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To cite this version:

Jean-Mathieu Magot. Homotopy invariants of vector fields in 3-manifolds. K-Theory and Homology [math.KT]. Université Grenoble Alpes, 2016. English. NNT : 2016GREAM070 . tel-01681440v1

HAL Id: tel-01681440

https://theses.hal.science/tel-01681440v1

Submitted on 11 Jan 2018 (v1), last revised 12 Jan 2018 (v2)

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Communauté UNIVERSITÉ Grenoble Alpes

THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE GRENOBLE

Spécialité : **mathématiques**

Arrêté ministériel : 7 août 2006

Présentée par

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Thèse dirigée par **Christine LESCOP** et
codirigée par **Jean-Baptiste MEILHAN**

préparée au sein de l'**Institut Fourier**
dans **l'École Doctorale MSTII**

Invariants homotopiques de champs de vecteurs en dimension 3

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Remerciements

En premier lieu, toute ma gratitude va à Christine Lescop et à Jean-Baptiste Meilhan qui ont très patiemment dirigé mes travaux un lustre durant. Je les remercie en particulier pour leur disponibilité en toutes circonstances, la minutie de leurs relectures, leurs encouragements. Si j'ai pu mener à bien ces recherches, c'est d'abord à eux que je le dois.

Mes remerciements vont également à Gwénaël Massuyeau et à Michael Eisermann qui ont accepté d'être rapporteurs de cette thèse et dont les remarques et suggestions ont permis d'en enrichir le manuscrit. Je remercie aussi vivement Thomas Fiedler, Louis Funar et Pierre Vogel de remplir le rôle d'examinateur, pour ainsi dire au pied levé.

Mes années de doctorat n'auraient pas été ce qu'elles ont été sans les longues et passionnées discussions avec Julien Korinman. Ma reconnaissance va aussi à Delphine Moussard pour m'avoir introduit d'abord dans le petit monde des doctorants de l'UJF, puis de la topologie. Je n'oublie pas ceux qui m'ont aidé à faire vivre le *séminaire compréhensible*, Clément Debin, Thibaut Delcroix, Guillaume Idelon-Riton, Simon Schmidt, Binbin Xu ... et les autres, moins assidus. Pour avoir partagé des doutes, des angoisses et une même crise de légitimité, merci à Jordane Granier et à Kevin Corbineau.

Une pensée reconnaissante encore aux membres ou proches de VasKho, Benjamin Audoux, Paolo Bellingeri, Emmanuel Wagner qui m'ont toujours chaleureusement accueilli et à qui je dois également une grande partie de ma culture mathématique.

Plus largement, j'aimerais remercier les autres membres de la communauté mathématique qu'il m'a été donné de croiser lors d'invitations, de conférences, d'écoles, Peter Ozsváth, Michael Polyak, et bien d'autres qu'il serait bien sûr trop fastidieux d'énumérer ici.

Introduction générale

Conventions

Tout au long de ce manuscrit, les 3-variétés compactes peuvent avoir du bord, sauf mention explicite du contraire. Toutes les variétés sont implicitement munies de structures riemanniennes. Les énoncés et les preuves sont indépendants des structures choisies.

Si M est une variété orientée et si A est une sous-variété de M, on note TM, respectivement TA, le fibré tangent de M, respectivement de A, et on note NA le fibré orthogonal de A dans M. Ce dernier est canoniquement isomorphe au fibré normal de A dans M. Les fibres de NA sont orientées de sorte que $NA \oplus TA = TM$ pour chacune d'entre elles. Les bords des variétés sont orientés selon la convention de la normale extérieure en premier.

Si A et B sont des sous-variétés d'une variété orientée M, leur intersection est orientée de sorte que sur chaque fibre $N(A \cap B) = NA \oplus NB$. De plus, si A et B sont de dimensions complémentaires, *ie* si $\dim(A) + \dim(B) = \dim(M)$, on pose $\varepsilon_{A\cap B}(x) = 1$ si $x \in A \cap B$ est tel que $T_xA \oplus T_xB = T_xM$ et $\varepsilon_{A\cap B}(x) = -1$ sinon. Si A et B sont des sous-variétés compactes transverses d'une variété orientée M et si A et B sont complémentaires, l'*intersection algébrique de A et de B* vaut

$$\langle A, B\rangle_M = \sum_{x\in A\cap B} \varepsilon_{A\cap B}(x).$$

Soient L_1 et L_2 deux chaînes rationnelles d'une n-variété orientée M. Supposons que L_1 et L_2 bordent des chaînes rationnelles Σ_1 et Σ_2, respectivement. Si L_1 est transverse à Σ_2, si L_2 est transverse à Σ_1 et si $\dim(L_1) + \dim(L_2) = n - 1$, alors le *nombre d'enlacement de L_1 et L_2 dans M* vaut

$$lk_M(L_1, L_2) = \langle \Sigma_1, L_2\rangle_M = (-1)^{n-\dim(L_2)}\langle L_1, \Sigma_2\rangle_M.$$

Présentation du chapitre I

Contexte

Dans [Gom98], R. Gompf définit un invariant homotopique des champs de plans orientés des 3-variétés fermées orientées, noté θ_G. Cet invariant est défini pour les champs de plans orientés ξ de toute 3-variété fermée orientée M dont la première classe de Chern $c_1(\xi)$ est un élément de torsion de $H^2(M;\mathbb{Z})$. Cet invariant apparaît notamment dans la construction d'une graduation absolue pour les groupes d'homologie de Heegaard-Floer. Puisque les normales unitaires positives des champs de plans orientés d'une 3-variété riemannienne M sont exactement les sections du fibré unitaire UM, il est aussi possible de voir l'invariant de Gompf comme un invariant homotopique des champs de vecteurs unitaires, aussi appelés *combings*. Dans ce cadre, l'invariant θ_G est défini pour les *combings de torsion* de toute 3-variété fermée orientée M, c'est-à-dire des combings X tels que la classe d'Euler $e_2(X^\perp)$ du fibré $X^\perp$ est un élément de torsion de $H^2(M;\mathbb{Z})$.

Dans [Les15b], C. Lescop propose une définition de θ_G à l'aide d'une construction de Pontrjagin sur les combings. Dans le chapitre I, on utilise une approche similaire afin de définir les nombres de Pontrjagin à partir des pseudo-parallélisations qui généralisent la notion de parallélisation. Cela permet de définir une extension relative de l'invariant de Gompf pour toutes les 3-variétés compactes orientées. Enfin, la dernière partie de ce chapitre est dédiée à l'étude des variations de θ_G lors de chirurgies lagrangiennes. Il en ressort que l'invariant de Gompf étendu peut être vu comme un invariant de type fini de degré 2. Afin de mener à bien ces calculs, les pseudo-parallélisations se montrent particulièrement indiquées. Elles sont, en quelque sorte, compatibles avec les chirurgies lagrangiennes alors que les parallélisations ne le sont pas.

Premières définitions et énoncés du chapitre I

Un *combing* (X,σ) d'une 3-variété compacte orientée M est un couple formé d'une section X du fibré tangent unitaire UM et d'une section jamais nulle σ de la restriction $X^\perp_{|\partial M}$ du fibré normal $X^\perp$ à ∂M. Pour plus de concision, la section σ pourra être oubliée de la notation. Pour tout combing (X,σ), le triplet $\rho(X) = (X_{|\partial M}, \sigma, X_{|\partial M} \wedge \sigma)$ définit une trivialisation de $TM_{|\partial M}$. Un combing d'une 3-variété compacte orientée M peut donc aussi être vu comme un couple (X,ρ) où X est une section de UM et où ρ est une trivialisation de $TM_{|\partial M}$ dont X est le premier vecteur.

Deux combings (X, σ_X) et (Y, σ_Y) d'une 3-variété compacte orientée M sont dits *transverses* lorsque le graphe $X(M)$ est transverse à $Y(M)$ et à $-Y(M)$ dans UM. Les combings (X, σ_X) et (Y, σ_Y) sont dits ∂-compatibles lorsque $X_{|\partial M} = Y_{|\partial M}$, $\sigma_X = \sigma_Y$, $X(\mathring{M})$ est transverse à $Y(\mathring{M})$ et à $-Y(\mathring{M})$ dans UM, et

$$\overline{X(\mathring{M}) \cap Y(\mathring{M})} \cap \partial M = \emptyset.$$

Si (X, σ_X) et (Y, σ_Y) sont ∂-compatibles, on définit deux entrelacs $L_{X=Y}$ et $L_{X=-Y}$ comme suit. Premièrement, si P_M désigne la projection de UM sur M, on pose

$$L_{X=-Y} = P_M(X(M) \cap (-Y)(M)).$$

Deuxièmement, il existe un entrelacs $L_{X=Y}$ dans $\mathring{M}$ tel que

$$P_M(X(M) \cap Y(M)) = \partial M \sqcup L_{X=Y}.$$

Si (X, σ) est un combing d'une 3-variété compacte orientée M, sa *classe d'Euler relative* $e_2^M(X^\perp, \sigma)$ dans $H^2(M, \partial M; \mathbb{Z})$ est une obstruction à étendre σ en une section jamais nulle de $X^\perp$. Cette obstruction est telle que son dual de Poincaré $P(e_2^M(X^\perp, \sigma))$ est représentée par le lieu d'annulation d'une section générique de $X^\perp$ prolongeant σ. Ce lieu d'annulation est orienté car il est coorienté par l'orientation de $X^\perp$. Lorsque M est fermée, la *classe d'Euler* $e_2(X^\perp)$ de X est simplement cette obstruction à trouver une section jamais nulle de $X^\perp$.

Un combing (X, σ) d'une 3-variété compacte orientée M est un *combing de torsion* si $e_2^M(X^\perp, \sigma)$ est un élément de torsion de $H^2(M, \partial M; \mathbb{Z})$, c'est-à-dire si $\left[e_2^M(X^\perp, \sigma)\right] = 0$ dans $H^2(M, \partial M; \mathbb{Q})$.

Soient M_1 et M_2 des 3-variétés compactes orientées. Les variétés M_1 et M_2 ont leurs *bords identifiés* si un collier de ∂M_1 et un collier de ∂M_2 sont identifiés. Dans ce cas, $TM_{1|\partial M_1} = \mathbb{R}n_1 \oplus T\partial M_1$ est identifié à $TM_{2|\partial M_2} = \mathbb{R}n_2 \oplus T\partial M_2$ de sorte que la normale extérieure n_1 de M_1 soit identifiée avec la normale extérieure n_2 de M_2.

Si τ_1 et τ_2 sont des parallélisations de deux 3-variétés compactes orientées M_1 et M_2 de bords identifiés, telles que τ_1 et τ_2 coïncident sur $\partial M_1 \simeq \partial M_2$, alors la *première classe relative de Pontrjagin de* τ_1 *et* τ_2 est un élément $p_1(\tau_1, \tau_2)$ de $\mathbb{Z}$ correspondant à l'obstruction de Pontrjagin à étendre une trivialisation particulière de $TW \otimes \mathbb{C}$, $\tau(\tau_1, \tau_2)$, définie sur le bord d'un cobordisme W de signature zéro entre M_1 à M_2 (*cf.* partie I.2.1 ou [Les15b, Subsection 4.1]). Dans le cas d'une parallélisation τ d'une 3-variété fermée orientée M, on définit une version absolue. Le *nombre de Pontrjagin* $p_1(\tau)$ *de* τ est le nombre de Pontrjagin relatif $p_1(\tau_\emptyset, \tau)$

où $\tau_\emptyset$ est la parallélisation de l'ensemble vide. Pour deux parallélisations τ_1 et τ_2 d'une 3-variété fermée orientée, on obtient donc

$$p_1(\tau_1, \tau_2) = p_1(\tau_2) - p_1(\tau_1).$$

Dans [Les15b], C. Lescop montre le théorème suivant qui donne une définition des *nombres de Pontrjagin* des combings de torsion dans les 3-variétés fermées orientées.

Théorème ([Les15b, Theorem 1.2 & Subsection 4.3])**.** *Soit M une 3-variété fermée orientée. Il existe une unique application*

$$p_1 \ : \ \{\textit{classes d'homotopie des combings de torsion de } M\} \longrightarrow \mathbb{Q}$$

telle que :

(i) pour tout combing X tel que X s'étend en une parallélisation τ de M :

$$p_1([X]) = p_1(\tau),$$

(ii) si X et Y sont deux combings transverses de M, alors

$$p_1([Y]) - p_1([X]) = 4 \cdot lk(L_{X=Y}, L_{X=-Y}).$$

De plus, l'application p_1 coïncide avec l'invariant de Gompf : pour tout combing de torsion X,

$$p_1([X]) = \theta_G(X^\perp).$$

Dans le chapitre I, nous étudions les variations des nombres de Pontrjagin des combings de torsion des 3-variétés compactes orientées lors de chirurgies lagrangiennes. Ces chirurgies sont définies comme suit.

