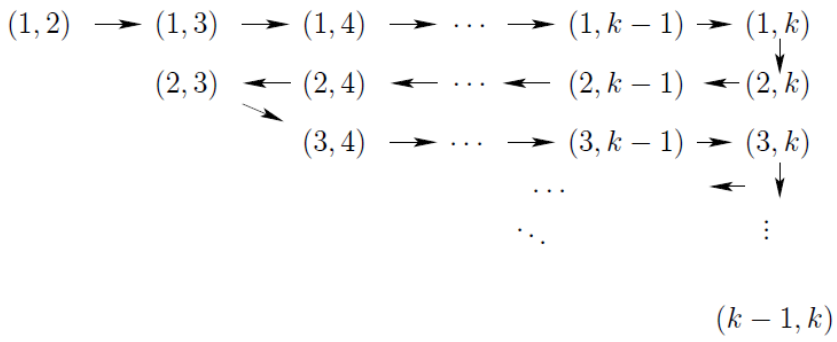

Figure 2.1: Sequence of the 2-color sets

algorithm to color the edges such that these color sets appear on vertices of each part of B one by one, until there is no edge left. This will complete the proof.

According to the sequence of the 2-color sets in Figure 2.1, we have a sequence of colors.

$$1, 2, 1, 3, \ldots, 1, k-1, 1, k, k, 2, k-1, 2, \ldots, 4, 2, 3, 2, 3, 4, \ldots, 3, k-1, 3, k, k, 4, \ldots \quad (2.7)$$

Now we give a partial vertex-distinguishing k-edge-coloring of B through an algorithm.

Algorithm

Input: A 2-regular balanced bipartite graph B.

Output: A vertex-distinguish k-edge-coloring of B.

Step 1. Let $c(i)$ be the i-th number of color sequence (2.7). Set $i = 1$.

Step 2. Choose an uncolored cycle arbitrarily, denoted by $u_1u_2 \ldots u_{2t}u_1$ in clockwise order. For any vertex u, denote the successor and the predecessor of u by u^+ and u^-, respectively. Let $u = u_1$, $v = u_2$. Color the edge uv with $c(i)$. Set $i = i + 1$ and color edge vv^+ with $c(i)$.

Step 3. ***While*** $v \neq u$, i.e., the cycle is not yet well-colored, ***do***

If $c(i) = k$, color uu^- with $c(i + 1)$, which equals to k obviously.

Set $u = u^-$, $i = i + 2$.

If $c(i)$ is odd and $c(i) = c(i + 1) + 1$, color uu^- with $c(i + 1)$ and u^-u^{--} with

$c(i+2)$. Set $u = u^{--}$, $i = i+3$.

Else, color vv^+ with $c(i+1)$. Set $v = v^+$, $i = i+1$.

End while

Step 4. If there exists uncolored cycles, back to **Step 2**.

In the coloring process of the algorithm above, we call a *jump* appears if the coming two uncolored edges will receive two colors in different rows according to Figure 2.1. If the cycle can be colored with no jump, then we can color it just follow the "**Else**" part of the **Step 3**. This can be seen in Figure 2.2. If not, then either the coming four uncolored edges will receive colors $i,\ k,\ k$ and $i+1$ (which induces a contradiction that two adjacent edges receive the same color k) or the coming four uncolored edges will receive colors $j,\ j+1,\ j$ and $j+1$ (which induces that $(j, j+1)$ appear on two vertices in the same part). The two cases will be avoided by the first "**If**" and the second "**If**" of **Step 3**, respectively. One can see this in Figure 2.3.

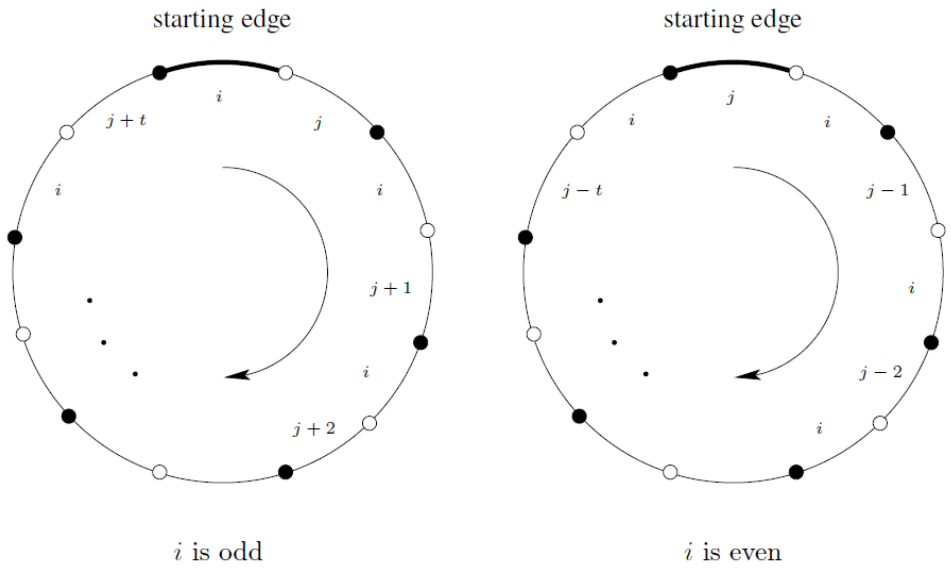

Figure 2.2: Coloring of a cycle with no jump

Thus we give a PVDPE coloring of B. □

The proof of Theorem 2.3 (3) is complete.

(1) Let $n \leq 7$. Then either $d \in \{1, 2\}$ or $d \geq \frac{n-1}{2}$. If $d \in \{1, 2\}$, then by Theorem 2.3 (2) and (3) we have $\chi'_{vd}(D^n_d) = k_{d,n}$. If $d \geq \frac{n-1}{2}$, then by Theorem 2.1 and Claim

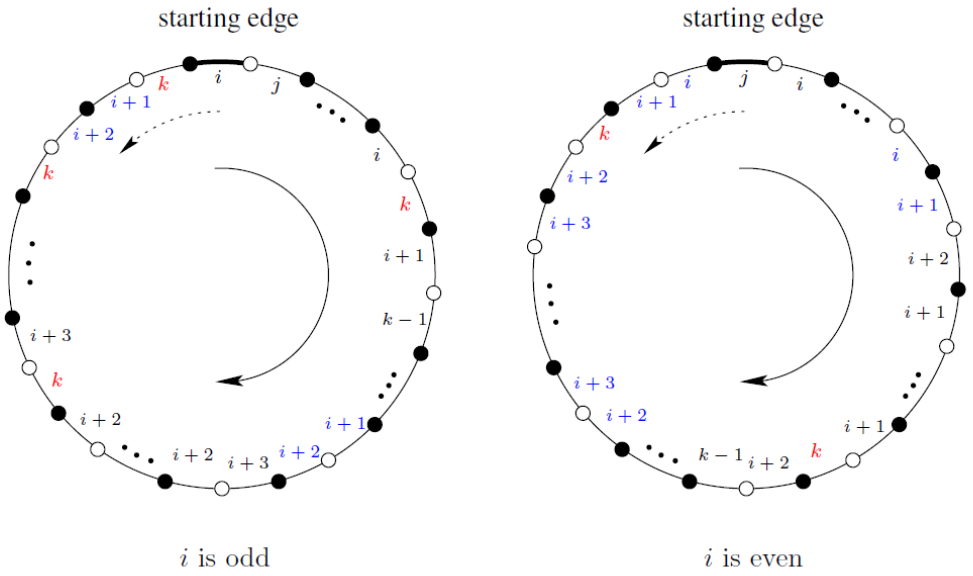

Figure 2.3: Coloring of cycles with jumps

2.4 we have $\chi'_{vd}(D_d^n) \leq \Delta(B_d^n) + t \leq d + 2 = k_{d,n}$ and thus $\chi'_{vd}(D_d^n) = k_{d,n}$. So $\chi'_{pvd}(D_d^n) = k_{d,n}$ for $n \leq 7$.

(4) Every D_d^n can also be defined as a union of a D_i^n and a D_{d-i}^n with the same vertex set. Note that every D_i^n has a VDPA coloring with $\chi'^2_{vd}(D_i^n)$ colors and every D_{d-i}^n has a proper arc coloring with $d-i$ colors and with the same color set for every vertex. Then we have that

$$\chi'^2_{vd}(D_d^n) \leq \chi'^2_{vd}(D_i^n) + d - i \tag{2.8}$$

for any $i \leq d$. Note that $\chi'^2_{vd}(D_i^n) = k_{2,n} = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$ and $d \leq k_{2,n} - 3$. Take $i = 2$, we have

$$\chi'^2_{vd}(D_d^n) \leq \chi'^2_{vd}(D_2^n) + d - 2 \leq 2k_{2,n} - 5 = \left\lceil \sqrt{1+8n} \right\rceil - 4. \tag{2.9}$$

(5) By Claim 2.4, we have $k_{d,n} = d + 2$ now and thus it suffices to show the following lemma for corresponding bipartite graph B_d^n.

Lemma 2.7. *If $k_{2,n} - 2 \leq d \leq k_2^{2n} + 1$, then $\chi'_{pvd}(B_d^n) \in \{d+2, d+3\}$.*

Proof. Clearly, $\chi'_{pvd}(B_d^n) \geq d+2$. Now we show that $\chi'_{pvd}(B_d^n) \leq d+3$.

Let f be a proper $(d+2)$-edge-coloring of B_d^n with color set $T = \{1, \ldots, d+2\}$ and let $Z \in \{X, Y\}$. The coloring f is *Z-distinguishing* if no two vertices in Z are incident with the same color set, i.e., $n_S^Z \leq 1$ for any $S \subseteq T$ with $|S| = d$. And f is *Z-semi-distinguishing* if no three vertices in Z are incident with the same color set, i.e., $n_S^Z \leq 2$ for any $S \subseteq T$ with $|S| = d$. The following result holds.

Claim 2.5. *Let $d \geq k_{2,n} - 2$. Then B_d^n has an X-distinguishing and Y-semi-distinguishing proper edge coloring with color set T.*

Proof. We first show that B_d^n has an X-distinguishing proper edge coloring with color set T by using the result of list-edge-colorings of bipartite graphs.

Given an undirected graph G with edge set $E(G)$. A *list-edge-assignment* L of G is an assignment of lists of distinct colors to the edges of G. We call G is *k-list-edge-colorable* if for any list-edge-assignment L with $L(e) \geq k$ for any $e \in E(G)$ there exists a proper edge coloring f with $f(e) \in L(e)$ for any $e \in E(G)$. Recall that Galvin [44] has proved that every bipartite graph B is $\Delta(B)$-list-edge-colorable.

Since $\binom{d+2}{d} \geq n$, there exist at least n pairwise distinct color sets with d colors. It follows that every vertex x of X can be assigned a color set $S(x) \subseteq T$ with $|S(x)| = d$ and $S(x_1) \neq S(x_2)$ for any two distinct vertices x_1 and x_2 in X. Let L be a list-edge-assignment of B with $L(e) = S(x)$ for any e incident with x. Since $\Delta(B) = d$, then by the result of Galvin [44] we can get a proper edge coloring f_1 of B such that every edge incident with x is colored by one color in $S(x)$. It follows directly that $F_1(x) = S(x)$ for any vertex x in X. Thus f_1 is an X-distinguishing proper edge coloring.

Then we show that B_d^n has an X-distinguishing Y-semi-distinguishing proper edge coloring with $d+2$ colors.

Define an *optimal X-distinguishing coloring* to be an X-distinguishing proper edge coloring with minimal value of $\sum_S (n_S^Y)^2$. Then by the same method used in

the proof of Claim 2.3, we can get that the following claim holds. Since the proof is very similar to that of Claim 2.3, we omit the details here.

Claim 2.6. *Assume we have an optimal X-distinguishing coloring f of B_d^n and $S \subseteq [\alpha \diamond \beta]$. Then we can change the coloring by interchanging α and β on some edges such that we get a new optimal X-distinguishing coloring f' in which both the pair of values n_S^Y and $n_{i_{\alpha\beta}S}^Y$ (differing by one) and one other pair of values $n_{S'}$ and $n_{i_{\alpha\beta}S'}$ (both in X or both in Y, differing by one) are interchanged, and all other $n_{S^*}^X$ and $n_{S^*}^Y$ remain the same. Moreover, we have $|n_S^X - n_{i_{\alpha\beta}S}^X| \leq 1$ and $|n_S^Y - n_{i_{\alpha\beta}S}^Y| \leq 1$ for any $S \in [\alpha \diamond \beta]$.*

Now by the similar analysis used in the proof of Lemma 2.5 we can show that $|n_s^Y - n_{S'}^Y| \leq 2$ for any $S, S' \subseteq \{1, \ldots, d+2\}$ with $|S| = |S'|$. This implies that every optimal X-distinguishing coloring is an X-distinguishing Y-semi-distinguishing proper edge coloring for B_d^n with $d \geq k_{2,n} - 2$. □

Given an X-distinguishing Y-semi-distinguishing proper edge coloring f^* of B_d^n with $d+2$ colors. If f^* is a PVDPE coloring, then we are done. Now assume that there exist two vertices $u, v \in Y$ with $F(u) = F(v)$. Since $\binom{d+2}{d} \geq n$, there exists $S \subseteq T$ with $|S| = d$ and $n_S^Y = 0$. Let $T \backslash S = \{\alpha, \beta\}$. By Claim 2.3, we know that every bad vertex should be incident with two edges colored with α and β respectively. Otherwise we get a color set S' with $n_{S'}^Y = 2$, but $d(S, S') = 1$ which contradicts Claim 2.3.

Then for any two vertices $u, v \in Y$ with $F(u) = F(v)$, we recolor the α-colored edge incident with u with a new color α' and keep the colors of the other edges incident with u. Besides, the colors of the edges incidents with v unchanged. Then we get a new proper edge coloring of B_d^n with $d+3$ colors and one can check that it is a PVDPE coloring. □

The proof of Theorem 2.3 (5) is complete.

(6) By Claim 2.4, we have $\pi'(B_d^n) = d+2$ and $\chi'_{pvd}(B_d^n) \geq d+2$. Recall that

Balister et al. [15] have showed that $\chi'_{vd}(G) = \pi(G)$ for every vdec-graph G with $\Delta(G) \leq \sqrt{2|V(G)|} + 4$ and $\delta(G) \geq 5$. Note that

$$k_{2,2n} = \min\{k \in \mathbf{Z} : \binom{k}{2} \geq 2n\} = \left\lceil \frac{1+\sqrt{1+16n}}{2} \right\rceil$$

and

$$k_{2,2n} + 4 \geq 2\sqrt{n} + 4 = \sqrt{4n} + 4 = \sqrt{2|V(B_d^n)|} + 4 \geq 5.$$

We have $\chi'_{vd}(B_d^n) = \pi(B_d^n) = d + 2$. Clearly, $\chi'_{pvd}(B_d^n) \leq \chi'_{vd}(B_d^n)$. Thus $\chi'_{pvd}(B_d^n) = d + 2$.

(7) Clearly, $k_{d,n} \geq d + 1$ for $d \in \{n-1, n\}$. Since $\binom{d+1}{d} = d + 1 \geq n$, we have $k_{d,n} = d + 1$. By Corollary 2.1, we have $\chi'^2_{vd}(D_{n-1}^n) = n$. By Lemma 2.2, we have $\chi'^2_{vd}(D) \leq n+1$ if loops are allowed in D and thus $\chi'^2_{vd}(D_n^n) = n+1$. Here, we will give explicit VPPA colorings for D_{n-1}^n and D_n^n with n and $n+1$ colors, respectively. Note that D_n^n consists of a D_{n-1}^n and n loops (each vertex has a loop). Then the VDPA coloring of D_n^n follows from the VDPA coloring of d_{n-1}^n and all the loops colored by one new color. So we only consider D_{n-1}^n in the following. Let $\{1, \ldots, n\}$ be the vertex set of D_{n-1}^n and let $\overrightarrow{ij}$ be the arc of D_{n-1}^n with tail i and head j.

We will first use the existence of diagonal Latin square of order n on $1, 2, \ldots, n$ to offer a VDPA coloring for D_{n-1}^n with n colors.

A *Latin square* A_n of order n on $1, \ldots, n$ is an array of n rows and n columns such that every row and column consists of $1, \ldots, n$. Let a_{ij} be the element of A_n on the ith row and jth column. Call A_n is *diagonal* if $a_{ii} = i$ for $1 \leq i \leq n$. Hilton [53] showed that a diagonal A_n exists for any $n \geq 3$. Let A_n^* be a diagonal Latin square. Then we can color the arc $\overrightarrow{ij}$ of D_{n-1}^n with a_{ij} of A_n^*. It follows that $S^+(v_i) = S^-(v_i) = \{1, \ldots, n\} \setminus \{i\}$ for any $1 \leq i \leq n$. This implies a VDPA coloring of D_{n-1}^n.

Since the diagonal Latin square is not easy to get, then we give a more intuitive VDPA coloring of D_{n-1}^n as follows.

Case 1. *n is odd.*

$$A_1 = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 3 & 4 & \dots & n & 1 \\ 3 & 4 & 5 & \dots & 1 & 2 \\ & & \vdots & & \vdots & \\ n-1 & n & 1 & \dots & n-3 & n-2 \\ n & 1 & 2 & \dots & n-2 & n-1 \end{pmatrix} \tag{2.10}$$

Based on the matrix A_1 above, we color the arc $\overrightarrow{ij}$ of D^n_{n-1} with a_{ij} of A_1. Then

$$S^+(v_i) = S^-(v_i) = \begin{cases} \{1,\dots,n\} \setminus \{2i-1\}, & for \;\; 1 \le i \le (n+1)/2; \\ \{1,\dots,n\} \setminus \{2i-(n+1)\}, & otherwise. \end{cases} \tag{2.11}$$

One can check that this is a VDPA coloring of D^n_{n-1}.

Case 2. *n is odd.*

$$A_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-2 & n-1 & n \\ 1 & n & 2 & 3 & \dots & n-3 & n-2 & n-1 \\ n & 1 & n-1 & 2 & \dots & n-4 & n-3 & n-2 \\ n-1 & n & 1 & n-2 & \dots & n-5 & n-4 & n-3 \\ & & \vdots & \vdots & & \vdots & & \\ 5 & 6 & 7 & 8 & \dots & 4 & 2 & 3 \\ 4 & 5 & 6 & 7 & \dots & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 & \dots & n & 1 & 2 \end{pmatrix} \tag{2.12}$$

Based on the matrix A_2 above, we color the arc $\overrightarrow{ij}$ of D with a_{ij} of A_2. Note that each row of A_2 consists of numbers $1,\dots,n$. Note also that in the j-th column the number $j+1(mod\ n)$ does not appear and the number $n-j+2(mod\ n)$ appears

twice. Then

$$\begin{cases} S^+(v_i) = \{1, \ldots, n\} \setminus \{n - i + 2(mod\ n)\},\ for\ 1 \leq i \leq n. \\ S^-(v_i) = \{1, \ldots, n\} \setminus \{i + 1(mod\ n)\},\ for\ 1 \leq i \leq n. \end{cases} \tag{2.13}$$

One can check that this is a VDPA coloring of D^n_{n-1}.

The proof of Theorem 2.3 (7) is complete.

Now we finish the proof of Theorem 2.3. □

2.5 Equitable vertex-distinguishing proper arc colorings of digraphs

In 2008, Rudašová and Soták [83] showed that $\chi'_{evd}(G) = \chi'_{vd}(G)$. We study the analogous problem for digraphs and show that both $\chi'^2_{vd}(D)$ and $\chi'^2_{svd}(D)$ will not be changed if the coloring is, in addition, required to be equitable.

Theorem 2.4. *Let D be a vdac-digraph. Then $\chi'^2_{evd}(D) = \chi'^2_{vd}(D)$ and $\chi'^2_{esvd}(D) = \chi'^2_{svd}(D)$.*

Proof. The proofs of $\chi'_{esvd}(D) = \chi'_{svd}(D)$ and $\chi'_{evd}(D) = \chi'_{vd}(D)$ are similar. So we only present the proof of $\chi'_{evd}(D) = \chi'_{vd}(D)$ in this section. Note that $\chi'_{evd}(D) = \chi'_{epvd}(B^{(D)})$. It suffices to show the following result on balanced bipartite graphs.

Lemma 2.8. *Let B be a pvdec-bipartite-graph. Then $\chi'_{epvd}(B) = \chi'_{pvd}(B)$.*

Proof. Note that $\chi'_{epvd}(B) \geq \chi'_{pvd}(B)$ and it suffices to show $\chi'_{epvd}(B) \leq \chi'_{pvd}(B)$. We get it by showing that B has an equitable PVDPE coloring using $\chi'_{pvd}(B)$ colors.

Recall that B is a pvdec-bipartite-graph. Among all PVDPE colorings of B with $k = \chi'_{pvd}(B)$ colors, choose one ψ with minimum $\sum_{a,b} |e_a - e_b|$ and then choose two colors α and β in ψ such that $e_\alpha - e_\beta$ is maximum. Assume the opposite that ψ is not equitable. Then $e_\alpha - e_\beta \geq 2$. For any $\gamma \in \{1, \ldots, k\} \backslash \{\alpha, \beta\}$, we have $e_\beta \leq e_\gamma \leq e_\alpha$ by the choices of α and β.

Since $e_\alpha > e_\beta$, there exists a vertex u with $S(u) \cap \{\alpha, \beta\} = \alpha$ and $n_{i_{\alpha,\beta}S(u)} = 0$. Actually, if $n_{i_{\alpha,\beta}S(u)} \neq 0$ for each $S(u)$ with $S(u) \cap \{\alpha, \beta\} = \alpha$ then $e_\alpha \leq e_\beta$, a contradiction.

Let $P_1 = v_1 \dots v_1'$ be an (α, β)-Kempe-path with one end vertex $u = v_1$ and, without loss of generality, the other end vertex $v_1' \in X$. If there exists $v_2 \in X$ with $S(v_2) = i_{\alpha,\beta}S(v_1')$, then we can let $P_2 = v_2 \dots v_2'$ be the (α, β)-Kempe-path with two end vertices v_2 and v_2'. Continue this process until there exists a Kempe-path $P_t = v_t \dots v_t'$ such that $i_{\alpha,\beta}S(v_t')$ unused in the part where v_t' lies in. This process will terminate in finite steps since the number of vertices is finite. Denote the union of these Kempe-paths by H.

We can distinguish two cases for v_t'.

Case 3. *$S(v_t') \cap \{\alpha, \beta\} = \alpha$ and $e_\alpha - e_\beta = 1$ in H.*

Case 4. *$S(v_t') \cap \{\alpha, \beta\} = \beta$ and $e_\alpha - e_\beta = 0$ in H.*

Since $e_\alpha \geq e_\beta + 2$, there exists such a union of Kempe-paths of Case 1, without loss of generality, say H^*. We construct a new coloring ψ' of B as follows: interchange the colors α and β on the edges of H^*. This new coloring ψ' is still a VDPA coloring since the color sets of the internal vertices are not changed, and for the starting vertex u and ending vertex v, $S(u), S(v)$ are changed to $i_{\alpha,\beta}S(u), i_{\alpha,\beta}S(v)$ with $n_{i_{\alpha,\beta}S(u)} = n_{i_{\alpha,\beta}S(v)} = 0$, respectively.

Now consider the sum $\sum_{a,b} |e_a - e_b|$ for the coloring ψ'. Compare it with the original sum for ψ, the following facts hold:

- $|e_\alpha - e_\beta| + |e_\beta - e_\alpha|$ will decrease 4;
- $|e_\alpha - e_\gamma| + |e_\gamma - e_\alpha| + |e_\beta - e_\gamma| + |e_\gamma - e_\beta|$ will decrease 4 if $e_\beta < e_\gamma < e_\alpha$ and it remains the same if $e_\gamma \in \{e_\alpha, e_\beta\}$.

Hence, the sum will decrease at least 4, a contradiction to the choice of ψ. □

The proof of Theorem 2.4 is complete. □

At the end of this section, we give a simple proof for the results of $\chi_e'(G)$ and $\chi_e'(D)$, where G is an arbitrary undirected graph and D is an arbitrary digraph.

Although the fact blow seems somewhat trivial and we are almost sure that more than one researcher have proved it before, for the completeness of this part, we give the sketch of its proof here.

Fact 2.5. *Let G be an undirected graph. Then $\chi'_e(G) = \chi'(G)$.*

Proof. Among all proper $\chi'(G)$-edge-colorings of G, choose one named ψ with minimum value of $\sum_{a,b} |e_a - e_b|$, and then choose two colors α and β with maximum value of $e_\alpha - e_\beta$ in ψ. Assume the opposite that ψ is not equitable, then $e_\alpha - e_\beta \geq 2$. For any $\gamma \in \{1, \ldots, k\} \backslash \{\alpha, \beta\}$, we have $e_\beta \leq e_\gamma \leq e_\alpha$ by the choices of α and β. Note that interchanging the colors on an (α, β)-Kempe-path implies a new proper edge coloring. It follows from $e_\alpha - e_\beta \geq 2$ that there exists an (α, β)-Kempe-path with more edges colored by α than colored by β. Then we interchange the colors of the edges of such an (α, β)-Kempe-path. Now consider the sum $\sum_{a,b} |e_a - e_b|$ for the resulting coloring. Compare it with the original sum for ψ, the following facts hold.

