Geometry of rationally connected varieties

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Table des matières

Introduction

Un des objectifs ultimes de la géométrie algébrique est la classification des variétés algébriques. Par le théorème des résolutions de singularités de Hironaka, toute variété complexe est birationnellement équivalente à une variété lisse. Rappelons que deux variétés algébriques X et X' sont dites birationnellement équivalentes s'il existe un isomorphisme d'un ouvert de X dans un ouvert de X'. Le but de la géométrie birationnelle est donc de classifier les variétés algébriques à équivalence birationnelle près.

Regardons d'abord le cas des courbes. Toute courbe est birationnellement équivalente à une unique courbe projective lisse. Soit C une courbe projective lisse. Le genre de C est défini par $g(C) = h^0(C, \omega_C)$, où ω_C est le fibré cotangent de C. Si on fixe le genre g, alors il existe une famille de courbes projectives lisses de genre g. La famille est paramétrée par une variété algébrique $\mathscr{M}_g$, qui s'appelle la variété des modules des courbes lisses projectives de genre g (voir par exemple [HM98, §2]).

En dimension supérieure, on introduit les plurigenres et la dimension de Kodaira qui généralisent la notion de genre d'une courbe. Soit X une variété projective complexe lisse de dimension n. Notons ω_X le fibré canonique de X, qui est le déterminant du fibré cotangent de X. Soit K_X un diviseur canonique, *i.e.* un diviseur tel que $\omega_X \cong \mathscr{O}_X(K_X)$. Alors, pour un nombre naturel m, on définit le m-plurigenre de X par

$$P_m(X) = h^0(X, \omega_X^{\otimes m}).$$

La dimension de Kodaira de X est définie par

$$\kappa(X) = \limsup_{m \to \infty} \frac{\log P_m(X)}{\log m}.$$

La dimension de Kodaira prend ses valeurs dans $\{-\infty, 0, 1, ..., n\}$. La variété X est dite de type général si $\kappa(X) = n$.

Exemple 0.0.1. Soit C une courbe projective lisse, alors

$$\begin{cases} \text{si } g(C) = 0, \text{ alors } \kappa(C) = -\infty, \ C \cong \mathbb{P}^1 \text{ et } -K_C \text{ est ample,} \\ \text{si } g(C) = 1, \text{ alors } \kappa(C) = 0, \ C \text{ est une courbe elliptique et } K_C = 0, \\ \text{si } g(C) \geqslant 2, \text{ alors } \kappa(C) = 1 \text{ et } K_C \text{ est ample} \end{cases}$$

Dans une classe d'équivalence birationnelle, il y a beaucoup de variétés projectives lisses. Il est naturel de se demander si nous pouvons choisir un bon représentant dans une classe d'équivalence birationnelle. Le cas des surfaces complexes est complètement traité dans les années 1930 par les géomètres italiens. Soit X une surface complexe projective lisse. S'il existe une (-1)-courbe C dans X (*i.e.* $C \cong \mathbb{P}^1$ et le nombre d'intersection $C^2 = -1$), alors par le théorème de Castelnuovo (voir par exemple [Har77, Thm. V.5.7]), il existe une fibration (*i.e.* un morphisme propre surjective à fibres connexes) $X \to X_1$ contractant exactement C de sorte que X_1 est encore lisse. En répétant cette opération, nous pouvons obtenir une suite de morphismes birationnels

$$X = X_0 \to X_1 \to \cdots \to X_k = X^*$$

telle que X^* ne contienne aucune (-1)-courbe. Ce processus s'appelle un programme des modèles minimaux (MMP en abrégé) pour la surface X. De plus, nous avons

- ou bien X^* est une surface réglée,
- ou bien $X^* \cong \mathbb{P}^2$,
- ou bien K_{X^*} est nef.

Rappelons qu'un diviseur de Cartier D dans une variété projective normale X est dit nef si pour tout morphisme g d'une courbe C projective lisse dans X, le diviseur g^*D est de degré positif ou nul sur C. Pour des surfaces dont le diviseur canonique est nef, nous avons le théorème suivant (voir par exemple [Mat02, Thm. 1-5-6]).

Théorème 0.0.2 (Théorème d'Abondance pour les surfaces)**.** *Soit Y une surface complexe projective lisse telle que K_Y soit nef. Alors il existe un nombre entier $m > 0$ tel que le faisceau $\mathscr{O}_Y(mK_Y)$ soit engendré par ses sections globales. Le système linéaire de $\mathscr{O}_Y(mK_Y)$ induit une fibration $f : Y \to Z$, applée la fibration d'Iitaka (voir par exemple [Iit82, §2.9]), telle que* $\dim Z = \kappa(Y)$.

Soit Y une surface projective complexe avec K_Y nef. Selon sa dimension de Kodaira, nous avons trois possibilités :

1. Si $\kappa(Y) = 0$, alors Y est une surface $K3$, une surface d'Enriques ou un quotient étale d'une surface abélienne.
2. Si $\kappa(Y) = 1$, alors la fibration d'Iitaka $f : Y \to Z$ définie dans le théorème précédent induit une structure de surface elliptique minimale sur Y.
3. La surface Y est de type général.

De plus, sauf dans le dernier cas, il existe une classification détaillée de Y (voir par exemple [BHPV04, §VI]).

D'après les travaux de Fano, Iskovskikh, Iitaka, Ueno, Shokurov, Reid, etc, on conjecture qu'on puisse aussi établir des MMP pour les variétés de dimensions supérieures. Cependant, plusiers obstacles apparaissent. Une des difficultés est de généraliser les contractions de (-1)-courbes dans une surface. L'idée de Mori est de remplacer les (-1)-courbes par les rayons extrêmaux de $\overline{\mathrm{NE}}(X)$ qui ont intersection strictement négative avec K_X ([Mor82]). Rappelons que $\overline{\mathrm{NE}}(X)$ est le cône convexe fermé dans $N_1(X)$ engendré par les 1-cycles effectifs dans X, où $\mathrm{N}_1(X)$ est l'espace vectoriel réel engendré par les classes numériques de courbes dans X. Il introduit aussi la notion d'espace fibré de Mori, qui est la généralisation des surfaces réglées en grandes dimensions. Une variété projective lisse Y est un espace fibré de Mori s'il existe une contraction d'un rayon extrêmal strictement K_Y-négatif $f : Y \to Z$ telle que $\dim Z < \dim Y$.

Un autre problème est qu'il existe des variétés de dimension 3 qui ne possèdent pas de modèle minimal lisse. Il faut donc aussi considérer les variétés singulières. À la suite des travaux de Kawamata, Reid, Shokurov, etc, nous avons le théorème du cône suivant pour les variétés peu singulières (voir par exemple [Thm. 3-2-1 et Thm. 4-2-1]KMM87).

Théorème 0.0.3 (Théorème du Cône)**.** *Soit X une variété complexe $\mathbb{Q}$-factorielle projective à singularités Kawamata log terminales. Alors*

1. *Il existe un ensemble au plus dénombrable de courbes rationnelles $C_j \subseteq X$, avec $j \in J$, telles que $K_X \cdot C_j < 0$ pour tout j et*

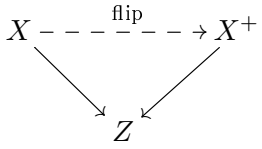
$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geqslant 0} + \sum_{j \in J} \mathbb{R}^+[C_j].$$

2. *Pour tout $\epsilon > 0$ et tout diviseur ample H, il existe un sous-ensemble fini I de J tel que*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X + \epsilon H) \geqslant 0} + \sum_{i \in I} \mathbb{R}^+[C_i].$$

3. *Soit $R \subseteq \overline{\mathrm{NE}}(X)$ un rayon extrêmal dont l'intersection avec K_X est strictement négative. Alors il existe une unique fibration projective $c_R : X \to Z$ telle qu'une courbe $C \subseteq X$ soit contractée par c_R si et seulement si $[C] \subseteq R$. Le morphisme c_R est appelé la contraction de R.*

Même si nous avons le théorème de contraction fourni par le théorème ci-dessus, il y a encore un obstacle. Considérons une contraction extrêmale $c : X \to Z$. Si le lieu exceptionnel de c est de codimension au moins 2 dans X, alors la base Z est trop singulière pour lui appliquer le théorème du cône. En effet, dans ce cas, le diviseur K_Z n'est jamais $\mathbb{Q}$-Cartier. L'idée de Mori pour ce problème est d'introduire un autre type de transformation birationnelle, le flip, pour X.



L'existence des flips est démontrée par Mori ([Mor88, Thm. 0.2.5]) en dimension 3, par Shokurov en dimension 4 et par Birkar, Cascini, Hacon et M^{c}Kernan ([BCHM10, Cor. 1.4.1]) dans le cas général.

Soit X une variété complexe projective lisse. Alors par le théorème du cône et l'existence des flips, nous avons une suite de transformations birationnelles de X

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots$$

On conjecture qu'une telle suite est toujours finie et que si X^* est le dernier terme, alors ou bien K_{X^*} est nef, ou bien X^* est un espace fibré de Mori. La conjecture de terminaison de MMP est équivalente à la non-existence d'une suite infinie de flips. Cette conjecture est démontrée par Shokurov dans le cas de dimension 3 ([Sho86, Cor. 2.17]) et par Kawamata, Matsuda et Matsuki dans le cas de dimension 4 ([KMM87, Thm. 5-1-15]). En dimension au moins 5, la conjecture est encore ouverte.

Une autre conjecture dans le MMP est la conjecture d'Abondance.

Conjecture 0.0.4. *(Conjecture d'Abondance) Soit Y une variété complexe $\mathbb{Q}$-factorielle projective à singularités Kawamata log terminales telle que K_Y soit nef. Alors il existe un nombre entier $m > 0$ tel que le faisceau $\mathscr{O}_Y(mK_Y)$ soit engendré par ses sections globales. Le système linéaire de $\mathscr{O}_Y(mK_Y)$ induit une fibration $f : Y \to Z$ telle que* $\dim Z = \kappa(Y)$.

La conjecture d'abondance est seulement démontrée en dimensions deux et trois ([Kaw92]). Ces deux dernières conjectures suggèrent que toute variété complexe X admet, à équivalence birationnelle près, une fibration $f : X \to Y$ telle que la fibre générale F vérifie que ou bien K_F est ample, ou bien $K_F = 0$, ou bien $-K_F$ est ample. Remarquons que cette classification des fibres ressemble à celle de l'Exemple 0.0.1 pour les courbes. Ces trois types extrêmaux des variétés forment la base de la classification des variétés. Les variétés du troisième type sont appelées variétés de Fano.

Pour les variétés projectives lisses X telle que $K_X = 0$, nous avons le théorème de classification de Beauville-Bogomolov suivant.

Théorème 0.0.5 ([Bea83, Thm. 1])**.** *Soit X une variété projective complexe lisse telle que K_X soit numériquement triviale. Alors il existe un revêtement fini étale $X' \to X$ tel que*

$$X' \cong T \times \prod_i V_i \times \prod_j W_j,$$

où T est un tore, V_i est une variété de Calabi-Yau simplement connexe et W_j est une variété symplectique irréductible.

Les variétés symplectiques irréductibles sont la généralisation des surfaces K3 en grandes dimensions. Dans le Chapitre 4, nous regardons les fibrations lagrangiennes sur des variétés complexes projectives symplectiques irréductibles. Soit $f : X \to Y$ une fibration d'une variété complexe projective symplectique irréductible de dimension $2n$ sur une variété projective normale de dimension strictement comprise entre 0 et $2n$. Alors la fibration f est toujours une fibration lagrangienne. De plus, la base Y est une variété de Fano de dimension n, à singularités Kawamata log terminales, dont le nombre de Picard est 1 ([Mat99, Thm. 2]). Dans ce chapitre, nous étudions le cas où $n = 2$ et prouvons qu'il existe au plus deux possibilités pour Y.

Le Chapitre 5 se consacre à l'étude des variétés de Fano complexes peu singulières dont tout diviseur effectif est nef. Une variété X de ce type est un objet minimal dans sa classe d'équivalence birationnelle : si $f : X \to Y$ est un morphisme birationnel, alors f est un isomorphisme ([Dru14, Lem. 4.4]). Il est connu que le nombre de Picard $\rho(X)$ de X est au plus équal à $\dim X$ ([Wiś91, Thm. 2.2], [Dru14, Lem. 4.7]). Druel classifie ces variétés qui vérifient $\rho(X) = \dim X$ ([Dru14, Thm. 1.1]). Dans le Chapitre 5, nous donnons une classification dans le cas où $\rho(X) = \dim X - 1$.

Une autre approche du problème de classification est plus géométrique. Il s'agit de comprendre la géométrie intrinsèque des variétés en regardant des courbes, particulièrement des courbes rationnelles, dans des variétés.

Soit $f : \mathbb{P}^1 \to X$ une courbe rationnelle dans une variété lisse X. Alors cette courbe est dite libre si f^*T_X est un fibré vectoriel nef, *i.e.* un fibré vectoriel qui n'admet pas de quotient fibré en droites de degré strictement negatif, où T_X est le fibré tangent de X. La courbe est dite très libre si f^*T_X est un fibré vectoriel ample, *i.e.* un fibré vectoriel qui n'admet pas de quotient fibré en droites de degré négatif ou nul. Il est intéressant de considérer des variétés qui contiennent beaucoup de courbes rationnelles. Une variété projective complexe X est dite uniréglée si elle est recouverte par des courbes rationnelles. Le critère suivant est important pour comprendre la géométrie des courbes rationnelles dans une variété uniréglée lisse.

Théorème 0.0.6 ([Kol96, Thm. IV.1.9])**.** *Soit X une variété complexe projective lisse. Alors X est uniréglée si et seulement si elle contient une courbe rationnelle libre.*

De plus, si X est une variété complexe projective uniréglée lisse, alors les courbes rationnelles libres recouvrent un sous-ensemble dense de X. Remarquons que, si $f : \mathbb{P}^1 \to X$ est une courbe libre, alors $-f^*K_X$ est un diviseur ample sur $\mathbb{P}^1$.

Corollaire 0.0.7. *Soit X une variété complexe projective lisse uniréglée. Alors, pour tout entier $m > 0$, $h^0(X, \mathscr{O}_X(mK_X)) = 0$.*

En particulier, la dimension de Kodaira d'une variété complexe projective lisse uniréglée est $-\infty$. Il est conjecturé que la réciproque du corollaire est

aussi vraie. Remarquons que cette conjecture résulte de la conjecture du terminaison de MMP et la conjecture d'abondance. En effet, soit X une variété complexe projective lisse dont la dimension de Kodaira est $-\infty$. Si la conjecture de terminaison de MMP est vraie, alors soit X^* un résultat d'un MMP de X. On a $\kappa(X^*) = -\infty$. Si la conjecture d'abondance est vraie, alors X^* est un espace fibré de Mori. Comme les fibres générales d'une fibration de Mori sont des variétés de Fano, et les variétés de Fano peu singulières sont rationnellement connexes (voir ci-dessous), X^* est uniréglée. Cela implique que X est aussi uniréglée.

Une variété complexe projective X est dite rationnellement connexe par chaînes si pour deux points généraux de X, il existe une suite finie de courbes rationnelles qui les joint. La variété X est dite rationnellement connexe si pour deux points généraux de X, il existe une courbe rationnelle passant par eux. Si la variété X est lisse, alors elle est rationnellement connexe par chaînes si et seulement si elle est rationnellement connexe ([KMM92b, 2.1]). Les variétés de Fano lisses sont rationnellement connexes ([Cam92, 3.2], [KMM92a, 3.3]).

Théorème 0.0.8 ([Kol96, Thm. IV.3.7]). *Une variété complexe projective est rationnellement connexe si et seulement si elle contient une courbe rationnelle très libre.*

Corollaire 0.0.9. *Soit X une variété complexe projective lisse rationnellement connexe. Alors X ne porte pas de pluri-forme non nulle, i.e.*

$$h^0(X, (\Omega^1_X)^{\otimes m}) = 0$$

pour tout entier $m > 0$.

Mumford conjecture que la réciproque du corollaire ci-dessus est aussi vraie. Dans le cas de la dimension 3, la conjecture est vraie ([Kol96, Cor. 5.7.1]).

Il est naturel de croire que les courbes rationnelles dans une variété peu singulière ont des comportements similaires à ceux des courbes rationnelles dans une variété lisse. Hacon et M^{c}Kernan démontrent qu'une variété complexe projective à singularités Kawamata log terminales est rationnellement connexe par chaînes si et seulement si elle est rationnellement connexe ([HM07, Cor. 1.8]). Il est encore vrai qu'une variété complexe de Fano à singularités Kawamata log terminales est rationnellement connexe ([Zha06, Thm. 1], [HM07, Cor. 1.13]). Il y a aussi une version du Corollaire 0.0.9 pour les variétés singulières.

Théorème 0.0.10 ([GKP14, Thm. 3.3]). *Soit X une variété complexe projective rationnellement connexe localement factorielle à singularités Kawamata log terminales. Alors pour tout entier $m > 0$, on a $h^0(X, (\Omega^1_X)^{[\otimes m]}) = 0$, où $(\Omega^1_X)^{[\otimes m]}$ est le double dual de $(\Omega^1_X)^{\otimes m}$.*

Les conditions sur les singularités dans le théorème ci-dessus sont optimales. Il existe des variétés rationnellement connexes X à singularités Kawamata log terminales telles que mK_X soit linéairement équivalent à un diviseur effectif pour un certain entier $m > 0$ ([Kol08, Example 43] ou [Tot12, Example 10]). De plus, il existe des variétés rationnellement connexes X à singularités canoniques non localement factorielles telles que $h^0(X, (\Omega^1_X)^{[\otimes 2]})$ soit non nulle ([GKP14, Example 3.7]).

Dans le Chapitre 6 et le Chapitre 7 de cette thèse, nous nous concentrons sur la classification des variétés complexes projectives rationnellement connexes peu singulières en petites dimensions qui portent des pluri-formes non nulles.

Chapitre 1

Notations et Préliminaires

Nous allons donner des notations et allons rappeler quelques résultats préliminaires. Une variété définie sur un corps k est un schéma géométriquement intègre, séparé et de type fini sur k. Sauf dans la section 4.4.3, nous travaillons sur $\mathbb{C}$, le corps des nombres complexes. Un morphisme $f : X \to Y$ entre deux variétés normales est appelé une fibration si f est propre, surjectif et à fibres connexes. Si $f : X \to Y$ est un morphisme propre surjectif et si la variété X est lisse, alors le discriminant de f est défini comme le sous-schéma réduit de Y au-dessus de quelle f n'est pas lisse.

1.1 Diviseurs sur les variétés normales

Soit X une variété projective normale de dimension n. Nous allons utiliser les notations suivantes :

$\mathrm{Cl}(X)$:= le groupe des classes d'équivalence linéaire de diviseurs de Weil sur X.

$\mathrm{Pic}(X)$:= le groupe de Picard de X, *i.e.* le groupe des classes d'équivalence linéaire de diviseurs de Cartier sur X.

$\mathrm{Z}_1(X)$:= le groupe abélien libre engendré par les courbes projectives dans X.

Un élément D dans $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ est appelé un $\mathbb{Q}$-diviseur (de Weil). Il est dit $\mathbb{Q}$-Cartier si un multiple non nul de D est un diviseur de Cartier. Deux $\mathbb{Q}$-diviseurs D et D' sont dits $\mathbb{Q}$-linéairement équivalents s'il existe un entier non nul r tel que rD et rD' soient deux diviseurs de Weil entiers linéairement équivalents. Si c'est le cas, on note $D \sim_{\mathbb{Q}} D'$. La variété X est dite localement factorielle si le morphisme naturel $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ est surjectif. Elle est dite $\mathbb{Q}$-factorielle si tout diviseur de Weil dans X est $\mathbb{Q}$-Cartier.

Si C est une courbe projective et D est un diviseur de Cartier dans X, alors nous définissons le nombre d'intersection de C et D par $C \cdot D = \deg f^*D$, où $f : \bar{C} \to C$ et la normalisation. Nous pouvons alors introduire

un accouplement

$$\cdot : \mathrm{Z}_1(X) \times \mathrm{Pic}(X) \to \mathbb{Z}.$$

Soit $\equiv$ l'équivalence numérique pour cet accouplement. Alors, en prenant le quotient par rapport aux équivalences numériques, nous obtenons une forme bilinéaire non-dégénérée

$$\cdot : \mathrm{N}_1(X) \times \mathrm{N}^1(X) \to \mathbb{R},$$

où $\mathrm{N}_1(X) = (\mathrm{Z}_1(X)/\equiv) \otimes \mathbb{R}$ et $\mathrm{N}^1(X) = (\mathrm{Pic}(X)/\equiv) \otimes \mathbb{R}$. Ces deux espaces vectoriels ont la même dimension $\rho(X)$ et nous l'appelons le nombre de Picard de X.

Dans le cas des surfaces, une courbe C incluse dans le lieu lisse d'une surface S est appelée une $(-k)$-courbe si elle isomorphe à $\mathbb{P}^1$ et $C \cdot C = -k$.

Soit $D = \sum_{i=1}^{r} a_i D_i$ un $\mathbb{Q}$-diviseur non nul, où les D_i sont des diviseurs premiers et les a_i sont des nombres rationnels non nuls. Alors D est dit effectif si $a_i \geqslant 0$ pour tout i. Il est dit $\mathbb{Q}$-effectif s'il est $\mathbb{Q}$-linéairement équivalent à un $\mathbb{Q}$-diviseur effectif. Le diviseur D_{red} est défini par

$$D_{\mathrm{red}} = \sum_{i=1}^{r} D_i.$$

On appelle $\min\{a_i\}$ le plus petit coefficient de D. Notons $[D] = \sum_{i=1}^{r} [a_i] D_i \in \mathrm{Cl}(X)$, où $[a_i]$ est la partie entière de a_i. Le diviseur D est dit à croisements normaux simples ou snc («simple normal crossing» en anglais) si tous les D_i sont lisses et toutes leurs intersections sont transverses.

Soit $\overline{\mathrm{NE}}(X)$ le cône fermé convexe dans $\mathrm{N}_1(X)$ engendré par les classes de courbes.

Définition 1.1.1. *Un élément D dans $\mathrm{N}^1(X)$ est dit nef si $\alpha \cdot D \geqslant 0$ pour toute classe $\alpha \in \overline{\mathrm{NE}}(X)$. Notons $\mathrm{Nef}(X)$ le cône dans $\mathrm{N}^1(X)$ des classes nefs.*

Soit $\mathrm{Amp}(X)$ le cône convexe dans $\mathrm{N}^1(X)$ engendré par les classes de diviseurs amples. Un $\mathbb{Q}$-diviseur $\mathbb{Q}$-Cartier D est dit $\mathbb{Q}$-ample (ou ample) si mD est un diviseur de Cartier ample pour un nombre entier $m > 0$. Par le théorème suivant, le cône $\mathrm{Nef}(X)$ est l'adhérence de $\mathrm{Amp}(X)$ et $\mathrm{Amp}(X)$ est l'intérieur de $\mathrm{Nef}(X)$.

Théorème 1.1.2 (Critère de Kleiman [Kle66, Prop. IV.2.2])**.** *Un diviseur de Cartier $D \in \mathrm{Pic}(X)$ est ample si et seulement si $\alpha \cdot D > 0$ pour toute classe $\alpha \in \overline{\mathrm{NE}}(X) \backslash \{0\}$.*

Soit $f : X \to Y$ un morphisme projectif entre deux variétés normales et soit D un $\mathbb{Q}$-diviseur $\mathbb{Q}$-Cartier sur X. Alors D est dit f-relativement ample ou f-ample si la restriction de D sur chaque composante irréductible

de chaque fibre de f est ample. De manière équivalente, D est f-ample si et seulement si $\alpha \cdot D > 0$ pour toute classe $\alpha \in \overline{\mathrm{NE}}(X/Y)\backslash\{0\}$, où $\overline{\mathrm{NE}}(X/Y)$ est le cône convexe fermé dans $N_1(X)$ engendré par les courbes contractées par f (voir par exemple [KM98, Thm. 1.44]).

Soit $\mathrm{Psef}(X) \subseteq \mathrm{N}^1(X)$ le cône des diviseurs pseudo-effectifs, *i.e.* le cône convexe fermé engendré par des diviseurs effectifs. Comme les diviseurs très amples sont effectifs, nous avons $\mathrm{Nef}(X) \subseteq \mathrm{Psef}(X)$.

Soit $f : Y \to X$ un morphisme surjectif entre deux variétés normales projectives. Soit D un élément dans $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Alors, nous définissons le tiré-en-arrière de D par f par

$$f^*D = \frac{1}{r} f^*(rD) \in \mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q},$$

où r est un entier non nul tel que $rD \in \mathrm{Pic}(X)$. De plus, si f est equidimensionnel et si D est un diviseur de Weil entier, alors f^*D coïncide avec l'adhérence de $(f|_{Y_0})^*(D|_{X_0})$ dans Y, qui est aussi un diviseur de Weil entier, où X_0 est le lieu lisse de X et $Y_0 = f^{-1}(X_0)$.

Soit $f : X \dashrightarrow Y$ une application birationnelle et soit $D \in \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. La transformée stricte de D par f, f_*D, est définie comme étant l'adhérence de $f|_U(U \cap D)$, où U est le plus grand ouvert de X tel que $f|_U$ soit un isomorphisme.

1.2 Faisceaux réflexifs

1.2.1 Propriétés de base

Soient X une variété normale et $\mathscr{F}$ un faisceau cohérent sur X. Le faisceau dual de $\mathscr{F}$ est défini par $\mathscr{F}^* = \mathscr{H}om(\mathscr{F}, \mathscr{O}_X)$. Le faisceau $\mathscr{F}$ est dit réflexif si le morphisme naturel $\mathscr{F} \to \mathscr{F}^{**}$ est un isomorphisme. En particulier, $\mathscr{F}^{**}$ est un faisceau réflexif et est appelé l'enveloppe réflexive de $\mathscr{F}$. Nous avons le critère de réflexivité suivant.

Proposition 1.2.1 ([Har80, Prop 1.6]). *Soit $\mathscr{F}$ un faisceau cohérent sur une variété normale X. Alors $\mathscr{F}$ est réflexif si et seulement s'il est sans torsion et pour tout ouvert U et tout fermé $Y \subseteq U$ tel que $\mathrm{codim}_U Y \geqslant 2$, le morphisme naturel $\mathscr{F}(U) \to \mathscr{F}(U\backslash Y)$ est un isomorphisme.*

Pour un faisceau cohérent $\mathscr{F}$ sur une variété normale X, notons $\mathscr{F}^{[\otimes m]} = (\mathscr{F}^{\otimes m})^{**}$ et $\mathscr{F}^{[\wedge m]} = (\mathscr{F}^{\wedge m})^{**}$. Si $\mathscr{G}$ est un autre faisceau cohérent sur X, alors nous notons $\mathscr{F}[\otimes]\mathscr{G} = (\mathscr{F} \otimes \mathscr{G})^{**}$. Soit D un diviseur de Weil sur X. Définissons le faisceau $\mathscr{O}_X(D)$ par, pour tout ouvert U de X,

$$\mathscr{O}_X(D)(U) = \{f \text{ fonction rationnelle de } X \mid \mathrm{div}(f) + D \text{ est effectif}\} \cup \{0\},$$

où $\mathrm{div}(f)$ est le diviseur associée à f. Remarquons que $\mathscr{O}_X(D)$ est réflexif et que si deux diviseurs D et D' sont linéairement équivalents, alors les

faisceaux $\mathscr{O}_X(D)$ et $\mathscr{O}_X(D')$ sont isomorphes (voir [Rei80, Appendix to §1]). Nous obtenons donc un morphisme de groupe

$$\mathrm{Cl}(X) \to \{\text{faisceaux réflexifs de rang 1}\}/\cong$$

.

Proposition 1.2.2 (voir [Rei80, Appendix to §1]). *Le morphisme défini ci-dessus est un isomorphisme.*

Soit Ω^1_X le faisceau des formes de Kähler sur X. Soit $\Omega^{[r]}_X$ l'enveloppe réflexive de $\Omega^r_X = (\Omega^1_X)^{\wedge r}$ pour tout $r \geqslant 0$. Si $\dim X = n$, alors on note $\omega_X = \Omega^{[n]}_X$ et on l'appelle le faisceau canonique de X. Soit K_X un diviseur de Weil tel que $\mathscr{O}_X(K_X) \cong \omega_X$. Le diviseur K_X est appelé un diviseur canonique de X. La variété X est dite Gorenstein si elle est Cohen-Macaulay et K_X est un diviseur de Cartier. Elle est dite $\mathbb{Q}$-Gorenstein si K_X est $\mathbb{Q}$-Cartier. Un élément σ dans $H^0(X, (\Omega^1_X)^{[\otimes m]})$ est appelé une m-pluri-forme (réflexive) sur X, où $m > 0$ est un entier.

1.2.2 Pente des faisceaux cohérents

Soit X une variété complexe projective normale $\mathbb{Q}$-factorielle. Soit $\alpha \in \mathrm{N}_1(X)$ une classe de courbe et soit $\mathscr{F}$ faisceau réflexif de rang un. Alors par Proposition 1.2.2, il existe un diviseur de Weil D tel que $\mathscr{O}_X(D) \cong \mathscr{F}$. On défini le nombre d'intersection $\alpha \cdot \mathscr{F}$ par $\alpha \cdot \mathscr{F} = \alpha \cdot D$.

Une classe $\alpha \in \mathrm{N}_1(X)$ est dite mobile si $\alpha \cdot D \geqslant 0$ pour tout diviseur effectif D ([BDPP13, Thm. 0.2]). Soit $\mathscr{F}$ un faisceau cohérent sur X et soit $\mathscr{H}$ un sous-faisceau cohérent. Alors, le plus petit sous-faisceau cohérent $\mathscr{H}^{sat}$ contenant $\mathscr{H}$ tel que le quotient $\mathscr{F}/\mathscr{H}^{sat}$ soit un faisceau sans torsion est appelé la saturation de $\mathscr{H}$. Le sous-faisceau $\mathscr{H}$ est dit saturé si $\mathscr{H} = \mathscr{H}^{sat}$.

Soit α une classe mobile. Alors pour tout faisceau cohérent $\mathscr{F}$ de rang strictement positif sur X, nous définissons la pente de $\mathscr{F}$ par rapport à α par

$$\mu_\alpha(\mathscr{F}) := \frac{\alpha \cdot \det \mathscr{F}}{\operatorname{rang} \mathscr{F}},$$

où $\det \mathscr{F} = (\mathscr{F}^{\wedge(\operatorname{rang}\mathscr{F})})^{**}$. De plus, soit

$$\mu^{max}_\alpha(\mathscr{F}) = \sup\{\mu_\alpha(\mathscr{G}) \mid \mathscr{G} \subseteq \mathscr{F} \text{ un faisceau coherent}\}.$$

Pour tout faisceau cohérent $\mathscr{F}$ de rang strictement positif, il existe un sous-faisceau cohérent saturé $\mathscr{H} \subseteq \mathscr{F}$ tel que $\mu^{max}_\alpha(\mathscr{F}) = \mu_\alpha(\mathscr{H})$ ([GKP14, App. A]).

Proposition 1.2.3 ([GKP14, App. A]). *Soit X une variété projective normale complexe $\mathbb{Q}$-factorielle. Soit α une classe mobile. Alors pour tous faisceaux cohérents $\mathscr{F}$ et $\mathscr{G}$ de rang strictement positif, on a*

$$\mu_\alpha((\mathscr{F} \otimes \mathscr{G})^{**}) = \mu_\alpha(\mathscr{F}) + \mu_\alpha(\mathscr{G})$$

et

$$\mu_\alpha^{max}((\mathscr{F}\otimes\mathscr{G})^{**}) = \mu_\alpha^{max}(\mathscr{F}) + \mu_\alpha^{max}(\mathscr{G}).$$

Si A est un diviseur ample dans X, alors A^{n-1} est une classe mobile, où n est la dimension de X. Pour simplicité, on note $\mu_A = \mu_{A^{n-1}}$ et $\mu_A^{max} = \mu_{A^{n-1}}^{max}$.

1.3 Singularités terminales, canoniques et Kawamata log terminales

1.3.1 Définition des singularités

Rappelons quelques définitions sur les singularités du MMP. Les notions sont expliquées en détail dans le livre [KM98]. Une paire (X, Δ) consiste en une variété normale complexe X et un $\mathbb{Q}$-diviseur effectif Δ. La paire (X, Δ) est dite à croisements normaux simples (ou snc en abrégé) si X est lisse et le support de Δ est un diviseur à croisements normaux simples.

Par le théorème de Hironaka, il existe un morphisme birationnel $f : Y \to X$ tel que, E, le lieu exceptionnel de f, soit de codimension pure 1 et que la paire $(Y, E + f_*^{-1}\Delta)$ soit snc. Un tel morphisme est appelé une résolution des singularités pour la paire (X, Δ). Soient $E_1, ..., E_k$ les composantes irréductibles de E. Si nous supposons que le $\mathbb{Q}$-diviseur $K_X + \Delta$ est $\mathbb{Q}$-Cartier, alors il existe des nombres rationnels $a(f, X, \Delta, E_1), ..., a(f, X, \Delta, E_k)$ tels que

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum_{i=1}^{k} a(f, X, \Delta, E_i)E_i.$$

Remarquons que si $f' : Y' \to X$ est une autre résolution des singularités pour la paire (X, Δ) qui se factorise à travers $Y' \to Y$, et si E_i' est la transformée stricte de E_i dans X', alors pour tout $i \in \{1, ..., k\}$, nous avons $a(f, X, \Delta, E_i) = a(f', X, \Delta, E_i')$. Par conséquent, $a(f, X, \Delta, E_i) = a(X, \Delta, E_i)$ est indépendant du choix de résolution des singularités (voir par exemple [KM98, Remark 2.23]). Ce nombre est appelé la discrépance de E_i (par rapport à (X, Δ)). La discrépance de (X, Δ), $\mathrm{discrep}(X, \Delta)$, est définie par

$$\inf\{a(X, \Delta, E) \mid E \text{ un diviseur exceptionnel d'une résolution } f : Y \to X\}$$

Définition 1.3.1. *Soit (X, Δ) une paire telle que $K_X + \Delta$ soit $\mathbb{Q}$-Cartier. Alors la paire est dite*

- *terminale si* $\mathrm{discrep}(X, \Delta) > 0$ *;*
- *canonique si* $\mathrm{discrep}(X, \Delta) \geqslant 0$ *;*
- *Kawamata log terminale (ou klt en abrégé) si* $\mathrm{discrep}(X, \Delta) > -1$ *et* $\lfloor \Delta \rfloor \leqslant 0$.

Pour qu'une paire (X, Δ) soit terminale, (resp. canonique, klt), il suffit qu'il existe une résolution $f : Y \to X$ telle que si $K_Y \equiv f^*(K_X + \Delta) + \sum_{i=1}^{k} a(X, \Delta, E_i)E_i$ alors $a(X, \Delta, E_i) > 0$ (resp. $a(X, \Delta, E_i) \geqslant 0$, $a(X, \Delta, E_i) > -1$ et $\lfloor \Delta \rfloor \leqslant 0$) pour tout $i \in \{1, ..., k\}$ ([KMM87, Lem. 0-2-12]).

Une variété normale complexe X est dite à singularités terminales (resp. canoniques, klt) si la paire $(X, 0)$ est une paire terminale (resp. canonique, klt).

Exemple 1.3.2. Soit X une variété complexe normale de dimension n et soit x un point dans X. La variété X est dite à *singularités quotient* au voisinage de x s'il existe un voisinage analytique U de x tel que U est analytiquement isomorphe à V/G, où V est une variété lisse et G est un groupe fini agissant sur V. Les singularités quotients sont klt. Si X est une surface, alors elle est à singularités klt si et seulement si elle est à singularités quotients [KM98, Prop. 4.18].

1.3.2 Singularités de surfaces

Considérons le cas où X est une surface normale. Il existe une résolution des singularités minimale $r : \widetilde{X} \to X$ telle que $K_{\widetilde{X}} \cdot C \geqslant 0$ pour toute courbe C contractée par r. Si $g : Y \to X$ est une résolution des singularités, alors il existe un morphisme $h : Y \to \widetilde{X}$ tel que $g = r \circ h$ ([Kol07, Thm. 2.16]). Si X est à singularités klt, alors elle est à singularités quotients. En particulier, elle est à singularités $\mathbb{Q}$-factorielles rationnelles ([KM98, §4.2]).

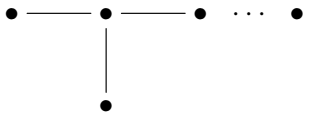
Rappelons la notion de graphe dual ([KM98, Def. 4.6]). Soit $C = \bigcup C_k$ une collection de courbes projectives dans une surface lisse S. Le graphe dual Γ de C est défini comme suit :

1. Les sommets de Γ représentent les courbes C_k.
2. Deux sommets $C_k \neq C_l$ sont connectés par $C_k \cdot C_l$ arêtes.

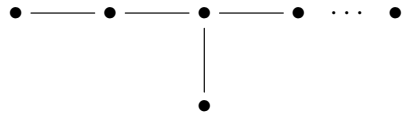
Soit (X, ξ) un germe de surface à singularités canoniques telle que ξ soit l'unique point singulier de X. Les singularités canoniques de surfaces sont aussi appelées les singularités Du Val (voir par exemple [KM98, §4]). Les singularités Du Val sont classifiées. Ce sont les singularités de type A_i, D_i et E_i (voir ci-dessous). Ainsi, les singularités canoniques de surfaces sont aussi appelées singularités ADE. Soit $r : \widetilde{X} \to X$ la résolution minimale de X et soit $C = \bigcup C_k$ le lieu exceptionnel de r. Alors C_k est une (-2)-courbe pour tout k. Nous avons la classification suivante :

1. (X, ξ) est de type A_i, où $i \in \mathbb{N}^*$. Analytiquement, elle est isomorphe à la singularité définie par l'équation $x^2+y^2+z^{i+1} = 0$. Le lieu exceptionnel de r possède i composantes, et son graphe dual est une chaîne comme ci-dessous

2. (X, ξ) est de type D_i, où $i \geqslant 4$. Analytiquement, elle est isomorphe à la singularité définie par l'équation $x^2 + y^2 z + z^{i-1} = 0$. Le lieu exceptionnel de r possède i composantes, et son graphe dual est comme ci-dessous



3. (X, ξ) est de type E_i, où $i \in \{6, 7, 8\}$. Le lieu exceptionnel de r possède i composantes, et son graphe dual est comme ci-dessous



Si $i = 6$, alors (X, ξ) est analytiquement isomorphe à la singularité définie par l'équation $x^2+y^3+z^4 = 0$. Si $i = 7$, alors elle est analytiquement isomorphe à la singularité définie par l'équation $x^2 + y^3 + yz^3 = 0$. Si $i = 8$, alors elle est analytiquement isomorphe à la singularité définie par l'équation $x^2 + y^3 + z^5 = 0$.

En particulière, les singularités ADE sont des singularités d'hypersurfaces donc sont Gorenstein.

1.4 Variétés rationnellement connexes, variétés de Fano

Soit X une variété complexe projective. Alors elle est dite uniréglée si par un point général x, il existe une courbe rationnelle passant par x. Elle est dite rationnellement connexe si pour deux points généraux x, x' de X, il existe une courbe rationnelle passant par eux. (voir par exemple [Kol96, §IV])

Une variété normale projective complexe X est appelée une variété de Fano si $-K_X$ est un diviseur $\mathbb{Q}$-Cartier et $\mathbb{Q}$-ample. S'il existe un $\mathbb{Q}$-diviseur effectif Δ dans X tel que la paire (X, Δ) soit klt et que $-(K_X + \Delta)$ soit $\mathbb{Q}$-ample, alors la paire (X, Δ) est dite log Fano. Une fibration $f : X \to Y$ entre deux variétés normales est dite de Fano si $-K_X$ est de $\mathbb{Q}$-Cartier et f-ample.

Une courbe projective complexe lisse est une variété de Fano si et seulement si elle est une courbe rationnelle. Si une surface projective complexe lisse est une variété de Fano, alors elle est aussi une surface rationnelle, *i.e.* elle est birationnelle à $\mathbb{P}^2$. En dimension supérieure, ce n'est plus vrai : il existe des variétés de Fano projectives complexes lisses de dimension $n \geqslant 3$ qui ne sont pas birationnelles à $\mathbb{P}^n$. Cependant, les variétés de Fano lisses sont toujours rationnellement connexes ([Cam92], [KMM92a]). Le théorème

suivant affirme que les variétés de Fano à singularités klt sont aussi rationnellement connexes ([Zha06, Thm. 1], [HM07, Cor. 1.13]).

Théorème 1.4.1. *Soit (X, Δ) une paire log Fano. Alors X est rationnellement connexe.*

1.5 Programme des Modèles Minimaux

La motivation du programme des modèles minimaux, ou MMP («Minimal Model Program» en anglais), est de trouver, pour une variété X, un modèle birationnel X^* tel que ou bien K_{X^*} soit nef, ou bien il existe une «fibration de Mori» $f : X \to Z$ (voir Proposition-Définition 1.5.3 ci-dessous) vers une variété de dimension strictement plus petite. Un des théorèmes les plus importants est le théorème du cône. Avant de présenter ce théorème, rappelons quelques notions de base de la géométrie des cônes.

Soit V un espace vectoriel réel de dimension finie. Un ensemble $\mathscr{C} \subseteq V$ est appelé un cône si $\mathscr{C}$ est stable pour la multiplication par tout réel strictement positif. Le cône est dit convexe si pour tous $v, v' \in \mathscr{C}$, la somme $v + v'$ est aussi dans $\mathscr{C}$. Un sous-ensemble $\mathscr{F}$ de $\mathscr{C}$ est appelé une face de $\mathscr{C}$ s'il est convexe vérifiant la propriété suivante : si un segment ouvert $]v, v'[$ tracé dans $\mathscr{C}$ rencontre $\mathscr{F}$, alors le segment fermé $[v, v']$ est inclus dans $\mathscr{F}$. Un rayon extrêmal de $\mathscr{C}$ est une face de dimension 1. Si $(\mathscr{C}_i)_{i\in I}$ est un ensemble au plus dénombrable de cônes dans V, alors la somme $\sum_{i\in I} \mathscr{C}_i$ est définie comme le plus petit cône convexe fermé contenant tous les $\mathscr{C}_i$.

Soit $\mathscr{C} \subseteq V$ un cône convexe fermé et soit $f \in \mathrm{Hom}(V, \mathbb{R})$ une forme linéaire. Alors $\mathscr{C}_{f\geqslant 0}$ est le cône défini par

$$\{v \in \mathscr{C} \mid f(v) \geqslant 0\}.$$

Une face $\mathscr{F}$ est dite strictement f-négative si $f(v) < 0$ pour tout $v \in \mathscr{F}\backslash\{0\}$. Dans le cas où $V = \mathrm{N}_1(X)$ pour une variété normale X et f est induite par l'intersection avec une classe $D \in \mathrm{N}^1(X)$, nous notons $\mathscr{C}_{D\geqslant 0} = \mathscr{C}_{f\geqslant 0}$. Une face $\mathscr{F}$ est dite strictement D-négative si elle est strictement f-négative.

Le théorème du cône est comme suit (voir par exemple [KMM87, Thm. 3-2-1 et Thm. 4-2-1]).

Théorème 1.5.1 (Théorème du Cône)**.** *Soit (X, Δ) une paire klt telle que X soit projective $\mathbb{Q}$-factorielle. Alors*

1. *Il existe un ensemble au plus dénombrable de courbes rationnelles $C_j \subseteq X$, avec $j \in J$, telles que $C_j \cdot (K_X + \Delta) < 0$ pour tout j et*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta)\geqslant 0} + \sum_{j\in J} \mathbb{R}^+[C_j].$$

2. *Pour tout $\epsilon > 0$ et tout diviseur ample H, il existe un sous-ensemble fini I de J tel que*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta+\epsilon H)\geqslant 0} + \sum_{i\in I} \mathbb{R}^+[C_i].$$

3. *Soit $F \subseteq \overline{\mathrm{NE}}(X)$ une face strictement $(K_X + \Delta)$-négative. Alors il existe une unique fibration projective $c_F : X \to Z$ telle qu'une courbe $C \subseteq X$ soit contractée par c_F si et seulement si $[C] \subseteq F$. Le morphisme c_F est appelé la contraction de F.*
4. *Soit c_F la contraction d'une face strictement $(K_X + \Delta)$-négative $F \subseteq \overline{\mathrm{NE}}(X)$. Soit L un diviseur de Cartier dans X tel que $C \cdot L = 0$ pour toute courbe C dont la classe est dans F. Alors il existe un diviseur de Cartier L_Z dans Z tel que $\mathscr{O}_X(L) \cong \mathscr{O}_X(c_Z^* L_Z)$.*

Remarque 1.5.2. Soit (X, Δ) une paire klt log Fano. Alors par le théorème ci-dessus, le cône $\overline{\mathrm{NE}}(X)$ est polyédral.

Soit (X, Δ) une paire klt telle que X soit projective $\mathbb{Q}$-factorielle. Supposons qu'il existe un rayon extrêmal $R \subseteq \overline{\mathrm{NE}}(X)$ qui est strictement $(K_X+\Delta)$-négatif. Alors c_R est appelée une contraction extrêmale. Soit $Ex(c_R)$ son lieu exceptionnel. Alors nous avons la proposition suivante ([KM98, Prop. 2.5]).

Proposition-Définition 1.5.3. *Soit Z une composante irréductible du lieu exceptionnel $Ex(c_R)$. Si Z est de codimension 1, alors $Z = Ex(c_R)$. Suivant la codimension de $Ex(c_R)$, il y a trois possibilités :*

1. *Si $Ex(c_R)$ contient un diviseur, alors c_R est dite divisorielle.*
2. *Si $Ex(c_R)$ est de codimension au moins 2, alors c_R est appelée une contraction petite.*
3. *Si $Ex(c_R) = X$, alors c_R est dite de type fibré et appelée une fibration de Mori. Dans ce cas, X est appelé un espace fibré de Mori.*

Si $c_R : X \to Z$ est de type fibré ou divisorielle, alors Z est aussi $\mathbb{Q}$-factorielle et $\rho(Z) = \rho(X) - 1$ (voir par exemple [KM98, Prop. 3.36]). Si c_R est une contraction petite, alors Z n'est jamais $\mathbb{Q}$-factoriel. Pour le dernier cas, introduisons la notion de flip ([KM98, Def. 3.33]).

Définition 1.5.4. *Soit (X, Δ) une paire telle que $K_X + \Delta$ soit $\mathbb{Q}$-Cartier. Soit $f : X \to Z$ un morphisme birationnel tel que $-(K_X + \Delta)$ soit f-ample et que $Ex(f)$ soit de codimension au moins 2. Une variété X^+ avec une fibration birationnelle $f^+ : X^+ \to Z$ est appelée un $(K_X + \Delta)$-flip de f si*

1. *$K_{X^+} + \Delta^+$ est $\mathbb{Q}$-Cartier, où Δ^+ est la transformée stricte de Δ dans X^+,*
2. *$K_{X^+} + \Delta^+$ est f^+-ample,*

3. $Ex(f^+)$ est de codimension au moins 2.

Par abus de language, nous appelons aussi l'application birationnelle $\varphi : X \dashrightarrow X^+$ un $(K_X + \Delta)$-flip.

Si $f : X \dashrightarrow X'$ est une contraction extrêmale divisorielle ou un flip associée à une contraction extrêmale petite, alors X' est au plus aussi singulière que X. Plus précicément :

Proposition 1.5.5 ([KM98, Cor. 3.42]). *Soit (X, Δ) une paire klt (resp. canonique, terminale) telle que X soit $\mathbb{Q}$-factorielle. Soit $f : X \to Z$ une contraction extrêmale petite. Si X^+ est un $(K_X + \Delta)$-flip de f, alors la paire (X^+, Δ^+) est klt (resp. canonique, terminale), où Δ^+ est la transformée stricte de Δ dans X^+.*

Proposition 1.5.6 ([KM98, Cor. 3.43]). *Soit (X, Δ) une paire klt telle que X soit $\mathbb{Q}$-factorielle. Soit $f : X \to X'$ une contraction extrêmale divisorielle et soit E le lieu exceptionnel. Alors X' est $\mathbb{Q}$-factorielle et (X', Δ') est aussi une paire klt, où Δ' est la transformée stricte de Δ dans X'. De plus, si (X, Δ) est terminale (resp. canonique) et si $E \not\subset \Delta$, alors (X', Δ') est terminale (resp. canonique).*

Dans le cas d'une fibration de Mori, nous avons la proposition suivante.

Proposition 1.5.7 ([KMM87, Lem. 5-1-5] et [Fuj99, Thm. 0.2]). *Soit (X, Δ) une paire klt telle que X soit $\mathbb{Q}$-factorielle. Si $f : X \to Z$ est une fibration de Mori associée à une face strictement $(K_X + \Delta)$-négative, alors Z est $\mathbb{Q}$-factorielle klt.*

Soit (X, Δ) une paire klt telle que X soit projective $\mathbb{Q}$-factorielle. Le MMP pour la paire (X, Δ) est une suite de transformations birationnelles que nous allons obtenir par récurrence,

$$(X, \Delta) = (X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots (X_i, \Delta_i) \dashrightarrow \cdots$$

Supposons qu'une paire klt (X_i, Δ_i) est construite pour certain entier naturel i et que X_i est projective $\mathbb{Q}$-factorielle. Si $K_{X_i} + \Delta_i$ est nef, alors (X_i, Δ_i) est un *modèle minimal* et le programme est terminé. Nous prenons $(X^*, \Delta^*) = (X_i, \Delta_i)$ comme le résultat du programme.

Si $K_{X_i} + \Delta_i$ n'est pas nef, alors il existe un rayon extrêmal strictement $(K_{X_i} + \Delta_i)$-négatif $R \subseteq \overline{\mathrm{NE}}(X_i)$. Soit $c_R : X_i \to Z$ la contraction de R. Si c_R est une fibration de Mori, alors le programme est terminé et nous prenons $(X^*, \Delta^*) = (X_i, \Delta_i)$ comme le résultat du programme. Si c_R est une contraction de type divisoriel, alors nous posons $(X_{i+1}, \Delta_{i+1}) = (Z, (c_R)_* \Delta_i)$. Si c_R est une contraction petite, alors il existe un flip $(X_i, \Delta_i) \dashrightarrow (X_{i+1}, \Delta_{i+1})$ par [BCHM10, Cor. 1.4.1]. Dans les deux derniers cas, (X_{i+1}, Δ_{i+1}) est une paire klt et X_{i+1} est projective $\mathbb{Q}$-factorielle.

On conjecture que la suite de transformations birationnelles ci-dessus est toujours finie et que nous pouvons obtenir un modèle (X^*, Δ^*) tel que ou bien $K_{X^*} + \Delta^*$ soit nef, ou bien il existe une fibration de Mori $f : X \to Z$. Cette conjecture de terminaison de MMP est équivalente à la non-existence de suite infinie de flips. Dans le cas $\dim X = 2$, la conjecture est vraie. En effet, si $\dim X = 2$, alors il n'y a que des contractions divisorielles dans la suite du programme. Dans le cas où $\dim X = 3, 4$, la conjecture est aussi vraie ([Sho86, Cor. 2.17], [KMM87, Thm. 5-1-15]). En plus grandes dimensions, elle reste encore ouverte.

Chapitre 2

Résumé des résultats principaux

2.1 Fibrations lagrangiennes sur les variétés symplectiques de dimension quatre

Dans le Chapitre 4, nous étudions la base d'une fibration lagrangienne d'une variété symplectique de dimension 4. Rappelons d'abord quelques définitions sur les variétés symplectiques.

Définition 2.1.1. *Soit M une variété projective lisse de dimension $2n$ définie sur un corps k. Une 2-forme fermée $\Omega \in H^0(M, \Omega_M^2)$ est dite symplectique si elle est partout non-dégénérée. La variété M est dite symplectique irréductible si elle est simplement connexe et qu'il existe une forme symplectique Ω telle que $H^0(M, \Omega_M^2) \cong k\Omega$.*

Les variétés symplectiques irréductibles sont l'analogue des surfaces K3 en grandes dimensions. Une variété complexe symplectique irréductible est aussi appelée une variété hyperkählerienne. On remarque que, si M est une variété symplectique irréductible, alors $\Omega^{\wedge n}$ est un élément dans $H^0(M, \Omega_M^{2n})$ qui est partout non nul. Par conséquent, $\Omega_M^{2n} \cong \mathscr{O}_M$. Beauville démontre le théorème suivant.

Théorème 2.1.2 ([Bea83, Thm. 1])**.** *Soit X une variété projective complexe lisse telle que $K_X \equiv 0$. Alors il existe un revêtement fini étale $X' \to X$ tel que*

$$X' \cong T \times \prod_i V_i \times \prod_j W_j,$$

où T est un tore, V_i est une variété de Calabi-Yau simplement connexe et W_j est une variété symplectique irréductible.

Exemple 2.1.3. Nous rassemblons deux exemples classiques des variétés symplectiques irréductibles. Ce sont les variétés symplectiques irréductibles de type $K3^{[n]}$ et de type Kummer généralisés ([Bea83, §6 et §7]).

- *Type* $K3^{[n]}$. Une surface projective complexe est symplectique irréductible si et seulement si elle est une surface $K3$. Soit S une surface $K3$. Alors le schéma d'Hilbert des sous-schémas de S de dimension 0 et de longueur n, $\mathrm{Hilb}^n(S)$, est une variété symplectique irréductible de dimension $2n$.

- *Type Kummer généralisés.* Soient A un tore complexe de dimension 2 et $0 \in A$ le point zero. Soit $\mathrm{Hilb}^{n+1}(A)$ le schéma d'Hilbert des sous-schémas de A de dimension 0 et de longueur $n+1$ et soit $S^{n+1}(A)$ le produit symétrique de $n+1$ copies de A. Alors, il existe des morphismes naturels,

$$\mathrm{Hilb}^{n+1}(A) \longrightarrow S^{n+1}(A) \xrightarrow{+} A,$$

où $+$ désigne la sommation de A. Soit $K^{n+1}(A) \subseteq \mathrm{Hilb}^{n+1}(A)$ la fibre au-dessus de $0 \in A$ de la composition des morphismes ci-dessus. Alors K^{n+1} est une variété symplectique irréductible de dimension $2n$.

Soit M une variété projective symplectique irréductible définie sur un corps k de dimension $2n$. Soit Ω une forme symplectique. Une sous-variété N est dite lagrangienne si elle est de dimension n et si la restriction de Ω sur la partie lisse de N est une 2-forme identiquement nulle. Une fibration $f : M \to X$ est dite lagrangienne si toute composante irreductible de toute fibre de f est une sous-variété lagrangienne.

Les fibrations lagrangiennes sont un objet important pour la géométrie des variétés symplectiques irréductibles. Il est conjecturé par plusieurs mathématiciens (Beauville, Bogomolov, Hassett-Tschinkel, Huybrechts, Sawon, Tyurin, ...) que toute variété symplectique irréductible peut être déformée en une variété symplectique irréductible qui admet une fibration lagrangienne (voir par exemple [Bea11, §1.6]). Ici nous supposons l'existence d'une fibration lagrangienne $f : M \to X$ et nous nous intéressons à X. Matsushita démontre le résultat suivant ([Mat99, Thm. 2]).

Théorème 2.1.4. *Soit $f : M \to X$ une fibration d'une variété complexe projective symplectique irréductible lisse M de dimension $2n$ dans une variété projective normale X. Supposons que $0 < \dim X < 2n$. Alors nous avons les propriétés suivantes :*

1. *f est une fibration lagrangienne ;*
2. *X est une variété de Fano de dimension n $\mathbb{Q}$-factorielle à singularités klt telle que $\rho(X) = 1$.*

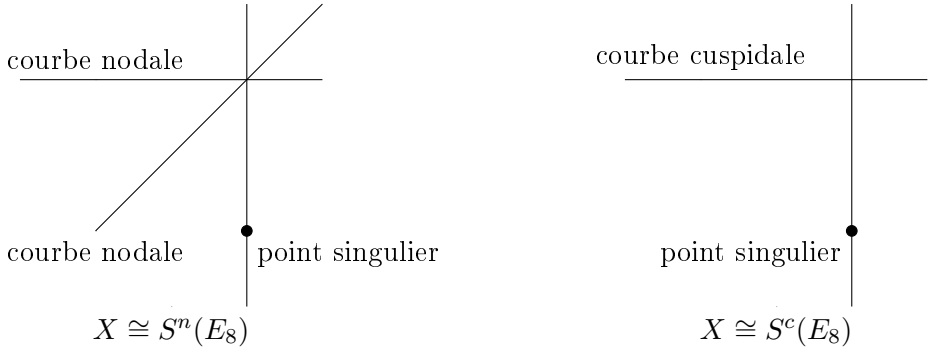
Dans les exemples connus de fibrations lagrangiennes, la base est toujours isomorphe à un espace projectif. En effet, si nous supposons que la base est lisse, alors Hwang prouve le théorème suivant.