Un *corps en anses d'homologie de genre $g \in \mathbb{N}$*, ou $\mathbb{Q}$HH, est une 3-variété compacte orientée à bord dont l'homologie à coefficients dans $\mathbb{Q}$ est la même que celle d'un corps en anses standard de genre g. Le *lagrangien* d'une telle variété A est défini comme

$$\mathfrak{L}_A := \ker\left(i^A_* : H_1(\partial A; \mathbb{Q}) \longrightarrow H_1(A; \mathbb{Q})\right)$$

où i^A est l'inclusion de ∂A dans A. Une donnée de chirurgie lagrangienne dans une 3-variété compacte orientée M est un triplet (A, B, h), aussi noté simplement $({}^B/_A)$, où $A \subset M$, où B et A sont des $\mathbb{Q}$HH et où $h : \partial A \to \partial B$ est une $LP_\mathbb{Q}$-*identification*, *ie* un homéomorphisme tel que $h_*(\mathfrak{L}_A) = \mathfrak{L}_B$. Effectuer une $LP_\mathbb{Q}$-*chirurgie* de donnée (A, B, h) dans M consiste à construire la variété :

$$M\,({}^B/_A) = \left(M \setminus \mathring{A}\right) \bigcup_h B.$$

Si (M, X) est une 3-variété compacte orientée munie d'un combing, si (A, B, h) est une donnée de chirurgie lagrangienne dans M, et si X_B est un combing de B qui coïncide avec X sur $\partial A \simeq \partial B$, alors (A, B, h, X_B), ou plus succinctement $({}^B/_A, X_B)$, est une *donnée de chirurgie lagrangienne dans* (M, X). Effectuer la *chirurgie lagrangienne* associée à la donnée (A, B, h, X_B) dans (M, X) consiste à construire la variété $M\,({}^B/_A)$ munie du combing :

$$X({}^B/_A) = \begin{cases} X & \text{sur } M \setminus \mathring{A}, \\ X_B & \text{sur } B. \end{cases}$$

Le premier des résultats principaux du chapitre I est une formule de variation pour les nombres de Pontrjagin. Dans le cas de variétés fermées, cette formule se présente comme suit.

Théorème (Theorem I.1)**.** *Soit (M, X) une 3-variété fermée orientée munie d'un combing. Soient $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in\{1,2\}}$ deux chirurgies lagrangiennes disjointes dans (M, X) (ie A_1 et A_2 sont disjoints). Pour tout $I \subset \{1, 2\}$, soit (M_I, X^I) la variété coiffée obtenue en effectuant les chirurgies associées à $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in I}$. (NB : Pour tous I, $J \subset \{1, 2\}$, les combings X^I et X^J coïncident sur $(M\setminus\cup_{i\in I\cup J}A_i)\cup_{i\in I\cap J} B_i$.) Si $\{X^I\}_{I\subset\{1,2\}}$ est une famille de combings de torsion des $\{M_I\}_{I\subset\{1,2\}}$, alors*

$$\sum_{I\subset\{1,2\}} (-1)^{|I|} p_1([X^I]) = -2 \cdot lk_M \left(L_{\{X^I\}}({}^{B_1}/_{A_1}), L_{\{X^I\}}({}^{B_2}/_{A_2})\right),$$

où le membre de droite est défini comme suit. Pour tout $i \in \{1, 2\}$, soit

$$H_1(A_i; \mathbb{Q}) \xleftarrow{i_*^{A_i}} \frac{H_1(\partial A_i; \mathbb{Q})}{\mathfrak{L}_{A_i}} = \frac{H_1(\partial B_i; \mathbb{Q})}{\mathfrak{L}_{B_i}} \xrightarrow{i_*^{B_i}} H_1(B_i; \mathbb{Q})$$

la suite d'isomorphismes induits par les inclusions. Il existe une unique classe d'homologie $L_{\{X^I\}}({}^{B_i}/_{A_i})$ dans $H_1(A_i; \mathbb{Q})$ telle que pour toute section σ_i de $X^{\perp}_{|\partial A_i}$ ne s'annulant jamais :

$$L_{\{X^I\}}({}^{B_i}/_{A_i}) = i_*^{A_i} \circ (i_*^{B_i})^{-1} \left(P(e_2^{B_i}(X^{\perp}_{B_i}, \sigma_i))\right) - P(e_2^{A_i}(X^{\perp}_{|A_i}, \sigma_i)),$$

où P désigne des isomorphismes de dualités de Poincaré de $H^2(A_i, \partial A_i; \mathbb{Q})$ dans $H_1(A_i; \mathbb{Q})$ ou de $H^2(B_i, \partial B_i; \mathbb{Q})$ dans $H_1(B_i; \mathbb{Q})$. Les classes $L_{\{X^I\}}({}^{B_1}/_{A_1})$ et $L_{\{X^I\}}({}^{B_2}/_{A_2})$ sont nulles dans $H_1(M; \mathbb{Q})$ et l'application

$$lk_M : \ker\big(H_1(A_1; \mathbb{Q}) \to H_1(M; \mathbb{Q})\big) \times \ker\big(H_1(A_2; \mathbb{Q}) \to H_1(M; \mathbb{Q})\big) \longrightarrow \mathbb{Q}$$

est bien définie.

Exemple. Considérons $\mathbb{S}^3$ munie d'une parallélisation τ prolongeant la parallélisation standard de la boule unité. Dans cette boule, considérons un entrelacs de Hopf positif ainsi qu'un voisinage tubulaire $A_1 \sqcup A_2$ de cet entrelacs. Soit X le combing $\tau(e_1)$, où $e_1 = (1,0,0) \in \mathbb{S}^2$, et soient $B_1 = A_1$ et $B_2 = A_2$. Identifions A_1 et A_2 avec $\mathbb{D}^2 \times \mathbb{S}^1$ et considérons une application lisse $g : \mathbb{D}^2 \to \mathbb{S}^2$ telle que $g(\partial\mathbb{D}^2) = e_1$, et telle que $-e_1$ est une valeur régulière de g de degré 1 possèdant un unique antécédent ω. Enfin, pour tout $i \in \{1,2\}$, soit X_{B_i} le combing :

$$X_{B_i} : \begin{cases} \mathbb{D}^2 \times \mathbb{S}^1 \longrightarrow UM \\ \quad (z,u) \longmapsto \tau((z,u), g(u)). \end{cases}$$

Dans cette configuration, $L_{X(\{B_i/A_i\}_{i\in\{1,2\}})=-X} = L_{X_{B_1}=-X_{|A_1}} \cup L_{X_{B_2}=-X_{|A_2}}$, et, pour $i \in \{1,2\}$, $L_{X_{B_i}=-X_{|A_i}} = \mathbb{S}^1 \times \{\omega\}$, en utilisant l'identification de A_i avec $\mathbb{D}^2 \times \mathbb{S}^1$. On verra dans la proposition I.1.10 que $L_{\{X^I\}}({}^{B_i}/_{A_i}) = 2[L_{X_{B_i}=-X_{|A_i}}]$, pour tout $i \in \{1,2\}$. D'où,

$$\sum_{I\subset\{1,2\}} (-1)^{|I|} p_1([X^I]) = -8.$$

En général, si $({}^B/_A)$ est une chirurgie lagrangienne dans une 3-variété compacte orientée, une trivialisation de $TM_{|(M\setminus\mathring{A})}$ peut ne pas s'étendre en une parallélisation de $M({}^B/_A)$. Il s'ensuit qu'une chirurgie lagrangienne ne peut être considérée comme un mouvement local sur les 3-variétés compactes orientées parallélisées. Cela rend le calcul des variations des nombres de Pontrjagin des combings de torsion sous chirurgies lagrangiennes difficile puisque ces nombres de Pontrjagin sont définis relativement aux nombres de Pontrjagin des parallélisations.

Cependant, si M est une 3-variété compacte orientée, et si ρ est une trivialisation de $TM_{|\partial M}$, alors l'obstruction à trouver une parallélisation de M qui prolonge ρ est un élément de $H^2(M, \partial M; {}^{\mathbb{Z}}/_{2\mathbb{Z}})$ – donc son dual de Poincaré est un élément $[\gamma]$ de $H_1(M; {}^{\mathbb{Z}}/_{2\mathbb{Z}})$ – et il est possible de contourner une telle obstruction grâce à la notion de *pseudo-parallélisation*, développée par C. Lescop. L'exposé formel est repoussé à la sous-section 1.3 (voir aussi [Les10]), pour l'instant, mentionnons simplement qu'une pseudo-parallélisation $\bar{\tau}$ d'une 3-variété compacte orientée M est un triplet $(N(\gamma); \tau_e, \tau_d)$ où $N(\gamma)$ est un voisinage tubulaire parallélisé d'un entrelacs γ dans $\mathring{M}$, τ_e est une parallélisation de $M \setminus N(\gamma)$ et $\tau_d : N(\gamma) \times \mathbb{R}^3 \to TN(\gamma)$ est une parallélisation de $N(\gamma)$ telle qu'il existe une section E_1^d de UM :

$$E_1^d : \begin{cases} m \in M \setminus \mathring{N}(\gamma) \longmapsto \tau_e(m, e_1) \\ \qquad m \in N(\gamma) \longmapsto \tau_d(m, e_1) \end{cases}$$

où $e_1 = (1,0,0)$. Enfin, soulignons que $\bar{\tau}$ détermine une seconde section E_1^g de UM qui coïncide avec E_1^d sur $M \setminus N(\gamma)$. Les sections E_1^d et E_1^g sont appelées *sections*

siamoises de $\bar{\tau}$ et l'entrelacs γ est l'*entrelacs de la pseudo-parallélisation* $\bar{\tau}$.

À une pseudo-parallélisation, C. Lescop a aussi montré qu'il est possible d'associer une *trivialisation complexe* à homotopie près, *cf.* Définition I.2.1. Il en résulte une extension naturelle de la notion de premier nombre de Pontrjagin relatif des parallélisations aux pseudo-parallélisations. De plus, comme dans le cas des parallélisations, une pseudo-parallélisation $\bar{\tau}$ d'une 3-variété compacte orientée M admet des *pseudo-sections* $\bar{\tau}(M \times \{v\})$, pour tout $v \in \mathbb{S}^2$. Dans le cas particulier où $v = e_1$ la pseudo-section $\bar{\tau}(M \times \{e_1\})$ de $\bar{\tau}$ peut s'écrire :

$$\bar{\tau}(M \times \{e_1\}) = \frac{E_1^d(M) + E_1^g(M)}{2}.$$

Un combing (X, σ) de M est *compatible avec* $\bar{\tau}$ si (X, σ) est ∂-compatible avec $(E_1^d, E^e_{2|\partial M})$ et $(E_1^g, E^e_{2|\partial M})$, où E_2^e est le second vecteur de τ_e, et si de plus

$$L_{E_1^d = X} \cap L_{E_1^g = -X} = \emptyset \quad \text{et} \quad L_{E_1^g = X} \cap L_{E_1^d = -X} = \emptyset.$$

Lorsque (X, σ) et $\bar{\tau}$ sont compatibles, alors $\rho(X) = \bar{\tau}_{|\partial M}$ et on obtient deux moyennes d'entrelacs orientés disjoints dans $\mathring{M}$:

$$L_{\bar{\tau}=X} = \frac{L_{E_1^d=X} + L_{E_1^g=X}}{2} \quad \text{et} \quad L_{\bar{\tau}=-X} = \frac{L_{E_1^d=-X} + L_{E_1^g=-X}}{2}.$$

Les pseudo-parallélisations permettent de revisiter la définition des nombres de Pontrjagin et de l'étendre aux combings de torsion des 3-variétés compactes orientées comme suit. Pour toute variété W, notons $P_{\mathbb{S}^2}$ la projection standard de $W \times \mathbb{S}^2$ sur $\mathbb{S}^2$.

Lemme (Lemma I.3)**.** *Soit (X, σ) un combing de torsion d'une 3-variété compacte orientée M, soit $\bar{\tau}$ une pseudo-parallélisation de M. Notons E_1^d et E_1^g les sections siamoises de $\bar{\tau}$. Si $\bar{\tau}$ et (X, σ) sont compatibles, alors l'expression*

$$4 \cdot lk_M(L_{\bar{\tau}=X}, L_{\bar{\tau}=-X}) - lk_{\mathbb{S}^2}(e_1 - (-e_1), P_{\mathbb{S}^2} \circ \tau_d^{-1} \circ X(L_{E_1^d = -E_1^g}))$$

ne dépend que de la classe d'homotopie de (X, σ). On la note $p_1(\bar{\tau}, [X])$, et $p_1([X], \bar{\tau})$ désigne son opposé.

Théorème (Theorem I.4)**.** *Soient (X_1, σ_1) et (X_2, σ_2) des combings de torsion de deux 3-variétés compactes orientées M_1 et M_2 dont les bords sont identifiés l'un à l'autre. Supposons que (X_1, σ_1) et (X_2, σ_2) coïncident sur le bord. Pour $i \in \{1, 2\}$, soit $\bar{\tau}_i$ une pseudo-parallélisation de M_i telle que $\bar{\tau}_i$ et (X_i, σ_i) sont compatibles. L'expression*

$$p_1([X_1], [X_2]) = p_1([X_1], \bar{\tau}_1) + p_1(\bar{\tau}_1, \bar{\tau}_2) + p_1(\bar{\tau}_2, [X_2])$$

ne dépend que des classes d'homotopie de (X_1, σ_1) et de (X_2, σ_2) et définit le premier nombre de Pontrjagin relatif de (X_1, σ_1) et de (X_2, σ_2). *De plus, si M_1 et M_2 sont fermées, alors*

$$p_1([X_1], [X_2]) = p_1([X_2]) - p_1([X_1]).$$

Remarque. Le théorème I.1, dont nous reprenons les notations et hypothèses dans cette remarque, montre qu'il n'est pas possible de définir naïvement le nombre $p_1([X_{|A_1}], [X_{\{1\}_{|B_1}}])$ comme $p_1([X]) - p_1([X_{\{1\}}])$ où X prolonge $X_{|A_1}$ sur la variété sans bord M, et $X_{\{1\}}$ prolonge $X_{\{1\}_{|B_1}}$ de la même manière à $M({}^{B_1}/_{A_1})$. En effet, ce théorème I.1 et l'exemple qui le suit montrent que l'expression $(p_1([X]) - p_1([X_{\{1\}}]))$ dépend de la variété coiffée (M, X) dans laquelle $(A_1, X_{|A_1})$ est plongé. Elle dépend même de la coiffure X qui étend à M la coiffure $X_{|A_1}$ de A_1 pour la variété M fixée de cet exemple, puisque

$$\big(p_1([X]) - p_1([X_{\{1\}}])\big) - \big(p_1([X_{\{2\}}]) - p_1([X_{\{1,2\}}])\big) = -8$$

pour cet exemple.

