



# Local study of sub-Finslerian control systems

Entisar Abdul-Latif Ali

## ► To cite this version:

Entisar Abdul-Latif Ali. Local study of sub-Finslerian control systems. Optimization and Control [math.OC]. Université Grenoble Alpes, 2017. English. NNT : 2017GREAM005 . tel-01681328v1

**HAL Id: tel-01681328**

**<https://theses.hal.science/tel-01681328v1>**

Submitted on 11 Jan 2018 (v1), last revised 12 Jan 2018 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## THÈSE

Pour obtenir le grade de

### **DOCTEUR DE LA COMMUNAUTÉ UNIVERSITÉ GRENOBLE ALPES**

Spécialité : Mathématiques

Arrêté ministériel : 25 mai 2016

Présentée par

**ENTISAR ABDUL-LATIF ALI**

Thèse dirigée par **Grégoire CHARLOT**, Maître de conférences,  
UGA

préparée au sein du **Laboratoire Institut Fourier**  
dans l'**École Doctorale Mathématiques, Sciences et  
technologies de l'information, Informatique**

**Etude locale de systèmes contrôles de type  
sous-finslerien.**

**Local study of sub-Finslerian control  
systems**

Thèse soutenue publiquement le **31 janvier 2017**,  
devant le jury composé de :

**Monsieur GREGOIRE CHARLOT**

MAITRE DE CONFERENCES, UNIVERSITE GRENOBLE ALPES,  
Directeur de thèse

**Monsieur YACINE CHITOUR**

PROFESSEUR, UNIVERSITE PARIS-SUD, Rapporteur

**Monsieur FREDERIC JEAN**

PROFESSEUR, ENSTA ParisTech - PALAISEAU, Rapporteur

**Monsieur HERVE PAJOT**

PROFESSEUR, UNIVERSITE GRENOBLE ALPES, Président

**Monsieur MARIO SIGALOTTI**

CHARGE DE RECHERCHE, INRIA SACLAY- ÎLE DE-FRANCE,  
Examineur

**Monsieur CONSTANTIN VERNICOS**

MAITRE DE CONFERENCES, UNIVERSITE DE MONTPELLIER,  
Examineur









## Remerciements

Mes remerciements vont d'abord à mon directeur de thèse Grégoire Charlot, pour m'avoir proposé de travailler dans ce domaine de recherche passionnant et très actif. Il a su trouver un sujet très riche, et orienter mes recherches au cours des années de questions assez simples vers des objectifs plus ambitieux. Lors de ces années, vous avez toujours été à mes côtés pour me soutenir, me proposer des idées géniales et m'encourager.

Vous m'avez aidé à améliorer mon Français. Votre passion, votre optimisme et les discussions entre nous m'ont été très profitables. Pour tout cela je vous remercie mille fois.

C'est un grand honneur pour moi que Yacine Chitour et Frédéric Jean aient accepté d'être rapporteurs de ma thèse. Je les remercie chaleureusement pour l'intérêt qu'ils ont porté à mes travaux, ainsi que Hervé Pajot, Mario Sigalotti et Constantin Vernicos qui ont accepté de faire partie de mon jury de thèse.

Je remercie également l'Institut Fourier pour son accueil pendant ces années, et pour tout. Je remercie tout particulièrement tous les administratifs, Zilora, Géraldine Rahal, et les secrétaires de l'IF qui m'ont aidée à résoudre beaucoup de questions.

Je remercie tout particulièrement mes soeurs, pour leur soutien. Et finalement, c'est à mon mari Bashar que je dois le plus, pour son soutien et sa présence à mes côtés dans les moments difficiles. Si je suis arrivée jusqu'ici, c'est en grande partie grâce à lui. Et à mes petites filles, Juna et Mayar, pour tout !



# Contents

<b>Introduction</b>	<b>9</b>
<b>1 Sub-Finslerian Structures</b>	<b>13</b>
1.1 The Point of View of Differential Geometry . . . . .	13
1.1.1 The constant rank case . . . . .	13
1.1.2 A more general definition . . . . .	14
1.1.3 The Problems we are considering in this thesis . . . . .	14
1.2 The Point of View of control . . . . .	15
1.3 General Properties . . . . .	16
1.3.1 Controllability . . . . .	16
1.3.2 Existence of Minimizers . . . . .	16
1.3.3 Pontryagin Maximum Principle . . . . .	17
1.3.4 Switching . . . . .	18
1.3.5 Switching in $\mathbf{R}^2$ . . . . .	18
1.3.6 Switching in $\mathbf{R}^3$ . . . . .	19
1.3.7 Singular extremals . . . . .	19
1.4 Normal forms in $\mathbf{R}^2$ and $\mathbf{R}^3$ . . . . .	20
1.4.1 Thom Transversality Theorem . . . . .	21
1.4.2 Generic properties in $\mathbf{R}^2$ . . . . .	22
1.4.3 Normal form on 2D . . . . .	25
1.4.4 Normal form in dimension 3 . . . . .	26
1.5 Optimal syntheses . . . . .	27
<b>2 Optimal Synthesis in <math>\mathbf{R}^2</math></b>	<b>29</b>
2.1 Initial conditions and their parametrization . . . . .	29
2.1.1 Different types of extremals . . . . .	29
2.1.2 Privileged coordinates and nilpotent approximation . . . . .	30
2.1.3 Parametrization of the initial conditions . . . . .	31



2.2	Describing the optimal synthesis . . . . .	32
2.2.1	Symbols of extremals . . . . .	32
2.2.2	Symmetries . . . . .	32
2.3	$(NF_1)$ case . . . . .	33
2.3.1	Synthesis in the first quadrant . . . . .	34
2.4	$(NF_{2a})$ case . . . . .	35
2.5	$(NF_{2b})$ case . . . . .	37
2.6	$(NF_3)$ case . . . . .	39
<b>3</b>	<b>Optimal Synthesis in <math>\mathbf{R}^3</math></b>	<b>41</b>
3.1	Nilpotent case . . . . .	42
3.2	Extremals with $ \lambda_z  \gg 1$ . . . . .	44
3.2.1	Hamiltonian equations . . . . .	44
3.2.2	Conjugate locus . . . . .	46
3.2.3	Cut locus . . . . .	48
3.3	Extremals with only one control switching several times . . . .	52
3.3.1	Cut locus generated by extremals with $\lambda_z(0) \sim 0$ . . .	56

# Introduction

Sub-Finslerian geometry is a generalization of both sub-Riemannian and Finslerian geometry. To define such a structure you need to fix a connected manifold  $M$ , a distribution  $\Delta$  on  $M$  and a norm on  $\Delta$ . An admissible curve for such a structure is, as in sub-Riemannian geometry, an absolutely continuous curve almost everywhere tangent to the distribution and its length is, as in Finsler geometry, the integral of the norm of its speed. Under a good condition of non integrability of  $\Delta$  any two points of  $M$  are connected by an admissible curve and the distance between these two points is the infimum of the lengths of the admissible curves joining the two points.

From the eighties, the mathematical activity around sub-Riemannian geometry is very important and increasing. The list of publications is very large and a survey is not the object of this thesis. Let focus on some publications that motivated the same kind of work in sub-Riemannian geometry as the one we are presenting here for sub-Finslerian metrics.

At the end of the eighties, G. Ben Arous and R. Léandre published a series of articles (see [9, 17, 18, 8]) with new results on the asymptotics of the heat kernel in sub-Riemannian geometry on the model of the one of Molčanov [19] in Riemannian geometry. These results, giving information outside the cut locus in absence of abnormal extremals, motivated several works on the local synthesis of sub-Riemannian structures in the contact case (see [2, 15, 4]) and the Martinet case (see [11, 10]) in dimension three and the quasi-contact case (see [14]) in dimension four.

In this thesis, we make the same work in the sub-Finslerian context, when the norm is a maximum norm, in dimension two and three. The local optimal synthesis is investigated, with a focus on the evaluation of the first conjugate locus and of the cut locus.

In the case of the dimension two, the notion of distribution is larger than the classic one: we define it locally by the data of two vector fields, that

can become parallel along a submanifold of  $M$ . The study we produce here of such structures uses techniques developed in [12] to define objects that are very useful to analyze the dynamics, but also techniques developed in [2, 15, 4] in the sub-Riemannian context to compute jets of the extremals with large initial conditions.

This thesis is organized in three parts. The first part contains definitions and theorems in geometry and control which are used throughout the thesis. It also present results that are commune to the two cases treated in the following parts. The second part deals with the local optimal synthesis in dimension 2: we construct a normal form and describe the generic local synthesis. The third part concerns the dimension 3 when the distribution is contact: we build a normal form and describe the local synthesis.

In chapter 1, in section 1.1 and 1.2 we define the notion of sub-Finslerian structure and present the case of maximum norm we treat in this manuscript. In section 1.3, we recall classical results about controllability, existence of local minimizers, and the Pontryagin Maximum Principle which is a necessary condition theorem on curves to be optimal, that is to realize the distance. In the same section, we also present the notion of switching and the tools to study it, in particular switching functions and their properties, and the notion of singular control. In section 1.4 we recall the notion of genericity, present a list of generic properties in dimension 2 and construct a normal form in the two cases we are studying : the generic case in dimension 2 and the contact case in dimension 3, generic in a weaker sense.

In chapter 2, we study the local optimal synthesis in dimension 2. It may look quite simple but the zoology is quite rich. In section 2.1 we present the nilpotent cases. They play the role of model of the dynamics, as do the euclidean case in Riemannian geometry. In section 2.2 we describe the effect on the invariants of the choice of the vector fields used to define the control system. For exampel the effect of choosing  $-G_1$  instead of  $G_1$ . In sections 2.3 to 2.6, we describe the synthesis in different cases appearing in the normal form. Conditions on the invariants are given for the existence of a local cut locus.

In chapter 3, we study the local optimal synthesis in dimension 3 when the distribution is contact. In section 3.1, we study the nilpotent case. This last one was partially studied in the article [13]. Here we define and describe the first conjugate locus, and we describe the Maxwell set and the cut locus. The cut time and the Maxwell time in the nilpotent case give a good estimation of

the cut time in the generic cases for the extremals with large initial condition. In section 3.2 we describe the conjugate and the cut loci in the generic case (in a restricted sense with respect to the one in the study of the dimension 2). In section 3.3 we describe the synthesis corresponding to extremals that switch several times on only one control, and, when it happens, the cut locus generated by these extremals.



# Chapter 1

## Sub-Finslerian Structures

In this chapter, we define the structures we are going to study with both differential and control points of view in sections 1.1 and 1.2, we recall Chow-Rashevsky theorem and Filippov theorem in subsections 1.3.1 and 1.3.2. In subsection 1.3.3 we give the formulation of Pontryagin Maximum Principal, an important tool. In subsections 1.3.4 to 1.3.7 we discuss switching and give first necessary conditions to the existence of singular curves. In section 1.4 we recall Thom transversality theorem and use it to construct normal forms in dimension 2 and 3.

### 1.1 The Point of View of Differential Geometry

#### 1.1.1 The constant rank case

In general, the data of a sub-Finsler structure is the data of a triplet  $(M, \Delta, \mu)$  where  $M$  is a manifold,  $\Delta$  is a distribution on  $M$ , that is a sub-bundle of  $TM$ , and  $\mu$  is a norm on  $\Delta$ .

Once defined such a structure, one can define admissible curves as absolutely continuous curves such that  $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$  a.e.

For such a curve defined on  $[0, T]$ , we define its length:

$$\ell(\gamma) = \int_0^T \mu(\dot{\gamma}(t)) dt.$$

If any two points of  $M$  are linked by an admissible curve, that is if the structure is controllable, we can define a distance on  $M$  by

$$d(q, q') = \inf\{\ell(\gamma) \mid \gamma \text{ admissible}, \gamma(0) = q, \gamma(T) = q'\}$$

### 1.1.2 A more general definition

It happens that it may be natural to consider dynamical systems where  $\Delta$  is no more a sub-bundle of constant rank but is a more general structure.

For example, the Grushin structure on  $\mathbf{R}^2$  where  $\Delta$  is defined as  $\text{span}\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\}$  has such property.

One way to define geometrically such a structure is the following: a sub-Finslerian structure is now defined by  $(M, E, \pi, f, \mu)$  where  $M$  is a manifold,  $\pi : E \rightarrow M$  is a vector bundle,  $f : E \rightarrow TM$  is a morphism of vector bundle (in particular if  $v \in E_q$  then  $f(v) \in T_qM$ ) and  $\mu$  is a norm (as defined below) on  $E$ . In general, one add the assumption that the map  $f_*$ , induced by  $f$  on  $\Gamma(E)$  by  $f_*(\sigma) = f \circ \sigma$ , is one-to-one from  $\Gamma(E)$  to  $\text{Vec}(M)$ . We denote  $\Delta_q = f(E_q)$ . In this context, a curve is said admissible if  $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$  for a.e.  $t$ . For  $v \in \Delta_q$ ,  $\mu(v) := \inf\{\mu(X) \mid X \in E_q, f(X) = v\}$ . The definitions of length and distance are unchanged.

The dimension of the fiber of  $E$  is called rank of the structure.

**Definition 1** (Maximum norm). In  $\mathbf{R}^n$  a maximum norm is a norm such that exists a linear coordinate system  $(x_1, \dots, x_n)$  s.t

$$|(x_1, \dots, x_n)| = \max\{|x_i|, i \leq n\}$$

In the sub-Finslerian context, we say that the norm  $\mu$  is a maximum norm if  $\forall q \in M$  exists a linear coordinate system  $(v_1, \dots, v_n)$  on  $E_q$  (or  $\Delta_q$ ) such that  $\forall X = (v_1, \dots, v_n) \in E_q$  then  $\mu(X) = \max_{i=1, \dots, n}\{|v_i|\}$ .

### 1.1.3 The Problems we are considering in this thesis

In this thesis, we studied locally two class of SF structures defined with a maximum norm:

1. SF structure of rank 2 on 2d manifolds,
2. SF structure of rank 2 on 3d manifolds for contact distributions.

In the first case, the distribution may have non constant rank, if the map  $f$  is not one-to-one even if  $f_*$  is one-to-one.

In the second case, we assume that  $f$  is one-to-one and moreover we assume that  $\Delta$ , which is a sub-bundle, satisfies  $[\Delta, \Delta] = TM$  locally.

In both cases, since we are interested in the local study of such structures, we may assume that  $M = \mathbf{R}^n$  ( $n=2$  or  $3$ ) or even a neighborhood of  $0$  in  $\mathbf{R}^n$ .

Moreover, we consider only maximum norms.

## 1.2 The Point of View of control

To define with a control point of view the previous structures, we proceed as follows. Let give a manifold  $M$  (or  $\mathbf{R}^n$  if we have only local) and  $(F_1, \dots, F_k)$  be  $k$  vector fields defined on  $M$ . The control system we are going to consider is

$$\dot{q}(t) = \sum_{i=1}^k u_i(t) F_i(q(t)), \quad (1.1)$$

where the functions  $u_i$  are measurable functions such that

$$\max_{i=1, \dots, k} \{|u_i(t)|\} \leq 1, \text{ for a.e. } t. \quad (1.2)$$

An admissible curve is defined by the fact that it is an absolutely continuous curve s.t  $\dot{\gamma}(t) \in \text{span}\{F_1, \dots, F_k\}$  for a.e.  $t$ . It is said to be optimal between  $q$  and  $q'$  if  $\gamma(0) = q, \gamma(T) = q'$  and  $T$  is the infimum of time necessary to join the two points under the constraint  $\max\{|u_i|\} \leq 1$ .

We define  $\Delta_q = \text{span}\{F_i(q)\}$  and the maximum norm associate to this family on  $\Delta_q$  by

$$|X| = \inf_u \left\{ \max_{i=1, \dots, k} \left\{ |u_i| \mid X = \sum_{i=1}^m u_i F_i(q) \right\} \right\}$$

We define the length of an admissible curve by

$$\ell(\gamma) = \int_0^T |\dot{\gamma}(t)| dt$$

and if  $|\dot{\gamma}(t)| = 1$  for a.e.  $t$ ,  $\ell(\gamma)$  is equal to the time necessary to follow the curve. If the system is controllable, we can define the distance as

$$d(q, q') = \inf\{\ell(\gamma) \mid \gamma \text{ admissible and } \gamma(0) = q, \gamma(T) = q'\}.$$



### 1.3 General Properties

We introduce some general properties as the local controllability by Chow-Rashevsky theorem, the existence of minimizer by Filippov theorem, the Pontryagin Maximum Principle theorem, conditions on switching controls and the definition of switching functions and singular extremals.

Recall that in the following we are going to consider only the two cases: the distributions of rank 2 in  $\mathbf{R}^2$ ; the contact distributions in  $\mathbf{R}^3$  that is subbundle  $\Delta$  of  $T\mathbf{R}^3$  such that  $\Delta^2 = T\mathbf{R}^3$ .

#### 1.3.1 Controllability

For such systems the classical theorem concerning local controllability is

**Theorem 2** (Chow, Rashevsky, 1938). *Let  $M$  be a smooth manifold and  $X_1, \dots, X_m$  be  $m$  smooth vector fields on  $M$ . Assume that*

$$\text{Lie}\{X_1, \dots, X_m\}(q) = T_q M, \forall q \in M$$

*then the control system*

$$\dot{q} = \sum_{i=1}^m u_i X^i(q)$$

*is locally controllable in any time at every point of  $M$ .*

In the systems we are going to consider,  $\Delta_q$  or  $\Delta_q^2$  or  $\Delta_q^3$  is equal to  $T_q M$  hence the system is controllable.

#### 1.3.2 Existence of Minimizers

Once answered the question of controllability, it is natural to try to check that the distance is realized that is that between two given points the distance is realized by a minimizer. The classical result is Filippov theorem, which in our context take the form

**Theorem 3** (Filippov, in the SF context). *Let  $M$  be a locally compact bracket generating sub-Finslerian manifold. Since at each point  $q$  the unit ball  $B_q \subset T_q M$  is convex and compact, and since for all  $R > 0$  the set of points of distance less or equal to  $R$  has compact closure, then for all  $q' \in M$  exists a minimizer realizing the distance:*

$$d(q, q') = \min\{\ell(\gamma), \gamma \text{ admissible}, \gamma(0) = q, \gamma(T) = q'\}.$$

See [1] for the proof.

### 1.3.3 Pontryagin Maximum Principle

The PMP gives necessary conditions for a curve to be a minimizer. In our context, it gives:

**Theorem 4** (Pontryagin Maximum Principle). *Consider the control system (1.1) subject to (1.2). Let define the Hamiltonian as follows. For every  $(q, \lambda, u, \lambda_0) \in T^*M \times [-1, 1]^k \times \mathbf{R}_-$ , the Hamiltonian is defined by*

$$H(q, \lambda, u, \lambda_0) := u_1 < \lambda, F_1(q) > + u_2 < \lambda, F_2(q) > + \lambda_0.$$

*If  $q(\cdot) : [0, T] \rightarrow M$  is a time optimal trajectory corresponding to a control  $u(\cdot) : [0, T] \rightarrow [-1, 1]^2$ , then there exist a never vanishing Lipschitz continuous covector  $\lambda(\cdot) : t \in [0, T] \rightarrow \lambda(t) \in T_q^*(t)M$  and a constant  $\lambda_0 \leq 0$  such that for a.e  $t \in [0, T]$ , we have*

1.  $\lambda(t) \neq 0, \forall t$ ;
2.  $\dot{q}(t) = \frac{\partial H}{\partial \lambda}(q(t), \lambda(t), u(t), \lambda_0)$ , for a.e.  $t$ ;
3.  $\dot{\lambda}(t) = -\frac{\partial H}{\partial q}(q(t), \lambda(t), u(t), \lambda_0)$ , for a.e.  $t$ ;
4.  $H(q(t), \lambda(t), u(t), \lambda_0) = H_M(q(t), \lambda(t), \lambda_0)$ , for a.e.  $t$ ; where

$$H_M(q(t), \lambda(t), \lambda_0) = \max\{H(q, \lambda, u, \lambda_0) : u \in [-1, 1]^2\}$$

5.  $H_M(q(t), \lambda(t), \lambda_0) = 0$ .

**Definition 5.** The map  $\lambda : [0, T] \rightarrow T_{x(t)}^*M$  is called covector. A trajectory  $x(\cdot)$  satisfying conditions 1, 2, 3, 4 and 5 is called an extremal (resp. a couple  $(x(\cdot), \lambda(\cdot))$  is called an extremal pair). If  $(x(\cdot), \lambda(\cdot))$  satisfies moreover  $\lambda_0 = 0$  (resp.  $\lambda_0 < 0$ ), then it is called an abnormal extremal (resp. a normal extremal).

An extremal is said to be nontrivial if it does not correspond to controls a.e vanishing. Notice that a trivial extremal is an abnormal extremal.

*Remark 6.* In all the situations we are going to study, except at isolated points,  $\Delta^2 = TM$ . It is a well known fact that in this case there is no non trivial abnormal extremal hence we can assume  $\lambda_0 = -1$ .

### 1.3.4 Switching

In this section we are interested to give properties of optimal trajectories and to determine when the controls switch from  $+1$  to  $-1$  or viceversa and when they may assume values in  $] -1, +1[$ . Moreover, we would like to predict which kind of switching can happen by using properties of the vector fields  $F_i$ . Along an extremal, if  $\langle \lambda(t), F_i(x(t)) \rangle > 0$  (resp.  $< 0$ ) then  $u_i(t) = 1$  (resp.  $u_i(t) = -1$ ). It is a direct consequence of the PMP. This motivates the introduction of the switching functions  $\phi_i$ :

**Definition 7.** For an extremal triplet  $(q(\cdot), \lambda(\cdot), u(\cdot))$ , define the switching functions

$$\phi_i(t) = \langle \lambda(t), F_i(q(t)) \rangle, i = 1, 2.$$

Thanks to  $\lambda_0 = -1$ , the  $\phi_i$  functions satisfy

$$u_1(t)\phi_1(t) + u_2(t)\phi_2(t) = 1, \quad \text{for a.e. } t.$$

*Remark 8.* Notice that the  $\phi_i(\cdot)$  are at least Lipschitz continuous.

A direct consequence of the maximality condition is

**Proposition 9.** If  $\phi_i(t) > 0$  (resp.  $\phi_i(t) < 0$ ) then  $u_i(t) = 1$  (resp.  $u_i(t) = -1$ ).

If  $\phi_i(t) = 0$  and  $\dot{\phi}_i(t) > 0$  (resp.  $\dot{\phi}_i(t) < 0$ ) then  $\phi_i$  changes sign at time  $t$  and the control  $u_i$  switches from  $-1$  to  $+1$  (resp. from  $+1$  to  $-1$ ).

Defining  $\phi_3(t) = \langle \lambda(t), [F_1, F_2](q(t)) \rangle$  then one computes easily that

$$\dot{\phi}_1(t) = -u_2(t)\phi_3(t) \quad \text{and} \quad \dot{\phi}_2(t) = u_1(t)\phi_3(t), \quad \text{for a.e. } t.$$

### 1.3.5 Switching in $\mathbf{R}^2$

In the case  $\mathbf{R}^2$ , let define the following sets. We denote  $\Delta_A$  the set of points where  $F_1$  and  $F_2$  are parallel,  $\Delta_1$  the set of points where  $F_1$  is parallel to  $[F_1, F_2]$  and  $\Delta_2$  the set of points where  $F_2$  is parallel to  $[F_1, F_2]$ .

Outside  $\Delta_A$ , one can define the functions  $f_1$  and  $f_2$  by

$$[F_1, F_2] = f_2 F_1 - f_1 F_2.$$

Then, outside  $\Delta_A$ , at a time  $t$  where  $\phi_1(t) = 0$  we get

$$\dot{\phi}_1(t) = -u_2(t)\phi_3(t) = u_2(t)f_1(q(t))\phi_2(q(t))$$

and if  $q(t) \notin \Delta_1$  then  $f_1(q(t)) \neq 0$  and the sign of  $\dot{\phi}_1(t)$  is given by the sign of  $f_1(q(t))$ . And at time where  $\phi_2(t) = 0$  then the sign of  $\dot{\phi}_2(t)$  is given by the sign of  $f_2(q(t))$ . Hence, outside  $\Delta_A \cup \Delta_1 \cup \Delta_2$ , the signs of the functions  $f_1$  and  $f_2$  determine the possible switches of the control functions.

### 1.3.6 Switching in $\mathbf{R}^3$

In the case of  $\mathbf{R}^3$ , the hypothesis that we are considering the contact case allows to claim that  $(F_1, F_2, [F_1, F_2])$  form a frame of  $T\mathbf{R}^3$  at each point. Hence, if we denote  $F_3 = [F_1, F_2]$ ,  $F_4 = [F_1, F_3]$ , and  $F_5 = [F_2, F_3]$  then we can define the six functions  $f_{ij}$ ,  $i = 1, 2$ ,  $j = 1, 2, 3$ , by

$$F_4 = f_{41}F_1 + f_{42}F_2 + f_{43}F_3, \quad F_5 = f_{51}F_1 + f_{52}F_2 + f_{53}F_3.$$

and define  $\phi_4 = \langle \lambda, F_4 \rangle$  and  $\phi_5 = \langle \lambda, F_5 \rangle$ . Then one computes easily that

$$\dot{\phi}_3 = u_1(f_{41}\phi_1 + f_{42}\phi_2 + f_{43}\phi_3) + u_2(f_{51}\phi_1 + f_{52}\phi_2 + f_{53}\phi_3).$$

### 1.3.7 Singular extremals

**Definition 10.** A nontrivial extremal trajectory  $q(\cdot)$  is said to be  $u_i$ -singular if  $\phi_i(\cdot) = 0$  along it.

Let us introduce definitions to describe different types of controls

**Definition 11.** We call *bang* an extremal trajectory corresponding to constant controls with value 1 or  $-1$  and *bang-bang* an extremal which is a finite concatenation of bangs.

A time  $t$  is said to be a *switching time* if  $u$  is not bang in any neighborhood of  $t$ . Similarly,  $t$  is said to be a  $u_i$ -switching time if  $u_i$  is not constant in any neighborhood of  $t$ .

*Remark 12.* Along a  $u_i$ -singular arc  $\phi_i \equiv 0$  which implies  $\dot{\phi}_i \equiv 0$  and  $\phi_3 \equiv 0$ .

In  $\mathbf{R}^2$  this implies that  $f_i(q(t)) \equiv 0$  or  $q(t) \in \Delta_A$  hence that  $q(t) \in \Delta_A \cup \Delta_i$ .

In  $\mathbf{R}^3$ , for example for a  $u_1$ -singular, along which  $\phi_2 \equiv 1$ , this implies that

$$0 = u_1(q(t))f_{42}(q(t)) + f_{52}(q(t)).$$

Hence on a domain where  $|f_{52}| > |f_{42}|$ , no  $u_1$ -singular can run.

**Proposition 13.** *In  $\mathbf{R}^2$ , under the generic assumption that  $\Delta_A$ ,  $\Delta_1$  and  $\Delta_2$  are submanifolds traversal by pair (see section 1.4) then*

1. *The support of a  $u_i$ -singular trajectory is always contained in the set  $\Delta_i$ .*
2. *At each point of an arc of  $\Delta_1$ , if  $G_1(q(t))$  and  $G_2(q(t))$  point on the same side of  $\Delta_1$  where  $f_1 > 0$ , then a  $u_1$ -singular extremal can run on  $\Delta_1$ . If  $G_1(q(t))$  and  $G_2(q(t))$  point on opposite side or if  $G_1$  point on the side of  $\Delta_1$  where  $f_1 < 0$  then no  $u_1$ -singular can run on it.*
3. *At each point of an arc of  $\Delta_2$ , if  $G_1(q(t))$  and  $-G_2(q(t))$  point on the same side of  $\Delta_2$  where  $f_2 > 0$ , then a  $u_2$ -singular extremal can run on  $\Delta_2$ . If  $G_1(q(t))$  and  $-G_2(q(t))$  point on opposite side or if  $G_1$  point on the side of  $\Delta_1$  where  $f_2 < 0$  then no  $u_2$ -singular can run on it.*
4. *Let consider a  $u_i$ -singular  $q(\cdot)$  satisfying 2 or 3. If it does not intersect  $\Delta_A$  and if at each time  $G_1(q(t))$  and  $G_2(q(t))$  are not tangent to  $\Delta_i$  then  $q(\cdot)$  is a local minimizer that is at each time  $t$  exists  $\epsilon$  such that  $q(\cdot)$  realizes the SF-distance between  $q(t_1)$  and  $q(t_2)$  for any  $t_1$  and  $t_2$  in  $]t - \epsilon, t + \epsilon[$ .*

**Proof.** See [6].

**Definition 14.** If a connected part of  $\Delta_1$  (resp.  $\Delta_2$ ) satisfies the point 2 (resp. point 3) of Proposition 13 at each point, then it is called a turnpike. If it does not at each point, it is called an anti-turnpike (see [12]).

*Remark 15.* Along a  $u_i$ -singular extremal the control  $u_i$  is completely determined by the fact that the dynamics should be tangent to  $\Delta_i$ .

## 1.4 Normal forms in $\mathbf{R}^2$ and $\mathbf{R}^3$

In this section we give generic properties of sub-Finslerian structures by Thom Transversality Theorem and some of its corollaries, and a normal form in dimension 2 and dimension 3.

### 1.4.1 Thom Transversality Theorem

In the following, and in particular in dimension 2, we are going to use deeply the following theorem in order to describe generic properties of couples of vector fields. In the following one should think of  $M$  as  $\mathbf{R}^2$  or  $\mathbf{R}^3$ ,  $N$  as the set of couple of vectors on  $\mathbf{R}^2$  or  $\mathbf{R}^3$ ,  $C^\infty(M, N)$  as the set of couple of vector fields on  $\mathbf{R}^2$  or  $\mathbf{R}^3$ , and  $J^k(M, N)$  the set of  $k$ -jets of couple of vector fields.

**Theorem 16** (Thom Transversality Theorem, [16], Page 82). *Let  $M, N$  be smooth manifolds and  $k \geq 1$  an integer. If  $S_1, \dots, S_r$  are smooth submanifolds of  $J^k(M, N)$  then the set*

$$\{f \in C^\infty(M, N) : J^k f \pitchfork S_i \text{ for } i = 1, 2, \dots, r\},$$

*is residual in the  $C^\infty$ -Whitney topology.*

**Corollary 17.** *Assume that  $\text{codim } S_i > \dim M$  for  $i = 1, \dots, r$  and  $k \geq 1$ . Then the set*

$$\{f \in C^\infty(M, N) : J^k f(M) \cap S_i = \emptyset \text{ for } i = 1, \dots, r\},$$

*is residual in the  $C^\infty$ -Whitney topology.*

**Corollary 18.** *For every  $f$  in the residual set defined in Theorem 16, the inverse images  $\tilde{S}_i := (J^k f)^{-1}(S_i)$  is a smooth submanifold of  $M$  and  $\text{codim } S_i = \text{codim } \tilde{S}_i$  for  $i = 1, \dots, r$ .*

*Remark 19.* Let  $\varphi$  be a diffeomorphism of  $M$  and  $\phi$  be a diffeomorphism of  $N$ . The map

$$\sigma_{\varphi, \phi} : \begin{cases} C^\infty(M, N) & \longrightarrow & C^\infty(M, N) \\ f & \longmapsto & \varphi \circ f \circ \phi \end{cases}$$

induces a diffeomorphism  $\sigma_{\varphi, \phi}^*$  of  $J^k(M, N)$  which sends submanifolds of  $J^k(M, N)$  on submanifolds of  $J^k(M, N)$ . Moreover,  $f$  is in the residual set defined in Theorem 16, if and only if  $\sigma_{\varphi, \phi}(f)$  is in the residual set

$$\{g \in C^\infty(M, N) : J^k g \pitchfork \sigma_{\varphi, \phi}^*(S_i) \text{ for } i = 1, \dots, r\}.$$

This remark is important to facilitate the presentation of the proofs of the generic properties given in the next section.

**Definition 20.** In the following, we will say that a property of maps is generic if it is true on a residual set defined as in Thom's theorem.

### 1.4.2 Generic properties in $\mathbf{R}^2$

In this part, since we are interested in local properties, we consider control systems in a neighborhood of 0 in  $\mathbf{R}^2$  of the type

$$\dot{q} = u_1 F_1(q) + u_2 F_2(q) \quad (1.3)$$

where  $F_1$  and  $F_2$  are smooth vector fields and  $u_1$  and  $u_2$  are the control functions satisfying the following constraints

$$|u_1| \leq 1 \text{ and } |u_2| \leq 1. \quad (1.4)$$

As discussed before, the controls often take values  $\pm 1$  hence it is natural to introduce the vector fields  $G_1 = F_1 + F_2$  and  $G_2 = F_1 - F_2$ .

Now we give a list of generic properties for couples of vector fields on 2d-manifolds. We list ten generic properties, these properties depending on the position of the velocity vectors  $G_1$  and  $G_2$  and some properties of these vectors. We can locally consider a couple of vector fields as the data of a map as the following

$$f : \begin{cases} U \subset \mathbf{R}^2 & \rightarrow \mathbf{R}^2 \times \mathbf{R}^2 \\ (x, y) & \mapsto (g_1(x, y), g_2(x, y), g_3(x, y), g_4(x, y)) \end{cases}$$

and we define a  $k$ -jet of such a map as the data of a map at the initial point  $(x_0, y_0)$  as the following

$$J^k f : \begin{cases} U \subset \mathbf{R}^2 & \rightarrow \mathbf{R}_k[x, y]^4 \\ (x, y) & \mapsto (P_1(x, y), \dots, P_4(x, y)) \end{cases}$$

where  $P_i$  ( $1 \leq i \leq 4$ ) is the Taylor series of order  $k$  of a functions  $g_i$  at  $(x_0, y_0)$ .

Now we describe submanifolds of  $\mathbf{R}_k[x, y]^4$  in coordinates, by writing:

$$P_h(x, y) = \sum_{i=0}^k \sum_{j=0}^{k-i} p_{1,i,j} x^i y^j \text{ for } h = 1, \dots, 4.$$

In the following the  $g_i$  are the coordinates of the  $G_j$  in a local coordinate system.

Here we give the generic properties of vector fields  $(F_1, F_2)$  on  $M$

*Generic property 1 (GP1): The set of points where  $G_1 = G_2 = 0$  is empty, for generic couples of vector fields  $(F_1, F_2)$  on  $M$ .*

Indeed in coordinates such points correspond to jets with  $p_{1,0,0} = p_{2,0,0} = p_{3,0,0} = p_{4,0,0} = 0$  which form a submanifold of  $\mathbf{R}_k[x, y]^4$  of codimension 4. Hence, by corollary 17, the property is proven.

Thanks to remark 19, up to a permutation between  $\pm F_1$  and  $\pm F_2$ , we will assume in the following that  $G_1 \equiv (1, 0)$  locally and hence assume that  $g_1 \equiv 1$  and  $g_2 \equiv 0$  hence that satisfy  $p_{1,0,0} = 1$  and  $p_{1,i,j} = p_{2,0,0} = p_{2,i,j} = 0$  when  $i \neq 0$  or  $j \neq 0$ . As a consequence we are reduced to apply Thom's theorem and its corollaries for the situation

$$f : \begin{cases} U \subset \mathbf{R}^2 & \rightarrow \mathbf{R}^2 \\ (x, y) & \mapsto (g_3(x, y), g_4(x, y)) \end{cases}$$

and a  $k$ -jet at  $(x_0, y_0)$  of such a map as the data of a map

$$J^k f : \begin{cases} U \subset \mathbf{R}^2 & \rightarrow \mathbf{R}_k[x, y]^2 \\ (x, y) & \mapsto (P_3(x, y), P_4(x, y)) \end{cases}$$

where  $P_i$  ( $i = 3, 4$ ) is the Taylor series of order  $k$  of a  $g_i$  at  $(x_0, y_0)$ .

*Generic property 2 (GP2): The set of points where  $G_2 = 0$  is a discrete set. The same holds for the set where  $F_1 = 0$  or the set where  $F_2 = 0$ , for generic couples of vector fields  $(F_1, F_2)$  on  $M$ .*

In fact these points correspond to jets with  $p_{3,0,0} = p_{4,0,0} = 0$  which is a submanifold of  $\mathbf{R}_k[x, y]^2$  of codimension 2. Hence, thanks to corollary 18, the set of points where  $G_2 = 0$  is a discrete set and it is generically a submanifold of  $M$  of codimension 2.

*Generic property 3 (GP3): the set of points  $\Delta_A$  where  $G_1$  is parallel to  $G_2$  is an imbedded submanifold of codimension 1, for generic couples of vector fields  $(F_1, F_2)$  on  $M$ .*

Indeed, assuming  $G_1 = (1, 0)$ ,  $\Delta_A$  is exactly the set of points where  $p_{4,0,0} = 0$ . It is an imbedded submanifold of  $\mathbf{R}_k[x, y]^2$  of codimension 1. Thanks to (GP1) and to corollary 18, we can conclude that generically  $\Delta_A$  is an imbedded submanifold of  $M$  of codimension 1.

*Generic property 4 (GP4): the set  $\Delta_1$  of points where  $F_1$  is parallel to  $[F_1, F_2]$  is an imbedded submanifold of codimension 1. The same holds for  $\Delta_2$  where  $F_2$  is parallel to the bracket  $[F_1, F_2]$ , for generic couples of vector fields  $(F_1, F_2)$  on  $M$ .*

In order to prove (GP4), we will compute the bracket  $[F_1, F_2]$  and we will describe  $\Delta_1$  in coordinates.  $[F_1, F_2] = -\frac{1}{2}[G_1, G_2]$  hence has coordinates



$-\frac{1}{2}p_{3,1,0}$  and  $-\frac{1}{2}p_{4,1,0}$  and  $F_1$  has coordinates  $\frac{1}{2}(1 + p_{3,0,0})$  and  $\frac{1}{2}p_{4,0,0}$ . Hence  $\Delta_1$  is the set of points where

$$\begin{vmatrix} -\frac{1}{2}p_{3,1,0} & \frac{1}{2}(1 + p_{3,0,0}) \\ -\frac{1}{2}p_{4,1,0} & \frac{1}{2}p_{4,0,0} \end{vmatrix} = 0.$$

The differential of this determinant is not degenerate hence the set of  $\mathbf{R}_k[x, y]^2$  satisfying this equality is clearly an imbedded submanifold of codimension 1. Hence generically  $\Delta_1$  is the preimage of an immersed submanifold of codimension 1 which, thanks to corollary 18, permits to conclude that  $\Delta_1$  is an immersed submanifold of codimension 1.

*Generic property 5 (GP5): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , the sets  $(\Delta_A \cap \Delta_1)$ ,  $(\Delta_A \cap \Delta_2)$  and  $(\Delta_1 \cap \Delta_2)$  are discrete.*

Assuming  $G_1 = (1, 0)$ , the set  $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$  is the set of points where  $(F_1, F_2)$  is free and  $[F_1, F_2] = 0$  that is

$$\begin{aligned} p_{4,0,0} &\neq 0, \\ p_{3,1,0} &= 0 \\ p_{4,1,0} &= 0. \end{aligned}$$

This set is an immersed submanifold of codimension 2 of  $\mathbf{R}_k[x, y]^2$  hence, by corollary 18, the set  $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$  is generically a discrete set.

The set  $(\Delta_A \cap \Delta_2) \setminus \Delta_1$  is a set of points where  $F_2 = 0$ . By (GP2) it is a discrete set. The same holds for  $(\Delta_A \cap \Delta_1) \setminus \Delta_2$  which is a set of points where  $F_1 = 0$ .

The set  $\Delta_A \cap \Delta_1 \cap \Delta_2$  is the union of the subset where  $F_1 \neq 0$  and  $F_1 \parallel F_2 \parallel [F_1, F_2]$  and a subset where  $F_1 = 0$ . The second is discrete. Since  $G_1 = (1, 0)$ , the first set is also defined by  $G_1 \parallel G_2 \parallel [G_1, G_2]$  that is  $p_{4,0,0} = 0$  and  $p_{4,1,0} = 0$ . Hence, thanks to corollary 18, the set where  $F_1 \neq 0$  and  $F_1 \parallel F_2 \parallel [F_1, F_2]$  is a submanifold of codimension 2 that is a discrete set.

*Generic property 6 (GP6): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , the set of points where  $G_1 \parallel G_2 \parallel [G_1, G_2] \parallel [G_1, [G_1, G_2]]$  is empty.*

The set where  $G_1 \parallel G_2 \parallel [G_1, G_2] \parallel [G_1, [G_1, G_2]]$  is such that  $p_{4,0,0} = p_{4,1,0} = p_{4,2,0} = 0$ . Hence, thanks to corollary 18, it is a submanifold of codimension 3 that is an empty set.

*Generic property 7 (GP7): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , at the points  $q$  where  $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$  one gets  $G_1(q) \in T_q \Delta_A$ .*

The property  $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$  implies that  $p_{4,0,0} = p_{4,1,0} = 0$ . If  $p_{4,0,1} \neq 0$  then  $\Delta_A$  can be written  $p_{4,0,1}y = o(x)$  that is  $\Delta_A$  is tangent to the  $x$  axis and  $G_1 \in T_q\Delta_A$ . Hence the set of points where  $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$  and  $G_1(q) \notin T_q\Delta_A$  corresponds to jets with  $p_{4,0,0} = p_{4,1,0} = p_{4,0,1} = 0$  which is a submanifold of codimension 3. Hence generically, at the points  $q$  where  $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$ , one has  $G_1(q) \in T_q\Delta_A$ .

One can even detail more the generic properties: using the Thom transversality theorem and its corollaries, we can prove that generically

*Generic property 8 (GP8): along  $\Delta_1 \setminus (\Delta_2 \cup \Delta_A)$ , the points where  $G_1$  or  $G_2$  is tangent to  $\Delta_1$  are isolated. The same holds true for  $\Delta_2 \setminus (\Delta_1 \cup \Delta_A)$ .*

*Generic property 9 (GP9): at points of  $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$ , neither  $G_1$  nor  $G_2$  are tangent to  $\Delta_1$  or  $\Delta_2$ .*

*Generic property 10 (GP10): along  $\Delta_A \setminus (\Delta_1 \cup \Delta_2)$ , the set of points where  $G_1 = 0$  or  $G_2 = 0$  is discrete.*

### 1.4.3 Normal form on 2D

We have used the generic properties established in the previous to show:

**Theorem 21** (Normal form). *For generic couples of vector fields  $(F_1, F_2)$  on a 2d manifold  $M$ , up to an exchange between  $\pm F_1$  and  $\pm F_2$ , at each point  $q$  of the manifold there exist a unique coordinate system  $(x, y)$  centered at  $q$  such that one of the following normal form holds:*

$$(NF_1) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= \partial_y + x(a_{10} + a_{20}x + a_{11}y + o(x, y))\partial_x + x(b_{10} + b_{20}x + b_{11}y + o(x, y))\partial_y, \text{ and } q \notin \Delta_A. \end{aligned}$$

$$(NF_2) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= (a_0 + a_{10}x + a_{01}y + o(x, y))\partial_x + x(1 + x(b_{20} + O(x, y)))\partial_y, \\ &\text{with } 0 \leq a_0 \leq 1, \text{ and } q \in \Delta_A \setminus \Delta_1. \end{aligned}$$

$$(NF_3) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= (a_0 + o(1))\partial_x + (b_{01}y + \tfrac{1}{2}x^2 + b_{11}xy + b_{02}y^2 + o(x^2, y^2))\partial_y \text{ with } \\ &b_{01} > 0 \text{ and } 0 < a_0 < 1, \text{ and } q \in \Delta_A \cap \Delta_1 \cap \Delta_2 \text{ and } G_1(q) \in T_q\Delta_A. \end{aligned}$$

Moreover, for  $(NF1)$ ,  $(NF2)$  one of the following subcases holds:

$(NF_{1a})$   $(NF_1)$  holds with  $a_{10} - b_{10} \neq 0$  and  $a_{10} + b_{10} \neq 0$ . It corresponds to  $q \notin \Delta_A \cup \Delta_1 \cup \Delta_2$ .

$(NF_{1b})$   $(NF_1)$  holds with  $a_{10} - b_{10} = 0$  and  $a_{10} + b_{10} \neq 0$ . It corresponds to  $q \in \Delta_1 \setminus (\Delta_A \cup \Delta_2)$ .

$(NF_{1c})$   $(NF_1)$  holds with  $a_{10} - b_{10} \neq 0$  and  $a_{10} + b_{10} = 0$ . It corresponds to  $q \in \Delta_2 \setminus (\Delta_A \cup \Delta_1)$ .

$(NF_{1d})$   $(NF_1)$  holds with  $a_{10} = b_{10} = 0$ . It corresponds to  $q \in (\Delta_1 \cap \Delta_2) \setminus \Delta_A$ .

$(NF_{2a})$   $(NF_2)$  holds with  $0 \leq a_0 < 1$ . It corresponds to  $q \in \Delta_A \setminus (\Delta_1 \cup \Delta_2)$ .

$(NF_{2b})$   $(NF_2)$  holds with  $a_0 = 1$ . It corresponds to  $q \in (\Delta_A \cap \Delta_2) \setminus \Delta_1$  that is to  $q \in \Delta_A \setminus \Delta_1$  such that  $F_2(q) = 0$ .