Théorème 2.1.5 ([Hwa08, Thm. 1.2]). *Soit f une fibration lagrangienne d'une variété complexe projective symplectique irréductible lisse M de dimension $2n$ dans une variété projective lisse X. Alors $X \cong \mathbb{P}^n$.*

Si la variété symplectique irréductible se déforme en une variété symplectique irréductible du type $K3^{[n]}$ ou du type Kummer généralisés, alors la base est toujours isomorphe à $\mathbb{P}^n$ ([Mat13, Cor. 1.1], [Mar14, Thm. 1.3], [BM14, Thm. 1.5] et [Yos12, Prop. 3.38]). Cependant, il existe d'autres types de variétés symplectiques irréductibles. O'Grady construit deux nouveaux exemples de variétés symplectiques irréductibles dans [O'G99] et [O'G03]. Dans le Chapitre 4, nous démontrons le théorème suivant.

Théorème 2.1.6. *Soit f une fibration d'une variété complexe projective symplectique irréductible lisse M de dimension 4 sur une surface projective normale X. Alors, nous avons deux possibilités. Soit $X \cong \mathbb{P}^2$, soit $X \cong S^n(E_8)$.*

La surface $S^n(E_8)$ est l'une des deux surfaces de Fano qui contiennent un unique point singulier qui est canonique du type E_8. La construction de ces deux surfaces est comme suit. Soit C une courbe rationnelle cubique singulière dans $X_1 = \mathbb{P}^2$. Soit x l'un des points d'inflection lisses de C et soit L la droite tangente à C en x. En éclatant le point x, nous obtenons une surface X_2. Nous construisons une surface rationnelle X_9 par récurrence de la manière suivante. Supposons que X_i est construite pour un $i \geqslant 2$. Soit $X_{i+1} \to X_i$ l'éclatement du point d'intersection de la transformée stricte de C et le diviseur exceptionnel de $X_i \to X_{i-1}$. Alors X_9 contient exactement huit (-2)-courbes : la transformée stricte de L et les transformées strictes des diviseurs exceptionnels de $X_8 \to X_1$. Soit $X_9 \to X$ le morphisme birationnnel qui contracte exactement ces courbes. Alors X est une surface de Fano qui contient un unique point singulier qui est canonique du type E_8. Si C est une courbe cuspidale, alors le system linéaire $|-K_X|$ contient une courbe rationnelle cuspidale dans le lieu lisse de X. On note $S^c(E_8)$ cette surface. Si C est une courbe nodale, alors le système linéaire $|-K_X|$ contient deux courbes rationnelles nodales dans le lieu lisse de X. On note $S^n(E_8)$ cette surface.



courbes rationnelles dans $|-K_X|$

La preuve du Théorème 2.1.6 consiste en plusieures étapes. Nous allons d'abord démontrer le théorème suivant, qui est une généralisation d'un résultat de Kollár ([Kol86b, Cor. 3.9]).

Théorème 2.1.7. *Soit $g : V \to W$ une fibration projective équidimensionnelle entre variétés complexes normales quasi-projectives. Supposons que V est lisse, que $\omega_V \cong \mathscr{O}_V$, et que W est Cohen-Macaulay $\mathbb{Q}$-Gorenstein. Soit L un diviseur de Weil $\mathbb{Q}$-Cartier dans W, et soit $E = g^*L$. Alors $R^i g_* \mathscr{O}_V(E)$ est réflexif pour tout $i \geqslant 0$.*

En particulier, si $f : M \to X$ est une fibration lagrangienne, alors le Théorème 2.1.7 implique que $R^i f_* \mathscr{O}_M$ est réflexif. Par la méthode de Matsushita ([Mat05]), nous obtenons $R^i f_* \mathscr{O}_M \cong \Omega_X^{[i]}$. Ensuite, nous démontrons le théorème d'annulation suivant.

Theorem 2.1.8. *Soit $f : M \to X$ une fibration lagrangienne d'une variété complexe projective symplectique irréducible lisse M sur une variété projective normale X. Soit H un diviseur de Weil $\mathbb{Q}$-ample dans X et soit $D = f^*H$. Alors, pour tout $j > 0$ et pour tout $i \geqslant 0$, nous avons*

$$h^j(X, R^i f_*(\mathscr{O}_M(D))) = 0 \text{ et } h^i(M, \mathscr{O}_M(D)) = h^0(X, \Omega_X^{[i]}[\otimes]\mathscr{O}_X(H)).$$

Dans le cas où $\dim M = 4$, nous en déduisons

$$h^0(X, \mathscr{O}_X(H)) - h^0(X, \Omega_X^{[1]}[\otimes]\mathscr{O}_X(H)) + h^0(X, \mathscr{O}_X(H + K_X)) = 3$$

pour tout diviseur de Weil $\mathbb{Q}$-ample H dans X. Enfin, nous montrons le théorème de classification suivant qui implique le Théorème 2.1.7.

Théorème 2.1.9. *Soit X une surface de Fano complexe à singularités klt dont le lieu lisse est algébriquement simplement connexe. Supposons que $\rho(X) = 1$ et*

$$h^0(X, \mathscr{O}_X(H)) - h^0(X, \Omega_X^{[1]}[\otimes]\mathscr{O}_X(H)) + h^0(X, \mathscr{O}_X(H + K_X)) = 3$$

pour tout diviseur de Weil $\mathbb{Q}$-ample H dans X. Alors soit $X \cong \mathbb{P}^2$, soit $X \cong S^n(E_8)$.

Rappelons qu'une variété X est dite algébriquement simplement connexe si tout revêtement étale fini $X' \to X$, avec X' irréductible, est un isomorphism.

2.2 Variétés de Fano dont tout diviseur effectif est nef

Soit X une variété projective normale. Alors nous avons l'inclusion

$$\mathrm{Nef}(X) \subseteq \mathrm{Psef}(X).$$

Il est naturel de regarder des variétés X telles que $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Les exemples les plus simples sont les variétés de nombre Picard 1 et les espaces homogènes. Si X est une variété torique lisse, alors $\mathrm{Nef}(X) = \mathrm{Psef}(X)$ si et seulement si X est un produit d'espaces projectifs ([FS09, Prop. 5.3]). Le cas où X est une variété horosphérique est traité dans [Li13].

Exemple 2.2.1 ([Ful11, Prop. 1.5]). Soit C une coubre projective lisse et soit $\mathscr{E}$ un fibré vectoriel sur C. Soit $X = \mathbb{P}(\mathscr{E})$ le fibré projectif associé à $\mathscr{E}$. Alors $\mathrm{Nef}(X) = \mathrm{Psef}(X)$ si et seulement si $\mathscr{E}$ est semistable.

Dans le Chapitre 5, nous nous intéressons aux variétés de Fano X à singularités canoniques telles que $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Une telle variété est un objet minimal dans sa classe d'équivalence birationnelle : si $f : X \to Y$ est un morphisme birationnel, alors f est un isomorphisme ([Dru14, Lem. 4.4]). De plus, le nombre de Picard de X est toujours au plus égal à la dimension de X. Si $\rho(X)$ est maximal, alors Druel montre le théorème suivant ([Dru14, Thm. 1.1] et [Dru14, Prop. 10.4]).

Théorème 2.2.2. *Soit X une variété de Fano localement factorielle à singularités canoniques. Supposons que $\rho(X) = \dim X$ et que $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Alors $X \cong X_1 \times \cdots \times X_k$, où pour tout $i = 1, ..., k$, ou bien $X_i \cong \mathbb{P}^1$, ou bien $\dim X_i \geqslant 3$ et X_i est un revêtement double de $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, ramifié le long d'un diviseur de degré $(2, ..., 2)$.*

Nous étudions le cas où le nombre de Picard de X est égal à la dimension moins 1. Plus précicément, nous classifions les variétés de Fano X telles que

1. X est localement factorielle, à singularités canoniques ;
2. X est lisse en codimension 2 ;
3. $\mathrm{Nef}(X) = \mathrm{Psef}(X)$.

Dans le cas des surfaces, il n'est pas difficile de voir que $X \cong \mathbb{P}^2$. Dans le cas de dimension 3, nous avons le résultat suivant.

Théorème 2.2.3. *Soit X une variété de Fano de dimension trois localement factorielle à singularités canoniques isolées. Supposons que* $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. *Alors*

1. $X \cong \mathbb{P}^1 \times \mathbb{P}^2$.
2. *X est un revêtement double de $\mathbb{P}^1 \times \mathbb{P}^2$, ramifié le long d'un diviseur premier de degré $(2,2)$.*
3. *X est un revêtement double de $\mathbb{P}^1 \times \mathbb{P}^2$, ramifié le long d'un diviseur premier de degré $(2,4)$.*
4. *X est une hypersurface dans $\mathbb{P}^2 \times \mathbb{P}^2$ de degré $(1,1)$.*
5. *X est une hypersurface dans $\mathbb{P}^2 \times \mathbb{P}^2$ de degré $(1,2)$.*
6. *X est une hypersurface dans $\mathbb{P}^2 \times \mathbb{P}^2$ de degré $(2,2)$.*
7. *X est un revêtement double d'une hypersurface lisse Y dans $\mathbb{P}^2 \times \mathbb{P}^2$ de degré $(1,1)$, ramifié le long d'un diviseur premier D, où D est l'intersection de Y et une hypersurface de degré $(2,2)$ dans $\mathbb{P}^2 \times \mathbb{P}^2$.*

Plus généralement, nous démontrons le théorème suivant.

Théorème 2.2.4. *Soit X une variété de Fano de dimension $n \geqslant 3$ localement factorielle à singularités canoniques. Supposons que X est lisse en codimension 2, que $\rho(X) = \dim X - 1$ et que* $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. *Alors $X \cong X_1 \times X_2$, où X_1 est l'une des variétés du Théorème 2.2.2 et X_2 est l'une des variétés dans le Théorème 2.2.3 et l'Exemple 2.2.5.*

Exemple 2.2.5. Dans la liste suivante, nous donnons quelques exemples des variétés de Fano X.

1. X est un revêtement double de $(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$, ramifié le long d'un diviseur premier de degré $(2,...,2,2)$ ou $(2,...,2,4)$.
2. Soit $Z = (\mathbb{P}^1)^s \times \mathbb{P}^2 \times (\mathbb{P}^1)^r$ tel que $r, s > 0$ et $r + s = n - 2$. Soit Y_1 un revêtement double de $(\mathbb{P}^1)^s \times \mathbb{P}^2$, ramifié le long d'un diviseur premier de degré $(2,...,2,2)$. Soit $Y = Y_1 \times (\mathbb{P}^1)^r$ et soit $g : Y \to \mathbb{P}^2 \times (\mathbb{P}^1)^r$ la projection naturelle. Soit X un revêtement double de Y, ramifié le long d'un diviseur premier D, qui est le tiré-en-arrière par g d'un diviseur de degré $(2,2,...,2)$ dans $\mathbb{P}^2 \times (\mathbb{P}^1)^r$.
3. Soit W un diviseur lisse de degré $(1,1)$ dans $B_1 \times B_2$, où $B_1 \cong B_2 \cong \mathbb{P}^2$. Soit X un revêtement double de $(\mathbb{P}^1)^{n-3} \times W$, ramifié le long d'un diviseur premier D, qui est l'intersection de $(\mathbb{P}^1)^{n-3} \times W$ et un diviseur de degré $(2,...,2,2,2)$ dans $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$.
4. Soient $B_1 \cong B_2 \cong \mathbb{P}^2$. Soit Y_1 un revêtement double de $(\mathbb{P}^1)^{n-3} \times B_1$, ramifié le long d'un diviseur premier de degré $(2,..,2,2)$. Soit $Y = Y_1 \times B_2$. Soit p la projection naturelle de Y dans $B_1 \times B_2$. Soit W une hypersurface normale dans $B_1 \times B_2$ de degré $(1,2)$, où $k \in \{1,2\}$. Soit $X = p^*W$.

5. Soient $B_1 \cong B_2 \cong \mathbb{P}^2$. Supposons que $n \geqslant 5$. Soit $V_1 = (\mathbb{P}^1)^r \times B_1$ et soit $V_2 = B_2 \times (\mathbb{P}^1)^s$, où $r, s > 0$ et $r+s = n-3$. Soit Y_1 un revêtement double de V_1, ramifié le long d'un diviseur premier de degré $(2, ..., 2, 2)$ et soit Y_2 un revêtement double de V_2, ramifié le long d'un diviseur premier de degré $(2, 2, ..., 2)$. Soit $Y = Y_1 \times Y_2$ et soit p la projection naturelle de Y dans $B_1 \times B_2$. Soit W une hypersurface normale dans $B_1 \times B_2$ de degré $(1, 1)$. Soit $X = p^*W$.

Expliquons l'idée des preuves des théorèmes. Soit X une variété qui vérifie les conditions du Théorème 2.2.4. Considérons d'abord le cas où $\dim X = 3$. Alors le cône $\overline{\mathrm{NE}}(X)$ possède exactement deux rayons extrêmaux R_1, R_2. Soit $f_i : X \to B_i$ la contraction de R_i pour $i = 1, 2$. Nous pouvons démontrer que soit $B_i \cong \mathbb{P}^1$, soit $B_i \cong \mathbb{P}^2$. De plus, le produit $f_1 \times f_2$ est fini sur son image. Par conséquent, X est un revêtement fini de $\mathbb{P}^1 \times \mathbb{P}^2$ ou un revêtement fini d'une hypersurface de $\mathbb{P}^2 \times \mathbb{P}^2$. D'après une étude des morphismes finis entre variétés de Fano de dimension 3, nous pouvons conclure la preuve du Théorème 2.2.3.

Ensuite, nous traitons le cas où $n = \dim X \geqslant 4$. Nous allons démontrer le Théorème 2.2.4 par récurrence sur la dimension de X. Nous montrons d'abord qu'il existe toujours une fibration $g : X \to \mathbb{P}^1$. Pour cela, nous nous ramenons au cas où $n = 4$.

Il existe une fibration $h : X \to Y$ telle que $h \times g : X \to Y \times \mathbb{P}^1$ soit un morphisme fini surjectif. Nous pouvons démontrer que Y vérifie aussi les conditions du Théorème 2.2.4. Par récurrence, supposons que le Théorème 2.2.4 est vrai en dimension strictement plus petite que n. Alors, $h \times g$ induit un morphisme fini surjectif $f : X \to Z$ tel que $Z \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times \mathbb{P}^2$ ou $Z \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times W$, où W est une hypersurface normale dans $\mathbb{P}^2 \times \mathbb{P}^2$. Enfin, nous déduisons le Theorem 2.2.4 en étudiant les morphismes finis entres des variétés de Fano.

2.3 Pluri-formes sur les variétés singulières rationnellement connexes

Soit X une variété complexe rationnellement connexe. Si X est lisse, alors, par un point général x de X, il passe une courbe rationnelle très libre $f : \mathbb{P}^1 \to X$. Par définition, f^*T_X est un fibré ample, où T_X est le fibré tangent de X. En particulier, pour toute pluri-forme $\sigma \in H^0(X, (\Omega^1_X)^{\otimes m})$, le tiré-en-arrière $f^*\sigma$ est identiquement nul. Par conséquent, la pluri-forme σ est nulle en point x. Comme x est un point général, cela implique le théorème suivant ([Kol96, Cor IV.3.8]).

Théorème 2.3.1. *Soit X une variété projective complexe lisse rationnellement connexe. Alors X ne porte pas de pluri-forme non nulle, i.e. pour tout $m > 0$, on a $h^0(X, (\Omega^1_X)^{\otimes m}) = 0$.*

Mumford conjecture que la réciproque du théorème est aussi vraie.

Conjecture 2.3.2. *Soit X une variété projective complexe lisse telle que $h^0(X, (\Omega^1_X)^{\otimes m}) = 0$ pour tout $m > 0$. Alors X est une variété rationnellement connexe.*

Il est naturel de se demander s'il existe des annulations analogues lorsque X est une variété projective complexe rationnellement connexe singulière. Greb, Kebekus, Kovács et Peternell démontrent que les variétés projectives complexes rationnellement connexes à singularités klt ne portent pas de formes différentielles réflexives.

Théorème 2.3.3. *Soit (X, Δ) une paire klt. Supposons que X est rationnellement connexe. Alors $h^0(X, \Omega^{[m]}_X) = 0$ pour tout $m > 0$.*

Quant aux pluri-formes, la situation est plus délicate. Greb, Kekebus et Peternell démontrent le théorème suivant ([GKP14, Thm. 3.3]).

Théorème 2.3.4. *Soit X une variété projective complexe localement factorielle à singularités canoniques. Si X est rationnellement connexe, alors $h^0(X, (\Omega^1_X)^{[\otimes m]}) = 0$ pour tout $m > 0$.*

Le Théorème 2.3.1 n'est plus vrai pour les variétés à singularités klt. Il existe des variétés rationnellement connexes à singularités klt dont le diviseur canonique est $\mathbb{Q}$-effectif ([Kol08, Example 43] ou [Tot12, Example 10]). La condition que X soit localement factorielle dans ce théorème est aussi indispensable. Il existe des variétés complexes projectives rationnellement connexes à singularités canoniques qui portent des pluri-formes non nulles.

Example 2.3.5 ([GKP14, Example 3.7]). Soit $X_0 \xrightarrow{\pi_0} \mathbb{P}^1$ une surface réglée lisse. Soient $q_1, ..., q_4$ quatre points distincts dans $\mathbb{P}^1$. Pour chaque point q_i, on effectue la suite suivante de transformations birationnelles :

(i) On éclate un point x_i dans la fibre de π_0 au-dessus de q_i. Alors on obtient deux (-1)-courbes au-dessus de q_i, qui se rencontrent transversalement en point x'_i.

(ii) On éclate le point x'_i. Alors il y a deux (-2)-courbes et une (-1)-courbes au dessus de q_i. La (-1)-courbe est de multiplicité 2 dans la fibre au-dessus q_i.

(iii) On contracte les deux (-2)-courbes. Cela produit deux points singuliers de la surface au-dessus de q_i. Les singularités sont canoniques de type A_1. La fibre au-dessus de q_i est alors irréductible et de multiplicité 2.

Nous obtenons une surface rationnelle à singularités canoniques $X \xrightarrow{\pi} \mathbb{P}^1$. De plus, nous avons $h^0(X, (\Omega^1_X)^{[\otimes 2]}) \neq 0$.

La raison pour laquelle la surface X construite dans l'Exemple 2.3.5 porte des 2-pluri-formes non nulles est la suivante. Soit X° le lieu lisse de X. Comme $\Omega^1_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(-2)$, il existe deux sections rationnelles σ_1 et σ_2 telles que les uniques pôles de σ_1 (resp. de σ_2) soient les points q_1 et q_2 (resp. q_3 et q_4). De plus, les pôles de σ_1 et σ_2 sont simples. Soit $\sigma = (\pi|_{X^\circ})^*(\sigma_1 \otimes \sigma_2)$. Alors σ est une section rationnelle de $(\Omega^1_{X^\circ})^{\otimes 2}$. Montrons que σ est une section régulière. Par symétrie, il suffit de montrer que σ n'a pas de pôle le long de la fibre de $\pi|_{X^\circ}$ au-dessus de q_1. Soit t une coordonnée locale de $\mathbb{P}^1$ telle que q_1 soit défini par $t = 0$ et telle que

$$\sigma_1 \otimes \sigma_2 = \frac{1}{t}\mathrm{d}t \otimes \mathrm{d}t.$$

Soient (a, b) des coordonnées locales de X° telles que π s'exprime par $(a, b) \mapsto a^2$. Alors

$$\sigma = \pi^*(\frac{1}{t}\mathrm{d}t \otimes \mathrm{d}t) = \frac{1}{a^2}\mathrm{d}a^2 \otimes \mathrm{d}a^2 = 4\mathrm{d}a \otimes \mathrm{d}a.$$

Cela implique que σ est un élément non nul de $H^0(X^\circ, (\Omega^1_{X^\circ})^{\otimes 2})$. Donc $h^0(X, (\Omega^1_X)^{[\otimes 2]}) \neq 0$.

2.3.1 Le cas des surfaces

Dans le Chapitre 6, nous étudions les surfaces complexes projectives à singularités canoniques rationnellement connexes qui portent des pluri-formes non nulles. Nous proposons deux méthodes pour construire de telles surfaces et démontrons que toute telle surface peut être construite par ces méthodes.

La première méthode au-dessous est analogue à la construction dans l'Exemple 2.3.5. Nous construisons la surface par une suite de transformations birationnelles explicites à partir d'une surface réglée rationnelle lisse.

Construction 2.3.6. Soit $X_0 \xrightarrow{\pi_0} \mathbb{P}^1$ une surface réglée lisse. Prenons r points distincts $q_1, q_2, ..., q_r$ dans $\mathbb{P}^1$ avec $r \geqslant 4$. On effectue les transformations birationnelles suivantes des surfaces :

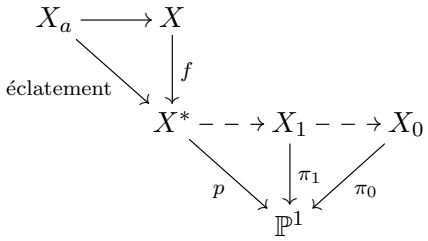
(i) Pour tout point q_i, on effectue la même suite de transformations birationnelles que celle de l'Exemple 2.3.5. On obtient une surface rationnelle à singularités canoniques $X_1 \xrightarrow{\pi_1} \mathbb{P}^1$. Les fibres multiples de π_1 sont alors $\pi_1^*q_1, \pi_1^*q_2, ..., \pi_1^*q_r$.

(ii) On effectue un nombre fini de fois la transformation birationnelle suivante : on éclate un point lisse dans une fibre multiple au-dessus de $\mathbb{P}^1$, ensuite, on contracte la transformée stricte de la fibre initiale. On obtient une surface $X^* \xrightarrow{p} \mathbb{P}^1$.

(iii) A partir de X^*, on effectue une suite d'éclatements de points lisses. On obtient une surface $X_a \longrightarrow \mathbb{P}^1$.

(iv) On contracte des chaînes de (-2)-courbes qui sont exceptionnelles par rapport à $X_a \to X^*$.

À la fin, nous obtenons une surface rationnelle à singularités canoniques $X \xrightarrow{\pi} \mathbb{P}^1$.



Théorème 2.3.7. *Soit X une surface complexe projective rationnellement connexe à singularités canoniques qui porte des pluri-formes non nulles. Alors X peut être construite par la méthode de la Construction 2.3.6*

Remarquons que si $X \xrightarrow{\pi} \mathbb{P}^1$ est une surface obtenue par la Construction 2.3.6, alors les fibres multiples de π sont $\pi^*q_1, \pi^*q_2, ..., \pi^*q_r$. Comme dans l'Exemple 2.3.5, ces fibres multiples sont exactement la source des pluri-formes non nulles.

Théorème 2.3.8. *Soit X une surface complexe projective rationnellement connexe à singularités canoniques qui porte des pluri-formes non nulles. Si X^* est le résultat d'un MMP pour X, alors X^* est un espace fibré de Mori au-dessus de $\mathbb{P}^1$. Notons $p : X^* \to \mathbb{P}^1$ la fibration de Mori. Soit r le nombre des fibres multiples de p. Alors $r \geqslant 4$ et*

$$h^0(X, (\Omega^1_X)^{[\otimes m]}) = h^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r))$$

pour tout $m > 0$. En particulier, à m fixé, la dimension des m-pluri-formes ne dépend que du nombre des fibres multiples de p.

Avec les mêmes notations que dans le théorème précédent, par réduction semi-stable, il existe un revêtement fini galoisien $\gamma : B \to \mathbb{P}^1$ tel que la fibration $Z \to B$ soit à fibres réduites, où Z est la normalisation du produit fibré $X^* \times_{\mathbb{P}^1} B$. On montre le théorème suivant.

Théorème 2.3.9. *Soit X une surface complexe projective rationnellement connexe à singularités canoniques qui porte des pluri-formes non nulles et soit X^* un résultat d'un MMP. Soit π la composition $X \to X^* \to \mathbb{P}^1$. Alors il existe une courbe projective lisse B de genre strictement positif et un diagramme commutatif*

$$\begin{array}{ccc} Y & \xrightarrow{\Gamma} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\gamma} & \mathbb{P}^1 \end{array}$$

tel que γ et Γ soit des morphismes finis galoisiens de groupe de Galois $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. De plus, Y est la normalisation du produit fibré $B \times_{\mathbb{P}^1} X$ et

Γ *est étale en codimension* 1. *Nous avons*

$$h^0(X,(\Omega^1_X)^{[\otimes m]}) = h^0(Y,(\Omega^1_Y)^{[\otimes m]})^G = h^0(B,(\Omega^1_B)^{\otimes m})^G$$

pour tout $m > 0$.

Remarquons que $h^0(B,(\Omega^1_B)^{\otimes 2})^G \neq 0$. En effet, comme G est un groupe commutatif et $h^0(B,\Omega^1_B)$ est non nulle, il existe une forme non nulle $\sigma \in H^0(B,\Omega^1_B)$ qui est un vecteur propre commun des éléments de G. Puisque tout élémént de G est d'ordre 2, on obtient que $\sigma^{\otimes 2}$ est un élément non nul dans $H^0(B,(\Omega^1_B)^{\otimes 2})^G$.

Ce théorème implique que toute surface rationnellement connexe complexe à singularités canoniques qui porte des pluri-formes non nulles peut être construite par la méthode suivante.

Construction 2.3.10. Soit Y une surface complexe projective à singularités canoniques et soit G un sous-groupe fini de $\mathrm{Aut}(Y)$, le groupe des automorphismes de Y, dont l'action sur Y est galoisienne et libre en codimension 1. Supposons qu'il existe une fibration G-équivariante π' de Y sur une courbe lisse B de genre strictement positif telle que $B/G \cong \mathbb{P}^1$ et que les fibres générales de π' soient isomorphes à $\mathbb{P}^1$. Soit $X = Y/G$. Alors X est une surface rationnellement connexe à singularités canoniques qui porte des pluri-formes non nulles.

À la fin de ce chapitre, on démontre le théorème suivant, qui donne une réponse négative à une question de Debarre : "existe-t-il une surface projective complexe à singularités canoniques dont le lieu lisse n'est pas rationnellement connexe telle que $h^0(X,(\Omega^1_X)^{[\otimes m]}) = 0$ pour tout $m > 0$?"

Théorème 2.3.11. *Soit X une surface projective complexe à singularités canoniques. Alors le lieu lisse de X est rationnellement connexe si et seulement si $h^0(X,(\Omega^1_X)^{[\otimes m]}) = 0$ pour tout $m > 0$.*

2.3.2 Le cas de la dimension trois

Dans le Chapitre 7, nous étudions les variétés complexes rationnellement connexes X de dimension 3 qui portent des pluri-formes non nulles. Dans le cas de dimension 3, la situation est plus compliquée. Supposons que X est $\mathbb{Q}$-factorielle et à singularités canoniques. Par un programme des modèles minimaux, nous obtenons une suite de transformations birationnelles

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k = X^*,$$

où X^* est un espace fibré de Mori. Comme $\dim X = 3$, il peut y avoir des flips dans la suite précédente. En particulier, la composition $X \dashrightarrow X^*$ n'est pas toujours un morphisme. Cependant, pour tout $m > 0$, il existe une injection naturelle

$$H^0(X,(\Omega^1_X)^{[\otimes m]}) \hookrightarrow H^0(X^*,(\Omega^1_{X^*})^{[\otimes m]}).$$

Donc X^* porte des pluri-formes non nulles.

Supposons dans un premier temps que $X \cong X^*$ est un espace fibré de Mori. Alors le nombre de Picard de X est au moins 2 par le théorème suivant. En particulier, la variété X est fibrée soit au-dessus de $\mathbb{P}^1$, soit au-dessus d'une surface.

Théorème 2.3.12. *Soit X une variété de Fano à singularités klt telle que $\rho(X) = 1$. Alors, pour tout $m > 0$, nous avons $h^0(X, (\Omega^1_X)^{[\otimes m]}) = 0$.*

Dans le théorème suivant, nous démontrons que, comme dans le cas des surfaces, il existe une fibration équidimensionnelle q de X sur une variété $\mathbb{Q}$-factorielle Z, telle que les pluri-formes de X viennent de la base Z. Soient $D_1, ..., D_k$ les diviseurs premiers dans Z tels que q^*D_i n'est pas réduit. Soit m_i le plus petit coefficient de q^*D_i. Définissons un $\mathbb{Q}$-diviseur Δ dans Z par

$$\Delta = \sum_i \frac{m_i - 1}{m_i} D_i.$$

Théorème 2.3.13. *Soit X un espace fibré de Mori $\mathbb{Q}$-factoriel à singularités canoniques de dimension 3. Supposons que $h^0(X, (\Omega^1_X)^{[\otimes m]}) \neq 0$ pour un entier $m > 0$. Alors il existe une fibration $q : X \to Z$ telle que $\dim Z > 0$ et telle que $K_Z + \Delta$ soit $\mathbb{Q}$-effectif, où Δ est défini comme ci-dessus. Nous avons deux possibilités.*

1. *La base Z est une surface et q est une fibration de Mori. Soit l un entier strictement positif tel que $l(K_Z + \Delta)$ soit linéairement équivalent à un diviseur de Cartier effectif. Alors les sections globales de $\mathcal{O}(l(K_Z + \Delta))$ induisent des $(2l)$-pluri-formes sur X.*
2. *La base Z est isomorphe à $\mathbb{P}^1$. Pour tout $m > 0$, nous avons*

$$H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m + \sum_i [\frac{m(m_i - 1)}{m_i}] D_i)),$$

où $[\ \ ]$ désigne la partie entière.

Considérons le cas général où X n'est plus un espace fibré de Mori. Soit $X \dashrightarrow X^*$ le résultat d'un MMP. Alors, par ce qui précède, il existe une fibration $q : X^* \to Z$ telle que les pluri-formes de X^* viennent de Z. Malheureusement, l'application $X \dashrightarrow Z$ n'est pas toujours régulière. Même si elle l'est, elle n'est pas toujours équidimensionnelle. Il est donc difficile de prédire quelles pluri-formes de X^* peuvent se relever sur X. Toutefois, si nous supposons que X est à singularités terminales, alors $Z \cong \mathbb{P}^1$ et $X \dashrightarrow Z$ est toujours un morphisme par le théorème suivant, qui ressemble au Théorème 2.3.8 pour les surfaces.

Théorème 2.3.14. *Soit X une variété complexe rationnellement connexe à singularités terminales de dimension 3 qui porte des pluri-formes non nulles.*

Alors il existe une fibration $q : X \to \mathbb{P}^1$ dont les fibres générales sont des surfaces rationnellement connexes lisses. Soient $z_1, ..., z_r$ les points dans $\mathbb{P}^1$ tels que le diviseur de Cartier q^*z_i soit non réduit. Soit m_i le plus petit coefficient de q^*z_i. Alors pour tout $m > 0$,

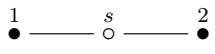
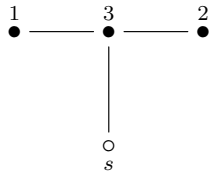
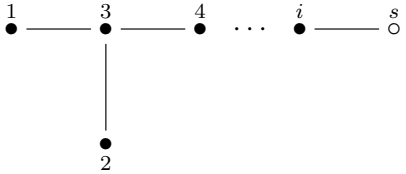
$$h^0(X, (\Omega^1_X)^{[\otimes m]}) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + \sum_{i=1}^{r}[\frac{m(m_i - 1)}{m_i}])).$$

Chapitre 3

Geometry of klt surfaces

In this chapter, we will prove some results on geometry of complex surfaces with klt singularities. In particular, we prove the following theorem.

Theorem 3.0.1. *Let $p : X \to B$ be a Mori fibration from a klt quasi-projective surface to a smooth curve. Let b be a point in B, and let $C = \mathrm{Supp}(p^*b)$. Assume that $p^*b = 2C$. Then X has canonical singularities along p^*b. Let $r : \widetilde{X} \to X$ be the minimal resolution of singularities along the fiber p^*b and let $\tilde{p} = p \circ r$. We have the following table of possibilities for p^*b, and each of these possibilities can occur. In the table, the dual graph is the one of the support of $\tilde{p}^*b \subseteq \widetilde{X}$, the point with label s corresponds to $\widetilde{C}$ and the other points correspond to the r-exceptional divisors, where $\widetilde{C}$ is the strict transform of C in $\widetilde{X}$.*

Type of fiber	Dual graph
$(A_1 + A_1)$	
(D_3)	
(D_i) with $i > 3$	

In the table of the theorem above, we see that a multiple fiber of type $(A_1 + A_1)$ is a multiple fiber which contains exactly two singular points and both of them are of type A_1. A multiple fiber of type (D_i) with $i \geqslant 3$ is a

multiple fiber which contains exactly one singular point which is of type D_i (Note that the singularity D_3 is the same as A_3).

3.1 Preparatory lemmas

The lemmas in this chapter may be well known for experts. For the convenience of readers, we give account of the proofs here.

Lemma 3.1.1. *Let $\varphi : X \dashrightarrow X'$ be an extremal divisorial contraction or a flip. Assume that X has $\mathbb{Q}$-factorial canonical singularities. If Y is an irreducible closed subvariety in X' such that it is the center of a divisor E over X' which has discrepancy 0, then φ^{-1} is a morphism around the generic point of Y.*

Proof. Assume the opposite. Then the discrepancy of E in X is strictly smaller than the one in X' (see [KM98, Lem. 3.38]). This implies that X does not have canonical singularities along the center of E in X. This is a contradiction. □

Lemma 3.1.2. *Let X be a normal projective $\mathbb{Q}$-factorial surface. Let C, D be two different projective curves in X. Assume that both C and D pass through a smooth point x of X. Then $C \cdot D \geqslant 1$.*

Proof. Let $r : \widetilde{X} \to X$ be the minimal resolution of X. Then by the projection formula, we obtain

$$C \cdot D = (r^{-1})_* C \cdot r^* D \geqslant (r^{-1})_* C \cdot (r^{-1})_* D.$$

Since r is minimal and X is smooth at x, r^{-1} is an isomorphism around x. Hence both $(r^{-1})_* C$ and $(r^{-1})_* D$ pass through the smooth point y, where $\{y\} = r^{-1}(\{x\})$. Since $\widetilde{X}$ is smooth, we have $(r^{-1})_* C \cdot (r^{-1})_* D \geqslant 1$ by [Har77, Prop. V.1.4]. Hence $C \cdot D \geqslant 1$. □

Lemma 3.1.3. *Let Y be a klt quasi-projective surface. Let $E_i \subseteq Y$ $(i = 1, ..., r)$ be smooth projective K_Y-non-negative rational curves forming an snc divisor $\bigcup_{i=1}^r E_i$. Assume that the intersection matrix $(E_i \cdot E_j)_{1 \leqslant i,j \leqslant r}$ is negative definite, and that there are rational numbers $-1 < a_i \leqslant 0$ such that $(K_Y - \sum_{i=1}^r a_i E_i) \cdot E_j = 0$ for every $j \in \{1, ..., r\}$. Let $\{C_k\}$ be any subset of $\{E_i\}$. Then there is a birational morphism $Y \to X$ contracting the C_k's and no other curves. Moreover, X has klt singularities.*

Proof. We may assume that $\{C_k\}$ is not empty. By renumbering if necessary, we may assume that $\{C_k\} = \{E_1, ..., E_s\}$ with $1 \leqslant s \leqslant r$. Note that the intersection matrix $(E_i \cdot E_j)_{1 \leqslant i,j \leqslant s}$ is negative definite. Hence there are real numbers $b_1, ..., b_s$ such that $(K_Y - \sum_{i=1}^s b_i E_i) \cdot E_j = 0$ for all $1 \leqslant j \leqslant s$. Since $K_Y \cdot E_i$ is an integer for all i and $E_i \cdot E_j$ is integer for all $1 \leqslant i, j \leqslant s$,

the b_i's are rational numbers. On the one hand, since $K_Y \cdot E_i \geqslant 0$ for all i, we have $b_i \leqslant 0$ for all $1 \leqslant i \leqslant s$ by [KM98, Cor. 4.2]. On the other hand, we have that $(K_Y - \sum_{i=1}^{s} a_i E_i) \cdot E_j = \sum_{i=s+1}^{r} a_i E_i \cdot E_j \geqslant 0$ for all $1 \leqslant j \leqslant s$. Hence by [KM98, Cor. 4.2], we have $-1 < a_i \leqslant b_i$ for all $1 \leqslant i \leqslant s$. Note that the pair $(Y, -\sum_{i=1}^{s} b_i E_i)$ is klt. Hence by the same argument as in [KM98, Prop. 4.10], there is a birational morphism $g : Y \to X$ contracting the C_k's and no other curves. Moreover, X has klt singularities since the pair $(Y, -\sum_{i=1}^{s} b_i E_i)$ is klt and $g^* K_X = K_Y - \sum_{i=1}^{s} b_i E_i$. □

Remark 3.1.4. One of the application of Lemma 3.1.3 is as follows. Let $g : Y \to Z$ be a partial minimal resolution of singularities of a klt surface Z. That is, if $z \in Z$ is a point such that g^{-1} is not an isomorphism around z, then z is singular, and g is the minimal resolution of the singularity at z. Let $\{E_i\}$ be the set of the g-exceptional curves. Since Z has klt singularities, the E_i's satisfy the conditions in the lemma. Hence for any subset $\{C_k\}$ of $\{E_i\}$, there is a birational morphism $Y \to X$ contracting exactly the C_k's. The morphism $g : Y \to Z$ factors through $Y \to X$. Moreover, if Z has canonical singularities, then so has X (in this case $a_i = 0$ for all $i \in \{1, ..., r\}$, and $b_j = 0$ for all $j \in \{1, ..., s\}$ by the calculation in the proof of the lemma).

Similarly, we can prove the following lemma, which shows that we can contract some connected collection of (-2)-curves in a surface.

Lemma 3.1.5. *Let $E = \bigcup_{1 \leqslant k \leqslant i} E_k$ be a connected collection of (-2)-curves in a smooth quasi-projective surface V whose dual graph is the same as the one of the support of the exceptional set of a minimal resolution for a canonical surface singularity. Then there exists a morphism $c : V \to W$ such that W has canonical singularities and c contracts exactly E.*

Proof. We have $K_V \cdot E_k = 0$ for every k. The intersection matrix $\{E_k \cdot E_j\}$ is negative definite by [KM98, Lem. 3.40]. Thus there is a contraction $c : V \to W$ contracting exactly E by Lemma 3.1.3. Since V is smooth and $K_V = c^* K_W$, we obtain that W has canonical singularities. □

Recall that a variety X is called algebraically simply connected if any finite étale morphism $\tau : X' \to X$, with X' irreducible, is an isomorphism.

Lemma 3.1.6. *Let X be a variety. Let U be a non-empty Zariski open subset of X. If U is algebraically simply connected, then so is X.*

Proof. Let $c : X' \to X$ be a finite étale cover such that X' is irreducible. Let $U' = c^{-1}(U)$. Then U' is irreducible. Hence, $c|_{U'}$ is an isomorphism for U is algebraically simply connected. This implies that c is an isomorphism. Thus X is algebraically simply connected. □

Lemma 3.1.7 ([KM99, Lem. 3.3])**.** *Let X be a quasi-projective surface with canonical singularities. Let $f : X \to Y$ be a divisorial contraction in the*

MMP. Assume that the smooth locus of X is algebraically simply connected. Then the smooth locus of Y is algebraically simply connected.

Proof. Let $r : \widetilde{X} \to X$ and $s : \widetilde{Y} \to Y$ be minimal resolutions of X and Y. Then there is a natural morphism $\tilde{f} : \widetilde{X} \to \widetilde{Y}$. Moreover, $\widetilde{X}$ can be obtained from $\widetilde{Y}$ by a sequence of blow-ups.

Let E (resp. D) be the exceptional set of r (resp. of s). Then $\widetilde{X}\backslash E$ is algebraically simply connected, and we only have to prove that $\widetilde{Y}\backslash D$ is algebraically simply connected. Since f^{-1} is an isomorphism around the singular points of Y (See Lemma 3.1.1), we have $\tilde{f}^{-1}(D) \subseteq E$. Hence, $\widetilde{X}\backslash E \subseteq \tilde{f}^{-1}(\widetilde{Y}\backslash D)$. By Lemma 3.1.6, we obtain that $\tilde{f}^{-1}(\widetilde{Y}\backslash D)$ is algebraically simply connected. Note that $\tilde{f}^{-1}(\widetilde{Y}\backslash D)$ can be obtained from $\widetilde{Y}\backslash D$ by a sequence of blow-ups. Hence, $\widetilde{Y}\backslash D$ is also algebraically simply connected. □

3.2 $\mathbb{P}^1$-fibrations over a curve

In this section, we will give some properties of fibered surfaces $X \to B$ such that B is a smooth curve, and general fibers of the fibration are smooth rational curves. The fibration $X \to B$ is called a $\mathbb{P}^1$-bundle if it is smooth.

3.2.1 Basic properties

Lemma 3.2.1. *Let $p : X \to B$ be a projective fibration such that X is normal and B is smooth. Assume that every fiber of p is reduced and irreducible, and there is a smooth fiber isomorphic to $\mathbb{P}^1$. Then p is a $\mathbb{P}^1$-bundle.*

Proof. Note that general fibers of p are smooth rational curves. By [Kol96, Exercise II.2.8.6.3], we obtain that every fiber of p is a rational curve. Thus by [Kol96, Thm. II.2.8], p is a $\mathbb{P}^1$-bundle. □

Recall that a Hirzebruch surface Σ_k is a smooth rational ruled surface isomorphic to $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-k))$. In particular, if $k > 0$, then there is a unique rational curve C in Σ_k such that $C^2 = -k$ (See [Har77, §V.2]).

Lemma 3.2.2. *Let $\widetilde{X} = \Sigma_k$ be a Hirzebruch surface with $k > 0$, and let C be the unique rational curve in $\widetilde{X}$ such that $C \cdot C = -k$. Let $p : \widetilde{X} \to \mathbb{P}^1$ be the $\mathbb{P}^1$-bundle, and let F be a fiber of p. Let $\pi : \widetilde{X} \to X$ be the morphism contracting C. Then $K_X^2 = 8 + \frac{(k-2)^2}{k}$, $-K_X \cdot \pi_* F = 1 + \frac{2}{k}$, and $\pi_* F \cdot \pi_* F = \frac{1}{k}$. In particular, $K_X^2 > -K_X \cdot \pi_* F$*

Proof. We have $-K_{\widetilde{X}} = -\pi^* K_X + \frac{k-2}{k} C$, $K_{\widetilde{X}} \cdot K_{\widetilde{X}} = 8$, and $C \cdot F = 1$. Hence $K_X \cdot K_X = 8 + \frac{(k-2)^2}{k}$, and $-K_X \cdot \pi_* F = 1 + \frac{2}{k}$. We also obtain that $\pi_* F \cdot \pi_* F = \frac{1}{k}$. □

Proposition 3.2.3. *Let $p : X \to B$ be a Mori fibration from a quasi-projective surface with klt singularities to a smooth curve. Let b be a point of B. Then we have the following properties.*

1. *X is smooth along the support of p^*b if and only if p^*b is reduced;*
2. *if X has canonical singularities along C, then the coefficient of C in p^*b is at most equal to 2.*

Proof. 1. Since p is a Mori fibration, every fiber of p is irreducible. Note that X is Cohen-Macaulay since it is a normal surface. Therefore the Cartier divisor p^*b is also Cohen-Macaulay. Hence it is generically reduced if and only if it is a reduced subscheme. Moreover, since B is a smooth curve, the morphism p is a flat morphism.

First we assume that p^*b is reduced. Then by Lemma 3.2.1, X is smooth along p^*z. Conversely, assume that X is smooth along p^*b. Then by adjunction formula, we have

$$2h^1(C, \mathscr{O}_C) - 2 = (K_X + C) \cdot C = K_X \cdot C < 0.$$

Therefore, $K_X \cdot C = -2$ which is equal to $K_X \cdot p^*b$. Hence p^*b is reduced.

2. Assume that X has canonical singularities along C. Let $\alpha \in \mathbb{N}$ be the coefficient of C in p^*z. Then

$$-2 = K_S \cdot p^*z = \alpha K_S \cdot C.$$

However, since K_S is a Cartier divisor, $K_S \cdot C \in \mathbb{Z}$. Thus $-2 \in \alpha\mathbb{Z}$ which means $\alpha \leqslant 2$. □

In the remainder of this section, our aim is to prove the Theorem 3.0.1. We will prove the theorem in two steps. We will first show that the condition that $p^*b = 2C$ implies that X has canonical singularities along C. Then we will prove that each of the possibilities in the table of the theorem can occur.

3.2.2 Singularities and fiber multiplicities

In this subsection, we will prove the following proposition.

Proposition 3.2.4. *Let $p : X \to B$ be a Mori fibration from a klt quasi-projective surface to a smooth curve. Let b be a point in B, and let $C = \mathrm{Supp}(p^*b)$. Assume that $p^*b = 2C$. Then X has canonical singularities along p^*b. More precisely, we have exactly two possibilities*

1. *there are two singular points on C, and both of them are of type A_1;*
2. *there is one singular point on C, and the singularity is of type D_i for some $i \geqslant 3$.*

We will first prove some lemmas.

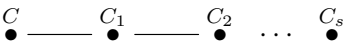
Lemma 3.2.5. *Let $p : X \to B$ be a projective fibration from a smooth quasi-projective surface to a smooth curve B such that general fibers of p are smooth rational curves. Let b be a point in B. Then the divisor p^*b has snc support. Moreover, the dual graph of the support of p^*b is a tree.*

We recall that a tree is an undirected connected graph without simple cycles.

Proof. Let $X \to X'$ be the result of a p-relative MMP. Then X' is smooth, and there is a natural morphism $p' : X \to B$. Since general fibers of p are rational curves, p' is a Mori fibration. Hence, by Proposition 3.2.3, p'^*b is a smooth rational curve for X' is smooth. Since X can be obtained by a sequence of blow-ups from X', the divisor p^*b has snc support, and the dual graph of the support of p^*b is a tree. □

Lemma 3.2.6. *Let $p : X \to B$ be a projective fibration from a smooth quasi-projective surface to a smooth curve B such that general fibers of p are smooth rational curves. Let b be a point in B. Assume that there is exactly one (-1)-curve C in p^*b, then the following properties hold.*

1. *One of the curves in p^*b which meet C is a (-2)-curve.*
2. *Assume that there is a chain of rational curves $E = \sum_{i=1}^{s} C_i$ in the fiber p^*b such that the dual graph of $C + (\sum_{i=1}^{s} C_i)$ is as follows.*

   Assume further that $E + C$ and $((p^*b)_{\mathrm{red}} - E - C)$ intersect only along C_s. Then all of the C_i's are (-2)-curves.
3. *The multiplicity of C in p^*b is larger than one.*

Proof. If p^*b has one component, then it is a 0-curve. If p^*b has two components, then both of them are (-1)-curves. Since there is exactly one (-1)-curve in p^*b by assumption, we obtain that p^*b has at least three components.

1. Let $X \to Y$ be the birational morphism which contracts exactly C. Then Y is smooth, and there is a fibration $q : Y \to B$ induced by p. Note that q^*b has at least two components. Hence, from the MMP, we know that there is at least one (-1)-curve D in q^*b. Since there is exactly one (-1)-curve C in p^*b, the strict transform of D in X meets C, and is a (-2)-curve.

2. We will prove by induction on i. If $i = 1$, then from the first property, we know that C_1 is a (-2)-curve. Assume that $C_1, ..., C_j$ are all (-2)-curves for some $j \geqslant 1$. We will prove that C_{j+1} is a (-2)-curve. Set $C_0 = C$. There is a birational morphism $h : X \to X'$ which contracts exactly $C_0, ..., C_{j-1}$. Let $p' : X' \to \mathbb{P}^1$ be the fibration induced by p. Then the strict transform of C_j in X' becomes the unique (-1)-curve in p'^*b since $E + C$ and $((p^*b)_{\mathrm{red}} - E - C)$ intersect only along C_s. By the first property, we obtain that the strict transform of C_{j+1} in X' is a (-2)-curve. This shows that C_{j+1} is a (-2)-curve since h is an isomorphism around C_{j+1}. This completes the induction.

3. We will prove by induction on the number k of components of the fiber p^*b. If $k = 3$, then the multiplicity of C is 2. Assume that the lemma is true for $k = l$, where $l \geqslant 3$ is an integer.

Now we assume that $k = l + 1$. Let $X \to Y$ be the birational morphism which contracts exactly C. Then X is the blow-up of a point y in Y. Let $q : Y \to \mathbb{P}^1$ be the fibration induced by p. If there are more than one (-1)-curves in q^*b, then they intersect at the point y. In this case, the multiplicity of C is larger than one. If there is exactly one (-1)-curve D in q^*b, then the multiplicity m of D in q^*b is larger than one. Note that the multiplicity of C in p^*b is not less than m since y lies on D. This completes the induction and the proof of the lemma. □

Lemma 3.2.7. *Let $p : X \to B$ be a projective fibration from a smooth surface to a smooth curve. Let b be a point in B. Let $C_1, ..., C_r$ be the components of p^*b. Then the intersection matrix $(C_i \cdot C_j)$ is negative. Moreover, $(\sum a_i C_i)^2 = 0$ if and only if $\sum a_i C_i$ is proportional to the fiber. In particular, if C is an irreducible component of p^*b such that $C \cdot C = 0$, then p^*b is irreducible with support C.*

Proof. See [Rei97, §A.7]. □

Lemma 3.2.8. *Let $p : X \to B$ be a Mori fibration from a klt quasi-projective surface to a smooth curve B. Let b be a point in B. Assume that X is not smooth along p^*b. Then there are at most 2 singular points of X on p^*b.*

Proof. Let $r : \widetilde{X} \to X$ be the minimal resolution of X. Let $\tilde{p} = p \circ r$. Let D be the support of p^*b. Since p is a Mori fibration, D is irreducible. Let $\widetilde{D}$ be the strict transform of D in $\widetilde{X}$. Then $\widetilde{D}$ is the only $K_{\widetilde{X}}$-negative curve in the fiber $\tilde{p}^*b$. In fact, $\widetilde{D}$ is a (-1)-curve by Lemma 3.2.5 and Lemma 3.2.7. Let $\widetilde{X} \to S$ be the divisorial contraction which contracts $\widetilde{D}$. Let $q : S \to \mathbb{P}^1$ be the fibration induced by $\tilde{p}$. Then S is smooth, and q^*b has snc support by Lemma 3.2.5. This implies that $\mathrm{Supp}\,(\tilde{p}^*b) - \widetilde{D}$ has at most two connected components. Thus there are at most 2 singular points of X on p^*b. □

Lemma 3.2.9. *Let $p : X \to B$ be a Mori fibration from a klt quasi-projective surface to a smooth curve. Let b be a point in B, and let C be the support of p^*b. Assume that $p^*b = 2C$, and that there are two singular points of X on C. Then X has canonical singularities along C, and the two singular points are of type A_1.*

Proof. Let $r : \widetilde{X} \to X$ be the minimal resolution and let $\tilde{p} : \widetilde{X} \to B$ be the fibration induced by p. If $\widetilde{C}$ is the strict transform of C in $\widetilde{X}$, then it is the unique (-1)-curve in $\tilde{p}^*b$. Let $\widetilde{X} \to Y$ be the morphism which contracts exactly $\widetilde{C}$. Let $q : Y \to B$ be the fibration induced by $\tilde{p}$.

Since there are two singular points of X on C, and the multiplicity of the fiber p^*b is two, we can decompose $\tilde{p}^*b - 2\widetilde{C}$ as $D + D' + R$ with D, D'

irreducible such that $\widetilde{C} \cdot D = \widetilde{C} \cdot D' = 1$, $D \cdot D' = 0$, and $\widetilde{C} \cdot R = 0$. By Lemma 3.2.6.1, either D or D' is a (-2)-curve. Without loss of generality, we may assume that D is a (-2)-curve. Then

$$0 = \tilde{p}^*b \cdot D = 2\widetilde{C} \cdot D + D^2 + D' \cdot D + R \cdot D = R \cdot D.$$

In particular, the intersection of R and D is empty.

Let $Y \to Z$ be the morphism contracting exactly the strict transform of D and $s : Z \to B$ be the fibration induced by q. Let E be the strict transform of D' in Z. Then there is at least one component in s^*b which has negative intersection number with K_Z. That is, there is at least one component in s^*b which has self-intersection number larger than -2. Since R does not meet D, we obtain that E is the unique curve in s^*b which can have self intersection number larger than -2. Hence $E^2 \geqslant -1$. Since the multiplicity of E in s^*b is one, by Lemma 3.2.6.3, we have $E^2 \neq -1$. However, by Lemma 3.2.7, we have $E^2 \leqslant 0$. Hence, $E^2 = 0$ and D' is a (-2)-curve. For the same reason as before, we obtain that $R \cdot D' = 0$.

Hence, $R = 0$ and $\tilde{p}^*b = D + D' + 2\widetilde{C}$. This proves the lemma. □

Now we can prove Proposition 3.2.4.

Proof of Proposition 3.2.4. Since p is a Mori fibration, C is irreducible. By Proposition 3.2.3, X is not smooth along C for $p^*b = 2C$.

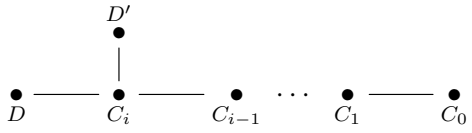
Let t be the number of singular points of X on the support of p^*b. Then $t \leqslant 2$ by Lemma 3.2.8. Let $r : \widetilde{X} \to X$ be the minimal resolution, and let $\widetilde{C}$ be the strict transform of C in $\widetilde{X}$. Then $\widetilde{C}$ is a (-1)-curve. Let $\tilde{p} = p \circ r$. Let $E = \tilde{p}^*b - 2\widetilde{C}$.

First assume that there are exactly two singular points on C. Then the singularities at these two points are canonical of type A_1 by Lemma 3.2.9.

Now assume that there is only one singular point on C. Let $C_0 = \widetilde{C}$, and let $E_0 = E$. We will show that there is a positive integer i such that we can decompose

$$\tilde{p}^*b = 2(C_0 + C_1 + \cdots + C_i) + D + D' + R \qquad (*)$$

such that all of the C_j's with $j > 0$ are (-2)-curves, that both D and D' are smooth rational curves and that R is disjoint from $C_0 + \cdots + C_i$. Furthermore, we have $D \cdot D' = 0$, $C_i \cdot D = C_i \cdot D' = 1$, $C_j \cdot C_{j+1} = 1$ for $0 \leqslant j \leqslant i-1$, and $C_j \cdot C_k = 0$ if $k - j > 1$. In particular, the dual graph of $\tilde{p}^*b - R$ is as follows.



We will construct the decomposition $(*)$ by induction. Since there is only one singular point over C, and $\tilde{p}^*b$ is an snc divisor, the support of E_0 intersects C_0 transversally at a point. Moreover since

$$C_0 \cdot E_0 = C_0 \cdot \tilde{p}^*b - 2C_0^2 = 2,$$

we can decompose E_0 into $2C_1 + E_1$, where C_1 is a smooth rational curve and $C_0 \cdot E_1 = 0$. From Lemma 3.2.6.1, we know that C_1 is a (-2)-curve. We have

$$C_1 \cdot E_0 = C_1 \cdot (\tilde{p}^*b - 2C_0) = -2C_1 \cdot C_0 = -2$$

and

$$E_0^2 = (\tilde{p}^*b - 2C_0)^2 = 4C_0^2 = -4.$$

Hence we obtain that

$$E_1^2 = (E_0 - 2C_1)^2 = -4 \text{ and } C_1 \cdot E_1 = C_1 \cdot (E_0 - 2C_1) = 2.$$

Assume that we can decompose

$$\tilde{p}^*b = 2(C_0 + C_1 + \cdots + C_k) + E_k$$

for some $k > 0$ such that $C_1, ..., C_k$ are (-2)-curves, $E_k^2 = -4$, $C_k \cdot E_k = 2$ and E_k is disjoint from $(C_0 + \cdots + C_{k-1})$.

The condition $C_k \cdot E_k = 2$ implies that C_k and E_k intersect at one or two points. If they intersect at one point, then we can decompose E_k into $2C_{k+1} + E_{k+1}$ such that C_{k+1} is a smooth rational curve, $C_k \cdot C_{k+1} = 1$ and $E_{k+1} \cdot C_k = 0$. Then

$$\tilde{p}^*b = 2(C_0 + C_1 + \cdots + C_{k+1}) + E_{k+1}.$$

From Lemma 3.2.6.2, we know that C_{k+1} is a (-2)-curve. We also have

$$C_{k+1} \cdot E_k = C_{k+1} \cdot (\tilde{p}^*b - 2(C_0 + \cdots C_k)) = -2C_{k+1} \cdot C_k = -2.$$

Thus

$$E_{k+1}^2 = (E_k - 2C_{k+1})^2 = -4 \text{ and } C_{k+1} \cdot E_{k+1} = C_{k+1} \cdot (E_k - 2C_{k+1}) = 2.$$

We are in the same situation as before. In this case, we repeat the same procedure.