Le théorème précédent mène au corollaire suivant dans le cas fermé. Cela permet, en particulier, d'établir un lien entre les deux généralisations indépendantes des nombres de Pontrjagin pour les pseudo-parallélisations et pour les combings de torsion des 3-variétés fermées orientées.

Corollaire (Corollary I.5)**.** *Soit X un combing de torsion d'une 3-variété fermée orientée M. Soit $\bar{\tau} = (N(\gamma); \tau_e, \tau_d)$ une pseudo-parallélisation de M. Notons E_1^d et E_1^g les sections siamoises de $\bar{\tau}$. Si X est transverse à E_1^d et à E_1^g et si les intersections $L_{E_1^d = X} \cap L_{E_1^g = -X}$ et $L_{E_1^g = X} \cap L_{E_1^d = -X}$ sont vides, alors*

$$\begin{aligned} p_1([X]) = p_1(\bar{\tau}) &+ 4 \cdot lk_M(L_{\bar{\tau}=X}, L_{\bar{\tau}=-X}) \\ &- lk_{\mathbb{S}^2}\left(e_1 - (-e_1)\ ,\ P_{\mathbb{S}^2} \circ \tau_d^{-1} \circ X(L_{{E_1}^d = -{E_1}^g})\right). \end{aligned}$$

Un second corollaire peut être extrait du théorème I.4 lorsque l'on dispose de parallélisations. Le cas fermé avec des parallélisations n'étant rien d'autre que le théorème de C. Lescop reproduit plus haut.

Corollaire (Corollary I.6). *Soient (X_1, σ_1) et (X_2, σ_2) des combings de torsion de deux 3-variétés compactes orientées M_1 et M_2 dont les bords sont identifiés, tels que (X_1, σ_1) et (X_2, σ_2) coïncident sur le bord. Si, pour tout $i \in \{1, 2\}$, $\tau_i = (E_1^i, E_2^i, E_3^i)$ est une parallélisation de M_i telle que (X_i, σ_i) et $(E_1^i, E_{2|\partial M_i}^i)$ sont ∂-compatibles, alors*

$$p_1([X_1], [X_2]) = p_1(\tau_1, \tau_2) + 4 \cdot lk_{M_2}(L_{E_1^2 = X_2}\ ,\ L_{E_1^2 = -X_2}) - 4 \cdot lk_{M_1}(L_{E_1^1 = X_1}\ ,\ L_{E_1^1 = -X_1}).$$

Enfin, pour des combings de torsion définis sur une même 3-variété compacte orientée, on obtient la formule de variation suivante, comme dans le cas clos.

Proposition (Proposition I.7). *Si (X, σ) et (Y, σ) sont des combings de torsion ∂-compatibles d'une 3-variété compacte orientée M, alors*

$$p_1([X], [Y]) = 4 \cdot lk_M(L_{X=Y}, L_{X=-Y}).$$

Soit M une 3-variété compacte orientée. Pour toute section σ de $TM_{|\partial M}$, soit $\mathrm{Spin}^c(M, \sigma)$ l'ensemble des *Spin^c-structures sur M relatives à σ*, *ie* l'ensemble des classes d'homotopie (relative au bord de M) sur $M \setminus \mathcal{B}$ des combings (X, σ) de M, où $\mathcal{B}$ est une boule dans $\mathring{M}$ (voir [DM05], pour une présentation détaillée des Spin^c-structures). Notre invariant relatif p_1 permet de distinguer les combings d'une Spin^c-structure fixée à homotopie (relative au bord) près, ce qui généralise une propriété de l'invariant de Gompf déjà connue dans le cas clos. Je remercie Gwénaël Massuyeau pour m'avoir suggéré cet énoncé.

Théorème (Theorem I.8). *Soit (X, σ) et (Y, σ) des combings de torsion ∂-compatibles d'une 3-variété compacte orientée M représentant la même Spin^c-structure. les combings X et Y sont homotopes relativement au bord si et seulement si $p_1([X], [Y]) = 0$.*

Le résultat clef pour la preuve du théorème I.4 est la généralisation suivante de la reformulation de la variation des nombres de Pontrjagin des parallélisations comme une intersection algébrique triple de chaînes.

Proposition (Proposition I.9). *Soient τ et $\bar{\tau}$ deux pseudo-parallélisations d'une 3-variété compacte orientée M qui coïncident sur le bord et dont les entrelacs sont disjoints. Pour tout $v \in \mathbb{S}^2$, il existe une 4-chaîne $C_4(\tau, \bar{\tau}; v)$ de $[0, 1] \times UM$ transverse au bord de $[0, 1] \times UM$ telle que*

$$\partial C_4(\tau, \bar{\tau}; v) = \{1\} \times \bar{\tau}(M \times \{v\}) - \{0\} \times \tau(M \times \{v\}) - [0, 1] \times \tau(\partial M \times \{v\})$$

et pour tous x, y et z dans $\mathbb{S}^2$ à distances de e_1 deux à deux distinctes :

$$p_1(\tau, \bar{\tau}) = 4 \cdot \langle C_4(\tau, \bar{\tau}; x), C_4(\tau, \bar{\tau}; y), C_4(\tau, \bar{\tau}; z) \rangle_{[0,1] \times UM},$$

pour tout triplet ($C_4(\tau, \bar{\tau}; x)$, $C_4(\tau, \bar{\tau}; y)$, $C_4(\tau, \bar{\tau}; z)$) de chaînes deux à deux transverses vérifiant les hypothèses précédentes.

La formule générale pour les variations des nombres de Pontrjagin des combings de torsion des 3-variétés compactes orientées se présente comme suit.

Théorème (Theorem I.10). *Soient (M, X) une 3-variété compacte orientée munie d'un combing et $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in\{1,2\}}$ deux chirurgies lagrangiennes disjointes dans (M, X). Pour tout $I \subset \{1, 2\}$, soit (M_I, X^I) la variété coiffée obtenue en effectuant les chirurgies associées à $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in I}$. Si $\{X^I\}_{I\subset\{1,2\}}$ est une famille de combings de torsion des $\{M_I\}_{I\subset\{1,2\}}$, alors*

$$p_1\left([X^{\{2\}}], [X^{\{1,2\}}]\right) - p_1\left([X], [X^{\{1\}}]\right) = -2 \cdot lk_M\left(L_{\{X^I\}}({}^{B_1}/_{A_1}), L_{\{X^I\}}({}^{B_2}/_{A_2})\right),$$

où le membre de droite est défini comme suit. Pour tout $i \in \{1, 2\}$, soit

$$H_1(A_i;\mathbb{Q}) \xleftarrow{i_*^{A_i}} \frac{H_1(\partial A_i;\mathbb{Q})}{\mathfrak{L}_{A_i}} = \frac{H_1(\partial B_i;\mathbb{Q})}{\mathfrak{L}_{B_i}} \xrightarrow{i_*^{B_i}} H_1(B_i;\mathbb{Q})$$

la suite d'isomorphismes induits par les inclusions. Il existe une unique classe d'homologie $L_{\{X^I\}}({}^{B_i}/_{A_i})$ dans $H_1(A_i;\mathbb{Q})$ telle que pour toute section σ_i de $X^{\perp}_{|\partial A_i}$ ne s'annulant jamais :

$$L_{\{X^I\}}({}^{B_i}/_{A_i}) = i_*^{A_i} \circ (i_*^{B_i})^{-1}\left(P(e_2^{B_i}(X^{\perp}_{B_i}, \sigma_i))\right) - P(e_2^{A_i}(X^{\perp}_{|A_i}, \sigma_i)),$$

où P désigne des isomorphismes de dualités de Poincaré de $H^2(A_i, \partial A_i;\mathbb{Q})$ dans $H_1(A_i;\mathbb{Q})$ ou de $H^2(B_i, \partial B_i;\mathbb{Q})$ dans $H_1(B_i;\mathbb{Q})$. Les classes $L_{\{X^I\}}({}^{B_1}/_{A_1})$ et $L_{\{X^I\}}({}^{B_2}/_{A_2})$ sont nulles dans $H_1(M;\mathbb{Q})$ et l'application

$$lk_M : \ker\big(H_1(A_1;\mathbb{Q}) \to H_1(M;\mathbb{Q})\big) \times \ker\big(H_1(A_2;\mathbb{Q}) \to H_1(M;\mathbb{Q})\big) \longrightarrow \mathbb{Q}$$

est bien définie.

Une conséquence directe de cette formule de variation est que l'invariant de Gompf étendu pour les combings de torsion des 3-variétés compactes orientées est un invariant de type fini de degré 2 relativement aux chirurgies lagrangiennes.

Corollaire (Corollary I.11). *Soit k un entier supérieur ou égal à 3. Soient (M, X) une 3-variété compacte orientée munie d'un combing et $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in\{1,\dots,k\}}$ une famille de chirurgies lagrangiennes disjointes dans (M, X). Pour tout $I \subset \{1, \dots, k\}$, soit (M_I, X^I) la variété coiffée obtenue en effectuant les chirurgies lagrangiennes associées à $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in I}$. Si $\{X^I\}_{I\subset\{1,\dots,k\}}$ est une famille de combings de torsion des $\{M_I\}_{I\subset\{1,\dots,k\}}$, alors*

$$\sum_{I\subset\{2,\dots,k\}} (-1)^{|I|}\, p_1\left([X^I], [X^{I\cup\{1\}}]\right) = 0.$$

Si $\partial M = \emptyset$, on obtient

$$\sum_{I\subset\{1,\dots,k\}} (-1)^{|I|}\, p_1\left([X^I]\right) = 0.$$

Présentation du chapitre II

Contexte

Dans [Kon94], M. Kontsevich a esquissé une définition d'un invariant des sphères d'homologie rationnelle * Z à valeurs dans un espace de diagrammes de Jacobi en utilisant des formes différentielles définies sur des espaces de configurations. G. Kuperberg et D. Thurston ont prouvé que Z est un invariant de type fini universel des sphères d'homologie entière relativement aux chirurgies de Torelli (voir [KT99]). Dans [Les04a], C. Lescop a détaillé la construction de l'invariant Z.

L'invariant Z est gradué, et les auteurs sus-mentionnés ont décrit le terme de plus petit degré de Z, appelé invariant Θ. Plus précisément, l'invariant Θ est le plus simple invariant des sphères d'homologie rationnelle parallélisées défini à partir des espaces de configurations. Dans [KT99], G. Kuperberg et D. Thurston ont montré que, pour toute sphère d'homologie entière M munie d'une parallélisation τ, la valeur $\Theta(M,\tau)$ se décompose en $6 \cdot \lambda(M) + {}^1\!/_4 \cdot p_1(\tau)$ où λ est l'invariant de Casson et où $p_1(\tau)$ est le nombre de Pontrjagin de τ.

Dans [Les04b], C. Lescop a généralisé cette formule aux sphères d'homologie rationnelle, et dans [Les15b], elle a montré que l'invariant Θ était aussi un invariant des classes d'homotopie de combings dans les sphères d'homologie rationnelle, pour lequel cette formule s'étend naturellement : pour toute sphère d'homologie rationnelle M munie d'une classe d'homotopie de combings $[X]$, on a

$$\Theta(M,[X]) = 3 \cdot \lambda_{\mathrm{cw}}(M) + {}^1\!/_4 \cdot p_1([X]) \qquad (\star)$$

où λ_{cw} est la généralisation de Walker de l'invariant de Casson normalisée de sorte que $\lambda_{\mathrm{cw}}(M) = 2\lambda(M)$ pour toute sphère d'homologie entière M, et où p_1 est l'extension des nombres de Pontrjagin pour les classes d'homotopie des combings de torsion des 3-variétés fermées orientées définie au début du chapitre I.

Enfin, dans [Les15a], C. Lescop a donné une formule combinatoire pour l'invariant Θ à l'aide de diagrammes de Heegaard puis elle a prouvé combinatoirement l'invariance de cette formule (voir [Les16]). Dans le prolongement de ce travail, le chapitre II présente une preuve combinatoire de la formule $(\star)$.

$*$. Dans ce manuscrit, nous définissons les sphères d'homologie rationnelle comme les 3-variétés fermées *orientées* ayant la même homologie à coefficients rationnels que $\mathbb{S}^3$.

Premières définitions et énoncés du chapitre II

Dans [Wal92], K. Walker a défini une extension λ_{cw} de l'invariant de Casson pour les sphères d'homologie rationnelle. Cet invariant peut se définir comme suit.