**Proof.** see [6]

#### 1.4.4 Normal form in dimension 3

Since we consider only points  $q$  where the distribution is contact then  $G_1$ ,  $G_2$  and  $[G_1, G_2] = -2[F_1, F_2]$  form a basis of  $T_q\mathbf{R}^3$ . Hence, we can build a coordinate system centered at  $q$ , by the following way. Let denote  $e^{tX}$  the flow at time  $t$  of a vector field  $X$ . We can define

$$\Xi : (x, y, z) \longmapsto e^{xG_1}e^{yG_2}e^{z[G_1, G_2]}q,$$

which to  $(x, y, z)$  associates the point reached by starting at  $q$  and following  $[G_1, G_2]$  during time  $z$ , then  $G_2$  during time  $y$  and finally  $G_1$  during time  $x$ . The map  $\Xi$  is smooth and satisfies

$$\frac{\partial \Xi}{\partial x}(x, y, z) = G_1, \quad \frac{\partial \Xi}{\partial y}(0, y, z) = G_2, \quad \text{and} \quad \frac{\partial \Xi}{\partial z}(0, 0, z) = [G_1, G_2].$$

As a consequence  $\Xi$  is not degenerate at  $(0, 0, 0)$  and defines a coordinate system in a neighborhood of  $q$ . Such coordinates are called *normal coordinates* and  $G_1$  and  $G_2$  satisfy

$$\begin{aligned} G_1(x, y, z) &= \partial_x, \\ G_2(x, y, z) &= x\epsilon_x(x, y, z)\partial_x + (1 + x\epsilon_y(x, y, z))\partial_y + x(1 + \epsilon_z(x, y, z))\partial_z \end{aligned}$$

where  $\epsilon_x, \epsilon_y, \epsilon_z$  are smooth functions satisfying  $\epsilon_x(0, 0, z) = \epsilon_y(0, 0, z) = \epsilon_z(0, 0, z) = 0$ . Hence we can give the following expressions of  $G_2$

$$\begin{aligned} G_2(x, y, z) = & (a_{200}x^2 + a_{110}xy + x\theta_x(x, y, z))\partial_x \\ & + (1 + b_{200}x^2 + b_{110}xy + x\theta_y(x, y, z))\partial_y \\ & + (x + c_{200}x^2 + c_{110}xy + c_{300}x^3 \\ & + c_{210}x^2y + c_{120}xy^2 + x\theta_z(x, y, z))\partial_z \end{aligned}$$

where  $\theta_x, \theta_y$  and  $\theta_z$  are smooth functions with Taylor series of respective order 1, 1, 2 is null with  $x$  and  $y$  of order 1 and  $z$  of order 2 and  $\theta_x(0, 0, z) = \theta_y(0, 0, z) = \theta_z(0, 0, z) = 0$ .

It is well known that for a contact distribution there is no abnormal.

## 1.5 Optimal syntheses

Several considerations are common to the two cases.

First, the importance of the nilpotent approximation. In both cases, the notions associated with the nilpotent approximation, as the orders of the coordinates or privileged coordinates, are fundamental to drive the computations correctly.

Second, the fact that  $|\dot{\phi}_i| \leq |\phi_3|$  implies that, if we fix a constant  $K$ , the extremals with  $|\phi_3| < K$  cannot see the two controls switching in short time. As a consequence, we observe five types of extremals: the ones whose both controls switch; the ones that do not switch; the ones that switch only once, the ones that are singular; the ones such that one control switches several times. The two last one correspond to the existence of singular extremals in the nilpotent approximation.

Third, to study the extremals with both controls switching, and the corresponding cut points, the technique is quite similar in both dimensions. In this case, one coordinate ( $y$  or  $z$  later) has order more than one and the dual coordinate  $\lambda_y$  or  $\lambda_z$  of the covector is large. The technique consists in both cases in computing jets with respect to the inverse  $r$  of this coordinate. Finally, we are able to write an exponential map which is smooth by part. And to give a description of the conjugate locus, the front, and the cut locus.



# Chapter 2

## Optimal Synthesis in $\mathbb{R}^2$

In this section, we study the local geometry of Finslerian and sub-Finslerian structures associated to the maximum norm in dimension 2: short extremals, cut locus, generalized conjugate locus, switching locus, small spheres.

For this purpose we use ideas developed by Ugo Boscain, Thomas Chambrion and Grégoire Charlot in [12] where the study of SF-structures defined with a maximum norm is started. As we will see, even if it looks apriori quite simple, it happens that the zoology is quite rich even considering only generic cases. We establish some properties of the minimizing trajectories and we present the synthesis of the nilpotent case. We compute the jets of the geodesics, the switching and conjugate times and the switching and conjugate loci. We calculate the cut locus.

Of course, the general situation cannot be completely described since singular cases may have very special behavior. For example in the case  $F_1 = \frac{\partial}{\partial x}$  and  $F_2 = \frac{\partial}{\partial y}$  then any admissible trajectory with  $u_1 \equiv 1$  and  $\int_0^1 u_2(t)dt = 0$  joins optimally  $(0, 0)$  to  $(1, 0)$ . Hence in the following, we will consider only "generic" situations in the sense given in the Thom Transversality Theorem. And we use deeply the normal form presented in the previous chapter.

### 2.1 Initial conditions and their parametrization

#### 2.1.1 Different types of extremals

One proves easily that in the  $(NF_1)$  case, then  $\max(|\lambda_x(0)|, |\lambda_y(0)|) = 1$ . Hence, in this case, the set of initial conditions  $\lambda$  is compact and, since the

variations of the  $\phi_i$  is bounded, extremals switching in short time or singular extremals should have a  $\phi_i$  null or close to zero. Which implies that only one control may switch in short time.

In the  $(NF_2)$  and  $(NF_3)$  cases, then  $|\lambda_x(0)| = 1$  and there is no condition on  $\lambda_y$ . Hence the set of initial condition is not compact. This allows to consider initial conditions with  $|\lambda_y| \gg 1$  and hence will appear optimal extremals along which the two controls are switching.

In the  $(NF_{2a})$  and  $(NF_3)$  cases,  $\phi_1(0) = \frac{1+a_0}{2}$  and  $\phi_2(0) = \frac{1-a_0}{2}$ . Hence, if one considers a compact set of initial conditions, the corresponding extremals do not switch in short time. And they are not singular. As a consequence, to consider the extremals switching at least once, one should consider initial conditions with  $|\lambda_y(0)| \gg 1$ .

In the  $(NF_{2b})$  case, since  $\phi_2(0) = 0$  hence, even if one considers a compact set of initial conditions, the corresponding extremals may switch in short time.

### 2.1.2 Privileged coordinates and nilpotent approximation

In the computations, we use the notion of privileged coordinates and nilpotent approximation. For the definition of these objects, we refer to [7]. The coordinates we constructed in the normal form are always privileged coordinates. What is important to understand here is that, using these notions, a good notion of weight of the coordinates is introduced, giving informations on the increasing of these coordinates with the times along extremals. And that the expressions of extremals for the nilpotent approximation are very good approximations of the true extremals of the true SF-metric.

In the  $(NF_1)$  case,  $x$  and  $y$  have weight 1 and  $\partial_x$  and  $\partial_y$  have weight  $-1$  as operators of derivation. In the  $(NF_2)$  case  $x$  has weight 1 and  $y$  has weight 2,  $\partial_x$  has weight  $-1$  and  $\partial_y$  have weight  $-2$ . In the  $(NF_3)$  case,  $x$  has weight 1 and  $y$  has weight 3,  $\partial_x$  has weight  $-1$  and  $\partial_y$  have weight  $-3$ .

In privileged coordinates, along an extremal, in the  $(NF_1)$  case  $x$  and  $y$  are  $O(t)$  (and may be not  $o(t)$ ), in the  $(NF_2)$  case  $x = O(t)$  and  $y = O(t^2)$  and in the  $(NF_3)$  case  $x = O(t)$  and  $y = O(t^3)$ .

In the following,  $o_k(x, y)$  will denote a function whose valuation at 0 has order larger than  $k$  respectively to the weights of  $x$  and  $y$ . For example  $x^7$

has always weight 7 and  $y^3$  has weight 3 in the  $(NF_1)$  case but 9 in the  $(NF_3)$  case.

With this notion of weights, we define the nilpotent approximation of our normal forms in the three cases

$$\begin{aligned}
(NF_1) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= \partial_y; \\
(NF_2) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= a_0 \partial_x + x \partial_y; \\
(NF_3) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= a_0 \partial_x + \frac{1}{2} x^2 \partial_y;
\end{aligned}$$

which corresponds to an approximation to order -1. In the computation, when computing developments with respect to the parameter  $r_0 = \frac{1}{\lambda_y(0)}$ , that is for  $|\lambda_y(0)| \gg 1$ , we need the approximation to order 0 for  $(NF_{2a})$  and  $(NF_3)$ , and the approximation to order 1 for  $(NF_{2b})$

$$\begin{aligned}
(NF_{2a}) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= (a_0 + a_{10}x) \partial_x + x(1 + b_{20}x) \partial_y; \\
(NF_{2b}) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= (1 + a_{10}x + a_{01}y + a_{20}x^2) \partial_x + x(1 + b_{20}x + b_{30}x^2) \partial_y; \\
(NF_3) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= (a_0 + a_{10}x) \partial_x + \left( \frac{x^2}{2} + b_{01}y + b_{30}x^3 \right) \partial_y;
\end{aligned}$$

### 2.1.3 Parametrization of the initial conditions

In the case  $(NF_2)$ , in order to estimate the extremals with  $|\lambda_y(0)| \gg 1$ , we proceed to the following change of coordinates and time: we denote  $r = \frac{1}{\lambda_y}$ ,  $p = \frac{\lambda_x}{\lambda_y}$  and, if  $t$  denote the time, we denote  $s = t\lambda_y = \frac{t}{r}$ . Then, since  $\lambda_x(0) = \pm 1$  then  $p(0) = \pm r_0$  and the other initial parameter is  $r_0$  assumed to be close to 0.

In the case  $(NF_3)$ , in order to estimate the extremals that switch, that is with  $|\lambda_y(0)| \gg 1$ , we proceed to the following change of coordinates and time: we set  $r = \text{sign}(\lambda_y(0)) \frac{1}{\sqrt{|\lambda_y|}}$ , and set  $s = \frac{t}{r}$ .



These changes of coordinates and time are motivated by the behavior of the extremals in the nilpotent cases, the order chosen for  $s$  corresponding to what appears for cut time in the nilpotent case.

## 2.2 Describing the optimal synthesis

### 2.2.1 Symbols of extremals

As we will see in the following, the local optimal curves will be finite concatenations of bang arcs and  $u_i$ -singular arcs. In order to facilitate the presentation, a bang arc following  $\pm G_i$  will be symbolized by  $[[\pm G_i]]$ , a  $u_1$ -singular arc with control  $u_2 \equiv 1$  will be symbolized by  $[[S_1^+]]$ , a  $u_1$ -singular arc with control  $u_2 \equiv -1$  will be symbolized by  $[[S_1^-]]$ , and we will combined this symbols in such a way that  $[[ -G_1, G_2, S_2^+ ]]$  symbolizes the concatenation of a bang arc following  $-G_1$  with a bang arc following  $G_2$  and a  $u_2$ -singular arc with control  $u_1 \equiv 1$ .

### 2.2.2 Symmetries

We consider the effect of the change of role between  $\pm F_1$  and  $\pm F_2$ , and their effects on the values of the  $f_i$ 's and on the parameters. In particular, these changings imply changing  $G_1$  and  $G_2$  and hence changing the coordinates  $x$  and  $y$ .

Let consider an example :  $\bar{F}_1 = -F_1$  and  $\bar{F}_2 = F_2$ . The SF structure defined by  $(\bar{F}_1, \bar{F}_2)$  is exactly the same but the  $f_i$ 's and  $G_i$  are changed. One easily checks that

$$[\bar{F}_1, \bar{F}_2] = [-F_1, F_2] = -[F_1, F_2] = -(f_2 F_1 - f_1 F_2) = f_2 \bar{F}_1 - (-f_1) \bar{F}_2$$

hence that  $\bar{f}_1 = -f_1$ ,  $\bar{f}_2 = f_2$ ,  $\bar{G}_1 = -G_2$  and  $\bar{G}_2 = -G_1$ .

Let us consider changes that replace  $G_1$  by  $\pm G_1$  and  $G_2$  by  $\pm G_2$ . These changes are interesting from a calculus point of view. Effectively, if we computed the jet of a bang-bang extremals with symbol  $[[G_1, G_2]]$  and of its switching times, then we are able to get the other expressions for the bang-bang extremals with symbols  $[[\pm G_1, \pm G_2]]$ . This avoids repetition. For example, if one gets the expression of an extremal with symbol  $[[G_1, G_2]]$  as function of the initial conditions, one gets the expression of an extremal

with symbol  $[-G_1, G_2]$  by respecting the effect on the coordinates and the invariants  $a_0, a_{10}$ , etc. of the corresponding change of role of  $F_1$  and  $F_2$ .

More details are given in [6].

## 2.3 $(NF_1)$ case

At points  $q$  where  $(NF_1)$  holds, one can compute that

$$\begin{aligned} f_1(x, y) &= \frac{1}{2} ((a_{10} - b_{10}) + (2(a_{20} - b_{20}) - b_{10}(a_{10} - b_{10}))x + (a_{11} - b_{11})y \\ &\quad + (3(a_{30} - b_{30}) - b_{10}(a_{20} - b_{20}) - (2b_{20} - b_{10}^2)(a_{10} - b_{10}))x^2 \\ &\quad + (2(a_{21} - b_{21}) - b_{11}(a_{10} - b_{10}) - b_{10}(a_{11} - b_{11}))xy + (a_{12} - b_{12})y^2) + o_2(x, y), \\ f_2(x, y) &= -\frac{1}{2} ((a_{10} + b_{10}) + (2(a_{20} + b_{20}) - b_{10}(a_{10} + b_{10}))x + (a_{11} + b_{11})y \\ &\quad + (3(a_{30} + b_{30}) - b_{10}(a_{20} + b_{20}) - (2b_{20} - b_{10}^2)(a_{10} + b_{10}))x^2 \\ &\quad + (2(a_{21} + b_{21}) - b_{11}(a_{10} + b_{10}) - b_{10}(a_{11} + b_{11}))xy + (a_{12} + b_{12})y^2) + o_2(x, y). \end{aligned}$$

Hence, with respect to what have been proved upper, the only authorized switches are

$a_{10} - b_{10}$	$-a_{10} - b_{10}$	$u_1$ -switch	$u_2$ -switch
$> 0$	$> 0$	$-1 \rightarrow +1$	$-1 \rightarrow +1$
$> 0$	$< 0$	$-1 \rightarrow +1$	$+1 \rightarrow -1$
$< 0$	$> 0$	$+1 \rightarrow -1$	$-1 \rightarrow +1$
$< 0$	$< 0$	$+1 \rightarrow -1$	$+1 \rightarrow -1$

*Remark 22* (Generic invariants). Let remark that generically, in the  $(NF_1)$  case, the base point  $q$  is not in  $\Delta_A$  and one of the following situation occurs

- $|a_{10}| \neq |b_{10}|$ ,  $q \notin \Delta_1 \cup \Delta_2$ ,
- $a_{10} - b_{10} = 0$  and  $a_{10} + b_{10} \neq 0$ , that is  $q \in \Delta_1 \setminus \Delta_2$  and
  - $a_{20} - b_{20} \neq 0$  and  $a_{11} - b_{11} \neq 0$ , and  $G_1$  and  $G_2$  are not tangent to  $\Delta_1$  at  $q$ ,
  - $a_{20} - b_{20} = 0$  and  $a_{30} - b_{30} \neq 0$  and  $a_{11} - b_{11} \neq 0$ , and  $G_1$  is tangent to  $\Delta_1$  at  $q$ ,
  - $a_{20} - b_{20} \neq 0$  and  $a_{11} - b_{11} = 0$  and  $a_{12} - b_{12} \neq 0$ , and  $G_2$  is tangent to  $\Delta_1$  at  $q$ ,

- $a_{10} + b_{10} = 0$  and  $a_{10} - b_{10} \neq 0$ , that is  $q \in \Delta_2 \setminus \Delta_1$  and
  - $a_{20} + b_{20} \neq 0$  and  $a_{11} + b_{11} \neq 0$ , and  $G_1$  and  $G_2$  are not tangent to  $\Delta_2$  at  $q$ ,
  - $a_{20} + b_{20} = 0$  and  $a_{30} + b_{30} \neq 0$  and  $a_{11} + b_{11} \neq 0$ , and  $G_1$  is tangent to  $\Delta_2$  at  $q$ ,
  - $a_{20} + b_{20} \neq 0$  and  $a_{11} + b_{11} = 0$  and  $a_{12} + b_{12} \neq 0$ , and  $G_2$  is tangent to  $\Delta_1$  at  $q$ ,
- $a_{10} = b_{10} = 0$ ,  $|a_{20}| \neq |b_{20}|$  and  $|a_{11}| \neq |b_{11}|$ , and  $q \in \Delta_1 \cap \Delta_2$  and  $G_1$  and  $G_2$  are neither tangent to  $G_1$  nor to  $G_2$ .

### 2.3.1 Synthesis in the first quadrant

One first remark for constructing the optimal synthesis is the following. Any short extremal stays in one of the four quadrants delimited by the axis. This property is a consequence of the fact that only one control can switch in short time in this case. Then an extremal switching only on  $u_1$  and with control  $u_2$  equal to 1 enters the domain  $\{(x, y) | x \geq 0 \text{ and } y \leq 0\}$  and cannot leave it.

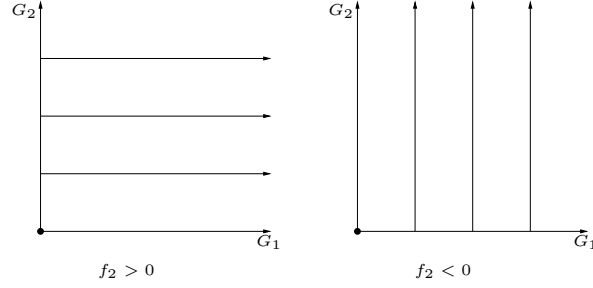
Hence we just have to understand the synthesis in the first quadrant and then obtain the rest of the synthesis by changing the roles of  $G_1$  and  $-G_1$ , and/or the roles of  $G_1$  and  $-G_2$  in order to get it in the other quadrants.

In the first quadrant, only  $u_2$  can switch. In the following cases

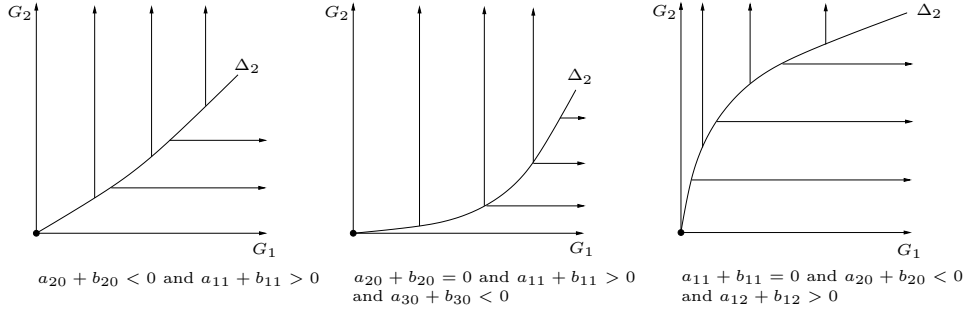
- $a_{10} + b_{10} \neq 0$  ( $q \notin \Delta_2$ ),
- $a_{10} + b_{10} = 0$  and  $(a_{20} + b_{20})(a_{11} + b_{11}) > 0$ ,
- $a_{10} + b_{10} = 0$  and  $(a_{20} + b_{20}) = 0$  and  $(a_{30} + b_{30})(a_{11} + b_{11}) > 0$ ,
- $a_{10} + b_{10} = 0$  and  $(a_{11} + b_{11}) = 0$  and  $(a_{12} + b_{12})(a_{20} + b_{20}) > 0$ ,

then  $\Delta_2$  do not enter the quadrant. Hence  $f_2$  has constant sign inside and hence, depending on the sign of  $f_2$  in the quadrant, one gets the switching rule and only one switch of  $u_2$  can occur and the possible symbols are  $[[G_1]]$ ,  $[[G_2]]$  and  $[[G_2, G_1]]$  if  $f_2 > 0$  and  $[[G_1]]$ ,  $[[G_2]]$  and  $[[G_1, G_2]]$  if  $f_2 < 0$ . The possible pictures are given in Figure 2.1.

In the other cases,  $\Delta_2$  enter the quadrant. Then, depending on the fact that  $G_1$  or  $G_2$  is or not tangent to  $\Delta_2$ , depending on the fact that  $\Delta_2$  is or

Figure 2.1: The syntheses when  $\Delta_2$  does not enter the quadrant

is not a turnpike, we get one of the following symbols. If  $\Delta_2$  is a turnpike then the possible symbols are  $[[G_1]]$ ,  $[[G_2]]$ ,  $[[S_2^+, G_1]]$ ,  $[[S_2^+, G_2]]$  and there is no local cut locus. If  $\Delta_2$  is not a turnpike then the possible symbols are  $[[G_1]]$ ,  $[[G_2]]$ ,  $[[G_2, G_1]]$ ,  $[[G_1, G_2]]$  and there is a local cut locus, which has the same equation as  $\Delta_2$ , up to the order that determines the form of  $\Delta_2$ . The synthesis are presented in the Figures 2.2 and 2.3.

Figure 2.2: The syntheses when  $\Delta_2$  is a turnpike

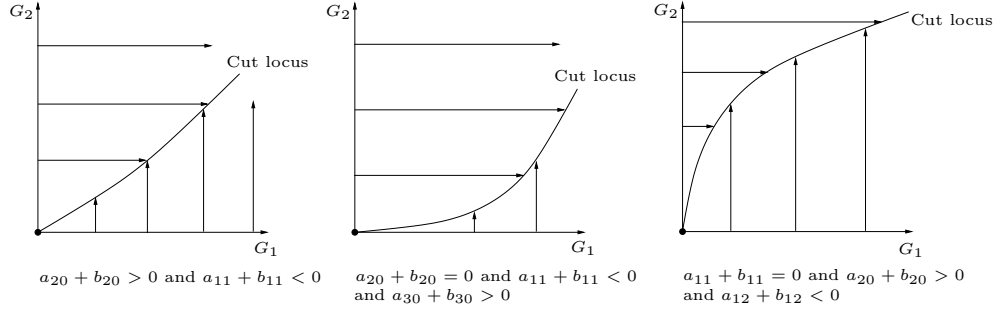
## 2.4 $(NF_{2a})$ case

Recall that the normal form  $(NF_{2a})$  gives

$$G_1(x, y) = \partial_x, \quad G_2(x, y) = (a_0 + a_{10}x + o_1(x, y))\partial_x + (x + b_{20}x^2 + o(x, y))\partial_y,$$

with  $0 \leq a_0 < 1$ .

Such a point is neither in  $\Delta_1$  nor  $\Delta_2$ . Hence no singular extremal will appear in the study of the local synthesis.

Figure 2.3: The syntheses when  $\Delta_2$  enters the quadrant but is not a turnpike

As explained before, if we want to study extremals that switch in short time, we need to consider  $\phi_3$  large that is  $|\lambda_y|$  large.

The weight of the coordinates, together with the computation made in the subriemannian context which are similar, motivates the change of coordinates  $r = \frac{1}{\lambda_y}$ ,  $p = r\lambda_x$  and the change of time  $s = t/r$ .

With the new variables  $(x, y, p, r)$  and the new time  $s$ , the Hamiltonian equations become

$$\begin{aligned}
 x' &= r \frac{\partial H}{\partial \lambda_x}(x, y, p, 1), \\
 y' &= r \frac{\partial H}{\partial \lambda_y}(x, y, p, 1), \\
 p' &= -r \frac{\partial H}{\partial x}(x, y, p, 1) + rp \frac{\partial H}{\partial y}(x, y, p, 1), \\
 r' &= r^2 \frac{\partial H}{\partial y}(x, y, p, 1).
 \end{aligned}$$

Now, looking for the solutions as taylor series in  $r_0$ , that is under the form

$$\begin{aligned}
 x(r_0, s) &= x_1(s)r_0 + x_2(s)r_0^2 + o(r_0^2), & p(r_0, s) &= p_1(s)r_0 + p_2(s)r_0^2 + o(r_0^2), \\
 y(r_0, s) &= y_2(s)r_0^2 + y_3(s)r_0^3 + o(r_0^3), & r(r_0, s) &= r_0 + r_2(s)r_0^2 + o(r_0^2),
 \end{aligned}$$

one finds the equations

$$\begin{aligned}
 x'_1 &= \frac{u_1+u_2}{2} + \frac{u_1-u_2}{2}a_0, & x'_2 &= \frac{u_1-u_2}{2}a_{10}x_1, \\
 y'_2 &= \frac{u_1-u_2}{4}x_1, & y'_3 &= \frac{u_1-u_2}{2}(b_{20}x_1^2 + x_2), \\
 p'_1 &= -\frac{u_1-u_2}{2}x_1, & p'_2 &= -\frac{u_1-u_2}{2}(a_{10}p_1 + 2b_{20}x_1), \\
 r'_2 &= 0,
 \end{aligned}$$

Using these equations, we are able to compute the jets with respect to  $r_0$  of the four types of extremals: depending on the sign of  $p(0) = \pm 1$  and of  $r_0$ .

For each of these types we can compute the functions  $x_1, x_2, y_2, y_3, p_1, p_2$  and  $r_2 \equiv 0$  of the variable  $s$  for the first bang. We can then compute the jets of  $\phi_1$  and  $\phi_2$  for the first bang and look for the first switching time under the form  $s_1 = s_{10} + s_{11}r_0$  and then repeat the procedure for the second bang and so on. Finally, if we denote  $\delta_p = p(0)$  and  $\delta_r = \text{sign}(r_0)$  then the controls during the first bang are  $u_1 = u_2 = \delta_p$ . The first time of switch is

$$s_1 = \delta_r(1 - \delta_r a_0) - \delta_p(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20})r_0 + o(r_0)$$

and corresponds to  $\phi_2(s_1) = 0$  if  $\delta_r = 1$  or  $\phi_1(s_1) = 0$  if  $\delta_r = -1$ . The second bang corresponds to  $u_1 = \delta_p \delta_r$  and  $u_2 = -\delta_p \delta_r$  and the second switch is at

$$s_2 = \delta_r(3 - \delta_r a_0) - \delta_p((1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20})r_0 + o(r_0)$$

where  $\phi_1(s_2) = 0$  if  $\delta_r = 1$  and  $\phi_2(s_2) = 0$  if  $\delta_r = -1$ . At this time

$$\begin{aligned} x(s_2) &= \delta_p(\delta_r + a_0)r_0 - \delta_r(\delta_r + a_0)(-\delta_r a_{10} + b_{20} + \delta_r a_0 b_{20})r_0^2 + o(r_0^2), \\ y(s_2) &= 2\delta_r r_0^2 - \delta_p \frac{4}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})r_0^3 + o(r_0^3). \end{aligned}$$

The third bang corresponds to  $u_1 = u_2 = -1$  if  $\delta_p = 1$  and to  $u_1 = u_2 = 1$  if  $\delta_p = -1$ . The third switching time satisfies  $s_3 = \delta_r(5 - \delta_r a_0) + O(r_0)$  and the corresponding time  $t_3$  is larger than the cut time as we will see.

Being a little careful, we are able to compute the cut locus. Details on the techniques of computations are given in [6].

The upper part of the cut locus ( $y > 0$ ) satisfies

$$x_{cut} = -\frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2), \quad y_{cut} = 2\rho^2,$$

when the lower part satisfies

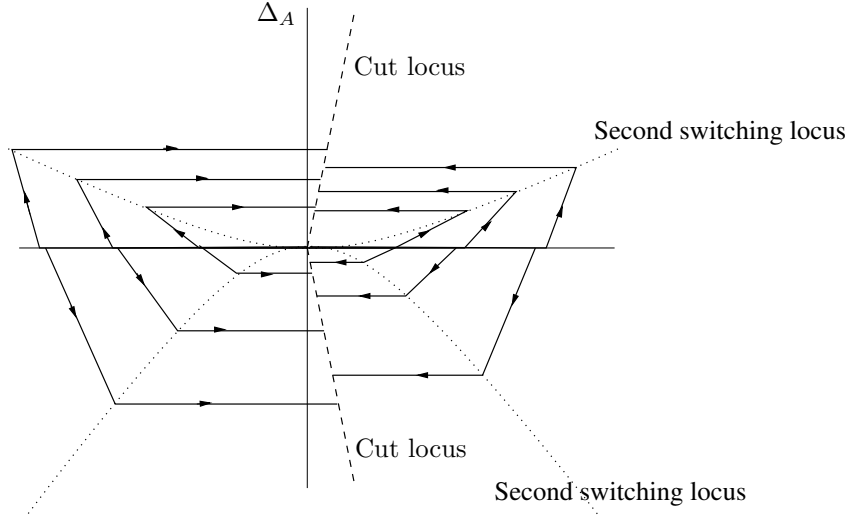
$$x_{cut} = -\frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2), \quad y_{cut} = -2\rho^2.$$

The synthesis is represented in the Figure 2.4

## 2.5 ( $NF_{2b}$ ) case

Recall that the normal form ( $NF_{2b}$ ) gives  $G_1(x, y) = \partial_x$ , and

$$G_2(x, y) = (1 + a_{10}x + a_{01}y + a_{20}x^2 + o_2(x, y))\partial_x + (x + b_{20}x^2 + b_{30}x^3 + o_3(x, y))\partial_y.$$

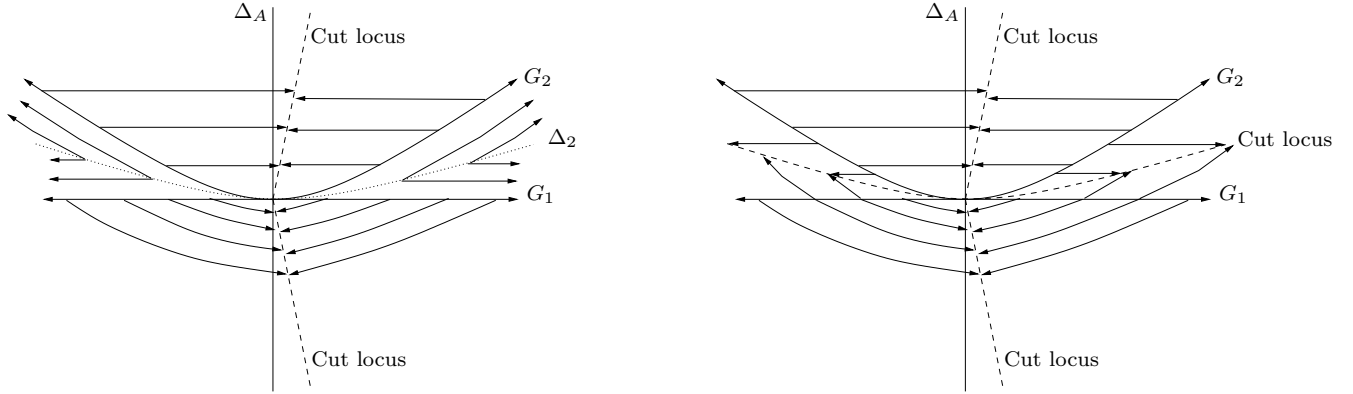
Figure 2.4: The optimal synthesis in the  $(NF_{2a})$  case

In this case, the extremals with initial condition  $|\lambda_y(0)| \gg 1$  are the limit when  $a_0$  goes to 1 of the extremal presented in the case  $(NF_{2a})$ . If  $\lambda_y(0) \gg 1$  then the symbol starts with  $[[G_2, -G_1]]$  or with  $[[ -G_2, G_1]]$  and if  $-\lambda_y(0) \gg 1$  then the symbol starts with  $[[G_1, -G_2]]$  or with  $[[ -G_1, G_2]]$ .

What is new in this limit case is that now an extremal can start by following  $\pm G_2$  and the extremal having the previous symbols do not enter the domain between the two integral curves of  $G_1$  and  $G_2$ , or between the integral curve of  $-G_1$  and  $-G_2$ . In [6], we put in evidence the invariants that determine the synthesis in these domains and we can have moreover the symbols

- $[[G_1, G_2]]$  and  $[[ -G_1, -G_2]]$ ,
- or  $[[G_2, G_1]]$  and  $[[ -G_2, -G_1]]$ ,
- or  $[[S_2^+, G_1]]$  and  $[[S_2^+, G_2]]$  and  $[[S_2^-, -G_1]]$  and  $[[S_2^-, -G_2]]$ ,
- or  $[[G_1, G_2]]$  and  $[[G_2, G_1]]$  and  $[[ -G_2, -G_1]]$  and  $[[ -G_1, -G_2]]$ , and there is a new branch of the cut locus.

The picture 2.5 illustrates the two last cases.

Figure 2.5: Two different syntheses in the  $(NF_{2b})$  case

## 2.6 $(NF_3)$ case

In this case we use the same techniques as in case  $(NF_{2a})$  and we find the final result for the cut locus

$$\begin{aligned}
 x_{cut} &= - \left( \frac{a_0}{3(1 + \sqrt{1 - a_0^2})} b_{01} + 4b_{30} \right) \rho^2 + o(\rho^2), \\
 y_{cut} &= - \frac{\sqrt{2}((1 + a_0)^{\frac{3}{2}} - (1 - a_0)^{\frac{3}{2}})}{3a_0} \rho^3.
 \end{aligned}$$

The cut locus appears to be a cusp whose tangent at the singular point is the tangent to  $\Delta_A$ , see Figure 2.6.

For more details see [6]



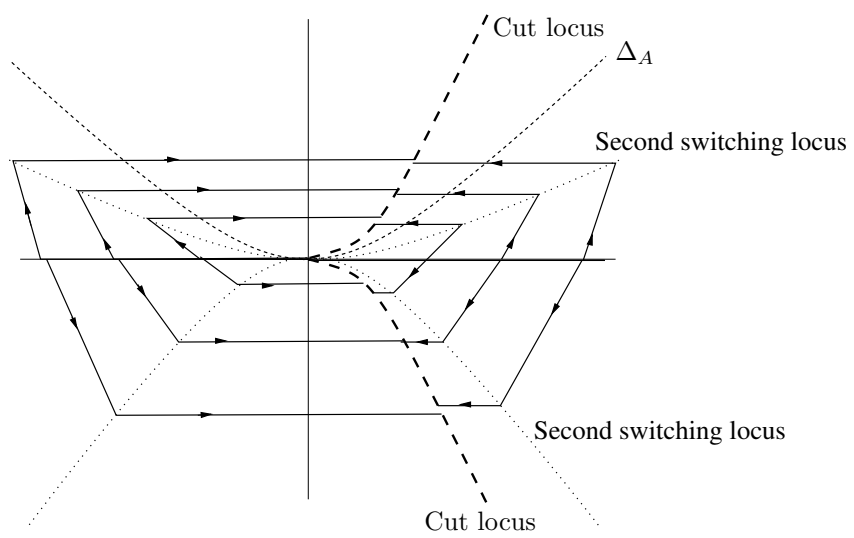


Figure 2.6: The synthesis in the  $(NF_3)$  case

# Chapter 3

## Optimal Synthesis in $\mathbf{R}^3$

In this section, we study the local synthesis of SF metrics for contact distribution in  $\mathbf{R}^3$ . As seen before, we say that a property is generic for this class of sub-Finslerian metrics if it is true on a residual set of such metrics for the  $C^\infty$ -Whitney topology. Genericity is usually proven using Thom transversality theorem. But, for this residual set of metrics, we are going to consider the local geometry only at points in the complementary of a set included in a finite union of codimension 1 submanifolds. For example, we consider only contact points and generically the set of points where the distribution is not contact is the Martinet surface which has codimension 1. We may also ask that an invariant appearing in the normal form is not null, which happens also outside a codimension 1 submanifold. All along our work we will assume only a finite number of such assumptions.

As in the  $(NF_{2b})$  case, two types of very different extremals should be studied to describe all the optimal synthesis: the ones corresponding to the unbounded part of the set of initial condition for which both controls can switch and do switch; other ones, satisfying  $\phi_3$  and  $\phi_i$  ( $i = 1$  or  $2$ ) close to 0, along which the control  $u_i$  may switch several times in short time, the other one being constant.

For the part of the synthesis corresponding to the non compact part, we use the techniques developed in the subriemannian context by Agrachev et al to compute jets of the extremals with respect to the parameter  $r_0$  introduced in the section 1.5.

For the part of the synthesis, we develop new techniques on switching functions in order to describe the extremals for which only one control switches several times.

### 3.1 Nilpotent case

As in sub-Riemannian geometry (see [7, 3]), the nilpotent approximation plays an important role as "good estimation" of the real situation. The nilpotent approximation at  $(0, 0, 0)$  of  $G_1, G_2$  given in the normal form is

$$\widehat{G}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{G}_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}$$

It is a left invariant sub-Finslerian metric defined on the Heisenberg group with the representation

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' + xy').$$

We define the Hamiltonian for the nilpotent case and thus we get the differential equations in the non singular case.

The singular extremals are those corresponding to  $\lambda_z \equiv 0$ . There are infinitely many. Effectively, any curve defined on the interval  $[0, 1]$  with  $u_1 \equiv 1$  and  $\int_0^1 u_2(t)dt = 0$  reaches the point  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{8})$  optimally.

By solving these Hamiltonian equations, we get the non singular extremals with  $\lambda_z(0) \neq 0$ . As in dimension 2 we are able to compute explicitly the consecutive bang arcs and switching times. For example, along the fifth bang of the extremals starting with speed  $G_2$ , the controls satisfy  $u_1 = 1$  and  $u_2 = -1$  and we have

$$\begin{aligned} x(s) &= 0, & p_x(s) &= 8r + \alpha_1 r - sr, & \phi_1(s) &= \frac{9+\alpha_1-s}{2}, \\ y(s) &= -8r + sr, & p_y(s) &= p_y(0) = r, & \phi_2(s) &= \frac{7+\alpha_1-s}{2}. \\ z(s) &= 4r^2, \end{aligned}$$

One shows that any extremal with  $\lambda_z(0) \neq 0$  is optimal until  $s = 8$  or  $t = 8r$  and is no more optimal after. The behavior of the extremals in the nilpotent case is illustrated in Figure 3.1.

Figure 3.2 shows the conjugate locus and three points of view of the part of the sphere that is reached by non singular extremals.

Now let us concentrate again on the extremals. One can consider the *exponential map* which to  $(r, \alpha, s)$  where  $\alpha \in [-1, 1]$ ,  $r > 0$ ,  $s \geq 0$  associates the end point of the geodesic with initial condition  $\lambda_x = \alpha$ ,  $\lambda_y = 1$  and  $\lambda_z = \frac{1}{r}$  for the time  $t = rs$ . This map is smooth at points with  $-1 < \alpha < 1$ ,

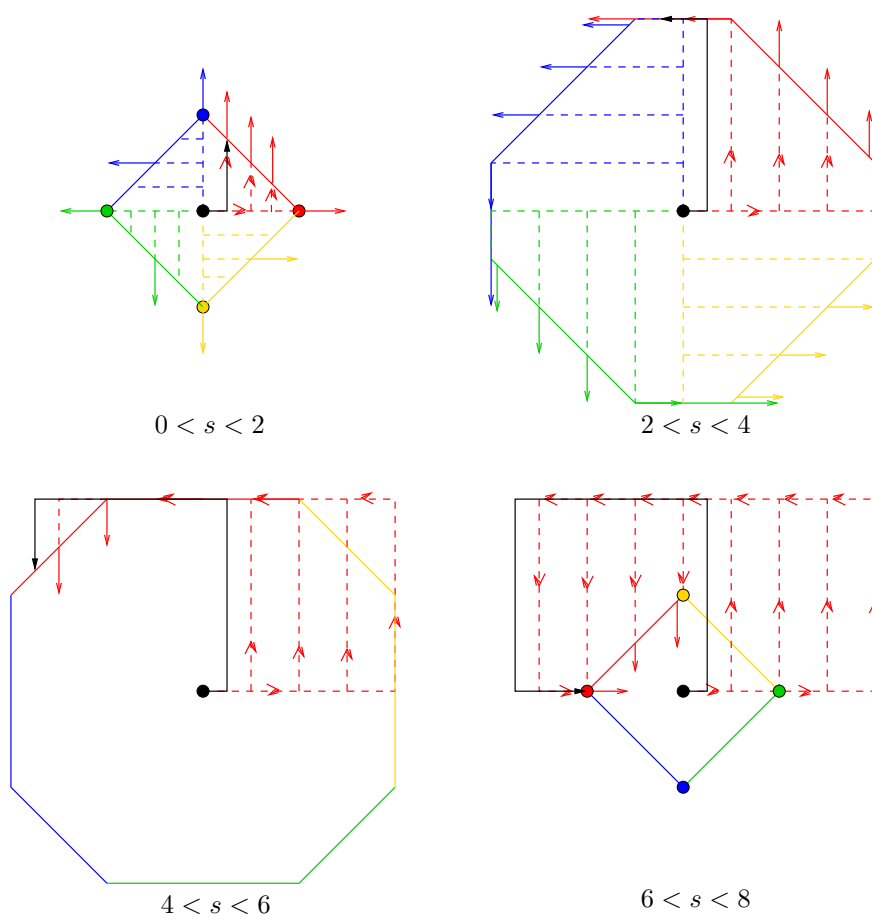


Figure 3.1: Evolution of the front at  $r$  fixed.

$s_i(p_x, r) < s < s_{i+1}(p_x, r)$  for a certain  $i$  where  $s_j(p_x, r)$  is the  $j^{th}$  switching time of the geodesic with initial condition  $p_x, p_y = 1$  and  $r$ . The same can be done for  $\lambda_y = -1$  or  $\lambda_x = \pm 1$  and  $\lambda_y \in [-1, 1]$ . Since it is smooth for  $-r < p_x < r$  and  $s \neq s_i \forall i$ , we can compute its jacobian. It happens that it is null during the two first bangs, and that it has opposite sign to that of  $r$  during the third and fourth bangs. It is again null during the fifth bang. As we will see later for  $r$  small in the generic cases, the jacobian will not be null during the third and fourth bangs also. In the nilpotent case, for  $t \in ]rs_4, rs_5[$ ,  $Jac(t) = 0$ . We define the conjugate time  $t$  as the infimum of the time  $\tau$  such that the jacobian takes positive and negative values before  $\tau$ .

Since in the nilpotent case, the conjugate time is  $t_5$ , the first conjugate locus is the set of points where a geodesic switches for the fifth time. The first conjugate locus is

$$\{(2\delta r, 0, \pm 4r^2) | r \in \mathbf{R}, \delta \in ]-1, 1[ \} \cup \{(0, 2\delta r, \pm 4r^2) | r \in \mathbf{R}, \delta \in ]-1, 1[ \}.$$

The Maxwell set, whose point are reached by several optimal extremal is exactly the same set.

Figure 3.2 shows the conjugate locus and three points of view of the part of the sphere that is reached by non singular extremals.

## 3.2 Extremals with $|\lambda_z| \gg 1$

### 3.2.1 Hamiltonian equations

The Hamiltonian dynamics is given by

$$\begin{aligned} \dot{x} &= \frac{u_1 + u_2}{2} + \frac{u_1 - u_2}{2}(a_{200}x^2 + a_{110}xy + \theta_x), \\ \dot{y} &= \frac{u_1 - u_2}{2}(1 + b_{200}x^2 + b_{110}xy + \theta_y), \\ \dot{z} &= \frac{u_1 - u_2}{2}(x + c_{200}x^2 + c_{110}xy + c_{300}x^3 + c_{210}x^2y + c_{120}y^2x + \theta_z), \\ \dot{\lambda}_x &= -\frac{u_1 - u_2}{2}(\lambda_x(2a_{200}x + a_{110}y) + \lambda_y(2b_{200}x + b_{110}y) \\ &\quad + \lambda_z(1 + 2c_{200}x + 3c_{300}x^2 + c_{110}y + 2c_{210}xy + c_{120}y^2)), \\ \dot{\lambda}_y &= -\frac{u_1 - u_2}{2}(a_{110}x\lambda_x + b_{110}x\lambda_y + \lambda_z(c_{110}x + c_{210}x^2 + 2c_{120}xy)), \\ \dot{\lambda}_z &= \frac{u_1 - u_2}{2}\lambda_zx(c_{201}x + c_{111}y), \end{aligned}$$

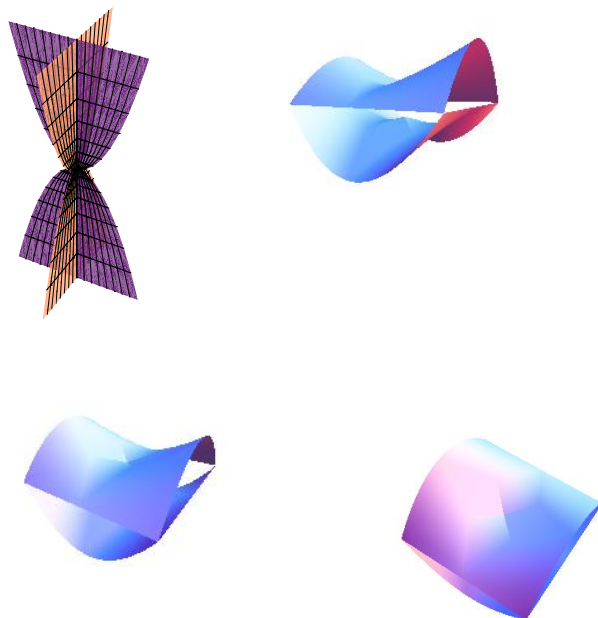


Figure 3.2: The conjugate locus and three points of view of the non singular part of the sphere in the nilpotent case

where

$$\begin{aligned} u_1(t) &= \text{sign}(\phi_1(t)), & u_2(t) &= \text{sign}(\phi_2(t)), \\ \phi_1(t) &= \lambda(t)F_1(q(t)), & \phi_2(t) &= \lambda(t)F_2(q(t)). \end{aligned}$$

From now  $\dot{x}$  denotes  $\frac{dx}{ds}$ . Using the change of coordinates  $p_x = \frac{\lambda_x}{\lambda_y}$ ,  $p_y = \frac{\lambda_y}{\lambda_y}$ ,  $r = \frac{1}{\lambda_y}$  and the change of time  $s = \frac{t}{r}$ , we get the new dynamical system

$$\begin{aligned}\dot{x} &= \frac{u_1 + u_2}{2}r + \frac{u_1 + u_2}{2}r(a_{200}x^2 + a_{110}xy + \theta_x), \\ \dot{y} &= \frac{u_1 - u_2}{2}r(1 + b_{200}x^2 + b_{110}xy + \theta_y), \\ \dot{z} &= \frac{1}{2}r(\theta_z(u_1 + u_2) + (u_1 - u_2)(x + c_{200}x^2 + c_{300}x^3 + c_{110}xy + c_{210}x^2y + c_{120}xy^2)), \\ \dot{p}_x &= -\frac{u_1 - u_2}{2}r(1 + 2c_{200}x + p_x(2a_{200}x + a_{110}y) + p_y(2b_{200}x + b_{110}y) + 3c_{300}x^2 \\ &\quad + c_{110}y + 2c_{210}xy + c_{120}y^2), \\ \dot{p}_y &= -\frac{u_1 - u_2}{2}r(c_{110}x + a_{110}p_x + b_{110}p_y + c_{210}x^2 + 2c_{120}xy), \\ \dot{r} &= \frac{u_1 - u_2}{2}r^2x(c_{201}x + c_{111}y).\end{aligned}$$

where, denoting  $p = (p_x, p_y, 1)$ ,

$$\begin{aligned}\phi_1(t) &= \frac{1}{r}pF_1(q(t)), & \phi_2(t) &= \frac{1}{r}pF_2(q(t)) \\ u_1(t) &= \text{sign}(\phi_1(t)), & u_2(t) &= \text{sign}(\phi_2(t)).\end{aligned}$$

Since the set of initial condition is a square for  $(p_x, p_y)$ , we define in fact four Hamiltonian flows for each initial speed  $(G_1, -G_1, G_2, -G_2)$ . For example, for the geodesics with initial speed equal to  $G_2$  we have  $p_y(0) = r$  and  $p_x = \alpha_1 r$  with  $\alpha_1 \in [-1, 1]$ . The new Hamiltonian flow as for variables  $(r_0, \alpha_1, s)$  where  $r_0 = r(0)$ ,  $p_x(0) = \alpha_1 r$  and  $s = \frac{t}{r}$ .

