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Moduli spaces of anti-invariant vector bundles over curves and conformal blocks

Hacem Zelaci

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de
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Par

Hacen ZELACI

Espaces de Modules de Fibrés Vectoriels Anti-invariants sur les Courbes et Blocs Conformes

Dirigée par Pr. Christian PAULY

Soutenue le 29 Septembre 2017.

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Moduli Spaces of Anti-invariant Vector Bundles over Curves
and Conformal Blocks

August 22, 2017

Abstract

by Hacen ZELACI

Moduli Spaces of Anti-invariant Vector Bundles over Curves and Conformal Blocks

Let X be a smooth irreducible projective curve with an involution σ . In this dissertation, we study the moduli spaces of invariant and anti-invariant vector bundles over X under the induced action of σ . We introduce the notion of σ -quadratic modules and use it, with GIT, to construct these moduli spaces, and then we study some of their main properties. It turns out that these moduli spaces correspond to moduli spaces of parahoric \mathcal{G} -torsors on the quotient curve X/σ , for some parahoric Bruhat-Tits group schemes \mathcal{G} , which are twisted in the anti-invariant case.

We study the Hitchin system over these moduli spaces and use it to derive a classification of their connected components using dominant maps from Prym varieties. We also study the determinant of cohomology line bundle on the moduli spaces of anti-invariant vector bundles. In some cases this line bundle admits some square roots called *Pfaffian* of cohomology line bundles. We prove that the spaces of global sections of the powers of these line bundles (spaces of generalized theta functions) can be canonically identified with the conformal blocks for some twisted affine Kac-Moody Lie algebras of type $A^{(2)}$.

Résumé

Espaces de Modules des Fibrés Vectoriels Anti-invariants sur les Courbes et Blocs Conformés.

Soit X une courbe projective lisse et irréductible munie d'une involution σ . Dans cette thèse, nous étudions les fibrés vectoriels invariants et anti-invariants sur X sous l'action induite par σ . On introduit la notion de modules σ -quadratiques et on l'utilise, avec GIT, pour construire ces espaces de modules, puis on en étudie certaines propriétés. Ces espaces de modules correspondent aux espaces de modules de \mathcal{G} -torseurs *parahoriques* sur la courbe X/σ , pour certains schémas en groupes parahoriques \mathcal{G} de type Bruhat-Tits, qui sont twistés dans le cas des anti-invariants.

Nous développons les systèmes de Hitchin sur ces espaces de modules et on les utilise pour dériver une classification de leurs composantes connexes en les dominant par des variétés de Prym. On étudie aussi le fibré déterminant sur les espaces de modules des fibrés vectoriels anti-invariants. Dans certains cas, ce fibré en droites admet certaines racines carrées appelées fibrés *Pfaffiens*. On montre que les espaces des sections globales des puissances de ces fibrés en droites (les espaces des fonctions theta généralisées) peuvent être canoniquement identifier avec les blocs conformés associés aux algèbres de Kac-Moody affines twistées de type $A^{(2)}$.

To my parents: Ammar & Zohra

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"...and he [prophet Solomon] said: "My Lord, enable me to be grateful for Your favor which You have bestowed upon me and upon my parents and to do righteousness of which You approve. And admit me by Your mercy into [the ranks of] Your righteous servants.""

Quran [27,19]

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Introduction

Moduli spaces are one of the fundamental constructions of algebraic geometry. They arise in connection with classification problems. Roughly speaking a moduli space for a collection of objects A and an equivalence relation \sim between these objects is a classification space such that each point corresponds to one, and only one, equivalence class of objects. Therefore, set theoretically, the moduli space is defined as the set of equivalence classes of objects A/\sim .

The study of moduli problems is a central topic in algebraic geometry. After the development of GIT theory, moduli spaces of vector bundles over curves were constructed in the 70's by Mumford, Narasimhan and Seshadri. Since then, these moduli spaces have been intensively studied by many mathematicians.

Moduli problems of line bundles over complex curves have been studied in the 19th century by Weierstrass, Riemann, Abel, Jacobi and others. The Jacobian of a curve builds a bridge between the geometry of curves and the theory of abelian varieties. The analogue of the Jacobian for a cover of two curves, called the Prym variety, have attracted the attention of many mathematicians since Mumford's seminal article in the 70's. Prym varieties are defined as the identity components of the kernel of the norm map attached to some cover $X \rightarrow Y$. In the case of degree 2 covers, they have a special description as the identity component of the locus of isomorphism classes of line bundles L over X such that

$$\sigma^*L \cong L^{-1},$$

where σ is the involution on X that interchanges the two sheets.

In this dissertation, we study higher rank vector bundles with such an *anti-invariance* propriety. Consider the moduli space $\mathcal{U}_X(r,0)$ of stable vector bundles of rank r and degree 0 over a smooth projective curve X with an involution σ . This involution induces by pullback an involution on $\mathcal{U}_X(r,0)$. Let E be a stable vector bundle, we say that E is *anti-invariant* if there exists an isomorphism

$$\psi : \sigma^*E \xrightarrow{\sim} E^*,$$

where E^* is the dual vector bundle. We say that the anti-invariant vector bundle E is σ -*symmetric* (resp. σ -*alternating*) if $\sigma^*\psi = {}^t\psi$ (resp. $\sigma^*\psi = -{}^t\psi$). We denote by $\mathcal{U}_X^{\sigma,+}(r)$ and $\mathcal{U}_X^{\sigma,-}(r)$ the loci of σ -symmetric and σ -alternating anti-invariant vector bundles respectively. We will see that these varieties correspond to moduli spaces of the form $\mathcal{M}_Y(\mathcal{G})$, i.e. moduli spaces of \mathcal{G} -torsors over Y for some particular type of group schemes \mathcal{G} called *parahoric Bruhat-Tits* group schemes (see for example [PR08a], [Hei10] and [BS14]).

Parahoric group schemes are special case of *integral models* of a semisimple algebraic group G . An integral model of G over X is a smooth affine group scheme whose generic fiber is isomorphic to G . They have been introduced by Bruhat and Tits in their seminal work [BT72], [BT84]. For an integral model \mathcal{G} , there is a finite number of points, called ramification points, over which the fiber of \mathcal{G} is not semisimple. An integral model \mathcal{G} is

parahoric if, for a ramification point p , the fiber $\mathcal{G}_{\mathcal{O}_p}$ is a parahoric subgroup of $\mathcal{G}_{\mathcal{K}}$ ([BT84], Définition 5.2.6), where \mathcal{O}_p is the completion of the local ring at p and $\mathcal{K} = \text{Frac}(\mathcal{O}_p)$. The name "parahoric" is a portmanteau of "parabolic" and "Iwahori".

Roughly speaking, parahoric subgroups are the natural generalizations of the parabolic subgroups to groups defined over local fields. For example, the parabolic subgroups P of G are those such that G/P is proper. Similarly, a parahoric subgroup can be defined as a subgroup \mathcal{P} of $G(\mathcal{K})$ such that $G(\mathcal{K})/\mathcal{P}$ is ind-proper (i.e direct limit of proper varieties). They can also be defined as the stabilizer of some self-dual periodic lattice chain (In some cases one should take the intersection with the kernel of the Kottwitz homomorphism, cf. [PR08b] §4).

The main problem considered in this thesis is the study of the moduli spaces of anti-invariant vector bundles. By studying the deformations of anti-invariant vector bundles, we identify the fibers of the tangent bundles to $\mathcal{U}_X^{\sigma, \pm}(r)$ with eigenspaces associated to the eigenvalues ∓ 1 with respect to the canonical involution on $H^1(X, \text{End}(E))$. This involution is induced by the anti-invariance structure of E . Based on that, we use Lefschetz fixed point theorem to drive formulas for their dimensions. Namely we prove that

$$\dim(\mathcal{U}_X^{\sigma, \pm}(r)) = \frac{1}{2}(g_X - 1) \pm \frac{rn}{2},$$

where g_X is the genus of X and $2n$ is the degree of the ramification divisor of $X \rightarrow X/\sigma$.

After that, we introduce the definitions of semistability and S -equivalence of anti-invariant vector bundles (which is closely related to the semistability of orthogonal and symplectic bundles). Using a twisted notion of quadratic modules ([Sor93]), which we call σ -quadratic modules, we construct the moduli space $\mathcal{M}_X^{\sigma, +}(r)$ that parameterizes the S -equivalence classes of semistable σ -symmetric anti-invariant vector bundles of rank r over X . The same method can be used to construct the moduli space $\mathcal{M}_X^{\sigma, -}(r)$ of semistable σ -alternating vector bundles, using the σ -alternating modules rather than the σ -quadratic ones.

Next, we consider the irreducibility problem. To study the connected components of $\mathcal{U}_X^{\sigma, \pm}(r)$, we use the Hitchin system and the theory of the nilpotent cone to establish dominant maps on these moduli spaces from some Prym varieties (we obtain similar results as in [BNR89]).

Hitchin systems are algebraically integrable systems defined on the cotangent space of the moduli space of stable G -bundles on a Riemann surface. They lie at the crossroads between algebraic geometry, Lie theory and the theory of integrable systems. They have been introduced and studied by Hitchin ([Hit87]) in the case of classical algebraic groups (GL_r , Sp_{2m} and SO_r). Let $\mathcal{M}_X(G)$ be the moduli space of stable G -bundles over X , the tangent space to $\mathcal{M}_X(G)$ at a point $[E]$ can be identified with

$$H^1(X, \text{Ad}(E)) \cong H^0(X, \text{Ad}(E) \otimes K_X)^*,$$

where $\text{Ad}(E)$ is the adjoint bundle associated to E , which is a bundle of Lie algebras isomorphic to $\mathfrak{g} = \text{Lie}(G)$. Hence, by Serre duality, the fiber of the cotangent bundle at E is $H^0(X, \text{Ad}(E) \otimes K_X)$. If we consider a basis of the invariant polynomials on \mathfrak{g} under the adjoint action, we get a map

$$T_E^* \mathcal{M}_X(G) = H^0(X, \text{Ad}(E) \otimes K_X) \longrightarrow \bigoplus_{i=1}^k H^0(X, K_X^{d_i}),$$

where the $\{d_i\}$ are the degrees of these invariant polynomials. Hitchin ([Hit87]) has shown that these two spaces have the same dimension.

In the case $G = \mathrm{GL}_r$, a basis of the invariant polynomials is given by the coefficients of the characteristic polynomial. If E is a stable vector bundle, then this gives rise to a map

$$\mathcal{H}_E : H^0(X, \mathrm{End}(E) \otimes K_X) \longrightarrow \bigoplus_{i=1}^r H^0(X, K_X^i) =: W,$$

which associates to each Higgs field $\phi : E \rightarrow E \otimes K_X$, the coefficients of its characteristic polynomial. The associated map

$$\mathcal{H} : T^* \mathcal{M}_X(\mathrm{GL}_r) \longrightarrow W$$

is called the *Hitchin morphism*. By choosing a basis of W , \mathcal{H} is represented by $d = r^2(g_X - 1) + 1$ functions f_1, \dots, f_d . Hitchin has proved that this system is *algebraically completely integrable*, i.e. its generic fiber is an open subset of an abelian variety of dimension d , f_1, \dots, f_d are Poisson-commute, $f_1 \wedge \dots \wedge f_d$ is generically nonzero and the vector fields $\mathcal{X}_{f_1}, \dots, \mathcal{X}_{f_d}$ associated to f_1, \dots, f_d (defined using the canonical 2-form on $T^* \mathcal{M}_X(\mathrm{GL}_r)$) are linear.

Moreover, consider the map

$$\Pi : T^* \mathcal{U}_X(r, 0) \rightarrow \mathcal{U}_X(r, 0) \times W$$

whose first factor is the canonical projection and the second factor is \mathcal{H} . Then, it is proved in [BNR89] that Π is dominant.

We start by describing the basis of the Hitchin morphism on the spaces $\mathcal{U}_X^{\sigma,+}(r)$ and $\mathcal{U}_X^{\sigma,-}(r)$, i.e. we define two subspaces $W^{\sigma,+}$ and $W^{\sigma,-}$ of W such that $\dim(W^{\sigma,\pm}) = \dim(\mathcal{U}_X^{\sigma,\pm}(r))$ and the map Π induces, by restriction, maps

$$\Pi : T^* \mathcal{U}_X^{\sigma,\pm} \longrightarrow \mathcal{U}_X^{\sigma,\pm} \times W^{\sigma,\pm}.$$

Using the nilpotent cone theory, we show that these maps are still dominant.

The space $W^{\sigma,+}$ is actually a vector subspace of W . However, in the ramified case, $W^{\sigma,-}$ is not a vector subspace. It is in fact an affine subvariety given by some quadratic equations. In the unramified case, the two spaces coincide.

Moreover, we study the smoothness of the spectral curves (see section 3.1) when the spectral data are in $W^{\sigma,\pm}$; for general point in $W^{\sigma,+}$, the associated spectral curve is smooth, and for general point in $W^{\sigma,-}$, the associated spectral curve is singular with just nodes as singularity. In both cases, the involution σ lifts to an involution on the spectral curve. Based on a result of Beauville, Narasimhan and Ramanan ([BNR89]) we show that the Prym varieties on these general spectral curves (or their normalizations in the singular case), with respect to these lifting of σ , dominate our moduli spaces of anti-invariant vector bundles.

Using these results, we deduce a complete classification of the connected components of the loci $\mathcal{U}_X^{\sigma,+}(r)$ and $\mathcal{U}_X^{\sigma,-}(r)$.

We also consider the case of trivial determinant anti-invariant vector bundles, denoted $\mathcal{S}\mathcal{U}_X^{\sigma,\pm}(r)$. This case turns out to be a slightly different. For example, we will show that $\mathcal{S}\mathcal{U}_X^{\sigma,-}(r)$ has a big number of connected components in the ramified case.

To sum up, by studying the Hitchin system on the moduli spaces of anti-invariant vector bundles, we deduce the following classification of the connected components :

- If π is ramified, then
 - $\mathcal{U}_X^{\sigma,+}(r)$ and $\mathcal{SU}_X^{\sigma,+}(r)$ are connected.
 - $\mathcal{U}_X^{\sigma,-}(r)$ has two connected components, when r is even (and empty otherwise).
 - $\mathcal{SU}_X^{\sigma,-}(r)$ has 2^{2n-1} connected components (when r is even), where $2n$ is the number of fixed points of σ .
- If π is unramified, then
 - $\mathcal{U}_X^{\sigma,+}(r) \cong \mathcal{U}_X^{\sigma,-}(r)$ and each one has two connected components.
 - $\mathcal{SU}_X^{\sigma,+}(r)$ is connected.
 - $\mathcal{SU}_X^{\sigma,-}(r)$ is connected if r is even, and empty otherwise.

We consider also the Hitchin system on the moduli spaces of stable σ -invariant vector bundles. A vector bundle E is called σ -invariant if $\sigma^*E \cong E$. The moduli space of these vector bundles has a lot of connected components (at least in the ramified case) which are parameterized by some topological type naturally attached to the linearizations on the considered bundles at the ramifications points. By an elementary computation, we deduce the number of all these types.

The σ -invariant vector bundles are a special case of the (π, G) -bundles, as called by Seshadri ([Ses70], [BS14]). In our case of vector bundles (i.e. $G = \mathrm{GL}_r$) they correspond (in some sense) to parabolic vector bundles on the quotient curve $Y := X/\sigma$, where the parabolic structure is over the branch locus of the double cover $X \rightarrow Y$, and this structure is encoded by the type of the σ -invariant bundles.

As in the anti-invariant case, we describe explicitly the base of the Hitchin map for any type of σ -invariant vector bundles. Moreover, for each type, we show that the invariant locus of the Jacobian varieties of the general spectral curve (or its normalization) dominate the moduli space of the σ -invariant vector bundles.

We should mention that the Hitchin systems for parabolic vector bundles have been already studied (cf. for example [LM] where the case of smooth spectral curves is considered). However, we consider the general case where the spectral data define singular spectral curves. We show that by considering the normalizations of these singular spectral curves we still get dominance results as in the smooth case.

Very recently, Baraglia, Kamgarpour and Varma have studied the complete integrability of the Hitchin system over the moduli spaces of parahoric \mathcal{G} -bundles, for a non-twisted parahoric group scheme \mathcal{G} . This can be thought of as a generalization of the parabolic bundles case. As far as we know, the Hitchin system for the twisted parahoric \mathcal{G} -torsors has not been considered before. Our study however treats the special case of twisted parahoric group schemes of type A .

The next problem that we consider is the study of line bundles over these moduli spaces. The question that arises naturally is whether the restriction of the determinant bundle to $\mathcal{SU}_X^{\sigma,\pm}$ is primitive. Since using the Hitchin system we showed that $\mathcal{U}_X^{\sigma,-}(r)$ is dominated by a Prym variety of some unramified double cover and since the restriction of the polarization of the Jacobian to this Prym variety has a square root, this let us conjecture that the restriction of the determinant bundle to $\mathcal{SU}_X^{\sigma,-}(r)$ has a square root too. We will show that this is true and in fact these square roots are parameterized by the

σ -invariant theta characteristic over the curve X . We call these square roots *Pfaffian of cohomology* line bundles.

On the other hand, the restriction of the determinant bundle to $\mathcal{SU}_X^{\sigma,+}(r)$ is primitive in the ramified case. This can be seen also using the results obtained from the study of the Hitchin system.

In the étale case, the two spaces $\mathcal{U}_X^{\sigma,+}(r)$ and $\mathcal{U}_X^{\sigma,-}(r)$ are isomorphic, as we have mentioned above, and the Pfaffian line bundle exists also in this case.

We also consider the spaces of generalized theta functions of the powers of the determinant and Pfaffian line bundles on $\mathcal{SU}_X^{\sigma,+}(r)$ and $\mathcal{SU}_X^{\sigma,-}(r)$ respectively. Using the results of Kumar and Mathieu ([Kum87], [Mat88]), we show that these vector spaces can be canonically identified with the conformal blocks of the twisted affine Kac-Moody Lie algebras, called *twisted conformal blocks*. In particular, we prove a special case of a conjecture by Pappas and Rapoport ([PR08b] Conjecture 3.7)

The twisted conformal blocks have been defined by Frenkel and Szeczeny [FS04] in the framework of vertex algebras. However, in our case, one can define them in the usual way; roughly, giving a ramified cover $X \rightarrow Y$ of degree $d = 2$ or 3 , a level l , a simple Lie algebra \mathfrak{g} and a set of dominant weights $(\lambda_p)_{p \in \text{Ram}(X/Y)}$ (labeled by the ramification divisor) of an affine twisted Kac-Moody Lie algebra $\hat{\mathcal{L}}(\mathfrak{g}, \tau)$ associated to an automorphism τ of \mathfrak{g} of order d . Then the twisted conformal blocks associated to this data is defined as the dual of the space of coinvariant of the product of the irreducible integral representations of level l associated to λ_p , with respect to the algebra $\mathfrak{g}(X \setminus R)^\tau$. See section 4.1 for more details.

Plan of the thesis.

In the first chapter, we will start by studying the σ -invariant vector bundles. As we have mentioned, this is a special case of the (Γ, G) -bundles where in our case Γ is just $\mathbb{Z}/2$. This theory has been studied by C.S. Seshadri ([Ses70], [Ses10] and [BS14]), also J.E. Andersen and J. Grove ([AG06]) has studied the invariant vector bundles of rank 2 under the action of an automorphism of the curve. We start by classifying their connected components and count their dimensions. We also spell out their identification with the parahoric \mathcal{G} -bundles over Y . We use a result by Balaji and Seshadri to count differently the dimensions of these connected components in the case of special linear group.

The second chapter will be reserved to the σ -anti-invariant vector bundles. We start by giving some basic fact and count the dimensions of these moduli spaces. Then we show how to identify such bundles with the parahoric \mathcal{G} -torsors over the quotient curve. We use this identification to deduce some results about the moduli stacks of the anti-invariant vector bundles by applying some results of Heinloth ([Hei10]). We also construct the associated moduli spaces by introducing the σ -quadratic and σ -alternating modules.

In the third chapter we study the Hitchin system over the moduli spaces of anti-invariant vector bundles as well as the invariant ones. We prove that these Hitchin systems are still algebraically integrable in some cases. We use these systems to classify the connected components of $\mathcal{U}_X^{\sigma,\pm}(r)$ and $\mathcal{SU}_X^{\sigma,\pm}(r)$. This was in fact our motivation to consider these

algebraic systems. Based on the results of Laumon ([Lau88]), we show that in the anti-invariant case, the Prym varieties over a general spectral curves dominates our moduli spaces, and in the invariant case, the invariant locus in the Jacobian varieties dominates the moduli of σ -invariant bundles.

This chapter corresponds to a preprint (arXiv:1612.06910) and it has been already submitted to a journal.

The last chapter will be devoted to the study of line bundles over the moduli spaces of anti-invariant vector bundles with trivial determinant and their global sections called generalized theta functions. We prove that the restriction of the determinant bundle to these moduli spaces admit square roots in some cases. We prove an identification of the generalized theta functions and the twisted conformal blocks associated to some twisted affine Kac-Moody Lie algebras (of type $A^{(2)}$ with the notation of [Kac90]). We also count the dimension of the space of generalized theta function of level 1 of the Pfaffian line bundle by establishing an analogue of a result of Beauville, Narasimhan and Ramanan ([BNR89]).

Chapter 1

Invariant Vector Bundles

The ground field is always assumed to be \mathbb{C} . We denote by X a smooth irreducible projective curve of genus $g_X \geq 2$, together with a non trivial involution $\sigma : X \rightarrow X$. We denote by $\pi : X \rightarrow X/\sigma =: Y$ the quotient map, g_Y the genus of Y and J_X, J_Y their respective Jacobians.

1.1 Invariant line bundles

Let $R \subset X$ be the ramification divisor of $\pi : X \rightarrow Y$. Since X is smooth, all of the ramification points are simple and their number is $2n$, for some non-negative integer n . Moreover by Hurwitz formula we have

$$g_X = 2g_Y + n - 1.$$

Denote by $\Delta \in \text{Pic}^n(Y)$ the line bundle on Y such that $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \Delta^{-1}$. If $\eta \in \ker(\pi^*)$, then

$$\begin{aligned} \pi_*\pi^*\eta &= \pi_*\mathcal{O}_X \\ \Rightarrow \det(\pi_*\mathcal{O}_X) \otimes \eta^2 &= \det(\pi_*\mathcal{O}_X), \end{aligned}$$

hence $\ker(\pi^*) \subset J_Y[2]$, where for an abelian variety A , we denote by $A[r]$ the r -torsion points of A . From [Mum74], we know that if π is unramified, then $\ker(\pi^*) = \{0, \Delta\}$, and in this case $\Delta \in J_Y[2]$, and if π is ramified, then π^* is injective.

Consider the endomorphism $u = 1 - \sigma^*$ of J_X , and let $P_0 = \text{Im}(u) = \ker(2 - u)_0$. P_0 is called the *Prym variety* of the cover $\pi : X \rightarrow Y$. However, in this thesis, by a *Prym variety* of a cover of curves $q : \bar{X} \rightarrow \bar{Y}$ we mean (unless otherwise explicitly mentioned) the kernel of the norm map $\text{Nm} : J_{\bar{X}} \rightarrow J_{\bar{Y}}$ attached to q , which may be non-connected (hence it is not an abelian variety). Recall that the norm map Nm is defined, at the level of Weil divisors, by associating to $\sum_i n_i p_i$ the divisor $\sum_i n_i q(p_i)$.

The abelian variety P_0 is connected of dimension

$$g_X - g_Y = g_Y + n - 1.$$

Let $e_2 : J_Y[2] \times J_Y[2] \rightarrow \{\pm 1\}$ the bilinear skew-symmetric form induced by the principle polarization. If π is unramified, we set

$$G = \{\eta \in J_Y[2] \mid e_2(\eta, \Delta) = 1\},$$

and $G = J_Y[2]$ if not (i.e $G = (\text{Ker}(\pi^*))^\perp$ with respect to e_2). Let

$$H = \{(L, \pi^*L^{-1}) \mid L \in G\}.$$

In fact, H is the kernel of the morphism

$$\pi^* \otimes i : J_Y \times P_0 \rightarrow J_X,$$

where i is the inclusion $P_0 \hookrightarrow J_X$.
 Moreover, we have (see loc. cit.)

$$J_X \simeq J_Y \times P_0/H.$$

Let $p = \dim(P_0)$, and q such that $|\ker(\pi^*)| = 2^{g_Y - q}$, so we get $|G| = 2^{g_Y + q}$.
 If L is a σ -invariant line bundle, i.e. $\sigma^*L \xrightarrow{\sim} L$, then we claim that $L \in J_Y \times P_0[2]/H$.
 Indeed, write $L = \pi^*M \otimes F$ for some $(M, F) \in J_Y \times P_0$, since L is σ -invariant, then F is σ -invariant too. But $\sigma^*F = F^{-1}$, so $F^2 = \mathcal{O}_X$. The converse is obvious. Hence, if J_X^σ denote the locus of σ -invariant line bundles, then

$$J_X^\sigma \simeq J_Y \times P_0[2]/H.$$

Note that $\text{card}(P_0[2]) = 2^{2p}$ and $\text{card}(\pi^*G) = 2^{2q}$, we conclude that

$$J_X^\sigma \simeq \pi^*J_Y \times (\mathbb{Z}/2\mathbb{Z})^{2(p-q)}.$$

So the number of connected components of J_X^σ is $2^{2(n-1)}$ when $n \geq 1$ and 1 when $n = 0$.
 In particular, if π is unramified, we have $J_X^\sigma = \pi^*J_Y$.

We are going now to describe explicitly these $2^{2(n-1)}$ connected components of J_X^σ .
 First, we recall an important lemma (due to Kempf, see [DN89]).

Lemma 1.1.1. (*Kempf's Lemma*) *Let E be a vector bundle on X , with a linearization $\phi : \sigma^*E \xrightarrow{\sim} E$, i.e. $\phi \circ \sigma^*\phi = \text{id}$. Then (E, ϕ) descends to Y (i.e. $E \cong \pi^*F$ for some vector bundle F on Y and ϕ is the canonical associated linearization) if and only if ϕ acts as the identity on the fiber E_p , for any $p \in R$.*

As a consequence of this Lemma, we have the following

Corollary 1.1.2. *The canonical line bundle K_X of X descends to Y .*

Proof. By differentiating the involution $\sigma : X \rightarrow X$ we get a linear isomorphism $d\sigma : K_X^{-1} \rightarrow \sigma^*K_X^{-1}$. Since $\sigma^2 = \text{id}$, we deduce

$$d\sigma \circ \sigma^*(d\sigma) = \text{id}.$$

Hence $d\sigma$ is a linearization of K_X^{-1} . Moreover if t is a local parameter near a ramification point $p \in R$, then $\sigma(t) = -t$, hence $d\sigma = -1$ over p . By Lemma 1.1.1 we deduce that K_X descends to Y . \square

By Hurwitz formula we have $\mathcal{O}_X(R) = K_X \otimes \pi^*K_Y^{-1}$, it follows that $\mathcal{O}_X(R)$ descends to Y . Furthermore, using the relative duality (see e.g. [Har77] Ex III.6.10), we deduce that $\mathcal{O}_X(R) = \pi^*\Delta$, hence $K_X = \pi^*(K_Y \otimes \Delta)$.

Remark 1.1.3. Suppose that π is ramified. Let L be a line bundle on Y , then π^*L has a canonical linearization. We call it the *positive* linearization, (because it equals $+\text{id}$ over each $p \in R$). Its opposite is called the *negative* linearization. Moreover, fixing a linearization ϕ on a line bundle M induces an involution on the spaces $H^i(X, M)$ (for $i = 0, 1$) defined by associating to a local section s the section $\phi(\sigma^*s)$. In the case of π^*L , we have, with respect to the positive linearization, the following identifications

$$H^0(X, \pi^*L)_+ \cong H^0(Y, L), \quad H^0(X, \pi^*L)_- \cong H^0(Y, L \otimes \Delta^{-1}),$$

where, for a vector space V with an involution, we denote by V_+ (resp. V_-) the eigenspace associated to the eigenvalue $+1$ (resp. -1).

If $\pi : X \rightarrow Y$ is étale, then $K_X = \pi^*K_Y = \pi^*(K_Y \otimes \Delta)$. We define the positive linearization on K_X to be the linearization attached to $K_Y \otimes \Delta$.

We will describe explicitly the connected components of J_X^σ . Consider $S \subset R$ a subset of cardinality $2s$, then for all $M \in \text{Pic}^{-s}(Y)$, one has $\pi^*M(S) \in J_X^\sigma$, and it lies in π^*J_Y if and only if $S = \emptyset$ or $S = R$ (Kempf's lemma). Moreover, $\pi^*M(S)$ and $\pi^*N(T)$ belong to the same connected component if and only if their difference is in the identity component π^*J_Y . In other words, $\pi^*L(S-T) \in \pi^*J_Y$, (for some L in $\text{Pic}(Y)$), hence $S = T$ or $S = T^c$, where $T^c = R - T$.

The number of such subset S up to complementary is given by

$$\frac{1}{2} \sum_{k=0}^n \binom{2n}{2k} = \frac{1}{2} 2^{2n-1} = 2^{2(n-1)}.$$

Therefore, the connected components of J_X^σ are classified by the even cardinality subsets of R up to complementary.

The case of degree 1 line bundles is almost the same, the σ -invariant locus is denoted by $\text{Pic}^1(X)^\sigma$. If π is unramified, then $\text{Pic}^1(X)^\sigma = \emptyset$, so we assume that $n \geq 1$. Let $p \in X$ be a ramification point, the translation map given by

$$\begin{aligned} T_p : J_X^\sigma &\longrightarrow \text{Pic}^1(X)^\sigma \\ L &\longrightarrow L(p) = L \otimes \mathcal{O}(p). \end{aligned}$$

is an isomorphism. In particular $\text{Pic}^1(X)^\sigma$ contains the same number of connected components as J_X^σ .

As before, let $S \subset R$ be a subset of cardinality $2s + 1$ and $M \in \text{Pic}^{-s}(Y)$. It is clear that $\pi^*M(S) \in \text{Pic}^1(X)^\sigma$, and if $\pi^*M(S)$ and $\pi^*N(T)$ are in the same connected component, then $\pi^*L(S-T) \in \pi^*J_Y$ which implies, as we have seen, that $S = T$ or T^c .

To finish, it is easy to see that the number of such subset of odd cardinality up to complementary is again $2^{2(n-1)}$.

We mention that these line bundles has been already studied by Beauville in [Bea13].

Another method to identify the connected components of J_X^σ is to observe that $\text{P}_0[2] = J_X^\sigma[2]$, and in fact $\text{P}_0[2]$ intersects all the connected components of J_X^σ . Hence

$$\pi_0(J_X^\sigma) = \text{P}_0[2]/\pi^*J_Y[2].$$

1.2 Invariant vector bundles

A vector bundle E on X is called σ -invariant if there exists an isomorphism

$$\varphi : \sigma^*E \xrightarrow{\sim} E.$$

The isomorphism φ is called *linearization* if $\varphi \circ \sigma^*\varphi = \text{id}_E$. In fact, a linearization corresponds to a lifting of the involution σ to an involution $\tilde{\sigma} : E \rightarrow E$, such that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\sigma}} & E \\ \varphi^{-1} \searrow & & \downarrow \\ \sigma^*E & \xrightarrow{\quad} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & X \end{array}$$

commutes. Using the linearization φ we obtain a linear involution on the space $H^i(X, E)$ for $i = 0, 1$, given locally by

$$s \longrightarrow \varphi(\sigma^* s).$$

We denote their eigenspaces by $H^i(X, E)_\pm$.

Remark 1.2.1. If E is σ -invariant and stable, then it has only 2 linearizations; φ and $-\varphi$.

Suppose that E is a σ -invariant *stable* vector bundle and $\varphi : \sigma^* E \rightarrow E$ a linearization. We define the *type* of E to be

$$\tau = (\varphi_p)_{p \in R} \pmod{\pm I_r},$$

with $\varphi_p \in \text{End}(E_p)$. We denote usually by k_p the multiplicity of the eigenvalue -1 of φ_p , and most of the time we identify the type τ with the associated vector $(k_p)_{p \in R}$. Note that the vectors $(k_p)_p$ and $(r - k_p)_p$ represent the same type (due to multiplication by -1). Moreover, we have the following relation between the type and the degree d of E

$$\sum_{p \in R} k_p \equiv d \pmod{2}.$$

Indeed, define F to be the kernel of

$$0 \rightarrow F \rightarrow E \rightarrow \bigoplus_{p \in R} (E_p)_- \rightarrow 0.$$

F is called negative elementary transformation of E . By Kempf's Lemma, it follows that F descends to Y , hence

$$d - \sum_{p \in R} k_p = \text{deg}(F) \equiv 0 \pmod{2}.$$

One can also deduce this relation by looking at the determinant of E , which is σ -invariant.

Denote by $\mathcal{U}_X^{\sigma, \tau}(r, d) \subset \mathcal{U}_X(r, d)$ the locus of classes $[E] \in \mathcal{U}_X(r, 0)$ such that E is σ -invariant stable vector bundle of type τ . Note that $\mathcal{U}_X^{\sigma, \tau}(r, d)$ is smooth. In fact there is a more general result

Lemma 1.2.2. *Let Z be a smooth variety with an involution τ . Then the fixed locus Z^τ is smooth closed subvariety of Z .*

In fact the action can be linearized locally around any point $z \in Z^\tau$. This is true in more general context (see Edixhoven [Edi92]).

By an elementary calculation, we get the number of all possible types:

$$\pi_0(\mathcal{U}_X^\sigma(r, d)) = \begin{cases} \frac{1}{4}((r+1)^{2n} - 1) + 1 & \text{if } r \equiv d \equiv 0 \pmod{2} \\ \frac{1}{4}((r+1)^{2n} - 1) & \text{if } r \equiv d+1 \equiv 0 \pmod{2} \\ \frac{1}{4}(r+1)^{2n} & \text{if } r \equiv 1 \pmod{2}. \end{cases}$$

To prove the existence of stable σ -invariant vector bundles of a given type, we use cyclic covers.

Lemma 1.2.3. *Let $\tau = (k_p)_{p \in R} \in \mathbb{N}^{2n}$ such that $\sum_{p \in R} k_p \equiv d \pmod{2}$. Then there exists a stable σ -invariant vector bundle $(E, \phi) \in \mathcal{U}_X^\sigma(r, d)$ of type τ .*

Proof. Let $\beta \in J_X[r]$ be a primitive r -torsion line bundle over X which descends to Y . Denote by $q : X_\beta \rightarrow X$ the cyclic unramified cover of degree r of X defined by β (see section 3.1). By Lemma 2.3.1, the involution $\sigma : X \rightarrow X$ lifts to an involution $\tilde{\sigma}$ on X_β . Moreover, the fixed locus of this involution equals $q^{-1}(R)$. In particular, there are $2rn$ fixed points by $\tilde{\sigma}$. Let $R = \{p_1, \dots, p_{2n}\}$ and for each i we choose an order on the fiber $q^{-1}(p_i) = \{p_{i1}, \dots, p_{ir}\}$. Now for a type $\tau = (k_p)_{p \in R}$ of σ -invariant vector bundles, we define a type $\tilde{\tau} = (k_{p_{ij}})_{i,j}$ of $\tilde{\sigma}$ -invariant line bundles on X_β as follows :

$$k_{p_{ij}} = \begin{cases} -1 & \text{if } 1 \leq j \leq k_i \\ +1 & \text{otherwise .} \end{cases}$$

It is easy to see that the direct image of a $\tilde{\sigma}$ -invariant line bundle of type $\tilde{\tau}$ is a σ -invariant vector bundle of type τ . Moreover, by Proposition 2.3.2, we deduce that for general $\tilde{\sigma}$ -invariant line bundle on X_β , q_*L is in fact stable. \square

1.3 Infinitesimal study

Recall that a deformation of E over $\text{Spec}(\mathbb{C}[\varepsilon])$ ($\varepsilon^2 = 0$) is defined to be a locally free coherent sheaf \mathcal{E} on $X_\varepsilon = X \times \text{Spec}(\mathbb{C}[\varepsilon])$ together with a homomorphism $\mathcal{E} \rightarrow E$ of $\mathcal{O}_{X_\varepsilon}$ -module such that the induced map $\mathcal{E} \otimes_{\mathcal{O}_X} \rightarrow E$ is an isomorphism. Canonically, the set of deformation of E over $\text{Spec}(\mathbb{C}[\varepsilon])$ is isomorphic to $H^1(X, \text{End}(E))$, where $\text{End}(E) \cong E \otimes E^*$ stands for the sheaf of endomorphisms of E .

By definition, a deformation is locally free, so it is flat, thus taking the tensor product with \mathcal{E} of the exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{\varepsilon} \mathcal{O}_{X_\varepsilon} \rightarrow \mathcal{O}_X \rightarrow 0$$

we obtain the exact sequence

$$0 \rightarrow E \xrightarrow{\varepsilon} \mathcal{E} \rightarrow E \rightarrow 0.$$

Assume now that E is stable σ -invariant vector bundle of rank r and degree d . Let τ be the type of E . We want to identify the tangent space to $\mathcal{U}_X^{\sigma, \tau}(r, d)$ at E . The tangent space to the moduli space $\mathcal{U}_X(r, d)$ at a smooth point E is given by

$$\text{T}_E \mathcal{U}_X(r, d) \cong H^1(X, \text{End}(E)).$$

The linearization $\varphi : \sigma^*E \rightarrow E$ induces a linear involution f on $H^1(X, E \otimes E^*)$ defined locally by associating to local section $x \otimes \lambda$ of $E \otimes E^*$ the section

$$f(x \otimes \lambda) = \varphi(\sigma^*(x)) \otimes \sigma^*({}^t\varphi(\lambda)).$$

Clearly, this involution does not depend on the choice of φ .

Given $\eta = (\eta_{ij})_{i,j} \in H^1(X, E \otimes E^*)$, it corresponds to an infinitesimal deformation

$$0 \rightarrow E \rightarrow \mathcal{E} \rightarrow E \rightarrow 0$$

over X_ε . In fact if we set

$$g_{ij} = \phi_i \circ (\text{id} + \varepsilon \eta_{ij}) \circ \phi_j^{-1},$$

where $\phi_i : E|_{U_i} \rightarrow U_i \times \mathbb{C}^r$ are some local trivializations of E , then $(g_{ij})_{i,j}$ are transition functions of \mathcal{E} (we will prove this in §2.5, Lemma 2.5.1 below).

Now $\eta \in T_E \mathcal{U}_X^{\sigma, \tau}(r, d)$ if and only if \mathcal{E} is σ -invariant, and \mathcal{E} is σ -invariant if and only if it has σ -invariant transition functions. Since we can choose ϕ_i to be σ -invariant, i.e. $\sigma^* \phi_i = \phi_i \circ \varphi$, we deduce that \mathcal{E} is σ -invariant iff η is invariant with respect to f . Thus

$$T_E \mathcal{U}_X^{\sigma, \tau}(r, d) \cong H^1(X, E \otimes E^*)_+.$$

In particular we deduce the dimension of $\mathcal{U}_X^{\sigma, \tau}(r, d)$.

Proposition 1.3.1. *The dimension of the locus of σ -invariant vector bundles of fixed type τ is given by*

$$\dim(\mathcal{U}_X^{\sigma, \tau}(r, d)) = r^2(g_Y - 1) + 1 + \sum_{p \in R} k_p(r - k_p),$$

where $(k_p)_{p \in R}$ are the integers associated to τ .

Proof. To calculate the dimension of $H^1(X, E \otimes E^*)_+$ we use Lefschetz fixed point theorem (cf. Appendix F), to simplify the notations we let

$$h_{\pm}^1 = \dim_{\mathbb{C}}(H^1(X, E \otimes E^*)_{\pm}).$$

We have

$$\begin{cases} h_+^1 + h_-^1 = r^2(g_X - 1) + 1 & \text{(By Riemann-Roch Formula)} \\ h_+^1 - h_-^1 = 1 - \frac{1}{2} \sum_{p \in R} \text{Tr}(f_p) & \text{(By Lefschetz fixed point theorem)} \end{cases},$$

we have used the fact that $h^0(X, E \otimes E^*)_+ = 1$ (the identity $E \rightarrow E$ is σ -invariant).

By the very definition, $f_p = \varphi_p \otimes \varphi_p$, it follows that the multiplicity of the eigenvalue -1 of f_p is $2k_p(r - k_p)$, hence $\text{Tr}(f_p) = (r - 2k_p)^2$, so we have

$$\begin{cases} h_+^1 + h_-^1 = 2r^2(g_Y - 1) + r^2n + 1 \\ h_+^1 - h_-^1 = 1 - r^2n + 2 \sum_{p \in R} k_p(r - k_p) \end{cases}.$$

It follows

$$\dim(\mathcal{U}_X^{\sigma, \tau}(r, d)) = h_+^1 = r^2(g_Y - 1) + 1 + \sum_{p \in R} k_p(r - k_p).$$

□

In particular, since $\det : \mathcal{U}_X^{\sigma, \tau}(r, 0) \rightarrow \text{Pic}^{\sigma, \tilde{\tau}}(X)$ is surjective, where $\tilde{\tau} = ((-1)^{k_p})_{p \in R} \pmod{\pm 1}$, we have

$$\begin{aligned} \dim(\mathcal{SU}_X^{\sigma, \tau}(r)) &= \dim(\mathcal{U}_X^{\sigma, \tau}(r, 0)) - g_Y \\ &= (r^2 - 1)(g_Y - 1) + \sum_{p \in R} k_p(r - k_p). \end{aligned}$$

Remark 1.3.2. The dimension of the locus of σ -invariant vector bundle $\mathcal{U}_X^{\sigma}(r, d)$ is the maximum of these dimensions :

$$\dim(\mathcal{U}_X^{\sigma}(r, d)) = \begin{cases} r^2(g_Y - 1) + n \frac{r^2}{2} + 1 & r \equiv 0 \pmod{2} \\ r^2(g_Y - 1) + n \frac{r^2 - 1}{2} + 1 & r \equiv 1 \pmod{2} \end{cases}.$$

These dimensions correspond to the following types (called *maximal types*)

$$\mathfrak{max} = \begin{cases} \{\tau = (\varphi_p)_p \pmod{\pm I_r} \mid k_p = r/2, \forall p \in R\} & r \equiv 0 \pmod{2} \\ \{\tau = (\varphi_p)_p \pmod{\pm I_r} \mid k_p = (r+1)/2 \text{ or } k_p = (r-1)/2\} & r \equiv 1 \pmod{2} \end{cases}.$$

In the odd case, the cardinal of \mathfrak{max} is $2^{2(n-1)}$.

1.4 Moduli space of σ -invariant vector bundles

We start here by recalling some results from [BS14]. A σ -group scheme over X is a group scheme \mathcal{G} over X with a lifting of the action of σ to \mathcal{G} as group scheme automorphism. Denote by $\mathcal{M}_X(\mathcal{G})$ the moduli stack of \mathcal{G} -torsors over X .

Definition 1.4.1 ((σ, \mathcal{G}) -bundle). Let \mathcal{G} be a σ -group scheme over X . A (σ, \mathcal{G}) -bundle is a \mathcal{G} -bundle E over X with a lifting of the action of $\sigma : X \rightarrow X$ to the total space of E (denoted also by σ) such that for each $x \in E$ and $g \in \mathcal{G}$, we have $\sigma(x \cdot g) = \sigma(x) \cdot \sigma(g)$.

By definition, the action of σ on E is not a \mathcal{G} -morphism. But it gives an isomorphism of total spaces (by the universal property of the fiber product)

$$\begin{array}{ccc} E \times_X X & \xrightarrow{\varphi} & E, \\ & \searrow & \swarrow \\ & & X \end{array}$$

which verifies

$$\varphi(v \cdot g) = \varphi(v) \cdot \sigma(g),$$

for $g \in \mathcal{G} \times_X X$ and $v \in E \times_X X$. This is again not a \mathcal{G} -morphism, but we can associate to it canonically a \mathcal{G} -isomorphism (over the identity of X)

$$E \xrightarrow{\sim} E^\sigma,$$

where $E^\sigma = (E \times_X X) \times^{\mathcal{G}} \mathcal{G}$, here \mathcal{G} acts on itself via σ .

Definition 1.4.2. (Parahoric group scheme) A smooth affine group scheme \mathcal{G} over a curve X is said to be *parahoric Bruhat-Tits group scheme* if there is a finite subset $R \subset X$ such that if \mathcal{O}_x is the completion of the local ring at $x \in R$ then $\mathcal{G}_{\mathcal{O}_x}$ is a parahoric group scheme over $\mathrm{Spec}(\mathcal{O}_x)$ (in the sens of Bruhat-Tits, [BT84] Définition 5.2.6) for each $x \in R$ and the fibers \mathcal{G}_y is semisimple for all $y \in X \setminus R$.

In the following lemma, we show the correspondence between σ -invariant vector bundles and (σ, \mathcal{G}) -bundles.

Lemma 1.4.3. *Giving a σ -invariant vector bundle (E, ϕ) of type τ is the same as giving $(\sigma, \mathcal{H}_\tau)$ -bundle, for some σ -group scheme \mathcal{H}_τ over X attached to τ .*

Proof. Fix a σ -linearized vector bundle (F_τ, ϕ_τ) of type τ and let $\mathcal{H}_\tau = \mathrm{Aut}(F_\tau)$. The linearization ϕ_τ induces an action on \mathcal{H}_τ given by

$$g \longrightarrow \sigma_\tau(g) = \phi_\tau \circ \sigma^* g \circ \phi_\tau^{-1},$$

this makes \mathcal{H}_τ a σ -group scheme over X . Now let (E, ϕ) be a σ -invariant vector bundle of type τ , then the frame bundle $\tilde{E} := \mathrm{Isom}(F_\tau, E)$ is clearly a $(\sigma_\tau, \mathcal{H}_\tau)$ -bundle, where the action of σ is given as follows: for a local isomorphism $f \in \tilde{E}|_U$, we have $\sigma(f) = \phi \circ \sigma^* f \circ \phi_\tau^{-1}$. Conversely, giving $(\sigma, \mathcal{H}_\tau)$ -bundle \tilde{E} , we have a commutative diagram

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{\sigma}} & \tilde{E} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & X, \end{array}$$

which gives us an isomorphism

$$\tilde{E} \times_X X \xrightarrow{\sim} \tilde{E}.$$

Thus $E = \tilde{E}(\mathbb{C}^r)$ is a σ -invariant vector bundle. □

Let $\mathcal{U}_X^{\sigma,\tau}(r, d)$ be the moduli stack of σ -invariant vector bundles over X of type τ . In the paper [BS14], they identify $\mathcal{U}_X^{\sigma,\tau}(r, d)$ with the stack of \mathcal{G}_τ -torsors over Y

$$\mathcal{U}_X^{\sigma,\tau}(r, d) \cong \mathcal{M}_Y(\mathcal{G}_\tau)$$

for some parahoric Bruhat-Tits group scheme \mathcal{G}_τ associated to the type τ .