- $|e_\alpha - e_\beta| + |e_\beta - e_\alpha|$ will decrease 4;
- $|e_\alpha - e_\gamma| + |e_\gamma - e_\alpha| + |e_\beta - e_\gamma| + |e_\gamma - e_\beta|$ will decrease 4 if $e_\beta < e_\gamma < e_\alpha$ and it remains the same if $e_\gamma \in \{e_\alpha, e_\beta\}$.

This contradicts the minimality of $\sum_{a,b} |e_a - e_b|$. Thus ψ is equitable. □

Note that $\chi'^2_e(D) = \chi'_e(B^D) = \chi'(B^D)$. The analogous result for digraphs holds directly.

Fact 2.6. *Let D be a digraph. Then $\chi'^2_e(D) = \chi'^2(D) = \Delta(D)$.*

2.6 Conclusion

In this chapter, the (semi-)VDPA coloring of digraphs is introduced. Many results on $\chi'^2_{vd}(D)$, where D is a vdac-digraph, have been obtained. We give upper bounds for $\chi'^2_{vd}(D)$ and $\chi'^2_{svd}(D)$ respectively. In particular, the value of $\chi'^2_{vd}(D)$ is obtained for some regular digraph D. Moreover, we show that the values of $\chi'^2_{vd}(D)$ and $\chi'^2_{svd}(D)$ will not be changed if the coloring, in addition, required to be equitable.

For further consideration, it would be interesting to consider the *strong* 2-type VDPA coloring f of vdac-digraphs, here "strong" we mean that the color sets in $\{F^+(v), F^+(v), F^-(u), F^-(v)\}$ are pairwise distinct for every two vertices u and v. One can check that it is equal to the VDPE colorings of balance bipartite graphs if loops are allowed in the vdac-digraphs.

Despite few analysis on $\chi'^1_{vd}(D)$, it is also an interesting problem and would be difficult too. Analogous to Corollary 2.1, it seems that $\chi'^1_{vd}(D) \leq n$ also holds. Directed cycles are trivial examples supporting this conjecture.

Chapter 3

Short Cycles in Digraphs with Forbidden Subdigraphs

3.1 Introduction

An *oriented* graph is a digraph without loops, parallel arcs or directed 2-cycles. Let D be an oriented graph with vertex set $V(D)$ and arc set $E(D)$. For a subset S of $V(D)$, denote by $D|_S$ the oriented graph induced by S. For a subdigraph P of D, denote by $D\backslash P$ the oriented graph induced by $V(D)\backslash V(P)$. For a vertex v of D, denote by $N_D^+(v)$ and $d_D^+(v) = |N_D^+(v)|$ its outneighborhood and outdegree in D, respectively. For better presentation, we use $\langle u, v\rangle$ to denote an arc uv of D, and for an arc $\langle u, v\rangle$, u is its *tail* and v is its *head*. Vertices u and v are *independent* if neither $\langle u, v\rangle$ nor $\langle v, u\rangle$ is an arc. Let $k \geq 3$ be a positive integer. Define a *quasi-k-cycle* be an oriented graph that can be obtained by reversing the direction of one arc of a cycle of length k. Or simply, we use *quasi-cycle* for quasi-k-cycle when the context is clear. Let n, l, r with $n \geq l \geq 2$ and $n \geq r$ be three positive integers.

3.2 Caccetta-Häggkvist conjecture with induced forbidden subdigraphs

In particular, characterizing some forbidden subdigraphs is another meaningful way to consider this conjecture. For the case $l = 3$ of Conjecture 1.4, Lichiardopol [69] verified it with one induced forbidden subdigraph I_3; it was noted in [80] that it holds with one induced forbidden subdigraph $\vec{K}_{1,2}$, it was also noted that Seymour verified it with one induced forbidden subdigraph $\vec{K}_{2,1}$; and as a corollary of a result in [80], it is true with one induced forbidden subdigraph $\vec{P}_3$. The four oriented graphs mentioned here can be found in Figure 3.1.

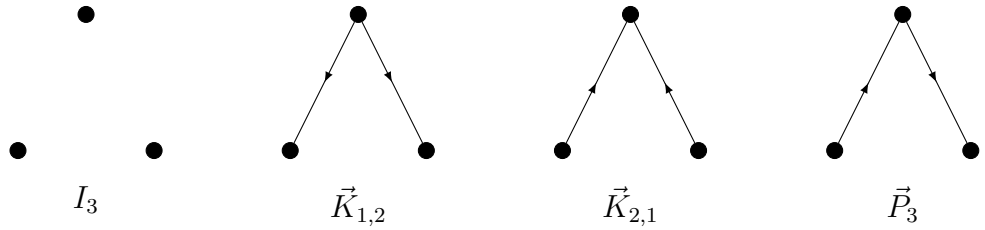

Figure 3.1: Some oriented graphs on three vertices.

Besides, by deeply considering the nature of the conjectured extremal configurations, Razborov [80] verified the case $l = 3$ with three well defined induced forbidden subdigraphs (see Figure 3.2) as follows.

Theorem 3.1 (Razborov [80])**.** *Let D be an oriented graph on n vertices without directed triangles. If D contains none of the oriented graphs of Figure 3.2 as an induced subdigraph, then D contains a vertex with outdegree at most $\frac{n-1}{3}$.*

We generalize Theorem 3.1 to the case $l \geq 4$ by the following theorem.

Theorem 3.2. *Let D be an oriented graph on n vertices without cycles of lengths at most l. If D contains none of the oriented graphs of Figure 3.3 as an induced subdigraph, then D contains a vertex with outdegree at most $\frac{n-1}{l}$.*

Proof. Let D be the oriented graph defined in Theorem 3.2. Assume that Caccetta-Häggkvist conjecture holds for all the proper induced subdigraphs of D. Throughout

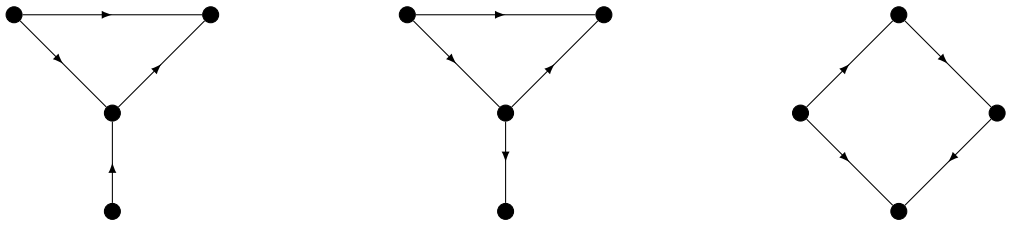

Figure 3.2: Forbidden oriented graphs for Theorem 3.1.

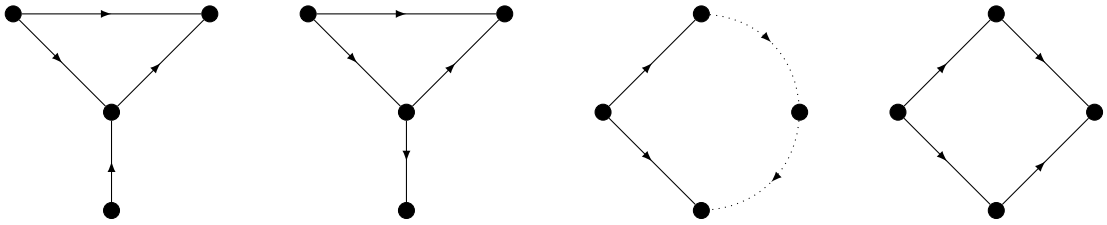

Figure 3.3: Forbidden oriented graphs for Theorem 3.2.

the proof, the calculations of the *densities* of the oriented graphs in Figure 3.4 will play an important role. We first give the definitions.

For an oriented graph $O^A(u,v)$ with $u,v \in V(D)$ in Figure 3.4, let W_{O^A} be the set of vertices in $V(D)\backslash\{u,v\}$ such that $D|_{\{u,v,w\}}$ is isomorphic to $O^A(u,v)$ for any $w \in W_{O^A}$. Define the *density* of $O^A(u,v)$ as follows.

$$d(O^A(u,v),D) = \frac{|W_{O^A}|}{n-2}. \tag{3.1}$$

For $\hat{O}^A(u,v)$ with $u,v \in V(D)$ in Figure 3.4, let $W_{\hat{O}^A}$ be the set of pairs of vertices in $V(D)\backslash\{u,v\}$ such that $D|_{\{u,v,w,w'\}}$ is isomorphic to $\hat{O}^A(u,v)$ for any $\{w,w'\} \in W_{\hat{O}^A}$. Define the density of $\hat{O}^A(u,v)$ as follows.

$$d(\hat{O}^A(u,v),D) = \frac{|W_{\hat{O}^A}|}{\binom{n-2}{2}}. \tag{3.2}$$

Similarly, we can define the densities for other oriented graphs listed in Figure 3.4,

i.e., $d(I^A(u,v),D)$, $d(P_3^A(u,v),D)$, $d(\vec{K}_{2,1}^N(u,v),D)$ and $d(\vec{P}_3^N(u,v),D)$. Besides, let

$$d(\alpha(v)) = \frac{d_D^+(v)}{n-1} \tag{3.3}$$

for a vertex v of D. Note that now it suffices to show $\alpha(v) \leq 1/l$ for some vertex v of D.

In sake of convenience, we write H for $d(H,D)$ in the following when no confusion occurs.

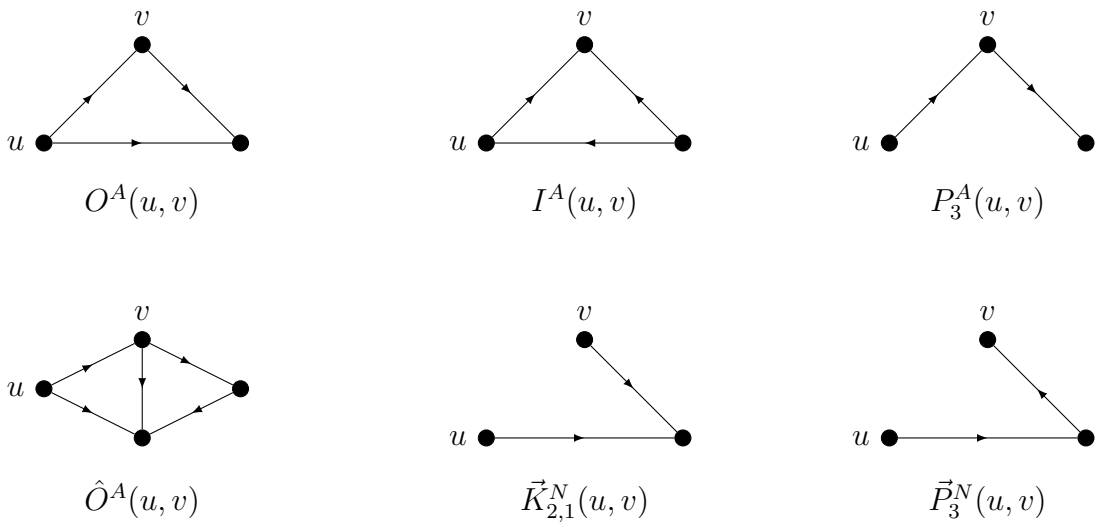

Figure 3.4: Oriented graphs needed in the proof of Theorem 3.2.

We call an arc $\langle u,v\rangle \in E(D)$ *critical* if $O^A(u,v)$ is minimal over all arcs going out from u. Note that v is a vertex in $N_D^+(u)$ with smallest outdegree in $D|_{N_D^+(u)}$. The following two claims will be used later. Since the proofs can be found in [80], we omit the details here.

Claim 3.1 (Razborov [80])**.** *Let $\langle u,v\rangle$ and $\langle v,w\rangle$ be two critical arcs. Then u and w are independent, and $\vec{K}_{2,1}^N(u,w) = 0$.*

Claim 3.2 (Razborov [80])**.** *Let $\langle u,v\rangle$ be a critical arc. Then $\hat{O}^A(u,v) = 0$.*

We first show a relationship between $O^A(u,v)$ and $\vec{P}_3^N(v,w)$, where $\langle u,v\rangle$ and $\langle v,w\rangle$ are two critical arcs.

Claim 3.3. *If $\langle u, v\rangle$ and $\langle v, w\rangle$ are two critical arcs, then*

$$\begin{aligned} O^A(u,v) &\leq \frac{\vec{P}_3^N(u,w)}{l} - \frac{1}{l(n-2)}, \\ I^A(v,w) &\leq \frac{\vec{P}_3^N(u,w)}{l} - \frac{1}{l(n-2)}. \end{aligned} \tag{3.4}$$

Proof. The proof of the second inequality is similar to that of the first one. So we only show the first inequality. By Claim 3.1, u and w are independent and thus $\vec{P}_3^N(u,w)$ exists. Let H be the set of vertices which contribute to $\vec{P}_3^N(u,w)$ and let $h = |H|$. Then $\vec{P}_3^N(u,w) = \frac{h}{n-2}$. Applying the inductive assumption to $D|_H$, there is a vertex $v^* \in H$ that has outdegree at most $\frac{h-1}{l}$. Now we show that

$$O^A(u,v^*) \leq \frac{h-1}{l(n-2)} = \frac{\vec{P}_3^N(u,w)}{l} - \frac{1}{l(n-2)} \tag{3.5}$$

from which the claim follows since $O^A(u,v) \leq O^A(u,v^*)$ due to the criticality of $\langle u, v\rangle$. Note that it suffices to show that every vertex x contributing to $O^A(u,v^*)$ belongs to H, that is $\langle x, w\rangle \in E(D)$. If x and w are independent, then $\{u, v^*, x, w\}$ induces an out-pendant. If $\langle w, x\rangle \in E(D)$, then $x \in \vec{K}_{2,1}^N(u,w)$ which contradicts to Claim 3.1 that $\vec{K}_{2,1}^N(u,w) = 0$. □

Now we consider a path consisting of $l-1$ critical arcs.

Claim 3.4. *For any $i \in \{1, \ldots, l-1\}$ and for any $\{v_1, \ldots, v_l\} \subseteq V(D)$ satisfying that $\langle v_i, v_{i+1}\rangle$ is a critical arc and $\langle v_i, v_j\rangle \notin E(D)$ for any $j \neq i+1$, we have*

$$\sum_{i=1}^{l} \alpha(v_i) + (O^A(v_1,v_2) + I^A(v_1,v_2)) - (O^A(v_{l-1},v_l) + I^A(v_{l-1},v_l)) \leq 1. \tag{3.6}$$

Proof. It is equal to show that

$$\begin{aligned}\sum_{i=1}^{l}\alpha(v_i) + (O^A(v_1,v_2)+I^A(v_1,v_2)) - (O^A(v_{l-1},v_l)+I^A(v_{l-1},v_l))\\ +\sum_{j=2}^{l-2}O^A(v_j,v_{j+1}) - \sum_{j=2}^{l-2}O^A(v_j,v_{j+1})\\ +\sum_{j'=2}^{l-1}I^A(v_{j'},v_{j'+1}) - \sum_{j'=2}^{l-1}I^A(v_{j'},v_{j'+1})\\ \leq 1.\end{aligned} \tag{3.7}$$

By Inequality (3.4) in Claim 3.3, we have

$$\begin{aligned}O^A(u,v) &\leq \vec{P}_3^N(u,w) - (l-1)O^A(u,v) - \frac{1}{n-2},\\ I^A(u,v) &\leq \vec{P}_3^N(u,w) - (l-1)I^A(u,v) - \frac{1}{n-2}.\end{aligned} \tag{3.8}$$

Thus, it suffices to show that

$$\begin{aligned}\sum_{i=1}^{l}\alpha(v_i) + I^A(v_1,v_2) + 2\sum_{j=1}^{l-2}\vec{P}_3^N(v_j,v_{j+2})\\ -(l-1)O^A(v_1,v_2) - l\sum_{k=2}^{l-2}O^A(v_k,v_{k+1}) - O^A(v_{l-1},v_l)\\ -l\sum_{k'=1}^{l-2}I^A(v_{k'},v_{k'+1}) - I^A(v_{l-1},v_l)\\ \leq 1+\frac{2(l-2)}{n-2}.\end{aligned} \tag{3.9}$$

Now we re-calculate all quantities in the left-hand side of Inequality (3.9) in $V(D)\backslash\{v_1,\ldots,v_l\}$. Denote these re-calculated quantities with $\widetilde{\alpha},\ldots,\widetilde{K_{2,1}^A}$, respectively. Note that D has no cycle of length at most l and $\langle v_i,v_j\rangle \notin E(D)$ for any $j \neq i+1$. By the definitions of the terms in Inequality (3.9) and the re-calculated terms, we have the following three facts.

Fact 3.1. *For any $i \in \{1,\dots,l-1\}$ and any $j \in \{1,\dots,l-2\}$ we have*

$$\begin{aligned}
\alpha(v_i) &= \frac{n-l}{n-1}\widetilde{\alpha}(v_i) + \frac{1}{n-1};\\
\alpha(v_l) &= \frac{n-l}{n-1}\widetilde{\alpha}(v_l);\\
\vec{P}_3^N(v_j, v_{j+2}) &= \frac{n-l}{n-2}\widetilde{\vec{P}_3^N}(v_j, v_{j+2}) + \frac{1}{n-2};\\
O^A(u,w) &= \frac{n-l}{n-2}\widetilde{O^A}(u,w);\\
I^A(u,w) &= \frac{n-l}{n-2}\widetilde{I^A}(u,w).
\end{aligned} \tag{3.10}$$

Fact 3.2. *Let x be an arbitrary vertex in $V(D)\backslash\{v_1,\dots,v_l\}$. If x contributes to $\widetilde{I^A}(v_1,v_2)$, then x contributes to no term in $\{\widetilde{\alpha}(v_i) : i = 1,\dots,l\}$ and no term in $\{\widetilde{\vec{P}_3^N}(v_j,v_{j+2}) : j = 1,\dots,l-2\}$.*

Fact 3.3. *Let x be an arbitrary vertex in $V(D)\backslash\{v_1,\dots,v_l\}$. Assume that x does not contribute to $\widetilde{I^A}(v_1,v_2)$. By Claim 3.2, we have that x contributes to at most two terms in $\{\alpha(v_i) : i = 1.\dots,l\}$. If x contributes to two terms in $\{\widetilde{\alpha}(v_i) : i = 1.\dots,l\}$, then x contributes to one term in $\{\widetilde{O^A}(v_j,v_{j+1}) : i = 1,\dots,l-2\}$ for some j. If x contributes to one term in $\{\widetilde{\alpha}(v_i) : i = 1.\dots,l\}$, then x contributes to at most one term in $\{\widetilde{\vec{P}_3^N}(v_j,v_{j+2}) : j = 1,\dots,l-2\}$.*

Since $\sum_{i=1}^{l}\alpha(v_i) > 1$ by our assumption, we have

$$\begin{aligned}
\frac{n-l}{n-2}\sum_{i=1}^{l}\widetilde{\alpha}(v_i) &= \frac{n-1}{n-2}\sum_{i=1}^{l}\alpha(v_i) - \frac{l-1}{n-2}\\
&= \sum_{i=1}^{l}\alpha(v_i) + \frac{\sum_{i=1}^{l}\alpha(v_i) - (l-1)}{n-2}\\
&\geq \sum_{i=1}^{l}\alpha(v_i) - \frac{l-2}{n-2}.
\end{aligned} \tag{3.11}$$

Thus,

$$\sum_{i=1}^{l}\alpha(v_i) \leq \frac{n-l}{n-2}\sum_{i=1}^{l}\widetilde{\alpha}(v_i) + \frac{l-2}{n-2}. \tag{3.12}$$

Now for (9) it also suffices to show that

$$\begin{aligned}\frac{n-l}{n-2}(\sum_{i=1}^{l}\widetilde{\alpha}(v_i)+\widetilde{I^A}(v_1,v_2)+2\sum_{j=1}^{l-2}\widetilde{\vec{P}_3^N}(v_j,v_{j+2}))+\frac{3(l-2)}{n-2}\\ -\frac{n-l}{n-2}((l-1)\widetilde{O^A}(v_1,v_2)+l\sum_{k=2}^{l-2}\widetilde{O^A}(v_k,v_{k+1})+\widetilde{O^A}(v_{l-1},v_l))\\ -\frac{n-l}{n-2}(l\sum_{k'=1}^{l-2}\widetilde{I^A}(v_{k'},v_{k'+1})+\widetilde{I^A}(v_{l-1},v_l))\\ \leq 1+\frac{2(l-2)}{n-2}.\end{aligned} \tag{3.13}$$

That is,

$$\begin{aligned}\sum_{i=1}^{l}\widetilde{\alpha}(v_i)+\widetilde{I^A}(v_1,v_2)+2\sum_{j=1}^{l-2}\widetilde{\vec{P}_3^N}(v_j,v_{j+2})\\ -(l-1)\widetilde{O^A}(v_1,v_2)-l\sum_{k=2}^{l-2}\widetilde{O^A}(v_k,v_{k+1})-\widetilde{O^A}(v_{l-1},v_l)\\ -l\sum_{k'=1}^{l-2}\widetilde{I^A}(v_{k'},v_{k'+1})-\widetilde{I^A}(v_{l-1},v_l)\\ \leq 1.\end{aligned} \tag{3.14}$$

By Facts 2 and 3, it suffices to consider the case that x contributes to both $\widetilde{\alpha}(v_i)$ and $\widetilde{\vec{P}_3^N}(v_i,v_{i+2})$ for some $2\leq i\leq l-2$. Since D has no induced F_4, then x contributes to at least one of $\{\widetilde{O^A}(v_i,v_{i+1}),\widetilde{I^A}(v_{i+1},v_{i+2})\}$. Thus the Inequality (3.14) follows and the proof is complete. □

By the definition of critical arcs, there exists a critical arc going out of u for any vertex u. So a cycle consisting of critical arcs exists. Let $C=u_1\dots u_p$ be such one of minimal length.

Claim 3.5. *Let u_i and u_j be any two vertices of C. Then $\langle u_i,u_j\rangle\in E(D)$ if and only if $j=i+1$ (modulo p).*

Proof. The sufficiency is obvious. For the necessity, assume the opposite that $\langle u_i,u_j\rangle\in$

$E(D)$ for some $j \neq i+1$. If $\langle u_i, u_j \rangle$ is not a critical arc, then $O^A(u_i, u_j) \neq \emptyset$. Note that $\{u_i, u_j, u_{j-1}, u\}$ induces an in-pendant for any $u \in O^A(u_i, u_j)$. So $\langle u_i, u_j \rangle$ is a critical arc. Replacing $u_i u_{i+1} \dots u_j$ by $\langle u_i, u_j \rangle$ yields a cycle consisting of less number of critical arcs, a contradiction to the minimality of C. □

We now prove Theorem 3.2. Sum up the quantities in the left side of Inequalities (3.6) along the cycle C. Note that the terms O^A and I^A will get canceled. Therefore,

$$\sum_{i=1}^{p} \alpha(u_i) \leq p/l. \tag{3.15}$$

It follows that there exists at least one vertex u_j with $\alpha(u_j) \leq 1/l$. The proof of Theorem 3.2 is complete. □

3.3 Caccetta-Häggkvist conjecture with forbidden subdigraphs

In [80], Razborov also proved that the Caccetta-Häggkvist conjecture holds for digraphs with three forbidden subdigraphs (not necessarily induced).

Theorem 3.3 (Razborov [80])**.** *Let D be an oriented graph on n vertices without directed triangles. If D contains none of the oriented graphs of Figure 3.5 as a subdigraph (not necessarily induced), then D contains a vertex with outdegree at most $\frac{n-1}{3}$.*

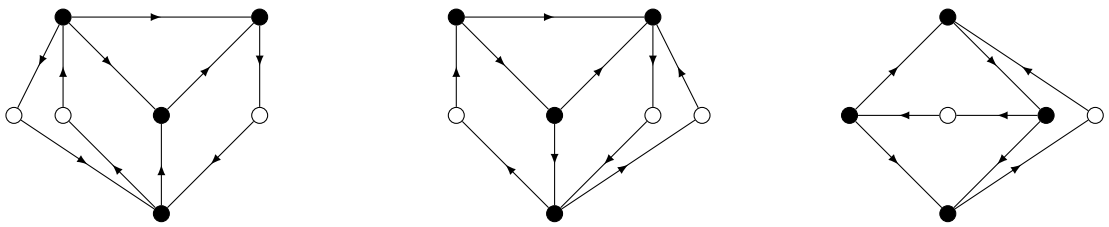

Figure 3.5: Forbidden oriented graphs for Theorem 3.3.

To generalize Theorem 3.3 to the case $l \geq 4$, or equivalently, to drop the restriction of being induced of Theorem 3.2, we introduce an operation as follows.