In order to compute jets of the Hamiltonian flow we write as Taylor series in  $r_0$  and we compute three orders of the smooth differential equations. These equations are integrable hence we can compute jets of switching functions and hence jets of switching times. Finally, we are able to compute the jets of the different bangs of the extremals. For more details we refer to [5]. If we restrict the computation to  $x, y, z$  as functions of  $(r_0, \alpha, s)$  for the four Hamiltonian flows, we get four exponential maps that we denote  $\text{Exp}_\beta$  where  $\beta = -1, 1, -2$  or  $2$  depending on if the initial velocity is  $-G_1, G_1, -G_2, G_2$ . In [20], M. Sigalotti proves, studying second order optimality conditions, that this family of extremals cannot be optimal after the fifth switch.

### 3.2.2 Conjugate locus

For these exponential maps, one can compute their jacobian for each bang arc. One finds

- $Jac(\text{Exp}_{\pm 2}) = 0$  for  $0 < s < s_2, s \neq s_1$ ,
- $Jac(\text{Exp}_{\pm 2}) = -8r_0^3 + o(r_0^3)$  for  $s_2 < s < s_3$ ,
- $Jac(\text{Exp}_{\pm 2}) = -8r_0^3 + o(r_0^3)$  for  $s_3 < s < s_4$ ,
- $Jac(\text{Exp}_{\pm 2}) = 32(2c_{120} - c_{110}^2)r_0^5 + o(r_0^5)$  for  $s_4 < s < s_5$ ,
- $Jac(\text{Exp}_{\pm 2}) = 8r_0^3 + o(r_0^3)$  for  $s_5 < s < s_6$ ,

and

- $Jac(\text{Exp}_{\pm 1}) = 0$  if  $0 < s < s_1$  or  $s_1 < s < s_2$ ,
- $Jac(\text{Exp}_{\pm 1}) = -4r_0^3 + o(r_0^3)$  if  $s_2 < s < s_3$ ,
- $Jac(\text{Exp}_{\pm 1}) = -8r_0^3 + o(r_0^3)$  if  $s_3 < s < s_4$ ,
- $Jac(\text{Exp}_{\pm 1}) = 64(3c_{300} - 2b_{200} - 2c_{200}^2)r_0^5 + o(r_0^5)$  if  $s_4 < s < s_5$ ,
- $Jac(\text{Exp}_{\pm 1}) = 8r_0^3 + o(r_0^3)$  if  $s_5 < s < s_6$ .

We can now state the following proposition introducing two important invariants that will be determinant in the form of the cut locus.

**Proposition 23.** *Let  $G_1$  and  $G_2$  as in the normal form given in section 2.*

- *If  $C_1 = 3c_{300} - 2b_{200} - 2c_{200}^2 > 0$  then the fourth switching time  $t_4$  is the first conjugate time for geodesic with initial velocity  $\pm G_1$ . If  $C_1 < 0$  then it is the fifth  $t_5$ .*
- *If  $C_2 = 2c_{120} - c_{110}^2 > 0$  then the fourth switching time  $t_4$  is the first conjugate time for geodesics with initial velocity  $\pm G_2$ . If  $C_2 < 0$  then it is the fifth  $t_5$ .*

Using the expansion of the exponential maps, we can give the expressions of the upper part of the first conjugate locus for the four exponential maps.

For  $\text{Exp}_{\pm 1}$ , if  $C_1 > 0$

$$\begin{aligned}
 x_{conj} &= \pm(\alpha_2 - 1)r_0 + (4c_{110} - c_{200}(\alpha_2 - 1)^2)r_0^2 + o(r_0^2), \\
 y_{conj} &= -8c_{200}r_0^2 \pm 4(b_{110} + 6c_{110}c_{200} - 2c_{210} \\
 &\quad + (4b_{200} + 12c_{200}^2 - 6c_{300})\alpha_2)r_0^3 + o(r_0^3), \\
 z_{conj} &= 4r_0^2 \mp 8(c_{110} + 2c_{200}\alpha_2)r_0^3 + o(r_0^3),
 \end{aligned}$$



and if  $C_1 < 0$

$$\begin{aligned} x_{conj} &= \pm(1 + \alpha_2)r_0 + (4c_{110} - c_{200}(1 + \alpha_2)^2)r_0^2 + o(r_0^2), \\ y_{conj} &= -8c_{200}r_0^2 \pm 4(b_{110} + 6c_{110}c_{200} - 2c_{210} \\ &\quad + (4b_{200} + 12c_{200}^2 - 6c_{300})\alpha_2)r_0^3 + o(r_0^3), \\ z_{conj} &= 4r_0^2 \mp 8(c_{110} + 2c_{200}\alpha_2)r_0^3 + o(r_0^3), \end{aligned}$$

and for  $Exp_{\pm 2}$ , if  $C_2 > 0$

$$\begin{aligned} x_{conj} &= 4c_{110}r_0^2 \pm 4(b_{110} + 6c_{110}c_{200} - 2c_{210} \\ &\quad + \alpha_1(2c_{120} - 3c_{110}^2))r_0^3 + o(r_0^3), \\ y_{conj} &= \pm(-1 + \alpha_1)r_0 - \frac{1}{2}(16c_{200} + c_{110}(\alpha_1 - 1)^2)r_0^2 + o(r_0^2), \\ z_{conj} &= 4r_0^2 \pm 4(4c_{200} - c_{110}(1 + \alpha_1))r_0^3 + o(r_0^3), \end{aligned}$$

and if  $C_2 < 0$

$$\begin{aligned} x_{conj} &= 4c_{110}r_0^2 \pm 4(b_{110} + 6c_{110}c_{200} - 2c_{210} \\ &\quad + \alpha_1(2c_{120} - 3c_{110}^2))r_0^3 + o(r_0^3), \\ y_{conj} &= \pm(1 + \alpha_1)r_0 - \frac{1}{2}(16c_{200} + c_{110}(1 + \alpha_1)^2)r_0^2 + o(r_0^2), \\ z_{conj} &= 4r_0^2 \pm 4(4c_{200} + c_{110}(1 - \alpha_1))r_0^3 + o(r_0^3). \end{aligned}$$

From the nilpotent case we have that the loss of optimality may come during the fourth bang or the fifth bang.

### 3.2.3 Cut locus

The idea to compute the cut locus associated to extremals with  $\lambda_z \gg 1$  is to look close to the parameters (initial conditions plus time) of the Maxwell set for the nilpotent approximation. Then to compute an suspension at  $z4\rho^2$  for  $\rho$  small of the exponential maps (fourth and fifth bangs of the four exponential maps). And finally to identify the self intersections in the for cases

- $C_1 > 0$  and  $C_2 > 0$ ,
- $C_1 < 0$  and  $C_2 > 0$ ,
- $C_1 > 0$  and  $C_2 < 0$ ,

- $C_1 < 0$  and  $C_2 < 0$ .

The second and third cases being equivalent, we present only one.

**Cut when  $C_1 > 0$  and  $C_2 > 0$ :** in this case, the extremals lose optimality before the fifth bang. The self intersections of the front is between fourth bang fronts of the exponential maps. For example, the fourth bang front of  $\exp_1$  intersects the fourth bang front of  $\exp_2$  for  $\alpha_2$  between  $-1$  and  $1 + o(\rho^2)$  and  $\alpha_1$  between  $1$  and  $1 + o(\rho^2)$ . This is illustrated in Figure 3.3.

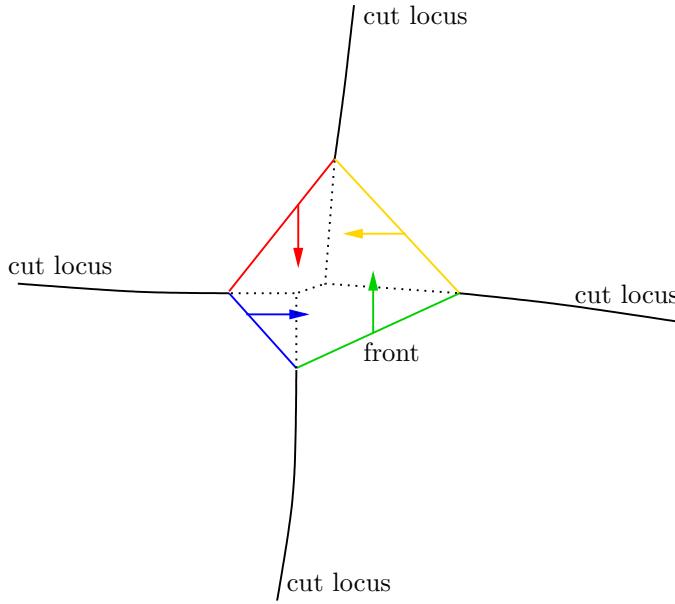


Figure 3.3: Closure of the cut locus at  $z$  fixed.

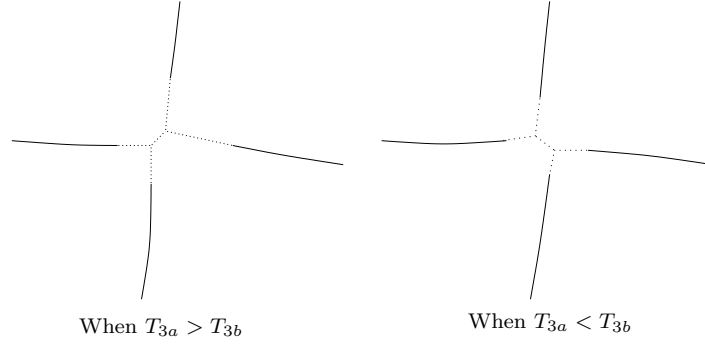
The optimal synthesis at  $z = 4\rho^2$  closes at  $t \equiv 8\rho$ . Let define two times  $t_a = 8\rho + T_{3a}\rho^3$  and  $t_b = 8\rho + T_{3b}\rho^3$  where

$$T_{3a} = \frac{4}{3}(a_{110} - 3b_{110} + 6b_{200} + 3c_{110}^2 - 4c_{120} + 6c_{110}c_{200} + 12c_{200}^2 + 6c_{210} - 12c_{300})$$

and

$$T_{3b} = \frac{4}{3}(a_{110} + 3b_{110} + 6b_{200} + 3c_{110}^2 - 4c_{120} + 18c_{110}c_{200} + 12c_{200}^2 - 6c_{210} - 12c_{300}).$$

Then, if  $T_{3a} < T_{3b}$  then the optimal synthesis closes by the intersection of the front of  $\exp_2$  and of  $\exp_{-2}$ , when if  $T_{3b} < T_{3a}$  then the optimal synthesis closes by the intersection of the front of  $\exp_1$  and of  $\exp_{-1}$ . See Figure 3.4.

Figure 3.4: Closure of the cut locus at  $z$  fixed

Finally we can present the upper part of the cut locus when  $C_1 > 0$  and  $C_2 > 0$  in Figure 3.5

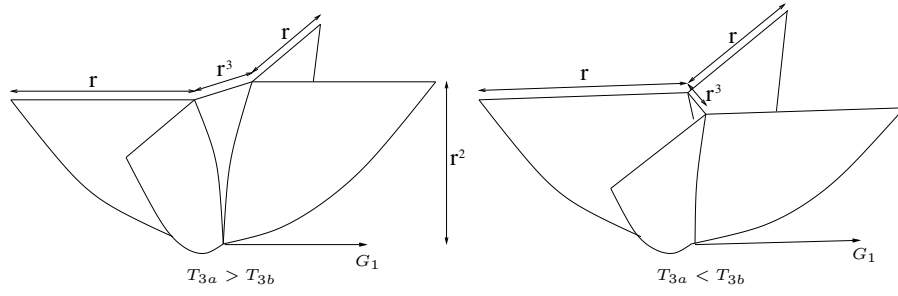
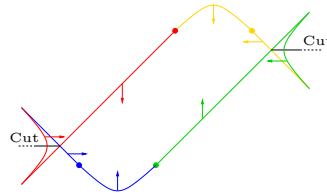


Figure 3.5: The upper part of the cut locus

**Cut when  $C_1 > 0$  and  $C_2 < 0$ :** in this case, the intersections of  $\exp_{\pm 1}$  and  $\exp_{\pm 2}$  still happen before the fifth bang when the intersections of  $\exp_{\pm 1}$  and  $\exp_{\mp 2}$  do not happen before the fifth bang. See Figure 3.6.

Figure 3.6: The front before  $t = 8\rho$  when  $C_1 > 0$  and  $C_2 < 0$

Different type of sequences of intersection can occur, depending on the invariants but in any cases, the intersection of the cut locus with  $z = 4\rho^2$  as only one branch which is continuous and smooth by parts. See Figure 3.7

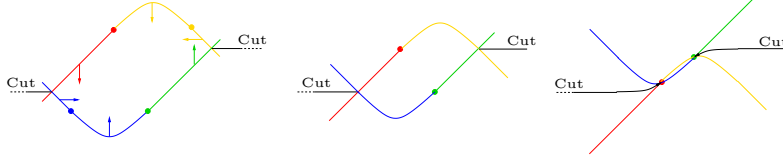


Figure 3.7: Picture of the front at times with  $T_2 = 0$  and  $T_3 < T_{3c}$ ,  $T_3 = T_{3c}$  and  $T_3 = T_{3g}$

And finally we can give a picture of the complete upper part of the cut locus

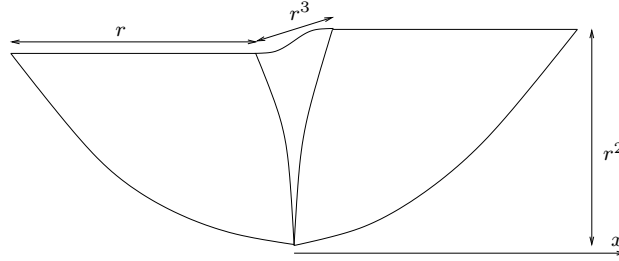


Figure 3.8: Picture of the cut locus when  $C_1 > 0$  and  $C_2 < 0$

**Cut when  $C_1 < 0$  and  $C_2 < 0$ :** finally, in this last case, the first intersection is always between two fifth bang fronts. Again different sequences of self intersections may occur giving rise, here, to two different geometries for the cut locus: one or five branches. For the computations we refer to the [5].

The picture of the evolution of the front when 5 branches appear is given in Figure 3.9.

When it gives rise to only one branch, we present it in Figure 3.10.

Finally we can give the picture of the cut locus in this two cases in Figure 3.11.

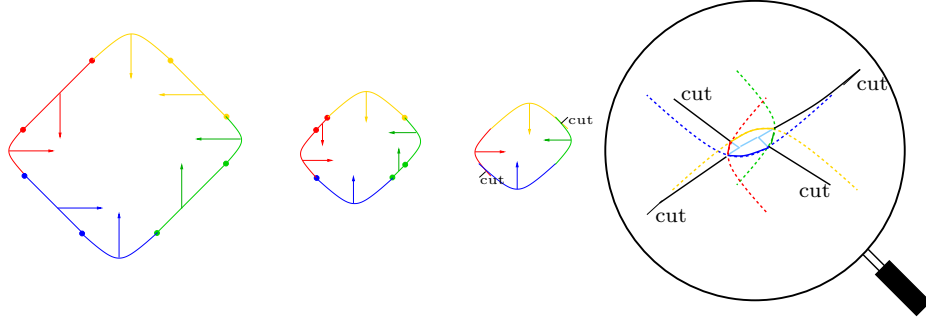


Figure 3.9: Evolution of the front when giving rise to five branches

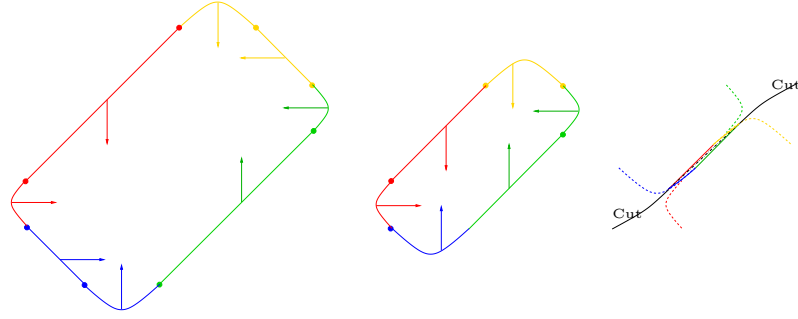


Figure 3.10: Evolution of the front when giving rise to only one branch

### 3.3 Extremals with only one control switching several times

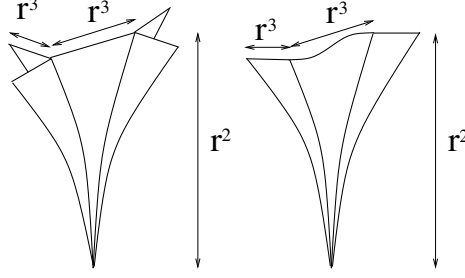
For  $|\lambda_z|$  large enough the dynamics is described in the previous sections. We can now choose a constant  $\Lambda_z > 0$  large enough and assume that the extremal we consider verify  $|\lambda_z| < \Lambda_z$ . As seen before, along an extremal

$$\dot{\phi}_3 = u_1(f_{41}\phi_1 + f_{42}\phi_2 + f_{43}\phi_3) + u_2(f_{51}\phi_1 + f_{52}\phi_2 + f_{53}\phi_3),$$

and, with  $|\phi_1| \leq 1$  and  $|\phi_2| \leq 1$ , we get

$$|\dot{\phi}_3| \leq |f_{41}| + |f_{42}| + |f_{51}| + |f_{52}| + (|f_{53}| + |f_{43}|)|\phi_3| \leq 4M' + 2M'\Lambda_z$$

where  $M'$  is a local bound of the  $f_{ij}$ . This implies that, for the extremals we are considering, the possibility of switching in short time implies that the corresponding switching function starts close to 0. Which implies that in short time only one control switches. And if in short time a control switches twice hence  $\phi_3$  should change sign and hence starts close to 0 that is  $\lambda_z$  should starts close to 0.

Figure 3.11: Possible cut loci when  $C_1 < 0$  and  $C_2 < 0$ 

In the following, we will be interested only in finding extremals that switch at least twice (on the same control) since the ones that switch only once are yet obtained with initial conditions with large  $|\lambda_z|$ .

We will consider only extremals with  $u_1 \equiv 1$ , the study of the other ones being equivalent. Along such an extremal

$$\ddot{\phi}_2 = u_1 \dot{\phi}_3 = \dot{\phi}_3$$

and since  $u_1 \equiv 1$  one gets

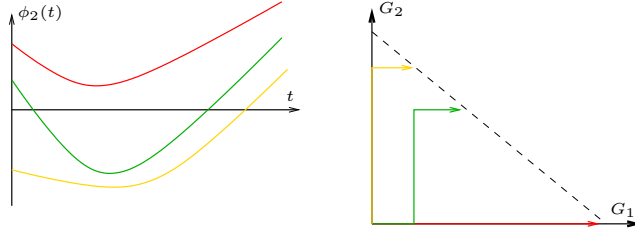
$$\ddot{\phi}_2 = (f_{41} + u_2 f_{51})\phi_1 + (f_{42} + u_2 f_{52})\phi_2 + (f_{43} + u_2 f_{53})\phi_3.$$

Since  $\phi_3(t) = O(t)$ ,  $\phi_2 = O(t)$  and  $\phi_1(t) = 1 + O(t)$  we get that

$$\ddot{\phi}_2(t) = (f_{41} + u_2 f_{51}) + O(t).$$

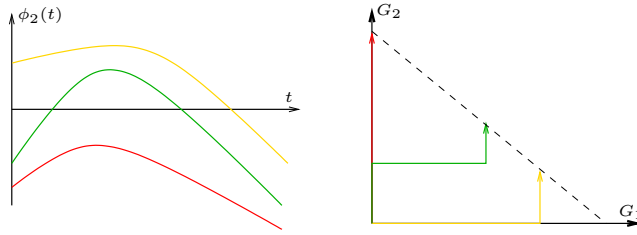
In the following we assume that we are considering a point where  $f_{41} + f_{51} \neq 0$  and  $f_{41} - f_{51} \neq 0$ . We consider then the four following cases

1. If  $|f_{51}| < f_{41}$  then  $f_{41} + u_2 f_{51} > 0$  for all  $u_2 \in [0, 1]$  and  $\ddot{\phi}_2(t) > 0$  for all  $t$ . As a consequence the only possible behaviors of the control  $u_2$  are (see Figure 3.12)
  - (a)  $u_2 \equiv 1$ ,
  - (b)  $u_2 = -1$  during a first interval of time and switches to 1,
  - (c)  $u_2 = 1$  during a first interval of time, then  $-1$  during a second one, and finally switches to 1.

Figure 3.12: Extremals when  $|f_{51}| < f_{41}$ 

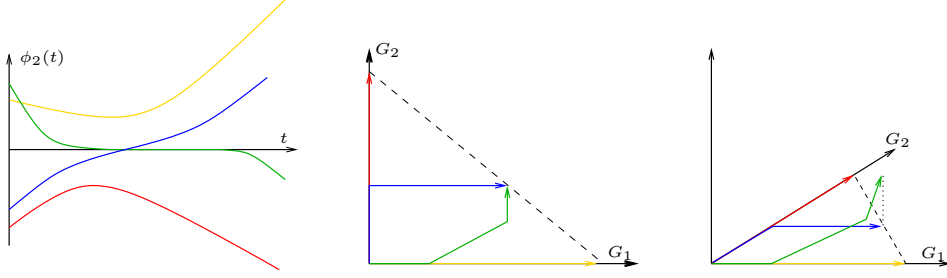
2. If  $|f_{51}| < -f_{41}$  then  $f_{41} + u_2 f_{51} < 0$  for all  $u_2 \in [0, 1]$  and  $\ddot{\phi}_2(t) < 0$  for all  $t$ . As a consequence the only possible behaviors of the control  $u_2$  are (see Figure 3.13)

- (a)  $u_2 \equiv -1$ ,
- (b)  $u_2 = 1$  during a first interval of time and switches to  $-1$ ,
- (c)  $u_2 = -1$  during a first interval of time, then  $1$  during a second one, and finally switches to  $-1$ .

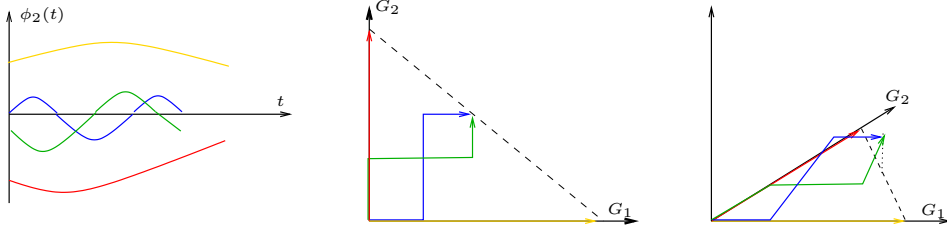
Figure 3.13: Extremals when  $|f_{51}| < -f_{41}$ 

3. If  $|f_{41}| < f_{51}$  then  $f_{41} + f_{51} > 0$  hence  $\ddot{\phi}_2(t) > 0$  when  $\phi_2(t) > 0$  and  $f_{41} - f_{51} < 0$  hence  $\ddot{\phi}_2(t) < 0$  when  $\phi_2(t) < 0$ . In that case the possible behaviors of the control  $u_2$  are (see Figure 3.14)

- (a)  $u_2$  is constant and equal to  $\pm 1$ ,
- (b)  $u_2$  is equal to  $1$  or  $-1$  during a first interval of time and switches to  $-1$  or  $1$ ,
- (c)  $u_2$  is equal to  $1$  or  $-1$  during a first interval of time, then  $\phi_2 = 0$  during a second interval and  $u_2(t) = -\frac{f_{41}(q(t))}{f_{51}(q(t))} + O(t)$ , and finally  $u_2$  switches to  $1$  or  $-1$ .


 Figure 3.14: Extremals when  $|f_{41}| < f_{51}$ 

4. If  $|f_{41}| < -f_{51}$  then  $f_{41} + f_{51} < 0$  hence  $\ddot{\phi}_2(t) < 0$  when  $\phi_2(t) > 0$  and  $f_{41} - f_{51} > 0$  hence  $\ddot{\phi}_2(t) > 0$  when  $\phi_2(t) < 0$ . In that case the list of possible behaviors may be very large. In the following we analyze more deeply to prove that (see Figure 3.15)
- (a)  $u_2$  is constant and equal to  $\pm 1$ ,
  - (b)  $u_2$  is constant and equal to  $\pm 1$  during a first interval of time and switches to  $\mp 1$ ,
  - (c)  $u_2$  is constant and equal to  $\pm 1$  during a first interval of time and switches to  $\mp 1$ , and finally switches a gain to  $\pm 1$ .


 Figure 3.15: Extremals when  $|f_{41}| < -f_{51}$ 

Apriori, the analysis proposed here is not sufficient to ensure that no extremal with more than 2 switches on the same control can be optimal. We refer to the article [5] where it is proven that if a short extremal has four bangs on the same control it cannot be optimal.

The case  $|f_{41}| < -f_{51}$  is the only one where appears a cut locus.

The switching law of the other extremals with at least two switches on the same control, with  $u_1 \equiv -1$  or  $u_2 \equiv 1$  or  $u_2 \equiv -1$  can be obtain by working



on the effect of the symmetries exchanging the roles of  $G_1$  and  $-G_1$  on one part and  $G_2$  and  $-G_2$  on the other part.

### 3.3.1 Cut locus generated by extremals with $\lambda_z(0) \sim 0$

As a consequence of the previous computations, we can describe the part of the local cut locus generated by the extremals with  $\lambda_0(0) \sim 0$ .

- if  $(a_{200} + b_{200} < 0$  or  $a_{110} + b_{110} > 0)$  and  $(b_{110} - a_{110} < 0$  or  $b_{200} - a_{200} < 0)$  then this part of the local cut locus is empty.
- if  $a_{200} + b_{200} > 0$  and  $a_{110} + b_{110} < 0$  and  $(b_{110} - a_{110} < 0$  or  $b_{200} - a_{200} < 0)$  then this part of the cut locus writes

$$\{(x, -2\frac{a_{200} + b_{200}}{a_{110} + b_{110}}x + o(x), z) \mid 0 \leq z \leq -2\frac{a_{200} + b_{200}}{a_{110} + b_{110}}x^2 + o(x^2)\}$$

- if  $(a_{200} + b_{200} < 0$  or  $a_{110} + b_{110} > 0)$  and  $b_{110} - a_{110} > 0$  and  $b_{200} - a_{200} > 0$  then this part of the cut locus writes

$$\{(x, -2\frac{a_{200} - b_{200}}{a_{110} - b_{110}}x + o(x), z) \mid 0 \geq z \geq -2\frac{a_{200} - b_{200}}{a_{110} - b_{110}}x^2 + o(x^2)\}$$

- if  $a_{200} + b_{200} > 0$  and  $a_{110} + b_{110} < 0$  and  $b_{110} - a_{110} > 0$  and  $b_{200} - a_{200} > 0$  then this part of the local cut locus is the union of the two previous sets.

Finally we can propose the picture of this part of the cut locus in Figure 3.16

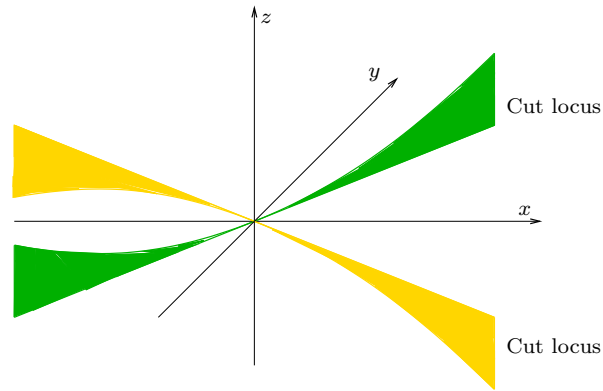


Figure 3.16: Part of the cut locus generated by the extremal with  $\lambda_z(0) \sim 0$  when  $|f_{41}| < -f_{51}$  and  $|f_{52}| < f_{42}$



# Bibliography

- [1] A. Agrachev, D. Barilari, and U. Boscain. *Introduction to Riemannian and sub-Riemannian geometry (Lecture Notes)*. [http://people.sissa.it/agrachev/agrachev\\_files/notes.html](http://people.sissa.it/agrachev/agrachev_files/notes.html).
- [2] A. Agrachev, B. Bonnard, M. Chyba, and I. Kupka. Sub-Riemannian sphere in Martinet flat case. *ESAIM Control Optim. Calc. Var.*, 2:377–448, 1997.
- [3] A. A. Agrachev, El-H. Chakir El-A., and J. P. Gauthier. Sub-Riemannian metrics on  $\mathbf{R}^3$ . In *Geometric control and non-holonomic mechanics (Mexico City, 1996)*, volume 25 of *CMS Conf. Proc.*, pages 29–78. Amer. Math. Soc., Providence, RI, 1998.
- [4] Andrei Agrachev and Jean-Paul Gauthier. On the subanalyticity of Carnot-Caratheodory distances. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(3):359–382, 2001.
- [5] E. Ali and G. Charlot. Local contact sub-finslerian geometry for maximum norms in dimension 3. *preprint*.
- [6] E. Ali and G. Charlot. Local (sub) finslerian geometry for the maximum norm in dimension 2. *preprint*.
- [7] André Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 1–78. Birkhäuser, Basel, 1996.
- [8] G. Ben Arous. Développement asymptotique du noyau de la chaleur hypoelliptique hors du cut-locus. *Ann. Sci. École Norm. Sup. (4)*, 21(3):307–331, 1988.
- [9] G. Ben Arous and R. Léandre. Décroissance exponentielle du noyau de la chaleur sur la diagonale. II. *Probab. Theory Related Fields*, 90(3):377–402, 1991.
- [10] B. Bonnard, M. Chyba, and E. Trelat. Sub-riemannian geometry, one-parameter deformation of the martinet flat case. *J. Dyn. Control Syst.* 4, No.1, 59-76 (1998)., 4:59–76, 1998.

- [11] Bernard Bonnard and Monique Chyba. Méthodes géométriques et analytiques pour étudier l'application exponentielle, la sphère et le front d'onde en géométrie sous-riemannienne dans le cas Martinet. *ESAIM Control Optim. Calc. Var.*, 4:245–334 (electronic), 1999.
- [12] Ugo Boscain, Thomas Chambrion, and Grégoire Charlot. Nonisotropic 3-level quantum systems: complete solutions for minimum time and minimum energy. *Discrete Contin. Dyn. Syst. Ser. B*, 5(4):957–990, 2005.
- [13] E. Breuillard and E. Le Donne. On the rate of convergence to the asymptotic cone for nilpotent groups and subfinsler geometry. *Proc. Natl. Acad. Sci. USA*, 110(48):19220–19226, 2013.
- [14] Grégoire Charlot. Quasi-contact s-r metrics : normal form in  $\mathbb{R}^{2n}$ , wave front and caustic in  $\mathbb{R}^4$ . *Acta App. Math.*, 74:217–263, 2002.
- [15] El-Houcine Chakir El Alaoui, J.-P. Gauthier, and I. Kupka. Small sub-riemannian balls on  $\mathbb{R}^3$ . *J. Dyn. Control Syst.*, 2(3):359–421, 1996.
- [16] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.
- [17] Rémi Léandre. Majoration en temps petit de la densité d'une diffusion dégénérée. *Probab. Theory Related Fields*, 74(2):289–294, 1987.
- [18] Rémi Léandre. Minoration en temps petit de la densité d'une diffusion dégénérée. *J. Funct. Anal.*, 74(2):399–414, 1987.
- [19] S. A. Molčanov. Diffusion processes, and Riemannian geometry. *Uspehi Mat. Nauk*, 30(1(181)):3–59, 1975.
- [20] M. Sigalotti. Some computations for 2nd variations in sub-finsler geometry. *preprint*.

# Local (sub) Finslerian geometry for the maximum norms in dimension 2\*

Entisar A.-L. Ali♣ and G. Charlot♣

♣ *Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France*  
*Dyala University, Irak*

entisar.ali@univ-grenoble-alpes.fr

♣ *Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France*  
gregoire.charlot@univ-grenoble-alpes.fr

February 14, 2017

## Abstract

We consider specific sub-Finslerian structures in the neighborhood of 0 in  $\mathbb{R}^2$ , defined by fixing a family of vector fields  $(F_1, F_2)$  and considering the norm defined on the non constant rank distribution  $\Delta = \text{vect}\{F_1, F_2\}$  by

$$|G| = \inf_u \{\max\{|u_1|, |u_2|\} \mid G = u_1 F_1 + u_2 F_2\}.$$

If  $F_1$  and  $F_2$  are not proportionnal at  $p$  then we obtain a Finslerian structure; if not, the structure is sub-Finslerian on a distribution with non constant rank. We are interested in the study of the local geometry of these Finslerian and sub-Finslerian structures: generic properties, normal form, short geodesics, cut locus, switching locus and small spheres.

## 1 Introduction

From the 80's, the interest for the sub-Riemannian geometry increases with a lot of contributions in several domains as PDEs, analysis, probability, geometry and control. One of the question was to understand the local geometry of sub-riemannian metrics, as the singularities of small spheres, local cut locus, local conjugate locus and so on, motivated in particular by new results on the heat kernel in the sub-Riemannian context, see [10, 22, 23, 9]. The contact and the Martinet cases were deeply studied (see [1, 12, 11, 19, 2]). The quasi-contact case in dimension 4 also (see [15]). These results allowed to give new results on the asymptotics of the heat kernel at cut and conjugate loci in the 3D contact and 4D quasi-contact cases ([7, 6]).

In this article, we start the same work for Finslerian and sub-Finslerian metrics associated with a maximum norm: let consider a manifold  $M$ , a vector bundle  $\pi : E \rightarrow M$  with fibers of same dimension as  $M$  endowed with a maximum norm, and a morphism of vector bundles  $f : E \rightarrow TM$  such that the map from  $\Gamma(E) \rightarrow \text{Vec}(M)$  defined by  $\sigma \mapsto f \circ \sigma$  is injective. An admissible curve is a curve  $\gamma$  in  $M$  such that exists a lift  $\sigma$  in  $E$  with  $\dot{\gamma}(t) = f(\sigma(t))$  a.e. The length of such a curve is the infimum of the  $\int_0^T |\sigma(t)| dt$  for all possible such  $\sigma$  and the distance between two points  $q_0$  and

---

\*This research has been supported by ANR-15-CE40-0018.

$q_1$  is the infimum of the lengths of the curves joining  $q_0$  and  $q_1$ . Remark that the map  $f$  itself is not assumed to be injective everywhere: at points where  $f$  is injective the structure is Finslerian when at points where it is not it is sub-Finslerian.

Here we concentrate our attention on the local study of such structures in dimension 2, that is when  $M$  and the fibers of  $E$  have dimension 2.

Equivalently, with a control point of view and since we are interested in local properties, we consider control systems in a neighborhood of 0 in  $\mathbb{R}^2$  of the type

$$\dot{q} = u_1 F_1(q) + u_2 F_2(q) \quad (1)$$

where  $F_1$  and  $F_2$  are smooth vector fields and  $u_1$  and  $u_2$  are control functions satisfying

$$|u_1| \leq 1 \text{ and } |u_2| \leq 1. \quad (2)$$

Up to reparameterization, minimizing the distance in the geometric context is equivalent to minimizing the time of transfer in the control context.

We are interested in the study of the time optimal synthesis of such systems. Of course, the general situation cannot be completely described since singular cases may have very special behaviour. For example, in the case  $F_1 = \partial_x$  and  $F_2 = \partial_y$ , any admissible trajectory with  $u_1 \equiv 1$  and  $\int_0^1 u_2(t)dt = 0$  joins optimally  $(0,0)$  to  $(1,0)$ . Hence in the following, we will consider only "generic" situations as defined in section 2.1.

Few works exist concerning sub-Finsler geometry since it is a new subject. Let mention the works [17, 18] for dimension 3, considering norms which are assumed to be smooth outside the zero section. In [14], the sphere of a left invariant sub-Finsler structure associated to a maximum norm in the Heisenberg group is described. In the preprint [5], the authors describe the extremals (and discuss in particular their number of switches before the loss of optimality) for the Heisenberg, Grushin and Martinet distributions. In the preprint [4], we describe, in the 3D generic contact case, the small sphere and the local cut locus.

The paper is organised as follows.

In section 2 we recall Thom's transversality theorem and some of its corollaries, define what we mean by generic, give generic properties of the couples of vector fields on 2 dimensional manifolds and give a normal form for the generic couples.

In section 3, we give first general results about the optimal synthesis; recalling classical results as Chow-Rashevski, Filippov and Pontryagin theorems; analysing the possibilities for extremals to switch or to be singular depending on their initial condition; giving details on the weights of coordinates in the normal form and on the associated nilpotent approximation.

In section 4, we present the local synthesis in all the generic cases presented in the normal form of section 2.

## 2 Normal form

In this section, the goal is to give a list of properties of generic couples  $(F_1, F_2)$  and to construct a normal form for the couple  $(G_1, G_2)$  defined by  $G_1 = F_1 + F_2$  and  $G_2 = F_1 - F_2$ . As we will see,  $\pm G_1$  and  $\pm G_2$  are the velocities of a large class of the minimizers of the optimal control system defined by (1) and (2).

In order to do that we use the Thom's transversality theorem and some of its corollaries.

## 2.1 Generic properties of couples of smooth vector fields on 2d-manifolds

### 2.1.1 Thom's transversality theorem

Denote  $J^k(M, N)$  the set of  $k$ -jets of maps from  $M$  to  $N$ .

**Theorem 1** (Thom Transversality Theorem, [21], Page 82). *Let  $M, N$  be smooth manifolds and  $k \geq 1$  an integer. If  $S_1, \dots, S_r$  are smooth submanifolds of  $J^k(M, N)$  then the set*

$$\{f \in C^\infty(M, N) : J^k f \pitchfork S_i \text{ for } i = 1, 2, \dots, r\},$$

*is residual in the  $C^\infty$ -Whitney topology.*

**Corollary 2.** *Assume that  $\text{codim } S_i > \dim M$  for  $i = 1, \dots, r$  and  $k \geq 1$ . Then the set*

$$\{f \in C^\infty(M, N) : J^k f(M) \cap S_i = \emptyset \text{ for } i = 1, \dots, r\},$$

*is residual in the  $C^\infty$ -Whitney topology.*

**Corollary 3.** *For every  $f$  in the residual set defined in Theorem 1, the inverse images  $\tilde{S}_i := (J^k f)^{-1}(S_i)$  is a smooth submanifold of  $M$  and  $\text{codim } S_i = \text{codim } \tilde{S}_i$  for  $i = 1, \dots, r$ .*

*Remark 4.* Let  $\varphi$  be a diffeomorphism of  $M$  and  $\phi$  be a diffeomorphism of  $N$ . The map

$$\sigma_{\varphi, \phi} : \begin{cases} C^\infty(M, N) & \longrightarrow & C^\infty(M, N) \\ f & \longmapsto & \varphi \circ f \circ \phi \end{cases}$$

induces a diffeomorphism  $\sigma_{\varphi, \phi}^*$  of  $J^k(M, N)$  which sends submanifolds of  $J^k(M, N)$  on submanifolds of  $J^k(M, N)$ . Moreover,  $f$  is in the residual set defined in theorem 1, if and only if  $\sigma_{\varphi, \phi}(f)$  is in the residual set

$$\{g \in C^\infty(M, N) : J^k g \pitchfork \sigma_{\varphi, \phi}^*(S_i) \text{ for } i = 1, \dots, r\}.$$

This remark is important to facilitate the presentation of the proofs of the generic properties given in the next section.

**Definition 5.** In the following, we will say that a property of maps is generic if it is true on a residual set defined as in Thom's theorem.

### 2.1.2 First generic properties

We want to give a list of generic properties for couples of vector fields on 2d-manifolds.

In order to use Thom transversality theorem, we work locally in coordinates. Locally one can consider a couple of vector fields as the data of a map

$$g : \begin{cases} U \subset \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \times \mathbb{R}^2 \\ (x, y) & \mapsto & ((g_1(x, y), g_2(x, y)), (g_3(x, y), g_4(x, y))) \end{cases}$$

and the  $k$ -jet at  $q = (0, 0) \in U$  of  $g$  as the data of the map

$$J^k g : \begin{cases} \mathbb{R}^2 & \rightarrow & \mathbb{R}_k[x, y]^4 \\ (x, y) & \mapsto & (P_1(x, y), \dots, P_4(x, y)) \end{cases}$$



where  $P_i$  ( $1 \leq i \leq 4$ ) is the Taylor series of order  $k$  of  $g_i$  at  $q$ .

In order to describe submanifolds of  $\mathbb{R}_k[x, y]^4$  in coordinates, we write:

$$P_1(x, y) = \sum_{i=0}^k \sum_{j=0}^{k-i} p_{1,i,j} x^i y^j, \dots, P_4(x, y) = \sum_{i=0}^k \sum_{j=0}^{k-i} p_{4,i,j} x^i y^j.$$

In the following  $(g_1, g_2)$  are the coordinates of  $G_1$  and  $(g_3, g_4)$  the coordinates of  $G_2$  in a local coordinate system.

*Generic property 1 (GP1): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , the set of points where  $G_1 = G_2 = 0$  is empty.*

Indeed in coordinates such points correspond to jets with  $p_{1,0,0} = p_{2,0,0} = p_{3,0,0} = p_{4,0,0} = 0$  which form a submanifold of  $\mathbb{R}_k[x, y]^4$  of codimension 4. Hence, thanks to corollary 2, the property is proven.

Let call  $J_N^k$  the set of  $k$ -jets such that  $P_1 \equiv 1$  and  $P_2 \equiv 0$ . Once assumed that we choose a coordinate system such that  $G_1 = (1, 0)$  then  $J^k g$  is in  $J_N^k$ .

Assume that a set  $S$  of  $J^k(\mathbb{R}^2, \mathbb{R}^4)$  is defined has the zero level of a finite number of functions  $h_i$ ,  $i = 1 \dots k$ , which differentials form a free family when restricted to  $TJ_N^k$ . Then locally the differentials of the functions  $h_i$  form a free family and hence, close to  $J_N^k \cap S$ , the set  $S$  is locally a submanifold. In this context, the codimension of  $S$  in  $J^k(\mathbb{R}^2, \mathbb{R}^4)$  is equal to the codimension of  $S' = S \cap J_N^k$  in  $J_N^k$ .

Thanks to remark 4, up to a permutation between  $\pm F_1$  and  $\pm F_2$  and a good choice of coordinates, we will assume in all the following that  $G_1 \equiv (1, 0)$  locally that is  $g_1 \equiv 1$  and  $g_2 \equiv 0$ . It corresponds to jets in  $J_N^k$ . As a consequence, if a set  $S$  is defined by a finite number of functions  $h_i$ ,  $i = 1 \dots k$ , which differentials form a free family when restricted to  $TJ_N^k$ , then to apply Thom's theorem and its corollaries we are reduced to apply them to the map

$$g : \begin{cases} U \subset \mathbb{R}^2 & \rightarrow \mathbb{R}^2 \\ (x, y) & \mapsto (g_3(x, y), g_4(x, y)) \end{cases}$$

and the set  $S' = S \cap J_N^k$  seen as a submanifold of  $J^k(\mathbb{R}^2, \mathbb{R}^2)$ .

*Generic property 2 (GP2): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , the set of points where  $G_2 = 0$  is a discrete set. The same holds for the set where  $F_1 = 0$  or the set where  $F_2 = 0$ .*

Indeed such points correspond to jets with  $p_{3,0,0} = p_{4,0,0} = 0$  which is a submanifold of  $\mathbb{R}_k[x, y]^2$  of codimension 2. Hence, thanks to corollary 3, the set where  $G_2 = 0$  is generically a submanifold of  $M$  of codimension 2 that is a discrete set. For  $F_2 = 0$  the equations are  $p_{3,0,0} = 1$  and  $p_{4,0,0} = 0$  and for  $F_1 = 0$  the equations are  $p_{3,0,0} = -1$  and  $p_{4,0,0} = 0$ .

*Generic property 3 (GP3): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , the set  $\Delta_A$  of points where  $G_2$  is parallel to  $G_1$  is an imbedded submanifold of codimension 1.*

Indeed  $\Delta_A$  is exactly the set of points where  $g_4 = 0$ , corresponding to jets with  $p_{4,0,0} = 0$ . This last set is an imbedded submanifold of  $\mathbb{R}_k[x, y]^2$  of codimension 1. Thanks to (GP1) and to corollary 3, we can conclude that generically  $\Delta_A$  is an imbedded submanifold of codimension 1.