More precisely, consider a σ -group scheme \mathcal{H} over X . Let $\mathcal{G} = \text{Res}_{X/Y}(\mathcal{H})^\sigma$ be the invariant subgroup scheme of the Weil restriction of \mathcal{H} with respect to $\pi : X \rightarrow Y$, i.e. the scheme that represents the functor $\pi_*(\mathcal{H})^\sigma$ (this is representable because $\pi_*(\mathcal{H})$ is, see [BLR90] Theorem 4 and Proposition 6).

Theorem 1.4.4. [BS14] *Let $\mathcal{M}_X(\sigma, \mathcal{H})$ be the moduli stack of (σ, \mathcal{H}) -bundles over X , then we have an isomorphism*

$$\mathcal{M}_X(\sigma, \mathcal{H}) \xrightarrow{\sim} \mathcal{M}_Y(\mathcal{G})$$

given by the invariant direct image π_*^σ .

To apply this in our situation, let \mathcal{H}_τ be the σ -group scheme defined in Lemma 1.4.3. Then the group scheme \mathcal{G}_τ is the σ -invariant Weil restriction of \mathcal{H}_τ

$$\mathcal{G}_\tau = \text{Res}_{X/Y}(\mathcal{H}_\tau)^\sigma.$$

Moreover, in loc. cit. the associated coarse moduli space is constructed and the main result assures that it is irreducible normal projective variety.

Remark 1.4.5. Since we deal with GL_r -bundles, the parahoric group scheme \mathcal{G}_τ is of parabolic type ([Ses10]), which implies that the moduli of σ -invariant vector bundles of type τ is isomorphic to the moduli space of parabolic vector bundles with parabolic structures, related to τ , at the branch points of $X \rightarrow Y$. Indeed giving a σ -invariant vector bundle E of rank r , degree d and type τ , is the same as giving a vector bundle F of rank r on Y of degree $\nu = d - \sum_{p \in R} k_p$, with a vector subspace G_p of $F_{\pi(p)}$ of dimension k_p for each $p \in R$. To obtain F from E one can take the negative elementary transformation of E along the eigenspaces $(E_p)_-$. Conversely, E can be constructed from F as the positive elementary transformation along the subspaces G_p . One verifies easily that the dimension of $\mathcal{U}_Y(r, \nu) \times \prod_{p \in R} \text{Gr}(k_p, F_{\pi(p)})$ is the same as $\mathcal{U}_X^{\sigma,\tau}(r, 0)$, where $\text{Gr}(k_p, F_{\pi(p)})$ is the Grassmannian parameterizing k_p dimensional subspaces of $F_{\pi(p)}$.

However, for a general reductive group G , the situation is more subtle.

Let's spell out the definition of the semistability of σ -invariant vector bundles and give some properties.

Definition 1.4.6. We say that a σ -invariant vector bundle (E, ϕ) of rank r and degree d is semi-stable (resp. stable) if for every σ -invariant sub-bundle F of E one has

$$\mu(F) \leq \mu(E) \quad (\text{resp. } \mu(F) < \mu(E)),$$

where $\mu(E) = \text{deg}(E)/\text{rk}(E)$ is the slope.

Lemma 1.4.7. *A σ -invariant vector bundle (E, ϕ) is semi-stable if and only if the vector bundle E is semi-stable.*

Proof. the "if" part is obvious.

For a subbundle $F \subset E$ we denote by

$$s(E, F) = \deg(E)\mathrm{rk}(F) - \deg(F)\mathrm{rk}(E).$$

Remark that $\mu(F) \leq \mu(E)$ if and only if $s(E, F) \geq 0$, for any non-zero subbundle F of E . Let F be any subbundle of a semi-stable σ -invariant vector bundle (E, ϕ) , let P be the image of $\sigma^*F \oplus F \rightarrow E$, and $N \subset E$ such that

$$0 \rightarrow N \rightarrow \sigma^*F \oplus F \rightarrow P \rightarrow 0.$$

Claim. *The two sub-bundles P and N are σ -invariant.*

It is clear that $\phi(\sigma^*P) \subset P$, hence $\phi|_P : \sigma^*P \rightarrow P$ is a linearization. For N , as N is the kernel of the map $\sigma^*F \oplus F \rightarrow P$, and this map is clearly σ -equivariant for the action of σ , so the action of σ on $\sigma^*F \oplus F$ induces an action of N , thus it is σ -invariant. Now we can calculate

$$\begin{aligned} s(E, F) &= \deg(E)\mathrm{rk}(F) - \deg(F)\mathrm{rk}(E) \\ &= \frac{1}{2}\deg(E)(\mathrm{rk}(P) + \mathrm{rk}(N)) - \frac{1}{2}(\deg(P) + \deg(N))\mathrm{rk}(E) \\ &= \frac{1}{2}(s(E, P) + s(E, N)) \\ &\geq 0. \end{aligned}$$

□

Recall the definition of semi-stability of (σ, GL_r) -bundle (see [BS14] for example).

Definition 1.4.8. A (σ, GL_r) -bundle is semi-stable (resp. stable) if for any maximal parabolic subgroup $P \subset \mathrm{GL}_r$ and every σ -invariant reduction of structure group $s : X \rightarrow E(\mathrm{GL}_r/P)$ we have

$$\deg(s^*E(\mathfrak{gl}_r/\mathfrak{p})) \geq 0 \quad (\text{resp. } > 0)$$

where \mathfrak{gl}_r and \mathfrak{p} denote the Lie algebras of GL_r and P respectively.

Proposition 1.4.9. *A (σ, GL_r) -bundle is (semi-)stable if and only if $E(\mathbb{C}^r)$ is (semi-)stable σ -invariant vector bundle.*

Proof. (Adapted from [HM04])

Let E be a (σ, GL_r) -bundle. Suppose that $E(\mathbb{C}^r)$ is semi-stable and let $P \subset \mathrm{GL}_r$ be a maximal parabolic subgroup, and $s : X \rightarrow E/P$ a σ -invariant reduction of the structure group. The parabolic subgroup P corresponds to a flag

$$\{0\} \subset V \subset \mathbb{C}^r.$$

Denote $F = (s^*E)(V)$.

Claim. (1) $s^*E(\mathfrak{gl}/\mathfrak{p}) \cong F^* \otimes (E(\mathbb{C}^r)/F)$.

(2) F is σ -invariant.

Proof of the claim. 1. c.f Proposition 1 of [HM04].

2. Since P stabilizes V , F is well defined, and since s is σ -invariant, s^*E is a (σ, P) -bundle. Thus, F is a σ -invariant vector subbundle of $E(\mathbb{C}^r)$.

□

Thus $\mu(F) \leq \mu(E(\mathbb{C}^r))$, which is equivalent to $\mu(F) \leq \mu(E(\mathbb{C}^r)/F)$. Using the first point of the claim, we deduce

$$\deg(s^*E(\mathfrak{g}/\mathfrak{p})) \geq 0.$$

Hence E is semi-stable as principal bundle.

Conversely, assume that E is a semi-stable (σ, GL_r) -bundle. Let F be a σ -invariant vector subbundle of $E(\mathbb{C}^r)$. By completing the transition functions of F to transition functions of E , we see that F is of the form $s^*E(V)$ for some reduction s to some maximal parabolic subgroup $P \subset \mathrm{GL}_r$, as F is σ -invariant, s is σ -invariant too. We deduce that

$$\deg(s^*E(\mathfrak{g}/\mathfrak{p})) \geq 0.$$

As before, this implies that $\mu(F) \leq \mu(E(\mathbb{C}^r)/F)$, hence $\mu(F) \leq \mu(E)$, which means that $E(\mathbb{C}^r)$ is a semi-stable σ -invariant vector bundle. For the stability, one just has to replace the inequalities by strict ones. □

As an application, we consider here the case $G = \mathrm{SL}_r$ and we apply the main theorem of [BS14] to compute the dimension of the associated moduli space. Denote by $\mathrm{T} \cong (\mathbb{G}_m)^{r-1}$ its maximal torus and SU_r its maximal compact subgroup (of Hermitian matrices). Denote by

$$\langle \cdot, \cdot \rangle : X^*(\mathrm{SL}_r) \times Y^*(\mathrm{SL}_r) \rightarrow \mathbb{Z}$$

the canonical bilinear form on the spaces of characters $X^*(\mathrm{SL}_r)$ and of 1-parameter subgroups $Y^*(\mathrm{SL}_r)$.

Fix a type $\tau = (k_p)_{p \in R} \bmod \pm 1$, such that the $k_p \geq 0$ are all even (because the vector bundles have trivial determinant).

We associate to each k_p the matrix

$$A_p = \mathrm{diag}(\underbrace{-1, \dots, -1}_{k_p \text{ times}}, +1, \dots, +1),$$

and a 1-parameter subgroup $\tilde{\theta}_p \in Y^*(\mathrm{T})$

$$\tilde{\theta}_p = (\underbrace{1, \dots, 1}_{k_p \text{ times}}, 0, \dots, 0).$$

Finally let

$$\theta_p = \frac{1}{2} \tilde{\theta}_p \in Y^*(\mathrm{T}) \otimes \mathbb{Q}.$$

(see [BS14] Lemma 2.2.8). Thus we can represent θ_p by

$$(\underbrace{1/2, \dots, 1/2}_{k_p \text{ times}}, 0, \dots, 0).$$

The root system associated to the adjoint representation of T is given by

$$\mathcal{R} = \{\lambda_{i,j} : T \rightarrow \mathbb{G}_m \mid \lambda_{i,j}(X) = x_i/x_j, i \neq j\}$$

As an element of \mathbb{Z}^r , $\lambda_{i,j}$ is equal to $(0, \dots, 1, \dots, -1, \dots, 0)$ or $(0, \dots, -1, \dots, 1, \dots, 0)$ (depending on whether $i < j$ or $j < i$).

We choose the set of simple root to be

$$\mathcal{S} = \{\lambda_{i,i+1} \mid i = 1, \dots, r-1\}.$$

So that the set of positive roots are

$$\mathcal{R}^+ = \{\lambda_{i,j} \in R \mid i < j\}.$$

Note that the maximal root is given by

$$\lambda_{1,r} = \lambda_{1,2} \cdots \lambda_{r-1,r}.$$

We will count the dimension of this moduli space by applying the main theorem of [BS14].

Theorem 1.4.10. *Let $SU_X^{\sigma,\tau}(r)$ be the moduli space of σ -invariant vector bundles with trivial determinant and of fixed type $\tau = (k_p)_p \bmod \pm 1$ as above. Then the dimension of $SU_X^{\sigma,\tau}(r)$ is given by*

$$(r^2 - 1)(g_Y - 1) + \sum_{p \in R} k_p(r - k_p).$$

Proof. To apply the main theorem of [BS14], we need to calculate the numbers $e(\theta_p)$ defined by

$$e(\theta_p) = \dim_{\mathbb{R}}(\mathrm{SU}_r) - |\mathcal{S}| - \#\{\lambda \in \mathcal{R} \mid \langle \theta_p, \lambda \rangle = \pm 1 \text{ or } 0\}.$$

It is easy to see that for any $\lambda_{i,j}$, one has

$$\langle \theta_p, \lambda_{i,j} \rangle = \pm \frac{1}{2} \text{ or } 0.$$

In fact $\langle \theta_p, \lambda_{i,j} \rangle$ is just the dot product in \mathbb{Q}^r of the two vectors θ_p and $\lambda_{i,j}$. The number of $\lambda_{i,j}$ such that $\langle \theta_p, \lambda_{i,j} \rangle = 0$ is

$$(r - k_p)^2 + k_p^2 - r.$$

It follows

$$\begin{aligned} e(\theta_p) &= r^2 - 1 - (r - 1) - ((r - k_p)^2 + k_p^2 - r) \\ &= r^2 - r - (r^2 - r - 2rk_p + 2k_p^2) \\ &= 2k_p(r - k_p). \end{aligned}$$

Finally we get

$$\begin{aligned} \dim(\mathcal{M}_Y(\mathcal{G}_\tau)) &= \dim(\mathrm{SL}_r)(g_Y - 1) + \frac{1}{2} \sum_{p \in R} e(\theta_p) \\ &= (r^2 - 1)(g_Y - 1) + \sum_{p \in R} k_p(r - k_p). \end{aligned}$$

□

Chapter 2

Anti-invariant Vector Bundles

2.1 Anti-invariant vector bundles

Fix an integer $r \geq 2$. Let E be a vector bundle E over X of rank r . E is called σ -anti-invariant (or simply anti-invariant) vector bundle if there exists an isomorphism

$$\psi : \sigma^* E \xrightarrow{\sim} E^*.$$

If E is stable, then this isomorphism is unique up to a scalar. Take an isomorphism $\psi : \sigma^* E \xrightarrow{\sim} E^*$, by pulling back with σ and taking the dual we get ${}^t(\sigma^* \psi) : \sigma^* E \xrightarrow{\sim} E^*$. So, there exists a non-zero $\lambda \in \mathbb{C}$, such that ${}^t(\sigma^* \psi) = \lambda \psi$. By applying again σ^* and taking the dual on this last equality, we deduce $\lambda^2 = 1$, thus $\lambda = \pm 1$.

Denote by $\tilde{\psi}$ the non-degenerated bilinear form canonically associated to ψ defined as the composition

$$\tilde{\psi} : \sigma^* E \otimes E \xrightarrow{\psi \otimes \text{id}} E^* \otimes E \xrightarrow{\text{Tr}} \mathcal{O}_X,$$

where Tr is the trace map. Sometimes we use this bilinear form instead of ψ .

Definition 2.1.1. We say that (E, ψ) is σ -symmetric (resp. σ -alternating) if $\lambda = 1$ (resp. $\lambda = -1$). We denote by $\mathcal{U}_X^{\sigma, \pm}(r) \subset \mathcal{U}_X(r, 0)$ the locus of isomorphism classes of stable σ -symmetric (resp. σ -alternating) vector bundles E . The case of trivial determinant is denoted $\mathcal{SU}_X^{\sigma, \pm}(r)$.

Observation. If π is ramified and $r \equiv 1 \pmod{2}$, then ψ is necessarily σ -symmetric.

Proof. Indeed, let p be a ramification point, then $\psi : \sigma^* E \rightarrow E^*$ induces an isomorphism $\psi_p : E_p \rightarrow E_p^*$ which is symmetric or alternating. But since r is odd, ψ_p is necessarily symmetric. \square

Note also that in the special case of rank 2, the σ -anti-invariant vector bundles with trivial determinant are the same as the σ -invariant vector bundles.

Remark 2.1.2. A stable σ -alternating vector bundle does not necessarily have a trivial determinant (not like the symplectic case). Moreover, the determinant map $\det : \mathcal{U}_X^{\sigma, \pm}(r) \rightarrow \text{Prym}(X/Y)$ is surjective (see Proposition 2.3.3).

Assume for the moment that π is ramified. Let (E, ψ) be a stable σ -alternating vector bundle, then over a ramification point $p \in R$, $\psi_p : E_p \rightarrow E_p^*$ is an antisymmetric isomorphism. If we assume that E has trivial determinant and ψ as well, then the Pfaffian $\text{pf}(\psi_p)$ of ψ_p is equal to ± 1 . For such anti-invariant vector bundle, we associate a type

$$\tau = (\text{pf}(\psi_p))_{p \in R} \pmod{\pm 1}.$$

We will see in the next chapter that these types classifies the connected components of the locus $\mathcal{SU}_X^{\sigma, -}(r)$ of stable σ -alternating vector bundles with trivial determinant.

Let (E, ψ) be a σ -symmetric (resp. σ -alternating) anti-invariant vector bundle, such that E is polystable vector bundle. It is easy to see that E can be decomposed as

$$E = \left(\bigoplus_{i=1}^a F_i^{\oplus f_i} \right) \oplus \left(\bigoplus_{j=1}^b G_j^{\oplus g_j} \right) \oplus \left(\bigoplus_{k=1}^c (H_k \oplus \sigma^* H_k^*)^{\oplus h_k} \right)$$

with F_i, G_j and H_k stable vector bundles (mutually non isomorphic), such that

- F_i are σ -symmetric (resp. σ -alternating).
- G_j are σ -alternating (resp. σ -symmetric).
- H_k are not σ -anti-invariant.

In particular, one remarks that $g_j \geq 2$ for all j .

The isomorphism ψ can be decomposed as well in the form

$$\psi = \bigoplus_{i=1}^a \alpha_i \oplus \bigoplus_{j=1}^b \beta_j \oplus \bigoplus_{k=1}^c \gamma_k$$

where α_i (resp. β_j, γ_k) are σ -symmetric (resp. σ -alternating) isomorphism on $F_i^{\oplus f_i}$ (resp. $G_j^{\oplus g_j}, (H_k \oplus \sigma^* H_k^*)^{\oplus h_k}$).

Let's treat the case of line bundles. Consider a line bundle L such that $\sigma^* L \cong L^{-1}$. Because we have $L \otimes \sigma^* L \cong \pi^* \text{Nm}(L)$, it follows that $\pi^* \text{Nm}(L) \cong \mathcal{O}_X$, hence two cases may happen:

1. If π is ramified, then π^* is injective, so $\text{Nm}(L) = \mathcal{O}_Y$.
2. If π is étale, then the kernel of π^* is $\{\mathcal{O}_Y, \Delta\}$, so either $\text{Nm}(L) = \mathcal{O}_Y$ or $\text{Nm}(L) = \Delta$.

Lemma 2.1.3. *If L is a line bundle such that $\text{Nm}(L) = \mathcal{O}_X$ then L is σ -symmetric. Assume that π is étale, then if $\text{Nm}(L) = \Delta$ then L is σ -alternating.*

Proof. The line bundle $L \otimes \sigma^* L$ has a canonical linearization given by transposition. And the line bundle $\pi^* \text{Nm}(L)$ has the canonical linearization (which we have called positive in the ramified case). These two linearizations are the same via the isomorphism

$$L \otimes \sigma^* L \cong \pi^* \text{Nm}(L).$$

Assume that $\text{Nm}(L) = \mathcal{O}_Y$, the isomorphism $\sigma^* L \cong L^{-1}$ is in fact a global section of $L \otimes \sigma^* L$, which is unique up to scalar multiplication. Then by Remark 1.1.3, we have

$$\begin{aligned} H^0(X, L \otimes \sigma^* L)_+ &= H^0(X, \pi^* \text{Nm}(L))_+ \\ &= H^0(Y, \text{Nm}(L)) = \mathbb{C}. \end{aligned}$$

This implies that L is σ -symmetric.

If π is étale and $\text{Nm}(L) = \Delta$, then it is clear that L is anti-invariant, and again by Remark 1.1.3 we have

$$\begin{aligned} H^0(X, L \otimes \sigma^* L)_- &= H^0(X, \pi^* \text{Nm}(L))_- \\ &= H^0(Y, \text{Nm}(L) \otimes \Delta) = \mathbb{C}. \end{aligned}$$

Hence L is σ -alternating. □

Note that in the étale case and odd rank, the determinant of a stable σ -alternating vector bundle belongs to $\text{Nm}^{-1}(\Delta)$.

2.2 Bruhat-Tits parahoric \mathcal{G} -torsors

Let $\mathcal{G} = \text{Res}_{X/Y}(\mathbf{SL}_r)^\sigma$ be the invariant subgroup scheme of the Weil restriction of \mathbf{SL}_r , where \mathbf{SL}_r is the constant group scheme $X \times \mathbf{SL}_r$ over X and the action of σ on \mathbf{SL}_r is given by

$$\sigma(x, g) = (\sigma(x), {}^t g^{-1}).$$

Fix a σ -alternating vector bundle with trivial determinant (F_τ, ψ_τ) of type τ . Define $\mathcal{P}_\tau = \text{Aut}(F_\tau)$. It is a group scheme over X which is étale locally isomorphic to \mathbf{SL}_r . The isomorphism $\psi_\tau : \sigma^* F_\tau \rightarrow F_\tau^*$ induces an involution, denoted σ^τ , on \mathcal{P}_τ given by

$$\alpha \longrightarrow {}^t \psi_\tau^{-1} \circ \sigma^*(\alpha^{-1}) \circ {}^t \psi_\tau.$$

So $(\sigma^\tau, \mathcal{P}_\tau)$ is a σ -group scheme over X . Finally define the group scheme

$$\mathcal{H}_\tau = \text{Res}_{X/Y}(\mathcal{P}_\tau)^{\sigma^\tau}.$$

Proposition 2.2.1. *The group schemes \mathcal{G} and \mathcal{H}_τ are smooth affine separated group schemes of finite type which are parahoric Bruhat-Tits group schemes. If $r \geq 3$, \mathcal{G} and \mathcal{H}_τ are not generically constant. The set of $y \in Y$ such that \mathcal{G}_y and $(\mathcal{H}_\tau)_y$ are not semi-simple is exactly the set of branch points of the double cover $\pi : X \rightarrow Y$.*

Proof. For the first part, we refer to [BLR90] Section 7.6, Proposition 5. As well as [Edi92] Proposition 3.5. Moreover, by [PR08b] §4, taking $I = \{0\}$, we deduce that $\mathcal{G}(\mathcal{O}_p)$ is a parahoric subgroup of $\mathcal{G}(\mathcal{K}_p)$, where here \mathcal{O}_p is the completion of the local ring at the branch point $p \in Y$, and \mathcal{K}_p its fraction field. Further we will see (cf. subsection 4.2.2) that for every $p \in B$, the flag variety $\mathcal{G}(\mathcal{K}_p)/\mathcal{G}(\mathcal{O}_p)$ (resp. $\mathcal{H}_\tau(\mathcal{K}_p)/\mathcal{H}_\tau(\mathcal{O}_p)$) is a direct limit of symplectic (resp. special orthogonal) Grassmannian which is proper, hence these flag varieties are ind-proper. So $\mathcal{G}(\mathcal{O}_p)$ (resp. $\mathcal{H}_\tau(\mathcal{O}_p)$) is parahoric subgroup of $\mathcal{G}(\mathcal{K}_p)$ (resp. $\mathcal{H}_\tau(\mathcal{K}_p)$).

We can calculate the fibers of \mathcal{G} explicitly. Let $x \in X \setminus R$ (recall that R is the divisor of ramification points). Denote by y its image in Y . By definition, we have

$$\mathcal{G}_y = \mathbf{SL}_r(\pi^{-1}(y))^\sigma = (\mathbf{SL}_r \times \mathbf{SL}_r)^\sigma,$$

where $\sigma(g, h) = ({}^t h^{-1}, {}^t g^{-1})$. So

$$\mathcal{G}_y = \{(g, {}^t g^{-1}) \mid g \in \mathbf{SL}_r\} \cong \mathbf{SL}_r.$$

Now, take $p \in B$, $\pi^{-1}(p)$ is, scheme theoretically, a double point, let us see it as $\text{Spec}(\mathbb{C}[\varepsilon])$, with $\varepsilon^2 = 0$, this gives

$$\mathcal{G}_p = \mathbf{SL}_r(\pi^{-1}(p))^\sigma = \mathbf{SL}_r(\mathbb{C}[\varepsilon])^\sigma,$$

where the action of σ on $\mathbb{C}[\varepsilon]$ is given by $\varepsilon \rightarrow -\varepsilon$. So \mathcal{G}_p is the group of elements (g, h) such that

$$\begin{aligned} g + \varepsilon h &= {}^t(g - \varepsilon h)^{-1} = ({}^t g - \varepsilon {}^t h)^{-1} \\ &= {}^t g^{-1} + \varepsilon {}^t g^{-1} {}^t h {}^t g^{-1}, \end{aligned}$$

and

$$\det(g + \varepsilon h) = 1.$$

In other words $g = {}^t g^{-1}$, ${}^t gh = {}^t(gh)$ and $g + \varepsilon h$ has determinant 1. So $g \in \text{SO}_r(\mathbb{C})$, and h is an $r \times r$ matrix such that ${}^t gh$ is symmetric. The last condition is equivalent to

$$\det(I_r + \varepsilon {}^t gh) = 1 + \varepsilon \text{Tr}({}^t gh) = 1.$$

Hence $\text{Tr}({}^t gh) = 0$. It follows that \mathcal{G}_p is isomorphic to $\text{SO}_r(\mathbb{C}) \times \text{Sym}_r^0(\mathbb{C})$ with group law given by $(g, h)(k, l) = (gk, gl + hk)$, where $\text{Sym}_r^0(\mathbb{C})$ is the additive group of symmetric traceless matrices. We have a non split exact sequence:

$$0 \rightarrow \text{Sym}_r^0(\mathbb{C}) \rightarrow \mathcal{G}_p \rightarrow \text{SO}_r(\mathbb{C}) \rightarrow 1.$$

Note that \mathcal{G}_p is not semi-simple.

Assume now that r is even. With the exact same computation as above we get

$$(\mathcal{H}_\tau)_p \cong \text{SL}_r \text{ for } p \in Y \text{ not a branch point,}$$

and for a branch point p we have

$$0 \rightarrow \text{ASym}_{r,p}^0 \rightarrow (\mathcal{H}_\tau)_p \rightarrow \text{Sp}_r \rightarrow 0,$$

where

$$\text{ASym}_{r,p} = \{h \in M_r \mid \text{Tr}(h) = 0, M_p h = {}^t h M_p = -{}^t(M_p h)\},$$

where $M_p = ({}^t \psi_\tau^{-1})_p$ and Sp_r is the symplectic group over \mathbb{C} . \square

Let $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$ (resp. $\mathcal{S}\mathcal{U}_X^{\sigma,\tau}(r)$) be the stack defined by associating to a \mathbb{C} -algebra R the groupoid of (E, δ, ψ) , where E is a σ -symmetric (resp. σ -alternating of type τ) vector bundle over $X_R = X \times \text{Spec}(R)$, δ a trivialization of $\det(E)$ and a σ -symmetric (resp. σ -alternating of type τ) isomorphism $\psi : \sigma^* E \xrightarrow{\sim} E^*$ which is compatible (in the obvious sense) with δ .

Proposition 2.2.2. *Let $\mathcal{M}_Y(\mathcal{G})$ (resp. $\mathcal{M}_Y(\mathcal{H}_\tau)$) be the stack of right \mathcal{G} -torsors (resp. \mathcal{H}_τ -torsors) on Y , then $\mathcal{M}_Y(\mathcal{G})$ (resp. $\mathcal{M}_Y(\mathcal{H}_\tau)$) is a smooth algebraic stack, locally of finite type, which is isomorphic to $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$ (resp. $\mathcal{S}\mathcal{U}_X^{\sigma,\tau}(r)$).*

Proof. The first part of the theorem is proved in [Hei10] Proposition 1.

By Theorem 1.4.4, $\mathcal{M}_Y(\mathcal{G}) \cong \mathcal{M}_X(\sigma, \mathbf{SL}_r)$. So it is sufficient to show $\mathcal{M}_X(\sigma, \mathbf{SL}_r) \cong \mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$. Let S be a \mathbb{C} -algebra, and (E, δ, ψ) be an element of $\mathcal{M}_X(\sigma, \mathbf{SL}_r)(S)$. Consider the automorphism of the frame bundle $\tilde{E} := \text{Isom}(\mathcal{O}_{X_S}^{\oplus r}, E)$ given by

$$\tilde{\psi}(f) = {}^t(\psi \circ \sigma^* f)^{-1},$$

for $f \in \tilde{E}$ (we identify $\sigma^*(\mathcal{O}_{X_S}^{\oplus r}) \cong \mathcal{O}_{X_S}^{\oplus r}$ using the canonical linearization). Since $\sigma^* \psi = {}^t \psi$, we get $\tilde{\psi} \circ \tilde{\psi}(f) = f$, thus

$$\tilde{\psi}^2 = \text{id},$$

so $\tilde{\psi}$ is a lifting of the action of σ to \tilde{E} , and any other lifting differs by an involution of $\mathcal{O}_{X_S}^{\oplus r}$. Moreover, for $g \in \mathbf{SL}_r(\mathcal{O}_{X_S})$, we have

$$\tilde{\psi}(f \cdot g) = \tilde{\psi}(f) \cdot \sigma(g),$$

where $\sigma(g) = {}^t g^{-1}$. Thus \tilde{E} is (σ, \mathbf{SL}_r) -bundle.

Conversely, a \mathcal{G} -bundle E over Y_S gives, by Theorem 1.4.4, a (σ, \mathbf{SL}_r) -bundle over X_S denoted again by E . Let $\tilde{\psi}$ be the action of σ on E . Then

$$E(\mathbb{C}^r) := E \times^{\mathbf{SL}_r} \mathbb{C}^r$$

is σ -anti-invariant vector bundle. Let U be a σ -invariant open subset of X_S such that $E(\mathbb{C}^r)|_U$ is trivial and fix a σ -invariant trivialization $\varphi : \mathcal{O}_U^{\oplus r} \rightarrow E(\mathbb{C})|_U$. Define

$\psi|_U = {}^t\tilde{\psi}(\varphi)^{-1} \circ \sigma^*\varphi^{-1}$, then ψ is a σ -symmetric isomorphism $\sigma^*E(\mathbb{C}^r)|_U \rightarrow E(\mathbb{C}^r)^*$. Gluing such local isomorphisms, we get an isomorphism $\psi : \sigma^*E(\mathbb{C}^r) \rightarrow E(\mathbb{C}^r)^*$. Hence we get an element of $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)(S)$.

Now, let (E, ψ) be a σ -alternating vector bundle over X_S . Consider the bundle

$$\tilde{E} = \text{Isom}(F_\tau, E).$$

It is an \mathcal{H}_τ -bundle. Moreover, ψ induces an automorphism $\tilde{\psi}$ on \tilde{E} given by

$$\tilde{\psi}(f) = {}^t\psi^{-1} \circ {}^t(\sigma^*f)^{-1} \circ {}^t\psi_\tau.$$

Clearly this is an involution which makes \tilde{E} a $(\sigma^\tau, \mathcal{P}_\tau)$ -bundle.

Conversely, a $(\sigma^\tau, \mathcal{P}_\tau)$ -bundle gives, with exactly the same method as before, a σ -alternating vector bundle. \square

Proposition 2.2.3. *We have $\pi_1(\mathcal{G}_{\bar{\eta}}) = 1$ and $\pi_1((\mathcal{H}_\tau)_{\bar{\eta}}) = 1$.*

Proof. We treat just the case of \mathcal{G} . The other one is similar.

Since $\pi : X \rightarrow Y$ is generically unramified, $X_{\bar{\eta}}$ is two points (to see this, note that $K(X)$ is quadratic extension of $K(Y)$, so $\overline{K(Y)} = \overline{K(X)}$, and there is two embeddings of $K(Y) \hookrightarrow \overline{K(X)}$ inducing the canonical inclusion $K(Y) \subset \overline{K(Y)}$, using the $\text{Gal}(K(X)/K(Y))$, this gives the two points). So by definition $\mathcal{G}_{\bar{\eta}}$ is the invariant part of the action of $\tilde{\sigma}$ on $\text{SL}_r(\bar{\eta}) \times \text{SL}_r(\bar{\eta})$, hence it can be identified with $\text{SL}_r(\bar{\eta})$, thus $\pi_1(\mathcal{G}_{\bar{\eta}}) = 1$. \square

Corollary 2.2.4. *The stacks $\mathcal{M}_Y(\mathcal{G})$ and $\mathcal{M}_Y(\mathcal{H}_\tau)$ are connected.*

Proof. This follows from Proposition 2.2.3 applied to [Hei10] Theorem 2. \square

We will give another proof of this result using the Hitchin system. More precisely, we construct dominant rational maps from some Prym varieties to the loci of stable anti-invariant bundles $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$ and $\mathcal{S}\mathcal{U}_X^{\sigma,-}(r)$.

2.3 The existence problem

Here we construct examples of stable anti-invariant vector bundles. Let $\beta \in J_X[r]$ a primitive r -torsion point of the Jacobian which descends to Y , so in particular we assume that the genus g_Y of Y is at least 1. Denote by $q : X_\beta \rightarrow X$ the associated cyclic étale cover of X of degree r which can be defined as a spectral curve associated to the spectral data $(0, \dots, 0, 1)$ (see section 3.1). Denote by ι a generator of the Galois group $\text{Gal}(X_\beta/X)$.

Lemma 2.3.1. *The involution $\sigma : X \rightarrow X$ lifts to an involution $\tilde{\sigma} : X_\beta \rightarrow X_\beta$. Moreover, if r is even, there are two such lifting of σ such that one of them has no fixed points, we denote it by $\tilde{\sigma}_-$.*

Proof. The curve X_β is a spectral curve given by the equation $x^r - 1 = 0$ in the ruled surface $\mathbb{P}(\mathcal{O}_X \oplus \beta^{-1})$. As in the proof of Proposition 3.2.1, the positive linearization on β gives an involution $\tilde{\sigma}$ on X_β that lifts σ . If r is even, then the negative linearization gives also a lifting of σ . One remarks that $q(\text{Fix}(\tilde{\sigma})) \subset \text{Fix}(\sigma)$, hence if $\pi : X \rightarrow Y$ is étale, then $X_\beta \rightarrow X_\beta/\tilde{\sigma}$ is étale too. However, if r is even, the negative linearization has no fixed point because 0 is not a root of $x^r - 1 = 0$. \square

Proposition 2.3.2. *The line bundles of degree 0 on X_β such that q_*L is not stable are those with non-trivial stabilizer subgroup of $\langle \iota \rangle$.*

Proof. This is true for any Galois cover, it is proved in the (unpublished) paper of Beauville "On the stability of the direct image of a generic vector bundle".

Let $L \in \text{Pic}^0(X_\beta)$ such that q_*L is not stable. Let $F \hookrightarrow q_*L$ be a stable subbundle of degree 0, it follows

$$q^*F \hookrightarrow q^*q_*L = L \oplus \iota^*L \oplus \cdots \oplus (\iota^{r-1})^*L,$$

hence q^*F is of the form $\bigoplus_{j \in J} (\iota^j)^*L$ for some $J \subsetneq \{0, \dots, r-1\}$. In particular both F and q_*L are semi-stable. On the other hand, The adjunction formula gives a non-zero map $q^*F \rightarrow (\iota^k)^*L$ for any k . As q^*F is semi-stable of degree 0, this map is surjective. Hence $\bigoplus_{j \in J} (\iota^j)^*L \rightarrow (\iota^k)^*L$ is surjective for any k . It follows that there exists $k \in \{1, \dots, r-1\}$ such that $(\iota^k)^*L \cong L$. So ι^k is in the stabilizer of L .

Conversely, let L such that $(\iota^k)^*L \cong L$ for some $0 < k < r$. Then the vector bundle $\iota^*L \oplus \cdots \oplus (\iota^k)^*L$ is ι -invariant, so it descends to a vector bundle, say F , on X . As $\deg(F) = 0$, by adjunction, we deduce that $F \hookrightarrow q_*L$, hence q_*L is not stable. \square

Now we can construct some stable anti-invariant vector bundles.

Proposition 2.3.3. *1. There exist stable σ -symmetric anti-invariant vector bundles. If r is even or π is étale, then there exist stable σ -alternating vector bundles.*

2. The determinant maps

$$\det : \mathcal{U}_X^{\sigma,+}(r) \rightarrow P^+ = \text{Nm}^{-1}(\mathcal{O}_Y),$$

$$\det : \mathcal{U}_X^{\sigma,-}(r) \rightarrow P^- = \begin{cases} \text{Nm}^{-1}(\mathcal{O}_Y) & r \equiv 0 \pmod{2} \\ \text{Nm}^{-1}(\Delta) & r \equiv 1 \pmod{2} \text{ and } \pi \text{ étale} \end{cases},$$

are surjective.

Proof. 1. We denote $Y_\beta = X_\beta/\tilde{\sigma}$ and $Z_\beta = X_\beta/\tilde{\sigma}_-$ if r is even. By Proposition 2.3.2 we deduce that a general element in $\text{Nm}_{X_\beta/Y_\beta}^{-1}(\mathcal{O}_{Y_\beta})$ has a stable direct image which is σ -symmetric. Let Δ_β (resp. Δ'_β) be the 2-torsion point attached to $X_\beta \rightarrow Z_\beta$ (resp. $X_\beta \rightarrow Y_\beta$), then a general element in $\text{Nm}_{X_\beta/Z_\beta}^{-1}(\Delta_\beta)$ has a stable direct image which is σ -alternating. If r is odd and $\pi : X \rightarrow Y$ is étale, also a general element in $\text{Nm}_{X_\beta/Y_\beta}^{-1}(\Delta'_\beta)$ has a stable direct image which is again σ -alternating. Note that being σ -symmetric or σ -alternating here is due to Lemma 2.1.3.

2. If π is ramified, or π is étale and r is odd, then the second point is clear due to taking the tensor product of a fixed anti-invariant vector bundle by elements of P^\pm . Assume that π is étale and r is even, taking the tensor product by elements of P^\pm does not make the determinant surjective, so we need to prove the existence of stable vector bundles whose determinants are in both connected components of P^\pm . But one remarks that $\text{Nm}_{X_\beta/X} : P^\pm \rightarrow P^\pm$ is surjective, where $P^+ = \text{Nm}_{X_\beta/Y_\beta}^{-1}(\mathcal{O}_{Y_\beta})$ and $P^- = \text{Nm}_{X_\beta/Y_\beta}^{-1}(\Delta_\beta)$. Since we have $\det(q_*L) = \text{Nm}_{X_\beta/X}(L) \otimes \beta^{r(r-1)/2}$, we deduce that the image of the determinant map intersects the two connected components of $P^+ = P^-$. Taking now the tensor product with elements of the identity component of P^+ gives the result. \square

2.4 Moduli space of anti-invariant vector bundles

2.4.1 σ -quadratic modules

This subsection is devoted to the study of the moduli of σ -quadratic modules, which will be used later in the construction of the moduli space of σ -symmetric anti-invariant vector bundles. Our main reference here is [Sor93].

Let W be a finite dimension vector space with an involution σ , and H a vector space. A σ -quadratic form is a linear map $q : H \rightarrow H^* \otimes W$ such that for all $x, y \in H$

$$q(x)(y) = \sigma(q(y)(x)).$$

A σ -quadratic module with values in W is a pair (H, q) as above. A map between two σ -quadratic modules (H, q) and (H', q') is a linear map $f : H \rightarrow H'$ such that

$$q = ({}^t f \otimes \text{id}) \circ q' \circ f.$$

For a vector subspace $V \subset H$, we define its orthogonal to be

$$V^{\perp\sigma} = \{x \in H \mid q(x, y) = 0 \ \forall y \in V\}.$$

A σ -isotropic (resp. totally σ -isotropic) subspace V of (H, q) is a vector subspace such that $V \cap V^{\perp\sigma} \neq 0$ (resp. $V \subset V^{\perp\sigma}$). We will mainly use the notion of totally σ -isotropic as we will see later on.

Definition 2.4.1. The σ -quadratic module (H, q) is called semi-stable (resp. stable) if for any non-zero totally σ -isotropic vector subspace $V \subset H$ we have

$$\dim(V) + \dim(V^{\perp\sigma}) \leq \dim(H) \quad (\text{resp. } <).$$

Remark that a semi-stable σ -quadratic module is necessarily injective.

Denote by $\Gamma(H, W)^\sigma$ the vector space of σ -quadratic forms $q : H \rightarrow H^* \otimes W$, and let $P(H, W)^\sigma = \mathbb{P}\Gamma(H, W)^\sigma$. The group $\text{SL}(H)$ acts linearly in a natural way on $\Gamma(H, W)^\sigma$ by associating to q the $({}^t g^{-1} \otimes \text{id}) \circ q \circ g^{-1}$. This action induces clearly an action on $P(H, W)^\sigma$.

Proposition 2.4.2. A σ -quadratic module (H, q) is semi-stable (resp. stable) if and only if the point $[q] \in P(H, W)^\sigma$ is semi-stable (resp. stable) with respect to the action of $\text{SL}(H)$.

Proof. We use Hilbert-Mumford criterion ([Pot97] Theorem 6.5.5) and we use also their notation for the weight. Assume that q is semi-stable σ -quadratic form on H , let λ be a non trivial one parameter subgroup of $\text{SL}(H)$. Consider the eigenvalue decomposition of

$$H = \bigoplus_{i=1}^s H_i,$$

where the restriction of $\lambda(t)$ to H_i equals $t^{-m_i} \text{id}$, we assume also that $m_1 < \dots < m_s$. Since $\lambda(t) \in \text{SL}(H)$, we have

$$\sum_{i=1}^s m_i \dim(H_i) = 0.$$

Note that since λ is not trivial, there exists k such that $m_k < 0 \leq m_{k+1}$. Now q decomposes as $q = (q_{ij})_{ij}$, where $q_{ij} : H_i \rightarrow H_j$. It follows that the Hilbert-Mumford weight of q is equal to

$$\mu(\lambda, q) = -\min\{m_i + m_j \mid \forall (i, j) \text{ such that } q_{ij} \neq 0\}.$$

Suppose that $\mu(\lambda, q) < 0$ and let $V = \bigoplus_{i=1}^k H_i$. Then

$$V \oplus \bigoplus_{i \in I} H_i \subset V^{\perp \sigma},$$

where $I = \{i \geq k+1 \mid m_j + m_i \leq 0 \text{ for all } j \leq k\}$. In particular V is totally σ -isotropic. Let $l = \max(I)$, so we get

$$\begin{aligned} m_{l+1} \left(\dim(V) + \dim(V^{\perp \sigma}) \right) &\geq m_{l+1} \sum_{i=1}^k \dim(H_i) + m_{l+1} \sum_{i=1}^l \dim(H_i) \\ &> - \sum_{i=1}^k m_i \dim(H_i) + m_{l+1} \sum_{i=1}^l \dim(H_i) \\ &= \sum_{i=k+1}^s m_i \dim(H_i) + m_{l+1} \sum_{i=1}^l \dim(H_i) \\ &\geq m_{l+1} \sum_{i=1}^s \dim(H_i) = m_{l+1} \dim(H), \end{aligned}$$

which contradicts the semistability of q , hence $\mu(\lambda, q) \geq 0$.

Conversely, assume that for any 1-parameter subgroup λ we have $\mu(\lambda, q) \geq 0$. Let $V \subset H$ be a totally σ -isotropic subspace with respect to q , and denote by H_1 a complementary subspace of V in $V^{\perp \sigma}$, and by H_2 a complementary subspace of $V^{\perp \sigma}$ in H , so we have $H = V \oplus H_1 \oplus H_2$. Consider the integers

$$\begin{aligned} m_1 &= 2\dim(H) - 2\dim(V) - \dim(H_1), \\ m_2 &= \dim(H) - 2\dim(V) - \dim(H_1), \\ m_3 &= -2\dim(V) - \dim(H_1). \end{aligned}$$

Then we have $m_3 < m_2 < m_1$ and

$$m_1 \dim(V) + m_2 \dim(H_1) + m_3 \dim(H_2) = 0.$$

Let's consider the 1-parameter subgroup λ of $\text{SL}(H)$ associated to the decomposition $H = V \oplus H_1 \oplus H_2$ with characters given by the weights m_1 , m_2 and m_3 (respecting the order of the decomposition). It follows that λ acts on q by the matrix

$$\begin{pmatrix} 0 & 0 & t^{-m_1-m_3} \\ 0 & t^{-2m_2} & t^{-m_2-m_3} \\ t^{-m_1-m_3} & t^{-m_2-m_3} & t^{-2m_3} \end{pmatrix}.$$

By definition, we deduce that

$$\mu(\lambda, q) = -\min\{-2m_2, -m_1 - m_3\} = 2m_2,$$

and by hypothesis we have $\mu(\lambda, q) \geq 0$. Hence $m_2 \geq 0$, which is exactly

$$\dim(V) + \dim(V^{\perp \sigma}) \leq \dim(H).$$

□

Let (H, q) be a semi-stable and non-stable σ -quadratic module, there exists a minimal totally σ -isotropic subspace H_1 of H such that $\dim(H_1) + \dim(H_1^{\perp\sigma}) = \dim(H)$. We repeat this procedure after replacing H by $H_1^{\perp\sigma}/H_1$ with its reduced σ -quadratic form. So we construct a filtration

$$0 \subset H_1 \subset H_2 \subset \cdots \subset H_k \subset H,$$

of totally σ -isotropic subspaces such that

- (i) $H_i/H_{i-1} \subset H_{i-1}^{\perp\sigma}/H_{i-1}$ are minimal totally σ -isotropic such that

$$\dim(H_i/H_{i-1}) + \dim(H_i/H_{i-1})^{\perp\sigma} = \dim(H_{i-1}^{\perp\sigma}/H_{i-1}).$$

- (ii) $H_k^{\perp\sigma}/H_k$ is stable.

We define the σ -quadratic graded module associated to (H, q) to be

$$\text{gr}(H, q) = H_k^{\perp\sigma}/H_k \bigoplus_{i=1}^{k-1} (H_i/H_{i-1}) \oplus (H_i/H_{i-1})^{\perp\sigma},$$

with the induced form. The integer k is called the length of the graded σ -quadratic module. Two σ -quadratic modules are said S -equivalent if they have isomorphic graded modules.

Proposition 2.4.3. *Let $Q(H, W)^\sigma = P(H, W)^{\sigma, ss} // SL(H)$ be the geometric quotient of the subspace of semi-stable points $P(H, W)^{\sigma, ss}$ by $SL(H)$. Then a point of $Q(H, W)^\sigma$ represents an S -equivalence class of σ -quadratic modules.*

Proof. The proof is the same as that of [Sor93] Proposition 2.5. We prove it in two steps:

1. First we prove $\text{gr}(H, q)$ is in the closure of the orbit of q by showing that there exists a 1-parameter subgroup λ of $SL(H)$ such that $\text{gr}(H, q) = \lim_{t \rightarrow 0} \lambda(t) \cdot q$. We prove this by induction on k . If $k = 0$, that's (H, q) is stable, there is nothing to prove. Assume the result for $k - 1$. Let (H, q) be a semi-stable σ -quadratic module with a graded module of length k . Choose a *minimal* totally σ -isotropic subspace $H_1 \subset H_1^{\perp\sigma} \subset H$. Let H_2 and H_3 be (any) complements of H_1 in $H_1^{\perp\sigma}$ and $H_1^{\perp\sigma}$ in H respectively. Then we have the following decomposition of q

$$\begin{array}{c} H_1 & H_2 & H_3 \\ H_1^* \otimes W & \begin{pmatrix} 0 & 0 & \alpha \end{pmatrix} \\ H_2^* \otimes W & \begin{pmatrix} 0 & q' & \beta \end{pmatrix} \\ H_3^* \otimes W & \begin{pmatrix} \sigma^* \alpha^* & \sigma^* \beta^* & \gamma \end{pmatrix} \end{array},$$

for some σ -quadratic module q' on H_2 and some maps α, β and γ (this last verifies $\sigma^* \gamma^* = \gamma$). Clearly the graded module associated to q' is of length $k - 1$ and we can apply the induction hypothesis to obtain a 1-parameter subgroup λ' of $SL(H_2)$ such that $\lim_{t \rightarrow 0} \lambda'(t) \cdot q' = \text{gr}(q')$. Finally define λ to be the 1-parameter subgroup of $SL(H)$ given by

$$t \longrightarrow \begin{array}{c} H_1 & H_2 & H_3 \\ H_1 & \begin{pmatrix} t & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \\ H_2 & & \\ H_3 & & \end{array}.$$

We see immediately that $\lim_{t \rightarrow 0} \lambda(t) \cdot q = \text{gr}(H, q)$.