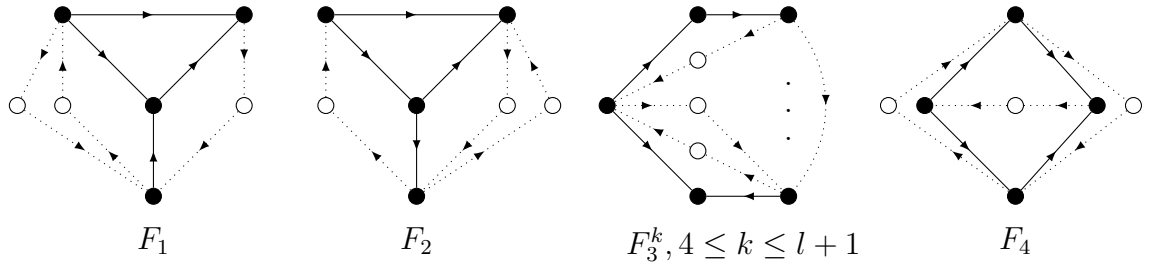

Figure 3.6: Forbidden oriented graphs for Theorem 3.4.

Operation 1. *Let u and v be two arbitrary independent vertices in an oriented graph without cycles of lengths at most l.*
(1) *If there is one path between u and v, assume w.l.o.g. that from u to v, of length $s \leq l-1$, then add a path from v to u of length $l-s+1$ (add $l-s$ vertices);*
(2) *If there is no path between u and v, then add a path from u to v of length $s \geq 2$ and a path from v to u of length $s' \geq 2$ such that $s+s' = l+1$ (add $l-2$ vertices).*

Denote by F_1, F_2, F_3^k $(4 \leq k \leq l+1)$ and F_4 (see Figure 3.6) the four oriented graphs generated from the four oriented graphs in Figure 3.3 by this operation, respectively. The new added vertices are denoted by empty circles and the new added paths are denoted by dotted lines.

Now we generalize Theorem 3.3 to the case $l \geq 4$ as follows.

Theorem 3.4. *Let D be an oriented graph on n vertices without cycles of lengths at most l. If D contains none of the oriented graphs of Figure 3.6 as a subdigraph, then D contains a vertex with outdegree at most $\frac{n-1}{l}$.*

Proof. Assume that Conjecture 1.4 holds for all oriented graphs that contain at least one oriented graph in Figure 3.6. Let D be an arbitrary oriented graph without cycles of lengths at most l. It suffices to show that D contains a vertex v with outdegree at most $\frac{n-1}{l}$. We can assume w.l.o.g. that D is maximal, i.e., adding any new arcs to D destroys the C_k-freeness for some $3 \leq k \leq l$.

If D contains no induced subdigraphs in Figure 3.3, then we are done by Theorem 3.2. Now assume that D contains at least one induced subdigraph in Figure 3.3.

By the maximality of D and our construction of F_1, F_2, F_3^k and F_4, D contains

at least one oriented graph in Figure 3.6. So a vertex v with outdegree at most $\frac{n-1}{l}$ exists by our assumption at the beginning. □

So far, we complete the generalization of Theorems 3.1 and 3.3 by Theorems 3.2 and 3.4, respectively. As a supplement, we also obtain the following forbidden subdigraph condition for Conjecture 1.4. It is somewhat trivial but very interesting and a simple proof is presented.

Proposition 3.1. *Let D be an oriented graph on n vertices without cycles of lengths at most l. If D contains no (induced) quasi-k-cycle for any $3 \leq k \leq l+1$, then D contains a vertex with outdegree at most $\frac{n-1}{l}$.*

Proof. Assume that D has minimum outdegree at least $\frac{n}{l}$. It follows that D has a cycle and furthermore by assumption the cycle has length more than l. So D has a path of length at least $l-1$ and let $P = v_1 \dots v_l$ be a path of length $l-1$. By the minimum outdegree condition, we have

$$\sum_{i=1}^{l} d_D^+(v) \geq n. \tag{3.16}$$

Note that $\sum_{i=1}^{l} d_P^+(v) = l-1$. Thus,

$$\sum_{i=1}^{l} d_{D\setminus P}^+(v) \geq n-l+1. \tag{3.17}$$

It follows that there exist two vertices in $V(P)$ having a common outneighbor in $V(D)\setminus V(P)$, which implies a quasi-k-cycle for some $3 \leq k \leq l+1$, a contradiction. □

Note that Proposition 3.1 has an equivalent statement: Every digraph on n vertices with minimum outdegree r contains either a cycle of length at most $\lceil \frac{n}{r} \rceil$ or a quasi cycle of length at most $\lceil \frac{n}{r} \rceil + 1$.

3.4 Conclusion

In this chapter, we consider the famous Caccetta-Häggkvist conjecture. Motivated by the result of Razborov [80] on $l = 3$ of Conjecture 1.4 with forbidden subdigraphs. We generalize this result by showing that Caccetta-Häggkvist conjecture (Conjecture 1.4) holds for $l \geq 4$ with four given forbidden subdigraphs. It is worth noting that the definition "density" is the key quantity in flag algebras, which was introduced by Razborov [79] in 2007. For more details, we refer the readers to [79, 80].

Chapter 4

Vertex-Disjoint Cycles in Bipartite Tournaments

4.1 Introduction

We write $u \to L$ if $u \to v$ for every $v \in L$ and write $L \to u$ if $v \to u$ for every $v \in L$. Define a $\{k, l\}$-cycle to be a cycle of length either k or l. As mentioned in Section 1.4, vertex-disjoint cycles in graphs and digraphs have attracted much attention. The work of this chapter on vertex-disjoint cycles in bipartite tournaments is mainly motivated by the results on vertex-disjoint cycles in tournaments.

4.2 Vertex-disjoint cycles in bipartite tournaments

In 2001, Chen et al. [35] showed the following theorem.

Theorem 4.1 (Chen et al. [35])**.** *Let T be a k-connected tournament with at least $5k-3$ vertices and $k \geq 2$. Then T contains k vertex-disjoint cycles.*

Here, we give a similar result for bipartite tournaments as follows.

Theorem 4.2. *Let $BT = (X, Y; E)$ be a k-connected bipartite tournament with $\min\{|X|, |Y|\} \geq 4k-3$ and $k \geq 2$. Then BT contains k vertex-disjoint cycles.*

As a direct corollary, we have the following result.

Corollary 4.1. *Every k-connected balanced (hamiltonian) bipartite tournament with at least $8k-6$ vertices contains k vertex-disjoint cycles.*

Proof of Theorem 4.2. To the contrary, let $k \geq 2$ be the smallest positive integer such that there exists a k-connected bipartite tournament with $\min\{|X|, |Y|\} \geq 4k-3$ that does not contain k vertex-disjoint cycles. By the minimality of k and the fact that every 1-connected bipartite tournament has a cycle, BT has $k-1$ vertex-disjoint cycles. By Lemma 1.2, BT has $k-1$ vertex-disjoint cycles of length 4, say, $Q_1, \ldots, Q_{k-1}$. Let

$$H = BT - \bigcup_{j=1}^{k-1} V(Q_j). \tag{4.1}$$

Note that H has no cycle. Let $H_1, H_2, \ldots, H_{2i-1}, H_{2i}, \ldots, H_{2m-1}, H_{2m}$, $m \geq 1$, $1 \leq i \leq m$, be the vertex-disjoint subsets defined in Lemma 1.3. Assume w.l.o.g that $H_{2i-1} \subseteq X$ and $H_{2i} \subseteq Y$.

Let M be the set of the first appeared k monochromatic vertices according to the sequence $H_1, \ldots, H_{2m}$, without loss of generality (or simply, w.l.o.g.), assume that $M \subseteq X$. Let N be the set of the last appeared k vertices (in Y) according to the sequence $H_1, \ldots, H_{2m}$. Since

$$|V(H)| = |V(BT)| - \sum_{j=1}^{k-1} |V(Q_j)| \geq 8k - 6 - 4(k-1) \geq 4k-2 \tag{4.2}$$

and there is no arc from H_p to H_q for $p > q$, we have $M \to N$. Since BT is k-connected, there exist k vertex-disjoint paths from N to M. Clearly, these paths plus the appropriate arcs from M to N form k vertex-disjoint cycles. The proof of Theorem 4.2 is complete. □

4.3 Vertex-disjoint cycles of given lengths in bipartite tournaments

In 2010, Lichiardopol [68] considered the vertex-disjoint cycles of given length in tournaments and proposed the following conjecture.

Conjecture 4.1 (Lichiardopol [68])**.** Let T be a tournament with $\delta^+(T) \geq (q-1)r-1$. Then T contains r vertex-disjoint q-cycles.

Motivated by the conjecture, we consider the analogous problem for bipartite tournaments, i.e., vertex-disjoint cycles of given length(s) in bipartite tournaments. The following results have been proved.

Theorem 4.3. *Let BT be a bipartite tournament with $\delta^+(BT) \geq qr-1$. Then BT contains r vertex-disjoint cycles either of length $2q$ for even q or of length in $\{2q, 2q+2\}$ for odd q.*

Theorem 4.4. *Let BT be a bipartite tournament with $\delta^+(BT) \geq qr-1$ and let $t_1, \ldots, t_r \in [4, 2q]$ be any r even integers. Then BT contains r vertex-disjoint cycles of lengths $t'_1, \ldots, t'_r$ such that $t'_i = t_i$ for $t_i = 0 \pmod 4$ and $t'_i \in \{t_i, t_i+2\}$ for $t_i = 2 \pmod 4$, where $1 \leq i \leq r$.*

We leave the proofs of Theorems 4.3 and 4.4 at the end of this section.

In 1981, Bermond and Thomassen [23] conjectured that every digraph with minimum outdegree at least $2r-1$ contains at least r vertex-disjoint cycles. This is trivially true for $r = 1$. Thomassen [93] and Lichiardopol, Por and Sereni [71] proved it for $r = 2$ and $r = 3$, respectively. In 2010, Bessy, Lichiardopol and Sereni [25] verified it for regular tournaments. Recently, Bang-Jensen et al. [17] verified it for tournaments. Take $q = 2$ in Theorem 4.3, then the Bermond-Thomassen conjecture will be verified for bipartite tournaments.

Corollary 4.2. *Let BT be a bipartite tournament with $\delta^+(BT) \geq 2r-1$. Then BT contains r vertex-disjoint 4-cycles.*

We give some preliminary results as follows.

Theorem 4.5 (Jackson [59])**.** *Let BT be a strong bipartite tournament with $\delta^+(BT) \geq s$ and $\delta^-(BT) \geq t$. Then BT contains a cycle of length at least $2(s+t)$.*

The following fact will be used later.

Fact 4.1. *$F_{4\cdot k}$ contains a $4k'$-cycle for all $1 \leq k' \leq k$.*

Theorem 4.6 (Beineke and Little [22]). *Let C be a $2s$-cycle of a bipartite tournament BT. If the sub-bipartite-tournament induced on C is not isomorphic to $F_{4\cdot k}$, where $k = s/2$, then BT contains a $2s'$-cycle for all $2 \leq s' \leq s$.*

We now prove Theorems 4.3 and 4.4, respectively.

Proof of Theorem 4.3. We can assume w.l.o.g. that BT is strong. In fact, if not, then we can choose a strong component with minimum outdegree at least $qr - 1$. Then $\delta^-(BT) \geq 1$ and BT has a cycle of length at least $2qr$ by Theorem 4.5. Thus BT has a $2q$-cycle for even q or a $(2q+2)$-cycle for odd q by Theorem 4.6 and Fact 4.1.

By induction on r. It obviously holds for $r = 1$. Assume that $r \geq 2$ and every bipartite tournament with minimum outdegree at least $q(r-1) - 1$ contains $r - 1$ vertex-disjoint cycles either of length $2q$ for even q or of lengths in $\{2q, 2q+2\}$ for odd q. We distinguish two cases.

Case 1. *q is even.*

Note that BT has a $2q$-cycle. Denote it by C and let $BT' = BT \backslash C$. Then

$$\delta^+(BT') \geq qr - 1 - q = q(r-1) - 1. \tag{4.3}$$

By hypothesis BT' has $r - 1$ vertex-disjoint $2q$-cycles. These cycles plus C form r vertex-disjoint $2q$-cycles of BT.

Case 2. *q is odd.*

If BT has a $2q$-cycle C, then the same as Inequality (4.3) we have $\delta^+(BT') \geq q(r-1) - 1$ for $BT' = BT \backslash C$. Thus by hypothesis BT' has $r - 1$ vertex-disjoint $\{2q, 2q+2\}$-cycles. These cycles plus C form r vertex-disjoint $\{2q, 2q+2\}$-cycles of BT.

Now assume that BT has no $2q$-cycle. We will show that BT has r vertex-disjoint $(2q+2)$-cycles.

Note that BT has a cycle of length at least $2qr$. Let C be the maximum cycle of BT. Then $|C| \geq 2qr$. Since BT contains no $2q$-cycle, by Theorem 4.6, we have that C induces a $F_{4\cdot k}$ for $k = |C|/4$. Recall that $F_{4\cdot k} = F(K, L, M, N)$ with $|K| = |L| = |M| = |N| = k$.

If $|C| \geq r(2q+2)$, then $F_{4\cdot k}$ contains at least r vertex-disjoint $(2q+2)$-cycles and thus the result holds. Now assume that $|C| < r(2q+2)$. We will get a contradiction by showing that BT has a cycle longer than C.

Note that

$$d_C^+(v) = \frac{|C|}{4} < \frac{r(q+1)}{2} \tag{4.4}$$

for any $v \in V(C)$ and

$$\delta^+(BT) \geq qr - 1 > \frac{r(q+1)}{2} > d_C^+(v). \tag{4.5}$$

Thus $X \cap (BT\backslash C) \neq \emptyset$, $Y \cap (BT\backslash C) \neq \emptyset$ and every vertex of C has at least one outneighbor in $BT\backslash C$.

For any $x \in X \cap (BT\backslash C)$, assume w.l.o.g. that $y_1 \to x$ for some $y_1 \in L$ and assume that there exists $y_2 \in L$ with $x \to y_2$. Since $|C| \geq 2q+2$, then $BT[C]$ has a path P of length $2q-2$ from y_2 to y_1. Now $y_2Py_1xy_2$ is a $2q$-cycle, a contradiction. This, and by symmetry, implies that

- $x \to L$ or $L \to x$ for any $x \in X \cap (BT\backslash C)$;
- $x \to N$ or $N \to x$ for any $x \in X \cap (BT\backslash C)$;
- $y \to K$ or $K \to y$ for any $y \in Y \cap (BT\backslash C)$;
- $y \to M$ or $M \to y$ for any $y \in Y \cap (BT\backslash C)$.

For any $y \in N$, since $d_C^+(y) = k < qr - 1$, there exists $x_K \in X \cap (BT\backslash V(C))$ with $N \to x_K$. If $L \to x_K$, then since BT is strong there exists a path from x_K to C. Assume that $P = x_K v_1 \ldots v_p$ is a shortest one. Since $N \to x_K$ and $L \to x_K$, we have $v_1 \notin L \cup N$ and $p \geq 2$. If $v_p \in K \cup M$, then w.l.o.g. assume that $v_p \in K$. Let P' be

a Hamilton path of $BT[C]$ from v_p to a vertex $y' \in N$. Then $x_K P v_p P' y' x_K$ is a cycle longer than C, a contradiction. If $v_p \in L \cup N$, then $p \geq 3$ and assume w.l.o.g. that $v_p \in L$. Let x'', y'' be two vertices in K and N respectively and let P'' be a Hamilton path of $BT[C]\backslash\{x''\}$ from v_p to y''. Then $x_K P v_p P'' y'' x_K$ is a cycle longer than C, a contradiction. This, and by symmetry, shows that

- $x_K \to L$ and $N \to x_K$ for some $x_K \in X \cap (BT\backslash C)$;
- $x_M \to N$ and $L \to x_M$ for some $x_M \in X \cap (BT\backslash C)$;
- $y_L \to M$ and $K \to y_L$ for some $y_L \in Y \cap (BT\backslash C)$;
- $y_N \to K$ and $M \to y_N$ for some $y_N \in Y \cap (BT\backslash C)$.

Let $x_1 \in K$, $x_2, x_2' \in M$, $y_1, y_1' \in L$ and $y_2 \in N$. Let P^* be a Hamilton path of $BT[C]\backslash\{x_1, x_2, y_1, y_2\}$ from x_2' to y_1'. Then

$$y_1' x_M y_2 x_K y_1 x_2 y_N x_1 y_L x_2' P^* y_1' \tag{4.6}$$

is a cycle longer than C, a contradiction.

The proof of Case 2 is complete.

The proof of Theorem 4.3 is complete. □

Proof of Theorem 4.4. By Theorem 4.6 and Fact 4.1, the sub-bipartite-tournament induced on any $2q$-cycle either contains a $2q'$-cycle for any $2 \leq q' \leq q$ or contains a $2q'$-cycle for any even q' with $2 \leq q' \leq q$. Then the result follows directly from Theorem 4.3. □

4.4 Conclusion

In this chapter, we consider vertex-disjoint cycles and vertex-disjoint cycles of given lengths in bipartite tournaments. Let BT be a bipartite tournament with $\delta^+(BT) \geq qr - 1$ and let $t_1, \ldots, t_r \in [4, 2q]$ be any r even integers. We show that BT contains

r vertex-disjoint cycles of length $t'_1, \ldots, t'_r$ such that $t'_i = t_i$ for $t_i = 0$ $(mod\ 4)$ and $t'_i \in \{t_i, t_i + 2\}$ for $t_i = 2$ $(mod\ 4)$, where $1 \leq i \leq r$.

Chapter 5

Cycle Factors in Regular Bipartite Tournaments

5.1 Introduction

Let $D = (V(D), E(D))$ be a digraph and let k be a positive integer. For an arc $\overrightarrow{uv}$ of D, we write $u \to v$ and say u dominates v (or v is dominated by u), and we call u and v the *tail* and the *head* of the arc, respectively. For two vertex-disjoint subsets P and Q of $V(D)$, we write $P \to Q$ if every arc of D between P and Q goes from P to Q, and we write $P \nrightarrow Q$ if there exists an arc of D between P and Q that goes from Q to P. Let R be a subset of $V(D)$. We use $N_R^+(P)$ (resp. $N_R^-(P)$) to denote the set of vertices of R which are dominated by (resp. dominate) at least one vertex of P. For convenience, we write $v \to P$ for $\{v\} \to P$, $v \nrightarrow P$ for $\{v\} \nrightarrow P$, $P + v$ for $P \cup \{v\}$, $P - v$ for $P \backslash \{v\}$, $P + u - v$ (or $P - v + u$) for $P \cup \{u\} \backslash \{v\}$, $N_D^+(v)$ for $N_D^+(\{v\})$, $N_D^-(v)$ for $N_D^-(\{v\})$, $d_D^+(v)$ for $|N_D^+(v)|$ and $d_D^-(v)$ for $|N_D^-(v)|$.

Recall that a *k-cycle-factor* is a cycle factor consisting of k cycles. Here we call the two cycles of a 2-*cycle factor* are *complementary*.

Denote by

$$B = (X, Y; E)$$

a bipartite tournament with bipartition (X, Y), vertex set $V(B) = X \cup Y$ and arc set

$E(B)$. It is well known that B has a cycle factor if and only if B contains a perfect matching from X to Y and a perfect matching from Y to X. By Hall's Theorem, B has a perfect matching from X to Y if and only if $|N_B^+(P)| \geq |P|$ for any $P \subseteq X$.

It is well known that every tournament has a Hamilton path. Let T be a tournament with order n. We say T is a *transitive tournament* if T has a Hamilton path $v_1 \to v_2 \to \cdots \to v_n$ such that $v_i \to v_j$ for any $1 \leq i < j \leq n$. Define $\delta^+(T) = \min_{v \in V(T)}\{d_T^+(v)\}$, $\delta^-(T) = \min_{v \in V(T)}\{d_T^-(v)\}$. Let T_7 be the tournament of order 7 containing no transitive subtournament of order 4. Let $\mathcal{T}_7^*$ be the set of 3-regular tournaments of order 7. Denote by $\mathcal{D}_{3,2}$ the set of the 2-regular 3-partite tournaments with exactly two vertices in each partite set.

The problem of 2-cycle-factors in 2-connected tournaments was completely solved by Reid [81] and Song [89]. Moreover, Li and Shu [67] characterized strong tournaments that have a 2-cycle-factor. Thus the problem of 2-cycle-factors in tournaments has been almost completely solved.

Theorem 5.1 (Reid [81] and Song [89]). *Let T be a 2-connected tournament with $|V(T)| \geq 6$. Then T has a 2-cycle-factor of cycle-lengths t and $|V(T)| - t$ for all $3 \leq t \leq |V(T)| - 3$, unless T is isomorphic to T_7.*

Theorem 5.2 (Li and Shu [67]). *Let T be a strong tournament with $|V(T)| \geq 6$ and $\max\{\delta^-(T), \delta^+(T)\} \geq 3$. Then T has a 2-cycle-factor, unless T is isomorphic to T_7.*

The problem of k-cycle-factors in highly connected tournaments was posed by Bollobás (see Reid [82]).

Problem 5.1 (Bollobás). *Let k be a positive integer. What is the least integer $h(k)$ such that all but a finite number of $h(k)$-connected tournaments contain a k-cycle-factor?*

Reid showed that $h(k)$ exists and $h(k) \leq 3k - 4$. Chen et al. [35] proved in 2001 that $h(k) = k$.

Theorem 5.3 (Chen et al. [35]). *Let T be a k-connected tournament with $|V(T)| \geq 8k$. Then T has a k-cycle-factor.*

The problem of k-cycle-factors of given cycle-lengths in highly connected tournaments was posed by Song [89].

Problem 5.2 (Song [89]). *Let k, n be two positive integers and let $n_1, \ldots, n_k$ be k integers with $n = n_1 + \ldots + n_k$ and $n_i \geq 3$ for $i = 1, \ldots, k$. What is the least integer $h'(k)$ such that all but a finite number of $h'(k)$-connected tournaments contain a k-cycle-factor of given cycle-lengths $n_1, \ldots, n_k$?*

Note that $h'(1) = h(1) = 1$. Song showed that $h'(2) = h(2) = 2$. It is clear that $h'(k) \geq h(k)$. Song conjectured that $h'(k) = h(k)$. By Theorem 5.3, Song's conjecture is $h'(k) = k$. Recently, Kühn et al. [64] gave an upper bound of $h'(k)$.

Theorem 5.4 (Kühn et al. [64]). *Let T be a $10^{10}k^4 \log k$-connected tournament on n vertices and let $n_1, \ldots, n_k$ be k integers with $n = n_1 + \ldots + n_k$ and $n_i \geq 3$ for $i = 1, \ldots, k$. Then T has a k-cycle-factor of cycle-lengths $n_1, \ldots, n_k$.*

5.2 2-cycle-factors in regular bipartite tournaments

The problem of 2-cycle-factors in regular bipartite tournaments was investigated by Song [88], Zhang and Song [106], Zhang et al. [105], and Zhang and Wang [107].

Theorem 5.5 (Zhang and Song [106]). *Let B be a k-regular bipartite tournament with $k \geq 2$. Then B has a 2-cycle-factor.*

Theorem 5.6 (Zhang et al. [105]). *Let B be a k-regular bipartite tournament with $k \geq 2$. Then for any $uv \in E(B)$, B has a 2-cycle-factor such that one cycle contains uv and has length 4.*

Theorem 5.7 (Zhang and Wang [107]). *Let B be a k-regular bipartite tournament with $k \geq 2$. Then for any $uv \in E(B)$ and for any $w \in V(B)\backslash\{u, v\}$, B has a 2-cycle-factor such that one cycle contains uv and has length 4 and the other cycle contains w, unless B is isomorphic to a special digraph (defined in [107]).*

Volkmann [94] characterized the regular c-partite tournaments with $c \geq 3$ that have a 2-cycle-factor.

Theorem 5.8 (Volkmann [94]). *Let D be a regular c-partite tournament with $c \geq 3$ and $|V(D)| \geq 6$. Then D has a 2-cycle-factor, unless D is isomorphic to some digraph in $\mathcal{T}_7^* \cup \mathcal{D}_{3,2}$.*

Thus, all regular multipartite tournaments containing a 2-cycle-factor have been characterized.

Theorem 5.9 (Volkmann [94]). *Let D be a regular c-partite tournament. If $c = 2$ and $|V(D)| \geq 8$ or $c \geq 3$ and $|V(D)| \geq 6$, then D has a 2-cycle-factor, unless D is isomorphic to some digraph in $\mathcal{T}_7^* \cup \mathcal{D}_{3,2}$.*

Corollary 5.1 (Volkmann [94]). *Let D be a regular multipartite tournament with $|V(D)| \geq 8$. Then D has a 2-cycle-factor.*

In [105], it was conjectured that every k-regular bipartite tournament not isomorphic to $F_{4\cdot k}$ has a 2-cycle-factor of all possible cycle-lengths.

Conjecture 5.1 (Zhang et al. [105]). Let B be a k-regular bipartite tournament not isomorphic to $F_{4\cdot k}$. Then B has a 2-cycle-factor of even cycle-lengths t and $|V(B)|-t$ for all $4 \leq t \leq |V(B)| - 4$.

By Theorem 5.6, Conjecture 5.1 is true for $t = 4$. In this chapter, we show that Conjecture 5.1 is true for $t = 6$.