*Generic property 4 (GP4): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , the set  $\Delta_1$  of points where  $F_1$  is parallel to  $[F_1, F_2]$  is an imbedded submanifold of codimension 1. The same holds for  $\Delta_2$  where  $F_2$  is parallel to  $[F_1, F_2]$ .*

In order to prove (GP4), compute  $[F_1, F_2]$  and describe  $\Delta_1$  in coordinates.  $[F_1, F_2] = -\frac{1}{2}[G_1, G_2]$  hence has coordinates  $-\frac{1}{2}p_{3,1,0}$  and  $-\frac{1}{2}p_{4,1,0}$  and  $F_1$  has coordinates  $\frac{1}{2}(1 + p_{3,0,0})$  and  $\frac{1}{2}p_{4,0,0}$ . Hence  $\Delta_1$  corresponds to jets satisfying

$$\begin{vmatrix} -\frac{1}{2}p_{3,1,0} & \frac{1}{2}(1 + p_{3,0,0}) \\ -\frac{1}{2}p_{4,1,0} & \frac{1}{2}p_{4,0,0} \end{vmatrix} = 0.$$

The differential of this determinant is not degenerate hence the set of  $\mathbb{R}_k[x, y]^2$  satisfying this equality is an imbedded submanifold of codimension 1. Hence, generically,  $\Delta_1$  is the preimage of an immersed submanifold of codimension 1 which, thanks to corollary 3, permits to conclude that  $\Delta_1$  is an immersed submanifold of codimension 1.

*Generic property 5 (GP5): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , the sets  $(\Delta_A \cap \Delta_1)$ ,  $(\Delta_A \cap \Delta_2)$  and  $(\Delta_1 \cap \Delta_2)$  are discrete.*

Since  $G_1 = (1, 0)$ , the set  $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$  is the set of points where  $(F_1, F_2)$  is free and  $[F_1, F_2] = 0$  that is

$$\begin{aligned} p_{4,0,0} &\neq 0, \\ p_{3,1,0} &= 0 \\ p_{4,1,0} &= 0. \end{aligned}$$

This set is an immersed submanifold of codimension 2 of  $\mathbb{R}_k[x, y]^2$  hence, thanks to corollary 3, the set  $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$  is generically a discrete set.

The set  $(\Delta_A \cap \Delta_2) \setminus \Delta_1$  is a set of points where  $F_2 = 0$ . By (GP2) it is a discrete set. The same holds for  $(\Delta_A \cap \Delta_1) \setminus \Delta_2$  which is a set of points where  $F_1 = 0$ .

The set  $\Delta_A \cap \Delta_1 \cap \Delta_2$  is the union of the subset where  $F_1 \neq 0$  and  $F_1 \parallel F_2 \parallel [F_1, F_2]$  with a subset where  $F_1 = 0$ . The second is discrete. The first set is also defined by  $G_1 \parallel G_2 \parallel [G_1, G_2]$  that is  $p_{4,0,0} = 0$  and  $p_{4,1,0} = 0$ . Hence, thanks to corollary 3, the set where  $F_1 \neq 0$  and  $F_1 \parallel F_2 \parallel [F_1, F_2]$  is a submanifold of codimension 2 that is a discrete set.

*Generic property 6 (GP6): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , the set of points where  $G_1 \parallel G_2 \parallel [G_1, G_2] \parallel [G_1, [G_1, G_2]]$  is empty.*

The set where  $G_1 \parallel G_2 \parallel [G_1, G_2] \parallel [G_1, G_2]$  is such that  $p_{4,0,0} = p_{4,1,0} = p_{4,2,0} = 0$ . Hence, thanks to corollary 3, it is a submanifold of codimension 3 that is an empty set.

*Generic property 7 (GP7): for generic couples of vector fields  $(F_1, F_2)$  on  $M$ , at the points  $q$  where  $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$  one gets  $G_1(q) \in T_q \Delta_A$ .*

The property  $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$  implies that  $p_{4,0,0} = p_{4,1,0} = 0$ . If  $p_{4,0,1} \neq 0$  then  $\Delta_A$  can be written  $p_{4,0,1}y = o(x)$  that is  $\Delta_A$  is tangent to the  $x$  axis and  $G_1 \in T_q \Delta_A$ . Hence the set of points where  $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$  and  $G_1(q) \notin T_q \Delta_A$  corresponds to jets with  $p_{4,0,0} = p_{4,1,0} = p_{4,0,1} = 0$  which is a submanifold of codimension 3. Hence generically, at the points  $q$  where  $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$ , one has  $G_1(q) \in T_q \Delta_A$ .

One can even detail more the generic properties: using Thom transversality theorem and its corollaries, we can prove that generically

*Generic property 8 (GP8): along  $\Delta_1 \setminus (\Delta_2 \cup \Delta_A)$ , the points where  $G_1$  or  $G_2$  is tangent to  $\Delta_1$  are isolated. The same holds true for  $\Delta_2 \setminus (\Delta_1 \cup \Delta_A)$ .*

*Generic property 9 (GP9): at points of  $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$ , neither  $G_1$  nor  $G_2$  are tangent to  $\Delta_1$  or  $\Delta_2$ .*

*Generic property 10 (GP10): along  $\Delta_A \setminus (\Delta_1 \cup \Delta_2)$ , the set of points where  $G_2 = 0$  or  $G_2 = \pm G_1$  is discrete.*

## 2.2 Normal form

Thanks to the generic properties established in the previous section, we can prove :

**Theorem 6** (Normal form). *For generic couples of vector fields  $(F_1, F_2)$  on a 2d manifold  $M$ , up to an exchange between  $\pm F_1$  and  $\pm F_2$ , at each point  $q$  of the manifold  $G_1 \neq 0$  and it exists a unique coordinate system  $(x, y)$  centred at  $q$  such that one of the following normal form holds:*

$$(NF_1) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= \partial_y + x(a_{10} + a_{20}x + a_{11}y + o(x, y))\partial_x + x(b_{10} + b_{20}x + b_{11}y + o(x, y))\partial_y, \\ \text{and } q &\notin \Delta_A. \end{aligned}$$

$$(NF_2) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= (a_0 + a_{10}x + a_{01}y + o(x, y))\partial_x + x(1 + x(b_{20} + O(x, y)))\partial_y, \\ \text{with } 0 \leq a_0 &\leq 1, \text{ and } q \in \Delta_A \setminus \Delta_1. \end{aligned}$$

$$(NF_3) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= (a_0 + o(1))\partial_x + (b_{01}y + \frac{1}{2}x^2 + b_{11}xy + b_{02}y^2 + o(x^2, y^2))\partial_y, \\ \text{with } b_{01} > 0 \text{ and } 0 < a_0 < 1, & q \in \Delta_A \cap \Delta_1 \cap \Delta_2 \text{ and } G_1(q) \in T_q \Delta_A. \end{aligned}$$

For  $(NF1)$  and  $(NF2)$  one of the following subcases holds:

- $(NF_{1a})$   $(NF_1)$  holds with  $a_{10} - b_{10} \neq 0$  and  $a_{10} + b_{10} \neq 0$ . It corresponds to  $q \notin \Delta_A \cup \Delta_1 \cup \Delta_2$ .
- $(NF_{1b})$   $(NF_1)$  holds with  $a_{10} - b_{10} = 0$  and  $a_{10} + b_{10} \neq 0$ . It corresponds to  $q \in \Delta_1 \setminus (\Delta_A \cup \Delta_2)$ .
- $(NF_{1c})$   $(NF_1)$  holds with  $a_{10} - b_{10} \neq 0$  and  $a_{10} + b_{10} = 0$ . It corresponds to  $q \in \Delta_2 \setminus (\Delta_A \cup \Delta_1)$ .
- $(NF_{1d})$   $(NF_1)$  holds with  $a_{10} = b_{10} = 0$ . It corresponds to  $q \in (\Delta_1 \cap \Delta_2) \setminus \Delta_A$ .
- $(NF_{2a})$   $(NF_2)$  holds with  $0 \leq a_0 < 1$ . It corresponds to  $q \in \Delta_A \setminus (\Delta_1 \cup \Delta_2)$ .
- $(NF_{2b})$   $(NF_2)$  holds with  $a_0 = 1$ . It corresponds to  $q \in (\Delta_A \cap \Delta_2) \setminus \Delta_1$  that is to  $q \in \Delta_A \setminus \Delta_1$  such that  $F_2(q) = 0$ .

Such coordinate system is called the normal coordinate system associated with  $F_1$  and  $F_2$ .

**Proof.** We assume that all the generic properties given before are satisfied. Thanks to  $(GP1)$ , and thanks to the fact that we are working locally, we can assume that  $G_1$  is not zero.

Thanks to  $(GP3)$ , we know that  $\Delta_A$  is a submanifold of dimension 1. Let start by considering a point  $q$  outside  $\Delta_A$ . Let define the map  $\varphi$  which to  $(x, y)$  in a neighborhood  $U$  of  $(0, 0)$  in  $\mathbb{R}^2$  associates the point reached by starting at  $q$  and following  $G_2$  during time  $y$  and then  $G_1$  during time  $x$  that is

$$\varphi : \begin{cases} U & \rightarrow M \\ (x, y) & \mapsto e^{xG_1}e^{yG_2}q \end{cases}$$

Since  $\partial_x \varphi(0,0) = G_1(q)$  and  $\partial_y \varphi(0,0) = G_2(q)$ ,  $\varphi$  is a local diffeomorphism hence defines a local coordinate system. One proves easily that at each point of coordinates  $(x,y)$  the vector  $G_1(x,y) = (1,0)$ . Moreover, along the  $y$  axis, since  $\varphi(0,y) = e^{yG_2}q$  then  $G_2(0,y) = (0,1)$ . This implies the normal form  $(NF_1)$ . With the normal form  $(NF_1)$ , one gets that

$$\begin{aligned} [F_1, F_2](0) &= -\frac{1}{2}[G_1, G_2](0) = -\frac{1}{2}(a_{10}, b_{10}), \\ F_1(0) &= \frac{1}{2}(G_1(0) + G_2(0)) = \left(\frac{1}{2}, \frac{1}{2}\right), \\ F_2(0) &= \frac{1}{2}(G_1(0) - G_2(0)) = \left(\frac{1}{2}, -\frac{1}{2}\right) \end{aligned}$$

which implies that

$$[F_1, F_2](0) = -\frac{a_{10} + b_{10}}{2}F_1(0) - \frac{a_{10} - b_{10}}{2}F_2(0).$$

The subcases follow immediately.

Assume now that  $q \in \Delta_A \setminus \Delta_1$ . Hence  $G_1(q)$  and  $G_2(q)$  are parallel and since we assume that  $G_1(q)$  is not 0, we can assume up to a change of role that  $G_2(q) = \alpha G_1(q)$  with  $\alpha \in [0, 1]$ . Since  $q \notin \Delta_1$ ,  $G_1(q)$  and  $[G_1, G_2](q)$  are not parallel. This implies that  $G_1$  is not tangent to  $\Delta_A$ . As a consequence, one can choose a local parameterization  $\gamma(t)$  of  $\Delta_A$  such that  $\gamma(0) = q$  and  $\dot{\gamma}(t)$  has second coordinate 1 in the basis  $(G_1(\gamma(t)), [G_1, G_2](\gamma(t)))$ . We can now define the map  $\varphi$  which to  $(x, y)$  in a neighborhood  $U$  of  $(0,0)$  in  $\mathbb{R}^2$  associates the point reached by starting at  $\gamma(y)$  and following  $G_1$  during time  $x$  that is

$$\varphi : \begin{cases} U & \rightarrow M \\ (x, y) & \mapsto e^{xG_1}\gamma(y) \end{cases}$$

In this coordinate system,  $\Delta_A$  is the  $y$  axis,  $G_1(x, y) = (1, 0)$  and the second coordinate of  $G_2$  is null at  $x = 0$  hence it is the product of the function  $(x \mapsto x)$  with a smooth function  $g$ . Moreover, thanks to the property of  $\gamma$ ,  $g(0, y) = 1$  which implies that  $g(x, y) = 1 + xh(x, y)$  with  $h$  a smooth function. This is exactly  $(NF_2)$ . If  $0 \leq a_0 < 1$  then  $F_1(q)$  and  $F_2(q)$  are not null and since they are parallel but not parallel to  $[F_1, F_2](q)$  then  $q \in \Delta_A \setminus (\Delta_1 \cup \Delta_2)$ . If  $a_0 = 1$  then  $F_2(q) = 0$  and  $q \in (\Delta_A \cap \Delta_2) \setminus \Delta_1$ .

The case where  $q \in (\Delta_A \cap \Delta_1) \setminus \Delta_2$  can be treated by exchanging the roles of  $G_1$  and  $G_2$  since in this case  $G_2(q) \neq 0$ .

Let assume finally that  $q \in \Delta_A \cap \Delta_1 \cap \Delta_2$ . Thanks to  $(GP6)$  and  $(GP7)$  at such a point  $G_1$  and  $[G_1, [G_1, G_2]]$  are not parallel. Hence we can define the map  $\varphi$  which to  $(x, y)$  in a neighborhood  $U$  of  $(0,0)$  in  $\mathbb{R}^2$  associates the point reached by starting at  $q$  and following  $[G_1, [G_1, G_2]]$  during time  $y$  and then  $G_1$  during time  $x$  that is

$$\varphi : \begin{cases} U & \rightarrow M \\ (x, y) & \mapsto e^{xG_1}e^{y[G_1, [G_1, G_2]]}q \end{cases}$$

The fact that  $G_2$  and  $[G_1, G_2]$  are parallel to  $G_1$  implies  $b_0 = 0$  and  $b_{10} = 0$ . The fact that, along the  $y$  axis,  $[G_1, [G_1, G_2]] = (0, 1)$  implies in particular that  $b_{20} = \frac{1}{2}$  which finishes the proof.

### 3 General facts about the computation of the optimal synthesis

#### 3.1 Local controllability and existence of minimizers

In the three cases of the normal form  $(NF_1)$ ,  $(NF_2)$  and  $(NF_3)$  one checks that

$$\text{span}(F_1, F_2, [F_1, F_2], [F_1, [F_1, F_2]], [F_2, [F_1, F_2]]) = \mathbb{R}^2.$$

Hence, as a consequence of Chow-Rashevski theorem (see [3, 25, 16]), generically such a control system is locally controllable that is locally, for any two points, always exists an admissible curve joining the two points.

Moreover, since at each point the set of admissible velocities is convex and compact, thanks to Filippov theorem (see [3, 20]), locally for any two points, always exists at least a minimizer.

#### 3.2 Pontryagin Maximum Principle (PMP)

The Pontryagin Maximum Principle (PMP for short, see [3, 24]) gives necessary conditions for a curve to be a minimizer of a control problem. For our problem it takes the following form.

**Theorem 7 (PMP).** *Let define the Hamiltonian*

$$H(q, \lambda, u, \lambda_0) = u_1 \lambda \cdot F_1(q) + u_2 \lambda \cdot F_2(q) + \lambda_0$$

where  $q \in \mathbb{R}^2$ ,  $\lambda \in T^*\mathbb{R}^2$ ,  $u \in \mathbb{R}^2$  and  $\lambda_0 \in \mathbb{R}$ . For any minimizer  $(q(t), u(t))$ , there exist a never vanishing Lipschitz covector  $\lambda : t \mapsto \lambda(t) \in T_{q(t)}^*\mathbb{R}^2$  and a constant  $\lambda_0 \leq 0$  such that

- $\dot{q}(t) = \frac{\partial H}{\partial \lambda}(q(t), \lambda(t), u(t), \lambda_0),$
- $\dot{\lambda}(t) = -\frac{\partial H}{\partial q}(q(t), \lambda(t), u(t), \lambda_0),$
- $0 = H(q(t), \lambda(t), u(t), \lambda_0) = \max_v \{H(q, \lambda, v, \lambda_0) \mid |v_i| \leq 1 \text{ for } i = 1, 2\}.$

If  $\lambda_0 = 0$ ,  $q$  is said abnormal, if not  $q$  is said normal. It may be both. A solution of the PMP is called an extremal.

**Proposition 8.** *For a generic SF metric on a 2D manifold defined with a maximum norm, there is no abnormal extremal. Hence we can fix  $\lambda_0 = -1$ . This is our choice in the following.*

**Proof.** It is a classical fact that an abnormal extremal should correspond to a covector  $\lambda \neq 0$  orthogonal to  $F_1$ ,  $F_2$  and  $[F_1, F_2]$ . This implies that along the trajectory the three vectors are parallel. But generically this happens only on a discrete set, which forbids to get a non trivial curve. ■

#### 3.3 Switchings

In this section, we follow the ideas of [13].

**Definition 9.** For an extremal triplet  $(q(\cdot), \lambda(\cdot), u(\cdot))$ , define the *switching functions*

$$\phi_i(t) = \langle \lambda(t), F_i(q(t)) \rangle, i = 1, 2,$$

and the function  $\phi_3(t) = \langle \lambda(t), [F_1, F_2](q(t)) \rangle$ .

Thanks to  $\lambda_0 = -1$ , the  $\phi_i$  functions satisfy

$$u_1(t)\phi_1(t) + u_2(t)\phi_2(t) = 1, \quad \text{for a.e. } t.$$

A direct consequence of the maximality condition is

**Proposition 10.** *If  $\phi_i(t) > 0$  (resp.  $\phi_i(t) < 0$ ) then  $u_i(t) = 1$  (resp.  $u_i(t) = -1$ ).*

*If  $\phi_i(t) = 0$  and  $\dot{\phi}_i(t) > 0$  (resp.  $\dot{\phi}_i(t) < 0$ ) then  $\phi_i$  changes sign at time  $t$  and the control  $u_i$  switches from  $-1$  to  $+1$  (resp. from  $+1$  to  $-1$ ).*

**Definition 11.** We call *bang* an extremal trajectory corresponding to constant controls with value  $1$  or  $-1$  and *bang-bang* an extremal which is a finite concatenation of bangs. We call  $u_i$ -*singular* an extremal corresponding to a null switching function  $\phi_i$ . A time  $t$  is said to be a *switching time* if  $u$  is not bang in any neighborhood of  $t$ .

**Definition 12.** Outside  $\Delta_A$ , let define the functions  $f_1$  and  $f_2$  by

$$[F_1, F_2](q) = f_2(q)F_1(q) - f_1(q)F_2(q).$$

It is clear that

$$\Delta_1 \setminus \Delta_A = f_1^{-1}(0), \quad \Delta_2 \setminus \Delta_A = f_2^{-1}(0).$$

**Proposition 13** (Switching rules). *Outside  $\Delta_A \cup \Delta_1 \cup \Delta_2$  the possible switches of the controls are*

- *if  $f_1 > 0$  then  $u_1$  can only switch from  $-1$  to  $+1$  when  $\phi_1$  goes to  $0$ ,*
- *if  $f_1 < 0$  then  $u_1$  can only switch from  $+1$  to  $-1$  when  $\phi_1$  goes to  $0$ ,*
- *if  $f_2 > 0$  then  $u_2$  can only switch from  $-1$  to  $+1$  when  $\phi_2$  goes to  $0$ ,*
- *if  $f_2 < 0$  then  $u_2$  can only switch from  $+1$  to  $-1$  when  $\phi_2$  goes to  $0$ .*

**Proof.** The fact that  $\dot{\phi}_1(t) = -u_2 \cdot \lambda \cdot [F_1, F_2]$  and  $\dot{\phi}_2(t) = u_1 \cdot \lambda \cdot [F_1, F_2]$  implies that, outside  $\Delta_A \cup \Delta_1 \cup \Delta_2$ ,

$$\dot{\phi}_1(t) = u_2(t) (f_1(q(t))\phi_2(t) - f_2(q(t))\phi_1(t)) = -u_2(t)\phi_3(t), \quad (3)$$

$$\dot{\phi}_2(t) = u_1(t) (f_2(q(t))\phi_1(t) - f_1(q(t))\phi_2(t)) = u_1(t)\phi_3(t). \quad (4)$$

Now, if  $\phi_1(t) = 0$  then  $|\phi_2(t)| = 1$  which implies  $u_2(t)\phi_2(t) = 1$  and hence  $\dot{\phi}_1(t)$  and  $f_1(q(t))$  have same sign and the sign of  $f_1(q(t))$  determines the switch.

The same holds true for  $f_2$ ,  $\phi_2$  and  $u_2$ . ■

As a consequence, on each connected component of the complement of  $\Delta_A \cup \Delta_1 \cup \Delta_2$ , each control  $u_i$  can take only values  $-1$  and  $+1$  and can switch only once from  $-1$  to  $+1$  if  $f_i > 0$  or from  $+1$  to  $-1$  if  $f_i < 0$ .

**Proposition 14.** *At any point  $q$  outside  $\Delta_A$ , exists a  $\tau > 0$  such that for any extremal issued from  $q$  and of length less than  $\tau$ , only one of the two controls may switch.*

**Proof.** If  $\phi_1(t) = 0$  then  $|\phi_2(t)| = 1$ . Hence, if  $\phi_1(t) = 0$  and  $\phi_2(t') = 0$  then  $\phi_1$  passes from value 0 to  $\pm 1$  in time  $t' - t$  which implies that  $|\dot{\phi}_1|$  takes values larger than  $\frac{1}{|t' - t|}$ . But, since  $\dot{\phi}_1(t) = -u_2(f_2(q(t))\phi_1(q(t)) - f_1(q(t))\phi_2(q(t)))$ , we have  $|\dot{\phi}_1(t)| \leq |f_1(q(t))| + |f_2(q(t))|$ . As a consequence, if locally  $|f_1 + f_2| < M$  then  $|t' - t|$  cannot be smaller than  $1/M$ .

**Proposition 15.** *At any point  $q$  outside  $\Delta_A$ , consider the normal coordinate system centered at  $q$ . Any local extremal stays in one of the following domains :  $\mathbb{R}_+ \times \mathbb{R}_+$ ,  $\mathbb{R}_+ \times \mathbb{R}_-$ ,  $\mathbb{R}_- \times \mathbb{R}_+$  or  $\mathbb{R}_- \times \mathbb{R}_-$ .*

**Proof.** Thanks to previous proposition, only one control may switch in short time. Assume that  $u_1 \equiv 1$ . Then at each time  $u_1 F_1 + u_2 F_2 = F_1 + u_2 F_2$  hence the dynamics takes the form  $\alpha G_1 + (1 - \alpha)G_2$  with  $\alpha \in [0, 1]$ . This dynamics leaves invariant the set  $\mathbb{R}_+ \times \mathbb{R}_+$ , hence the extremal does not leave this set. By the same argument one proves that if  $u_1 \equiv -1$  then the extremal stays in  $\mathbb{R}_- \times \mathbb{R}_-$ , if  $u_2 \equiv 1$  then the extremal stays in  $\mathbb{R}_+ \times \mathbb{R}_-$  and that if  $u_2 \equiv -1$  then the extremal stays in  $\mathbb{R}_- \times \mathbb{R}_+$ .

### 3.4 Initial conditions and their parameterization

One proves easily that in the  $(NF_1)$  case,  $\max(|\lambda_x(0)|, |\lambda_y(0)|) = 1$ . Hence the set of initial conditions  $\lambda$  is compact and extremals switching in short time or singular extremals should have a  $\phi_i$  null or close to zero. Moreover only one control can switch in short time (see Proposition 14).

In the  $(NF_2)$  and  $(NF_3)$  cases  $|\lambda_x(0)| = 1$  and there is no condition on  $\lambda_y$ . Hence the set of initial condition is not compact. This allows to consider initial conditions with  $|\lambda_y| \gg 1$  and hence will appear optimal extremals along which the two controls switch. It is not in contradiction with the Proposition 14 since in this case the base point belongs to  $\Delta_A$ .

In the  $(NF_{2a})$  and  $(NF_3)$  cases,  $\phi_1(0) = \pm \frac{1+a_0}{2}$  and  $\phi_2(0) = \pm \frac{1-a_0}{2}$ . Hence, if one consider a compact set of initial conditions, the corresponding extremals do not switch in short time. And are not singular. As a consequence, to consider the extremal switching at least once, one should consider initial conditions with  $|\lambda_y(0)| \gg 1$ .

Let us give an idea of how to estimate the  $|\lambda_y(0)|$  corresponding to a  $u_1$ -switch at small time  $t$  and the consequence in terms of choice of change of coordinates.

In the  $(NF_2)$  case,  $\phi_1(0) = \frac{1+a_0}{2} \geq \frac{1}{2}$ . Hence, if along an extremal the control  $u_1$  switches for  $t$  small hence on gets, since  $x(t) = O(t)$  and  $y(t) = O(t^2)$ ,

$$0 = \lambda(t).F_1(x(t), y(t)) = \frac{1+a_0}{2} + \lambda_y(0) \frac{x(t)}{2} + O(t)$$

and it implies that if an extremal sees its control  $u_1$  switching at  $t$  then  $\lambda_y(0)$  should be like  $\frac{1}{t}$ . Hence, in order to make estimations of the corresponding extremals, it is natural to choose as small parameter  $r_0 = \frac{1}{\lambda_y(0)}$ , to make the change of coordinate  $r = \frac{1}{\lambda_y}$ , the change of time  $s = \frac{t}{r}$  and the change of coordinate  $p_x = r\lambda_x$ . This is what we do in the subsections 4.2 and 4.3.

In the  $(NF_3)$  case,  $\phi_1(0) = \frac{1+a_0}{2} \geq \frac{1}{2}$ . Hence, if along an extremal the control  $u_1$  switches for  $t$  small hence on gets, since  $x(t) = O(t)$  and  $y(t) = O(t^3)$ ,

$$0 = \lambda(t).F_1(x(t), y(t)) = \frac{1+a_0}{2} + \lambda_y(0) \frac{x^2(t)}{4} + O(t)$$

and it implies that if an extremal sees its control  $u_1$  switching at  $t$  then  $\lambda_y(0)$  should be like  $\frac{1}{t^2}$ . Hence, in order to make estimations of the corresponding extremals, it is natural to choose as small parameter  $r_0$  such that  $\lambda_y(0) = \pm \frac{1}{r_0^2}$ , to make the change of coordinate  $r = \frac{\pm 1}{\sqrt{|\lambda_y|}}$  and the change of time  $s = \frac{t}{r}$ . This is what we do in the subsection 4.4.

### 3.5 Weights, orders and nilpotent approximation

The definitions of privileged coordinates and nilpotent approximation are too long to be given here. We refer to [8]. The coordinates we constructed in the normal form are privileged coordinates.

In the  $(NF_1)$  case,  $x$  and  $y$  have weight 1 and  $\partial_x$  and  $\partial_y$  have weight  $-1$  as operators of derivation. In the  $(NF_2)$  case  $x$  has weight 1 and  $y$  has weight 2,  $\partial_x$  has weight  $-1$  and  $\partial_y$  have weight  $-2$ . In the  $(NF_3)$  case,  $x$  has weight 1 and  $y$  has weight 3,  $\partial_x$  has weight  $-1$  and  $\partial_y$  have weight  $-3$ .

In privileged coordinates, one way to understand the weights of the variables naturally is to estimate how they vary with time in small time along an admissible curve. As seen before, in the  $(NF_1)$  case  $x$  and  $y$  are  $O(t)$  (and may be not  $o(t)$ ), in the  $(NF_2)$  case  $x = O(t)$  and  $y = O(t^2)$  and in the  $(NF_3)$  case  $x = O(t)$  and  $y = O(t^3)$ .

In the following,  $o_k(x, y)$  will denote a function whose valuation at 0 has order larger than  $k$  respectively to the weights of  $x$  and  $y$ . For example  $x^7$  has always weight 7 and  $y^3$  has weight 3 in the  $(NF_1)$  case but 9 in the  $(NF_3)$  case.

With this notion of weights, we define the nilpotent approximation of our normal forms in the three cases

$$\begin{aligned} (NF_1) \quad G_1(x, y) &= \partial_x, \\ G_2(x, y) &= \partial_y, \\ (NF_2) \quad G_1(x, y) &= \partial_x, \\ G_2(x, y) &= a_0 \partial_x + x \partial_y, \\ (NF_3) \quad G_1(x, y) &= \partial_x, \\ G_2(x, y) &= a_0 \partial_x + \frac{1}{2} x^2 \partial_y, \end{aligned}$$

which corresponds to an approximation to order -1. In the following, when we will compute developments with respect to the parameter  $r_0$ , that is for  $|\lambda_y(0)| \gg 1$ , we will need the approximation to order 0 for  $(NF_{2a})$  and  $(NF_3)$ , and the approximation to order 1 for  $(NF_{2b})$

$$\begin{aligned} (NF_{2a}) \quad G_1(x, y) &= \partial_x, \\ G_2(x, y) &= (a_0 + a_{10}x) \partial_x + x(1 + b_{20}x) \partial_y, \\ (NF_{2b}) \quad G_1(x, y) &= \partial_x, \\ G_2(x, y) &= (1 + a_{10}x + a_{01}y + a_{20}x^2) \partial_x + x(1 + b_{20}x + b_{30}x^2) \partial_y, \\ (NF_3) \quad G_1(x, y) &= \partial_x, \\ G_2(x, y) &= (a_0 + a_{10}x) \partial_x + \left( \frac{x^2}{2} + b_{01}y + b_{30}x^3 \right) \partial_y, \end{aligned}$$

In the  $(NF_1)$  case, we will need the approximation to order 2 in order to compute the cut locus,



when present:

$$\begin{aligned} (NF_1) \quad G_1(x, y) &= \partial_x, \\ G_2(x, y) &= x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2)\partial_x + \\ &\quad + (1 + x(b_{10} + b_{20}x + b_{11}y + b_{30}x^2 + b_{21}xy + b_{12}y^2))\partial_y, \end{aligned}$$

### 3.6 Symbols of extremals

As we will see in the following, the local extremals will be finite concatenations of bang arcs and  $u_i$ -singular arcs. In order to facilitate the presentation, a bang arc following  $\pm G_i$  will be symbolized by  $[[\pm G_i]]$ , a  $u_1$ -singular arc with control  $u_2 \equiv 1$  will be symbolized by  $[[S_1^+]]$ , a  $u_1$ -singular arc with control  $u_2 \equiv -1$  will be symbolized by  $[[S_1^-]]$ , and we will combined these symbols in such a way that  $[[ -G_1, G_2, S_2^+ ]]$  symbolizes the concatenation of a bang arc following  $-G_1$  with a bang arc following  $G_2$  and a  $u_2$ -singular arc with control  $u_1 \equiv 1$ .

### 3.7 Symmetries

One can change the roles of the vectors  $F_1$  and  $F_2$  and look at the effect on the functions  $f_i$  or on the invariants appearing in the normal form. For this last part, one should be careful that changing the role of  $F_1$  and  $F_2$  implies changing  $G_1$  and  $G_2$  and hence changing the coordinates  $x$  and  $y$ .

Let first look at the effect on the functions  $f_i$  on an example :  $\bar{F}_1 = -F_1$  and  $\bar{F}_2 = F_2$ . If we define the control system with  $(\bar{F}_1, \bar{F}_2)$ , it defines the same SF structure. We compute easily that

$$[\bar{F}_1, \bar{F}_2] = [-F_1, F_2] = -[F_1, F_2] = -(f_2F_1 - f_1F_2) = f_2\bar{F}_1 - (-f_1)\bar{F}_2$$

hence  $\bar{f}_1 = -f_1$  and  $\bar{f}_2 = f_2$ . With this choice  $\bar{G}_1 = -G_2$  and  $\bar{G}_2 = -G_1$ . Of course, with such a change on the vectors  $G_1$  and  $G_2$  the change on the invariants is not so trivial to compute.

In the following we consider changes that send  $G_1$  to  $\pm G_1$  and  $G_2$  to  $\pm G_2$ . These changes are interesting from a calculus point of view. Effectively, once computed the jet of a bang-bang extremals with symbol  $[[G_1, G_2]]$  and of its switching times, we are able to get the expressions for the bang-bang extremals with symbols  $[[\pm G_1, \pm G_2]]$ . No use to repeat the computations. For example, if one gets the expression of an extremal with symbol  $[[G_1, G_2]]$  as function of the initial conditions, one gets the expression of an extremal with symbol  $[[ -G_1, G_2 ]]$  by respecting the effect on the coordinates and the invariants  $a_0, a_{10}$ , etc. of the corresponding change of role of  $F_1$  and  $F_2$ .

#### 3.7.1 $\bar{G}_1 = -G_1$ and $\bar{G}_2 = G_2$

Let consider the change  $\bar{F}_1 = -F_2$  and  $\bar{F}_2 = -F_1$ . Then  $\bar{G}_1 = -G_1$  and  $\bar{G}_2 = G_2$ . With this choice,

$$[\bar{F}_1, \bar{F}_2] = [-F_2, -F_1] = -[F_1, F_2] = -(f_2F_1 - f_1F_2) = (-f_1)\bar{F}_1 - (-f_2)\bar{F}_2$$

hence  $\bar{f}_1 = -f_2$  and  $\bar{f}_2 = -f_1$ . Moreover,  $[\bar{G}_1, \bar{G}_2] = -[G_1, G_2]$  and  $[\bar{G}_1, [\bar{G}_1, \bar{G}_2]] = [G_1, [G_1, G_2]]$ .

We can now consider the effect of this change of role on the coordinates and on the invariants in the three cases of the normal form

$(NF_1)$  In this case,  $\bar{x} = -x$  and  $\bar{y} = y$ , hence  $\partial_{\bar{x}} = -\partial_x$  and  $\partial_{\bar{y}} = \partial_y$  and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (a_{10}\bar{x} - a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} + (1 - b_{10}\bar{x} + b_{20}\bar{x}^2 - b_{11}\bar{x}\bar{y} + o_2(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF<sub>2</sub>) In this case,  $\bar{x} = -x$  and  $\bar{y} = -y$ , hence  $\partial_{\bar{x}} = -\partial_x$  and  $\partial_{\bar{y}} = -\partial_y$  and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_0 + a_{10}\bar{x} - a_{01}\bar{y} - a_{20}\bar{x}^2 + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x} - b_{20}\bar{x}^2 + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF<sub>3</sub>) In this case,  $\bar{x} = -x$  and  $\bar{y} = y$ , hence  $\partial_{\bar{x}} = -\partial_x$  and  $\partial_{\bar{y}} = \partial_y$  and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_0 + a_{10}\bar{x} + o_1(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x}^2/2 + b_{01}\bar{y} - b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

### 3.7.2 $\bar{G}_1 = G_1$ and $\bar{G}_2 = -G_2$

Let consider the change  $\bar{F}_1 = F_2$  and  $\bar{F}_2 = F_1$ . Then  $\bar{G}_1 = G_1$  and  $\bar{G}_2 = -G_2$ . With this choice,

$$[\bar{F}_1, \bar{F}_2] = [F_2, F_1] = -[F_1, F_2] = -(f_2F_1 - f_1F_2) = (f_1)\bar{F}_1 - (f_2)\bar{F}_2$$

hence  $\bar{f}_1 = f_2$  and  $\bar{f}_2 = f_1$ . Moreover,  $[\bar{G}_1, \bar{G}_2] = -[G_1, G_2]$  and  $[\bar{G}_1, [\bar{G}_1, \bar{G}_2]] = -[G_1, [G_1, G_2]]$ .

We can know consider the effect of this change of role on the coordinates and on the invariants in the three cases of the normal form

(NF<sub>1</sub>) In this case,  $\bar{x} = x$  and  $\bar{y} = -y$ , hence  $\partial_{\bar{x}} = \partial_x$  and  $\partial_{\bar{y}} = -\partial_y$  and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_{10}\bar{x} - a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{x}} + (1 + b_{10}\bar{x} + b_{20}\bar{x}^2 - b_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF<sub>2</sub>) In this case,  $\bar{x} = x$  and  $\bar{y} = -y$ , hence  $\partial_{\bar{x}} = \partial_x$  and  $\partial_{\bar{y}} = -\partial_y$  and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_0 - a_{10}\bar{x} + a_{01}\bar{y} - a_{20}\bar{x}^2 + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x} + b_{20}\bar{x}^2 + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF<sub>3</sub>) In this case,  $\bar{x} = x$  and  $\bar{y} = -y$ , hence  $\partial_{\bar{x}} = \partial_x$  and  $\partial_{\bar{y}} = -\partial_y$  and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_0 - a_{10}\bar{x} + o_1(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x}^2/2 - b_{01}\bar{y} + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

### 3.7.3 $\bar{G}_1 = -G_1$ and $\bar{G}_2 = -G_2$

Let consider the change  $\bar{F}_1 = -F_1$  and  $\bar{F}_2 = -F_2$ . Then  $\bar{G}_1 = -G_1$  and  $\bar{G}_2 = -G_2$ . With this choice,

$$[\bar{F}_1, \bar{F}_2] = [-F_1, -F_2] = [F_1, F_2] = (f_2F_1 - f_1F_2) = (-f_2)\bar{F}_1 - (-f_1)\bar{F}_2$$

hence  $\bar{f}_1 = -f_1$  and  $\bar{f}_2 = -f_2$ . Moreover,  $[\bar{G}_1, \bar{G}_2] = [G_1, G_2]$  and  $[\bar{G}_1, [\bar{G}_1, \bar{G}_2]] = -[G_1, [G_1, G_2]]$ .

We can know consider the effect of this change of role on the coordinates and on the invariants in the three cases of the normal form

(NF<sub>1</sub>) In this case,  $\bar{x} = -x$  and  $\bar{y} = -y$ , hence  $\partial_{\bar{x}} = -\partial_x$  and  $\partial_{\bar{y}} = -\partial_y$ . Moreover

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_{10}\bar{x} + a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{x}} + (1 - b_{10}\bar{x} + b_{20}\bar{x}^2 + b_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF<sub>2</sub>) In this case,  $\bar{x} = -x$  and  $\bar{y} = y$ , hence  $\partial_{\bar{x}} = -\partial_x$  and  $\partial_{\bar{y}} = \partial_y$ . Moreover

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (a_0 - a_{10}\bar{x} + a_{01}\bar{y} + a_{20}\bar{x}^2 + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x} - b_{20}\bar{x}^2 + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF<sub>3</sub>) In this case,  $\bar{x} = -x$  and  $\bar{y} = -y$ , hence  $\partial_{\bar{x}} = -\partial_x$  and  $\partial_{\bar{y}} = -\partial_y$ . Moreover

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (a_0 - a_{10}\bar{x} + o_1(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x}^2/2 - b_{01}\bar{y} - b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

## 4 The generic local optimal synthesis

We present for generic couples  $(F_1, F_2)$  the local synthesis issued from a point  $q$ . The coordinates  $(x, y)$ , centred at  $q$ , are those which have been constructed in the corresponding normal form in section 2.

### 4.1 $(NF_1)$ case

At points  $q$  where  $(NF_1)$  holds, one can compute that

$$\begin{aligned}
 f_1(x, y) &= \frac{1}{2}(a_{10} - b_{10}) \\
 &\quad + (2(a_{20} - b_{20}) - b_{10}(a_{10} - b_{10}))\frac{x}{2} + (a_{11} - b_{11})\frac{y}{2} \\
 &\quad + (3(a_{30} - b_{30}) - b_{10}(a_{20} - b_{20}) - (2b_{20} - b_{10}^2)(a_{10} - b_{10}))\frac{x^2}{2} \\
 &\quad + (2(a_{21} - b_{21}) - b_{11}(a_{10} - b_{10}) - b_{10}(a_{11} - b_{11}))\frac{xy}{2} + (a_{12} - b_{12})\frac{y^2}{2} + o_2(x, y), \\
 f_2(x, y) &= -\frac{1}{2}(a_{10} + b_{10}) \\
 &\quad - (2(a_{20} + b_{20}) - b_{10}(a_{10} + b_{10}))\frac{x}{2} - (a_{11} + b_{11})\frac{y}{2} \\
 &\quad - (3(a_{30} + b_{30}) - b_{10}(a_{20} + b_{20}) - (2b_{20} - b_{10}^2)(a_{10} + b_{10}))\frac{x^2}{2} \\
 &\quad - (2(a_{21} + b_{21}) - b_{11}(a_{10} + b_{10}) - b_{10}(a_{11} + b_{11}))\frac{xy}{2} - (a_{12} + b_{12})\frac{y^2}{2} + o_2(x, y).
 \end{aligned}$$

Hence, thanks to Proposition 13, if  $a_{10} - b_{10} > 0$  (resp.  $< 0$ ) then  $u_1$  is bang-bang and the only possible switch is  $-1 \rightarrow +1$  (resp.  $+1 \rightarrow -1$ ) and if  $a_{10} + b_{10} < 0$  (resp.  $> 0$ ) then  $u_2$  is bang-bang and the only possible switch is  $-1 \rightarrow +1$  (resp.  $+1 \rightarrow -1$ ).

*Remark 16* (Generic invariants). Let remark that generically, in the  $(NF_1)$  case, one of the following situation occurs

- $|a_{10}| \neq |b_{10}|$  ( $NF_{1a}$ ),
- $a_{10} = b_{10} \neq 0$  and  $a_{20} - b_{20} \neq 0$  and  $a_{11} - b_{11} \neq 0$ ,
- $a_{10} = b_{10} \neq 0$  and  $a_{20} - b_{20} = 0$  and  $a_{30} - b_{30} \neq 0$  and  $a_{11} - b_{11} \neq 0$ ,
- $a_{10} = b_{10} \neq 0$  and  $a_{20} - b_{20} \neq 0$  and  $a_{11} - b_{11} = 0$  and  $a_{12} - b_{12} \neq 0$ ,
- $a_{10} = -b_{10} \neq 0$  and  $a_{20} + b_{20} \neq 0$  and  $a_{11} + b_{11} \neq 0$ ,
- $a_{10} = -b_{10} \neq 0$  and  $a_{20} + b_{20} = 0$  and  $a_{30} + b_{30} \neq 0$  and  $a_{11} + b_{11} \neq 0$ ,
- $a_{10} = -b_{10} \neq 0$  and  $a_{20} + b_{20} \neq 0$  and  $a_{11} + b_{11} = 0$  and  $a_{12} + b_{12} \neq 0$ .
- $a_{10} = b_{10} = 0$  and  $a_{20} + b_{20} \neq 0$  and  $a_{11} + b_{11} \neq 0$ .

#### 4.1.1 Singular extremals

We consider now the properties of singular extremals and their support.

**Proposition 17.** *Under the generic assumption that  $\Delta_A$ ,  $\Delta_1$  and  $\Delta_2$  are submanifolds transversal by pair then*

1. *The support of a  $u_i$ -singular is included in  $\Delta_i$ .*
2. *A  $u_1$ -singular extremal can follow  $\Delta_1$  being optimal only if, at each point  $q(t)$  of the singular,  $G_1(q(t))$  and  $G_2(q(t))$  are pointing on the same side of  $\Delta_1$  (or one is tangent to  $\Delta_1$ ) where  $f_1 > 0$ .*
3. *A  $u_2$ -singular extremal can follow  $\Delta_2$  being optimal only if, at each point  $q(t)$  of the singular,  $G_1(q(t))$  and  $-G_2(q(t))$  are pointing on the same side of  $\Delta_2$  (or one is tangent to  $\Delta_2$ ) where  $f_2 > 0$ .*
4. *Let consider a  $u_i$ -singular  $q(\cdot)$  satisfying 2 or 3. If it does not intersect  $\Delta_A$  and if at each time  $G_1(q(t))$  and  $G_2(q(t))$  are not tangent to  $\Delta_i$  then  $q(\cdot)$  is a local minimizer that is at each time  $t$  exists  $\epsilon$  such that  $q(\cdot)$  realizes the SF-distance between  $q(t_1)$  and  $q(t_2)$  for any  $t_1$  and  $t_2$  in  $]t - \epsilon, t + \epsilon[$ .*

**Proof.**

1. Outside  $\Delta_A \cup \Delta_i$ ,  $\phi_i$  has isolated zero hence any  $u_i$ -singular should live in  $\Delta_A \cup \Delta_i$ . Moreover, since generically the set of points of  $\Delta_A$  where the dynamics is tangent to  $\Delta_A$  is isolated, a  $u_i$ -singular crosses  $\Delta_A$  only at isolated times, which are consequently also in  $\Delta_i$ .
2. Same proof as point 3.
3. If a  $u_2$ -singular  $q(\cdot)$  has  $u_1 = 1$  then its speed is  $F_1(q(t)) + u_2(t)F_2(q(t))$  which is tangent to  $\Delta_2$ . But  $u_2 \in [-1, 1]$  hence either  $|u_2(t)| = 1$  and  $G_1$  or  $G_2$  are tangent to  $\Delta_2$  or  $|u_2(t)| < 1$  and  $G_2(q(t)) = F_1(q(t)) - F_2(q(t))$  and  $G_1(q(t)) = F_1(q(t)) + F_2(q(t))$  point on opposite side.

Now, assume that  $f_2 < 0$  in the domain where points  $G_1(q(t))$ . With the expression given before, this corresponds to  $(a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} > 0)$  or  $(a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} = 0$  and  $a_{30} + b_{30} > 0)$ . Thanks to the previous results, if the extremal leave  $\Delta_2$  at time  $t_0$  it starts by a bang. Assume for example that this bang follows  $G_1$ . Then during this bang

$$\dot{\phi}_2 = u_1 \phi_3 = u_1(f_2 \phi_1 - f_1 \phi_2)$$

and hence

$$\ddot{\phi}_2 = u_1(\partial_x f_2 \phi_1 + f_2 \dot{\phi}_1 - \partial_x f_1 \phi_2 - f_1 \dot{\phi}_2)$$

which implies  $\ddot{\phi}_2(t_0) = u_1(t_0)\partial_x f_2(q(t_0))\phi_1(t_0) = \partial_x f_2(q(t_0))$  since  $u_1(t_0) = 1$  and  $\phi_1(t_0) = 1$ . But if  $a_{20} + b_{20} > 0$  then  $\partial_x f_2(q(t_0)) = -(a_{20} + b_{20}) + O(t_0) < 0$ . Then, since  $\phi_2(t_0) = \dot{\phi}_2(t_0) = 0$  and  $\ddot{\phi}_2(t_0) < 0$ , for  $t$  just after  $t_0$   $\phi_2(t) < 0$  which is in contradiction with the fact that the trajectory follows  $G_1$  just after  $t_0$ .

The same proof can be done for the other cases: a  $u_2$ -singular with  $u_1 = 1$  and switching to  $G_2$ , or  $u_2$ -singulars with  $u_1 = -1$ . Hence in this case, no extremal following  $\Delta_2$  can leave  $\Delta_2$ .