2. We show here that the orbit of a σ -quadratic graded module (H, q) is closed. Again we use induction on the length k . If $k = 0$, then q is stable. For every 1-parameter subgroup of $\mathrm{SL}(H)$, let $q_0 = \lim_{t \rightarrow 0} \lambda(t) \cdot q$. Since q is stable, its orbit is proper. So by the valuative criterion of properness, we deduce that q_0 is in the orbit of q . Assume now the result for $k - 1$, let λ be a 1-parameter subgroup and assume that the limit $q_0 = \lim_{t \rightarrow 0} \lambda(t) \cdot q$ exists. Let $H = H_1 \oplus H_2 \oplus H_3$ be a decomposition as above. So q can be written

$$\begin{array}{c} H_1 & H_2 & H_3 \\ H_1^* \otimes W & \left(\begin{array}{ccc} 0 & 0 & \alpha \\ 0 & q' & \beta \\ \sigma^* \alpha^* & \sigma^* \beta^* & \gamma \end{array} \right) \\ H_2^* \otimes W & & \\ H_3^* \otimes W & & \end{array}.$$

Denote $H_i(t) = \lambda(t)(H_i)$, and $\alpha_t = \lambda(t) \cdot \alpha$. The subspace $H_1(t)$ is totally σ -isotropic with respect to $q_t = \lambda(t) \cdot q$ and the module $H_1 \rightarrow H_3^* \otimes W$ is stable. We can assume that $\lambda(t)$ (for all $t \in \mathbb{C}^*$) stabilizes H_1 and $H_1^{\perp\sigma} = H_1 \oplus H_2$. Hence we can write $\lambda(t)^{-1}$ in the form

$$\begin{array}{c} H_1 & H_2 & H_3 \\ H_1 & \left(\begin{array}{ccc} f(t) & g(t) & h(t) \\ 0 & u(t) & v(t) \\ 0 & 0 & w(t) \end{array} \right) \\ H_2 & & \\ H_3 & & \end{array}.$$

Moreover, without changing q_t , we can assume that $\det(f(t)) = \det(u(t)) = \det(w(t)) = 1$. It follows that $\alpha_t = {}^t f(t) \alpha w(t)$. Since α is stable, and since α_t has a limit by assumption, it follows, by properness, that $f(t)$ and $w(t)$ have limits f_0 and w_0 . Moreover, By the induction hypothesis, we deduce that $u(t)$ has a limit u_0 . Now we can explicitly calculate β_t and γ_t in function of $g(t)$, $h(t)$ and $v(t)$ (with the coefficients of q_t) and we deduce the existence of limits of $g(t)$, $h(t)$ and $h(t)$. This ends the proof. □

2.4.2 Semistability of anti-invariant bundles

Let (E, ψ) be an anti-invariant vector bundle over X . We say that a subbundle F of E is σ -isotropic if the induced map $\psi : \sigma^* F \rightarrow F^*$ is identically zero.

Definition 2.4.4. Let (E, ψ) be an anti-invariant vector bundle over X . We say that it is *semi-stable* (resp. *stable*) if for every σ -isotropic sub-bundle F of E , one has

$$\mu(F) \leq 0 \text{ (resp. } \mu(F) < 0).$$

Proposition 2.4.5. (E, ψ) is semi-stable if and only if E is semi-stable vector bundle.

Proof. We follow the same lines of the proof of [Ram81] 4.2, page 155.

The "if" part is obvious. Conversely, take F to be any sub-bundle of E . Define $F^{\perp\sigma}$ to be the kernel of the surjective morphism:

$$E \xrightarrow{\sim} \sigma^* E^* \rightarrow \sigma^* F^*.$$

Note that $F^{\perp\sigma}$ have the same degree as F , and F is σ -isotropic if and only if $F \subset F^{\perp\sigma}$. Then, the sub-bundle N of E generated by $F \cap F^{\perp\sigma}$ is σ -isotropic. Indeed, we have

$N \subset F$, so $F^{\perp\sigma} \subset N^{\perp\sigma}$, interchanging F and $F^{\perp\sigma}$ we get $F \subset N^{\perp\sigma}$, hence $N \subset N^{\perp\sigma}$. Let M be the image of $F \oplus F^{\perp\sigma}$ in E . We have $M = N^{\perp\sigma}$, to see this, note that $N^{\perp\sigma}$ contains F and $F^{\perp\sigma}$, so it contains M , but this two bundles have the same rank. Moreover we have

$$0 \rightarrow N \rightarrow F \oplus F^{\perp\sigma} \rightarrow M \rightarrow 0,$$

which implies also

$$0 \rightarrow M^{\perp\sigma} \rightarrow F \oplus F^{\perp\sigma} \rightarrow N^{\perp\sigma} \rightarrow 0,$$

we deduce that they have the same degree too. Hence they are equal.

Therefore, $\deg(N) = \deg(F)$, but $\deg(N) \leq 0$ because it is σ -isotropic and (E, ψ) is semi-stable by hypothesis, so E is semi-stable as a vector bundle. \square

Let E be a σ -symmetric anti-invariant vector bundle, the following lemma generalizes the isotropic filtration of self-dual vector bundle.

Lemma 2.4.6. *There exists a filtration of E of the form*

$$0 = F_0 \subset F_1 \subset \dots \subset F_k \subseteq F_k^{\perp\sigma} \subset F_{k-1}^{\perp\sigma} \subset \dots \subset F_0^{\perp\sigma} = E,$$

where F_i are degree 0 sub-bundles of E (which are of course σ -isotropic) such that F_i/F_{i-1} is stable vector bundle of rank ≥ 1 for $i = 1, \dots, k$.

Proof. The proof is similar to that of Lemma 1.9 of [Hit05].

The proof is a constructive one, we consider the set of all σ -isotropic subbundles of E , which contains 0 and E . If E is stable anti-invariant vector bundle, then it has no σ -isotropic proper sub-bundle of degree 0, and the filtration is $0 \subset 0^{\perp\sigma} = E$. Otherwise, let F_1 be a σ -isotropic sub-bundle of E of degree 0 and smallest rank (it is a stable vector bundle, because otherwise, a proper sub-bundle of F_1 of degree 0 would be a σ -isotropic sub-bundle of E , contradicting the minimality of $\text{rk}(F_1)$). Now, we repeat this procedure on E/F_1 instead of E . \square

Lemma 2.4.7. *Consider the above filtration, then we have*

$$\sigma^*(F_{i-1}^{\perp\sigma}/F_i^{\perp\sigma}) \cong (F_i/F_{i-1})^*, \quad \sigma^*(F_k^{\perp\sigma}/F_k) \cong (F_k^{\perp\sigma}/F_k)^*,$$

for $i = 1, \dots, k$.

Proof. For $i = 1$, this is just the definition of $F_1^{\perp\sigma}$. Let $i > 1$, and consider

$$0 \subset F_{i-1} \subset F_i \subset F_i^{\perp\sigma} \subset F_{i-1}^{\perp\sigma} \subset E.$$

We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_{i-1}^{\perp\sigma} & \xrightarrow{i} & E & \longrightarrow & \sigma^* F_{i-1}^* \longrightarrow 0 \\ & & & & \searrow p_1 & & \uparrow p_2 \\ & & & & & & \sigma^* F_i^* \\ & & & & & & \uparrow \\ & & & & & & \sigma^*(F_i/F_{i-1})^* \end{array} .$$

Since the composition $p_2 \circ p_1 \circ i$ is identically zero, it follows that $p_1 \circ i$ factorizes through $\sigma^*(F_i/F_{i-1})^*$. The resulting map $F_{i-1}^{\perp\sigma} \rightarrow \sigma^*(F_i/F_{i-1})^*$ is nonzero map because otherwise

$F_{i-1}^{\perp\sigma} \subset F_i^{\perp\sigma}$, thus $F_i/F_{i-1} = 0$ which contradicts the definition of the above filtration. Its kernel contains $F_i^{\perp\sigma}$, so we obtain a nonzero map

$$F_{i-1}^{\perp\sigma}/F_i^{\perp\sigma} \rightarrow \sigma^*(F_i/F_{i-1})^*.$$

But this two bundles are stable of the same rank and degree, so the last map has to be an isomorphism.

For $i = k$, we have a nonzero map

$$F_k^{\perp\sigma}/F_k \rightarrow \sigma^*(F_k^{\perp\sigma}/F_k)^*,$$

otherwise $F_k = F_k^{\perp\sigma}$. So the same argument as before gives the result. \square

The above lemma proves that the bundle

$$\mathrm{gr}^\sigma(E) = \bigoplus_{i=1}^k \left(F_i/F_{i-1} \oplus F_{i-1}^{\perp\sigma}/F_i^{\perp\sigma} \right) \oplus (F_k^{\perp\sigma}/F_k)$$

is an anti-invariant vector bundle. Moreover, it is σ -symmetric (resp. σ -alternating) if E is σ -symmetric (resp. σ -alternating).

Definition 2.4.8. The vector bundle $\mathrm{gr}^\sigma(E)$ is called the σ -graded bundle associated to (E, ψ) . Two σ -anti-invariant vector bundles E and F are said to be S -equivalent if their associated σ -graded bundles are isomorphic.

Example 2.4.9. We give an example of two non-isomorphic σ -symmetric anti-invariant vector bundles which are S -equivalent. Let M be an element of $\mathrm{Prym}_{X/Y}$, and $\phi : \sigma^*M \xrightarrow{\sim} M^*$. The vector bundle $M^{\oplus 2}$ with the σ -symmetric isomorphism

$$\psi = \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix}$$

is a σ -symmetric anti-invariant vector bundle. Now for $\eta \in \mathrm{Ext}^1(M, M)_- \cong H^1(X, \mathcal{O}_X)_-$, where the involution on this vector space is given by pullback by σ . Consider the associated extension of M by M

$$0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0.$$

Note that in rank 2 taking the dual does not change the extension class in $H^1(X, \mathcal{O}_X)$ because of the formula $E^* \cong E \otimes \det(E)^{-1}$.

Since η is a -1 eigenvector, E is anti-invariant. Indeed, by pulling back by σ we get the extension

$$\begin{array}{ccccccc} 0 & \longrightarrow & \sigma^*M & \longrightarrow & \sigma^*E & \longrightarrow & \sigma^*M \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & M^{-1} & \longrightarrow & E \otimes M^{-2} & \longrightarrow & M^{-1} \longrightarrow 0. \end{array}$$

But $E \otimes M^{-2}$ is isomorphic E^* . Moreover, if $\eta \neq 0$ then E is not isomorphic to $M^{\oplus 2}$ (see subsection 2.5 for more details about the deformations of anti-invariant vector bundles). However, clearly E and $M^{\oplus 2}$ are S -equivalent as σ -anti-invariant vector bundles.

2.4.3 Construction of the moduli space

σ -symmetric case

Fix an ample σ -linearized line bundle $(\mathcal{O}(1), \eta)$ of degree 1 over X (in the étale case, there are no such bundle, so one has to take degree 2 instead of degree 1, but this doesn't produce any difference). We follow the method of [Sor93] to construct this moduli space. Let ν be some big integer such that for any semi-stable coherent sheaf E over X of rank r and degree 0, we have $H^1(X, E(\nu)) = 0$ and $E(\nu)$ is generated by global sections. Let $\mathcal{F} = \mathcal{O}_X^m(-\nu)$ where $m = r\nu + r(1 - g_X)$. Denote $H = \mathbb{C}^m$.

Consider the functor

$$\text{Quot}^\sigma : (\text{algebraic varieties}) \rightarrow (\text{sets})$$

which associates to a variety T the set of isomorphism classes of $(E, q, \bar{\phi})$, where E is coherent quotient sheaf $q : p_1^* \mathcal{F} \rightarrow E$ over $X \times T$ flat over T , and $\bar{\phi}$ is class, modulo \mathbb{C}^* , of σ -symmetric isomorphism $\sigma^* E \cong E^*$ (σ acts only on X), such that, for each $t \in T$, E_t is a semi-stable, σ -symmetric and locally free of rank r and q induces an isomorphism $H \rightarrow H^0(X, E_t(\nu))$. Two triplets $(E, q, \bar{\phi})$ and $(F, p, \bar{\psi})$ are isomorphic if there exists an isomorphism $f : E \rightarrow F$ such that $p = f \circ q$ and $\psi \circ \sigma^* f = {}^t f^{-1} \circ \phi$ (for some $\phi \in \bar{\phi}$ and $\psi \in \bar{\psi}$).

Let $[E, q, \bar{\phi}] \in \text{Quot}^\sigma(\mathbb{C})$, consider the diagram

$$\begin{array}{ccccc} H \otimes \mathcal{O}_X & \xrightarrow{\sigma^* q} & \sigma^* E(\nu) & \longrightarrow & 0 \\ & & \downarrow \phi & & \\ 0 & \longrightarrow & E^*(\nu) & \xrightarrow{{}^t q} & H^* \otimes \mathcal{O}_X(2\nu). \end{array}$$

The composition $h = {}^t q \circ \phi \circ \sigma^* q$ gives, at the level of global sections, a σ -quadratic form $H \rightarrow H^* \otimes W$, where $W = H^0(X, \mathcal{O}_X(2\nu))$ with an involution induced by the linearization on $\mathcal{O}(1)$. Hence we get a point $\bar{h} \in P(H, W)^\sigma$. This actually defines a transformation $\mathfrak{h} : \text{Quot}^\sigma \rightarrow P(H, W)^\sigma$, where $P(H, W)^\sigma$ is seen as a functor by associating to a variety T the space $P(H_T, W_T)^\sigma$, where $H_T = H^0(X \times T, \mathcal{O}_{X \times T}^{\oplus m})$ and $W_T = H^0(X \times T, \mathcal{O}_{X \times T}(2\nu))$.

Proposition 2.4.10. *Let (E, ψ) be a σ -symmetric vector bundle, and h its corresponding point of $\Gamma(H, W)^\sigma$, then the following are equivalent:*

- (a) *The bundle E is semi-stable.*
- (b) *h is semi-stable with respect to the action of $SL(H)$.*

Moreover, (E, ψ) is stable if and only if h is stable.

Proof. Assume that (E, ψ) is semi-stable, let $V \subset H$ be a totally σ -isotropic. Denote by F and F' the subsheaves of E generated by V and $V^{\perp\sigma}$ respectively. By Proposition 2.4.5 the induced vector bundle is semi-stable, hence by [Pot97] Proposition 7.1.1, for all subsheaf F of E , one has

$$\frac{h^0(F(m))}{\text{rk}(F)} \leq \frac{h^0(E(m))}{\text{rk}(E)},$$

for $m \geq \nu$ large enough. By applying this to F and F' , and then summing up, we deduce

$$h^0(F(\nu)) + h^0(F'(\nu)) \leq h^0(E(m)),$$

which is the same as

$$\dim(V) + \dim(V^{\perp\sigma}) \leq \dim(H).$$

Hence (H, h) is semi-stable. So by Proposition 2.4.2, h is semi-stable with respect to the action of $\mathrm{SL}(H)$.

Conversely, suppose that h is semi-stable, then by Proposition 2.4.2, (H, h) is also semi-stable. Let F be a σ -isotropic subbundle of E , $V = H^0(F(\nu))$ and $V' = H^0(F^{\perp\sigma}(\nu))$. We have $V' \subset V^{\perp\sigma}$. Indeed, we have a commutative diagram

$$\begin{array}{ccccccc} V' & \longrightarrow & H^0(E(\nu)) & \xrightarrow{\sim} & H^0(\sigma^*E^*(\nu)) & \longrightarrow & H^* \otimes W \\ & & \searrow & & \downarrow & & \downarrow \\ & & & & H^0(\sigma^*F^*(\nu)) & \longrightarrow & V^* \otimes W. \end{array}$$

Since F is totally σ -isotropic, the composition $V' \rightarrow V^* \otimes W$ is identically zero. Hence $V' \subset V^{\perp\sigma}$. Since we have also $V \subset V'$, we deduce that V is totally σ -isotropic subspace of H . So we get

$$\dim(V) + \dim(V') \leq \dim(V) + \dim(V^{\perp\sigma}) \leq \dim(H).$$

It follows

$$\begin{aligned} \dim(V) + \dim(V') &= \deg(F) + \mathrm{rk}(F)\nu + \deg(F^{\perp\sigma}) + \mathrm{rk}(F^{\perp\sigma})\nu + r(1 - g_X) \\ &= 2\deg(F) + r\nu + r(1 - g_X) \\ &\leq r\nu + r(1 - g_X) = \dim(H). \end{aligned}$$

Hence $\deg(F) \leq 0$. This proves that E is semi-stable. \square

Now, let $i \geq 0$ and denote by $H_i = H \otimes H^0(\mathcal{O}_X(i))$, $W_i = H^0(\mathcal{O}_X(2\nu + 2i))$. For a σ -quadratic module (H, h) , we denote by (H_i, h_i) the σ -quadratic module obtained as follows: taking the tensor product with $\mathcal{O}(i)$ we obtain

$$H \otimes \mathcal{O}(i) \longrightarrow H^* \otimes \mathcal{O}(i) \otimes W \longrightarrow H^* \otimes \mathcal{O}(i) \otimes W \otimes H^0(\mathcal{O}(i))^* \otimes H^0(\mathcal{O}(i)).$$

Then at the level of global sections we deduce

$$H_i \longrightarrow H_i^* \otimes W \otimes H^0(\mathcal{O}(i))^2 \longrightarrow H_i \otimes W_i,$$

and the composition is denoted h_i .

Let $Z \subset P(H, W)^\sigma$ be the locus of σ -quadratic forms \bar{h} such that

$$\mathrm{rk}(h_i) \leq r(\nu + i - g_X + 1), \quad \forall i \geq 0.$$

It is clear that Z contains the image of $\mathfrak{H}(\mathbb{C})$. Moreover we have the following

Theorem 2.4.11. *Let $\mathcal{Q}^\sigma \subset Z$ be the open of semi-stable points, then \mathcal{Q}^σ represents the functor Quot^σ .*

Proof. We need to prove that \mathfrak{H} induces an isomorphism of functor between Quot^σ and the functor of points of \mathcal{Q}^σ . The main point is to show this for the \mathbb{C} valued points. By Proposition 2.4.10, we deduce that the image of $\mathfrak{H}(\mathbb{C})$ is contained in $\mathcal{Q}^\sigma(\mathbb{C})$. Giving a point $\bar{h} \in \mathcal{Q}^\sigma$, fix a representative h of \bar{h} . Taking the tensor product with $\mathcal{O}_X(-\nu)$ gives

$$H \otimes \mathcal{O}_X(-\nu) \xrightarrow{h} H^* \otimes W \otimes \mathcal{O}_X(-\nu) \xrightarrow{ev} H^* \otimes \mathcal{O}_X(\nu).$$

Let $F = \text{Ker}(ev \circ h)$ and $E = H \otimes \mathcal{O}_X(-\nu)/F$. E doesn't depend on the chosen representative of \bar{h} and we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & H \otimes \mathcal{O}_X(-\nu) & \longrightarrow & E & \longrightarrow & 0 \\ & & & & \downarrow ev \circ h & & & & \\ 0 & \longrightarrow & \sigma^* E^* & \longrightarrow & H^* \otimes \mathcal{O}_X(\nu) & \xrightarrow{p} & \sigma^* F^* & \longrightarrow & 0. \end{array}$$

By definition, $ev \circ h$ vanishes over F , hence it factorizes through E giving an injective map $f : E \rightarrow H^* \otimes \mathcal{O}_X(\nu)$, since h is σ -symmetric, we deduce that $p \circ ev \circ h = \sigma^*({}^t h \circ {}^t ev \circ {}^t p) = 0$, so the map f gives a σ -symmetric morphism $\psi : \sigma^* E \rightarrow E^*$, which is clearly injective.

Let s be the rank of E and d its degree. By what we have just said we deduce $d \leq 0$. From the condition defining Z , we deduce that for all i

$$d + s(\nu + i + 1 - g_X) \leq \text{rk}(q_i) \leq r(\nu + i + 1 - g_X),$$

so in particular we deduce that $r \geq s$. But since q is semi-stable, the map $q : H \rightarrow H^* \otimes W$ is injective, hence $H^0(F(\nu)) = 0$. Thus the map $H \rightarrow H^0(E(\nu))$ is injective and we deduce

$$r(\nu + 1 - g_X) \leq d + s(\nu + 1 - g_X),$$

hence $d \geq 0$, thus $d = 0$. It follows that $s \geq r$, and so $r = s$. Hence ψ is surjective, thus (E, ψ) is a σ -symmetric vector bundles.

Using the universal family over \mathcal{Q}^σ , one can make the above construction functorial which gives an inverse to \mathfrak{H} . \square

Consider the functor

$$\text{Bun}_X^{\sigma,+}(r) : (\text{algebraic varieties}) \longrightarrow (\text{sets}),$$

that associates to a variety T the set of isomorphism classes of families (\mathcal{E}, ψ) of rank r σ -symmetric anti-invariant vector bundles over X parameterized by T , such that \mathcal{E}_t is semi-stable for all $t \in T$.

Theorem 2.4.12. *Consider the good quotient $\mathcal{M}_X^{\sigma,+}(r) = \text{Quot}^\sigma(\mathbb{C}) // \text{SL}(H)$. Then $\mathcal{M}_X^{\sigma,+}(r)$ is a coarse moduli space for the functor $\text{Bun}_X^{\sigma,+}(r)$, which is a projective variety, and its underlying set consists of S -equivalence classes of semi-stable σ -symmetric anti-invariant vector bundles.*

Proof. Consider a family (\mathcal{E}, ψ) of σ -symmetric semi-stable bundles parameterized by a variety T , then for ν big enough, $p_{2*}\mathcal{E}(\nu)$ and $p_{2*}(\sigma^*\mathcal{E}^*(\nu))$ are locally free, so by choosing local trivializations, we deduce a unique, up to an action of $\text{SL}(H)$, map to \mathcal{Q}^σ . Thus we get a morphism $T \rightarrow \mathcal{M}_X^{\sigma,+}(r)$. This is obviously functorial in T .

A point $a \in \mathcal{M}_X^{\sigma,+}(r)$, corresponds by \mathfrak{H} to a point of $\mathcal{Q}^\sigma // \text{SL}(H)$, this transformation respects the graded gr. Hence, using Proposition 2.4.3, we deduce that a represents an S -equivalence class of semi-stable σ -symmetric vector bundles. \square

σ -alternating case

The construction of the moduli space $\mathcal{M}_X^{\sigma,-}(r)$ of semi-stable σ -alternating vector bundles follows the same method as the σ -symmetric case, using σ -alternating modules rather than quadratic ones. A module $q : H \rightarrow H^* \otimes W$ is σ -alternating if

$$q(x)(y) = -\sigma(q(y)(x)).$$

Similar results about semistability, filtrations and S -equivalence of σ -alternating forms can be checked in this case too. We omit the details.

By Proposition 2.4.5 we have canonical forgetful maps

$$\mathcal{M}_X^{\sigma,+}(r) \rightarrow \bar{\mathcal{U}}_X(r, 0),$$

$$\mathcal{M}_X^{\sigma,-}(r) \rightarrow \bar{\mathcal{U}}_X(r, 0),$$

where $\bar{\mathcal{U}}_X(r, 0)$ is the moduli space of semi-stable vector bundles of rank r and degree 0 over X . The images of these maps are obviously $\bar{\mathcal{U}}_X^{\sigma,\pm}(r)$. A natural question arises: what are the degrees of these maps?

Remark 2.4.13. Note that the involution $E \rightarrow \sigma^*E^*$ is well defined on $\bar{\mathcal{U}}_X(r, 0)$, since we have $\text{gr}(\sigma^*E^*) = \sigma^*(\text{gr}(E))^*$.

Proposition 2.4.14. *The forgetful maps $\mathcal{M}_X^{\sigma,+}(r) \rightarrow \bar{\mathcal{U}}_X^{\sigma,+}(r)$ and $\mathcal{M}_X^{\sigma,-}(r) \rightarrow \bar{\mathcal{U}}_X^{\sigma,-}(r)$ are injective. In particular they are bijective.*

Proof. We treat the σ -symmetric case. Let (E, ψ) be a σ -symmetric vector bundle, suppose that E is stable, so $\text{Aut}_{\text{GL}_r}(E) = \mathbb{C}^*$ and $\psi : \sigma^*E \rightarrow E^*$ is unique up to scalar multiplication. The action of $\text{Aut}_{\text{GL}_r}(E)$ on these σ -symmetric forms is given by

$$f \cdot \psi = ({}^t f)\psi(\sigma^* f).$$

If $f = \xi \text{Id}_E$, with $\xi \in \mathbb{C}^*$, then this action is simply given by $\psi \rightarrow \xi^2 \psi$. It follows that this action is transitive, hence (E, ψ) and $(E, \lambda \psi)$ are isomorphic as σ -symmetric vector bundles.

If E is strictly semi-stable, using the decomposition of such polystable anti-invariant vector bundles given at the end of section 2.1, we can assume that E is of the form $F^{\oplus d}$ or $(G \oplus \sigma^*G^*)^{\oplus d}$ for stable anti-invariant vector bundle F and stable non-anti-invariant vector bundle G . Now, the set of σ -symmetric isomorphisms $\psi : \sigma^*E \rightarrow E^*$ is equal to the locus of symmetric matrices of $\text{GL}_d(\mathbb{C})$ in both cases. Hence it is sufficient to use the fact that non-degenerated symmetric matrices can be decomposed in the form ${}^t M \times M$. This shows that all the σ -symmetric isomorphisms on E define the same point in $\mathcal{M}_X^{\sigma,+}(r)$. \square

The case of vector bundles with trivial determinant is slightly different. For simplicity we consider the forgetful maps just on the stable loci

$$\mathcal{SM}_X^{\sigma,\pm,s}(r) \rightarrow \mathcal{SU}_X^{\sigma,\pm}(r).$$

Here $\mathcal{SM}_X^{\sigma,\pm,s}(r)$ is the locus of stable σ -symmetric or σ -alternating vector bundles in the moduli space $\mathcal{SM}_X^{\sigma,\pm}(r)$.

Proposition 2.4.15. *We have two cases:*

- (1) *If r is odd, then the forgetful map $\mathcal{SM}_X^{\sigma,+}(r) \rightarrow \mathcal{SU}_X^{\sigma,+}(r)$ is injective.*
- (2) *If r is even, the forgetful map $\mathcal{SM}_X^{\sigma,\pm}(r) \rightarrow \mathcal{SU}_X^{\sigma,\pm}(r)$ is of degree 2.*

Proof. Let (E, ψ) be a σ -symmetric vector bundle with a trivialization of its determinant. Suppose that E is stable. As $\text{Aut}_{\text{GL}_r}(E) = \mathbb{C}^*$, we see that $\text{Aut}_{\text{SL}_r}(E) = \mu_r$, where μ_r is the group of r^{th} roots of unity. Remark that the map $\mu_r \rightarrow \mu_r$, given by $\xi \mapsto \xi^2$ is a bijection if r is odd, and it is two-to-one on its image if r is even.

- (1) If r is odd, since E is stable, $\psi : \sigma^*E \rightarrow E^*$ is unique up to scalar multiplication, as $\det(\psi) = 1$, the number of such isomorphisms is exactly r . The action of $\text{Aut}_{\text{SL}_r}(E)$ on these r σ -symmetric forms is given by

$$f \cdot \psi = ({}^t f)\psi(\sigma^* f).$$

As $f = \xi \text{Id}_E$, for $\xi \in \mu_r$, then we conclude as in the proof of Proposition 2.4.14 that this action is transitive.

- (2) Assume r is even, with the same argument as above, we see that the action has two different orbits. So E admits two non equivalent σ -symmetric forms. The same argument applies for the σ -alternating case. □

Remark 2.4.16. Note that the above Proposition is similar to the situation of forgetful map of orthogonal bundles. See [Ser08].

2.5 Tangent space and dimensions

The tangent space to the moduli space $\mathcal{U}_X(r, 0)$ at a smooth point E is canonically given by

$$\text{T}_E \mathcal{S}\mathcal{U}_X(r) \cong H^1(X, \text{End}(E)),$$

where $\text{End}(E) \cong E^* \otimes E$ stands for the sheaf of endomorphisms of E .

We want to identify the tangent spaces to $\mathcal{U}_X^{\sigma, \pm}(r)$ at a point E . Before that recall that a deformation of E over $\text{Spec}(\mathbb{C}[\varepsilon])$ ($\varepsilon^2 = 0$) is defined to be a locally free coherent sheaf \mathcal{E} on $X_\varepsilon = X \times \text{Spec}(\mathbb{C}[\varepsilon])$ together with a homomorphism $\mathcal{E} \rightarrow E$ of $\mathcal{O}_{X_\varepsilon}$ -module, such that the induced map $\mathcal{E} \otimes \mathcal{O}_X \rightarrow E$ is an isomorphism. Canonically, the set of deformations of E over $\text{Spec}(\mathbb{C}[\varepsilon])$ is isomorphic to $H^1(X, \text{End}(E))$. As by definition, a deformation is locally free, so it is flat, thus taking the tensor product of the exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{\varepsilon} \mathcal{O}_{X_\varepsilon} \rightarrow \mathcal{O}_X \rightarrow 0$$

with \mathcal{E} we get

$$0 \rightarrow E \xrightarrow{\varepsilon} \mathcal{E} \rightarrow E \rightarrow 0.$$

Let E be a σ -symmetric anti-invariant vector bundle and $\psi : \sigma^*E \cong E^*$. Suppose that E is given by the transition functions $f_{ij} = \varphi_i \circ \varphi_j^{-1} : U_{ij} \rightarrow \text{GL}_r$, where the $\varphi_i : E_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}^r$ are local trivializations of E . The covering $\{U_i\}_i$ of X is chosen to be σ -invariant, i.e. $\sigma(U_i) = U_i$ (to get such covering, just pullback a covering of Y that trivializes both $\pi_* \mathcal{O}_X$ and $\pi_* E$ over Y). Note that we can choose $\{\varphi_i\}$ such that the diagram

$$\begin{array}{ccc} \sigma^* \varphi_i : \sigma^* E_{U_i} & \longrightarrow & U_i \times \mathbb{C}^r \\ \psi \downarrow & & \downarrow \sigma \times I_r \\ {}^t \varphi_i^{-1} : E_{U_i}^* & \longrightarrow & U_i \times \mathbb{C}^r \end{array}$$

commutes. Indeed, by taking an *étale* neighborhood U of each point $x \in X$, such that $\sigma(U) = U$, we can construct a frame (e_1, \dots, e_r) of $E|_U$ on which the pairing $\tilde{\psi} : E \otimes \sigma^* E \rightarrow \mathcal{O}_X$ is represented by the trivial matrix I_r . To construct such a frame, we apply the Gram-Schmidt process. As in this procedure, we need to calculate some square roots, that's the reason why we have to work on the *étale* topology. Moreover, we should mention that

if we start with a frame (u_1, \dots, u_r) near x , it may happen that $\tilde{\psi}(u_i \otimes \sigma^* u_i)_x = 0$, in this case, we just replace u_i with $u_i + u_j$, for some $j > i$ such that $\tilde{\psi}((u_i + u_j) \otimes \sigma^*(u_i + u_j))_x \neq 0$. Taking such trivializations, we get transition functions f_{ij} such that $\sigma^* f_{ij} = {}^t f_{ij}^{-1}$. We know that the extension \mathcal{E} (which corresponds to some $\eta = \{\eta_{ij}\}_{ij} \in H^1(X, E \otimes E^*)$) is given by transition functions of the form

$$f_{ij} + \varepsilon g_{ij} : U_{ij} \rightarrow \mathrm{GL}_r(\mathbb{C}[\varepsilon]).$$

We want to find the relation between these transition functions and η . First of all, in order that $\{f_{ij} + \varepsilon g_{ij}\}_{ij}$ represents a 1-cocycle, we must have the two conditions

$$\begin{cases} g_{ii} = 0 \\ g_{ij} f_{ji} + f_{ij} g_{jk} f_{ki} + f_{ik} g_{ki} = 0 \end{cases}.$$

Now let $\eta = \{\eta_{ij}\}_{ij} \in H^1(X, E \otimes E^*)$, which verifies $\eta_{ii} = 0$ and

$$\eta_{ij} + \eta_{jk} + \eta_{ki} = 0.$$

Each η_{ij} can be seen as local morphism

$$\begin{array}{ccc} \eta_{ij} : E|_{U_{ij}} & \xrightarrow{\quad} & E|_{U_{ij}} \\ & \searrow \varphi_j & \swarrow \varphi_i \\ & U_{ij} \times \mathbb{C}^r & \end{array}$$

Denote by $g_{ij} = \varphi_i \circ \eta_{ij} \circ \varphi_j^{-1}$, we can rewrite the above conditions on η in the form

$$g_{ii} = 0,$$

$$\varphi_i^{-1} \circ g_{ij} \circ \varphi_j + \varphi_j^{-1} \circ g_{jk} \circ \varphi_k + \varphi_k^{-1} \circ g_{ki} \circ \varphi_i = 0.$$

Composing by φ_i from the left and φ_i^{-1} from the right, we get

$$\begin{aligned} g_{ij} \circ \varphi_j \circ \varphi_i^{-1} + \varphi_i \circ \varphi_j^{-1} \circ g_{jk} \circ \varphi_k \circ \varphi_i^{-1} + \varphi_i \circ \varphi_k^{-1} \circ g_{ki} &= 0 \\ \Leftrightarrow g_{ij} f_{ji} + f_{ij} g_{jk} f_{ki} + f_{ik} g_{ki} &= 0. \end{aligned}$$

Lemma 2.5.1. $f_{ij} + \varepsilon g_{ij} = \varphi_i \circ (id + \varepsilon \eta_{ij}) \circ \varphi_j^{-1}$ are transition functions of \mathcal{E} .

Proof. Let $\eta = \{\eta_{ij}\}_{ij} \in H^1(X, E \otimes E^*)$, locally the extension \mathcal{E} is trivial, that's

$$\mathcal{E}|_{U_{i\varepsilon}} \cong E|_{U_i} \oplus \varepsilon E|_{U_i}, \quad x \mapsto (\varpi(x), x - s_i \circ \varpi(x)),$$

where $\varpi : \mathcal{E} \rightarrow E$ and s_i is a local section of ϖ on the local open set U_i , and $U_{i\varepsilon} = U_i \times \mathrm{Spec}(\mathbb{C}[\varepsilon])$. This isomorphism is $\mathcal{O}_{X_\varepsilon}$ -linear.

Composing with the trivialization

$$\varphi_i + \varepsilon \varphi_i : E|_{U_i} \oplus \varepsilon E|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i} \oplus \varepsilon \mathcal{O}_{U_i},$$

we get a trivialization

$$\phi_i : \mathcal{E}|_{U_{i\varepsilon}} \xrightarrow{\sim} \mathcal{O}_{U_{i\varepsilon}},$$

given by

$$\phi_i = \varphi_i \circ \varpi + \varepsilon \varphi_i (id - s_i \circ \varpi).$$

Remark that

$$\phi_i^{-1} = s_i \circ \varphi_i^{-1} - \varepsilon s_i(id - s_i \circ \varpi) s_i \circ \varphi_i^{-1}.$$

So, we calculate the transition functions of \mathcal{E}

$$\begin{aligned} \phi_i \circ \phi_j^{-1} &= (\varphi_i \circ \varpi + \varepsilon \varphi_i(id - s_i \circ \varpi))(s_j \circ \varphi_j^{-1} - \varepsilon s_j(id - s_j \circ \varpi) s_j \circ \varphi_j^{-1}) \quad (\text{because } \varepsilon^2 = 0) \\ &= f_{ij} + \varepsilon(\varphi_i(id - s_i \circ \varpi) s_j \circ \varphi_j^{-1} - \varphi_i \circ \varpi \circ s_j(id - s_j \circ \varpi) s_j \circ \varphi_j^{-1}) \\ &= f_{ij} + \varepsilon(\varphi_i(s_j - s_i) \varphi_j^{-1}) \\ &= f_{ij} + \varepsilon(\varphi_i \eta_{ij} \varphi_j^{-1}) \\ &= f_{ij} + \varepsilon g_{ij}. \end{aligned}$$

□

Now η is in the tangent space to $\mathcal{U}_X^{\sigma,+}(r)$ at E if and only if the corresponding extension \mathcal{E} is σ -symmetric anti-invariant vector bundle on X_ε , where σ extended to an involution on X_ε by taking $\sigma(\varepsilon) = \varepsilon$.

On the transition functions, this means that

$$\sigma^*(f_{ij} + \varepsilon g_{ij}) = {}^t(f_{ij} + \varepsilon g_{ij})^{-1},$$

which gives¹

$$\sigma^* f_{ij} = {}^t f_{ij}^{-1},$$

and

$$\begin{aligned} \sigma^* g_{ij} &= -{}^t f_{ij}^{-1} {}^t g_{ij} {}^t f_{ij}^{-1} \\ \Leftrightarrow \sigma^* \varphi_i \circ \sigma^* \eta_{ij} \circ \sigma^* \varphi_j^{-1} &= -{}^t \varphi_i^{-1} \circ {}^t \eta_{ij} \circ {}^t \varphi_j \\ \Leftrightarrow \sigma^* \varphi_i \circ \sigma^* \eta_{ij} \circ \sigma^* \varphi_j^{-1} &= -\sigma^* \varphi_i \circ \psi^{-1} \circ {}^t \eta_{ij} \circ \psi \circ \sigma^* \varphi_j^{-1} \\ \Leftrightarrow \sigma^* \eta_{ij} &= -\psi^{-1} \circ {}^t \eta_{ij} \circ \psi \\ \Leftrightarrow \sigma^*(\eta_{ij} \circ {}^t \psi^{-1}) &= -\psi^{-1} \circ {}^t \eta_{ij} = -{}^t(\eta_{ij} \circ {}^t \psi^{-1}). \end{aligned}$$

Thus

$$\eta \circ {}^t \psi^{-1} \in H^1(X, E \otimes \sigma^* E)_-.$$

where $H^1(X, E \otimes \sigma^* E)_-$ is the proper subspace associated to the eigenvalue -1 of the involution of $H^1(X, E \otimes \sigma^* E)$ given by

$$\xi \rightarrow \sigma^*({}^t \xi).$$

Consider the case of $\mathcal{U}_X^{\sigma,-}(r)$. Assume that r is even and π is ramified (we will show later that $\mathcal{U}_X^{\sigma,+}(r) \cong \mathcal{U}_X^{\sigma,-}(r)$ in the étale case). Fix a point E of $\mathcal{U}_X^{\sigma,-}(r)$. In this case, $\tilde{\psi}$ can be represented with respect to some frame by the matrix

$$J_r = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}.$$

Such frame gives a set of trivializations $\{\varphi_i\}_i$ such that

$$(\sigma \times J_r) \circ \sigma^* \varphi_i = {}^t \varphi_i^{-1} \circ \sigma^* \psi.$$

¹Recall that $(f + \varepsilon g)^{-1} = f^{-1} - \varepsilon f^{-1} g f^{-1}$ in $\text{GL}_r(\mathbb{C}[\varepsilon])$, and $\det(f + \varepsilon g) = \det(f)(1 + \varepsilon \text{Tr}(f^{-1} g))$.

So the associated transition functions $\{f_{ij}\}$ verify

$$\sigma^* f_{ij} = -J_r {}^t f_{ij}^{-1} J_r.$$

It follows that the deformation \mathcal{E} is in the tangent space $T_E \mathcal{U}_X^{\sigma, -}(r)$ if and only if we have

$$\begin{aligned} \sigma^*(f_{ij} + \varepsilon g_{ij}) &= -J_r {}^t (f_{ij} + \varepsilon g_{ij})^{-1} J_r \\ &= -J_r {}^t f_{ij}^{-1} J_r + \varepsilon J_r {}^t f_{ij}^{-1} {}^t g_{ij} {}^t f_{ij}^{-1} J_r. \end{aligned}$$

Thus

$$\begin{aligned} \sigma^* \varphi_i \circ \sigma^* \eta_{ij} \circ \sigma^* \varphi_j^{-1} &= J_r {}^t \varphi_i^{-1} \circ {}^t \eta_{ij} \circ {}^t \varphi_j J_r \\ \Leftrightarrow \sigma^* \eta_{ij} &= -\psi^{-1} \circ {}^t \eta_{ij} \circ \psi \\ \Leftrightarrow \sigma^*(\eta_{ij} \circ {}^t \psi^{-1}) &= {}^t (\eta_{ij} \circ {}^t \psi^{-1}). \end{aligned}$$

Finally

$$\eta \circ {}^t \psi^{-1} \in H^1(X, E \otimes \sigma^* E)_+.$$

We have showed so far

Theorem 2.5.2. *With the above notations, we have*

- (a) *The tangent space to $\mathcal{U}_X^{\sigma, +}(r)$ at a point E is isomorphic to $H^1(X, E \otimes \sigma^* E)_-$. In particular we have*

$$\begin{aligned} \dim(\mathcal{U}_X^{\sigma, +}(r)) &= \frac{r^2}{2}(g_X - 1) + \frac{nr}{2} \\ &= r^2(g_Y - 1) + n \frac{r(r+1)}{2}. \end{aligned}$$

- (b) *The tangent space to $\mathcal{U}_X^{\sigma, -}(r)$ at a point E is isomorphic to $H^1(X, E \otimes \sigma^* E)_+$. In particular we have*

$$\begin{aligned} \dim(\mathcal{U}_X^{\sigma, -}(r)) &= \frac{r^2}{2}(g_X - 1) - \frac{nr}{2} \\ &= r^2(g_Y - 1) + n \frac{r(r-1)}{2}. \end{aligned}$$

Proof. We need just to calculate the dimensions. Let E be a σ -anti-invariant stable vector bundle, denote by $F = E \otimes \sigma^* E$. First we have

$$h^1(X, F) = h_+^1 + h_-^1 = r^2(g_X - 1) + 1, \quad (2.1)$$

where we denote for simplicity $h_{\pm}^0 = h^0(X, F)_{\pm}$, $h_{\pm}^1 = h^1(X, F)_{\pm}$.

Let $\varsigma : \sigma^* F \rightarrow F$ be the canonical linearization which equals to the transposition $(\sigma^*(s \otimes \sigma^* t) \rightarrow t \otimes \sigma^* s)$.

Applying Lefschetz fixed point formula (see Appendix F, also [AB68]), we obtain

$$h_+^1 - h_-^1 = h_+^0 - h_-^0 - \sum_{p \in R} \frac{\text{Tr}(\varsigma_p)}{\det(id - d_p \sigma)}.$$

It is clear that $d_p \sigma : T_p X \rightarrow T_p X$ is equal to $-id$ (see Lemma 1.1.2), and the trace of the involution $\varsigma_p : F_p \rightarrow F_p$ is equal to

$$\dim(F_p)_+ - \dim(F_p)_-.$$

But, $F_p = E_p \otimes E_p = \text{Sym}^2 E_p \oplus \bigwedge^2 E_p$, and $h_+^0 = 1$ if ψ is σ -symmetric, $h_-^0 = 1$ if ψ is σ -alternating. Hence

$$\begin{aligned} h_+^1 - h_-^1 &= -\frac{1}{2} \left(\sum_{p \in R} \frac{r(r+1)}{2} - \frac{r(r-1)}{2} \right) + 1 \\ &= -nr + 1 \quad \text{if } \psi \text{ is } \sigma\text{-symmetric.} \\ h_+^1 - h_-^1 &= -\frac{1}{2} \left(\sum_{p \in R} \frac{r(r+1)}{2} - \frac{r(r-1)}{2} \right) - 1 \\ &= -nr - 1 \quad \text{if } \psi \text{ is } \sigma\text{-alternating.} \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we deduce

$$\begin{aligned} h_-^1 &= \frac{r^2}{2}(g_X - 1) + \frac{nr}{2} \quad \text{if } \psi \text{ est } \sigma\text{-symmetric.} \\ h_+^1 &= \frac{r^2}{2}(g_X - 1) - \frac{nr}{2} \quad \text{if } \psi \text{ est } \sigma\text{-alternating.} \end{aligned}$$

The other equalities are consequences of Hurwitz formula. \square

In particular, one deduces

$$\begin{aligned} \dim(\mathcal{SU}_X^{\sigma,+}(r)) &= (r^2 - 1)(g_Y - 1) + n \frac{(r+2)(r-1)}{2}, \\ \dim(\mathcal{SU}_X^{\sigma,-}(r)) &= (r^2 - 1)(g_Y - 1) + n \frac{(r+1)(r-2)}{2}. \end{aligned}$$

Remark 2.5.3. Another method to compute the dimensions is to consider the map

$$H^1(X, E \otimes E^*) \rightarrow H^1(X, \mathcal{O}_X).$$

This map is not equivariant with respect to the action of σ . In fact, the image by this map of $H^1(E \otimes E^*)_-$ when E is σ -symmetric (resp. $H^1(E \otimes E^*)_+$ when E is σ -alternating) is always included in $H^1(X, \mathcal{O}_X)_-$ (with respect to the canonical linearization on \mathcal{O}_X).

Chapter 3

Hitchin systems

Hitchin in [Hit87] has defined and studied some integrable systems related to the moduli space of stable G -bundles over X , where $G = \mathrm{GL}_r, \mathrm{Sp}_{2m}$ and SO_r . Let $\mathcal{M}_X(G)$ be this moduli space, the tangent space to $\mathcal{M}_X(G)$ at a point $[E]$ can be identified with

$$H^1(X, \mathrm{Ad}(E)) \cong H^0(X, \mathrm{Ad}(E) \otimes K_X)^*,$$

where $\mathrm{Ad}(E)$ is the adjoint bundle associated to E , which is a bundle of Lie algebras isomorphic to $\mathfrak{g} = \mathrm{Lie}(G)$. By Serre duality, the fiber of the cotangent bundle is $H^0(X, \mathrm{Ad}(E) \otimes K_X)$. By considering a basis of the invariant polynomials under the adjoint action on \mathfrak{g} , one gets a map

$$T_E^* \mathcal{M}_X(G) = H^0(X, \mathrm{Ad}(E) \otimes K_X) \longrightarrow \bigoplus_{i=1}^k H^0(X, K_X^{d_i}),$$

where the $(d_i)_i$ are the degrees of these invariant polynomials. Hitchin has shown that these two spaces have the same dimension.