Theorem 5.10. *Let B be a k-regular bipartite tournament not isomorphic to $F_{4\cdot k}$ and $k \geq 3$. Then B has a 2-cycle-factor of cycle-lengths 6 and $|V(B)| - 6$.*

Our proof of Theorem 5.10 is heavily based on Lemma 5.1.

Lemma 5.1. *Let B be a k-regular bipartite tournament not isomorphic to $F_{4\cdot k}$ and $k \geq 3$. Then B has a cycle C of length 6 such that $B - C$ has a cycle factor.*

The following theorems are needed in the proofs of Theorem 5.10 and Lemma 5.1.

Theorem 5.11 (Häggkvist and Manoussakis [47]). *A bipartite tournament is Hamiltonian if and only if it has a cycle factor and is strong.*

Theorem 5.12 (Häggkvist and Manoussakis [47]). *Let B be a bipartite tournament containing a cycle factor, B is not strong if and only if B has a m-cycle factor $C_1, C_2, \ldots, C_m, m \geq 2$, such that $C_i \to C_j$ for $i < j$.*

Theorem 5.13 (Amar and Manoussakis [9]). *Let uv be any arc of a k-regular bipartite tournament B not isomorphic to $F_{4\cdot k}$. Then there are cycles of all even length m, $4 \leq m \leq 4k$, through uv.*

We now prove Theorem 5.10 and Lemma 5.1. The main tool of the proofs is Hall's Theorem.

Proof of Theorem 5.10. By Lemma 5.1, B has a cycle C of length 6 such that $R = B - C$ has a cycle factor. It suffices to show that R is Hamiltonian or that there is a cycle $C^* \neq C$ such that $|C^*| = 6$ and $B - C^*$ is Hamiltonian. If R is strong, then R is Hamiltonian for every k by Theorem 5.11. If $k = 3$, then $|R| = 6$ and R has a cycle factor, where each cycle has length at least 4. Hence this cycle factor is a Hamilton cycle. Now assume that R is not strong and $k \geq 4$.

Case 1. $k = 4$.

Let $B = (X, Y; E)$ and let

$$C = x_1 \to y_1 \to x_2 \to y_2 \to x_3 \to y_3 \to x_1,$$

where $\{x_1, x_2, x_3\} \subseteq X$ and $\{y_1, y_2, y_3\} \subseteq Y$. Note that R is not strong and $|R| = 10$. By Theorem 5.12, R has a 2-cycle-factor, say C_1 and C_2, such that $C_1 \to C_2$. Moreover, one cycle has length 4 and the other cycle has length 6. Now let

$$1 \to 2 \to 3 \to 4 \to 1,$$

$$5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 5$$

be the two cycles of the 2-cycle-factor, where $\{1, 3, 5, 7, 9\} \subseteq X$ and $\{2, 4, 6, 8, 10\} \subseteq Y$.

Suppose first that $|C_1| = 4$ and $|C_2| = 6$. Each vertex of C_1 dominates one vertex of C_1 and three vertices of C_2. Hence since B is 4-regular, we have $C \to C_1$ and every vertex of C_2 has at least two outneighbors in C. Assume w.l.o.g. that $6 \to x_1$. Let

$$C' = x_1 \to y_1 \to 3 \to 4 \to 5 \to 6 \to x_1.$$

Note that $|C'| = 6$. If $10 \to x_2$, then $B - C'$ has a Hamilton cycle $x_2 \to y_2 \to x_3 \to y_3 \to 1 \to 2 \to 7 \to 8 \to 9 \to 10 \to x_2$. If $x_2 \to 10$, then since $\{1, 3, 9\} \to 10$ and $x_2 \to \{y_2, 2, 4\}$ we have $10 \to 7$ and $8 \to x_2$. Then $B - C'$ has a Hamilton cycle $x_2 \to y_2 \to x_3 \to y_3 \to 1 \to 2 \to 9 \to 10 \to 7 \to 8 \to x_2$.

Now suppose that $|C_1| = 6$ and $|C_2| = 4$. Each vertex of C_2 is dominated by one vertex of C_2 and three vertices of C_1. Hence since B is 4-regular, we have $C_2 \to C$ and every vertex of C_1 has at least two inneighbors in C. Assume w.l.o.g. that $y_1 \to 5$. Similarly to the above analysis, we can show that

$$C'' = x_1 \to y_1 \to 5 \to 6 \to 3 \to 4 \to x_1$$

is a cycle of length 6 such that $B - C''$ is Hamiltonian.

Case 2. $k \geq 5$.

Let $C_1, C_2, \ldots, C_m, m \geq 2$, be cycles of R as given in Theorem 5.12. Let $|C_i| = n_i$ for $i = 1, \ldots, m$. Note that $|V(B)| = 4k = \sum_{i=1}^{m} n_i + 6$. If $n_1 \leq n_2 + \ldots + n_m$, then for every vertex v of C_1 such that $d^+_{C_1}(v)$ is maximal we have

$$\begin{aligned} k \geq |N^+_R(v)| &\geq \frac{n_1}{4} + \frac{n_2 + \ldots + n_m}{2} \\ &= \frac{n_1 + n_2 + \ldots + n_m}{4} + \frac{n_2 + \ldots + n_m}{4} \\ &\geq \frac{n_1 + n_2 + \ldots + n_m}{4} + \frac{n_1 + n_2 + \ldots + n_m}{8} \\ &= \frac{4k - 6}{4} + \frac{4k - 6}{8} \\ &= k + (\frac{k}{2} - \frac{9}{4}), \end{aligned}$$

a contradiction to $k \geq 5$. On the other hand, if $n_1 \geq n_2 + \ldots + n_m$, then using similar

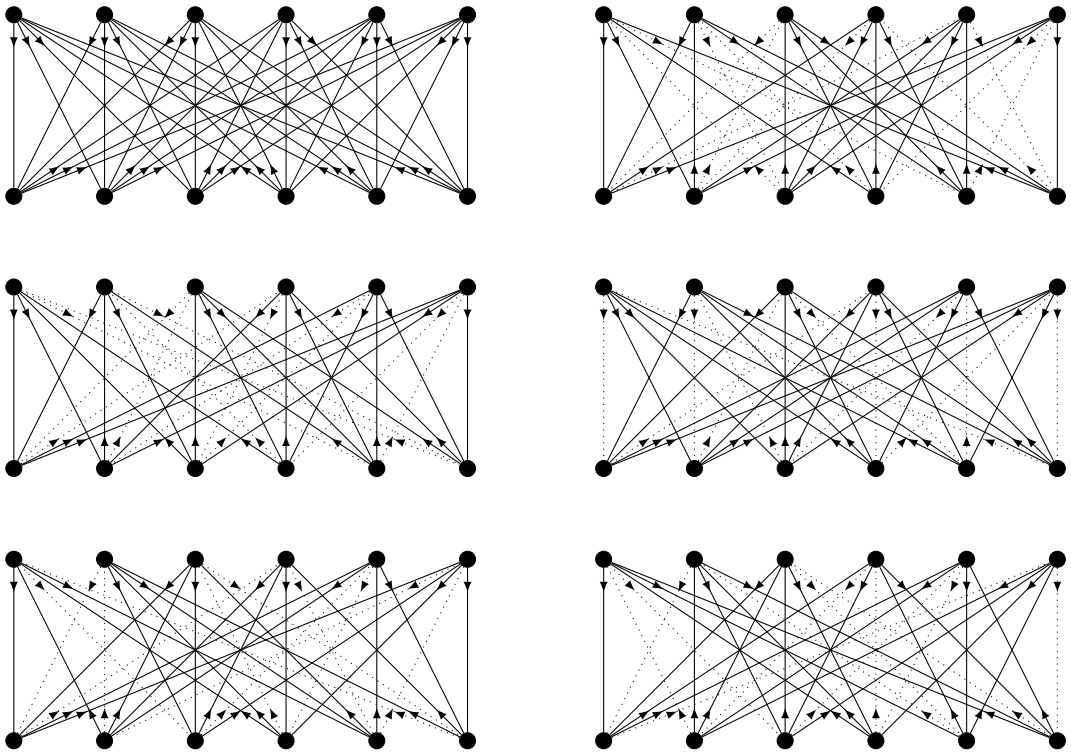

Figure 5.1: Non-isomorphic 3-regular bipartite tournaments with 2-cycle-factors.

arguments we can obtain a contradiction by considering $|N_R^-(v)|$, where v is now a vertex of C_m such that $d_{C_m}^-(v)$ is maximal. □

Proof of Lemma 5.1. Let $B = (X, Y; E)$. For $k = 3$, by considering the arcs from X to Y, there is a one-to-one correspondence between 3-regular bipartite tournaments and 3-regular bipartite graphs on 12 vertices. Using the utility "genbg", Mckay [73] verified that there are exactly 6 non-isomorphic 3-regular bipartite graphs on 12 vertices. The corresponding 6 non-isomorphic 3-regular bipartite tournaments together with the 2-cycle-factors (except the first one $F_{4.3}$) are presented in Figure 5.1. Alternatively, our method in the proof for $k \geq 4$ below can be extended to a proof for $k = 3$. From now on, we assume that $k \geq 4$.

For convenience, we say a cycle C^* of B is *good* if C^* has length 6 and $B - C^*$ has a cycle factor. Assume the opposite that Lemma 5.1 is not true. Then B has no good cycle. By Theorem 5.13, we can let

$$C = x_1 \to y_1 \to x_2 \to y_2 \to x_3 \to y_3 \to x_1$$

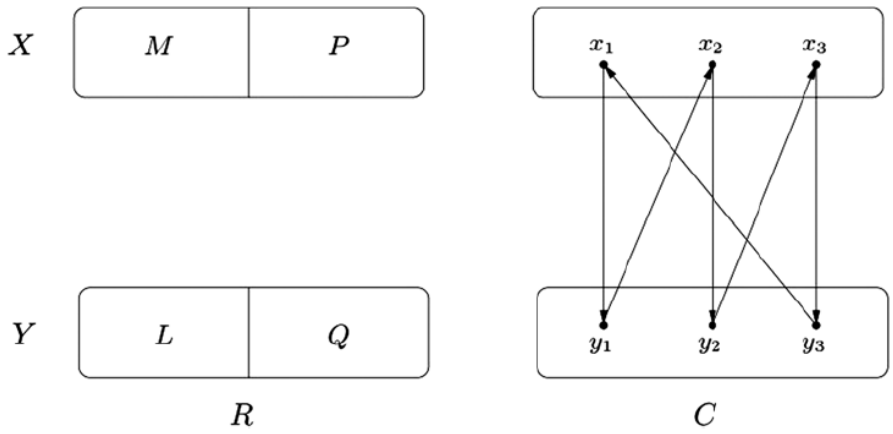

Figure 5.2: $B = (X, Y; E)$.

be a cycle of length 6 in B, where $\{x_1, x_2, x_3\} \subseteq X$ and $\{y_1, y_2, y_3\} \subseteq Y$. Let

$$R = B - C.$$

Then R has no cycle factor. By Hall's Theorem, there exists a subset P either of $X - \{x_1, x_2, x_3\}$ or of $Y - \{y_1, y_2, y_3\}$ such that $|P| > |N_R^+(P)|$. Assume w.l.o.g. that

$$P \subseteq X - \{x_1, x_2, x_3\}. \tag{5.1}$$

Define Q, M and L as follows (see Figure 5.2).

$$Q = N_R^+(P),\ M = X - (P \cup \{x_1, x_2, x_3\}),\ L = Y - (Q \cup \{y_1, y_2, y_3\}). \tag{5.2}$$

Claim 5.1. $k \geq |P| > |Q| \geq k - 3$.

Proof. It follows directly from the facts that B is k-regular, $|P| > |N_R^+(P)| = |Q|$ and $Q \subseteq Y - \{y_1, y_2, y_3\}$. □

It suffices to consider the following six possible cases:

(A) $|P| = k$ and $|Q| = k - 1$;

(B) $|P| = k$ and $|Q| = k - 2$;

(C) $|P| = k$ and $|Q| = k - 3$;

(D) $|P| = k - 1$ and $|Q| = k - 2$;

(E) $|P| = k - 1$ and $|Q| = k - 3$;

(F) $|P| = k - 2$ and $|Q| = k - 3$.

Since the proof of Case (C) is simple, the proofs of Cases (E) and (F) are heavily based on that of Case (D), the proof of Case (B) is heavily based on that of Cases (D) and (E), and the proof of Case (A) is heavily based on that of Cases (C), (D), (E) and (F), we consider the six cases above in the following order: (C), (D), (E), (B), (F) and (A).

Case 1. *$|P| = k$ and $|Q| = k - 3$.*

By using regularity on degrees, we have $P \to Q \cup \{y_1, y_2, y_3\}$. So $P + x_1 \subseteq N_B^-(y_1)$ and $d_B^-(y_1) \geq k + 1$, a contradiction.

Case 2. *$|P| = k - 1$ and $|Q| = k - 2$.*

Let $B[C]$ denote the subdigraph of B induced by $V(C)$ and let

$$E_3 = E(B[C]) - E(C). \tag{5.3}$$

We distinguish four cases.

Case 2.1. *Every arc of E_3 has a tail in $\{x_1, x_2, x_3\}$.*

It follows that

$$x_1 \to y_2,\ x_2 \to y_3,\ x_3 \to y_1.$$

If $L \to x_1$, then $x_2 \to L$ and $d_B^+(x_2) \geq |L + y_2 + y_3| = k + 1$. So $L \nrightarrow x_1$ and there exists $l \in L$ such that $x_1 \to l$. If $x_2 \to Q$, then $L \to x_2$, $x_1 \to L$ and $d_B^+(x_1) \geq |L + y_1 + y_2| = k + 1$. So $x_2 \nrightarrow Q$ and there exists $q \in Q$ such that $q \to x_2$. If $P \to y_i$ for some $i \in \{1, 2, 3\}$, then $P + x_i + x_{i+2} \to y_i$ and $d_B^-(y_i) \geq k + 1$, where

$x_4 = x_1$ and $x_5 = x_2$. So $P \nrightarrow y_i$ and $N_P^+(y_i) \neq \emptyset$ for any $i \in \{1,2,3\}$. Since $L \to P$ and $|L| = k-1$, we have $N_P^+(y_i) \subseteq N_P^-(y_{i+1})$, where $y_4 = y_1$. Thus for $i \in \{1,2,3\}$ we have

$$N_P^+(y_i) \cap N_P^-(y_{i+1}) \neq \emptyset.$$

Let

$$p_i \in N_P^+(y_i) \cap N_P^-(y_{i+1}).$$

Note that p_1, p_2, p_3 are different and $\{p_1, p_2, p_3\} \to Q$. Let

$$C' = l \to p_3 \to q \to x_2 \to y_3 \to x_1 \to l$$

and let $R' = B - C'$. We show that C' is a good cycle, i.e., for any $P' \subseteq X - \{p_3, x_1, x_2\}$ or $P' \subseteq Y - \{l, q, y_3\}$ we have $|N_{R'}^+(P')| \geq |P'|$. By Claim 5.1, it is obvious for $|P'| \geq k+1$ and for $|P'| \leq k-3$.

Case 2.1.1. $P' \subseteq X - \{p_3, x_1, x_2\}$.

For any $\{u, v\} \subseteq M + x_3$ and any $l' \in L - l$, since $l' \to P$ we have $l' \nrightarrow \{u, v\}$, $l' \in N_{R'}^+(\{u, v\})$ and

$$L - l \subseteq N_{R'}^+(\{u, v\}). \tag{5.4}$$

For any $w \in P - p_3$, since $L \to w$ and $|(Q-q) \cup \{y_1, y_2\}| = k-1$ we have $|N^-_{(Q-q)\cup\{y_1,y_2\}}(w)| \leq 1$ and

$$|N^+_{(Q-q)\cup\{y_1,y_2\}}(w)| \geq k-2. \tag{5.5}$$

If $|P'| = k-2$, then since either $|P' \cap (M + x_3)| \geq 2$ or $P' \cap (P - p_3) \neq \emptyset$ for $k \geq 4$ we have

$$|N_{R'}^+(P')| \geq \min_{w \in P - p_3} \{|L - l|, |N^+_{(Q-q)\cup\{y_1,y_2\}}(w)|\} \geq k-2.$$

If $|P'| = k$, then since $|P' \cap (M + x_3)| \geq 2$ and $P' \cap (P - p_3) \neq \emptyset$ for $k \geq 4$ we have

$$|N_{R'}^+(P')| \geq |L - l| + \min_{w \in P - p_3} |N^+_{(Q-q)\cup\{y_1,y_2\}}(w)| \geq k.$$

Now let $|P'| = k-1$. If $P' \cap (P - p_3) = \emptyset$, then $P' = M + x_3$ and

$$|N^+_{R'}(P')| \geq |L - l + y_1| = k - 1.$$

If $P' \cap (P - p_3) \neq \emptyset$, then $|P' \cap (M + x_3)| \geq 1$. If $|P' \cap (M + x_3)| = 1$, then $\{p_1, p_2\} \subseteq P'$ and

$$|N^+_{R'}(P')| \geq |N^+_{R'}(\{p_1, p_2\})| = |Q - q| + |\{y_1, y_2\}| = k - 1.$$

If $|P' \cap (M + x_3)| \geq 2$, then

$$|N^+_{R'}(P')| \geq |L - l| + \min_{w \in P - p_3} |N^+_{(Q-q)\cup\{y_1,y_2\}}(w)| \geq k - 1.$$

Case 2.1.2. $P' \subseteq Y - \{l, q, y_3\}$.

Note that $|P - p_3| = k - 2$ and

$$L - l \to P - p_3.$$

Since $\{p_3, x_1\} \to y_1$ and $y_1 \to x_2$, we have

$$|N^+_{R'}(y_1)| = k - 1.$$

Since $\{p_3, x_1, x_2\} \to y_2$, we have

$$|N^+_{R'}(y_2)| = k.$$

If $|P'| = k - 2$, then $P' \cap ((L - l) \cup \{y_1, y_2\}) \neq \emptyset$ and

$$|N^+_{R'}(P')| \geq \min\{|P - p_3|,\ |N^+_{R'}(y_1)|, |N^+_{R'}(y_2)|\} \geq k - 2.$$

Let $|P'| = k - 1$. If $P' \cap \{y_1, y_2\} \neq \emptyset$, then

$$|N^+_{R'}(P')| \geq \min\{|N^+_{R'}(y_1)|, |N^+_{R'}(y_2)|\} = k - 1.$$

If $P' \cap \{y_1, y_2\} = \emptyset$, then $P' \cap (L - l) \neq \emptyset$ and $P - p_3 \subseteq N^+_{R'}(P')$. Since $x_3 \to \{y_1, y_3\}$, we have $x_3 \nrightarrow P'$, $x_3 \in N^+_{R'}(P')$ and

$$|N^+_{R'}(P')| \geq |P - p_3 + x_3| = k - 1.$$

Now let $|P'| = k$. Then $P' \cap (L - l) \neq \emptyset$ and $P - p_3 \subseteq N^+_{R'}(P')$. If $M \to P'$, then $l \to M$ and $d^+_B(l) \geq |M| + |P| \geq k + 1$ for $k \geq 4$. Thus $M \nrightarrow P'$ and there exists $m \in M$ such that $m \in N^+_{R'}(P')$. Since $x_3 \to y_3$, we have $x_3 \nrightarrow P'$ and $x_3 \in N^+_{R'}(P')$. Thus

$$|N^+_{R'}(P')| \geq |P - p_3| + |\{m, x_3\}| = k.$$

So C' is a good cycle, a contradiction.

Case 2.2. *There are exactly two arcs of E_3 which have a tail in $\{x_1, x_2, x_3\}$.*

Assume w.l.o.g. that

$$x_1 \to y_2,\ x_2 \to y_3,\ y_1 \to x_3.$$

Since $\{x_1, x_2\} \to y_2$ and $|P| = k - 1$, we have $P \nrightarrow y_2$ and there exists $p_1 \in P$ such that $y_2 \to p_1$. Note that $p_1 \to Q + y_1 + y_3$. Similarly, there exists $p_2 \in P$ such that $y_3 \to p_2$ and $p_2 \to Q + y_1 + y_2$. Note that $N^+_L(x_i) \neq \emptyset$, as otherwise, $L \to x_i$, $x_j \to L$ for some $j \in \{1, 2\}$ and $d^+_B(x_j) \geq k + 1$. Note also that $N^-_Q(x_1) \neq \emptyset$, as otherwise, $x_1 \to Q + y_1 + y_2$, $L \to x_1$, $x_2 \to L + y_2 + y_3$ and $d^+_B(x_2) \geq k + 1$. Let

$$l \in N^+_L(x_3),\ q \in N^-_Q(x_1),$$

$$C' = l \to p_2 \to q \to x_1 \to y_2 \to x_3 \to l$$

and $R' = B - C'$. We show that C' is a good cycle, i.e., for any $P' \subseteq X - \{p_2, x_1, x_3\}$ or $P' \subseteq Y - \{l, q, y_2\}$ we have $|N^+_{R'}(P')| \geq |P'|$. By Claim 5.1, it is obvious for $|P'| \geq k + 1$ and for $|P'| \leq k - 3$.

Case 2.2.1. $P' \subseteq X - \{p_2, x_1, x_3\}$.

As in Case 2.1 (see Equations (4) and (5)), for any $\{u, v\} \subseteq M + x_2$ and for any $w \in P - p_2$,

$$L - l \subseteq N^+_{R'}(\{u, v\}),$$

$$|N^+_{(Q-q)\cup\{y_1,y_3\}}(w)| \geq k - 2.$$

Thus $|N^+_{R'}(P')| \geq |P'|$ for $|P'| \in \{k-2, k\}$. Now let $|P'| = k - 1$. If $p_1 \in P'$, then since $p_1 \to (Q - q) \cup \{y_1, y_3\}$ we have

$$|N^+_{R'}(P')| \geq |N^+_{R'}(p_1)| = k - 1.$$

If $p_1 \notin P'$, then $|P' \cap (M + x_2)| \geq 2$ and $L - l \subseteq N^+_{R'}(P')$. If $P' \cap (P - p_2) = \emptyset$, then $P' = M + x_2$, $y_3 \in N^+_{R'}(P')$ and

$$|N^+_{R'}(P')| \geq |L - l + y_3| = k - 1.$$

If $P' \cap (P - p_2) \neq \emptyset$, then

$$|N^+_{R'}(P')| \geq |L - l| + \min_{w\in P-p_2} |N^+_{(Q-q)\cup\{y_1,y_3\}}(w)| \geq k - 1.$$

Case 2.2.2. $P' \subseteq Y - \{l, q, y_2\}$.

Note that $|P - p_2| = k - 2$ and

$$L - l \to P - p_2.$$

For any $q' \in Q - q$, since $p_2 \to Q - q$ we have

$$|N^+_{R'}(q')| \geq k - 2.$$

Moreover, since $p_1 \to Q - q$ we have

$$|N^+_{M+x_2}(q')| \geq 1.$$

Since $\{p_2, x_1\} \to y_1$ and $y_1 \to x_3$, we have

$$|N^+_{R'}(y_1)| = k - 1.$$

Similarly, since $y_3 \to \{p_2, x_1\}$ and $\{p_1, x_3\} \to y_3$ we have

$$|N^+_{R'}(y_3)| = k - 2,\ |N^+_{M+x_2}(y_3)| \geq 1.$$

So $|N^+_{R'}(P')| \geq |P'|$ for $|P'| = k - 2$. Let $|P'| = k - 1$. If $y_1 \in P'$, then

$$|N^+_{R'}(P')| \geq |N^+_{R'}(y_1)| = k - 1.$$

If $y_1 \notin P'$, then $P' \cap (L - l) \neq \emptyset$, $P' \cap (Q - q + y_3) \neq \emptyset$ and

$$|N^+_{R'}(P')| \geq |P - p_2| + \min_{q' \in Q - q} \{|N^+_{M+x_2}(q')|, |N^+_{M+x_2}(y_3)|\} \geq k - 1.$$

Now let $|P'| = k$. Then $P' \cap (L - l) \neq \emptyset$ and $P - p_2 \subseteq N^+_{R'}(P')$. If $M \to P'$, then $l \to M$ and $d^+_B(l) \geq |M| + |P| \geq k + 1$ for $k \geq 4$. So $M \nrightarrow P'$ and there exists $m \in M$ such that $m \in N^+_{R'}(P')$. Since $x_2 \to y_2$, we have $x_2 \nrightarrow P'$ and $x_2 \in N^+_{R'}(P')$. Thus

$$|N^+_{R'}(P')| \geq |P - p_2| + |\{m, x_2\}| \geq k.$$

So C' is a good cycle, a contradiction.

Case 2.3. *There is exactly one arc of E_3 which has a tail in $\{x_1, x_2, x_3\}$.*

Assume w.l.o.g. that

$$x_1 \to y_2,\ y_3 \to x_2,\ y_1 \to x_3.$$

Since $\{x_1, x_2\} \to y_2$ and $|P| = k - 1$, we have $P \nrightarrow y_2$ and there exists $p^* \in P$ such that

$$y_2 \to p^*,\ p^* \to Q + y_1 + y_3.$$

Since $\{y_1, y_3\} \to x_2$ and $|L| = k-1$, we have $L \nrightarrow x_2$ and there exists $l \in L$ such that $x_2 \to l$. Now we distinguish two cases.