Now, assume that  $\Delta_2$  is such that  $G_1$  and  $-G_2$  point in the same side where  $f_2 < 0$  at  $q$  and that the  $u_2$ -singular is optimal. Consider the normal coordinate system centered at  $q$  and the domain  $\mathbb{R}_+ \times \mathbb{R}_+$ . One can show, with the previous analysis, that the only possible extremals issued from  $q$  and entering the domain are the singular arc following  $\Delta_2$  and the bang-bang extremals starting with symbol  $[[G_1, G_2]]$  or  $[[G_2, G_1]]$ .

Let us prove that these last ones do not switch again before crossing  $\Delta_2$ . If an extremal starts with  $[[G_2, G_1]]$ , switching for the first time at  $t = \epsilon$  and hence at  $y = \epsilon$  then along the second bang  $x = t - \epsilon$ ,  $y = \epsilon$ ,  $\lambda \equiv (1, 1)$  and one computes easily that for  $t > \epsilon$

$$\phi_2(t) = -\frac{1}{2}((a_{20} + b_{20})(t - \epsilon)^2 + (a_{11} + b_{11})(t - \epsilon)\epsilon + o_2(\epsilon, (t - \epsilon))).$$

If  $(a_{20} + b_{20})(a_{11} + b_{11}) < 0$  then the second time of switch satisfies  $t - \epsilon = -\frac{a_{11}+b_{11}}{a_{20}+b_{20}}\epsilon + o(\epsilon)$  and hence the second switching locus has the form  $(-\frac{a_{11}+b_{11}}{a_{20}+b_{20}}\epsilon, \epsilon)$ . But  $\Delta_2$  satisfies that  $x = -\frac{1}{2}\frac{a_{11}+b_{11}}{a_{20}+b_{20}}y + o(y)$  and hence the second bang crosses  $\Delta_2$  before ending. In the case  $a_{20} + b_{20} = 0$  hence  $(a_{11} - b_{11})(a_{30} + b_{30}) < 0$  and one shows that the second switching locus has the form  $(\sqrt{-\frac{a_{11}+b_{11}}{a_{30}+b_{30}}}\epsilon, \epsilon)$  and  $\Delta_2$  satisfies that  $x = \sqrt{-\frac{a_{11}+b_{11}}{3(a_{30}+b_{30})}}y + o(y)$  hence again the second bang crosses  $\Delta_2$  before ending. The same kind of computations show the same result when  $a_{11} + b_{11} = 0$  and  $(a_{20} + b_{20})(a_{12} + b_{12}) < 0$ . The same holds for extremal starting by  $[[G_1, G_2]]$ .

Finally, the different extremals with symbol  $[[G_1, G_2]]$  do not intersect each other after their first switch hence they cannot lose optimality by crossing each other. Idem for those with symbol  $[[G_2, G_1]]$ . Hence they can lose optimality by crossing the singular extremal or extremals with the other symbol. Anyway, this implies that optimal extremals are coming back to  $\Delta_2$ . But this is not possible since in this case an extremal with symbol  $[[G_1, G_2, S_2^+]]$  would exist which is not the case since the switching is coming strictly after the crossing with  $\Delta_2$ .

Hence, the  $u_2$ -singular is not optimal.

4. It is a consequence of the analysis coming further but we can give a quick idea: in this case, if  $q$  is a point on  $\Delta_1$  and if we construct normal coordinates centered at  $q$ , then the only local extremals entering the domains  $\{xy < 0\}$  are the one starting by a  $u_1$ -singular and switching or not locally only once to  $u_1 = \pm 1$ . Hence the  $u_1$ -singular is locally optimal.

*Remark 18.* For what concerns the point 4, assume that  $q$  is a point where  $G_1$  or  $G_2$  is tangent to  $\Delta_1$  and  $\Delta_1 \cap \{xy < 0\}$  is such that at each point  $G_1$  and  $G_2$  are transverse to  $\Delta_1$  and point in the domain  $\{f_1 > 0\}$ . Then, starting from  $q$ , a  $u_1$ -singular can run on  $\Delta_1 \cap \{xy < 0\}$  and is locally optimal. The same arguments than those exposed at point 4 work.

**Definition 19.** If a connected part of  $\Delta_1$  (resp.  $\Delta_2$ ) is such that at each point  $G_1$  and  $G_2$  (resp.  $G_1$  and  $-G_2$ ) point on the same side where  $f_1 > 0$  (resp.  $f_2 > 0$ ), it is called a turnpike. If it does not at each point, it is called an anti-turnpike (see [13]).

*Remark 20.* Along a  $u_i$ -singular extremal the control  $u_i$  is completely determined by the fact that the dynamics should be tangent to  $\Delta_i$ .

#### 4.1.2 Optimal synthesis in the domain $\mathbb{R}_+ \times \mathbb{R}_+$

Let consider a point  $q$  and the normal coordinate system  $(x, y)$  centered at  $q$ . The dynamics entering  $\mathbb{R}_+^* \times \mathbb{R}_+^*$  is with  $u_1 \equiv 1$  since  $u_2$  switches (Propositions 14 and 15). Three different cases can be identified.

**1st. case.**  $\Delta_2 \cap (\mathbb{R}_+ \times \mathbb{R}_+ \setminus \{0\})$  is empty locally. No  $u_2$ -singular enters the domain. It corresponds to the case  $(NF_{1a})$  where  $|a_{10}| \neq |b_{10}|$  and to the cases  $(NF_{1c})$  and  $(NF_{1d})$  where  $a_{10} + b_{10} = 0$  and

- $(a_{20} + b_{20})(a_{11} + b_{11}) > 0$ ,
- or  $a_{20} + b_{20} = 0$  and  $(a_{30} + b_{30})(a_{11} + b_{11}) > 0$ ,
- or  $a_{11} + b_{11} = 0$  and  $(a_{20} + b_{20})(a_{12} + b_{12}) > 0$ .

Only one  $u_2$ -switch can occur along the extremal. One has  $f_2 > 0$  in the domain if

- $a_{10} + b_{10} < 0$ ,
- or  $a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} < 0$ ,
- or  $a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} = 0$  and  $a_{11} + b_{11} < 0$ ,

and in this case the possible extremals of the domain have symbol  $[[G_1]]$  or  $[[G_2]]$  or  $[[G_2, G_1]]$ . One has  $f_2 < 0$  in the domain if

- $a_{10} + b_{10} > 0$ ,
- or  $a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} > 0$ ,
- or  $a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} = 0$  and  $a_{11} + b_{11} > 0$ .

and in this case the possible extremals of the domain have symbol  $[[G_1]]$  or  $[[G_2]]$  or  $[[G_1, G_2]]$ .

In this case 1, the picture of the synthesis is given in the Figure 1.

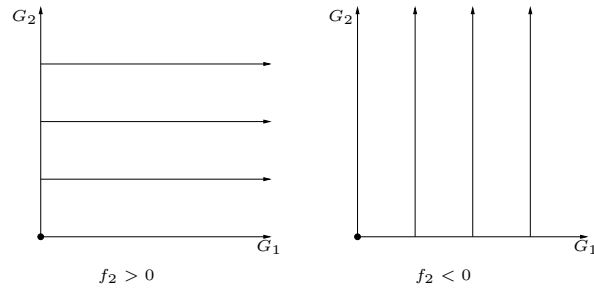


Figure 1: The syntheses when  $f_2 \neq 0$  in  $(\mathbb{R}_+ \times \mathbb{R}_+) \setminus \{0\}$

**2nd. case.**  $\Delta_2 \cap (\mathbb{R}_+^* \times \mathbb{R}_+^*)$  is not empty locally and is a turnpike. It corresponds to the cases where  $a_{10} + b_{10} = 0$  and

- $a_{20} + b_{20} < 0$  and  $a_{11} + b_{11} > 0$ ,
- or  $a_{20} + b_{20} = 0$  and  $a_{11} + b_{11} > 0$  and  $a_{30} + b_{30} < 0$ ,

- or  $a_{11} + b_{11} = 0$  and  $a_{20} + b_{20} < 0$  and  $a_{12} + b_{12} > 0$ .

Then  $f_2 > 0$  locally along  $\{x > 0, y = 0\}$  and  $f_2 < 0$  along  $\{x = 0, y > 0\}$ . Hence no bang-bang extremal with symbol  $[[G_1, G_2]]$  or  $[[G_2, G_1]]$  exists and any extremal entering the domain starts with a  $u_2$ -singular arc. If it switches to  $G_1$  then it enters the domain  $(\mathbb{R}_+^* \times \mathbb{R}_+^*) \cap \{f_2 > 0\}$  which is invariant by  $G_1$  hence it does not switch anymore. If it switches to  $G_2$  it enters the domain  $(\mathbb{R}_+^* \times \mathbb{R}_+^*) \cap \{f_2 < 0\}$  which is invariant by  $G_2$  hence it does not switch anymore.

As a consequence, the only possible symbols for extremals are  $[[G_1]]$ ,  $[[G_2]]$ ,  $[[S_2^+, G_1]]$  and  $[[S_2^+, G_2]]$ .

In this case 2, the picture of the synthesis is given in the Figure 2.

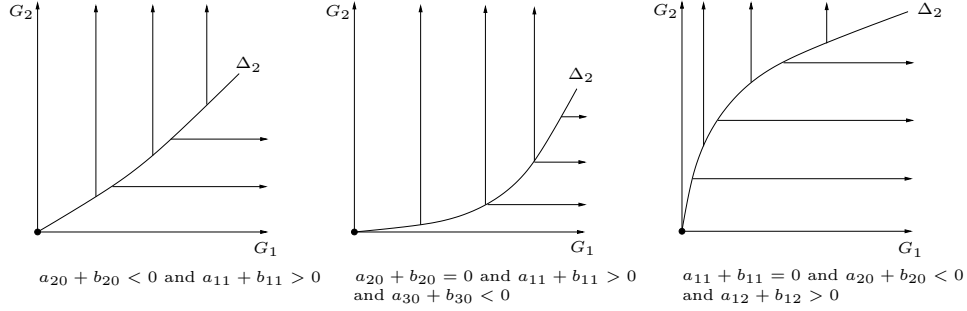


Figure 2: The syntheses when  $a_{10} + b_{10} = 0$  and  $\Delta_2$  is a turnpike

**3rd. case.**  $\Delta_2 \cap (\mathbb{R}_+^* \times \mathbb{R}_+^*)$  is not empty locally and is a anti-turnpike. It corresponds to the cases where  $a_{10} + b_{10} = 0$  and

- $a_{20} + b_{20} > 0$  and  $a_{11} + b_{11} < 0$ ,
- or  $a_{20} + b_{20} = 0$  and  $a_{11} + b_{11} < 0$  and  $a_{30} + b_{30} > 0$ ,
- or  $a_{11} + b_{11} = 0$  and  $a_{20} + b_{20} > 0$  and  $a_{12} + b_{12} < 0$ .

Then, as seen in Proposition 17, no  $u_2$ -singular is extremal. hence the possible beginning of symbols entering the domain are  $[[G_1, G_2]]$  and  $[[G_2, G_1]]$ . In order to complete the synthesis in this case, we have to compute the cut time and cut locus. In fact the two kind of extremals intersect before their second switching time. Let prove it.

Let fix an  $\epsilon_2 > 0$  and consider at time  $t > \epsilon_2$  the extremal with symbol  $[[G_2, G_1]]$  switching at time  $\epsilon_2$ . One computes easily that  $x(t) = t - \epsilon_2$  and  $y(t) = \epsilon_2$ . For an  $\epsilon_1 > 0$  and the extremal

with symbol  $[[G_1, G_2]]$  switching at time  $\epsilon_1$ , one gets by integrating the equations that

$$\begin{aligned}
x(t) &= \epsilon_1 + a_{10}\epsilon_1(t - \epsilon_1) + a_{20}\epsilon_1^2(t - \epsilon_1) + \frac{1}{2}(a_{10}^2 + a_{11})\epsilon_1(t - \epsilon_1)^2 \\
&\quad + a_{30}\epsilon_1^3(t - \epsilon_1) + \frac{1}{2}(3a_{10}a_{20} + a_{21} + a_{11}b_{10})\epsilon_1^2(t - \epsilon_1)^2 \\
&\quad + \frac{1}{3}\left(\frac{1}{2}a_{10}^3 + \frac{3}{2}a_{10}a_{11} + a_{12}\right)\epsilon_1(t - \epsilon_1)^3 \\
y(t) &= (t - \epsilon_1) + b_{10}\epsilon_1(t - \epsilon_1) + b_{20}\epsilon_1^2(t - \epsilon_1) + \frac{1}{2}(a_{10}b_{10} + b_{11})\epsilon_1(t - \epsilon_1)^2 \\
&\quad + b_{30}\epsilon_1^3(t - \epsilon_1) + \frac{1}{2}(a_{20}b_{10} + b_{10}b_{11} + 2a_{10}b_{20} + b_{21})\epsilon_1^2(t - \epsilon_1)^2 \\
&\quad + \frac{1}{3}\left(\frac{1}{2}(a_{10}^2 + a_{11})b_{10} + a_{10}b_{11} + b_{12}\right)\epsilon_1(t - \epsilon_1)^3
\end{aligned}$$

Let assume first that  $a_{20} + b_{20} > 0$  and  $a_{11} + b_{11} < 0$ . Along the first front (depending on  $\epsilon_2$ )  $x + y = t$  when along the second  $x + y = t + \epsilon_1(t - \epsilon_1)((a_{20} + b_{20})\epsilon_1 + \frac{1}{2}(a_{11} + b_{11}))$  hence they are transverse at

$$\epsilon_1 = \frac{t}{1 - \frac{2(a_{20} + b_{20})}{a_{11} + b_{11}}}$$

and they intersect at a point such that  $y = -2\frac{a_{20} - b_{20}}{a_{11} - b_{11}}x + o(x)$ . As seen previously, the switching locus for extremals with symbol  $[[G_2, G_1]]$  satisfies  $y = -\frac{a_{20} - b_{20}}{a_{11} - b_{11}}x + o(x)$  hence it stops to be optimal before switching. The same holds true for the extremals with symbol  $[[G_1, G_2]]$ . Finally the cut locus satisfies

$$y_{cut} = -2\frac{a_{20} - b_{20}}{a_{11} - b_{11}}x_{cut} + o(x_{cut})$$

and is tangent to  $\Delta_2$ .

The same computations can be done when  $G_1$  or  $G_2$  is tangent to  $\Delta_2$ . Then one computes that the extremals lose optimality by crossing the cut before the second switch and that

- if  $a_{20} + b_{20} = 0$  then

$$y_{cut} = -3\frac{a_{30} + b_{30}}{a_{11} + b_{11}}x_{cut}^2 + o(x_{cut}^2),$$

- if  $a_{11} + b_{11} = 0$  then

$$x_{cut} = -\frac{1}{2}\frac{a_{12} + b_{12}}{a_{20} + b_{20}}y_{cut}^2 + o(y_{cut}^2).$$

In all cases the cut is tangent to  $\Delta_2$  and the contact is of order 2 when  $(a_{20} + b_{20})(a_{11} + b_{11}) = 0$ .

In this case 3, the picture of the synthesis is given in the Figure 3.

*Remark 21.* Using the symmetries presented in section 3.7, one can obtain from the optimal synthesis in the domain  $\mathbb{R}_+ \times \mathbb{R}_+$  the optimal synthesis in the three other domains.



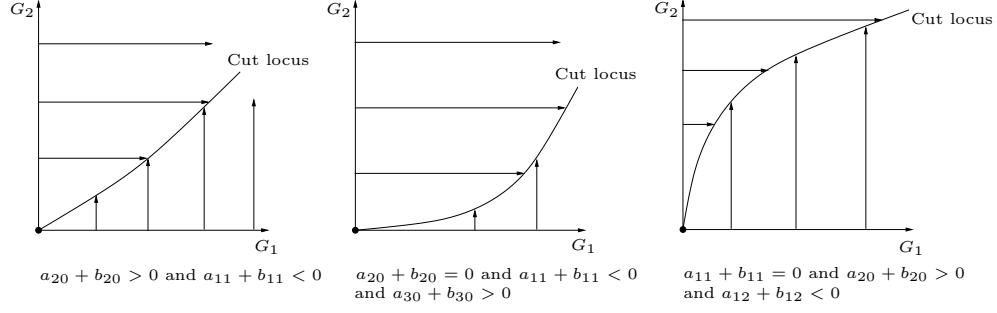


Figure 3: The syntheses when  $a_{10} + b_{10} = 0$  and  $\Delta_2$  is not a turnpike

#### 4.1.3 Optimal synthesis in the domain $\mathbb{R}_- \times \mathbb{R}_-$

The dynamics entering  $\mathbb{R}_-^* \times \mathbb{R}_-^*$  is with  $u_1 \equiv -1$  since  $u_2$  switches (Propositions 14 and 15). Three different cases can be identified.

**1st. case.**  $\Delta_2 \cap (\mathbb{R}_- \times \mathbb{R}_- \setminus \{0\})$  is empty locally. No  $u_2$ -singular enters the domain. It corresponds to the case  $(NF_{1a})$  where  $|a_{10}| \neq |b_{10}|$  and to the cases  $(NF_{1c})$  and  $(NF_{1d})$  where  $a_{10} + b_{10} = 0$  and

- $(a_{20} + b_{20})(a_{11} + b_{11}) > 0$ ,
- or  $a_{20} + b_{20} = 0$  and  $(a_{30} + b_{30})(a_{11} + b_{11}) < 0$ ,
- or  $a_{11} + b_{11} = 0$  and  $(a_{20} + b_{20})(a_{12} + b_{12}) < 0$ .

Only one  $u_2$ -switch can occur along the extremal. One has  $f_2 > 0$  in the domain if

- $a_{10} + b_{10} < 0$ ,
- or  $a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} > 0$ ,
- or  $a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} = 0$  and  $a_{11} + b_{11} > 0$ ,

and in this case the possible extremals of the domain have symbol  $[-G_1]$  or  $[-G_2]$  or  $[-G_1, -G_2]$ . One has  $f_2 < 0$  in the domain if

- $a_{10} + b_{10} > 0$ ,
- or  $a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} < 0$ ,
- or  $a_{10} + b_{10} = 0$  and  $a_{20} + b_{20} = 0$  and  $a_{11} + b_{11} < 0$ .

and in this case the possible extremals of the domain have symbol  $[-G_1]$  or  $[-G_2]$  or  $[-G_2, -G_1]$ .

**2nd. case.**  $\Delta_2 \cap (\mathbb{R}_-^* \times \mathbb{R}_-^*)$  is not empty locally and is a turnpike. It corresponds to the cases where  $a_{10} + b_{10} = 0$  and

- $a_{20} + b_{20} < 0$  and  $a_{11} + b_{11} > 0$ ,
- or  $a_{20} + b_{20} = 0$  and  $a_{11} + b_{11} > 0$  and  $a_{30} + b_{30} > 0$ ,
- or  $a_{11} + b_{11} = 0$  and  $a_{20} + b_{20} < 0$  and  $a_{12} + b_{12} < 0$ .

In this case, the possible symbols for extremals are  $[-G_1]$ ,  $[-G_2]$ ,  $[S_2^-, -G_1]$  and  $[S_2^-, -G_2]$ .

**3rd. case.**  $\Delta_2 \cap (\mathbb{R}_+^* \times \mathbb{R}_-^*)$  is not empty locally and is a anti-turnpike. It corresponds to the cases where  $a_{10} + b_{10} = 0$  and

- $a_{20} + b_{20} > 0$  and  $a_{11} + b_{11} < 0$ ,
- or  $a_{20} + b_{20} = 0$  and  $a_{11} + b_{11} < 0$  and  $a_{30} + b_{30} < 0$ ,
- or  $a_{11} + b_{11} = 0$  and  $a_{20} + b_{20} > 0$  and  $a_{12} + b_{12} > 0$ .

Then the only optimal symbols are  $[-G_1]$ ,  $[-G_2]$ ,  $[-G_1, -G_2]$  and  $[-G_2, -G_1]$ . Moreover

- if  $a_{20} + b_{20} > 0$  and  $a_{11} + b_{11} < 0$ , the cut locus satisfies

$$y_{cut} = -2 \frac{a_{20} + b_{20}}{a_{11} + b_{11}} x_{cut} + o(x_{cut}),$$

- if  $a_{20} + b_{20} = 0$  then

$$y_{cut} = -3 \frac{a_{30} + b_{30}}{a_{11} + b_{11}} x_{cut}^2 + o(x_{cut}^2),$$

- if  $a_{11} + b_{11} = 0$  then

$$x_{cut} = -\frac{1}{2} \frac{a_{12} + b_{12}}{a_{20} + b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases the cut is tangent to  $\Delta_2$  and the contact is of order 2 when  $(a_{20} + b_{20})(a_{11} + b_{11}) = 0$ .

#### 4.1.4 Optimal synthesis in the domain $\mathbb{R}_+ \times \mathbb{R}_-$

The dynamics entering  $\mathbb{R}_+^* \times \mathbb{R}_-^*$  is with  $u_2 \equiv 1$  since  $u_1$  switches (Propositions 14 and 15). Three different cases can be identified.

**1st. case.**  $\Delta_1 \cap (\mathbb{R}_+ \times \mathbb{R}_- \setminus \{0\})$  is empty locally. No  $u_1$ -singular enters the domain. It corresponds to the case  $(NF_{1a})$  where  $|a_{10}| \neq |b_{10}|$  and to the cases  $(NF_{1b})$  and  $(NF_{1d})$  where  $a_{10} - b_{10} = 0$  and

- $(a_{20} - b_{20})(a_{11} - b_{11}) < 0$ ,
- or  $a_{20} - b_{20} = 0$  and  $(a_{30} - b_{30})(a_{11} - b_{11}) < 0$ ,
- or  $a_{11} - b_{11} = 0$  and  $(a_{20} - b_{20})(a_{12} - b_{12}) > 0$ .

Only one  $u_1$ -switch can occur along the extremal. One has  $f_1 > 0$  in the domain if

- $a_{10} - b_{10} > 0$ ,
- or  $a_{10} - b_{10} = 0$  and  $a_{20} - b_{20} > 0$ ,
- or  $a_{10} - b_{10} = 0$  and  $a_{20} - b_{20} = 0$  and  $a_{11} + b_{11} < 0$ ,

and in this case the possible extremals of the domain have symbol  $[G_1]$  or  $[-G_2]$  or  $[-G_2, G_1]$ . One has  $f_1 < 0$  in the domain if

- $a_{10} - b_{10} < 0$ ,

- or  $a_{10} - b_{10} = 0$  and  $a_{20} - b_{20} < 0$ ,
- or  $a_{10} - b_{10} = 0$  and  $a_{20} - b_{20} = 0$  and  $a_{11} - b_{11} > 0$ .

and in this case the possible extremals of the domain have symbol  $[[G_1]]$  or  $[[ -G_2]]$  or  $[[G_1, -G_2]]$ .

**2nd. case.**  $\Delta_1 \cap (\mathbb{R}_+^* \times \mathbb{R}_-^*)$  is not empty locally and is a turnpike. It corresponds to the cases where  $a_{10} - b_{10} = 0$  and

- $a_{20} - b_{20} > 0$  and  $a_{11} - b_{11} > 0$ ,
- or  $a_{20} - b_{20} = 0$  and  $a_{11} - b_{11} > 0$  and  $a_{30} - b_{30} > 0$ ,
- or  $a_{11} - b_{11} = 0$  and  $a_{20} - b_{20} > 0$  and  $a_{12} - b_{12} < 0$ .

In this case, the possible symbols for extremals are  $[[G_1]]$ ,  $[[ -G_2]]$ ,  $[[S_1^+, G_1]]$  and  $[[S_1^+, -G_2]]$ .

**3rd. case.**  $\Delta_1 \cap (\mathbb{R}_+^* \times \mathbb{R}_-^*)$  is not empty locally and is a anti-turnpike. It corresponds to the cases where  $a_{10} - b_{10} = 0$  and

- $a_{20} - b_{20} < 0$  and  $a_{11} - b_{11} < 0$ ,
- or  $a_{20} - b_{20} = 0$  and  $a_{11} - b_{11} < 0$  and  $a_{30} - b_{30} < 0$ ,
- or  $a_{11} - b_{11} = 0$  and  $a_{20} - b_{20} < 0$  and  $a_{12} - b_{12} > 0$ .

Then the only optimal symbols are  $[[G_1]]$ ,  $[[ -G_2]]$ ,  $[[G_1, -G_2]]$  and  $[[ -G_2, G_1]]$ . Moreover

- if  $a_{20} - b_{20} < 0$  and  $a_{11} + b_{11} < 0$ , the cut locus satisfies

$$y_{cut} = -2 \frac{a_{20} - b_{20}}{a_{11} - b_{11}} x_{cut} + o(x_{cut}),$$

- if  $a_{20} - b_{20} = 0$  then

$$y_{cut} = -3 \frac{a_{30} - b_{30}}{a_{11} - b_{11}} x_{cut}^2 + o(x_{cut}^2),$$

- if  $a_{11} - b_{11} = 0$  then

$$x_{cut} = -\frac{1}{2} \frac{a_{12} - b_{12}}{a_{20} - b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases the cut is tangent to  $\Delta_1$  and the contact is of order 2 when  $(a_{20} - b_{20})(a_{11} - b_{11}) = 0$ .

#### 4.1.5 Optimal synthesis in the domain $\mathbb{R}_- \times \mathbb{R}_+$

The dynamics entering  $\mathbb{R}_-^* \times \mathbb{R}_+^*$  is with  $u_2 \equiv -1$  since  $u_1$  switches (Propositions 14 and 15). Three different cases can be identified.

**1st. case.**  $\Delta_1 \cap (\mathbb{R}_- \times \mathbb{R}_+ \setminus \{0\})$  is empty locally. No  $u_1$ -singular enters the domain. It corresponds to the case  $(NF_{1a})$  where  $|a_{10}| \neq |b_{10}|$  and to the cases  $(NF_{1b})$  and  $(NF_{1d})$  where  $a_{10} - b_{10} = 0$  and

- $(a_{20} - b_{20})(a_{11} - b_{11}) < 0$ ,
- or  $a_{20} - b_{20} = 0$  and  $(a_{30} - b_{30})(a_{11} - b_{11}) > 0$ ,

- or  $a_{11} - b_{11} = 0$  and  $(a_{20} - b_{20})(a_{12} - b_{12}) < 0$ .

Only one  $u_1$ -switch can occur along the extremal. One has  $f_1 > 0$  in the domain if

- $a_{10} - b_{10} > 0$ ,
- or  $a_{10} - b_{10} = 0$  and  $a_{20} - b_{20} < 0$ ,
- or  $a_{10} - b_{10} = 0$  and  $a_{20} - b_{20} = 0$  and  $a_{11} + b_{11} > 0$ ,

and in this case the possible extremals of the domain have symbol  $[[ -G_1 ]]$  or  $[[ G_2 ]]$  or  $[[ -G_1, G_2 ]]$ . One has  $f_1 < 0$  in the domain if

- $a_{10} - b_{10} < 0$ ,
- or  $a_{10} - b_{10} = 0$  and  $a_{20} - b_{20} > 0$ ,
- or  $a_{10} - b_{10} = 0$  and  $a_{20} - b_{20} = 0$  and  $a_{11} - b_{11} < 0$ ,

and in this case the possible extremals of the domain have symbol  $[[ -G_1 ]]$  or  $[[ G_2 ]]$  or  $[[ G_2, -G_1 ]]$ .

**2nd. case.**  $\Delta_1 \cap (\mathbb{R}_-^* \times \mathbb{R}_+^*)$  is not empty locally and is a turnpike. It corresponds to the cases where  $a_{10} - b_{10} = 0$  and

- $a_{20} - b_{20} > 0$  and  $a_{11} - b_{11} > 0$ ,
- or  $a_{20} - b_{20} = 0$  and  $a_{11} - b_{11} > 0$  and  $a_{30} - b_{30} < 0$ ,
- or  $a_{11} - b_{11} = 0$  and  $a_{20} - b_{20} > 0$  and  $a_{12} - b_{12} > 0$ .

In this case, the possible symbols for extremals are  $[[ -G_1 ]]$ ,  $[[ G_2 ]]$ ,  $[[ S_1^-, -G_1 ]]$  and  $[[ S_1^-, G_2 ]]$ .

**3rd. case.**  $\Delta_1 \cap (\mathbb{R}_-^* \times \mathbb{R}_+^*)$  is not empty locally and is a anti-turnpike. It corresponds to the cases where  $a_{10} - b_{10} = 0$  and

- $a_{20} - b_{20} < 0$  and  $a_{11} - b_{11} < 0$ ,
- or  $a_{20} - b_{20} = 0$  and  $a_{11} - b_{11} < 0$  and  $a_{30} - b_{30} > 0$ ,
- or  $a_{11} - b_{11} = 0$  and  $a_{20} - b_{20} < 0$  and  $a_{12} - b_{12} < 0$ .

Then the only optimal symbols are  $[[ -G_1 ]]$ ,  $[[ G_2 ]]$ ,  $[[ -G_1, G_2 ]]$  and  $[[ G_2, -G_1 ]]$ . Moreover

- if  $a_{20} - b_{20} < 0$  and  $a_{11} + b_{11} < 0$ , the cut locus satisfies

$$y_{cut} = -2 \frac{a_{20} - b_{20}}{a_{11} - b_{11}} x_{cut} + o(x_{cut}),$$

- if  $a_{20} - b_{20} = 0$  then

$$y_{cut} = -3 \frac{a_{30} - b_{30}}{a_{11} - b_{11}} x_{cut}^2 + o(x_{cut}^2),$$

- if  $a_{11} - b_{11} = 0$  then

$$x_{cut} = -\frac{1}{2} \frac{a_{12} - b_{12}}{a_{20} - b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases the cut is tangent to  $\Delta_1$  and the contact is of order 2 when  $(a_{20} - b_{20})(a_{11} - b_{11}) = 0$ .

## 4.2 $(NF_{2a})$ case

Recall that the normal form  $(NF_{2a})$  gives

$$G_1(x, y) = \partial_x, \quad G_2(x, y) = (a_0 + a_{10}x + o_1(x, y))\partial_x + (x + b_{20}x^2 + o(x, y))\partial_y,$$

with  $0 \leq a_0 < 1$ .

Such a point is neither in  $\Delta_1$  nor  $\Delta_2$ . Hence no singular extremal will appear in the study of the local synthesis.

One can compute easily that, for any extremal starting at 0,  $\phi_1(0) = \frac{1}{2}\lambda_x(0)(1 + a_0)$  and  $\phi_2(0) = \frac{1}{2}\lambda_x(0)(1 - a_0)$ . With  $H = 0$  it gives  $|\lambda_x(0)| = 1$ . Hence, since  $\dot{\phi}_1 = -u_2\phi_3$  and  $\dot{\phi}_2 = u_1\phi_3$ , if we want to study extremals that switch in short time, we need to consider  $\phi_3$  large that is  $|\lambda_y|$  large.

Moreover, since along an extremal issued from 0  $|\dot{x}(t)| \leq 1$  for  $t$  small, one gets easily that  $|x(t)| \leq t$  and  $|y(t)| \leq t^2$  for  $t$  small enough. Hence  $\phi_1(t) = \frac{1+a_0}{2}\lambda_x(0) + x(t)\lambda_y(0) + o(t, x(t)\lambda_y(0))$  and  $\phi_2(t) = \frac{1-a_0}{2}\lambda_x(0) + x(t)\lambda_y(t) + o(t, x(t)\lambda_y(t))$ . This implies that if one wants to consider an extremal switching at time  $\tau$  small, he should consider initial conditions  $\lambda_y(0) \sim \frac{1}{\tau}$ . Inversing the point of view, if we consider an initial condition  $\lambda_y(0) = \frac{1}{r_0}$  with  $r_0$  small, the switching time should be of order 1 in  $r_0$ . This motivates the following change of coordinates on the fibers of the cotangent:  $r = \frac{1}{\lambda_y}$ ,  $p = r\lambda_x$  and the change of time  $s = t/r$ .

### 4.2.1 equations of the dynamics

With the new variables  $(x, y, p, r)$  and the new time  $s$ , the Hamiltonian equations become

$$\begin{aligned} x' &= r \frac{\partial H}{\partial \lambda_x}(x, y, p, -1), \\ y' &= r \frac{\partial H}{\partial \lambda_y}(x, y, p, -1), \\ p' &= -r \frac{\partial H}{\partial x}(x, y, p, -1) + rp \frac{\partial H}{\partial y}(x, y, p, -1), \\ r' &= r^2 \frac{\partial H}{\partial y}(x, y, p, -1). \end{aligned}$$

Now, looking for the solutions as taylor series in  $r_0$ , that is under the form

$$\begin{aligned} x(r_0, s) &= x_1(s)r_0 + x_2(s)r_0^2 + o(r_0^2), & p(r_0, s) &= p_1(s)r_0 + p_2(s)r_0^2 + o(r_0^2), \\ y(r_0, s) &= y_2(s)r_0^2 + y_3(s)r_0^3 + o(r_0^3), & r(r_0, s) &= r_0 + r_2(s)r_0^2 + o(r_0^2), \end{aligned}$$

one finds the equations

$$\begin{aligned} x'_1 &= \frac{u_1+u_2}{2} + \frac{u_1-u_2}{2}a_0, & x'_2 &= \frac{u_1-u_2}{2}a_{10}x_1, \\ y'_2 &= \frac{u_1-u_2}{4}x_1, & y'_3 &= \frac{u_1-u_2}{2}(b_{20}x_1^2 + x_2), \\ p'_1 &= -\frac{u_1-u_2}{2}x_1, & p'_2 &= -\frac{u_1-u_2}{2}(a_{10}p_1 + 2b_{20}x_1), \\ r'_2 &= 0, \end{aligned}$$

### 4.2.2 Computation of the jets

Using these equations, we are able to compute the jets with respect to  $r_0$  of four types of extremals: depending on the sign of  $p(0) = \pm 1$  and of  $r_0$ . For each of these types we can compute the functions  $x_1, x_2, y_2, y_3, p_1, p_2$  and  $r_2 \equiv 0$  of the variable  $s$  for the first bang. We can then compute the jets of  $\phi_1$  and  $\phi_2$  for the first bang and look for the first switching time under the form  $s_1 = s_{10} + s_{11}r_0$  and then repeat the procedure for the second bang and so on. Finally, if we denote  $\delta_p = \text{sign}(p(0))$  and  $\delta_r = \text{sign}(r_0)$  then the controls during the first bang are  $u_1 = u_2 = \delta_p$ . The first time of switch is

$$s_1 = \delta_r(1 - \delta_r a_0) - \delta_p(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20})r_0 + o(r_0)$$

and corresponds to  $\phi_2(s_1) = 0$  if  $\delta_r = 1$  or  $\phi_1(s_1) = 0$  if  $\delta_r = -1$ . The second bang corresponds to  $u_1 = \delta_p \delta_r$  and  $u_2 = -\delta_p \delta_r$  and the second switch is at

$$s_2 = \delta_r(3 - \delta_r a_0) - \delta_p((1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20})r_0 + o(r_0)$$

where  $\phi_1(s_2) = 0$  if  $\delta_r = 1$  and  $\phi_2(s_2) = 0$  if  $\delta_r = -1$ . At this time

$$\begin{aligned} x(s_2) &= \delta_p(\delta_r + a_0)r_0 - \delta_r(\delta_r + a_0)(-\delta_r a_{10} + b_{20} + \delta_r a_0 b_{20})r_0^2 + o(r_0^2), \\ y(s_2) &= 2\delta_r r_0^2 - \delta_p \frac{4}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})r_0^3 + o(r_0^3). \end{aligned}$$

The third bang corresponds to  $u_1 = u_2 = -1$  if  $\delta_p = 1$  and to  $u_1 = u_2 = 1$  if  $\delta_p = -1$ . The third switching time satisfies  $s_3 = \delta_r(5 - \delta_r a_0) + O(r_0)$  and the corresponding time  $t_3$  is larger than the cut time as we will see later.

Let us analyze a little the situation in terms of cut locus for these extremals: if we consider the extremals with  $\delta_p = \delta_r = 1$ , they all start following  $G_1$ , without loosing optimality. Then they switch to  $G_2$  at  $t = r_0(1 - a_0) + o(r_0)$ . During this second bang, they do not intersect one each other since they are all following  $G_2$  with a different initial condition on  $\{x > 0, y = 0\}$ . Then they switch to  $-G_1$  but at a different  $y$  hence again they cannot intersect. The loss of optimality cannot come from an intersection with extremals with  $\delta_r = -1$  since these last one live in  $\{y \leq 0\}$ . As we will see in the following, the loss of optimality will come from the intersection with an extremal with  $-\delta_p = \delta_r = 1$  during the third bang. Of course, the same occurs for extremals with  $\delta_r = -1$ .

Let fix a small parameter  $\rho > 0$ . Since the dynamics during the third bang of all the extremals is given by  $\pm G_1 = \pm \partial_x$ ,  $y$  is constant during these third bangs. Hence, for the extremals with  $\delta_r = 1$ , we can look for the  $r_0$ , as a jet in  $\rho$ , such that  $y = 2\rho^2$  during the third bang, and for the extremals with  $\delta_r = -1$ , we can look for the  $r_0$ , as a jet in  $\rho$ , such that  $y = -2\rho^2$  during the third bang. The result is

$$r_0 = \delta_r \rho + \delta_r \delta_p \frac{1}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2)$$

which allows to compute

$$t_2 = (3 - \delta_r a_0)\rho - \delta_r \delta_p \frac{3a_{10} - a_0^2 a_{10} + \delta_r 6b_{20} - 3a_0 b_{20} + a_0^3 b_{20}}{3}\rho^2 + o(\rho^2).$$

Hence we can compute  $x(t) = x(t_2) + (t - t_2)$  for this  $r_0$  that is

$$x(t) = -\delta_p t + \delta_p 4\rho - \frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2).$$

We are now in situation to complete the computation of the jet of the cut locus: an extremal intersects an extremal of same length at the time  $t_{cut} = 4\rho + o(\rho^2)$  which is less than  $t_3 = (5 - \delta_r a_0)\rho$  hence  $t_{cut}$  is the cut time. When  $\delta_r = 1$  the cut point satisfies

$$x_{cut} = -\frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2), \quad y_{cut} = 2\rho^2,$$

and when  $\delta_r = -1$  the cut point satisfies

$$x_{cut} = -\frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2), \quad y_{cut} = -2\rho^2.$$

Finally, if one wants to describe the sphere at time  $t$  small, one have that the first switching time is

$$t_1 = \delta_r(1 - \delta_r a_0)r_0 - \delta_p(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20})r_0^2 + o(r_0^2)$$

and hence, at  $t$  small, the  $r_0$  corresponding to a first switching point is

$$r_1 = \frac{t}{\delta_r(1 - \delta_r a_0)} + \delta_r \delta_p \frac{\delta_r a_{10} + b_{20}(1 - \delta_r a_0)}{(1 - a_0)^2} t^2 + o(t^2).$$

The second switching time is

$$t_2 = \delta_r(3 - \delta_r a_0)r_0 - \delta_p((1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20})r_0^2 + o(r_0^2)$$

which implies that, at  $t$  small, the  $r_0$  corresponding to a second switching point is

$$r_2 = \frac{t}{\delta_r(3 - \delta_r a_0)} + \delta_p \frac{(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20}}{\delta_r(3 - \delta_r a_0)^3} t^2 + o(t^2).$$

And the cut time is

$$t_{cut} = 4\delta_r(r_0 - \delta_r \delta_p \frac{1}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})r_0^2) + o(r_0^2)$$

which implies that at  $t$  small the  $r_0$  corresponding to a cut point is

$$r_{cut} = \frac{\delta_r}{4}(t + \frac{\delta_p}{12}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})t^2) + o(t^2).$$

### 4.3 $(NF_{2b})$ case

Recall that the normal form  $(NF_{2b})$  gives  $G_1(x, y) = \partial_x$ , and

$$G_2(x, y) = (1 + a_{10}x + a_{01}y + a_{20}x^2 + o_2(x, y))\partial_x + (x + b_{20}x^2 + b_{30}x^3 + o_3(x, y))\partial_y.$$

In this case, the extremals with initial condition  $|\lambda_y(0)| \gg 1$  are the limit when  $a_0$  goes to 1 of the extremal presented in the case  $(NF_{2a})$ . If  $\lambda_y(0) \gg 1$  then the symbol starts with  $[[G_2, -G_1]]$  or with  $[[ -G_2, G_1]]$  and if  $-\lambda_y(0) \gg 1$  then the symbol starts with  $[[G_1, -G_2]]$  or with  $[[ -G_1, G_2]]$ .

But  $F_2(0) = 0$  then for all extremals  $\phi_2(0) = 0$ . Hence, an extremal may also, depending on the invariants, have symbol starting by  $[[G_2, G_1]]$ ,  $[[G_1, G_2]]$ ,  $[[S_2^+, G_1]]$  or  $[[S_2^+, G_2]]$  if  $\lambda_x(0) = 1$ , and,  $[[ -G_2, -G_1]]$ ,  $[[ -G_1, -G_2]]$ ,  $[[S_2^-, -G_1]]$  or  $[[S_2^-, -G_2]]$  if  $\lambda_x(0) = -1$ .

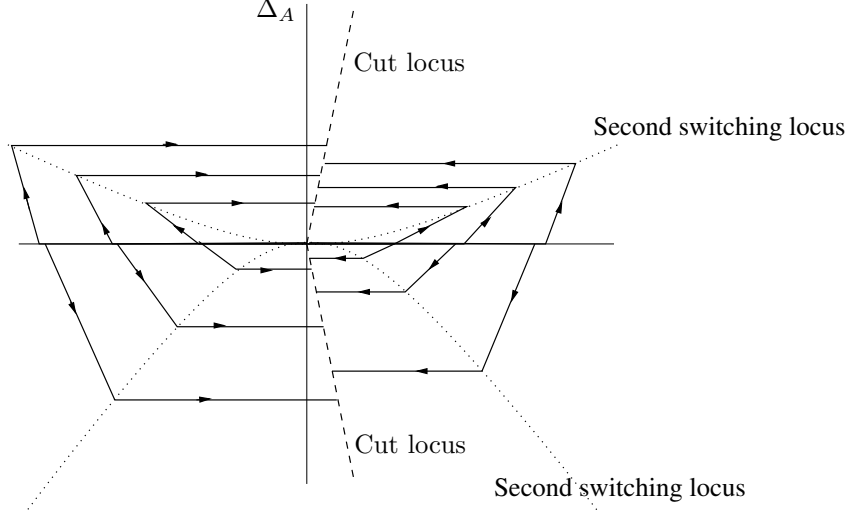


Figure 4: The optimal synthesis in the  $(NF_{2a})$  case

If  $\lambda_x(0) = 1$  then at least for small time  $u_1(t) = 1$  and  $x(t) = t + o(t)$  and  $y(t) = o(t)$ . Then, computing  $\phi_2$  one finds  $\phi_2(t) = -\lambda_x(t)\frac{a_{10}}{2} - \lambda_y(t)\frac{x(t)}{2} + o(t) = -(\frac{a_{10}+\lambda_y(0)}{2})t + o(t)$ . Hence if  $\lambda_y(0) > -a_{10}$  then, since  $\phi_2(0) < 0$  for small time, the extremal starts by a bang following  $G_2$ . If  $\lambda_y(0) < -a_{10}$  then  $\phi_2(0) > 0$  for small time and the extremal starts by a bang following  $G_1$ .

If  $\lambda_x(0) = -1$  then at least for small time  $u_1(t) = -1$  and  $x(t) = -t + o(t)$  and  $y(t) = o(t)$ . Then  $\phi_2(t) = (\frac{a_{10}-\lambda_y(0)}{2})t + o(t)$ . Hence if  $\lambda_y(0) > a_{10}$  then, since  $\phi_2(0) < 0$  for small time, the extremal starts by a bang following  $-G_1$ . If  $\lambda_y(0) < a_{10}$  then  $\phi_2(0) > 0$  for small time and the extremal starts by a bang following  $-G_2$ .

In coordinates, one can compute that

$$\det(F_2, [F_1, F_2])(x, y) = \frac{1}{4}((a_{10}b_{20} - a_{20})x^2 + a_{01}y) + o_2(x, y)$$

where  $x$  has weight 1 and  $y$  has weight 2. Since generically at such points (which are isolated points)  $a_{01} \neq 0$  then an equation for  $\Delta_2$  is given by

$$y = \frac{a_{20} - a_{10}b_{20}}{a_{01}}x^2 + o(x^2).$$

Remark that generically  $\frac{a_{20}-a_{10}b_{20}}{a_{01}}$  is neither 0 nor  $\frac{1}{2}$ . Moreover

$$f_2(x, y) = \frac{\det(F_2, [F_1, F_2])(x, y)}{\det(F_2, F_1)(x, y)} = \frac{((a_{10}b_{20} - a_{20})x^2 + a_{01}y) + o_2(x, y)}{2x}.$$

Recall that an equation of the support of the integral curve of  $G_1$  passing by 0 is  $y = 0$  and that an equation for the support of the integral curve of  $G_2$  passing by 0 is  $y = \frac{x^2}{2} + o(x^2)$ .

If  $\frac{a_{20}-a_{10}b_{20}}{a_{01}} < 0$  or if  $\frac{a_{20}-a_{10}b_{20}}{a_{01}} > \frac{1}{2}$  then  $\Delta_2$  does not enter the domain  $\mathcal{D} = \{x > 0, 0 < y < \frac{x^2}{2}\}$  and along it  $G_1$  and  $G_2$  point on the same side of  $\Delta_2$  hence  $\Delta_2$  is not a turnpike. In these cases:



- if  $a_{10}b_{20} - a_{20} > 0$  then  $f_2 > 0$  in  $\mathcal{D}$  and the new extremals, that are not described as limit of the case  $NF_{2a}$ , have symbol  $[[G_2, G_1]]$ .
- if  $a_{10}b_{20} - a_{20} < 0$  then  $f_2 < 0$  in  $\mathcal{D}$  and the new extremals, that are not described as limit of the case  $NF_{2a}$ , have symbol  $[[G_1, G_2]]$ .