In the case $G = \mathrm{GL}_r$, a basis of the invariant polynomials is given by the coefficients of the characteristic polynomial. If E is a stable vector bundle, then this gives rise to a map

$$\mathcal{H}_E : H^0(X, \mathrm{End}(E) \otimes K_X) \longrightarrow \bigoplus_{i=1}^r H^0(X, K_X^i) =: W,$$

which associates to each Higgs field ϕ , the coefficients of its characteristic polynomial. The associated map

$$\mathcal{H} : T^* \mathcal{M}_X(\mathrm{GL}_r) \longrightarrow W$$

is called the *Hitchin morphism*. By choosing a basis of W , \mathcal{H} is represented by $d = r^2(g_X - 1) + 1$ functions f_1, \dots, f_d . Hitchin has proved that this system is *algebraically completely integrable*, i.e. its generic fiber is an open set in an abelian variety of dimension d , and the vector fields $\mathcal{X}_{f_1}, \dots, \mathcal{X}_{f_d}$ associated to f_1, \dots, f_d (defined using the canonical 2-form on $T^* \mathcal{M}_X(\mathrm{GL}_r)$) are linear.

Moreover, let $\mathcal{U}_X(r, 0)$ be the moduli space of stable vector bundles of rank r and degree 0 on X . Consider the map

$$\Pi : T^* \mathcal{U}_X(r, 0) \rightarrow \mathcal{U}_X(r, 0) \times W$$

whose first factor is the canonical projection and the second factor is \mathcal{H} . Then it is proved in [BNR89] that Π is dominant.

The main topic of this chapter is the study of the Hitchin systems for the anti-invariant and the invariant loci. We use these systems to identify the connected components of $\mathcal{U}_X^{\sigma,+}(r)$ and $\mathcal{U}_X^{\sigma,-}(r)$. The irreducibility of the invariant locus (of a fixed type) is already

know in more general setting (see [BS14]).

We stress that in this chapter we always assume, unless otherwise stated, that the vector bundles are stable.

3.1 Generalities on spectral curves and Hitchin systems

In this section we recall the general theory of spectral curves. Our main reference is [BNR89].

Let L be any line bundle over a smooth projective curve X . Consider the ruled surface over X given by

$$\bar{q} : \mathbb{S} = \mathbb{P}(\mathcal{O}_X \oplus L^{-1}) \rightarrow X,$$

where for a vector bundle \mathcal{E} we denote $\text{Sym}^\bullet(\mathcal{E})$ the symmetric algebra and

$$\mathbb{P}(\mathcal{E}) = \text{Proj}(\text{Sym}^\bullet(\mathcal{E})).$$

Hence a point in \mathbb{S} lying over $x \in X$ corresponds to a hyperplane in the fiber $(\mathcal{O}_X \oplus L^{-1})_x$. It follows that the total space of L denoted $|L|$ is contained in \mathbb{S} .

Let $\mathcal{O}(1)$ be the relatively ample line bundle over \mathbb{S} . It is well known that $\bar{q}_*\mathcal{O}(1) \cong \mathcal{O}_X \oplus L^{-1}$. Hence $\mathcal{O}(1)$ has a canonical section, denoted by y , corresponding to the direct summand \mathcal{O}_X . Also by the projection formula $\bar{q}_*(\bar{q}^*L \otimes \mathcal{O}(1))$ is isomorphic to $L \oplus \mathcal{O}_X$, so it has also a canonical section which we denote by x .

Let

$$s = (s_1, \dots, s_r) \in \bigoplus_{i=1}^r H^0(X, L^i) =: W_L$$

be an r -tuple of global sections of L^i and consider the global section

$$x^r + (\bar{q}^*s_1)yx^{r-1} + \dots + (\bar{q}^*s_r)y^r \in H^0(\mathbb{S}, \bar{q}^*K_X^r \otimes \mathcal{O}(r)). \quad (3.1)$$

We denote by \tilde{X}_s its zero scheme which is a curve. We say that \tilde{X}_s is the *spectral curve* associated to $s \in W_L$. Denote $q : \tilde{X}_s \rightarrow X$ the restriction of \bar{q} to \tilde{X}_s . It is clear that \tilde{X}_s is finite cover of degree r of X and its fiber over $p \in X$ is given by the homogeneous equation in \mathbb{P}^1

$$x^r + s_1(p)x^{r-1}y \dots + s_r(p)y^r = 0.$$

Lemma 3.1.1. *The set of elements $s \in W_L$ corresponding to smooth spectral curves \tilde{X}_s is open. In particular it is dense whenever it is not empty.*

Proof. Assume that \tilde{X}_s is integral (i.e. reduced and irreducible, which is true for general $s \in W$, see [BNR89] Remark 3.1) and let

$$P(x, t) = x^r + s_1(t)x^{r-1} + \dots + s_r(t) = 0$$

be the equation of \tilde{X}_s locally over a point $p \in X$, where t is a local parameter near p . Then, by the Jacobian criterion of smoothness, \tilde{X}_s is singular at a point $\lambda \in \tilde{X}_s$ over p if and only if

$$\frac{\partial P}{\partial x}(\lambda, 0) = \frac{\partial P}{\partial t}(\lambda, 0) = 0,$$

i.e.

$$\begin{aligned} r\lambda^{r-1} + (r-1)s_1(0)\lambda^{r-2} + \dots + s_{r-1}(0) &= 0, \\ s'_1(0)\lambda^{r-1} + s'_2(0)\lambda^{r-2} + \dots + s'_r(0) &= 0. \end{aligned}$$

Clearly these two equations give a closed condition on $s = (s_1, \dots, s_r) \in W_L$. Hence the set of $s \in W_L$ corresponding to smooth curves \tilde{X}_s is open. \square

Remark 3.1.2. We remark that the criterion of smoothness given in [BNR89] Remark 3.5, is not correct. In fact the criterion assumes that the singular point is located at $\lambda = 0$.

Remark 3.1.3. An alternative way to construction \tilde{X}_s is as follows: consider the symmetric \mathcal{O}_X -algebra $\mathrm{Sym}^\bullet(L^{-1})$. Define the ideal

$$\mathfrak{J} = \left\langle \bigoplus_i s_i(L^{-r}) \right\rangle \subset \mathrm{Sym}^\bullet(L^{-1}),$$

where $s_i \in H^0(X, L^i)$ is seen here as an embedding $s_i : L^{-r} \rightarrow L^{-r+i}$. Then \tilde{X}_s can be defined as $\mathrm{Spec}(\mathrm{Sym}^\bullet(L^{-1})/\mathfrak{J})$.

Suppose that \tilde{X}_s is smooth and let $\tilde{S} = \mathrm{Ram}(\tilde{X}_s/X) \subset \tilde{X}_s$ be the ramification divisor of $q : \tilde{X}_s \rightarrow X$.

Recall that

$$q_*\mathcal{O}_{\tilde{X}_s} \cong \mathcal{O}_X \oplus L^{-1} \oplus \dots \oplus L^{-(r-1)},$$

hence, by duality of finite flat morphisms (see e.g. [Har77] Ex III.6.10)

$$q_*\left(\mathcal{O}_{\tilde{X}_s}(\tilde{S})\right) \cong \left(q_*\mathcal{O}_{\tilde{X}_s}\right)^* \cong \mathcal{O}_X \oplus L \oplus \dots \oplus L^{r-1}.$$

In particular, using the fact that for any line bundle M over \tilde{X}_s

$$\det(q_*M) = \det(q_*\mathcal{O}_{\tilde{X}_s}) \otimes \mathrm{Nm}_{\tilde{X}_s/X}(M),$$

where $\mathrm{Nm}_{\tilde{X}_s/X} : \mathrm{Pic}(\tilde{X}_s) \rightarrow \mathrm{Pic}(X)$ is the norm map, we deduce

$$\deg(\tilde{S}) = r(r-1)\deg(L).$$

Furthermore, by Hurwitz formula, we have $K_{\tilde{X}_s} = q^*K_X(\tilde{S})$. Thus, by the projection formula we get

$$q_*K_{\tilde{X}_s} \cong K_X \oplus K_X L \oplus \dots \oplus K_X L^{r-1}.$$

It follows that the genus $g_{\tilde{X}_s}$ of \tilde{X}_s is

$$g_{\tilde{X}_s} = \deg(L) \frac{r(r-1)}{2} + r(g_X - 1) + 1.$$

Recall that for a stable vector bundle E , the Hitchin map

$$\mathcal{H}_E : H^0(X, E \otimes E^* \otimes L) \rightarrow W_L$$

is defined by

$$s \longrightarrow \mathcal{H}_E(s) = \left((-1)^i \mathrm{Tr} \left(\bigwedge^i s \right) \right)_i,$$

where Tr is the trace map.

We recall a very important result from [BNR89].

Proposition 3.1.4. *Let \tilde{X}_s be an integral (resp. smooth) spectral curve over X associated to $s \in W_L$. Then there is a one-to-one correspondence between torsion-free $\mathcal{O}_{\tilde{X}_s}$ -modules of rank 1 (resp. $\mathrm{Pic}(\tilde{X}_s)$) and the isomorphism classes of pairs (E, ϕ) where E is a rank r vector bundle and $\phi : E \rightarrow E \otimes L$ is a morphism such that $\mathcal{H}_E(\phi) = s$*

Maybe the most important case of spectral curves is when $L = K_X$. We denote simply by W the space W_{K_X} . In this case, the genus $g_{\tilde{X}_s}$ of \tilde{X}_s is $g_{\tilde{X}_s} = r^2(g_X - 1) + 1$, which coincides with the dimension of the moduli space $\mathcal{U}_X(r, 0)$ of stable vector bundles of rank r and degree 0 over X . In [BNR89] it is proved that the map

$$\Pi : T^*\mathcal{U}_X(r, 0) \rightarrow \mathcal{U}_X(r, 0) \times W$$

is dominant. Moreover, the fiber $\mathcal{H}^{-1}(s)$ of a general point $s \in W$ is isomorphic to an open subset of $\text{Pic}^m(\tilde{X}_s)$, where $m = r(r-1)(g_X - 1)$. We claim that this is still true for the classical algebraic groups Sp_{2m} et SO_r . Consider the moduli spaces $\mathcal{M}_X(\text{Sp}_{2m})$ and $\mathcal{M}_X(\text{SO}_r)$ of Sp_{2m} -bundles and SO_r -bundles respectively which are *stable as vector bundles*. Define

$$W_{\text{Sp}_{2m}} = \bigoplus_{i=1}^m H^0(X, K_X^{2i}),$$

and

$$W_{\text{SO}_r} = \begin{cases} \bigoplus_{i=1}^{r/2-1} H^0(X, K_X^{2i}) \oplus H^0(X, K_X^{r/2}) & r \equiv 0 \pmod{2} \\ \bigoplus_{i=1}^{(r-1)/2} H^0(X, K_X^{2i}) & r \equiv 1 \pmod{2} \end{cases}.$$

For general $s \in W_{\text{Sp}_{2m}}$ the curve \tilde{X}_s is smooth, and for general $s \in W_{\text{SO}_r}$ the associated \tilde{X}_s is nodal curve. In this case we denote \hat{X}_s its normalisation. In both cases, the involution of the ruled surface \mathbb{S} that sends x to $-x$ induces an involution on \tilde{X}_s , we denote it by ι . Remark that in the singular case, ι lifts to an involution on \hat{X} without fixed points.

Recall that Hitchin ([Hit87]) has proved that the map Π induces maps

$$\begin{aligned} T^*\mathcal{M}_X(\text{Sp}_{2m}) &\longrightarrow \mathcal{M}_X(\text{Sp}_{2m}) \times W_{\text{Sp}_{2m}}, \\ T^*\mathcal{M}_X(\text{SO}_r) &\longrightarrow \mathcal{M}_X(\text{SO}_r) \times W_{\text{SO}_r}. \end{aligned} \tag{3.2}$$

Moreover, the generic fiber in the case of symplectic bundles is isomorphic to an open set of a translate of the Prym variety of $\tilde{X}_s \rightarrow \tilde{X}_s/\iota$. In the case of orthogonal bundles, the generic fiber is an open dense of the Prym variety of $\hat{X}_s \rightarrow \hat{X}_s/\iota$. We refer to [Hit87] for more details.

Proposition 3.1.5. *The restrictions of Π given in (3.2) are dominant. Moreover, for general $s \in W_{\text{Sp}_{2m}}$ (resp. $s \in W_{\text{SO}_r}$), if \mathcal{P} is a translation of the Prym variety of $\tilde{X}_s \rightarrow \tilde{X}_s/\iota$ (resp. $\hat{X}_s \rightarrow \hat{X}_s/\iota$), then the pushforward map*

$$\mathcal{P} \dashrightarrow \mathcal{M}_X(\text{Sp}_{2m}) \text{ (resp. } \mathcal{M}_X(\text{SO}_r))$$

is dominant.

Proof. Laumon has proved in [Lau88] that the nilpotent cone

$$\Lambda_G \subset T^*\mathcal{M}_X(G)$$

is Lagrangian, for any reductive algebraic group G . In particular, for $G = \text{Sp}_{2m}$ (resp. $G = \text{SO}_r$), we deduce that the locus of G -bundles E such that

$$\mathcal{H}_E : H^0(X, \text{Ad}(E) \otimes K_X) \rightarrow W_{\text{Sp}_{2m}} \text{ (resp. } W_{\text{SO}_r})$$

is dominant, forms an open dense subset of $\mathcal{M}_X(G)$. Indeed, we have

$$\dim(\Lambda_G) = \dim(\mathcal{M}_X(G)),$$

and the restriction of the canonical projection $T^*\mathcal{M}_X(G) \rightarrow \mathcal{M}_X(G)$ to Λ_G is surjective (because $(E, 0) \in \Lambda_G$ for any G -bundle E). Hence by dimension theorem, it follows that there exists an open dense subset of $\mathcal{M}_X(G)$ over which Λ_G is reduced to the zero section of $T^*\mathcal{M}_X(G)$. This open subset is by definition the set of very stable bundles E , for which, the map \mathcal{H}_E is dominant.

It follows that the restrictions of Π given in (3.2) are dominant maps. Hence for general $s \in W_{\mathrm{Sp}_{2m}}$ (resp. $s \in W_{\mathrm{SO}_r}$), we get a dominant maps

$$\mathcal{H}^{-1}(s) \longrightarrow \mathcal{M}_X(\mathrm{Sp}_{2m}) \quad (\text{resp. } \mathcal{M}_X(\mathrm{SO}_r)).$$

Furthermore, if S is the ramification of $\tilde{X}_s/\iota \rightarrow X$ (resp. $\hat{X}_s/\iota \rightarrow X$), $\mathcal{P} = Nm^{-1}(\mathcal{O}(S))$, where Nm is the norm map attached to the cover $\tilde{X}_s \rightarrow \tilde{X}_s/\iota$ (resp. $\hat{X}_s \rightarrow \hat{X}_s/\iota$), then, by [Hit87], $\mathcal{H}^{-1}(s)$ is an open dense of \mathcal{P} . Thus the pushforward map

$$\mathcal{P} \dashrightarrow \mathcal{M}_X(\mathrm{Sp}_{2m}) \quad (\text{resp. } \mathcal{M}_X(\mathrm{SO}_r))$$

is dominant rational map.

Remark that in the symplectic case, the involution ι has some fixed points, this implies that \mathcal{P} is irreducible. While in the orthogonal case, ι is étale, hence \mathcal{P} has two connected components, each one of them dominates a connected component of $\mathcal{M}_X(\mathrm{SO}_r)$. In particular we deduce a cohomological criterion identifying the two connected components of $\mathcal{M}_X(\mathrm{SO}_r)$. More explicitly, take an even theta characteristic κ of X , then the two components are distinguished by the parity of $h^0(X, E \otimes \kappa)$. This is the same as the criterion given by the Stiefel-Whitney class (see for example [Bea06]). \square

3.2 The Hitchin system for anti-invariant vector bundles

For $s \in W$, we denote by $q : \tilde{X}_s \rightarrow X$ the associated spectral cover of X , and by $\tilde{S} = \mathrm{Ram}(\tilde{X}_s/X)$ its ramification divisor.

Fix the positive linearizations on K_X and \mathcal{O}_X (see Remark 1.1.3). Recall that this linearization equals id over the ramification points. We denote these linearizations by

$$\eta : \sigma^*K_X \rightarrow K_X, \quad \nu : \sigma^*\mathcal{O}_X \rightarrow \mathcal{O}_X.$$

The linearization η induces an involution on the space of global sections of K_X^i for each $i \geq 1$. We define

$$W^{\sigma,+} = \bigoplus_{i=1}^r H^0(X, K_X^i)_+.$$

Proposition 3.2.1. *Consider an r -tuple of global sections $s = (s_1, \dots, s_r) \in W^{\sigma,+}$ and let \tilde{X}_s be the associated spectral curve over X . Then the involution $\sigma : X \rightarrow X$ lifts to an involution $\tilde{\sigma}$ on \tilde{X}_s and $\mathcal{O}(\tilde{S})$ descends to $\tilde{Y}_s := \tilde{X}_s/\tilde{\sigma}$.*

Proof. We have an isomorphism

$$\mathcal{O}_X \oplus K_X^{-1} \xrightarrow{\iota_{\nu \otimes \eta}} \sigma^*(\mathcal{O}_X \oplus K_X^{-1}),$$

which induces an involution $\bar{\sigma}$ on $\mathbb{S} = \mathbb{P}(\mathcal{O}_X \oplus K_X^{-1})$. Let $\tilde{X}_s \subset \mathbb{P}(\mathcal{O}_X \oplus K_X^{-1})$ be the spectral curve associated to s .

Recall that the canonical section y of $\mathcal{O}(1)$ is identified with the identity section of $\bar{q}_*\mathcal{O}(1) \cong \mathcal{O}_X \oplus K_X^{-1}$, therefore it is $\bar{\sigma}$ -invariant. The section x is by definition the canonical section of $\bar{q}^*K_X \otimes \mathcal{O}(1)$. In fact it can be seen as the canonical section of $\bar{q}^*K_X \rightarrow |K_X|$, where $|K_X|$

is the total space of K_X . Hence x is invariant with respect to the positive linearization. As by definition $\eta^{\otimes k}(\sigma^*(s_k)) = s_k$, we deduce that

$$\bar{\sigma}((\bar{q}^* s_k) y^k x^{r-k}) = (\bar{q}^* s_k) y^k x^{r-k}.$$

Thus the section defining \tilde{X}_s

$$x^r + (\bar{q}^* s_1) y x^{r-1} + \cdots + (\bar{q}^* s_r) y^r \in H^0(\mathbb{S}, \bar{q}^* K_X^r \otimes \mathcal{O}(r))$$

is $\bar{\sigma}$ -invariant. Hence $\bar{\sigma}(\tilde{X}_s) = \tilde{X}_s$, so $\bar{\sigma}$ induces an involution on \tilde{X}_s which we denote by $\tilde{\sigma}$.

Remark that $\bar{\sigma}$ acts trivially on the fibers of $\bar{q} : \mathbb{S} \rightarrow X$ over the ramification points of $\pi : X \rightarrow Y$. Thus the ramification locus of $\tilde{\sigma}$ is $q^{-1}(R)$.

By Hurwitz formula we have $\mathcal{O}(\tilde{S}) = K_{\tilde{X}_s} \otimes q^* K_X^{-1}$. We also know by Lemma 1.1.2 that $K_{\tilde{X}_s}$ (resp. K_X) descends to \tilde{Y}_s (resp. Y). Moreover, $K_{\tilde{X}_s} = \tilde{\pi}^* K_{\tilde{Y}_s}(\tilde{R})$ (resp. $K_X = \pi^* K_Y(R)$), where $\tilde{R} = \text{Ram}(\tilde{X}_s/\tilde{Y}_s)$, and we have used the notation of the commutative diagram

$$\begin{array}{ccc} \tilde{X}_s & \xrightarrow{q} & X \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \tilde{Y}_s & \xrightarrow{\tilde{q}} & Y, \end{array}$$

since $\mathcal{O}(\tilde{R}) = q^* \mathcal{O}(R)$, it follows that

$$\begin{aligned} \mathcal{O}(\tilde{S}) &= K_{\tilde{X}_s} \otimes q^* K_X^{-1} \\ &= \tilde{\pi}^* K_{\tilde{Y}_s} \otimes q^*(\pi^* K_Y^{-1}) \otimes \mathcal{O}(\tilde{R}) \otimes q^* \mathcal{O}(-R) \\ &= \tilde{\pi}^* \left(K_{\tilde{Y}_s} \otimes \tilde{q}^* K_Y^{-1} \right). \end{aligned}$$

Since by Hurwitz formula $K_{\tilde{Y}_s} \otimes \tilde{q}^* K_Y^{-1} = \mathcal{O}(S)$, where $S = \text{Ram}(\tilde{Y}_s/Y)$, we deduce that $\mathcal{O}(\tilde{S}) = \tilde{\pi}^* \mathcal{O}(S)$. \square

We keep the notations of the last proposition hereafter.

Remark 3.2.2. Remark that for $s \in W^{\sigma,+}$, \tilde{Y}_s is a spectral cover of Y associated to some spectral data of the line bundle $L = K_Y \otimes \Delta$ over Y . This is because the sections s_i descend to Y .

Lemma 3.2.3. *Let F be a σ -linearized vector bundle, and consider the positive linearization on K_X . Then the Serre duality isomorphism*

$$H^1(X, F^*) \xrightarrow{\sim} H^0(X, F \otimes K_X)^*$$

is anti-equivariant with respect to the induced involutions on the two spaces.

Proof. If F is a σ -linearized vector bundle, we have an equivariant perfect pairing:

$$H^0(X, F) \otimes H^1(X, F^* \otimes K_X) \rightarrow H^1(X, K_X) \xrightarrow{\sim} \mathbb{C}.$$

As the fixed linearization is the positive one, it follows by Remark 1.1.3 that

$$\begin{aligned} H^1(X, K_X)_- &= H^1(X, \pi^*(K_Y \otimes \Delta))_- \\ &= H^1(Y, K_Y \otimes \Delta \otimes \Delta^{-1}) \\ &= H^1(Y, K_Y) = \mathbb{C}. \end{aligned}$$

So

$$H^1(X, K_X) = H^1(X, K_X)_-.$$

Since the above pairing is equivariant, we get the result. \square

Let ς be the canonical linearization on $E \otimes \sigma^* E$ given by the transposition, then the linearization $\varsigma \otimes \eta$ on $E \otimes \sigma^* E \otimes K_X$ induces an involution on $H^0(X, E \otimes \sigma^* E \otimes K_X)$ which we denote by f . By the above proposition, one gets an isomorphism

$$\begin{aligned} T^* \mathcal{U}_X^{\sigma,+}(r) &\xrightarrow{\sim} H^0(X, E \otimes \sigma^* E \otimes K_X)_+, \\ T^* \mathcal{U}_X^{\sigma,-}(r) &\xrightarrow{\sim} H^0(X, E \otimes \sigma^* E \otimes K_X)_-, \end{aligned}$$

We denote by \mathcal{H}_i the i^{th} component of the Hitchin map

$$\mathcal{H}_E : H^0(X, E \otimes \sigma^* E \otimes K_X) \rightarrow W.$$

Proposition 3.2.4. *Let E be σ -anti-invariant stable vector bundle and $\psi : \sigma^* E \cong E^*$ be an isomorphism.*

1. *If ψ is σ -symmetric, then \mathcal{H}_i induces a map*

$$\mathcal{H}_i : H^0(X, E \otimes \sigma^* E \otimes K_X)_+ \rightarrow H^0(X, K_X^i)_+.$$

2. *If ψ is σ -alternating, then \mathcal{H}_i induces a map*

$$\mathcal{H}_i : H^0(X, E \otimes \sigma^* E \otimes K_X)_- \rightarrow H^0(X, K_X^i)_+.$$

Proof. Let f be the involution on $H^0(X, E \otimes \sigma^* E \otimes K_X)$ defined above. Let $\phi \in H^0(X, E \otimes \sigma^* E \otimes K_X)$, locally we can write $\phi = \sum_k s_k \otimes \sigma^*(t_k) \otimes \alpha_k$, where α_k (resp. s_k, t_k) are local sections of K_X (resp. E). We can see the section ϕ as a map $E \rightarrow E \otimes K_X$ which is defined locally by

$$x \longrightarrow \phi(x) = \sum_k \langle \psi(\sigma^*(t_k)), x \rangle s_k \otimes \alpha_k.$$

Thus $\bigwedge^i \phi$ is defined locally by

$$\begin{aligned} \bigwedge^i \phi(x_1 \wedge \cdots \wedge x_i) &= i! \phi(x_1) \wedge \cdots \wedge \phi(x_i) \\ &= i! \left(\sum_{k_1} \langle \psi(\sigma^*(t_{k_1})), x_1 \rangle s_{k_1} \otimes \alpha_{k_1} \right) \wedge \cdots \wedge \left(\sum_{k_i} \langle \psi(\sigma^*(t_{k_i})), x_i \rangle s_{k_i} \otimes \alpha_{k_i} \right) \\ &= i! \sum_{k_1, \dots, k_i} \langle \psi(\sigma^*(t_{k_1})), x_1 \rangle \cdots \langle \psi(\sigma^*(t_{k_i})), x_i \rangle s_{k_1} \wedge \cdots \wedge s_{k_i} \otimes \left(\bigotimes_{j=1}^i \alpha_{k_j} \right) \\ &= i! \sum_{k_1 < \cdots < k_i} \det \left(\langle \psi(\sigma^*(t_{k_j})), x_l \rangle \right)_{j,l} s_{k_1} \wedge \cdots \wedge s_{k_i} \otimes \left(\bigotimes_{j=1}^i \alpha_{k_j} \right) \\ &= i! \sum_{k_1 < \cdots < k_i} \left\langle \left(\bigwedge^i \psi \right) (\sigma^*(t_{k_1}) \wedge \cdots \wedge \sigma^*(t_{k_i})), x_1 \wedge \cdots \wedge x_i \right\rangle s_{k_1} \wedge \cdots \wedge s_{k_i} \\ &\quad \otimes \left(\bigotimes_{j=1}^i \alpha_{k_j} \right). \end{aligned}$$

For the last equality, we use the canonical isomorphism $\bigwedge^k E^* \cong (\bigwedge^k E)^*$ given by the determinant. It follows that (locally) we have

$$\bigwedge^i \phi = i! \sum_{k_1 < \dots < k_i} s_{k_1} \wedge \dots \wedge s_{k_i} \otimes \sigma^*(t_{k_1}) \wedge \dots \wedge \sigma^*(t_{k_i}) \otimes \bigotimes_{j=1}^i \alpha_{k_j}.$$

1. Suppose that ψ is σ -symmetric, hence for any local section s and t of E , one has

$$\langle \psi(\sigma^*(t)), s \rangle = \nu(\sigma^* \langle \psi(\sigma^*(s)), t \rangle).$$

Hence

$$\begin{aligned} \mathcal{H}_i(f(\phi)) &= (-1)^i \text{Tr}(t(\sigma^*(\bigwedge^i \phi))) \\ &= (-1)^i i! \sum_{k_1 < \dots < k_i} \left\langle \bigwedge^i \psi(\sigma^*(s_{k_1}) \wedge \dots \wedge \sigma^*(s_{k_i})), t_{k_1} \wedge \dots \wedge t_{k_i} \right\rangle \bigotimes_{j=1}^i \eta(\sigma^*(\alpha_{k_j})) \\ &= (-1)^i i! \sum_{k_1 < \dots < k_i} \det(\langle \psi(\sigma^*(s_{k_l})), t_{k_{l'}} \rangle)_{1 \leq l, l' \leq i} \bigotimes_{j=1}^i \eta(\sigma^*(\alpha_{k_j})) \\ &= (-1)^i i! \sum_{k_1 < \dots < k_i} \nu \left(\sigma^* \left\langle \bigwedge^i \psi(\sigma^*(t_{k_1}) \wedge \dots \wedge \sigma^*(t_{k_i})), s_{k_1} \wedge \dots \wedge s_{k_i} \right\rangle \right) \bigotimes_{j=1}^i \eta(\sigma^*(\alpha_{k_j})) \\ &= \eta^{\otimes i}(\sigma^*(\mathcal{H}_i(\phi))). \end{aligned}$$

Thus, if $f(\phi) = \phi$, then $\eta^{\otimes i}(\sigma^*(\mathcal{H}_i(\phi))) = \mathcal{H}_i(\phi)$.

2. If ψ is σ -alternating, so we have

$$\langle \psi(\sigma^*(t)), s \rangle = -\nu(\sigma^* \langle \psi(\sigma^*(s)), t \rangle).$$

By the above calculation, it follows that

$$\mathcal{H}_i(f(\phi)) = (-1)^i \eta^{\otimes i}(\sigma^*(\mathcal{H}_i(\phi))).$$

On the other hand, it is clear that

$$\mathcal{H}_i(-\phi) = (-1)^i \mathcal{H}_i(\phi),$$

so if $f(\phi) = -\phi$, then

$$\eta^{\otimes i}(\sigma^*(\mathcal{H}_i(\phi))) = \mathcal{H}_i(\phi).$$

□

We claim that $\dim(W^{\sigma,+}) = \dim(\mathcal{U}_X^{\sigma,+}(r))$. Indeed we have

$$\begin{aligned} H^0(X, K_X^i) &\cong H^0(Y, \pi_* K_X^i) \\ &= H^0(Y, K_Y^i \otimes \Delta^i) \oplus H^0(Y, K_Y \otimes \Delta^{i-1}). \end{aligned}$$

As the fixed linearization on K_X is the positive one, by Remark 1.1.3, we obtain

$$H^0(X, K_X^i)_+ \cong H^0(Y, K_Y^i \otimes \Delta^i),$$

hence

$$h^0(X, K_X^i)_+ = (2i - 1)(g_Y - 1) + in.$$

It follows that

$$\begin{aligned}\dim(W^{\sigma,+}) &= \sum_{i=1}^r (2i-1)(g_Y-1) + in \\ &= r^2(g_Y-1) + \frac{r(r+1)}{2}n.\end{aligned}$$

Remark 3.2.5. We use Riemann-Roch and Lefschetz fixed point theorem (see Appendix F) to calculate the dimension of $W^{\sigma,+}$ by a second method. Two cases should be distinguished (for simplicity of notations, we denote by $h_{\pm}^0(i)$ the dimensions of $H^0(K_X^i)_{\pm}$).

1. $i = 1$: as the Serre duality is anti-equivariant for the positive linearization on K_X , we deduce that $h^1(X, K_X)_- = h^0(X, \mathcal{O}_X)_+ = 1$ (see remark 3.2.3). Hence

$$\begin{cases} h_+^0(1) + h_-^0(1) &= g_X \\ h_+^0(1) - h_-^0(1) &= -1 + \frac{1}{2} \sum_{k=1}^{2n} 1 = n-1. \end{cases}$$

So we get $h_+^0(1) = g_Y - 1 + n$.

2. $i \geq 2$: in this case $H^1(X, K_X^i) = 0$, it follows

$$\begin{cases} h_+^0(i) + h_-^0(i) &= (2i-1)(g_X-1) \\ h_+^0(i) - h_-^0(i) &= \frac{1}{2} \sum_{k=1}^{2n} 1 = n. \end{cases}$$

Hence $h_+^0(i) = \frac{1}{2}((2i-1)(g_X-1) + n) = (2i-1)(g_Y-1) + in$.

Finally, we get

$$\sum_{i=1}^r h^0(X, K_X^i)_+ = r^2(g_Y-1) + \frac{r(r+1)}{2}n.$$

To study the irreducibility of $\mathcal{U}_X^{\sigma,+}(r)$ and $\mathcal{U}_X^{\sigma,-}(r)$, we will use the notion of *very stable* vector bundles, which has been introduced in [Lau88]. Let E be a stable vector bundle, and let $\phi : E \rightarrow E \otimes K_X$ be a Higgs field. We say that ϕ is *nilpotent* if the composition of the maps

$$E \xrightarrow{\phi} E \otimes K_X \xrightarrow{\phi \otimes id} E \otimes K_X^2 \rightarrow \cdots \rightarrow E \otimes K_X^{r-1} \xrightarrow{\phi \otimes id} E \otimes K_X^r$$

is identically zero.

Definition 3.2.6. We say that a vector bundle E is *very stable* if E has no nilpotent Higgs field other than 0.

If E is a very stable vector bundle, then the Hitchin morphism

$$\mathcal{H}_E : H^0(X, E \otimes E^* \otimes K_X) \rightarrow W$$

is dominant. Indeed, by definition, $\mathcal{H}_E^{-1}(0) = \{0\}$, but the two spaces have the same dimension, this implies that \mathcal{H}_E is dominant.

One of the main results of [Lau88] is that the locus of very stable vector bundles is an open dense subscheme of the moduli space of vector bundles.

Definition 3.2.7. We say that a σ -symmetric (resp. σ -alternating) anti-invariant vector bundle E is *very stable* if E has no nilpotent Higgs field

$$\phi \in H^0(E \otimes \sigma^*E \otimes K_X)_+ \quad (\text{resp. } \phi \in H^0(E \otimes \sigma^*E \otimes K_X)_-)$$

other than 0.

Let $T^*\mathcal{U}_X(r, 0)$ be the cotangent bundle of $\mathcal{U}_X(r, 0)$. This bundle is invariant with respect to the involution $E \rightarrow \sigma^*E^*$ on $\mathcal{U}_X(r, 0)$. In fact, this is true more generally for any variety Z with an involution τ . To see this consider the differential of τ , it gives a linear isomorphism

$$d\tau : TZ \longrightarrow \tau^*TZ,$$

but $\tau^2 = id_Z$, this implies that $d\tau \circ \tau^*d\tau = id$. Thus $d\tau$ is a linearization on TZ , hence ${}^t d\tau$ is a linearization on T^*Z .

In particular, in our case, the involution $E \rightarrow \sigma^*E^*$ of $\mathcal{U}_X(r, 0)$ lifts to an involution on $T^*\mathcal{U}_X(r, 0)$. If we identify $T^*\mathcal{U}_X(r, 0) \cong H^0(X, E \otimes \sigma^*E \otimes K_X)$ using $\psi : \sigma^*E \cong E^*$ and Serre duality, then this lifting is the involution f on $H^0(X, E \otimes \sigma^*E \otimes K_X)$ given by $f(\phi) = {}^t(\sigma^*\phi)$ in the σ -symmetric case, and $f(\phi) = -{}^t(\sigma^*\phi)$ in the σ -alternating case. Moreover, by section 2.5, the fixed locus of this involution is the cotangent bundle $T^*\mathcal{U}_X^{\sigma,+}(r)$ (resp. $T^*\mathcal{U}_X^{\sigma,-}(r)$). Hence we can consider both $T^*\mathcal{U}_X^{\sigma,\pm}(r)$ as closed subspaces of $T^*\mathcal{U}_X(r, 0)$. Moreover, the tautological symplectic form on $T^*\mathcal{U}_X(r, 0)$ restricts to the tautological forms on $T^*\mathcal{U}_X^{\sigma,\pm}(r)$.

Following the notations of [Lau88], let $\Lambda_{X,r} \subset T^*\mathcal{U}_X(r, 0)$ be the nilpotent cone, that's the set of (E, ϕ) with ϕ nilpotent Higgs field. Set

$$\Lambda_{X,r}^{\sigma,+} = \Lambda_{X,r} \cap T^*\mathcal{U}_X^{\sigma,+}(r), \quad \Lambda_{X,r}^{\sigma,-} = \Lambda_{X,r} \cap T^*\mathcal{U}_X^{\sigma,-}(r).$$

Theorem 3.2.8. *The nilpotent cone $\Lambda_{X,r}^{\sigma,+}$ (resp. $\Lambda_{X,r}^{\sigma,-}$) is Lagrangian in $T^*\mathcal{U}_X^{\sigma,+}(r)$ (resp. $T^*\mathcal{U}_X^{\sigma,-}(r)$). In particular the locus of very stable anti-invariant vector bundles is dense in $\mathcal{U}_X^{\sigma,+}(r)$ (resp. $\mathcal{U}_X^{\sigma,-}(r)$).*

Proof. We prove the σ -symmetric case, the σ -alternating is absolutely the same. If V is symplectic space, then the restriction of a Lagrangian subspace $L \subset V$ to a symplectic subspace $F \subset V$ is an isotropic subspace of F , this implies that $\Lambda_{X,r}^{\sigma,+}$ is an isotropic subspace of $T^*\mathcal{U}_X^{\sigma,+}(r)$. In particular its dimension is at most $\dim(\mathcal{U}_X^{\sigma,+}(r))$. But it is clear that $\mathcal{U}_X^{\sigma,+}(r) \subset \Lambda_{X,r}^{\sigma,+}$, by seeing any anti-invariant vector bundle E as the trivial pair $(E, 0) \in \Lambda_{X,r}^{\sigma,+}$. This implies that

$$\dim(\Lambda_{X,r}^{\sigma,+}) = \frac{1}{2} \dim(T^*\mathcal{U}_X^{\sigma,+}(r)),$$

hence $\Lambda_{X,r}^{\sigma,+}$ is Lagrangian of $T^*\mathcal{U}_X^{\sigma,+}(r)$. □

3.2.1 σ -symmetric case

The ramified case

Suppose that $\pi : X \rightarrow Y$ is ramified and denote $m = r(r-1)(g_X - 1)$. Recall that $\deg(\tilde{S}) = 2m$, where $\tilde{S} = \text{Ram}(\tilde{X}_s/X)$. We fix the positive linearization on $\mathcal{O}(\tilde{S})$.

For general $s \in W^{\sigma,+}$, consider the subvariety $\mathcal{P}^+ \subset \text{Pic}^m(\tilde{X}_s)$ of isomorphism classes of line bundles L such that

$$\widetilde{\text{Nm}}(L) \cong \mathcal{O}_{\tilde{Y}_s}(S),$$

where $S = \text{Ram}(\tilde{Y}_s/Y)$, and $\widetilde{\text{Nm}} : \text{Pic}^0(\tilde{X}_s) \rightarrow \text{Pic}^0(\tilde{Y}_s)$ the norm map attached to $\tilde{\pi} : \tilde{X}_s \rightarrow \tilde{Y}_s$.

By Proposition 3.2.1, $\tilde{\pi}^* \mathcal{O}_{\tilde{Y}_s}(S) = \mathcal{O}_{\tilde{X}_s}(\tilde{S})$, it follows that for each $L \in \mathcal{P}^+$, we have

$$\tilde{\sigma}^* L \cong L^{-1}(\tilde{S}).$$

In particular any $M \in \mathcal{P}^+$ gives by tensor product an isomorphism

$$\mathcal{P}^+ \xrightarrow{\sim} \text{Prym}(\tilde{X}_s/\tilde{Y}_s).$$

Lemma 3.2.9. *For general $s \in W^{\sigma,+}$, we have*

$$\dim(\mathcal{P}^+) = \dim(\mathcal{U}_X^{\sigma,+}(r)).$$

Proof. For general $s \in W^{\sigma,+}$, the curve \tilde{X}_s is smooth and its genus is $g_{\tilde{X}_s} = r^2(g_X - 1) + 1$. Indeed, by Lemma 3.1.1, and because $W^{\sigma,+}$ is irreducible, it suffices to prove that there exists an $s \in W^{\sigma,+}$ such that \tilde{X}_s is smooth. To do so, take $s_r \in H^0(X, K_X^r)_+$ to be a general section that has just simple roots which are different from the ramification points (i.e. outside a finite union of hyperplanes of sections vanishing at points of R). This is possible because the hyperplane $H^0(K_X^r(-p)) \subset H^0(K_X^r)$ contains $H^0(K_X^r)_-$, so necessarily it does not contain $H^0(K_X^r)_+$, and this for every $p \in R$. Then using the proof of Proposition 3.1.1, we deduce that the spectral curve attached to $s = (0, \dots, 0, s_r)$ is smooth.

Moreover, if \tilde{X}_s is smooth then, by Lemma 1.2.2, we deduce that $\tilde{R} = \text{Ram}(\tilde{X}_s/\tilde{Y}_s) = q^{-1}(R)$ has no multiple points (i.e. reduced divisor). Furthermore we have $\deg(\tilde{R}) = 2rn$, so we get

$$\begin{aligned} \dim(\mathcal{P}^+) &= g_{\tilde{X}_s} - g_{\tilde{Y}_s} \\ &= \frac{1}{2}(g_{\tilde{X}_s} - 1 + rn) && \text{(by Riemann-Roch)} \\ &= \frac{r^2}{2}(g_X - 1) + \frac{rn}{2}. \end{aligned}$$

□

Remark 3.2.10. We can prove directly that if a ramification point in \tilde{R} is double then it is singular point. Indeed, assume that $a \in \tilde{R}$ is a multiple point over $p \in R$, and let t be a local parameter in a neighborhood of p . Then \tilde{X}_s is given locally over p by the equation

$$P(x, t) = x^r + s_1(t)x^{r-1} + \dots + s_r(t) = 0.$$

In particular, because a is multiple point, we have

$$P(a, 0) = \frac{\partial P}{\partial x}(a, 0) = 0.$$

Now, as $s_i \in H^0(X, K_X^i)_+$, it follows that $s_i(-t) = s_i(t)$, so for each i , we can write locally near p

$$s_i(t) = a_0 + t^2 a_2 + \dots.$$

This implies that $P(x, t) = P_0(x) + t^2 P_2(x) + \dots$, where P_i are polynomials in x . Hence

$$\frac{\partial P}{\partial t}(a, 0) = 0.$$

By the Jacobian criterion of smoothness, we deduce that \tilde{X}_s is singular at a .

Remark 3.2.11. We can calculate $\dim(\mathcal{P}^+)$ using the fact that \tilde{Y}_s is spectral curve over Y . Indeed, since the s_1, \dots, s_r descend to Y , we see that \tilde{Y}_s is a spectral curve over Y with respect to the line bundle $K_Y \otimes \Delta$. Hence by subsection 3.1, we deduce that

$$\begin{aligned} g_{\tilde{Y}_s} &= \deg(K_Y \otimes \Delta) \frac{r(r-1)}{2} + r(g_Y - 1) + 1 \\ &= (2g_Y - 2 + n) \frac{r(r-1)}{2} + r(g_Y - 1) + 1 \\ &= r^2(g_Y - 1) + n \frac{r(r-1)}{2} + 1. \end{aligned}$$

It follows that

$$\begin{aligned} \dim(\mathcal{P}^+) &= g_{\tilde{X}_s} - g_{\tilde{Y}_s} \\ &= r^2(g_X - 1) + 1 - \left(r^2(g_Y - 1) + n \frac{r(r-1)}{2} + 1 \right) \\ &= r^2(g_Y - 1) + \frac{r(r+1)}{2}n. \end{aligned}$$

Theorem 3.2.12. *Suppose that $\pi : X \rightarrow Y$ is ramified. Then for general $s \in W^{\sigma,+}$, the rational pushforward map*

$$q_* : \mathcal{P}^+ \dashrightarrow \mathcal{U}_X^{\sigma,+}(r)$$

is dominant. In particular $\mathcal{U}_X^{\sigma,+}(r)$ is irreducible.

Proof. First, by the duality for finite flat morphisms, this map is well defined, more precisely, one has

$$\begin{aligned} \sigma^* q_* L &\cong q_*(\tilde{\sigma}^* L) \\ &\cong q_*(L^{-1} \otimes \mathcal{O}(\tilde{S})) \\ &\cong (q_* L)^*, \end{aligned}$$

the last isomorphism is the duality for finite flat morphisms (see for example [Har77] Ex. III.6.10). The isomorphism $\psi : \sigma^*(q_* L) \rightarrow (q_* L)^*$ is defined using the pairing $\tilde{\psi} : q_* L \otimes \sigma^*(q_* L) \rightarrow \mathcal{O}_X$, which is defined as follows: for $v, w \in H^0(U, q_* L) = H^0(q^{-1}(U), L)$, we put

$$\tilde{\psi}(v \otimes \tilde{\sigma}^* w) \cdot \xi = \langle v, \tilde{\sigma}^* w \rangle,$$

where $\xi \in H^0(\tilde{X}_s, \mathcal{O}(\tilde{S}))$ is the canonical section equals the derivative of $q : \tilde{X}_s \rightarrow X$, and $\langle, \rangle : L \otimes \tilde{\sigma}^* L \rightarrow \mathcal{O}(\tilde{S})$ is an isomorphism. This last isomorphism is $\tilde{\sigma}$ -symmetric for the positive linearization on $\mathcal{O}(\tilde{S})$. Indeed, it is a global section of $\tilde{\sigma}^* L^{-1} \otimes L^{-1} \otimes \mathcal{O}(\tilde{S})$ ($\cong \mathcal{O}_{\tilde{X}_s}$) and because we fixed the positive linearization on $\mathcal{O}(\tilde{S})$ we deduce

$$\begin{aligned} H^0(\tilde{X}_s, \tilde{\sigma}^* L^{-1} \otimes L^{-1} \otimes \mathcal{O}(\tilde{S}))_+ &= H^0(\tilde{X}_s, \tilde{\pi}^* \widetilde{\text{Nm}}(L^{-1}) \otimes \tilde{\pi}^*(\mathcal{O}(S)))_+ \\ &= H^0(\tilde{X}_s, \tilde{\pi}^*(\mathcal{O}(-S)) \otimes \tilde{\pi}^*(\mathcal{O}(S)))_+ \\ &= H^0(\tilde{Y}_s, \mathcal{O}_{\tilde{Y}_s}) \\ &= \mathbb{C}. \end{aligned}$$

Further ξ is $\tilde{\sigma}$ -invariant global section of $\mathcal{O}(\tilde{S})$ with respect to the positive linearization. Hence ψ is σ -symmetric.