Case 2.3.1. $y_1 \to P - p^*$.

It follows that $P - p^* \to Q + y_2 + y_3$, $L + y_1 \to P - p^*$ and $P \to Q$. We first show the following claim.

Claim 5.2. *In Case 2.3.1, we have $L \nrightarrow x_1$.*

Proof. Assume the opposite that $L \to x_1$. It follows that

$$L \to P + x_1,\ M + x_2 + x_3 \to L,\ x_1 \to Q + y_1 + y_2,$$

$$Q \to M + x_2 + x_3,\ P + x_1 \to Q,\ M \to L + y_1.$$

Let $l' \in L$, $p \in P - p^*$, $q \in Q$ and $m \in M$. Then we have

$$M - m \to L - l - l',\ L - l - l' \to P - p - p^*,$$

$$P - p - p^* \to Q - q,\ Q - q \to M - m,$$

and

$$|L - l - l'| = |P - p - p^*| = |Q - q| = |M - m| = k - 3.$$

It follows that the subdigraph induced by $(L-l-l') \cup (P-p-p^*) \cup (Q-q) \cup (M-m)$ has a Hamilton path $l^* \overrightarrow{H} m^*$ starting with $l^* \in L-l-l'$ and ending with $m^* \in M-m$. Then

$$C_1' = m \to l \to p \to q \to x_2 \to y_2 \to m$$

is a good cycle since $B - C_1'$ has a Hamilton cycle $l' \to p^* \to y_3 \to x_1 \to y_1 \to x_3 \to l^* \overrightarrow{H} m^* \to l'$, a contradiction. □

Let

$$C' = l \to p^* \to y_1 \to x_3 \to y_3 \to x_2 \to l$$

and let $R' = B - C'$. We show that C' is a good cycle, i.e., for any $P' \subseteq X - \{p^*, x_2, x_3\}$ or $P' \subseteq Y - \{l, y_1, y_3\}$ we have $|N^+_{R'}(P')| \geq |P'|$. By Claim 5.1, it is obvious for $|P'| \geq k+1$ and for $|P'| \leq k-3$.

Suppose first that $P' \subseteq X - \{p^*, x_2, x_3\}$. Since $y_3 \to M + x_1$, then for any $v \in M + x_1$ we have $|N^+_{R'}(v)| \geq k-2$. Note that $|Q + y_2| = k-1$ and

$$P - p^* \to Q + y_2.$$

So $|N^+_{R'}(P')| \geq |P'|$ for $|P'| = k-2$. Let $|P'| = k-1$. As in Cases 2.1 and 2.2, for any $\{u, v\} \subseteq M + x_1$,

$$L - l \subseteq N^+_{R'}(\{u, v\}).$$

If $P' \cap (P - p^*) \neq \emptyset$, then

$$|N^+_{R'}(P')| \geq |Q + y_2| = k-1.$$

If $P' \cap (P - p^*) = \emptyset$, then $P' = M + x_1$ and

$$|N^+_{R'}(P')| \geq |L - l + y_2| = k-1.$$

Now let $|P'| = k$. Then $|P' \cap (M + x_1)| \geq 2$, $P' \cap (P - p^*) \neq \emptyset$ and

$$|N^+_{R'}(P')| \geq |L - l| + |Q + y_2| \geq k.$$

Now suppose that $P' \subseteq Y - \{l, y_1, y_3\}$. Note that $|P - p^*| = k-2$ and

$$L - l \to P - p^*.$$

Since $P \to Q$ and $|P| = k-1$, then for any $q \in Q$ we have $|N^-_{M+x_1}(q)| \leq 1$ and

$$|N^+_{R'}(q)| = |N^+_{M+x_1}(q)| \geq k-2.$$

Since $(P - p^*) \cup \{x_1, x_2\} \to y_2$, we have $y_2 \to M$ and

$$|N^+_{R'}(y_2)| = |N^+_{M+x_1}(y_2)| = k - 2.$$

So $|N^+_{R'}(P')| \geq |P'|$ for $|P'| = k - 2$. Let $|P'| = k - 1$. If $P' \cap (L - l) \neq \emptyset$, then $P' \cap (Q + y_2) \neq \emptyset$ and

$$|N^+_{R'}(P')| \geq |P - p^*| + \min_{q \in Q}\{|N^+_{M+x_1}(q)|, |N^+_{M+x_1}(y_2)|\} \geq k - 1.$$

If $P' \cap (L - l) = \emptyset$, then $P' = Q + y_2$. If $x_1 \notin N^+_{R'}(Q + y_2)$, then $x_1 \to Q + y_2$, $x_1 \to Q + y_1 + y_2$ and $L \to x_1$, a contradiction to Claim 5.2. Thus

$$|N^+_{R'}(Q + y_2)| \geq |N^+_M(y_2)| + |\{x_1\}| = k - 1.$$

Now let $|P'| = k$. Then $P' \cap (L - l) \neq \emptyset$ and $P - p^* \subseteq N^+_{R'}(P')$. If $m \notin N^+_{R'}(P')$ for some $m \in M$, then $m \to P'$ and $d^+_B(m) \geq |P' + y_1| = k + 1$. So $M \subseteq N^+_{R'}(P')$. Since $x_1 \to y_1$, we have $x_1 \nrightarrow P'$, $x_1 \in N^+_{R'}(P')$ and

$$|N^+_{R'}(P')| \geq |P - p^*| + |M + x_1| \geq k.$$

So C' is a good cycle, a contradiction.

Case 2.3.2. $y_1 \nrightarrow P - p^*$.

It follows that there exists $p \in P - p^*$ such that $p \to y_1$. Let

$$C'' = l \to p \to y_1 \to x_3 \to y_3 \to x_2 \to l$$

and let $R'' = B - C''$. We show that C'' is a good cycle, i.e., for any $P'' \subseteq X - \{p, x_2, x_3\}$ or $P'' \subseteq Y - \{l, y_1, y_3\}$ we have $|N^+_{R''}(P'')| \geq |P''|$. By Claim 5.1, it is obvious for $|P''| \geq k + 1$ and for $|P''| \leq k - 3$. We first show the following claim.

Claim 5.3. *In Case 2.3.2, we have the following two statements.*
(1) *For any $p \in P - p^*$ with $p \to y_1$, we have $|N^+_{R''}(P - p + m)| \geq k - 1$ for any*

$m \in M$.

(2) $M + x_1 \nrightarrow L - l + y_2$.

Proof. (1) Assume the opposite that $|N^+_{R''}(P - p + m)| \leq k - 2$ for some $m \in M$. Since $|N^+_{R''}(P-p+m)| \geq |N^+_{R''}(p^*)| = k-2$, we have $|N^+_{R''}(P-p+m)| = k-2$ and $N^+_{R''}(P - p + m) = N^+_{R''}(p^*) = Q$. It follows that

$$y_2 \to (P-p) \cup \{m, x_3\},\ P - p \to Q + y_1 + y_3,\ y_1 \to M + x_2 + x_3,$$

$$L - l \to P + m,\ x_1 \to (L-l) \cup \{y_1, y_2\},\ l \to P + x_1,$$

$$x_2 \to L + y_2,\ x_3 \to L + y_3,\ M - m \to L - l,$$

$$(L-l) \cup \{y_1, y_2\} \to m,\ m \to Q + l + y_3.$$

Since $|N^+_Q(p)| \geq k-3$, there exists $q \in Q$ such that $p \to Q - q$. Thus $P + m \to Q - q$. Let $l' \in L - l$. Similarly to the proof of Claim 5.2, the subdigraph induced by $(L - l - l') \cup (P - p - p^*) \cup (Q - q) \cup (M - m)$ has a Hamilton path $l^* \overrightarrow{H} m^*$ starting with $l^* \in L - l - l'$ and ending with $m^* \in M - m$. Then

$$C''_1 = m \to l \to p \to y_1 \to x_2 \to y_2 \to m$$

is a good cycle since $B - C''_1$ has a Hamilton cycle $l' \to p^* \to q \to x_3 \to y_3 \to x_1 \to l^* \overrightarrow{H} m^* \to l'$, a contradiction.

(2) Assume the opposite that $M + x_1 \to L - l + y_2$. It follows that

$$x_1 \to (L-l) \cup \{y_1, y_2\},\ M + x_1 + x_2 \to y_2,$$

$$l \to x_1,\ M + x_2 + x_3 \to l,\ y_2 \to P + x_3,$$

$$P \to Q + y_1 + y_3,\ M \to L + y_2,\ Q + y_1 + y_3 \to M.$$

Let $q \in Q$, $m \in M$, $l' \in L - l$ and $p' \in P - p - p^*$. If $x_2 \to Q$, then since $x_2 \to \{l, y_2\}$

we have

$$L - l \to x_2,\ M \to L.$$

Similarly to the proof of Claim 5.2, the subdigraph induced by $(Q-q)\cup(M-m)\cup(L-l-l')\cup(P-p-p')$ has a Hamilton path $q^*\overrightarrow{H_1}p^*$ starting with $q^* \in Q-q$ and ending with $p^* \in P-p$. Then

$$C_2'' = l' \to p' \to q \to m \to y_2 \to x_3 \to l$$

is a good cycle since $B - C_2''$ has a Hamilton cycle $y_3 \to x_1 \to y_1 \to x_2 \to l \to p \to q^*\overrightarrow{H_1}p^* \to y_3$, a contradiction. So $x_2 \nrightarrow Q$ and there exists $q^{**} \in Q$ with $q^{**} \to x_2$. Let $q' \in Q - q^{**}$. Similarly to the proof of Claim 5.2, the subdigraph induced by $(M-m)\cup(L-l-l')\cup(P-p-p^*)\cup(Q-q')$ has a Hamilton path $m^*\overrightarrow{H_2}q^{**}$ starting with $m^* \in M-m$ and ending with $q^{**} \in Q-q'$. Then

$$C_3'' = y_2 \to p^* \to y_1 \to x_3 \to y_3 \to m \to y_2$$

is a good cycle since $B - C_3''$ has a Hamilton cycle $x_2 \to l \to x_1 \to l' \to p \to q' \to m^*\overrightarrow{H_2}q^{**} \to x_2$, a contradiction. □

Suppose first that $P'' \subseteq X - \{p, x_2, x_3\}$. Note that $|L-l| = k-2$ and for any $\{u,v\} \subseteq M + x_1$,

$$L - l \subseteq N^+_{R''}(\{u,v\}).$$

For any $w \in P-p$, since $l \to P-p$ we have

$$|N^+_{Q+y_2}(w)| \geq k-2.$$

So $|N^+_{R''}(P'')| \geq |P''|$ for $|P''| \in \{k-2, k\}$. Now let $|P''| = k-1$. If $P'' \cap (P-p) \neq \emptyset$, then $P'' \cap (M+x_1) \neq \emptyset$. If $|P'' \cap (M+x_1)| \geq 2$, then

$$|N^+_{R''}(P'')| \geq |L-l| + \min_{w \in P-p} |N^+_{Q+y_2}(w)| \geq k-1.$$

If $|P'' \cap (M + x_1)| = 1$, then $P'' = P - p + x_1$ or $P'' = P - p + m$ for some $m \in M$. By Claim 5.3 (1), it suffices to consider that $P'' = P - p + x_1$. Note that

$$|N^+_{R''}(P - p + x_1)| \geq |N^+_{R''}(p^*) \cup N^+_{R''}(x_1)| \geq |Q + y_2| = k - 1.$$

If $P'' \cap (P - p) = \emptyset$, then $P'' = M + x_1$ and

$$|N^+_{R''}(M + x_1)| \geq |L - l + y_2| = k - 1.$$

Now suppose that $P'' \subseteq Y - \{l, y_1, y_3\}$. Note that $|P - p| = k - 2$ and

$$L - l \to P - p.$$

For any $\{u, v\} \subseteq Q + y_2$, since $\{u, v\} \nrightarrow p$ we have

$$|N^+_{R''}(\{u, v\})| \geq k - 2.$$

First let $|P''| = k - 2$. Then either $P'' \cap (L - l) \neq \emptyset$ or $|P'' \cap (Q + y_1)| \geq 2$. Thus

$$|N^+_{R''}(P'')| \geq \min\{|P - p|, |N^+_{R''}(\{u, v\})|\} = k - 2.$$

Then let $|P''| = k - 1$. If $P'' \cap (L - l) = \emptyset$, then $P'' = Q + y_2$. Note that

$$|N^+_{R''}(Q + y_2)| \geq |N^+_{R''}(y_2)| \geq k - 2.$$

If $|N^+_{R''}(Q + y_2)| = k - 2$, then $|N^+_{R''}(y_2)| = k - 2$ and $N^+_{R''}(Q + y_2) = N^+_{R''}(y_2)$. Since $x_1 \notin N^+_{R''}(y_2)$, we have $x_1 \notin N^+_{R''}(Q + y_2)$, $x_1 \to Q + y_1 + y_2$, $L \to x_1$ and $M \to L$. For any $m \in M$, since $m \to L$ we have $m \nrightarrow Q + y_2$ and $m \in N^+_{R''}(Q + y_2)$. So $M \subseteq N^+_{R''}(Q + y_2)$, $N^+_{R''}(Q + y_2) = M$ and $P - p \to Q + y_2$, a contradiction to $y_2 \to p^*$. Thus

$$|N^+_{R''}(Q + y_2)| \geq k - 1.$$

If $P'' \cap (L-l) \neq \emptyset$, then $P'' \cap (Q+y_2) \neq \emptyset$ and $P-p \subseteq N^+_{R''}(P'')$. Since $|P-p| = k-2$, it suffices to show that

$$M + x_1 \nrightarrow P''.$$

Assume the opposite that $M + x_1 \to P''$. Then $M + x_1 \to P'' \cap (Q + y_2)$. Since $p^* \to Q$ and $x_2 \to y_2$, we have $P'' \cap (Q + y_2) \to p$. Since $L \to p$ and $|L| = k - 1$, we have $|P'' \cap (Q + y_2)| = 1$. Then $P'' = L - l + y_2$ or $P'' = L - l + q$ for some $q \in Q$. If $P'' = L - l + y_2$, then by Claim 5.3 (2) we have $M + x_1 \nrightarrow P''$. Let $P'' = L - l + q$. If $x_1 \to L - l + q$, then $d^+_B(x_1) \geq |L - l + q| + |\{y_1, y_2\}| = k + 1$. So $x_1 \nrightarrow P''$ and $M + x_1 \nrightarrow P''$. Thus for $|P''| = k - 1$,

$$|N^+_{R''}(P'')| \geq |P''|.$$

Now let $|P''| = k$. Then $P'' \cap (L - l) \neq \emptyset$ and $P - p \subseteq N^+_{R''}(P'')$. Since $x_1 \to y_1$, we have $x_1 \nrightarrow P''$ and $x_1 \in N^+_{R''}(P'')$. Note that $|P - p + x_1| = k - 1$. It suffices to show that $M \nrightarrow P''$. Assume the opposite that $M \to P''$. Then $l \to M \cup P$ and $d^+_B(l) \geq |M| + |P| \geq k + 1$, a contradiction.

So C'' is a good cycle, a contradiction.

Case 2.4. *No arc of E_3 has a tail in $\{x_1, x_2, x_3\}$.*

It follows that

$$y_1 \to x_3,\ y_2 \to x_1,\ y_3 \to x_2.$$

Now we distinguish two cases.

Case 2.4.1. $P \to \{y_1, y_2, y_3\}$.

Since $x_i \to y_i$ for $i \in \{1, 2, 3\}$, we have $\{y_1, y_2, y_3\} \to M$. Let $m \in M$. There exists $l \in L$ such that $m \to l$, as otherwise, $L \to m$ and $d^-_B(m) \geq |L| + |\{y_1, y_2, y_3\}| = k + 2$. Let $p \in P$,

$$C' = m \to l \to p \to y_1 \to x_2 \to y_2 \to m$$

and $R' = B - C'$. We show that C' is a good cycle, i.e., for any $P' \subseteq X - \{m, p, x_2\}$ or $P' \subseteq Y - \{l, y_1, y_2\}$ we have $|N^+_{R'}(P')| \geq |P'|$. By Claim 5.1, it is obvious for

$|P'| \geq k+1$ and for $|P'| \leq k-3$. Let

$$E'_3 = E(B[C']) - E(C').$$

Note that $p \to y_2$ and $y_1 \to m$. This implies that there exists an arc of E'_3 which has a tail in $\{m, p, x_2\}$ and an arc of E'_3 which has a tail in $\{l, y_1, y_2\}$. By the proofs of Case 2.2 and Case 2.3, it suffices to consider that $|P'| \in \{k-2, k\}$.

Suppose first that $P' \subseteq X - \{m, p, x_2\}$. For any $\{u, v\} \subseteq (M-m) \cup \{x_1, x_3\}$ and for any $p' \in P - p$, since $L \to P$ we have

$$L - l \subseteq N^+_{R'}(\{u, v\}),\ |N^+_{Q+y_3}(p')| = k-2.$$

Thus $|N^+_{R'}(P')| \geq |P'|$ for $|P'| \in \{k-2, k\}$, where $k \geq 4$.

Now suppose that $P' \subseteq Y - \{l, y_1, y_2\}$. Note that $|P-p| = k-2$ and $L-l \to P-p$. First let $|P'| = k-2$. If $P' \cap (L-l) \neq \emptyset$, then $|N^+_{R'}(P')| \geq |P-p| = k-2$. If $P' \cap (L-l) = \emptyset$, then $P' \subseteq Q + y_3$. If $|N^+_{R'}(P')| = k-3$, then $P' \to p$ and $d^-_B(p) \geq |L| + |P'| \geq k+1$ for $k \geq 4$, a contradiction. So $|N^+_{R'}(P')| \geq k-2$. Now let $|P'| = k$. Then $P' \cap (l-l) \neq \emptyset$ and $P - p \subseteq N^+_{R'}(P')$. Since $x_1 \to y_1$, we have $x_1 \nrightarrow P'$ and $x_1 \in N^+_{R'}(P')$. Note that $|P - p + x_1| = k-1$. It suffices to show that $M - m + x_3 \nrightarrow P'$. If $M - m + x_3 \to P'$, then $l \to M - m + x_3$ and $d^+_B(l) \geq |M - m + x_3| + |P| \geq k+1$ for $k \geq 4$, a contradiction.

So C' is a good cycle, a contradiction.

Case 2.4.2. $P \nrightarrow \{y_1, y_2, y_3\}$.

Assume w.l.o.g. that $y_1 \to p$ for some $p \in P$. Then $p \to Q + y_2 + y_3$. Since $\{y_2, y_3\} \to x_1$, we have $L \nrightarrow x_1$ and there exists $l \in L$ such that $x_1 \to l$. Let

$$C'' = l \to p \to y_2 \to x_3 \to y_3 \to x_1 \to l$$

and let $R'' = B - C''$. We show that C'' is a good cycle, i.e., for any $P'' \subseteq X - \{p, x_1, x_3\}$ or $P'' \subseteq Y - \{l, y_2, y_3\}$ we have $|N^+_{R''}(P'')| \geq |P''|$. By Claim 5.1, it is

obvious for $|P''| \geq k+1$ and for $|P''| \leq k-3$. Let

$$E_3'' = E(B[C'']) - E(C'').$$

Note that $p \to y_3$ and $y_2 \to x_1$. This implies that there exists an arc of E_3'' which has a tail in $\{p, x_1, x_3\}$ and an arc of E_3'' which has a tail in $\{l, y_2, y_3\}$. By the proofs of Case 2.2 and Case 2.3, it suffices to consider that $|P''| \in \{k-2, k\}$.

Suppose first that $P'' \subseteq X - \{p, x_1, x_3\}$. For any $\{u, v\} \subseteq M + x_2$ and for any $w \in P - p$, since $L \to P$ we have

$$|N^+_{Q+y_1}(w)| \geq k-2,\ L - l \subseteq N^+_{R'}(\{u, v\}).$$

So $|N^+_{R''}(P'')| \geq |P''|$ for $|P''| \in \{k-2, k\}$.

Now suppose that $P'' \subseteq Y - \{l, y_2, y_3\}$. Note that $|P-p| = k-2$ and $L-l \to P-p$. Let $|P''| = k-2$. If $P'' \cap (L-l) \neq \emptyset$, then $|N^+_{R''}(P'')| \geq k-2$. If $P'' \cap (L-l) = \emptyset$, then $P'' \subseteq Q + y_1$. If $|N^+_{R''}(P'')| = k-3$, then $P'' \to p$ and $d^-_B(p) \geq |L| + |P''| \geq k+1$ for $k \geq 4$. So $|N^+_{R''}(P'')| \geq k-2$. Now let $|P''| = k$. Then $P'' \cap (l-l) \neq \emptyset$ and $P - p \subseteq N^+_{R''}(P'')$. Since $x_2 \to y_2$, we have $x_2 \nrightarrow P''$ and $x_2 \in N^+_{R''}(P'')$. Note that $|P - p + x_2| = k-1$. It suffices to show that $M \nrightarrow P''$. If $M \to P''$, then $l \to M$ and $d^+_B(l) \geq |M| + |P| \geq k+1$ for $k \geq 4$, a contradiction.

So C'' is a good cycle, a contradiction.

Case 3. $|P| = k-1$ *and* $|Q| = k-3$.

Note that $|L| = k$. Then $N^+_L(x_i) \neq \emptyset$ and $N^-_L(x_i) \neq \emptyset$ for $i \in \{1, 2, 3\}$. Also, $N^+_L(x_i) \cap N^-_L(x_{i+1}) \neq \emptyset$, where $x_4 = x_1$, as otherwise, $N^-_L(x_{i+1}) \subseteq N^-_L(x_i)$ and every vertex of $N^-_L(x_{i+1})$ has outdegree at least $|P + x_i + x_{i+1}| = k+1$. Let

$$l \in N^+_L(x_1) \cap N^-_L(x_2),$$

$$C' = x_1 \to l \to x_2 \to y_2 \to x_3 \to y_3 \to x_1$$

and $R' = B - C'$. Note that now $|N^+_{R'}(P)| = |Q + y_1| = k-2$. By the proof of Case

2, B has a good cycle, a contradiction.

Case 4. $|P| = k$ *and* $|Q| = k-2$.

For any $p \in P$, since $|P-p| = k-1$ then by the proofs of Case 2 and Case 3 we have $|N_R^+(P-p)| \geq k-1$. So $|Q| = |N_R^+(P)| \geq |N_R^+(P-p)| \geq k-1$, a contradiction to $|Q| = k-2$.

Case 5. $|P| = k-2$ *and* $|Q| = k-3$.

As in Case 3, $N_L^+(x_i) \neq \emptyset$ and $N_L^-(x_i) \neq \emptyset$ for $i \in \{1,2,3\}$. Assume w.l.o.g. that

$$|N_L^+(x_1)| \geq \max\{|N_L^+(x_2)|, |N_L^+(x_3)|\}.$$

If $|N_L^+(x_1)| = |N_L^+(x_2)| = |N_L^+(x_3)| = 1$, then $\{x_1, x_2, x_3\} \to Q$ and every vertex of Q has indegree at least $|P| + |\{x_1, x_2, x_3\}| \geq k+1$, a contradiction. So we have

$$|N_L^+(x_1)| \geq 2.$$

Let l be a vertex of $N_L^+(x_1)$ such that it has minimum number of inneighbors in $\{x_1, x_2, x_3\}$ and let $p \in P$. Let

$$C' = l \to p \to y_2 \to x_3 \to y_3 \to x_1 \to l$$

and let $R' = B - C'$. We show that C' is a good cycle, i.e., for any $P' \subseteq X - \{p, x_1, x_3\}$ or $P' \subseteq Y - \{l, y_2, y_3\}$ we have $|N_{R'}^+(P')| \geq |P'|$. By Claim 5.1, it is obvious for $|P'| \geq k+1$ and for $|P'| \leq k-3$. By the proofs of Case 2 and Case 3, it suffices to consider that $|P'| \in \{k-2, k\}$. We first show the following claim.

Claim 5.4. *In Case 5, if $k = 4$ then we have the following three statements.*

(1) $Q + y_1 \nrightarrow M + x_2$.