If  $\frac{a_{20}-a_{10}b_{20}}{a_{01}} > 0$  and  $\frac{a_{20}-a_{10}b_{20}}{a_{01}} < \frac{1}{2}$  then  $\Delta_2$  enters  $\mathcal{D}$  and along it  $G_1$  and  $G_2$  point on opposite sides of  $\Delta_2$ . In this case:

- if  $a_{10}b_{20} - a_{20} > 0$  then, along  $\Delta_2 \cap \mathcal{D}$ ,  $G_1$  points in direction of  $f_2 > 0$  and  $\Delta_2$  is a turnpike. Then, the only extremals entering the domain  $\mathcal{D}$  start with a singular arc and have symbols  $[[S_2^+]]$ ,  $[[S_2^+, G_1]]$  or  $[[S_2^+, G_2]]$ .
- if  $a_{10}b_{20} - a_{20} < 0$  then, along  $\Delta_2 \cap \mathcal{D}$ ,  $G_1$  points in direction of  $f_2 < 0$  and  $\Delta_2$  is not a turnpike. In this case the symbols start with  $[[G_1, G_2]]$  and  $[[G_2, G_1]]$ . One can compute, with the same techniques that in section 4.2.2, the switching times and the second switching locus for extremals that enter the domain  $\mathcal{D}$ , that is for extremal with initial condition  $\lambda_y(0) = -a_{10} + \delta\epsilon$  with  $\epsilon > 0$  small and  $\delta = \pm 1$ . If  $\delta < 0$  then the symbol is  $[[G_1, G_2, G_1]]$  and the switching times are  $t_1 = \frac{\epsilon}{a_{20}-a_{10}b_{20}}$  and  $t_2 = t_1 + \frac{2\epsilon}{a_{01}-2a_{20}+2a_{10}b_{20}}$ , the second switching locus being

$$x(\epsilon) = \frac{a_{01}\epsilon}{(a_{20} - a_{10}b_{20})(a_{01} - 2a_{20} + 2a_{10}b_{20})}, \quad y(\epsilon) = \frac{2(a_{01} - a_{20} + a_{10}b_{20})\epsilon^2}{(a_{20} - a_{10}b_{20})(a_{01} - 2a_{20} + 2a_{10}b_{20})^2}.$$

If  $\delta > 0$  then the symbol is  $[[G_2, G_1, G_2]]$  and the switching times are  $t_1 = \frac{2\epsilon}{a_{01}-2a_{20}+2a_{10}b_{20}}$  and  $t_2 = t_1 + \frac{\epsilon}{a_{20}-a_{10}b_{20}}$ , the second switching locus being

$$x(\epsilon) = \frac{a_{01}\epsilon}{(a_{20} - a_{10}b_{20})(a_{01} - 2a_{20} + 2a_{10}b_{20})}, \quad y(\epsilon) = \frac{2\epsilon^2}{(a_{01} - 2a_{20} + 2a_{10}b_{20})^2}.$$

One prove easily that all these extremals cut  $\Delta_2$  before the second switching. Moreover they cannot be optimal after the second switching (by considerations on the jacobian). Hence the only optimal symbols entering the domain  $\mathcal{D}$  are  $[[G_1, G_2]]$  and  $[[G_2, G_1]]$ .

#### 4.4 $(NF_3)$ case

Recall that in the  $(NF_3)$  case,  $x$  has weight 1 and  $y$  has weight 3. Hence we can write

$$G_1(x, y) = \partial_x \quad G_2(x, y) = (a_0 + a_{10}x + o(x, y))\partial_x + \left(\frac{x^2}{2} + b_{01}y + b_{30}x^3 + o_3(x, y)\right)\partial_y$$

with  $b_{0,1} \neq 0$  and  $0 < a_0 < 1$ , where  $o_k(x, y)$  has the meaning given in subsection 3.5. As in the  $(NF_{2b})$  case, for any extremal starting at 0,

$$\phi_1(0) = \frac{1}{2}\lambda_x(0)(1 + a_0) \quad \text{and} \quad \phi_2(0) = \frac{1}{2}\lambda_x(0)(1 - a_0).$$

And for the same reasons, if we want to study extremals that switch in short time, we need to consider  $|\lambda_y|$  large.

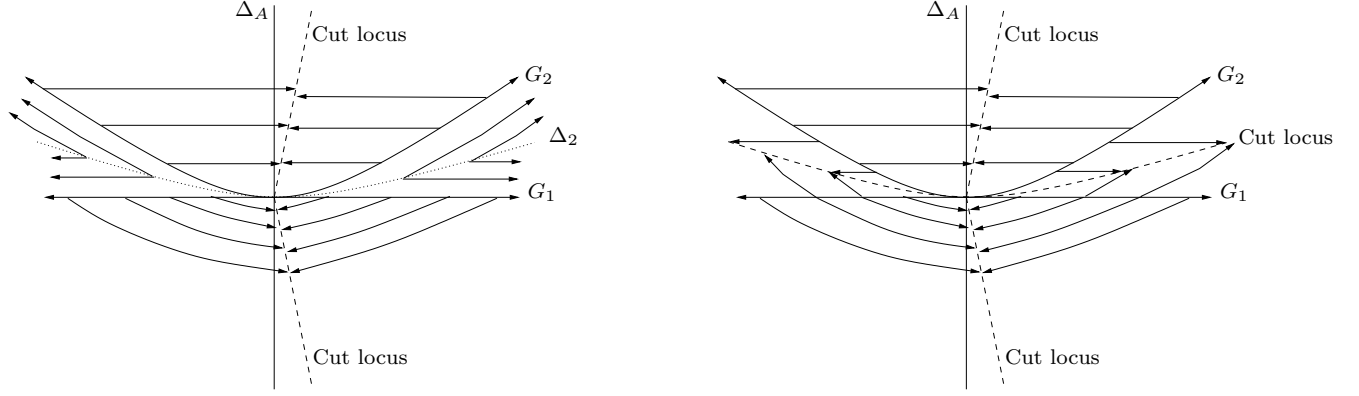


Figure 5: Two different syntheses in the  $(NF_{2b})$  case

The set of initial condition is  $\{(\lambda_x(0), \lambda_y(0)) \mid \lambda_x(0) = \pm 1\}$ . We parameterize the upper part of this set by setting  $\lambda_y(0) = \frac{1}{r_0^2}$  and the lower part by  $\lambda_y(0) = -\frac{1}{r_0^2}$ .

As explained in subsection 3.4, in order to compute extremals with  $\lambda_y(0) \gg 1$  we make the change of coordinates  $r = \frac{1}{\sqrt{\lambda_y}}$ ,  $X = \frac{x}{r}$ ,  $Y = \frac{y}{r^3}$  and the change of time  $s = \frac{t}{r}$ .

Now, looking for the solutions as taylor series in  $r_0$ , that is under the form

$$\begin{aligned} X(r_0, s) &= X_0(s) + r_0 X_1(s) + o(r_0), & \lambda_x(r_0, s) &= \lambda_{x0}(s) + r_0 \lambda_{x1}(s) + o(r_0), \\ Y(r_0, s) &= Y_0(s) + r_0 Y_1(s) + o(r_0), & r(r_0, s) &= r_0 + r_0^2 r_2(s) + o_2(r_0) \end{aligned}$$

one finds the equations

$$\begin{aligned} X'_0(s) &= \frac{1}{2}(u_1 + u_2) + \frac{a_0}{2}(u_1 - u_2), & X'_1(s) &= \frac{(u_1 - u_2)}{4}(2a_{10} - b_{01})X_0(s), \\ Y'_0(s) &= \frac{1}{4}(u_1 - u_2)X_0^2(s), & Y'_1(s) &= \frac{(u_1 - u_2)}{4}(2b_{30}X_0^3(s) + 2X_0(s)X_1(s) - b_{01}Y_0(s)), \\ \lambda'_{x0}(s) &= -\frac{1}{2}(u_1 - u_2)X_0(s), & \lambda'_{x1}(s) &= -\frac{(u_1 - u_2)}{2}(a_{10}\lambda_{x0}(s) + 3b_{30}X_0^2(s) + X_1(s)), \\ r'_2(s) &= \frac{b_{01}}{4}(u_1 - u_2), \end{aligned}$$

For an initial condition  $\lambda_x(0) = 1$ , one find  $\phi_1(0) > 0$  and  $\phi_2(0) > 0$ , hence  $u_1(0) = u_2(0) = 1$ . One can integrate the equations and look for the first switching time as a Taylor series  $s^1 = s_0^1 + r_0 s_1^1 + o(r_0)$  and compute  $\phi_2(r_0, s_0^1 + r_0 s_1^1 + o(r_0))$  in order to compute

$$s_0^1 = \sqrt{2}\sqrt{1 - a_0} \quad \text{and} \quad s_1^1 = -a_{10} - 2b_{30}(1 - a_0).$$

At the switching time

$$\begin{aligned} X(s^1) &= \sqrt{2}\sqrt{1 - a_0} - (a_{10} + 2b_{30})(1 - a_0)r_0, & \lambda_x(s^1) &= 1, \\ Y(s^1) &= 0, & r(s^1) &= r_0. \end{aligned}$$

After this first switch  $\phi_1(0) > 0$  and  $\phi_2(0) < 0$ , hence  $u_1(0) = 1$  and  $u_2(0) = -1$ . We can compute and look for the next switching time and one finds that  $\phi_1$  goes to 0 at  $s^2 = s_0^2 + r_0 s_1^2 + o(r_0)$  with

$$s_0^2 = s_0^1 + \sqrt{2} \frac{\sqrt{1 + a_0} - \sqrt{1 - a_0}}{a_0}, \quad s_1^2 = s_1^1 + \frac{b_{01}((1 - a_0)^{\frac{3}{2}} - \sqrt{1 + a_0}(1 - 2a_0)) - 12b_{30}a_0^2\sqrt{1 + a_0}}{3a_0^2\sqrt{1 + a_0}}.$$

At the second switching time

$$\begin{aligned}
X(s^2) &= (\sqrt{2}\sqrt{1+a_0}) + \left( \frac{3a_{10}a_0\sqrt{1+a_0}+b_{01}((1-a_0)^{\frac{3}{2}}-(1+a_0)^{\frac{3}{2}})-6b_{30}a_0(1+a_0)^{\frac{3}{2}}}{3a_0\sqrt{1+a_0}} \right) r_0, \\
Y(s^2) &= \left( \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}{3a_0} \right) - \left( \frac{2b_{01}(1-a_0+a_0^2-(1-a_0)^{\frac{3}{2}}\sqrt{1+a_0})+12a_0^2b_{30}}{3a_0^2} \right) r_0, \\
\lambda_x(s^2) &= -1, \\
r(s^2) &= r_0 + \left( \frac{(\sqrt{1+a_0}-\sqrt{1-a_0})b_{01}}{\sqrt{2}a_0} \right) r_0^2.
\end{aligned}$$

After this second switch,  $\phi_1(0) < 0$  and  $\phi_2(0) < 0$ , hence  $u_1(0) = u_2(0) = -1$ . One can compute the third switch as being  $s^3 = s_0^3 + r_0s_1^3 + o(r_0)$  with

$$s_0^3 = s_0^2 + 2\sqrt{2}\sqrt{1+a_0}, \quad s_1^3 = s_1^2 - \frac{2((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})b_{01}}{3a_0\sqrt{1+a_0}}.$$

At this time  $X(s^3) = -\sqrt{2}\sqrt{1+a_0} + O(r_0)$  and we will see that this third switching time comes after the cut time.

The same computations can be done for the extremals starting with  $\lambda_x(0) = -1$ . We use the notation  $\bar{z}$  for variables  $z$  corresponding to these extremals. During the first bang the controls are  $\bar{u}_1 = \bar{u}_2 = -1$ , during the second  $\bar{u}_1 = 1$  and  $\bar{u}_2 = -1$  and during the third one  $\bar{u}_1 = \bar{u}_2 = 1$ . The switching times are  $\bar{s}^1$  and  $\bar{s}^2$  satisfying

$$\begin{aligned}
\bar{s}_0^1 &= \sqrt{2}\sqrt{1+a_0}, & \bar{s}_1^1 &= -a_{10} + 2b_{30}(1+a_0), \\
\bar{s}_0^2 &= \bar{s}_0^1 + \sqrt{2}\frac{\sqrt{1+a_0}-\sqrt{1-a_0}}{a_0}, & \bar{s}_1^2 &= \bar{s}_1^1 + \frac{b_{01}((1+a_0)^{\frac{3}{2}}-\sqrt{1-a_0}(1+2a_0))+12b_{30}a_0^2\sqrt{1-a_0}}{3a_0^2\sqrt{1-a_0}}.
\end{aligned}$$

And at the second switching time

$$\begin{aligned}
\bar{X}(\bar{s}^2) &= -(\sqrt{2}\sqrt{1-a_0}) + \left( \frac{-3a_{10}a_0\sqrt{1-a_0}+b_{01}((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})-6b_{30}a_0(1-a_0)^{\frac{3}{2}}}{3a_0\sqrt{1-a_0}} \right) \bar{r}_0 + o(\bar{r}_0), \\
\bar{Y}(\bar{s}^2) &= \left( \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}{3a_0} \right) - \left( \frac{2b_{01}(1+a_0+a_0^2-\sqrt{1-a_0}(1+a_0)^{\frac{3}{2}})-12a_0^2b_{30}}{3a_0^2} \right) \bar{r}_0 + o(\bar{r}_0), \\
\bar{\lambda}_x(\bar{s}^2) &= -1, \\
\bar{r}(\bar{s}^2) &= \bar{r}_0 + \left( \frac{(\sqrt{1+a_0}-\sqrt{1-a_0})b_{01}}{\sqrt{2}a_0} \right) \bar{r}_0^2 + o(\bar{r}_0^2).
\end{aligned}$$

One can compute that at the third switching time  $\bar{X}(\bar{s}^3) = \sqrt{2}\sqrt{1-a_0} + O(r_0)$ .

We are now ready to compute the cut locus. As one can estimate easily, an extremal starting with  $\lambda_x(0) > 0$  intersects an extremal starting with  $\lambda_x(0) < 0$ , both during their third bang. Moreover, since  $Y(s^2) = \bar{Y}(\bar{s}^2) + o(r_0)$  one have that  $\bar{r}_0 = r_0 + o(r_0)$ .

Let fix a  $\rho$  and look for the extremals that intersect at  $y = \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3$ . We write  $r_0 = \rho + R_{cut}\rho^2 + o(\rho^2)$  and look for  $R_{cut}$  such that  $r_0Y(s^2) = \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3 + o(\rho^4)$ . We find

$$R_{cut} = \frac{\sqrt{2}((-2a_0^2 + (2+a_0)(-1+\sqrt{1-a_0^2}))b_{01} + 6a_0^2b_{30})}{3a_0((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}.$$

For  $\bar{r}_0 = \rho + \bar{R}_{cut}\rho^2 + o(\rho^2)$  one finds  $\bar{R}_{cut} = \frac{\sqrt{2}((-2a_0^2+(2-a_0)(-1+\sqrt{1-a_0^2}))b_{01}-6a_0^2b_{30})}{3a_0((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}$ . With these values, we can compute the second switching times  $t^2 = rs^2 = t_1^2\rho + t_2^2\rho^2 + o(\rho^3)$  and  $\bar{t}_2 = \bar{r}\bar{s}^2 = \bar{t}_1^2\rho + \bar{t}_2^2\rho^2 + o(\rho^3)$  with

$$\begin{aligned} t_1^2 &= \sqrt{2} \left( \sqrt{1-a_0} + \frac{\sqrt{1+a_0} - \sqrt{1-a_0}}{a_0} \right) \\ t_2^2 &= -a_{10} + \frac{(-5+2a_0-6a_0^2+a_0^3)\sqrt{1+a_0} - (-5-3a_0+a_0^2+3a_0^3)\sqrt{1-a_0}}{3a_0^2\sqrt{1+a_0}(2+\sqrt{1-a_0^2})}b_{01} \\ &\quad + \frac{2(-4+a_0-2a_0^2+a_0^3+(-1+a_0)\sqrt{1-a_0^2})}{3+a_0^2}b_{30} \\ \bar{t}_1^2 &= \sqrt{2} \left( \sqrt{1+a_0} + \frac{\sqrt{1+a_0} - \sqrt{1-a_0}}{a_0} \right) \\ \bar{t}_2^2 &= -a_{10} - \frac{(-5+3a_0+a_0^2-3a_0^3)\sqrt{1+a_0} + (5+2a_0+6a_0^2+a_0^3)\sqrt{1-a_0}}{3a_0^2\sqrt{1-a_0}(2+\sqrt{1-a_0^2})}b_{01} \\ &\quad + \frac{2(4+a_0+2a_0^2+a_0^3)+(1+a_0)\sqrt{1-a_0^2}}{3+a_0^2}b_{30} \end{aligned}$$

and the  $x$  coordinates of the point of second switching under the form  $x = x_1\rho + x_2\rho^2 + o(\rho^3)$  and  $\bar{x} = \bar{x}_1\rho + \bar{x}_2\rho^2 + o(\rho^3)$  with

$$\begin{aligned} x_1 &= \frac{2\sqrt{2}(1+3a_0^2-(1-a_0^2)^{\frac{3}{2}})}{a_0((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}, \\ x_2 &= -\frac{5+a_0+5a_0^2-(5+a_0)\sqrt{1-a_0^2}}{3a_0^2}b_{01} - 4b_{30}, \\ \bar{x}_1 &= -\frac{2\sqrt{2}(1+3a_0^2-(1-a_0^2)^{\frac{3}{2}})}{a_0((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}, \\ \bar{x}_2 &= \frac{5-a_0+5a_0^2+(-5+a_0)\sqrt{1-a_0^2}}{3a_0^2}b_{01} - 4b_{30}. \end{aligned}$$

One find easily that the cut locus is at  $x_c = \frac{x_1+\bar{x}_1}{2}\rho + \frac{x_2+\bar{x}_2}{2}\rho^2 + o(\rho^2)$  that is

$$\begin{aligned} x_{cut}^+ &= -\left( \frac{a_0}{3(1+\sqrt{1-a_0^2})}b_{01} + 4b_{30} \right) \rho^2 + o(\rho^2), \\ y_{cut}^+ &= \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3. \end{aligned}$$

When  $-\lambda_y(0) \gg 1$ , then we set  $r = \frac{1}{\sqrt{-\lambda_y}}$ . Equations are changed but the final result is very similar

$$\begin{aligned} x_{cut}^- &= -\left( \frac{a_0}{3(1+\sqrt{1-a_0^2})}b_{01} + 4b_{30} \right) \rho^2 + o(\rho^2), \\ y_{cut}^- &= -\frac{\sqrt{2}((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3. \end{aligned}$$

Finally, the cut locus appears to be a cusp whose tangent at the singular point is the tangent to  $\Delta_A$ , see Figure 6.

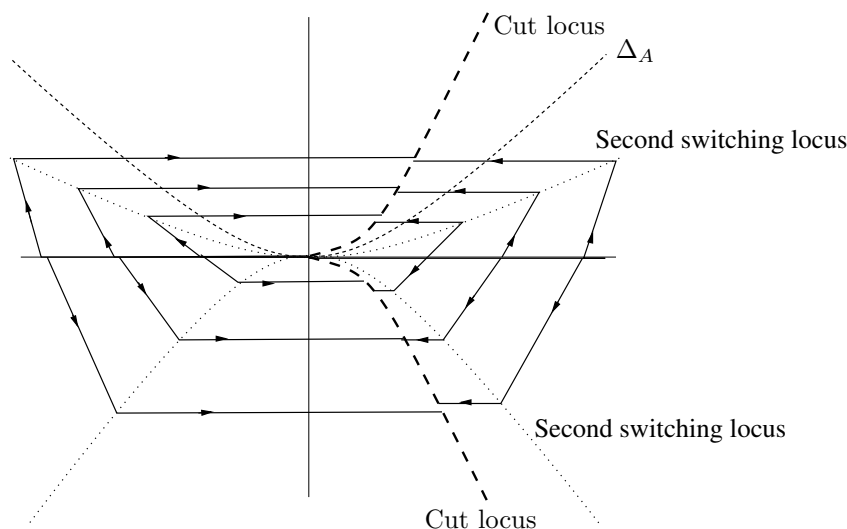


Figure 6: The synthesis in the  $(NF_3)$  case

## References

- [1] A. Agrachev, B. Bonnard, M. Chyba, and I. Kupka. Sub-Riemannian sphere in Martinet flat case. *ESAIM Control Optim. Calc. Var.*, 2:377–448, 1997.
- [2] Andrei Agrachev and Jean-Paul Gauthier. On the subanalyticity of Carnot-Caratheodory distances. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(3):359–382, 2001.
- [3] Andrei A. Agrachev and Yuri L. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.
- [4] E. Ali and G. Charlot. Local contact sub-finslerian geometry for maximum norms in dimension 3. *preprint*.
- [5] D. Barilari, U. Boscain, E. Le Donne, and M. Sigalotti. Sub-finsler structures from the time-optimal control viewpoint for some nilpotent distributions. *arXiv:1506.04339*, 2016.
- [6] Davide Barilari, Ugo Boscain, Grégoire Charlot, and Robert W. Neel. On the heat diffusion for generic riemannian and sub-riemannian structures. *IMRN*, 2016.
- [7] Davide Barilari, Ugo Boscain, and Robert W. Neel. Small-time heat kernel asymptotics at the sub-Riemannian cut locus. *J. Differential Geom.*, 92(3):373–416, 2012.
- [8] André Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 1–78. Birkhäuser, Basel, 1996.

- [9] G. Ben Arous. Développement asymptotique du noyau de la chaleur hypoelliptique hors du cut-locus. *Ann. Sci. École Norm. Sup. (4)*, 21(3):307–331, 1988.
- [10] G. Ben Arous and R. Léandre. Décroissance exponentielle du noyau de la chaleur sur la diagonale. II. *Probab. Theory Related Fields*, 90(3):377–402, 1991.
- [11] B. Bonnard, M. Chyba, and E. Trelat. Sub-riemannian geometry, one-parameter deformation of the martinet flat case. *J. Dyn. Control Syst.* 4, No.1, 59-76 (1998)., 4:59–76, 1998.
- [12] Bernard Bonnard and Monique Chyba. Méthodes géométriques et analytiques pour étudier l’application exponentielle, la sphère et le front d’onde en géométrie sous-riemannienne dans le cas Martinet. *ESAIM Control Optim. Calc. Var.*, 4:245–334 (electronic), 1999.
- [13] Ugo Boscain, Thomas Chambrion, and Grégoire Charlot. Nonisotropic 3-level quantum systems: complete solutions for minimum time and minimum energy. *Discrete Contin. Dyn. Syst. Ser. B*, 5(4):957–990, 2005.
- [14] E. Breuillard and E. Le Donne. On the rate of convergence to the asymptotic cone for nilpotent groups and subfinsler geometry. *Proc. Natl. Acad. Sci. USA*, 110(48):19220–19226, 2013.
- [15] Grégoire Charlot. Quasi-contact s-r metrics : normal form in  $\mathbb{R}^{2n}$ , wave front and caustic in  $\mathbb{R}^4$ . *Acta App. Math.*, 74:217–263, 2002.
- [16] W. L. Chow. ber systeme von linearen partiellen differentialgleichungen erster ordnung. *Math. Ann.*, 117:98–105, 1939.
- [17] Jeanne Clelland and Christopher Moseley. Sub-finsler geometry in dimension three. *Differ. Geom. Appl.*, 24(6):628–651, 2006.
- [18] Jeanne Clelland, Christopher Moseley, and George Wilkens. Geometry of sub-finsler engel manifolds. *Asian J. Math.*, 11(4):699–726, 2007.
- [19] El-Houcine Chakir El Alaoui, J.-P. Gauthier, and I. Kupka. Small sub-riemannian balls on  $\mathbb{R}^3$ . *J. Dyn. Control Syst.*, 2(3):359–421, 1996.
- [20] A.F. Filippov. On some questions in the theory of optimal regulation: existence of a solution of the problem of optimal regulation in the class of bounded measurable functions. *Vestnik Moskov. Univ. Ser. Mat. Meh. Astr.Fiz.Him.*, 2:2532, 1959.
- [21] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.
- [22] Rémi Léandre. Majoration en temps petit de la densité d’une diffusion dégénérée. *Probab. Theory Related Fields*, 74(2):289–294, 1987.
- [23] Rémi Léandre. Minoration en temps petit de la densité d’une diffusion dégénérée. *J. Funct. Anal.*, 74(2):399–414, 1987.
- [24] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. “Nauka”, Moscow, fourth edition, 1983.

- [25] P.K. Rashevsky. About connecting two points of complete nonholonomic space by admissible curve. *Uch. Zap. Ped. Inst. Libknehta*, 2:83–94, 1938.

# Local contact sub-Finslerian geometry for maximum norms in dimension 3\*

Entisar A.-L. Ali♣ and G. Charlot♣

♣ *Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France*  
*Dyala University, Irak*

entisar.ali@univ-grenoble-alpes.fr

♣ *Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France*  
gregoire.charlot@univ-grenoble-alpes.fr

February 14, 2017

## Abstract

Local geometry of sub-Finslerian structures in dimension 3 associated with a maximum norm are studied in the contact case. A normal form is given. Short extremals, local switching conjugate and cut loci, and small spheres are described in the generic case.

Keywords: local optimal synthesis, sub-Finslerian geometry, contact distribution.

## 1 Introduction

From a geometric point of view the sub-Finslerian (SF) structure we are interested in here is a triplet  $(M, \Delta, |\cdot|_\infty)$  where  $M$  is a connected manifold,  $\Delta$  is a sub-bundle of the tangent bundle, and  $|\cdot|_\infty$  is a maximum norm on  $\Delta$ . With such a structure we can define

**Definition 1.** Let  $\gamma : [0, T] \rightarrow M$  be a curve in  $M$ . It is said admissible if  $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$  a.e. The length of an admissible  $\gamma$  is defined as

$$\ell(\gamma) := \int_0^T |\dot{\gamma}(t)|_\infty dt$$

and the distance between two point  $p$  and  $q$  in  $M$  as the infimum of the lengths of the curves that join  $p$  to  $q$

$$d(p, q) = \inf \{ \ell(\gamma) \mid \dot{\gamma}(t) \in \Delta_{\gamma(t)} \text{ a.e., } \gamma(0) = p, \gamma(T) = q \}.$$

If  $\text{Lie}_q(\Delta) = T_q M$  for any  $q$  then locally, for any couple of points  $(q_1, q_2)$ , exists an admissible curve joining  $q_1$  and  $q_2$ . The distance between  $q_1$  and  $q_2$  is defined as the infimum of the lengths of the admissible curves joining the two points.

---

\*This research has been supported by ANR-15-CE40-0018.



When one is only interested in local issues, we can define the structure by the data of  $k$  linearly independent vector fields  $F_1, \dots, F_k$  and by the standard maximum norm defined on  $\text{span}(F_1, \dots, F_k)$  by

$$|G| = \max_i \{|u_i| \mid G = \sum_i u_i F_i\}.$$

From a control point of view, we are considering the dynamics

$$\dot{q} = \sum_{i=1}^k u_i F_i(q), \quad (1)$$

with the constraints

$$|u_i| \leq 1, \forall i \leq k, \quad (2)$$

and we are interested in the optimal synthesis for the problem of minimizing time. In this situation  $\Delta = \text{span}\{F_1, \dots, F_k\}$ . In this article, we are interested only in the local version of this problem, that is to understand the local synthesis for small time (or small distance). Moreover we fix our attention on the case of constant rank of smallest dimension namely  $n = 3, k = 2$ . In the following we work in the neighborhood of  $0 \in \mathbb{R}^3$ .

We say that a property is generic for this class of sub-Finslerian metrics if it is true on a residual set of such metrics for the  $C^\infty$ -Whitney topology. Genericity is usually proven using Thom transversality theorem. One proves easily that generically the set of points  $q$  where a distribution  $\Delta$  of dimension 2, on a manifold of dimension 3, satisfies  $[\Delta, \Delta]_q = \Delta_q$  is a sub-manifold of dimension 2 called the Martinet surface. Outside this surface, the distribution is contact:  $[\Delta, \Delta]_q = T_q M$ . We are interested in describing local objects, such as optimal trajectories, cut locus, conjugate locus, switching locus and small spheres at contact points.

Few publications exist about sub-Finslerian geometry since it is a new subject. Let mention the works [22, 23] for dimension 3, considering norms which are smooth outside the zero section. In [19], the sphere of a left invariant sub-Finsler structure associated to a maximum norm in the Heisenberg group is described. In the preprint [7], the authors describe the extremals (and discuss in particular their number of switches before the loss of optimality) for the Heisenberg, Grushin and Martinet distributions. In the preprint [6], the authors describe, in the 2D generic case, the small spheres and the local cut locus. In this last preprint, the distribution is not supposed of constant rank and it can be related to the almost-riemannian case, see [2, 18, 17, 14, 16].

The work we propose here is a continuation of what has been done in sub-Riemannian geometry at the end of the nineties for codimension one distributions in the contact, quasi-contact and Martinet cases (see [1, 15, 13, 24, 4, 20]). These works, in addition to the interest of understanding the local geometry, were in particular motivated by results on the heat kernel asymptotics in the sub-Riemannian context (see [12, 26, 27, 11]). They allowed recently to give new results on the asymptotics (see [9, 8]).

In section 2, we construct a normal form for couples  $(F_1, F_2)$  defining contact distribution  $\Delta$ . In section 3, we establish some properties of the minimizing trajectories and construct exponential maps. In section 4 we present the optimal synthesis of the nilpotent case. In section 5, we present the jets of the extremals, the switching and conjugate times and the switching and conjugate loci for extremals "following" the bracket  $[F_1, F_2]$ . In section 6, we calculate the cut locus generated by these extremals, similar to the cut locus in the 3D contact sub-Riemannian case. In section 7

we discuss the optimal synthesis linked to extremals with only one control switching several times, very different from the sub-Riemannian case. In section 8, we discuss the stability of the conjugate and cut loci constructed in the previous sections.

## 2 Normal form

In this section, the goal is to construct a normal form for the couple  $(G_1, G_2)$  defined by  $G_1 = F_1 + F_2$  and  $G_2 = F_1 - F_2$ . As we will see later,  $\pm G_1$  and  $\pm G_2$  are the velocities of the non singular extremals of the optimal control system defined by (1) and (2).

Since we consider only points  $q$  where the distribution is contact then  $G_1$ ,  $G_2$  and  $[G_1, G_2] = -2[F_1, F_2]$  form a basis of  $T_q\mathbb{R}^3$ . Hence, we can build a coordinate system centered at  $q$ , by the following way. Let denote  $e^{tX}$  the flow at time  $t$  of a vector field  $X$ . We can define

$$\Xi : (x, y, z) \longmapsto e^{xG_1}e^{yG_2}e^{z[G_1, G_2]}q,$$

which to  $(x, y, z)$  associates the point reached by starting at  $q$  and following  $[G_1, G_2]$  during time  $z$ , then  $G_2$  during time  $y$  and finally  $G_1$  during time  $x$ . The map  $\Xi$  is smooth and satisfies

$$\frac{\partial \Xi}{\partial x}(x, y, z) = G_1, \quad \frac{\partial \Xi}{\partial y}(x, y, z) = G_2, \quad \text{and} \quad \frac{\partial \Xi}{\partial z}(x, y, z) = [G_1, G_2].$$

As a consequence  $\Xi$  is not degenerate at  $(0, 0, 0)$  and defines a coordinate system in a neighborhood of  $q$ . Such coordinates are called *normal coordinates* and  $G_1$  and  $G_2$  satisfy

$$\begin{aligned} G_1(x, y, z) &= \partial_x, \\ G_2(x, y, z) &= x\epsilon_x(x, y, z)\partial_x + (1 + x\epsilon_y(x, y, z))\partial_y + x(1 + \epsilon_z(x, y, z))\partial_z \end{aligned}$$

where  $\epsilon_x$ ,  $\epsilon_y$ ,  $\epsilon_z$  are smooth functions satisfying  $\epsilon_x(0, 0, z) = \epsilon_y(0, 0, z) = \epsilon_z(0, 0, z) = 0$ . Hence we can give the following expressions of  $G_2$

$$\begin{aligned} G_2(x, y, z) &= (a_{200}x^2 + a_{110}xy + x\theta_x(x, y, z))\partial_x \\ &\quad + (1 + b_{200}x^2 + b_{110}xy + x\theta_y(x, y, z))\partial_y \\ &\quad + (x + c_{200}x^2 + c_{110}xy + c_{300}x^3 \\ &\quad + c_{210}x^2y + c_{120}xy^2 + x\theta_z(x, y, z))\partial_z \end{aligned}$$

where  $\theta_x$ ,  $\theta_y$  and  $\theta_z$  are smooth functions such that  $\theta_x(0, 0, z) = \theta_y(0, 0, z) = \theta_z(0, 0, z) = 0$  and whose Taylor series of respective order 1, 1, 2 are null with  $x$  and  $y$  of order 1 and  $z$  of order 2.

## 3 General facts about the computation of the optimal synthesis

In the following of the paper we are going to study the local geometry for a generic class of 3D sub-Finslerian metric defined by a maximum norm, that is for a residual set for the Whitney  $C^\infty$  topology on the set of such metrics. But, for this residual set of metrics, we are going to consider the local geometry only at points in the complementary of a set included in a finite union of codimension 1 submanifolds.

For example, we consider only contact points and generically the set of points where the distribution is not contact is the Martinet surface which has codimension 1. We may also ask that an invariant appearing in the normal form is not null, which happens also outside a codimension 1 submanifold. All along the paper we will assume only a finite number of such assumptions.

### 3.1 Controllability and existence of minimizers

The contact hypothesis is

$$\text{span}(F_1, F_2, [F_1, F_2]) = \mathbb{R}^3.$$

Hence, as a consequence of Chow-Rashevski theorem (see [5, 28, 21]), such a control system is locally controllable that is locally, for any two points, always exists an admissible curve joining the two points.

Moreover, since at each point the set of admissible velocities is convex and compact (in the control version), thanks to Filippov theorem (see [5, 25]), locally for any two points, always exists at least a minimizer.

### 3.2 Pontryagin Maximum Principle (PMP) and Switching Function

The Pontryagin Maximum Principle (PMP) gives necessary conditions for a curve to be a minimizer of the SF distance. For our problem it takes the following form.

**Theorem 2** (PMP). *Let define the Hamiltonian:*

$$H(q, \lambda, u, \lambda_0) = u_1 \lambda \cdot F_1(q) + u_2 \lambda \cdot F_2(q) + \lambda_0$$

where  $q \in \mathbb{R}^3$ ,  $\lambda \in T^*\mathbb{R}^3$ ,  $u \in \mathbb{R}^2$  and  $\lambda_0 \in \mathbb{R}_-$ . For any minimizer  $(q(t), u(t))$  there exist a never vanishing Lipschitz continuous covector  $\lambda : t \mapsto \lambda(t) \in T^*\mathbb{R}^3$  and a constant  $\lambda_0 \leq 0$  such that for a.e.  $t \in [0, T]$  we have

- i.  $\dot{q}(t) = \frac{\partial H}{\partial \lambda}(q(t), \lambda(t), u(t), \lambda_0)$ ,
- ii.  $\dot{\lambda}(t) = -\frac{\partial H}{\partial q}(q(t), \lambda(t), u(t), \lambda_0)$ ,
- iii.  $H(q(t), \lambda(t), u(t), \lambda_0) = \max_v \{H(q(t), \lambda(t), v, \lambda_0) \mid \max_{i=1,2} |v_i| \leq 1\}$ ,
- iv.  $H(q(t), \lambda(t), u(t), \lambda_0) = 0$ .

If  $\lambda_0 = 0$  then  $q$  is said abnormal, if not  $q$  is said normal. It may be both. A solution of the PMP is called an extremal.

*Remark 3.* It is well known that for a contact distribution there is no abnormal extremal. In the following we fix  $\lambda_0 = -1$ .

In the following, we will have to consider the vector fields  $F_3 = [F_1, F_2]$ ,  $F_4 = [F_1, [F_1, F_2]]$  and  $F_5 = [F_2, [F_1, F_2]]$ . We can now define

**Definition 4.** For an extremal triplet  $(q(\cdot), \lambda(\cdot), u(\cdot))$ , we define the functions

$$\phi_i(t) = \langle \lambda(t), F_i(q(t)) \rangle, i = 1 \cdots 5.$$

The functions  $\phi_1$  and  $\phi_2$  are called the switching functions.

**Proposition 5.** For  $i = 1, 2$

- 1. If  $\phi_i(t) > 0$  (resp  $\phi_i(t) < 0$ ) then  $u_i(t) = 1$  (resp  $u_i(t) = -1$ ).

2. If  $\phi_i(t) = 0$  and  $\dot{\phi}_i(t) > 0$  (resp  $\dot{\phi}_i(t) < 0$ ) then  $\phi_i$  changes sign at time  $t$  and the control  $u_i$  switches from  $-1$  to  $+1$  (resp from  $+1$  to  $-1$ ).

Proof: Point 1. is a direct consequence of the maximality condition of the PMP. Point 2. is a direct consequence of point 1.

*Remark 6.* One can compute easily that along a bang arc

$$\dot{\phi}_1 = -u_2\phi_3 \quad \text{and} \quad \dot{\phi}_2 = u_1\phi_3.$$

and moreover, since  $(F_1, F_2, F_3)$  is a frame of the tangent space, we can define the function  $f_{ij}$  for  $i = 4, 5$  and  $j = 1, 2, 3$  by setting

$$F_4 = [F_1, [F_1, F_2]] = f_{41}F_1 + f_{42}F_2 + f_{43}[F_1, F_2],$$

$$F_5 = [F_2, [F_1, F_2]] = f_{51}F_1 + f_{52}F_2 + f_{53}[F_1, F_2].$$

Now, along an extremal, one computes easily that

$$\dot{\phi}_3 = u_1\phi_4 + u_2\phi_5 \tag{3}$$

$$= u_1(f_{41}\phi_1 + f_{42}\phi_2 + f_{43}\phi_3) + u_2(f_{51}\phi_1 + f_{52}\phi_2 + f_{53}\phi_3) \tag{4}$$

**Definition 7.** We call *bang* an extremal trajectory corresponding to constant controls with value 1 or  $-1$  and *bang-bang* an extremal which is a finite concatenation of bangs. We call  *$u_i$ -singular* an extremal corresponding to a null switching function  $\phi_i(\cdot)$ . A time  $t$  is said to be a *switching time* if  $u$  is not bang in any neighborhood of  $t$ .

*Remark 8.* Notice that the switching functions  $\phi_i(\cdot)$  are at least Lipschitz continuous. Moreover thanks to condition 4 of PMP and  $\lambda_0 = -1$  we have that  $u_1(t)\phi_1(t) + u_2(t)\phi_2(t) = 1$ , for all  $t$  which implies

$$|\phi_1(t)| + |\phi_2(t)| = 1.$$

*Remark 9.* Along a  $u_1$ -singular,  $\phi_1 \equiv 0$ ,  $\phi_3 \equiv 0$  and  $|\phi_2| \equiv 1$ . If  $\phi_2 \equiv \pm 1$  then  $u_2 \equiv \pm 1$  and, thanks to equation (4), we get that

$$u_1 f_{42} \pm f_{52} \equiv 0.$$

which determines entirely the control  $u_1$ .

### 3.3 Change of coordinates

We first concentrate our attention on extremals with initial  $|\lambda_z|$  very large corresponding to short cut times (as we will see later).

Following the techniques used in the 3d-contact case in sub-Riemannian geometry (see Agrachev et al [3]), one can make the following change of coordinates and time

$$r = \frac{1}{\lambda_z}, \quad s = \frac{t}{r}, \quad p_x = r\lambda_x, \quad p_y = r\lambda_y.$$

Denoting  $p = (p_x, p_y, 1)$  and  $q = (x, y, z)$  one gets the equations for the extremals

$$\begin{aligned}\frac{dq}{ds} &= r(u_1 F_1(q) + u_2 F_2(q)), \\ \frac{dp}{ds} &= r(-p(u_1 dF_1(q) + u_2 dF_2(q)) + (p(u_1 \frac{\partial F_1(q)}{\partial z} \\ &\quad + u_2 \frac{\partial F_2(q)}{\partial z}))p), \\ \frac{dr}{ds} &= r^2 p(u_1 \frac{\partial F_1(q)}{\partial z} + u_2 \frac{\partial F_2(q)}{\partial z}).\end{aligned}$$

### 3.4 Exponential map and conjugate locus

The set of initial condition is determined by

$$H = u_1 \lambda(0) F_1(0) + u_2 \lambda(0) F_2(0) - 1 = 0$$

which implies  $\max\{|\lambda_x(0)|, |\lambda_y(0)|\} = 1$ . This implies that  $\max\{|p_x(0)|, |p_y(0)|\} = r(0)$ .

If an extremal is not singular, then it starts by a first bang and hence by the speed  $\pm G_1$  or  $\pm G_2$ . Assume  $r_0 > 0$ . If the first bang follows  $\pm G_1$  then  $p_x(0) = \pm r_0$  and we define  $\alpha_2$  by setting  $p_y(0) = \mp r_0 \alpha_2$  with  $\alpha_2 \in ]-1, 1]$ . If the first bang follows  $\pm G_2$  then  $p_y(0) = \pm r_0$  and we define  $\alpha_1$  by setting  $p_x(0) = \pm r_0 \alpha_1$  with  $\alpha_1 \in ]-1, 1]$ . With this convention, among the extremals starting with  $r_0$  fixed and following  $\pm G_1$  (resp  $\pm G_2$ ), the last one to switch is the one with initial condition  $\alpha_2 = 1$  (resp.  $\alpha_1 = 1$ ).

We can hence define 4 exponential maps corresponding to the 4 initial speed  $\pm G_1$  and  $\mp G_2$  and describing the bang-bang extremals. For these maps, depending on  $r_0$ ,  $\alpha_i$  and  $s$ , when  $\alpha_i \neq 1$  and when  $s$  is not a switching time of the extremal with initial condition  $(r_0, \alpha_i)$ , one can compute the jacobian with respect to the parameters  $(r_0, \alpha_i, s)$ .

Recall that we denote by  $t$  the time and  $s$  the new time after reparameterization.

**Definition 10.** The first conjugate time along an extremal is the infimum of the times  $t$  such that exist  $t_1$  and  $t_2$  with  $0 < t_1 < t_2 < t$  such that  $Jac(t_1)Jac(t_2) < 0$ . The first conjugate point along an extremal is the point reached at first conjugate time and the first conjugate locus is the set of the first conjugate points.

The cut locus is the set of points where an extremal curve loses optimality.

The Maxwell set is the set of points where two optimal curves meet.

The sphere at time  $t$  is the collection of all end points at time  $t$  of the optimal extremals.

*Remark 11.* With this definition, it will happen that the Maxwell set is not always included in the cut locus (which is very different from the Riemannian case).

## 4 Nilpotent case

This part of the paper is not entirely new since this case has been studied in [7, 19]

As in sub-Riemannian geometry (see [10, 3]), the nilpotent approximation plays an important role as "good estimation" of the real situation. The nilpotent approximation at  $(0, 0, 0)$  of  $G_1$ ,  $G_2$

given in the normal form is

$$\widehat{G}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{G}_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}$$

It is a left invariant sub-Finslerian metric defined on the Heisenberg group with the representation

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' + xy').$$

It is the tangent space in the sense of Gromov. See [10].

The Hamiltonian for the nilpotent case is

$$H = \frac{u_1 + u_2}{2} \lambda_x + \frac{u_1 - u_2}{2} (\lambda_y + x \lambda_z) - 1.$$

Thus the differential equations are given by

$$\begin{aligned} \dot{x} &= \frac{u_1 + u_2}{2}, & \dot{\lambda}_x &= -\frac{u_1 - u_2}{2} \lambda_z, \\ \dot{y} &= \frac{u_1 - u_2}{2}, & \dot{\lambda}_y &= 0, \\ \dot{z} &= \frac{u_1 - u_2}{2} x, & \dot{\lambda}_z &= 0, \end{aligned}$$

which implies that  $\lambda_y$  and  $\lambda_z$  are constants.

Before entering the computations, one can think that, thanks to the PMP, most of the optimal trajectories will be concatenations of bang arcs of  $\pm G_1$  and  $\pm G_2$ . Moreover, one shows relatively easily that the extremals are solutions of an isoperimetric problem, the  $z$  coordinate being a certain area defined from the projection on the  $(x, y)$ -plane of the trajectory, as it is in the Heisenberg case in subriemannian geometry. Hence it seems clear that many optimal curves project on squares. As we will see, a large class of optimal curves satisfy this property but many others, the singular ones, do not satisfy it which is very different to the subriemannian case.

#### 4.1 Extremals with $\lambda_z \neq 0$

Changing the variables and time for

$$r = \frac{1}{\lambda_z}, \quad s = \frac{t}{r}, \quad p_x = r \lambda_x, \quad p_y = r \lambda_y,$$

and denoting  $\dot{g}$  the derivate of a function  $g$  with respect to  $s$  we have

$$\begin{aligned} \dot{x} &= r \frac{u_1 + u_2}{2}, & \dot{p}_x &= -r \frac{u_1 - u_2}{2}, \\ \dot{y} &= r \frac{u_1 - u_2}{2}, & \dot{p}_y &= 0, \\ \dot{z} &= r \frac{u_1 - u_2}{2} x, & \dot{r} &= 0. \end{aligned}$$

Let present, for example, the computation of extremals with  $\lambda_z \equiv \lambda_z(0) > 0$ ,  $\lambda_y \equiv \lambda_y(0) = 1$ ,  $\lambda_x \in ]-1, 1]$ . In  $x, y, z, p_x, p_y, r, s$  coordinates, one gets  $p_y = r$ ,  $p_x = r\alpha$  with  $\alpha \in ]-1, 1]$  and  $\phi_1(s) = \frac{p_x(s) + p_y + x(s)}{2r}$  and  $\phi_2(s) = \frac{p_x(s) - p_y - x(s)}{2r}$ . We denote  $s_1, s_2$ , etc. the sequence of switching times along an extremal. During the first bang, since  $\phi_1(0) = \frac{\alpha_1 r + p_y}{2r} > 0$  hence  $u_1 = 1$ , and since  $\phi_2(0) = \frac{\alpha_1 r - p_y}{2r} \leq 0$  and  $\dot{\phi}_2(0) = -\frac{u_1}{2} \lambda_z < 0$ , the controls satisfy  $u_1 = 1$  and  $u_2 = -1$ . Moreover

$$\begin{aligned} x(s) &= 0, & p_x(s) &= r\alpha_1 - rs, & \phi_1(s) &= \frac{\alpha_1 - s + 1}{2}, \\ y(s) &= rs, & p_y(s) &= p_y(0) = r, & \phi_2(s) &= \frac{\alpha_1 - s - 1}{2}. \\ z(s) &= 0, \end{aligned}$$

The first switching time  $s_1$  corresponds to  $\phi_1(s_1) = 0$  hence  $s_1 = 1 + \alpha_1$ .