Since there are only finitely many components in $\tilde{p}^*b$, we can find a positive integer i such that

$$\tilde{p}^*b = 2(C_0 + C_1 + \cdots + C_i) + E_i$$

and C_i intersects E_i at two different points. As in the proof of Lemma 3.2.9, we can write $E_i = D + D' + R$, where D, D' are smooth rational curves such

that $D \cdot D' = 0$, $R \cdot C_i = 0$, $D \cdot C_i = 1$, and $D' \cdot C_i = 1$. Hence we obtain the decomposition $(*)$

Now we will prove that $R = 0$. There is a birational morphism $\widetilde{X} \to W$ which contracts exactly $C_1, ..., C_{i-1}$. The surface W is smooth and there is a fibration $q : W \to B$ induced by $\tilde{p}$. We still denote by C_i, D, D' and R their strict transformations in W. Then C_i is the unique (-1)-curve in q^*b and $q^*b = 2C_i + D + D' + R$. Similarly to the proof of Lemma 3.2.9, we can prove that $D^2 = D'^2 = -2$, and $R = 0$. Hence, the singularity of the singular point on p^*b is canonical of type D_{i+2}. □

Remark 3.2.10. We would like to thank Professor Höring for suggesting the following simple proof for the first part of Theorem 3.0.1. Let $r : \widetilde{X} \to X$ be the minimal resolution. Then we can contract all (-2)-curves in $\widetilde{X}$ which are r-exceptional and obtain a surface with canonical singularities $\bar{X} \xrightarrow{\varphi} X$. Then, in order to prove that X has canonical singularities along p^*b, we only need to prove that φ is an isomorphism along p^*b. Assume the opposite. Let $\bar{C}$ be the strict transform of C in $\bar{X}$ and let $q : \bar{X} \to B$ be the natural fibration. Then q^*b is reducible and $-K_{\bar{X}} \cdot \bar{C} > -K_X \cdot C = 1$. Since $\bar{X}$ has canonical singularities, $-K_{\bar{X}} \cdot \bar{C}$ is an integer. Thus this number is at least 2. Moreover, by [Kol96, Exercise II.1.3.4], this implies that the curve $\bar{C}$ can deform in $\bar{X}$. This is a contradiction since $\bar{C}$ is an irreducible component of a reducible fiber of q.

3.2.3 Construction of singular fibers.

In this subsection, we will provide methods for constructions of all singular fibers in the table of Theorem 3.0.1.

We will first construct the fiber of type $(A_1 + A_1)$.

Construction 3.2.11. Let $p' : X' \to B$ be a smooth Mori fibration from a quasi-projective surface to a smooth curve. Let b be a point in B. Perform the following sequence of birational transformations of the ruled surface :

(i) Blow up a point x in the fiber over b. Then we get two (-1)-curves which meet transversely at x'.

(ii) Blow up the point x'. Over b, we get two disjoint (-2)-curves and one (-1)-curve. The (-1)-curve appears in the fiber with multiplicity two.

(iii) Blow down the two (-2)-curves. We get two singular points on the fiber, each of them is of type A_1.

In the end, we obtain a Mori fiber surface $X \xrightarrow{p} B$ such that the fiber of p over b is of type $(A_1 + A_1)$.

In order to construct the remaining singular fibers, we will prove the following lemma.

Lemma 3.2.12. *Let $p : X \to B$ be a Mori fibration from a singular quasi-projective surface with canonical singularities to a smooth curve. Let b be a point in B, and let $C = \mathrm{Supp}(p^*b)$. Assume that $p^*b = 2C$. Let $x \in X$ be a smooth point over b and let W be the blow-up of X at x. Let E be the exceptional divisor of $W \to X$. Let D be the strict transform of C in W. Then we can blow down D and obtain another Mori fiber surface $T \xrightarrow{q} B$.*

Proof. Let h be the natural fibration from W to B. We have $C \cdot C = 0$, $K_X \cdot C = -1$, $K_W \cdot E = -1$, $E \cdot E = -1$ and $D \cdot E = 1$. Thus

$$K_W \cdot D = 0 \text{ and } D \cdot D = -1.$$

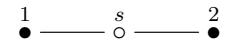
Let H be an ample divisor on W. Then there is a positive integer k such that $(H + kD) \cdot D = 0$. Let $A = H + kD$. Note that A is nef and big and D is the only curve which has intersection number 0 with A. Since $K_W \cdot D = 0$, for a large enough positive integer a, the divisor $aA - K_W$ is nef and big. Hence by the relative basepoint-free theorem (see [KM98, Thm. 3.24]), there is a positive integer b such that the natural morphism $h^*(h_*\mathscr{O}_W(bA)) \to \mathscr{O}_W(bA)$ is surjective. Let $c : W \to T$ be the fibration induced by $h^*(h_*\mathscr{O}_W(bA)) \to \mathscr{O}_W(bA)$. Then c contracts exactly D and the fibration h induces a fibration $q : T \to B$ which is also a Mori fibration. □

We can use the elementary transformation in Lemma 3.2.12 to construct multiple fibers of type (D_i), $i \geqslant 3$.

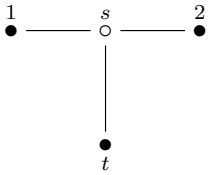
Lemma 3.2.13. *Let $p : X \to B$ be a Mori fibration from a quasi-projective surface with canonical singularities to a smooth curve. Let $b \in B$ and let $T \xrightarrow{q} B$ be the Mori fiber surface obtain by the method in Lemma 3.2.12. If the fiber of p is of type $(A_1 + A_1)$ over b, then the fiber of q is of type (D_3) over b. If the fiber of p is of type (D_i) over b, then the fiber of q is of type (D_{i+1}) over b for $i \geqslant 3$.*

Proof. Let $\widetilde{T} \to T$ be the minimal resolution and let $\tilde{q} : \widetilde{T} \to B$ be the natural fibration. We will compute the dual graph of the support of the fiber $\tilde{q}^*b$. Let W be the same as in Lemma 3.2.12. From the construction of T, we know that $\widetilde{T} \to T$ factors through $\widetilde{T} \to W$ and the last morphism is also the minimal resolution of W. Since $W \to X$ is a blow-up of a smooth point of X, the surface $\widetilde{T}$ can be obtained by blowing up the same point in $\widetilde{X}$, where $\widetilde{X} \to X$ is the minimal resolution. Let $\tilde{p} : \widetilde{X} \to B$ be the natural fibration.

Let C be the support of p^*b. If the fiber p^*b is of type $(A_1 + A_1)$, the dual graph of the support of $\tilde{p}^*b$ in $\widetilde{X}$ is

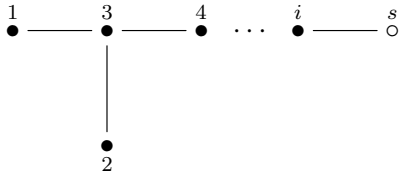


where s represents $\widetilde{C}$, the strict transform of C in $\widetilde{X}$. After blowing up the point on $\widetilde{C}$ as we mentioned above, the new graph is

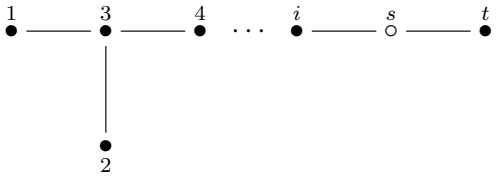


This graph is the dual graph of the support of $\tilde{q}^*b$ and the point with label t corresponds to the strict transform of the support of q^*b in $\widetilde{T}$. The graph shows that there is only one singular point of T on q^*b which is of type D_3. Hence the fiber q^*b is of type (D_3).

If p^*b is of type D_i, then from the proof of Proposition 3.2.4, we know that the dual graph of the support of $\tilde{p}^*b$ is



where the point with label s correspond to $\widetilde{C}$ (If $i = 3$, then s is just connected to the point with label 3). After blowing up the point on $\widetilde{C}$, we obtain the dual graph of the support of $\tilde{q}^*b$, which is



The point with label t corresponds to the strict transform of the support of q^*b in $\widetilde{T}$. This implies that q^*b is of type (D_{i+1}). □

Now we can complete the proof of Theorem 3.0.1.

Proof of Theorem 3.0.1. The table of fibers in the theorem follows from Proposition 3.2.4. By Construction 3.2.11 and Lemma 3.2.13, every type of fiber in the table occurs. □

Now we will show that every singular fiber can be obtained from a smooth fiber by the methods of Construction 3.2.11 and Lemma 3.2.12.

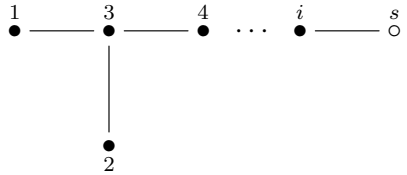
Lemma 3.2.14. *Let $p : X \to B$ be a Mori fibration from a quasi-projective surface with canonical singularities to a smooth curve. Let $b \in B$ such that p^*b is a singular fiber. Then this singular fiber can be obtained from a smooth ruled surface $X_1 \to B$ by the method of Construction 3.2.11 followed by a finite sequence of elementary birational transformations described in Lemma 3.2.12.*

Proof. Let $\widetilde{X} \to X$ be the minimal resolution and let $\tilde{p} : \widetilde{X} \to \mathbb{P}^1$ be the natural fibration. Let $\widetilde{X} \to Z$ be the result of a $\tilde{p}$-MMP. Then the natural

fibration $Z \to B$ is a smooth Mori fibration. Moreover, $\widetilde{S}$ can be obtained from Z by a sequence of blow-ups.

If p^*b is of type $(A_1 + A_1)$, then $\widetilde{X}$ can be obtained from Z by two blow-ups as in the first two step of Construction 3.2.11. Blow-down the two (-2)-curves in $\widetilde{X}$ we obtain X. In this case, we take $X_1 = Z$.

If p^*b is of type (D_i), then the dual graph of $\tilde{p}^*b$ is



We can blow down the curves which correspond to the points of labels $1, 2, ..., i-1$ (by Lemma 3.1.5) and the curve corresponding to the point of label s (for it is a (-1)-curve). By blowing down these curves, we obtain a fiber surface $p_Y : Y \to B$ such that p_Y^*b is irreducible. Hence p_Y is a Mori fibration. Moreover, the fiber p_Y^*b is of type (D_{i-1}) if $i > 3$ and of type $(A_1 + A_1)$ if $i = 3$. Note that the image of the curve corresponding to s in Y is a smooth point y. Hence if we perform the birational transformation in Lemma 3.2.12 for Y, starting by blowing up y, then we will obtain X.

By induction, we can conclude the lemma. □

Chapitre 4

Lagrangian fibrations on symplectic fourfolds

4.1 Introduction

A smooth variety M (defined over a field k) of dimension $2n$ equipped with an everywhere non-degenerate closed two form $\Omega \in H^0(M, \Omega_M^2)$ is called a *symplectic* variety. Note that in this case $\Omega^{\wedge n} \in H^0(M, \Omega_M^{2n})$ is non-zero everywhere. Hence $\Omega_M^{2n} \cong \mathscr{O}_M$. A subvariety $N \subseteq M$ of dimension n is said to be *Lagrangian* if the restriction of Ω on the smooth locus of N is identically zero. A fibration $f : M \to X$ from a symplectic variety to a normal variety X is called a *Lagrangian fibration* if every component of every fiber of f is a Lagrangian subvariety of M.

A simply connected complex projective manifold M is called a *projective irreducible symplectic manifold* if M has a symplectic 2-form Ω, and $H^0(M, \Omega_M^2) = \mathbb{C}\Omega$. Any non trivial fibration from a projective irreducible symplectic manifold M to a normal variety X is a Lagrangian fibration by the following theorem.

Theorem 4.1.1 ([Mat99, Thm. 2])**.** *Let $f : M \to X$ be a fibration from a complex projective irreducible symplectic manifold M of dimension $2n$ to a normal variety X. Assume that $0 < \dim X < 2n$. Then*

(1) f is a Lagrangian fibration.

(2) X is a $\mathbb{Q}$-factorial klt Fano variety of dimension n with Picard number 1.

Consider a Lagrangian fibration from a complex projective irreducible symplectic manifold M to a normal variety X. We are interested in the base variety X. In all examples of Lagrangian fibrations, X is always isomorphic to $\mathbb{P}^n$. It is natural to ask if this is true for all Lagrangian fibrations. On the one hand, if we assume that the base is smooth, then it is proved by Hwang (see [Hwa08, Thm. 1.2]) that the base is always the projective space.

On the other hand, if the irreducible symplectic manifold M is of $K3^{[n]}$ or generalized Kummer deformation type, then the answer is also positive (see [Mat13, Cor. 1.1], [Mar14, Thm. 1.3], [BM14, Thm. 1.5] and [Yos12, Prop. 3.38]). However, it is still unclear if the base is always smooth. Moreover there are also irreducible symplectic manifolds which are not of $K3^{[n]}$ or generalized Kummer deformation type (see [O'G99] and [O'G03]). Thus the answer to this question is still unknown in the general setting. In this chapter, we study the case when M has dimension 4. We prove the following theorem.

Theorem 4.1.2. *Let $f : M \to X$ be a Lagrangian fibration from a projective irreducible symplectic manifold M of dimension 4 to a normal surface X. Then either $X \cong \mathbb{P}^2$ or $X \cong S^n(E_8)$.*

The surface $S^n(E_8)$ is the unique Fano surface with exactly one singular point which is Du Val of type E_8, and two nodal rational curves in its anti-canonical system. For more details on this surface, see the end of section 4.3.1. We will prove the theorem in several steps. We will first prove the following theorem (see section 4.2.3), which is a generalisation of a result of Kollár (see [Kol86b, Cor. 3.9]).

Theorem 4.1.3. *Let $g : V \to W$ be a projective equidimensional fibration between complex normal quasi-projective varieties. Assume that V is smooth, that $\omega_V \cong \mathscr{O}_V$, and that W is Cohen-Macaulay and $\mathbb{Q}$-Gorenstein. Let H be a $\mathbb{Q}$-Cartier Weil divisor on W, and let $D = g^*H$. Then $R^i g_* \mathscr{O}_V(D)$ is reflexive for all $i \geqslant 0$.*

In particular, if $f : M \to X$ is a Lagrangian fibration, then $R^i f_* \mathscr{O}_M$ is reflexive. Similarly to the proof of [Mat05, Thm. 1.3], we can show that $R^i f_* \mathscr{O}_M \cong \Omega_X^{[i]}$. Afterwards, we will prove the following result (see section 4.2).

Theorem 4.1.4. *Let $f : M \to X$ be a Lagrangian fibration from a complex projective irreducible symplectic manifold M to a normal projective variety X. Let H be a $\mathbb{Q}$-ample integral Weil divisor in X, and let $D = f^*H$. Then for all $j > 0$ and $i \geqslant 0$, we have*

$$h^j(X, R^i f_*(\mathscr{O}_M(D))) = 0 \text{ and } h^i(M, \mathscr{O}_M(D)) = h^0(X, \Omega_X^{[i]}[\otimes]\mathscr{O}_X(H)).$$

As a corollary, we conclude that if $\dim M = 4$, then

$$h^0(X, \mathscr{O}_X(H)) - h^0(X, \Omega_X^{[1]}[\otimes]\mathscr{O}_X(H)) + h^0(X, \mathscr{O}_X(H + K_X)) = 3$$

for any $\mathbb{Q}$-ample Weil divisor H on X. In the end, we will prove the following classification result on rational surfaces and deduce Theorem 4.1.2 (see section 4.3).

Theorem 4.1.5. *Let X be a complex Fano surface with klt singularities whose smooth locus is algebraically simply connected. Assume that X has Picard number 1, and*

$$h^0(X, \mathscr{O}_X(H)) - h^0(X, \Omega_X^{[1]}[\otimes]\mathscr{O}_X(H)) + h^0(X, \mathscr{O}_X(H + K_X)) = 3$$

for any $\mathbb{Q}$-ample integral Weil divisor H. Then there are exactly two possibilities : either $X \cong \mathbb{P}^2$ or $X \cong S^n(E_8)$.

4.2 Cohomology properties

The aim of this section is to prove Theorem 4.1.4. One of the applications of this theorem is the following corollary.

Corollary 4.2.1. *Let $f : M \to X$ be a Lagrangian fibration from a projective irreducible symplectic manifold M of dimension $2n$ to a normal projective variety X. Let H be a $\mathbb{Q}$-ample integral Weil divisor in X. Then $\sum_{i=0}^{n}(-1)^i h^0(X, \Omega_X^{[i]}[\otimes]\mathscr{O}_X(H)) = n + 1$.*

Proof of Corollary 4.2.1. Let $D = f^*H$. By Theorem 4.1.4, we have

$$h^i(M, \mathscr{O}_M(D)) = h^0(X, \Omega_X^{[i]}[\otimes]\mathscr{O}_X(H))$$

for all $i \geqslant 0$. Hence we only have to prove that $\sum_{i=0}^{n}(-1)^i h^i(M, \mathscr{O}_M(D)) = n+1$. Note that Theorem 4.1.4 also shows that $h^i(M, \mathscr{O}_M(D)) = 0$ for $i > n$. Hence it is enough to prove that

$$\sum_{i=0}^{2n}(-1)^i h^i(M, \mathscr{O}_M(D)) = n + 1.$$

However, $\sum_{i=0}^{2n}(-1)^i h^i(M, \mathscr{O}_M(D))$ is the Euler characteristic $\chi(\mathscr{O}_M(D))$ of the Cartier divisor D. Thus it suffices to prove that $\chi(\mathscr{O}_M(D)) = n + 1$.

Let q_M be the Beauville-Bogomolov quadric form on M. Then $q_M(D) = 0$ since the intersection number D^{2n} is zero. Moreover, we have $\chi(\mathscr{O}_M(D)) = \sum \frac{a_i}{(2i)!} q_M(D)^i$, where the a_i's are complex numbers depending only on M (see [Huy99, 1.11]). Hence $\chi(\mathscr{O}_M(D)) = \chi(\mathscr{O}_M) = n + 1$. □

4.2.1 Notation and outline of the section

Let X and T be two schemes. Then X is called a T-scheme if there is a morphism $X \to T$. Let A be a ring. Then we may call X a A-scheme if X is a $(\operatorname{Spec} A)$-scheme. Note that a point in a scheme may not be a closed point in this section.

Let T be a scheme and let $f : X \to Y$ be a morphism of T-schemes. Let $\mathscr{F}, \mathscr{G}$ be two coherent sheaves on X and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a morphism of

sheaves. Let t be any scheme point of T and let κ be the residue field of t. There is a natural morphism $\iota : \operatorname{Spec}\kappa \to T$. We denote the fiber product of X, Y, $\mathscr{F}$, $\mathscr{G}$, f and α under the base change ι by X_t, Y_t, $\mathscr{F}_t$, $\mathscr{G}_t$, f_t and α_t. If U is an open subset of T, similarly we denote the fiber product of X, Y, $\mathscr{F}$, $\mathscr{G}$, f and α under the base change $U \hookrightarrow T$ by X_U, Y_U, $\mathscr{F}_U$, $\mathscr{G}_U$, f_U and α_U.

Outline of the proof of Theorem 4.1.4. We will first show that

$$R^i f_* \mathscr{O}_M(D) \cong \Omega_X^{[i]}[\otimes]\mathscr{O}_X(H)$$

for all $i \geqslant 0$ (see Proposition 4.2.6). Hence if $h^j(X, R^i f_* \mathscr{O}_M(D)) = 0$ for all $j > 0$ and $i \geqslant 0$, then the Leray spectral sequence calculating $H^i(M, \mathscr{O}_M(D))$ degenerates at E_2, and we have $h^i(M, \mathscr{O}_M(D)) = h^0(X, \Omega_X^{[i]}[\otimes]\mathscr{O}_X(H))$ for all $i \geqslant 0$. Thus, in order to prove Theorem 4.1.4, it is enough to prove that $h^j(X, R^i f_* \mathscr{O}_M(D)) = 0$ for $j > 0$ (Note that if D is Cartier, then this is true by [Kol86a, Thm. 2.1] for $\omega_M \cong \mathscr{O}_M$).

If we assume that $\dim M = 4$, then we can prove this vanishing by covering tricks and a theorem due to Kollár (See Remark 4.2.17). In higher dimension, we will prove by reduction modulo p. We can reduce to the case when $f : M \to X$, H and D are defined over an algebraic number field K. In this case, there is a $\mathbb{Z}$-algebra of finite type $A \subseteq K$, which is a localisation of the ring of integers of K, of Krull dimension 1 such that the coefficients of the equations defining $f : M \to X$, H and D are contained in A. Let $T = \operatorname{Spec} A$, and let η be its generic point. Then f induces a morphism $\varphi : \mathscr{M} \to \mathscr{X}$ of T-schemes such that φ_η coincides with f after a field extension. Let $\mathscr{D}$ be the divisor on $\mathscr{M}$ induced by D, and let $\mathscr{H}$ be the divisor on $\mathscr{X}$ induced by H. We will show that $h^j(\mathscr{X}_t, R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t)) = 0$ for all $j > 0$, $i \geqslant 0$ and general closed point $t \in T$. By a base change property, this implies that $h^j(\mathscr{X}_\eta, R^i(\varphi_\eta)_* \mathscr{O}_{\mathscr{M}_\eta}(\mathscr{D}_\eta)) = 0$. Then we obtain $h^j(X, R^i f_* \mathscr{O}_M(D)) = 0$.

We will divide this section into four parts. In the first part, we will develop some tools for the proof of Theorem 4.1.4. In the second part, we will show the reflexivity of some higher direct images. Note that for general closed $t \in T$, the varieties $\mathscr{X}_t$ and $\mathscr{M}_t$ are defined over a finite field. We will prove that $h^j(\mathscr{X}_t, R^i f_* \mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t)) = 0$ in the third part. In the last subsection we will complete the proof of Theorem 4.1.4.

4.2.2 Base change properties

We will prove some properties on cohomology and base change. The aim is to prove the following proposition.

Proposition 4.2.2. *Let $g : V \to W$ be a projective morphism of integral T-schemes of finite type, where T is an integral noetherian scheme. Let $\mathscr{F}$ be a coherent sheaf on V. Then there is a non-empty open subset T_0 of T such that there is a natural isomorphism $(R^i g_* \mathscr{F})|_{W_t} \cong R^i(g_t)_* \mathscr{F}_t$ for any*

point $t \in T_0$ and any $i \geqslant 0$, where $g_t : V_t \to W_t$ is the restriction of g over t, and $\mathscr{F}_t$ is the restriction of $\mathscr{F}$ on V_t.

$$\begin{array}{ccc} V_t & \longrightarrow & V \\ \downarrow{\scriptstyle g_t} & & \downarrow{\scriptstyle g} \\ W_t & \longrightarrow & W \\ \downarrow & & \downarrow \\ \{t\} & \longrightarrow & T \end{array}$$

This proposition is similar to [Har77, Thm. III.12.8]. In fact, our proof follows that of [Har77, Thm. III.12.8]. We will need two lemmas.

Lemma 4.2.3. *Let A be a noetherian ring. Let V be a projective A-scheme of finite type, and let $\mathscr{F}$ be a coherent sheaf on V. Then there is a complex, bounded from above, of finitely generated free A-modules $L^\bullet$ such that $H^i(L^\bullet) \cong H^i(V, \mathscr{F})$ for all $i \geqslant 0$.*

Proof. Let (U_i) be an affine open cover of V. Let $C^\bullet$ be the Čech complex of $\mathscr{F}$ with respect to (U_i). Then $H^i(C^\bullet) \cong H^i(V, \mathscr{F})$ for all $i \geqslant 0$ (See [Har77, Thm. III.4.5]). Since V is projective, $H^i(V, \mathscr{F})$ is a finitely generated A-module for all i (See [Har77, Thm. III.5.2]). Hence, by [Har77, Thm. III.12.3], there is a complex of finitely generated free A-modules $L^\bullet$ such that $H^i(L^\bullet) \cong H^i(C^\bullet)$ for all $i \geqslant 0$. Hence $H^i(L^\bullet) \cong H^i(V, \mathscr{F})$ for all $i \geqslant 0$. □

Lemma 4.2.4. *Let B be a noetherian ring, and let A be a B-algebra of finite type. Let $C^\bullet$ and $L^\bullet$ be two complexes, bounded from above, of A-modules such that C^j and L^j are B-flat for all j. Assume that there is a quasi-isomorphism $L^\bullet \to C^\bullet$. Then for any B-module R and any $i \geqslant 0$, there is a natural isomorphism $H^i(R \otimes_B L^\bullet) \cong H^i(R \otimes_B C^\bullet)$.*

Proof. It is enough to prove the lemma for finitely generated B-modules since every B-module is the direct limit of finitely generated modules and both $\otimes$ and H^i commute with direct limits.

We will prove the lemma by induction on i. For i large enough, C^i and L^i are both 0. Hence the lemma is true for large enough i. Assume that $H^{i+1}(R \otimes_B L^\bullet) \cong H^{i+1}(R \otimes_B C^\bullet)$ for any finitely generated B-module R, we will show that $H^i(R \otimes_B L^\bullet) \cong H^i(R \otimes_B C^\bullet)$ for any finitely generated B-module R. Fix a finitely generated B-module R, then there is a free B-module E and a finitely generated B-module K such that we have an exact sequence $0 \to K \to E \to R \to 0$. Since C^j and L^j are B-flat for all j, we get a commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K \otimes L^\bullet & \longrightarrow & E \otimes L^\bullet & \longrightarrow & R \otimes L^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K \otimes C^\bullet & \longrightarrow & E \otimes C^\bullet & \longrightarrow & R \otimes C^\bullet & \longrightarrow & 0 \end{array}$$

By taking the cohomology, we obtain the following diagram with exact rows,

$$\begin{array}{ccccccccc} H^i(K\otimes L^\bullet) & \longrightarrow & H^i(E\otimes L^\bullet) & \longrightarrow & H^i(R\otimes L^\bullet) & \longrightarrow & H^{i+1}(K\otimes L^\bullet) & \longrightarrow & H^{i+1}(E\otimes L^\bullet) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^i(K\otimes C^\bullet) & \longrightarrow & H^i(E\otimes C^\bullet) & \longrightarrow & H^i(R\otimes C^\bullet) & \longrightarrow & H^{i+1}(K\otimes C^\bullet) & \longrightarrow & H^{i+1}(E\otimes C^\bullet) \end{array}$$

Since E is free, and $L^\bullet$ and $C^\bullet$ are quasi-isomorphic, the morphism from $H^i(E\otimes L^\bullet)$ to $H^i(E\otimes C^\bullet)$ is an isomorphism. By induction hypothesis, the last two vertical morphisms in the last diagram are isomorphisms. Hence $H^i(R\otimes_B L^\bullet)\to H^i(R\otimes_B C^\bullet)$ is surjective. Since R is chosen arbitrarily, this implies that the first vertical morphism is also surjective. Hence

$$H^i(R\otimes_B L^\bullet)\cong H^i(R\otimes_B C^\bullet).$$

This completes the induction and the proof of the lemma □

We will now prove Proposition 4.2.2.

Proof of Proposition 4.2.2. Without loss of generality, we may assume that $T=\operatorname{Spec} B$ is affine. First we assume that $W=\operatorname{Spec} A$ is affine. Let (U_i) be a finite affine open cover of V. Let $C^\bullet$ be the Čech complex of $\mathscr{F}$ with respect to (U_i). Let $L^\bullet$ be the complex of finitely generated free A-modules given by Lemma 4.2.3. Then $H^i(L^\bullet)\cong H^i(V,\mathscr{F})$ for all $i\geqslant 0$. There is a non-empty open affine subset $T_0=\operatorname{Spec} B$ of T such that for any point $t\in T_0$ and for any $i\geqslant 0$, we have $H^i(A_t\otimes L^\bullet)\cong H^i(L^\bullet)\otimes A_t$ (See [Gro66, Cor. 9.4.3]), where A_t is the structure ring of the affine scheme W_t.

Since $V\to W$ is a separated morphism, the intersection of any two affine open subsets of V is still affine. Hence for any j, both C^j and L^j are finitely generated modules over finitely generated B-algebras. Moreover, both $C^\bullet$ and $L^\bullet$ are bounded. Hence by shrinking T_0, we may assume that both C^j and L^j are B-flat for any integer j (See [Gro65, Thm. 6.9.1]). By Lemma 4.2.4, this implies that

$$H^i(A_t\otimes C^\bullet)\cong H^i(A_t\otimes L^\bullet)\cong H^i(L^\bullet)\otimes A_t\cong H^i(C^\bullet)\otimes A_t.$$

Note that $A_t\otimes C^\bullet$ is the Čech complex of $\mathscr{F}_t$ with respect to $((U_i)_t)$, where $(U_i)_t$ is the fiber of U_i over $t\in T_0$. Hence, we have

$$H^i(V,\mathscr{F})\otimes A_t\cong H^i(C^\bullet)\otimes A_t\cong H^i(A_t\otimes C^\bullet)\cong H^i(V_t,\mathscr{F}_t).$$

Since $R^ig_*\mathscr{F}$ (resp. $R^i(g_t)_*\mathscr{F}_t$) is the coherent sheaf associated to $H^i(V,\mathscr{F})$ (resp. to $H^i(V_t,\mathscr{F}_t)$), we obtain that for all $t\in T_0$, $(R^ig_*\mathscr{F})|_{W_t}\cong R^i(g_t)_*\mathscr{F}_t$.

Now we treat the general case. We cover W by finitely many affine open subsets $W_1,...,W_k$. Let $V_j=g^{-1}(W_j)$, $\mathscr{F}_j=\mathscr{F}|_{V_j}$ and $g_j=g|_{V_j}$ for all j. Then for every $j=1,...,k$, there is a non-empty open subset T_j of T such that for any point $t\in T_j$, we have the base change property for t,

$\mathscr{F}_j$ and $g_j : V_j \to W_j$. Let T_0 be the open set $\bigcap_{j=1}^k W_j$. Then for any $t \in T_0$ and any $i \geqslant 0$, the natural morphism $(R^i g_* \mathscr{F})|_{W_t} \to R^i (g_t)_* \mathscr{F}_t$ is an isomorphism. □

As a corollary of the proposition, we obtain the following result.

Lemma 4.2.5. *Let $g : V \to W$ be a projective morphism of integral T-schemes of finite type, where T is an integral noetherian scheme. Let η be the generic point of T. Let $\mathscr{F}$ be a coherent sheaf on V. Let i, j be two non-negative integers. If $h^j(W_t, R^i(g_t)_* \mathscr{F}_t) = 0$ for general closed point $t \in T$, then $h^j(W_\eta, R^i(g_\eta)_* \mathscr{F}_\eta) = 0$.*

Proof. For a general point $t \in T$, we have $R^i(g_t)_* \mathscr{F}_t \cong (R^i g_* \mathscr{F})|_{W_t}$ by Proposition 4.2.2. Hence $H^j(W_t, (R^i g_* \mathscr{F})|_{W_t}) = \{0\}$ for general closed point $t \in T$. Since $R^i g_* \mathscr{F}$ is a coherent sheaf on W (See [Har77, Thm. III.5.2]), by shrinking T, we may assume that $R^i g_* \mathscr{F}$ is flat over T (See [Gro65, Thm. 6.9.1]). By [Har77, Cor. III.12.9], we obtain $R^j h_* (R^i g_* \mathscr{F}) = 0$, where h is the morphism from W to T. By [Har77, Cor. III.12.9] again, we have $h^j(W_\eta, (R^i g_* \mathscr{F})|_{W_\eta}) = 0$. From Proposition 4.2.2, we conclude that

$$h^j(W_\eta, R^i(g_\eta)_* \mathscr{F}_\eta) = 0.$$

□

4.2.3 Reflexivity of higher direct images

Consider a Lagrangian fibration f from a smooth complex projective symplectic variety M to a projective variety X. Let H be a Weil divisor on X, and let $D = f^*H$. We will show the that the sheaf $R^i f_* \mathscr{O}_M(D)$ is reflexive and is isomorphic to $\Omega^i_X[\otimes]\mathscr{O}_X(H)$ for all $i \geqslant 0$.

Proposition 4.2.6. *Let f be a projective Lagrangian fibration from a complex smooth quasi-projective symplectic variety M to a variety X with $\mathbb{Q}$-factorial klt singularities. Let H be a Weil divisor on X, and let $D = f^*H$. Then $R^i f_* \mathscr{O}_M(D)$ is isomorphic to $\Omega^i_X[\otimes]\mathscr{O}_X(H)$ for all $i \geqslant 0$.*

Moreover, let B be a $\mathbb{Z}$-subalgebra of finite type of $\mathbb{C}$. Then $T = \operatorname{Spec} B$ is integral. Let η be its generic point. Assume that $f : M \to X$, H and D are defined over B. There is a morphism $\varphi : \mathscr{M} \to \mathscr{X}$ of T-schemes given by the equations defining $f : M \to X$. There is a divisor $\mathscr{H}$ on $\mathscr{X}$ given by the equations defining H. Let $\mathscr{D} = \varphi^* \mathscr{H}$. Then we will prove that for general closed point $t \in T$, the sheaf $R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t)$ is reflexive and isomorphic to $R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}[\otimes]\mathscr{O}_{\mathscr{X}_t}(\mathscr{H}_t)$ for all $i \geqslant 0$ (see Corollary 4.2.8).

We will first prove Theorem 4.1.3 which implies that $R^i f_* \mathscr{O}_M(D)$ is reflexive.

Proof of Theorem 4.1.3. Set $m = \dim V$ and $k = \dim W$.

First we assume that W is Gorenstein and H is Cartier. Then by the projection formula, we have $R^i g_* \mathscr{O}_V(D) \cong R^i g_* \mathscr{O}_V \otimes \mathscr{O}_W(H)$ for all $i \geqslant 0$. Hence it is enough to prove that $R^i g_* \mathscr{O}_V$ is reflexive for all i. Let $W \hookrightarrow \bar{W}$ be a projective compactification of W such that $\bar{W}$ is normal. Let $\bar{g} : \bar{V} \to \bar{W}$ be a compactification of g such that $\bar{V}$ is smooth. As in the proof of [Kol86b, Cor. 3.9], the torsion-free part of $R^i \bar{g}_* \mathscr{O}_{\bar{V}}$ is $\mathscr{H}om(R^{m-k-i}\bar{g}_* \omega_{\bar{V}},\ \omega_{\bar{W}})$. Since $\mathscr{O}_V \cong \omega_V$, we have $(R^i \bar{g}_* \mathscr{O}_{\bar{V}})|_W \cong (R^i \bar{g}_* \omega_{\bar{V}})|_W$. Since $R^i \bar{g}_* \omega_{\bar{V}}$ is torsion-free by [Kol86a, Thm. 2.1], we obtain that $(R^i \bar{g}_* \mathscr{O}_{\bar{V}})|_W$ is torsion free. Hence we have

$$\begin{aligned} R^i g_* \mathscr{O}_V \cong (R^i \bar{g}_* \mathscr{O}_{\bar{V}})|_W &\cong \mathscr{H}om(R^{m-k-i}\bar{g}_* \omega_{\bar{V}},\ \omega_{\bar{W}})|_W \\ &= \mathscr{H}om(R^{m-k-i} g_* \omega_V,\ \omega_W). \end{aligned}$$

Since W is Gorenstein, ω_W is an invertible sheaf. This implies that the sheaf $\mathscr{H}om(R^{m-k-i} g_* \omega_V,\ \omega_W)$ is reflexive. Hence $R^i g_* \mathscr{O}_V$ is reflexive.

Now we will treat the general case. Let w be a closed point in W. Then the problem is local around w. Hence we may replace W by any open neighbourhood of w. As in [KM98, Def. 5.19], by shrinking W if necessary, there is a finite cover $p : W' \to W$ which is étale in codimension 1 such that both $K_{W'} = p^* K_W$ and $H' = p^* H$ are Cartier. Let V' be the normalisation of $V \times_W W'$. Let $g' : V' \to W'$ and $q : V' \to V$ be the natural projections. Write $D' = q^* D$.

$$\begin{array}{ccc} V' & \xrightarrow{q} & V \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ W' & \xrightarrow{p} & W \end{array}$$

Since p is étale in codimension 1, V is smooth, and g is equidimensional, the morphism q is also étale in codimension 1. By the Zariski purity theorem (See [Zar58, Prop. 2]), this implies that q is étale. Hence V' is smooth, and $\mathscr{O}_{V'} \cong \omega_{V'}$. Hence $Rg'_* \mathscr{O}_{V'}(D')$ is reflexive. Hence $p_*(Rg'_* \mathscr{O}_{V'}(D'))$ is reflexive since p is finite (See [Har80, Cor. 1.7]). On the one hand, since q is finite, from the Leray spectral sequence, we know that $R^i g_*(q_* \mathscr{O}_{V'}(D'))$ is isomorphic to $p_*(R^i g'_* \mathscr{O}_{V'}(D'))$, which is reflexive. On the other hand, since $\mathscr{O}_V$ is a direct summand of $q_* \mathscr{O}_{V'}$, the sheaf $\mathscr{O}_V(D)$ is a direct summand of $q_* \mathscr{O}_{V'}(D')$ by the projection formula. Hence $R^i g_* \mathscr{O}_V(D)$, as a direct summand of the reflexive sheaf $R^i g_*(q_* \mathscr{O}_{V'}(D'))$, is reflexive on W. □

Now we will complete the proof of Proposition 4.2.6.

Proof of Proposition 4.2.6. As in [Mat05, 2.12], there is an open subset U in X which has the following properties

(1) $\operatorname{codim} X \backslash U \geqslant 2$;

(2) $\bigwedge^i ((R^1 f_* \mathscr{O}_M)|_U) \cong (R^i f_* \mathscr{O}_M)|_U$ for all $i > 0$;

(3) $(R^1 f_* \mathscr{O}_M)|_U \cong \Omega^1_U$.

Since $R^i f_* \mathscr{O}_M$ is reflexive by Theorem 4.1.3, and codim $X \backslash U \geqslant 2$, we have $R^i f_* \mathscr{O}_M \cong \Omega_X^{[i]}$ for all $i > 0$ by Proposition 1.2.1. From the projection formula, $R^i f_* \mathscr{O}_M(D)|_{X_0}$ is isomorphic to $(\Omega_X^{[i]}[\otimes]\mathscr{O}_X(H))|_{X_0}$, where X_0 is the smooth locus of X. Since X is smooth in codimension 1, from Proposition 1.2.1, we conclude that $R^i f_* \mathscr{O}_M(D) \cong \Omega^i_X[\otimes]\mathscr{O}_X(H)$. □

Now let T be an integral affine noetherian scheme with the generic point η. Consider a morphism $g : V \to W$ of normal integral T-schemes of finite type. Let $\mathscr{F}$ be a coherent sheaf on V. In the remainder of this subsection, we will prove the following proposition.

Proposition 4.2.7. *With the notation as above, assume that $R^i(g_\eta)_* \mathscr{F}_\eta$ is reflexive. Then there is a non-empty open subset T_0 of T such that for any $t \in T_0$ and any $i \geqslant 0$, the sheaf $R^i(g_t)_* \mathscr{F}_t$ is reflexive.*

One of the applications of Proposition 4.2.7 is the following corollary.

Corollary 4.2.8. *Let $f : M \to X$ be a Lagrangian fibration from a complex projective symplectic manifold M to a normal variety X. Let H be a Weil divisor on X, and let $D = f^*H$. Let A be a subalgebra of $\mathbb{C}$ such that f, H and D are defined over A. Let $T = \operatorname{Spec} A$, and let η be its generic point. The equations defining $f : M \to X$, H and D induce a morphism $\varphi : \mathscr{M} \to \mathscr{X}$ of T-schemes, and two divisors $\mathscr{H}$ and $\mathscr{D}$ on $\mathscr{X}$ and $\mathscr{M}$ respectively. Then for general $t \in T$ and any $i \geqslant 0$, the sheaf $R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t)$ is reflexive, and $R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t) \cong R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}[\otimes]\mathscr{O}_{\mathscr{X}_t}(\mathscr{H}_t)$.*

Proof of Corollary 4.2.8. From Proposition 4.2.6, we know that $R^i f_* \mathscr{O}_M(D)$ is reflexive for all i. Note that φ_η, $\mathscr{H}_\eta$, and $\mathscr{D}_\eta$ coincide with f, H and D after a field extension. Since a field extension is a faithfully flat base change, by [Har77, Prop. III.9.3] and Lemma 4.2.9 below, $R^i(\varphi_\eta)_* \mathscr{O}_{\mathscr{M}_\eta}(\mathscr{D}_\eta)$ is also reflexive for all i. From Proposition 4.2.7, we obtain that for general $t \in T$ and any $i \geqslant 0$, $R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t)$ is reflexive.

By the projection formula, if $(\mathscr{X}_t)_0$ is the smooth locus of $\mathscr{X}_t$, then

$$(R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t))|_{(\mathscr{X}_t)_0} \cong (R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}[\otimes]\mathscr{O}_{\mathscr{X}_t}(\mathscr{H}_t))|_{(\mathscr{X}_t)_0}.$$

Since $\mathscr{X}_\eta$ is normal, $\mathscr{X}_t$ is normal for general $t \in T$ (See [Gro66, Prop. 9.9.4]). Hence for general $t \in T$, $R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t) \cong R^i(\varphi_t)_* \mathscr{O}_{\mathscr{M}_t}[\otimes]\mathscr{O}_{\mathscr{X}_t}(\mathscr{H}_t)$ by Proposition 1.2.1. □

Lemma 4.2.9. *Let $f : V \to W$ be a faithfully flat morphism of noetherian integral schemes. Let $\mathscr{F}$ be a coherent sheaf on W. Then $\mathscr{F}$ is reflexive if and only if $f^*\mathscr{F}$ is reflexive.*

Proof. As in the proof of [Har80, Prop 1.8], we have $f^*(\mathscr{F}^{**}) \cong (f^*\mathscr{F})^{**}$ for f is flat. Since f is faithfully flat, the natural morphism $\mathscr{F} \to \mathscr{F}^{**}$ is an isomorphism if and only if $f^*(\mathscr{F}) \to f^*(\mathscr{F}^{**})$ is. Hence $\mathscr{F}$ is reflexive if and only if $f^*\mathscr{F} \to (f^*\mathscr{F})^{**}$ is an isomorphism, that is, if and only if $f^*\mathscr{F}$ is reflexive. □

In order to prove Proposition 4.2.7, we will need several lemmas. The following lemma gives a criterion of reflexivity for coherent sheaves.

Lemma 4.2.10. *A coherent sheaf $\mathscr{F}$ on an integral noetherian scheme V is reflexive if and only if it fits in an exact sequence $0 \to \mathscr{F} \to \mathscr{J} \to \mathscr{K}$, where $\mathscr{J}$ and $\mathscr{K}$ are locally free.*

Proof. If we have such an exact sequence, then by [Har80, Prop. 1.1], $\mathscr{F}$ is reflexive since $\mathscr{K}$ is torsion-free.

Conversely, assume that $\mathscr{F}$ is reflexive. There are two locally free sheaves $\mathscr{J}$, $\mathscr{K}$ and an exact sequence $0 \to \mathscr{F}^{**} \to \mathscr{J} \to \mathscr{K}$ (See [Har80, Prop. 1.1]). Since $\mathscr{F}$ is reflexive, we have $\mathscr{F} \cong \mathscr{F}^{**}$. Hence we obtain an exact sequence $0 \to \mathscr{F} \to \mathscr{J} \to \mathscr{K}$. □

In the three lemmas below, we prove that reflexivity is a "constructible" property (Lemma 4.2.13).

Lemma 4.2.11. *Let $T = \operatorname{Spec} B$ be an integral affine noetherian scheme. Let $V \to T$ be an integral T-scheme of finite type. Let η be the generic point of T. Let $\alpha^\eta : \mathscr{F}^\eta \to \mathscr{G}^\eta$ be a morphism of coherent sheaves on V_η. Then there is an open subset T_0 of T and a morphism $\alpha : \mathscr{F} \to \mathscr{G}$ of coherent sheaves on V_{T_0} such that $\mathscr{F}_\eta = \mathscr{F}^\eta$, $\mathscr{G}_\eta = \mathscr{G}^\eta$, and $\alpha_\eta = \alpha^\eta$. Moreover, if $\mathscr{F}^\eta$ (resp. $\mathscr{G}^\eta$) is locally free, then $\mathscr{F}$ (resp. $\mathscr{G}$) can be chosen to be locally free.*

The idea of the proof is as follows. There are only finitely many equations which define α^η, $\mathscr{F}^\eta$ and $\mathscr{G}^\eta$. Then there is a localisation B_0 of B such that all of these equations are defined over B_0. These equations define $\alpha : \mathscr{F} \to \mathscr{G}$ over V_{T_0}, where $T_0 = \operatorname{Spec} B_0$.

Proof of Lemma 4.2.11. Let K be the fraction field of B. Note that V is a noetherian scheme. Hence V can be covered by finitely many open affine subsets $U_1, ..., U_k$. Then V_η is covered by open affine subsets $(U_1)_\eta, ..., (U_k)_\eta$. We can cover $U_i \cap U_j$ by finitely many open affine subsets $R_s^{ij} = \operatorname{Spec} C_s^{ij}$. Let A_i be the structure ring of U_i, and let $(A_i)_\eta$ be the structure ring of $(U_i)_\eta$. Then for all i, A_i is a B-algebra of finite type. Since $\mathscr{F}^\eta$ is a coherent sheaf, over each $(U_i)_\eta$, there is an $(A_i)_\eta$-module M_i^η of finite presentation such that $\mathscr{F}^\eta|_{(U_i)_\eta}$ is the coherent sheaf associated to M_i^η. For every i, we have an exact sequence

$$((A_i)_\eta)^{\oplus t_i} \to ((A_i)_\eta)^{\oplus r_i} \to M_i^\eta \to 0$$

where r_i and t_i are some non negative integers. The first morphism θ_i in the sequence is defined by finitely many equations. Since $\mathscr{F}^\eta$ is a coherent sheaf, there is an isomorphism $(\mu_s^{ij})^\eta$ from $M_i^\eta \otimes_B C_s^{ij}$ to $M_j^\eta \otimes_B C_s^{ij}$. Then $(\mu_s^{ij})^\eta$ is given by finitely many equations. There is an open affine subset $T_0 = \operatorname{Spec} B_0$ of T such that all of the equations defining the θ_i's and the $(\mu_s^{ij})^\eta$'s are defined over B_0. Hence, for every i, we have an exact sequence of $(A_i \otimes_B B_0)$-modules, induced by the previous one,

$$(A_i \otimes_B B_0)^{\oplus t_i} \to (A_i \otimes_B B_0)^{\oplus r_i} \to M_i \to 0.$$

The $(A_i \otimes_B B_0)$-module M_i induces a coherent sheaf on $(U_i)_{T_0} = U_i \times_T T_0$. Moreover, since all of the equations defining the $(\mu_s^{ij})^\eta$'s are defined over B_0, the function $(\mu_s^{ij})^\eta$ induces a transition function μ_s^{ij} from $M_i \otimes_B C_s^{ij}$ to $M_j \otimes_B C_s^{ij}$. The modules M_i and the isomorphisms μ_s^{ij} define a coherent sheaf $\mathscr{F}$ on V_{T_0} such that $\mathscr{F}_\eta$ is isomorphic to $\mathscr{F}^\eta$. If $\mathscr{F}^\eta$ is locally free, then $\mathscr{F}$ can also be constructed to be locally free (for example, we can take $t_i = 0$ for all i).

For the same reason as above, by shrinking T_0, we can construct a coherent sheaf $\mathscr{G}$ on V_{T_0} such that $\mathscr{G}_\eta$ is isomorphic to $\mathscr{G}^\eta$. Assume that $\mathscr{G}^\eta|_{(U_i)_\eta}$ is the sheaf associated to a $(A_i)_\eta$-module N_i^η of finite presentation. Then there is a morphism of $(A_i)_\eta$-modules $e_i^\eta : M_i^\eta \to N_i^\eta$ such that $\alpha|_{(U_i)_\eta}$ is associated to e_i^η. Since e_i^η is defined by finitely many equations, by shrinking T_0, we may assume that all of the equations defining the e_i^η's are defined over B_0. From these equations, we obtain morphisms $e_i : M_i \to N_i$. These morphisms induce a morphism $\alpha : \mathscr{F} \to \mathscr{G}$ such that α_η is the same as α^η. □

With the same method, we can obtain the following lemma.

Lemma 4.2.12. *Let $T = \operatorname{Spec} B$ be an integral affine noetherian scheme. Let $V \to T$ be a T-scheme of finite type. Let η be the generic point of T. Let $\mathscr{F}$, $\mathscr{G}$ be coherent sheaves on V. Assume that there is a morphism $\alpha^\eta : \mathscr{F}_\eta \to \mathscr{G}_\eta$ of coherent sheaves on V_η. Then there is an open subset T_0 of T and a morphism $\alpha : \mathscr{F}_{T_0} \to \mathscr{G}_{T_0}$ of coherent sheaves on V_{T_0} which extends α^η.*

Lemma 4.2.13. *Let V be a T-scheme of finite type, where T is an integral noetherian scheme. Let η be the generic point of T. Let $\mathscr{F}$ be a coherent sheaf on V such that $\mathscr{F}_\eta$ is a reflexive sheaf on V_η. Then for general $t \in T$, $\mathscr{F}_t$ is reflexive on V_t.*

Proof. We may assume that T is affine. Since $\mathscr{F}_\eta$ is reflexive, by Lemma 4.2.10 there are two locally free sheaves $\mathscr{J}^\eta$, $\mathscr{K}^\eta$ such that we have an exact sequence $0 \to \mathscr{F}_\eta \xrightarrow{\beta^\eta} \mathscr{J}^\eta \xrightarrow{\alpha^\eta} \mathscr{K}^\eta$. By Lemma 4.2.11, there is an open subset T_1 of T and a morphism $\alpha : \mathscr{J} \to \mathscr{K}$ of locally free sheaves on V_{T_1}, such that $\mathscr{J}_\eta = \mathscr{J}^\eta$, $\mathscr{K}_\eta = \mathscr{K}^\eta$, and $\alpha_\eta = \alpha^\eta$. From Lemma 4.2.12, by

shrinking T_1, we may assume that there is a morphism $\beta : \mathscr{F}_{T_1} \to \mathscr{J}$ such that $\beta_\eta = \beta^\eta$.

Consider the morphisms $\beta : \mathscr{F}_{T_1} \to \mathscr{J}$ and $\alpha : \mathscr{J} \to \mathscr{K}$. By [Gro66, Prop. 9.4.4], the set T_0 of points $t \in T_1$ such that the sequence $0 \to \mathscr{F}_t \to \mathscr{J}_t \to \mathscr{K}_t$ is exact is constructible. Moreover, we know that $\eta \in T_0$. Hence T_0 contains an open subset of T. For any $t \in T_0$, $\mathscr{J}_t$ and $\mathscr{K}_t$ are locally free. Hence for any $t \in T_0$, the sheaf $\mathscr{F}_t$ is reflexive on V_t by Lemma 4.2.10. This completes the proof of the lemma. □

Combining Lemma 4.2.13 and Proposition 4.2.2, we can prove Proposition 4.2.7.

Proof of Proposition 4.2.7. From Proposition 4.2.2, there is an open subset T_1 of T such that for any $t \in T_1$ and any $i \geqslant 0$, we have $R^i(g_t)_*\mathscr{F}_t \cong (R^i g_*\mathscr{F})|_{W_t}$. By hypothesis, $R^i(g_\eta)_*\mathscr{F}_\eta$ is reflexive. By Proposition 4.2.2, this implies that $(R^i g_*\mathscr{F})|_{W_\eta}$ is reflexive. Hence by Lemma 4.2.13, there is an open subset T_0 of T_1 such that for all $t \in T_0$, the sheaf $(R^i g_*\mathscr{F})|_{W_t}$ is reflexive. Hence for all $t \in T_0$, $R^i(g_t)_*\mathscr{F}_t$ is reflexive. □

4.2.4 Vanishing in positive characteristic

We will prove a vanishing lemma (Lemma 4.2.15) in positive characteristic. If Y is a variety defined over a perfect field of positive characteristic (for example, a finite field), then Y is called Frobenius-split if $\mathscr{O}_Y$ is a direct summand of $F_{abs*}\mathscr{O}_Y$, where $F_{abs} : Y \to Y$ is the absolute Frobenius morphism (See [BK05, §1.1] for more details).

Lemma 4.2.14. *Let Y be a normal projective variety. Let $\mathscr{F}$ be a reflexive sheaf on Y such that $\mathscr{F}^{[\otimes m]}$ is a very ample invertible sheaf for some $m > 0$. Let $\mathscr{G}$ be a coherent sheaf on Y. Then there is an integer l such that $h^i(Y, \mathscr{G}[\otimes]\mathscr{F}^{[\otimes t]}) = 0$ for all $t \geqslant l$ and $i > 0$.*

Proof. Since $\mathscr{F}^{[\otimes m]}$ is invertible, $\mathscr{F}^{[\otimes ms]} \cong (\mathscr{F}^{[\otimes m]})^{\otimes s}$ for all integer s (See Proposition 1.2.1). For any integer r between 0 and $m-1$, there is an integer $l_r > 0$ such that $H^i(Y, (\mathscr{G}[\otimes]\mathscr{F}^{[\otimes r]}) \otimes \mathscr{F}^{[\otimes ms]}) = \{0\}$ for all $i > 0$ and $s \geqslant l_r$ (See [Har77, Thm. III.5.2]). Let $l = m(l_0 + \cdots + l_{m-1})$, then for any integer $t \geqslant l$, we have

$$\mathscr{G}[\otimes]\mathscr{F}^{[\otimes t]} = (\mathscr{G}[\otimes]\mathscr{F}^{[\otimes (t - m[\frac{t}{m}])]}) \otimes \mathscr{F}^{[\otimes (m[\frac{t}{m}])]}$$

with $0 \leqslant t - m[\frac{t}{m}] \leqslant m-1$, and $[\frac{t}{m}] \geqslant l_0 + \cdots + l_{m-1}$. Hence for all $t \geqslant l$ and $i > 0$, we have $h^i(Y, \mathscr{G}[\otimes]\mathscr{F}^{[\otimes t]}) = 0$. □

Lemma 4.2.15. *Let $g : Y \to Z$ be a morphism of normal projective varieties over a field of characteristic $p > 0$. Let L be a Weil divisor on Z such that mL is an ample Cartier divisor for some $m > 0$. Assume that Y is Frobenius-split. Then $h^j(Z, R^i g_*\mathscr{O}_Y[\otimes]\mathscr{O}_Z(L)) = 0$ for all $j > 0$ and $i \geqslant 0$.*

Proof. Fix $j > 0$ and $i \geqslant 0$. Let $F_Y^e : Y \to Y$ and $F_Z^e : Z \to Z$ be the compositions of e absolute Frobenius morphisms for $e > 0$. Then $\mathscr{O}_Y$ is a direct summand of $(F_Y^e)_*\mathscr{O}_Y$ since Y is Frobenius-split. Hence we only have to prove $h^j(Z, R^i g_*((F_Y^e)_*\mathscr{O}_Y)[\otimes]\mathscr{O}_Z(L)) = 0$ for some $e > 0$. Since F_Y^e and F_Z^e are finite morphisms, from the Leray spectral sequence, we have

$$R^i g_*((F_Y^e)_*\mathscr{O}_Y) \cong (F_Z^e)_*(R^i g_*\mathscr{O}_Y).$$

Moreover, if Z_0 is the smooth locus of Z, then $L|_{Z_0}$ is a Cartier divisor, and $((F_Z^e)^*L)|_{Z_0} = (p^e L)|_{Z_0}$. Hence by the projection formula,

$$((F_Z^e)_*(R^i g_*\mathscr{O}_Y \otimes \mathscr{O}_Z(p^e L)))|_{Z_0} \cong (((F_Z^e)_*(R^i f_*\mathscr{O}_Y)) \otimes \mathscr{O}_Z(L))|_{Z_0}.$$

Since $(F_Z^e)_*(R^i g_*\mathscr{O}_Y[\otimes]\mathscr{O}_Z(p^e L))$ is a reflexive sheaf on Z (See [Har80, Cor. 1.7]), and Z is smooth in codimension 1, by Proposition 1.2.1, we obtain

$$(F_Z^e)_*(R^i g_*\mathscr{O}_Y[\otimes]\mathscr{O}_Z(p^e L)) \cong ((F_Z^e)_*(R^i g_*\mathscr{O}_Y))[\otimes]\mathscr{O}_Z(L).$$

Since

$$H^j(Z, R^i g_*((F_Y^e)_*\mathscr{O}_Y)[\otimes]\mathscr{O}_Z(L)) \cong H^j(Z, ((F_Z^e)_*(R^i g_*\mathscr{O}_Y))[\otimes]\mathscr{O}_Z(L)),$$

we have

$$H^j(Z, R^i g_*((F_Y^e)_*\mathscr{O}_Y)[\otimes]\mathscr{O}_Z(L)) \cong H^j(Z, (F_Z^e)_*(R^i g_*\mathscr{O}_Y[\otimes]\mathscr{O}_Z(p^e L)))$$

Since F_Z^e is a finite morphism, the Leray spectral sequence shows that the right-hand side is isomorphic to $H^j(Z, R^i g_*\mathscr{O}_Y[\otimes]\mathscr{O}_Y(p^e L))$. Thus it is enough to prove that

$$h^j(Z, R^i g_*\mathscr{O}_Y[\otimes]\mathscr{O}_Z(p^e L)) = 0.$$

for some $e > 0$. This is true by Lemma 4.2.14. □

4.2.5 Proof of Theorem 4.1.4

By the vanishing lemma in positive characteristic, we can prove Theorem 4.1.4 in the special case when everything is defined over an algebraic number field.

Lemma 4.2.16. *Let $f : M \to X$, H and D be as in Theorem 4.1.4. Assume further that everything is defined over an algebraic number field K. Then $h^j(X, R^i f_*\mathscr{O}_M(D)) = 0$ for all $j > 0$ and $i \geqslant 0$.*

Proof. Since X has $\mathbb{Q}$-factorial klt singularities (See Theorem 4.1.1), by Proposition 4.2.6, $R^i f_*\mathscr{O}_M(D)$ is reflexive on X for all $i \geqslant 0$.