(2) *If $P' \subseteq L - l$ and $|P'| = 2$, then $|N_{R'}^+(P')| \geq 2$.*

(3) *If $P' \subseteq Y - \{l, y_2, y_3\}$ and $|P'| = 4$, then $|N_{R'}^+(P')| \geq 4$.*

Proof. Note that $|N_L^+(x_1)| \geq 2$. Let

$$M = \{m, m', m''\},\ L = \{l, l', l'', l'''\},\ P = \{p, p'\},\ Q = \{q\}.$$

(1) Assume the opposite that $Q + y_1 \to M + x_2$, i.e., $\{q, y_1\} \to \{m, m', m'', x_2\}$. Then $\{p, p', x_1, x_3\} \to \{q, y_1\}$, $|N_L^+(x_1)| = 2$ and $\{y_2, y_3\} \to x_1$. Since $|N_L^+(x_2)| \leq |N_L^+(x_1)|$ and $\{q, y_1\} \to x_2$, we have $|N_L^+(x_2)| = 2$ and $x_2 \to \{y_2, y_3\}$. Since $\{p, p', x_2, x_3\} \to y_3$, we have $y_3 \to \{x_1, m, m', m''\}$. Since $\{p, p', x_2\} \to y_2$ and $y_2 \to \{x_1, x_3\}$, we have $|N_M^-(y_2)| = 1$. Assume w.l.o.g. that $m \to y_2$. Thus $m \nrightarrow L$ and assume w.l.o.g. that $l' \to m$. Now $\{l', q, y_1, y_3\} \to m$ and $m \to \{l, l'', l'''\}$. Since $N^+_{\{q,y_1,y_2,y_3\}}(x_3) = \{q, y_1, y_3\}$, we have $|N_L^-(x_3)| = 3$. Note that $|N_L^+(x_2)| = 2$. So we can assume w.l.o.g. that $x_2 \to l''$ and $l''' \to x_3$. Note also that $q \to \{m, m', m'', x_2\}$ and $\{m', m''\} \to L$. Then

$$C_1' = l'' \to p \to y_2 \to x_1 \to y_1 \to x_2 \to l''$$

is a good cycle since $B - C_1'$ has a Hamilton cycle $l' \to m \to l \to p' \to q \to m' \to l''' \to x_3 \to y_3 \to m''$, a contradiction.

(2) Assume w.l.o.g. that $P' = \{l', l''\}$. Suppose the opposite that $|N^+_{R'}(\{l', l''\})| \leq 1$. Since $\{l', l''\} \to p'$, we have $|N^+_{R'}(\{l', l''\})| = 1$. Then $\{l', l''\} \to \{p, p', x_1, x_3\}$ and $\{m, m', m'', x_2\} \to \{l', l''\}$. Since $|N_L^+(x_1)| \geq 2$, we have $x_1 \to \{l, l'''\}$. Since $|N_L^+(x_1)| \geq |N_L^+(x_2)|$, we have $\{l, l'''\} \to x_2$. Note that $\{p, p', x_3\} \to y_3$. So $|N^+_{\{m,m',m''\}}(y_3)| \geq 2$ and assume w.l.o.g. that $y_3 \to \{m, m'\}$. Since $l \to \{p, p', x_2\}$, we have $l \nrightarrow \{m, m'\}$ and assume w.l.o.g. that $m \to l$. Let

$$C_2' = m \to l \to p \to y_2 \to x_3 \to y_3 \to m$$

and let $R_2' = B - C_2'$. We show that C_2' is a good cycle, i.e., for any $P_2' \subseteq X - \{m, p, x_3\}$ and for $P_2' \subseteq Y - \{l, y_2, y_3\}$ we have $|N^+_{R_2'}(P_2')| \geq |P_2'|$. By Claim 5.1, it is obvious for $|P_2'| \geq 5$ and for $|P_2'| \leq 1$. As Case 2 and Case 3 were already considered, it suffices

to consider that $|P_2'| \in \{2, 4\}$. Since $\{p, p', x_1\} \to y_1$, we have

$$N^+_{R_2'}(y_1) \cap \{m', m''\} \neq \emptyset.$$

Since $y_1 \to x_2$, we have $|N^+_{R_2'}(y_1)| \geq 2$. Note that

$$\{m', m'', x_2\} \to \{l', l''\},\ p' \to \{q, y_1\},$$

$$x_1 \to \{l''', y_1\},\ \{l', l''\} \to \{p', x_1\},\ l''' \to \{p', x_2\}.$$

So $|N^+_{R_2'}(P_2')| \geq |P_2'|$ for $|P_2'| = 4$ and $P_2' \subseteq X - \{m, p, x_3\}$, and for $|P_2'| = 2$. Now assume that $|P_2'| = 4$ and $P_2' \subseteq Y - \{l, y_2, y_3\}$. Since $\{p', x_2\} \to y_2$ and $x_1 \to l$, we have $\{p', x_1, x_2\} \subseteq N^+_{R_2'}(P_2')$. If $N^+_{R_2'}(P_2') \cap \{m', m''\} = \emptyset$, then $\{m', m''\} \to P_2'$, $l \to \{m', m''\}$ and $d^+_B(l) \geq |\{m, m', p, p', x_2\}| = 5$, a contradiction. Thus $N^+_{R_2'}(P_2') \cap \{m', m''\} \neq \emptyset$ and

$$|N^+_{R_2'}(P_2')| \geq |\{p', x_1, x_2\}| + |N^+_{R_2'}(P_2') \cap \{m', m''\}| \geq 4.$$

So C_2' is a good cycle, a contradiction.

(3) Note that $P' \cap (L - l) \neq \emptyset$. Then $p' \in N^+_{R'}(P')$. Since $x_2 \to y_2$, we have $x_2 \nrightarrow P'$ and $x_2 \in N^+_{R'}(P')$. It suffices to show that

$$|N^+_{R'}(P') \cap M| \geq 2.$$

If $|N^+_{R'}(P') \cap M| = 0$, then $M \to P'$, $l \to M$ and $d^+_B(l) \geq |\{m, m', m'', p, p''\}| = 5$, a contradiction. Now let $|N^+_{R'}(P') \cap M| = 1$ and assume w.l.o.g. that $m \in N^+_{R'}(P')$. Then $\{m', m''\} \to P'$ and $l \to \{m', m''\}$. So $l \to \{m, m'', p, p'\}$ and $\{x_1, x_2, x_3\} \to l$. Since $\{p, p', x_1\} \to y_1$, we have $\{m', m''\} \nrightarrow y_1$, $y_1 \notin P'$ and $P' = \{l', l'', l''', q\}$. Since $|N^+_L(x_1)| \geq 2$, we can assume w.l.o.g. that $l' \in N^+_L(x_1)$. By the minimality of $|N^-_{\{x_1,x_2,x_3\}}(l)|$ and $\{x_1, x_2, x_3\} \to l$, we have $\{x_1, x_2, x_3\} \to l'$ and $d^-_B(l') \geq |\{m', m'', x_1, x_2, x_3\}| \geq 5$, a contradiction. □

Case 5.1. $P' \subseteq X - \{p, x_1, x_3\}$.

Note that $|Q + y_1| = k - 2$ and

$$P - p \to Q + y_1.$$

Let first $|P'| = k - 2$. If $P' \cap (P - p) \neq \emptyset$, then

$$|N^+_{R'}(P')| \geq |Q + y_1| = k - 2.$$

If $P' \cap (P - p) = \emptyset$, then $P' \subseteq M + x_2$. If $|N^+_{R'}(P')| = k - 3$, then $P' \to \{l, y_2, y_3\}$ and $d^-_B(y_3) \geq |P + x_3| + |P'| \geq k + 1$ for $k \geq 4$. So

$$|N^+_{R'}(P')| \geq k - 2.$$

Now let $|P'| = k$. If $P' \cap (P - p) \neq \emptyset$, then $Q + y_1 \subseteq N^+_{R'}(P')$. For any $l' \in L - l$, since $l' \to p$ we have $l' \nrightarrow P'$ and $l' \in N^+_{R'}(P')$. Thus $L - l \subseteq N^+_{R'}(P')$ and

$$|N^+_{R'}(P')| \geq |Q + y_1| + |L - l| \geq k.$$

If $P' \cap P = \emptyset$, then $P' = M + x_2$ and $L - l \subseteq N^+_{R'}(M + x_2)$. Since $|L - l| = k - 1$, it suffices to show that $Q + y_1 \nrightarrow M + x_2$. By Claim 4 (1), we can assume that $k \geq 5$. If $Q + y_1 \to M + x_2$, then $|N^+_L(x_2)| \geq k - 2$ and $P + x_1 + x_3 \to Q + y_1$. So $|N^+_L(x_1)| \leq 2$. Note that $|N^+_L(x_1)| \geq 2$. So $|N^+_L(x_1)| = 2$. Now

$$2 = |N^+_L(x_1)| \geq |N^+_L(x_2)| \geq k - 2,$$

implying that $k \leq 4$, a contradiction. Thus for $k \geq 5$,

$$|N^+_{R'}(P')| \geq k.$$

Case 5.2. $P' \subseteq Y - \{l, y_2, y_3\}$.

For any $v \in Q + y_1$, since $p \to Q + y_1$ we have

$$|N^+_{R'}(v)| \geq k - 2.$$

First let $|P'| = k - 2$. If $P' \cap (Q + y_1) \neq \emptyset$, then

$$|N^+_{R'}(P')| \geq \min_{v \in Q+y_1} |N^+_{R'}(v)| \geq k - 2.$$

If $P' \cap (Q + y_1) = \emptyset$, then $P' \subseteq L - l$. By Claim 4 (2), we can assume that $k \geq 5$. If $|N^+_{R'}(P')| = k - 3$, then $P' \to x_1$ and $x_2 \to P'$. Thus

$$2 = |L - P'| \geq |N^+_L(x_1)| \geq |N^+_L(x_2)| \geq |P'| = k - 2,$$

implying that $k \leq 4$, a contradiction. Thus for $k \geq 5$,

$$|N^+_{R'}(P')| \geq k - 2.$$

Now let $|P'| = k$. Then $P' \cap (L - l) \neq \emptyset$ and $P - p \subseteq N^+_{R'}(P')$. By Claim 4 (3), it suffices to consider that $k \geq 5$. Since $x_2 \to y_2$, we have $x_2 \nrightarrow P'$ and $x_2 \in N^+_{R'}(P')$. Since $l \to P$, we have $|N^+_M(l)| \leq 2$ and $|N^-_M(l)| \geq k-3$. Then $|N^+_{R'}(P') \cap M| \geq k-3$. Thus for $k \geq 5$,

$$|N^+_{R'}(P')| \geq |P - p + x_2| + |N^+_{R'}(P') \cap M| \geq k.$$

So C' is a good cycle, a contradiction.

Case 6. $|P| = k$ *and* $|Q| = k - 1$.

Note that $N^+_P(y_i) \neq \emptyset$ and $N^-_P(y_i) \neq \emptyset$ for $i \in \{1, 2, 3\}$. At least two sets of

$$N^+_P(y_1) \cap N^-_P(y_2),\ N^+_P(y_2) \cap N^-_P(y_3),\ N^+_P(y_3) \cap N^-_P(y_1)$$

are not empty. If not, assume w.l.o.g. that $N^+_P(y_1) \cap N^-_P(y_2) = N^+_P(y_2) \cap N^-_P(y_3) = \emptyset$,

then $N_P^+(y_1) \subseteq N_P^+(y_2) \subseteq N_P^+(y_3)$ and every vertex in $N_P^+(y_1)$ has indegree at least $|L| + |\{y_1, y_2, y_3\}| = k + 1$, a contradiction. Now assume w.l.o.g. that

$$N_P^+(y_1) \cap N_P^-(y_2) \neq \emptyset,\ N_P^+(y_2) \cap N_P^-(y_3) \neq \emptyset$$

and let

$$p \in N_P^+(y_1) \cap N_P^-(y_2),\ p' \in N_P^+(y_2) \cap N_P^-(y_3).$$

If $N_P^+(y_3) \cap N_P^-(y_1) \neq \emptyset$, then let $p'' \in N_P^+(y_3) \cap N_P^-(y_1)$. Let

$$C' = y_1 \to p \to y_2 \to p' \to y_3 \to p'' \to y_1$$

and let $R' = B - C'$. Since now $N_{R'}^+(L) = P - \{p, p', p''\}$, $|L| = k - 2$, $|P - \{p, p', p''\}| = k - 3$ and Case 5 was already considered, there exists a good cycle. So $N_P^+(y_3) \cap N_P^-(y_1) = \emptyset$. Let

$$C'' = x_1 \to y_1 \to p \to y_2 \to x_3 \to y_3 \to x_1$$

and let $R'' = B - C''$. We show that C'' is a good cycle, i.e., for any $P'' \subseteq X - \{p, x_1, x_3\}$ or $P'' \subseteq Y - \{y_1, y_2, y_3\}$ we have $|N_{R''}^+(P'')| \geq |P''|$. By Claim 5.1, it is obvious for $|P''| \geq k + 1$ and for $|P''| \leq k - 3$. As Cases 2, 3, 4 and 5 were already considered, it suffices to consider that $|P''| = k$.

Case 6.1. $P'' \subseteq X - \{p, x_1, x_3\}$.

Note that $P'' \cap (M + x_2) \neq \emptyset$. So $L \subseteq N_{R''}^+(P'')$. If $|N_{R''}^+(P'') \cap Q| = k - 3$, then there exist $\{q_1, q_2\} \subseteq Q$ such that $\{q_1, q_2\} \to P''$ and thus $\{p, x_1, x_3\} \to \{q_1, q_2\}$. Since $x_1 \to L + y_1$, we have $d_B^+(x_1) \geq |L + y_1| + |\{q_1, q_2\}| = k + 1$, a contradiction. So $|N_{R''}^+(P'') \cap Q| \geq k - 2$ and

$$|N_{R''}^+(P'')| \geq |L| + |N_{R''}^+(P'') \cap Q| \geq (k - 2) + (k - 2) \geq k.$$

Case 6.2. $P'' \subseteq Y - \{y_1, y_2, y_3\}$.

Since $P'' \cap L \neq \emptyset$, we have $P - p \subseteq N^+_{R''}(P'')$. Since $x_2 \to y_2$, we have $x_2 \nrightarrow P''$ and $x_2 \in N^+_{R''}(P'')$. Thus

$$|N^+_{R''}(P'')| \geq |P - p + x_2| = k.$$

So C'' is a good cycle, a contradiction.

The proof of Lemma 5.1 is complete. □

5.3 k-cycle-factors in regular bipartite tournaments

The k-cycle-factors of a highly connected tournament was posed by Bollobás (see [82]) and was proved by Chen et al. [35].

Theorem 5.14 (Chen et al. [35])**.** *Let T be a k-connected tournament with $|V(T)| \geq 8k$. Then T has a k-cycle-factor.*

Motivated by Theorem 5.14, we consider k-cycle-factors of a highly connected bipartite tournament. Häggkvist and Manoussakis [47] proved that a bipartite tournament is Hamiltonian if and only if it is strong and has a cycle-factor. Note that there are infinite families of highly connected bipartite tournament without Hamilton cycles and thus without a cycle-factor, i.e. it cannot be partitioned into cycles. Thus, we should assume that the considered bipartite tournaments are Hamiltonian. Let $f(k)$ be the smallest integer so that all but a finite number of $f(k)$-connected Hamiltonian bipartite tournaments have a k-cycle-factor. Note that $f(1) = 1$. We conjecture that $f(k)$ exists and $f(k) = k$ for general k. By Corollary 4.1, we propose the following conjecture.

Conjecture 5.2. Every k-connected Hamiltonian bipartite tournament with at least $8k - 6$ vertices has a k-cycle-factor.

By replacing the condition "Hamiltonian" by "regular", we get a weaker result.

Theorem 5.15. *Let $BT = (X, Y; E)$ be a k-connected regular bipartite tournament with $|V(BT)| \geq 8k - 6$. Then BT has a k-cycle-factor.*

Proof. By Corollary 3.1, BT contains at least k vertex-disjoint cycles. Let $C_1, \ldots, C_k$ be k vertex-disjoint cycles of BT such that $\sum_{i=1}^{k} |V(C_i)|$ is maximum. To the contrary, assume that $\sum_{i=1}^{k} |V(C_i)| < n$. Let $\Psi = \{C_1, \ldots, C_k\}$ and let

$$H = BT - \cup_{i=1}^{k} V(C_i). \tag{5.6}$$

Claim 5.5. *H has no cycle.*

Proof. Assume that H has a cycle C. Let C_i be an arbitrary cycle in Ψ. We show that either $C_i \to C$ or $C \to C_i$. Let $u \in V(C_i) \cap X$ and $v \in V(C) \cap Y$. Assume without loss of generality that $u \to v$. For any vertex $x \in V(C) \cap X$, if $x \to u^+_{C_i}$ then $u^+_{C_i} C_i u v x u^+_{C_i}$ is a cycle longer than C_i and we can get another k cycles containing more vertices, a contradiction. Thus, $u^+_{C_i} \to C$. Similarly, $u^{++}_{C_i} \to C$, $\ldots$, $u \to C$ and thus $C_i \to C$. If $v \to u$, then similarly $C \to C_i$. Now let

$$\Psi_1 = \{C_i \in \Psi : C_i \to C\}, \quad \Psi_2 = \{C_j \in \Psi : C \to C_j\}. \tag{5.7}$$

Assume first that $\Psi_1 = \emptyset$. Then $\Psi_2 = \Psi$. Since BT is connected, there is a path from Ψ to C and let P be one with minimal length. Then all the internal vertices of P are in $V(H) - V(C)$. Let w and z be the starting vertex and the ending vertex of P, respectively. Assume w.l.o.g. that $w \in C_j \cap X$. If $z \in Y$, then

$$wPzCz^-_C \to w^+_{C_j} C_j w \tag{5.8}$$

is a longer cycle than C_j. If $z \in X$, then

$$wPzCz^{--}_C w^+_{C_j} C_j w \tag{5.9}$$

is a longer cycle than C_j. For each case we can get another k cycles containing more vertices, a contradiction. If $\Psi_2 = \emptyset$, then similarly we can get a contradiction.

Now assume that $\Psi_1 \neq \emptyset$ and $\Psi_2 \neq \emptyset$. If $\Psi_1 \to \Psi_2$, then since BT is connected there is a path from Ψ_2 to $\Psi_1 \cup C$. Let P be one with minimal length and let w and

z be the starting vertex and the ending vertex of P, respectively. Note that $C \to \Psi_2$. By the minimality of P, all the internal vertices of P are in $V(H) - V(C)$. Assume w.l.o.g. that $w \in C_i \cap X$ and $z \in C_j$. If $z \in Y$, then

$$wPzC_jz^-_{C_j}w^+_{C_i}C_iw \tag{5.10}$$

plus other $k-1$ unused cycles form k cycles containing more vertices, a contradiction. If $z \in X$, then

$$wPzC_jz^{--}_{C_j}w^+_{C_i}C_iw \tag{5.11}$$

plus other $k-1$ unused cycles form k cycles containing more vertices, a contradiction. Thus $\Psi_1 \nrightarrow \Psi_2$ and $C_i \nrightarrow C_j$ for some $C_i \in \Psi_1$ and for some $C_j \in \Psi_2$. Then $u \to v$ for some $u \in C_j$ and for some $v \in C_i$. Note that $u^+_{C_j} \to v^-_{C_i}$. Then

$$u^+_{C_j}v^-_{C_i}ww^+_Cu^+_{C_j}, \tag{5.12}$$

$$uvC_iv^{--}_{C_i}w^-_Cw^{--}_Cu_{C^{++}_jC_j}u \tag{5.13}$$

are two cycles containing more vertices than C_i and C_j. We get another k cycles containing more vertices than Ψ, a contradiction. So H has no cycle. □

By Lemma 1.2, BT has vertex-disjoint subsets $H_1, H_2, \ldots, H_{2i-1}, H_{2i}, \ldots, H_{2m-1}, H_{2m}$, $m \geq 1$, $1 \leq i \leq m$, such that $H_{2i-1} \subseteq X$, $H_{2i} \subseteq Y$ and there is no arc from H_p to H_q for $p > q$. The following result holds.

Claim 5.6. *let $u \in H_s$, $v \in H_t$ and $s < t$. Then $d^+_\Psi(u) \geq d^+_\Psi(v)$ and $d^-_\Psi(u) \leq d^-_\Psi(v)$.*

Proof. Note that the result holds clearly if u and v are in the same color class. Assume w.l.o.g. that $u \in X$ and $v \in Y$. Let $w \in V(C_i)$ be an arbitrary outneighbor of v in Ψ. Then $w \in X$ and $w^-_{C_i} \in Y$. If $w^-_{C_i} \to u$, then

$$wC_iw^-_{C_i}uvw \tag{5.14}$$

is a longer cycle than C_i, a contradiction. Thus $d^+_\Psi(u) \geq d^+_\Psi(v)$. Similarly, we have

$d^-_\Psi(u) \leq d^-_\Psi(v)$. □

For any two vertices $u \in H_1$ and $v \in H_{2m}$, we have $d^+_\Psi(u) \geq d^+_\Psi(v)$ by Claim 5.6. Note that $d^+_H(u) > d^+_H(v)$. Thus, $d^+_{BT}(u) > d^+_{BT}(v)$, a contradiction to the assumption that BT is regular.

The proof of Theorem 5.15 is complete. □

5.4 Conclusion

In this chapter, we consider the 2-cycle-factors and more generally k-cycle-factors of regular bipartite tournaments. First, we prove that every k-regular bipartite tournament B not isomorphic to a special digraph and with $k \geq 3$ has a 2-cycle-factor of cycle-lengths 6 and $|V(B)| - 6$. This gives a support to Conjecture 5.1 stating that every k-regular bipartite tournament B not isomorphic to a special digraph $F_{4\cdot k}$ contains all 2-cycle-factors. Also, we show that every k-connected regular bipartite tournament has a k-cycle-factor.

Chapter 6

Universal Arcs and Directed Cuts

6.1 Introduction

The concept of *universal arc* was first proposed by Ádám [2] in 1999 and is relatively new. The main problem is whether every cycle-connected digraph contains a universal arc and only few results have been obtained till now. For the max cut and cut-cover problem, it is very old and have been considered extensively. Tournaments and Cayley digraphs are two types of digraphs which have attracted much attention in digraph theory due to its special structure. In this chapter, we consider universal arcs in tournaments and directed cuts in the Cayley digraph $X(\mathbb{Z}_2^k, S_k)$.

6.2 Universal arcs in tournaments

The main result is the following theorem.

Theorem 6.1. *Let T be a tournament on at least 3 vertices. Then*

(1) *T has a universal arc if and only if T is 1-connected;*

(2) *every arc of T is universal if and only if T is 2-connected or $T \in \mathbf{T}_{\mathbf{s}}^*$;*

(3) *every arc of C is universal if C is a longest cycle containing a given universal arc in T and T is 1-connected.*

Remark. (1) *The 1-connected digraph in Figure 6.1 shows that the result in Theorem 6.1 (1) does not hold for general digraphs.*

(2) *Thomassen [92] showed in 1980 that every arc of a 3-connected tournament is contained in a Hamilton cycle, which implies that every arc of a 3-connected tournament is universal. Theorem 6.1 (2) shows that a weaker condition guarantees this property.*
(3) *Note that the longest cycle containing a given universal arc could be non-Hamiltonian. So the result in Theorem 6.1 (3) is not trivial.*

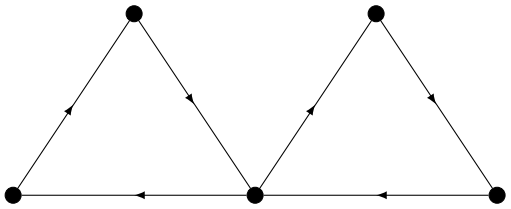

Figure 6.1: A 1-connected digraph with no universal arc.

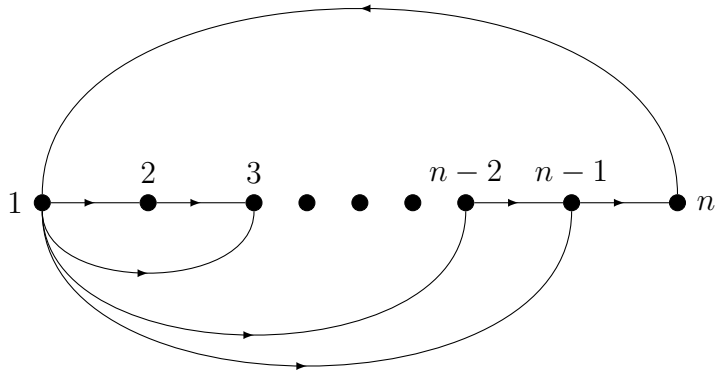

Figure 6.2: A 1-connected but not 2-connected tournament T_n with n universal arcs.

Proof. Let x, y be any two vertices of T. An (x, y)-path is a path from x to y. Let C be a cycle of T. A $(x, y)^{\overline{C}}$-path is a (x, y)-path with no internal vertex on C. For two vertices u, v of C, denote by u^+ and u^- the successor and predecessor of u on C, respectively; and denote by uCv the unique (u, v)-path on C. Let $[u, v]$ be the vertex set of uCv and let $d_C(u, v) = |[u, v]| - 1$ be the distance from u to v on C. For any vertex set $V' \subseteq V(T)$, we write $u \to V'$ if $u \to v$ for any vertex $v \in V'$ and write $V' \to u$ if $v \to u$ for any vertex $v \in V'$.

(1) Since every 1-connected tournament has a Hamilton cycle, then T has a universal arc. It suffices to show the converse statement. It follows from a more general result as follows.