Pour tous nombres réels p, q, r, on désigne par $\lfloor r \rfloor$ la partie entière de r et on pose

$$((r)) = \begin{cases} 0 & \text{si } r \in \mathbb{Z}, \\ r - \lfloor r \rfloor - 1/2 & \text{sinon} \end{cases} \quad \text{et} \quad s(q,p) = \mathrm{signe}(p) \sum_{k=1}^{|p|} \left(\left(\frac{k}{p} \right) \right) \left(\left(\frac{kq}{p} \right) \right).$$

Pour un nœud K dans une sphère d'homologie rationnelle M, notons $N(K)$ un voisinage tubulaire de K et Δ_K le polynome d'Alexander de $M \setminus \mathring{N}(K)$ normalisé de sorte que $\Delta_K(1) = 1$ et $\Delta_K(t^{-1}) = \Delta_K(t)$. Si ℓ désigne un des deux générateurs générateur du *lagrangien entier* de $M \setminus \mathring{N}(K)$, c'est-à-dire de

$$\ker(H_1(\partial(M \setminus \mathring{N}(K)); \mathbb{Z}) \longrightarrow H_1(M \setminus \mathring{N}(K); \mathbb{Z})/\mathcal{T})$$

où $\mathcal{T}$ est le sous groupe de torsion de $H_1(M \setminus \mathring{N}(K); \mathbb{Z})$, si a et b sont des classes d'homologie primitives dans $H_1(\partial(M \setminus \mathring{N}(K)); \mathbb{Z})$ telles que $\langle a, \ell \rangle \neq 0$ et $\langle b, \ell \rangle \neq 0$, et si $\{x, y\}$ est une base de $H_1(\partial(M \setminus \mathring{N}(K)); \mathbb{Z})$ telle que $\langle x, y \rangle = 1$ et $\ell = dy$ pour un certain $d \in \mathbb{Z}$, alors on pose,

$$\tau(a, b; \ell) = s(\langle x, b \rangle, \langle y, b \rangle) - s(\langle x, a \rangle, \langle y, a \rangle) + \frac{d^2 - 1}{12} \frac{\langle a, b \rangle}{\langle a, \ell \rangle \langle b, \ell \rangle}. \quad \text{([Wal92, p.81])}$$

Le nombre $\tau(a, b; \ell)$ est indépendant du choix de base $\{x, y\}$.

Théorème ([Wal92, Theorem 5.1]). *Il existe un unique invariant, λ_{cw}, des sphères d'homologie rationnelle à valeurs dans $\mathbb{Q}$ tel que :*

(i) $\lambda_{\mathrm{cw}}(\mathbb{S}^3) = 0$,

(ii) si K est un nœud dans une sphère d'homologie rationnelle M, si ℓ est un générateur du lagrangien entier de $M \setminus \mathring{N}(K)$ et si a et b sont des classes d'homologie primitives dans $H_1(\partial(M \setminus \mathring{N}(K)); \mathbb{Z})$ telles que $\langle a, \ell \rangle \neq 0$ et $\langle b, \ell \rangle \neq 0$, alors

$$\lambda_{\mathrm{cw}}(M_b) = \lambda_{\mathrm{cw}}(M_a) + \tau(a, b; \ell) + \frac{\langle a, b \rangle}{\langle a, \ell \rangle \langle b, \ell \rangle} \Delta''_K(1)$$

où M_a et M_b, désignent les résultats des remplissages de Dehn de $M \setminus \mathring{N}(K)$ où a et b bordent respectivement des disques.

Selon [Les15b], il apparaît que l'invariant Θ des sphères d'homologie rationnelles munies d'une classe d'homotopie de combings peut être décomposé à l'aide de l'invariant de Casson-Walker et des nombres de Pontrjagin des combings de torsion des 3-variétés fermées orientées.

Théorème (Theorem II.1)**.** *Soit M une sphère d'homologie rationnelle (orientée) et $[X]$ une classe d'homotopie de combings sur M. Si λ_{cw} est l'invariant de Casson-Walker et si $p_1([X])$ est le nombre de Pontrjagin de $[X]$, alors*

$$\Theta(M,[X]) = 3 \cdot \lambda_{\mathrm{cw}}(M) + {}^1\!/_4 \cdot p_1([X]).$$

Dans ce chapitre, nous présentons une preuve combinatoire de ce résultat à l'aide de la définition combinatoire de l'invariant Θ présentée dans [Les16]. La clef de voûte de notre preuve est la caractérisation suivante de l'invariant de Casson-Walker à l'aide de la théorie des invariants de type fini des sphères d'homologie rationnelle relativement aux chirurgies lagrangiennes (*cf.* partie 2).

Théorème (Theorem II.2)**.** *L'invariant de Casson-Walker, λ_{cw}, est l'unique invariant des sphères d'homologie rationnelle tel que*

(i) *λ_{cw} est additif pour la somme connexe,*

(ii) *λ_{cw} est un invariant de type fini de degré au plus 2 relativement aux chirurgies lagrangiennes,*

(iii) *$\lambda_{\mathrm{cw}}(M) = -\lambda_{\mathrm{cw}}(-M)$ pour toute sphère d'homologie rationnelle M où $-M$ désigne la variété M munie de l'orientation opposée,*

(iv) *$\lambda_{\mathrm{cw}}(\mathcal{P}) = -2$ où $\mathcal{P}$ est la sphère d'homologie de Poincaré obtenue à partir de $\mathbb{S}^3$ par chirurgie de coefficient -1 sur le noeud de trèfle gauche,*

et où (ii) signifie que pour tout triplet $\{{}^{B_i}\!/_{A_i}\}_{i\in\{1,2,3\}}$ de chirurgies lagrangiennes disjointes dans une sphère d'homologie rationnelle M, on a

$$\sum_{I\subset\{1,2,3\}} (-1)^{|I|} \cdot \lambda_{\mathrm{cw}}(M(\{{}^{B_i}\!/_{A_i}\}_{i\in I})) = 0.$$

La principale difficulté dans l'utilisation de cette caractérisation afin de prouver de façon combinatoire que $\Theta(M,[X]) = 3 \cdot \lambda_{\mathrm{cw}}(M) + {}^1\!/_4 \cdot p_1([X])$, est de montrer que $\frac{1}{3}(\Theta - \frac{p_1}{4})$ est bien un invariant de type fini de degré au plus 2. Une première étape est réalisée dans le chapitre I où il est prouvé que les nombres de Pontrjagin des combings ont un "comportement de type fini" (voir le théorème I.10). Dans le chapitre II, on utilisera le travail de décomposition des chirurgies lagrangiennes de D. Moussard en chirurgies dites *élémentaires* (sommes connexes particulières, chirurgies T_k et chirurgies borroméennes – voir théorème II.1.19 et [Mou12]), puis, on définira des diagrammes de Heegaard standardisés pour ces chirurgies afin de pouvoir effectuer les calculs des variations de Θ à l'aide de la formule combinatoire de C. Lescop.

Chapitre I

Variation formulas for an extended Gompf invariant

Introduction

Context

In [Gom98], R. Gompf defined a homotopy invariant θ_G of oriented 2-plane fields in 3-manifolds. This invariant is defined for oriented 2-plane fields ξ in a closed oriented 3-manifold M when the first Chern class $c_1(\xi)$ is a torsion element of $H^2(M;\mathbb{Z})$. This invariant appears, for instance, in the construction of an absolute grading for the Heegaard-Floer homology groups, see [GH11]. Note that the positive unit normal of an oriented 2-plane field of a Riemannian 3-manifold M is a section of its unit tangent bundle UM so that homotopy classes of oriented 2-plane fields of M are in one-to-one correspondence with homotopy classes of sections of UM. Thus, the invariant θ_G may be regarded as an invariant of homotopy classes of nowhere zero vector fields, also called *combings*. In that setting, the Gompf invariant is defined for *torsion combings* of closed oriented 3-manifolds M, *ie* combings X such that the Euler class $e_2(X^\perp)$ of the normal bundle $X^\perp$ is a torsion element of $H^2(M;\mathbb{Z})$.

In [Les15b], C. Lescop proposed an alternative definition of θ_G using a Pontrjagin construction from the combing viewpoint. In this chapter, we use a similar approach to show how to define Pontrjagin numbers for torsion combings by using pseudo-parallelizations, which are a generalization of parallelizations. This enables us to define a relative extension of the Gompf invariant for torsion combings in all compact oriented 3-manifolds with boundary. We also study the iterated variations under Lagrangian-preserving surgeries of this extended invariant and prove

that it has degree two for a suitable finite type invariant theory. In such a study, pseudo-parallelizations reveal decisive since they are, in some sense, compatible with Lagrangian-preserving surgeries while genuine parallelizations are not.

Conventions

In this manuscript, compact oriented 3-manifolds may have boundary unless otherwise mentioned. All manifolds are implicitly equipped with Riemannian structures. The statements and the proofs are independent of the chosen Riemannian structures.

If M is an oriented manifold and if A is a submanifold of M, let TM, resp. TA, denote the tangent bundles to M, resp. A, and let NA refer to the orthogonal bundle to A in M, which is canonically isomorphic to the normal bundle to A in M. The fibers of NA are oriented so that $NA \oplus TA = TM$ fiberwise and the boundaries of all compact manifolds are oriented using the outward first convention.

If A and B are transverse submanifolds of an oriented manifold M, their intersection is oriented so that $N(A \cap B) = NA \oplus NB$, fiberwise. Moreover, if A and B have complementary dimensions, *ie* if $\dim(A) + \dim(B) = \dim(M)$, let $\varepsilon_{A\cap B}(x) = 1$ if $x \in A \cap B$ is such that $T_xA \oplus T_xB = T_xM$ and $\varepsilon_{A\cap B}(x) = -1$ otherwise. If A and B are compact transverse submanifolds of an oriented manifold M with complementary dimensions, the *algebraic intersection of A and B in M* is

$$\langle A, B\rangle_M = \sum_{x\in A\cap B} \varepsilon_{A\cap B}(x).$$

Let L_1 and L_2 be two rational chains of an oriented n-manifold M. Assume that L_1 and L_2 bound two rational chains Σ_1 and Σ_2, respectively. If L_1 is transverse to Σ_2, if L_2 is transverse to Σ_1 and if $\dim(L_1) + \dim(L_2) = n - 1$, then the *linking number of L_1 and L_2 in M* is

$$lk_M(L_1, L_2) = \langle \Sigma_1, L_2\rangle_M = (-1)^{n-\dim(L_2)}\langle L_1, \Sigma_2\rangle_M.$$

Setting and statements

A *combing* (X, σ) of a compact oriented 3-manifold M is a section X of the unit tangent bundle UM together with a nonvanishing section σ of the restriction $X^{\perp}_{|\partial M}$ of the normal bundle $X^{\perp}$ to ∂M. For simplicity's sake, the section σ may be omitted in the notation of a combing. For any combing (X, σ), note that $\rho(X) = (X_{|\partial M}, \sigma, X_{|\partial M} \wedge \sigma)$, where $\wedge$ denotes the cross product, is a trivialization of $TM_{|\partial M}$. So, a combing of a compact oriented 3-manifold M may also be seen as

a pair (X, ρ) where X is a section of UM that is the first vector of a trivialization ρ of $TM_{|\partial M}$ together with this trivialization.

Two combings (X, σ_X) and (Y, σ_Y) of a compact oriented 3-manifold M are said to be *transverse* when the graph $X(M)$ is transverse to $Y(M)$ and $-Y(M)$ in UM. The combings (X, σ_X) and (Y, σ_Y) are said to be *∂-compatible* when $X_{|\partial M} = Y_{|\partial M}$, $\sigma_X = \sigma_Y$, $X(\mathring{M})$ is transverse to $Y(\mathring{M})$ and $-Y(\mathring{M})$ in UM, and

$$\overline{X(\mathring{M}) \cap Y(\mathring{M})} \cap UM_{|\partial M} = \emptyset.$$

When (X, σ_X) and (Y, σ_Y) are ∂-compatible, define two links $L_{X=Y}$ and $L_{X=-Y}$ as follows. First, let P_M denote the projection from UM to M and set

$$L_{X=-Y} = P_M(X(M) \cap (-Y)(M)).$$

Second, there exists a link $L_{X=Y}$ in $\mathring{M}$ such that

$$P_M(X(M) \cap Y(M)) = \partial M \sqcup L_{X=Y}.$$

If (X, σ) is a combing of a compact oriented 3-manifold M, its *relative Euler class* $e_2^M(X^\perp, \sigma)$ in $H^2(M, \partial M; \mathbb{Z})$ is an obstruction to extending the section σ as a nonvanishing section of $X^\perp$. This obstruction is such that its Poincaré dual $P(e_2^M(X^\perp, \sigma))$ is represented by the zero set of a generic section of $X^\perp$ extending σ. This zero set is oriented by its coorientation induced by the orientation of $X^\perp$. When M is closed, the *Euler class* $e_2(X^\perp)$ of X is just this obstruction to finding a nonvanishing section of $X^\perp$.

A combing (X, σ) of a compact oriented 3-manifold M is a *torsion combing* if $e_2^M(X^\perp, \sigma)$ is a torsion element of $H^2(M, \partial M; \mathbb{Z})$, *ie* if $\left[e_2^M(X^\perp, \sigma)\right] = 0$ in $H^2(M, \partial M; \mathbb{Q})$.