Conversely, giving a σ -symmetric anti-invariant stable vector bundle E and $\phi \in H^0(X, E \otimes \sigma^* E \otimes K_X)_+$ such that $\mathcal{H}_E(\phi) = s$, then the corresponding line bundle over \tilde{X}_s is in \mathcal{P}^+ . To see this, consider the exact sequence (see [BNR89] Remark 3.7)

$$0 \rightarrow L(-\tilde{S}) \rightarrow q^*(E) \xrightarrow{q^*\phi-x} q^*(E \otimes K_X) \rightarrow L \otimes q^*K_X \rightarrow 0. \quad (3.3)$$

By taking the dual, pulling back by $\tilde{\sigma}$ and then taking the tensor product by $\tilde{\sigma}^*q^*K_X$, we get the exact sequence

$$0 \rightarrow \tilde{\sigma}^*(L^{-1}) \rightarrow \tilde{\sigma}^*q^*(E^*) \xrightarrow{\tilde{\sigma}^*({}^t(q^*\phi))-\tilde{\sigma}^*x} \tilde{\sigma}^*(q^*(E^* \otimes K_X)) \rightarrow \tilde{\sigma}^*(L^{-1}(\tilde{S}) \otimes q^*K_X) \rightarrow 0. \quad (3.4)$$

Since ϕ is invariant, i.e. ${}^t(\sigma^*\phi) = \phi$, the middle maps of the exact sequences (3.3) and (3.4) are identified using the isomorphism $\psi : \sigma^*E \rightarrow E^*$, hence they have isomorphic kernels. This implies that $L(-\tilde{S}) \cong \tilde{\sigma}^*(L^{-1})$. Thus $L \in \mathcal{P}^+$.

Moreover, by Lemma 3.2.9 we deduce that whenever \tilde{X}_s is smooth we have $\dim(\mathcal{P}^+) = \dim(\mathcal{U}_X^{\sigma,+}(r))$.

Now, if $E \in \mathcal{U}_X^{\sigma,+}(r)$ is very stable then the map

$$\mathcal{H}_E : H^0(X, E \otimes \sigma^*E \otimes K_X)_+ \rightarrow W^{\sigma,+}$$

is dominant, it follows that the map

$$\Pi : T^*\mathcal{U}_X^{\sigma,+}(r) \rightarrow \mathcal{U}_X^{\sigma,+}(r) \times W^{\sigma,+}$$

is dominant too, because the locus of very stable vector bundles is dense inside $\mathcal{U}_X^{\sigma,+}(r)$ by Theorem 3.2.8. In particular, fixing a general $s \in W^{\sigma,+}$, we obtain a dominant morphism

$$\mathcal{H}^{-1}(s) \rightarrow \mathcal{U}_X^{\sigma,+}(r),$$

where $\mathcal{H} : T^*\mathcal{U}_X^{\sigma,+}(r) \rightarrow W^{\sigma,+}$ is the Hitchin morphism. But, by Proposition 3.1.4 and what we have said above, we deduce that $(E, \phi) \in \mathcal{H}^{-1}(s)$ if and only if $E \cong q_*L$ for some $L \in \mathcal{P}^+$. It follows that the rational map

$$q_* : \mathcal{P}^+ \dashrightarrow \mathcal{U}_X^{\sigma,+}(r)$$

is dominant.

As $\tilde{R} = q^{-1}(R)$, one deduces that $\tilde{X}_s \rightarrow \tilde{Y}_s$ is ramified. This implies the connectedness of \mathcal{P}^+ , hence the irreducibility of $\mathcal{U}_X^{\sigma,+}(r)$. \square

The étale case

Assume that the cover $\pi : X \rightarrow Y$ is étale. In this case, any σ -invariant vector bundle over X descends to Y by Kempf's Lemma. In particular, we have

$$K_X = \pi^*(K_Y \otimes \Delta) = \pi^*K_Y.$$

Recall that the linearization on K_X attached to $K_Y \otimes \Delta$ is called the positive linearization and that Serre duality is anti-equivariant with respect to this linearization.

Remark that $\mathcal{O}(\tilde{S}) = \tilde{\pi}^*\mathcal{O}(S) = \tilde{\pi}^*(\mathcal{O}(S) \otimes \tilde{\Delta})$, where $\tilde{\Delta} = \det(\tilde{\pi}_*\mathcal{O}_{\tilde{X}_s})^{-1}$. We fix the linearization on $\mathcal{O}(\tilde{S})$ attached to the $\mathcal{O}(S)$ and we continue calling it the positive linearization.

Theorem 3.2.13. *Suppose that $\pi : X \rightarrow Y$ is étale, then the pushforward rational map q_* induces a dominant map*

$$q_* : \mathcal{P}^+ \dashrightarrow \mathcal{U}_X^{\sigma,+}(r).$$

In particular $\mathcal{U}_X^{\sigma,+}(r)$ has two connected components.

Proof. Clearly $\tilde{X}_s \rightarrow \tilde{Y}_s$ is étale if and only if $X \rightarrow Y$ is. Hence \mathcal{P}^+ has two connected components. We show that it is impossible to produce the same stable vector bundle E as the direct image of two line bundles from the two connected components of \mathcal{P}^+ . To see this assume that we have L and L' , two line bundles each from a connected component of \mathcal{P}^+ , such that $q_*L \cong q_*L' \in \mathcal{U}_X^{\sigma,+}(r)$. Let M be a line bundle on \tilde{X}_s such that M descends to \tilde{Y}_s and $\widetilde{\text{Nm}}(M) = \mathcal{O}(S)$, in particular $M^2 \cong \mathcal{O}(\tilde{S})$. Let κ be an even theta characteristic on \tilde{X}_s such that $M^{-1} \otimes \kappa$ is the pullback of a theta characteristic κ' on Y , i.e. $M^{-1} \otimes \kappa = q^*(\pi^*(\kappa'))$, note that such a pair (M, κ) exists by Lemma 3.2.14 below. Then, by [BL04] Theorem 12.6.2, we know that

$$h^0(\tilde{X}_s, L \otimes M^{-1} \otimes \kappa) \equiv 0 \pmod{2}, \quad h^0(\tilde{X}_s, L' \otimes M^{-1} \otimes \kappa) \equiv 1 \pmod{2}$$

Using the projection formula, this gives

$$h^0(X, q_*L \otimes \pi^*\kappa') \equiv 0 \pmod{2} \quad \text{and} \quad h^0(X, q_*L' \otimes \pi^*\kappa') \equiv 1 \pmod{2}$$

a contradiction.

Moreover, if $E \in \mathcal{U}_X^{\sigma,+}(r)$, then the associated line bundle L over \tilde{X}_s constructed in the proof of Theorem 3.2.12, verifies $\tilde{\sigma}^*L \cong L^{-1}(\tilde{S})$. Since $\tilde{\pi} : \tilde{X}_s \rightarrow \tilde{Y}_s$ is étale, it follows that either $\widetilde{\text{Nm}}(L) = \mathcal{O}_{\tilde{Y}_s}(S)$, or $\widetilde{\text{Nm}}(L) = \mathcal{O}_{\tilde{Y}_s}(S) \otimes \tilde{\Delta}$. But ψ induces a $\tilde{\sigma}$ -symmetric isomorphism $\tilde{\sigma}^*L \rightarrow L^{-1}(\tilde{S})$, and because we have fixed the positive linearization on $\mathcal{O}_{\tilde{X}_s}(\tilde{S})$, it follows that $\widetilde{\text{Nm}}(L) = \mathcal{O}_{\tilde{Y}_s}(S)$. So $L \in \mathcal{P}^+$.

Now the image of the rational map $q_* : \mathcal{P}^+ \rightarrow \mathcal{U}_X^{\sigma,+}(r)$ has two connected components, which are dense, and by Mumford [Mum71], the map

$$E \rightarrow h^0(X, \pi_*E \otimes \kappa') \pmod{2}$$

is constant under deformation of E . Hence $\mathcal{U}_X^{\sigma,+}(r)$ can't be irreducible. It follows that it has two connected components. \square

Lemma 3.2.14. *Suppose that $\pi : X \rightarrow Y$ is étale and \tilde{X}_s is smooth. Then there exist an even theta characteristic κ on \tilde{X}_s , and a line bundle M that descends to \tilde{Y}_s and verifies $M^2 \cong \mathcal{O}(\tilde{S})$, such that $M^{-1} \otimes \kappa$ descends to a theta characteristic on Y .*

Proof. Recall that we denoted by $\tilde{\Delta} := \det\left(\tilde{\pi}_*\mathcal{O}_{\tilde{X}_s}\right)^{-1}$, note that $\tilde{\Delta}$ is non-trivial 2-torsion line bundle over \tilde{Y}_s . By [BL04] page 382, we know that there exists an even theta characteristic, say κ'' , on \tilde{Y}_s such that

$$h^0(\kappa'') \equiv h^0(\kappa'' \otimes \tilde{\Delta}) \equiv 0 \pmod{2}.$$

If we set $\kappa = \tilde{\pi}^*\kappa''$, we get by the projection formula

$$h^0(\kappa) = h^0(\kappa'') + h^0(\kappa'' \otimes \tilde{\Delta}) \equiv 0 \pmod{2},$$

hence κ is even theta characteristic. Moreover, let N be a line bundle on \tilde{Y}_s such that $N^2 \cong \mathcal{O}(S)$ (recall that $S := \text{Ram}(\tilde{Y}_s/Y)$ has an even degree), then by Hurwitz formula, we have

$$(N^{-1} \otimes \kappa'')^2 = (\tilde{q}^*\kappa')^2,$$

where κ' is a (any) theta characteristic on Y . It follows that there exists a 2-torsion line bundle α on \tilde{Y}_s such that

$$N^{-1} \otimes \kappa'' \otimes \alpha = \tilde{q}^* \kappa'.$$

It suffices to take $M = \tilde{\pi}^*(N \otimes \alpha^{-1})$. \square

Remark 3.2.15. The determinant induces a morphism $\det : \mathcal{U}_X^{\sigma,+}(r) \rightarrow P = \text{Nm}^{-1}(\mathcal{O}_Y)$. Note that $P = \mathcal{U}_X^{\sigma,+}(1)$. The composition of this map with the direct image q_* gives a map

$$\mathcal{P}^+ \rightarrow P.$$

Moreover, each connected component of \mathcal{P}^+ dominates a connected component of P (it is even surjective). Indeed, let $L \in \mathcal{P}^+$ and M be a line bundle in \mathcal{P}^+ such that $\text{Nm}_{\tilde{X}_s/X}(M) = \delta$, where $\delta = (q_* \mathcal{O}_{\tilde{X}_s})^{-1}$. Then $L \otimes M^{-1}$ is in the Prym variety of $\tilde{X}_s \rightarrow \tilde{Y}_s$, hence can be written as $\tilde{\sigma}^* \lambda \otimes \lambda^{-1}$, it follows that

$$\det(q_* L) = \text{Nm}_{\tilde{X}_s/X}(L) \otimes \delta^{-1} = \sigma^* \text{Nm}_{\tilde{X}_s/X}(\lambda) \otimes \text{Nm}_{\tilde{X}_s/X}(\lambda)^{-1}.$$

Then it suffices to recall that the image of the map $\lambda \rightarrow \tilde{\sigma}^* \lambda \otimes \lambda^{-1}$ equals the identity component of the Prym variety when λ runs $\text{Pic}^0(\tilde{X})$ and equals the other component when it runs $\text{Pic}^1(\tilde{X})$.

In particular, using Proposition 2.3.3, we deduce that for general line bundle L in both connected components of \mathcal{P}^+ , $q_* L$ is stable.

Trivial determinant case

In this section, nothing is assumed on the cover $\pi : X \rightarrow Y$, i.e. it may be ramified or not. Denote by $\mathcal{Q}^+ = \text{Nm}_{\tilde{X}_s/X}^{-1}(\delta)$, where $\delta = (\det(q_* \mathcal{O}_{\tilde{X}_s}))^{-1}$. For general $s \in W^{\sigma,+}$, \mathcal{Q}^+ is isomorphic to the Prym variety of the spectral cover $\tilde{X}_s \rightarrow X$. In particular it is connected.

Proposition 3.2.16. *For general $s \in W^{\sigma,+}$, $\mathcal{P}^+ \cap \mathcal{Q}^+$ is connected.*

Proof. Fixing an element in $\mathcal{P}^+ \cap \mathcal{Q}^+$ gives by tensor product an isomorphism

$$\mathcal{P} \cap \mathcal{Q} \xrightarrow{\sim} \mathcal{P}^+ \cap \mathcal{Q}^+.$$

So it is sufficient to prove the connectedness of $\mathcal{P} \cap \mathcal{Q}$ (recall that \mathcal{P} and \mathcal{Q} are the Prym varieties of $\tilde{\pi} : \tilde{X}_s \rightarrow \tilde{Y}_s$ and $q : \tilde{X}_s \rightarrow X$ respectively). The norm map $\widetilde{\text{Nm}} : J_{\tilde{X}_s} \rightarrow J_{\tilde{Y}_s}$ induces a homomorphism

$$\vartheta : \mathcal{Q} \rightarrow \underline{\mathcal{Q}},$$

which is just the restriction of $\widetilde{\text{Nm}}$ to \mathcal{Q} , here $\underline{\mathcal{Q}}$ is the Prym variety of $\tilde{q} : \tilde{Y}_s \rightarrow Y$. We have a commutative diagram

$$\begin{array}{ccccccc} \mathcal{Q} & \hookrightarrow & J_{\tilde{X}_s} & \xrightarrow{\sim} & \widehat{J}_{\tilde{X}_s} & \twoheadrightarrow & \widehat{\mathcal{Q}} \\ \uparrow \mu & & \uparrow \tilde{\pi}^* & & \uparrow \widetilde{\text{Nm}} & & \uparrow \hat{\vartheta} \\ \underline{\mathcal{Q}} & \hookrightarrow & J_{\tilde{Y}_s} & \xrightarrow{\sim} & \widehat{J}_{\tilde{Y}_s} & \twoheadrightarrow & \widehat{\underline{\mathcal{Q}}} \end{array}$$

where $\widehat{\mathcal{Q}}$ and $\widehat{\underline{\mathcal{Q}}}$ are the dual abelian varieties of \mathcal{Q} and $\underline{\mathcal{Q}}$ respectively and $\mu : \underline{\mathcal{Q}} \rightarrow \mathcal{Q}$ is the morphism defined by the factorization

$$\tilde{\pi}^*|_{\underline{\mathcal{Q}}} : \underline{\mathcal{Q}} \xrightarrow{\mu} \mathcal{Q} \hookrightarrow J_{\tilde{X}_s}.$$

We obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\varphi_{\mathcal{Q}}} & \widehat{\mathcal{Q}} \\ \mu \uparrow & & \uparrow \hat{\vartheta} \\ \underline{\mathcal{Q}} & \xrightarrow{\varphi_{\underline{\mathcal{Q}}}} & \underline{\widehat{\mathcal{Q}}}, \end{array} \quad (3.5)$$

where $\varphi_{\mathcal{Q}} : \mathcal{Q} \rightarrow \widehat{\mathcal{Q}}$ (resp. $\varphi_{\underline{\mathcal{Q}}} : \underline{\mathcal{Q}} \rightarrow \underline{\widehat{\mathcal{Q}}}$) is the restriction of the principal polarization of $J_{\tilde{X}_s}$ (resp. $J_{\tilde{Y}_s}$) to \mathcal{Q} (resp. $\underline{\mathcal{Q}}$). By [BNR89] Remark 2.7, as the spectral covers are always ramified, the types of these two polarizations are $(1, \dots, 1, \underbrace{r, \dots, r}_{g_X})$ and $(1, \dots, 1, \underbrace{r, \dots, r}_{g_Y})$ respectively. Hence the degree of these two restrictions is r^{2g_X} and r^{2g_Y} respectively.

Assume that $X \rightarrow Y$ is ramified. Then μ is injective. By Diagram (3.5) it follows that

$$\text{card}(\text{Ker}(\varphi_{\mathcal{Q}} \circ \mu)) = \text{card}(\text{Ker}(\varphi_{\underline{\mathcal{Q}}} \circ \hat{\vartheta})).$$

It is easy to see that $\text{card}(\text{Ker}(\varphi_{\mathcal{Q}} \circ \mu)) = r^{2g_Y}$. Indeed if $L \in \text{Ker}(\varphi_{\mathcal{Q}} \circ \mu)$, then $\tilde{\pi}^*L = q^*M$ for some $M \in J_X[r]$, but this implies that M descends to Y (here we use Kempf's Lemma (1.1.1) to prove that if q^*M descends to \tilde{Y}_s then M descends to Y). Hence $L = \tilde{q}^*N$ for some $N \in J_Y[r]$, i.e. $\text{Ker}(\varphi_{\mathcal{Q}} \circ \mu) = \tilde{q}^*J_Y[r]$. As \tilde{q}^* is injective (c.f. [BNR89] Remark 3.10) this implies the result.

But we also have $\text{Ker}(\varphi_{\underline{\mathcal{Q}}}) = \tilde{q}^*J_Y[r]$, so $\text{card}(\text{Ker}(\varphi_{\underline{\mathcal{Q}}})) = r^{2g_Y}$. This proves that $\hat{\vartheta}$ is injective. By general theory of abelian varieties (see for example [BL04] Proposition 2.4.3), $\text{Ker}(f)$ and $\text{Ker}(\hat{f})$ have the same number of connected components for any surjective morphism $f : A \rightarrow B$ between abelian varieties. Since ϑ is clearly surjective, it follows that $\text{Ker}(\vartheta)$ is connected, and by definition, the kernel of ϑ is precisely $\mathcal{P} \cap \mathcal{Q}$.

If $X \rightarrow Y$ is étale, then so is $\tilde{\pi} : \tilde{X}_s \rightarrow \tilde{Y}_s$. In this case μ has degree 2. Let $L \in \text{Ker}(\varphi_{\mathcal{Q}} \circ \mu)$, then as above $\tilde{\pi}^*L = q^*M$, for some $M \in J_X[r]$, hence M descends to Y , say $M = \pi^*N$, as $M^r = \mathcal{O}_X$, we get $N^r = \mathcal{O}_Y$ or $N^r = \Delta$, recall that Δ is the 2-torsion line bundle attached to $X \rightarrow Y$. Denote the set of r^{th} roots of Δ by T . It follows that $L = \tilde{q}^*N$ or $L = \tilde{q}^*N \otimes \tilde{\Delta}$ for $N \in J_Y[r] \cup T$, where $\tilde{\Delta} \in J_{\tilde{Y}_s}[2]$ is the line bundle attached to $\tilde{X}_s \rightarrow \tilde{Y}_s$. Note that $\tilde{\Delta} = \tilde{q}^*\Delta$. Since $L \in \underline{\mathcal{Q}}$, we deduce

- if r is even, then multiplication by Δ is an involution of $J_Y[r]$ and T , so $L \in \tilde{q}^*J_Y[r]$ (because $\tilde{q}^*T \cap \underline{\mathcal{Q}} = \emptyset$). This implies that $\text{Ker}(\varphi_{\mathcal{Q}} \circ \mu) = \tilde{q}^*J_Y[r]$, hence $\text{card}(\text{Ker}(\varphi_{\mathcal{Q}} \circ \mu)) = r^{2g_Y}$. It follows that $\hat{\vartheta}$ is injective. So $\mathcal{P} \cap \mathcal{Q}$ is connected.
- if r is odd, then multiplication by Δ is an isomorphism $J_Y[r] \cong T$, and since $q^*\Delta = \tilde{\Delta}$ we can assume that $N \in J_Y[r]$. As $\text{Nm}_{\tilde{Y}_s/Y}(\tilde{q}^*N \otimes \tilde{\Delta}) = \Delta^r = \Delta$, then $\tilde{q}^*N \otimes \tilde{\Delta} \notin \underline{\mathcal{Q}}$. It follows $L \in \tilde{q}^*J_Y[r]$, hence $\text{card}(\text{Ker}(\varphi_{\mathcal{Q}} \circ \mu)) = r^{2g_Y}$, and so we deduce again the connectedness of $\mathcal{P} \cap \mathcal{Q}$.

□

Theorem 3.2.17. *The pushforward map*

$$q_* : \mathcal{P}^+ \cap \mathcal{Q}^+ \dashrightarrow \text{SU}_X^{\sigma,+}(r)$$

is dominant. In particular $\text{SU}_X^{\sigma,+}(r)$ is irreducible.

Proof. Let P_0 be the identity component of the Prym variety of $X \rightarrow Y$, then it is clear that the map

$$\mathcal{SU}_X^{\sigma,+}(r) \times P_0 \longrightarrow \mathcal{U}_{X,0}^{\sigma,+}(r)$$

is surjective, where $\mathcal{U}_{X,0}^{\sigma,+}(r)$ is the connected component of $\mathcal{U}_X^{\sigma,+}(r)$ which is over P_0 by the determinant map. It follows by Theorems 3.2.12 and 3.2.13 (in the ramified and étale cases respectively) that for general $E \in \mathcal{U}_{X,0}^{\sigma,+}(r)$, there exists $L \in \mathcal{P}^+$ such that $q_*L = E$. Let $\lambda \in P_0$ be an r^{th} root of $\det(E)^{-1}$. It follows by the projection formula that $q_*(L \otimes q^*\lambda) = E \otimes \lambda \in \mathcal{SU}_X^{\sigma,+}(r)$. Note that $L \otimes q^*\lambda \in \mathcal{P}^+$ because $\widetilde{\text{Nm}}(q^*\lambda) = \mathcal{O}_{\tilde{Y}_s}$. Hence a general $E \in \mathcal{SU}_X^{\sigma,+}(r)$ can be written as a direct image of some $L \in \mathcal{P}^+$. But since

$$\det(q_*L) = \delta^{-1} \otimes \text{Nm}_{\tilde{X}_s/X}(L),$$

where $\delta = \det(q_*\mathcal{O}_{\tilde{X}_s})^{-1}$, we deduce that if q_*L has trivial determinant then $\text{Nm}_{\tilde{X}_s/X}(L) = \delta$, thus $L \in \mathcal{Q}^+$. So we get a dominant rational map

$$\mathcal{P}^+ \cap \mathcal{Q}^+ \longrightarrow \mathcal{SU}_X^{\sigma,+}(r).$$

Now by Proposition 3.2.16, $\mathcal{P}^+ \cap \mathcal{Q}^+$ is connected. So $\mathcal{SU}_X^{\sigma,+}(r)$ is irreducible. \square

Remark 3.2.18. Remark that the map

$$\mathcal{SU}_X^{\sigma,+}(r) \times P \longrightarrow \mathcal{U}_X^{\sigma,+}(r)$$

is surjective, unless $\pi : X \rightarrow Y$ is étale and r is even, for which its image is one connected component. Indeed, the ramified case is clear, so assume that π is étale, then if r is odd, the map $[r] : P \rightarrow P$ is surjective, and if r is even, its image is the identity component $P_0 \subset P$. Now use Theorem 3.2.13 to deduce the result.

3.2.2 σ -alternating case

The ramified case

Suppose that $\pi : X \rightarrow Y$ is ramified and r is even. Let E be a σ -alternating stable vector bundle. Consider the involution f on the space $H^0(X, E \otimes \sigma^*E \otimes K_X)$ associated to the linearization $\varsigma \otimes \eta$ on $E \otimes \sigma^*E \otimes K_X$, where ς is the linearization on $E \otimes \sigma^*E$ equals the transposition.

Let $\phi \in H^0(X, E \otimes \sigma^*E \otimes K_X)_-$. Using the isomorphism $\psi : \sigma^*E \cong E^*$, we can see ϕ as a map $E \rightarrow E \otimes K_X$ (in fact we just identify ϕ and $\phi \circ ({}^t\psi)$ to simplify the notations). Then we have

Lemma 3.2.19. *The following diagram*

$$\begin{array}{ccc} \sigma^*\phi : & \sigma^*E & \longrightarrow & \sigma^*E \otimes \sigma^*K_X \\ & \uparrow \psi^{-1} & & \downarrow \psi \otimes \eta \\ {}^t\phi : & E^* & \longrightarrow & E^* \otimes K_X, \end{array}$$

commutes.

Proof. Write (locally)

$$\phi = \sum_k s_k \otimes \sigma^* t_k \otimes \alpha_k,$$

and let v be a local section of E^* , then we have

$$\begin{aligned} (\psi \otimes \eta) \circ (\sigma^* \phi) \circ \psi^{-1}(v) &= (\psi \otimes \eta) \left(\sum_k \langle (\sigma^* \psi)(t_k), \psi^{-1}(v) \rangle \sigma^* s_k \otimes \sigma^* \alpha_k \right) \\ &= (\psi \otimes \eta) \left(\sum_k - \langle ({}^t \psi)(t_k), \psi^{-1}(v) \rangle \sigma^* s_k \otimes \sigma^* \alpha_k \right) \\ &= -(\psi \otimes \eta) \left(\sum_k \langle t_k, v \rangle \sigma^* s_k \otimes \sigma^* \alpha_k \right) \\ &= - \sum_k \langle t_k, v \rangle \psi(\sigma^* s_k) \otimes \eta(\sigma^* \alpha_k) \\ &= - \left(\sum_k t_k \otimes \psi(\sigma^* s_k) \otimes \eta(\sigma^* \alpha_k) \right) (v) \\ &= - {}^t f(\phi)(v) \\ &= {}^t \phi(v). \end{aligned}$$

Thus

$${}^t \phi = (\psi \otimes \eta) \circ (\sigma^* \phi) \circ \psi^{-1}.$$

□

In particular, over a ramification point $p \in R$, we have

$${}^t \phi_p = \psi_p \cdot \phi_p \cdot \psi_p^{-1}. \quad (3.6)$$

Lemma 3.2.20. *Let J_r be the $r \times r$ matrix*

$$\begin{pmatrix} 0 & I_{r/2} \\ -I_{r/2} & 0 \end{pmatrix},$$

where $I_{r/2}$ is the identity matrix of size $r/2$. Let \mathcal{A} be the set of matrices A such that ${}^t A = J_r A J_r^{-1}$. Then the characteristic polynomial on \mathcal{A} that sends A to $\chi(A)$ is a square of a polynomial in the coefficients of A . In particular $\det(A)$ is a square too.

Proof. Let $B = A - \lambda I_r$, then

$${}^t B = {}^t A - \lambda I_r = J_r (A - \lambda I_r) J_r^{-1} = J_r B J_r^{-1},$$

it follows that ${}^t (J_r B) = -J_r B$, hence $J_r B$ is anti-symmetric matrix, thus

$$\chi(A) = \det(B) = \det(J_r B) = \text{pf}(J_r B)^2,$$

where, for an anti-symmetric matrix M , we denote by $\text{pf}(M)$ its Pfaffian. □

For $s \in W^{\sigma,+}$ and $p \in X$, fix an isomorphism $(K_X)_p \cong \mathbb{C}$ and let

$$P(x, p) = x^r + s_1(p)x^{r-1} + \cdots + s_r(p) \in \mathbb{C}[x].$$

Define

$$W^{\sigma,-} = \{s \in W^{\sigma,+} \mid P(x, p) \text{ is square for all } p \in R\} \subset W^{\sigma,+}.$$

Proposition 3.2.21. *The Hitchin morphism induces a map*

$$T^*\mathcal{U}_X^{\sigma,-}(r) \longrightarrow W^{\sigma,-}.$$

Moreover, for each $s \in W^{\sigma,-}$ the associated spectral curve \tilde{X}_s is singular and we have $\tilde{R} = q^{-1}(R) \subset \tilde{S} = \text{Ram}(\tilde{X}_s/X)$. And for general $s \in W^{\sigma,-}$, the singular locus of \tilde{X}_s is exactly \tilde{R} and all singular points are nodes.

Proof. The first part follows directly from equation (3.6) and Lemma 3.2.20.

Using Lemma 1.2.2 we deduce that the fixed locus of an involution on smooth curve is smooth, hence reduced (if it is not empty). This implies that for any $s \in W^{\sigma,-}$, the associated spectral curve \tilde{X}_s is singular at every point of \tilde{R} . To see that these are the only singularities for general $s \in W^{\sigma,-}$, it is sufficient to show that the set of spectral data with such property is not empty in $W^{\sigma,-}$. For this, just take the spectral data $s = (0, \dots, 0, \pi^*s_r) \in W^{\sigma,-}$, where s_r is a general section in $H^0(Y, K_Y^r \otimes \Delta^{r-1})$. Moreover, for general spectral data $s \in W^{\sigma,-}$ the divisor \tilde{R} has degree rn with just double points. Hence all singularity are nodes.

To see that $\tilde{R} \subset \tilde{S}$, recall that by [BNR89] Remark 3.3, the discriminant of the polynomial

$$x^r + q^*s_1x^{r-1} + \dots + q^*s_r$$

gives the ramification divisor $\tilde{S} = \text{Ram}(\tilde{X}_s/X)$. In other words, a point $a \in \tilde{X}_s$ (over $p \in X$) is in \tilde{S} if and only if a is a multiple root of $P(x, p)$. Hence we deduce $\tilde{R} \subset \tilde{S}$. \square

It is clear that $W^{\sigma,-}$ is not a linear subspace of $W^{\sigma,+}$. So this system is not algebraically integrable in the sense of Hitchin [Hit87].

Moreover, for general $s \in W^{\sigma,-}$, over each $p \in R$, the polynomial $P(x, p)$ is a square of a polynomial with *simple roots*. Thus the singularities are ordinary double points. The condition that the polynomial $P(x, p)$ is a square of a polynomial with simple roots is given by $r/2$ equations, hence it decreases the dimension of $W^{\sigma,+}$ by $r/2$, for each $p \in R$. More precisely, if $D = \mathbb{C}[x]_{\leq r}$ is the vector space of polynomials of degree at most r , and $\mathcal{S} \subset D$ is the locus of square polynomials. Then $W^{\sigma,-}$ can be defined as the pullback of $\bigoplus_{p \in R} \mathcal{S}$ via the map

$$W^{\sigma,+} \longrightarrow \bigoplus_{p \in R} D,$$

which sends $s \in W^{\sigma,+}$ to the polynomials $(P(x, p))_{p \in R}$. Since this map is a surjective linear map and because $\text{codim}_D(\mathcal{S}) = r/2$ we deduce

$$\begin{aligned} \dim(W^{\sigma,-}) &= \dim(W^{\sigma,+}) - 2n \frac{r}{2} \\ &= \frac{r^2}{2}(g_X - 1) + \frac{nr}{2} - nr \\ &= \frac{r^2}{2}(g_X - 1) - \frac{nr}{2} \\ &= \dim(\mathcal{U}_X^{\sigma,-}(r)). \end{aligned}$$

Let $s \in W^{\sigma,-}$ be a general spectral data such that the singular locus of \tilde{X}_s is precisely \tilde{R} and all singularities are nodes. Denote by $q : \hat{X}_s \rightarrow \tilde{X}_s$ its normalization, then the genus

$g_{\hat{X}_s}$ of \hat{X}_s is given by

$$\begin{aligned} g_{\hat{X}_s} &= (\text{arithmetic genus of } \tilde{X}_s) - (\text{number of singular points}) \\ &= r^2(g_X - 1) + 1 - \frac{1}{2}\deg(\tilde{R}) \\ &= r^2(g_X - 1) + 1 - rn. \end{aligned} \quad (3.7)$$

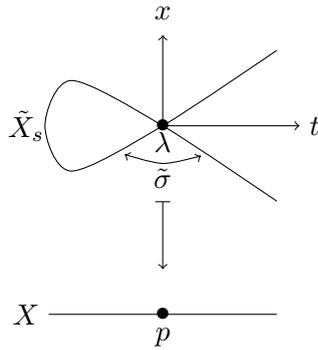
Lemma 3.2.22. *Let $\hat{\sigma}$ be the lifting of the involution $\tilde{\sigma}$ to \hat{X}_s . Then $\hat{\sigma}$ has no fixed points ($\hat{\sigma}$ interchanges the two points over each singular point). Moreover, we have*

$$\hat{X}_s/\hat{\sigma} \cong \tilde{X}_s/\tilde{\sigma} = \tilde{Y}_s.$$

Proof. If t is local parameter near $p \in R$ and x is a local parameter induced by the tautological section x of the pullback of K_X to $|K_X|$ in a neighborhood of a ramification point $\lambda \in \tilde{R}$ over p , then by definition, $\tilde{\sigma}$ sends $t \rightarrow -t$ and $x \rightarrow x$, and we can write the equation of \tilde{X}_s near λ as

$$x^2 + t^2 + (\text{higher terms}).$$

Then it is clear that $\tilde{\sigma}$ interchanges the two tangent lines at this singular point. Thus it interchanges the two branches of \tilde{X}_s over λ .



Now q induces a map $\hat{X}_s/\hat{\sigma} \rightarrow \tilde{X}_s/\tilde{\sigma}$ which is an isomorphism outside the branch points of $\tilde{X}_s \rightarrow \tilde{Y}_s$. But we see that it is a one-to-one also over this locus. Since $\tilde{X}_s/\tilde{\sigma}$ is smooth (this can be seen locally using the equation of \tilde{X}_s), we deduce that this bijection is an isomorphism. \square

Let $\hat{\pi} : \hat{X}_s \rightarrow \tilde{Y}_s$, and $\mathcal{P} = \text{Prym}(\hat{X}_s/\tilde{Y}_s)$, then we have

$$\begin{aligned} \dim(\mathcal{P}) &= g_{\tilde{Y}_s} - 1 \quad (\hat{\pi} \text{ is étale}) \\ &= \frac{1}{2}(g_{\hat{X}_s} - 1) \quad (\text{Riemann-Roch}) \\ &= \frac{r^2}{2}(g_X - 1) - \frac{rn}{2} \quad (\text{by formula (3.7)}) \\ &= \dim(\mathcal{U}_{\tilde{X}_s}^{\sigma_s^-}). \end{aligned}$$

Recall that we denoted by $\tilde{q} : \tilde{Y}_s \rightarrow Y$, $S = \text{Ram}(\tilde{Y}_s/Y)$. Let $\hat{\Delta} = \det(\hat{\pi}_* \mathcal{O}_{\hat{X}_s})^{-1}$ and $\hat{S} = \text{Ram}(\hat{X}_s/X)$. The line bundle $\mathcal{O}(\hat{S})$ descends to \tilde{Y}_s , and we have

$$\mathcal{O}(\hat{S}) = q^* \mathcal{O}(\tilde{S} - \tilde{R}) = \hat{\pi}^*(\mathcal{O}(S) \otimes \tilde{q}^* \Delta^{-1}),$$

this induces a linearization on $\mathcal{O}(\hat{S})$, which we call *positive*, and we fix it hereafter. The line bundles L on \hat{X}_s such that

$$\hat{\sigma}^*L \cong L^{-1}(\hat{S}),$$

with a $\hat{\sigma}$ -alternating isomorphism (see Lemma 2.1.3) are those with norm (with respect to $\hat{\pi}$) equals $\mathcal{O}(S) \otimes (q'^*\Delta^{-1}) \otimes \hat{\Delta}$. We denote this subvariety of line bundles by $\hat{\mathcal{P}}$. Denote by \hat{q} the map $\hat{X}_s \rightarrow X$. We have

Theorem 3.2.23. *Suppose that $X \rightarrow Y$ is ramified. For general $s \in W^{\sigma,-}$, the pushforward map*

$$\hat{q}_* : \hat{\mathcal{P}} \dashrightarrow \mathcal{U}_X^{\sigma,-}(r)$$

is dominant. In particular $\mathcal{U}_X^{\sigma,-}(r)$ has two irreducible components.

Proof. As in Theorem 3.2.12 we deduce that this map is well-defined. Moreover, using Theorem 3.2.8, we deduce that the map:

$$\Pi : T^*\mathcal{U}_X^{\sigma,-}(r) \rightarrow \mathcal{U}_X^{\sigma,-}(r) \times W^{\sigma,-}$$

is dominant. Hence, for general $s \in W^{\sigma,-}$, we get a dominant map

$$\mathcal{H}^{-1}(s) \longrightarrow \mathcal{U}_X^{\sigma,-}(r).$$

We claim that $\mathcal{H}^{-1}(s)$ is a non-empty open set of $\hat{\mathcal{P}}$. By Proposition 3.1.4, $\mathcal{H}^{-1}(s)$ is in bijection with an open set of isomorphism classes of rank one torsion-free $\mathcal{O}_{\hat{X}_s}$ -modules. Given such a torsion-free $\mathcal{O}_{\hat{X}_s}$ -module \mathcal{F} , we have $\hat{\sigma}^*\mathcal{F} \cong \mathcal{F}^*(\tilde{S})$ (follows from the exact sequence (3.3)). For general $s \in W^{\sigma,-}$, the divisor $\frac{1}{2}\tilde{R}$ is reduced, consider the line bundle $L = \mathfrak{q}^*(\mathcal{F}(-\frac{1}{2}\tilde{R}))$ on \hat{X}_s , it verifies

$$\begin{aligned} \hat{\sigma}^*L &\cong \mathfrak{q}^*(\hat{\sigma}^*\mathcal{F}(-\frac{1}{2}\tilde{R})) \\ &\cong \mathfrak{q}^*(\mathcal{F}^*(-\frac{1}{2}\tilde{R} + \tilde{S})) \\ &\cong (\mathfrak{q}^*\mathcal{F}(-\frac{1}{2}\tilde{R}))^{-1}(\hat{S}) \\ &\cong L^{-1}(\hat{S}). \end{aligned}$$

In fact the isomorphism $\hat{\sigma}^*L \cong L^{-1}(\hat{S})$ is induced by ψ , hence it is $\hat{\sigma}$ -alternating, thus $L \in \hat{\mathcal{P}}$.

Conversely, given $L \in \hat{\mathcal{P}}$ such that \hat{q}_*L is stable, then by duality of finite flat morphisms we deduce that \hat{q}_*L is σ -alternating anti-invariant vector bundle.

We have seen that the involution $\hat{\sigma}$ has no fixed point, hence $\hat{\mathcal{P}}$ has two connected components distinguished by the parity of

$$h^0(L \otimes \hat{q}^*\kappa) = h^0(\hat{q}_*L \otimes \kappa),$$

where κ is a theta characteristic over X . It follows that the image of the two connected components of $\hat{\mathcal{P}}$ can't intersect. Moreover, for some σ -invariant square root α of $\mathcal{O}(R)$, we have $\kappa = \alpha \otimes \pi^*\kappa'$, where κ' is a theta characteristic over Y . Since the σ -bilinear form $\tilde{\psi} : E \otimes \sigma^*E \rightarrow \mathcal{O}_X$ is σ -alternating, that's to say

$$\tilde{\psi}(s \otimes \sigma^*t) = -\nu(\sigma^*(\tilde{\psi}(t \otimes \sigma^*s))),$$

where $\nu : \sigma^* \mathcal{O}_X \rightarrow \mathcal{O}_X$ is the positive linearization. Taking the tensor product with α we get a bilinear form

$$(E \otimes \alpha) \otimes \sigma^*(E \otimes \alpha) \longrightarrow \mathcal{O}_X(R),$$

which induces a symmetric non-degenerate bilinear form

$$\pi_*(E \otimes \alpha) \otimes \pi_*(E \otimes \alpha) \longrightarrow \mathcal{O}_Y.$$

Hence, using the result of Mumford [Mum71], the map $\mathcal{U}_X^{\sigma,-}(r) \rightarrow \mathbb{Z}/2$ given by

$$E \longrightarrow h^0(\pi_*(E \otimes \alpha) \otimes \kappa') \pmod{2}$$

is constant under deformation of E . This implies that $\mathcal{U}_X^{\sigma,-}(r)$ has two connected components. \square

The étale case

The étale case is a little special. Assume that $\pi : X \rightarrow Y$ is étale. Recall that Δ denotes the 2-torsion line bundle associated to π . Then we have the following results

Theorem 3.2.24. *Choosing a line bundle on X of norm Δ induces by tensor product an isomorphism*

$$\mathcal{U}_X^{\sigma,+}(r) \xrightarrow{\sim} \mathcal{U}_X^{\sigma,-}(r),$$

and we have $W^{\sigma,+} = W^{\sigma,-}$. Furthermore Π induces dominant maps

$$T^* \mathcal{U}_X^{\sigma,+}(r) \longrightarrow \mathcal{U}_X^{\sigma,+}(r) \times W^{\sigma,+},$$

$$T^* \mathcal{U}_X^{\sigma,-}(r) \longrightarrow \mathcal{U}_X^{\sigma,-}(r) \times W^{\sigma,-}.$$

Moreover, for general $s \in W^{\sigma,+}$, the pushforward map q_* induces dominant rational maps

$$\mathcal{P}^+ \dashrightarrow \mathcal{U}_X^{\sigma,+}(r), \quad \mathcal{P}^- \dashrightarrow \mathcal{U}_X^{\sigma,-}(r),$$

where \mathcal{P}^+ , \mathcal{P}^- are different translates of the Prym variety of $\tilde{X}_s \rightarrow \tilde{X}_s/\tilde{\sigma}$. In particular both $\mathcal{U}_X^{\sigma,+}(r)$ and $\mathcal{U}_X^{\sigma,-}(r)$ have two connected components.

Proof. It is clear that $W^{\sigma,-} = W^{\sigma,+}$, and for general $s \in W^{\sigma,+}$, the associated spectral curve \tilde{X}_s is smooth and the attached involution $\tilde{\sigma}$ has no fixed points. Define

$$\mathcal{P}^- = \widetilde{\text{Nm}}^{-1}(\mathcal{O}(S) \otimes \tilde{\Delta}) \subset \text{Pic}^m(\tilde{X}_s).$$

Since we fixed the positive linearization on $\mathcal{O}(\tilde{S})$, it follows that the isomorphism $\tilde{\sigma}^* L \xrightarrow{\sim} L^{-1}(\tilde{S})$ is $\tilde{\sigma}$ -alternating for all $L \in \mathcal{P}^-$. Moreover, let ξ be a line bundle over X of norm Δ , and $E \in \mathcal{U}_X^{\sigma,+}(r)$. As $\sigma^* \xi \rightarrow \xi^{-1}$ is σ -alternating, then the isomorphism $\sigma^*(E \otimes \xi) \xrightarrow{\sim} (E \otimes \xi)^*$ is σ -alternating too. Since $q^* \Delta = \tilde{\Delta}$, we have

$$\widetilde{\text{Nm}}(q^* \xi) = q^* \text{Nm}(\xi) = \tilde{\Delta},$$

it follows that $q^* \xi$ induces by tensor product an isomorphism $\mathcal{P}^+ \cong \mathcal{P}^-$. This with Theorem 3.2.13 ends the proof of Theorem 3.2.24. \square

Remark 3.2.25. In this case the determinant induces a morphism $\det : \mathcal{U}_X^{\sigma,-}(r) \rightarrow P = \text{Nm}^{-1}(\mathcal{O}_Y)$ if r is even, and $\det : \mathcal{U}_X^{\sigma,-}(r) \rightarrow P' := \text{Nm}^{-1}(\Delta)$ if r is odd.

Trivial determinant case

Here we study the case of trivial determinant σ -alternating vector bundles, denoted $SU_{\hat{X}}^{\sigma,-}(r)$.

The étale case can be easily deduced by what has been established so far. More precisely by Theorem 3.2.24 and Remark 3.2.25 we deduce the following

Proposition 3.2.26. *Assume that $X \rightarrow Y$ is étale, then we have*

1. *If r is odd, then $SU_{\hat{X}}^{\sigma,-}(r) = \emptyset$.*
2. *If r is even, then $SU_{\hat{X}}^{\sigma,-}(r) \cong SU_{\hat{X}}^{\sigma,+}(r)$.*

From now on we assume that $\pi : X \rightarrow Y$ is ramified and r is even. We use the notations of subsection 3.2.2. In particular we have the following diagram

$$\begin{array}{ccccc}
 & & \hat{q} & & \\
 & & \curvearrowright & & \\
 \hat{X}_s & \xrightarrow{q} & \tilde{X}_s & \xrightarrow{q} & X \\
 & \searrow \hat{\pi} & \downarrow \tilde{\pi} & & \downarrow \pi \\
 & & \tilde{Y}_s & \xrightarrow{\tilde{q}} & Y.
 \end{array}$$

Lemma 3.2.27. *For general $s \in W^{\sigma,-}$, the pullback map $\hat{q}^* : \text{Pic}^0(X) \rightarrow \text{Pic}^0(\hat{X}_s)$ is injective.*

Proof. Note that for general $s \in W^{\sigma,-}$, the cover $\tilde{X}_s \rightarrow X$ does not factorize through an étale cover. So the same is true for $\hat{X}_s \rightarrow X$. Now apply [BL04] Proposition 11.4.3. \square

In particular we deduce that the Prym variety of $\hat{q} : \hat{X}_s \rightarrow X$ is connected.

Denote by \mathcal{Q} and $\underline{\mathcal{Q}}$ the Prym varieties of $\hat{X}_s \rightarrow X$ and $\tilde{Y}_s \rightarrow Y$ respectively, and by $\hat{\mathcal{Q}}$ and $\underline{\hat{\mathcal{Q}}}$ their dual abelian varieties. Let \mathcal{P} be the Prym variety of $\hat{\pi} : \hat{X}_s \rightarrow \tilde{Y}_s$.

Proposition 3.2.28. *For general $s \in W^{\sigma,-}$, the intersection $\mathcal{P} \cap \mathcal{Q}$ has 2^{2n-1} connected components.*

Proof. Let $\mu : \underline{\mathcal{Q}} \rightarrow \mathcal{Q}$ be the morphism induced by $\tilde{\pi}^* : \text{Pic}^0(\tilde{Y}_s) \rightarrow \text{Pic}^0(\hat{X}_s)$, and $\vartheta : \mathcal{Q} \rightarrow \underline{\hat{\mathcal{Q}}}$ the one induced by $\text{Nm}_{\hat{X}_s/\tilde{Y}_s} : \text{Pic}^0(\hat{X}_s) \rightarrow \text{Pic}^0(\tilde{Y}_s)$. As in the proof of Proposition 3.2.16, we deduce a commutative diagram

$$\begin{array}{ccc}
 \mathcal{Q} & \xrightarrow{\varphi_{\mathcal{Q}}} & \hat{\mathcal{Q}} \\
 \mu \uparrow & & \uparrow \hat{\vartheta} \\
 \underline{\mathcal{Q}} & \xrightarrow{\varphi_{\underline{\mathcal{Q}}}} & \underline{\hat{\mathcal{Q}}}.
 \end{array}$$

This implies that

$$\ker(\varphi_{\mathcal{Q}} \circ \mu) = \ker(\hat{\vartheta} \circ \varphi_{\underline{\mathcal{Q}}}).$$

Since $\hat{X}_s \rightarrow \tilde{Y}_s$ is étale, μ has degree 2. Moreover $\ker(\varphi_{\mathcal{Q}}) = \hat{q}^*(J_X[r])$, hence $\ker(\varphi_{\mathcal{Q}} \circ \mu) = (\hat{\pi}^*)^{-1}(\hat{q}^*(J_X[r]))$. Now since $\hat{\pi}$ is étale, for a line bundle $L \in \text{Pic}^0(X)$, \hat{q}^*L descends to \tilde{Y}_s if and only if it is $\hat{\sigma}$ -invariant, and since $\hat{\sigma}$ is the lifting of σ and \hat{q}^* is injective, \hat{q}^*L is $\hat{\sigma}$ -invariant if and only if L is σ -invariant. So we get

$$\ker(\varphi_{\mathcal{Q}} \circ \mu) = (\hat{\pi}^*)^{-1}(\hat{q}^*(J_X[r])^\sigma).$$

Now, the locus $(J_X[r])^\sigma$ has 2^{2n-2} isomorphic subloci which are parameterized by the types of σ -invariant line bundles over X and the identity sublocus is $\pi^*(J_Y[r])$ (see Chapter 1 for more details about σ -invariant bundles and their types). It follows that $\text{card}((J_X[r])^\sigma) = r^{2g_Y} 2^{2n-2}$. Since $\hat{\pi}^*$ has degree 2, we deduce that

$$\text{card}(\ker(\varphi_{\mathcal{Q}} \circ \mu)) = r^{2g_Y} 2^{2n-1}.$$

On the other hand, $\deg(\varphi_{\mathcal{Q}}) = r^{2g_Y}$. It follows that $\deg(\hat{\vartheta}) = 2^{2n-1}$. This implies that $\ker(\vartheta)$ has 2^{2n-1} connected components. But by definition $\ker(\vartheta) = \mathcal{P} \cap \mathcal{Q}$. \square

Recall that we have denoted by $\hat{S} = \text{Ram}(\hat{X}_s/X)$ and by $\hat{\mathcal{P}}$ the variety of line bundles L on \hat{X}_s such that

$$\hat{\sigma}^* L \cong L^{-1}(\hat{S}).$$

Let $\hat{\mathcal{Q}} = \text{Nm}_{\hat{X}_s/X}^{-1}(\hat{\delta})$, where $\hat{\delta} = \det(\hat{q}_* \mathcal{O}_{\hat{X}_s})^{-1}$.