Lemma 6.1. *If a digraph has a universal arc, then it is 1-connected.*

Proof. Let D be a digraph with a universal arc uv. It suffices to show that there is an (x, y)-path for any two vertices x and y of D. If $\{x, y\} \cap \{u, v\} \neq \emptyset$, then it follows directly from the definition of universal arcs. Suppose that $\{x, y\} \cap \{u, v\} = \emptyset$. Since x and uv are in a cycle, there is an (x, u)-path P_1. Similarly, there is a (v, y)-path P_2. If P_1 and P_2 are vertex disjoint, then xP_1uvP_2y is an (x, y)-path. If P_1 and P_2 have at least one common vertex, let w be the first appeared vertex on the path P_1, then xP_1wP_2y is an (x, y)-path. □

(2) Let T be a tournament satisfying that every arc is universal. Assume that T is not 2-connected and $T \notin \mathbf{T}_\mathbf{s}^*$. If T is not 1-connected, then the arc uu' is a non-universal arc where $u, u' \in V(T)$ satisfying that here exists no (u', u)-path. Now we have T is 1-connected and $T \notin \mathbf{T}_\mathbf{s}^*$. Then T has a cut vertex v such that one of the subtournaments induced by $N^+(v)$ and $N^-(v)$ (denote by T_x and T_y, respectively) is not 1-connected. Assume without loss of generality that T_x is not 1-connected. Then there exists no (x, x')-path for some two vertices $x, x' \in V(T_x)$ excluding v. Thus vx and x' are not in any cycle and vx is not a universal arc, a contradiction.

For the converse, it follows directly from the two lemmas below.

Lemma 6.2. *Every arc of a 2-connected tournament T is universal.*

Proof. Assume the opposite that T has a non-universal arc uv. Let C be a longest cycle containing uv in T. Since T has no 2-cycles, we have $|V(C)| \geq 3$. By assumption C is not hamiltonian. Let w be an arbitrary vertex in $V(T)\backslash V(C)$. For convenience, a cycle containing both uv and w is called *good* in the following. We will get a contradiction by showing that a good cycle exists.

If $w \to u$, then since T is 2-connected there exists a $(x, w)^{\overline{C}}$-path P for some $x \in V(C)\backslash\{u\}$. Now $wuvCxPw$ is a good cycle. If $v \to w$, then similarly since T is

2-connected there exists a $(w,y)^{\overline{C}}$-path P' for some $y \in V(C)\backslash\{v\}$. Now $uvwP'yCu$ is a good cycle. Assume from now on that $u \to w$ and $w \to v$.

We claim first that $w \to [v,z]$ and $[z^+,u] \to w$ for some $z \in V(C)\backslash\{u\}$. If not, then there exists $z' \in V(C)\backslash\{u\}$ with $z' \to w$ and $w \to z'^+$. Now $uvCz'wz'^+Cu$ is a cycle containing uv and is longer than C, a contradiction.

Since T is 2-connected, there exists a $(v_1,u_1)^{\overline{C}}$-path P^* for some $v_1 \in [v,z]$ and some $u_1 \in [z^+,u]$. If $w \in V(P^*)$, then $uvCv_1P^*u_1Cu$ is a good cycle. Now let $w \notin V(P^*)$. Assume that $d_C(u_1,u)$ is as small as possible and in addition $d_C(v,v_1)$ is as small as possible. By symmetry, we can assume without loss of generality that $d_C(u_1,u) \leq d_C(v,v_1)$. Now we distinguish two cases.

Case 7. $u_1 = u$.

Assume first that $v_1 \neq v$. Since T is 2-connected, there exists a $(x,y)^{\overline{C}}$-path P for some $x \in [v,v_1^-]$ and some $y \in [v_1^+,u]$. If P and P^* have a common veretex, say z, then the path $xPzP^*u$ contradicts the choice of P^*. So P and P^* are vertex disjoint and $y \in [v_1^+,u^-]$.

If $u \neq z^+$, then $u^- \to w$ and $uvCxPyCu^-wv_1P^*u$ is good cycle. Now let $u = z^+$. Since T is 2-connected, there exists a $(w',w)^{\overline{C}}$-path P' for some $w' \in V(C)\backslash\{u\}$. If $w' \neq u^-$, then $uvCw'P'wu^-u$ is a good cycle. Now let $w' = u^-$. If P' and P^* have a common vertex, without loss of generality let u' be the first vertex of P' contained in P^*, then replace u^-u of C by $u^-P'u'P^*u$ will yield a cycle containing uv longer than C. So P' and P^* are vertex disjoint. If P' and P are internally vertex disjoint, then $uvCxPyCu^-P'wv_1P^*u$ is a good cycle. If P' and P have a common internal vertex, without loss of generality let w'' the first vertex of P' contained in P (possibly $w'' = w$), then $uvCxPw''P'wv_1P^*u$ is a good cycle.

Now let $v_1 = v$. Since $u \to v$, then $|V(P^*)| \geq 3$. Let x be the successor of v on P^*.

If $x \to w$, then since T is 2-connected there exists a (w,w')-path P for some $w' \in V(C) \cup V(P^*)\backslash\{v\}$. Moreover, we can assume that P has no internal vertex in $V(C) \cup V(P^*)$. If $w' \in V(C)\backslash\{v\}$, then $uvxwPw'Cu$ is a good cycle. If $w' \in$

$V(P^*)\backslash\{v,x\}$, then $uvxwPw'P^*u$ is a good cycle. If $w'=x$, since either $w\to v^+$ or $v^+\to w$, then either $uvxwv^+Cu$ or uvv^+wPxP^*u is a good cycle.

If $w\to x$, then similarly since T is 2-connected there exists a (w'',w)-path P' for some $w''\in V(C)\cup V(P^*)\backslash\{u\}$. Moreover, we can assume that P' has no internal vertex in $V(C)\cup V(P^*)$. If $w''\in V(C)\backslash\{u\}$, then $uvCw''P'wxP^*u$ is a good cycle. Now we have that $w\to V(C)\backslash\{u\}$ and $w''\in V(P^*)\backslash\{u\}$. Then $uvP^*w''P'wv^+Cu$ is a good cycle.

Case 8. $u_1\neq u$.

It follows that $v_1\neq v$. If $u_1\neq z^+$, then $u_1^-\to w$. Since T is 2-connected, there exists a $(x,y)^{\overline{C}}$-path P for some $x\in[v,v_1^-]$ and some $y\in[v_1^+,u]$. If $y\in[u_1,u]$, then the path P contradicts the choice of P^*. So we have that $y\in[v_1^+,u_1^-]$. If P and P^* have a common vertex, say x', then the path $xPx'P^*u_1$ contradicts the choice of P^*. So P and P^* are vertex disjoint. Then $uvCxPyCu_1^-wv_1P^*u_1Cu$ is a good cycle. Now assume that $u_1=z^+$. Since T is 2-connected, there exists a $(x',y')^{\overline{C}}$-path P' for some $x\in[v,u_1^-]$ and some $y\in[u_1^+,u]$. Now the path P' contradicts the choice of P^*. The proof is complete. □

Lemma 6.3. *Every arc of a tournament $T\in\mathbf{T}_\mathbf{s}^*$ is universal.*

Proof. We first show the following claim.

Claim 6.1. *For any arc uv and any vertex w of a 1-connected tournament T, one of the following three statements hold.*
(1) *uv and w are in a common cycle;*
(2) *there exists a (v,w)-path excluding u;*
(3) *there exists a (w,u)-path excluding v.*

Proof. Since T is 1-connected, there is a (v,u)-path P. Since $u\to v$. we have $|V(P)|\geq 3$. Assume that uv and w are not in a common cycle. Then $w\notin V(P)$. For any vertex $w'\in V(P)-\{u,v\}$, if $w'\to w$ then $vPw'w$ is a (v,w)-path excluding u, if $w\to w'$ then $ww'Pu$ is a (w,u)-path excluding v. □

Recall that $T_y \to v$ and $v \to T_x$. Since T is 1-connected, there exist at least one arc goes from T_x to T_y. We claim that every arc between T_x and T_y goes from T_x to T_y. If not, then since both T_x and T_y are 1-connected we have v is not a cut vertex. Let x, y, z be arbitrary vertices in T_x, T_y and T, respectively.

First consider the arc vx. If $z \in T_y \cup \{v, x\}$, then one can easily see that vx and z are in a cycle. If $z \in T_x \backslash \{x\}$, then since T_x is 1-connected there is a (x, z)-path P in T_x and $vxPzyv$ is a cycle containing both vx and z. So vx is universal. Similar, we can show that any arc yv with $y \in T_y$ is universal.

Then consider an arbitrary arc xx' in T_x. If $z \in T_y \cup \{v, x, x'\}$, then one can see that xx' and z are in a cycle. If $z \in T_x \backslash \{x, x'\}$, then by Claim 6.1 we only need to consider the case that there is an (x', z)-path P' or a (z, x)-path P'' in T_x. Note that either $xx'P'zyvx$ or $xx'yvzP''x$ is a cycle containing both xx' and z. So xx' is universal.

Now consider the arc xy. If $z = v$, then xyz is a cycle containing both xy and z. If $z \in T_x$, note that there is a (z, x)-path P''' in T_x, then $xyvzP'''x$ is a cycle containing both xy and z. Similarly, we can prove the case $z \in T_y$. So xy is universal. □

(3) Let C be a longest cycle containing a given universal arc uv. By Theorem 6.1 (1), we have T is 1-connected. If $u' \to w$ and $w \to u'^+$ for some $u' \in V(C) \backslash \{u\}$. Replace $u'u'^+$ by $u'wu'^+$ will yield a cycle containing uv longer than C, a contradiction. Thus we have $V(T) \backslash V(C) = W_a \cup W_b \cup W_c$, where W_a, W_b and W_c are defined as follows.

- $W_a = \{w \in V(T) \backslash V(C) : \ V(C) \to w\}$.
- $W_b = \{w \in V(T) \backslash V(C) : \ w \to V(C)\}$.
- $W_c = \{w \in V(T) \backslash V(C) : \ [z^+, u] \to w \text{ and } w \to [v, z] \text{ for some } z \in V(C) \backslash \{u\}\}$.

Claim 6.2. *$W_a = W_b = \emptyset$ and $V(T) \backslash V(C) = W_c$.*

Proof. Denote by W_c^1 and W_c^2 the subsets of W_c with $z = v$ and $z \neq v$, respectively. Note that $V(C) \backslash \{v\} \to W_c^1$ and $W_c^2 \to v^+$.

Assume that there exists a vertex $w_a \in W_a$. Note that uv is a universal arc and there exists a cycle containing both uv and w. So there is a (w_a, u)-path P with $v \notin V(P)$. Since $V(C)\backslash\{v\} \to W_c^1$, we have $P \cap (W_b \cup W_c^2) \neq \emptyset$. Let x be the first vertex of P contained in $W_b \cup W_c^2$. Then $w_aPx \cap V(C) = \emptyset$. Now replace vv^+ by vw_aPxv^+. A longer cycle containing uv appears, a contradiction. Thus $W_a = \emptyset$. Similarly, we can show that $W_b = \emptyset$. □

Let $u'v'$ be an arbitrary arc of C distinct from uv. For any vertex $w \in V(C)$, the cycle C contains both $u'v'$ and w. For any vertex $w \in V(T)\backslash V(C)$, by Claim 6.2 we have $w \in W_c$. Then the cycle obtained from C by replacing uv with uwv is a cycle containing both $u'v'$ and w. So every arc of C distinct from uv is also a universal arc, which completes the proof.

The proof of Theorem 6.1 is complete. □

6.3 Directed cuts in a type of Cayley digraph

We deal with the Cayley digraph $X(\mathbb{Z}_2^k, S_k)$.

Theorem 6.2. $g(X(\mathbb{Z}_2^k, S_k)) \geq \begin{cases} \frac{(3k+1)2^k+1}{9}, & \textit{if } k \textit{ is odd;} \\ \frac{(3k+1)2^k-1}{9}, & \textit{if } k \textit{ is even.} \end{cases}$

By the result of Bai *et al.* [11] we have $c(X(\mathbb{Z}_2^k, S_k)) \leq h(2k+1)+1$. Here we give the precise value.

Theorem 6.3. $c(X(\mathbb{Z}_2^k, S_k)) = \begin{cases} 2, & \textit{if } k = 1; \\ 3, & \textit{if } k = 2; \\ 4, & \textit{if } k \geq 3. \end{cases}$

Proof. We first show the following lemma.

Lemma 6.4. *The independence number $\alpha(X)$ of X is 2^{k-2}.*

Proof. Let $V(X) = \{0, 1, \ldots, 2^k - 1\}$. For any $x \in V(X)$,

$$\{x, x + 2^{k-2}, x + 2^{k-1}, x + 3 \cdot 2^{k-2}\}$$

is a subset of $V(X)$ satisfying that every vertex is adjacent to the other three vertices. Since there are 2^{k-2} pairwise disjoint such sets in $V(X)$, we have $\alpha(X) \leq 2^{k-2}$. Note that

$$\{0, 3, 6, \ldots, 3(2^{k-2} - 1)\}$$

is an independent set of X with cardinality 2^{k-2}. Then we have $\alpha(X) = 2^{k-2}$. □

Let $I = \{0, 3, 6, \ldots, 3(2^{k-2} - 1)\}$ and $V_1 = \{i, i + 2^{k-2} : i \in I\}$. Note that $\overline{V_1} = \{i + 2^{k-1}, i + 3 \cdot 2^{k-2} : i \in I\}$. We consider the cut $E(V_1, \overline{V_1})$. For the subsets

$$A = \{x, x + 2^{k-2}\} \textit{ and } B = \{y, y + 2^{k-2}\},$$

where $x, y \in I$ and $x \neq y$, there are three possible arcs in the induced graph of $A \cup B$ with starting vertices in A. One is from x to $x + 2^{k-2}$, one is from x to $y + 2^{k-2}$ and the last one is from $x + 2^{k-2}$ to y. For the first kind, there are 2^{k-2} arcs since there are 2^{k-2} subsets like $\{x, x + 2^{k-2}\}$. For the second kind, we have two possible cases

$$y + 2^{k-2} - x = 2^{t_1}, \textit{ where } y + 2^{k-2} > x \textit{ and } 0 \leq t_1 \leq k - 3,$$

$$y + 2^{k-2} - x + 2^k = 2^{t_2}, \textit{ where } y + 2^{k-2} < x \textit{ and } 0 \leq t_2 \leq k - 3.$$

For the second case, we have $x - y = 2^{k-2} + 2^k - 2^{t_2} =$

$$y - x = 2^{t_1} - 2^{k-2}, \textit{ if } y + 2^{k-2} > x \textit{ and } 0 \leq t_1 \leq k - 3,$$

$$x - y = 2^{k-2} + 2^k - 2^{t_2}, \textit{ if } y + 2^{k-2} < x \textit{ and } 0 \leq t_2 \leq k - 3.$$

For the third kind, we also have two possible cases.

$$y - (x + 2^{k-2}) = 2^{t_3}, \textit{ where } y > (x + 2^{k-2}) \textit{ and } 0 \leq t_2 \leq k - 3,$$

$$y - (x + 2^{k-2}) + 2^k = 2^{t_4}, \textit{ where } y > (x + 2^{k-2}) \textit{ and } 0 \leq t_2 \leq k - 3.$$

Let $m_i = k-2-t_i$ for $i = 1, 2$. Since $x - y \equiv 0 \pmod 3$, we have

$$2^{k-2-m_1}(2^{m_1}-1) = 3 \cdot s_1 (1 \le m_1 \le k-2, 1 \le s_1 \le 2^{k-2}-1),$$

$$2^{k-2-m_2}(2^{m_2}+1) = 3 \cdot s_2 (1 \le m_2 \le k-2, 1 \le s_2 \le 2^{k-2}-1).$$

Note that for every $1 \le m \le k-2$, $2^m - 1$ can be divided by 3 only if m is even and $2^m + 1$ can be divided by 3 only if m is odd. So for every m satisfying $1 \le m \le k-2$ we can have a common solution of the two equations above

$$s = \frac{1}{3}(2^{k-2} + (-1)^{m+1} \cdot 2^{k-2-m})$$

Note that $x - y = 3s$ and $x, y \in I$ where $|I| = 2^{k-2}$. So there are $2^{k-2} - s$ pairs x and y satisfying that $x - y = 3s$ for any s in I, i.e., $2^{k-2} - s$ arcs of the second kind or the third kind for any s. The total number of the arcs of second and third kinds is

$$\sum_{m=1}^{k-2} \{2^{k-2} - \frac{1}{3}[2^{k-2} + (-1)^{m+1} \cdot 2^{k-2-m}]\}.$$

Thus we have

$$\begin{aligned}
|E(V_1, \overline{V_1})| &\ge k \cdot 2^{k-1} - 2^{k-2} - \sum_{m=1}^{k-2} [2^{k-2} - \frac{1}{3}(2^{k-2} + (-1)^{m+1} \cdot 2^{k-2-m})] \\
&= 2^{k-2}[(2k-1) - \frac{2}{3}(k-2) + \frac{1}{3}\sum_{m=1}^{k-2}((-1)^{m+1} \cdot 2^{-m})] \\
&= 2^{k-2}[\frac{4k}{3} + \frac{1}{3} + \frac{1}{3}(\sum_{m=1}^{k-2}((-1)^{m+1} \cdot 2^{-m})] \\
&= \begin{cases} \frac{(3k+1)2^k+1}{9}, & \text{if } k \text{ is odd;} \\ \frac{(3k+1)2^k-1}{9}, & \text{if } k \text{ is even.} \end{cases}
\end{aligned}$$

The proof of Theorem 6.2 is complete. □

Proof. One can easily verify that the result holds for $k = 1, 2$. It suffices to consider

the case $k \geq 3$. Our proof relies heavily on the following structure property of this type of Cayley digraph.

Lemma 6.5. *$X(\mathbb{Z}_2^{k+1}, S_{k+1})$ can be constructed by two copies of $X(\mathbb{Z}_2^k, S_k)$.*

Proof. We denote $j - i (\text{mod } 2^k)$ by $j - i$ in the following in sake of convenience. Let D_1 and D_2 be two copies of $X(\mathbb{Z}_2^k, S_k)$ with vertex sets $\{v_0, v_2, \ldots, v_{2^{k+1}-2}\}$ and $\{v_1, v_3, \ldots, v_{2^{k+1}-1}\}$, respectively. Note that $v_iv_j \in E(D_t)$, $t = 1, 2$, if and only if $j - i \in \{2 \cdot 2^{i-1} : i = 1, \ldots, k\}$. Let D be a digraph with vertex set $V(D_1) \cup V(D_2)$. Add an arc from v_i to v_j if $j - i = 1$ and use E^* to denote the set of arcs added in this way. Let

$$E(D) = E(D_1) \cup E(D_2) \cup E^*.$$

For an arbitrary arc $v_iv_j \in E(D)$, we can show that

$$j - i \in \{2^{i-1} : i = 1, \ldots, k + 1\}.$$

If $v_iv_j \in E(D_t)$ where $t \in \{1, 2\}$, then $j - i \in \{2 \cdot 2^{i-1} : i = 1, \ldots, k\}$ and $\{2 \cdot 2^{i-1} : i = 1, \ldots, k\} \subset \{2^{i-1} : i = 1, \ldots, k + 1\}$. If $v_iv_j \in E^*$, then $j - i = 2^0 \in \{2^{i-1} : i = 1, \ldots, k + 1\}$. Thus D is isomorphic to $X(\mathbb{Z}_2^{k+1}, S_{k+1})$. □

For convenience, we use X_2^3 to denote $X(\mathbb{Z}_2^3, S_3)$ in the following.

Lemma 6.6. $c(X_2^3) = 4$.

Proof. Let

$$V_1 = \{v_0, v_3, v_5, v_6\},\ V_2 = \{v_1, v_2, v_4, v_7\},$$

$$V_3 = \{v_1, v_3, v_6, v_7\},\ V_4 = \{v_0, v_2, v_5, v_7\}.$$

Then $\{E(V_i, \overline{V}_i)\colon\ i = 1, \ldots, 4\}$ is a 4-cut-cover of X_2^3. It suffices to prove that $c(X_2^3) \geq 4$. Assume that $c(X_2^3) \leq 3$. Denote the possible three subsets covering $E(X_2^3)$ by E_1, E_2 and E_3. Assign color 1 to the arcs in E_1, color 2 to the arcs in $E(2)\backslash E(1)$ and color 3 to the rest arcs. It follows a proper 3-arc-coloring of X_2^3, i.e., a coloring of arcs with 3 colors such that no two consecutive arcs receive a common

color. Without loss of generality, assume that $v_0v_4 \in E(1)$ and $v_4v_0 \in E(2)$. Let $d_c(v)$ be the number of colors used on the arcs incident with v, and let $d^i(v)$ be the number of arcs which are incident with v and are colored by i. Note that $d_c(v) \leq 3$.

Claim 6.3. *$d_c(v) = 3$ for each vertex v of X_2^3.*

Proof. Suppose that there is a vertex, without loss of generality v_0, with $d_c(v_0) \leq 2$. For each vertex v of X_2^3, note that neither its indegree nor its outdegree is zero, we have $d_c(v) \geq 2$. So $d_c(v_0) = 2$, $v_0v_1, v_0v_2 \in E(1)$ and $v_6v_0, v_7v_0 \in E(2)$. Since $v_0v_2v_4$ is a directed triangle, we have $v_2v_4 \in E(3)$. Similarly, we have $v_4v_6 \in E(3)$. It follows that two consecutive arcs have a common color, a contradiction. □

Claim 6.4. *$d^i(v) = 2$ for each vertex v of X_2^3 where $i = 1, 2, 3$.*

Proof. If not, then by Claim 1 and the fact that $d(v) = 6$ there is a vertex, without loss of generality v_0, with $d^i(v_0) = 3$ for some $i \in \{1, 2, 3\}$. Without loss of generality, suppose that $d^1(v_0) = 3$. Since $v_4v_0 \in E(2)$, we have $v_0v_1, v_0v_2, v_0v_4 \in E(1)$. Since $v_0v_2v_4$ is a directed triangle, we have $v_2v_4 \in E(3)$. Note that $v_6v_0 \notin E(1)$. If $v_6v_0 \in E(2)$, then $v_4v_6 \in E(3)$ and a contradiction to $v_2v_4 \in E(3)$. So $v_6v_0 \in E(3)$. Then $v_2v_6 \in E(2)$ since $v_0v_2v_6$ is a directed triangle. Now we have $v_0v_1 \in E(1)$, $v_2v_4 \in E(3)$ and $v_2v_6 \in E(2)$, implying that no color can be put on v_1v_2. □

Since $d^-(v) = d^+(v) = 3$ for any vertex v of X_2^3, then by Claim 6.3 and Claim 6.4 there exists two consecutive arcs having a common color. A contradiction to the fact that this is a proper 3-arc-coloring of X_2^3. □

Lemma 6.4 and Lemma 6.6 imply that for $k \geq 3$

$$c(k+1) \geq c(k) \geq c(3) = 4.$$

Note that the underlying graph of $X(\mathbb{Z}_2^{k+1}, S_{k+1})$ can be covered by two cuts. Thus $X(\mathbb{Z}_2^{k+1}, S_{k+1})$ can be covered by four cuts. This completes the proof of Theorem 6.3. □

6.4 Conclusion

In this chapter, we first completely characterize the universal arcs in tournaments. To be precise, we show that (1) T has a universal arc if and only if T is 1-connected; (2) every arc of T is universal if and only if T is 2-connected or $T \in \mathbf{T}_{\mathbf{s}}^*$, where $\mathbf{T}_{\mathbf{s}}^*$ is a special class of 1-connected tournaments; and (3) every arc of C is universal if C is a longest cycle containing a given universal arc in T and T is 1-connected. Then we consider the cuts in a type of Cayley digraph $X(\mathbb{Z}_2^k, S_k)$, where $\mathbb{Z}_2^k$ consists of all binary vectors with length k and $S_k = \{2^{i-1} : i = 1, \ldots, k\}$. We obtain a lower bound of the maximum number of arcs contained in a directed cut of $X(\mathbb{Z}_2^k, S_k)$ and the minimum number of directed cuts required to cover the arcs of $X(\mathbb{Z}_2^k, S_k)$.

Chapter 7

Conclusion and Perspective

In this Chapter, we summarize the main results of this thesis and conclude with some problems for further research.

7.1 Arc colorings

In Chapter 2, we mainly consider the (semi-)vertex-distinguishing proper arc coloring of digraphs. We give upper bounds for $\chi'^2_{vd}(D)$ and $\chi'^2_{svd}(D)$ respectively. In particular, the value of $\chi'^2_{vd}(D)$ is obtained for some regular digraph D. Moreover, we show that the values of $\chi'^2_{vd}(D)$ and $\chi'^2_{svd}(D)$ will not be changed if the coloring, in addition, required to be equitable. But the following two conjectures remain open for general cases.

Conjecture 7.1. Let D be a vdac-digraph. Then $\chi'^2_{vd}(D) = \pi(D)$.

Conjecture 7.2. $\chi'^2_{vd}(D^n_d) = \pi(D^n_d) = k_{d,n}$.

Besides, it will also be interesting to consider the 1-type VDPA colorings of digraphs. What is the minimum number of colors, denoted by $\chi'^1_{vd}(D)$, required for a 1-type VDPA colorings of a digraph D? Considering the cycles with large order, we have that $\chi'^1_{vd}(D)$ could be very far from the 1-type arc chromatic number $\chi'^1(D)$ of D. We conjecture that $\chi'^1_{vd}(D)$ is bounded by $|V(D)|$.

Conjecture 7.3. $\chi'^1_{vd}(D) \leq |V(D)|$.