Let M_1 and M_2 be two compact oriented 3-manifolds. The manifolds M_1 and M_2 are said to have *identified boundaries* if a collar of ∂M_1 in M_1 and a collar of ∂M_2 in M_2 are identified. In this case, $TM_{1|\partial M_1} = \mathbb{R}n_1 \oplus T\partial M_1$ is naturally identified with $TM_{2|\partial M_2} = \mathbb{R}n_2 \oplus T\partial M_2$ by an identification that maps the outward normal vector field n_1 to M_1 to the outward normal vector field n_2 to M_2.

If τ_1 and τ_2 are parallelizations of two compact oriented 3-manifolds M_1 and M_2 with identified boundaries such that τ_1 and τ_2 coincide on $\partial M_1 \simeq \partial M_2$, then the *first relative Pontrjagin number of* τ_1 *and* τ_2 is an element $p_1(\tau_1, \tau_2)$ of $\mathbb{Z}$ which corresponds to the Pontrjagin obstruction to extending a specific trivialization $\tau(\tau_1, \tau_2)$ of $TW \otimes \mathbb{C}$ defined on the boundary of a cobordism W from M_1 to M_2

with signature zero (see Subsection 2.1 or [Les15b, Subsection 4.1]). In the case of a parallelization τ of a closed oriented 3-manifold M, we get an absolute version. The *Pontrjagin number* $p_1(\tau)$ *of* τ is the relative Pontrjagin number $p_1(\tau_\emptyset, \tau)$ where $\tau_\emptyset$ is the parallelization of the empty set. Hence, for two parallelizations τ_1 and τ_2 of some closed oriented 3-manifolds,

$$p_1(\tau_1, \tau_2) = p_1(\tau_2) - p_1(\tau_1).$$

In [Les15b], using an interpretation of the variation of Pontrjagin numbers of parallelizations as an intersection of chains, C. Lescop showed that such a variation can be computed using only the first vectors of the parallelizations. This led her to the following theorem, which contains a definition of the *Pontrjagin numbers* for torsion combings of closed oriented 3-manifolds.

Theorem ([Les15b, Theorem 1.2 & Subsection 4.3])**.** *Let M be a closed oriented 3-manifold. There exists a unique map*

$$p_1 \ : \ \{\textit{homotopy classes of torsion combings of } M\} \longrightarrow \mathbb{Q}$$

such that :

(i) for any combing X on M such that X extends to a parallelization τ of M :

$$p_1([X]) = p_1(\tau),$$

(ii) if X and Y are two transverse torsion combings of M, then

$$p_1([Y]) - p_1([X]) = 4 \cdot lk(L_{X=Y}, L_{X=-Y}).$$

Furthermore, the map p_1 coincides with the Gompf invariant : for all torsion combing X,

$$p_1([X]) = \theta_G(X^\perp).$$

In this chapter, we study the variations of the Pontrjagin numbers of torsion combings of compact oriented 3-manifolds with respect to specific surgeries, called *Lagrangian-preserving surgeries*, which are defined as follows.

A *rational homology handlebody of genus* $g \in \mathbb{N}$, or $\mathbb{Q}$HH for short, is a compact oriented 3-manifold with the same homology with coefficients in $\mathbb{Q}$ as the standard genus g handlebody. Note that the boundary of a genus g rational homology handlebody is homeomorphic to the standard closed connected oriented surface of genus g. The *Lagrangian* of a $\mathbb{Q}$HH A is

$$\mathfrak{L}_A := \ker(i_*^A : H_1(\partial A; \mathbb{Q}) \longrightarrow H_1(A; \mathbb{Q}))$$

where i^A is the inclusion of ∂A into A. An *$LP_{\mathbb{Q}}$-surgery datum* in a compact oriented 3-manifold M is a triple (A, B, h), or $({}^B/_A)$ for short, where $A \subset M$, where B and A are $\mathbb{Q}$HH and where $h : \partial A \to \partial B$ is an identification homeomorphism, called *$LP_{\mathbb{Q}}$-identification*, such that $h_*(\mathfrak{L}_A) = \mathfrak{L}_B$. Performing the *$LP_{\mathbb{Q}}$-surgery* associated with the datum (A, B, h) in M consists in constructing the manifold :

$$M\,({}^B/_A) = \left(M \setminus \mathring{A}\right) \bigcup_h B.$$

If (M, X) is a compact oriented 3-manifold equipped with a combing, if (A, B, h) is an $\mathrm{LP}_{\mathbb{Q}}$-surgery datum in M, and if X_B is a combing of B that coincides with X on $\partial A \simeq \partial B$, then (A, B, h, X_B), or $({}^B/_A, X_B)$ for short, is an *$LP_{\mathbb{Q}}$-surgery datum in* (M, X). Performing the *$LP_{\mathbb{Q}}$-surgery* associated with the datum (A, B, h, X_B) in (M, X) consists in constructing the manifold $M\,({}^B/_A)$ equipped with the combing :

$$X({}^B/_A) = \begin{cases} X & \text{on } M \setminus \mathring{A}, \\ X_B & \text{on } B. \end{cases}$$

The first main result of this chapter is a variation formula for Pontrjagin numbers, which reads as follows for closed oriented 3-manifolds.

Theorem I.1. *Let (M, X) be a closed oriented 3-manifold equipped with a combing and let $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in\{1,2\}}$ be two disjoint $LP_{\mathbb{Q}}$-surgeries in (M, X) (ie A_1 and A_2 are disjoint). For all $I \subset \{1, 2\}$ let (M_I, X^I) be the combed manifold obtained by performing the surgeries associated to the data $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in I}$. Note that for all $I, J \subset \{1, 2\}$, X^I and X^J coincide on $(M \setminus \cup_{i\in I\cup J} A_i) \cup_{i\in I\cap J} B_i$. If $\{X^I\}_{I\subset\{1,2\}}$ is a family of torsion combings of the $\{M_I\}_{I\subset\{1,2\}}$, then*

$$\sum_{I\subset\{1,2\}} (-1)^{|I|} p_1([X^I]) = -2 \cdot lk_M \left(L_{\{X^I\}}({}^{B_1}/_{A_1}), L_{\{X^I\}}({}^{B_2}/_{A_2})\right),$$

where the right-hand side of the equality is defined as follows. For all $i \in \{1, 2\}$, let

$$H_1(A_i; \mathbb{Q}) \xleftarrow{i_*^{A_i}} \frac{H_1(\partial A_i; \mathbb{Q})}{\mathfrak{L}_{A_i}} = \frac{H_1(\partial B_i; \mathbb{Q})}{\mathfrak{L}_{B_i}} \xrightarrow{i_*^{B_i}} H_1(B_i; \mathbb{Q})$$

be the sequence of isomorphisms induced by the inclusions. There exists a unique homology class $L_{\{X^I\}}({}^{B_i}/_{A_i})$ in $H_1(A_i; \mathbb{Q})$ such that for any nonvanishing section σ_i of $X^{\perp}_{|\partial A_i}$:

$$L_{\{X^I\}}({}^{B_i}/_{A_i}) = i_*^{A_i} \circ (i_*^{B_i})^{-1} \left(P(e_2^{B_i}(X^{\perp}_{B_i}, \sigma_i))\right) - P(e_2^{A_i}(X^{\perp}_{|A_i}, \sigma_i)),$$

where P stands for Poincaré duality isomorphisms from $H^2(A_i, \partial A_i; \mathbb{Q})$ to $H_1(A_i; \mathbb{Q})$ or from $H^2(B_i, \partial B_i; \mathbb{Q})$ to $H_1(B_i; \mathbb{Q})$. Furthermore, the classes $L_{\{X^I\}}({}^{B_1}/_{A_1})$ and $L_{\{X^I\}}({}^{B_2}/_{A_2})$ are mapped to zero in $H_1(M; \mathbb{Q})$ and the map

$$lk_M : \ker\big(H_1(A_1; \mathbb{Q}) \to H_1(M; \mathbb{Q})\big) \times \ker\big(H_1(A_2; \mathbb{Q}) \to H_1(M; \mathbb{Q})\big) \longrightarrow \mathbb{Q}$$

is well-defined.

Example I.2. As an example, consider $\mathbb{S}^3$ equipped with a parallelization τ which extends the standard parallelization of the unit ball. In this ball, consider a positive Hopf link and let $A_1 \sqcup A_2$ be a tubular neighborhood of this link. Let X be the combing $\tau(e_1)$, where $e_1 = (1, 0, 0) \in \mathbb{S}^2$, and let $B_1 = A_1$ and $B_2 = A_2$. Identify A_1 and A_2 with $\mathbb{D}^2 \times \mathbb{S}^1$ and consider a smooth map $g : \mathbb{D}^2 \to \mathbb{S}^2$ such that $g(\partial\mathbb{D}^2) = e_1$, and such that $-e_1$ is a degree 1 regular value of g with a single preimage ω. Finally, for all $i \in \{1, 2\}$, let X_{B_i} be the combing :

$$X_{B_i} : \begin{cases} \mathbb{D}^2 \times \mathbb{S}^1 \longrightarrow UM \\ \quad (z, u) \longmapsto \tau((z, u), g(u)). \end{cases}$$

In this case, $L_{X(\{{}^{B_i}/_{A_i}\}_{i\in\{1,2\}})=-X} = L_{X_{B_1}=-X_{|A_1}} \cup L_{X_{B_2}=-X_{|A_2}}$, and, for $i \in \{1, 2\}$, using the identification of A_i with $\mathbb{D}^1 \times \mathbb{S}^1$, the link $L_{X_{B_i}=-X_{|A_i}}$ reads $\mathbb{S}^1 \times \{\omega\}$. As we will see in Proposition I.1.10, for all $i \in \{1, 2\}$, $L_{\{X^I\}}({}^{B_i}/_{A_i}) = 2[L_{X_{B_i}=-X_{|A_i}}]$. Eventually,

$$\sum_{I\subset\{1,2\}} (-1)^{|I|} p_1([X^I]) = -8.$$

In general, for an $\mathrm{LP}_{\mathbb{Q}}$-surgery datum $({}^{B}/_{A})$ in a compact oriented 3-manifold M, a trivialization of $TM_{|(M\setminus\mathring{A})}$ cannot be extended as a parallelization of $M({}^{B}/_{A})$. It follows that $\mathrm{LP}_{\mathbb{Q}}$-surgeries cannot be expressed as local moves on parallelized compact oriented 3-manifolds. This makes computing the variation of Pontrjagin numbers of torsion combings under $\mathrm{LP}_{\mathbb{Q}}$-surgeries tricky since Pontrjagin numbers of torsion combings are defined with respect to Pontrjagin numbers of parallelizations.

However, if M is a compact oriented 3-manifold and if ρ is a trivialization of $TM_{|\partial M}$, then the obstruction to finding a parallelization of M which coincides with ρ on ∂M is an element of $H^2(M, \partial M; {}^{\mathbb{Z}}/_{2\mathbb{Z}})$ – hence, its Poincaré dual is an element $[\gamma]$ of $H_1(M; {}^{\mathbb{Z}}/_{2\mathbb{Z}})$ – and it is possible to get around such an obstruction thanks to the notion of *pseudo-parallelization* developed by C. Lescop. Let us postpone the formal definition to Subsection 1.3 (see also [Les10]) and, for the time being, let us just mention that a pseudo-parallelization $\bar{\tau}$ of a compact oriented 3-manifold M is a triple $(N(\gamma); \tau_e, \tau_d)$ where $N(\gamma)$ is a framed tubular neighborhood of a link

γ in $\mathring{M}$, τ_e is a parallelization of $M \setminus N(\gamma)$ and $\tau_d : N(\gamma) \times \mathbb{R}^3 \to TN(\gamma)$ is a parallelization of $N(\gamma)$ such that there exists a section E_1^d of UM :

$$E_1^d : \begin{cases} m \in M \setminus \mathring{N}(\gamma) \longmapsto \tau_e(m, e_1) \\ \qquad m \in N(\gamma) \longmapsto \tau_d(m, e_1) \end{cases}$$

where $e_1 = (1, 0, 0)$. Let us finally mention that $\bar{\tau}$ also determines a section E_1^g of UM which coincides with E_1^d on $M \setminus \mathring{N}(\gamma)$. The sections E_1^d and E_1^g are the *Siamese sections* of $\bar{\tau}$ and the link γ is the *link of the pseudo-parallelization* $\bar{\tau}$.

To a pseudo-parallelization, C. Lescop showed that it is possible to associate a *complex trivialization* up to homotopy, see Definition I.2.1. This leads to a natural extension of the notion of first relative Pontrjagin numbers of parallelizations to pseudo-parallelizations. Furthermore, as in the case of parallelizations, a pseudo-parallelization $\bar{\tau}$ of a compact oriented 3-manifold M admits *pseudo-sections* $\bar{\tau}(M \times \{v\})$ which are 3-chains of UM, for all $v \in \mathbb{S}^2$. In the special case $v = e_1$ the pseudo-section $\bar{\tau}(M \times \{e_1\})$ of $\bar{\tau}$ can be written as :

$$\bar{\tau}(M \times \{e_1\}) = \frac{E_1^d(M) + E_1^g(M)}{2}.$$

A combing (X, σ) of M is said to be *compatible with* $\bar{\tau}$ if (X, σ) is ∂-compatible with $(E_1^d, E^e_{2|\partial M})$ and $(E_1^g, E^e_{2|\partial M})$, where E_2^e is the second vector of τ_e, and if

$$L_{E_1^d = X} \cap L_{E_1^g = -X} = \emptyset \quad \text{and} \quad L_{E_1^g = X} \cap L_{E_1^d = -X} = \emptyset.$$

If (X, σ) and $\bar{\tau}$ are compatible, then $\rho(X) = \tau_{|\partial M}$ and we get two disjoint rational combinations of oriented links in $\mathring{M}$:

$$L_{\bar{\tau} = X} = \frac{L_{E_1^d = X} + L_{E_1^g = X}}{2} \quad \text{and} \quad L_{\bar{\tau} = -X} = \frac{L_{E_1^d = -X} + L_{E_1^g = -X}}{2}.$$

Pseudo-parallelizations allow us to revisit the definition of Pontrjagin numbers and to generalize it to torsion combings of compact oriented 3-manifolds as follows. Let $P_{\mathbb{S}^2}$ denote the standard projection from $W \times \mathbb{S}^2$ to $\mathbb{S}^2$, for any manifold W.