Theorem 3.2.29. *For general $s \in W^{\sigma,-}$, the pushforward map*

$$\hat{q}_* : \hat{\mathcal{P}} \cap \hat{\mathcal{Q}} \longrightarrow \mathcal{SU}_X^{\sigma,-}(r)$$

is dominant. In particular we deduce that $\mathcal{SU}_X^{\sigma,-}(r)$ has 2^{2n-1} connected components.

Proof. Let \mathcal{P} be the Prym variety of $X \rightarrow Y$. It is clear that the map

$$\mathcal{SU}_X^{\sigma,-}(r) \times \mathcal{P} \longrightarrow \mathcal{U}_X^{\sigma,-}(r)$$

is surjective. By Theorem 3.2.23 we have that for general $E \in \mathcal{U}_X^{\sigma,-}(r)$ there exists $L \in \hat{\mathcal{P}}$ such that $\hat{q}_* L = E$. Let $\lambda \in \mathcal{P}$ be an r^{th} root of $\det(E)^{-1} \in \mathcal{P}$. It follows by the projection formula that $\hat{q}_*(L \otimes \hat{q}^* \lambda) = E \otimes \lambda \in \mathcal{SU}_X^{\sigma,-}(r)$. Note that $L \otimes \hat{q}^* \lambda \in \hat{\mathcal{P}}$ because $\text{Nm}_{\hat{X}_s/\hat{Y}_s}(\hat{q}^* \lambda) = \mathcal{O}_{\hat{Y}_s}$. Hence a general $E \in \mathcal{SU}_X^{\sigma,-}(r)$ can be written as a direct image of some $L \in \hat{\mathcal{P}}$. But since

$$\det(\hat{q}_* L) = \hat{\delta}^{-1} \otimes \text{Nm}_{\hat{X}_s/X}(L),$$

we deduce that if $\hat{q}_* L$ has trivial determinant then $\text{Nm}_{\hat{X}_s/X}(L) = \hat{\delta}$, thus $L \in \hat{\mathcal{Q}}$. So we get a dominant rational map

$$\hat{\mathcal{P}} \cap \hat{\mathcal{Q}} \longrightarrow \mathcal{SU}_X^{\sigma,-}(r).$$

Now we associate to any stable σ -alternating anti-invariant vector bundle E a topological invariant as follows: let $\psi : \sigma^* E \rightarrow E^*$ be a σ -alternating isomorphism with determinant 1 (since E is stable, there exist only r such isomorphisms). Then over any ramification point $p \in R$ take the Pfaffian $\text{pf}(\psi_p)$ of the restriction of ψ to the fiber E_p . Note that $\text{pf}(\psi_p)$ equals ± 1 . Then the type of E is defined as

$$\tau = (\text{pf}(\psi_p))_{p \in R} \pmod{\pm 1}.$$

It is clear that this is well-defined, i.e. it does not depend on the choice of ψ . Moreover the finite group $\mathcal{P}[r]$ of r -torsion elements of the Prym variety of $X \rightarrow Y$ acts equivariantly on $\mathcal{SU}_X^{\sigma,-}(r)$ and $\hat{\mathcal{P}} \cap \hat{\mathcal{Q}}$. Actually, the action of $\mathcal{P}[r]$ on $\mathcal{SU}_X^{\sigma,-}(r)$ induces a free action of $\mathcal{P}[r]/(\pi^* J_Y[r])$ on the set of types $\{\tau\}$ given by

$$\bar{\lambda} \cdot \tau = (v_p \cdot \tau_p)_{p \in R} \pmod{\pm 1},$$

where $\bar{\lambda} \in \mathcal{P}[r]/(\pi^* J_Y[r])$ and $v = (v_p)_p$ is the type of the σ -invariant line bundle $\lambda^{r/2} \in \mathcal{P}[2]$ (see Chapter 1 for the definition of these types). The number of types v is 2^{2n-2} . Since the map $\mathcal{SU}_X^{\sigma,-}(r) \times \mathcal{P} \longrightarrow \mathcal{U}_X^{\sigma,-}(r)$ is surjective, \mathcal{P} is connected and the space $\mathcal{U}_X^{\sigma,-}(r)$ has two connected components (Theorem 3.2.23), we deduce that $\mathcal{SU}_X^{\sigma,-}(r)$ has 2^{2n-1} connected components. \square

3.3 The Hitchin system for invariant vector bundles

We have seen in Remark 1.4.5 that σ -invariant vector bundles of fixed type τ correspond to parabolic vector bundles over Y with parabolic structures associated to τ at the ramification points. The Hitchin system for parabolic vector bundles has been studied in the *smooth* case by Logares and Martens [LM]. In this special case we have an explicit description of the fibers of the Hitchin map depending on the considered type, as well as a dominance result as in the case of anti-invariant vector bundles. We treat also the singular case.

We use results and notations of the previous section. We always suppose that the cover $X \rightarrow Y$ is ramified, the étale case is trivial. Fix the positive linearization on \mathcal{O}_X and the *negative* linearization on K_X (it will be clear after why we made this choice). Denote by

$$\rho : \sigma^* K_X \rightarrow K_X$$

this linearization. We have $\bar{\sigma}(x) = -x$, where $\bar{\sigma}$ is the involution on $\mathbb{S} = \mathbb{P}(\mathcal{O}_X \oplus K_X^{-1})$ induced by these linearizations and x is the tautological section of the pullback of K_X to \mathbb{S} .

Let

$$W^{\sigma, \mathfrak{m}} = \bigoplus_{i=1}^r H^0(K_X^i)_+.$$

The \mathfrak{m} in the notation refers to *maximal* types.

For simplicity, we assume in the sequel that the degree d of the σ -invariant vector bundles is 0.

Lemma 3.3.1. *Let $s \in W^{\sigma, \mathfrak{m}}$, \tilde{X}_s the associated spectral curve, then σ lifts to an involution $\tilde{\sigma}$ on \tilde{X}_s . Moreover, for general such s , we have*

- *If r is even, then this involution has no fixed point.*
- *If r is odd, this involution has just $2n$ fixed points.*

Proof. Consider $s = (s_1, \dots, s_r) \in W^{\sigma, \mathfrak{m}}$, so we have $s_i(\sigma(p)) = (-1)^i s_i(p)$ for each point $p \in X$. Let $p \in X$, and $x_0 = [x_0 : 1] \in (\tilde{X}_s)_p$, then the involution $\bar{\sigma}$ on $\mathbb{P}(\mathcal{O}_X \oplus K_X^{-1})$ attached to the fixed linearizations on \mathcal{O}_X and K_X^{-1} sends x_0 to $y_0 = [-x_0 : 1]$ in $(\tilde{X}_s)_{\sigma(p)}$, but y_0 is a solution to

$$x^r + s_1(\sigma(p))x^{r-1} + \dots + s_r(\sigma(p)) = 0.$$

Indeed

$$\begin{aligned} y_0^r + s_1(\sigma(p))y_0^{r-1} + \dots + s_r(\sigma(p)) &= (-x_0)^r + (-s_1(p))(-x_0)^{r-1} + \dots + (-1)^r s_r(p) \\ &= (-1)^r (x_0^r + s_1(p)x_0^{r-1} + \dots + s_r(p)) \\ &= 0. \end{aligned}$$

Hence $\bar{\sigma}(x_0)$ is in $(\tilde{X}_s)_{\sigma(p)}$, thus $\bar{\sigma}$ induces an involution on \tilde{X}_s which we are looking for. Note that 0 is the only fixed point of $\bar{\sigma}$ over a ramification point. One remarks that for odd i , $s_i(p) = 0$, for any $p \in R$. Suppose that r is odd, this implies that 0 is always in $(\tilde{X}_s)_p$, and for general $s \in W^{\sigma, \mathfrak{m}}$, it is a simple root of the equation above, hence there are just $2n$ fixed points in \tilde{X}_s . If r is even, we deduce that for general s , $\bar{\sigma}$ has no fixed point. \square

Denote as before $\tilde{\pi} : \tilde{X}_s \rightarrow \tilde{Y}_s := \tilde{X}_s/\tilde{\sigma}$. Using Riemann-Roch formula, we get for general $s \in W^{\sigma, m}$:

$$g_{\tilde{Y}_s} = \frac{1}{2}g_{\tilde{X}_s} + \frac{1}{2} - \frac{k}{2},$$

where k is the half of the number of fixed points of $\tilde{\sigma}$.

Let $\text{Pic}^m(\tilde{X}_s)^{\tilde{\sigma}}$ the locus of $\tilde{\sigma}$ -invariant line bundles of degree m over \tilde{X}_s . Since $g_{\tilde{X}_s} = r^2(g_X - 1) + 1$ and $g_X - 1 = 2(g_Y - 1) + n$, we deduce from Lemma 3.3.1

$$\begin{aligned} \dim(\text{Pic}^m(\tilde{X}_s)^{\tilde{\sigma}}) &= g_{\tilde{Y}_s} = \begin{cases} r^2(g_Y - 1) + n\frac{r^2}{2} + 1 & r \equiv 0 \pmod{2} \\ r^2(g_Y - 1) + n\frac{r^2-1}{2} + 1 & r \equiv 1 \pmod{2} \end{cases} \\ &= \dim(\mathcal{U}_X^\sigma(r, 0)), \end{aligned}$$

where the last equality is due to Proposition 1.3.1.

Remark 3.3.2. For general $s \in W^{\sigma, m}$, \tilde{X}_s is smooth. Indeed, taking $s = (0, \dots, 0, s_r) \in W^{\sigma, m}$, where $s_r \in H^0(K_X^r)_+$ is a general global section which vanishes at most with multiplicity one at every ramification point. Then, by the proof of Lemma 3.1.1, we deduce that \tilde{X}_s is smooth. Since $W^{\sigma, m}$ is irreducible, it follows that the set of $s \in W^{\sigma, m}$ such that \tilde{X}_s is smooth, is dense $W^{\sigma, m}$.

Let E be a stable σ -invariant vector bundle. Recall from subsection 1.3 that we have considered the involutions on $H^0(X, E \otimes E^* \otimes K_X)$ and $H^1(X, E \otimes E^*)$ induced by the canonical isomorphism $\sigma^*(E \otimes E^*) \rightarrow E \otimes E^*$ and the linearization ρ on K_X (note that this is independent of the choice of a linearization on E). By Lemma 3.2.3, Serre duality is equivariant with respect to these involutions, i.e.

$$H^1(X, E \otimes E^*)_+^* \xrightarrow{\sim} H^0(X, E \otimes E^* \otimes K_X)_+.$$

Further we have

Proposition 3.3.3. *The Hitchin morphism induces a map*

$$\mathcal{H}_E : H^0(X, E \otimes E^* \otimes K_X)_+ \longrightarrow W^{\sigma, m} = \bigoplus_{i=1}^r H^0(X, K_X^i)_+.$$

Moreover, we have an equality of dimensions

$$\dim(\mathcal{U}_X^\sigma(r, 0)) = \dim(W^{\sigma, m}).$$

Proof. By the proof of Proposition 3.2.4, we deduce that

$$\mathcal{H}_i(f(\phi)) = \rho^{\otimes i}(\sigma^*(\mathcal{H}_i(\phi))),$$

where f is the involution on $H^0(X, E \otimes E^* \otimes K_X)$ described above. Here, one should make a similar explicit local description of \mathcal{H}_i , this implies the first part of the lemma.

As we use the negative linearization on K_X , by Remark 1.1.3, it follows

$$\begin{cases} h^0(X, K_X^i)_+ = h^0(Y, K_Y^i \otimes \Delta^i) = (2i-1)(g_Y-1) + in & i \equiv 0 \pmod{2} \\ h^0(X, K_X^i)_+ = h^0(Y, K_Y^i \otimes \Delta^{i-1}) = (2i-1)(g_Y-1) + (i-1)n & i \equiv 1 \pmod{2}, i \geq 3. \\ h^0(X, K_X)_+ = h^0(Y, K_Y) = g_Y \end{cases}$$

Taking the sum, we get

$$\sum_{i=1}^r h^0(X, K_X^i)_+ = \begin{cases} r^2(g_Y-1) + n\frac{r^2}{2} + 1 & r \equiv 0 \pmod{2} \\ r^2(g_Y-1) + n\frac{r^2-1}{2} + 1 & r \equiv 1 \pmod{2} \end{cases}.$$

□

3.3.1 Smooth case

Recall from Remark 1.3.2 that we have defined a maximal type to be a type τ such that $\mathcal{U}_X^{\sigma,\tau}(r,0)$ has maximal dimension, and we have denoted the set of such types by $\mathfrak{M}\mathfrak{A}\mathfrak{X}$.

Theorem 3.3.4. *Let $s \in W^{\sigma,m}$ such that \tilde{X}_s is smooth. Then the direct image map induces a dominant map*

$$q_* : \text{Pic}^m(\tilde{X}_s)^{\tilde{\sigma}} \dashrightarrow \mathcal{U}_X^{\sigma,m}(r,0),$$

where $\mathcal{U}_X^{\sigma,m}(r,0)$ is the moduli space of σ -invariant vector bundles of type $\tau \in \mathfrak{M}\mathfrak{A}\mathfrak{X}$ (see Remark 1.3.2). Moreover, for each type $\tau \in \mathfrak{M}\mathfrak{A}\mathfrak{X}$, there exists a unique types $\tilde{\tau}$ of invariant line bundles in $\text{Pic}(\tilde{X}_s)$, such that we have a dominant map

$$q_* : \text{Pic}^m(\tilde{X}_s)^{\tilde{\sigma},\tilde{\tau}} \dashrightarrow \mathcal{U}_X^{\sigma,\tau}(r,0).$$

Proof. It is clear that L is $\tilde{\sigma}$ -invariant if and only if q_*L is σ -invariant.

By the proof of Theorem 3.2.12, for general σ -invariant vector bundle E of type $\tau \in \mathfrak{M}\mathfrak{A}\mathfrak{X}$, the restriction of the Hitchin map

$$\mathcal{H}_E : H^0(X, E \otimes E^* \otimes K_X)_+ \rightarrow W^{\sigma,m}$$

is dominant. This implies that the map

$$\Pi : T^*\mathcal{U}_X^{\sigma,m}(r,0) \longrightarrow \mathcal{U}_X^{\sigma,m}(r,0) \times W^{\sigma,m}$$

is dominant too. Moreover, if we fix a type $\tau \in \mathfrak{M}\mathfrak{A}\mathfrak{X}$, then there exists a corresponding type $\tilde{\tau}$ of $\tilde{\sigma}$ -invariant line bundles on \tilde{X}_s , such that

$$\text{Pic}^m(\tilde{X}_s)^{\tilde{\sigma},\tilde{\tau}} \dashrightarrow \mathcal{U}_X^{\sigma,\tau}(r,0)$$

is dominant. The type $\tilde{\tau}$ is constructed as follows: Suppose that r is odd (the even case is trivial since $\tilde{\sigma}$ has no fixed point by Lemma 3.3.1). Remark first that $\tilde{\tau} \in \{+1, -1\}^{2n}/\pm$. if $p \in R$ is such that $k_p = (r+1)/2$, then over such p , take -1 in $\tilde{\tau}$, and $+1$ over the rest of points in R (note that, because $s \in W^{\sigma,m}$ is general, over each $p \in R$ there is just one fixed point by $\tilde{\sigma}$, so we identify $\tilde{R} = \text{Fix}(\tilde{\sigma})$ with R in this case). If L is $\tilde{\sigma}$ -invariant line bundle over \tilde{X}_s of type $\tilde{\tau}$, then using the identification $(q_*L)_p \cong \bigoplus_{x \in q^{-1}(p)} L_x$, we see easily that the type of q_*L is τ . □

We prove in the next section that the only types corresponding to smooth spectral covers of X are the maximal ones.

3.3.2 General case

Now let τ be any type, for simplicity of notations we suppose that we have just one point $p \in R$ such that $k_p < \lfloor \frac{r}{2} \rfloor$, where $\lfloor \cdot \rfloor$ stands for the floor function. In fact we can always suppose $k_p \leq \lfloor r/2 \rfloor$ due to taking the tensor product by $\mathcal{O}_X(p)$. And for all other ramification point $a \neq p$, we assume that k_a is maximal (that's $k_a = r/2$ if r is even, and $k_a = (r \pm 1)/2$ if r is odd).

To get a vector bundle of such type as a direct image, we should have $r - 2k_p$ fixed points by $\tilde{\sigma}$ above p . Indeed, let x be a point of \tilde{X}_s above p , as $\tilde{\sigma}$ interchanges the two fibers L_x and $L_{\tilde{\sigma}(x)}$, its matrix over these two points is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, one should have $2k_p$ non-fixed points to get the -1 eigenvalue with multiplicity k_p . This intuition is proved in the following theorem

Theorem 3.3.5. *Let τ be a type as above, and denote by*

$$W^{\sigma,\tau} = H^0(K_X)_+ \oplus \cdots \oplus H^0(K_X^{2k_p+1})_+ \oplus \bigoplus_{i=2k_p+2}^r H^0(X, K_X^i(-(i-2k_p-1)p))_+.$$

Then, for any σ -invariant stable vector bundle E of type τ , the Hitchin map factorizes through $W^{\sigma,\tau}$ given a map

$$\mathcal{H}_E : H^0(E \otimes E^* \otimes K_X)_+ \longrightarrow W^{\sigma,\tau}.$$

Moreover,

$$\dim(\mathcal{U}_X^{\sigma,\tau}(r, 0)) = \dim(W^{\sigma,\tau}).$$

Proof. First we verify the dimensions. By Proposition 1.3.1, we have

$$\dim(\mathcal{U}_X^{\sigma,\tau}(r, 0)) = r^2(g_Y - 1) + 1 + k_p(r - k_p) + \begin{cases} (2n-1)\frac{r^2}{4} & \text{if } r \equiv 0 \pmod{2} \\ (2n-1)\frac{r^2-1}{4} & \text{if } r \equiv 1 \pmod{2} \end{cases}.$$

Recall that we have fixed the negative linearization on K_X . By Lefschetz fixed point formula we deduce that

$$h^0(X, K_X^k(-ip))_+ = (2k-1)(g_Y - 1) + \begin{cases} kn - \frac{i}{2} & k \equiv 0, i \equiv 0 \pmod{2} \\ kn - \frac{i+1}{2} & k \equiv 0, i \equiv 1 \pmod{2} \\ (k-1)n - \frac{i}{2} & k \equiv 1, i \equiv 0 \pmod{2} \\ (k-1)n - \frac{i-1}{2} & k \equiv 1, i \equiv 1 \pmod{2} \end{cases}. \quad (3.8)$$

So the dimension of $W^{\sigma,\tau}$ is given by

$$\dim(W^{\sigma,\tau}) = \dim(W^{\sigma,m}) - \sum_{i=1}^{r-2k_p-1} d(i),$$

where

$$d(i) = \begin{cases} \frac{i}{2} & \text{if } i \equiv 0 \pmod{2} \\ \frac{i+1}{2} & \text{if } i \equiv 1 \pmod{2} \end{cases}.$$

By a simple computation, we get

$$\sum_{i=1}^{r-2k_p-1} d(i) = \begin{cases} \frac{r^2}{4} - k_p(r - k_p) & \text{if } r \equiv 0 \pmod{2} \\ \frac{r^2-1}{4} - k_p(r - k_p) & \text{if } r \equiv 1 \pmod{2} \end{cases}.$$

It follows that

$$\begin{aligned} \dim(W^{\sigma,\tau}) &= \begin{cases} r^2(g_Y - 1) + 1 + (2n-1)\frac{r^2}{4} + k_p(r - k_p) & \text{if } r \equiv 0 \pmod{2} \\ r^2(g_Y - 1) + 1 + (2n-1)\frac{r^2-1}{4} + k_p(r - k_p) & \text{if } r \equiv 1 \pmod{2} \end{cases} \\ &= \dim(\mathcal{U}_X^{\sigma,\tau}(r, 0)). \end{aligned}$$

Now take $\phi \in H^0(X, E \otimes E^* \otimes K_X)_+$ and let $\varphi : \sigma^*E \xrightarrow{\sim} E$ be a linearization. Then the following diagram commutes

$$\begin{array}{ccc} \phi : & E & \longrightarrow & E \otimes K_X \\ & \uparrow \varphi & & \downarrow \sigma^*(\varphi \otimes \rho) \\ \sigma^*\phi : & \sigma^*E & \longrightarrow & \sigma^*E \otimes \sigma^*K_X, \end{array}$$

that's

$$\sigma^* \phi = \sigma^*(\varphi \otimes \rho) \circ \phi \circ \varphi.$$

In particular over p , one has

$$\phi_p = -A_p \phi_p A_p,$$

where $A_p = \text{diag}(\underbrace{-1, \dots, -1}_{k_p}, 1, \dots, 1)$ is $r \times r$ diagonal matrix.

This implies that the matrix ϕ_p is of the form

$$\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix},$$

where M and N are two matrices of type $k_p \times (r - k_p)$ and $(r - k_p) \times k_p$.

Now let t be a local parameter on the neighborhood of p , and denote $\phi(t) = \phi_p + t\phi'$ the restriction of ϕ to this neighborhood. Then we get

$$s_r(t) = \det(\phi(t)) = \det(\phi_p + t\phi').$$

But

$$\phi_p + t\phi' = \begin{pmatrix} t \times * & M' \\ N' & t \times * \end{pmatrix}, \quad (3.9)$$

where M' and N' are two matrices of type $k_p \times (r - k_p)$ and $(r - k_p) \times k_p$ respectively, which are not necessarily divisible by t . Using the development of the determinant as a sum of monomials in the entries of the matrix, we see that every monomial contains at least $r - 2k_p$ entries that belong to neither M' nor N' , i.e. divisible by t . So we deduce that $\det(\phi(t))$ is divisible by t^{r-2k_p} , hence

$$\det(\phi(t)) \in H^0(X, K_X^r(-(r - 2k_p)p))_+.$$

But we have

$$H^0(X, K_X^r(-(r - 2k_p - 1)p))_+ = H^0(X, K_X^r(-(r - 2k_p)p))_+,$$

because the first space is included in the second one and they have the same dimension by formula (3.8).

The general case is treated similarly. Let $i \geq 2k_p + 2$. Consider $\phi(t)$ as an element of $\mathcal{M}at_r(\mathbb{C}[[t]])$, and denote it by $\phi(t) = (a_{i,j}(t))_{i,j}$. As $\mathcal{H}_i(\phi)$ is by definition $(-1)^i \text{Tr}(\Lambda^i \phi(t))$. We just need to calculate the diagonal elements of the matrix $\Lambda^i \phi(t)$. Assume that

$$\Lambda^i \phi(t) = (\alpha_{\underline{k}, \underline{l}}(t)),$$

where \underline{k} and \underline{l} are i -tuples of strictly increasing integer in $\{1, \dots, r\}$. Then if $\underline{k} = (k_1 < \dots < k_i)$ we have

$$\alpha_{\underline{k}, \underline{k}}(t) = \det(a_{k_l, k_{l'}}(t))_{1 \leq l, l' \leq i}.$$

Hence, from the form of $\phi(t)$ given in (3.9), we deduce that $\mathcal{H}_i(\phi(t))$ is divisible by at least t^{i-2k_p} , hence

$$\mathcal{H}_i(\phi) \in H^0(X, K_X^i(-(i - 2k_p)p))_+ = H^0(X, K_X^i(-(i - 2k_p - 1)p))_+.$$

Thus

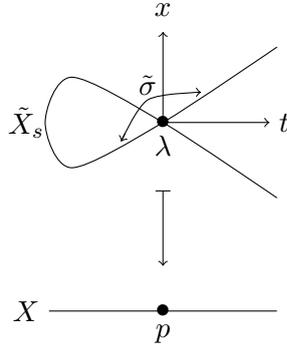
$$\mathcal{H}_E(\phi) \in W^{\sigma, \tau}.$$

□

Unfortunately, most of the types correspond to singular spectral curve $\tilde{X}_s \rightarrow X$. But for general $s \in W^{\sigma, \tau}$, where τ is as above, the corresponding \tilde{X}_s has just the point 0 over p which is singular with multiplicity $r - 2k_p$. Moreover, this singularity is ordinary (the tangents at this point are distinct), we can see that using the equation defining \tilde{X}_s and the generality of $s \in W^{\sigma, \tau}$. Hence the geometric genus of the normalization \hat{X}_s of \tilde{X}_s is equals to

$$g_{\hat{X}_s} = r^2(g_X - 1) + 1 - \frac{(r - 2k_p)(r - 2k_p - 1)}{2}.$$

Moreover, the involution $\tilde{\sigma}$ lifts to an involution $\hat{\sigma}$ on \hat{X}_s with $r - 2k_p$ fixed points if r is even and $r - 2k_p + 2n - 1$ if r is odd (Recall that we assumed for simplicity that we have just one point $p \in R$ with $k_p < [r/2]$). Indeed, if t is a local parameter in a local neighborhood of p and x is a local parameter near the ramification point $0 \in \tilde{R}$ over p , then by definition, $\tilde{\sigma}$ sends $t \rightarrow -t$ and $x \rightarrow -x$, thus it does not interchange the two branches near λ



If r is odd, $\hat{\sigma}$ fixes also the fixed points of $\tilde{\sigma}$ outside the singular points.

Let $\varepsilon(r) = r \pmod{2}$. It follows that the genus $g_{\hat{Y}_s}$ of $\hat{Y}_s = \hat{X}_s / \hat{\sigma}$ is given by

$$\begin{aligned} g_{\hat{Y}_s} &= \frac{1}{2} \left(g_{\hat{X}_s} + 1 - \frac{r - 2k_p}{2} - \varepsilon(r) \frac{2n - 1}{2} \right) \\ &= \frac{r^2}{2} (g_X - 1) + 1 - \frac{(r - 2k_p)^2}{4} + \varepsilon(r) \frac{2n - 1}{4} \\ &= r^2 (g_Y - 1) + 1 + n \frac{r^2}{2} - \frac{(r - 2k_p)^2}{4} + \varepsilon(r) \frac{2n - 1}{4} \\ &= r^2 (g_Y - 1) + 1 + (2n - 1) \frac{r^2 - \varepsilon(r)}{4} + k_p (r - k_p). \end{aligned}$$

This implies

$$g_{\hat{Y}_s} = \dim(\text{Pic}^m(\hat{X}_s)^{\hat{\sigma}}) = \dim(\mathcal{U}_X^{\sigma, \tau}(r, 0)).$$

So the generic fiber of the Hitchin morphism

$$\mathcal{H} : T^* \mathcal{U}_X^{\sigma, \tau}(r, 0) \longrightarrow W^{\sigma, \tau}$$

is of maximal dimension. So we get again the complete integrability of the Hitchin system in this case too.

Moreover, with the same method used so far proving the dominance results, we deduce

Theorem 3.3.6. *For each type τ , the pushforward map*

$$\text{Pic}^m(\hat{X}_s)^{\hat{\sigma}, \hat{\tau}} \dashrightarrow \mathcal{U}_X^{\sigma, \tau}(r, 0)$$

is dominant, for some type $\hat{\tau}$ of $\hat{\sigma}$ -invariant line bundles over \hat{X}_s .

Proof. First the type $\hat{\tau}$ is constructed as follows: If r is even, $\hat{\sigma}$ has $r - 2k_p$ fixed points which are all over p , then since k_p is chosen strictly smaller than $[r/2]$, we take the trivial type over these $r - 2k_p$ fixed points. If r is odd, then over any ramification point a such that $k_a = \frac{r+1}{2}$ the type is equal -1 , and over the rest of ramification points other than p the type is $+1$.

Now we deduce as in the proof of Theorem 3.2.12 that the map

$$\Pi : T^*\mathcal{U}_X^{\sigma,\tau}(r, 0) \longrightarrow \mathcal{U}_X^{\sigma,\tau}(r, 0) \times W^{\sigma,\tau}$$

is dominant. So for general $s \in W^{\sigma,\tau}$, the fiber $\mathcal{H}^{-1}(s)$ dominates $\mathcal{U}_X^{\sigma,\tau}(r, 0)$. Moreover $\mathcal{H}^{-1}(s)$ is identified, by Proposition 3.1.4, with a set of torsion-free rank one sheaves over \tilde{X}_s , which are $\tilde{\sigma}$ -invariant. Let x_0 be the singular point of \tilde{X}_s over p . Then twisting these torsion-free sheaves with

$$\mathcal{O}_{\tilde{X}_s} \left(-\frac{(r-2k_p)(r-2k_p-1)}{2} x_0 \right)$$

and pulling them back to \hat{X}_s , we identify $\mathcal{H}^{-1}(s)$ with the open subset of $\text{Pic}^m(\hat{X}_s)^{\hat{\sigma}, \hat{\tau}}$ of line bundles such that \hat{q}_*L is stable, where m is the degree of $(\hat{q}_*\mathcal{O}_{\hat{X}_s})^*$ ($m = r(r-1)(g_X - 1) - \frac{(r-2k_p)(r-2k_p-1)}{2}$). The result follows. \square

Chapter 4

Conformal Blocks

Recall that we have denoted by $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$ (resp. $\mathcal{S}\mathcal{U}_X^{\sigma,\tau}(r)$) the moduli stack of σ -symmetric (resp. σ -alternating of type τ) anti-invariant vector bundles with trivial determinant. When the type τ is the trivial one, we denote by $\mathcal{S}\mathcal{U}_X^{\sigma,-}(r)$ the corresponding moduli stack. The main topic of this chapter is the study of some line bundles over the moduli stacks $\mathcal{S}\mathcal{U}_X^{\sigma,\pm}(r)$ and the spaces of their global sections, called generalized theta functions. It turns out that the restriction of the determinant bundle to $\mathcal{S}\mathcal{U}_X^{\sigma,-}(r)$ has a square root associated to each σ -invariant theta characteristic κ over X , we denote \mathcal{P}_κ this line bundle, and call it the *Pfaffian of cohomology* line bundle. However, in the ramified case, this is not true for the stack $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$ of σ -symmetric bundles. In the étale case, the two stacks are isomorphic (for even rank, see Theorem 3.2.24). In this case also, the determinant bundle admits some square roots.

Our main result is the identification of the spaces of global sections of the powers of the Pfaffian line bundle \mathcal{P}_κ (resp. the determinant line bundle \mathcal{D}) on $\mathcal{S}\mathcal{U}_X^{\sigma,-}(r)$ (resp. $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$) with the *conformal blocks* spaces $\mathcal{V}_{\sigma,-}(k)$ and $\mathcal{V}_{\sigma,+}(k)$ associated to some twisted affine Kac-Moody Lie algebras respectively.

4.1 Preliminaries on twisted Kac-Moody algebras

In this first section, we recall briefly the construction of the twisted affine Kac-Moody Lie algebras and the attached conformal blocks. We use notations of [Kac90]. The definition of twisted conformal blocks is adapted from [Hon16] (which in turn adapted from [FS04], where a more general definition is given in the framework of vertex algebras).

Consider an outer automorphism τ of the Lie algebra $\mathfrak{sl}_r(\mathbb{C})$. It is an order two automorphism. The involution τ is extended to an automorphism of the affine Kac-Moody algebra $\widehat{\mathcal{L}}(\mathfrak{sl}_r) = \mathfrak{sl}_r(\mathcal{K}) \oplus \mathbb{C}K$, where $\mathcal{K} = \mathbb{C}((t))$ and K a central element, by sending $x \otimes g(t)$ to $\tau(x) \otimes g(-t)$ and fixes the center. Then the fixed subalgebra of this involution, denoted by $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \tau)$, is an affine Lie algebra of type $A_l^{(2)}$ (after adding a scaling element D), where $l = \lfloor r/2 \rfloor$, and it is called *twisted* affine Lie algebra. Let \mathfrak{g} be the finite simple Lie algebra of $\mathcal{L}(\mathfrak{sl}_r, \tau)$ (see [Kac90] §6.3 for a precise definition). Then \mathfrak{g} is of type C_l if r is odd, and is isomorphic to the fixed subalgebra $\mathfrak{sl}_r(\mathbb{C})^\tau$ if r is even.

Since we will be interested mainly in the following two involutions

$$\sigma^+(a(t)) = -{}^t a(-t), \quad \sigma^-(a(t)) = -J_r {}^t a(-t) J_r^{-1},$$

where

$$J_r = \begin{pmatrix} 0 & I_{r/2} \\ -I_{r/2} & 0 \end{pmatrix},$$

we give an explicit constructions of $\mathcal{L}(\mathfrak{sl}_r, \sigma^\pm)$.

Let $M_{i,j}$ be the canonical basis of the vector space of square matrices of size r . Let $\mathfrak{h} \subset \mathfrak{sl}_r(\mathbb{C})$ be the Cartan subalgebra of diagonal matrices and let $\alpha'_1, \dots, \alpha'_{r-1} \in \mathfrak{h}^*$ be the

simple roots defined by $\alpha_i = M_{i,i}^* - M_{i+1,i+1}^*$. Denote by E'_1, \dots, E'_{r-1} and F'_1, \dots, F'_{r-1} the Chevalley generators of $\mathfrak{sl}_r(\mathbb{C})$: $E'_i = M_{i,i+1}$, $F'_i = -{}^t E'_i$. Let $\alpha'_0{}^\vee = M_{1,1} - M_{r,r}$, $E'_0 = M_{1,r}$ and $F'_0 = -{}^t E'_0$. Then the Chevalley generators of $\widehat{\mathcal{L}}(\mathfrak{sl}_r)$ are given by

$$e_0 = t \otimes E'_0, \quad f_0 = t^{-1} \otimes F'_0,$$

and for $i \in \{1, \dots, r-1\}$

$$e_i = 1 \otimes E'_i, \quad f_i = 1 \otimes F'_i.$$

Recall the Lie bracket on $\widehat{\mathcal{L}}(\mathfrak{sl}_r)$ is given by

$$[g(t), h(t)] = [g, h] \otimes P(t)Q(t) + (g, h) \text{Res}\left(\frac{dP}{dt}Q\right)K,$$

where $g, h \in \mathfrak{sl}_r$, $P, Q \in \mathcal{K}$ and $(,)$ is the normalized Killing form on \mathfrak{sl}_r .

Moreover, by extending the linear forms α_i to $\mathfrak{h} \oplus \mathbb{C}K$ such that $\alpha_i(K) = 0$, then α_i are the simple roots of $\widehat{\mathcal{L}}(\mathfrak{sl}_r)$.

Case $\mathcal{L}(\mathfrak{sl}_r, \sigma^-)$. Let $r = 2l$. This is the algebra constructed in [Kac90] Page 128. We can assume, after conjugation, that σ^- sends E'_i to E'_{r-i} , $F'_i \rightarrow F'_{r-i}$ and $\alpha'_i \rightarrow \alpha'_{r-i}$. So let's define

- $\alpha_i{}^\vee = \alpha'_i{}^\vee + \alpha'_{r-i}{}^\vee$ ($1 \leq i \leq l-1$), $\alpha_l{}^\vee = \alpha'_l{}^\vee$ and $\alpha_0{}^\vee = -2\alpha'_0{}^\vee + \alpha'_1{}^\vee + \alpha'_{r-1}{}^\vee$.
- $E_i = E'_i + E'_{r-i}$ ($1 \leq i \leq l-1$), $E_l = E'_l$ and $E_0 = E'_{-\alpha'_0 + \alpha'_{r-1}} - E'_{-\alpha'_0 + \alpha'_1}$.
- $F_i = F'_i + F'_{r-i}$ ($1 \leq i \leq l-1$), $F_l = F'_l$ and $F_0 = -E'_{\alpha'_0 - \alpha'_{r-1}} + E'_{\alpha'_0 - \alpha'_1}$.

The Chevalley generators of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)$ are given by

$$e_i = 1 \otimes E_i, \quad f_i = 1 \otimes F_i \quad \text{for } i = 1, \dots, l.$$

$$e_0 = t \otimes E_0, \quad f_0 = t^{-1} \otimes F_0.$$

Consider the elements $\tilde{\alpha}_i{}^\vee = 2\alpha_i{}^\vee / (\alpha_i{}^\vee, \alpha_i{}^\vee) \in \mathfrak{h}$. Since the normalized bilinear form $(;)$ is non-degenerate on \mathfrak{h} it induces an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$. So let $\tilde{\alpha}_i$ be the images of $\tilde{\alpha}_i{}^\vee$ under this bijection. Then the simple roots of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)$ are given by

$$\alpha_0 = \frac{1}{2} \otimes \tilde{\alpha}_0,$$

$$\alpha_i = 1 \otimes \tilde{\alpha}_i, \quad i = 1, \dots, l.$$

The simple coroots are just $1 \otimes \alpha_i{}^\vee$, for $i = 1, \dots, l$. We denote them again by $\alpha_i{}^\vee$. For $i = 0$ the simple coroot is $2K + 1 \otimes \alpha_0{}^\vee$. We denote it also by $\alpha_0{}^\vee$.

In particular, the normalized bilinear form on $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)$ is given by

$$(P \otimes x; Q \otimes y) = \frac{1}{2} \text{Res}(t^{-1}PQ)(x; y),$$

where $(,)$ is the normalized bilinear form on $\mathfrak{sl}_r(\mathbb{C})$. The 2-cocycle on $\mathcal{L}(\mathfrak{sl}_r, \sigma^-)$ that defines $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)$ is given by

$$\psi(g(t), h(t)) = \frac{1}{2} \text{Res}\left(\text{Tr}\left(\frac{dg}{dt}h\right)\right).$$

Case $\mathcal{L}(\mathfrak{sl}_r, \sigma^+)$. We treat the case $r = 2l$ (the odd case is again treated in [Kac90]). We can assume, after conjugation, that σ^+ sends E'_i to $-E'_{r-i}$, $F'_i \rightarrow -F'_{r-i}$ and $\alpha'_i \rightarrow \alpha'_{r-i}$. So we define the following elements of \mathfrak{sl}_{2l} :

- $\beta_i^\vee = \alpha'_{l-i} + \alpha'_{l+i}$ ($1 \leq i \leq l-1$), $\beta_l^\vee = \alpha'_0$ and $\beta_0^\vee = 2\alpha'_l + \alpha'_{l-1} + \alpha'_{l+1}$.
- $E_i = E'_i - E'_{r-i}$ ($1 \leq i \leq l-1$), $E_l = E'_0$ and $E_0 = E'_{\alpha'_l + \alpha'_{l+1}} - E'_{\alpha'_l + \alpha'_{l-1}}$.
- $F_i = F'_i - F'_{r-i}$ ($1 \leq i \leq l-1$), $F_l = F'_0$ and $F_0 = -E'_{-\alpha'_l - \alpha'_{l+1}} + E'_{-\alpha'_l - \alpha'_{l-1}}$.

Remark that the affine node β_0 of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+)$ is then the node α_l with the notation of Table Aff 2 of [Kac90] Page 55. Thus when deleting this node the remaining diagram is of type D_l .

As before, we define the Chevalley generators of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+)$ by

$$e_i = 1 \otimes E_i, \quad f_i = 1 \otimes F_i \quad \text{for } i = 1, \dots, l.$$

$$e_0 = t \otimes E_0, \quad f_0 = t^{-1} \otimes F_0.$$

The simple coroots of the simple invariant Lie algebra ($= \mathfrak{so}_{2l}$) are given by

$$\tilde{\beta}_i^\vee = 2\beta_i^\vee / (\beta_i^\vee, \beta_i^\vee), \quad i = 0, \dots, l.$$

As above denote by $\tilde{\beta}_i$ the corresponding elements of \mathfrak{h}^* . Then the simple roots of $\widehat{\mathcal{L}}(\mathfrak{sl}_{2l}, \sigma^+)$ are given by

$$\beta_0 = 2K + 1 \otimes \tilde{\beta}_0,$$

$$\beta_i = 1 \otimes \tilde{\beta}_i, \quad i = 1, \dots, l.$$

From the construction of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+)$, it is clear that the Coxeter coefficients and their dual in this case are taken in the inverse order. We recall the dual Coxeter coefficients of the twisted Kac-Moody algebras $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$ in the following table.

	a_0^\vee	a_1^\vee	a_2^\vee	\cdots	a_{l-1}^\vee	a_l^\vee
$\widehat{\mathcal{L}}(\mathfrak{sl}_{2l}, \sigma^+)$	2	2	2	\cdots	1	1
$\widehat{\mathcal{L}}(\mathfrak{sl}_{2l}, \sigma^-)$	1	1	2	\cdots	2	2
$\widehat{\mathcal{L}}(\mathfrak{sl}_{2l+1}, \sigma^+)$	1	2	2	\cdots	2	2

(4.1)

Dual Coxeter coefficients.

Now, when we add a scaling elements to the above algebras, i.e. derivations D_\pm such that

$$[D_\pm, t^n \otimes x] = nt^n \otimes x,$$

then, by [Kac90] Theorem 8.5, both Kac-Moody algebras $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm) \oplus \mathbb{C}D_\pm$ are isomorphic to the Kac-Moody algebra $\mathfrak{g}(A)$, where A is the affine generalized Cartan matrix of type $A_{r-1}^{(2)}$. In particular, we deduce an isomorphism

$$\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+) \oplus \mathbb{C}D_+ \cong \widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-) \oplus \mathbb{C}D_-.$$

Moreover, the derivations D_\pm induces a weight decomposition of the algebras $\mathcal{L}(\mathfrak{sl}_r, \sigma^\pm) \oplus \mathbb{C}D_\pm$. The main observation is that the above isomorphism does not respect the decomposition $\mathcal{L}(\mathfrak{sl}_r, \sigma^\pm) \oplus \mathbb{C}K \oplus \mathbb{C}D_\pm$.

We will see in a moment that under the above isomorphism, the fundamental weight λ_0^+ of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+) \oplus \mathbb{C}D$ is sent to twice the fundamental weight λ_0^- .

Twisted conformal blocks

Let $\lambda_0^\pm, \dots, \lambda_l^\pm$ be the fundamental weights of the twisted affine Lie algebras $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$, i.e. λ_i^\pm are linear forms on the Cartan subalgebras such that

$$\lambda_i^+(\beta_j) = \lambda_i^-(\alpha_j) = \delta_{ij}, \quad i, j = 0, \dots, l.$$

Denote by $\mathfrak{g} \subset \mathcal{L}(\mathfrak{sl}_r, \sigma^\pm)$ the simple Lie algebra generated by e_i and f_i for $i = 1, \dots, l$. Note that \mathfrak{g} is of type D_l in the case of σ^+ when r is even, and it is of type C_l otherwise. Moreover, we have the identifications

$$\lambda_i^\pm = \check{\lambda}_i + a_i^\vee \lambda_0^\pm, \quad i = 1, \dots, l.$$

where $\check{\lambda}_i$ ($i = 1, \dots, l$) are the fundamental weights of \mathfrak{g} .

Remark 4.1.1. Remark that, for an even rank r , the weight λ_0^+ has level equals $a_0^\vee = 2$, while λ_0^- has level $a_0^\vee = 1$ (see Table 4.1).

Denote by $P^{\sigma, \pm}$ the set of dominant integral weights of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$. By [Kac90] §12.4 (see also [Hon16] Lemma 6.1), one deduces a bijection between $P^{\sigma, \pm}$ and the set

$$\tilde{P}^{\sigma, \pm} = \{(\lambda, k) | \lambda \in \mathring{P}, \langle \lambda, \varrho^\pm \rangle \leq k\},$$

where \mathring{P} is the set of dominant weights of \mathfrak{g} , and ϱ^- is the highest coroot of \mathfrak{g} when r is even, and ϱ^+ is twice the highest short coroot of \mathfrak{g} when r is odd.

For $\mu^\pm \in P^{\sigma, \pm}$, denote by $\mathcal{H}_{\mu^\pm}(k)$ the irreducible highest weight module of level k of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$ of highest weight μ^\pm . Let $\vec{\mu}^\pm = (\mu_1^\pm, \dots, \mu_{2n}^\pm)$ be a vector of elements of $P^{\sigma, \pm}$ parameterized by the points of R , and define

$$\mathcal{H}_{\vec{\mu}^\pm}(k) = \mathcal{H}_{\mu_1^\pm}(k) \otimes \dots \otimes \mathcal{H}_{\mu_{2n}^\pm}(k).$$

Finally, let $\mathcal{A}_R = H^0(X \setminus R, \mathcal{O}_X)$. By considering the associated Lorrent series at $p \in R$, we get an inclusion $\mathfrak{sl}_r(\mathcal{A}_R)^{\sigma^\pm} \subset \mathfrak{sl}_r(\mathcal{K}_p)^{\sigma^\pm}$. We can then define an action of $\mathfrak{sl}_r(\mathcal{A}_R)^{\sigma^\pm}$ on $\mathcal{H}_{\vec{\mu}^\pm}(k)$ as product of representations (i.e diagonal action). More explicitly, for $\alpha \in \mathfrak{sl}_r(\mathcal{A}_R)^{\sigma^\pm}$ and $X = X_1 \otimes \dots \otimes X_{2n}$, we have

$$\alpha \cdot X = \sum_i X_1 \otimes \dots \otimes \alpha \cdot X_i \otimes \dots \otimes X_{2n}.$$

Definition 4.1.2. The *conformal block* attached to the data $(X, \sigma, R, \vec{\mu}^\pm, \widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm), k)$ is defined by

$$\mathcal{V}_{\sigma, \pm}(k) = \left[(\mathcal{H}_{\vec{\mu}^\pm}(k))_{\mathfrak{sl}_r(\mathcal{A}_R)^{\sigma^\pm}} \right]^*,$$

where for a \mathfrak{g} -module V , we denote by $V_{\mathfrak{g}}$ the space of coinvariants of V , thus the largest quotient of V on which \mathfrak{g} acts trivially.