During the second bang, the controls satisfy  $u_1 = -1$  and  $u_2 = -1$  and

$$\begin{aligned} x(s) &= -sr + r + \alpha_1 r, & p_x(s) &= -r, & \phi_1(s) &= \frac{-s+1+\alpha_1}{2}, \\ y(s) &= r + \alpha_1 r, & p_y(s) &= p_y(0) = r, & \phi_2(s) &= \frac{s-3-\alpha_1}{2}. \\ z(s) &= 0, \end{aligned}$$

The second switching time  $s_2$  corresponds to  $\phi_2(s_2) = 0$  hence  $s_2 = 3 + \alpha_1$ .

Along the third bang, the controls satisfy  $u_1 = -1$  and  $u_2 = 1$  and

$$\begin{aligned} x(s) &= -2r, & p_x(s) &= -\alpha_1 r - 4r + sr, & \phi_1(s) &= \frac{-\alpha_1-5+s}{2}, \\ y(s) &= 4r + 2\alpha_1 r - sr, & p_y(s) &= p_y(0) = r, & \phi_2(s) &= \frac{-\alpha_1-3+s}{2}. \\ z(s) &= 2r(s - (3 + \alpha_1)), \end{aligned}$$

The third switching time  $s_3$  corresponds to  $\phi_1(s_3) = 0$  hence  $s_3 = 5 + \alpha_1$ .

During the fourth bang, the controls satisfy  $u_1 = 1$  and  $u_2 = 1$  and

$$\begin{aligned} x(s) &= -7r - \alpha_1 r + sr, & p_x(s) &= r, & \phi_1(s) &= \frac{-5-\alpha_1+s}{2}, \\ y(s) &= -r + \alpha_1 r, & p_y(s) &= p_y(0) = r, & \phi_2(s) &= \frac{7+\alpha_1-s}{2}. \\ z(s) &= 4r^2, \end{aligned}$$

The fourth switching time  $s_4$  corresponds to  $\phi_2(s_4) = 0$  hence  $s_4 = 7 + \alpha_1$ .

Along the fifth bang, the controls satisfy  $u_1 = 1$  and  $u_2 = -1$  and

$$\begin{aligned} x(s) &= 0, & p_x(s) &= 8r + \alpha_1 r - sr, & \phi_1(s) &= \frac{9+\alpha_1-s}{2}, \\ y(s) &= -r + \alpha_1 r + sr, & p_y(s) &= p_y(0) = r, & \phi_2(s) &= \frac{7+\alpha_1-s}{2}. \\ z(s) &= 4r^2, \end{aligned}$$

The fifth switching time  $s_5$  corresponds to  $\phi_1(s_5) = 0$  hence  $s_5 = 9 + \alpha_1$ .

The other extremals with  $\lambda_z \neq 0$  can be computed the same way and are very similar. Finally, extremals with  $\lambda_z > 0$  have projections in the  $(x, y)$ -plane which are squares and the  $z$ -coordinate after one turn of the square is equal to the area of the square. This implies that they are all optimal until the end of this turn that is until  $s = 8$  or  $t = \frac{8}{p_z}$ . After they lose optimality, crossing one each other transversally. As a consequence the cut time is  $s = 8$  or  $t = 8r$  and the cut locus is the vertical axis (as in the Heisenberg case in sub-riemannian geometry).

## 4.2 Extremal with $\lambda_z = 0$

What about the extremals with  $\lambda_z = 0$ ? For such an extremal,  $\lambda$  is constant and  $\phi_1 = \frac{\lambda_x + \lambda_y}{2}$  and  $\phi_2 = \frac{\lambda_x - \lambda_y}{2}$  are also constant. If both are not zero hence  $u_1$  and  $u_2$  are constants along the extremal, the corresponding curve is optimal and is an extremal. If  $\phi_1 \equiv 0$  and  $\phi_2 \equiv 1$  then the extremal is  $u_1$ -singular and the control  $u_1$  is not determined by the max condition of the PMP. In fact in this case, one proves easily that for any choice of  $u_1(\cdot)$  such that  $|u_1(t)| \leq 1$ , one gets for any  $T > 0$ , a minimizer from  $(0, 0, 0)$  to  $(\frac{\int_0^T u_1(t)dt + T}{2}, \frac{\int_0^T u_1(t)dt - T}{2}, z)$  where

$$z = \int_0^T \frac{(u_1(t) - 1)(\int_0^t u_1(\tau)d\tau + t)}{4} dt.$$

The proof comes from the fact that the projection of this point on the  $(x, y)$ -plane is on the segment between the two points  $(T, 0)$  and  $(0, -T)$ . The same kind of computation can be done for  $\phi_1 \equiv 0$  and  $\phi_2 \equiv -1$  or  $\phi_1 \equiv \pm 1$  and  $\phi_2 \equiv 0$ .

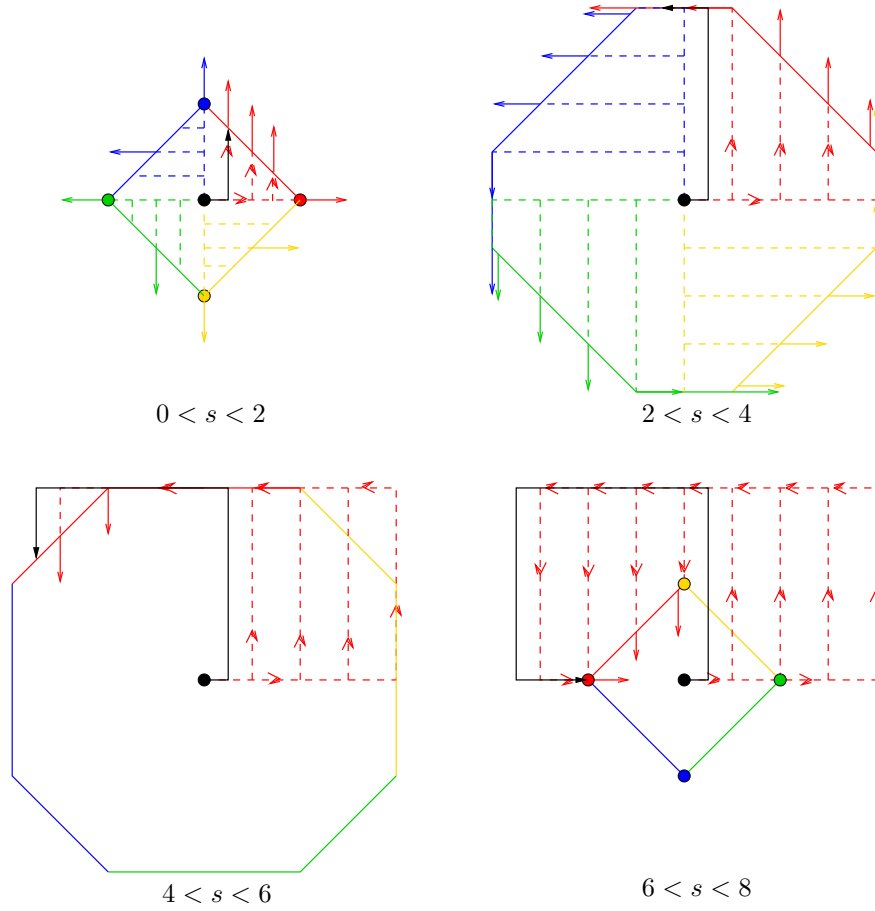


Figure 1: Evolution of the front at  $r \neq 0$  fixed. In red dot lines and in black the extremals with initial speed  $G_1$ , in full line the front at 4 different times, with four colors corresponding to the four possible initial speeds



### 4.3 Exponential map

Let us concentrate again on the extremals with  $\lambda_z \neq 0$ . One can consider the *exponential map* which to  $(r, \alpha, s)$  where  $\alpha \in [-1, 1[$ ,  $r > 0$ ,  $s \geq 0$  associates the end point of the extremal with initial condition  $\lambda_x = \alpha$ ,  $\lambda_y = 1$  and  $\lambda_z = \frac{1}{r}$  for the time  $t = rs$ . This map is smooth at points with  $-1 < \alpha < 1$ ,  $s_i(p_x, r) < s < s_{i+1}(p_x, r)$  for a certain  $i$  where  $s_j(p_x, r)$  is the  $j^{th}$  switching time of the extremal with initial condition  $p_x, p_y = 1$  and  $r$ . The same can be done for  $\lambda_y = -1$  or  $\lambda_x = \pm 1$  and  $\lambda_y \in [-1, 1]$ . Since it is smooth for  $-r < p_x < r$  and  $s \neq s_i \forall i$ , we can compute its jacobian. It happens that it is null during the two first bangs, and that it has opposite sign to the one of  $r$  during the third and fourth bangs. It is again null during the fifth bang. As we will see later for  $r$  small in the generic cases, the jacobian will not be null during the third and fourth bangs also. In the nilpotent case, the first conjugate time is  $t_5 = rs_5$  and for  $t \in ]rs_4, rs_5[$ ,  $Jac(t) = 0$ .

### 4.4 Geometric objects

Since the conjugate time is  $t_5$ , the first conjugate locus is the set of points where an extremal switches for the fifth time. The first conjugate locus is

$$\{(2\delta r, 0, \pm 4r^2) | r \in \mathbb{R}, \delta \in ]-1, 1[ \} \cup \{(0, 2\delta r, \pm 4r^2) | r \in \mathbb{R}, \delta \in ]-1, 1[ \}.$$

The Maxwell set is exactly the same set.

Figure 2 shows the conjugate locus and three points of view of the part of the sphere that is reached by non singular extremals.

## 5 Extremals with both controls switching

In this section, we present the computation of jets of extremals with large covector  $|\lambda|$  and of geometric objects attached to them: switching locus and conjugate locus. As in the nilpotent case, we can define a Hamiltonian flow which, to an initial condition  $(\lambda_x, \lambda_y, \lambda_z)$  (with  $\max(|\lambda_x|, |\lambda_y|) = 1$ ) associates the end point at time  $t$  of the solution of the dynamics

$$\begin{aligned} \dot{x} &= \frac{u_1 + u_2}{2} + \frac{u_1 - u_2}{2}(a_{200}x^2 + a_{110}xy + \theta_x), \\ \dot{y} &= \frac{u_1 - u_2}{2}(1 + b_{200}x^2 + b_{110}xy + \theta_y), \\ \dot{z} &= \frac{u_1 - u_2}{2}(x + c_{200}x^2 + c_{110}xy + c_{300}x^3 + c_{210}x^2y + c_{120}y^2x + \theta_z), \\ \dot{\lambda}_x &= -\frac{u_1 - u_2}{2}(\lambda_x(2a_{200}x + a_{110}y) + \lambda_y(2b_{200}x + b_{110}y) \\ &\quad + \lambda_z(1 + 2c_{200}x + 3c_{300}x^2 + c_{110}y + 2c_{210}xy + c_{120}y^2)), \\ \dot{\lambda}_y &= -\frac{u_1 - u_2}{2}(a_{110}x\lambda_x + b_{110}x\lambda_y + \lambda_z(c_{110}x + c_{210}x^2 + 2c_{120}xy)), \\ \dot{\lambda}_z &= \frac{u_1 - u_2}{2}\lambda_zx(c_{201}x + c_{111}y), \end{aligned}$$

$$\begin{aligned} u_1(t) &= \text{sign}(\phi_1(t)), & u_2(t) &= \text{sign}(\phi_2(t)), \\ \phi_1(t) &= \lambda(t)F_1(q(t)), & \phi_2(t) &= \lambda(t)F_2(q(t)). \end{aligned}$$

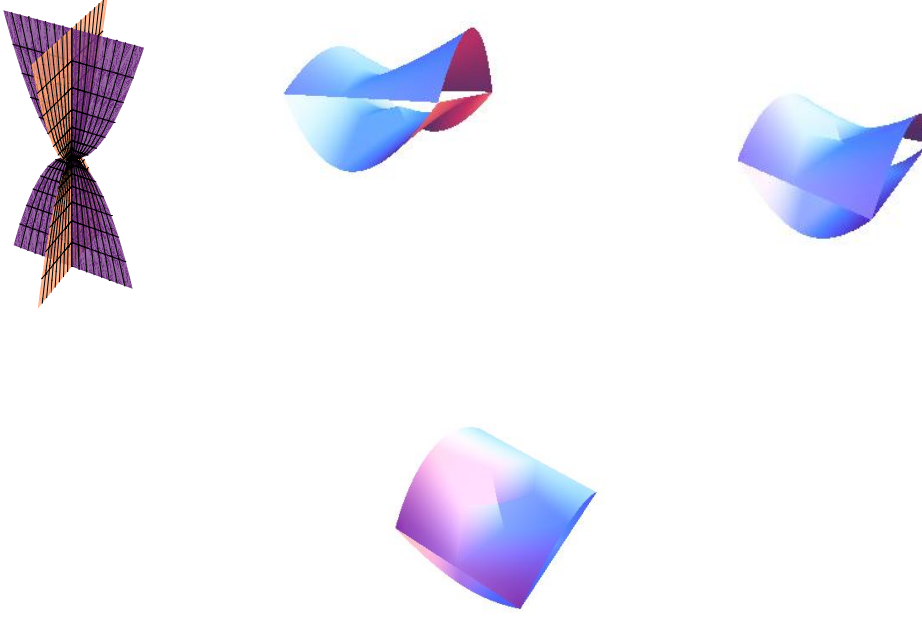


Figure 2: The conjugate locus and three points of view of the non singular part of the sphere in the nilpotent case

From now  $\dot{x}$  denotes  $\frac{dx}{ds}$ . Using the change of coordinates for  $(x, y, z, p, r, s)$ . we can define a new Hamiltonian flow by the dynamics

$$\begin{aligned}
\dot{x} &= \frac{u_1 + u_2}{2}r + \frac{u_1 + u_2}{2}r(a_{200}x^2 + a_{110}xy + \theta_x), \\
\dot{y} &= \frac{u_1 - u_2}{2}r(1 + b_{200}x^2 + b_{110}xy + \theta_y), \\
\dot{z} &= \frac{1}{2}r(\theta_z(u_1 + u_2) + (u_1 - u_2)(x + c_{200}x^2 + c_{300}x^3 + c_{110}xy + c_{210}x^2y + c_{120}xy^2)), \\
\dot{p}_x &= -\frac{u_1 - u_2}{2}r(1 + 2c_{200}x + p_x(2a_{200}x + a_{110}y) + p_y(2b_{200}x + b_{110}y) + 3c_{300}x^2 \\
&\quad + c_{110}y + 2c_{210}xy + c_{120}y^2), \\
\dot{p}_y &= -\frac{u_1 - u_2}{2}r(c_{110}x + a_{110}p_xx + b_{110}p_yx + c_{210}x^2 + 2c_{120}xy), \\
\dot{r} &= \frac{u_1 - u_2}{2}r^2x(c_{201}x + c_{111}y).
\end{aligned}$$

$$\begin{aligned}
\phi_1(t) &= \frac{1}{r}pF_1(q(t)), & \phi_2(t) &= \frac{1}{r}pF_2(q(t)) \\
u_1(t) &= \text{sign}(\phi_1(t)), & u_2(t) &= \text{sign}(\phi_2(t)).
\end{aligned}$$

Since the set of initial condition is a square for  $(p_x, p_y)$ , we define in fact four Hamiltonian flows for each initial speed  $(G_1, -G_1, G_2, -G_2)$ . For example, for the extremals with initial speed equal to  $G_2$  we have  $p_y(0) = r$  and  $p_x = \alpha r$  with  $\alpha \in ]-1, 1]$ . The new Hamiltonian flow as for variables  $(r_0, \alpha, s)$  where  $r_0 = r(0)$ ,  $p_x(0) = \alpha r$  and  $s = \frac{t}{r_0}$ . In order to compute jets of the Hamiltonian flow we write

$$\begin{aligned} x(r_0, \alpha, s) &= x_1(\alpha, s)r_0 + x_2(\alpha, s)r_0^2 + x_3(\alpha, s)r_0^3 + X_4(r_0, \alpha, s)r_0^4, \\ y(r_0, \alpha, s) &= y_1(\alpha, s)r_0 + y_2(\alpha, s)r_0^2 + y_3(\alpha, s)r_0^3 + Y_4(r_0, \alpha, s)r_0^4, \\ z(r_0, \alpha, s) &= z_2(\alpha, s)r_0^2 + z_3(\alpha, s)r_0^3 + z_4(\alpha, s)r_0^4 + Z_5(r_0, \alpha, s)r_0^5, \\ p_x(r_0, \alpha, s) &= p_{x1}(\alpha, s)r_0 + p_{x2}(\alpha, s)r_0^2 + p_{x3}(\alpha, s)r_0^3 + P_{x4}(r_0, \alpha, s)r_0^4, \\ p_y(r_0, \alpha, s) &= p_{y1}(\alpha, s)r_0 + p_{y2}(\alpha, s)r_0^2 + p_{y3}(\alpha, s)r_0^3 + P_{y4}(r_0, \alpha, s)r_0^4, \\ r(r_0, \alpha, s) &= r_0 + r_2(\alpha, s)r_0^2 + r_3(\alpha, s)r_0^3 + R_4(r_0, \alpha, s)r_0^4. \end{aligned}$$

where all the new functions are smooth functions of their variables. Using this dynamics we find the following. For the first order

$$\begin{aligned} \dot{x}_1 &= \frac{u_1 + u_2}{2}, & \dot{p}_{x1} &= \frac{-u_1 + u_2}{2}, \\ \dot{y}_1 &= \frac{u_1 - u_2}{2}, & \dot{p}_{y1} &= 0, \\ \dot{z}_2 &= \frac{u_1 - u_2}{2}x_1. \end{aligned}$$

For the second order

$$\begin{aligned} \dot{x}_2 &= 0, & \dot{p}_{x2} &= -\frac{u_1 - u_2}{2}(2c_{200}x_1 + c_{110}y_1), \\ \dot{y}_2 &= 0, & \dot{p}_{y2} &= -\frac{u_1 - u_2}{2}c_{110}x_1, \\ \dot{z}_3 &= \frac{u_1 - u_2}{2}(x_2 + x_1(c_{200}x_1 + c_{110}y_1)), & \dot{r}_2 &= 0. \end{aligned}$$

For the third order

$$\begin{aligned} \dot{x}_3 &= \frac{u_1 - u_2}{2}(a_{200}x_1^2 + a_{110}x_1y_1), \\ \dot{y}_3 &= \frac{u_1 - u_2}{2}(b_{200}x_1^2 + b_{110}x_1y_1), \\ \dot{z}_4 &= \frac{u_1 - u_2}{2}(c_{300}x_1^3 + 2c_{200}x_1x_2 + x_3 + c_{110}x_2y_1 + c_{110}x_1y_2 + c_{210}x_1^2y_1 + c_{120}x_1y_1^2), \\ \dot{p}_{x3} &= -\frac{u_1 - u_2}{2}(2a_{200}p_{x1}x_1 + 2b_{200}p_{y1}x_1 + 2c_{200}x_2 + 3c_{300}x_1^2 \\ &\quad + a_{110}p_{x1}y_1 + b_{110}p_{y1}y_1 + c_{110}y_2 + 2c_{210}x_1y_1 + c_{120}y_1^2), \\ \dot{p}_{y3} &= \frac{u_1 - u_2}{2}(-c_{110}x_2 - x_1(a_{110}p_{x1} + b_{110}p_{y1} + c_{210}x_1 + 2c_{120}y_1)), \\ \dot{r}_3 &= 0. \end{aligned}$$

Recall that the extremals we are interested in have initial condition

$$\begin{aligned} x(r_0, \alpha, 0) &= 0, & p_x(r_0, \alpha, 0) &= r_0p_{x1}(\alpha, 0), \\ y(r_0, \alpha, 0) &= 0, & p_y(r_0, \alpha, 0) &= r_0p_{y1}(\alpha, 0), \\ z(r_0, \alpha, 0) &= 0, & r(r_0, \alpha, 0) &= r_0. \end{aligned}$$

These equations are integrable hence we can compute jets of switching functions and hence jets of switching times. Finally, we are able to compute the jets of the different bangs of the extremals.

If we restrict the computation to  $x, y, z$  as functions of  $(r_0, \alpha, s)$  for the four Hamiltonian flows, we get four exponential maps that we denote  $\text{Exp}_\beta$  where  $\beta = -1, 1, -2$  or  $2$  depending on if the initial velocity is  $-G_1, G_1, -G_2, G_2$ .

It happens that all the extremals computed that way are turning extremals like in 3D contact sub-riemannian geometry. For example, if  $r_0 > 0$  and if the extremal starts with  $G_1$  then after it switches to  $G_2$ , then  $-G_1, -G_2, G_1$  and so on.

In [29], M. Sigalotti proves, studying second order optimality conditions, that this family of extremals cannot be optimal after the fifth switch.

For these exponential maps, one can compute their jacobian for each bang arc. One finds

- $Jac(\text{Exp}_{\pm 2}) = 0$  for  $0 < s < s_2, s \neq s_1$ ,
- $Jac(\text{Exp}_{\pm 2}) = -8r_0^3 + o(r_0^3)$  for  $s_2 < s < s_3$ ,
- $Jac(\text{Exp}_{\pm 2}) = -8r_0^3 + o(r_0^3)$  for  $s_3 < s < s_4$ ,
- $Jac(\text{Exp}_{\pm 2}) = 32(2c_{120} - c_{110}^2)r_0^5 + o(r_0^5)$  for  $s_4 < s < s_5$ ,
- $Jac(\text{Exp}_{\pm 2}) = 8r_0^3 + o(r_0^3)$  for  $s_5 < s < s_6$ ,

and

- $Jac(\text{Exp}_{\pm 1}) = 0$  if  $0 < s < s_1$  or  $s_1 < s < s_2$ ,
- $Jac(\text{Exp}_{\pm 1}) = -4r_0^3 + o(r_0^3)$  if  $s_2 < s < s_3$ ,
- $Jac(\text{Exp}_{\pm 1}) = -8r_0^3 + o(r_0^3)$  if  $s_3 < s < s_4$ ,
- $Jac(\text{Exp}_{\pm 1}) = 64(3c_{300} - 2b_{200} - 2c_{200}^2)r_0^5 + o(r_0^5)$  if  $s_4 < s < s_5$ ,
- $Jac(\text{Exp}_{\pm 1}) = 8r_0^3 + o(r_0^3)$  if  $s_5 < s < s_6$ .

We can now state the following proposition which shows that the sign of the Jacobian is an important invariant which determines the conjugate time.

**Proposition 12.** *Let  $G_1$  and  $G_2$  as in the normal form given in section 2.*

- *If  $C_1 = 3c_{300} - 2b_{200} - 2c_{200}^2 > 0$  then the fourth switching time  $t_4$  is the first conjugate time for extremal with initial velocity  $\pm G_1$ . If  $C_1 < 0$  then it is the fifth  $t_5$ .*
- *If  $C_2 = 2c_{120} - c_{110}^2 > 0$  then the fourth switching time  $t_4$  is the first conjugate time for extremals with initial velocity  $\pm G_2$ . If  $C_2 < 0$  then it is the fifth  $t_5$ .*

Still using the expressions given in Appendix, we can give the expressions of the upper part of the first conjugate locus for the four exponential maps.

For  $\text{Exp}_{\pm 1}$ , if  $C_1 > 0$

$$\begin{aligned} x_{conj} &= \pm(\alpha_2 - 1)r_0 + (4c_{110} - c_{200}(\alpha_2 - 1)^2)r_0^2 + o(r_0^2), \\ y_{conj} &= -8c_{200}r_0^2 \pm 4(b_{110} + 6c_{110}c_{200} - 2c_{210} \\ &\quad + (4b_{200} + 12c_{200}^2 - 6c_{300})\alpha_2)r_0^3 + o(r_0^3), \\ z_{conj} &= 4r_0^2 \mp 8(c_{110} + 2c_{200}\alpha_2)r_0^3 + o(r_0^3), \end{aligned}$$

and if  $C_1 < 0$

$$\begin{aligned} x_{conj} &= \pm(1 + \alpha_2)r_0 + (4c_{110} - c_{200}(1 + \alpha_2)^2)r_0^2 + o(r_0^2), \\ y_{conj} &= -8c_{200}r_0^2 \pm 4(b_{110} + 6c_{110}c_{200} - 2c_{210} \\ &\quad + (4b_{200} + 12c_{200}^2 - 6c_{300})\alpha_2)r_0^3 + o(r_0^3), \\ z_{conj} &= 4r_0^2 \mp 8(c_{110} + 2c_{200}\alpha_2)r_0^3 + o(r_0^3), \end{aligned}$$

and for  $Exp_{\pm 2}$ , if  $C_2 > 0$

$$\begin{aligned} x_{conj} &= 4c_{110}r_0^2 \pm 4(b_{110} + 6c_{110}c_{200} - 2c_{210} \\ &\quad + \alpha_1(2c_{120} - 3c_{110}^2))r_0^3 + o(r_0^3), \\ y_{conj} &= \pm(-1 + \alpha_1)r_0 - \frac{1}{2}(16c_{200} + c_{110}(\alpha_1 - 1)^2)r_0^2 + o(r_0^2), \\ z_{conj} &= 4r_0^2 \pm 4(4c_{200} - c_{110}(1 + \alpha_1))r_0^3 + o(r_0^3), \end{aligned}$$

and if  $C_2 < 0$

$$\begin{aligned} x_{conj} &= 4c_{110}r_0^2 \pm 4(b_{110} + 6c_{110}c_{200} - 2c_{210} \\ &\quad + \alpha_1(2c_{120} - 3c_{110}^2))r_0^3 + o(r_0^3), \\ y_{conj} &= \pm(1 + \alpha_1)r_0 - \frac{1}{2}(16c_{200} + c_{110}(1 + \alpha_1)^2)r_0^2 + o(r_0^2), \\ z_{conj} &= 4r_0^2 \pm 4(4c_{200} + c_{110}(1 - \alpha_1))r_0^3 + o(r_0^3). \end{aligned}$$

## 6 Local Cut Locus of extremals with $\lambda_z(0) \gg 1$

In the nilpotent case, the extremals with  $|\alpha| < 1$  reach the Maxwell set at the fourth switch when, for those with  $|\alpha| = 1$ , it is at the third switch. When  $C_1 \neq 0$  and  $C_2 \neq 0$  we will see that the cut is reached during the fourth or fifth bang.

From section 4, we can conclude that the loss of optimality may come during the fourth bang or the fifth bang. Moreover, in [29] the author proves that the extremals we are considering cannot be optimal after the fifth switch. Hence we can conclude that the cut locus comes from the intersection of two fourth bangs of different  $exp_i$ , the intersection of two fifth bangs of different  $exp_i$ , the intersection of a fourth bang and a fifth bang of two different  $exp_i$ .

In the following we compute, for the jets of order 3, 3 and 4 of  $x$ ,  $y$  and  $z$  in  $r_0$ , the possible intersections listed previously, and finally describe the possible pictures of the cut locus depending on the values of invariants of the structure appearing in the normal form. Finally we discuss the stability of the pictures.

### 6.1 Intersections of fourth bangs

#### 6.1.1 Intersection of an extremal starting with $\pm G_1$ with one starting with $\pm G_2$

As seen in the nilpotent case, an extremal starting with  $\pm G_1$  and  $|\alpha_2| < 1$  meets the Maxwell set at  $s = s_4$  and intersect at this time the extremal starting with  $\pm G_2$  and  $\alpha_1 = 1$ . Hence, we compute the jets of  $Exp_{\pm 1}$  close to the fourth switch time that is at  $s = 7 + \alpha_2 + T_2r_0 + T_3r_0^2$  and the jets

of  $Exp_{\pm 2}$  for  $r'_0 = r_0 + R_2 r_0^2 + R_3 r_0^3$ ,  $\alpha_1 = 1 - \alpha_{11} r_0 - \alpha_{12} r_0^2$  and at time  $s' = s \frac{r_0}{r'_0}$ . Asking that the corresponding points are the same, one gets

$$\begin{aligned} R_2 &= \mp 2c_{200}(1 + \alpha_2) \\ T_2 &= \mp 8c_{110} - c_{200}(1 + 14\alpha_2 + \alpha_2^2) \\ \alpha_{11} &= 0 \end{aligned}$$

and

$$\begin{aligned} R_3 &= \frac{(1 + \alpha_2)}{2}(3b_{110} + 6c_{110}c_{200} + 4c_{200}^2(1 + 3\alpha_2) \\ &\quad + 4b_{200}(-1 + \alpha_2) + 6c_{300}(1 - \alpha_2) - 6c_{210}) \\ T_3 &= \frac{16}{3}a_{110} + 20c_{110}^2 - a_{200} + 8b_{200} + 38c_{200}^2 - \frac{40}{3}c_{120} - 27c_{300} \\ &\quad + (8b_{110} + 2a_{200} + 9b_{200} + 48c_{110}c_{200} + 14c_{200}^2 - 15c_{300} - 16c_{210})\alpha_2 \\ &\quad + (-a_{200} + 14b_{200} + 42c_{200}^2 - 21c_{300})\alpha_2^2 \\ &\quad + (b_{200} + 2c_{200}^2 - c_{300})\alpha_2^3 \\ \alpha_{12} &= 4(1 + \alpha_2)(3c_{300} - 2b_{200} - 2c_{200}^2) = 4(1 + \alpha_2)C_1 \end{aligned}$$

We see here that in order the intersection exists,  $\alpha_1 = 1 - \alpha_{11} r_0 - \alpha_{12} r_0^2$  should be less or equal to 1 hence, since  $\alpha_{11} = 0$ , one should have  $\alpha_{12} > 0$  which implies  $C_1 > 0$ .

When  $C_1 > 0$ , once computed the corresponding points (depending on  $r_0$  and  $\alpha_2$ ) one can compute the suspension of this part of the cut locus by looking at its intersection with  $z = 4\rho^2$  for  $\rho$  small. One gets

$$\begin{aligned} x_{cut} &= \pm(-1 + \alpha_2)\rho + (3c_{110} - c_{200} + c_{110}\alpha_2 + c_{200}\alpha_2^2)\rho^2 \\ &\quad \pm \frac{1}{2}(a_{110} - 7c_{110}^2 - 2a_{200} + 2b_{200} - 8c_{110}c_{200} + 4c_{200}^2 + 12c_{120} - 4c_{300} \\ &\quad + (4a_{200} - a_{110} - 5b_{110} - c_{110}^2 - 6c_{110}c_{200} - 4c_{200}^2 + 4c_{120} + 10c_{210})\alpha_2 \\ &\quad + (6c_{210} - 3b_{110} - 2a_{200} - 2c_{110}c_{200})\alpha_2^2 + (4c_{300} - 2b_{200})\alpha_2^3)\rho^3 \\ y_{cut} &= -8c_{200}\rho^2 \pm (4b_{110} + 8c_{110}c_{200} + (8b_{200} - 12c_{300} + 8c_{200}^2)(\alpha_2 - 1))\rho^3 \\ z_{cut} &= 4\rho^2 \end{aligned}$$

### 6.1.2 Intersection of an extremal starting with $\pm G_2$ with one starting with $\mp G_1$

The same computations can be done for extremals starting by  $\pm G_2$  and intersecting  $\mp G_1$  and one gets that  $C_2$  should be positive. Hence

$$\begin{aligned}
x_{cut} &= 4c_{110}\rho^2 \pm (4b_{110} + 8c_{110}c_{200} - 8c_{210} + (6c_{110}^2 - 4c_{120})(1 - \alpha_1))\rho^3 \\
y_{cut} &= \pm(-1 + \alpha_1)\rho + (-c_{110} - 6c_{200} + c_{110}\alpha_1 - 2c_{200}\alpha_1)\rho^2 \\
&\quad \pm \frac{1}{24}(4a_{110} - 24b_{110} - 21c_{110}^2 - 312b_{200} - 144c_{110}c_{200} - 336c_{200}^2 \\
&\quad - 4c_{120} + 432c_{300} + 24c_{210} + (108b_{110} + 51c_{110}^2 - 72b_{200} \\
&\quad + 264c_{110}c_{200} - 48c_{200}^2 - 36c_{120} + 144c_{300} - 168c_{210})\alpha_1 \\
&\quad + 12(b_{110} - 27c_{110}^2 + 72c_{110}c_{200} + 36c_{120} - 48c_{210})\alpha_1^2 \\
&\quad + (4c_{120} - 4a_{110} - 3c_{110}^2)\alpha_1^3)\rho^3 \\
z_{cut} &= 4\rho^2
\end{aligned}$$

### 6.1.3 Intersection of the front starting with $G_1$ with the one starting with $-G_1$

Such a self-intersection of the front can take place only at  $s = 8 + O(r_0)$  as in the nilpotent case. In order to compute such intersection close to  $s = 8$ , we proceed as follows. We compute the intersection of these parts of the front with  $z = 4\rho^2$  for  $\rho^2$ . In order to do this, we fix  $t = 8\rho + T_2\rho^2 + T_3\rho^3$ , for each type of extremal fix  $\alpha_2 = 1 - \alpha_{21}\rho - \alpha_{22}\rho^2$  and find the  $r_0$  such that the corresponding point  $Exp_{\pm 1}(r_0, \alpha, t/r_0)$  satisfies  $z = 4\rho^2$ . For the extremals starting by  $\pm G_1$  one finds

$$\begin{aligned}
x_{sus} &= (4c_{110} \mp \alpha_{21})\rho^2 \mp (+4b_{110} + 4c_{110}^2 \pm 4c_{200}\alpha_{21} \\
&\quad + 2c_{110}(4c_{200} \pm \alpha_{21}) + \alpha_{22} - 8c_{120} - 8c_{210})\rho^3 \\
y_{sus} &= (-8c_{200} \mp \alpha_{21} \mp T_2)\rho^2 \pm \left(\frac{4}{3}a_{110} - \alpha_{21}^2 - \alpha_{22} + 8b_{200} \pm 2\alpha_{21}c_{110}\right. \\
&\quad \left. + \frac{8}{3}c_{120} \mp 4\alpha_{21}c_{200} + 16c_{110}c_{200} + 16c_{200}^2 - 16c_{300} - \alpha_{21}T_2 \pm 4c_{110}T_2 - T_3\right)\rho^3 \\
z_{sus} &= 4\rho^2
\end{aligned}$$

It is then easy to show that, in order to get a contact between these two fronts,  $T_2$  should be equal to 0 and  $\alpha_{21+} = -\alpha_{21-}$ . But, since both should be positive hence  $\alpha_{21+} = \alpha_{21-} = 0$  and this implies that  $T_3$  should be equal to

$$T_{3b} = \frac{4}{3}(a_{110} + 3b_{110} + 6b_{200} + 3c_{110}^2 - 4c_{120} + 18c_{110}c_{200} + 12c_{200}^2 - 6c_{210} - 12c_{300}).$$

At this time, with  $\alpha_{21+} = \alpha_{21-} = 0$ , the two fronts are segments belonging to the same line.

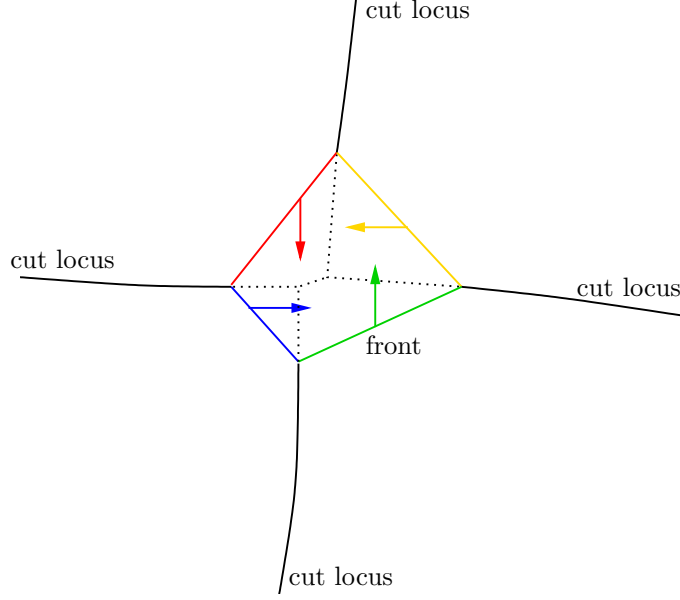


Figure 3: Closure of the cut locus at  $z$  fixed.

#### 6.1.4 Intersection of the front starting with $G_2$ with the one starting with $-G_2$

We proceed the same way. For the extremals starting by  $\pm G_2$  one finds

$$\begin{aligned}
 x_{sus} &= (4c_{110} \pm \alpha_{11} \pm T_2)\rho^2 \pm \left(-\frac{4}{3}a_{110} - 4c_{110}^2 + 8b_{200} \mp 4c_{200}\alpha_{11} \right. \\
 &\quad \left. - 2c_{110}(8c_{200} \pm \alpha_{11}) + \alpha_{12} + \frac{16}{3}c_{120} - 8c_{300} + T_3\right)\rho^3 \\
 y_{sus} &= -(8c_{200} \pm \alpha_{11})\rho^2 \pm (4b_{110} - 16b_{200} + 8c_{110}c_{200} - 16c_{200}^2 \\
 &\quad \mp 2c_{110}\alpha_{11} \pm 4c_{200}\alpha_{11} - \alpha_{12} + 24c_{300} - 8c_{210})\rho^3 \\
 z_{sus} &= 4\rho^2
 \end{aligned}$$

It is then easy to show that, in order to get a contact between these two fronts,  $T_2$  should be equal to 0 and  $\alpha_{11+} = -\alpha_{11-}$ . But, since both should be positive hence  $\alpha_{11+} = \alpha_{11-} = 0$  and this implies that  $T_3$  should be equal to

$$T_{3a} = \frac{4}{3}(a_{110} - 3b_{110} + 6b_{200} + 3c_{110}^2 - 4c_{120} + 6c_{110}c_{200} + 12c_{200}^2 + 6c_{210} - 12c_{300}).$$

## 6.2 Cut locus when $C_1 > 0$ and $C_2 > 0$

With the considerations given before, if  $C_1 > 0$ ,  $C_2 > 0$  and  $T_{3a} \neq T_{3b}$ , the intersection of the cut locus with  $\{z = 4\rho^2\}$  is constituted of 5 branches as in the Figure 3.

The four external branches comes from the intersection of the fourth bangs of  $\exp_{\pm 1}$  with  $\exp_{\pm 2}$  and of the fourth bangs of  $\exp_{\pm 1}$  with  $\exp_{\mp 2}$ , see Figure 3. The central branch is the intersection



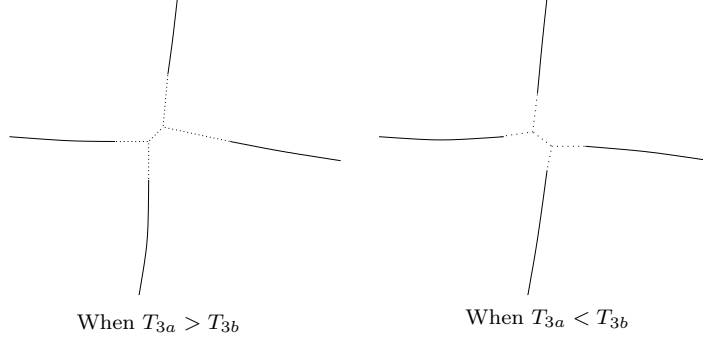


Figure 4: Closure of the cut locus at  $z$  fixed

of the fourth bangs of  $\exp_1$  with  $\exp_{-1}$  if  $T_{3b} < T_{3a}$  or of the fourth bangs of  $\exp_2$  with  $\exp_{-2}$  if  $T_{3a} < T_{3b}$ , see Figure 4.

After  $\min\{T_{3a}, T_{3b}\}$  all the extremals participating to the construction of this part of the cut locus have lost optimality.

Finally the picture of the cut depends on the sign of

$$T_{3a} - T_{3b} = -8(b_{110} + 2c_{110}c_{200} - 2c_{210}).$$

If  $T_{3a} > T_{3b}$  then the two points of the cut locus that connect three branches are with

$$\begin{aligned} x &= 4c_{110}\rho^2 \pm C\rho^3 + o(\rho^3) \\ y &= -8c_{200}\rho^2 \pm C\rho^3 + o(\rho^3) \\ z &= 4\rho^2 \end{aligned}$$

with  $C = 4(b_{110} + 2c_{110}c_{200} - 2c_{210})$ , when if  $T_{3a} < T_{3b}$  then the two points of the cut locus that connect three branches satisfy

$$\begin{aligned} x &= 4c_{110}\rho^2 \pm C\rho^3 + o(\rho^3) \\ y &= -8c_{200}\rho^2 \mp C\rho^3 + o(\rho^3) \\ z &= 4\rho^2 \end{aligned}$$

Finally we can present the upper part of the cut locus when  $C_1 > 0$  and  $C_2 > 0$  in Figure 5

### 6.3 Suspension of fifth bang front

At  $6 < s < 8$ , the part of the front corresponding to the fifth bang is close to  $(\pm(s-8)\rho, 0, 4\rho^2)$  for the front starting with  $\pm G_1$  and close to  $(0, \pm(s-8)\rho, 4\rho^2)$  for the front starting with  $\pm G_2$ . Hence the intersections come at  $s$  close to 8.

In order to compute these intersections we fix a small  $\rho$ , consider a time  $t = 8\rho + T_2\rho^2 + T_3\rho^3$ , and for each type of extremal find the  $r_0$  such that the corresponding point  $Exp_{\pm 1}(r_0, \alpha, t/r_0)$

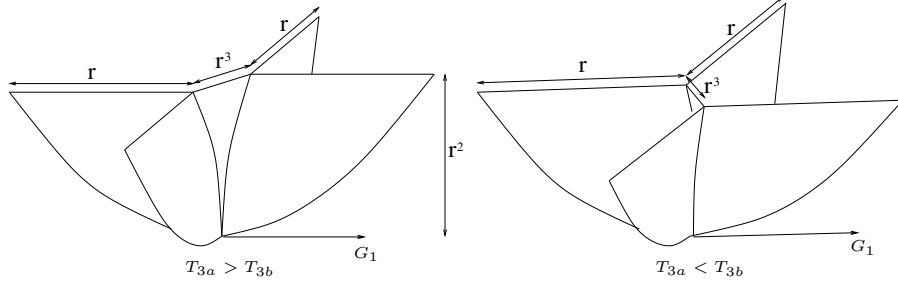


Figure 5: The upper part of the cut locus

satisfies  $z = 4\rho^2$ . For the extremals starting by  $\pm G_1$  one finds  $exp_{\pm 1}$

$$\begin{aligned} x_{\pm 1sus} &= (4c_{110} \pm T_2)\rho^2 \pm \left(\frac{16}{3}c_{120} - \frac{4}{3}a_{110} - 4c_{110}^2 - 16c_{110}c_{200} - 8c_{200}^2\right. \\ &\quad \left.+ 4c_{300} + T_3 - 4C_1\alpha_2^2\right)\rho^3 \\ y_{\pm 1sus} &= -8c_{200}\rho^2 \pm (4b_{110} + 8c_{110}c_{200} - 8c_{210} - 8C_1\alpha_2)\rho^3 \\ z_{\pm 1sus} &= 4\rho^2 \end{aligned}$$

For the extremals starting by  $\pm G_2$  one finds  $exp_{\pm 2}$

$$\begin{aligned} x_{\pm 2sus} &= 4c_{110}\rho^2 \pm (4b_{110} + 8c_{110}c_{200} - 8c_{210} + 4C_2\alpha_1)\rho^3 \\ y_{\pm 2sus} &= (-8c_{200} \pm T_2)\rho^2 \pm \left(\frac{4}{3}c_{120} - \frac{4}{3}a_{110} - 8b_{200} - 16c_{200}^2 - 2c_{110}^2\right. \\ &\quad \left.+ 16c_{300} + 4c_{110}(-4c_{200} \pm T_2) + T_3 - 2C_2\alpha_1^2\right)\rho^3 \\ z_{\pm 2sus} &= 4\rho^2 \end{aligned}$$

As one can see, the intersection of the fifth bang front at  $t$  with the plane  $z = 4\rho^2$  is the union of arc of parabolas. If we consider all these curves for  $\alpha_i \in [0, 1]$  we can observe that the tangents at  $\alpha = \pm 1$  are line with equations of the  $x + y = c$  or  $x - y = c$ . Moreover, this tangent at  $\alpha_2 = -1$  of the fifth bang front of  $exp_{\pm 1}$  is tangent to the fourth bang at the corresponding  $\alpha_1$  of  $exp_{\pm 2}$ , and the tangent at  $\alpha_1 = -1$  of the fifth bang front of  $exp_{\pm 2}$  is tangent to the fourth bang at the corresponding  $\alpha_2$  of  $exp_{\pm 1}$ .

Moreover remark that, at  $T_2 = 0$ , the intersection of the front with  $z = 4\rho^2$  still has a central symmetry at this order of jets, centred at

$$(x, y) = (4c_{110}\rho^2, -8c_{200}\rho^2).$$

#### 6.4 Cut locus when $C_1 > 0$ and $C_2 < 0$

If  $C_1 > 0$  and  $C_2 < 0$  then the picture of the front at  $t < 8\rho$  is as in the Figure 6. The fifth bang of  $exp_{\pm 1}$  do not participate to the optimal synthesis and the fourth bang front of  $exp_{\pm 1}$  intersect the fourth bang front of  $exp_{\pm 2}$ . The fifth bang front of  $exp_{\pm 2}$  is optimal.