Lemma I.3. *Let (X, σ) be a torsion combing of a compact oriented 3-manifold M, let $\bar{\tau}$ be a pseudo-parallelization of M, and let E_1^d and E_1^g be the Siamese sections of $\bar{\tau}$. If $\bar{\tau}$ and (X, σ) are compatible, then the expression*

$$4 \cdot lk_M(L_{\bar{\tau} = X}, L_{\bar{\tau} = -X}) - lk_{\mathbb{S}^2}\left(e_1 - (-e_1)\ ,\ P_{\mathbb{S}^2} \circ \tau_d^{-1} \circ X(L_{E_1{}^d = -E_1{}^g})\right)$$

depends only on the homotopy class of (X, σ). It will be denoted $p_1(\bar{\tau}, [X])$ and its opposite will be written $p_1([X], \bar{\tau})$.

Theorem I.4. *Let (X_1, σ_{X_1}) and (X_2, σ_{X_2}) be torsion combings of two compact oriented 3-manifolds M_1 and M_2 with identified boundaries such that (X_1, σ_{X_1}) and (X_2, σ_{X_2}) coincide on the boundary. For $i \in \{1, 2\}$, let $\bar{\tau}_i$ be a pseudo-parallelization of M_i such that $\bar{\tau}_i$ and (X_i, σ_{X_i}) are compatible. The expression*

$$p_1([X_1], [X_2]) = p_1([X_1], \bar{\tau}_1) + p_1(\bar{\tau}_1, \bar{\tau}_2) + p_1(\bar{\tau}_2, [X_2])$$

depends only on the homotopy classes of (X_1, σ_{X_1}) and (X_2, σ_{X_2}), and it defines the first relative Pontrjagin number of (X_1, σ_{X_1}) and (X_2, σ_{X_2}). *Moreover, if M_1 and M_2 are closed, then*

$$p_1([X_1], [X_2]) = p_1([X_2]) - p_1([X_1]).$$

Under the assumptions of Theorem I.1, we see that it would be impossible to naively define $p_1([X_{|A_1}], [X_{\{1\}|B_1}])$ as $p_1([X]) - p_1([X_{\{1\}}])$, where X extends $X_{|A_1}$ to the closed manifold M, and $X_{\{1\}}$ extends $X_{\{1\}|B_1}$ in the same way to $M({}^{B_1}/_{A_1})$. Indeed Theorem I.1 and the example that follows it show that the expression $\big(p_1([X]) - p_1([X_{\{1\}}])\big)$ depends on the combed manifold (M, X) into which $(A_1, X_{|A_1})$ has been embedded. It even depends on the combing X that extends the combing $X_{|A_1}$ of A_1 to M for the fixed manifold M of this example, since

$$\big(p_1([X]) - p_1([X_{\{1\}}])\big) - \big(p_1([X_{\{2\}}]) - p_1([X_{\{1,2\}}])\big) = -8$$

there.

Theorem I.4 translates as follows in the closed case and it bridges a gap between the two dissimilar generalizations of the Pontrjagin numbers of parallelizations for pseudo-parallelizations and for torsion combings in closed oriented 3-manifolds.

Corollary I.5. *Let X be a torsion combing of a closed oriented 3-manifold M and let $\bar{\tau} = (N(\gamma); \tau_e, \tau_d)$ be a pseudo-parallelization of M. Let E_1^d and E_1^g denote the Siamese sections of $\bar{\tau}$. If X is transverse to E_1^d and E_1^g and if $L_{E_1^d = X} \cap L_{E_1^g = -X}$ and $L_{E_1^g = X} \cap L_{E_1^d = -X}$ are empty, then*

$$\begin{aligned} p_1([X]) = p_1(\bar{\tau}) &+ 4 \cdot lk_M(L_{\bar{\tau}=X}, L_{\bar{\tau}=-X}) \\ &- lk_{\mathbb{S}^2}\left(e_1 - (-e_1)\ ,\ P_{\mathbb{S}^2} \circ \tau_d^{-1} \circ X(L_{{E_1}^d = -{E_1}^g})\right). \end{aligned}$$

Another special case is when genuine parallelizations can be used. The closed case with genuine parallelizations is nothing but C. Lescop's definition of the Pontrjagin number of torsion combings in closed oriented 3-manifolds stated above.

Corollary I.6. *Let (X_1, σ_1) and (X_2, σ_2) be torsion combings of two compact oriented 3-manifolds M_1 and M_2 with identified boundaries such that (X_1, σ_1) and*

(X_2, σ_2) coincide on the boundary. If, for $i \in \{1,2\}$, $\tau_i = (E_1^i, E_2^i, E_3^i)$ is a parallelization of M_i such that (X_i, σ_i) and $(E_1^i, E_{2|\partial M_i}^i)$ are ∂-compatible, then

$$p_1([X_1],[X_2]) = p_1(\tau_1,\tau_2) + 4 \cdot lk_{M_2}(L_{E_1^2=X_2}\ ,\ L_{E_1^2=-X_2}) - 4 \cdot lk_{M_1}(L_{E_1^1=X_1}\ ,\ L_{E_1^1=-X_1}).$$

Finally, for torsion combings defined on a fixed compact oriented 3-manifold, we have the following simple variation formula, as in the closed case.

Proposition I.7. *If (X, σ) and (Y, σ) are ∂-compatible torsion combings of a compact oriented 3-manifold M, then*

$$p_1([X],[Y]) = 4 \cdot lk_M(L_{X=Y}, L_{X=-Y}).$$

Let M be a compact oriented 3-manifold. For all section σ of $TM_{|\partial M}$, let $\mathrm{Spin}^c(M, \sigma)$ denote the *Spin^c-structures on M relative to σ*, *ie* the set of homotopy classes on $M \setminus \{\omega\}$ of combings (X, σ) of M, where ω is any point in $\mathring{M}$ (see [DM05], for a detailed presentation of Spin^c-structures). Thanks to Proposition I.7, it is possible to classify the torsion combings of a fixed Spin^c-structure up to homotopy, thus generalizing a property of the Gompf invariant in the closed case. I wish to thank Gwénaël Massuyeau for suggesting this statement.

Theorem I.8. *Let (X, σ) and (Y, σ) be ∂-compatible torsion combings of a compact oriented 3-manifold M which represent the same Spin^c-structure. The combings (X, σ) and (Y, σ) are homotopic relatively to the boundary if and only if $p_1([X],[Y]) = 0$.*

The key tool in the proof of Theorem I.4 is the following generalization of the interpretation of the variation of the Pontrjagin numbers of parallelizations as an algebraic intersection of three chains.

Proposition I.9. *Let τ and $\bar{\tau}$ be two pseudo-parallelizations of a compact oriented 3-manifold M that coincide on ∂M and whose links are disjoint. For any $v \in \mathbb{S}^2$, there exists a 4-chain $C_4(\tau, \bar{\tau}; v)$ of $[0,1] \times UM$ transverse to the boundary of $[0,1] \times UM$ such that*

$$\partial C_4(\tau, \bar{\tau}; v) = \{1\} \times \bar{\tau}(M \times \{v\}) - \{0\} \times \tau(M \times \{v\}) - [0,1] \times \tau(\partial M \times \{v\})$$

and for any x, y and z in $\mathbb{S}^2$ with pairwise different distances to e_1 :

$$p_1(\tau, \bar{\tau}) = 4 \cdot \langle C_4(\tau, \bar{\tau}; x), C_4(\tau, \bar{\tau}; y), C_4(\tau, \bar{\tau}; z) \rangle_{[0,1] \times UM}$$

for any triple of pairwise transverse $C_4(\tau, \bar{\tau}, x)$, $C_4(\tau, \bar{\tau}, y)$ and $C_4(\tau, \bar{\tau}, z)$ that satisfy the hypotheses above.

Our general variation formula for Pontrjagin numbers of torsion combings reads as follows for all compact oriented 3-manifolds.

Theorem I.10. *Let (M, X) be a compact oriented 3-manifold equipped with a combing, let $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in\{1,2\}}$ be two disjoint $LP_{\mathbb{Q}}$-surgeries in (M, X), and, for all $I \subset \{1, 2\}$, let $X^I = X(\{{}^{B_i}/_{A_i}\}_{i\in I})$. If $\{X^I\}_{I\subset\{1,2\}}$ is a family of torsion combings of the $M_I = M(\{{}^{B_i}/_{A_i}\}_{i\in I})$, then*

$$p_1([X^{\{2\}}], [X^{\{1,2\}}]) - p_1([X], [X^{\{1\}}]) = -2 \cdot lk_M \left(L_{\{X^I\}}({}^{B_1}/_{A_1}), L_{\{X^I\}}({}^{B_2}/_{A_2})\right),$$

where the right-hand side is defined as follows. For all $i \in \{1, 2\}$, let

$$H_1(A_i;\mathbb{Q}) \xleftarrow{i_*^{A_i}} \frac{H_1(\partial A_i;\mathbb{Q})}{\mathfrak{L}_{A_i}} = \frac{H_1(\partial B_i;\mathbb{Q})}{\mathfrak{L}_{B_i}} \xrightarrow{i_*^{B_i}} H_1(B_i;\mathbb{Q})$$

be the sequence of isomorphisms induced by the inclusions. There exists a unique homology class $L_{\{X^I\}}({}^{B_i}/_{A_i})$ in $H_1(A_i;\mathbb{Q})$ such that for any nonvanishing section σ_i of $X^{\perp}_{|\partial A_i}$:

$$L_{\{X^I\}}({}^{B_i}/_{A_i}) = i_*^{A_i} \circ (i_*^{B_i})^{-1} \left(P(e_2^{B_i}(X^{\perp}_{B_i}, \sigma_i))\right) - P(e_2^{A_i}(X^{\perp}_{|A_i}, \sigma_i)),$$

where P stands for Poincaré duality isomorphisms from $H^2(A_i, \partial A_i;\mathbb{Q})$ to $H_1(A_i;\mathbb{Q})$ or from $H^2(B_i, \partial B_i;\mathbb{Q})$ to $H_1(B_i;\mathbb{Q})$. Furthermore, the classes $L_{\{X^I\}}({}^{B_1}/_{A_1})$ and $L_{\{X^I\}}({}^{B_2}/_{A_2})$ are mapped to zero in $H_1(M;\mathbb{Q})$ and the map

$$lk_M : \ker\big(H_1(A_1;\mathbb{Q}) \to H_1(M;\mathbb{Q})\big) \times \ker\big(H_1(A_2;\mathbb{Q}) \to H_1(M;\mathbb{Q})\big) \longrightarrow \mathbb{Q}$$

is well-defined.

A direct consequence of this variation formula is that the extended Gompf invariant for torsion combings of compact oriented 3-manifolds is a degree two finite type invariant with respect to $\mathrm{LP}_{\mathbb{Q}}$-surgeries.

Corollary I.11. *Let (M, X) be a compact oriented 3-manifold equipped with a combing, let $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in\{1,\dots,k\}}$ be a family of disjoint $LP_{\mathbb{Q}}$-surgeries in (M, X), and, for all $I \subset \{1, \dots, k\}$, let (M_I, X^I) be the combed manifold obtained by performing the surgeries associated to the data $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in I}$. If $k \geqslant 3$, and if $\{X^I\}_{I\subset\{1,\dots,k\}}$ is a family of torsion combings of the $\{M_I\}_{I\subset\{1,2\}}$, then*

$$\sum_{I\subset\{2,\dots,k\}} (-1)^{|I|}\, p_1\left([X^I], [X^{I\cup\{1\}}]\right) = 0.$$

If $\partial M = \emptyset$, this reads

$$\sum_{I\subset\{1,\dots,k\}} (-1)^{|I|}\, p_1\left([X^I]\right) = 0.$$

1 More about ...

1.1 Lagrangian-preserving surgeries

Proposition I.1.1. *Let $({}^{B}/_{A})$ be an $LP_{\mathbb{Q}}$-surgery datum in a compact oriented 3-manifold M and let L_1 and L_2 be links in $M \setminus \mathring{A}$. If L_1 and L_2 are rationally null-homologous in M, then they are null-homologous in $M({}^{B}/_{A})$ and*

$$lk_{M({}^{B}/_{A})}(L_1, L_2) = lk_M(L_1, L_2).$$

Proof. For $i \in \{1, 2\}$, let Σ_i be a 2-chain in M such that $\partial\Sigma_i = L_i$ and Σ_i is transverse to ∂A. Since $({}^{B}/_{A})$ is an $\mathrm{LP}_{\mathbb{Q}}$-surgery, $[\Sigma_i \cap \partial A] = 0$ in $H_1(B; \mathbb{Q})$. Therefore, for all $i \in \{1, 2\}$, there exists a 2-chain Σ'_i in $M({}^{B}/_{A})$ such that

$$\partial\Sigma'_i = L_i \quad \text{and} \quad \Sigma'_i \cap (M({}^{B}/_{A}) \setminus \mathring{B}) = \Sigma_i \cap (M \setminus \mathring{A}).$$

As a consequence L_1 and L_2 are null-homologous in $M({}^{B}/_{A})$ and, since $L_2 \subset M \setminus \mathring{A}$, it follows that

$$lk_{M({}^{B}/_{A})}(L_1, L_2) = \langle \Sigma'_1, L_2 \rangle_{M({}^{B}/_{A})} = \langle \Sigma_1, L_2 \rangle_M = lk_M(L_1, L_2).$$

□

A *rational homology 3-sphere*, or a $\mathbb{Q}$HS for short, is a closed oriented 3-manifold with the same homology with rational coefficients as $\mathbb{S}^3$.