In [Hon16], it is shown that these are finite dimensional vector spaces and a formula for their dimensions is conjectured (a Verlinde formula for twisted conformal blocks).

4.2 Loop groups and uniformization theorem

4.2.1 Uniformization theorem

For a ramification point $p \in X$, denote by \mathcal{O}_p the completion of the local ring at p , \mathcal{K}_p its fraction field and \mathcal{V}_p a complementary vector subspace of \mathcal{O}_p in \mathcal{K}_p . Let $\mathcal{S}\mathcal{U}_X(r)$ denote the moduli stack of rank r vector bundles over X with a trivialization of its determinant. Let's fix the canonical linearization on \mathcal{O}_X , so we identify $\sigma^*\mathcal{O}_X$ and \mathcal{O}_X . Moreover, since all the types are isomorphic, we assume hereafter that $\tau = (+1, \dots, +1) \bmod \pm 1$ and denote the corresponding moduli stack by $\mathcal{S}\mathcal{U}_X^{\sigma^\pm}(r)$.

In [BL94], it is proved that

$$\mathcal{S}\mathcal{U}_X(r) \cong \mathbf{SL}_r(\mathcal{O}_p) \backslash \mathbf{SL}_r(\mathcal{K}_p) / \mathbf{SL}_r(\mathcal{A}_p),$$

where $\mathcal{A}_p = H^0(X - p, \mathcal{O}_X)$. Let t be a local parameter at p , then $\mathcal{K}_p \cong \mathbb{C}((t))$, $\mathcal{O}_p \cong \mathbb{C}[[t]]$.

Consider the two involutions σ^\pm on $\mathbf{SL}_r(\mathcal{K}_p)$ given by

$$\begin{aligned} g(t) &\rightarrow \sigma^+(g(t)) = {}^t g(-t)^{-1}, \\ g(t) &\rightarrow \sigma^-(g(t)) = J_r \cdot {}^t g(-t)^{-1} \cdot J_r^{-1}, \end{aligned}$$

where

$$J_r = \begin{pmatrix} 0 & I_{r/2} \\ -I_{r/2} & 0 \end{pmatrix},$$

$I_{r/2}$ is the identity matrix of size $r/2$. Note that

$$\text{pf}(J_r) = 1.$$

Let $\mathcal{Q} = \mathbf{SL}_r(\mathcal{O}_p) \backslash \mathbf{SL}_r(\mathcal{K}_p)$. In [PR08b], it is proved that

$$\mathcal{Q}^{\sigma^\pm} = \mathbf{SL}_r(\mathcal{O}_p)^{\sigma^\pm} \backslash \mathbf{SL}_r(\mathcal{K}_p)^{\sigma^\pm}.$$

Note that $\mathbf{SL}_r(\mathcal{O}_p)^{\sigma^\pm}$ is the maximal parahoric subgroup of $\mathbf{SL}_r(\mathcal{K}_p)^{\sigma^\pm}$ and, with the notations of loc. cit. this case corresponds to $I = \{0\}$. In fact, their involution is the conjugation of σ^\pm by the anti-diagonal matrix D_r with all entries equal 1. But this does not change much. Indeed, by taking a matrix A such that $D_r = {}^t A A$ (such matrix can be constructed easily), then conjugation by A realizes an isomorphism between $\mathbf{SL}_r(\mathcal{K})^{\sigma^\pm}$ and their invariant locus.

We denote in the sequel by \mathcal{Q}^{σ^\pm} the quotient $\mathbf{SL}_r(\mathcal{O}_p)^{\sigma^\pm} \backslash \mathbf{SL}_r(\mathcal{K}_p)^{\sigma^\pm}$, for some $p \in R$.

Theorem 4.2.1. *We have an isomorphism of stacks*

$$\mathcal{S}\mathcal{U}_X^{\sigma^\pm}(r) \cong \mathbf{SL}_r(\mathcal{O}_p)^{\sigma^\pm} \backslash \mathbf{SL}_r(\mathcal{K}_p)^{\sigma^\pm} / \mathbf{SL}_r(\mathcal{A}_p)^{\sigma^\pm}.$$

Moreover, the projections $\mathcal{Q}^{\sigma^\pm} \rightarrow \mathcal{S}\mathcal{U}_X^{\sigma^\pm}(r)$ are locally trivial for the fppf topology.

Proof. Let \mathcal{G} and \mathcal{H}_τ the invariant Weil restrictions of the constant group scheme \mathbf{SL}_r (see Proposition 2.2.1). Using the main Theorem of [Hei10], we deduce, for a ramification point $p \in X$ over a branch point $y \in Y$, that

$$\begin{aligned} \mathcal{M}_Y(\mathcal{G}) &\cong \mathcal{G}(\mathcal{O}_y) \backslash \mathcal{G}(\mathcal{K}_y) / H^0(Y \setminus y, \mathcal{G}) \\ &\cong \mathbf{SL}_r(\mathcal{O}_p)^{\sigma^\pm} \backslash \mathbf{SL}_r(\mathcal{K}_p)^{\sigma^\pm} / \mathbf{SL}_r(\mathcal{A}_p)^{\sigma^\pm}. \end{aligned}$$

$$\mathcal{M}_Y(\mathcal{H}_\tau) \cong \mathcal{H}_\tau(\mathcal{O}_y) \backslash \mathcal{H}_\tau(\mathcal{K}_y) / H^0(Y \setminus y, \mathcal{H}_\tau).$$

Since the type τ is assumed to be trivial, $H^0(Y \setminus y, \mathcal{H}_\tau) \cong \mathbf{SL}_r(\mathcal{A}_p)^{\sigma^-}$. Thus

$$\mathcal{M}_Y(\mathcal{H}_\tau) \cong \mathbf{SL}_r(\mathcal{O}_p)^{\sigma^-} \backslash \mathbf{SL}_r(\mathcal{K}_p)^{\sigma^-} / \mathbf{SL}_r(\mathcal{A}_p)^{\sigma^-}.$$

Now it suffices to apply Proposition 2.2.2. □

4.2.2 The Grassmannian viewpoint

Note that $\mathcal{Q}^{\sigma,+}$ is an ind-variety, which is a direct limit of a system of projective varieties $(\mathcal{Q}_N^{\sigma,+})_{N \geq 0}$, the $\mathcal{Q}_N^{\sigma,+}$ are the quotients $(S^0)^{\sigma^+} \backslash (S^N)^{\sigma^+}$, where S^N is the subscheme of $\mathbf{SL}_r(\mathcal{K})$ parameterizing matrices $A(t)$ such that $A(t)$ and $A(t)^{-1}$ have poles of order at most N . As we said above, since all the stacks $\mathcal{S}\mathcal{U}_X^{\sigma,\tau}(r)$ are isomorphic, so for simplicity we assume that τ is the trivial type. So let's denote $\mathcal{Q}^{\sigma,-}$ the quotient $\mathbf{SL}_r(\mathcal{O}_p)^{\sigma^-} \backslash \mathbf{SL}_r(\mathcal{K}_p)^{\sigma^-}$. This is again an ind-variety direct limit of $(\mathcal{Q}_N^{\sigma,-})_{N \geq 0}$, the $\mathcal{Q}_N^{\sigma,-}$ are the quotients $(S^0)^{\sigma^-} \backslash (S^N)^{\sigma^-}$.

By [BL94] Proposition 2.4, the varieties $\mathcal{Q}_N := S^0 \backslash S^N$ are identified with subvarieties (with the same underline topological spaces) of the Grassmannian $\mathrm{Gr}^t(rN, 2rN)$ of t -stable subspaces of dimension rN of $F_N^r := t^{-N} \mathcal{O}^{\oplus r} / t^N \mathcal{O}^{\oplus r}$.

Consider the σ -Hermitian forms $\Psi_{\pm} : \mathcal{K}^r \times \mathcal{K}^r \longrightarrow \mathcal{K}$ defined by

$$\Psi_+(v, w) = {}^t v \cdot \sigma(w) = \sum_{i=1}^r v_i \sigma(w_i),$$

$$\Psi_-(v, w) = {}^t v \cdot J_r \cdot \sigma(w),$$

where $v = {}^t(v_1, \dots, v_r)$ and $w = {}^t(w_1, \dots, w_r)$ are in \mathcal{K}^r . Then the groups $\mathrm{SL}_r(\mathcal{K})^{\sigma^{\pm}}$ can be defined as the loci of matrices $A \in \mathrm{SL}_r(\mathcal{K})$ which are unitary with respect to the forms Ψ_{\pm} , i.e. $\Psi_{\pm}(A \cdot v, A \cdot w) = \Psi_{\pm}(v, w)$ for all $v, w \in \mathcal{K}^r$.

Consider the forms $\tilde{\Psi}_{\pm}$ on $t^{-N} \mathcal{O}^r \subset \mathcal{K}^r$ defined as the composition

$$\tilde{\Psi}_{\pm} : t^{-N} \mathcal{O}^{\oplus r} \times t^{-N} \mathcal{O}^{\oplus r} \xrightarrow{\Psi_{\pm}} t^{-2N} \mathcal{O} \xrightarrow{\mathrm{Res}} \mathbb{C},$$

where $\mathrm{Res} : \mathcal{K} \rightarrow \mathbb{C}$ is the residue map. The forms $\tilde{\Psi}_{\pm}$ vanish on $t^N \mathcal{O}^{\oplus r} \subset t^{-N} \mathcal{O}^{\oplus r}$, hence they induce two forms, denoted again by $\tilde{\Psi}_{\pm}$, on F_N^r

$$\tilde{\Psi}_{\pm} : F_N^r \times F_N^r \longrightarrow \mathbb{C}.$$

Lemma 4.2.2. $\tilde{\Psi}_+$ is an anti-symmetric non-degenerate bilinear form on F_N^r , while $\tilde{\Psi}_-$ is a symmetric non-degenerate bilinear form.

Proof. Consider the canonical basis of the vector space F_r given by the classes of t^k for $k = -N, \dots, N-1$. It induces a canonical basis of F_N^r . Then for $v = (v_i)_i, w = (w_i)_i \in F_N^r$, the forms Ψ_{\pm} are given explicitly in this basis by

$$\Psi_+(v, w) = \sum_{i=1}^r \sum_{j=-N}^{N-1} (-1)^{-j-1} a_j^i b_{-j-1}^i,$$

$$\Psi_-(v, w) = \sum_{i=1}^r \sum_{j=-N}^{N-1} (-1)^{-j-1+\varepsilon(i)} a_j^i b_{-j-1}^{r-i},$$

where $\varepsilon(i)$ equals 1 if $i \in \{1, \dots, r/2\}$, and 0 otherwise, and $v_i = (a_j^i), w_i = (b_j^i)$ are in F_N^r . From this the result follows easily. \square

We have the following

Proposition 4.2.3. The spaces $\mathcal{Q}_N^{\sigma,\pm}$ are isomorphic to closed subvarieties (with the same underline topological subspaces) of the isotropic Grassmannian $\mathrm{Gr}_{\pm}^{t,\sigma}(rN, 2rN)$ which parameterizes $\tilde{\Psi}_{\pm}$ -isotropic t -stable vector subspaces of F_N^r of dimension rN .

Proof. We prove it for the symmetric case, the other one follows similarly.

The image of $\mathcal{O}^{\oplus r}$ in F_N^r is $\tilde{\Psi}_+$ -isotropic, hence, for every $A(t) \in (S^N)^{\sigma^+}$, the corresponding point in $\mathrm{Gr}^t(rN, 2rN)$ of the class of $A(t)$ in $\mathcal{Q}_N^{\sigma^+}$ is $\tilde{\Psi}_+$ -isotropic. Thus it is in $\mathrm{Gr}_+^{t,\sigma}(rN, 2rN)$.

Conversely, assume that we have a point W of the isotropic Grassmannian. Let $A(t) \in S^N$ be a representative of the corresponding class in $S^0 \backslash S^N$. We have for every $v, w \in \mathcal{O}^r$, $\Psi_+(A \cdot v, A \cdot w) \in \mathcal{O}$, to see this assume that for some $v, w \in \mathcal{O}^r$, the coefficient of t^{-k} of $\Psi_+(A \cdot v, A \cdot w)$ is nonzero ($k > 0$), then one deduces that $\mathrm{Res}(\Psi_+(A \cdot (t^k v), A \cdot w)) = \tilde{\Psi}_+(A \cdot (t^k v), A \cdot w) \neq 0$, contradiction. Now let $(e_i)_i$ be the canonical basis of the \mathcal{K} -vector space \mathcal{K}^r and let $B(t) = (\Psi_+(A \cdot e_i, A \cdot e_j))_{i,j}$, we see that $B(t) \in \mathrm{SL}_r(\mathcal{O}) = S^0$, and we have by definition $B(t) = {}^t A(t) A(-t)$. In particular we see that $B(t) = {}^t B(-t)$, hence $B(t) = {}^t C(-t) \cdot C(t)$ for some $C(t) \in \mathrm{SL}_r(\mathcal{O})$, and $C(t) A(t)$ is also a representative of W and it is of course in $(S^N)^{\sigma^+}$. In other words the corresponding point of W in $S^0 \backslash S^N$ is in $\mathcal{Q}_N^{\sigma^+}$. This proves the proposition. \square

Consider the variety $\mathcal{Q}_N = S^0 \backslash S^N$ which is as a topological space isomorphic to the Grassmannian $\mathrm{Gr}^t(rN, 2rN)$. Fix an identification of \mathcal{Q}_N as subspace of the homogeneous space $\mathrm{SL}_{2rN}(\mathbb{C})/\mathrm{P}_N$, where P_N is the parabolic subgroup of SL_{2rN} of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A, B and C are square $rN \times rN$ matrices. Let $\mathcal{O}_{\mathcal{Q}_N}(1)$ be the line bundle attached to the character $\chi : \mathrm{P} \rightarrow \mathbb{C}^*$ which sends a matrix as above to $\det(A^{-1})$. It is well known that the Picard group of \mathcal{Q}_N is infinite cyclic generated by $\mathcal{O}_{\mathcal{Q}_N}(1)$ (it is actually isomorphic to the character group of the maximal parabolic subgroup P_N).

Proposition 4.2.4. *The restriction of $\mathcal{O}_{\mathcal{Q}_N}(1)$ to $\mathcal{Q}_N^{\sigma;-}$ has a square root, which we denote by $\mathcal{O}_{\mathcal{Q}_N^{\sigma;-}}(1)$. It is in fact the generator of the Picard group of $\mathcal{Q}_N^{\sigma;-}$.*

Proof. By Proposition 4.2.3, the variety $\mathcal{Q}_N^{\sigma;-}$ is isomorphic to a subvariety of the classical Grassmannian $\mathrm{SO}_{2rN}(\mathbb{C})/\mathrm{P}'_N$, where $\mathrm{P}'_N = \mathrm{P}_N \cap \mathrm{SO}_{2rN}(\mathbb{C})$. The restriction of the character χ to P'_N is denoted again by χ . Now, consider the universal cover of $\mathrm{SO}_{2rn}(\mathbb{C})$, which is the Spin group $\mathrm{Spin}_{2rN}(\mathbb{C})$. It is a double cover of $\mathrm{SO}_{2rN}(\mathbb{C})$. Let $\tilde{\mathrm{P}}_N \subset \mathrm{Spin}_{2rN}(\mathbb{C})$ the inverse image of P'_N . Then, by [DSS92] Chapter 5, Theorem 3.3.1, the lifting of χ to $\tilde{\mathrm{P}}_N$ has a square root which we denote by χ_- . Since we have

$$\mathrm{Spin}_{2rn}(\mathbb{C})/\tilde{\mathrm{P}} \cong \mathrm{SO}_{2rn}/\mathrm{P}'_N,$$

we deduce that the line bundle over $\mathcal{Q}_N^{\sigma;-}$ attached to χ_- is the square root of the restriction of $\mathcal{O}_{\mathcal{Q}_N}(1)$.

The Picard group of $\mathrm{Spin}_{2rN}/\tilde{\mathrm{P}}_N$ is infinite cyclic isomorphic to the character group of $\tilde{\mathrm{P}}_N$, which is generated by χ_- . This implies the second claim. \square

Proposition 4.2.5. *The ind-varieties $\mathcal{Q}^{\sigma,\pm}$ are integral.*

Proof. We know already that $\mathcal{Q}^{\sigma,\pm}$ are connected, hence they are irreducible. Moreover, By [PR08b] Theorem 0.2, the flag varieties $\mathcal{Q}^{\sigma,\pm}$ are reduced.

Furthermore, we can show directly that the flag varieties $\mathcal{Q}^{\sigma,\pm}$ are reduced. We follow the same method as in [BL04]. First denote by $\mathbf{SL}_r(\mathcal{K})_0 = \mathbf{SL}_r(\mathbb{C}[t^{-1}])\mathbf{SL}_r(\mathcal{O})$. We know (cf. loc. cit.) that $\mathbf{SL}_r(\mathcal{K})_0$ is an open and that $g\mathbf{SL}_r(\mathcal{K})_0$, for $g \in \mathbf{SL}_r(\mathcal{K})$, cover $\mathbf{SL}_r(\mathcal{K})$. We deduce easily that $\mathbf{SL}_r(\mathcal{K})_0^{\sigma^\pm}$ is also an open and $g\mathbf{SL}_r(\mathcal{K})_0^{\sigma^\pm}$, for $g \in \mathbf{SL}_r(\mathcal{K})^{\sigma^\pm}$, covers $\mathbf{SL}_r(\mathcal{K})^{\sigma^\pm}$.

Now we claim the following

Claim. *The ind-varieties $\mathbf{SL}_r(\mathbb{C}[t^{-1}])^{\sigma^\pm}$ are reduced.*

Proof. Denote by $V_N^\pm \subset \mathbf{SL}_r(\mathbb{C}[t^{-1}])^{\sigma^\pm}$ the loci of matrices of degree (in t^{-1}) at most N . Then $\mathbf{SL}_r(\mathbb{C}[t^{-1}])^{\sigma^\pm}$ is a direct limit of V_N^\pm , hence it is sufficient to prove that V_N^\pm are reduced. In fact V_N^\pm are even smooth.

Let M_r be the vector space over \mathbb{C} of square matrices of size r . Denote by $M_r^{N+1} \subset M_r[t^{-1}]$ the subspace of polynomials of degree at most N with coefficients in M_r . One can easily check that V_N^+ (resp. V_N^-) is exactly the locus of elements of determinant 1 of the fiber over the identity matrix of the map $\alpha_\pm : M_r^{N+1} \rightarrow M_r^{2N+1}$ given by

$$\alpha_+(A(t)) = {}^t A(-t) \cdot A(t) \quad (\text{resp. } \alpha_-(A(t)) = {}^t A(-t) \cdot J_r \cdot A(t) \cdot J_r^{-1}).$$

Remark that the elements of these fibers have determinants in $\{\pm 1\}$.

For simplicity, we will consider just σ^+ case hereafter. The image of α^+ lives inside the vector subspace $(M_r^{2N+1})^\tau$ of τ -invariant matrices, where τ is the involution on M_r^{2N+1} defined by $\tau(A(t)) = {}^t A(-t)$. It is easy to see that the fiber of α^+ over I_r is given by

$$\frac{r(r-1)}{2}(2N+1) + r(N+1)$$

quadratic equations. They can be calculated explicitly: Let $X_{i,j}^d$ be the canonical dual basis of M_r^{N+1} , where i, j are the indices of the coefficients of the matrices and d corresponds to the powers of t^{-1} . Then the equations of the fiber can be written explicitly in the form

$$f_{ij}^d = \sum_{k=1}^r \sum_{n+m=d} (-1)^m X_{ik}^n X_{kj}^m - \delta_{ij} \delta_{d0},$$

where $1 \leq i \leq j \leq r$, n and m are between 0 and N .

Now since we have an action of $\mathrm{SO}_r(\mathbb{C})$ on this fiber, we can restrict ourselves to elements $A(t) \in M_r^{N+1}$ such that $A_0 = I_r$, we denote such locus by \tilde{M}_r^{N+1} . Now one can show, after some computations, that the Jacobian matrix associated to $(f_{ij}^d)_{i,j,d}$

$$\mathrm{Jac}((f_{ij}^d)_{i,j,d}) = \left(\frac{\partial f_{ij}^d}{\partial X_{kl}^m} \right)_{(i,j,d),(k,l,m)}$$

has in fact a full rank at each point of $V_N^+ \cap \tilde{M}_r^{N+1}$. Roughly speaking, to see this one can decompose it to a number of sub-matrices using the index i . For example, the first sub-matrix is the set of lines which contain the partial differentials of the polynomials f_{1j}^d . One remarks here that this sub-matrix is the only one which contains the variables X_{11}^d . So the resulting sub-matrices have full rank, and their lines are linearly independents. It follows by the Jacobian criterion of smoothness that the fiber of α^+ over the identity is smooth. Since this fiber is a fibration over $\{\pm 1\}$ by the determinant. We deduce that V_N^+ is smooth too. \square

Using the claim and [BL04] Lemma 6.3b), we deduce that $\mathcal{Q}^{\sigma,\pm}$ are reduced. \square

Example 4.2.6. We give an example in rank 2 of a non trivial element of V_N^+ . If N is odd, then one can take

$$\begin{pmatrix} 1 + t^{-N} & 1 + it^{-N} \\ -1 + it^{-N} & 1 - t^{-N} \end{pmatrix}.$$

One can use it to generate a bunch of examples in any rank.

4.2.3 Central extension

Consider the central extension of $\mathbf{SL}_r(\mathcal{K})$

$$0 \rightarrow \mathbb{G}_m \rightarrow \widehat{\mathbf{SL}}_r(\mathcal{K}) \rightarrow \mathbf{SL}_r(\mathcal{K}) \rightarrow 0.$$

The actions of σ^\pm lift to $\widehat{\mathbf{SL}}_r(\mathcal{K})$ giving a central extension of $\mathbf{SL}_r(\mathcal{K})^{\sigma^\pm}$

$$0 \rightarrow \mathbb{G}_m \rightarrow \widehat{\mathbf{SL}}_r(\mathcal{K})^{\sigma^\pm} \rightarrow \mathbf{SL}_r(\mathcal{K})^{\sigma^\pm} \rightarrow 0.$$

Indeed, let R be a \mathbb{C} -algebra, for $\gamma \in \mathbf{SL}_r(R((t)))$ let

$$\gamma = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$$

be its decomposition with respect to $R((t)) = \mathcal{V}_R \oplus R[[t]]$. Recall that \mathcal{V} is a complementary vector subspace of \mathcal{O} in \mathcal{K} .

By [BL94], an element of $\mathbf{SL}_r(R((t)))$ is given, locally on $\mathrm{Spec}(R)$, by a pair (γ, u) where $\gamma \in \mathbf{SL}_r(R((t)))$, $u \in \mathrm{Aut}(\mathcal{V}_R)$ such that $u \equiv a(\gamma) \pmod{\mathrm{End}^f(\mathcal{V}_R)}$, where $\mathrm{End}^f(\mathcal{V}_R) \subset \mathrm{End}(\mathcal{V}_R)$ is the set of finite rank endomorphisms of \mathcal{V}_R . By [BL94] Proposition 4.3, the map $\gamma \rightarrow \bar{a}(\gamma)$ is a group homomorphism from $\mathbf{SL}_r(R((t)))$ onto the group $\mathrm{Aut}^f(\mathcal{V}_R)$ of units of $\mathrm{End}(\mathcal{V}_R)/\mathrm{End}^f(\mathcal{V}_R)$. It follows that

$$\bar{a}(\gamma^{-1}) = \bar{a}(\gamma)^{-1},$$

hence

$$u^{-1} \equiv a(\gamma^{-1}) \pmod{\mathrm{End}^f(\mathcal{V}_R)}.$$

So, define the following actions on $\widehat{\mathbf{SL}}_r(\mathcal{K})$

$$\sigma^+ : (\gamma, u) \longrightarrow ({}^t\gamma(-t)^{-1}, {}^t u(-t)^{-1}),$$

$$\sigma^- : (\gamma, u) \longrightarrow (J_r {}^t\gamma(-t)^{-1} J_r^{-1}, J_r {}^t u(-t)^{-1} J_r^{-1}).$$

Clearly these are involutions which lift σ^\pm on $\mathbf{SL}_r(\mathcal{K})$.

The Lie algebra attached to $\widehat{\mathbf{SL}}_r(\mathcal{K})$ is given by the central extension

$$0 \rightarrow \mathbb{C} \rightarrow \widehat{\mathfrak{sl}}_r(\mathcal{K}) \rightarrow \mathfrak{sl}_r(\mathcal{K}) \rightarrow 0. \quad (4.2)$$

It is in fact isomorphic to the affine Lie algebra $\hat{\mathcal{L}}(\mathfrak{sl}_r) = \mathfrak{sl}_r(\mathcal{K}) \oplus \mathbb{C}$, with the Lie algebra structure given by

$$[(\alpha, u), (\beta, v)] = \left([\alpha; \beta], \mathrm{Res}\left(\mathrm{Tr}\left(\frac{d\alpha}{dt}\beta\right)\right) \right),$$

where Res stands for the *residue*. By pulling back the exact sequence (4.2) via the inclusions $\mathfrak{sl}_r(\mathcal{K})^{\sigma^\pm} \hookrightarrow \mathfrak{sl}_r(\mathcal{K})$ we get the central extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widehat{\mathfrak{sl}}_r(\mathcal{K}) & \longrightarrow & \mathfrak{sl}_r(\mathcal{K}) \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widehat{\mathfrak{sl}}_r(\mathcal{K})^{\sigma^\pm} & \longrightarrow & \mathfrak{sl}_r(\mathcal{K})^{\sigma^\pm} \longrightarrow 0, \end{array}$$

where σ^\pm act on $\widehat{\mathfrak{sl}}_r(\mathcal{K})$ by their actions on the first summand (which are given in Lemma 4.2.7 below). These are (after adding scaling elements) affine Kac-Moody Lie algebras of twisted type $A_{r-1}^{(2)}$. They are in fact the Lie algebras of the twisted groups $\widehat{\mathbf{SL}}_r(\mathcal{K})^{\sigma^\pm}$.

Lemma 4.2.7. *The Lie algebras associated to $\widehat{\mathbf{SL}}_r(\mathcal{K})^{\sigma^\pm}$ are the twisted affine Lie algebras of $\mathfrak{sl}_r(\mathcal{K})$ given by*

$$\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm) = \mathfrak{sl}_r(\mathcal{K})^{\sigma^\pm} \oplus \mathbb{C},$$

where the actions of σ^\pm on $\mathfrak{sl}_r(\mathcal{K})$ are given by

$$\sigma^+(g(t)) = -{}^t g(-t),$$

$$\sigma^-(g(t)) = -J_r {}^t g(-t) J_r^{-1}.$$

Proof. The proof is straightforward, we just remark that

$${}^t(I_r + \varepsilon\alpha)^{-1} = I_r - \varepsilon {}^t\alpha,$$

where $\varepsilon^2 = 0$. □

4.3 Determinant and Pfaffian line bundles

Let T be a locally noetherian \mathbb{C} -scheme. Denote by p_1 and p_2 the projection maps from $X \times T$ to X and T respectively. Let \mathcal{E} be a vector bundle over $X \times T$. The derived direct image complex $Rp_{2*}(\mathcal{E})$ is represented by a complex of vector bundles $0 \rightarrow F_0 \rightarrow F_1 \rightarrow 0$. The line bundle $\mathcal{D}_{\mathcal{E}} := \det(F_0)^{-1} \otimes \det(F_1)$ over T is independent of the choice of the representing complex and is called the *determinant of cohomology* of \mathcal{E} . The determinant of the universal family \mathcal{L} over $X \times \mathcal{S}\mathcal{U}_X(r)$ is called the *determinant bundle* over $\mathcal{S}\mathcal{U}_X(r)$.

Let κ be a σ -invariant theta characteristic over X .

Proposition 4.3.1. *Let (\mathcal{E}, ψ) be a family of σ -alternating vector bundles over X parameterized by T , with a σ -alternating non-degenerated form $\psi : \mathcal{E} \otimes \sigma^* \mathcal{E} \rightarrow \mathcal{O}_{X \times T}$. Let $\mathcal{E}_\kappa = \mathcal{E} \otimes p_1^* \kappa$. Then the determinant of cohomology line bundle $\mathcal{D}_{\mathcal{E}_\kappa}$ admits a square root $\mathcal{P}_{\mathcal{E}_\kappa}$ which we call Pfaffian of cohomology line bundle.*

Proof. Consider the family $\pi_* \mathcal{E}_\kappa$ over Y . It is equipped with a non-degenerated quadratic form with values in K_Y . Indeed, by the projection formula, ψ induces an isomorphism

$$\begin{aligned} \pi_* \mathcal{E}_\kappa &\cong \pi_*(\sigma^* \mathcal{E}_\kappa) \\ &\cong \pi_*(\mathcal{E}_\kappa^*(q_1^{-1}(R))) \otimes q_1^* K_Y \\ &\cong (\pi_* \mathcal{E}_\kappa)^* \otimes q_1^* K_Y, \end{aligned}$$

where the last isomorphism is the relative duality (see [Har77] Ex III.6.10) and $q_1 : Y \times T \rightarrow Y$ is the first projection. In fact the associated bilinear form is given by the composition

$$\pi_* \mathcal{E}_\kappa \otimes \pi_* \mathcal{E}_\kappa \rightarrow \pi_*(p_1^* K_X) = q_1^*(K_Y \otimes \Delta) \oplus q_1^* K_Y \rightarrow q_1^* K_Y.$$

Since we project on the -1 eigenspace of the linearization on K_X (recall that $\pi_* K_X = K_Y \Delta \oplus K_Y$) and because ψ is σ -alternating, we deduce that this bilinear form is symmetric.

We can apply now [LS97] Proposition 7.9 to get a square root of $\mathcal{D}_{\pi_* \mathcal{E}_\kappa}$. To finish the proof we just have to remark that

$$\mathcal{D}_{\mathcal{E}_\kappa} = \mathcal{D}_{\pi_* \mathcal{E}_\kappa}.$$

□

In particular, if we consider the universal family over $X \times \mathcal{S}\mathcal{U}_X^{\sigma,-}(r)$, we get, for each σ -invariant theta characteristic κ , a Pfaffian of cohomology line bundle \mathcal{P}_κ over $\mathcal{S}\mathcal{U}_X^{\sigma,-}(r)$.

On the other hand, consider the character $\chi : \widehat{\mathbf{SL}}_r(\mathcal{O}) \rightarrow \mathbb{G}_m$ which is just the second projection (recall that $\widehat{\mathbf{SL}}_r(\mathcal{O})$ splits). More precisely, a point of $\widehat{\mathbf{SL}}_r(\mathcal{O})$ can be represented locally on $\text{Spec}(R)$ by a pair (γ, u) , for $\gamma \in \text{SL}_r(R[[t]])$ and u an automorphism of \mathcal{V}_R such that $a(\gamma) \equiv u \pmod{\text{End}^f(\mathcal{V}_R)}$. So χ sends this point to $\det(a(\gamma)^{-1}u)$. To this character one may associate a line bundle \mathcal{L}_χ over \mathcal{Q} (see [BL94] §3). Moreover \mathcal{L}_χ is isomorphic to the pullback of the determinant bundle.

Lemma 4.3.2. *The restriction of the character χ to $\widehat{\mathbf{SL}}_r(\mathcal{O})^{\sigma-}$ has a square root which we denote by χ_- .*

Proof. With the notations of the proof of Proposition 4.2.4, one can see that $\widehat{\mathbf{SL}}_r(\mathcal{O})^{\sigma-}$ is the direct limit of the parabolic subgroups P'_N . So just take the direct limit in Proposition 4.2.4. \square

Let \mathcal{L}_- be the line bundle over $\mathcal{Q}^{\sigma,-}$ defined by the character χ_- and denote by $q : \mathcal{Q}^{\sigma,\pm} \rightarrow \mathcal{S}\mathcal{U}_X^{\sigma,\pm}(r)$ the quotient maps (there should be no confusion about which map is considered).

Denote by \mathcal{D} the determinant of cohomology line bundle over $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$ and by \mathcal{L} its pullback to $\mathcal{Q}^{\sigma,+}$.

Lemma 4.3.3. *The pullback of the Pfaffian line bundles \mathcal{P}_κ to $\mathcal{Q}^{\sigma,-}$ are independent of κ and they are all isomorphic to \mathcal{L}_- .*

Proof. It is known (see [Hei10]) that $\text{Pic}(\mathcal{Q}^{\sigma,-}) = \mathbb{Z}^N$, for some integer N . Since all the \mathcal{P}_κ are the square roots of the same line bundle \mathcal{D} , the result follows. \square

Remark that the line bundle \mathcal{D} over $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$ does not admit a square root in the ramified case. This can be seen using Theorem 3.2.17 and the fact that $\mathcal{P} \cap \mathcal{Q}$ is not principally polarized (see Theorem 3.2.17 for the notations).

If the cover $\pi : X \rightarrow Y$ is étale, then for each σ -invariant theta characteristic the universal family over $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$ is equipped with a non-degenerated quadratic form with values in K_X (this can be seen as in the proof of Proposition 4.3.1). Hence its determinant of cohomology admits a square root, a Pfaffian of cohomology bundle.

Moreover, using the functorial definition, one can show that the determinant bundle over the stack $\mathcal{U}_X^{\sigma,+}(r)$ has also a square root attached to any σ -invariant theta characteristic. We keep denoting these line bundles by \mathcal{P}_κ , and this should produce no confusion since we always explicitly mention the considered stack.

Lemma 4.3.4. *Assume that π is étale. Let κ be an even σ -invariant theta characteristic on X , then the Pfaffian line bundle \mathcal{P}_κ over $\mathcal{U}_{X,0}^{\sigma,+}(r)$ descends to the moduli space $\mathcal{U}_{X,0}^{\sigma,+}(r)$, where $\mathcal{U}_{X,0}^{\sigma,+}(r)$ is the connected component of $\mathcal{U}_X^{\sigma,+}(r)$ of vector bundles with determinant in the connected Prym variety P_0 of $\pi : X \rightarrow Y$.*

Proof. Let κ be a σ -invariant theta characteristic on X and $a = (E, q, \bar{\psi})$ be a point of $\text{Quot}^\sigma(\mathbb{C})$ (see subsection 2.4.3 for the notations). We assume that E is stable, then the stabilizer of a under the action of $\text{SL}(H)$ is just $\{\pm 1\}$. The action of this stabilizer on

$(\mathcal{P}_\kappa)_a$ is by definition multiplication by $g^{h^1(E \otimes \kappa)}$, for $g \in \{\pm 1\}$. Since $\mathcal{U}_{X,0}^{\sigma,+}(r)$ is connected, we have

$$h^1(E \otimes \kappa) = \begin{cases} 1 & \text{if } r \equiv h^0(\kappa) \equiv 1 \pmod{2} \\ 0 & \text{otherwise.} \end{cases}$$

This can be seen using Theorem 3.2.13. Since κ is even, it follows that -1 acts trivially on $(\mathcal{P}_\kappa)_a$, for any a . Using Kempf's Lemma we deduce the result. \square

Now we show the existence of the Pfaffian divisor. Assume that $\pi : X \rightarrow Y$ is étale. For a theta characteristic κ over X we denote by Θ_κ the divisor in $\mathcal{U}_X(r, 0)$ supported on vector bundles E such that $E \otimes \kappa$ has a non-zero global section.

Lemma 4.3.5. *There exists a theta characteristic κ_0 over Y , such that, if $\kappa = \pi^* \kappa_0$, then the restriction of the divisor $\Theta_\kappa \subset \mathcal{U}_X(r, 0)$ to $\mathcal{U}_{X,0}^{\sigma,+}(r)$ is again a divisor. Moreover there exists an effective divisor Ξ_κ in $\mathcal{U}_X^{\sigma,\pm}(r)$ such that $\mathcal{O}(\Xi_\kappa) = \mathcal{P}_\kappa$ and*

$$2\Xi_\kappa = \Theta_\kappa.$$

Proof. Using the Hitchin system (Precisely Theorem 3.2.13) we can reduce the question to $r = 1$. Now let κ_0 be a theta characteristic on Y such that the restriction of $\theta_\kappa = T_\kappa^* \theta$ to the (connected) Prym variety \mathbf{P} is a divisor, where $\theta \subset \text{Pic}^{g_X-1}(X)$ is the Riemann theta divisor and $T_\kappa : \text{Pic}^0(X) \rightarrow \text{Pic}^{g_X-1}(X)$ is the translation map by $\kappa = \pi^* \kappa_0$. Then we see easily that such κ gives the result. To construct such κ_0 we proceed as follows: Consider the direct image map $\mathbf{P} \rightarrow \mathcal{U}_Y(2, \Delta)$ from \mathbf{P} to the moduli space of semistable rank two vector bundles over Y of determinant Δ . Then consider the linear system $|\tilde{\theta} + \tilde{\theta}_\Delta|$, where $\tilde{\theta} \subset \text{Pic}^{g_Y-1}(Y)$ is the Riemann theta divisor and $\tilde{\theta}_\Delta$ is its translation by Δ . Then there is a canonical morphism $\varphi : \mathcal{U}_Y(2, \Delta) \rightarrow |\tilde{\theta} + \tilde{\theta}_\Delta|$, which sends a vector bundle E to the divisor $D(E) = \{L \in \text{Pic}^{g_Y-1}(Y) | h^0(E \otimes L) \geq 1\}$. Since $\pi_* L \cong \pi_* L \otimes \Delta$ for any $L \in \mathbf{P}$, one remarks that the composition $\varphi \circ \pi_*$ factorizes through the linear system $|\tilde{\theta} + \tilde{\theta}_\Delta|_+ \subset |\tilde{\theta} + \tilde{\theta}_\Delta|$ defined as the invariant locus with respect to taking the tensor product with Δ . Now it is sufficient to take κ_0 such that the associated hyperplane in $|\tilde{\theta} + \tilde{\theta}_\Delta|$ does not contain entirely the linear system $|\tilde{\theta} + \tilde{\theta}_\Delta|_+$, this is possible because the above linear system is base point free.

Moreover, as in [LS97] §7.10, whenever the restriction of Θ_κ to $\mathcal{U}_{X,0}^{\sigma,+}(r)$ is a divisor, there is an effective divisor Ξ_κ such that $2\Xi_\kappa = \Theta_\kappa|_{\mathcal{U}_{X,0}^{\sigma,+}(r)}$. In particular, \mathcal{P}_κ has a non-zero global section. \square

4.4 Generalized theta functions and conformal blocks

Assume in this section that the cover $\pi : X \rightarrow Y$ is ramified. We have formulated the uniformization theorem over a single ramification point. However we can use a bunch of points to uniformize our moduli stack. If we consider all the ramification points R , then we get the following

$$\mathcal{S}\mathcal{U}_X^{\sigma,\pm}(r) \cong \prod_{p \in R} \mathcal{Q}_p^{\sigma,\pm} / \mathbf{SL}_r(\mathcal{A}_R)^{\sigma^\pm},$$

where $\mathcal{Q}_p^{\sigma,\pm} = \mathbf{SL}_r(\mathcal{O}_p)^{\sigma^\pm} \backslash \mathbf{SL}_r(\mathcal{K}_p)^{\sigma^\pm}$, and $\mathcal{A}_R = H^0(X \setminus R, \mathcal{O}_X)$. Of course all $\mathcal{Q}_p^{\sigma,\pm}$ are isomorphic, but we emphasize on the fixed points.

Roughly speaking, this isomorphism can be seen as follows: choose a formal neighborhood D_p of each $p \in R$. Then giving a σ -symmetric vector bundle (E, ψ) of trivial determinant, we choose a σ -invariant local trivializations φ_p near each p and a σ -invariant trivialization φ_0 on $X \setminus R$. Then the corresponding point of the right hand side is just is the class of

$(\varphi_p \circ \varphi_0^{-1})_{p \in R}$. Conversely, giving a class of functions $(f_p)_{p \in R}$ of the RHS, we can construct a σ -symmetric vector bundle by gluing the trivial bundles on D_p and $X \setminus R$ using the functions f_p .

We have seen that the line bundle \mathcal{L}_- over $\mathcal{Q}_p^{\sigma,-}$ is isomorphic to $q^*\mathcal{P}_\kappa$ and that the line bundle \mathcal{L} over $\mathcal{Q}_p^{\sigma,+}$ is isomorphic to $q^*\mathcal{D}$. For $x \in R$, let $q_x : \prod_{p \in R} \mathcal{Q}_p^{\sigma,\pm} \rightarrow \mathcal{Q}_x^{\sigma,\pm}$ be the canonical projection. We define the line bundles

$$\mathcal{L}_- = \bigotimes_{p \in R} q_p^* \mathcal{L}_- \quad \text{and} \quad \mathcal{L} = \bigotimes_{p \in R} q_p^* \mathcal{L}$$

over $\prod_{p \in R} \mathcal{Q}_p^{\sigma,-}$ and $\prod_{p \in R} \mathcal{Q}_p^{\sigma,+}$ respectively. One can see that \mathcal{L}_- and \mathcal{L} are in fact the pullback via the projections $\prod_{p \in R} \mathcal{Q}_p^{\sigma,\pm} \rightarrow \mathcal{S}\mathcal{U}_X^{\sigma,\pm}(r)$ of the line bundles \mathcal{P}_κ and \mathcal{D} respectively. In particular, both of these line bundles have canonical $\mathbf{SL}(\mathcal{A}_R)^{\sigma^\pm}$ -linearizations. In fact these are the only ones due to the following

Proposition 4.4.1. *$\mathbf{SL}_r(\mathcal{A}_R)^{\sigma^\pm}$ are integral and they have only the trivial character.*

Proof. The proof is inspired from [LS97].

Using the local triviality of the projection $\prod_{p \in R} \mathcal{Q}_p^{\sigma,\pm} \rightarrow \mathcal{S}\mathcal{U}_X^{\sigma,\pm}(r)$ and Proposition 4.2.5 we deduce that $\mathbf{SL}_r(\mathcal{A}_R)^{\sigma^\pm}$ are reduced.

Now, since connected ind-groups are irreducible, it is sufficient to prove that $\mathbf{SL}_r(\mathcal{A}_R)^{\sigma^\pm}$ is connected. For a points $p_1, \dots, p_k \in X \setminus R$ we denote by $R_i = R \cup \{p_1, \sigma(p_1), \dots, p_i, \sigma(p_i)\}$. We claim the following

Claim. *We have an isomorphism*

$$\mathbf{SL}_r(\mathcal{A}_{R_i})^{\sigma^\pm} / \mathbf{SL}_r(\mathcal{A}_{R_{i-1}})^{\sigma^\pm} \cong (\mathcal{Q}_{p_i} \times \mathcal{Q}_{\sigma(p_i)})^{\sigma^\pm},$$

where the action of σ^\pm on the right hand side is given by $\sigma^\pm(f, g) = (\sigma^\pm(g), \sigma^\pm(f))$.

Proof. We have a canonical map $\mathbf{SL}_r(\mathcal{A}_{R_i})^{\sigma^\pm} \rightarrow (\mathcal{Q}_{p_i} \times \mathcal{Q}_{\sigma(p_i)})^{\sigma^\pm}$ which is clearly trivial on $\mathbf{SL}_r(\mathcal{A}_{R_{i-1}})^{\sigma^\pm}$. Hence we deduce a map $\mathbf{SL}_r(\mathcal{A}_{R_i})^{\sigma^\pm} / \mathbf{SL}_r(\mathcal{A}_{R_{i-1}})^{\sigma^\pm} \rightarrow (\mathcal{Q}_{p_i} \times \mathcal{Q}_{\sigma(p_i)})^{\sigma^\pm}$ which is actually injective. Now, by considering the uniformization over the two points $\{p_i, \sigma(p_i)\}$, we get

$$\mathcal{S}\mathcal{U}_X^{\sigma,\pm}(r) \cong (\mathcal{Q}_{p_i} \times \mathcal{Q}_{\sigma(p_i)})^{\sigma^\pm} / \mathbf{SL}_r(\mathcal{A}_{\{p_i, \sigma(p_i)\}})^{\sigma^\pm}.$$

Hence, for an \mathbb{C} -algebra S , giving a point of $(\mathcal{Q}_{p_i} \times \mathcal{Q}_{\sigma(p_i)})^{\sigma^\pm}(S)$ is the same as giving an anti-invariant (σ -symmetric or σ -alternating following \pm) vector bundle E over X_S and a trivialization $\delta : E|_{X_S^*} \rightarrow X_S^* \times \mathbb{C}^r$, where $X_S^* = X_S \setminus \{p_i, \sigma(p_i)\}$. For an S -algebra S' , let $T(S')$ be the space of σ^\pm -invariant trivializations of $E_{S'}$ over $X_{S, i-1} = X_S \setminus R_{i-1}$. Then $\mathbf{SL}_r(\mathcal{A}_{R_{i-1}})^{\sigma^\pm}$ acts on T , and in fact it is a bundle under that group. Moreover δ induces a map $\tilde{\delta} : T \rightarrow \mathbf{SL}_r(\mathcal{A}_{R_i})^{\sigma^\pm}$ by sending a trivialization ϕ to $\phi \circ \delta^{-1}$. Associating to (E, δ) the map $\tilde{\delta}$ gives an inverse to the above inclusion. \square

It is clear to see that $(\mathcal{Q}_{p_i} \times \mathcal{Q}_{\sigma(p_i)})^{\sigma^\pm} \cong \mathcal{Q}_{p_i} = \mathbf{SL}_r(\mathcal{O}_{p_i}) \backslash \mathbf{SL}_r(\mathcal{K}_{p_i})$ which is simply connected. So using the homotopy exact sequence, we deduce that

$$\pi_0(\mathbf{SL}_r(\mathcal{A}_{R_i})) = \pi_0(\mathbf{SL}_r(\mathcal{A}_{R_{i-1}})).$$

Now let $g \in \mathbf{SL}_r(\mathcal{A}_R)^{\sigma^\pm}$ and consider g as an element of $\mathbf{SL}_r(K)^{\sigma^\pm}$, where K is the function field of X . By [Tit79] (see also [PR08a] Section 4), we know that the special unitary groups

are simply connected and quasi-split. Steinberg ([Ste62]) has showed the Kneser-Tits property for quasi-split simply connected groups over any field (Recall that this property means that these groups are generated by the unipotent radicals of their standard parabolic subgroups). So applying that to $\mathrm{SL}_r(K_X)^{\sigma^\pm}$, we can assume that $g = \prod_i \exp(N_i)$, where N_i are nilpotent elements of $\mathfrak{sl}_r(K_X)^{\sigma^\pm}$. Let $\{p_1, \dots, p_k\}$ be the poles of N_i . For $t \in \mathbb{A}^1$, we let $g_t = \prod_i \exp(tN_i)$. Then for any $t \in \mathbb{A}^1$ we see that $g_t \in \mathrm{SL}_r(\mathcal{A}_{R_k})^{\sigma^\pm}$ and $t \rightarrow g_t$ is a path in $\mathrm{SL}_r(\mathcal{A}_{R_k})^{\sigma^\pm}$ that relates g to the identity. Hence $\mathbf{SL}_r(\mathcal{A}_{R_k})^{\sigma^\pm}$ is connected. So the same is true for $\mathbf{SL}_r(\mathcal{A}_R)^{\sigma^\pm}$ by what we have shown above.