There is a $\mathbb{Z}$-algebra of finite type $A \subseteq K$ of Krull dimension 1 such that the field of fraction of A is K, and the coefficients of equations defining

$f : M \to X$, H and D are contained in A. Let $T = \operatorname{Spec} A$, and let η be its generic point. Let $\varphi : \mathscr{M} \to \mathscr{X}$ be the morphism of T-schemes given by the equations defining $f : M \to X$. Fix some $i \geqslant 0$ and some $j > 0$.

Let $\mathscr{D}$ and $\mathscr{H}$ be the divisors on $\mathscr{M}$ and $\mathscr{X}$ given by the equations defining D and H. By shrinking T if necessary, we may assume that they are $\mathbb{Q}$-Cartier. Then for a general closed point $t \in T$, $\mathscr{M}_t$ is a smooth symplectic variety defined over the residue field of t, which is a finite field. Moreover, since $\mathscr{X}_\eta$ is normal, by [Gro66, Prop. 9.9.4], $\mathscr{X}_t$ is normal for general t. From [Dru12, Thm. 1.1 and Lem. 3.3], we know that $\mathscr{M}_t$ is Frobenius-split for general closed point $t \in T$. Hence, by Lemma 4.2.15, we have $h^j(\mathscr{X}_t, R^i(\varphi_t)_*\mathscr{O}_{\mathscr{M}_t}[\otimes]\mathscr{O}_{\mathscr{X}_t}(\mathscr{H}_t)) = 0$ for general closed point $t \in T$. By Corollary 4.2.8, this shows that $h^j(\mathscr{X}_t, R^i(\varphi_t)_*\mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t)) = 0$.

From Lemma 4.2.5, we obtain $h^j(\mathscr{X}_\eta, R^i(\varphi_\eta)_*\mathscr{O}_{\mathscr{M}_\eta}(\mathscr{D}_\eta)) = 0$. Note that $\varphi_\eta : \mathscr{M}_\eta \to \mathscr{X}_\eta$ coincides with $f : M \to X$ after a field extension. Since a field extension is a flat base change, by [Har77, Prop.III.9.3] we have $R^i f_*\mathscr{O}_M(D) \cong \mu^*(R^i(\varphi_\eta)_*\mathscr{O}_{\mathscr{M}_\eta}(\mathscr{D}_\eta))$, where $\mu : X \to \mathscr{X}_\eta$ is the natural morphism. By [Har77, Prop.III.9.3] again, we have

$$h^j(X, R^i f_*\mathscr{O}_M(D)) = h^j(\mathscr{X}_\eta, R^i(\varphi_\eta)_*\mathscr{O}_{\mathscr{M}_\eta}(\mathscr{D}_\eta)) = 0.$$

We also obtain that $h^j(X, \Omega_X^{[i]}[\otimes]\mathscr{O}_X(H)) = 0$ by Proposition 4.2.6. □

Now we will complete the proof of Theorem 4.1.4.

Proof of Theorem 4.1.4. Fix some $j > 0$ and some $i \geqslant 0$. Let $\bar{\mathbb{Q}}$ be the field of algebraic numbers over $\mathbb{Q}$. Then there is a $\bar{\mathbb{Q}}$-algebra $B \subseteq \mathbb{C}$ of finite type which contains all the coefficients of the equations defining $f : M \to X$, D and H. By Lemma 4.2.16, we may assume that $B \neq \bar{\mathbb{Q}}$.

Let $T = \operatorname{Spec} B$ and let η be its generic point. Then the equations induce a morphism $\varphi : \mathscr{M} \to \mathscr{X}$ of T-schemes, a divisor $\mathscr{H}$ on $\mathscr{X}$ and a divisor $\mathscr{D}$ on $\mathscr{M}$. Since $R^i\varphi_*\mathscr{O}_{\mathscr{M}}(\mathscr{D})$ is a coherent sheaf on $\mathscr{X}$, by shrinking T we may assume that this sheaf is flat over T (See [Gro65, Thm. 6.9.1]).

For general closed point $t \in T$, φ_t is a Lagrangian fibration from a smooth variety $\mathscr{M}_t$ to a normal variety $\mathscr{X}_t$. In addition, for general closed point t, we know that φ_t, $\mathscr{H}_t$ and $\mathscr{D}_t$ are all defined over an algebraic number field since the residue field of t is just $\bar{\mathbb{Q}}$. Hence by Lemma 4.2.16, we have $h^j(\mathscr{X}_t, R^i(\varphi_t)_*\mathscr{O}_{\mathscr{M}_t}(\mathscr{D}_t)) = 0$. By Lemma 4.2.5, this shows that $h^j(\mathscr{X}_\eta, R^i(\varphi_\eta)_*\mathscr{O}_{\mathscr{M}_\eta}(\mathscr{D}_\eta)) = 0$. Note that $\varphi_\eta : \mathscr{M}_\eta \to \mathscr{X}_\eta$ coincides with $f : M \to X$ after a field extension. Since a field extension is a flat base change, we have $H^j(X, R^i f_*\mathscr{O}_M(D)) = \{0\}$ (See [Har77, Prop.III.9.3]).

By Proposition 4.2.6 and the Leray spectral sequence, this proves the theorem. □

Remark 4.2.17. If we assume that $\dim M = 4$ and $\dim X = 2$, then we can prove Theorem 4.1.4 without using reduction modulo p. In fact, we only

have to prove that $h^j(X, R^i f_* \mathscr{O}_M(D)) = 0$ for all $j > 0$ and $i \geqslant 0$. Since $\omega_M \cong \mathscr{O}_M$, this is true by the following lemma.

Lemma 4.2.18. *Let $g : V \to W$ be a projective equidimensional fibration from a smooth projective variety to a normal projective surface. Let H be a $\mathbb{Q}$-ample Weil divisor on W, and let $D = g^*H$. Then*

$$h^j(X, R^i g_*(\mathscr{O}_V(D) \otimes \omega_V)) = 0$$

for all $j > 0$ and $i \geqslant 0$.

Proof. Let k be a positive integer such that kH is very ample. Then, since W is a surface, there is a prime Cartier divisor B on W such that

(1) $\mathscr{O}_W(B) \cong \mathscr{O}_W(kH)$

(2) B is smooth and is contained in the smooth locus of W

(3) the divisor $A = f^*B$ is smooth.

Let $p : W' \to W$ be the normalisation of the cyclic cover with respect to $\mathscr{O}_W(B) \cong (\mathscr{O}_W(H))^{\otimes k}$, and let $q : V' \to V$ be the cyclic cover with respect to $\mathscr{O}_V(A) \cong (\mathscr{O}_V(D))^{\otimes k}$ (See [KM98, Def. 2.52]). Then V' is smooth, W' is smooth along $p^{-1}(B)$, and there is a projective equidimensional fibration $g' : V' \to W'$ induced by g. We obtain the following commutative diagram.

$$\begin{array}{ccc} V' & \xrightarrow{q} & V \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ W' & \xrightarrow{p} & W \end{array}$$

Let $D' = q^*D$ and $H' = p^*H$. Then H' is linearly equivalent to the support of p^*B which is a Cartier divisor. Since H' is also ample, by the projection formula and [Kol86a, Thm. 2.1], we have

$$h^j(W', R^i g'_*(\mathscr{O}_{V'}(D') \otimes \omega_{V'})) = h^j(W', \mathscr{O}_{W'}(H') \otimes R^i g'_* \omega_{V'}) = 0$$

for all $j > 0$ and $i \geqslant 0$.

Since q and p are finite morphisms, by the Leray spectral sequence, we have

$$R^i g_*(q_*(\mathscr{O}_{V'}(D') \otimes \omega_{V'})) \cong p_*(R^i g'_*(\mathscr{O}_{V'}(D') \otimes \omega_{V'}))$$

and

$$h^j(W, p_*(R^i g'_*(\mathscr{O}_{V'}(D') \otimes \omega_{V'}))) = h^j(W', R^i g'_*(\mathscr{O}_{V'}(D') \otimes \omega_{V'})) = 0$$

for all $j > 0$ and $i \geqslant 0$. Thus

$$h^j(W, R^i g_*(q_*(\mathscr{O}_{V'}(D') \otimes \omega_{V'}))) = h^j(W, p_*(R^i g'_*(\mathscr{O}_{V'}(D') \otimes \omega_{V'}))) = 0$$

for all $j > 0$ and $i \geqslant 0$. Note that $q_* \omega_{V'} = \omega_V \otimes (\sum_{r=0}^{k-1} \mathscr{O}_V(rD))$. By the projection formula, we obtain that $\mathscr{O}_V(D) \otimes \omega_V$ is a direct summand of $q_*(\mathscr{O}_{V'}(D') \otimes \omega_{V'})$. Hence, $h^j(W, R^i g_*(\mathscr{O}_V(D) \otimes \omega_V)) = 0$ for all $j > 0$ and $i \geqslant 0$. □

4.3 Rational surfaces

In this section, we work over $\mathbb{C}$. A point in a variety will always refer to a closed point. The aim is to prove Theorem 4.1.5. Let X be a surface satisfying the conditions in Theorem 4.1.5 and let T be a minimal curve in X, that is, a curve which has minimal intersection number with $-K_X$. Then we have the following lemma.

Lemma 4.3.1. *Let X be a Fano surface with klt singularities whose smooth locus is algebraically simply connected. Assume that X has Picard number 1, and that $h^0(X, \mathscr{O}_X(H)) + h^0(X, \mathscr{O}_X(H + K_X)) \geqslant 3$ for all $\mathbb{Q}$-ample Weil divisor H. Let T be a minimal curve in X. Then the Weil divisor class group $\mathrm{Cl}(X)$ is generated by the class of T.*

Proof. Since $h^0(X, \mathscr{O}_X(-K_X)) + h^0(X, \mathscr{O}_X(K_X - K_X)) \geqslant 3$, we obtain that $h^0(X, \mathscr{O}_X(-K_X)) \geqslant 2$. Let D be a Weil divisor in X, and let a be the largest integer such that $(D - aT) \cdot T > 0$. Set $G = D - aT$. Then G is a $\mathbb{Q}$-ample Weil divisor since X has Picard number 1. By hypothesis, we have

$$h^0(X, \mathscr{O}_X(G)) + h^0(X, \mathscr{O}_X(K_X + G)) \geqslant 3.$$

Since $h^0(X, \mathscr{O}_X(-K_X)) \geqslant 1$, we have $h^0(X, \mathscr{O}_X(G)) \geqslant h^0(X, \mathscr{O}_X(K_X + G))$. Hence, $h^0(X, \mathscr{O}_X(G)) \geqslant 1$. However, by the choice of a, we have $0 < G \cdot T \leqslant T \cdot T$. Since T is a minimal curve, this implies that G is numerically equivalent to T. Hence D is numerically equivalent to $(a + 1)T$. By [AD14, Lem. 2.6], there is a smallest positive integer k such that $k(D - (a + 1)T)$ is linearly equivalent to the zero divisor.

There is a cyclic cover $X' \to X$ with respect to $\mathscr{O}_X(D-(a+1)T)^{\otimes k} \cong \mathscr{O}_X$, which is étale in codimension 1 (See [KM98, Def. 2.52]). Since the smooth locus of X is algebraically simply connected, we can only have $k = 1$. Thus, we obtain that D is linearly equivalent to $(a+1)T$. This shows that $\mathrm{Cl}(X) = \mathbb{Z} \cdot [T]$ □

Outline of the proof of Theorem 4.1.5. Let T be a minimal curve. Since $-K_X$ is ample, we have $(T+K_X)\cdot(-K_X) < T\cdot(-K_X)$. Thus $T+K_X$ cannot be linearly equivalent to a curve. This shows that $h^0(X, \mathscr{O}_X(T + K_X)) \leqslant 1$, and the equality holds if and only if $-K_X$ is linearly equivalent to T. Hence, by Lemma 4.3.1, the conditions in Theorem 4.1.5 imply that

1. either $h^0(X, \mathscr{O}_X(-K_X)) \geqslant 2$ and $\mathrm{Cl}(X) = \mathbb{Z} \cdot [K_X]$,
2. or $h^0(X, \mathscr{O}_X(T)) \geqslant 3$, $h^0(X, \mathscr{O}_X(T + K_X)) = 0$, and $\mathrm{Cl}(X) = \mathbb{Z} \cdot [T]$.

If we are in the first case, then we will prove that X has canonical singularities. After that, with the help of the classification of Fano surfaces with canonical singularities, we can show that $X \cong S^c(E_8)$ or $X \cong S^n(E_8)$ (see Section 4.3.1 for the definitions). In the second case, we will show directly that $X \cong \mathbb{P}^2$. Afterwards, we will exclude the case of $S^c(E_8)$. The complete proof of Theorem 4.1.5 will be given at the end of this section.

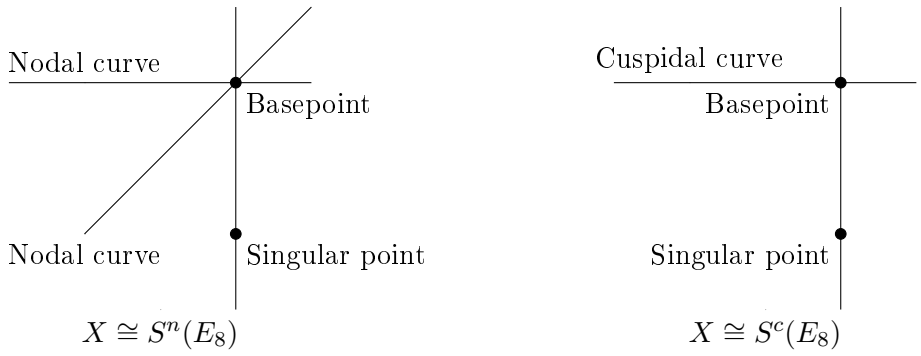
4.3.1 Classification of Fano surfaces with canonical singularities

We collect some classification results of Fano surfaces with canonical singularities (See [MZ88] for more details). Let X be a complex Fano surface with canonical singularities such that the smooth locus of X is simply connected. Assume that X has Picard number 1. Then there are at most two singular points in X. If X is smooth, then $X \cong \mathbb{P}^2$. If there is one singular point, then possible types of singularities of this point are

$$A_1,\ A_4,\ D_5,\ E_6,\ E_7,\ E_8.$$

If there are two singular points, then one is of type A_1, and the other is of type A_2. Except the case when there is a singular point of type E_8, for each type of singularities, there is exactly one isomorphism class for the surface. If the surface has a singular point of type E_8, then there are exactly two isomorphism classes $S^c(E_8)$ (See [KM99, Lem. 3.6.(1)]) and $S^n(E_8)$ (See [KM99, Lem. 3.6.(2)]) of such surfaces.

We recall some properties of the surfaces $S^c(E_8)$ and $S^n(E_8)$. These two surfaces can be constructed as follows. Choose a singular cubic rational curve C in $X_1 = \mathbb{P}^2$. Let x be one of the smooth inflection points of C. Recall that x is an inflection point if the line L in X_1 tangent to C at point x meets C at x with order at least 3 (See [EH00, §IV.1] for more details). Starting from $X_1 = \mathbb{P}^2$, we blow up the point x, and we get a surface X_2. We construct a rational surface X_9 by induction as follows. Assume that X_i is constructed ($i \geqslant 2$). Then $X_{i+1} \to X_i$ is the blow-up of the intersection point of the strict transform of C in X_i and the exceptional divisor of $X_i \to X_{i-1}$. There are exactly eight (-2)-curves in X_9 : the strict transform of L, and the strict transforms of all the exceptional curves of $X_8 \to X_1$. Let $X_9 \to X$ be the blow-down of all (-2)-curves in X_9. Then X is isomorphic to $S^c(E_8)$ if C is a cuspidal rational curve, or is isomorphic to $S^n(E_8)$ if C is a nodal curve. (This is why we use the terminology of these two surfaces. The index "c" is for cuspidal, and the index "n" is for nodal) The linear system $|-K_X|$ has dimension 1, and has a unique basepoint which is a smooth point of X. A general member of this linear system is a smooth elliptic curve in X. If $Y \to X$ is the blow up of the basepoint, then there is a fibration $p_Y : Y \to \mathbb{P}^1$, induced by $|-K_X|$, whose general fibers are elliptic curves. There is a unique rational curve in $|-K_X|$ which passes through the singular point of X. If $X \cong S^c(E_8)$ then there exist exactly two rational curves in $|-K_X|$. The one contained in the smooth locus of X is the strict transform of C in X. If $X \cong S^n(E_8)$, then there are exactly three rational curves in $|-K_X|$. Both of the two rational curves contained in the smooth locus of X are singular rational curves with a node (one of them is the strict transform of C in X).



Rational curves in $|-K_X|$

4.3.2 Construction of an intermediate surface

Consider a singular klt Fano surface X whose smooth locus is algebraically simply connected. In this subsection, we will construct a surface Z which is birational to X and with many nice properties. More precisely, we will prove the following proposition.

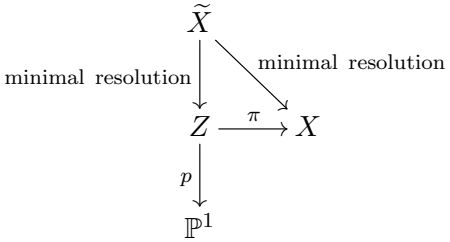
Proposition 4.3.2. *Let X be a singular rational surface with klt singularities whose smooth locus is algebraically simply connected. Then there is a rational surface Z with klt singularities such that*

(1) There is a fibration $p : Z \to \mathbb{P}^1$ whose general fibers are smooth rational curves.

(2) There is a birational morphism $\pi : Z \to X$ such that every exceptional curve of π is horizontal over $\mathbb{P}^1$ with respect to p.

(3) The minimal resolution $\widetilde{X} \to X$ factors through $\widetilde{X} \to Z$.

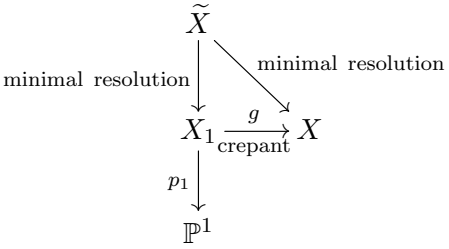
(4) There is at most one π-exceptional curve which is crepant over X, and if there is such a curve, then it is a section of p over $\mathbb{P}^1$.



Recall that an exceptional divisor is called crepant if its discrepancy is zero. This surface Z will be very useful for the proof of Theorem 4.1.5. In the following lemma, we prove Proposition 4.3.2 under the assumption that X has canonical singularities. For the proof, we use the classification of Fano surfaces with canonical singularities which have Picard number 1 (See [MZ88] or [KM99, §3]).

Lemma 4.3.3. *Let X be a singular rational surface with canonical singularities. Assume that the smooth locus of X is algebraically simply connected. Let $r : \widetilde{X} \to X$ be the minimal resolution of singularities. Then there is a rational surface $X_1 \xrightarrow{g} X$ with canonical singularities such that :*

1. *The minimal resolution $\widetilde{X} \to X$ factors through $\widetilde{X} \to X_1$ which is also the minimal resolution of X_1.*
2. *The morphism g contracts at most one curve.*
3. *There is a fibration $p_1 : X_1 \to \mathbb{P}^1$ whose general fibers are smooth rational curves.*
4. *If g contracts a curve C_1, then C_1 is a section of p_1 over $\mathbb{P}^1$, and the strict transform of C_1 in $\widetilde{X}$ is a $(-2)-$curve. In particular, C_1 is crepant over X.*



Proof. Let $f : X \to X'$ be the result of a MMP for X. Then X' has canonical singularities, and the smooth locus of X' is algebraically simply connected by Lemma 3.1.7. If X' has Picard number at least 2, then there is a Mori fibration $X' \to \mathbb{P}^1$ which induces a fibration $X \to \mathbb{P}^1$. In this case, we let $g : X_1 \to X$ be the identity map. In the following, we will assume that X' has Picard number 1.

If X' is not smooth, then by [MZ88, Lem. 10.1], we can extract a curve over X' and obtain a rational surface $X_1' \xrightarrow{g'} X'$ with canonical singularities such that

1. The minimal resolution $\widetilde{X}' \to X'$ factors through $\widetilde{X}' \to X_1'$ which is also the minimal resolution of X_1'.
2. The morphism g' contracts exactly one curve C_1'.
3. There is a fibration $p_1' : X_1' \to \mathbb{P}^1$ whose general fibers are smooth rational curves.
4. The curve C_1' is a section of p_1' over $\mathbb{P}^1$, and the strict transform of C_1' in $\widetilde{X}'$ is a $(-2)-$curve.

There is a natural morphism $\widetilde{X} \to \widetilde{X}'$. By contracting all exceptional curves of $\widetilde{X} \to X$, except maybe the strict transform of C_1' in $\widetilde{X}$, we obtain birational morphisms $\widetilde{X} \longrightarrow X_1 \xrightarrow{g} X$ (See Remark 3.1.4). Then X_1 has canonical singularities, and there is a natural morphism $f_1 : X_1 \to X_1'$. The composition $p_1 = p_1' \circ f_1$ gives a fibration from X_1 to $\mathbb{P}^1$. Hence X_1 satisfies the conditions in the lemma.

If X' is smooth, then $X' \cong \mathbb{P}^2$. Let $\theta : Y \to X'$ be the last step of the MMP for X. Then there is a natural morphism $\varphi : X \to Y$. Let $s : \widetilde{Y} \to Y$ be the minimal resolution of Y. Then $\widetilde{Y}$ can be obtained from X' (which is isomorphic to $\mathbb{P}^2$) by a sequence of blow-ups. Since X is not smooth, X is different from X'. Hence $Y \neq X'$, and $\widetilde{Y} \neq X'$. Since the singular locus of Y is contained in the exceptional divisor of $\theta : Y \to X'$, we obtain that every exceptional divisor of $\widetilde{Y} \to X'$ is over a point $x' \in X'$. Blowing up the point x' in X', we obtain a surface Z, which is isomorphic to the Hirzebruch surface Σ_1. Then $\widetilde{Y} \to X'$ factorise through $\widetilde{Y} \to Z$, and there is a fibration from Z to $\mathbb{P}^1$. Hence we obtain a fibration $\widetilde{Y} \to \mathbb{P}^1$ such that there is at most one curve in $\widetilde{Y}$ which is both exceptional for $s : \widetilde{Y} \to Y$ and horizontal over $\mathbb{P}^1$. Moreover, if this curve exists, then it is a section over $\mathbb{P}^1$ (it can only be the strict transform of the exceptional divisor of $Z \to X'$). By contracting all of the s-exceptional curves in $\widetilde{Y}$ which are not horizontal over $\mathbb{P}^1$ (See Remark 3.1.4), we get a rational surface $Y_1 \xrightarrow{h} Y$ which has at most canonical singularities such that

1. The minimal resolution $\widetilde{Y} \to Y$ factors through $\widetilde{Y} \to Y_1$ which is also the minimal resolution of Y_1.
2. The morphism h contracts at most one curve.
3. There is a fibration $q_1 : Y_1 \to \mathbb{P}^1$ whose general fibers are smooth rational curves.
4. If h contracts a curve B_1, then B_1 is a section of q_1 over $\mathbb{P}^1$, and the strict transform of B_1 in $\widetilde{Y}$ is a $(-2)-$curve.

There is a natural morphism $\widetilde{X} \to \widetilde{Y}$. By contracting all of the exceptional curves of $\widetilde{X} \to X$, except maybe the strict transform of B_1 in $\widetilde{X}$, we obtain birational morphisms $\widetilde{X} \longrightarrow X_1 \xrightarrow{g} X$ (See Remark 3.1.4). Then X_1 has canonical singularities, and there is a natural morphism $\varphi_1 : X_1 \to Y_1$. The composition $p_1 = q_1 \circ \varphi_1$ gives a fibration from X_1 to $\mathbb{P}^1$. Hence X_1 satisfies the conditions in the lemma. □

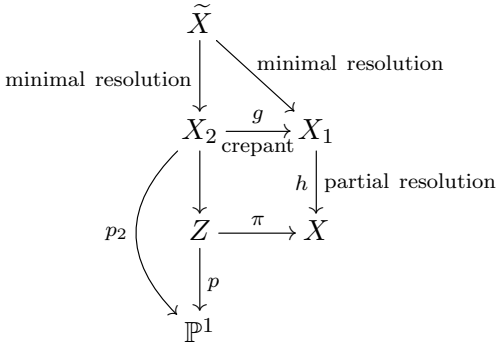
Now we can prove Proposition 4.3.2.

Proof of Proposition 4.3.2. Since X has isolated singularities, there is a birational morphism $h : X_1 \to X$ which resolves exactly the non-canonical singularities of X. We may also assume that this partial resolution is minimal. Then X_1 has at most canonical singularities, and none of the h-exceptional curves is crepant over X (See [KM98, Cor. 4.3]). Moreover, if $\widetilde{X} \to X$ is the minimal resolution, then there is a natural morphism $\widetilde{X} \to X_1$ which is also the minimal resolution of X_1. Since the smooth locus of X, which is algebraically simply connected, is isomorphic to an open subset of the smooth locus of X_1, the smooth locus of X_1 is algebraically simply connected by Lemma 3.1.6.

First we assume that X_1 is singular. Then by Lemma 4.3.3, there is a rational surface $X_2 \xrightarrow{g} X_1$ with canonical singularities such that :

1. The minimal resolution $\widetilde{X} \to X_1$ factors through $\widetilde{X} \to X_2$ which is also the minimal resolution of X_2.
2. The morphism g contracts at most one curve.
3. There is a fibration $p_2 : X_2 \to \mathbb{P}^1$ whose general fibers are smooth rational curves.
4. If g contracts a curve C_1, then C_1 is a section of p_2 over $\mathbb{P}^1$, and the strict transform of C_1 in $\widetilde{X}$ is a $(-2)-$curve.

Let $\varphi : X_2 \to X$ be the composition of $X_2 \xrightarrow{g} X_1 \xrightarrow{h} X$. Since g contracts at most one curve, and none of the h-exceptional curves is crepant over X, there is at most one curve in X_2 which is crepant over X. By Remark 3.1.4, we can contract all curves in X_2 which are both contracted by φ and by p_2. We obtain a normal surface Z such that $\widetilde{X} \to Z$ is the minimal resolution. Then Z has klt singularities. There is a natural fibration $p : Z \to \mathbb{P}^1$ induced by p_2. Moreover there is a natural birational morphism $\pi : Z \to X$ induced by φ.



Then there is at most one π-exceptional curve which is crepant over X since there is at most one curve in X_2 which is crepant over X. If this curve exists, then it is the strict transform of C_1 in Z. Hence it is a section over $\mathbb{P}^1$.

Now we assume that X_1 is a smooth rational surface. Then X_1 is different from $\mathbb{P}^2$ since X is singular. Hence, there is a fibration $p_1 : X_1 \to \mathbb{P}^1$. Let $g : X_2 \to X_1$ be the identity morphism and let $p_2 = p_1 \circ g : X_2 \to \mathbb{P}^1$. Let $\varphi = h \circ g : X_2 \to X$. We can construct $\pi : Z \to X$ as before such that Z satisfies all the conditions in the proposition. Moreover, in this case, none of the π-exceptional curves is crepant over X. □

4.3.3 Proof of Theorem 4.1.5 and Theorem 4.1.2

We will first prove two theorems on classification of rational surfaces (Theorem 4.3.4 and Theorem 4.3.5).

Theorem 4.3.4. *Let X be a Fano surface with klt singularities. Assume that $\mathrm{Cl}(X) = \mathbb{Z} \cdot [K_X]$ and $h^0(X, \mathscr{O}_X(-K_X)) \geqslant 2$. Then either $X \cong S^c(E_8)$ or $X \cong S^n(E_8)$.*

Proof. If X has canonical singularities, then K_X is a Cartier divisor. Hence $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$. This implies that X has one singular point of type E_8 by [MZ88, Lem. 6]. Hence either $X \cong S^c(E_8)$ or $X \cong S^n(E_8)$.

Now we will prove that X has canonical singularities. First note that X is singular. Let T be the curve given by a general member of $H^0(X, \mathscr{O}_X(-K_X))$. Then T is a minimal curve.

Let Z be the surface described in Proposition 4.3.2. Then there are birational morphisms $\widetilde{X} \longrightarrow Z \xrightarrow{\pi} X$ and a fibration $p : Z \to \mathbb{P}^1$, where $\widetilde{X} \to X$ is the minimal resolution of X.

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle p} & & \\ \mathbb{P}^1 & & \end{array}$$

Let F be a general fiber of the fibration $p : Z \to \mathbb{P}^1$. We denote the π-exceptional curves by $E_1, ..., E_s$. Since X has klt singularities, and the minimal resolution $\widetilde{X} \to X$ factors through $\widetilde{X} \to Z$, we obtain that $K_Z = \pi^* K_X + \sum_{i=1}^s a_i E_i$ with $-1 < a_i \leqslant 0$ for all i. Hence,

$$\pi^* T \cdot F = -\pi^* K_X \cdot F = -K_Z \cdot F + \sum_{i=1}^s a_i E_i \cdot F = 2 + \sum_{i=1}^s a_i E_i \cdot F.$$

Since all of the E_i's are horizontal over $\mathbb{P}^1$ with respect to the fibration p, we have $\sum a_i E_i \cdot F \leqslant 0$. Thus $\pi^* T \cdot F \leqslant 2$, and the equality holds if and only if all of the a_i's are equal to 0.

Moreover, we have $\pi^* T = \pi_*^{-1} T + \sum_{i=1}^s b_i E_i$ with $b_i \geqslant 0$ for all i. Hence

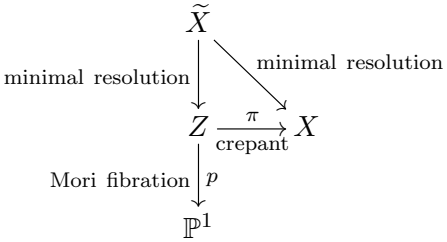
$$\pi_*^{-1} T \cdot F = \pi^* T \cdot F - \sum_{i=1}^s b_i E_i \cdot F \leqslant \pi^* T \cdot F \leqslant 2$$

for the same reason as before.

First assume that $\pi_*^{-1} T \cdot F < 2$. Then $\pi_*^{-1} T$ is either a section or a fiber of p. Hence T is a rational curve. Let $X \dashrightarrow \mathbb{P}^1$ be the rational map induced by a general 1-dimensional linear subsystem of the linear system $|-K_X|$ which contains T. Let Y be the normalisation of the graph of this rational map. Let $p_1 : Y \to X$ and $p_2 : Y \to \mathbb{P}^1$ be natural projections. Note that if a curve in Y is contained in the fiber of p_2, then it is not contracted by p_1 since the graph of $X \dashrightarrow \mathbb{P}^1$ is contained in $X \times \mathbb{P}^1$, and the normalisation map is finite. Hence every fiber of $p_2 : Y \to \mathbb{P}^1$ is reduced and irreducible for $-K_X$ is minimal. Let T' be the strict transform of T in Y. Then T' is smooth since Y is normal and T is a general member of $|-K_X|$. Since T is a

rational curve, we have $T' \cong \mathbb{P}^1$. Thus p_2 is a $\mathbb{P}^1$-bundle by Lemma 3.2.1. In this case, $Y \cong \Sigma_k$ is a Hirzebruch surface for some $k \geqslant 0$. Since X is singular, we obtain that $k \geqslant 2$ and that $p_1 : Y \to X$ is the morphism which contracts the special rational curve in Y with self intersection number $-k$. Thus $-K_X$ cannot be a minimal curve by Lemma 3.2.2 and we obtain a contradiction.

Hence we must have $\pi_*^{-1}T \cdot F = 2$. This shows that $a_i = b_i = 0$ for all i. By the construction of Z, this implies that there is exactly one π-exceptional curve D in Z. Since X has Picard number 1, Z has Picard number 2. Hence p is a Mori fibration, and every fiber of p is irreducible.



Moreover, since $b_i = 0$ for all i, the curve T is contained in the smooth locus of X. In particular, we have

$$-K_X \cdot T \geqslant 1.$$

In order to prove that X has canonical singularities, it is enough to prove that Z has canonical singularities since $\pi^* K_X = K_Z$. Note that if every fiber of p has multiplicity at most equal to 2, then by Theorem 3.0.1, Z has canonical singularities. Now we will assume by contradiction that there is a point $b \in \mathbb{P}^1$ such that $p^*b = mC$ with C reduced and $m > 2$. Since $a_i = 0$ for all i, by the projection formula we have

$$-K_X \cdot \pi_* C = -\pi^* K_X \cdot C = -K_Z \cdot C = \frac{2}{m} < 1 \leqslant -K_X \cdot T.$$

This is a contradiction since T is a minimal curve. Thus X has canonical singularities. This completes the proof of the theorem □

Theorem 4.3.5. *Let X be a Fano surface with klt singularities and let T be a minimal curve. Assume that* $\mathrm{Cl}(X) \cong \mathbb{Z} \cdot [T]$, $h^0(X, \mathscr{O}_X(T)) \geqslant 3$ *and* $h^0(X, \mathscr{O}_X(T + K_X)) = 0$. *Then* $X \cong \mathbb{P}^2$.

Proof. It is enough to prove that X is smooth. Assume that X is singular. Since

$$h^0(X, \mathscr{O}_X(T)) \geqslant 3,$$

through a general point of X, there passes at least two different minimal curve. Hence $T^2 \geqslant 1$ by Lemma 3.1.2. Let $\pi : Z \to X$ be the surface described in Proposition 4.3.2.

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle p} & & \\ \mathbb{P}^1 & & \end{array}$$

Let F be a general fiber of the fibration $p : Z \to \mathbb{P}^1$. We denote the π-exceptional curves by $E_1, ..., E_s$. As in the proof of Theorem 4.3.4, we have

$$-K_X \cdot \pi_* F = -\pi^* K_X \cdot F = -K_Z \cdot F + \sum_{i=1}^{s} a_i E_i \cdot F = 2 + \sum_{i=1}^{s} a_i E_i \cdot F \leqslant 2.$$

If $\pi_* F$ is a minimal curve, then every fiber of p is reduced and irreducible. Thus $Z \cong \Sigma_k$ is a Hirzebruch surface for some $k \geqslant 0$ by Lemma 3.2.1. Since X is singular, we obtain that $k \geqslant 2$ and that $\pi : Z \to X$ is the morphism which contracts the special rational curve in Z with self intersection number $-k$. Hence, by Lemma 3.2.2, we have

$$1 \leqslant T \cdot T = \pi_* F \cdot \pi_* F = \frac{1}{k}.$$

This is a contradiction.

If $\pi_* F$ is not a minimal curve in X, then there is an integer $\alpha > 1$ such that $\pi_* F$ is linearly equivalent to αT. Moreover, since $-K_X$ is $\mathbb{Q}$-ample, and $h^0(X, \mathscr{O}_X(T + K_X)) = 0$, there is an integer $\beta > 1$ such that $-K_X$ is linearly equivalent to βT. Hence, $-K_X \cdot \pi_* F = \alpha\beta T^2 \geqslant 4$, yielding a contradiction. □

We will prove that $S^c(E_8)$ does not satisfy the conditions of Theorem 4.1.5 (see Lemma 4.3.8). Let X be a surface isomorphic to $S^c(E_8)$. Let T be a minimal curve in X. Then $\mathscr{O}_X(T) \cong \mathscr{O}_X(-K_X)$. We will show that $h^0(X, \Omega_X^{[1]}[\otimes]\mathscr{O}_X(T)) \geqslant 1$. In order to do this, we will need the following results.

Lemma 4.3.6. *Let X be a surface isomorphic to $S^c(E_8)$. Then all of the elliptic curves in the linear system $|-K_X|$ are isomorphic.*

Proof. We can obtain X after some birational transform from $\mathbb{P}^2$ (See section 4.3.1). There is a unique cuspidal rational curve C' in the linear system $|-K_X|$ which is contained in the smooth locus of X. Let C be the strict transform of this curve in $\mathbb{P}^2$. If $z \in X$ is the unique basepoint of $|-K_X|$, then the image x of z in $\mathbb{P}^2$ is the unique smooth inflection point on C. We may assume that C is given by the equation $a_3 a_2^2 = a_1^3$, and that x is the point $[0 : 1 : 0]$, where $[a_1 : a_2 : a_3]$ are coordinates of $\mathbb{P}^2$. Let E be the strict transform in $\mathbb{P}^2$ of a smooth elliptic curve E' in the family induced by $|-K_X|$. Since the intersection of C' and E' is the point z, and the exceptional divisors of $X \to \mathbb{P}^2$ meet these two curves only at the point z, we obtain that the intersection of E and C is the point x.

Assume that E is given by the homogeneous equation

$$\lambda_1 a_1^3 + \lambda_2 a_2^3 + \lambda_3 a_3^3 + \lambda_4 a_1^2 a_2 + \lambda_5 a_2^2 a_3 + \lambda_6 a_3^2 a_1 \\ + \lambda_7 a_2^2 a_1 + \lambda_8 a_3^2 a_2 + \lambda_9 a_1^2 a_3 + \lambda_{10} a_1 a_2 a_3 = 0.$$

With affine coordinates $(b_1, b_3) = (\frac{a_1}{a_2}, \frac{a_3}{a_2})$, this equation becomes

$$\lambda_1 b_1^3 + \lambda_2 + \lambda_3 b_3^3 + \lambda_4 b_1^2 + \lambda_5 b_3 + \lambda_6 b_3^2 b_1 \\ + \lambda_7 b_1 + \lambda_8 b_3^2 + \lambda_9 b_1^2 b_3 + \lambda_{10} b_1 b_3 = 0.$$

The equation defining C becomes $b_3 = b_1^3$. If we replace b_3 by b_1^3 in the equation defining E, then we obtain

$$\lambda_1 b_1^3 + \lambda_2 + \lambda_3 b_1^9 + \lambda_4 b_1^2 + \lambda_5 b_1^3 + \lambda_6 b_1^7 + \lambda_7 b_1 + \lambda_8 b_1^6 + \lambda_9 b_1^5 + \lambda_{10} b_1^4 = 0.$$

This equation should have $b_1 = 0$ as a root with multiplicity 9 since E and C intersects at x with multiplicity 9. Hence we have $\lambda_1 + \lambda_5 = 0$, and $\lambda_k = 0$ for $k \neq 1, 3, 5$. Since E is smooth, $\lambda_1 \neq 0$. We may assume that $\lambda_1 = 1$, then E is given by $a_3 a_2^2 = a_1^3 + \lambda a_3^3$ with $\lambda \neq 0$. Hence the j-invariant of E is 0. Thus all of the elliptic curves in this family induced by $|-K_X|$ are isomorphic. □

Proposition 4.3.7. *Let X be a klt projective rational surface. Then $\rho(X)$, the Picard number of X, is euqal to $h^1(X, \Omega_X^{[1]})$.*

Proof. First assume that X is smooth. We have the following exponential exact sequence

$$H^1(X, \mathscr{O}_X) \to H^1(X, \mathscr{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X).$$

Since X is a smooth rational surface, we have $h^1(X, \mathscr{O}_X) = h^2(X, \mathscr{O}_X) = 0$. Hence $H^1(X, \mathscr{O}_X^*) \to H^2(X, \mathbb{Z})$ is an isomorphism. Since $H^1(X, \mathscr{O}_X^*)$ is isomorphic to the Picard group of X, we obtain that $\rho(X)$ is equal to $h^2(X, \mathbb{C})$. Since $h^2(X, \mathscr{O}_X) = 0$, we have $h^1(X, \Omega_X^1) = h^2(X, \mathbb{C})$ from the Hodge theory. Hence $h^1(X, \Omega_X^1) = \rho(X)$.

Now we assume that X is singular. Let $r : \widetilde{X} \to X$ be the minimal resolution of singularities. Then from the previous paragraph, we know that $h^2(\widetilde{X}, \mathbb{C})$ is equal to $\rho(\widetilde{X})$. Since X is $\mathbb{Q}$-factorial, we have $\rho(X) = \rho(\widetilde{X}) - l$, where l is the number of the r-exceptional curves. Hence $\rho(X) = h^2(\widetilde{X}, \mathbb{C}) - l$.

Since X has rational singularities, $R^1 r_* \mathscr{O}_{\widetilde{X}} = 0$. Hence the Leray spectral sequence for r gives an exact sequence

$$0 \to H^2(X, \mathbb{C}) \to H^2(\widetilde{X}, \mathbb{C}) \xrightarrow{p} H^0(X, R^2 r_* \mathbb{C}),$$

see [KM92, 12.1.3.2]. Moreover, from [KM92, Prop. 12.1.6], the image of p is free and generated exactly by the class of the the r-exceptional divisors. Hence we obtain that $h^2(X, \mathbb{C}) = h^2(\widetilde{X}, \mathbb{C}) - l$ and $\rho(X) = h^2(X, \mathbb{C})$.

Since X has quotient singularities (See [KM98, Prop. 4.18]), by [Ste77, Thm. 1.12], we have

$$\begin{aligned} h^2(X,\mathbb{C}) &= h^0(X,\Omega_X^{[2]}) + h^1(X,\Omega_X^{[1]}) + h^2(X,\mathscr{O}_X) \\ &= h^1(X,\Omega_X^{[1]}) + 2h^2(X,\mathscr{O}_X). \end{aligned}$$

Since X is a rational surface with rational singularities, from the Leray spectral sequence, we have

$$h^2(X,\mathscr{O}_X) = h^2(\widetilde{X},\mathscr{O}_{\widetilde{X}}) = 0.$$

Thus, $\rho(X) = h^2(X,\mathbb{C}) = h^1(X,\Omega_X^{[1]})$. □

Lemma 4.3.8. *Let $X \cong S^c(E_8)$. Then*

$$h^0(X,\mathscr{O}_X(-K_X)) - h^0(X,\Omega_X^{[1]}[\otimes]\mathscr{O}_X(-K_X)) + h^0(X,\mathscr{O}_X) \leqslant 2.$$

Proof. Let E be a smooth elliptic curve in $|-K_X|$. Let $c : Y \to X$ be the blow-up of the basepoint x of $|-K_X|$. Then we have a fibration $p : Y \to \mathbb{P}^1$ induced by $|-K_X|$. The general fibers of p are smooth elliptic curves. Since $X \cong S^c(E_8)$, this family of elliptic curves is isotrivial by Lemma 4.3.6. Let D be the strict transform of E in Y. Since D is a fiber, $\Omega_Y^1|_D$ is an extension of $\mathscr{O}_D$ by $\mathscr{O}_D$. Since the family is isotrivial, the Kodaira-Spencer map is zero. Hence $T_Y|_D \cong T_D \oplus p^*T_{\mathbb{P}^1}|_D$, where T_Y and T_D are the tangent sheaves of Y and D respectively. By taking the dual, we have $\Omega_Y^1|_D \cong \mathscr{O}_D \oplus \mathscr{O}_D$. Note that X is smooth along E and Y is smooth along D. From the natural morphism $c^*\Omega_X^1 \to \Omega_Y^1$ and the conormal exact sequences, we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c^*Q_E & \longrightarrow & (c^*\Omega_X^1)|_D & \longrightarrow & c^*\Omega_E^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow & & \\ 0 & \longrightarrow & Q_D & \longrightarrow & \Omega_Y^1|_D & \longrightarrow & \Omega_D^1 & \longrightarrow & 0 \end{array}$$

The morphisms θ_2 and θ_1 are injective, and the third is an isomorphism. By the snake lemma, we have $\operatorname{Coker}\theta_1 \cong \operatorname{Coker}\theta_2$. Note that Q_D is isomorphic to $\mathscr{O}_D$, and Q_E is isomorphic to $\mathscr{O}_E(-x)$. Hence $h^0(D,\operatorname{Coker}\theta_1) = 1$. We obtain that $h^0(D,\operatorname{Coker}\theta_2) = 1$. Hence

$$h^0(E,\Omega_X^1|_E) = h^0(D,(c^*\Omega_X^1)|_D) \geqslant h^0(D,\Omega_Y^1|_D) - h^0(D,\operatorname{Coker}\theta_2) = 1.$$

This shows that $\Omega_X^1|_E \cong \mathscr{O}_E \oplus \mathscr{O}_E(-x)$.

Moreover, since $\mathscr{O}_X(K_X)|_E \cong \mathscr{O}_E(-x)$, we have $(\Omega_X^1[\otimes]\mathscr{O}_X(-K_X))|_E \cong \mathscr{O}_E \oplus \mathscr{O}_E(x)$. Since X is smooth along E and $\mathscr{O}_X(E) \cong \mathscr{O}_X(-K_X)$, we have an exact sequence

$$0 \to \Omega_X^{[1]} \to \Omega_X^{[1]} \otimes \mathscr{O}_X(-K_X) \to (\Omega_X^1[\otimes]\mathscr{O}_X(-K_X))|_E \to 0.$$

This induces an exact sequence

$$\begin{aligned} 0 &\to H^0(X,\Omega_X^{[1]}) \to H^0(X,\Omega_X^{[1]}\otimes\mathscr{O}_X(-K_X)) \\ &\to H^0(E,(\Omega_X^1[\otimes]\mathscr{O}_X(-K_X))|_E) \to H^1(X,\Omega_X^{[1]}). \end{aligned}$$

Since X has Picard number 1, by Proposition 4.3.7, we have $h^1(X,\Omega_X^{[1]}) = 1$. Since $h^0(X,\Omega_X^{[1]}) = 0$ (See [GKKP11, Thm. 5.1]), and

$$h^0(E,(\Omega_X^1[\otimes]\mathscr{O}_X(-K_X))|_E) = h^0(E,\mathscr{O}_E\oplus\mathscr{O}_E(x)) = 2,$$

we obtain

$$h^0(X,\Omega_X^{[1]}\otimes\mathscr{O}_X(-K_X)) \geqslant 1.$$

Hence we have

$$h^0(X,\mathscr{O}_X(-K_X)) - h^0(X,\Omega_X^{[1]}[\otimes]\mathscr{O}_X(-K_X)) + h^0(X,\mathscr{O}_X) \leqslant 2.$$

□

Now we will complete the proof of Theorem 4.1.5.

Proof of Theorem 4.1.5. Let T be a minimal curve. From the discussion after Lemma 4.3.1, we know that one of the following properties holds

1. $h^0(X,\mathscr{O}_X(-K_X)) \geqslant 2$ and $\mathrm{Cl}(X) = \mathbb{Z}\cdot[K_X]$;
2. $h^0(X,\mathscr{O}_X(T)) \geqslant 3$, $h^0(X,\mathscr{O}_X(T+K_X)) = 0$, and $\mathrm{Cl}(X) = \mathbb{Z}\cdot[T]$

By Theorem 4.3.4, Theorem 4.3.5 and Lemma 4.3.8, we obtain that X is either isomorphic to $\mathbb{P}^2$ or isomorphic to $S^n(E_8)$. It remains to prove that both of these two surfaces satisfies the conditions of the theorem.

It is not hard to check that $\mathbb{P}^2$ satisfies the conditions. We will prove that if $X \cong S^n(E_8)$, then X satisfies the conditions in the theorem. In fact, we can show that the family of elliptic curves induced by $|-K_X|$ is not isotrivial by similar method of Lemma 4.3.6. Hence we have $h^0(E,(\Omega_X^1[\otimes]\mathscr{O}_X(-K_X))|_E) = 1$ in this case, where E is a smooth elliptic curve induced by a general member of $|-K_X|$. Since E is contained in the smooth locus of X, the composition of the natural morphisms

$$H^2(X,\mathbb{C}) \to H^2(\widetilde{X},\mathbb{C}) \to H^2(E,\mathbb{C})$$

is not zero, where $\widetilde{X} \to X$ is the minimal resolution. Hence, the natural morphism

$$H^1(X,\Omega_X^{[1]}) \to H^1(E,\Omega_E^1)$$

is not zero. Now consider the exact sequence

$$H^1(X,\Omega_X^{[1]}\otimes\mathscr{O}_X(-E)) \to H^1(X,\Omega_X^{[1]}) \to H^1(E,\Omega_X^{[1]}|_E).$$

Since the composition of

$$H^1(X, \Omega_X^{[1]}) \to H^1(E, \Omega_X^{[1]}|_E) \to H^1(E, \Omega_E^1)$$

is not zero, and $h^1(X, \Omega_X^{[1]}) = 1$, we obtain that the morphism

$$H^1(X, \Omega_X^{[1]} \otimes \mathscr{O}_X(-E)) \to H^1(X, \Omega_X^{[1]})$$

is zero. By taking the dual morphism, we obtain that the natural morphism $H^1(X, \Omega_X^{[1]}) \to H^1(X, \Omega_X^{[1]} \otimes \mathscr{O}_X(-K_X))$ is zero. Hence, as in the proof of Theorem 4.1.5, we obtain that

$$h^0(X, \Omega_X^{[1]} \otimes \mathscr{O}_X(-K_X)) = 0$$

in this case.

Note that every ample divisor T in X is linearly equivalent to aE for some $a > 0$. Consider the following exact sequences for all $k \geqslant 0$,

$$0 \to \mathscr{O}_X(kE) \to \mathscr{O}_X((k+1)E) \to \mathscr{O}_X((k+1)E)|_E \to 0$$

and

$$\begin{aligned} 0 \quad \to \quad & \mathscr{O}_X(kE)[\otimes]\Omega_X^{[1]} \to \mathscr{O}_X((k+1)E)[\otimes]\Omega_X^{[1]} \\ & \to (\mathscr{O}_X((k+1)E)[\otimes]\Omega_X^{[1]})|_E \to 0. \end{aligned}$$

By taking cohomologies, we obtain that, for all $k \geqslant 1$,

$$h^0(\mathscr{O}_X((k+1)E)) = h^0(\mathscr{O}_X(kE)) + k + 1$$

and

$$h^0(\mathscr{O}_X((k+1)E)[\otimes]\Omega_X^{[1]}) = h^0(\mathscr{O}_X(kE)[\otimes]\Omega_X^{[1]}) + 2k + 1.$$

By induction on k, we have

$$h^0(\mathscr{O}_X(kE)) = \frac{k^2+k+2}{2} \text{ and } h^0(\mathscr{O}_X(kE)[\otimes]\Omega_X^{[1]}) = k^2 - 1$$

for all $k \geqslant 1$. This shows that X satisfies the conditions of the theorem. □

Now we can deduce Theorem 4.1.2.

Proof of Theorem 4.1.2. By Corollary 4.2.1, we have

$$h^0(X, \mathscr{O}_X(H)) - h^0(X, \Omega_X^{[1]}[\otimes]\mathscr{O}_X(H)) + h^0(X, \mathscr{O}_X(H + K_X)) = 3$$

for any $\mathbb{Q}$-ample Weil divisor H on X. Since M is simply connected in codimension 1, the smooth locus of X is algebraically simply connected. Hence, by Theorem 4.1.5, either $X \cong \mathbb{P}^2$ or $X \cong S^n(E_8)$. □

Chapitre 5

Fano varieties with Nef = Psef

5.1 Introduction

Let X be a complex Fano variety. Then there is an inclusion

$$\mathrm{Nef}(X) \subseteq \mathrm{Psef}(X).$$

If $\mathrm{Nef}(X) = \mathrm{Psef}(X)$ and X has log canonical singularities, then $\rho(X)$, the Picard number of X, is at most equal to the dimension of X (see [Dru14, Lem. 4.9]). If we assume further that $\rho(X) = \dim X$ and that X has locally factorial canonical singularities, then X is a product of double covers of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. More precisely, Druel proves the following theorem (see [Dru14, Thm. 1.1] and [Dru14, Prop. 10.4]).

Theorem 5.1.1. *Let X be a Fano variety with locally factorial canonical singularities such that $\rho(X) = \dim X$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Then $X \cong X_1 \times \cdots \times X_k$ such that for all $i = 1, ..., k$, either $X_i \cong \mathbb{P}^1$, or $\dim X_i \geqslant 3$ and X_i is a double cover of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, branched along a prime divisor of degree $(2, ..., 2)$.*

In this chapter, we consider Fano varieties X with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. If the dimension of X is 2, then X is isomorphic to $\mathbb{P}^2$. In dimension 3, we prove the following theorem.

Theorem 5.1.2. *Let X be a Fano threefold with isolated locally factorial canonical singularities such that $\rho(X) = 2$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Then one of the following holds.*

1. *$X \cong \mathbb{P}^1 \times \mathbb{P}^2$.*
2. *X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$, branched along a prime divisor of degree $(2, 2)$.*
3. *X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$, branched along a prime divisor of degree $(2, 4)$.*

4. *X is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1,1)$.*
5. *X is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1,2)$.*
6. *X is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(2,2)$.*
7. *X is a double cover of a smooth hypersurface Y in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1,1)$, branched along a prime divisor D, which is the intersection of Y and a hypersurface of degree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.*

Note that if we assume that X is smooth, then Theorem 5.1.2 follows from the classification of smooth Fano threefold due to Mori and Mukai ([MM82]).

In higher dimensions, we will first introduce some examples. In the following examples, we provide methods of constructions of Fano varieties X with $\rho(X) = \dim X - 1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$ (see Construction 5.2.7 for more details).

Example 5.1.3. We will give two examples of Fano varieties X of dimension $n \geqslant 4$ which are finite covers of the product $M = (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$, of degree 2 or 4.

1. Let X be a double cover over M, branched along a prime prime divisor of degree $(2, ..., 2, k)$, where $k \in \{2, 4\}$.
2. We write $M = (\mathbb{P}^1)^s \times \mathbb{P}^2 \times (\mathbb{P}^1)^r$ such that $r, s > 0$ and $r+s = n-2$. Let N_1 be a double cover of $(\mathbb{P}^1)^s \times \mathbb{P}^2$, branched along a prime prime divisor of degree $(2, ..., 2, 2)$. Let $M_1 = N_1 \times (\mathbb{P}^1)^r$ and let $g : M_1 \to \mathbb{P}^2 \times (\mathbb{P}^1)^r$ be the natural projection. Let X be a double cover of M, branched along a prime divisor D which is the pullback by g of some divisor of degree $(2, 2, ..., 2)$ in $\mathbb{P}^2 \times (\mathbb{P}^1)^r$. Equivalently, X is isomorphic to the following fiber product

$$\begin{array}{ccc} X \cong M_1 \times_M M_2 & \longrightarrow & M_2 \\ \downarrow & & \downarrow{\scriptstyle p_2} \\ M_1 & \xrightarrow[p_1]{} & M \end{array}$$

   where p_1 is a double cover branched along a prime divisor of degree $(2, ..., 2, 2, 0, ..., 0)$ and p_2 is a double cover branched along a prime divisor of degree $(0, ..., 0, 2, 2, ..., 2)$.

Example 5.1.4. Let $B_1 \cong B_2 \cong \mathbb{P}^2$. We will give three examples of Fano varieties X of dimension $n \geqslant 4$ which are finite covers of the product $(\mathbb{P}^1)^{n-3} \times W$, of degree 2 or 4, where W is some normal ample hypersurface in $B_1 \times B_2$, which is a Fano threefold.

1. Let W be a smooth divisor of degree $(1,1)$ in $B_1 \times B_2$. Let X be a double cover of $(\mathbb{P}^1)^{n-3} \times W$, branched along a prime divisor D, which is the intersection of $(\mathbb{P}^1)^{n-3} \times W$ and some divisor of degree $(2, ..., 2, 2, 2)$ in $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$.

2. Let N_1 be a double cover of $(\mathbb{P}^1)^{n-3} \times B_1$, branched along a prime divisor of degree $(2, .., 2, 2)$. Let $M = N_1 \times B_2$ and let p be the natural projection from M to $B_1 \times B_2$. Let W be a normal hypersurface of degree $(1, k)$ in $B_1 \times B_2$, where $k \in \{1, 2\}$. Let $X = p^*W$. Equivalently, X is isomorphic to the following fiber product

$$\begin{array}{ccc} X \cong ((\mathbb{P}^1)^{n-3} \times W) \times_S M & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle p_2} \\ (\mathbb{P}^1)^{n-3} \times W & \xrightarrow[p_1]{} & S = (\mathbb{P}^1)^{n-3} \times B_1 \times B_2 \end{array}$$

where p_1 is the natural injection and p_2 is a double cover branched along a prime divisor of degree $(2, ..., 2, 2, 0)$.

3. Assume that $n \geqslant 5$. Let $S_1 = (\mathbb{P}^1)^r \times B_1$ and let $S_2 = B_2 \times (\mathbb{P}^1)^s$, where $r, s > 0$ and $r + s = n - 3$. Let N_1 be a double cover of S_1, branched along a prime divisor of degree $(2, ..., 2, 2)$ and let N_2 be a double cover of S_2, branched along a prime divisor of degree $(2, 2, ..., 2)$. Let $N = N_1 \times N_2$. Let p be the projection from N to $B_1 \times B_2$. Let W be a normal hypersurface of degree $(1, 1)$ in $B_1 \times B_2$. Let $X = p^*W$. Equivalently, X is isomorphic to the following fiber product

$$\begin{array}{ccc} X \cong ((\mathbb{P}^1)^{n-3} \times W) \times_S N & \longrightarrow & N \\ \downarrow & & \downarrow \\ (\mathbb{P}^1)^{n-3} \times W & \xrightarrow[p_1]{} & S = (\mathbb{P}^1)^{n-3} \times B_1 \times B_2 \end{array}$$

where p_1 is the natural injection.

The main objective of this chapter is to prove the following theorem.