Moreover, it will be interesting to consider the 1-type proper arc colorings of oriented planar graphs. In 1976, Steinberg proposed the following conjecture.

Conjecture 7.4 (Steinberg)**.** Every $\{C_4, C_5\}$-free planar graphs are 3-vertex-colorable.

This conjecture has received much attention and many progresses have been obtained. For more details, one can see [1, 27–30, 84]. Let G be an undirected graph and let $\chi'^1(G)$ be the maximum value of $\chi'^1(D)$, as D ranges over all the orientations of G. By Theorem 1.5 and Conjecture 7.4, we propose the following conjectures.

Conjecture 7.5. Let P be a $\{C_4, C_5\}$-free planar graph. Then $\chi'^1(P) \leq 3$ if and only if $\chi(P) \leq 3$.

Conjecture 7.6. Oriented $\{C_4, C_5\}$-free planar graphs are 3-arc-colorable.

Note that not every oriented planar graph is 3-arc-colorable. The graph in Figure 7.1 (in [102]) is an oriented planar graph but not 3-arc-colorable. Assume the opposite that it has a proper 3-arc-coloring. Note that both $\{x_1x_2, x_2x_3, x_3x_1\}$ and $\{y_1y_2, y_2y_3, y_3y_1\}$ should receive three distinct colors. Also, the arcs in $\{x_1z, x_2z, x_3z\}$ and the arcs in $\{zy_1, zy_2, zy_3\}$ should receive two distinct colors. Thus, there exists one arc in $\{x_1z, x_2z, x_3z\}$ and one arc in $\{zy_1, zy_2, zy_3\}$ having a common color, a contradiction.

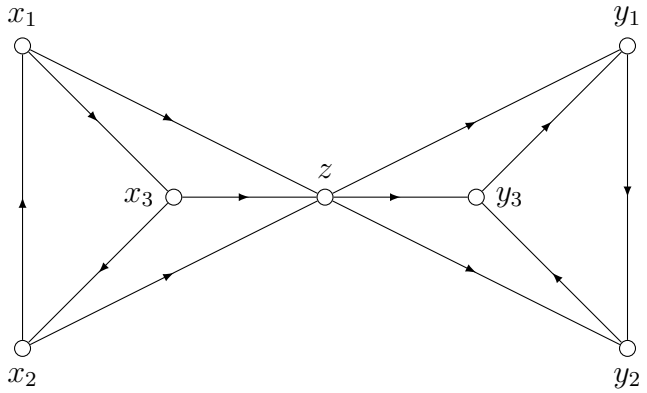

Figure 7.1: A non-3-arc-colorable oriented planar graph.

7.2 Short cycles

In Chapter 3, we consider the short cycles in digraphs. Especially, we focus on the Caccetta-Häggkvist conjecture (Conjecture 1.4) with forbidden subdigraphs. Motivated by the result of Razborov [80], which verifies the case $l = 3$ of the Caccetta-Häggkvist conjecture with three well defined (induced) forbidden subdigraphs, we generalize it by verifying Conjecture 1.4 for $l \geq 4$ with $l + 1$ well defined (induced) forbidden subdigraphs. It is natural to ask if less forbidden subdigraphs suffice for Conjecture 1.4. In particular, it seems that the forbidden subdigraph double-3-path in Theorem 3.2 (resp. F_4 in Theorem 3.4) could be removed. Also, the verification of Conjecture 1.4 for digraphs with no forbidden subdigraphs deserves further consideration.

7.3 Vertex-disjoint cycles

In Chapter 4, we consider vertex-disjoint cycles of given length in bipartite tournaments. Till now, the vertex-disjoint cycles of given length in tournaments and bipartite tournaments have been characterized. It is natural to consider the analogous problem in multipartite tournaments. We propose the following conjecture.

Conjecture 7.7. Let $D = (V_1, \ldots, V_k; E(D))$ be a k-partite tournament. If $d^+_{V_i}(v) \geq q_i r - 1$ for any $v \in V(D) \backslash V_i$, then D contains r vertex-disjoint $\sum_{i=1}^{k} q_i$-cycles.

7.4 Cycle factors

In Chapter 5, we consider cycle factors in bipartite tournaments. Note that Conjecture 5.1 remains unsolved for $8 \leq t \leq |V(B)| - 4$. It is natural to prove or disprove this conjecture for larger t.

It will also be interesting to consider under what conditions a bipartite tournament contains all k-cycle-factors. We propose the following conjecture.

Conjecture 7.8. For a positive integer k, there exists an integer $h(k)$ such that every

k-connected Hamiltonian bipartite tournament on at least $h(k)$ vertices contains all k-cycle-factors.

7.5 Universal arcs and directed cuts

In Chapter 6, we have completely characterized the universal arcs in tournaments. Recall that the Problem1.1 proposed by Ádám [2] in 1999 remains open. Also, recall that Hubenko conjectured in [58] that every arc of a maximal cycle of a cycle connected tournament is universal. These two problems deserve further consideration. Furthermore, analogous to the conception of "universal arc", we can define a *universal vertex* v of a digraph D to be one vertex such that xy and v are in a common cycle for any arc xy of D. It will be interesting to consider under what conditions a digraph contains a universal vertex.

Besides, in this chapter, we deal with directed cuts in a type of Cayley digraph $X(\mathbb{Z}_2^k, S_k)$. We have obtained a lower bound of $g(X(\mathbb{Z}_2^k, S_k))$ as follows.

$$g(X(\mathbb{Z}_2^k, S_k)) \geq \begin{cases} \frac{(3k+1)2^k+1}{9}, & \text{if } k \text{ is odd;} \\ \frac{(3k+1)2^k-1}{9}, & \text{if } k \text{ is even.} \end{cases}$$

We conjecture that the result is best possible, i.e., the inequality "$\geq$" can be replaced by "$=$".

Bibliography

[1] H.L. Abbott, B. Zhou, On small faces in 4-critical graphs, Ars Combin. 32 (1991) 203-207.

[2] A. Ádám, On some cyclic connectivity properties of digraphs, Acta Cybernet. 14 (1) (1999) 1-12.

[3] M. Aigner, E. Triesch, Z. Tuza, Irregular assignment and vertex-distinguishing edge-colorings of graphs, in: A. Barlotti et al. (Eds.), Combinatorics 90, Elsevier Science Publications, New York, 1999, 1-9.

[4] S. Akbari, H. Bidkhori, N. Nosrati, r-strong edge colorings of graphs, Discrete Mathmatics 306 (2006) 30053010.

[5] N. Alon, Disjoint directed cycles, J. Combin. Theory Ser. B 68 (1996) 167-178.

[6] N. Alon, B. Bollobás, A. Gyárfás, J. Lehel and A. Scott, Maximum directed cuts in acyclic digraphs, J. Graph Theory 55 (2007) 1-13.

[7] N. Alon and E. Halperin, Bipartite subgraphs of integer weighted graphs, Discrete Math. 181 (1998) 19-29.

[8] N. Alon, C. McDiarmid and M. Molloy, Edge-disjoint cycles in regular directed graphs, J. Graph Theory 22 (1996) 231-237.

[9] D. Amar and Y. Manoussakis, Cycles and paths of many lengths in bipartite digraphs, J. Combin. Theory Ser. B 50 (1990) 254-264.

[10] D. Amar and A. Raspaud, Covering the vertices of a digraph by cycles with precribed length, Discrete Math. 87 (1991) 111-118.

[11] Y. Bai, B. Li, S. Zhang, Covering the edges of digraphs in $D(3,3)$ and $D(4,4)$ with directed cuts, Discrete Math. 312 (2012) 1596-1601.

[12] P.N. Balister, Vertex-distinguishing edge colorings of random graphs, Random Structures & Algorithms 20 (1) (2002) 89-97.

[13] P.N. Balister, B. Bollobás and R.H. Schelp, Vertex-distinguishing colorings of graphs with $\Delta(G) = 2$, Discrete Math. 252 (2002) 17-29.

[14] P.N. Balister, E. Győri, J. Lehel and R.H. Schelp, Adjacent vertex distinguishing edge-colorings, AIAM J. Discrete Math. 21 (2007) 237-250.

[15] P.N. Balister, A. Kostochka, H. Li and R.H. Schelp, Balanced edge colorings, J. Combin. Theory Ser. B 90 (2004) 3-20.

[16] P.N. Balister, O.M. Riordan and R.H. Schelp, Vertex-distinguishing edge colorings of graphs, J. Graph Theory 42 (2003) 95-109.

[17] J. Bang-Jensen, S. Bessy and S. Thomassé, Disjoint 3-cycles in tournaments: A proof of the BermondCThomassen conjecture for tournaments, J. Graph Theory 75 (3) (2014) 284-302.

[18] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, in: Springer Monographs in Mathematics, Springer Verlag, London, 2008.

[19] C. Bazgan, A. Harkat-Benhamdine, H. Li and M. Woźniak, A note on the vertex-distinguishing proper coloring of graphs with large minimum degree, Discrete Math. 236 (2001) 37-42.

[20] C. Bazgan, A. Harkat-Benhamdine, H. Li and M. Woźniak, On the vertex-distinguishing proper edge-colorings of graphs, J. Combin. Theory Ser. B 75 (1999) 288-301.

[21] M. Behzad, G. Chartrand and C. Wall, On minimal regular digraphs with given girth, Fundamenta Mathematicae 69 (1970) 227-231.

[22] L.W. Beineke and C.H.C. Little, Cycles in bipartite tournaments, J. Combin. Theory Ser. B 32 (1982) 140-145.

[23] J.C. Bermond, C. Thomassen, Cycles in digraphs-a survey, J. Graph Theory 5 (1) (1981) 1-43.

[24] S. Bessy, F. Havet, and E. Birmelé, Arc-chromatic number of digraphs in which every vertex has bounded outdegree or bounded indegree, J. Graph Theory 53 (2006) 315-332.

[25] S. Bessy, N. Lichiardopol, J.S. Sereni, Two proofs of the Bermond-Thomassen conjecture for tournaments with bounded minimum in-degree, Discrete Math. 310 (3) (2010) 557-560.

[26] J.A. Bondy, Counting subgraphs: A new approach to the Caccetta-Häggkvist conjecture, Discrete Math. 165/166 (1997) 71-80.

[27] O.V. Borodin, To the paper of H.L. Abbott and B. Zhou on 4-critical planar graphs, Ars Combin. 43 (1996) 191-192.

[28] O.V. Borodin, Structural properties of plane graphs without adjacent triangles and an application to 3-colorings, J. Graph Theory 21 (2) (1996) 183-186.

[29] O.V. Borodin, A.N. Glebov, M. Montassier, A. Raspaud, Planar graphs without 5- and 7-cyles and without adjacent triangles adjacent are 3-colorable, J. Combin. Theory Ser. B 99 (2009) 268-273.

[30] O.V. Borodin, A.N. Glebov, A. Raspaud, Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable, J. Combin. Theory Ser. B 93 (2005) 303-311.

[31] A.C. Burris and R.H. Schelp, Vertex-distinguishing proper edge-colorings, J. Graph Theory 26 (2) (1997) 73-82.

[32] L. Caccetta and R. Häggkvist, On minimal digraphs with given girth, Congressus Numerantium 21 (1978) 181-187.

[33] P. Camion, Chemins et circuits hamiltoniens des graphes complets, C. R. Acad. Sci. Paris 249 (1959) 2151-2152.

[34] V. Chvátal and E. Szemerédi, Short cycles in directed graphs, J. Comb. Theory Ser. B 35 (1983) 323-327.

[35] G. Chen, R.J. Gould and H. Li, Partitioning vertices of a tournament into independent cycles, J. Combin. Theory Ser. B 50 (2001) 213-220.

[36] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, Acta Math. Acad. Sci. Hungar. 14 (1963) 423-439.

[37] Y. Dai, Y. Bu, An upper bound on the adjacent vertex-distinguishing chromatic number of graph, Math. Econ. 26 (2009) 107110.

[38] E. Dedo, D. Torri and N. Zagaglis Salvi, The observability of the Fibonacci and the Lucas cubes, Discrete Math. 255 (2002) 55-63.

[39] M. De Graaf, A. Schrijver and P. Seymour, Directed triangles in directed graphs, Discrete Math. 110 (1992) 279-282.

[40] G. Dirac, Some theorems on abstract graphs, Proceedings of the London Mathematical Society 2 (1952) 69-81.

[41] C.S. Edwards, Some extremal properties of bipartite subgraphs, Canadian J. Math. 25 (1973) 475-485.

[42] M El-Zahar, On circuits in graphs, Discrete Math. 50 (1984) 227-230.

[43] A. EI-Sahili, Functions and line digraphs, J. Graph Theory 44 (2003) 296-303.

[44] F. Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995) 153-158.

[45] A. Ghouila-Houri, Une condition suffisante d'existence d'un circuit Hamiltonien, C.R. Acad. Sci. Paris 25 (1960) 495-497.

[46] E. Győri, M. Horňák, C. Palmer and M. Woźniak, General neighbour-distinguishing index of a graph, Discrete Math. 308 (2008) 827-831.

[47] R. Häggkvist and Y. Manoussakis, Cycles and paths in bipartite tournaments with spanning configurations, Combinatorica 9 (1989) 51-56.

[48] C.C. Harner and R.C. Entringer, Arc colorings of digraphs, J. Combin. Theory Ser. B 13 (1972) 219-225.

[49] P. Hamburger, P. Haxell and A. Kostochka, On directed triangles in digraphs, Electronic J. Combinatorics 14 Note 19 (2007).

[50] Y.O. Hamidoune, A note on minimal directed graphs with given girth, J. Comb. Theory Ser. B 43 (1987) 343-348.

[51] H. Hatami, $\Delta + 300$ is a bound on the adjacent vertex distinguishing edge chromatic number, J. Comb. Theory Ser. B 95 (2005) 246-256.

[52] G. Hetyei, Cyclic connectivity classes of directed graphs, Acta Math. Acad. Paedagog. Nyházi. 17 (2) (2001) 47-59.

[53] A.J.W. Hilton, Embedding an incomplete diagonal Latin square in a complete diagonal Latin square, J. Combin. Theory Ser. A 15 (1973) 121-128.

[54] J. Hladký, D. Král and S. Norin, Counting flags in triangle-free graphs, Elctronic Notes in Discrete Math. 34 (2009) 621-625.

[55] C.T. Hoáng and B. Reed, A note on short cycles in digraphs, Discrete Math. 66 (1987) 103-107.

[56] H. Hocquard and M. Montassier, Adjacent vertex-distinguishing edge coloring of graphs with maximum degree Δ, J. Comb. Optimizaiton 26 (2013) 152-160.

[57] M. Horňák and R. Soták, Observability of complete multipartite graphs with equipotent parts, Ars Combin. 41 (1995) 289-301.

[58] A. Hubenko, On a cyclic connectivity property of directed graphs, Discrete Math. 308 (2008) 1018-1024.

[59] B. Jackson, Long paths and cycles in oriented graphs, J. Graph Theory 5 (1981) 145-157.

[60] R. de Joannis de Verclos, J. Sereni and J. Volec, Presented at **J.A. Bondy's 70th birthday**, Paris, 2014.

[61] P. Justesen, On Independent Circuits in Finite Graphs and a Conjecture of Erdös and Pósa, Ann. Disc. Math. 41 (1989) 299-306.

[62] L. Kelly, D. Kühn and D. Osthus, Cycles of given length in oriented graphs, J. Combin. Theory Ser. B 100 (2010) 251-264.

[63] D. Kühn, D. Osthus and D. Piguet, Embedding cycles of given length in oriented graphs, European J. Combinatorics 34 (2013) 495-501.

[64] D. Kühn, D. Osthus and T. Townsend, Proof of a tournament partition conjecture and an application to 1-factors with prescribed cycle lengths, Combinatorica, to appear.

[65] M. Laurent, Max-cut problem, in: annotated bibliographies in combinatorial optimization, Wiley, 1997, 241-259.

[66] J. Lehel, F. Maffray and M. Preissmann, Maximum directed cuts in digraphs with degree restriction, J. Graph Theory 61 (2009) 140-156.

[67] H. Li and J. Shu, The partition of a strong tournament, Discrete Math. 290 (2005) 211-220.

[68] N. Lichiardopol, Vertex-disjoint directed cycles of prescribed length in tournaments with given minimum out-degree and in-degree, Discrete Math. 310 (2010) 2567-2570.

[69] N. Lichiardopol, Proof of the Caccetta-Häggkvist conjecture for oriented graphs with positive minimum outdegree and of independent number two, Discrete Math. 313 (2013) 1540-1542.

[70] N. Lichiardopol, New lower bounds on the maximum number of arc-disjoint cycles in a digraph, preprint.

[71] N. Lichiardopol, A. Por, J.S. Sereni, A step toward the Bermond-Thomassen conjecture about disjoint cycles in digraphs, SIAM J. Discrete Math. 23 (2) (2009) 979-992.

[72] C.H.C. Little and H. Wang, Vertex-disjoint cycles in a directed graph, Australasian J. Combinatorics 11 (1995) 113-119.

[73] B. Mckay, Personal communication, http://cs.anu.edu.au/∼bdm/nauty/.

[74] H. Meyniel, Une condition suffisante d'existence d'un circuit hamiltonien dans un graphe orienté, J. Combin. Theory B 14 (1973) 137-147.

[75] T. Nishimura, Short cycles in digraphs, Discrete Math. 72 (1988) 295-298.

[76] O. Ore, Note on hamilton circuits, American Mathematical Monthly 67 (1960) 55.

[77] S. Poljak and V. Rödl, On the arc-chromatic number of a digraph, J. Combin. Theory Ser. B 31 (1981) 190-198.

[78] S. Poljak and Zs. Tuza, Maximum cuts and largest bipartite subgraphs, in: Combinatorial optimization, DIMACS Ser. in Discrete Mathematics and Theoretical Computer Science, volume 20, 1995, 181-244.

[79] A. Razborov, Flag algebras, J. Symbolic Logic 72 (2007) 1239-1282.

[80] A. Razborov, On the Caccetta-Häggkvist conjecture with forbidden subgraphs, J. Graph Theory 74 (2) (2012) 1-13.

[81] K.B. Reid, Two complementary circuits in two-connected tournaments, Ann. Discrete Math. 27 (1985) 321-334.

[82] K.B. Reid, Three problems on tournament, graph theory and its applications: east and west, Ann. New York Acad. Sci. 576 (1989) 466-473.

[83] J. Rudašová and R. Soták, Vertex-distinguishing proper edge colorings of some regular graphs, Discrete Math. 308 (2008) 795-802.

[84] D.P. Sanders, Y. Zhao, A note on the three color problem, Graphs Combin. 11 (1995) 91-94.

[85] J. Shen, Directed triangles in digraphs, J. Comb. Theory Ser. B 74 (1998) 405-407.

[86] J. Shen, On the girth of digraphs, Discrete Math. 211 (2000) 167-181.

[87] J. Shen, On the Caccetta-Häggkvist conjecture, Graphs Comb. 18 (2002) 645-654.

[88] Z.M. Song, Complementary cycles in bipartite tournaments, J. Nanjing Inst. Tech. 18 (1988) 32-38.

[89] Z.M. Song, Complementary cycles of all lengths in tournaments, J. Combin. Theory Ser. B 57 (1993) 18-25.

[90] B.D. Sullivan, A summary of results and problems related to the Caccette-Häggkvist conjecture, AIM Preprint 13 (2006): www.aimath.org/preprints.html.

[91] K. Taczuk and M. Woźiak, A note on the vertex-distinguishing index for some cubic graphs, Opuccula Mathematica 24 (2) (2004) 223-229.

[92] C. Thomassen, Hamiltonian-connected tournaments, J. Combin. Theory Ser. B 28 (1980) 142-163.

[93] C. Thomassen, Disjoint cycles in digraphs, Combinatorica 3 (3-4) (1983) 393-396.

[94] L. Volkmann, All regular multipartite tournaments that are cycle complementary, Discrete Math. 281 (2004) 255-266.

[95] L. Volkmann and S. Winzen, Every cycle-connected multipartite tournament has a universal arc, Discrete Math. 309 (2009) 1013-1017.

[96] H. Wang, On the maximum number of independent cycles in a graph, Discrete Mathmatices 205 (1999) 183-190.

[97] H. Wang, Independent directed triangles in a directed graph, Graphs and Combinatorics 16 (2000) 453-462.

[98] W. Wang and Y. Bu, On coloring problems, Handbook of Combinatorial Optimization (2013) 2095-2189.

[99] W. Wang and Y. Wang, Adjacent vertex-distinguishing edge colorings of K_4-minor free graphs, App. Math. Lett. 24 (2011) 2034-2037.

[100] D.B. West, Introduction to Graph Theory, Prentice-Hall, NJ, 1996.

[101] D. Woodall, Sufficient conditions for cycles in digraphs, Proc. London Math. Soc. 24 (1972) 739-755.

[102] C. Xu and S. Zhang, Personal communication.

[103] C. Xu, S. Zhang and Y. Lu, Covering digraphs with small indegrees or outdegrees by directed cuts, Discrete Math. 313 (2013) 1648-1654.

[104] D. Zhang and H. Wang, Disjoint directed quadrilaterals in a directed graph, J. Graph Theory 50 (2005) 91-104.

[105] K.M. Zhang, Y. Manoussakis and Z.M. Song, Complementary cycles containg a fixed arc in diregular bipartite tournaments, Discrete Math. 133 (1994) 325-328.

[106] K.M. Zhang and Z.M. Song, Complementary cycles containing a pair of fixed vertices in bipartite tournaments, Appl. Math. (in China) 3 (1988) 401-407.

[107] K.M. Zhang and J. Wang, Complementary cycles containing a fixed arc and a fixed vertex in bipartite tournaments, Ars Combin. 35 (1993) 265-269.

[108] Z. Zhang, L. Liu and J. Wang, Adjacent strong edge coloring of graphs, Appl. Math Lett. 15 (2002) 623-626.

Published and Submitted Papers

1. Covering the edges of digraphs in D(3,3) and D(4,4) with directed cuts, **Discrete Mathematics** 312 (2012) 1596-1601. (with Binlong Li and Shenggui Zhang)

2. Complementary cycles in regular bipartite tournaments, **Discrete Mathematics** 333 (2014) 14-27. (with Hao Li and Weihua He)

3. Vertex-disjoint cycles in bipartite tournaments, **Discrete Mathematics**, 2015, in press. (with Binlong Li and Hao Li)

4. Vertex-distinguishing proper arc colorings of digraphs, Submitted. (with Hao Li, Weihua He and Qiang Sun)

5. A note on the Caccetta-Häggkvist conjecture with forbidden subgraphs, Submitted. (with Hao Li, Weihua Yang and Weihua He)

6. Universal arcs in tournaments, Submitted. (with Hao Li, Binlong Li and Weihua He)

7. Hamiltonian cycles in spanning subgraphs of line graphs, Submitted. (with Hao Li, Weihua He and Weihua Yang)

8. A note on edge-disjoint Hamilton cycles in line graphs, Submitted. (with Hao Li, Weihua He and Weihua Yang)

9. Acyclic arc coloring in digraphs, Submitted. (with Hao Li and Weihua He)

Acknowledgements

First and foremost I would like to express my sincere gratitude to my advisor, the Directeur de Recherche of CNRS, Hao Li for his insightful suggestions and constant support. It has been an honor to be one of his Ph.D. students. His good research work on graph theory has indeed inspired me during my Ph.D. study. I thank him for giving me the opportunity to struggle, nurturing me as an independent researcher, and showing me the joy of doing good research. I appreciate all his contributions of time and ideas to make my Ph.D. experience productive and stimulating.

I also would like to thank Professor Shenggui Zhang, who first guided me into the research on graph theory. I am very grateful for his nice suggestions for my first scientific article and I really appreciate his kind support and encouragement, not only in my research but also in my personal life, throughout the past years.

Special thanks to my office mate and my friend, the Chargé de Recherche of CNRS, Réza Naserasr. Thanks for his help on my oral English. I appreciate a lot the helpful discussion with him on the problems of graph theory. Also, the members of GALAC group have contributed immensely to my personal and professional time in our lab LRI at University of Paris-Sud. Many thanks to all of them. Besides, I am indebted to many of my friends in Paris, especially to Weihua Yang, Weihua He, Chen Wang, Cong Zeng, Guangyu Li, Qiang Sun, Chuan Xu, Qian Zhang, Zhesheng Chen, Yiming Xiao, Ke Wang, etc. They made my life in Paris fruitful and beneficial.

It is my pleasure to thank the committee members of this thesis: Eric Angel, Cristina Bazgan, Yaojun Chen, Philippe Dague, Tomáš Kaiser and Réza Naserasr. I am grateful for their thoughtful and detailed comments. Especially, thank Tomáš for his careful reading of my thesis and his helpful suggestions on the writing.

I also wish to thank China Scholarship Council, who offered me the scholarship for my study at University of Paris-Sud.

Last but not the least, special thanks to my parents Guangqi Bai and Lin Cao, and to my brothers Yanhong Bai and Yanbin Bai. Words cannot express how grateful I am to them. I appreciate their love, continuous support and encouragement over the past years and the year to come. Even though they did not know exactly what I am doing. They provide me support and encouragement whenever and wherever needed. Thanks so much and I love them forever. To them, I dedicate this thesis.