Now let λ be a character of $\mathrm{SL}_r(\mathcal{A}_R)^{\sigma^\pm}$, seeing λ as a function, we consider its derivative at the identity which turns out to be a Lie algebras morphism from $\mathfrak{sl}_r(\mathcal{A}_R)^{\sigma^\pm}$ to the trivial algebra \mathbb{C} . However, the affine algebra $\mathfrak{sl}_r(\mathcal{A}_R)^{\sigma^\pm}$ equals the direct sum of two commutator subalgebras. Indeed, the algebra $\mathfrak{sl}_r(\mathcal{A}_R)$ equals to its commutator, and we have eigenspace decomposition with respect to σ^\pm

$$\mathfrak{sl}_r(\mathcal{A}_R) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1,$$

it follows

$$[\mathfrak{g}_{-1} \oplus \mathfrak{g}_1, \mathfrak{g}_{-1} \oplus \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus [\mathfrak{g}_1, \mathfrak{g}_{-1}] \oplus [\mathfrak{g}_1, \mathfrak{g}_1].$$

Hence $\mathfrak{sl}_r(\mathcal{A}_R)^{\sigma^\pm} = \mathfrak{g}_1 = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \oplus [\mathfrak{g}_1, \mathfrak{g}_1]$. So the derivative of λ at the identity is zero. Since λ is a group homomorphism, its derivative is identically zero everywhere. Since $\mathbf{SL}_r(\mathcal{A}_X)^{\sigma^\pm}$ is integral, we can write it as limit of integral varieties V_n and for n large $1 \in V_n$, so $\lambda|_{V_n} = 1$, hence $\lambda = 1$. \square

Fix an integer $k > 0$. For any dominant weight $\lambda^\pm \in P^{\sigma^\pm}$, there is a line bundle $\mathcal{L}(\lambda^\pm)$ over \mathcal{Q}^{σ^\pm} associated to the principal $\mathbf{SL}_r(\mathcal{O})^{\sigma^\pm}$ -bundle:

$$\mathbf{SL}_r(\mathcal{K})^{\sigma^\pm} \longrightarrow \mathcal{Q}^{\sigma^\pm},$$

defined using the character $e^{-\lambda^\pm}$ on $\mathbf{SL}_r(\mathcal{O})^{\sigma^\pm}$. Further, it is shown in [Kum87] that the space of global sections of powers of $\mathcal{L}(\lambda)$ is isomorphic to the dual of the irreducible highest integrable representation of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$ associated to λ^\pm .

We are mainly interested in the case where $\lambda^\pm = \lambda_0^\pm$. Denote by $\mathcal{H}_\pm(k)$ the highest weight representation of level k of $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^\pm)$ associated to the weight λ_0^\pm . It is called the basic representation of level k . So the above result of [Kum87] (see also [Mat88]) can be formulated as follows

Theorem 4.4.2 (Kumar, Mathieu). *1. The space $H^0(\mathcal{Q}^{\sigma^-}, q^* \mathcal{P}_\kappa^k)$ is canonically isomorphic, as $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^-)$ -module, to the dual of the basic representation $\mathcal{H}_-(k)$.*

2. The space $H^0(\mathcal{Q}^{\sigma^+}, q^ \mathcal{D}^k)$ is canonically isomorphic, as $\widehat{\mathcal{L}}(\mathfrak{sl}_r, \sigma^+)$ -module, to the dual of the basic representation $\mathcal{H}_+(k)$.*

Note that by Remark 4.1.1, when r is even, the weight λ_0^+ has level 2 while λ_0^- is of level 1. This explains why we have to take the determinant line bundle in σ^+ case and the Pfaffian line bundle in σ^- case.

The point that should be stressed here is that in [Kum87], Kumar has defined the ind-group $\mathbf{SL}_r(\mathcal{O}) \backslash \mathbf{SL}_r(\mathcal{K})$ using representation theory of Kac-Moody algebras. It is shown in [BL94] that this construction coincides with the usual functorial definition. Moreover,

Pappas and Rapoport have claimed in [PR08a] (page 3) that the constructions of Kumar coincide with their definitions of the Schubert varieties. In particular, we deduce in our spacial case that the ind-variety structure on the twisted flag varieties $\mathcal{Q}^{\sigma, \pm}$ are the same as those defined by Kumar.

Using the above results and assumptions, we deduce the following result which has been conjectured in a more general context by Pappas and Rapoport ([PR08a] Conjecture 3.7).

Proposition 4.4.3. *We have isomorphisms*

$$H^0(\mathcal{S}\mathcal{W}_X^{\sigma, -}(r), \mathcal{P}_\kappa^k) \cong \left(\prod_{p \in R} H^0(\mathcal{Q}_p^{\sigma, -}, \mathcal{L}_-^k) \right)^{\mathrm{sl}_r(\mathcal{A}_R)^{\sigma^-}},$$

$$H^0(\mathcal{S}\mathcal{W}_X^{\sigma, +}(r), \mathcal{D}^k) \cong \left(\prod_{p \in R} H^0(\mathcal{Q}_p^{\sigma, +}, \mathcal{L}^k) \right)^{\mathrm{sl}_r(\mathcal{A}_R)^{\sigma^+}}.$$

Proof. Since $\mathrm{SL}_r(\mathcal{A}_R)^{\sigma^\pm}$ and \mathcal{Q}^{σ^\pm} are integral, the result follows, using the Künneth formula, from [BL04] Proposition 7.4. \square

Now, Lemma 4.4.3 and Theorem 4.4.2 imply our main result

Theorem 4.4.4. *Let $k > 2$, we have*

1. *The space of global sections $H^0(\mathcal{S}\mathcal{W}_X^{\sigma, -}(r), \mathcal{P}_\kappa^k)$ is canonically isomorphic to the conormal block space $\mathcal{V}_{\sigma, -}(k)$.*
2. *The space of global sections $H^0(\mathcal{S}\mathcal{W}_X^{\sigma, +}(r), \mathcal{D}^k)$ is canonically isomorphic to the conormal block space $\mathcal{V}_{\sigma, +}(k)$.*

4.5 Application: An analogue of a result of Beauville-Narasimhan-Ramanan

Assume that $\pi : X \rightarrow Y$ is étale. Recall that we have constructed in section 3.2 a dominant rational map

$$q_* : \mathcal{P}' \dashrightarrow \mathcal{U}_{X,0}^{\sigma,+}(r),$$

where \mathcal{P}' is some translate of the connected Prym variety of some étale double cover and $\mathcal{U}_{X,0}^{\sigma,+}(r)$ is the connected component of the locus of *stable* σ -symmetric vector bundles E such that $\det(E) \in \mathbb{P}$, where \mathbb{P} is the (connected) Prym variety of $X \rightarrow Y$. We fix an identification of \mathcal{P}' with the Prym variety \mathcal{P} of the spectral cover $\tilde{X}_s \rightarrow \tilde{Y}_s$ for general spectral data $s \in W^{\sigma,+}$.

By Lemma 4.3.4, for even theta characteristic κ , the Pfaffian line bundle \mathcal{P}_κ descends to the moduli space $\mathcal{U}_{X,0}^{\sigma,+}(r)$. We denote it also by \mathcal{P}_κ .

Lemma 4.5.1. *There exists a theta characteristic κ on X such that*

$$\dim(H^0(\mathcal{U}_{X,0}^{\sigma,+}, \mathcal{P}_\kappa)) = 1.$$

Proof. Let $q : \tilde{X}_s \rightarrow X$ be a smooth spectral curve over X attached to a general $s \in W^{\sigma,+}$ (see section 3.2 for more details and notations). Let $\nu = g_{\tilde{X}_s} - 1$. First, the pullback of the determinant bundle via $q_* : J_{\tilde{X}_s}^\nu \rightarrow \mathcal{U}_X(r, r(g_X - 1))$ is the line bundle $\mathcal{O}(\tilde{\theta})$ attached to the Riemann theta divisor $\tilde{\theta}$ over $J_{\tilde{X}_s}^\nu$. Let $\mathcal{S} \subset \mathcal{P}' \subset J_{\tilde{X}_s}^\nu$ be the locus of line bundles L such that q_*L is semi-stable. By [BNR89] Proposition 5.1, the complement of the locus of line bundles $L \in J_{\tilde{X}_s}^\nu$ such that q_*L is semi-stable is contained properly in θ . Since the restriction of θ to \mathcal{P}' is still a divisor which is actually ample, we deduce (as in [BNR89] Proposition 5.1, b)) that the codimension of the complement of \mathcal{S} in \mathcal{P}' is at least 2. Since $q_* : \mathcal{S} \rightarrow \mathcal{U}_{X,0}^{\sigma,+}(r)$ is dominant (note that we need to make a translation by certain line bundle after taking the direct image in order to get degree 0 vector bundles), we deduce an injection

$$H^0(\mathcal{U}_{X,0}^{\sigma,+}(r), \mathcal{P}_\kappa) \hookrightarrow H^0(\mathcal{P}', \tilde{\xi}),$$

where $\tilde{\xi}$ is a line bundle defining a principal polarization on \mathcal{P}' . So $h^0(\mathcal{U}_{X,0}^{\sigma,+}(r), \mathcal{P}_\kappa)$ is at most 1.

But by Lemma 4.3.5, we know that for some κ , the line bundle \mathcal{P}_κ has a non-zero global section. This ends the proof. \square

Let ξ and $\tilde{\xi}$ be line bundles defining principal polarizations on P and \mathcal{P} respectively. We also denote by \mathcal{P}_κ the restriction of \mathcal{P}_κ to $SU_X^{\sigma,+}(r) \subset \mathcal{U}_{X,0}^{\sigma,+}(r)$.

Theorem 4.5.2. *We have an isomorphism*

$$H^0(P, r\xi)^* \cong H^0(SU_X^{\sigma,+}(r), \mathcal{P}_\kappa).$$

In particular we deduce

$$\dim(H^0(SU_X^{\sigma,+}(r), \mathcal{P}_\kappa)) = r^{g_Y-1}.$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{P}' \cap \mathcal{Q}' \times P & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow \\ SU_X^{\sigma,+}(r) \times P & \longrightarrow & \mathcal{U}_{X,0}^{\sigma,+}(r), \end{array}$$

where \mathcal{Q}' is some translation of the Prym variety of $\tilde{X}_s \rightarrow X$. Using [BNR89] Theorem 3, we deduce that the pullback of the line bundle \mathcal{P}_κ to $SU_X^{\sigma,+}(r) \times P$ is of the form $p_1^* \mathcal{P}_\kappa \otimes p_2^* \mathcal{O}(r\xi)$.

Now the rational map $\mathcal{P}' \cap \mathcal{Q}' \rightarrow SU_X^{\sigma,+}(r)$ is dominant (Theorem 3.2.17). It follows, by the same argument used in the proof above, that the map

$$H^0(SU_X^{\sigma,+}(r), \mathcal{P}_\kappa) \rightarrow H^0(\mathcal{P}' \cap \mathcal{Q}', \tilde{\xi})$$

is injective, where here we denote abusively by $\tilde{\xi}$ the restriction of $\tilde{\xi}$ to $\mathcal{P}' \cap \mathcal{Q}' \subset \mathcal{P}'$. Since the two abelian subvarieties P and $\mathcal{P}' \cap \mathcal{Q}'$ are a complementary pair inside \mathcal{P} , we obtain using [BNR89] Proposition 2.4 an isomorphism

$$H^0(\mathcal{P}' \cap \mathcal{Q}', \tilde{\xi}) \cong H^0(P, r\xi)^*.$$

Hence we deduce an injective map

$$H^0(SU_X^{\sigma,+}(r), \mathcal{P}_\kappa) \hookrightarrow H^0(P, r\xi)^*.$$

Moreover the group $\mathbb{P}[r]$ acts on $\mathbb{P}H^0(\mathbb{P}, r\xi)^*$ as well as on $\mathcal{S}\mathcal{U}_X^{\sigma,+}(r)$, hence it acts also on the linear system $\mathbb{P}H^0(\mathcal{S}\mathcal{U}_X^{\sigma,+}(r), \mathcal{P}_\kappa)$. Since the projective representation $\mathbb{P}H^0(\mathbb{P}, r\xi)^*$ is irreducible, the map $H^0(\mathcal{S}\mathcal{U}_X^{\sigma,+}(r), \mathcal{P}_\kappa) \hookrightarrow H^0(\mathbb{P}, r\xi)^*$, which is equivariant for these actions, is necessarily an isomorphism. \square

Appendix A

Anti-invariant vector bundles via representations

In this appendix, we study the anti-invariant vector bundles as representations of the fundamental group.

Narasimhan and Seshadri established a bijection between irreducible unitary representations of $\pi_1(X)$ modulo conjugation and the isomorphism classes of stable vector bundles of degree 0. So what can we say about σ -anti-invariant vector bundles?

We treat the étale case. So assume that the cover $\pi : X \rightarrow Y$ is unramified and denote the genus of Y by g . Fix $y_0 \in Y$ and let $\pi^{-1}(y_0) = \{x_1, x_2\}$. Let $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ be the generators of $\pi_1(Y, y_0)$, with the relation

$$\prod_i [\alpha_i, \beta_i] = 1.$$

Denote hereafter by $U(r)$ the complex unitary group of rank r . Suppose further that the line bundle $\Delta = \det(\pi_* \mathcal{O}_X)^{-1}$ is given by the representation $\rho_\Delta : \pi_1(Y, y_0) \rightarrow U(1)$ defined by

$$\begin{aligned} \alpha_i &\rightarrow 1, \beta_j \rightarrow 1, \quad \text{for } i = 1, \dots, g, j = 2, \dots, g, \\ \beta_1 &\rightarrow -1. \end{aligned}$$

With these assumptions, one can see that the map $\pi_* : \pi_1(X, x_1) \rightarrow \pi_1(Y, y_0)$ is injective and its image is equal to the kernel of ρ_Δ . Hence it is generated by

$$\begin{aligned} &\alpha_1, \dots, \alpha_g, \beta_1^2, \beta_2, \dots, \beta_g, \\ &\beta_1 \alpha_2 \beta_1^{-1}, \dots, \beta_1 \alpha_g \beta_1^{-1}, \beta_1 \beta_2 \beta_1^{-1}, \dots, \beta_1 \beta_g \beta_1^{-1}. \end{aligned}$$

Note that one can easily check that the relation between the commutators of these generators is satisfied.

Note also that $\pi_1(X, x_2)$ has the same image in $\pi_1(Y, y_0)$ as $\pi_1(X, x_1)$. In fact σ induces an isomorphism $\sigma^* : \pi_1(X, x_1) \cong \pi_1(X, x_2)$. Moreover if γ is the unique path (up to homotopy equivalence) on X that lifts β_1 and starting from x_2 (it is a path from x_2 to x_1), then conjugation by γ induces an isomorphism $C_\gamma : \pi_1(X, x_1) \xrightarrow{\sim} \pi_1(X, x_2)$. The composition of these two maps

$$\vartheta : \pi_1(X, x_1) \xrightarrow{C_\gamma} \pi_1(X, x_2) \xrightarrow{\sigma^*} \pi_1(X, x_1)$$

gives an automorphism of the group $\pi_1(X, x_1)$ whose square ϑ^2 is an inner automorphism, namely it is a conjugation by β_1^2 if identify $\pi_1(X, x_1)$ with its image in $\pi_1(Y, y_0)$.

Now consider the case of line bundles. We have the following result

Proposition A.0.1. *A line bundle L over X is anti-invariant if and only if the associated representation $\rho_L : \pi_1(X, x_1) \rightarrow U(1)$ is ϑ -equivariant, i.e. for all $\alpha \in \pi_1(X, x_1)$ we have*

$$\rho_L(\vartheta(\alpha)) = \rho_L(\alpha)^{-1}.$$

Proof. The representation associated to σ^*L and L^{-1} are $\rho_L \circ \vartheta$ and ρ_L^{-1} respectively. Hence we have an equivalence

$$\sigma^*L \cong L^{-1} \Leftrightarrow \rho_L(\vartheta(\alpha)) \equiv \rho_L(\alpha)^{-1}.$$

But, since $U(1)$ is abelian, two representation are conjugate if and only if they are equal. \square

Using this Proposition, we deduce

Corollary A.0.2. *The locus of anti-invariant line bundles has 4 connected components.*

Proof. Remark that the automorphism ϑ induces an automorphism on the image of $\pi_1(X, x_1)$ in $\pi_1(Y, y_0)$ given by conjugation by β_1 . Hence if ρ is an ϑ -equivariant representation, then we have

$$\rho(\beta_1 \alpha_i \beta_1^{-1}) = \rho(\alpha_i)^{-1}, \quad \rho(\beta_1 \beta_j \beta_1^{-1}) = \rho(\beta_j)^{-1}.$$

In particular, since $\vartheta(\beta_1^2) = \beta_1^2$, we have

$$\rho(\beta_1^2) = \rho(\beta_1^2)^{-1},$$

hence $\rho(\beta_1^2) \in \{\pm 1\}$. Moreover the equality

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \prod_{i=2}^g [\alpha_i, \beta_i] = 1,$$

implies

$$\beta_1 \alpha_1 \beta_1^{-1} = \prod_{i=2}^g [\alpha_i, \beta_i] \alpha_1.$$

Since $U(1)$ is abelian, it follows that $\rho(\beta_1 \alpha_1 \beta_1^{-1}) = \rho(\alpha_1)$. So we deduce that $\rho(\alpha_1) = \rho(\alpha_1)^{-1}$, hence $\rho(\alpha_1) \in \{\pm 1\}$. So the values of ρ at α_1 and β_1^2 classify the connected components. \square

Consider now the case of rank r vector bundles. Recall that by the theorem of Narasimhan-Seshadri, any stable vector bundle over X of rank r is uniquely associated to an equivalence class of irreducible unitary representation of $\pi_1(X, x_1)$. It is quite easy to see that if the representation ρ_E associated to E is ϑ -equivariant, then E is an anti-invariant vector bundle. In fact we have an equivalence

Theorem A.0.3. *Let E be a stable vector bundle and ρ_E its associated representation. Then E is σ -anti-invariant if and only if ρ_E is ϑ -equivariant, i.e for any $\alpha \in \pi_1(X, x_1)$, ρ_E verifies*

$$\rho_E(\vartheta(\alpha)) = {}^t \rho_E(\alpha)^{-1}.$$

In particular, we deduce the locus of stable anti-invariant vector bundles has 4 connected components.

Proof. Let $\text{Rep}^\vartheta(r)$ be the space of ϑ -equivariant irreducible unitary representations of $\pi_1(X, x_1)$. The real orthogonal group $O(r, \mathbb{R})$ (which equals the intersection of the complex one with the unitary group) acts on this space and two ϑ -equivariant representations are equivalent if and only if they are equivalent modulo a conjugation by an element of $O(r, \mathbb{R})$ (here use the fact that the action by conjugation of $U(r)$ on $\text{Rep}^\vartheta(r)$ is free modulo the center of $U(r)$). From what we have said above we deduce an injective map

$$\text{Rep}^\vartheta(r)/O(r, \mathbb{R}) \longrightarrow \mathcal{U}_X^\sigma(r),$$

where $\mathcal{U}_X^\sigma(r)$ is the locus of stable σ -anti-invariant vector bundles, i.e. the union $\mathcal{U}_X^{\sigma,+}(r) \cup \mathcal{U}_X^{\sigma,-}(r)$. Now we can calculate the dimension of the left hand side. Giving a ϑ -equivariant representation is the same as giving elements $\rho(\alpha_1), \dots, \rho(\alpha_g), \rho(\beta_1^2)\rho(\beta_2), \dots, \rho(\beta_g) \in U(r)$ subject to the conditions

$$\rho(\alpha_1) {}^t \rho(\alpha_1) \prod_{i=2}^g [\rho(\alpha_i), \rho(\beta_i)] = I_r, \quad {}^t \rho(\beta_1^2)^{-1} = \rho(\beta_1^2).$$

So $\rho(\beta_1^2) \in O(r, \mathbb{R})$. Hence this amounts to a *real* dimension equals $(2g-1)r^2 - r^2 + \frac{r(r-1)}{2}$. Since we take the equivalence classes modulo $O(r, \mathbb{R})$, we have to subtract $\frac{r(r-1)}{2}$. Thus the complex dimension is given by

$$\dim_{\mathbb{C}}(\text{Rep}^\vartheta) = r^2(g-1) = \dim(\mathcal{U}_X^{\sigma,+}(r)).$$

So we deduce that we have an isomorphism

$$\text{Rep}^\vartheta(r)/O(r, \mathbb{R}) \cong \mathcal{U}_X^\sigma(r).$$

Now we deduce that $\mathcal{U}_X^\sigma(r)$ has 4 connected components parameterized by the determinant of the values of the representation ρ at α_1 and β_1^2 . Indeed, let $\rho \in \text{Rep}^\vartheta(r)$, then from the conditions above we deduce that the $\det(\rho(\alpha_1)) = \pm 1$ and $\det(\rho(\beta_1^2)) = \pm 1$. These two equalities gives the required parameter. \square

Remark A.0.4. Note that, when the rank r is odd, the action of $\text{Rep}^\vartheta(1)$ on $\text{Rep}^\vartheta(r)$ by multiplication induces a transitive action of $\pi_0(\text{Rep}^\vartheta(1)) \cong \mu_2^2$ on $\pi_0(\text{Rep}^\vartheta(r)) \cong \mu_2^2$, where $\mu_2 = \{\pm 1\}$. This is actually what we have proved using the Hitchin systems in Theorem [3.2.24](#).

Appendix B

Stability of the pullback of stable vector bundles and application

Let $\pi : X \rightarrow Y$ be a ramified double cover, where X and Y are smooth irreducible projective curves. We will show that the pullback of a stable vector bundle over Y by π is stable. We are grateful to George Hitching for showing this result to us. However, we have slightly improved the proof. As an application, we construct examples of stable anti-invariant vector bundles.

Lemma B.0.1. *Let E be a vector bundle on Y , and $F \subset \pi^*E$ be a subbundle. Then F descends to Y iff the canonical linearization on π^*E gives a linearization on $F: \sigma^*F \xrightarrow{\sim} F$.*

Proof. The direct implication is clear. So let $F \subset \pi^*E$ such that the canonical linearization $\phi : \sigma^*(\pi^*E) \rightarrow \pi^*E$ restricted to a linearization of F

$$\phi : \sigma^*F \rightarrow F.$$

If p is a ramification point then $\phi = \text{id}$ on $(\pi^*E)_p$ (by lemma 1.1.1), so its restriction to F_p is also the identity. Thus, again by lemma 1.1.1, F descends to Y . \square

Theorem B.0.2. *Let $E \rightarrow Y$ be a stable vector bundle. Then π^*E is also stable.*

Proof. Let's denote by

$$s(E, F) = \deg(E)\text{rk}(F) - \deg(F)\text{rk}(E),$$

for two vector bundles $F \subset E$. It is clear that E is stable if and only if $s(E, F) > 0$ for all proper subbundle $F \subset E$. Note also that $s(\pi^*E, \pi^*F) = 2s(E, F)$ for any subbundle $F \subset E$ over Y .

Let $F \subset \pi^*E$ be any proper subbundle. Let $P \subset \pi^*E$ be the bundle generated by $F + \sigma^*F$, and N such that

$$0 \rightarrow N \rightarrow F \oplus \sigma^*F \rightarrow P \rightarrow 0.$$

Now, we claim that both P and N descend to Y . By lemma B.0.1, We have just to prove that the canonical linearization $\tilde{\sigma} : \sigma^*\pi^*E \rightarrow \pi^*E$ gives a linearization on these bundles. Let

$$F \oplus \sigma^*F \rightarrow \pi^*E$$

be the canonical map, it is clearly σ -equivariant, so its image P and its kernel N are both σ -invariant. Hence $P = \pi^*P'$ and $N = \pi^*N'$ for some sub-bundles P' and N' of E .

Let r_E and d_E (resp. $r_P, d_P, r_N, d_N, r_F, d_F$) be the rank and the degree of π^*E (resp. P, N, F). We have

$$\begin{aligned}
s(\pi^*E, P) + s(\pi^*E, N) &= r_P d_E - r_E d_P + r_N d_E - r_E d_N \\
&= (r_P + r_N) d_E - r_E (d_P + d_N) \\
&= 2r_F d_E - 2r_E d_F \\
&= 2s(\pi^*E, F).
\end{aligned}$$

But E is stable, so if P' or N' is proper (non trivial) sub-bundle of E , we deduce $s(\pi^*E, F) > 0$. So suppose that $P' = E$ and $N' = 0$, in this case we have

$$\pi^*E \cong F \oplus \sigma^*F,$$

as π is ramified, the vector bundle $F \oplus \sigma^*F$ descends to Y if and only if F descends, to see this, choose a ramification point $p \in R$, then from the commutative diagram

$$\begin{array}{ccc}
\sigma^*F \oplus F & \xrightarrow{\sim} & F \oplus \sigma^*F \\
\wr \downarrow & & \wr \downarrow \\
\sigma^*\pi^*E & \xrightarrow{\sim} & \pi^*E,
\end{array}$$

we deduce that over the ramification point p the linearization ϕ is trivial, so it induces a linearization $\sigma^*F \rightarrow F$. Thus by lemma B.0.1, F descends to Y .

Let F' be the vector bundle on Y such that $\pi^*F' = F$. As π^* is injective, we deduce that $E \cong F' \oplus F'$, which contradicts the stability of E . \square

Remark B.0.3. The last result is completely false if the cover is étale, a counterexample can easily be constructed: take any line bundle L on X of degree 1, then its direct image π_*L is rank two vector bundle of degree 1 on Y which is (by a result of Beauville) semi-stable. So it is stable. But we have

$$\pi^*\pi_*L \cong L \oplus \sigma^*L,$$

which is not stable.

Proposition B.0.4. *Let F be a stable orthogonal (resp. symplectic) vector bundle on Y , then $E = \pi^*F$ is a stable σ -anti-invariant vector bundle on X with a σ -symmetric (resp. σ -alternating) isomorphism*

$$\psi : \sigma^*E \xrightarrow{\sim} E^*.$$

Proof. Let $\phi : \sigma^*E \xrightarrow{\sim} E$ be the canonical linearization, which is the identity over the ramification points. Let

$$\xi : F \xrightarrow{\sim} F^*,$$

the orthogonal (resp. symplectic) isomorphism. Denote $\psi = \pi^*\xi \circ \phi$. We have to prove that ψ is σ -symmetric (resp. σ -alternating), i.e.

$$\sigma^*\psi = {}^t\psi \quad (\text{resp. } \sigma^*\psi = -{}^t\psi).$$

First we have a commutative diagram

$$\begin{array}{ccc}
\sigma^*E & \xrightarrow{\phi} & E \\
\sigma^*(\pi^*\xi) \downarrow & & \downarrow \pi^*\xi \\
\sigma^*E^* & \xrightarrow{{}^t(\sigma^*\phi)} & E^*,
\end{array}$$

which is easy to verify.

Hence we have

$$\begin{aligned} {}^t(\sigma^*\psi) &= {}^t(\sigma^*(\pi^*\xi) \circ \sigma^*\phi) \\ &= {}^t(\sigma^*\phi) \circ {}^t(\sigma^*(\pi^*\xi)) \\ &= {}^t(\sigma^*\phi) \circ \sigma^*(\pi^*({}^t\xi)) \\ &= \pm {}^t(\sigma^*\phi) \circ \sigma^*(\pi^*\xi) \\ &= \pm(\pi^*\xi) \circ \phi \\ &= \pm\psi. \end{aligned}$$

□

Appendix C

On the codimension of non very stable rank 2 vector bundles

In this appendix, we prove directly, without using the result of Laumon [Lau88], that the locus of stable and non very stable σ -symmetric anti-invariant vector bundles of rank 2 has codimension at least 1.

Let E be such vector bundle. Denote by $\psi : \sigma^*E \xrightarrow{\sim} E^*$ and

$$\phi \in H^0(X, E \otimes \sigma^*E \otimes K_X)_+$$

be a nilpotent Higgs field. In particular $\phi \in H^0(X, \text{End}_0(E) \otimes K_X)_+$. So we can suppose that $\det(E) \cong \mathcal{O}_X$. Then let L^{-1} be the kernel of $\phi : E \rightarrow E \otimes K_X$, which is a line bundle of degree $-d < 0$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^{-1} & \longrightarrow & E & \xrightarrow{p} & L & \longrightarrow & 0 \\ & & \downarrow 0 & & \downarrow \phi & & \downarrow & & \\ 0 & \longrightarrow & L^{-1} \otimes K_X & \xrightarrow{i} & E \otimes K_X & \longrightarrow & L \otimes K_X & \longrightarrow & 0. \end{array}$$

This implies that ϕ factorizes through L , that's to say, there exists $\phi' : L \rightarrow E \otimes K_X$ such that $\phi = \phi' \circ p$. Moreover, as $(\phi \otimes 1_{K_X}) \circ \phi = 0$, this implies that

$$(\phi \otimes 1_{K_X}) \circ \phi' = 0,$$

hence ϕ' factorizes through $\text{Ker}(\phi \otimes 1_{K_X}) = L^{-1} \otimes K_X$. Hence there exists $s \in H^0(X, L^{-2} \otimes K_X)$ such that

$$\phi = i \circ s \circ p.$$

In particular $d \leq g_X - 1$.

But, since ϕ is invariant, the following diagram

$$\begin{array}{ccc} \sigma^*\phi & : & \sigma^*E \longrightarrow \sigma^*E \otimes \sigma^*K_X \\ & \uparrow \psi^{-1} & \downarrow \psi \otimes \eta \\ {}^t\phi \otimes 1_{K_X} & : & E^* \longrightarrow E^* \otimes K_X, \end{array}$$

commutes. It follows that L^{-1} is ψ isotropic, i.e. ψ induces the zero map $\sigma^*L^{-1} \rightarrow L$. Furthermore, we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma^*L^{-1} & \longrightarrow & \sigma^*E & \xrightarrow{p} & \sigma^*L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \psi & & \downarrow & & \\ 0 & \longrightarrow & L^{-1} & \longrightarrow & E^* & \xrightarrow{i^*} & L & \longrightarrow & 0, \end{array}$$

which implies that ψ induces a non-zero map $\sigma^*L^{-1} \rightarrow L^{-1}$, hence it is an isomorphism. Thus L^{-1} (and so L) is σ -invariant line bundle.

Finally, giving a stable and non very stable vector bundle E of rank 2 with a non-zero nilpotent Higgs field ϕ , is the same as giving a σ -invariant line bundle L and an anti-invariant extension $\xi \in \text{Ext}^1(L, L^{-1})_-/\mathbb{C}^*$ (modulo a scalar) such that $\deg(L) = d \in \{1, \dots, g_X - 1\}$, with a non-zero section

$$s \in H^0(X, L^{-2} \otimes K_X)_+.$$

Fix the degree d . And for simplicity, we suppose that L descends to $M \in \text{Pic}^{d/2}(Y)$. Define

$$\Theta_d = \{[M] \in \text{Pic}^{d/2}(Y) \mid h^0(M^{-2}K_Y\Delta) > 0\}.$$

The dimension of Θ_d is given by

$$\min\{g_Y, -d + 2(g_Y - 1) + n\} =: \theta(d),$$

and the dimension of $\text{Ext}^1(L, L^{-1})_- \cong H^1(X, L^{-2})_-$ can be calculated easily using the fact that the fixed linearization on K_X is the positive one. We obtain

$$\dim(\text{Ext}^1(L, L^{-1})_-/\mathbb{C}^*) = h^1(Y, M^{-2}) - 1 = d + g_Y - 2.$$

Thus, the locus of such bundles has a dimension

$$\theta(d) + d + g_Y - 2 + h^0(Y, M^{-2} \otimes K_Y \otimes \Delta),$$

but for general $M \in \Theta_d$, $h^0(M^{-2} \otimes K_Y \otimes \Delta) = \min\{1, d - (g_Y - 2) - n\}$. So it follows that the dimension we are looking for is

$$\begin{aligned} \theta(d) + d + g_Y - 2 + \min\{1, d - (g_Y - 2) - n\} &= 3(g_Y - 1) + n \\ &= \dim(\mathcal{SU}_X^{\sigma,+}(r)). \end{aligned}$$

But this locus inside $T^*\mathcal{SU}_X^{\sigma,+}(2)$ is conic, that's to say stable by the canonical action of \mathbb{C}^* , hence over each point of $\mathcal{SU}_X^{\sigma,+}(2)$ the fiber of this locus has dimension at least 1. This implies that its image in $\mathcal{SU}_X^{\sigma,+}(2)$ has codimension at least 1.

Appendix D

Some Results on anti-invariant vector bundles

D.1 Anti-invariance of elementary transformations

Let (E, ψ) be an anti-invariant vector bundle, and fix a ramification point $p \in R$. After taking two elementary transformations (negative and then positive)

$$\begin{aligned} 0 \rightarrow F \rightarrow E \rightarrow \mathbb{C}_p \rightarrow 0, \\ 0 \rightarrow F \rightarrow E' \rightarrow \mathbb{C}_p \rightarrow 0, \end{aligned}$$

we ask if E' is σ -anti-invariant?

First, the restriction of ψ to F gives a map $\sigma^*F \rightarrow F^*$. Its co-rank at p is either 1 (for general transformation) or 2, and it is 2 if and only if

$$l \subset Q^*,$$

where $l = \ker(E_p^* \rightarrow F_p^*)$, and $Q^* = \{\phi \in E_p^* \mid \phi(\psi^{-1}(\phi)) = 0\}$ is the quadric associated to ψ^{-1} . Indeed, we have the diagram

$$F_p \xrightarrow{i} E_p \xrightarrow{\sim} E_p^* \rightarrow F_p^*.$$

In fact, the restriction of ψ_p to F_p is the composition of these maps. So the composed map is of co-rank 2 if and only if $l \subset \text{Im}(\psi_p \circ i)$. But by definition, the elements of l are those which vanish on $\text{Im}(i)$. It follows

$$l \subset \text{Im}(\psi_p \circ i) \implies l \subset Q^*.$$

Conversely, if we take $\phi \in l$ which is non-zero, so we have $\text{Im}(i) \subset \text{Ker}(\phi)$, but these two spaces are both lines, so $\text{Im}(i) = \text{Ker}(\phi)$, therefore

$$\phi(\psi_p^{-1}(\phi)) = 0 \implies \psi_p^{-1}(\phi) \in \text{Im}(i) \implies \phi \in \text{Im}(\psi_p \circ i).$$

Now, taking the first transformation such that the co-rank of ψ_p is 2, assuming that E' is stable (which is the general case), and looking to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \sigma^*F & \longrightarrow & \sigma^*E' & & \\ & & & & \downarrow & \searrow & \\ & & & & E'^* & \longrightarrow & F^* \longrightarrow 0. \end{array}$$

We see that $\sigma^*F \rightarrow F^*$ induces a map $\sigma^*E' \rightarrow E'^*$ if and only if

$$\ker(F_p \rightarrow E_p') \subset \ker(F_p \rightarrow F_p^*),$$

and

$$\mathrm{Im}(F_p \rightarrow F_p^*) \subset \mathrm{Im}(E_p'^* \rightarrow F_p^*).$$

This last two conditions are the same (we use here the fact that ${}^t\psi_p = \pm\psi_p$). So by choosing a line

$$L = \ker(F_p \rightarrow E_p') \subset \ker(F_p \rightarrow F_p^*),$$

we get a σ -anti-invariant E' . Moreover, all the elementary transformations of this kind are parameterized by $\mathbb{P}\ker(F_p \rightarrow F_p^*) = \mathbb{P}^1$.

D.2 Another description of anti-invariant vector bundles

Lemma D.2.1. *Giving a σ -symmetric anti-invariant vector bundle (E, ψ) over X is the same as giving a pair (E, φ) , where φ is a \mathcal{O}_Y -bilinear perfect form*

$$\varphi : \pi_* E \times \pi_* E \rightarrow \pi_* \mathcal{O}_X,$$

such that $\varphi(a \cdot u, v) = \varphi(u, \sigma^*(a) \cdot v)$ and $\varphi(u, v) = \sigma^*(\varphi(v, u))$ for all $u, v \in \pi_* E$, $a \in \pi_* \mathcal{O}_X$

Proof. Assume that we have (E, ψ) , then we take

$$\varphi = \pi_* \tilde{\psi},$$

where $\tilde{\psi} : E \otimes \sigma^* E \rightarrow \mathcal{O}_X$ is the σ -bilinear forms defined by ψ . We can easily check that φ verifies the conditions above.

Conversely, suppose that (E, φ) is given. First, consider the exact sequence

$$0 \rightarrow \pi^* \pi_* E \rightarrow E \oplus \sigma^* E \rightarrow E|_R \rightarrow 0,$$

where, as always, R denote the ramification divisor. So generically we have a canonical isomorphism between $\pi^* \pi_* E$ and $E \oplus \sigma^* E$. Notice that $\pi^* \varphi$ is \mathcal{O}_X -bilinear form, which verifies for all $a \in \pi^* \pi_* \mathcal{O}_X \cong \mathcal{O}_X \oplus \sigma^* \mathcal{O}_X$, which is of the form $a = a_+ + a_-$, we have

$$\pi^* \varphi(a \cdot u, v) = \pi^* \varphi(u, \sigma^*(a) \cdot v),$$

where $\sigma^*(a) = a_+ - a_-$, $a \cdot u = au_+ + \sigma^*(a)u_-$ (we decompose u according to $E \oplus \sigma^* E$) Taking $a_+ = 0$, $u = u_+$ and $v = v_+$ (resp. $u = u_-, v = v_-$) we get

$$a_- \pi^* \varphi(u, v) = \pi^* \varphi(a_- u, v) = \pi^* \varphi(u, -a_- v) = -a_- \pi^* \varphi(u, v).$$

So

$$\pi^* \varphi(u, v) = 0$$

for all $(u, v) \in E \times E$ (resp. $(u, v) \in \sigma^* E \times \sigma^* E$).

Hence, we take $\tilde{\psi} = \pi^* \varphi|_{E \otimes \sigma^* E}$. Notice that

$$\pi^* \varphi = \begin{pmatrix} 0 & \sigma^*({}^t\tilde{\psi}) \\ \tilde{\psi} & 0 \end{pmatrix}$$

according to the decomposition $(E \oplus \sigma^* E) \otimes (E \oplus \sigma^* E)$

□

D.3 Equality between two canonical maps

Studying the dominance of the Hitchin map, we had been led to consider the surjectivity of its differential. It turns out that it has a canonical identification with the dual of the differential of the pushforward map. Although that this has not been used in this dissertation, we would like to mention it here.

Let $s = (s_i) \in W = \bigoplus H^0(X, K_X^i)$ and $q : \tilde{X}_s \rightarrow X$ the associated spectral curve, let L be a line bundle over \tilde{X}_s such that $E := q_*L$ is stable vector bundle of rank r and degree 0. Denote by $\varphi : E \rightarrow E \otimes K_X$ the associated Higgs field. Consider the Hitchin map

$$\mathcal{H} : H^0(X, E \otimes E^* \otimes K_X) \longrightarrow W,$$

whose i^{th} component is defined by

$$\mathcal{H}_i(\phi) = \text{Tr}(\phi^i).$$

Theorem D.3.1. *With the above assumptions, the differential of the Hitchin map $d_\varphi \mathcal{H}$ is canonically identified with $(d_L(q_*))^*$.*

Proof. First by [Bea91], we have the identification

$$d_L q_* \cong H^1(\lambda),$$

where $\lambda : q_* \mathcal{O}_{\tilde{X}_s} \longrightarrow \text{End}(q_* L)$ is the canonical map coming from the $\mathcal{O}_{\tilde{X}_s}$ -module structure of L . Hence, by Serre duality

$$(d_L q_*)^* \cong H^1(\lambda)^* \cong H^0(\lambda^* \otimes id_{K_X}).$$

Moreover, by developing the formula $\mathcal{H}(\varphi + \varepsilon\psi) = \mathcal{H}(\varphi) + \varepsilon d_\varphi \mathcal{H}(\psi)$, we deduce easily

$$d_\varphi \mathcal{H}_i(\psi) = i \text{Tr}(\psi \circ \varphi^{i-1}).$$

On the other hand, the map

$$\lambda : \mathcal{O}_X \oplus K_X^{-1} \oplus \cdots \oplus K_X^{-r+1} \longrightarrow E \otimes E^*$$

is given explicitly by

$$\lambda(s_0, s_1, \dots, s_{r-1}) = s_0 \text{id} + \varphi(s_1) + \cdots + \varphi^{r-1}(s_{r-1}),$$

where φ is seen as a map $K_X^{-1} \rightarrow E \otimes E^*$. Hence

$$\lambda^*(\psi) = (\text{Tr}(\psi), \text{Tr}(\psi \circ \varphi), \dots, \text{Tr}(\psi \circ \varphi^{i-1})).$$

Thus $H^0(\lambda^* \otimes id_{K_X})(\psi) = (\text{Tr}(\psi \circ \varphi^{i-1}))_i$. So by taking the automorphism of W given by multiplication by $(1, 2, \dots, r)$, we get the identification. \square

Appendix E

Rank 2 case

In this appendix, we give another proof of the irreducibility of $\mathcal{U}_X^{\sigma,+}(2)$ and the fact that $\mathcal{U}_X^{\sigma,-}(2)$ has two connected components in the ramified case.

Assume that $\pi : X \rightarrow Y$ is ramified. In this case every vector bundle E over X with trivial determinant is an Sp_2 -bundle, that's $E \cong E^*$ with a symplectic form, and it admits a symmetric one if and only if it is polystable.

We see in this particular case that σ -invariant bundles are the same as σ -anti-invariant bundles. So let (E, ϕ) be a stable σ -invariant bundle with trivial determinant. The triviality of the determinant implies that the type of E must be of the form

$$\tau = (\mathbf{0}, \dots, \mathbf{0}),$$

$$\text{or } \tau = (A_1, \dots, A_{2n}), \text{ with } A_i \in \{+\mathbf{1}, -\mathbf{1}\},$$

where $\mathbf{0} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\pm\mathbf{1} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.

We have

$$\psi : \sigma^*E \xrightarrow{\phi} E \xrightarrow{q} E^*,$$

where q is a symplectic form. Let $\psi = q \circ \phi$. It is not difficult to see that if E has a type $(\mathbf{0}, \dots, \mathbf{0})$, then ψ is σ -symmetric, and it is σ -alternating otherwise.

In particular, one deduces that $\mathcal{S}\mathcal{U}_X^{\sigma,+}(2)$ is connected, and $\mathcal{S}\mathcal{U}_X^{\sigma,-}(2)$ has 2^{2n-1} connected components.

Moreover, we have surjective maps

$$\mathcal{S}\mathcal{U}_X^{\sigma,+}(2) \times P \rightarrow \mathcal{U}_X^{\sigma,+}(2),$$

$$\mathcal{S}\mathcal{U}_X^{\sigma,-}(2) \times P \rightarrow \mathcal{U}_X^{\sigma,-}(2).$$

This proves the irreducibility of $\mathcal{U}_X^{\sigma,+}(2)$.

The group $P[2]$ acts naturally on the set of connected components of $\mathcal{S}\mathcal{U}_X^{\sigma,-}(r)$ in the following way: for $\lambda \in P[2]$ of type $v = (\varepsilon_1, \dots, \varepsilon_{2n})$, where $\varepsilon_i \in \{\pm 1\}$, and for a type $\tau = (A_1, \dots, A_{2n})$ attached to a connected component of $\mathcal{S}\mathcal{U}_X^{\sigma,-}(r)$, we have

$$v \cdot \tau = (\varepsilon_1 A_1, \dots, \varepsilon_{2n} A_{2n}).$$

Furthermore this action is free modulo $\pi^*J_Y[2]$. Since $\mathrm{card}(P[2]/\pi^*J_Y[2]) = 2^{2n-2}$, we deduce that this action has two orbits. It follows that $\mathcal{U}_X^{\sigma,-}(2)$ has two connected components.

Appendix F

Lefschetz Fixed Point Formula

We have used the Lefschetz fixed point theorem in several places in this thesis, so we would like to explicitly write it down in this appendix. We translate this result into our spacial context. Before that let's introduce some notations.

Let X be a smooth projective curve, and denote by $\varrho : X \rightarrow X$ an arbitrary automorphism. Since X is smooth, any fixed point of ϱ is simple. Denote the fixed locus of ϱ by S . Consider now a ϱ -invariant vector bundle E of rank r and let $\psi : \varrho^*E \rightarrow E$ be an isomorphism.

The isomorphisms ϱ and ψ induce an automorphism on the spaces $H^i(X, E)$ for $i = 0, 1$. We denote these automorphism by ϱ_i .

Define the *Lefschetz number* attached to this data by

$$\begin{aligned} L(\varrho, \psi) &= \sum_i (-1)^i \text{Tr}(\varrho_i) \\ &= \text{Tr}(\varrho_0) - \text{Tr}(\varrho_1). \end{aligned}$$

Theorem F.0.1 ([AB68], Theorem 4.12). *With the above notations, we have*

$$L(\varrho, \psi) = \sum_{p \in S} \frac{\text{Tr}(\psi_p)}{\det(1 - d_p \varrho)},$$

where $d_p \varrho : T_p X \rightarrow T_p X$ denote the differential of ϱ at $p \in X$.

In particular, since we are mainly interested in involutions, we deduce the following corollary.

Corollary F.0.2. *Assuming that ϱ is an involution, we get*

$$h^0(X, E)_+ - h^0(X, E)_- - h^1(X, E)_+ + h^1(X, E)_- = \sum_{p \in S} \frac{\text{Tr}(\psi_p)}{\det(1 - d_p \varrho)}.$$

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