Theorem 5.1.5. *Let X be a Fano variety of dimension at least 3, with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Then $X \cong X_1 \times X_2$ such that*

- *X_1 is either a point or one of the varieties in Theorem 5.1.1 ;*
- *X_2 is either one of the varieties in the list of Theorem 5.1.2, or a variety constructed by the methods of Example 5.1.3 and Example 5.1.4.*

Note that this theorem is false without assuming the variety is smooth in codimension 2. In fact, even in dimension 2, there are surfaces X other than $\mathbb{P}^2$ which have Picard number 1 and locally factorial canonical singularities. They are surfaces $S^n(E_8)$ and $S^c(E_8)$ introduced in section 4.3 (see [MZ88, Lem. 6]).

Outline of the proof of the theorems. Let X be a variety satisfying the condition in Theorem 5.1.5. We first consider the case when $\dim X = 3$. In this case, the Mori cone $\overline{\mathrm{NE}}(X)$ has exactly two extremal rays R_1, R_2. Let $f_i : X \to B_i$ be the extremal contraction with respect to R_i for $i = 1, 2$. We can prove that B_i is either $\mathbb{P}^1$ or $\mathbb{P}^2$ and the product $f_1 \times f_2$ is finite onto its

image. Hence X is a finite cover of $\mathbb{P}^1 \times \mathbb{P}^2$ or a finite cover of a hypersurface of $\mathbb{P}^2 \times \mathbb{P}^2$. With the help of two results on finite morphisms between Fano threefolds, we can deduce the proof of Theorem 5.1.2.

After this, we consider the case when $n = \dim X \geqslant 4$. The proof of Theorem 5.1.5 is by induction on the dimension of X. We will first prove that there is a fibration $g : X \to \mathbb{P}^1$ since $n \geqslant 4$. We can reduce to the case when $n = 4$. The proof of four-dimensional case is given in section 5.5.

There is a Mori fibration $h : X \to Y$ such that $h \times g : X \to Y \times \mathbb{P}^1$ is a finite surjective morphism. We can prove further that Y also satisfies the condition in Theorem 5.1.5. By induction, we assume that Theorem 5.1.5 is true in dimension smaller than n. In particular, $h \times g$ induces a finite surjective morphism $f : X \to Z$ such that Z is either $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times W$, where W is a normal ample hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$. In the end, we will conclude the proof of Theorem 5.1.5 by studying some finite morphisms between Fano varieties.

In this chapter, we will work over $\mathbb{C}$. If X is a variety, then we denote by $(X)^k$ the product of k copies of X. Consider a Fano fibration $f : X \to Y$ between projective normal varieties such that X has $\mathbb{Q}$-factorial klt singularities. The relative Picard number of f is the dimension of the closed convex cone in $\mathrm{N}_1(X)$ generated by the curves contracted by f. This number is equal to $\rho(X) - \rho(Y)$ since f is a Fano fibration (see [KMM87, Lem. 3-2-5]). In particular, a Mori fibration is a Fano fibration whose relative Picard number is equal to one.

5.2 Double covers between normal varieties

We recall the notion of cyclic covers. Let Y be a normal variety and $\mathscr{L}$ a line bundle on Y. Assume that there is a positive integer k and a section $s \in H^0(Y, \mathscr{L}^{\otimes k})$. Let D be the zero locus of s. Then there is a cyclic cover $g : Z \to Y$ with respect to the isomorphism $\mathscr{L}^{\otimes k} \cong \mathscr{O}_Y(D)$ induced by s. The morphism g is branched exactly along D and $g^*D = k \cdot \mathrm{Supp}\,(g^*D)$ (see [KM98, Def. 2.50]). Note that, by construction, Z may not be normal. However, we have the following result.

Lemma 5.2.1. *Let Y be a normal Cohen-Macaulay variety and let D be a reduced divisor. Assume that there is a line bundle $\mathscr{L}$ on Y such that $\mathscr{L}^{\otimes k} \cong \mathscr{O}_Y(D)$ for some $k > 0$. If $f : Z \to Y$ is the corresponding cyclic cover, then Z is normal and Cohen-Macaulay.*

Proof. We may assume that $Y = \mathrm{Spec}\, A$ is affine. Then $Z \cong \mathrm{Spec}\, A[T]/(T^k - s)$, where T is an indeterminate and $s = 0$ is the equation defining D. Since Y is Cohen-Macaulay and Z is a Cartier divisor in $Y \times \mathbb{A}^1$, the variety Z is also Cohen-Macaulay. Hence, we only need to prove that Z is smooth in codimension 1. However, there is an open subset Y_0 of Y such

that $\operatorname{codim} Y\backslash Y_0 \geqslant 2$ and both Y_0 and $D|_{Y_0}$ are smooth. Thus $Z_0 = g^{-1}(Y_0)$ is smooth and $\operatorname{codim} Z\backslash Z_0 \geqslant 2$. This shows that Z is normal. □

Lemma 5.2.2. *Let $f : X \to Y$ be an equidimensional fibration between normal Cohen-Macaulay varieties. Assume that there is an open subset U of Y whose complement has codimension at least 2 such that f has reduced fibers over U. Let $g : Y' \to Y$ be a cyclic cover with Y' normal. Let X' be the fiber product $X \times_Y Y'$. Then X' is a normal Cohen-Macaulay variety.*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

Proof. Assume that g is the cyclic cover with respect to some isomorphism $\mathscr{L}^{\otimes k} \cong \mathscr{O}_Y(D)$, where D is a Cartier divisor and $\mathscr{L}$ is a line bundle on Y. Since Y' is normal, D is reduced. Then the natural morphism $X' \to X$ is the cyclic cover with respect to the isomorphism $(f^*\mathscr{L})^{\otimes k} \cong \mathscr{O}_X(f^*D)$. Since f is equidimensional and has reduced fibers over U, f^*D is a reduced divisor in X. Since X is Cohen-Macaulay, by Lemma 5.2.1, X' is normal Cohen-Macaulay. □

We will prove some properties on double covers.

Lemma 5.2.3. *Let $f : X \to Y$ be a double cover between normal varieties with Y locally factorial. Let D be the codimension 1 part of the discriminant of f. Then there is a line bundle $\mathscr{L}$ on Y such that $\mathscr{L}^{\otimes 2} \cong \mathscr{O}_Y(D)$. If $g : Z \to Y$ is the corresponding cyclic cover, then X is the normalisation of Z. Moreover, if Y is Cohen-Macaulay, then X is smooth if and only if both D and Y are smooth.*

Proof. Let Y_0 be the largest open subset contained in the smooth locus of Y such that $X_0 = f^{-1}(Y_0)$ and $D|_{Y_0}$ are smooth. Then $\operatorname{codim} Y\backslash Y_0 \geqslant 2$ and there is a line bundle $\mathscr{L}_0$ on Y_0 such that $f|_{X_0}$ is the cyclic cover with respect to $\mathscr{L}_0^{\otimes 2} \cong (\mathscr{O}_Y(D))|_{Y_0}$ (See [CD89, §0.1]). Since Y is locally factorial, there is a line bundle $\mathscr{L}$ on Y such that $\mathscr{L}|_{Y_0} \cong \mathscr{L}_0$. Then $\mathscr{L}^{\otimes 2} \cong \mathscr{O}_Y(D)$. Let $g : Z \to Y$ be the corresponding cyclic cover. Then X and Z are isomorphic over Y_0. Since X is normal, and both X and Z are finite over Y, we obtain that X is the normalisation of Z.

If Y is Cohen-Macaulay, then $X \cong Z$ by Lemma 5.2.1. Thus X is smooth if and only if D and Y are smooth (see [KM98, Lem. 2.51]). □

Lemma 5.2.4. *Let $f : X \to Y$ be a double cover between normal varieties. Assume that Y is $\mathbb{Q}$-factorial. Let $D \subseteq Y$ be the codimension 1 part of the discriminant of f. Then X is Fano if and only if $-K_Y - \frac{1}{2}D$ is $\mathbb{Q}$-ample.*

Proof. Since Y is $\mathbb{Q}$-factorial, $-K_X$ is $\mathbb{Q}$-linearly equivalent to $-f^*(K_Y + \frac{1}{2}D)$. Since f is finite, we obtain that $-K_X$ is $\mathbb{Q}$-ample if and only if $-K_Y - \frac{1}{2}D$ is $\mathbb{Q}$-ample. □

Lemma 5.2.5. *Let $f : X \to Y$ be an equidimensional Fano fibration of relative dimension 1 between normal projective varieties with $\mathbb{Q}$-factorial klt singularities. Then the relative Picard number of f is 1 if and only if f^*D is irreducible for every prime divisor D in Y.*

Proof. If the relative Picard number of f is 1, then f^*D is irreducible for every prime divisor D in Y.

Assume that the relative Picard number of f is at least 2. Since f is a Fano fibration and X has $\mathbb{Q}$-factorial klt singularities, we can run a MMP for X over Y by [BCHM10, Cor. 1.3.3]. We obtain the following sequence of birational maps over Y

$$\begin{array}{ccccccc} X_0 & - - \to & X_1 & - - \to & \cdots & - - \to & X_k \\ \downarrow & \swarrow & & & & \swarrow & \\ Y & & & & & & \end{array}$$

such that $X = X_0$ and that X_k is a Mori fiber space. Let $X_k \to B$ be the Mori fibration. Then there is a natural morphism $B \to Y$. Since f has relative dimension 1, the natural morphism $B \to Y$ is birational. On the one hand, since f is equidimensional, for any prime divisor in X, its image by f has codimension at most 1 in Y. Hence, the morphism $B \to Y$ does not contract any divisor. On the other hand, since Y is $\mathbb{Q}$-factorial, the exceptional locus of $B \to Y$ is either empty or pure of codimension 1. Hence $B \to Y$ is an isomorphism.

Since the relative Picard number of X is larger than 1, there is some X_i in the previous sequence such that $X_i \to X_{i+1}$ is a divisorial contraction. In particular, if $f_i : X_i \to Y$ is the natural fibration, then there is a prime divisor D in Y such that f_i^*D is reducible. Hence f^*D is also reducible. □

Lemma 5.2.6. *Let $g : Y \to Z$ be an equidimensional fibration between smooth projective varieties. Let $r : V \to Z$ be a Mori fibration with V smooth. Let X be the fiber product $Y \times_Z V$. Let $f : X \to Y$ and $\pi : X \to V$ be the natural fibrations.*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

Assume that X is a smooth Fano variety and that the discriminants of g and r do not have common components. Then the relative Picard number of f is 1.

Proof. Assume the opposite. Then by Lemma 5.2.5, there is a prime divisor D in Y such that f^*D is reducible. Let $R = f^{-1}(D)$ and let R_1, R_2 be two

different components of R. Let α be a general point in D. Let $E = g(D)$ and let $\beta = g(\alpha) \in E$. Then E is irreducible. Since g is equidimensional, the codimension of E is at most 1. Since $X = Y \times_Z V$, the fiber C' of r over β is isomorphic fiber C of f over α. Thus C' is reducible for C is reducible. Since β is a general point in E, we obtain that E is a divisor and is contained in the discriminant of r.

Let $R' = r^{-1}(E)$. Then, by Lemma 5.2.5, it is irreducible since r is a Mori fibration. We have the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi|_R} & R' \\ {\scriptstyle f|_R}\downarrow & & \downarrow{\scriptstyle r|_{R'}} \\ D & \xrightarrow[g|_D]{} & E \end{array}$$

Since $X = Y \times_Z V$, the fibration π is equidimensional. Since R' is irreducible, we obtain that the projections $\pi|_{R_1} : R_1 \to R'$ and $\pi|_{R_2} : R_2 \to R'$ are surjective. In particular, general fibers of $\pi|_R$ are reducible.

Let a be a general point of R' and let $b = r(a) \in E$. Since $X = Y \times_Z V$, the fiber of $\pi|_R$ over a is the same as the fiber of $g|_D$ over b. Since the discriminants of g and r do not have common components and b is a general point on E, the fiber of $g|D$ over b is irreducible. Hence the fiber of $\pi|_R$ over a is irreducible. This is a contradiction. □

Thanks to this lemma, we can show that we can construct smooth Fano varieties X such that $\rho(X) = n-1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$ with the methods in Example 5.1.3 and Example 5.1.4. For simplicity, we only look at the case of Example 5.1.4.2. The other cases are similar.

Construction 5.2.7. We will construct a smooth Fano variety X, of dimension n, with the methods of Example 5.1.4.2, such that $\rho(X) = n - 1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$.

Let W be a smooth divisor of degree $(1, k)$ in $B_1 \times B_2$, where $k \in \{1, 2\}$. Then the Picard number of W is 2 by the Lefschetz theorem (see [Laz04, Example 3.1.25]). We write

$$(\mathbb{P}^1)^{n-3} \times W = \mathbb{P}^1 \times T \text{ and } (\mathbb{P}^1)^{n-3} \times B_1 = \mathbb{P}^1 \times R.$$

There is a natural equidimensional fibration $T \to R$. We can choose a double cover $N_1 \to \mathbb{P}^1 \times R$, branched along some smooth divisor D of degree $(2, ..., 2, 2)$, such that the discriminant of $N_1 \to R$ and the discriminant of $T \to R$ do not have common components. Moreover, we choose D such that q^*D is smooth, where $q : \mathbb{P}^1 \times T \to \mathbb{P}^1 \times R$ is the natural fibration.

Let X be the pullback of $(\mathbb{P}^1)^{n-3} \times W$ by $N_1 \times B_2 \to (\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. Then we have

$$X \cong ((\mathbb{P}^1)^{n-3} \times W) \times_{(\mathbb{P}^1)^{n-3} \times B_1} N_1 = (\mathbb{P}^1 \times T) \times_{\mathbb{P}^1 \times R} N_1 \cong T \times_R N_1.$$

Now we will show that $\rho(X) = n-1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. By [Dru14, Lem. 3.4], the Picard number of N_1 is the same as the Picard number of $\mathbb{P}^1 \times R$. Thus the natural projection $N_1 \to R$ is a Mori fibration. Note that the natural morphism $\varphi : X \to \mathbb{P}^1 \times T$ is a double cover branched along q^*D. Since q^*D and $\mathbb{P}^1 \times T$ are smooth, X is smooth by Lemma 5.2.3. Moreover, X is a Fano variety by Lemma 5.2.4. Hence, by Lemma 5.2.6, $X \to T$ has relative Picard 1. Since T has Picard number $n-2$, we have $\rho(X) = n-1$.

Let E be a pseudo-effective divisor in X. Since $\rho(X) = n-1$, E is numerically equivalent to the pullback of a $\mathbb{Q}$-divisor of degree $(a_1, ..., a_{n-3}, b_1, b_2)$ in $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. Let F_1 be a fiber of the natural projection from X to the product $(\mathbb{P}^1)^{n-4} \times W$ (here, $(\mathbb{P}^1)^{n-4}$ is the product of the last $(n-4)$ $\mathbb{P}^1$'s of $(\mathbb{P}^1)^{n-3}$). Then the intersection number $F_1 \cdot E$ is at least zero. Thus $a_1 \geqslant 0$. Similarly, we obtain that $a_2, ..., a_{n-3} \geqslant 0$. Let F' be the fiber of the natural projection $X \to (\mathbb{P}^1)^{n-3} \times B_1$. Then $F' \cdot E \geqslant 0$. This shows that $b_2 \geqslant 0$. Similarly, we have $b_1 \geqslant 0$. Hence E is nef. Thus we obtain that $\mathrm{Nef}(X) = \mathrm{Psef}(X)$.

5.3 Fibrations on varieties of Theorem 5.1.5

Let X be a $\mathbb{Q}$-factorial klt Fano variety. Then for every face V of $\overline{\mathrm{NE}}(X)$, there is a fibration $f : X \to Y$ which contracts exactly the curves whose classes are in V. Conversely, if $f : X \to Y$ is a fibration, then the curves in the fibers of f generates a face V of $\overline{\mathrm{NE}}(X)$ (See for example [KMM87, §3-2]).

Lemma 5.3.1. *Let X be a Fano variety with $\mathbb{Q}$-factorial klt (resp. locally factorial canonical) singularities such that $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Let $f : X \to Y$ be any fibration. Then Y is also a Fano variety with $\mathbb{Q}$-factorial klt (resp. locally factorial canonical) singularities such that $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$.*

Proof. By [Dru14, Lem. 4.2], we have $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$. If X is with locally factorial canonical singularities, then so is Y by [Dru14, Cor. 4.8]. If X has $\mathbb{Q}$-factorial klt singularities, then the same argument of the proof of [Dru14, Cor. 4.8] shows that Y also has $\mathbb{Q}$-factorial klt singularities. By [FG12, Thm. 3.1], $-K_Y$ is big. Since $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$, this implies that $-K_Y$ is ample. □

The objective of this section is to prove the two following lemmas.

Lemma 5.3.2. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Assume that there is no fibration from X to $\mathbb{P}^1$. Let $f : X \to Y$ be a fibration. Then Y is also a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$.*

Lemma 5.3.3. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Assume that there is a fibration $f_1 : X \to W$ and a projective morphism $f_2 : X \to (\mathbb{P}^1)^r$ such that the product $f_1 \times f_2 : X \to W \times (\mathbb{P}^1)^r$ is a finite surjective morphism. Then W is also a Fano variety with locally factorial canonical singularities such that W is smooth in codimension 2, $\rho(W) = \dim W - 1$ and $\mathrm{Nef}(W) = \mathrm{Psef}(W)$.*

We will first prove some preliminary results.

Lemma 5.3.4. *Let X be a $\mathbb{Q}$-factorial klt Fano variety such that $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Let $f : X \to Y$ be a fibration. If the relative dimension of f is 1, then f is a Mori fibration.*

Proof. There is a Mori fibration $X \to Z$ over Y by [Dru14, Lem. 4.4]. The variety Z is a $\mathbb{Q}$-factorial klt Fano variety such that $\mathrm{Nef}(Z) = \mathrm{Psef}(Z)$ by Lemma 5.3.1. Since f has relative dimension 1, the natural fibration $Z \to Y$ is birational. Since $\mathrm{Nef}(Z) = \mathrm{Psef}(Z)$, we have $Z \cong Y$ by [Dru14, Lem. 4.4]. This implies that f is a Mori fibration. □

Lemma 5.3.5. *Let $f : X \to Y$ be a Mori fibration such that X has locally factorial canonical singularities. If X is smooth in codimension k and f is equidimensional of relative dimension 1, then Y is also smooth in codimension k.*

Proof. Let $y \in Y$ be a point. The problem is local around y. Without loss of generality, we may assume that Y is affine. Since f is a Fano fibration of relative dimension 1, by the main theorem of [AW93], the linear system $|-K_X|$ is basepoint-free. Let Z be a general divisor in the linear system $|-K_X|$. Then Z is smooth in codimension k. Moreover, since f is equidimensional, by shrinking Y if necessary, we may assume further that the morphism $f|_Z : Z \to Y$ is finite surjective of degree 2.

Note that Y has klt singularities (see [Fuj99, Cor. 3.5]). Thus it is Cohen-Macaulay. Moreover, it is locally factorial (see for example [Dru14, Lem. 4.6]). Since Z is smooth in codimension k, by Lemma 5.2.3, Y is smooth in codimension k. □

Remark that this lemma is not true without assuming that X is locally factorial. There is a Mori fibration $f : X \to Y$ such that X is a threefold with terminal singularities and that Y is singular, see [MP08, Example-Definition 1.1.1]

Lemma 5.3.6. *Let X be a Fano variety with $\mathbb{Q}$-factorial klt singularities such that $\rho(X) = \dim X - 1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Let $f : X \to Y$ be a Mori fibration. Then the dimension of Y is equal to $\rho(Y)$ or $\rho(Y) + 1$.*

Proof. By Lemma 5.3.1, Y is a $\mathbb{Q}$-factorial klt Fano variety with $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$. In particular, we have $\rho(Y) \leqslant \dim Y$ by [Dru14, Lem. 4.9]. Since f is a Mori fibration, we have $\rho(Y) = \rho(X) - 1$. Hence $\dim Y \leqslant \dim X - 1 = \rho(X) = \rho(Y) + 1$. Thus $\dim Y$ is equal to $\rho(Y)$ or $\rho(Y) + 1$. □

Lemma 5.3.7. *Let X be a $\mathbb{Q}$-factorial klt Fano variety such that $\mathrm{Nef}(X) = \mathrm{Psef}(X)$ and $\rho(X) = \dim X$. Then there is a fibration from X to $\mathbb{P}^1$.*

Proof. Let V be a face in $\overline{\mathrm{NE}}(X)$ of codimension 1 and let $f : X \to B$ be the corresponding contraction. Then $\dim B = \rho(B) = 1$ by [Dru14, Lem. 4.9.2]. This shows that $B \cong \mathbb{P}^1$. □

Lemma 5.3.8. *Let X be a Fano variety with $\mathbb{Q}$-factorial klt singularities such that $\rho(X) = \dim X - 1 \geqslant 2$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Assume that there is no fibration from X to $\mathbb{P}^1$. Then every Mori fibration $f : X \to Y$ is equidimensional of relative dimension 1.*

Proof. We will prove the lemma by induction on the dimension of X. If $\dim X = 3$, then $\dim Y = 2$. In this case, the fibration f is equidimensional. Assume that the lemma is true if $\dim X = n$ for some $n \geqslant 3$.

Now we assume that $\dim X = n+1$. Suppose that there is a fiber F of f such that $\dim F \geqslant 2$. Assume that f is the fibration which corresponds to an extremal ray R of $\overline{\mathrm{NE}}(X)$. Let R_1 be another extremal ray such that R and R_1 generates a face V of dimension 2. Let $g : X \to Z$ be the Mori fibration which corresponds to R_1 and let $X \to W$ be the fibration corresponding to V. By the rigidity lemma, we obtain the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow{p} & W \end{array}$$

By Lemma 5.3.1, Z is a $\mathbb{Q}$-factorial klt Fano variety such that $\mathrm{Nef}(Z) = \mathrm{Psef}(Z)$. By Lemma 5.3.6, $\rho(Z)$ is either $\dim Z$ or $\dim Z - 1$. Since there is no fibration from Z to $\mathbb{P}^1$, we have $\rho(Z) = \dim Z - 1$ by Lemma 5.3.7. Since $\rho(Z) = \rho(X) - 1 = n - 1$, we obtain that $\dim Z = n$. By induction hypothesis, the morphism p is equidimensional of relative dimension 1. Since none of the curves in F is contracted by g, the dimension of $E = g(F)$ is also larger than 1. Moreover, since $h(f(F))$ is a point, $p(E)$ is also a point. We obtain a contradiction. This completes the induction and the proof of the lemma. □

Now we will prove Lemma 5.3.2 and Lemma 5.3.3.

Proof of Lemma 5.3.2. We will prove the lemma by induction on the relative Picard number k of f. If f is a Mori fibration, then by Lemma 5.3.8, f is equidimensional of relative dimension 1. Hence $\rho(Y) = \dim(Y) - 1$. Moreover, Y is a Fano variety with locally factorial canonical singularities such that

$\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$ by Lemma 5.3.1 and is smooth in codimension 2 by Lemma 5.3.5.

Assume that the lemma is true for $k \leqslant l$ for some $l \geqslant 1$. Now we consider the case when $k = l + 1$. There is a Mori fibration $g : X \to Z$ over Y since $\mathrm{Nef}(X) = \mathrm{Psef}(X)$ (see [Dru14, Lem. 4.4]). Hence, by induction, Z is a Fano variety with locally factorial canonical singularities such that Z is smooth in codimension 2, $\rho(Z) = \dim Z - 1$ and $\mathrm{Nef}(Z) = \mathrm{Psef}(Z)$. Moreover, the relative Picard number of $Z \to Y$ is l. Thus, by induction hypotheses, Y is a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$. This completes the proof of the Lemma. □

Proof of Lemma 5.3.3. We will prove the lemma by induction on r. If $r = 1$, then the morphism f_1 is equidimensional of relative dimension 1 since $f_1 \times f_2$ is finite. By Lemma 5.3.4, f_1 is a Mori fibration. By Lemma 5.3.1 and Lemma 5.3.5, we obtain that W is a Fano variety with locally factorial canonical singularities such that W is smooth in codimension 2, $\rho(W) = \dim W - 1$ and $\mathrm{Nef}(W) = \mathrm{Psef}(W)$. Assume that the lemma is true for $r \leqslant k$ where $k \geqslant 1$ is an integer.

Now we assume that $r = k + 1$. We write $(\mathbb{P}^1)^{k+1} = (\mathbb{P}^1)^k \times \mathbb{P}^1$. Let $q_{k+1} : (\mathbb{P}^1)^{k+1} \to (\mathbb{P}^1)^k$ be the natural projection onto the product of the first k $\mathbb{P}^1$'s and let $p_{k+1} : (\mathbb{P}^1)^{k+1} \to \mathbb{P}^1$ be the natural projection onto the last factor. Let $\pi : X \to Y$ be the Stein factorisation of the product $f_1 \times q_{k+1} : X \to W \times (\mathbb{P}^1)^k$. Then the product $\pi \times p_{k+1} : X \to Y \times \mathbb{P}^1$ is also a finite surjective morphism. Hence, by induction hypotheses, Y is a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$. There is a natural fibration $g_1 : Y \to W$ induced by $X \to W$. There is a morphism $g_2 : Y \to (\mathbb{P}^1)^k$ induced by $X \to (\mathbb{P}^1)^k$. Since $X \to W \times (\mathbb{P}^1)^k \times \mathbb{P}^1$ is finite surjective, we obtain that $g_1 \times g_2 : Y \to W \times (\mathbb{P}^1)^k$ is also finite surjective. By induction hypotheses, W is a Fano variety with locally factorial canonical singularities such that W is smooth in codimension 2, $\rho(W) = \dim W - 1$ and $\mathrm{Nef}(W) = \mathrm{Psef}(W)$. This completes the proof of the lemma.

$$\begin{array}{ccccc} X & \xrightarrow[\text{surjective}]{\text{finite}} & Y \times \mathbb{P}^1 & \xrightarrow[\text{surjective}]{\text{finite}} & W \times (\mathbb{P}^1)^{k+1} \\ & & \downarrow & & \\ & & Y & \xrightarrow[\text{surjective}]{\text{finite}} & W \times (\mathbb{P}^1)^k \end{array}$$

□

5.4 Finite morphisms between Fano threefolds

In this section, we will prove some results on finite morphisms between Fano threefolds. As a corollary, we prove Theorem 5.1.2. Recall that if X is

a Fano threefold with Gorenstein canonical singularities, then

$$h^0(X, \mathscr{O}_X(-K_X)) = -\frac{1}{2}K_X^3 + 3.$$

In particular, $-K_X^3$ is a positive even integer (see [Rei83, §4.4]). We will first prove some lemmas.

Lemma 5.4.1. *Let $f : X \to Y$ be a Fano fibration of relative dimension 1 between normal quasi-projective varieties. Assume that X is smooth in codimension 2. Then there is an open subset U of Y whose complement has codimension at least 2 such that f has reduced fibers over U.*

Proof. By taking general hyperplane sections in Y, we can reduce to the case when X is a smooth surface and Y is a smooth curve.

Let y be a point in Y and let $E = f^*y$. Then $-K_X \cdot E = 2$. Since K_X is Cartier and f is a Fano fibration, this implies that E has at most two components. If E has two components C and D, then both of them are reduced. If E has one component C, then E is reduced for X is smooth (by a similar argument of Proposition 3.2.3). □

Lemma 5.4.2. *Let X be a normal threefold and let $B_1 \cong B_2 \cong \mathbb{P}^2$. Assume that there are two equidimensional fibrations $f_1 : X \to B_1$ and $f_2 : X \to B_2$ such that $f_1 \times f_2 : X \to B_1 \times B_2$ is finite onto its image W. Assume that W is of degree (p, q) such that $p, q \in \{1, 2\}$. Then W is normal.*

Proof. Since W is Cohen-Macaulay, we only need to prove that it is smooth in codimension 1. Let Z be the singular locus of W. Let $p_i : W \to B_i$ be the natural projection for $i = 1, 2$. Then p_1 is equidimensional. Since W is Cohen-Macaulay and B_1 is smooth, this implies that p_1 is flat.

Since general fibers of $f_1 : X \to B_1$ are irreducible, general fibers of p_1 are irreducible. Since W is reduced, general fibers of p_1 are reduced. Since $q \in \{1, 2\}$, we obtain that general fibers of p_1 are smooth rational curves. Since p_1 is flat, this implies that $p_1(Z)$ is a proper subvariety of B_1.

By symmetry, we can obtain that $p_2(Z)$ is also a proper subvariety of B_2. Thus $\operatorname{codim} Z \geqslant 2$. This completes the proof of the lemma. □

Lemma 5.4.3. *Let W be an ample normal hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$. Assume that W is a Fano variety with $\mathbb{Q}$-factorial klt singularities. Then the natural morphisms from W to $\mathbb{P}^2$ are equidimensional.*

Proof. Since W is ample, it has Picard number 2 by the Lefschetz theorem (see [Laz04, Example 3.1.25]). Thus, if $g : W \to \mathbb{P}^2$ is a natural projection, then it is a Mori fibration. Since W is a $\mathbb{Q}$-factorial klt threefold, this implies that g is equidimensional. □

Lemma 5.4.4. *Let $f : X \to Y$ be a finite surjective morphism between normal $\mathbb{Q}$-factorial varieties. Assume that $-K_X$ is big. Then $-K_Y$ is a big divisor.*

Proof. See [Dru14, Lem. 8.1]. □

Lemma 5.4.5. *Let $f : X \to Y$ be an equidimensional Fano fibration of relative dimension 1 such that X has Gorenstein canonical singularities and Y is smooth. Then f a conic bundle. That is, at least locally on Y, X can be embedded in $\mathbb{P}^2 \times Y$ such that every fiber of f is a conic in $\mathbb{P}^2$.*

Proof. Since $-K_X$ is relatively ample, $R^i f_*(\mathscr{O}_X(-K_X)) = 0$ for all $i > 0$ by the Kawamata-Viehweg vanishing theorem (see [KMM87, Thm. 1-2-5]). Since f is equidimensional of relative dimension 1, for every $y \in Y$, we have $h^2(X_y, \mathscr{O}_{X_y}(-K_X|_{X_y})) = 0$. By [Har77, Thm. III.12.11], we obtain that, for all $y \in Y$,

$$R^1 f_*(\mathscr{O}_X(-K_X)) \otimes k(y) \to H^1(X_y, \mathscr{O}_{X_y}(-K_X|_{X_y}))$$

is an isomorphism, where $k(y)$ is the residue field of y. Hence

$$h^1(X_y, \mathscr{O}_{X_y}(-K_X|_{X_y})) = 0$$

for all $y \in Y$. By [Har77, Thm. III.12.11] again, we obtain that

$$f_*(\mathscr{O}_X(-K_X)) \otimes k(y) \to H^0(X_y, \mathscr{O}_{X_y}(-K_X|_{X_y}))$$

is an isomorphism for all $y \in Y$.

Note that general fibers of f are smooth rational curves. Hence the Euler characteristic of the restriction of $\mathscr{O}_X(-K_X)$ on a general fiber of f is 3. Since f is equidimensional, X is Cohen-Macaulay and Y is smooth, the morphism f is flat. Since $\mathscr{O}_X(-K_X)$ is locally free on X, it is flat over Y. Thus the Euler characteristic of the restriction of $\mathscr{O}_X(-K_X)$ on every fiber of f is 3 (see the proof of [Har77, Thm. III.9.9]). This shows that $f_*(\mathscr{O}_X(-K_X))$ is a locally free sheaf of rank 3 by [Har77, Cor. III.12.9].

The problem is local on Y. Thus, we may assume that Y is affine and $f_*(\mathscr{O}_X(-K_X))$ is a free sheaf of rank 3. By the main theorem of [AW93], $\mathscr{O}_X(-K_X)$ is f-relatively generated. Since the sheaf $f_*(\mathscr{O}_X(-K_X))$ has rank 3, $f^* f_*(\mathscr{O}_X(-K_X))$ induces a morphism, over Y,

$$\varphi : X \to \mathbb{P}^2 \times Y.$$

Since $f_*(\mathscr{O}_X(-K_X)) \otimes k(y) \to H^0(X_y, \mathscr{O}_{X_y}(-K_X|_{X_y}))$ is an isomorphism and the invertible sheaf $\mathscr{O}_{X_y}(-K_X|_{X_y})$ is very ample for general $y \in Y$, the morphism φ is birational onto its image. Since $-K_X$ is f-relatively ample, the morphism φ is in fact an isomorphism onto its image.

Since $-K_X$ has degree 2 on every fiber of f, we obtain that $\varphi(X_y) \subseteq \mathbb{P}^2 \times \{y\}$ is a conic for every $y \in Y$. This completes the proof of the lemma. □

Lemma 5.4.6. *Let Z_1, Z_2 be two surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let Z be a normal hypersurface of degree $(1,1,1,1)$ in $Z_1 \times Z_2$. Assume that there is a finite morphism $f : X \to Z$ of degree $d \leqslant 2$ such that*

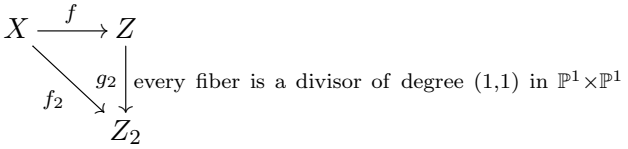
- *X is Fano threefold with Gorenstein canonical singularities;*
- *the natural morphisms $f_1 : X \to Z_1$ and $f_2 : X \to Z_2$ induced by f are equidimensional fibrations;*
- *$K_X \equiv \frac{1}{d} f^* K_Z$.*

Then f is an isomorphism.

Proof. Assume that f is of degree 2. Let D be the codimension 1 part of the discriminant of $f : X \to Z$. Then K_X is linearly equivalent to $f^*(K_Z) + f^{-1}(D)$. Since K_X and K_Z are Cartier divisors, we obtain that $f^{-1}(D)$ is Cartier. Since f is finite, this implies that D is $\mathbb{Q}$-Cartier. Then K_X is $\mathbb{Q}$-linearly equivalent to $f^*(K_Z + \frac{1}{2}D)$. Thus we have $D \equiv -K_Z$ (it is here we use the condition $K_X \equiv \frac{1}{d} f^* K_Z$). In particular, D is $\mathbb{Q}$-ample.

There are two natural morphisms from Z to $\mathbb{P}^1 \times Z_2$ induced by the natural projections from $Z_1 \times Z_2$ to $\mathbb{P}^1 \times Z_2$. Let $\varphi : Z \to \mathbb{P}^1 \times Z_2$ be one of them. Since Z is of degree $(1,1,1,1)$, we obtain that φ is birational. Moreover, φ is not an isomorphism since the Picard number of Z is 4 by the Lefschetz theorem ([Laz04, Example 3.1.25]). Thus its exceptional locus is non empty. Moreover, it is pure of codimension 1 since $\mathbb{P}^1 \times Z_2$ is smooth. Hence the morphism φ contracts at least one divisor. Since the natural fibration $g_2 : Z \to Z_2$ factors through φ, we obtain that there is a prime divisor H in Z_2 such that $g_2^* H$ is reducible.

The fibration g_2 is equidimensional since f_2 is. Every fiber of g_2 is isomorphic to a divisor of degree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Hence it is either a reduced fiber isomorphic to $\mathbb{P}^1$ or the union of two smooth rational curves which intersect at one point. This shows that $g_2^* H$ has two components R_1, R_2 and every fiber of g_2 over H has two components.



The divisor $f_2^* H$ has at least two components. Since X has Gorenstein canonical singularities and f_2 is equidimensional of relative dimension 1, every fiber of f_2 has at most two components by Lemma 5.4.5. We obtain that $f_2^* H$ also has two components S_1, S_2. Assume that $f(S_1) = R_1$ and $f(S_2) = R_2$. Let b be any point on H and let G be the fiber of g_2 over b. Then G has exactly two irreducible components by the previous paragrah. We write $G = G_1 \cup G_2$, where $G_1 = G \cap R_1$ and $G_2 = G \cap R_2$. Let F be the fiber of f_2 over b. Since the natural morphism $F \to G$ is surjective, F is reducible. By Lemma 5.4.5 again, the fiber F is reduced and is the union of

two smooth rational curves F_1 and F_2 which meet at one point. In particular, G is not contained in D. We assume that $F_i \subseteq S_i$ for $i = 1, 2$.

Both $f|_{F_1} : F_1 \to G_1$ and $f|_{F_2} : F_2 \to G_2$ are finite morphisms of degree 2. Since $G_1 \backslash G_2$ is simply connected, it contains a point which is in the branch locus of $f|_{F_1} : F_1 \to G_1$. Since g_2, being an equidimensional fibration from a Cohen-Macaulay variety to a smooth variety, is flat, $G_1 \backslash G_2$ is contain in Z_{ns}, the smooth locus of Z. Moreover, over Z_{ns}, the morphism f is branched exactly along $Z_{ns} \cap D$ by the Zariski purity theorem (see [Zar58, Prop. 2]). Thus D meets $G_1 \backslash G_2$ at at least one point. By symmetry, D meets $G_2 \backslash G_1$ at at least one point. However, since $D \equiv -K_Z$ and g_2 is a Fano fibration, the intersection number $D \cdot G$ is equal to 2. Thus we obtain that G and D meet at exactly two points which are smooth points of G.

Since b is chosen arbitrarily, we obtain that $D \cap R_1 \cap R_2$ is empty. However, since $R_1 \cap R_2$ is a subscheme of dimension 1 and D is $\mathbb{Q}$-ample, $D \cap R_1 \cap R_2$ is not empty. We obtain a contradiction. □

5.4.1 Finite covers of $\mathbb{P}^1 \times \mathbb{P}^2$

Proposition 5.4.7. *Let X be a Fano threefold with Gorenstein canonical singularities. Assume that there are two fibrations $f_1 : X \to \mathbb{P}^1$ and $f_2 : X \to \mathbb{P}^2$ such that $f = f_1 \times f_2 : X \to \mathbb{P}^1 \times \mathbb{P}^2$ is finite. Assume that K_X is numerically equivalent to the pullback of some $\mathbb{Q}$-divisor in $\mathbb{P}^1 \times \mathbb{P}^2$ by f. Then $d = \deg f$ is at most 2. Moreover, if $d = 2$, then f is a double cover which is branched along a divisor of degree $(2, k)$, where $k \in \{2, 4\}$.*

Proof. Let A_1, A_2 be two divisors in X such that $\mathscr{O}_X(A_1) \cong f_1^* \mathscr{O}_{\mathbb{P}^1}(1)$ and $\mathscr{O}_X(A_2) \cong f_2^* \mathscr{O}_{\mathbb{P}^2}(1)$. By assumption, there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1 A_1 + a_2 A_2$.

We will first prove that $d \leqslant 2$. Note that

$$A_1 A_2^2 = d \text{ and } A_1^2 A_2 = A_1^3 = A_2^3 = 0.$$

Since f_2 is a Fano fibration, general fibers of f_2 are smooth rational curves. This implies that $-K_X \cdot A_2^2 = 2$. Hence, $a_1 = \frac{2}{d}$. Moreover, we have

$$(-K_X)^3 = 3a_1 a_2^2 A_1 A_2^2 = 3a_1 a_2^2 d = 6a_2^2.$$

Since $-K_X$ is a Cartier divisor, we obtain that $6a_2^2 \in \mathbb{Z}$. Hence $a_2 \in \mathbb{Z}$.

We have $(-K_X - a_2 A_2) \equiv \frac{2}{d} A_1$. Hence the Cartier divisor $-K_X - a_2 A_2$ is numerically trivial on every fiber of f_1. Since f_1 is a Fano fibration, we obtain that $-K_X - a_2 A_2$ is linearly equivalent to some integral multiple of A_1 by [KMM87, Lem. 3-2-5]. Thus $\frac{2}{d} \in \mathbb{Z}$ and $d \leqslant 2$.

If $d = 1$, then $X \cong \mathbb{P}^1 \times \mathbb{P}^2$. If $d = 2$, then assume that f is branched along a divisor D of degree (p, q). Then by Lemma 5.2.4, X is Fano if and only if $p \leqslant 3$ and $q \leqslant 5$. Since f is a double cover, by Lemma 5.2.3, there

is a line bundle $\mathscr{L}$ on $\mathbb{P}^1 \times \mathbb{P}^2$ such that $\mathscr{L}^{\otimes 2} \cong \mathscr{O}(D)$. Hence p, q are even. Moreover, since f_1, f_2 have connected fibers, we have $p, q > 0$. Hence f is branched along a divisor of degree $(2,2)$ or $(2,4)$. This completes the proof of the proposition. □

Lemma 5.4.8. *Let X be a Fano threefold with Gorenstein canonical singularities. Assume that there is a fibration $f_1 : X \to \mathbb{P}^1$ and a projective surjective morphism $f_2 : X \to \mathbb{P}^2$ such that general fibers of f_2 have two connected components. Assume that $f = f_1 \times f_2 : X \to \mathbb{P}^1 \times \mathbb{P}^2$ is finite. Let A_1, A_2 be two divisors in X such that $\mathscr{O}_X(A_1) \cong f_1^* \mathscr{O}_{\mathbb{P}^1}(1)$ and $\mathscr{O}_X(A_2) \cong f_2^* \mathscr{O}_{\mathbb{P}^2}(1)$. Assume that there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1 A_1 + a_2 A_2$. Then f is of degree 2 or 4 and $a_1, a_2 \in \mathbb{Z}$.*

Proof. Since general fibers of f_2 have two connected components, the degree of f is an even integer. Set $\deg f = 2d$. Then we have

$$A_1 A_2^2 = 2d \text{ and } A_1^2 A_2 = A_1^3 = A_2^3 = 0.$$

Note that each component of a general fiber of f_2 is a smooth rational curve. This implies that

$$-K_X \cdot A_2^2 = 2 \times 2 = 4.$$

Hence, $a_1 = \frac{2}{d}$. Moreover, we have

$$(-K_X)^3 = 3 a_1 a_2^2 A_1 A_2^2 = 6 a_1 a_2^2 d = 12 a_2^2.$$

Since $(-K_X)^3$ is an even integer, we obtain that $6a_2^2 \in \mathbb{Z}$. This implies that $a_2 \in \mathbb{Z}$.

We have $(-K_X - a_2 A_2) \equiv \frac{2}{d} A_1$. As in the proof of Proposition 5.4.7, we obtain that $\frac{2}{d} \in \mathbb{Z}$. Hence $d \leqslant 2$ and $a_1 \in \mathbb{Z}$. □

5.4.2 Finite covers of hypersurfaces of $\mathbb{P}^2 \times \mathbb{P}^2$

Proposition 5.4.9. *Let X be a Fano threefold with Gorenstein canonical singularities. Assume that there are two equidimensional fibrations $f_1 : X \to B_1$ and $f_2 : X \to B_2$ such that $B_1 \cong B_2 \cong \mathbb{P}^2$ and that $f_1 \times f_2 : X \to B_1 \times B_2$ is finite onto its image W. Assume that K_X is numerically equivalent to the pullback of some $\mathbb{Q}$-divisor in $B_1 \times B_2$ by $f_1 \times f_2$. Let $f : X \to W$ be the natural morphism. Then there are exactly four possibilities :*

(1) f is an isomorphism, X is a hypersurface of degree $(1,1)$;

(2) f is an isomorphism, X is a hypersurface of degree $(2,1)$ or $(1,2)$;

(3) f is an isomorphism, X is a hypersurface of degree $(2,2)$;

(4) W is a smooth hypersurface of degree $(1,1)$, f is a double cover branched along a divisor which is the intersection of W and a divisor of degree $(2,2)$ in $B_1 \times B_2$.

Proof. Let A_1, A_2 be two divisors in X such that $\mathscr{O}_X(A_1) \cong f_1^*\mathscr{O}_{\mathrm{B}_1}(1)$ and $\mathscr{O}_X(A_2) \cong f_2^*\mathscr{O}_{B_2}(1)$. By assumption, there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1A_1 + a_2A_2$. Set $d = \deg f$. Assume that W is of degree (p, q) in $B_1 \times B_2$. We have

$$A_1^2A_2 = dq \text{ and } A_1A_2^2 = dp.$$

Moreover, $A_1^3 = A_2^3 = 0$. Since general fibers of f_1 and f_2 are smooth rational curves, we have $-K_X \cdot A_1^2 = -K_X \cdot A_2^2 = 2$. Hence

$$a_1 = \frac{2}{dp} \text{ and } a_2 = \frac{2}{dq}.$$

We have

$$-K_X^3 = 3a_1a_2^2A_1A_2^2 + 3a_1^2a_2A_1^2A_2 = \frac{24}{d^2}(\frac{1}{q^2} + \frac{1}{p^2}).$$

Since $-K_X^3$ a positive even integer, the number

$$r = \frac{12}{d^2}(\frac{1}{q^2} + \frac{1}{p^2})$$

is a positive integer. Without loss of generality, we may assume that $q \leqslant p$.

If $d = 1$, then r is an integer if and only if $q, p \in \{1, 2\}$. The morphism $f : X \to W$ is the normalisation map. However, W is normal by Lemma 5.4.2. Thus f is an isomorphism.

If $d = 2$, then we can only have $q = p = 1$. Since $f_1 : X \to B_1$ is equidimensional, so is the induced projection $p_1 : W \to B_1$. Hence every fiber of p_1 is a line in $\mathbb{P}^2$, which is smooth. Since W is Cohen-Macaulay and $\mathbb{P}^2$ is smooth, the morphism p_1 is flat. Hence p_1 is a smooth morphism and W is smooth. Since W is ample in $\mathbb{P}^2 \times \mathbb{P}^2$, the natural morphism $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) \to \mathrm{Pic}(W)$ is an isomorphism by the Lefschetz theorem (see [Laz04, Example 3.1.25]). Similarly to the last paragraph of the proof of Proposition 5.4.7, we obtain that f is branched along a divisor which is the intersection of W and a divisor of degree $(2, 2)$ in $B_1 \times B_2$.

If $d \geqslant 3$, then we have

$$1 \leqslant r \leqslant \frac{4}{3}(\frac{1}{q^2} + \frac{1}{p^2}).$$

Hence either $q = 1$ or $q, p \leqslant 2$. If $q = p = 1$, then $r = \frac{24}{d^2}$. In this case, r is an integer if and only $d = 1$ or 2. This is impossible since $d > 2$. If $q = 1$ and $p \geqslant 2$, then the previous inequality implies that $r \leqslant \frac{5}{3}$. Thus $r = 1$ and p^2 divides 12. In this case, we can only have $p = 2$ and $1 = r = \frac{12}{d^2} \cdot \frac{5}{4} = \frac{15}{d^2}$. This is impossible. If $q = p = 2$, then $r = \frac{6}{d^2}$ is an integer if and only if $d = 1$. This is also a contradiction.

Hence $d \leqslant 2$. This completes the proof of the proposition. □

Lemma 5.4.10. *Let X be a Fano threefold with Gorenstein canonical singularities. Let $B_1 \cong B_2 \cong \mathbb{P}^2$. Assume that there are two equidimensional projective morphisms $f_1 : X \to B_1$ and $f_2 : X \to B_2$ such that general fibers of f_i have two connected components for $i = 1, 2$ and $f_1 \times f_2 : X \to B_1 \times B_2$ is finite onto its image W. Let $g_i : X \to Z_i \to B_i$ be the Stein factorisation of f_i for $i = 1, 2$. Let A_1, A_2 be two divisors in X such that $\mathscr{O}_X(A_1) \cong f_1^*\mathscr{O}_{B_1}(1)$ and $\mathscr{O}_X(A_2) \cong f_2^*\mathscr{O}_{B_2}(1)$. Assume that*

- *W is a normal hypersurface with isolated $\mathbb{Q}$-factorial klt singularities of degree (p, q), where $p, q \in \{1, 2\}$;*
- *Z_1 and Z_2 are smooth ;*
- *there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1A_1 + a_2A_2$.*

Let $f : X \to W$ be the natural morphism. Then the degree of f is 4 and $a_1 = a_2 = p = q = 1$.

Proof. Let h be the natural morphism $Z_1 \times Z_2 \to B_1 \times B_2$ and let $Z = h^*W$. We will first show that that Z is normal. Since Z_1 is smooth, the double cover $Z_1 \to B_1$ is a cyclic cover with respect to some isomorphism $(\mathscr{L}_1)^{\otimes 2} \cong \mathscr{O}_{B_1}(D_1)$, where D_1 is a smooth divisor (see Lemma 5.2.3). Let $p_1 : W \to B_1$ be the natural projection. Let $V \subseteq Z_1 \times B_2$ be the pullback of W by $Z_1 \times B_2 \to B_1 \times B_2$. Then $V \cong W \times_{B_1} Z_1$.

Since W is a Fano threefold with $\mathbb{Q}$-factorial klt singularities, p_1 is an equidimensional Fano fibration by Lemma 5.4.3. Moreover, since W has isolated singularities, by Lemma 5.4.1, we obtain that p_1 has reduced fibers over some open subset of B_1 whose complement has codimension at least 2. Thus V is normal and Cohen-Macaulay by Lemma 5.2.2.

$$\begin{array}{ccccc} Z & & \longrightarrow & & Z_2 \\ \downarrow & & & & \downarrow \\ V & \longrightarrow & W & \xrightarrow[p_2]{} & B_2 \\ \downarrow & & \downarrow{\scriptstyle p_1} & & \\ Z_1 & \longrightarrow & B_1 & & \end{array}$$

Let $p_2 : W \to B_2$ and $r_2 : V \to B_2$ be the natural projections. Then there is an open subset U of B_2 such that $\operatorname{codim} B_2 \backslash U \geqslant 2$ and p_2 has reduced fibers over U by Lemma 5.4.3. Note that $V \to W$ is branched along $p_1^*D_1 \subseteq W$ and every component of $p_1^*D_1 \subseteq W$ is horizontal over B_2. Hence there is an open subset U' of U such that $\operatorname{codim} B_2 \backslash U' \geqslant 2$ and r_2 has reduced fibers over U'. Since Z is the pullback of V by $Z_1 \times Z_2 \to Z_1 \times B_2$, we have $Z \cong V \times_{B_2} Z_2$. As in the previous paragraph, Lemma 5.2.2 implies that Z is normal.

Hence Z is the image of X in $Z_1 \times Z_2$. Let $g : X \to Z$ be the natural morphism. Let $d = \deg f$. Since $Z \to W$ is of degree 4, we obtain that $d \in 4\mathbb{Z}$. We have

$$A_1^2A_2 = qd \text{ and } A_1A_2^2 = pd.$$

Note that every component of general fibers of f_i is a smooth rational curve for $i = 1, 2$. Hence we have

$$-K_X A_1^2 = -K_X A_2^2 = 4.$$

This shows that

$$a_1 = \frac{4}{pd}, \text{ and } a_2 = \frac{4}{qd}.$$

We have

$$-K_X^3 = 3a_1 a_2^2 A_1 A_2^2 + 3a_1^2 a_2 A_1^2 A_2 = \frac{192}{d^2}(\frac{1}{p^2} + \frac{1}{q^2}).$$

Since $-K_X^3$ is a positive even integer, we obtain that

$$r = \frac{96}{d^2}(\frac{1}{p^2} + \frac{1}{q^2})$$

is an integer. Since $d \in 4\mathbb{Z}$, this implies that $d = 4$ or $d = 8$.

Since $g : X \to Z$ is finite surjective, we obtain that $-K_Z$ is big by Lemma 5.4.4. Note that Z_1 and Z_2 are smooth. By the adjunction formula and the ramification formula, $-K_Z$ is big if and only if $Z_i \to B_i$ is branched along a smooth conic for $i = 1, 2$ and $p = q = 1$. Hence $Z_1 \cong Z_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, Z is of degree $(1, 1, 1, 1)$ in $Z_1 \times Z_2 \cong (\mathbb{P}^1)^4$ and $K_X \equiv \frac{4}{d} g^* K_Z$. Hence by Lemma 5.4.6, we obtain that $X \cong Z$ and $d = 4$. Thus $a_1 = a_2 = 1$. □

Lemma 5.4.11. *Let X be a Fano threefold with Gorenstein canonical singularities. Let $B_1 \cong B_2 \cong \mathbb{P}^2$. Assume that there are two equidimensional projective morphisms $f_1 : X \to B_1$ and $f_2 : X \to B_2$ such that general fibers of f_i have i connected components for $i = 1, 2$ and that $f_1 \times f_2 : X \to B_1 \times B_2$ is finite onto its image W. Assume that*

- *W is a normal hypersurface with isolated $\mathbb{Q}$-factorial klt singularities of degree (p, q), where $p, q \in \{1, 2\}$;*
- *If $X \to Z_2 \to B_2$ is the Stein factorisation of f_2, then Z_2 is smooth ;*
- *K_X is numerically equivalent to the pullback of some $\mathbb{Q}$-divisor in $B_1 \times B_2$ by $f_1 \times f_2$.*

Then the degree of f is 2 and $q = 1$.

Proof. Let $Z = h^* W$, where h is the natural morphism $B_1 \times Z_2 \to B_1 \times B_2$. Then $Z \cong W \times_{B_2} Z_2$. Since W has isolated klt singularities, $W \to B_2$ has reduced fibers over some open subset of B_2 whose complement has codimension at least 2. By Lemma 5.2.3, $Z_2 \to B_2$ is a cyclic cover. Hence by Lemma 5.2.2, Z is normal. Thus Z equal to the image of X in $B_1 \times Z_2$. Moreover, since $-K_X$ is ample, $-K_Z$ is big by Lemma 5.4.4. Thus $q = 1$ and $Z_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a double cover of B_2, branched along some smooth conic. Moreover, if $d = \deg f$, then $d \in 2\mathbb{Z}$.

$$\begin{array}{ccc} Z & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z_2 & \longrightarrow & B_2 \end{array}$$

Let $f : X \to W$ be the natural morphism. Let A_1, A_2 be two divisors in X such that $\mathscr{O}_X(A_1) \cong f_1^*\mathscr{O}_{B_1}(1)$ and $\mathscr{O}_X(A_2) \cong f_2^*\mathscr{O}_{B_2}(1)$. By assumption, there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1A_1 + a_2A_2$. We have

$$A_1^2A_2 = qd = d \text{ and } A_1A_2^2 = pd.$$

Note that every component of a general fiber of f_i is a smooth rational curve for $i = 1, 2$. Hence we have

$$-K_XA_1^2 = 2 \text{ and } -K_XA_2^2 = 4.$$

This shows that

$$a_1 = \frac{4}{pd} \text{ and } a_2 = \frac{2}{d}.$$

We have

$$-K_X^3 = 3a_1a_2^2A_1A_2^2 + 3a_1^2a_2A_1^2A_2 = \frac{48}{d^2}(1 + \frac{2}{p^2}).$$

Since $-K_X^3$ is a positive even integer, we obtain that

$$r = \frac{24}{d^2}(1 + \frac{2}{p^2})$$

is an integer. Since $d \in 2\mathbb{Z}$, this implies that $d = 2$ or 6.

Assume that $d = 6$. Then $a_1 = \frac{2}{3p}$ and $a_2 = \frac{1}{3}$. The hypersurface Z is of degree $(p, 1, 1)$ in $B_1 \times Z_2 \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $X \to Z$ is of degree 3. Let φ, ψ be the natural projections from X to $\mathbb{P}^1$ induced by $X \to Z_2$. Let F and G be general fibers of φ and ψ respectively. Then $\mathscr{O}_X(F + G) \cong \mathscr{O}_X(A_2)$. We have

$$A_1^2F = 3,\ A_1FG = 3p,\ G^2F = 0$$

and

$$-K_X \equiv \frac{2}{3p}A_1 + \frac{1}{3}F + \frac{1}{3}G.$$

By the adjunction formula, we have

$$\begin{aligned} K_F^2 &= (K_X + F)^2F = (\frac{2}{3p}A_1 + \frac{1}{3}G)^2F \\ &= \frac{4}{9p^2}A_1^2F + \frac{4}{9p}A_1GF = \frac{4}{3p^2} + \frac{4}{3} = \frac{4}{3p^2}(p^2 + 1). \end{aligned}$$

Since F is a surface with canonical singularities, $K_F^2 \in \mathbb{Z}$. This implies that 3 divides $(p^2 + 1)$ which is impossible since $p \in \mathbb{Z}$.

Thus we have $d = 2$. □

5.4.3 Proof of Theorem 5.1.2

Theorem 5.1.2 follows from Proposition 5.4.7 and Proposition 5.4.9.

Proof of Theorem 5.1.2. Let R_1, R_2 be the two extremal rays of the cone $\overline{\mathrm{NE}}(X)$. Let $f_i : X \to B_i$ be the Mori fibration corresponding R_i for $i = 1, 2$. Then by Lemma 5.3.1, B_i is a Fano variety with Picard number 1 for $i = 1, 2$. By symmetry, we may assume that $\dim B_1 \leqslant \dim B_2$. Since $R_1 \cap R_2 = \{0\}$, the morphism $f_1 \times f_2 : X \to B_1 \times B_2$ is finite onto its image.

Assume first that $B_1 = \mathbb{P}^1$. Then B_2 is of dimension 2 since $f_1 \times f_2$ is finite. By Lemma 5.3.3, we have $B_2 \cong \mathbb{P}^2$. Since X has Picard number 2, by Proposition 5.4.7, X is one of the threefolds in Proposition 5.4.7. Since X is locally factorial, if f is of degree two, then f is branched along a prime divisor by [Dru14, Lem. 3.7]. Hence X is one of the threefolds of $1 - 3$ of Theorem 5.1.2.