Proposition I.1.2. *Let $({}^{B}/_{A})$ be an $LP_{\mathbb{Q}}$-surgery in a compact oriented 3-manifold M. If M is a $\mathbb{Q}$HS, then $M({}^{B}/_{A})$ is a $\mathbb{Q}$HS.*

Proof. Let M be a $\mathbb{Q}$HS and let A be a $\mathbb{Q}$HH of genus $g \in \mathbb{N}$. Using the Mayer-Vietoris sequence associated to $M = A \cup (M \setminus \mathring{A})$ shows that $M \setminus \mathring{A}$ is a $\mathbb{Q}$HH of genus g and that the inclusions of ∂A into A and into $M \setminus \mathring{A}$ induce an isomorphism

$$H_1(\partial A; \mathbb{Q}) \simeq H_1(A; \mathbb{Q}) \oplus H_1(M \setminus \mathring{A}; \mathbb{Q}).$$

For details, see [Mou12, Sublemma 4.6]. It follows that $M({}^{B}/_{A}) \setminus \mathring{B}$ is also a genus g $\mathbb{Q}$HH and, since $({}^{B}/_{A})$ is an $\mathrm{LP}_{\mathbb{Q}}$-surgery, the inclusions of ∂B into B and into $M({}^{B}/_{A}) \setminus \mathring{B}$ induce an isomorphism

$$H_1(\partial B; \mathbb{Q}) \simeq H_1(B; \mathbb{Q}) \oplus H_1(M({}^{B}/_{A}) \setminus \mathring{B}; \mathbb{Q}).$$

Using this isomorphism in the Mayer-Vietoris sequence associated to the splitting $M({}^{B}/_{A}) = B \cup (M({}^{B}/_{A}) \setminus \mathring{B})$ shows that $M({}^{B}/_{A})$ is a $\mathbb{Q}$HS. □

Lemma I.1.3. *If A is a compact connected orientable 3-manifold with connected boundary and if the map $i_*^A : H_1(\partial A;\mathbb{Q}) \to H_1(A;\mathbb{Q})$ induced by the inclusion of ∂A into A is surjective, then A is a rational homology handlebody.*

Proof. First, for such a manifold A, we have $H_0(A;\mathbb{Q}) \simeq \mathbb{Q}$ and $H_3(A;\mathbb{Q}) \simeq 0$. Second, using the hypothesis on i_*^A in the exact sequence associated to $(A, \partial A)$, we get that $H_1(A, \partial A;\mathbb{Q}) = 0$. Using Poincaré duality and the universal coefficient theorem, it follows that

$$H_2(A;\mathbb{Q}) \simeq H^1(A, \partial A;\mathbb{Q}) \simeq \mathrm{Hom}(H_1(A, \partial A;\mathbb{Q}), \mathbb{Q}) = 0.$$

Moreover, we get the following exact sequence from the exact sequence associated to $(A, \partial A)$:

$$0 \to H_2(A, \partial A;\mathbb{Q}) \to H_1(\partial A;\mathbb{Q}) \to H_1(A;\mathbb{Q}) \to 0.$$

It follows that $\dim(H_2(A, \partial A;\mathbb{Q}))+\dim(H_1(A;\mathbb{Q})) = \dim(H_1(\partial A;\mathbb{Q})) = 2g$, where g denotes the genus of ∂A. However,

$$H_2(A, \partial A;\mathbb{Q}) \simeq H^1(A;\mathbb{Q}) \simeq \mathrm{Hom}(H_1(A;\mathbb{Q}), \mathbb{Q}),$$

hence $\dim(H_1(A;\mathbb{Q})) = g$. □

Proposition I.1.4. *Let A be a compact submanifold with connected boundary of a $\mathbb{Q}HS$ M, let B be a compact oriented 3-manifold and let $h : \partial A \to \partial B$ be a homeomorphism. If the surgered manifold $M({}^B/_A)$ is a $\mathbb{Q}HS$ and if*

$$lk_{M({}^B/_A)}(L_1, L_2) = lk_M(L_1, L_2)$$

for all disjoint links L_1 and L_2 in $M \setminus \mathring{A}$, then $({}^B/_A)$ is an $LP_{\mathbb{Q}}$-surgery.

Proof. Using the Mayer-Vietoris exact sequences associated to $M = A \cup (M \setminus \mathring{A})$ and $M({}^B/_A) = B \cup (M({}^B/_A) \setminus \mathring{B})$, we get that the maps $i_*^A : H_1(\partial A;\mathbb{Q}) \longrightarrow H_1(A;\mathbb{Q})$ and $i_*^B : H_1(\partial B;\mathbb{Q}) \longrightarrow H_1(B;\mathbb{Q})$ induced by the inclusions of ∂A and ∂B in A and B are surjective. Using Lemma I.1.3, it follows that A and B are rational homology handlebodies. Moreover, A and B have the same genus since $h : \partial A \to \partial B$ is a homeomorphism.

Let $P_{\mathfrak{L}_A}$ and $P_{\mathfrak{L}_B}$ denote the projections from $H_1(\partial A;\mathbb{Q})$ onto $\mathfrak{L}_A$ and $\mathfrak{L}_B$, respectively, with kernel $\mathfrak{L}_{M\setminus\mathring{A}}$. Consider a collar $[0,1] \times \partial A$ of ∂A such that $\{0\} \times \partial A \simeq \partial A$ and note that for all 1-cycles x and y of ∂A :

$$\langle P_{\mathfrak{L}_A}(y), x\rangle_{\partial A} = lk_M(\{1\}\times y, \{0\}\times x) = lk_{M({}^B/_A)}(\{1\}\times y, \{0\}\times x) = \langle P_{\mathfrak{L}_B}(y), x\rangle_{\partial B},$$

so that $P_{\mathfrak{L}_B} = P_{\mathfrak{L}_A}$ and $h_*(\mathfrak{L}_A) = \mathfrak{L}_B$. □

1.2 Combings

Proposition I.1.5. *If X and Y are ∂-compatible combings of a compact oriented 3-manifold M, then*

$$L_{X=Y} = L_{Y=X} \quad \textit{and} \quad L_{X=Y} = -L_{-X=-Y}.$$

Proof. First, by definition, the link $L_{X=Y}$ is the projection of the intersection of the sections $X(\mathring{M})$ and $Y(\mathring{M})$. This intersection is oriented so that

$$NX(\mathring{M}) \oplus NY(\mathring{M}) \oplus T(X(\mathring{M}) \cap Y(\mathring{M}))$$

orients UM, fiberwise. Since the normal bundles $NX(\mathring{M})$ and $NY(\mathring{M})$ have dimension 2, the isomorphism permuting them is orientation-preserving so that $L_{X=Y} = L_{Y=X}$. Second, $(-X)(\mathring{M}) \cap (-Y)(\mathring{M})$ is the image of $X(\mathring{M}) \cap Y(\mathring{M})$ under the map ι from UM to itself which acts on each fiber as the antipodal map. This map reverses the orientation of UM as well as the coorientations of $X(\mathring{M})$ and $Y(\mathring{M})$, *ie*

$$N(-X)(M) = -\iota(NX(M)), \; N(-Y)(M) = -\iota(NY(M)).$$

Since $N(-X)(M) \oplus N(-Y)(M) \oplus T((-X)(M) \cap (-Y)(M))$ has the orientation of UM

$$T((-X)(M) \cap (-Y)(M)) = -\iota(T(X(M) \cap Y(M))).$$

Hence, $L_{X=Y} = -L_{-X=-Y}$. □

Definition I.1.6. Let M be a compact oriented 3-manifold and let L be a link in $\mathring{M}$. Define the *blow up of M along L* as the 3-manifold $Bl(M, L)$ constructed from M in which L is replaced by its unit normal bundle in M. The 3-manifold $Bl(M, L)$ inherits a canonical differential structure. See [Les15b, Definition 3.5] for a detailed description.

Lemma I.1.7. *Let X and Y be ∂-compatible combings of a compact oriented 3-manifold M. There exists a 4-chain $\bar{F}(X, Y)$ of UM with boundary :*

$$\partial \bar{F}(X, Y) = Y(M) - X(M) + UM_{|L_{X=-Y}}.$$

Proof. To construct the desired 4-chain, start with the partial homotopy from X to Y

$$\tilde{F}(X, Y) : \begin{cases} [0, 1] \times (M \setminus L_{X=-Y}) \longrightarrow UM \\ \qquad\qquad (s, m) \longmapsto \big(m, \mathcal{H}_X^Y(s, m)\big) \end{cases}$$

where $\mathcal{H}_X^Y(s, m)$ is the unique point of the shortest geodesic arc from $X(m)$ to $Y(m)$ such that

$$d_{\mathbb{S}^2}(X(m), \mathcal{H}_X^Y(s, m)) = s \cdot d_{\mathbb{S}^2}(X(m), Y(m))$$

where $d_{\mathbb{S}^2}$ denotes the usual distance on $\mathbb{S}^2$. Next, extend the map

$$(s,m) \longmapsto \mathcal{H}_X^Y(s,m)$$

on the blow up of M along $L_{X=-Y}$. The section X induces a map

$$X : NL_{X=-Y} \longrightarrow -Y^{\perp}(L_{X=-Y})$$

which is a diffeomorphism on a neighborhood of $\{0\} \times L_{X=-Y}$ since X and Y are ∂-compatible combings. Furthermore, this diffeomorphism is orientation-preserving by definition of the orientation on $L_{X=-Y}$. So, for $n \in UN_mL_{X=-Y}$, $\mathcal{H}_X^Y(s,n)$ can be defined as the unique point at distance $s\pi$ from $X(m)$ on the unique half great circle from $X(m)$ to $Y(m)$ through $T_mX(n)$. Thanks to transversality again, $\{\mathcal{H}_X^Y(s,n) \mid s \in [0,1],\ n \in UN_mL_{X=-Y}\}$ is a whole sphere $\mathbb{S}^2$ for any fixed $m \in L_{X=-Y}$, so that

$$\partial\tilde{F}(X,Y)([0,1] \times Bl(M, L_{X=-Y})) = Y(M) - X(M) + \partial_{int}$$

where $\partial_{int} \simeq L_{X=-Y} \times \mathbb{S}^2$ (see [Les15b, Proof of Proposition 3.6] for the orientation of ∂_{int}). Finally, let $\bar{F}(X,Y) = \tilde{F}(X,Y)([0,1] \times Bl(M, L_{X=-Y}))$. $\square$

If X and Y are ∂-compatible combings of a compact oriented 3-manifold M and if σ is a nonvanishing section of $X^{\perp}_{|\partial M}$, let $\mathcal{H}_{X,\sigma}^{-Y}$ denote the map from $[0,1] \times (M \setminus L_{X=Y})$ to UM such that, for all (s,m) in $[0,1] \times \partial M$, $\mathcal{H}_{X,\sigma}^{-Y}(s,m)$ is the unique point at distance $s\pi$ from $X(m)$ on the unique geodesic arc starting from $X(m)$ in the direction of $\sigma(m)$ to $-X(m) = -Y(m)$ and, for all (s,m) in $[0,1] \times (\mathring{M} \setminus L_{X=Y})$, $\mathcal{H}_{X,\sigma}^{-Y}(s,m)$ is the unique point on the shortest geodesic arc from $X(m)$ to $-Y(m)$ such that

$$d_{\mathbb{S}^2}(X(m), \mathcal{H}_{X,\sigma}^{-Y}(s,m)) = s \cdot d_{\mathbb{S}^2}(X(m), -Y(m)).$$

As in the previous proof, $\mathcal{H}_{X,\sigma}^{-Y}$ may be extended as a map from $[0,1] \times Bl(M, L_{X=Y})$ to UM. In the case of $X = Y$, for all section σ of $X^{\perp}$, nonvanishing on ∂M, let $L_{\sigma=0}$ denote the oriented link $\{m \in M \mid \sigma(m) = 0\}$ and define a map $\mathcal{H}_{X,\sigma}^{-X}$ as the map from $[0,1] \times (M \setminus L_{\sigma=0})$ to UM such that, for all (s,m) in $[0,1] \times (M \setminus L_{\sigma=0})$, $\mathcal{H}_{X,\sigma}^{-X}(s,m)$ is the unique point at distance $s\pi$ from $X(m)$ on the unique geodesic arc starting from $X(m)$ in the direction of $\sigma(m)$ to $-X(m)$. Note that $L_{\sigma=0} \cap \partial M = \emptyset$, and $[L_{\sigma=0}] = P(e_2^M(X, \sigma_{|\partial M}))$. Here again, $\mathcal{H}_{X,\sigma}^{-X}$ may be extended as a map from $[0,1] \times Bl(M, L_{\sigma=0})$ to UM.