Let consider the closure of the cut, that it when  $t = 8\rho + T_2\rho^2 + T_3\rho^3$ . We can identify the following subcases

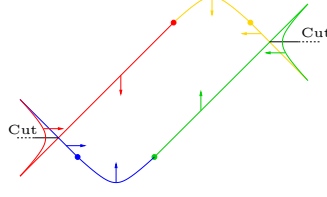


Figure 6: The front before  $t = 8\rho$  when  $C_1 > 0$  and  $C_2 < 0$

- When  $4b_{110} + 8c_{110}c_{200} - 8c_{210} - 4C_2 < 0$  then all the fifth bang of  $\exp_2$  satisfies  $x < 4c_{110}\rho^2$  when all the fifth bang of  $\exp_{-2}$  satisfies  $x > 4c_{110}\rho^2$ . This implies that the sequel of the self intersections of the front is the following : first the fourth bang front of  $\exp_{\pm 1}$  intersect the fourth bang front of  $\exp_{\pm 2}$ ; then at time  $T_2 = 0$ ,  $T_3 = T_{3c} = T_{3b} + \frac{4}{3}C_2 - \frac{8}{3}c_{110}^2 < T_{3b}$  the fourth bang of  $\exp_{\pm 1}$  intersects the fifth bang of  $\exp_{\pm 2}$ ; finally the fourth bang of  $\exp_1$  intersects the fourth bang of  $\exp_{-1}$  at  $T_2 = 0$  and  $T_3 = T_{3b}$ . See Figure 7.

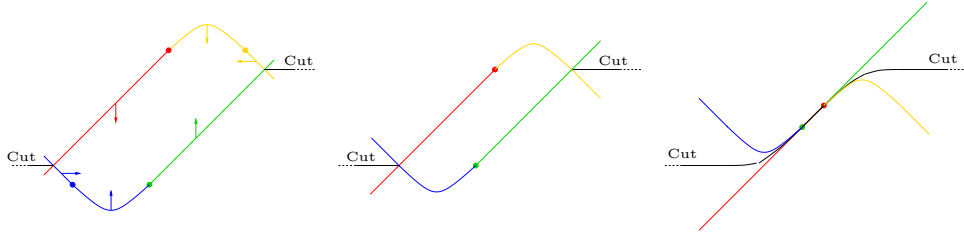


Figure 7: Picture of the front at times with  $T_2 = 0$  and  $T_3 < T_{3c}$ ,  $T_3 = T_{3c}$  and  $T_3 = T_{3b}$

- When  $4b_{110} + 8c_{110}c_{200} - 8c_{210} < 0$  and  $4b_{110} + 8c_{110}c_{200} - 8c_{210} - 4C_2 > 0$  then the relative position of the two parabola of the fifth bang of  $\exp_2$  and  $\exp_{-2}$  implies that the sequel of the self intersections of the front is the following : first the fourth bang front of  $\exp_{\pm 1}$  intersect the fourth bang front of  $\exp_{\pm 2}$ ; then at time  $T_2 = 0$ ,  $T_3 = T_{3c} = T_{3b} + \frac{4}{3}C_2 - \frac{8}{3}c_{110}^2 < T_{3b}$  the fourth bang of  $\exp_{\pm 1}$  intersects the fifth bang of  $\exp_{\pm 2}$ ; finally the fifth bang of  $\exp_2$  intersects the fifth bang of  $\exp_{-2}$  at a time with  $T_2 = 0$  and  $T_3 = T_{3g}$  between  $T_{3c}$  and  $T_{3b}$ . See Figure 8.

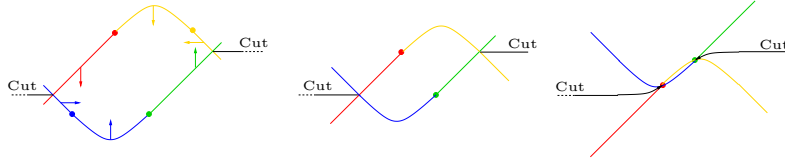


Figure 8: Picture of the front at times with  $T_2 = 0$  and  $T_3 < T_{3c}$ ,  $T_3 = T_{3c}$  and  $T_3 = T_{3g}$

- When  $4b_{110} + 8c_{110}c_{200} - 8c_{210} > 0$  and  $4b_{110} + 8c_{110}c_{200} - 8c_{210} + 4C_2 < 0$  then the relative position of the two parabola of the fifth bang of  $\exp_2$  and  $\exp_{-2}$  implies that the sequel of the self intersections of the front is the following : first the fourth bang front of  $\exp_{\pm 1}$  intersect the fourth bang front of  $\exp_{\pm 2}$ ; then at time  $T_2 = 0$  and  $T_3 = T_{3d} = T_{3a} + 2C_2 - 2c_{110}^2 < T_{3a}$

the fourth bang of  $\exp_{\pm 2}$  intersects the fifth bang of  $\exp_{\mp 2}$ ; finally the fifth bang of  $\exp_2$  intersects the fifth bang of  $\exp_{-2}$ . The picture is similar to the one of Figure 7.

- When  $4b_{110} + 8c_{110}c_{200} - 8c_{210} + 4C_2 > 0$  then all the fifth bang of  $\exp_2$  satisfies  $x > 4c_{110}\rho^2$  when all the fifth bang of  $\exp_{-2}$  satisfies  $x < 4c_{110}\rho^2$ . This implies that the sequel of the self intersections of the front is the following : first the fourth bang front of  $\exp_{\pm 1}$  intersect the fourth bang front of  $\exp_{\pm 2}$ ; then at time  $T_2 = 0$  and  $T_3 = T_{3d} = T_{3a} + 2C_2 - 2c_{110}^2 < T_{3a}$  the fourth bang of  $\exp_{\pm 2}$  intersects the fifth bang of  $\exp_{\mp 2}$ ; finally the fourth bang of  $\exp_2$  intersects the fourth bang of  $\exp_{-2}$  at  $T_2 = 0$  and  $T_3 = T_{3a}$ . The picture is similar to the one of Figure 8.

In the four cases, the cut locus has only one branch, which is continuous and piecewise smooth. And the proportions are those given in the Figure 9.

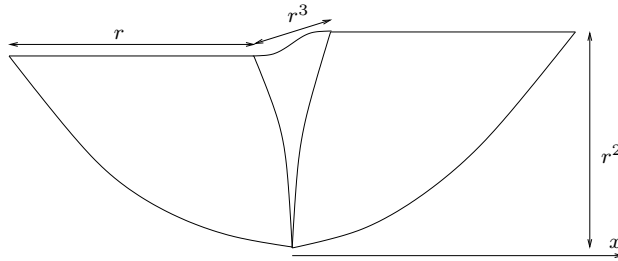


Figure 9: Picture of the cut locus when  $C_1 > 0$  and  $C_2 < 0$

### 6.5 Cut locus when $C_1 < 0$ and $C_2 > 0$

The same kind of computations can be done in this case as in the previous case. For the picture of the cut locus we refer to the same figure 8 where the  $x$ -axis should be replaced by the  $y$ -axis.

### 6.6 Intersections of fifth bangs

In the case  $C_1 < 0$  and  $C_2 < 0$ , the fifth bang front self intersect before losing optimality. As before this happen for  $t \sim 8\rho$  and we write  $t = 8\rho + T_2\rho^2 + T_3\rho^3$ .

As seen before, each fifth bang front is a part of parabola. For  $T_2 < 0$ , or  $T_2 = 0$  and  $T_3$  small enough, the four parabolas are not intersecting, are positioned as in the figure 10 and they are linked by the part of the front constituted of fourth bangs, and the front do not self intersect.

One way to build the optimal part of the front is to consider the expressions of the fifth bangs and of the four bangs, to consider them for all the values of  $\alpha_i \in [-1, 1]$  and to keep only the part which constitutes the boundary of the "central" domain (see Figure 10). The dynamics with respect to  $T_3$  of each of these expressions consist only on translations of  $\pm T_3$  along  $x$  or  $y$ . Hence to identify the optimal part of these expressions, we just have to understand what are the consecutive intersections when  $T_3$  varies.

- The first intersection is of the fifth bang front of  $\exp_{\pm 1}$  with the one of  $\exp_{\pm 2}$  at  $T_2 = 0$  and  $T_3 = T_{3e}$  or with the one of  $\exp_{\mp 2}$  at  $T_2 = 0$  and  $T_3 = T_{3f}$ .

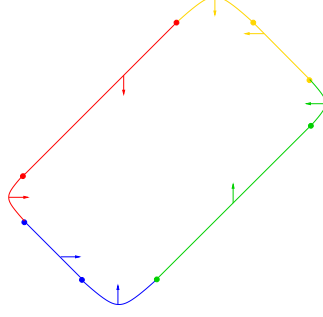


Figure 10: The front before  $t = 8\rho$  when  $C_1 < 0$  and  $C_2 < 0$

When writing the intersection of the fifth fronts, that is for example that  $x_{1sus}(\alpha_2 = -1) = x_{2sus}(\alpha_1 = 1)$  and  $y_{1sus}(\alpha_2 = -1) = y_{2sus}(\alpha_1 = 1)$ , one finds

$$T_{3e} = \frac{4}{3}(a_{110} + 3b_{110} - 6b_{200} + 18c_{110}c_{200} + 2c_{120} + 6c_{300} - 6c_{210})$$

and

$$T_{3f} = \frac{4}{3}(a_{110} - 3b_{110} - 6b_{200} + 6c_{110}c_{200} + 2c_{120} + 6c_{300} + 6c_{210}).$$

After that time, the fifth bang fronts that connected self intersect, until a next event.

Case 1 The next event can be that all the front corresponding to the fifth bang of  $\exp_{\pm 1}$  (resp.  $\exp_{\pm 2}$ ) is no more optimal. This comes from the fact that

- the entire arc of parabolas of the fifth bang front of  $\exp_{\pm 1}$  crossed the parabolas of  $\exp_{\pm 2}$  which occurs if  $2|C_1| < |C_2|$ .
- the entire arc of parabolas of the fifth bang front of  $\exp_{\pm 2}$  crossed the parabolas of  $\exp_{\pm 1}$  which occurs if  $2|C_1| > |C_2|$ ,

see figure 11. The corresponding time can be computed in the following way. Assume that  $T_{3e} < T_{3f}$  and hence that the first event was the contact of the fifth bang front of  $\exp_1$  with the one of  $\exp_2$  at one of their extremity. Then, the second event will happen at  $T_3$  such that one of the other extremities, let call it  $q(T_3)$  crosses the other parabola at  $p(T_3)$ , see Figure **FIGURE**. Thanks to the dynamics with respect to  $T_3$ ,  $p(T_3)$  and  $q(T_3)$  belongs for all  $T_3$  at the line  $x + y = c + T_3$  where  $c \in \mathbb{R}$ . Together with the expressions of the parabolas one find that the corresponding time is  $T_3 = T_{3e} + \tau_3$  with

$$\tau_3 = 8\sqrt{2C_1C_2}.$$

If  $T_{3f} < T_{3e}$  then it happens at  $T_3 = T_{3f} + \tau_3$ .

Case 2 An other event, that can occurs after the first intersection, is the other contact between fifth bang fronts occurs. If  $T_{3e} < T_{3f}$  then this event is at  $T_3 = T_{3f}$  and if  $T_{3f} < T_{3e}$  it is at  $T_3 = T_{3e}$ . See Figure 12

Case 1.1 In the case 1, the next event can be the closure of the synthesis by the contact of the four bangs. If  $T_{3e} < T_{3f}$  the fourth bang fronts of  $\exp_1$  and  $\exp_{-1}$  can intersect at time  $T_{3b}$ . If

$T_{3f} < T_{3e}$  the fourth bang fronts of  $\exp_2$  and  $\exp_{-2}$  can intersect at time  $T_{3a}$ . This case occurs only if the arc of parabolas of  $\exp_{\pm 1}$  from one part, and the arc of parabolas of  $\exp_{\pm 2}$  from the other part, do not intersect at any time  $T_3$ .

Case 1.2 In the case 1, another possibility is that the four bang front loses entirely its optimality. If  $2|C_1| < |C_2|$  it correspond to the time at which an extremity of the fifth bang front of  $\exp_2$  touches the fifth bang front of  $\exp_{-2}$ . If  $2|C_1| > |C_2|$  it correspond to the time at which an extremity of the fifth bang front of  $\exp_1$  touches the fifth bang front of  $\exp_{-1}$ . These times can be computed by translating in the calculus these intersection and we get in the different cases

- If  $T_{3e} < T_{3f}$  and  $|C_2| < 2|C_1|$  then  $T_3 = T_{3g} = -K_1 + 2C_1(1 + \alpha_g^2)$  with  $\alpha_g = -1 + \frac{1}{C_1}(b_{110} + 2c_{110}c_{200} - 2c_{210})$  and

$$K_1 = \frac{16}{3}c_{120} - \frac{4}{3}a_{110} - 4c_{110}^2 - 16c_{110}c_{200} - 8c_{200}^2 + 4c_{300}.$$

- If  $T_{3e} < T_{3f}$  and  $|C_2| > 2|C_1|$  then  $T_3 = T_{3h} = -K_2 + 2C_2(1 + \alpha_h^2)$  with  $\alpha_h = 1 - \frac{2}{C_2}(b_{110} + 2c_{110}c_{200} - 2c_{210})$  and

$$K_2 = \frac{4}{3}c_{120} - \frac{4}{3}a_{110} - 2c_{110}^2 - 16c_{110}c_{200} - 16c_{200}^2 + 16c_{300} - 8b_{200}.$$

- If  $T_{3e} > T_{3f}$  and  $|C_2| < 2|C_1|$  then  $T_3 = T_{3i} = -K_1 + 2C_1(1 + \alpha_i^2)$  with  $\alpha_i = 1 + \frac{1}{C_1}(b_{110} + 2c_{110}c_{200} - 2c_{210})$ .
- If  $T_{3e} > T_{3f}$  and  $|C_2| > 2|C_1|$  then  $T_3 = T_{3j} = -K_2 + 2C_2(1 + \alpha_j^2)$  with  $\alpha_j = -1 - \frac{2}{C_2}(b_{110} + 2c_{110}c_{200} - 2c_{210})$ .

After the fourth bang front lost optimality the optimal synthesis finishes by the last self intersection of the fifth bang front.

Case 2.2 In case two, after  $\max\{T_{3e}, T_{3f}\}$ , the optimal synthesis closes as follows. If  $|C_2| < 2|C_1|$ , then the next event is the loss of optimality of the entire fifth bang front of  $\exp_{\pm 2}$ , and the optimal synthesis finishes by the intersection of the parabolas of  $\exp_{\pm 1}$ . If  $|C_2| > 2|C_1|$ , then the next event is the loss of optimality of the entire fifth bang front of  $\exp_{\pm 1}$ , and the optimal synthesis finishes by the intersection of the parabolas of  $\exp_{\pm 2}$ .

## 6.7 Cut locus when $C_1 < 0$ and $C_2 < 0$

Thanks to the description of the different steps that can occur along the dynamics of the front, we can conclude by claiming

- If  $|T_{3e} - T_{3f}| < \tau_3$  then the cut locus has 5 smooth branches as in Figure 12.
- If not it has only one branch which is continuous and smooth by arcs, see Figure 11.

Finally we can give the picture of the cut locus in this two cases in Figure 13.

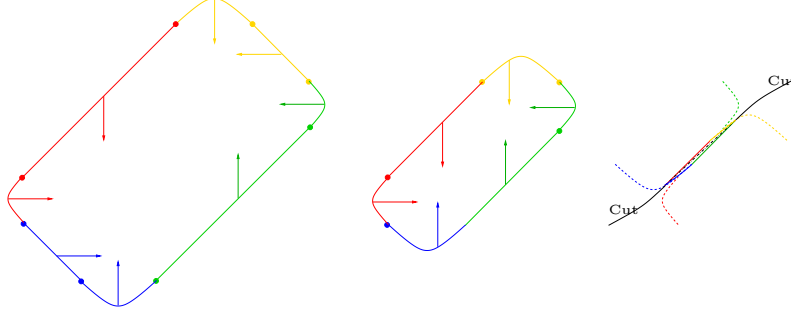


Figure 11: Evolution of the front when  $|T_{3e} - T_{3f}| > \tau_3$

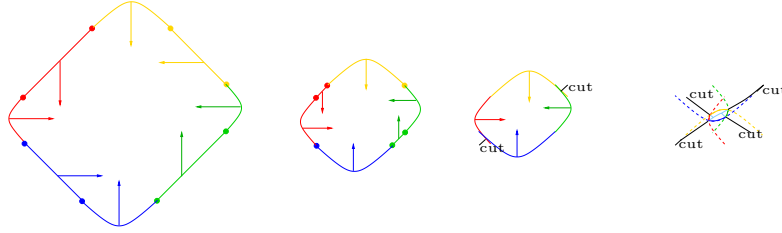


Figure 12: Evolution of the front when  $|T_{3e} - T_{3f}| < \tau_3$

## 6.8 Singularities and stability, open question

All the computations we made in this section for the cut locus or conjugate locus are stable except for extremals with initial conditions  $|\lambda_x| = |\lambda_y| = 1$ . Effectively, under the codimension 1 assumption that both  $C_1 \neq 0$  and  $C_2 \neq 0$ , except for these initial conditions, the cut points correspond to transversal self intersections of the wave front.

For the initial conditions  $|\lambda_x| = |\lambda_y| = 1$ , a further study should be done in order to find a good notion of stability, which is itself not clear, and to study it in this case. In the case  $C_1 > 0$  and  $C_2 > 0$ , the corresponding singularity in the sub-Riemannian contact case, corresponding to the extremity of the cut locus, is a cusp  $\mathcal{A}_3$  (in the classification of Arnol'd) and it is stable as smooth or lagrangian singularity. We may propose the conjecture that a good theory of stability should find in our context that the singularity is stable. If this conjecture is valid then the pictures of the cut locus are stable and valid not only for the jet of the dynamics we have computed but also for the true dynamics.

## 7 Extremals with only one control switching several times

For  $|\lambda_z|$  large enough the dynamics is described in the previous sections. We can now choose a constant  $\Lambda_z > 0$  large enough and considering only the extremal satisfying  $|\lambda_z| < \Lambda_z$ . As seen before, along an extremal

$$\dot{\phi}_3 = u_1(f_{41}\phi_1 + f_{42}\phi_2 + f_{43}\phi_3) + u_2(f_{51}\phi_1 + f_{52}\phi_2 + f_{53}\phi_3).$$

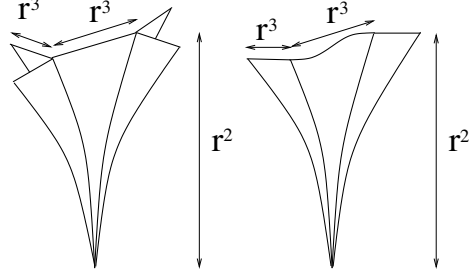


Figure 13: Possible cut loci when  $C_1 < 0$  and  $C_2 < 0$

One computes easily that

$$\begin{aligned} f_{41} &= -\frac{a_{110}+2a_{200}+b_{110}+2b_{200}}{4}, & f_{51} &= \frac{a_{110}-2a_{200}+b_{110}-2b_{200}}{4}, \\ f_{42} &= -\frac{a_{110}+2a_{200}-b_{110}-2b_{200}}{4}, & f_{52} &= \frac{a_{110}-2a_{200}-b_{110}+2b_{200}}{4}, \\ f_{43} &= \frac{1}{2}c_{110} + c_{200}, & f_{53} &= -\frac{1}{2}c_{110} + c_{200}. \end{aligned}$$

With  $|\phi_1| \leq 1$  and  $|\phi_2| \leq 1$ , we get

$$|\dot{\phi}_3| \leq |f_{41}| + |f_{42}| + |f_{51}| + |f_{52}| + (|f_{53}| + |f_{43}|)|\phi_3| \leq 4M' + 2M'\Lambda_z$$

where  $M'$  is a local bound of the  $f_{ij}$ . This implies that, for the extremals we are considering, the possibility of switching in short time implies that the corresponding switching function starts close to 0. Which implies that in short time only one control switches. And if in short time a control switches twice hence  $\phi_3$  should change sign and hence starts close to 0 that is  $\lambda_z$  should starts close to 0.

In the following, we will be interested only in finding extremals that switch at least twice (on the same control) since the ones that switch only once are yet obtained with initial conditions with large  $|\lambda_z|$ .

We will consider only extremals with  $u_1 \equiv 1$ , the study of the other ones being equivalent. Along such an extremal

$$\ddot{\phi}_2 = u_1 \dot{\phi}_3 = \dot{\phi}_3$$

and since  $u_1 \equiv 1$  one gets

$$\ddot{\phi}_2 = (f_{41} + u_2 f_{51})\phi_1 + (f_{42} + u_2 f_{52})\phi_2 + (f_{43} + u_2 f_{53})\phi_3.$$

Since  $\phi_3(t) = O(t)$ ,  $\phi_2 = O(t)$  and  $\phi_1(t) = 1 + O(t)$  we get that

$$\ddot{\phi}_2(t) = (f_{41} + u_2 f_{51}) + O(t).$$

In the following we assume that we are considering a point where  $f_{41} + f_{51} \neq 0$  and  $f_{41} - f_{51} \neq 0$ . We consider then the four following cases

1. If  $|f_{51}| < f_{41}$  then  $f_{41} + u_2 f_{51} > 0$  for all  $u_2 \in [0, 1]$  and  $\ddot{\phi}_2(t) > 0$  for all  $t$ . As a consequence the only possible behaviours of the control  $u_2$  are (see Figure 14)

- (a)  $u_2 \equiv 1$ ,



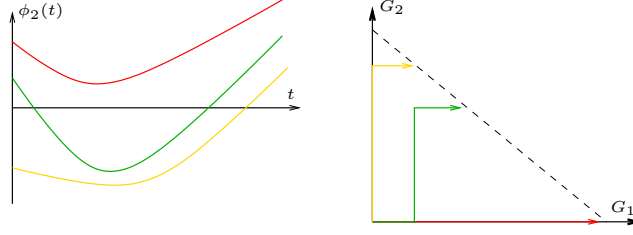


Figure 14: Extremals when  $|f_{51}| < f_{41}$

- (b)  $u_2 = -1$  during a first interval of time and switches to 1,
  - (c)  $u_2 = 1$  during a first interval of time, then  $-1$  during a second one, and finally switches to 1.
2. If  $|f_{51}| < -f_{41}$  then  $f_{41} + u_2 f_{51} < 0$  for all  $u_2 \in [0, 1]$  and  $\ddot{\phi}_2(t) < 0$  for all  $t$ . As a consequence the only possible behaviours of the control  $u_2$  are (see Figure 15)
- (a)  $u_2 \equiv -1$ ,
  - (b)  $u_2 = 1$  during a first interval of time and switches to  $-1$ ,
  - (c)  $u_2 = -1$  during a first interval of time, then 1 during a second one, and finally switches to  $-1$ .

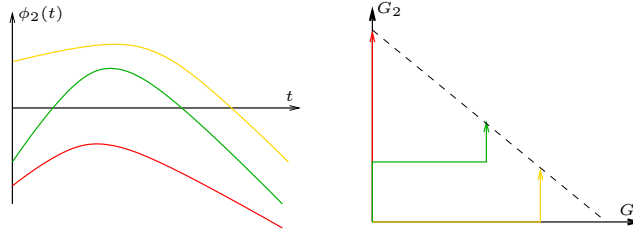


Figure 15: Extremals when  $|f_{51}| < -f_{41}$

3. If  $|f_{41}| < f_{51}$  then  $f_{41} + f_{51} > 0$  hence  $\ddot{\phi}_2(t) > 0$  when  $\phi_2(t) > 0$  and  $f_{41} - f_{51} < 0$  hence  $\ddot{\phi}_2(t) < 0$  when  $\phi_2(t) < 0$ . In that case the possible behaviours of the control  $u_2$  are (see Figure 16)
- (a)  $u_2$  is constant and equal to  $\pm 1$ ,
  - (b)  $u_2$  is equal to 1 or  $-1$  during a first interval of time and switches to  $-1$  or 1,
  - (c)  $u_2$  is equal to 1 or  $-1$  during a first interval of time, then  $\phi_2 = 0$  during a second interval where  $u_2(t) = -\frac{f_{41}(q(t))}{f_{51}(q(t))} + O(t)$ , and finally  $u_2$  switches to 1 or  $-1$ .
4. If  $|f_{41}| < -f_{51}$  then  $f_{41} + f_{51} < 0$  hence  $\ddot{\phi}_2(t) < 0$  when  $\phi_2(t) > 0$  and  $f_{41} - f_{51} > 0$  hence  $\ddot{\phi}_2(t) > 0$  when  $\phi_2(t) < 0$ . In that case the list of possible behaviours may be very large. In the following we analyse more deeply to prove that the possible behaviours are

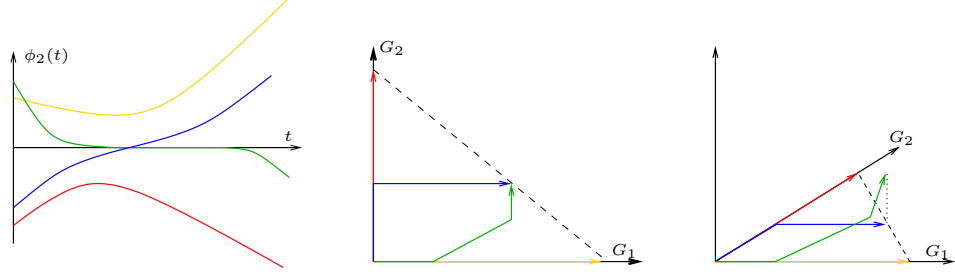


Figure 16: Extremals when  $|f_{41}| < f_{51}$

- (a)  $u_2$  is constant and equal to  $\pm 1$ ,
- (b)  $u_2$  is constant and equal to  $\pm 1$  during a first interval of time and switches to  $\mp 1$ ,
- (c)  $u_2$  is constant and equal to  $\pm 1$  during a first interval of time and switches to  $\mp 1$ , and finally switches again to  $\pm 1$ .

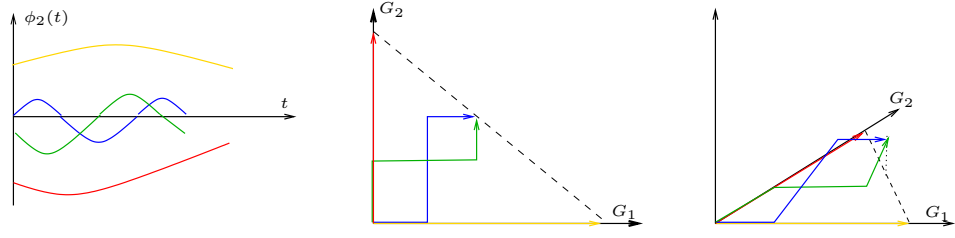


Figure 17: Extremals when  $|f_{41}| < -f_{51}$

A more precise description of the optimal ones is given in the following analysis. In particular, in this case, appears a cut locus.

### 7.1 Extremals when $|f_{41}| < -f_{51}$

In the following we prove that, in the case  $|f_{41}| < -f_{51}$ , an extremal with  $u_1 \equiv 1$  with four bangs is not optimal.

An easy computation shows that

$$\begin{aligned} f_{41}(0) &= -\frac{1}{2}(a_{200} + b_{200} + \frac{a_{110} + b_{110}}{2}) \\ f_{51}(0) &= -\frac{1}{2}(a_{200} + b_{200} - \frac{a_{110} + b_{110}}{2}) \end{aligned}$$

The hypothesis  $|f_{41}| < -f_{51}$  is equivalent to  $a_{200} + b_{200} > 0$  and  $\frac{a_{110} + b_{110}}{2} < 0$ .

Consider the three following extremals from  $(0, 0, 0)$  to  $(x, y, z)$ . The first one, denoted  $\epsilon$ , has  $u_2 = 1$  during time  $\epsilon_1$  then  $u_2 = -1$  during time  $\epsilon_2$  and finally  $u_2 = 1$  during time  $\epsilon_3$ . The second one, denoted  $\theta(t)$ , has  $u_2 = -1$  during time  $\theta_1$  then  $u_2 = 1$  during time  $\theta_2$  and finally  $u_2 = -1$  during time  $\theta_3$ . The last one, denoted  $\gamma(t)$ , has  $u_2 = -1$  during time  $\gamma_1$  then  $u_2 = 1$  during time

$\gamma_2$  then  $u_2 = -1$  during time  $\gamma_3$  and finally  $u_2 = 1$  during time  $\gamma_4$ . One prove easily that, denoting  $s_\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3$ ,  $s_\theta = \theta_1 + \theta_2 + \theta_3$  and  $s_\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ ,

$$\begin{aligned}
x_\epsilon(s_\epsilon) &= \epsilon_1 + \epsilon_3 + a_{200}\epsilon_1^2\epsilon_2 + a_{110}\epsilon_1\frac{\epsilon_2^2}{2} + o(s_\epsilon^3), \\
y_\epsilon(s_\epsilon) &= \epsilon_2 + b_{200}\epsilon_1^2\epsilon_2 + b_{110}\epsilon_1\frac{\epsilon_2^2}{2} + o(s_\epsilon^3), \\
z_\epsilon(s_\epsilon) &= \epsilon_1\epsilon_2 + o(s_\epsilon^2), \\
\\
x_\theta(s_\theta) &= \theta_2 + a_{200}\theta_2^2\theta_3 + a_{110}\theta_2\theta_3(\theta_1 + \frac{\theta_3}{2}) + o(s_\theta^3), \\
y_\theta(s_\theta) &= \theta_1 + \theta_3 + b_{200}\theta_2^2\theta_3 + b_{110}\theta_2\theta_3(\theta_1 + \frac{\theta_3}{2}) + o(s_\theta^3), \\
z_\theta(s_\theta) &= \theta_2\theta_3 + o(s_\theta^2), \\
\\
x_\gamma(s_\gamma) &= \gamma_2 + \gamma_4 + a_{200}\gamma_2^2\gamma_3 + a_{110}\gamma_2\gamma_3(\gamma_1 + \frac{\gamma_3}{2}) + o(s_\gamma^3), \\
y_\gamma(s_\gamma) &= \gamma_1 + \gamma_3 + b_{200}\gamma_2^2\gamma_3 + b_{110}\gamma_2\gamma_3(\gamma_1 + \frac{\gamma_3}{2}) + o(s_\gamma^3), \\
z_\gamma(s_\gamma) &= \gamma_2\gamma_3 + o(s_\gamma^2),
\end{aligned}$$

but since at  $s_\theta$  and  $s_\gamma$  the extremals are supposed to be at  $(x, y, z)$  then one gets

$$\begin{aligned}
x + y &= s_\theta + (a_{200} + b_{200})\theta_2^2\theta_3 + \frac{a_{110} + b_{110}}{2}\theta_2\theta_3(\theta_1 + \frac{\theta_3}{2}) + o(s_\theta^3) \\
&= s_\theta + (a_{200} + b_{200})\theta_2 z + \frac{a_{110} + b_{110}}{2}z(2\theta_1 + \theta_3) \\
x + y &= s_\gamma + (a_{200} + b_{200})\gamma_2^2\gamma_3 + \frac{a_{110} + b_{110}}{2}\gamma_2\gamma_3(\gamma_1 + \frac{\gamma_3}{2}) + o(s_\gamma^3) \\
&= s_\gamma + (a_{200} + b_{200})\gamma_2 z + \frac{a_{110} + b_{110}}{2}z(2\gamma_1 + \gamma_3).
\end{aligned}$$

Hence we deduce

$$\begin{aligned}
\frac{s_\gamma - s_\theta}{z} &= (a_{200} + b_{200})(\theta_2 - \gamma_2) + \frac{a_{110} + b_{110}}{2}(2\theta_1 + \theta_3 - (2\gamma_1 + \gamma_3)) + o(x + y) \\
&= (a_{200} + b_{200})(\theta_2 - \gamma_2) + \frac{a_{110} + b_{110}}{2}(\theta_1 - \gamma_1) + o(x + y)
\end{aligned}$$

since  $\theta_1 + \theta_3 = y + o(x + y)$  and  $\gamma_1 + \gamma_3 = y + o(x + y)$ .

Now, we should analyse the relation between  $\gamma_2$  and  $\gamma_3$ . One can prove that along the curve  $\gamma$ , during the second bang,  $\dot{\phi}_2 = f_{41} - f_{51} + o(t) = -\frac{a_{110} + b_{110}}{2} + o(t)$  and during the second bang  $\ddot{\phi}_2 = f_{41} + f_{51} + o(t) = -(a_{200} + b_{200}) + o(t)$ . One proves easily that, since  $\phi_2 = 0$  at the extremity of each of these intervalles, this implies that

$$\frac{\gamma_3}{\gamma_2} = -\frac{a_{200} + b_{200}}{\frac{a_{110} + b_{110}}{2}} + o(x + y),$$

hence exists  $\lambda > 0$  such that  $\gamma_3 = \lambda(a_{200} + b_{200}) + o((x+y)^2)$  and  $\gamma_2 = -\lambda \frac{a_{110} + b_{110}}{2} + o((x+y)^2)$ . As a consequence

$$\begin{aligned} \lambda \frac{s_\gamma - s_\theta}{z} &= \gamma_3(\theta_2 - \gamma_2) - \gamma_2(\theta_1 - \gamma_1) + o((x+y)^2) \\ &= \gamma_3(x - \gamma_2) - \gamma_2(\gamma_3 - \theta_3) + o((x+y)^2) \\ &= \gamma_3x + \gamma_2 \frac{z}{x} - 2\gamma_2\gamma_3 + o((x+y)^2) \\ &= \gamma_3x + \frac{z}{\gamma_3} \frac{z}{x} - 2z + o((x+y)^2) \end{aligned}$$

hence

$$\lambda \frac{s_\gamma - s_\theta}{z^2} = \frac{\gamma_3x}{z} + \frac{z}{x\gamma_3} - 2 + o(1)$$

hence  $s_\gamma - s_\theta$  is strictly positive except maybe when  $\gamma_3 \sim \theta_3$  and  $\gamma_2 \sim x$ .

But comparing with the curve  $\epsilon$  we get that  $s_\gamma - s_\epsilon > 0$  except maybe when  $\gamma_2 \sim \epsilon_1$  and  $\gamma_3 \sim y$ . Finally we can conclude that such an extremal  $\gamma$  is not optimal. The same proof can be done for the extremals with four bangs following first  $G_1$ , then  $G_2$ , then  $G_1$  and finally  $G_2$ . And no extremal with three switches on the same control can be optimal.

Comparing the curves  $\epsilon$  and  $\theta$  one gets

$$\frac{s_\epsilon - s_\theta}{z(1 - \frac{z}{xy})} = (a_{200} + b_{200})x + \frac{a_{110} + b_{110}}{2}y + o(x+y).$$

Hence, since  $a_{200} + b_{200} > 0$  and  $\frac{a_{110} + b_{110}}{2} < 0$  the curve  $\epsilon$  is optimal for  $y > -2\frac{a_{200} + b_{200}}{a_{110} + b_{110}}x + o(x)$  and we find that there is a cut locus which is tangent at 0 to the plane

$$(a_{200} + b_{200})x + \frac{a_{110} + b_{110}}{2}y = 0.$$

## 7.2 Other extremals generating cut locus

One show easily that, for extremals with  $u_1 \equiv 1$ , there is also cut locus only if  $|f_{41}| < -f_{51}$ , that is if  $a_{200} + b_{200} > 0$  and  $a_{110} + b_{110} < 0$ , and the tangent plane is the same.

In the cases  $u_2 \equiv 1$  and  $u_2 \equiv -1$  then there is cut locus only if  $|f_{52}| < f_{42}$ , that is if  $b_{110} - a_{110} > 0$  and  $b_{200} - a_{200} > 0$ . In this last case the tangent plane at 0 is

$$(a_{200} - b_{200})x + \frac{a_{110} - b_{110}}{2}y = 0.$$

## 7.3 Cut locus generated by extremals with $\lambda_z(0) \sim 0$

As a consequence of the previous computations, we can describe the part of the local cut locus generated by the extremals with  $\lambda_0(0) \sim 0$ .

- if  $(a_{200} + b_{200} < 0$  or  $a_{110} + b_{110} > 0)$  and  $(b_{110} - a_{110} < 0$  or  $b_{200} - a_{200} < 0)$  then this part of the local cut locus is empty.
- if  $a_{200} + b_{200} > 0$  and  $a_{110} + b_{110} < 0$  and  $(b_{110} - a_{110} < 0$  or  $b_{200} - a_{200} < 0)$  then this part of the cut locus writes

$$\{(x, -2\frac{a_{200} + b_{200}}{a_{110} + b_{110}}x + o(x), z) \mid 0 \leq z \leq -2\frac{a_{200} + b_{200}}{a_{110} + b_{110}}x^2 + o(x^2)\}$$

- if  $(a_{200} + b_{200} < 0$  or  $a_{110} + b_{110} > 0)$  and  $b_{110} - a_{110} > 0$  and  $b_{200} - a_{200} > 0$  then this part of the cut locus writes

$$\{(x, -2\frac{a_{200} - b_{200}}{a_{110} - b_{110}}x + o(x), z) \mid 0 \geq z \geq -2\frac{a_{200} - b_{200}}{a_{110} - b_{110}}x^2 + o(x^2)\}$$

- if  $a_{200} + b_{200} > 0$  and  $a_{110} + b_{110} < 0$  and  $b_{110} - a_{110} > 0$  and  $b_{200} - a_{200} > 0$  then this part of the local cut locus is the union of the two previous sets.

Finally we can propose the picture of this part of the cut locus in Figure 18

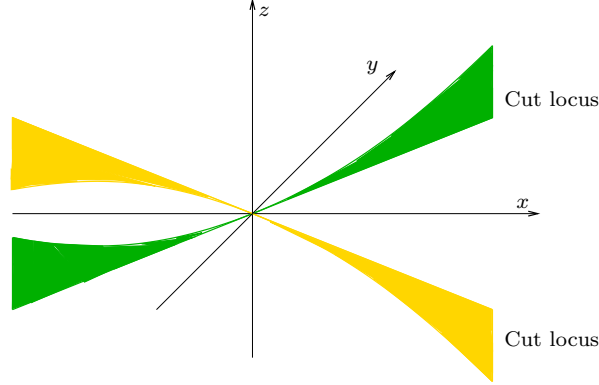


Figure 18: Part of the cut locus generated by the extremal with  $\lambda_z(0) \sim 0$  when  $|f_{41}| < -f_{51}$  and  $|f_{52}| < f_{42}$

## References

- [1] A. Agrachev, B. Bonnard, M. Chyba, and I. Kupka. Sub-Riemannian sphere in Martinet flat case. *ESAIM Control Optim. Calc. Var.*, 2:377–448, 1997.
- [2] A. A. Agrachev, U. Boscain, G. Charlot, R. Ghezzi, and M. Sigalotti. Two-dimensional almost-Riemannian structures with tangency points. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(3):793–807, 2010.
- [3] A. A. Agrachev, El-H. Chakir El-A., and J. P. Gauthier. Sub-Riemannian metrics on  $\mathbf{R}^3$ . In *Geometric control and non-holonomic mechanics (Mexico City, 1996)*, volume 25 of *CMS Conf. Proc.*, pages 29–78. Amer. Math. Soc., Providence, RI, 1998.
- [4] Andrei Agrachev and Jean-Paul Gauthier. On the subanalyticity of Carnot-Caratheodory distances. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(3):359–382, 2001.
- [5] Andrei A. Agrachev and Yuri L. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.
- [6] E. Ali and G. Charlot. Local (sub) finslarian geometry for the maximum norm in dimension 2. *preprint*.

- [7] D. Barilari, U. Boscain, E. Le Donne, and M. Sigalotti. Sub-finsler structures from the time-optimal control viewpoint for some nilpotent distributions. *arXiv:1506.04339*, 2016.
- [8] Davide Barilari, Ugo Boscain, Grégoire Charlot, and Robert W. Neel. On the heat diffusion for generic riemannian and sub-riemannian structures. *IMRN*, 2016.
- [9] Davide Barilari, Ugo Boscain, and Robert W. Neel. Small-time heat kernel asymptotics at the sub-Riemannian cut locus. *J. Differential Geom.*, 92(3):373–416, 2012.
- [10] André Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 1–78. Birkhäuser, Basel, 1996.
- [11] G. Ben Arous. Développement asymptotique du noyau de la chaleur hypoelliptique hors du cut-locus. *Ann. Sci. École Norm. Sup. (4)*, 21(3):307–331, 1988.
- [12] G. Ben Arous and R. Léandre. Décroissance exponentielle du noyau de la chaleur sur la diagonale. II. *Probab. Theory Related Fields*, 90(3):377–402, 1991.
- [13] B. Bonnard, M. Chyba, and E. Trelat. Sub-riemannian geometry, one-parameter deformation of the martinet flat case. *J. Dyn. Control Syst.* 4, No.1, 59-76 (1998)., 4:59–76, 1998.
- [14] Bernard Bonnard, Grégoire Charlot, Roberta Ghezzi, and Gabriel Janin. The Sphere and the Cut Locus at a Tangency Point in Two-Dimensional Almost-Riemannian Geometry. *J. Dynam. Control Systems*, 17(1):141–161, 2011.
- [15] Bernard Bonnard and Monique Chyba. Méthodes géométriques et analytiques pour étudier l’application exponentielle, la sphère et le front d’onde en géométrie sous-riemannienne dans le cas Martinet. *ESAIM Control Optim. Calc. Var.*, 4:245–334 (electronic), 1999.
- [16] U. Boscain, G. Charlot, and R. Ghezzi. Normal forms and invariants for 2-dimensional almost-Riemannian structures. *Differential Geom. Appl.*, 31(1):41–62, 2013.
- [17] U. Boscain, G. Charlot, R. Ghezzi, and M. Sigalotti. Lipschitz classification of almost-riemannian distances on compact oriented surfaces. *Journal of Geometric Analysis*, pages 1–18. 10.1007/s12220-011-9262-4.
- [18] Ugo Boscain, Thomas Chambrion, and Grégoire Charlot. Nonisotropic 3-level quantum systems: complete solutions for minimum time and minimum energy. *Discrete Contin. Dyn. Syst. Ser. B*, 5(4):957–990, 2005.
- [19] E. Breuillard and E. Le Donne. On the rate of convergence to the asymptotic cone for nilpotent groups and subfinsler geometry. *Proc. Natl. Acad. Sci. USA*, 110(48):19220–19226, 2013.
- [20] Grégoire Charlot. Quasi-contact s-r metrics : normal form in  $\mathbb{R}^{2n}$ , wave front and caustic in  $\mathbb{R}^4$ . *Acta App. Math.*, 74:217–263, 2002.
- [21] W. L. Chow. ber systeme von linearen partiellen differentialgleichungen erster ordnung. *Math. Ann.*, 117:98–105, 1939.
- [22] Jeanne Clelland and Christopher Moseley. Sub-finsler geometry in dimension three. *Differ. Geom. Appl.*, 24(6):628–651, 2006.

- [23] Jeanne Clelland, Christopher Moseley, and George Wilkens. Geometry of sub-finsler engel manifolds. *Asian J. Math.*, 11(4):699–726, 2007.
- [24] El-Houcine Chakir El Alaoui, J.-P. Gauthier, and I. Kupka. Small sub-riemannian balls on  $\mathbb{R}^3$ . *J. Dyn. Control Syst.*, 2(3):359–421, 1996.
- [25] A.F. Filippov. On some questions in the theory of optimal regulation: existence of a solution of the problem of optimal regulation in the class of bounded measurable functions. *Vestnik Moskov. Univ. Ser. Mat. Meh. Astr.Fiz.Him.*, 2:2532, 1959.
- [26] Rémi Léandre. Majoration en temps petit de la densité d’une diffusion dégénérée. *Probab. Theory Related Fields*, 74(2):289–294, 1987.
- [27] Rémi Léandre. Minoration en temps petit de la densité d’une diffusion dégénérée. *J. Funct. Anal.*, 74(2):399–414, 1987.
- [28] P.K. Rashevsky. About connecting two points of complete nonholonomic space by admissible curve. *Uch. Zap. Ped. Inst. Libknehta*, 2:83–94, 1938.
- [29] M. Sigalotti. Some computations for 2nd variations in sub-finsler geometry. *preprint*.

# Résumé

Dans cette thèse, j'étudie la géométrie locale des structures finslériennes et sous-finslériennes associées à la norme infini en dimension 2 et 3 : géodésiques généralisées courtes, lieu de coupure, lieu conjugué généralisé, lieu de "saut", petites sphères.

Pour définir une telle structure au voisinage d'un point  $p$  de  $\mathbb{R}^n$ , on se donne une famille de champs de vecteurs  $(F_1, \dots, F_k)$  et on considère la norme définie sur la distribution  $\Delta = \text{vect}\{F_1, \dots, F_k\}$  par  $|G| = \inf\{\max\{|u_i|\} \mid G = \sum_i u_i F_i\}$ .

En dimension 2, pour  $k = 2$ , si  $F_1$  et  $F_2$  ne sont pas proportionnels en  $p$  alors on obtient une structure finslérienne. Sinon, la structure est sous-finslérienne sur une distribution de rang non constant. Nous décrivons les objets géométriques décrits plus haut pour l'ensemble des couples génériques  $(F_1, F_2)$ .

En dimension 3, nous avons étudié la géométrie locale pour les distributions de contact.

---

# Abstract

In this thesis, I study the local geometry of Finslerian and sub-Finslerian structures associated to the maximum norm in dimension 2 and 3 : short generalized geodesics, cut locus, generalized conjugate locus, switching locus, small spheres.

To define such a structure in the neighborhood of a point  $p$  of  $\mathbb{R}^n$ , we fix a family of vector fields  $(F_1, \dots, F_k)$  and consider the norm defined on the distribution  $\Delta = \text{vect}\{F_1, \dots, F_k\}$  by  $|G| = \inf\{\max\{|u_i|\} \mid G = \sum_i u_i F_i\}$ .

In dimension 2, for  $k = 2$ , if  $F_1$  and  $F_2$  are not proportionnal at  $p$  then we obtain a Finslerian structure. If not, the structure is sub-Finslerian on a distribution with non constant rank. We describe the geometric objects for the set of all generic couples  $(F_1, F_2)$ .

In dimension 3, we studied the local geometry for contact distributions.