Assume that $\dim B_1 = 2$. Then $\dim B_2 = 2$. By Lemma 5.3.2, we have $B_1 \cong B_2 \cong \mathbb{P}^2$. Then X is one of the threefold in Proposition 5.4.9. Since X is locally factorial, as in the previous paragraph, we can conclude that X is one of the threefolds of $4 - 7$ of Theorem 5.1.2. □

The following observation will be useful for the proof of Theorem 5.1.5.

Lemma 5.4.12. *Let X be a threefold in $4 - 7$ of Theorem 5.1.2. Let W be the image of the natural morphism $X \to \mathbb{P}^2 \times \mathbb{P}^2$. Then W is a normal variety with isolated locally factorial canonical singularities.*

Proof. If the natural morphism $X \to W$ is an isomorphism, then there is nothing to prove. If $X \to W$ is of degree 2, then W is smooth of degree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ by Proposition 5.4.9. □

5.5 Case of dimension 4

We will study Fano fourfolds X with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = 3$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Consider the cone $\overline{\mathrm{NE}}(X)$. For every face V_2 of dimension 2 of $\overline{\mathrm{NE}}(X)$, there are exactly two faces V_1 and V_3, of dimension 2, whose intersections with V_2 are extremal rays of $\overline{\mathrm{NE}}(X)$. The aim of this section is to prove the following proposition.

Proposition 5.5.1. *If X is a Fano fourfold with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = 3$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$, then there is a fibration from X to $\mathbb{P}^1$.*

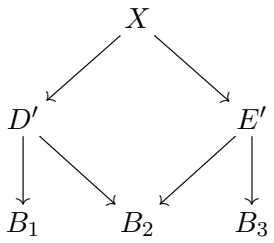
We will assume the existence of Fano fourfolds which satisfy the following condition $(*)$ and we will obtain a contradiction.

(∗) X is a Fano fourfold with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = 3$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Moreover, there is no fibration from X to $\mathbb{P}^1$.

We will first show that if X is a variety satisfying (∗), then it is a finite cover of the intersection of two hypersurfaces of degree $(0, 2, 2)$ and $(2, 2, 0)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Lemma 5.5.2. *Let X be a Fano fourfold satisfying the conditions in Proposition 5.5.1. Assume that there is no fibration from X to $\mathbb{P}^1$. Let V_1, V_2, V_3 be three distinct faces of dimensions 2 in $\overline{\mathrm{NE}}(X)$ such that $V_1 \cap V_2$ and $V_2 \cap V_3$ are extremal rays. Let f_i be the fibration from X to B_i corresponding to V_i ($i = 1, 2, 3$). Then $B_i = \mathbb{P}^2$ for all i and the morphism $f_1 \times f_2 \times f_3 : X \to \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ is finite onto its image Z. Moreover, Z is the intersection of hypersurfaces D and E of degree $(r, s, 0)$ and $(0, a, b)$ such that $r, s, a, b \in \{1, 2\}$.*

Proof. For all i, since the base B_i has Picard number 1 and there is no fibration from X to $\mathbb{P}^1$, by Lemma 5.3.2, we obtain that $B_i \cong \mathbb{P}^2$.



Let $g : X \to Y$ be the Mori fibration corresponding to the extremal ray $R_{12} = V_1 \cap V_2$. Then g is equidimensional of relative dimension 1 by Lemma 5.3.8. By Lemma 5.3.2, Y is a Fano threefold with isolated locally factorial canonical singularities such that $\rho(Y) = 2$ and $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$. Hence Y is one of the threefolds in $4 - 7$ of Theorem 5.1.2. Let D' be the image of $f_1 \times f_2 : X \to B_1 \times B_2$. Since $f_1 \times f_2$ factorises through $g : X \to Y$, D' is a hypersurface of degree (r, s) in $B_1 \times B_2$ with $r, s \in \{1, 2\}$ by Theorem 5.1.2. By the Lefschetz theorem, the natural map $\mathrm{Pic}(B_1 \times B_2) \to \mathrm{Pic}(D')$ is an isomorphism (see [Laz04, Example 3.1.25]). Moreover, D' has isolated locally factorial canonical singularities by Lemma 5.4.12.

Let $h : X \to D'$ be the natural morphism. Since $R_{12} \cap V_3 = \{0\}$, the morphism $h \times f_3 : X \to D' \times B_3$ is finite onto its image Z. Let $D = D' \times B_3$. Then D is a hypersurface of degree $(r, s, 0)$ in $B_1 \times B_2 \times B_3$. Since B_3 is smooth, the product D is locally factorial (this follows from the fact that the ring of polynomials with coefficients in a unique factorization domain is still a unique factorization domain). Hence Z is a Cartier divisor in D.

Note that the natural morphism $\mathrm{Pic}(D') \times \mathrm{Pic}(B_3) \to \mathrm{Pic}(D)$ is an isomorphism since $B_3 = \mathbb{P}^2$ is a simply connected manifold (see [Har77, Ex. III.12.6]). Hence the natural morphism

$$\mathrm{Pic}(B_1) \times \mathrm{Pic}(B_2) \times \mathrm{Pic}(B_3) \to \mathrm{Pic}(D)$$

is an isomorphism. Since Z is a Cartier divisor in D, we obtain that there is a hypersurface E of degree (c, a, b) in $B_1 \times B_2 \times B_3$ such that Z is the intersection of E and D.

Note that $V_3 \cap V_2$ is an extremal ray R_{23}. As in the first paragraph, we obtain that the image of the morphism $f_2 \times f_3 : X \to B_2 \times B_3$ is a proper subvariety. Thus the image of the natural morphism $Z \to B_2 \times B_3$ is also a proper subvariety. This implies that $c = 0$.

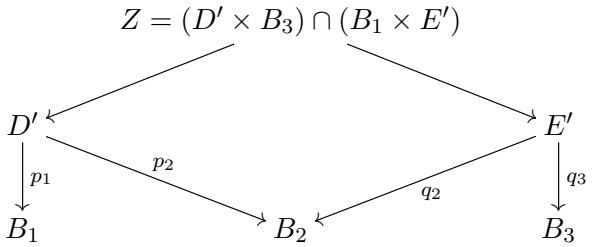
Let $E' \subseteq B_2 \times B_3$ be the image of $f_2 \times f_3$. Since $f_2 \times f_3$ factors through $X \to Z$, we obtain that E' is of degree (a, b) in $B_2 \times B_3$. As in the second paragraph of the proof, we have $a, b \in \{1, 2\}$ and $E \cong B_1 \times E'$. This completes the proof of the lemma. □

Lemma 5.5.3. *Let X be a Fano fourfold satisfying the conditions in Proposition 5.5.1. Assume that there is no fibration from X to $\mathbb{P}^1$. Then the cone $\overline{\mathrm{NE}}(X)$ is not simplicial.*

Proof. Assume the opposite. Since the cone $\overline{\mathrm{NE}}(X)$ is simplicial, it has three faces V_1, V_2, V_3 of dimension 2 and three extremal rays $V_1 \cap V_2$, $V_2 \cap V_3$ and $V_3 \cap V_1$. As in the proof of Lemma 5.5.2, for every face V_i, there is a fibration $f_i : X \to B_i$ corresponding V_i, where $B_i \cong \mathbb{P}^2$ $(i = 1, 2, 3)$. Let Z be the image of $f_1 \times f_2 \times f_3 : X \to B_1 \times B_2 \times B_3$. Then Z is the intersection of hypersurfaces of degree $(r, s, 0)$ and $(0, a, b)$ and that $r, s, a, b \in \{1, 2\}$. Since $V_3 \cap V_1 \neq \{0\}$, we obtain that the image of $f_1 \times f_3 : X \to B_1 \times B_3$ is a proper subvariety. This shows that the intersection number of six divisors in $B_1 \times B_2 \times B_3$ of degree $(1, 0, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, $(0, 0, 1)$, $(r, s, 0)$ and $(0, a, b)$ respectively is zero. Hence $as = 0$. This is a contradiction. □

Lemma 5.5.4. *Let $W = B_1 \times \mathrm{B}_2 \times B_3$, where $B_i = \mathbb{P}^2$ for $i = 1, 2, 3$. Let D' and E' be normal $\mathbb{Q}$-factorial klt hypersurfaces in $B_1 \times B_2$ and $B_2 \times B_3$ of degree $(a, 1)$ and (b, c) with $a, b, c \in \{1, 2\}$. Let $D = D' \times B_3$ and $E = B_1 \times E'$. In other words, D and E are hypersurfaces of degree $(a, 1, 0)$ and $(0, b, c)$ in W. Let Z be the intersection of D and E. If f_1, f_3 are the natural projections from Z to B_1, B_3 respectively, then $f = f_1 \times f_3 : Z \to B_1 \times B_3$ is not finite.*

Proof. Let p_1, p_2 (resp. q_2, q_3) be the natural projections from D' (resp. E') to B_1 and B_2 (resp. B_2 and B_3). Since D' and E' are $\mathbb{Q}$-factorial klt Fano varieties, the morphisms p_1, p_2, q_2, q_3 are equidimensional by Lemma 5.4.3. Thus $p_1 : D' \to B_1$ is a well-defined family of cycles in $B_2 = \mathbb{P}^2$ (see [Kol96, Def. I.3.11]). Since D' is of degree $(a, 1)$, the cycles in this family are lines in B_2. Thus there exist a natural morphism ch from B_1 to the Grassmannian of lines in B_2 (see [Kol96, Thm. I.3.21]), which is isomorphic to $\mathbb{P}^2$. Since $a \neq 0$, the morphism ch is not constant. Note that B_1 has Picard number 1. This shows that ch is finite. Hence it is surjective and for any line L in B_2, there is a point x in B_1 such that the fiber of p_1 over x is mapped to L by the projection p_2.



In order to prove the lemma, it is enough to find two points $x \in B_1$ and $z \in B_3$ such that $f^{-1}(\{(x,z)\})$ contains a curve. First we assume that $b = 1$. Let L be a line in B_2. From the previous paragraph, there is a point $x \in B_1$ such that $p_1^{-1}(\{x\})$ is mapped to L by p_2. Similarly, there is a point $z \in B_3$ such that $q_3^{-1}(\{z\})$ is mapped to L by q_2. Hence the curve $\{(x,y,z) \in W \mid y \in L\}$ is contained in $f^{-1}(\{(x,z)\})$. Thus, f is not finite.

Now we assume that $b = 2$. Since q_3 is equidimensional, $q_3 : E' \to B_3$ is a well-defined family of cycles in $B_2 \cong \mathbb{P}^2$. Hence there is a morphism from B_3 to the Chow variety of conics in $\mathbb{P}^2$. Since B_3 has Picard number 1 and $c \neq 0$, this morphism is finite onto its image. Note that there is a hyperplane H in the Chow variety of conics in $\mathbb{P}^2$ which parametrize the singular conics. Hence there is a point $z \in B_3$ such that the fiber of q_3 over z is the union of two lines L and L'. There is a point $x \in B_1$ such that $p_1^{-1}(\{x\})$ is mapped to L by p_2. This shows that the curve $\{(x,y,z) \in W \mid y \in L\}$ is contained in $f^{-1}(\{(x,z)\})$. Hence f is not finite. □

Lemma 5.5.5. *With the notation in Lemma 5.5.2, the numbers a, b, r, s are all equal to 2. Moreover, if we write $D = D' \times B_3$ and $E = B_1 \times E'$, then the natural projections from X to D' and E' are equidimensional Mori fibrations.*

Proof. By Lemma 5.5.3, the cone $\overline{\mathrm{NE}}(X)$ is not simplicial. Hence $V_1 \cap V_3 = \{0\}$. This implies that $f_1 \times f_3 : X \to B_1 \times B_3$ is finite surjective. Since $f_1 \times f_3$ factors through $X \to Z$, the natural morphism $Z \to B_1 \times B_2$ is finite surjective. By Lemma 5.4.12, D' and E' are Fano varieties with isolated locally factorial canonical singularities. Thus by Lemma 5.5.4, we have $s = a = 2$.

Note that there is a face V_4 of dimension 2 of $\overline{\mathrm{NE}}(X)$ such that $V_2 \neq V_4$ and $V_3 \cap V_4$ is an extremal ray. Let $f_4 : X \to B_4$ be the fibration corresponding to V_4. Then, as in Lemma 5.5.2, $B_4 = \mathbb{P}^2$ and

$$f_2 \times f_3 \times f_4 : X \to B_2 \times B_3 \times B_4$$

is finite onto its image T. Moreover, T is the intersection of hypersurfaces of degree of $(a,b,0)$ and $(0,p,q)$ such that $p, q \in \{1,2\}$. As in the previous paragraph, we have $b = p = 2$. Similarly, we can obtain that $r = 2$.

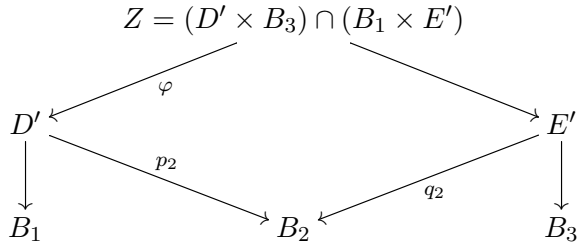
As in the proof of Lemma 5.5.2, D' is in fact equal to the image of $f_1 \times f_2 : X \to \mathbb{P}^2 \times \mathbb{P}^2$. Let $g : X \to Y$ be the Mori fibration corresponding the extremal ray $V_1 \cap V_2$. Then $X \to Y \to D'$ is just the Stein factorisation

of $X \to D'$. Since D' is of degree $(2,2)$ and Y is one of the varieties in $4-7$ of Theorem 5.1.2, we obtain that $Y \to D'$ is an isomorphism. Hence $X \to D'$ is an equidimensional Mori fibration by Lemma 5.3.8. Similarly, $X \to E'$ is also an equidimensional Mori fibration. □

Lemma 5.5.6. *With the notation in Lemma 5.5.2, the image Z of $f_1 \times f_2 \times f_3 : X \to B_1 \times B_2 \times B_3$ is normal.*

Proof. By Lemma 5.5.5, we have $Z = D \cap E$, where $D \cong D' \times B_3$ and $E \cong B_1 \times E'$ are hypersurfaces of degree $(2,2,0)$ and $(0,2,2)$ respectively in $B_1 \times B_2 \times B_3$.

By Lemma 5.4.12, E' is a Fano threefold with isolated singularities. Hence, by Lemma 5.4.1, there is an open subset U of B_2 whose complement is of codimension at least 2 such that the natural projection $q_2 : E' \to B_2$ has reduced fibers over U.



Consider the natural projection $\varphi : Z \to D'$. Let α be a point in D'. Then the fiber of φ over α is a conic in B_3. Moreover, if β is the image of α by the natural projection $p_2 : D' \to B_2$, then the fiber of q_2 over β is isomorphic to the fiber of φ over α. Hence general fibers of φ are smooth conics in B_3 and the fibers of φ over $p_2^{-1}(U) \subseteq D'$ are reduced.

Note that D' is a Fano threefold with locally factorial canonical singularities by Lemma 5.4.12. Thus $p_2 : D' \to B_2$ is equidimensional by Lemma 5.4.3. Since $\operatorname{codim} B_2 \backslash U \geqslant 2$, the complement of $p_2^{-1}(U)$ in D' has codimension at least 2. Let

$$V = \{z \in Z \mid \varphi^{-1}(\{\varphi(z)\}) \text{ is singular at } z\}.$$

Then $\operatorname{codim} V \geqslant 2$. Since Z is Cohen-Macaulay and φ is equidimensional, φ is flat over the smooth locus of D'. Since D' is smooth in codimension 1, this implies that Z is smooth in codimension 1. Thus Z is normal for it is Cohen-Macaulay. □

Now we can prove Proposition 5.5.1.

Proof of Proposition 5.5.1. Assume the opposite. With the same notation as in Lemma 5.5.2, there are fibrations $f_i : X \to B_i$ such that $f_1 \times f_2 \times f_3 : X \to B_1 \times B_2 \times B_3$ is finite onto its image Z. By Lemma 5.5.5, Z is the intersection of $D' \times B_3$ and $B_1 \times E'$ such that D' is normal of degree $(2,2)$ in

$B_1 \times B_2$ and E' is normal of degree $(2,2)$ in $B_2 \times B_3$. Moreover, $g : X \to D'$ is an equidimensional Mori fibration.

By Lemma 5.5.6, Z is normal. Let $p_i : Z \to B_i$ be the natural fibration and let H_i be a divisor in Z such that $\mathscr{O}_Z(H_i) \cong p_i^* \mathscr{O}_{B_i}(1)$ for $i = 1, 2, 3$. Then

$$-K_Z \equiv H_1 - H_2 + H_3.$$

Since X is a Fano variety, $-K_Z$ is big by Lemma 5.4.4. Moreover, since $\overline{\mathrm{NE}}(X)$ is not simplicial by Lemma 5.5.3, as in the proof of Lemma 5.5.5, $f_1 \times f_3 : X \to B_1 \times B_3$ is finite. We obtain that $p_1 \times p_3 : Z \to B_1 \times B_3$ is also finite. Hence $H_1 + H_3$ is an ample divisor and the intersection number $-K_Z(H_1 + H_3)^3$ is positive. However, we have

$$\begin{aligned} -K_Z(H_1 + H_3)^3 &= (H_1 + H_3 - H_2)(3H_1^2 H_3 + 3H_1 H_3^2) \\ &= 6H_1^2 H_3^2 - 3H_1^2 H_2 H_3 - 3H_1 H_2 H_3^2 = 0. \end{aligned}$$

We obtain a contradiction. □

5.6 Finite morphisms between Fano varieties.

In this section, we will study some finite morphisms between Fano varieties.

5.6.1 Finite morphisms over $(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$

Lemma 5.6.1. *Let Y be a variety constructed by the method of Example 5.1.3.2. Set $n = \dim Y$. Assume that Y satisfies the conditions of Theorem 5.1.5. Let X be a Fano variety with Gorenstein canonical singularities such that there is a finite surjective morphism $f : X \to Y$. Assume that the projection from X to $(\mathbb{P}^1)^{n-2}$ induced by f has connected fibers. Then f is an isomorphism.*

Proof. From the construction of Y, we have two double covers

$$Y \xrightarrow{\pi_1} T \xrightarrow{\pi_2} (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2,$$

where $\pi_2 : T \to (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ is branched along some prime divisor of degree $(2, ..., 2, 0, ..., 0, 2)$. In particular, $T \cong (\mathbb{P}^1)^r \times T'$, where $0 < r < n-2$ and T' is a double cover over $(\mathbb{P}^1)^{n-2-r} \times \mathbb{P}^2$, branched along a divisor of degree $(2, ..., 2, 2)$. Note that $Y \to T' \to (\mathbb{P}^1)^{n-2-r} \times \mathbb{P}^2$ is the Stein factorisation. By Lemma 5.3.3, we obtain that T' is smooth in codimension 2. Thus T is smooth in codimension 2.

We write

$$(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2 = (\mathbb{P}^1)^{n-3} \times (\mathbb{P}^1 \times \mathbb{P}^2) = (\mathbb{P}^1)^{n-3} \times Z.$$

Let p be the natural morphism from Y to $(\mathbb{P}^1)^{n-3}$. Let F be the fiber of p over a general point $s \in (\mathbb{P}^1)^{n-3}$. By restricting π_1 and π_2 over the point s, we obtain two double covers

$$F \xrightarrow{\pi_1|_F} V \xrightarrow{\pi_2|_V} \{s\} \times Z,$$

where V is the fiber of the natural projection $T \to (\mathbb{P}^1)^{n-3}$. Note that $V \cong \mathbb{P}^1 \times V'$, where V' is a general fiber of $T \to (\mathbb{P}^1)^{n-2}$. Moreover, the natural morphism $V \to \mathbb{P}^2$ is a double cover branched along a conic. Since T is smooth in codimension 2, V' is smooth. Thus $V' \to \mathbb{P}^2$ is a double cover branched along some smooth conic by Lemma 5.2.3.

Hence $V \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Assume that $F \to V$ is branched along a divisor D. Then D is the pullback of some divisor of degree $(2,2)$ in Z by construction. Thus D is a divisor of degree $(2,2,2)$ in $V \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Let G be the fiber of the natural morphism $X \to (\mathbb{P}^1)^{n-3}$ over s. Then $f|_G : G \to F$ is also a finite surjective morphism. Since s is a general point, G is a Fano variety with Gorenstein canonical singularities. By [Dru14, Thm. 1.3], $f|_G$ is of degree 1. Hence f is of degree 1 and is an isomorphism. □

Lemma 5.6.2. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Set $n = \dim X$ and assume that $n \geqslant 3$. Assume that Theorem 5.1.5 is true in dimension smaller than n. Assume that there is a fibration $X \to (\mathbb{P}^1)^{n-2}$ and a fibration $X \to \mathbb{P}^2$ such that the product morphism $f : X \to (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ is finite surjective. Then X is one of the varieties in Theorem 5.1.5.*

Proof. Let $Z = (\mathbb{P}^1)^{n-3}$ be the product of the last $n-3$ factors in the product $(\mathbb{P}^1)^{n-2}$. Note that if $X \to Y$ is the Stein factorisation of the projection

$$X \to Z \times \mathbb{P}^2 = (\mathbb{P}^1)^{n-3} \times \mathbb{P}^2,$$

then Y is a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$ by Lemma 5.3.3. Moreover, since $X \to (\mathbb{P}^1)^{n-2}$ has connected fibers, by Theorem 5.1.5 in dimension $n-1$, Y is of the form $(\mathbb{P}^1)^{s-1} \times Y_1$, where $s \geqslant 1$ and Y_1 is either a variety constructed by the methods of Example 5.1.3 or a variety in the list of Theorem 5.1.2. Thus $Y \to Z \times \mathbb{P}^2$ is of degree 1, 2 or 4. Moreover, by Lemma 5.3.3, Y_1 is a Fano variety with locally factorial canonical singularities such that Y_1 is smooth in codimension 2, $\rho(Y_1) = \dim Y_1 - 1$ and $\mathrm{Nef}(Y_1) = \mathrm{Psef}(Y_1)$. Let $f_1 : X \to \mathbb{P}^1$ be the projection to the first factor of $(\mathbb{P}^1)^{n-2}$.

Let A_1 is be a fiber of f_1. Let $A_2, ..., A_{n-2}$ be the fibers of X over the other $\mathbb{P}^1$. Let H be the pullback of a line in $\mathbb{P}^2$ by the natural projection

$X \to \mathbb{P}^2$. Note that K_X is numerically equivalent to the pullback of some $\mathbb{Q}$-divisor in $(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ since $\rho(X) = n-1$. Hence there are positive rational numbers $a_1, ..., a_{n-2}, b$ such that $-K_X \equiv a_1 A_1 + \cdots + a_{n-2} A_{n-2} + bH$.

$$\begin{array}{ccccccc} X & \longrightarrow & V = \mathbb{P}^1 \times Y & \longrightarrow & Y \cong (\mathbb{P}^1)^{s-1} \times Y_1 & \longrightarrow & Z \times \mathbb{P}^2 \\ & \searrow^{f_1} & \downarrow & & & & \\ & & \mathbb{P}^1 & & & & \end{array}$$

We will discuss on three case.

Case 1. Assume that the morphism $Y \to Z \times \mathbb{P}^2$ is of degree 4. Then Y_1 is a variety which can be constructed by the method of Example 5.1.3.2 and X is finite over $(\mathbb{P}^1)^s \times Y_1$. The natural morphism $X \to (\mathbb{P}^1)^s$ is a fibration since $X \to (\mathbb{P}^1)^{n-2}$ is. Let F be a general fiber of $X \to (\mathbb{P}^1)^s$. Then F is a Fano variety with Gorenstein canonical singularities. Since $X \to (\mathbb{P}^1)^{n-2}$ has connected fibers, the natural morphism $F \to (\mathbb{P}^1)^{n-2-s}$ given by $X \to Y_1 \to (\mathbb{P}^1)^{n-2-s}$ is a fibration. By Lemma 5.6.1, the natural finite morphism $F \to Y_1$ is an isomorphism. Hence $X \cong (\mathbb{P}^1)^s \times Y_1$ in this case.

Case 2. Assume that the morphism $Y \to Z \times \mathbb{P}^2$ is of degree 2. Let G be a general fiber of the natural projection $X \to Z$. Then G is a Fano threefold with Gorenstein canonical singularities. Moreover, the restriction $f_1|_G : G \to \mathbb{P}^1$ is a fibration, general fibers of the natural morphism $G \to \mathbb{P}^2$ have two connected components and the product $G \to \mathbb{P}^1 \times \mathbb{P}^2$ is finite. Note that $K_G \equiv a_1 A_1|_G + bH|_G$. By Lemma 5.4.8, $G \to \mathbb{P}^1 \times \mathbb{P}^2$ is of degree 2 or 4 and $b \in \mathbb{Z}$. Hence f is of degree 2 or 4.

The Cartier divisor $-K_X - bH$ is numerically trivial on the fibers of the Fano fibration $X \to (\mathbb{P}^1)^{n-2}$. Thus it is linearly equivalent to the pullback of some Cartier divisor in $(\mathbb{P}^1)^{n-2}$ by [KMM87, Lem. 3-2-5]. This implies that $a_1, ..., a_{n-2} \in \mathbb{Z}$.

Note that Y_1 is a double cover of $(\mathbb{P}^1)^r \times \mathbb{P}^2$, branched along some prime divisor of degree $(2, ..., 2, k)$ such that $k \in \{2, 4\}$ and $r + s = n - 2$. If $\deg f = 2$, then $X \cong (\mathbb{P}^1)^s \times Y_1$. If $\deg f = 4$, then X is a double cover of $V = \mathbb{P}^1 \times Y = (\mathbb{P}^1)^s \times Y_1$.

There is a natural double cover $h : V \to (\mathbb{P}^1)^s \times (\mathbb{P}^1)^r \times \mathbb{P}^2$, induced by $Y_1 \to (\mathbb{P}^1)^r \times \mathbb{P}^2$, branched along a prime divisor of degree $(0, ..., 0, 2, ..., 2, k)$. By the ramification formula, K_V is linearly equivalent to the pullback by h of a divisor of degree $(-2, ..., -2, -1, ..., -1, -3 + k/2)$ in $(\mathbb{P}^1)^s \times (\mathbb{P}^1)^r \times \mathbb{P}^2$. Since $\rho(V) = n-1$, we may assume that the double cover $X \to V$ is branched along a divisor $D \subseteq V$ which is numerically equivalent to the pullback by h of a divisor of degree $(d_1, ..., d_s, d'_1, ..., d'_r, e)$ in $(\mathbb{P}^1)^s \times (\mathbb{P}^1)^r \times \mathbb{P}^2$. By the ramification formula again, K_X is linearly equivalent to the pullbakc of a divisor of degree $(-2 + d_1/2, ..., -2 + d_s/2, -1 + d'_1/2, ..., -1 + d'_r/2, -3 + k/2 + e/2)$ in $(\mathbb{P}^1)^s \times (\mathbb{P}^1)^r \times \mathbb{P}^2$. Since $a_1, ..., a_{n-2}, b \in \mathbb{Z}$ and $-K_X$ is ample, we obtian that $d_1, ..., d_s, e \in \{0, 2\}$ and that $d'_1 = ... = d'_r = 0$. Since $X \to (\mathbb{P}^1)^{n-2}$ has connected fibers, we obtain that $e > 0$. Hence $e = 2$. Since $b > 0$, we obtain

that $k = 2$. Moreover, since $X \to \mathbb{P}^2$ has connected fibers, $d_1, ..., d_s$ are not all zero. Note that V is simply connected and locally factorial. Thus D is in fact linearly equivalent to the pullback of a divisor of degree $(d_1, ..., d_s, 0, ..., 0, e)$ in $(\mathbb{P}^1)^s \times \mathbb{P}^2 \times (\mathbb{P}^1)^r$. Note that D is a prime divisor by [Dru14, Lem. 3.7]. Thus we obtain that $X \cong X_1 \times X_2$, where $X_1 = (\mathbb{P}^1)^t$ with $s > t \geqslant 0$ and X_2 is a variety which can be constructed by the method of Example 5.1.3.2.

Case 3. Assume $Y \to Z \times \mathbb{P}^2$ is an isomorphism. Let G be a general fiber of the natural projection $X \to Z$. Then G is a Fano threefold with Gorenstein canonical singularities. Moreover, $f_1|_G : G \to \mathbb{P}^1$ and the natural morphism $G \to \mathbb{P}^2$ are fibrations, and the product $h : G \to \mathbb{P}^1 \times \mathbb{P}^2$ is finite. By Proposition 5.4.7, h is of degree at most 2. Hence f is of degree at most 2.

If the degree of f is 1, then f is an isomorphism. If $\deg f = 2$, then by Lemma 5.2.3 and Lemma 5.2.4, $f : X \to (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ is a double cover, branched along a prime divisor of degree $(d_1, ..., d_{n-2}, k)$ such that $d_1, ..., d_{n-2} \in \{0, 2\}$ and that $k \in \{0, 2, 4\}$. Since $X \to (\mathbb{P}^1)^{n-2}$ and $X \to \mathbb{P}^2$ have connected fibers, we obtain that $k \neq 0$ and $d_1, ..., d_{n-2}$ are not all zero. Hence $X \cong X_1 \times X_2$, where $X_1 \cong (\mathbb{P}^1)^t$ for some $t \geqslant 0$ and X_2 is a double cover of $(\mathbb{P}^1)^{n-2-t} \times \mathbb{P}^2$, branched along an prime divisor of degree $(2, ..., 2, 2)$ or $(2, ..., 2, 4)$. □

5.6.2 Finite morphisms over hypersurfaces of $(\mathbb{P}^1)^{n-3} \times \mathbb{P}^2 \times \mathbb{P}^2$

Lemma 5.6.3. *Let Y be a variety constructed by the method of Example 5.1.3.2 of dimension n. Then there is a finite surjective morphism, of degree 4, $Y \to (\mathbb{P}^1)^{n-2} \times B_1$ with $B_1 \cong \mathbb{P}^2$. Let X be a hypersurface in $Y \times B_2$, where $B_2 \cong \mathbb{P}^2$. Assume that the image of X in $B_1 \times B_2$ by the natural projection is a normal hypersurface W of degree (p, q) with isolated $\mathbb{Q}$-factorial klt singularities, where $p, q \in \{1, 2\}$. Then X is normal and $-K_X$ is not big.*

Proof. There is a natural finite surjective morphism $X \to D$, where

$$D = (\mathbb{P}^1)^{n-2} \times W \subseteq (\mathbb{P}^1)^{n-2} \times B_1 \times B_2$$

is a normal hypersurface. By the construction of Y, the natural morphism $Y \times B_2 \to (\mathbb{P}^1)^{n-2} \times B_1 \times B_2$ is the composition of the following two double covers

$$Y \times B_2 \xrightarrow{g} V \times B_2 \xrightarrow{h} (\mathbb{P}^1)^{n-2} \times B_1 \times B_2,$$

where $V \to (\mathbb{P}^1)^{n-2} \times B_1$ is a double cover, branched along a prime divisor of degree $(0, ..., 0, 2, ..., 2, 2)$.

Note that the natural morphism $D \to (\mathbb{P}^1)^{n-2} \times B_1$ is an equidimensional Fano fibration since $W \to B_1$ is equidimensional by Lemma 5.4.3. Since W has isolated singularities, D is smooth in codimension 2. There is an open subset U in $(\mathbb{P}^1)^{n-2} \times B_1$ whose complement has codimension at least 2 such that $D \to (\mathbb{P}^1)^{n-2} \times B_1$ has reduced fibers over U by Lemma 5.4.1. Note

that $V \to (\mathbb{P}^1)^{n-2} \times \mathrm{B}_1$ is a cyclic cover branched along some prime divisor and that

$$h^*D \cong D \times_{(\mathbb{P}^1)^{n-2} \times B_1} V.$$

Hence, h^*D is normal Cohen-Macaulay by Lemma 5.2.2. Moreover, $h^*D \to V$ also has reduced fibers over some open subset of V whose complement has codimension at least 2. Similarly, by Lemma 5.2.2 again,

$$g^*(h^*D) = (h^*D) \times_V Y$$

is also normal. Hence, $X = g^*(h^*D)$ and it is normal.

$$\begin{array}{ccccc} X = g^*(h^*D) & \longrightarrow & h^*D & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & V & \longrightarrow & (\mathbb{P}^1)^{n-2} \times B_1 \end{array}$$

Let $A_1, ..., A_{n-2}$ be the fibers of the natural projections from X to the factors $\mathbb{P}^1$. Let H_i be a divisor in X which is the pullback of a line in B_i for $i = 1, 2$. Then by the adjunction formula, we have

$$-K_X \equiv A_1 + \cdots + A_{n-2} + (1-p)H_1 + (3-q)H_2.$$

Since $p \geqslant 1$, $-K_X$ is not big. □

Lemma 5.6.4. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Set $n = \dim X$ and assume that $n \geqslant 4$. Assume that Theorem 5.1.5 is true in dimension smaller than n. Assume that there are fibrations $X \to (\mathbb{P}^1)^{n-3}$, $X \to B_1$ and $X \to B_2$ such that $B_1 \cong B_2 \cong \mathbb{P}^2$ and that the product morphism $X \to (\mathbb{P}^1)^{n-3} \times B_1 \times B_2$ is finite onto its image. Then X is one of the varieties in Theorem 5.1.5*

Proof. Let W be the image of $X \to B_1 \times B_2$ and let $X \to W' \to W$ be the Stein factorisation. Since the natural morphism $X \to (\mathbb{P}^1)^{n-3} \times W'$ is finite surjective, by Lemma 5.3.3, W' is a Fano threefold of Theorem 5.1.2. Thus by Lemma 5.4.12, W is a normal hypersurface of degree (p, q), with isolated locally factorial canonical singularities, such that $p, q \in \{1, 2\}$. Let $f : X \to (\mathbb{P}^1)^{n-3} \times W$ be the natural morphism.

Let $A_1, ..., A_{n-3}$ be the fibers of the natural fibrations from X to $\mathbb{P}^1$. Let H_i be the pullback of a line in B_i by the fibration from X to B_i for $i = 1, 2$. Since the Picard number of X is $n - 1$, there are positive rational numbers $a_1, ..., a_{n-3}, b_1, b_2$ such that $-K_X \equiv a_1A_1 + \cdots a_{n-3}A_{n-3} + b_1H_1 + b_2H_2$. Let F be the fiber of $X \to (\mathbb{P}^1)^{n-3}$ over a general point $\alpha \in (\mathbb{P}^1)^{n-3}$, then $-K_F \equiv (b_1H_1 + b_2H_2)|_F$. Moreover, F is a Fano threefold with isolated Gorenstein canonical singularities and the natural morphism $F \to W$ is finite.

Let $X \to Y$ be the Stein factorisation of $X \to (\mathbb{P}^1)^{n-3} \times B_1$. Then $X \to Y$ is a Mori fibration of relative dimension 1 (see Lemma 5.3.4). By Lemma Lemma 5.3.1, Y is a Fano variety with locally factorial canonical singularities such that $\rho(Y) = \dim Y - 1$ and $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$. The fibration $X \to Y$ is equidimensional since $(\mathbb{P}^1)^{n-3} \times W \to (\mathbb{P}^1)^{n-3} \times B_1$ is by Lemma 5.4.3. Hence by Lemma 5.3.5, Y is smooth in codimension 2. Thus, by induction, Y is one of the varieties in Theorem 5.1.5. In particular, $Y \to (\mathbb{P}^1)^{n-3} \times B_1$ is of degree 1, 2 or 4.

If it is of degree 4, then $Y = (\mathbb{P}^1)^r \times X'$ where $r \geqslant 0$ and X' is a variety which can be constructed by the method of Example 5.1.3.2. Let S be a general fiber of $X \to (\mathbb{P}^1)^r$. Then S is a Fano variety. If T is the image of S in $X' \times \mathrm{B}_2$, then by Lemma 5.6.3, T is normal and $-K_T$ is not big. Thus S can not be a Fano variety by Lemma 5.4.4. This is a contradiction. Hence the degree of $Y \to (\mathbb{P}^1)^{n-3} \times B_1$ is 1 or 2. Similarly, fibers of $X \to (\mathbb{P}^1)^{n-3} \times B_2$ have at most two connected components.

We will discuss on four cases.

Case 1. Assume that the projection $X \to (\mathbb{P}^1)^{n-3} \times B_i$ is a fibration for $i = 1, 2$. Then the natural projections from F to B_1 and B_2 are fibrations. By Proposition 5.4.9, the finite morphism $F \to W$ is of degree 1 or 2.

If this degree is 1, then $f : X \to (\mathbb{P}^1)^{n-3} \times W$ is an isomorphism. If the degree of $F \to W$ is 2, then f is a double cover. Moreover, W is smooth of degree $(1, 1)$ in $B_1 \times B_2$ by Proposition 5.4.9. Note that there is a natural isomorphism $\mathrm{Pic}((\mathbb{P}^1)^{n-3} \times B_1 \times B_2) \cong \mathrm{Pic}((\mathbb{P}^1)^{n-3} \times W)$. By Lemma 5.2.3 and Lemma 5.2.4, f is branched along a prime divisor D which is the intersection of $(\mathbb{P}^1)^{n-3} \times W$ and a divisor of degree $(d_1, ..., d_{n-3}, k, l)$ in $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$ such that $d_1, ..., d_{n-3}, k, l \in \{0, 2\}$. Since the two morphisms $F \to B_1$ and $f \to B_2$ are fibrations, $k, l \neq 0$. Hence $X \cong X_1 \times X_2$, where $X_1 = (\mathbb{P}^1)^r$ with some $r \geqslant 0$ and X_2 is a variety which can be constructed by the method of Example 5.1.4.1.

Case 2. Assume that $X \to (\mathbb{P}^1)^{n-3} \times B_2$ is a fibration and general fibers of $X \to (\mathbb{P}^1)^{n-3} \times B_1$ have two connected components. We recall that $X \to Y$ is the Stein factorisation of $X \to (\mathbb{P}^1)^{n-3} \times B_1$. Then $Y = (\mathbb{P}^1)^r \times Y_1$, where $r \geqslant 0$ and Y_1 is a variety which can be constructed by the method of Example 5.1.3.1 or a variety of Theorem 5.1.2.2 and Theorem 5.1.2.3. If $F \to Z_1 \to B_1$ is the Stein factorisation, then Z_1 is isomorphic to the fiber of $Y \to (\mathbb{P}^1)^{n-3}$ over α. Since Y is smooth in codimension 2, Z_1 is smooth. Thus F satisfies the conditions in Lemma 5.4.11. This implies that $p = 1$ and $F \to W$ is of degree 2. Thus $X \to (\mathbb{P}^1)^{n-3} \times W$ is of degree 2.

Let V be the pullback of $(\mathbb{P}^1)^{n-3} \times W$ by the natural morphism $Y \times B_2 \to (\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. As in the proof of Lemma 5.6.3, V is normal. Since $X \to (\mathbb{P}^1)^{n-3} \times W$ is of degree 2, we obtain that $X \cong V$. Since X is a Fano variety, we obtain that the double cover $Y_1 \to (\mathbb{P}^1)^{n-3-r} \times B_1$ is branched along some divisor of degree $(2, ..., 2, 2)$. Hence $X \cong X_1 \times X_2$, where $X_1 = (\mathbb{P}^1)^r$ and X_2 is a variety which can be constructed by the method of

Example 5.1.4.2.

Case. 3. Assume that $X \to (\mathbb{P}^1)^{n-3} \times B_1$ is a fibration and general fibers of $X \to (\mathbb{P}^1)^{n-3} \times B_2$ have two connected components. By symmetry, we can reduce to the second case.

Case 4. Assume that general fibers of the two projections $X \to (\mathbb{P}^1)^{n-3} \times B_1$ and $X \to (\mathbb{P}^1)^{n-3} \times B_2$ have two connected components. As in the second case, if $F \to Z_i \to B_i$ is the Stein factorisation for $i = 1, 2$, then Z_1 and Z_2 are smooth. Hence F satisfies the conditions in Lemma 5.4.10. The morphism $F \to W$ is of degree 4 and $p = q = b_1 = b_2 = 1$. Hence the morphism $X \to (\mathbb{P}^1)^{n-3} \times W$ is of degree 4. Moreover, $-K_X - b_1H_1 - b_2H_2$ is numerically trivial on the fibers of the Fano fibration $X \to (\mathbb{P}^1)^{n-3}$. Thus we obtain that $a_1, ..., a_{n-3} \in \mathbb{Z}$.

We recall that $X \to Y$ is the Stein factorisation of $X \to (\mathbb{P}^1)^{n-3} \times B_1$. Then we have $Y = (\mathbb{P}^1)^r \times Y_1$, where $r \geqslant 0$ and Y_1 is a variety which can be constructed by the method of Example 5.1.3.1 or a variety of Theorem 5.1.2.2 and Theorem 5.1.2.3. Let V be the pullback of $(\mathbb{P}^1)^{n-3} \times W$ by the natural morphism $Y \times B_2 \to (\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. Note that $(\mathbb{P}^1)^{n-3} \times W \to (\mathbb{P}^1)^{n-3} \times B_1$ has reduced fibers over some open subset of $(\mathbb{P}^1)^{n-3} \times B_1$ whose complement has codimension at least 2, for W is smooth in codimension 2 (see Lemma 5.4.1). Thus V is normal by the same argument as in the proof of Lemma 5.6.3. Hence $X \to V$ is a double cover.

$$\begin{array}{ccccccc} X & \longrightarrow & V & \longrightarrow & (\mathbb{P}^1)^{n-3} \times W & \longrightarrow & (\mathbb{P}^1)^{n-3} \times B_2 \\ & & \downarrow & & \downarrow & & \\ & & Y & \longrightarrow & (\mathbb{P}^1)^{n-3} \times B_1 & & \end{array}$$

Similarly, there is an open subset U of $(\mathbb{P}^1)^{n-3} \times B_2$ whose complement has codimension at least 2 such that the $(\mathbb{P}^1)^{n-3} \times W \to (\mathbb{P}^1)^{n-3} \times B_2$ has reduced fibers over U. The branch locus of $V \to (\mathbb{P}^1)^{n-3} \times W$ is the intersection of $(\mathbb{P}^1)^{n-3} \times W$ and a divisor of degree $(0, ..., 0, 2, ..., 2, 2, 0)$ in $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. Thus this branch locus is horizontal over $(\mathbb{P}^1)^{n-3} \times B_2$. Hence, the natural morphism $V \to (\mathbb{P}^1)^{n-3} \times B_2$ is a fibration and it has reduced fibers over some open subset $U' \subseteq U$ such that $\operatorname{codim} U \backslash U' \geqslant 2$. Thus the pullback of any prime divisor in $(\mathbb{P}^1)^r \times B_2$ by the fibration $V \to (\mathbb{P}^1)^r \times B_2$ is reduced.

Assume that $X \to V$ is branched along some divisor D. Then D is a prime divisor by [Dru14, Lem. 3.7] for X is locally factorial. Since $a_1, ..., a_{n-3} \in \mathbb{Z}_{>0}$ and $b_1 = b_2 = 1$, by the adjunction formula and the ramification formula, we obtain that $Y_1 \to (\mathbb{P}^1)^{n-3-r} \times \mathrm{B}_1$ is branched along some divisor of degree $(2, .., 2, 2)$ and D is numerically equivalent to the pullback of a divisor E of degree $(d_1, ..., d_r, 2)$ in $(\mathbb{P}^1)^r \times B_2$ by the natural projection $V \to (\mathbb{P}^1)^r \times B_2$ such that $d_1, ..., d_r \in \{0, 2\}$. Since the pullback of any prime divisor in $(\mathbb{P}^1)^r \times B_2$ by the fibration $V \to (\mathbb{P}^1)^r \times B_2$ is reduced, we have $D = f^*(f(D))$ and D is in fact linearly equivalent to the pullback of E.

Since the projection from X to B_2 is a fibration, we obtain that $r \geqslant 1$ and that $d_1, ..., d_r$ are not all equal to 0. Hence we obtain that $X \cong X_1 \times X_2$, where $X_1 = (\mathbb{P}^1)^t$ for some $r > t \geqslant 0$ and X_2 is a variety which can be constructed by the method of Example 5.1.4.3. □

5.7 Proof of Theorem 5.1.5

We will first prove the following lemma.

Lemma 5.7.1. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1 \geqslant 3$ and $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. Then there is a fibration from X to $\mathbb{P}^1$.*

Proof. We will argue by contradiction. Set $n = \dim X$. Assume that there is no fibration from X to $\mathbb{P}^1$. Let V be a face of $\overline{\mathrm{NE}}(X)$ of dimension $n-4$. Then there is a morphism $X \to Y$ corresponding to V. By Lemma 5.3.2, Y is a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$. Note that $\rho(Y) = (n-1) - (n-4) = 3$. Hence $\dim Y = 4$. By Proposition 5.5.1, there is a fibration from Y to $\mathbb{P}^1$ which induces a fibration from X to $\mathbb{P}^1$. This is a contradiction. □

Now we can prove Theorem 5.1.5.

Proof of Theorem 5.1.5. We will prove by induction on the dimension of X. If $\dim X = 3$, then the theorem follows from Theorem 5.1.2. Assume that the theorem is true in dimension smaller than n, where $n \geqslant 4$ is an integer. Now we consider the case of $\dim X = n$.

By Lemma 5.7.1, there is a fibration $f_1 : X \to \mathbb{P}^1$. There is an extremal ray R in $\overline{\mathrm{NE}}(X)$ such that the class of any curve in X contracted by f_1 is not contained in R. Let $f_2 : X \to Y$ be the fibration corresponding to R. Then $f = f_1 \times f_2 : X \to \mathbb{P}^1 \times Y$ is a finite surjective morphism. By assumption, Theorem 5.1.5 is true in dimension smaller than n. Hence Y is one of the varieties in Theorem 5.1.5 by Lemma 5.3.3. We have the following two possibilities.

Case 1. The variety Y is a finite cover of $(\mathbb{P}^1)^{n-3} \times \mathbb{P}^2$ and f induces a finite surjective morphism $g : X \to (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ such that the projections from X to $\mathbb{P}^1$ and the projection g_2 from X to $\mathbb{P}^2$ induced by g are fibrations. Let $X \xrightarrow{g_1} Z \xrightarrow{\pi} (\mathbb{P}^1)^{n-2}$ be the Stein factorisation of the natural projection induced by g. Then Z is one of the varieties in Theorem 5.1.1 by Lemma 5.3.1 and Lemma 5.3.6.

If π is of degree at least 2, then we can write $Z \cong Z_1 \times Z_2$ such that Z_1 is a double cover of $(\mathbb{P}^1)^r$ for some $r \geqslant 3$, branched along some prime divisor of degree $(2, ..., 2)$, and Z_2 is one of the varieties of Theorem 5.1.1, of dimension $n-2-r$. Let $h_1 : X \to Z_1$ be the natural fibration. Let

$h_2 : X \to X'$ be the Stein factorisation of the natural projection $X \to Z_2 \times \mathbb{P}^2$. Then, by Lemma 5.3.3, X' is a Fano variety with locally factorial canonical singularities such that X' is smooth in codimension 2, $\rho(X') = \dim X' - 1$ and $\mathrm{Nef}(X') = \mathrm{Psef}(X')$. Hence it is one of the varieties in Theorem 5.1.5 by induction hypotheses. Let F be a general fiber of h_2. Then $h_1|_F : F \to Z_1$ is a finite surjective morphism. Since F is a Fano variety with canonical Gorenstein singularities, h_1 is an isomorphism by [Dru14, Thm. 1.3]. Hence $X \cong Z_1 \times X'$ and it is one of the varieties in Theorem 5.1.5.

If π is of degree 1, then $Z = (\mathbb{P}^1)^{n-2}$. By Lemma 5.6.2, X is one of the varieties in Theorem 5.1.5.

Case 2. The variety Y is a finite cover of $(\mathbb{P}^1)^{n-4} \times W$, where W is a normal $\mathbb{Q}$-factorial divisor of degree (p, q) in $B_1 \times B_2 \cong \mathbb{P}^2 \times \mathbb{P}^2$ such that $p, q \in \{1, 2\}$. Moreover, f induces a finite surjective morphism $g : X \to (\mathbb{P}^1)^{n-3} \times W$ such that the projections from X to $\mathbb{P}^1$ and the projections from X to B_1 and B_2 induced by g are fibrations. Let $X \xrightarrow{g_1} Z \xrightarrow{\pi} (\mathbb{P}^1)^{n-3}$ be the Stein factorisation of the natural projection induced by g. As in the first case, if π is of degree at least 2, then $X \cong Z_1 \times X'$, where Z_1 is a double cover of $(\mathbb{P}^1)^r$ for some $r \geqslant 3$, branched along some prime divisor of degree $(2, ..., 2)$, and

$$X \to X' \to (\mathbb{P}^1)^{n-3-r} \times W$$

is the Stein factorisation. By Lemma 5.3.3 and induction hypotheses, X' is one of the varieties in Theorem 5.1.5.

If π is of degree 1, then $Z = (\mathbb{P}^1)^{n-3}$. By Lemma 5.6.4, X is one of the varieties in Theorem 5.1.5. □

Chapitre 6

Pluri-forms on rationally connected surfaces

6.1 Introduction

It is known that for a smooth projective rationally connected variety X, we have $h^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for $m > 0$, see [Kol96, Cor. IV.3.8]. In [GKKP11, Thm. 5.1], it is shown that if a pair (X, D) is klt and X is rationally connected, then $H^0(X, \Omega^{[m]}_X) = \{0\}$ for $m > 0$, where $\Omega^{[m]}_X$ is the reflexive hull of Ω^m_X. By [GKP14, Thm. 3.3], if X is factorial, rationally connected and has canonical singularities, then $H^0(X, (\Omega^1_X)^{[\otimes m]}) = \{0\}$ for $m > 0$, where $(\Omega^1_X)^{[\otimes m]}$ is the reflexive hull of $(\Omega^1_X)^{\otimes m}$. However, this is not true without the assumption of being factorial, see [GKP14, Example 3.7]. In this chapter, our aim is to classify rationally connected surfaces with canonical singularities which have non-zero reflexive pluri-forms. We will give two methods to construct such surfaces (see Construction 6.1.2 and Construction 6.1.6) and we will also prove that every such surface can be constructed by both of these methods (see Theorem 6.1.3 and Theorem 6.1.5). This gives an affirmative answer to [GKP14, Remark and Question 3.8]

The following example is the one given in [GKP14, Example 3.7].

Example 6.1.1. Let $\pi' : X' \to \mathbb{P}^1$ be any smooth ruled surface. Choose four distinct points q_1, q_2, q_3, q_4 in $\mathbb{P}^1$. For each point q_i, perform the birational transformations of Construction 3.2.11. Then we get a rationally connected surface $\pi : X \to \mathbb{P}^1$ with canonical singularities such that $H^0(X, (\Omega^1_X)^{[\otimes 2]}) \neq \{0\}$.

We will prove that every projective rationally connected surface X with canonical singularities and having non-zero pluri-forms can be constructed by a similar method (see Construction 6.1.2 below) from a smooth ruled surface over $\mathbb{P}^1$.

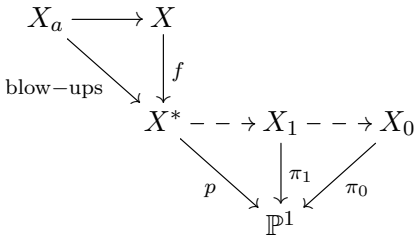
Construction 6.1.2. Take a smooth ruled surface $X_0 \xrightarrow{\pi_0} \mathbb{P}^1$ and choose distinct points $q_1, \ldots, q_r$ in $\mathbb{P}^1$ with $r \geqslant 4$. We perform a sequence of birational transformations as follows.

(i) For each q_i, perform the birational transformations of Construction 3.2.11. We get a fiber surface $\pi_1 : X_1 \to \mathbb{P}^1$. The non-reduced fibers of π_1 are $\pi_1^* q_1, \ldots, \pi_1^* q_r$.

(ii) Perform finitely many times this birational transformation : blow up a smooth point on a non-reduced fiber and then blow down the strict transform of the initial fiber. We obtain another fiber surface $p : X^* \to \mathbb{P}^1$ (see Lemma 3.2.12).

(iii) Starting from X^*, perform a sequence of blow-ups of smooth points, we get a surface X_a.

(iv) Blow down some chains of exceptional (-2)-curves for $X_a \to X^*$ (this is always possible by Lemma 3.1.5), we obtain a rational surface X.



Theorem 6.1.3. *The surface obtained by Construction 6.1.2 is a rationally connected surface which carries non-zero pluri-forms. Conversely, if X is a projective rationally connected surface with canonical singularities such that $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, then X can be constructed by the method described in Construction 6.1.2.*

Note that if X is a rational surface obtained by Construction 6.1.2, then there is a fibration $\pi : X \to \mathbb{P}^1$ induced by π_0. This fibration has multiple fibers over the points $q_1, \ldots, q_r$ that we have chosen at the beginning of the construction. In fact, these multiple fibers are exactly the source of non-zero forms on X by the theorem below.

Theorem 6.1.4. *Let X be a projective rationally connected surface with canonical singularities and having non-zero pluri-forms. If X^* is the result of a MMP, then X^* is a Mori fiber space over $\mathbb{P}^1$. Let $p : X^* \to \mathbb{P}^1$ be the Mori fibration. If r is the number of points over which p has non-reduced fibers, then we have $r \geqslant 4$ and*

$$H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r))$$

for $m > 0$. In particular, for fixed m, the dimension of m-pluri-forms depends only on the number of multiple fibers of p.

We note that both in Theorem 6.1.4 and in Construction 6.1.2, we meet a surface named X^*. We will see later (in the proof of Theorem 6.1.3 in §6.4) that, by choosing a good MMP, these two surfaces are identical. The points $q_1, \ldots, q_r$ are exactly the points over which $p : X^* \to \mathbb{P}^1$ has multiple fibers. By the semistable reduction, we can find a Galois cover $\gamma : E \to \mathbb{P}^1$ such that $Z \to E$ has only reduced fibers, where Z is the normalisation of $X^* \times_{\mathbb{P}^1} E$. Let Y be the normalisation of $X \times_{\mathbb{P}^1} E$. The following theorem shows that we can always choose a finite Galois cover γ which has degree 4 and the pluri-forms on X are exactly the G-invariant pluri-forms on Y, where G is the Galois group of γ.

Theorem 6.1.5. *Let X be a projective rationally connected surface with canonical singularities and having non-zero pluri-forms. Let π be the composition of $X \to X^* \to \mathbb{P}^1$. Then there is a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow[\text{4:1 cover}]{\Gamma} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\text{4:1 cover}]{\gamma} & \mathbb{P}^1 \end{array}$$

such that E is a smooth curve of positive genus and Y is a projective surface with canonical singularities. Both γ and Γ are Galois covers with Galois group $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and Γ is étale in codimension 1. Moreover, for all $m \geqslant 0$, we have $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(Y, (\Omega^1_Y)^{[\otimes m]})^G \cong H^0(E, (\Omega^1_E)^{\otimes m})^G$.

Note that Y is not rationally connected since E is not rationally connected. This theorem shows that every projective rationally connected surface with canonical singularities which has non-zero pluri-forms can be constructed by the following method.

Construction 6.1.6. Let Y be a projective surface with canonical singularities and let G be a finite subgroup of $\mathrm{Aut}(Y)$ whose action is étale in codimension 1. Assume that there is a G-invariant fibration π' from Y to a smooth curve E of positive genus such that $E/G = \mathbb{P}^1$ and that general fibers of π' are smooth rational curves. Let $X = Y/G$. Then X is rationally connected (see [GHS03, Thm. 1.1]) and $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$.

As a corollary, we obtain the following result.

Theorem 6.1.7. *If S is a Fano surface which has canonical singularities, then for all $m > 0$, $h^0(S, (\Omega^1_S)^{[\otimes m]}) = 0$.*

At the end of the chapter, we prove the following theorem.

Theorem 6.1.8. *Let X be a projective surface with canonical singularities. Then $h^0(X, (\Omega^1_X)^{[\otimes m]}) = 0$ for all positive m if and only if the smooth locus of X is rationally connected.*

Throughout this chapter, we will work over $\mathbb{C}$. If $p : X \to B$ is a fibration from a normal variety to a smooth curve and the non-reduced fibers of p are $p^*z_1, \ldots, p^*z_r$, then the *ramification divisor* of p is the divisor defined by

$$R = p^*z_1 + \cdots + p^*z_r - (p^*z_1 + \cdots + p^*z_r)_{\mathrm{red}}.$$

6.2 Vanishing theorem for Fano varieties with Picard number 1

In this section, we will prove the following theorem.

Theorem 6.2.1. *Let X be a $\mathbb{Q}$-factorial klt Fano variety with Picard number 1. Then $h^0(X, (\Omega^1_X)^{[\otimes m]}) = 0$ for any $m > 0$.*

Proof. We may assume that $\dim X > 1$. We will argue by contradiction. Assume that there is a positive integer m such that $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$. Let H be an ample divisor on X.

Since $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, we have an injective morphism of sheaves

$$\mathscr{O}_X \hookrightarrow (\Omega^1_X)^{[\otimes m]}.$$

This shows that

$$\mu^{max}_H((\Omega^1_X)^{[\otimes m]}) \geqslant \mu_H(\mathscr{O}_X) = 0.$$

By Proposition 1.2.3, we have $\mu^{max}_H(\Omega^{[1]}_X) = m^{-1}\mu^{max}_H((\Omega^1_X)^{[\otimes m]}) \geqslant 0$.

Therefore, there is a non-zero saturated coherent sheaf $\mathscr{F} \subseteq \Omega^{[1]}_X$ such that $\mu_H(\mathscr{F}) \geqslant 0$. Observe that $\operatorname{rank}\mathscr{F} < \dim X$, otherwise $\mathscr{F} = \Omega^{[1]}_X$ and $\det\mathscr{F} \cong K_X$. Thus $\mu_H(\mathscr{F}) < 0$, a contradiction.

We have two possibilities, either $\mu_H(\mathscr{F}) > 0$ or $\mu_H(\mathscr{F}) = 0$.

Case 1. Assume that $\mu_H(\mathscr{F}) > 0$. Since X has Picard number 1, $\det\mathscr{F}$ is ample and its Kodaira-Iitaka dimension is $\dim X$. However this contradicts Bogomolov-Sommese vanishing theorem (see [Gra15, Cor. 1.3]).

Case 2. Assume that $\mu_H(\mathscr{F}) = 0$. Let $\mathscr{G} = \det\mathscr{F}$. Then $\mathscr{G} \cdot H^{(\dim X - 1)} = 0$. Since X is $\mathbb{Q}$-factorial and klt, by [AD14, Lem. 2.6], there exists an integer l such that $(\mathscr{G}^{\otimes l})^{**}$ is isomorphic to $\mathscr{O}_X$. Let m be the smallest positive integer such that $(\mathscr{G}^{\otimes m})^{**} \cong \mathscr{O}_X$. Let $q : Z \to X$ be normalisation of the cyclic cover with respect to $(\mathscr{G}^{\otimes m})^{**} \cong \mathscr{O}_X$ (see [KM98, Def. 2.52]). Then $(q^*\mathscr{G})^{**} \cong \mathscr{O}_Z$. Since q is étale in codimension 1, Z is also klt by [Kol97, Prop. 3.16] and $-K_Z = q^*(-K_X)$ is ample. Thus Z is rationally connected by [HM07, Cor. 1.3 and 1.5]. And there are natural injective morphisms

$$(q^*\mathscr{G})^{**} \hookrightarrow (q^*\Omega^{[\operatorname{rank}\mathscr{F}]}_X)^{**} \hookrightarrow \Omega^{[\operatorname{rank}\mathscr{F}]}_Z.$$

Hence we have an injection $\mathscr{O}_Z \hookrightarrow \Omega^{[\operatorname{rank}\mathscr{F}]}_Z$, but this contradicts [GKKP11, Thm. 5.1]. □

6.3 Proof of Theorem 6.1.4

We will first prove Theorem 6.1.4. Let X be a projective rationally connected surface with canonical singularities which carries non-zero pluri-forms. Run a MMP for X. We will get a sequence of divisorial contractions

$$X = X_0 \to X_1 \to \cdots \to X_n = X^*.$$

Note that K_{X^*} is not nef. Thus the surface X^* is a Mori fiber surface over $\mathbb{P}^1$ by Theorem 6.2.1. Let $p : X^* \to \mathbb{P}^1$ be the Mori fibration. Let $f : X \to X^*$ be the composition of the sequence of the birational morphisms above and let $\pi = p \circ f : X \to \mathbb{P}^1$. Then for any $m \in \mathbb{N}$, there is an injection $H^0(X, (\Omega^1_X)^{[\otimes m]}) \hookrightarrow H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]})$ by the following lemma.

Lemma 6.3.1. *If $\varphi : Y \dashrightarrow Y'$ is a birational map between normal projective varieties such that φ^{-1} does not contract any divisor, then we have a natural injection $H^0(Y, (\Omega^1_Y)^{[\otimes m]}) \hookrightarrow H^0(Y', (\Omega^1_{Y'})^{[\otimes m]})$ for any integer $m > 0$.*

Proof. Since the birational map $\varphi^{-1} : Y' \dashrightarrow Y$ does not contract any divisor, it induces an isomorphism from an open subset W' of Y' onto an open subset W of Y such that $\operatorname{codim} Y' \backslash W' \geqslant 2$. By Proposition 1.2.1, we have $H^0(Y', (\Omega^1_{Y'})^{[\otimes m]}) \cong H^0(W', (\Omega^1_{W'})^{[\otimes m]})$ for any $m > 0$. Moreover, since W is an open subset of Y, we obtain $H^0(Y, (\Omega^1_Y)^{[\otimes m]}) \hookrightarrow H^0(W, (\Omega^1_W)^{[\otimes m]}) \cong H^0(W', (\Omega^1_{W'})^{[\otimes m]}) \cong H^0(Y', (\Omega^1_{Y'})^{[\otimes m]})$ for all $m > 0$. □

6.3.1 Source of non-zero reflexive pluri-forms

In this subsection, we will find out the source of non-zero pluri-forms on X^*. Let U be the smooth locus of X^*. Then the morphism of locally free sheaves on U

$$p^*\Omega^1_{\mathbb{P}^1} \to \Omega^1_U$$

factors through

$$\varphi : p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_U(R) \to \Omega^1_U$$

where

$$R = \sum_{z \in \mathbb{P}^1} p^*z - (p^*z)_{\mathrm{red}}$$

is the ramification divisor of p. Let V be the largest subset of U such that for any point $x \in V$, the evaluation of φ at x is injective. Then $\operatorname{codim} X^* \backslash V \geqslant 2$. By Proposition 1.2.1, this implies that

$$H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(V, (\Omega^1_V)^{\otimes m})$$

for any $m \in \mathbb{N}$.

Consider the exact sequence of sheaves on V

$$0 \to p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_V(R) \to \Omega^1_V \to \mathscr{G} \to 0$$

where $\mathscr{G}$ is isomorphic to $\Omega^1_{V/\mathbb{P}^1}/\text{torsion}$. Note that $p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_V(R) \to \Omega^1_V$ is an injection of vector bundles over V. Hence, $\mathscr{G}$ is locally free. Then there is a filtration (see Lemma 6.3.4 at the end of this subsection) over V

$$(\Omega^1_V)^{\otimes m} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{F}_s$$

such that for every $i \in \{0, \ldots, s-1\}$ and

$$\mathscr{F}_i/\mathscr{F}_{i+1} \cong \mathscr{G}^{\otimes a_i} \otimes (p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_V(R))^{\otimes(m-a_i)}$$

where $0 < a_i \leqslant m$ is an integer. Moreover, we have

$$\mathscr{F}_s \cong (p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_V(R))^{\otimes m} \cong (p^*\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes \mathscr{O}_V(mR).$$

Lemma 6.3.2. *With the notation above, there is a natural isomorphism*

$$H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m) \otimes p_*\mathscr{O}_{X^*}(mR))$$

for all $m \geqslant 0$.

Proof. Fix some $m \geqslant 0$. For a general point $z \in \mathbb{P}^1$, the support C of the fiber p^*z is isomorphic to $\mathbb{P}^1$ and is contained in V. Since p is smooth along C, we have

$$\mathscr{G}|_C \cong \mathscr{O}_C(-2) \text{ and } (p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_V(R))|_C \cong \mathscr{O}_C.$$

Thus $(\mathscr{F}_i/\mathscr{F}_{i+1})|_C$ is isomorphic $\mathscr{O}_C(-2a_i)$ for $i < s$. Since $a_i > 0$, we obtain that $h^0(V, \mathscr{F}_i/\mathscr{F}_{i+1}) = 0$ and $H^0(V, \mathscr{F}_i) \cong H^0(V, \mathscr{F}_{i+1})$ for $i < s$. This implies that

$$H^0(V, (\Omega^1_V)^{\otimes m}) \cong H^0(V, (p^*\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes \mathscr{O}_V(mR)).$$

By Proposition 1.2.1, the isomorphism above induces an isomorphism

$$H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(X^*, (p^*\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes \mathscr{O}_{X^*}(mR)).$$

We note that the right hand side of the last isomorphism is isomorphic to $H^0(\mathbb{P}^1, p_*((p^*\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes \mathscr{O}_{X^*}(mR)))$. By the projection formula, it is isomorphic to $H^0(\mathbb{P}^1, (\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes p_*\mathscr{O}_{X^*}(mR))$. Hence

$$H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m) \otimes p_*\mathscr{O}_{X^*}(mR)).$$

□

Note that $p_*\mathscr{O}_{X^*}(mR)$ is a torsion-free sheaf of rank 1 on $\mathbb{P}^1$. Thus it is an invertible sheaf and there is a $k \in \mathbb{Z}$ such that $\mathscr{O}_{\mathbb{P}^1}(k)$ is isomorphic to $p_*\mathscr{O}_{X^*}(mR)$. In the following lemma we will compute the integer k.

Lemma 6.3.3. *Assume that the non-reduced fibers of $p : X^* \to \mathbb{P}^1$ are over $z_1, \ldots, z_r$. Then for $m \in \mathbb{N}$, we have*

$$p_*\mathscr{O}_{X^*}(mR) \cong \mathscr{O}_{\mathbb{P}^1}([\frac{m}{2}](z_1 + \cdots + z_r)) \cong \mathscr{O}_{\mathbb{P}^1}([\frac{m}{2}]r),$$

where $[\ \ ]$ is the integer part. In particular,

$$H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r)).$$

Proof. Since any two points in $\mathbb{P}^1$ are linearly equivalent, the problem is local around every point z_i. We may assume that $r = 1$ for simplicity. From Proposition 3.2.3, we know that $p^*z_1 = 2R$. We may assume that $p_*\mathscr{O}_{X^*}(mR) \cong \mathscr{O}_{\mathbb{P}^1}(k \cdot z_1)$ and we have to prove that $k = [\frac{m}{2}]$.

Note that $\gamma \in H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(k \cdot z_1))$ is just a rational function on $\mathbb{P}^1$ which can only have pole at z_1 with multiplicity at most k. Its pull-back to X is a rational function which can only have pole along R with multiplicity at most $2k$. Thus k is the largest integer such that $2k \leqslant m$, *i.e.* $k = [\frac{m}{2}]$. □

In the following lemma, we will prove the existence of the filtration on $(\Omega^1_V)^{[\otimes m]}$.

Lemma 6.3.4. *Let $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ be an exact sequence of locally free sheaves on a variety X. Then for any $m > 0$, there is a positive integer s and a filtration*

$$\mathscr{F}^{\otimes m} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{F}_s = \mathscr{E}^{\otimes m}$$

such that for every $i \in \{0, \ldots, s-1\}$, the quotient $\mathscr{F}_i/\mathscr{F}_{i+1}$ is isomorphic to $\mathscr{G}^{\otimes a_i} \otimes \mathscr{E}^{\otimes(m-a_i)}$, where $0 < a_i \leqslant m$ is an integer.

Proof. We will prove by induction. If $m = 1$, then the assertion is true ($s = 1$ and $\mathscr{F}_1 = \mathscr{E}$). Assume that the assertion is true for $m = k$. Then we have a filtration

$$\mathscr{F}^{\otimes k} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{F}_s = \mathscr{E}^{\otimes k}$$

such that $\mathscr{F}_i/\mathscr{F}_{i+1}$ is isomorphic to $\mathscr{G}^{\otimes a_i} \otimes \mathscr{E}^{\otimes(k-a_i)}$, where $a_i > 0$ is an integer.

Consider the following exact sequence

$$0 \to \mathscr{E} \otimes \mathscr{F}^{\otimes k} \to \mathscr{F}^{\otimes(k+1)} \to \mathscr{G} \otimes \mathscr{F}^{\otimes k} \to 0.$$

By hypothesis of induction, we have a filtration

$$\mathscr{E} \otimes \mathscr{F}^{\otimes k} = \mathscr{E} \otimes \mathscr{F}_0 \supseteq \mathscr{E} \otimes \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{E} \otimes \mathscr{F}_s = \mathscr{E}^{\otimes(k+1)}.$$

Since $\mathscr{G}\otimes\mathscr{F}^{\otimes k}$ is the quotient of $\mathscr{F}^{\otimes(k+1)}$ by $\mathscr{E}\otimes\mathscr{F}^{\otimes k}$, we have a filtration, induced by the filtration on $\mathscr{F}^{\otimes k}$ above,

$$\mathscr{F}^{\otimes(k+1)}=\mathscr{H}_0\supseteq\mathscr{H}_1\supseteq\cdots\supseteq\mathscr{H}_s\supseteq\mathscr{E}\otimes\mathscr{F}^{\otimes k}$$

such that $\mathscr{H}_i/\mathscr{H}_{i+1}$ is isomorphic to $\mathscr{G}\otimes(\mathscr{G}^{\otimes a_i}\otimes\mathscr{E}^{\otimes(k-a_i)})$ and $\mathscr{H}_s/(\mathscr{E}\otimes\mathscr{F}^{\otimes k})\cong\mathscr{G}\otimes\mathscr{E}^{\otimes k}$. Hence we obtain a fibration

$$\mathscr{F}^{\otimes(k+1)}=\mathscr{H}_0\supseteq\mathscr{H}_1\supseteq\cdots\supseteq\mathscr{H}_s\supseteq\mathscr{E}\otimes\mathscr{F}_0\supseteq\cdots\supseteq\mathscr{E}\otimes\mathscr{F}_s=\mathscr{E}^{\otimes(k+1)}$$

which satisfies the conditions in the lemma. □

6.3.2 Back to the initial variety

We have studied X^* and now we have to reverse the MMP and pull back pluri-forms to the initial variety X. Our aim is to prove that

$$H^0(X,(\Omega^1_X)^{[\otimes m]})\cong H^0(X^*,(\Omega^1_{X^*})^{[\otimes m]}).$$

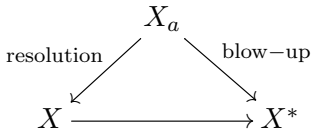
By Lemma 3.1.1, every exceptional divisor of $f:X\to X^*$ is over a smooth point of X^*. Now we can prove the isomorphism we mentioned at the beginning of this subsection.

Lemma 6.3.5. *The injection $H^0(X,(\Omega^1_X)^{[\otimes m]})\hookrightarrow H^0(X^*,(\Omega^1_{X^*})^{[\otimes m]})$ is an isomorphism.*

Proof. Let $X_a\to X$ be a projective birational morphism which is the minimal resolution for the singular points of X lying over smooth points of X^*. Then there is a natural injection

$$H^0(X_a,(\Omega^1_{X_a})^{[\otimes m]})\to H^0(X,(\Omega^1_X)^{[\otimes m]}).$$

By Lemma 3.1.1, f^{-1} is an isomorphism around the singular points of X^*. Hence all exceptional divisors of $X_a\to X^*$ are over smooth points of X^*. This implies that X_a can be obtain from X^* by a sequence of blow-ups of smooth points (see [Har77, Cor. V.5.4]).



There is a natural isomorphism $H^0(X_a,(\Omega^1_{X_a})^{[\otimes m]})\cong H^0(X^*,(\Omega^1_{X^*})^{[\otimes m]})$, which implies that $H^0(X,(\Omega^1_X)^{[\otimes m]})\cong H^0(X^*,(\Omega^1_{X^*})^{[\otimes m]})$. □

We can conclude Theorem 6.1.4.

Proof of Theorem 6.1.4. By Theorem 6.2.1, we have a Mori fibration $p:X^*\to\mathbb{P}^1$. Lemma 6.3.3 and Lemma 6.3.5 show that

$$H^0(X,(\Omega^1_X)^{[\otimes m]})\cong H^0(X^*,(\Omega^1_{X^*})^{[\otimes m]})\cong H^0(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(-2m+[\frac{m}{2}]r)).$$

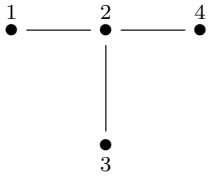
□

6.4 Proof of Theorem 6.1.3

We will prove Theorem 6.1.3 in this section. If X is a projective rationally connected surface with canonical singularities such that $h^0(X, (\Omega_X)^{[\otimes m]}) \neq 0$ for some $m > 0$ and X^* is the result of a MMP, then X and X^* are isomorphic around the singular locus of X^* by Lemma 3.1.1. The proof of Lemma 6.3.5 gives us an idea of how to reconstruct X from X^*. First we construct the surface X_a (the surface defined in the proof of Lemma 6.3.5) which can be obtained from X^* by a sequence of blow-ups of smooth points. Then we blow down some exceptional (-2)-curves for $X_a \to X^*$ and we obtain X. Note that these are just the birational transformations mentioned in step (iii) and (iv) of Construction 6.1.2.

In order to contract the (-2)-curves in the transformation above, we want to use Lemma 3.1.5. Thus, we have to study the structure of the exceptional set of $X_a \to X^*$.

Lemma 6.4.1. *Denote a germ of smooth surface by $(0 \in S)$. Let $h : S' \to S$ be the composition of a sequence of blow-ups of smooth points over $0 \in S$. Let D be the support of $h^{-1}0$. Then any (-2)-curve in D meets at most 2 other (-2)-curves. In another word, the dual graph of D cannot contain a subgraph as below such that each vertex of the subgraph corresponds a (-2)-curve.*



Proof. Assume the opposite. We know that we can reverse the process of blow-ups of smooth points by running a MMP relatively to S. Thus these four curves will be successively contracted during the MMP. The first one contracted cannot be the curve corresponding to the point with label 2, since after the contraction, the dual graph of the remaining curves is a tree by Lemma 3.2.5. Without loss of generality, we may assume that the curve corresponding to the point with label 1 is the first one contracted.

If the curve corresponding to the point 3 (or 4) is contracted secondly, then the self-intersection number of the curve corresponding to the point 2 becomes at least 0. If the curve corresponding to the point 2 is contracted secondly, a further contraction will also produce a curve with self-intersection number at least 0.

However, this curve of self intersection at least 0 is over $0 \in S$, it must have negative self-intersection number by the negativity theorem (see [KM98, Lem. 3.40]). This leads to a contradiction. □

Note that, in the lemma above, the dual graph of D is without simple

cycles. Hence this lemma shows that every connected collection of (-2)-curves in D has a dual graph as below

This is the dual graph of the exceptional set of the minimal resolution for the singularity of type A_i. By Lemma 3.1.5, it is possible to contract such a chain of (-2)-curves.

Now we can prove Theorem 6.1.3.

Proof of Theorem 6.1.3. First let X be a projective rationally connected surface with canonical singularities which carries non-zero pluri-forms. We will prove that X can be constructed by the method of Construction 6.1.2. Let $f : X \to X^*$ be the result of a MMP and let X_a be the surface defined in the proof of Lemma 6.3.5. The surface X can be obtained from X_a by a contraction of chains of (-2)-curves by Lemma 6.3.5, Lemma 6.4.1 and Lemma 3.1.5. By the proof of Lemma 6.3.5, X_a can be obtained from X^* by a sequence of blow-ups of smooth points. Since $X^* \to \mathbb{P}^1$ is a Mori fibration and X^* has canonical singularities, X^* can be obtain from a smooth ruled surface $X_0 \to \mathbb{P}^1$ by the method of step (i) and (ii) of Construction 6.1.2 (see Lemma 3.2.14). Thus X can be constructed by the method of Construction 6.1.2.

Now, let X be a surface constructed by the method of Construction 6.1.2. We will prove that X carries non-zero pluri-forms. Since $X^* \to \mathbb{P}^1$ is a Mori fibration, by running a f-relative MMP, we obtain that X^* is the result of this MMP (This is why we use the same notation X^*). By Lemma 6.3.3, we know that X^* carries non-zero pluri-forms. By Lemma 6.3.5, this shows that X carries non-zero pluri-forms. □

6.5 Proof of Theorem 6.1.5

We would like to prove Theorem 6.1.5 in this section. In [GKP14, Remark and Question 3.8], for X in Example 6.1.1, we can find a smooth elliptic curve E, a smooth ruled surface Y (which is $\widetilde{X}$ in [GKP14]) such that $\mathbb{P}^1$ is the quotient of E by $\mathbb{Z}/2\mathbb{Z}$ and X is the quotient of Y by the same group. In this section, we would like to construct such a surface Y for any rationally connected surface X with canonical singularities and having non-zero pluri-forms.

We will first construct the curve E.

Proposition 6.5.1. *Let $q_1, \ldots, q_r$ be r different points on $\mathbb{P}^1$ with $r \geqslant 4$, then there exist a smooth curve E, a $4:1$ Galois cover $\gamma : E \to \mathbb{P}^1$ with Galois group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that γ is exactly ramified over the q_i's and the degrees of ramification are all equal to 2.*

Proof. Since $r \geqslant 4$, we can find an elliptic curve D and a $2:1$ cover $\alpha : D \to \mathbb{P}^1$ such that α is ramified exactly over q_1, q_2, q_3, q_4. Let $\alpha^{-1}(\{q_i\}) = \{s_i, t_i\}$ for $i > 4$ and let $\alpha^{-1}(\{q_i\}) = \{s_i\}$ for $i = 1, 2$.

If $r > 4$, then $\mathscr{O}_D((r-4)s_1)^{\otimes 2}$ is isomorphic to $\mathscr{O}_D(\sum_{i>4} s_i + \sum_{i>4} t_i)$. Thus we can construct a ramified $2:1$ cyclic cover of E, with respect to the line bundle $\mathscr{O}_D((r-4)s_1)$,

$$\beta : E \to D$$

such that E is smooth and β is ramified exactly over $\{s_i, t_i \mid i > 4\}$ (see [KM98, Def. 2.50]).

If $r = 4$, then $\mathscr{O}_D(s_1 - s_2)^{\otimes 2} \cong \mathscr{O}_D$ and we can construct a $2:1$ cyclic cover of E, with respect to the non-trivial invertible sheaf $\mathscr{O}_D(s_1 - s_2)$,

$$\beta : E \to D$$

such that E is a smooth elliptic curve and β is étale.

Finally, in both cases, the composition

$$\gamma = \alpha \circ \beta : E \to \mathbb{P}^1$$

is a $4:1$ cover which is exactly ramified over the q_i's and the degrees of ramification are all equal to 2.

We will show that γ is a Galois cover with Galois group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For simplicity, we assume that $r > 4$. For the case of $r = 4$, the argument is similar. We only need to prove that we can lift the action of $Aut(D/\mathbb{P}^1) = \mathbb{Z}/2\mathbb{Z}$ on D to E. There is a natural action of $Aut(D/\mathbb{P}^1)$ on $\mathscr{O}_D$. Let $\mathscr{I} = \mathscr{O}_D((4-r)s_1)$. Then $\mathscr{I}$ can be regarded as an ideal sheaf on D. Since s_1 is invariant under the action of $Aut(D/\mathbb{P}^1)$ on D, the sheaf $\mathscr{I}$, as a subsheaf of $\mathscr{O}_D$, is stable under the action of $Aut(D/\mathbb{P}^1)$ on $\mathscr{O}_D$. This gives an action of $Aut(D/\mathbb{P}^1)$ on $\mathscr{I}$. Hence $Aut(D/\mathbb{P}^1)$ acts (diagonally) on the sheaf $\mathscr{O}_D \oplus \mathscr{I}$. There is a rational function h on D such that the divisor associated to h is

$$\sum_{i>4} s_i + \sum_{i>4} t_i + 2(4-r)s_1.$$

By multiplying h, we have a morphism from $\mathscr{I}^{\otimes 2}$ to $\mathscr{O}_D$. This morphism gives an $\mathscr{O}_D$-algebra structure on $\mathscr{O}_D \oplus \mathscr{I}$. By construction, with this $\mathscr{O}_D$-algebra structure, the sheaf $\mathscr{O}_D \oplus \mathscr{I}$ is isomorphic to $\beta_* \mathscr{O}_E$ as $\mathscr{O}_D$-algebras. Note that since

$$\sum_{i>4} s_i + \sum_{i>4} t_i = \alpha^*(\sum_{i>4} q_i) \text{ and } 2(4-r)s_1 = \alpha^*((4-r)q_1),$$

the rational function h is the pullback of a rational function on $\mathbb{P}^1$. Hence h is invariant under the action of $Aut(D/\mathbb{P}^1)$ on $\mathscr{O}_D$. This shows that the action of $Aut(D/\mathbb{P}^1)$ on $\mathscr{O}_D \oplus \mathscr{I}$ is compatible with the $\mathscr{O}_D$-algebra structure induced by h. Thus we obtain an extension of the action of $Aut(D/\mathbb{P}^1)$ on $\mathscr{O}_D$ to $\mathscr{O}_E$. This proves that we can lift the action of $Aut(D/\mathbb{P}^1)$ on D to E.

Since γ is a Galois cover, we have $E/G = \mathbb{P}^1$. □

Remark 6.5.2. What we want in the lemma above is to construct a finite morphism $\gamma : E \to \mathbb{P}^1$ which is exactly ramified over the q_i's and all of the ramified degrees are equal to 2. Note that the finite cover γ we constructed above is of degree four and the one in [GKP14, Remark and Question 3.8] is of degree two. However, if r is odd, then the Hurwitz's theorem (see [Har77, Cor. IV.2.4]) shows that it is not possible to have a $2:1$ cover which satisfies the condition.

Now we will prove Theorem 6.1.5.

Proof of Theorem 6.1.5. Let $q_1, \ldots, q_r$ be all of the points in $\mathbb{P}^1$ over which $p : X^* \to \mathbb{P}^1$ has multiple fibers. Let $\gamma : E \to \mathbb{P}^1$ be $4:1$ cover constructed in Lemma 6.5.1. Let Z be the normalisation of the fiber product $X^* \times_{\mathbb{P}^1} E$. Let $q : Z \to E$ and $\Theta : Z \to X^*$ be the natural projections. Then Θ is étale over the smooth locus of X^* and q has only reduced fibers.

We know that we can reconstruct X from X^* (see §6.4). Since $\Theta : Z \to X^*$ is étale over the smooth locus of X^*, and the exceptional divisors of $X \to X^*$ is over the smooth locus of X^*, every operation we do with X^* can be done in the analogue way with Z. After the operations, the surface $Y \xrightarrow{g} Z$ we obtained is just the normalisation of $X \times_{\mathbb{P}^1} E$. We have a commutative diagram as below

$$\begin{array}{ccc} Y & \xrightarrow{\Gamma} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{\Theta} & X^* \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{\gamma} & \mathbb{P}^1 \end{array} \qquad \pi' \text{ and } \pi \text{ as the outer arrows}$$

Then Γ is étale over the smooth locus of X and $X = Y/G$ where G is the Galois group of γ. The sheaf $\Gamma_*(\Omega_Y^{[\otimes m]})$ is a G-sheaf on X (the action of G on X is trivial) which is reflexive (see [Har80, Prop. 1.7]). Then $(\Gamma_*(\Omega_Y^{[\otimes m]}))^G$ is also reflexive (see [GKKP11, Lem. B.4]) and is isomorphic to $\Omega_X^{[\otimes m]}$ since Γ is étale over the smooth locus of X. Thus we have

$$H^0(Y, (\Omega_Y^1)^{[\otimes m]})^G \cong H^0(X, (\Omega_X^1)^{[\otimes m]}).$$

Moreover, for any $m \geqslant 0$, the natural morphism

$$H^0(Y, (\Omega_Y^1)^{[\otimes m]}) \to H^0(Z, (\Omega_Z^1)^{[\otimes m]})$$

is an isomorphism by the same argument as in the proof of Lemma 6.3.5. Since every fiber of q is reduced and general fibers of q are smooth rational curves, by the same argument as Lemma 6.3.2, we have

$$H^0(Y, (\Omega_Y^1)^{[\otimes m]}) \cong H^0(E, (\Omega_E^1)^{\otimes m}).$$

Hence we obtain isomorphisms

$$H^0(X,(\Omega^1_X)^{[\otimes m]}) \cong H^0(Y,(\Omega^1_Y)^{[\otimes m]})^G \cong H^0(E,(\Omega^1_E)^{\otimes m})^G.$$

This completes the proof of the Theorem. □

Now we want to compute the dimension of $H^0(X,(\Omega^1_X)^{[\otimes m]})$ in function of multiple fibers of $X^* \to \mathbb{P}^1$ with the formula above. We will first prove the following lemma.

Lemma 6.5.3. *Let R be the ramification divisor of the finite morphism $\gamma : W \to V$, then we have $(\gamma_*\mathscr{O}_W(R))^G \cong \mathscr{O}_V$.*

Proof. We have $H^0(U,(\gamma_*\mathscr{O}_W(R))^G) \cong H^0(\gamma^{-1}(U),\mathscr{O}_W(R))^G$ for any open set $U \subseteq V$. Let θ be a rational function on W such that θ represents an non-zero element in $H^0(\gamma^{-1}(U),\mathscr{O}_W(R))^G$. Since θ is G-invariant, it can also be regarded as a rational function on U. Since θ can only have simple poles at the support of R on $\gamma^{-1}(U)$, it cannot have any pole on U. Thus $(\gamma_*\mathscr{O}_W(R))^G \cong \mathscr{O}_V$. □

With the notation in the proof of Theorem 6.1.5, we have

$$(\Omega^1_E)^{\otimes m} \cong (\gamma^*\mathscr{O}_{\mathbb{P}^1}(-2m+[\frac{m}{2}](q_1+\cdots+q_r))) \otimes \mathscr{O}_E((m-2[\frac{m}{2}])R_\gamma).$$

By the projection formula, we have

$$\gamma_*(\Omega^1_E)^{\otimes m} \cong \mathscr{O}_{\mathbb{P}^1}(-2m+[\frac{m}{2}]r) \otimes \gamma_*\mathscr{O}_E((m-2[\frac{m}{2}])R_\gamma).$$

By taking the G-invariant part, we obtain

$$(\gamma_*(\Omega^1_E)^{\otimes m})^G \cong \mathscr{O}_{\mathbb{P}^1}(-2m+[\frac{m}{2}]r) \otimes (\gamma_*\mathscr{O}_E((m-2[\frac{m}{2}])R_\gamma))^G.$$

The lemma above implies that $(\gamma_*(\Omega^1_E)^{\otimes m})^G \cong (\mathscr{O}_{\mathbb{P}^1}(-2m+[\frac{m}{2}]r))$. Hence

$$H^0(E,(\Omega^1_E)^{\otimes m})^G \cong H^0(\mathbb{P}^1,(\gamma_*(\Omega^1_E)^{\otimes m})^G) \cong H^0(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(-2m+[\frac{m}{2}]r)).$$

This shows that $H^0(X,(\Omega^1_X)^{[\otimes m]}) \cong H^0(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(-2m+[\frac{m}{2}]r))$ and we recover the same formula as in Theorem 6.1.4

Now we will prove Theorem 6.1.7.

Proof of Theorem 6.1.7. Assume the opposite. If $S \to S^*$ is the result of a MMP for S, then S^* is a Mori fiber space. By Theorem 6.2.1, S^* has Picard number 2 and we have a Mori fibration $S^* \to \mathbb{P}^1$. Let p be the composition of $S \to S^* \to \mathbb{P}^1$. By Theorem 6.1.5, there is a smooth curve B of positive genus and a finite morphism $B \to \mathbb{P}^1$ such that the natural morphism $S_B \to S$ is étale in codimension 1, where S_B is the normalisation of $S \times_{\mathbb{P}^1} B$. Hence S_B

is a Fano surface. Moreover it has canonical singularities (see [KM98, Prop. 5.20]). So it is rationally connected by [HM07, Cor. 1.3 and 1.5]. Hence B is also rationally connected, which is a contradiction since its genus is positive. □

Example 6.5.4. We will give some examples. Let $h(m,r) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r))$. This is just the number of m-pluri-forms as a function of the number r of multiple fibers of $X^* \to \mathbb{P}^1$.

If $r = 4$, then $h(m,4) = 1$ if $m > 0$ is even and $h(m,4) = 0$ if m is odd.

If $r = 5$, then $h(2,5) = 2$, $h(3,5) = 0$ and $h(m,5) > 0$ if $m \geqslant 4$.

If $r \geqslant 6$, then $h(m,r) > 0$ for $m \geqslant 2$.

6.6 Rational connectedness and pluri-forms

In this section, we will prove Theorem 6.1.8. We recall that a quasi-projective variety is called rationally connected if for any two general point, there is a projective rational curve passing through them. A smooth quasi-projective variety is rationally connected if and only if it contains a very free curve (see [Kol96, Thm. IV.3.7]). We will first prove some lemmas.

Lemma 6.6.1. *Let A be a set of points in $\mathbb{P}^1$. If the cardinality of A is at most 3, then there is a finite cover $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\gamma^* b = 2(\gamma^* b)_{\mathrm{red}}$ for any $b \in A$.*

Proof. If A has at most two elements, then there is a double cover $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ which satisfies the property of the lemma.

Now we assume that $A = \{a, b, c\}$. There is a double cover $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ which is branched along a and b. Then $\varphi^{-1}(\{c\})$ has two elements x, y. There is a double cover $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ which is branched along x and y. Let $\gamma = \varphi \circ \psi$. Then γ satisfies the property of the lemma. □

Lemma 6.6.2. *Let $p : X \to \mathbb{P}^1$ be a Mori fibration, where X is a surface with canonical singularities. Assume that p has multiple fiber over at most three points of $\mathbb{P}^1$. Then for any open subset U of the smooth locus of X such that* $\operatorname{codim} X \backslash U \geqslant 2$, *it is rationally connected.*

Proof. Let A be the set of points of $\mathbb{P}^1$ over which p has multiple fibers. Then there is a finite cover $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\gamma^* b = 2(\gamma^* b)_{\mathrm{red}}$ for any $b \in A$ by Lemma 6.6.1. Let Z be the normalisation of the fiber product $X \times_{\mathbb{P}^1} \mathbb{P}^1$ with respective to γ. Let $g : Z \to \mathbb{P}^1$ be the natural fibration. Then g has only reduced fibers. Let $\Gamma : Z \to X$ be the natural projection. Then Γ is étale in codimension 1. In particular, Z has canonical singularities [KM98, Prop. 5.20].

$$\begin{array}{ccc} Z & \xrightarrow{\Gamma} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} \\ \mathbb{P}^1 & \xrightarrow{\gamma} & \mathbb{P}^1 \end{array}$$

Let $W = \Gamma^{-1}(U)$. Then $\operatorname{codim} Z \backslash W \geqslant 2$. Let $f : Z \to Y$ be the result of a g-relative MMP for Z. Then we obtain a Mori fibration $q : Y \to \mathbb{P}^1$ which has reduced fibers. Hence it is smooth by Proposition 3.2.3.

Note that there is an open subset V of Y contained in $f(W)$ such that $\operatorname{codim} Y \backslash V \geqslant 2$ and that f^{-1} induces an isomorphism from V onto its image. Since $\operatorname{codim} Y \backslash V \geqslant 2$, and Y is smooth rationally connected, we obtain that V is rationally connected by [Kol96, Exercise IV.3.7.1]. Hence W is rationally connected and so is U. □

Now we will prove Theorem 6.1.8.

Proof of Theorem 6.1.8. If the smooth locus of X is rationally connected, then for all positive m, we have $h^0(X, (\Omega^1_X)^{[\otimes m]}) = 0$.

Now we assume that $h^0(X, (\Omega^1_X)^{[\otimes m]}) = 0$ for all positive m and we will show that the smooth locus of X is rationally connected. Let $f : X \to X^*$ be the result of a MMP for X. Since $h^0(X, (\Omega^1_X)^{[\otimes m]}) = 0$ for all positive m, X^* does not carry any non-zero pluri-form neither by the same argument as the proof of Lemma 6.3.5. By the abundance theorem for surfaces (see [AFKM92, Thm. 11.1.3]), we obtain that K_{X^*} is not nef. Hence there is a Mori fibration $p : X^* \to B$, where B is either a curve or a point. Since X^* does not carry any non-zero pluri-form, B cannot be a curve of positive genus. Thus B is either isomorphic to $\mathbb{P}^1$ or a point.

There is a smooth open subset U of X^* such that $\operatorname{codim} X^* \backslash U \geqslant 2$ and that f^{-1} induces an isomorphism from U onto its image. Moreover, we may choose U in order that $f^{-1}(U)$ is contained in the smooth locus of X. In order to show that the smooth locus of X is rationally connected, we only need to show that U is rationally connected. We will discuss on two cases.

Case 1. Assume that $B \cong \mathbb{P}^1$. Then the Mori fibration p has at most three multiple fibers by the same argument of the proof of Lemma 6.3.2. Thus, by Lemma 6.6.2, U is rationally connected.

Case 2. Assume that B is a point. Then X^* has Picard number 1 and is a Fano surface. Then the smooth locus of X^* is rationally connected by [KM99, Cor. 1.6]. Since $\operatorname{codim} X^* \backslash U \geqslant 2$, we obtain that U is rationally connected by [Kol96, Exercise IV.3.7.1]

□

Chapitre 7

Pluri-forms on rationally connected threefolds

7.1 Introduction

In Chapter 6, we classify all rationally connected surfaces X with canonical singularities such that $h^0(X, (\Omega^1_X)^{[\otimes m]}) \neq 0$ for some $m > 0$. In this chapter, we will study the case of threefolds. We are interested in the structure of rationally connected threefolds X with klt singularities which carry non-zero pluri-forms and we try to find out the source of these forms. If $f : \bar{X} \to X$ is the $\mathbb{Q}$-factorial model (see [BCHM10, Cor. 1.4.3]), then f is an isomorphism in codimension 1 and any pluri-form of X lifts to $\bar{X}$. Hence we can reduce to the case when X is $\mathbb{Q}$-factorial. If the threefold has terminal singularities, we prove the following result which is similar to Theorem 6.1.4.

Theorem 7.1.1. *Let X be a projective rationally connected threefold with $\mathbb{Q}$-factorial terminal singularities. Then $h^0(X, (\Omega^1_X)^{[\otimes m]}) \neq 0$ for some $m > 0$ if and only if there is a fibration $p : X \to \mathbb{P}^1$ whose general fibers are smooth rationally connected surfaces such that $\sum_{z\in\mathbb{P}^1} \frac{m(p,z)-1}{m(p,z)} \geqslant 2$, where $m(p, z)$ is the smallest positive coefficient in the divisor p^*z. Moreover, if it is the case, then for all $m > 0$, we have*

$$h^0(X, (\Omega^1_X)^{[\otimes m]}) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + \sum_{z\in\mathbb{P}^1} [\frac{(m(p, z) - 1)m}{m(p, z)}])).$$

In particular, the dimension of m-pluri-forms depends only on the ramification of p.

Construction 7.1.2. Thanks to Theorem 7.1.1, every rationally connected threefold X with $\mathbb{Q}$-factorial terminal singularities which carries non-zero pluri-forms can be constructed as follows. There is a fibration $q : T \to B$ from a normal threefold T to a smooth curve B which has positive genus

such that $m(q,b)=1$ for all $b\in B$. There is a finite group G which acts on T and B such that q is G-equivariant and that $B/G\cong\mathbb{P}^1$ and $T/G\cong X$. For more details, see §7.4.

If the threefold X has canonical singularities, we also obtain some necessary conditions for existence of non-zero pluri-forms.

Theorem 7.1.3. *Let X be a projective rationally connected threefold with $\mathbb{Q}$-factorial canonical singularities which carries non-zero pluri-forms. Let $X\dashrightarrow X^*$ be the result of a minimal model program. Then there is always an equidimensional fibration $q:X^*\to Z$ with $\dim Z>0$ such that $K_Z+\Delta$ is $\mathbb{Q}$-effective. The divisor Δ is defined by $\Delta=\sum_i\frac{m(q,D_i)-1}{m(q,D_i)}D_i$, where the D_i's are all the prime divisors in Z such that q^*D_i is not reduced and $m(q,D_i)$ is the smallest coefficient in q^*D_i. More precisely, we have two possibilities :*

(1) The variety Z is a surface and $q:X^*\to Z$ is a Mori fibration such that there is a positive integer l such that $l(K_Z+\Delta)$ is an effective divisor. Moreover, the sections of $\mathscr{O}_Z(l(K_Z+\Delta))$ lift to sections of $\Omega^{[\otimes 2l]}_{X^*}$.

(2) The variety Z is isomorphic to $\mathbb{P}^1$ and for any $m>0$, we have

$$H^0(X^*,(\Omega^1_{X^*})^{[\otimes m]})\cong H^0(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(-2m+\sum_{z\in\mathbb{P}^1}[\frac{(m(q,z)-1)m}{m(q,z)}])).$$

The difficulty in the case of canonical singularities is that the rational map $X\dashrightarrow Z$ is not always regular. And even it is, it may not be equidimensional. If klt singularities are permitted for our threefolds, then there exist other structures. We give an example of rationally connected threefold of general type in Example 7.2.6.

Outline of the chapter. The main objective of this chapter is to prove Theorem 7.1.1. We will do this in several steps. First, we consider a projective rationally connected threefold X with $\mathbb{Q}$-factorial canonical singularities which carries non-zero pluri-forms. Since it is rationally connected and has canonical singularities, its canonical divisor K_X is not pseudo-effective. Hence if $f:X\dashrightarrow X^*$ is the result of a MMP for X, then K_{X^*} is not pseudo-effective neither. Thus there is a Mori fibration $p:X^*\to Z$ where Z is a normal variety of dimension less than 3. However, since X carries non-zero pluri-forms, we have $h^0(X^*,(\Omega^1_{X^*})^{[\otimes m]})\neq 0$ for some $m>0$. This implies that $\dim Z>0$ by Theorem 6.2.1. Hence either $\dim Z=1$ or $\dim Z=2$. We will treat these two cases separately in §7.4 and §7.5. If $\dim Z=1$, then $Z\cong\mathbb{P}^1$ and we will show that

$$K_{\mathbb{P}^1}+\sum_{z\in\mathbb{P}^1}\frac{m(p,z)-1}{m(p,z)}z$$

is $\mathbb{Q}$-effective, which is the same condition as in Theorem 7.1.1. If $\dim Z=2$, then we will define the $\mathbb{Q}$-divisor Δ on Z as in Theorem 7.1.3. We will prove

that either $K_Z + \Delta$ is $\mathbb{Q}$-effective or there is a fibration $X^* \to \mathbb{P}^1$ such that we can reduce to the situation of §7.4. In the last section, we assume that the variety X has terminal singularities. In this case, there is always a fibration from X^* to $\mathbb{P}^1$ and we are always in the situation of §7.4. This fibration induces a dominant rational map from X to $\mathbb{P}^1$. In the end, we will prove that this rational map is regular and complete the proof of Theorem 7.1.1.

Throughout this chapter, we will work over $\mathbb{C}$. Let $p : X \to Z$ be a morphism and D be a prime $\mathbb{Q}$-Cartier Weil divisor in Z. We define the number $m(p, D)$ by

$$\min\{\text{coefficient of } E \text{ in } p^*D \mid E \text{ is an irreducible component of } p^*D \text{ which dominates } D\}.$$

By generic smoothness, there are only finitely many divisors D such that $m(p, D) > 1$. Moreover, if $p : X \to Z$ is a morphism from a normal variety to a smooth curve, then we define the ramification divisor R as

$$\sum_{z \in Z} p^*z - (p^*z)_{\mathrm{red}}.$$

7.2 Examples

We will give some examples of rationally connected varieties which carry non-zero pluri-forms. First we will give an example of such varieties which have terminal singularities. For the construction of this example, we use the method of Theorem 7.4.2.

Example 7.2.1. Let $C_1 = \{[a : b : c] \in \mathbb{P}^2 \mid a^3+(a+c)b^2+c^3 = 0\}$ be a smooth elliptic curve in $\mathbb{P}^2$. Let $X_1 = \{([a : b : c], [x : y : z : t]) \in \mathbb{P}^2 \times \mathbb{P}^3 \mid a^3 + (a + c)b^2 + c^3 = 0,\ (a^2+2b^2+c^2)x^2+(a^2+3b^2+c^2)y^2+(a^2+4b^2+3c^2)z^2+(a^2+5b^2+6c^2)t^2 = 0\}$. Then X_1 is a smooth threefold and we have an induced fibration $p_1 : X_1 \to C_1$ such that all fibers of p_1 are smooth quadric surfaces which are Fano surfaces. Moreover, X_1 has Picard number 2 by the Lefschetz theorem (see [Laz04, Exmp. 3.1.23]). Hence p_1 is a Mori fibration. Moreover, since p_1 is smooth, we have $H^0(X_1, (\Omega^1_{X_1})^{\otimes 2}) \cong H^0(C_1, (\Omega^1_{C_1})^{\otimes 2})$ by Lemma 7.3.2. Let G be the group $\mathbb{Z}/2\mathbb{Z}$ and let $g \in G$ be the generator. We have an action of G on $\mathbb{P}^2 \times \mathbb{P}^3$ defined by $g \cdot ([a : b : c], [x : y : z : t]) = ([a : -b : c], [-x : -y : z : t])$. This action induces an action of G on X_1 and an action of G on C_1 such that p_1 in G-equivariant. We have $C_1/G = \mathbb{P}^1$. The action of G on X_1 is free in codimension 2 and it has exactly 16 fixed points. Since $G = \mathbb{Z}/2\mathbb{Z}$, the action of G on X_1 satisfies the condition of the first theorem in [Rei87, §5]. Thus $X = X_1/G$ is a threefold which has $\mathbb{Q}$-factorial terminal singularities. We have a Fano fibration $p : X \to \mathbb{P}^1$ induced by p_1. Moreover, the Picard number of X is not larger than the Picard number of X_1. Hence

p is a Mori fibration. Since general fibers of p are smooth quadric surfaces which are rationally connected, X is rationally connected by [GHS03, Thm. 1.1]. In addition, $H^0(X, (\Omega^1_X)^{[\otimes 2]}) \cong H^0(X_1, (\Omega^1_{X_1})^{\otimes 2})^G$ since $X_1 \to X$ is étale in codimension 1. Moreover $H^0(C_1, (\Omega^1_{C_1})^{\otimes 2})^G \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \cong \mathbb{C}$ (see Lemma 6.5.3). Hence we have $H^0(X, (\Omega^1_X)^{[\otimes 2]}) \cong \mathbb{C}$.

Remark 7.2.2. If X is a normal rationally connected threefold and if X^* is the result of a MMP for X, then $H^0(X, (\Omega^1_X)^{[\otimes m]})$ may be strictly included in $H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]})$ for some $m > 0$. For example, let $X^* \to \mathbb{P}^1$ be the variety described in the example above. Then X^* carries non-zero pluri-forms. Let $X \to X^*$ be a resolution of singularities of X^*. Then X^* is the result of a MMP for X since X^* has terminal singularities. However, since X is rationally connected and smooth, we have $H^0(X, (\Omega^1_X)^{[\otimes m]}) = \{0\}$ for any $m > 0$ (see [Kol96, Cor. IV.3.8]).

We will give two examples of rationally connected threefolds with canonical singularities which carry non-zero pluri-forms. In both cases, there is always a Mori fibration from the threefold X to a surface Z. In Example 7.2.3, the base Z is a surface with klt singularities such that $K_Z \sim_{\mathbb{Q}} 0$. In Example 7.2.5, the base Z is isomorphic to $\mathbb{P}^2$ which does not carry any non-zero pluri-form.

Example 7.2.3. Let C be the curve in $\mathbb{P}^2$ defined by $C = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2 \mid x_1^3 + x_2^3 + x_3^3 = 0\}$. There exists an action of group $G = \mathbb{Z}/3\mathbb{Z}$ on C defined by $g \cdot [x_1 : x_2 : x_3] = [\xi x_1 : x_2 : x_3]$ where g is a generator of G and ξ is a primitive 3rd root of unity. Let $Z_1 = C \times C$. Then there is an induced action of G on Z_1 which acts diagonally. We have $Z = Z_1/G$ is a klt rationally connected surface such that K_Z is $\mathbb{Q}$-linearly equivalent to the zero divisor (see [Tot12, Example 10]). Let $X_1 = \mathbb{P}^1 \times Z_1$ and define an action of G on $\mathbb{P}^1$ by $g \cdot [y_1 : y_2] \to [\xi y_1 : y_2]$ where $[y_1 : y_2]$ are homogeneous coordinates of $\mathbb{P}^1$. Then there is an induced action of G on the smooth threefold X_1 which acts diagonally. Since $G = \mathbb{Z}/3\mathbb{Z}$ and this action is free in codimension 2, this action satisfies the condition of [Rei80, Thm. 3.1]. Hence $X = X_1/G$ has canonical singularities. We also have a Mori fibration $X \to Z$ whose general fibers are isomorphic to $\mathbb{P}^1$. Since Z is rationally connected, so is X by [GHS03, Thm. 1.1]. Moreover, since K_Z is $\mathbb{Q}$-effective, we have $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$ (see §4.2).

In the example above, the non-zero pluri-forms come from K_Z, the canonical divisor of the base surface. In the following example, $-K_Z$ is ample. However X still carries non-zero pluri-forms. These forms come from the multiple fibers of the fibration $X \to Z$. Before giving the example, we will first introduce a method to construct non-reduced fibers.

Construction 7.2.4. We want to construct a fibration p from a normal threefold T with canonical singularities to a smooth surface S such that $p^*C = 2(p^*C)_{\mathrm{red}}$, where C is a smooth curve in S.

Let $T_0 = \mathbb{P}^1 \times S$ where S is a smooth projective surface. Let p_1, p_2 be the natural projections $T_0 \to \mathbb{P}^1$ and $T_0 \to S$. Let $C \subseteq S$ be a smooth curve and let $C_0 = \{z\} \times C$ be a section of p_2 over C in T_0, where z is a point of $\mathbb{P}^1$. Let $E_0 = p_2^* C$, which is a smooth divisor in T_0. We will perform a sequence of birational transformations of T_0.

First we blow up C_0, we obtain a morphism $T_1 \to T_0$ with exceptional divisor E_1. Denote still by E_0 the strict transform of E_0 in T_1. If F is a fiber of $T_1 \to S$ over a point of C, then $F = F_1 \cup F_0$ where $F_i \subseteq E_i$ are rational curves and $K_{T_1} \cdot F_i = -1$, $E_i \cdot F_i = -1$.

Now we blow up C_1, the intersection of E_0 and E_1. We obtain a morphism $T_2 \to T_1$ with exceptional divisor E_2. Denote still by E_0, E_1 the strict transforms of E_0 and E_1 in T_2. If F is any set-theoretic fiber of $T_2 \to S$ over a point of C, then $F = F_0 \cup F_1 \cup F_2$ where $F_i \subseteq E_i$ are rational curves and $K_{T_2} \cdot F_i = 0$, $E_i \cdot F_i = -2$ for $i = 0, 1$, $K_{T_2} \cdot F_2 = -1$, $E_2 \cdot F_2 = -1$. Let q be the fibration $T_2 \to S$.

We will blow down E_1 and E_0 in T_2. Let H be an ample $\mathbb{Q}$-divisor such that $(H + E_0) \cdot F_0 = 0$. Since $E_0 \cdot F_1 = 0$, there is a positive rational number b such that $(H + E_0 + bE_1) \cdot F_1 = 0$. Moreover $(H + E_0 + bE_1) \cdot F_0 = 0$ since $E_1 \cdot F_0 = 0$. Let A be a sufficiently ample divisor on S. Then the $\mathbb{Q}$-divisor $D = q^*A + H + E_0 + bE_1$ is nef and big. Moreover, any curve B which has intersection number 0 with D must be contracted by $T_2 \to S$ since A is sufficiently ample. The curve B is also contained in $E_0 \cup E_1$ since H is ample. Since $K_{T_2} \cdot F_0 = K_{T_2} \cdot F_1 = 0$, there exists a large integer k such that $kD - K_{T_2}$ is nef and big. Then by the basepoint-free theorem (see [KM98, Thm. 3.3]), there is a large enough integer a such that aD is Cartier and $|aD|$ is basepoint-free. The linear system $|aD|$ induces a contraction which contracts E_0 and E_1.

By contracting E_0 and E_1, we obtain a threefold T. And there is an induced fibration $p : T \to S$ such that $p^*C = 2(p^*C)_{\mathrm{red}}$. Note that $T_2 \to T$ is a resolution of singularities and $K_{T_2} = f^* K_T$. Hence V has canonical singularities.

Now we will construct the example.

Example 7.2.5. Let $C = \{[x : y : z] \in \mathbb{P}^2 \mid x^6 + y^6 + z^6 = 0\}$ be a smooth curve in $\mathbb{P}^2$. Then $2K_{\mathbb{P}^2} + C$ is linearly equivalent to the zero divisor. Let $X_0 = \mathbb{P}^1 \times \mathbb{P}^2$. By the method of Construction 7.2.4, we can construct a fiber space $p : X \to \mathbb{P}^2$ such that $p^*C = 2(p^*C)_{\mathrm{red}}$. Then X has canonical singularities. Since general fibers of p are isomorphic to $\mathbb{P}^1$, X is rationally connected by [GHS03, Thm. 1.1]. Moreover, since $2(K_{\mathbb{P}^2} + \frac{1}{2}D)$ is an effective divisor and p is equidimensional, we have $H^0(X, (\Omega^1_X)^{[\otimes 4]}) \neq \{0\}$ (see §4.2).

Note that in the three examples above, the variety X we constructed is a rationally connected variety with canonical singularities. The divisor K_X is not pseudo-effective and we always have a Mori fibration from X to a variety Z. The non-zero pluri-forms of X come from the base Z. However,

in the following example, the variety we construct is a rationally connected threefold with klt singularities whose canonical divisor is ample. Some non-zero pluri-forms come from its canonical divisor.

Example 7.2.6. Let X_1 be the Fermat hypersurface in $\mathbb{P}^4$ defined $x_0^6+x_1^6+\cdots+x_4^6=0$. Then X_1 is a smooth threefold such that K_{X_1} is ample. Moreover, the Picard number of X_1 is 1 by the Lefschetz theorem (see [Laz04, Exmp. 3.1.23]). Let G be the group $\mathbb{Z}/6\mathbb{Z}$ with generator g. Define an action of G on $\mathbb{P}^4$ by $g\cdot[x_0:\cdots:x_4]=[\omega x_0:\omega x_1:x_2:x_3:x_4]$, where ω is a primitive 6th root of unity. This action induces an action of G on X_1 which is free in codimension 1. Denote the quotient X_1/G by X and the natural morphism $X_1\to X$ by π. Then X has $\mathbb{Q}$-factorial klt singularities but does not have canonical singularities (see [Rei80, Thm 3.1]). Now we will prove that X is rationally connected. First we will prove X is uniruled. If w is a general point in X, then there is a point $w_1\in X_1$ such that $\pi(w_1)=w$ and the first two coordinates of w_1 are all non-zero. There are two hyperplanes in $\mathbb{P}^4$ passing through w_1, $H_1=\{a_2x_2+a_3x_3+a_4x_4=0\}$ and $H_2=\{b_2x_2+b_3x_3+b_4x_4=0\}$, such that the intersection $C_1=H_1\cap H_2\cap X_1$ is a smooth curve of genus 10. Moreover, there are exactly 6 points on C_1, given by $X_1\cap\{x_2=x_3=x_4=0\}$, which are fixed under the action of G. Hence $\pi|_{C_1}:C_1\to\pi(C_1)=C$ is ramified exactly at those 6 points with degree 6. By the ramification formula $g(C)=1+2^{-1}\times6^{-1}\times(2\times10-2-6\times5)=0$, where $g(C)$ is the genus of C. Hence C is a smooth rational curve. This implies that X is uniruled. Since X has Picard number 1, from the lemma below, we conclude that X is rationally connected.

Lemma 7.2.7. *Let X be a normal projective variety which is $\mathbb{Q}$-factorial. Assume that the Picard number of X is 1. Then X is rationally connected if and only if it is uniruled.*

Proof. Assume that X is uniruled and let $\pi:X\dashrightarrow Z$ be the MRC fibration for X (see for example [Kol96, Thm. V.5.2]). We will argue by contradiction. Assume that $\dim X>\dim Z>0$. Then there is a Zariski open subset Z_0 of Z such that π is regular and proper over Z_0. Let C be a curve contained in some fiber of $\pi|_{Z_0}$. Let D be a non-zero effective Cartier divisor on Z_0. Let $E_0=\pi^*D$. Then E_0 is a non-zero effective divisor on $X_0=\pi^{-1}Z_0$. Let E be the closure of E_0 in X, then E is a non-zero effective $\mathbb{Q}$-Cartier Weil divisor on X. However, the intersection number of E and C is zero, which contradicts the fact that X has Picard number 1. □

7.3 Preliminaries

In this section, we will prove some preparatory lemmas.

Lemma 7.3.1. *Let $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ be a complex of coherent sheaves on a normal variety X. Assume that there is an open subset W of X with* $\operatorname{codim} X\backslash W \geqslant 2$ *such that the induced sequence*

$$0 \to \mathscr{A}|_W \to \mathscr{B}|_W \to \mathscr{C}|_W \to 0$$

is an exact sequence of locally free sheaves on W. If $h^0(X, \mathscr{A}^{\otimes r}[\otimes]\mathscr{C}^{\otimes t}) = 0$ for all $t > 0$ and $r \geqslant 0$, then $H^0(X, \mathscr{B}^{[\otimes m]}) \cong H^0(X, \mathscr{A}^{[\otimes m]})$ for all $m > 0$.

Proof. On W, we have $H^0(W, \mathscr{A}|_W^{\otimes r} \otimes \mathscr{C}|_W^{\otimes t}) = \{0\}$ for all $t > 0$ and $r \geqslant 0$ by Proposition 1.2.1. Fix $m > 0$, by Lemma 6.3.4, we have a filtration

$$\mathscr{A}|_W^{\otimes m} = \mathscr{R}_0 \subseteq \cdots \subseteq \mathscr{R}_s = \mathscr{B}|_W^{\otimes m}$$

such that, for all $0 \leq i \leq s-1$, the quotient $\mathscr{R}_{i+1}/\mathscr{R}_i$ is isomorphic to $\mathscr{A}|_W^{\otimes m-a_i} \otimes \mathscr{C}|_W^{\otimes a_i}$ for some $a_i > 0$. Since $h^0(X, \mathscr{A}|_W^{\otimes r} \otimes \mathscr{C}|_W^{\otimes t}) = 0$ for all $t > 0$ and $r \geqslant 0$, we have $H^0(W, \mathscr{R}_i) \cong H^0(W, \mathscr{R}_{i+1})$ for $0 \leqslant i \leqslant s-1$. Thus

$$H^0(W, \mathscr{B}|_W^{\otimes m}) \cong H^0(W, \mathscr{A}|_W^{\otimes m}).$$

By Proposition 1.2.1, we have $H^0(X, \mathscr{B}^{[\otimes m]}) \cong H^0(X, \mathscr{A}^{[\otimes m]})$. □

One of the applications of the lemma above is the following, which gives a relation between pluri-forms and fibrations.

Lemma 7.3.2. *Let $p : X \to Z$ be a morphism between normal varieties. Assume that general fibers of p do not carry any non-zero pluri-form. Then for all integer $m > 0$, we have*

$$H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(X, ((p^*\Omega_Z^1)^{sat})^{[\otimes m]}),$$

where $(p^*\Omega_Z^1)^{sat}$ is the saturation of the image of $p^*\Omega_Z^1$ in $\Omega_X^{[1]}$.

Proof. We have an exact sequence of coherent sheaves $0 \to \mathscr{F} \to \Omega_X^{[1]} \to \mathscr{G} \to 0$ on X, where $\mathscr{F} = (p^*\Omega_Z^1)^{sat}$ and $\mathscr{G}$ is a torsion-free sheaf. In particular, $\mathscr{G}$ is locally free in codimension 1. Let V be the smooth locus of Z and let $W = p^{-1}(V)$. If U is the largest open subset of W on which $\mathscr{F}$, Ω_X^1 and $\mathscr{G}$ are locally free, then $\operatorname{codim} X\backslash U \geqslant 2$ and we have an exact sequence $0 \to \mathscr{F}|_U \to \Omega_U^1 \to \mathscr{G}|_U \to 0$ of locally free sheaves on U.

If F is a general fiber of $p|_U$, then $\mathscr{F}|_F$ is the direct sum of $\mathscr{O}_F$ and $\mathscr{G}|_F$ is isomorphic to Ω_F^1. Since general fibers of p do not carry any non-zero pluri-form, neither does F by Proposition 1.2.1. Hence we have $H^0(U, \mathscr{F}|_U^{\otimes r} \otimes \mathscr{G}|_U^{\otimes t}) = \{0\}$ for all $t > 0$ and $r \geqslant 0$. By Lemma 7.3.1, we have an isomorphism from $H^0(U, \mathscr{F}|_U^{\otimes m})$ to $H^0(U, (\Omega_U^1)^{\otimes m})$ for all $m > 0$. By Proposition 1.2.1, we have $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(X, \mathscr{F}^{[\otimes m]})$ for all $m > 0$. □

Proposition 7.3.3. *Let X be a projective threefold which has canonical singularities. Let X^* be the result of a MMP for X and denote the birational map $X \dashrightarrow X^*$ by f. Let Γ be the normalisation of the graph of f. Then there is a natural isomorphism $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]})$ for all $m > 0$.*

Proof. Since there is a natural birational projection $p_1 : \Gamma \to X$, by Lemma 6.3.1, there is an injection from $H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]})$ to $H^0(X, (\Omega^1_X)^{[\otimes m]})$. Let $p_2 : \Gamma \to X^*$ be the natural projection and let σ_X be a non-zero element in $H^0(X, (\Omega^1_X)^{[\otimes m]})$. Since X, Γ and X^* are birational, σ_X induces a rational section σ_Γ of $(\Omega^1_\Gamma)^{[\otimes m]}$ and an element σ_{X^*} of $H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]})$ (see Lemma 6.3.1). In order to prove that σ_Γ is a regular section it is enough to prove that σ_Γ does not have pole along any p_1-exceptional divisor. Let E be an exceptional divisor for p_1. Let $C \subseteq E$ be a curve which is exceptional for p_1. Then C is not contracted by p_2 since the graph of f is included in $X \times X^*$ and the normalisation map is finite. Hence $Y = p_2(E) \subseteq X^*$ is a curve. Note that f^{-1} is not a morphism around the generic point of Y. By Lemma 3.1.1, X^* has terminal singularities around the generic point of Y. Hence X^* is smooth around the generic point of Y since the codimension of Y in X^* is 2 (see [KM98, Cor. 5.18]). Moreover, since T, X and X^* are birational, the form σ_Γ is just the pullback of σ_{X^*} by p_2. Hence σ_Γ does not have pole along E. □

7.4 Fibrations over curves

In this section, we will give a method to construct rationally connected varieties which carry non-zero pluri-forms (Theorem 7.4.2). Together with Theorem 7.1.1, we will see that every rationally connected threefold with $\mathbb{Q}$-factorial terminal singularities which carries non-zero pluri-forms can be constructed by this method.

By Lemma 7.3.2, if general fibers of an equidimensional fibration $p : X \to Z$ do not carry any non-zero pluri-form, then for all $m > 0$, we have $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X, ((p^*\Omega^1_Z)^{sat})^{[\otimes m]})$. Moreover,

$$H^0(X, ((p^*\Omega^1_Z)^{sat})^{[\otimes m]}) \cong H^0(Z, p_*((p^*\Omega^1_Z)^{sat})^{[\otimes m]})).$$

Hence we would like to know what $p_*((p^*\Omega^1_Z)^{sat})^{[\otimes m]})$ is. In the case when Z is a smooth curve, this is not difficult to compute. Note that if R is the ramification divisor of p, then $((p^*\Omega^1_Z)^{sat})^{[\otimes m]} \cong (p^*\Omega^1_Z)^{\otimes m} \otimes \mathscr{O}_X(mR)$ for $m > 0$. By the projection formula, we have

$$p_*((p^*\Omega^1_Z)^{sat})^{[\otimes m]}) \cong (\Omega^1_Z)^{\otimes m} \otimes p_*\mathscr{O}_X(mR).$$

Hence it is enough to compute $p_*\mathscr{O}_X(mR)$.

Lemma 7.4.1. *Let $p : X \to Z$ be a fibration from a normal variety to a smooth curve. Let z be a point in Z. Let $D = p^*z - (p^*z)_{\mathrm{red}}$. Then $p_*\mathscr{O}_X(mD) \cong \mathscr{O}_Z([\frac{m(m(p,z)-1)}{m(p,z)}]z)$ for any $m \geqslant 0$.*

Proof. The problem is local and we may assume that Z is affine. We know that $p_*\mathscr{O}_W(mD)$ is an invertible sheaf. If θ is a section of $p_*\mathscr{O}_W(mD)$, then it is a rational function on Z whose pullback on W is a rational function which can only have pole along p^*z. Hence θ can only have pole at z. Let d be the degree of the pole, then the pullback of θ in V is a section of $\mathscr{O}_W(mD)$ if and only if $m(p,z)d \leqslant m(m(p,z)-1)$, that is, $d \leqslant [\frac{m(m(p,z)-1)}{m(p,z)}]$. □

This relation between ramification divisor and pluri-forms gives us an idea of how to construct rationally connected varieties which carry non-zero pluri-forms. Note that if $p : X \to \mathbb{P}^1$ is a fibration such that general fibers of p are rationally connected, then X is rationally connected by [GHS03, Thm. 1.1]. Moreover, if general fibers of p do not carry any non-zero pluri-form, then from the discussion above X carries non-zero pluri-forms if and only if $(\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes p_*\mathscr{O}_X(mR)$ has non-zero sections for some $m > 0$, where R is the ramification divisor of p. However, $\Omega^1_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(-2)$ and $p_*\mathscr{O}_X(mR) \cong \mathscr{O}_{\mathbb{P}^1}(\sum_{z\in\mathbb{P}^1}[\frac{(m(p,z)-1)m}{m(p,z)}])$ since any two points in $\mathbb{P}^1$ are linearly equivalent. Hence X carries non-zero pluri-forms if and only if $\sum_{z\in\mathbb{P}^1}\frac{m(p,z)-1}{m(p,z)} \geqslant 2$.

Now we will try to construct this kind of varieties by taking quotients. Let T be a normal projective variety, let B be a smooth projective curve and let G be a finite commutative group. Assume that there are actions of G on T and B. Assume that there is a G-equivariant fibration $q : T \to B$. Let $p_T : T \to T/G$, $p_B : B \to B/G$ be natural projections and let $p : T/G \to B/G$ be the induced fibration. Let S_i, $i = 1, ..., r$ be all the G-orbits in B whose cardinality is less than the cardinality of G. Let z_i be the image of S_i under the map $B \to B/G$. Then the z_i's are the points in B/G over which $B \to B/G$ is ramified. Let G_i be the stabilizer of a point b_i in S_i. Then G_i acts on the set-theoretic fiber $q^{-1}\{b_i\} = F_i$. If A_i is a component in F_i and the stabilizer of a general point in A_i has cardinality d_i, then p_T is ramified along $G_i \cdot A_i$ of degree d_i, where $G_i \cdot A_i$ is the orbit of A_i under G_i. In this case, $p_T(A_i)$ has coefficient $\frac{e_if_i}{d_i}$ in p^*z_i, where e_i is the coefficient of A_i in q^*b_i and f_i is the cardinality of G_i. Denote $\min\{\frac{e_if_i}{d_i} \mid A_i \text{ a component in } F_i\}$ by s_i. Then s_i is equal to $m(p, z_i)$. We have the following theorem.

Theorem 7.4.2. *Let T be a projective normal variety, B be a projective smooth curve with positive genus and G be a finite commutative group. Assume they satisfy the following conditions :*

1. *There is a fibration q from T to B such that for any general fiber F_q of q, F_q is rationally connected and does not carry any non-zero pluri-form. Moreover, $m(q, b) = 1$ for all $b \in B$.*

2. *There exist actions of G on B and on T such that q is G-equivariant.*
3. *The quotient B/G is isomorphic to $\mathbb{P}^1$.*

Then the quotient $X = T/G$ is a normal projective rationally connected variety and there is a fibration $p : X \to \mathbb{P}^1$. Moreover, if we define s_i, $i = 1, ..., r$ as above, then X carries non-zero pluri-forms if and only if $\Sigma_{i=1}^{r} \frac{s_i-1}{s_i} \geqslant 2$. More precisely,

$$H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + \sum_{i=1}^{r}[\frac{(s_i-1)m}{s_i}]))$$

for $m > 0$.

Proof. The fibration $p : X \to \mathbb{P}^1$ is induced by q. Since general fibers of q are rationally connected, general fibers of p are also rationally connected. Hence X is rationally connected by [GHS03, Thm. 1.1]. If F_p is a general fiber of p, then F_p does not carry any non-zero pluri-form neither. Hence, $H^0(X, (\Omega^1_X)^{[\otimes m]})$ is isomorphic to $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m) \otimes p_*\mathscr{O}_X(mR))$ for $m > 0$, where R is the ramification divisor of p. However, since $m(q, b) = 1$ for all $b \in B$, we have $\sum_{i=1}^{r}[\frac{(s_i-1)m}{s_i}] = \sum_{z\in\mathbb{P}^1}[\frac{(m(p,z)-1)m}{m(p,z)}]$ by the definition of s_i. Finally, we have $p_*\mathscr{O}_X(mR) \cong \mathscr{O}_{\mathbb{P}^1}(\sum_{i=1}^{r}[\frac{(s_i-1)m}{s_i}])$ and $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + \sum_{i=1}^{r}[\frac{(s_i-1)m}{s_i}]))$ by Lemma 7.4.1. □

Remark 7.4.3. Conversely, let $p : X \to \mathbb{P}^1$ be a fibration such that general fibers of p are rationally connected and do not carry non-zero pluri-form. If X carries non-zero pluri-forms, then X can be constructed by the method described above. In fact, by the discussion above, we have $\sum_{z\in\mathbb{P}^1} \frac{(m(p,z)-1)}{m(p,z)} \geqslant 2$. In particular, there are at least three points in $\mathbb{P}^1$ such that the multiplicity of p is larger than 1 over these points. Let $z_1, ..., z_r$ be all the points in $\mathbb{P}^1$ such that $m(p, z_i) > 1$ for all i. Since $r \geqslant 3$, there is a smooth curve B and a Galois cover $p_B : B \to \mathbb{P}^1$ with Galois group G such that p_B is ramified exactly over the z_i's and the degree of ramification is m_i at each point over z_i (see [KO82, Lem. 6.1]). Let T be the normalisation of the fiber product $X \times_{\mathbb{P}^1} B$. Then we obtain a natural fibration $q : T \to B$ such that $m(q, b) = 1$ for all $b \in B$. Moreover, G acts naturally on T and $T/G \cong X$.

Proposition 7.4.4. *Let X be a rationally connected threefold which has $\mathbb{Q}$-factorial canonical singularities. Assume that X carries non-zero pluri-forms. Let X^* be the result of a MMP for X and assume that there is a Mori fibration $p : X^* \to \mathbb{P}^1$. Let Y be the normalisation of the graph of $X \dashrightarrow X^*$. Then Y can be constructed by the method of Theorem 7.4.2.*

Proof. By Theorem 6.1.7 and Lemma 7.3.2, we know that every element in the space $H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]})$ comes from the base $\mathbb{P}^1$, so does every element in $H^0(X, (\Omega^1_X)^{[\otimes m]})$ by Lemma 6.3.1. We have a rational map $X \dashrightarrow$

$\mathbb{P}^1$ induced by p. By Proposition 7.3.3, we know that $H^0(X,(\Omega^1_X)^{[\otimes m]}) \cong H^0(Y,(\Omega^1_Y)^{[\otimes m]})$. Moreover, we have a natural fibration $g : Y \to \mathbb{P}^1$ which is the composition of $Y \to X^* \to \mathbb{P}^1$. If F_g is a general fiber of g, then there is a birational morphism $F_g \to S$, where S is a general fiber of $p : X^* \to \mathbb{P}^1$. Since S does not carry any non-zero pluri-form by Theorem 6.1.7, neither does F_g by Lemma 6.3.1. From Remark 7.4.3, we know that Y can be constructed by the method of Theorem 7.4.2. □

7.5 Mori fibrations and non-zero pluri-form

In this section, we study relations between Mori fibrations and non-zero pluri-forms. We will consider Mori fibrations from a normal threefold to a normal surface.

7.5.1 General properties

Consider a Mori fibration $p : X \to Z$ from a normal rationally connected threefold to a normal variety Z of positive dimension. Assume that X has $\mathbb{Q}$-factorial klt singularities. Let $D_1, ..., D_k$ be all prime Weil divisors in Z such that $m_i = m(p, D_i) > 1$. Let $\Delta = \sum_{i=1}^k \frac{m_i-1}{m_i} D_i$. Then $\det((p^*\Omega^1_Z)^{sat}) \cong \mathscr{O}_X(p^*(K_Z + \Delta))$ (see Remark 7.5.5). Assume that X carries non-zero pluri-forms and general fibers of p do not carry any non-zero pluri-form. If $\dim Z = 1$, then $K_Z + \Delta$ is an effective $\mathbb{Q}$-divisor of degree $-2 + \sum_{i=1}^k \frac{m_i-1}{m_i}$ on $\mathbb{P}^1$ (see §3). The aim of this subsection is to prove something analogue in the case when $\dim Z = 2$. We prove that if $K_Z + \Delta$ is not pseudo-effective, then there will be a fibration form Z to $\mathbb{P}^1$. In this case, we have an induced fibration $X \to \mathbb{P}^1$ and we are in the same situation as in §3 (see Lemma 7.5.10). In order to do this, we will run a MMP for the pair (Z, Δ). To this end, we prove the following proposition which implies that the pair (Z, Δ) is klt.

Proposition 7.5.1. *Let $p : X \to Z$ be a Mori fibration from a $\mathbb{Q}$-factorial klt quasi-projective variety X to a normal variety Z. Let $D_1, ..., D_k$ be pairwise distinct prime Weil divisors in Z such that $p^*D_i = m_i(p^*D_i)_{\mathrm{red}}$ with $m_i \geqslant 2$. Then the pair $(Z, \sum_{i=1}^k \frac{m_i-1}{m_i} D_i)$ is klt.*

Proof. Let $D = D_1 + \cdots + D_k$. By [KMM87, Lem. 5-1-5], Z is $\mathbb{Q}$-factorial. Note that the problem is local in Z, we may assume that Z is affine.

We will construct by induction a finite morphism $c_k : Z_k \to Z$ which is ramified over D such that $c_k^* D_i = m_i(c_k^* D_i)_{\mathrm{red}}$ for all i. Let k_1 be the smallest positive integer such that $k_1 D_1$ is a Cartier divisor. By taking the k_1-th root of the function defining the Cartier divisor $k_1 D_1$, we can construct a finite morphism $c_{1,1} : Z_{1,1} \to Z$ which is étale in codimension 1. Moreover, $c_{1,1}^* D_1$ is a reduced Cartier divisor (see [Mor88, Prop.-Def. 1.11]). By taking the m_1-th root of the function defining the Cartier divisor $c_{1,1}^* D_1$, we can

find a finite morphism $c_{1,2} : Z_1 \to Z_{1,1}$ which is ramified exactly over D_1 with ramification degree m_1 (see [Laz04, §4.1B]). Let $c_1 = c_{1,2} \circ c_{1,1} : Z_1 \to Z$. Then $c_1^* D_1 = m_1 (c_1^* D_1)_{\mathrm{red}}$. Assume that we have constructed a finite morphism $c_j : Z_j \to Z$ such that $c_j^* D_i = m_i (c_j^* D_i)_{\mathrm{red}}$ for any $1 \leqslant i \leqslant j$ and $c_j^* D_i$ is reduced for any $i > j$, where $1 \leqslant j < k$. Let $E = c_j^* D_{j+1}$. Then E is a reduced divisor. By the same method above, we can construct a finite morphism $d : Z_{j+1} \to Z_j$ which is ramified exactly over E with ramification degree m_{j+1}. Let $c_{j+1} : Z_{j+1} \to Z$ be the composition $c_j \circ d$. Then $c_{j+1}^* D_i = m_i (c_{j+1}^* D_i)_{\mathrm{red}}$ for any $1 \leqslant i \leqslant j+1$ and $c_{j+1}^* D_i$ is reduced for any $i > j+1$. By induction, we can construct the finite morphism c_k.

We have $K_{Z_k} = c_k^* (K_Z + \sum_{i=1}^k \frac{m_i - 1}{m_i} D_i)$. Let X_k be the normalisation of $X \times_Z Z_k$. Then the natural projection $c_X : X_k \to X$ is étale in codimension 1 and $K_{X_k} = c_X^* K_X$. Hence X_k is klt by [KM98, Prop. 5.20]. Moreover, $X_k \to Z_k$ is a Fano fibration since $X \to Z$ is a Mori fibration. Hence there is a $\mathbb{Q}$-divisor Δ_k such that the pair (Z_k, Δ_k) is klt by [Fuj99, Cor. 4.7]. Since K_{Z_k} is $\mathbb{Q}$-Cartier, Z_k is klt (see [KM98, Cor. 2.35]). So the pair $(Z, \sum_{i=1}^k \frac{m_i - 1}{m_i} D_i)$ is also klt by [KM98, Prop. 5.20]. □

In the remaining of this subsection, our aim is to prove the following theorem.

Theorem 7.5.2. *Let $p : X \to Z$ be a Mori fibration from a projective normal threefold to a normal projective surface. Assume that X has $\mathbb{Q}$-factorial klt singularities. Let $D_1, ..., D_k$ be all prime Weil divisors in Z such that $m_i = m(p, D_i) > 1$. Let $\Delta = \sum_{i=1}^k \frac{m_i - 1}{m_i} D_i$. If X carries non-zero pluri-forms and $K_Z + \Delta$ is not pseudo-effective, then the result Z' of any MMP for the pair (Z, Δ) has Picard number 2.*

First we would like to illustrate the idea of the proof in a simple case. Let $f : (Z, \Delta) \to (Z', \Delta')$ be the result of a MMP for the pair (Z, Δ). Assume in a first stage that $Z = Z'$. We will argue by contradiction. Assume that Z' has Picard number 1. Since general fibers of p are isomorphic to $\mathbb{P}^1$, they don't carry any non-zero pluri-form. By Lemma 7.3.2, the non-zero pluri-forms of X come from $(p^* \Omega_Z^1)^{sat}$. Since $K_Z + \Delta$ is not pseudo-effective, we can prove that there is a rank 1 coherent subsheaf $\mathscr{H}$ of $(p^* \Omega_Z^1)^{sat}$ such that $\mathscr{H}^{[\otimes l]}$ is an invertible sheaf which has non-zero global sections for some positive integer l. Next, we can prove that $\mathscr{H}^{[\otimes l]}$ is isomorphic to $\mathscr{O}_X$ under the assumption that Z has Picard number 1 (by using Lemma 7.5.8). Let $W \to X$ be the normalisation of the cyclic cover with respect to the isomorphism $\mathscr{H}^{[\otimes l]} \cong \mathscr{O}_X$ (see [KM98, Def. 2.52]). Let $W \to V \to Z$ be the Stein factorisation, then V will be a klt Fano variety (by Lemma 7.5.7). Moreover $H^0(V, \Omega_V^{[1]}) \neq \{0\}$ (we use Lemma 7.5.9). This contradicts [GKKP11, Thm. 5.1].

However, in general Z' is different from Z and the fibration $X \to Z'$ is not equidimensional. The idea of the proof will be the same as above but details will be more complicated. We will work over an open subset Z_0 of Z

such that $f|_{Z_0}$ is an isomorphism and $\operatorname{codim} Z' \backslash f(Z_0) \geqslant 2$. For the complete proof of the theorem, we will need several lemmas.

Lemma 7.5.3. *Let $p : X \to Z$ be an equidimensional morphism between smooth varieties. Let n be the dimension of X and d be the dimension of Z. Let D be a prime divisor in Z and E be a prime divisor in X such that E is a component of p^*D. Assume that the coefficient of E in p^*D is k. Then for any general point $x \in E$, there is an open neighbourhood $U \subseteq X$ of x such that $(p^*\Omega_Z^d)^{sat}|_U \cong \mathscr{O}_X(p^*K_Z + (k-1)E)|_U$, where $(p^*\Omega_Z^d)^{sat}$ is the saturation of $p^*\Omega_Z^d$ in Ω_X^d.*

Proof. We may assume that E is smooth around x and D is smooth around $p(x)$. There exist local coordinates $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_d)$ of X and Z around x and $p(x)$ such that E is defined by $\{a_1 = 0\}$, D is defined by $\{b_1 = 0\}$ and p is given by $(a_1, a_2, ..., a_n) \mapsto (a_1^k, a_2, ..., a_d)$. With these coordinates, the natural morphism $p^*\Omega_Z^1 \to \Omega_X^1$ is given by

$$(\mathrm{d}b_1, \mathrm{d}b_2, ..., \mathrm{d}b_d) \mapsto (ka_1^{k-1}\mathrm{d}a_1, \mathrm{d}a_2, ..., \mathrm{d}a_d)$$

and the image of the natural morphism $p^*\Omega_Z^d \to \Omega_X^d$ is generated by $p^*(\mathrm{d}b_1 \wedge \mathrm{d}b_2 \wedge \cdots \wedge \mathrm{d}b_d) = (ka_1^{k-1}\mathrm{d}a_1) \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_d$. Hence $(p^*\Omega_Z^d)^{sat}$ is generated by $\mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_d$. Since $\{a_1 = 0\}$ defines the divisor E, we have $(p^*\Omega_Z^d)^{sat}|_U \cong \mathscr{O}_X(p^*K_Z + (k-1)E)|_U$ for some open neighbourhood U of x. □

Lemma 7.5.4. *Let $p : X \to Z$ be an equidimensional morphism between normal varieties. Assume that Z is $\mathbb{Q}$-factorial. Let $D_1, ..., D_r$ be all the prime divisors in Z such that $m(p, D_i)$ is larger than 1. Write $\det((p^*(\Omega_Z^1))^{sat}) \cong \mathscr{O}_X(M)$ where M is a divisor on X. Then $M - (p^*(K_Z + \sum_{i=1}^r \frac{m(p,D_i)-1}{m(p,D_i)} D_i))$ is $\mathbb{Q}$-effective.*

Proof. By Proposition 1.2.1, we only have to prove the assertion on an open subset of X whose complement is of codimension at least 2. Hence we may assume that both X and Z are smooth and that $\sum_{i=1}^r D_i$ is smooth. In this case, by Lemma 7.5.3, the divisor M is linearly equivalent to $p^*K_Z + \sum_{i=1}^r \sum_{j=1}^{s_i}(n_{i,j} - 1)E_{i,j}$ where the $E_{i,1}, ..., E_{i,s_i}$ are the components of p^*D_i and $n_{i,j}$ is the coefficient of $E_{i,j}$ in p^*D_i. Since $m(p, D_i)$ is the smallest integer among $n_{i,1}, ..., n_{i,s_i}$, we have $n_{i,j} - 1 \geqslant \frac{m(p,D_i)-1}{m(p,D_i)} \cdot n_{i,j}$. Thus

$$\sum_{i=1}^r \sum_{j=1}^{s_i}(n_{i,j} - 1)E_{i,j} \geqslant \sum_{i=1}^r \frac{m(p, D_i) - 1}{m(p, D_i)} p^*D_i.$$

□

Remark 7.5.5. With the notation above, if in addition p is a Mori fibration, then p^*D_i is irreducible for all i since p has relative Picard number 1. In this case we obtain that

$$\det((p^*(\Omega^1_Z))^{sat}) \cong \mathscr{O}_X(p^*(K_Z + \sum_{i=1}^{r} \frac{m(p,D_i)-1}{m(p,D_i)} D_i)).$$

Lemma 7.5.6. *Let $p : X \to Z$ be an equidimensional fibration between normal varieties. Let $(p^*\Omega^1_Z)^{sat}$ be the saturation of the image of $p^*\Omega^1_Z$ in $\Omega^{[1]}_X$. Then $p_*(((p^*\Omega^1_Z)^{sat})^{[\wedge r]}) \cong \Omega^{[r]}_Z$ for $r > 0$.*

Proof. Let D be the sum of the divisors in Z over which p does not have reduced fibers. Applying Proposition 1.2.1 several times, we can suppose without loss of generality that Z is smooth and D is a snc divisor. Let $\mathscr{M} = p_*(((p^*\Omega^1_Z)^{sat})^{[\wedge r]})$

First we will prove that there is a natural injection from Ω^r_Z to $\mathscr{M}$. We have an injection from $p^*\Omega^1_Z$ to $(p^*\Omega^1_Z)^{sat}$. Hence the natural morphism from $(p^*\Omega^1_Z)^{\wedge r}$ to $((p^*\Omega^1_Z)^{sat})^{[\wedge r]}$ is generically injective. Since $(p^*\Omega^1_Z)^{\wedge r}$ is without torsion, the natural morphism from $(p^*\Omega^1_Z)^{\wedge r}$ to $((p^*\Omega^1_Z)^{sat})^{[\wedge r]}$ is injective. Since $p(X) = Z$, we have an injection from $p_*((p^*\Omega^1_Z)^{\wedge r})$ to $\mathscr{M}$. By the projection formula, this implies that Ω^r_Z is a subsheaf of $\mathscr{M}$.

Now we will prove that $\Omega^r_Z \cong \mathscr{M}$. Let $W = p^{-1}(Z \backslash D)$. Then the morphism $p|_W : W \to Z \backslash D$ is smooth in codimension 1. Thus,

$$((p^*\Omega^1_Z)^{sat})^{[\wedge r]}|_W \cong p^*\Omega^r_Z|_W$$

(see the proof of Lemma 7.5.3). Then we obtain $\mathscr{M}|_{Z\backslash D} \cong \Omega^r_{Z\backslash D}$ by the projection formula. Let U be any open set in Z and let β be any element of $\mathscr{M}(U)$, that is, a section of $\mathscr{M}|_U$. Then β is a rational section of $\Omega^r_Z|_U$ which can only have pole along D since $\mathscr{M}|_{Z\backslash D} \cong \Omega^r_{Z\backslash D}$. However, by the definition of $\mathscr{M}$, β induces a regular section of Ω^r_X on the smooth locus of $p^{-1}(U)$. This implies that β does not have pole along D. Thus $\mathscr{M}(U) = \Omega^r_Z(U)$. Hence $\Omega^r_Z \cong \mathscr{M}$. □

Lemma 7.5.7. *Let $p : X \to Z$ be an equidimensional fibration from a normal variety X to a smooth variety Z. Let $D_1, ..., D_r$ be all the prime divisors in Z such that the multiplicity $m(p, D_i)$ is larger than 1. If $c_X : X_1 \to X$ is a finite morphism which is étale in codimension 1 and $X_1 \xrightarrow{p_1} Z_1 \xrightarrow{c_Z} Z$ is the Stein factorisation, then $K_{Z_1} \leqslant c_Z^*(K_Z + \sum_{i=1}^r \frac{m(p,D_i)-1}{m(p,D_i)} D_i)$. That is, there is an effective $\mathbb{Q}$-divisor Δ_1 on Z_1 such that*

$$K_{Z_1} + \Delta_1 = c_Z^*(K_Z + \sum_{i=1}^{r} \frac{m(p,D_i)-1}{m(p,D_i)} D_i).$$

Proof. If D is a prime divisor in Z, then $p_1^*c_Z^*D = c_X^*p^*D$. Let E be any irreducible component of c_Z^*D. Since c_X is étale in codimension 1, by comparing the coefficients of E in $p_1^*c_Z^*D = c_X^*p^*D$, the coefficient m_E of E in c_Z^*D is not larger than $m(p,D)$. This implies that $(m_E-1) \leqslant m_E \cdot \frac{m(p,D)-1}{m(p,D)}$. Now from Lemma 7.5.3, we obtain $K_{Z_1} = c_Z^*K_Z + \Sigma_{j=1}^s(m_j-1)E_j$, where the E_j's are all prime divisors along which c_Z^* is ramified and m_j is the degree of ramification of c_Z along E_j. By the discussion above, we have $(m_j-1) \leqslant m_j \cdot \frac{m(p,p(E_j))-1}{m(p,p(E_j))}$. Hence $K_{Z_1} \leqslant c_Z^*(K_Z + \sum_{i=1}^r \frac{m(p,D_i)-1}{m(p,D_i)}D_i)$. □

Lemma 7.5.8. *Let $p : X \to Z$ be a Mori fibration from a normal threefold X to a smooth surface Z. Let $D_1, ..., D_r$ be all the prime divisors in Z such that the multiplicity $m_i = m(p, D_i)$ is larger than 1. Assume that there is a projective $\mathbb{Q}$-factorial variety V such that we have an open embedding $j : Z \to V$ with $\operatorname{codim} V\backslash Z \geqslant 2$. Assume further that the pair $(V, \sum_{i=1}^r \frac{m_i-1}{m_i}\bar{D}_i)$ is klt, where $\bar{D}_i$ is the closure of $\bar{D}_i$ in V. If $\mathscr{F}$ is a rank 1 reflexive subsheaf of $\Omega_X^{[1]}$ such that $\mathscr{F}^{[\otimes m_0]} \cong \mathscr{O}_X(p^*D_0)$ for some positive integer m_0 and some divisor D_0 on Z, then $\bar{D}_0$, the closure of D_0 in V, is not ample.*

Proof. Assume the opposite. First we assume further that $m_0 = 1$ and $\mathscr{O}_Z(D_0)$ has non-zero global sections. Let $E = p^*D_0$ which is effective. Then $\mathscr{F} = \mathscr{O}_X(E)$. Since general fibers of p are isomorphic to $\mathbb{P}^1$ which does not carry any non-zero pluri-form, by Lemma 7.3.2, the injection $\mathscr{O}_X(E) \hookrightarrow \Omega_X^{[1]}$ factorises through $(p^*\Omega_Z)^{sat}$. Hence by the projection formula, we have an injection from $\mathscr{O}_Z(D_0)$ to $p_*((p^*\Omega_Z^1)^{sat})$. However, by Lemma 7.5.6, we have $p_*((p^*\Omega_Z^1)^{sat}) = \Omega_Z^1$. By Proposition 1.2.1, we get an injection from $\mathscr{O}_V(\bar{D}_0)$ to $\Omega_V^{[1]}$. Hence by Bogomolov-Sommese theorem (see [Gra15, Cor. 1.3]), the Kodaira dimension of $\mathscr{O}_V(\bar{D}_0)$ is not larger than 1. Hence $\bar{D}_0$ is not ample. We obtain a contradiction.

Now we will treat the general case. We will show that we can reduce to the previous case. By replacing m_0 with a large multiple, we may assume that $\bar{D}_0$ is very ample. We may also assume that both $\bar{D}_0$ and p^*D_0 are prime and the pair $(V, \sum_{i=0}^r \frac{m_i-1}{m_i}\bar{D}_i)$ is klt. Let $c_X : X_1 \to X$ be the normalisation of the ramified cyclic cover with respect to $\mathscr{F}$, m_0 and p^*D_0 (see [KM98, Def. 2.52]). Then c_X is ramified over p^*D_0 with degree m and $(c_X^*\mathscr{F})^{**} \cong \mathscr{O}_{X_1}(E)$, where $E = (c_X^*p^*D_0)_{\text{red}}$. Moreover, over the smooth locus of X, there is an injection of sheaves from $c_X^*\Omega_X^1$ to $\Omega_{X_1}^1$. Hence by Proposition 1.2.1, we have an injection $\mathscr{O}_{X_1}(E) \hookrightarrow \Omega_{X_1}^{[1]}$.

Let $X_1 \xrightarrow{p_1} Z_1 \xrightarrow{c_Z} Z$ be the Stein factorisation. Let $c_V : V_1 \to V$ be the normalisation of V in the function field of X_1. Then we obtain an open embedding $j_1 : Z_1 \to V_1$ such that $\operatorname{codim} V_1\backslash Z_1 \geqslant 2$. If F_p is a general fiber of p and if F_{p_1} is a general fiber of p_1 which is mapped to F_p, then $F_{p_1} \to F_p$ is étale. Since p is a Mori fibration, F_p is a smooth rational curve which is simply connected. Hence $F_{p_1} \to F_p$ is an isomorphism and $F_{p_1} \cong \mathbb{P}^1$. Hence

c_Z is of degree m_0. Let $H = (c_Z^* D_0)_{\mathrm{red}}$. Since $c_Z \circ p_1 : X_1 \to Z$ has connected fibers over D_0, we have $c_Z^* D_0 = m_0 H$. Hence $E_1 = p_1^* H$. Let $\bar{H}$ be the closure of H in V_1. Then it is ample since c_V is finite.

Note that $c_X|_{X_1 \setminus E}$ is étale in codimension 1 and $c_Z^* D_0 = m_0 H$. By Lemma 7.5.7, we conclude that there is an effective $\mathbb{Q}$-divisor Δ_1' in Z_1 such that $K_{Z_1} + \Delta_1' = p^*(K_Z + \sum_{i=0}^r \frac{m_i - 1}{m_i} D_i)$. Hence if Δ_1 is the closure of Δ_1' in V_1, then $K_{V_1} + \Delta_1 = p^*(K_V + \sum_{i=0}^r \frac{m_i - 1}{m_i} \bar{D}_i)$. This implies that (V_1, Δ_1) is klt by [KM98, Prop. 5.20 and Cor. 2.35].

Hence $p_1 : X_1 \to Z_1$ satisfies the conditions in the lemma. There is an injection from $\mathscr{O}_{X_1}(E)$ to $\Omega_{X_1}^{[1]}$ and $\mathscr{O}_{X_1}(E) \cong \mathscr{O}_{X_1}(p_1^* H)$ such that $\bar{H}$ is ample in V_1. We are in the same situation as in the first case. This leads to a contradiction. □

Lemma 7.5.9. *Let $p : X \to Z$ be an equidimensional fibration such that general fibers of p do not carry any non-zero pluri-form. Assume that Z is smooth and there is an open embedding $j : Z \to V$ such that* $\operatorname{codim} V \setminus Z \geqslant 2$. *Assume that $H^0(V, \Omega_V^{[r]}) = \{0\}$ for all $r > 0$. Then $H^0(X, \Omega_X^{[r]}) = \{0\}$ for all $r > 0$.*

Proof. Assume the opposite. Let $\mathscr{F} = (p^* \Omega_Z^1)^{sat}$ be the saturation of the image of $p^* \Omega_Z^1$ in $\Omega_X^{[1]}$. Since general fibers of p do not carry any non-zero pluri-form, we have $H^0(X, \Omega_X^{[r]}) \cong H^0(X, \mathscr{F}^{[\wedge r]})$ by Lemma 7.3.2. Hence there is an injection from $\mathscr{O}_X$ to $\mathscr{F}^{[\wedge r]}$. By taking the direct image, we have an injection from $\mathscr{O}_Z$ to $p_*(\mathscr{F}^{[\wedge r]})$. By Lemma 7.5.6, $p_*(\mathscr{F}^{[\wedge r]}) \cong \Omega_Z^r$. This implies that $H^0(V, \Omega_V^{[r]}) \neq \{0\}$ by Proposition 1.2.1, which is a contradiction. □

Now we are ready to prove Theorem 7.5.2.

Proof of Theorem 7.5.2. We will argue by contradiction. Let $f : Z \to Z'$ be the result of a $(K_Z + \Delta)$-MMP. Assume that Z' has Picard number 1. Set $\Delta' = f_* \Delta$. Then $K_{Z'} + \Delta'$ is not pseudo-effective neither. Thus $-(K_{Z'} + \Delta')$ is ample. We know that there is a smooth open subset $Z_0' \subseteq Z'$ with $\operatorname{codim} Z' \setminus Z_0' \geqslant 2$ such that f^{-1} is an isomorphism from Z_0' onto its image. Let Z_0 be $f^{-1}(Z_0')$, which is an open subset in Z. Let $X_0 = p^{-1}(Z_0)$.

Since $\operatorname{codim} Z' \setminus Z_0' = 2$, there is a projective curve C_0' in Z_0' such that it is an ample divisor in Z'. Let C_0 be the strict transform of C_0' in Z_0. Let α be the class of the curve $p^* C_0 \cap H$, where H is a very ample divisor in X. Then the class α is movable and $p_* \alpha$ proportional to the class of C_0.

Since general fibers of p do not carry any non-zero pluri-form, by Lemma 7.3.2, we obtain that $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(X, ((p^* \Omega_Z^1)^{sat})^{[\otimes m]})$ for any $m > 0$, where $(p^* \Omega_Z^1)^{sat}$ is the saturation of the image of $(p^* \Omega_Z^1)$ in $\Omega_X^{[1]}$. Hence $\mu_\alpha^{max}((p^* \Omega_Z^1)^{sat}) \geqslant 0$ and there is a coherent sheaf $\mathscr{H}$ saturated in $(p^* \Omega_Z^1)^{sat}$ such that $\mu_\alpha(\mathscr{H}) \geqslant 0$.

However, since $p|_{X_0}$ is a Mori fibration and Z_0 is smooth, the $\mathbb{Q}$-Cartier divisor which associates to the determinant of $(p^*\Omega^1_Z)^{sat}|_{X_0}$ is equal to $p^*(K_Z+\Delta)|_{X_0}$ by Remark 7.5.5. Hence

$$\det\left((p^*\Omega^1_Z)^{sat}\right)\cdot\alpha = p^*(K_Z+\Delta)\cdot\alpha = (p\circ f)^*(K_{Z'}+\Delta')\cdot\alpha < 0.$$

This implies that $\mathscr{H}\neq(p^*\Omega^1_Z)^{sat}$. If $\mathscr{J}$ is the quotient $(p^*\Omega^1_Z)^{sat}/\mathscr{H}$, then $\operatorname{rank}\mathscr{J}=\operatorname{rank}\mathscr{H}=1$ and $\mathscr{H}[\otimes]\mathscr{J}=\det\left((p^*\Omega^1_Z)^{sat}\right)$. Thus $\mathscr{J}\cdot\alpha<0$.

Since $H^0(X,((p^*\Omega^1_Z)^{sat})^{[\otimes m]})\neq\{0\}$ for some $m>0$, we have

$$H^0(X,\mathscr{H}^{\otimes s}[\otimes]\mathscr{J}^{\otimes t})\neq\{0\}$$

for some $s,t\geqslant 0$ by Proposition 1.2.1 and Lemma 6.3.4. In this case, we have $s>t$ for $(\mathscr{H}[\otimes]\mathscr{J})\cdot\alpha<0$ and $\mathscr{J}\cdot\alpha<0$.

Let F_p be a general fiber of p. Then the class of F_p is movable. Hence $(\mathscr{H}^{[\otimes s]}[\otimes]\mathscr{J}^{[\otimes t]})\cdot F_p\geqslant 0$. Moreover, we have

$$(\mathscr{H}[\otimes]\mathscr{J})\cdot F_p=\det\left((p^*\Omega^1_Z)^{sat}\right)\cdot F_p=0.$$

This implies that $\mathscr{H}\cdot F_p\geqslant 0$ since $s>t$. However, since the restriction of $(p^*\Omega^1_Z)^{sat}$ on F_p is isomorphic to $\mathscr{O}_{F_p}\oplus\mathscr{O}_{F_p}$, we have $\mathscr{H}\cdot F_p=0$.

Let k be the smallest positive integer such that $\mathscr{H}^{[\otimes k]}$ is invertible. Then there is a Cartier divisor L in Z such that $\mathscr{H}^{[\otimes k]}\cong\mathscr{O}_X(p^*L)$. Let $L'=f_*\mathrm{L}$. Then $L'\cdot C_0'\geqslant 0$ since $\mu_\alpha(\mathscr{H})\geqslant 0$. Note that if $L'\cdot C_0'>0$, then L' is ample on Z' since Z' has Picard number 1. Hence by Lemma 7.5.8, we can only have $L'\cdot C_0'=0$ and L' is numerically equal to the zero divisor since Z' has Picard number 1. By [AD14, Lem. 2.6], there is a positive integer k' such that $k'L'$ is linearly equivalent to the zero divisor. Hence $\mathscr{O}_Z(k'L)|_{Z_0}\cong\mathscr{O}_{Z_0}$ and $\mathscr{H}^{[\otimes kk']}|_{X_0}\cong\mathscr{O}_{X_0}$. Let l be the smallest positive integer such that $\mathscr{H}^{[\otimes l]}|_{X_0}\cong\mathscr{O}_{X_0}$. Let $c:W_0\to X_0$ be the normalisation of the cyclic cover with respect to the isomorphism $\mathscr{H}^{[\otimes l]}|_{X_0}\cong\mathscr{O}_{X_0}$ (see [KM98, Def. 2.52]). Then c is étale in codimension 1. We have $\mathscr{O}_{W_0}\cong(c^*\mathscr{H})^{**}$ and there is an injection $(c^*\mathscr{H})^{**}\hookrightarrow(c^*\Omega^1_{X_0})^{**}\cong\Omega^{[1]}_{W_0}$. Hence $H^0(W_0,\Omega^{[1]}_{W_0})\neq\{0\}$.

Let $W_0\to V_0\to Z_0'$ be the Stein factorisation. Let $h:V\to Z'$ be the normalisation of Z' in the function field of W_0. Then there is an open embedding from V_0 to V such that $\operatorname{codim}V\backslash V_0\geqslant 2$. By Lemma 7.5.7, we have $K_{V_0}\leqslant(h^*(K_{Z'}+\Delta'))|_{V_0}$. Since $\operatorname{codim}V\backslash V_0\geqslant 2$, there is an effective $\mathbb{Q}$-divisor Δ_V in V such that $K_V+\Delta_V=h^*(K_{Z'}+\Delta')$. Thus the pair (V,Δ_V) is klt by [KM98, Prop. 5.20]. Moreover, $-(K_V+\Delta_V)$ is ample since $-(K_{Z'}+\Delta')$ is ample, hence V is rationally connected by [HM07, Cor. 1.13]. By [GKKP11, Thm. 5.1], we have $H^0(V,\Omega^{[1]}_V)=0$. However, since general fibers of $W_0\to V_0$ are isomorphic to $\mathbb{P}^1$ which does not carry any non-zero pluri-form, we conclude that $H^0(W_0,\Omega^{[1]}_{W_0})=\{0\}$ from Lemma 7.5.9. This is a contradiction. □

7.5.2 Proof of Theorem 7.1.3

The object of this subsection is to prove Theorem 7.1.3. We will study Mori fibrations $X \to Z$ such that X is a projective threefold with $\mathbb{Q}$-factorial canonical singularities and Z is a projective normal surface. Assume that X carries non-zero pluri-forms. Then by Theorem 7.5.2, either the result Z' of any $(K_Z + \Delta)$-MMP has Picard number 2 or $(K_Z + \Delta)$ is pseudo-effective, where Δ the $\mathbb{Q}$-divisor defined in Theorem 7.5.2. We will study the first case in Proposition 7.5.10 and the second case in Proposition 7.5.12.

Proposition 7.5.10. *Let $p : X \to Z$ be a Mori fibration from a projective threefold to a projective surface such that X has $\mathbb{Q}$-factorial canonical singularities. Let Δ be the divisor in Z defined in Theorem 7.5.2. Assume that $K_Z + \Delta$ is not pseudo-effective. Let $f : Z \to Z'$ be the result of a $(K_Z+\Delta)$-MMP and let Δ' be the strict transform of Δ in Z'. Then there is a $(K_{Z'}+\Delta')$-Mori fibration $\pi' : Z' \to \mathbb{P}^1$. Let $\pi = \pi' \circ f$ and $q = \pi \circ p : X \to \mathbb{P}^1$. Then we have $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X, ((q^*\Omega^1_{\mathbb{P}^1})^{sat})^{[\otimes m]})$ for any $m > 0$, where $(q^*\Omega^1_{\mathbb{P}^1})^{sat}$ is the saturation of the image of $q^*\Omega^1_{\mathbb{P}^1}$ in $\Omega^{[1]}_X$.*

Proof. Let $\mathscr{H} = (q^*\Omega^1_{\mathbb{P}^1})^{sat}$ and let $\mathscr{F} = (p^*\Omega^{[1]}_Z)^{sat}$. Then we have an exact sequence of coherent sheaves $0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{J} \to 0$, where $\mathscr{J}$ is a torsion-free sheaf such that $\det \mathscr{F} = \mathscr{H}[\otimes]\mathscr{J}$. Let α be a class of movable curves in X whose image in Z is not zero and is proportional to the class of general fibers of $\pi : Z \to \mathbb{P}^1$. Then $\mathscr{H} \cdot \alpha = 0$. Moreover, we have

$$(\mathscr{H}[\otimes]\mathscr{J}) \cdot \alpha = \det \mathscr{F} \cdot \alpha = p^*(K_Z + \Delta) \cdot \alpha$$

by Remark 7.5.5. This intersection number is negative since for a general fiber F_π of $\pi : \mathrm{Z} \to \mathbb{P}^1$, we have $(K_Z + \Delta) \cdot F_\pi = (K_{Z'} + \Delta') \cdot (f_* F_\pi) < 0$. Hence $\mathscr{J} \cdot \alpha < 0$.

There is an open subset U of X with $\operatorname{codim} X \backslash U \geqslant 2$ such that we have an exact sequence of locally free sheaves over U, $0 \to \mathscr{H}|_U \to \mathscr{F}|_U \to \mathscr{J}|_U \to 0$. Since $\mu_\alpha(\mathscr{H}^{\otimes s} \otimes \mathscr{J}^{\otimes t}) < 0$ if $t > 0$, we have $H^0(U, \mathscr{H}|_U^{\otimes s} \otimes \mathscr{J}|_U^{\otimes t}) = \{0\}$ if $t > 0$ by Proposition 1.2.1. Hence by Lemma 7.3.2 and Lemma 7.3.1, we have $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X, (\mathscr{F})^{[\otimes m]}) \cong H^0(X, (\mathscr{H})^{[\otimes m]})$ for any $m > 0$. □

Example 7.5.11. We will give an example of this kind of threefolds. Let $Z = \mathbb{P}^1 \times \mathbb{P}^1$. Denote by p_1, p_2 the two natural projections from Z to $\mathbb{P}^1$. Let z_1,...,z_r be $r \geqslant 4$ different points in $\mathbb{P}^1$ and let $C_i = p_1^* z_i$ for $i = 1, ..., r$. Let $X_0 = \mathbb{P}^1 \times Z$. By the method of Construction 7.2.4, we can construct a Mori fibration $\pi : X \to Z$ such that $m(\pi, C_i) = 2$ for $i = 1, ..., r$. Note that $K_Z + \frac{1}{2}(C_1 + \cdots + C_r)$ is not pseudo-effective since it has negative intersection number with general fibers of p_1. Moreover, we have

$$\begin{aligned} H^0(X, (\Omega^1_X)^{[\otimes 2]}) &\cong H^0(\mathbb{P}^1, (\Omega^1_{\mathbb{P}^1})^{\otimes 2} \otimes \mathscr{O}_{\mathbb{P}^1}(C_1 + \cdots + C_r)) \\ &\cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-4 + r)) \neq \{0\}. \end{aligned}$$

Now we will treat the second case. Note that this is the case for Example 7.2.3 and Example 7.2.5.

Proposition 7.5.12. *Let $p : X \to Z$ be a Mori fibration from a projective threefold to a projective surface. Assume that X has $\mathbb{Q}$-factorial klt singularities. Assume that $(K_Z + \Delta)$ is pseudo-effective, where Δ is the $\mathbb{Q}$-divisor defined in Theorem 7.5.2. Then X carries non-zero pluri-forms.*

Proof. By the abundance theorem for log surface (see. [AFKM92, Thm. 11.1.3]), $(K_Z+\Delta)$ is $\mathbb{Q}$-effective. Hence there is a positive integer l such that $l(K_Z + \Delta)$ is an effective Cartier divisor. This implies that $h^0(X, (\Omega^1_X)^{[\otimes 2l]})$ is non-zero by the Lemma 7.5.4. □

We can now conclude Theorem 7.1.3.

Proof of Theorem 7.1.3. Let $X^* \to Z$ be a Mori fibration. If Z is a curve, then we are in the second case of the theorem. Assume that $\dim Z = 2$. If $K_Z + \Delta$ is $\mathbb{Q}$-effective, then we are in the situation of Proposition 7.5.12. If $K_Z + \Delta$ is not $\mathbb{Q}$-effective, then by Proposition 7.5.10, there is a fibration $p : X^* \to \mathbb{P}^1$ such that $H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(X, ((p^*\Omega^1_{\mathbb{P}^1})^{sat})^{[\otimes m]})$ for any $m > 0$. By Lemma 7.4.1 we have $H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + \sum_{z\in\mathbb{P}^1}[\frac{(m(p,z)-1)m}{m(p,z)}]))$ for any $m > 0$. □

7.6 Proof of Theorem 7.1.1

In this section, we will complete the proof of Theorem 7.1.1. First we will show that if X is a rationally connected projective threefold with $\mathbb{Q}$-factorial *terminal* singularities which carries non-zero pluri-forms, then there is a dominant rational map from X to $\mathbb{P}^1$.

Lemma 7.6.1. *Let X be a rationally connected projective threefold with $\mathbb{Q}$-factorial terminal singularities such that $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$. Let $f : X \dashrightarrow X^*$ be the result of a MMP for X. Then there is a fibration $p : X^* \to \mathbb{P}^1$.*

Proof. Note that X^* is a Mori fiber space. Then we have a Mori fibration $q : X^* \to Z$, where Z is a normal rationally connected variety. If $\dim Z = 1$, then we are done.

By Lemma 6.3.1, we know that X^* carries non-zero pluri-forms. Hence $\dim Z > 0$ by Theorem 6.2.1. Assume that $\dim Z = 2$. Then Z has canonical singularities by [MP08, Cor. 1.2.8]. Hence K_Z is not pseudo-effective by [Kol96, Cor. 1.11]. Moreover, by Lemma 5.4.1, $m(q, D) = 1$ for any effective divisor D on Z. Hence, by Theorem 7.5.2, if $Z \to Z'$ is the result of a MMP for Z, then Z' has Picard number 2. Hence we have a Mori fibration $Z' \to \mathbb{P}^1$. Let p be the composition of $X^* \to Z \to Z' \to \mathbb{P}^1$. Then p is a fibration from X^* to $\mathbb{P}^1$ □

With the notation as above, note that a general fiber F' of $p : X^* \to \mathbb{P}^1$ is a smooth rationally connected surface. Hence F' does not carry any non-zero pluri-form. Let U be the largest open subset in X over which $f : X \dashrightarrow X^*$ is regular. Then $\operatorname{codim} X\backslash U \geqslant 2$, $\operatorname{codim} X^*\backslash f(U) \geqslant 2$ and the rational map $p : X \dashrightarrow \mathbb{P}^1$ is regular over U. If F is a general fiber of $U \to \mathbb{P}^1$, then $f(F) \subseteq F'$, where F' is a general fiber of p. Moreover $\operatorname{codim} F'\backslash f(F) \geqslant 2$. Hence $f(F)$ does not carry any non-zero pluri-form and neither does F by Lemma 6.3.1. The following lemma shows that the rational map $X \dashrightarrow \mathbb{P}^1$ is regular. Moreover, general fibers of $X \to \mathbb{P}^1$ are birational to the ones of $X^* \to \mathbb{P}^1$ which are rationally connected. Hence general fibers of $X \to \mathbb{P}^1$ are rationally connected.

Lemma 7.6.2. *Let X be a projective threefold with $\mathbb{Q}$-factorial terminal singularities. Assume that there is a non constant rational map $p : X \dashrightarrow \mathbb{P}^1$ which is regular over U such that $\operatorname{codim} X\backslash U \geqslant 2$. Assume that general fibers of $U \to \mathbb{P}^1$ do not carry any pluri-form. If $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, then p is regular.*

Proof. Let Γ be the normalisation of the graph of p. Let $p_1 : \Gamma \to X$, $p_2 : \Gamma \to \mathbb{P}^1$ be the natural projections. Then there is a natural injection from $H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]})$ to $H^0(X, (\Omega^1_X)^{[\otimes m]})$ by Lemma 6.3.1. Let σ be a non-zero element in $H^0(X, (\Omega^1_X)^{[\otimes m]})$. Then σ induces a rational section σ_Γ of $(\Omega^1_\Gamma)^{[\otimes m]}$ on Γ. Let E be a p_1-exceptional divisor. Then there is a curve in E which is contracted by p_2 since $\dim E = 2 > 1$. Hence this curve is not contracted by p_1 since the graph of p is included in $X \times \mathbb{P}^1$ and the normalisation map is finite. Thus $p_1(E)$ is a curve in X and X is smooth around the generic point of $p_1(E)$ since X is smooth in codimension 2 (see [KM98, Cor. 5.18]). Hence σ_Γ does not have pole along E. This implies that we have an isomorphism from $H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]})$ to $H^0(X, (\Omega^1_X)^{[\otimes m]})$ induced by p_1.

Note that $p_1^{-1}|_U$ induces an isomorphism from U onto its image. If F_U is the fiber of $p|_U$ over a general point z, then $p_1^{-1}(F_U)$ is an open subset of F_Γ, where F_Γ is the fiber of $p_2 : \Gamma \to \mathbb{P}^1$ over z. Since F_U does not carry any non-zero pluri-form, neither does F_Γ. By Lemma 7.3.2, this implies that $H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]}) \cong H^0(\Gamma, ((p_2^*\Omega^1_{\mathbb{P}^1})^{sat})^{[\otimes m]})$.

We will first prove that p is regular in codimension 2. Assume the opposite. Then there is a divisor D in Γ which is exceptional for $p_1 : \Gamma \to X$ and the codimension of $p_1(D)$ in X is 2. Since X is smooth in codimension 2 (see [KM98, Cor. 5.18]), X is smooth around the generic point of $p_1(D)$. Thus there is a smooth quasi-projective curve C in D such that Γ is smooth along C and C is contracted to a smooth point of X by p_1. Note that C is horizontal over $\mathbb{P}^1$ under the projection $p_2 : \Gamma \to \mathbb{P}^1$ for the same reason as before. Let σ be a non-zero element in $H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]})$. By the exact sequence of locally free sheaves $\Omega^1_\Gamma|_C \to \Omega^1_C \to 0$, σ induce an element σ_C in $H^0(C, (\Omega^1_C)^{\otimes m})$. On the one hand, C is horizontal over $\mathbb{P}^1$ and σ is non-zero

in $H^0(\Gamma, ((p_2^*\Omega^1_{\mathbb{P}^1})^{sat})^{[\otimes m]})$, we have $\sigma_C \neq 0$. On the other hand, since C is contracted to a smooth point in X, we obtain $\sigma_C = 0$ for σ is the pullback of certain element in $H^0(X, (\Omega^1_X)^{[\otimes m]})$. This is a contradiction.

Now we will prove that p is regular. Let F_1 and F_2 be two different fibers of $U \to \mathbb{P}^1$. Then their closures in X are two Weil divisors and their intersection is included in a closed subset of codimension at most 3. Hence their intersection is empty since X is $\mathbb{Q}$-factorial. This implies that p is regular. □

Together with Lemma 7.3.2 and Lemma 7.4.1, we can conclude Theorem 7.1.1.

Proof of Theorem 7.1.1. By Lemma 7.6.1 and 7.6.2, there is a fibration $p : X \to \mathbb{P}^1$ such that general fibers of p do not carry any non-zero pluri-form. Lemma 7.3.2 shows that all pluri-forms on X come from the base $\mathbb{P}^1$. Finally, we obtain the formula in the theorem from Lemma 7.4.1. □

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