In order to simplify notations, if A is a submanifold of a compact oriented 3-manifold M, we may implicitly use a parallelization of M to write $UM_{|A}$ as $A \times \mathbb{S}^2$.

Proposition I.1.8. *If (X,σ) and (Y,σ) are ∂-compatible combings of a compact oriented 3-manifold M, then, in $H_3(UM;\mathbb{Z})$,*

$$[L_{X=-Y}\times\mathbb{S}^2]=[X(M)-Y(M)]$$
$$[L_{X=Y}\times\mathbb{S}^2]=[X(M)-(-Y)(M)+\mathcal{H}^{-Y}_{X,\sigma}([0,1]\times\partial M)].$$

Proof. The first identity is a direct consequence of Lemma I.1.7. The second one can be obtained using a similar construction. Namely, construct a 4-chain $\bar{F}(X,-Y)$ using the partial homotopy from X to $-Y$:

$$\tilde{F}(X,-Y):\begin{cases}[0,1]\times(M\setminus L_{X=Y})\longrightarrow UM\\ \qquad\qquad (s,m)\longmapsto\left(m,\mathcal{H}^{-Y}_{X,\sigma}(s,m)\right).\end{cases}$$

As in the proof of Lemma I.1.7, $\tilde{F}(X,-Y)$ can be extended to $[0,1]\times Bl(M,L_{X=Y})$. Finally, we get a 4-chain $\bar{F}(X,-Y)$ of UM with boundary :

$$\partial\bar{F}(X,-Y)=(-Y)(M)-X(M)-\mathcal{H}^{-Y}_{X,\sigma}([0,1]\times\partial M)+UM_{|L_{X=Y}}.$$

□

Proposition I.1.9. *Let X be a combing of a compact oriented 3-manifold M and let $P:H^2(M,\partial M;\mathbb{Z})\to H_1(M;\mathbb{Z})$ be the Poincaré duality isomorphism. If M is closed, then, in $H_3(UM;\mathbb{Z})$,*

$$[P(e_2(X^\perp))\times\mathbb{S}^2]=[X(M)-(-X)(M)],$$

where $[P(.)\times S^2]$ abusively denotes the homology class of the preimage of a representative of $P(.)$ under the bundle projection $UM\to M$. In general, if σ is a section of $X^\perp$ such that $L_{\sigma=0}\cap\partial M=\emptyset$ then, in $H_3(UM;\mathbb{Z})$,

$$[P(e_2^M(X^\perp,\sigma_{|\partial M}))\times\mathbb{S}^2]=[X(M)-(-X)(M)+\mathcal{H}^{-X}_{X,\sigma}([0,1]\times\partial M)].$$

Proof. Recall that $P(e_2^M(X^\perp,\sigma_{|\partial M}))=[L_{\sigma=0}]$. Perturbing X by using σ, construct a section Y homotopic to X that coincides with X on ∂M and such that $[L_{X=Y}]=P(e_2^M(X^\perp,\sigma_{|\partial M}))$. Using Proposition I.1.8,

$$[L_{X=Y}\times\mathbb{S}^2]=[X(M)-(-Y)(M)+\mathcal{H}^{-Y}_{X,\sigma_{|\partial M}}([0,1]\times\partial M)],$$

so that

$$[P(e_2^M(X^\perp,\sigma_{|\partial M}))\times\mathbb{S}^2]=[X(M)-(-X)(M)+\mathcal{H}^{-X}_{X,\sigma}([0,1]\times\partial M)].$$

□

Proposition I.1.10. *If (X,σ) and (Y,σ) are ∂-compatible combings of a compact oriented 3-manifold M, then, in $H_1(M;\mathbb{Z})$,*

$$\begin{aligned} 2\cdot[L_{X=-Y}] &= P(e_2^M(X^\perp,\sigma)) - P(e_2^M(Y^\perp,\sigma)),\\ 2\cdot[L_{X=Y}] &= P(e_2^M(X^\perp,\sigma)) + P(e_2^M(Y^\perp,\sigma)). \end{aligned}$$

Proof. Extend σ as a section $\bar\sigma$ of $X^\perp$. Using Propositions I.1.5, I.1.8 and I.1.9, we get, in $H_3(UM;\mathbb{Z})$,

$$\begin{aligned} 2\ \cdot\ [L_{X=-Y}\times\mathbb{S}^2] &= [L_{X=-Y}\times\mathbb{S}^2] - [L_{-X=Y}\times\mathbb{S}^2]\\ &= [X(M)-Y(M)] - [(-X)(M)-(-Y)(M)]\\ &= [X(M)-Y(M)-(-X)(M)+(-Y)(M)\\ &\quad + \mathcal{H}^{-X}_{X,\bar\sigma}([0,1]\times\partial M) - \mathcal{H}^{-X}_{X,\bar\sigma}([0,1]\times\partial M)]\\ &= [X(M)-(-X)(M)+\mathcal{H}^{-X}_{X,\bar\sigma}([0,1]\times\partial M)]\\ &- [Y(M)-(-Y)(M)+\mathcal{H}^{-X}_{X,\bar\sigma}([0,1]\times\partial M)]\\ &= [P(e_2^M(X^\perp,\sigma))\times\mathbb{S}^2] - [P(e_2^M(Y^\perp,\sigma))\times\mathbb{S}^2], \end{aligned}$$

$$\begin{aligned} 2\cdot[L_{X=Y}\times\mathbb{S}^2] &= [L_{X=Y}\times\mathbb{S}^2] - [L_{-X=-Y}\times\mathbb{S}^2]\\ &= [X(M)-(-Y)(M)+\mathcal{H}^{-Y}_{X,\bar\sigma_{|\partial M}}([0,1]\times\partial M)]\\ &- [(-X)(M)-Y(M)+\mathcal{H}^{Y}_{-X,\bar\sigma_{|\partial M}}([0,1]\times\partial M)]\\ &= [X(M)-(-Y)(M)+\mathcal{H}^{-X}_{X,\sigma}([0,1]\times\partial M)]\\ &- [(-X)(M)-Y(M)+\mathcal{H}^{Y}_{-Y,\sigma}([0,1]\times\partial M)]\\ &= [X(M)-(-X)(M)+\mathcal{H}^{-X}_{X,\sigma}([0,1]\times\partial M)]\\ &- [(-Y)(M)-Y(M)+\mathcal{H}^{Y}_{-Y,\sigma}([0,1]\times\partial M)]\\ &= [P(e_2^M(X^\perp,\sigma))\times\mathbb{S}^2] - [P(e_2^M((-Y)^\perp,\sigma))\times\mathbb{S}^2]\\ &= [P(e_2^M(X^\perp,\sigma))\times\mathbb{S}^2] + [P(e_2^M(Y^\perp,\sigma))\times\mathbb{S}^2]. \end{aligned}$$

□

Remark I.1.11. If M is a compact oriented 3-manifold and if σ is a trivialization of $TM_{|\partial M}$, then the set $\mathrm{Spin}^c(M,\sigma)$ is a $H^2(M,\partial M;\mathbb{Z})$-affine space and the map

$$c:\begin{cases}\mathrm{Spin}^c(M,\sigma)\longrightarrow H^2(M,\partial M;\mathbb{Z})\\ \qquad [X]^c\longmapsto e_2^M(X^\perp,\sigma)\end{cases}$$

is affine over the doubling map. Moreover, $[X]^c-[Y]^c\in H^2(M,\partial M;\mathbb{Z})\simeq H_1(M;\mathbb{Z})$ is represented by $L_{X=-Y}$, hence $2\cdot[L_{X=-Y}] = P(e_2^M(X^\perp,\sigma)) - P(e_2^M(Y^\perp,\sigma))$. See [DM05, Section 1.3.4] for a detailed presentation using this point of view. Both Proposition I.1.10 and Corollary I.1.12 below are already-known results. For instance, Corollary I.1.12 is also present in [Les15b] (Lemma 2.16).

Corollary I.1.12. *If X and Y are transverse combings of a closed oriented 3-manifold M, then, in $H_1(M;\mathbb{Z})$,*

$$2\cdot[L_{X=-Y}] = P(e_2(X^\perp)) - P(e_2(Y^\perp)),$$
$$2\cdot[L_{X=Y}] = P(e_2(X^\perp)) + P(e_2(Y^\perp)).$$

1.3 Pseudo-parallelizations

A *pseudo-parallelization* $\bar{\tau}=(N(\gamma);\tau_e,\tau_d)$ of a compact oriented 3-manifold M is a triple where

- γ is a link in $\mathring{M}$,
- $N(\gamma)$ is a tubular neighborhood of γ given with a product struture :

$$N(\gamma)\simeq[a,b]\times\gamma\times[-1,1],$$

- τ_e is a genuine parallelization of $M\setminus\mathring{N(\gamma)}$,
- τ_d is a genuine parallelization of $N(\gamma)$ such that

$$\tau_d=\begin{cases}\tau_e & \text{on } \partial([a,b]\times\gamma\times[-1,1])\setminus\{b\}\times\gamma\times[-1,1]\\ \tau_e\circ\mathcal{T}_\gamma & \text{on } \{b\}\times\gamma\times[-1,1]\end{cases}$$

where, using a smooth increasing map $\theta:[-1,1]\to[-\pi,\pi]$ constant equal to π on the interval $[-1,-1+\varepsilon]$, and such that $\theta(-x)=-\theta(x)$, where $\varepsilon\in]0,1/2[$, and the rotations $R_{e_1,\pi+\theta(u)}$ of axis e_1 and angle $\pi+\theta(u)$,

$$\mathcal{T}_\gamma:\begin{cases}([a,b]\times\gamma\times[-1,1])\times\mathbb{R}^3\longrightarrow([a,b]\times\gamma\times[-1,1])\times\mathbb{R}^3\\ ((t,c,u),v)\longmapsto((t,c,u),R_{e_1,\pi+\theta(u)}(v)).\end{cases}$$

Note that a pseudo-parallelization whose link is empty is a parallelization.

Lemma I.1.13. *If M is a compact oriented 3-manifold with boundary and if ρ is a trivialization of $TM_{|\partial M}$, there exists a pseudo-parallelization $\bar{\tau}$ of M that coincides with ρ on ∂M.*

Proof. The obstruction to extending the trivialization ρ as a parallelization of M can be represented by an element $[\gamma]\in H_1(M;\pi_1(SO(3)))$ where γ is a link in M. It follows that ρ can be extended on $M\setminus\mathring{N(\gamma)}$ where $N(\gamma)$ is a tubular neighborhood of γ. Finally, according to [Les10, Lemma 10.2], it is possible to extend ρ as a pseudo-parallelization on each torus of $N(\gamma)$. □

Unlike the case of a parallelized rational homology 3-sphere, in the case of a rational homology 3-sphere equipped with a pseudo-parallelization, an $\mathrm{LP}_{\mathbb{Q}}$-surgery can be seen as a local move thanks to Lemma I.1.13 above.

Before we move on to the definition of pseudo-sections *ie* the counterpart of sections of parallelizations for pseudo-parallelizations, we need the following.

Definition I.1.14. Let $\bar{\tau} = (N(\gamma); \tau_e, \tau_d)$ be a pseudo-parallelization of a compact oriented 3-manifold. An *additional inner parallelization* is a map τ_g such that

$$\tau_g : \begin{cases} [a,b] \times \gamma \times [-1,1] \times \mathbb{R}^3 \longrightarrow TN(\gamma) \\ \qquad\qquad ((t,c,u),v) \longmapsto \tau_d\left((t,c,u), P_{\mathbb{R}^3} \circ \mathcal{T}_\gamma^{-1}((t,c,u), \mathcal{F}(t,u)(v))\right) \end{cases}$$

where $P_{\mathbb{R}^3} : [a,b] \times \gamma \times [-1,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ is the standard projection, and where, choosing $\varepsilon \in \,]0, 1/2[$, $\mathcal{F}$ is a map such that

$$\mathcal{F} : \begin{cases} [a,b] \times [-1,1] \longrightarrow SO(3) \\ \qquad\qquad (t,u) \longmapsto \begin{cases} \mathrm{Id}_{SO(3)} & \text{for } |u| > 1-\varepsilon \\ R_{e_1,\pi+\theta(u)} & \text{for } t < a+\varepsilon \\ R_{e_1,-\pi-\theta(u)} & \text{for } t > b-\varepsilon \end{cases} \end{cases}$$

which exists since $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$ and which is well-defined up to homotopy since $\pi_2(SO(3)) = 0$.

From now on, we will always consider pseudo-parallelizations together with an additional inner parallelization. Finally, note that if $\bar{\tau} = (N(\gamma); \tau_e, \tau_d, \tau_g)$ is a pseudo-parallelization of a compact oriented 3-manifold together with an additional inner parallelization, then :

- the parallelizations τ_e, τ_d and τ_g agree on $\partial N(\gamma) \setminus \{b\} \times \gamma \times [-1,1]$,
- $\tau_g = \tau_e \circ \mathcal{T}_\gamma^{-1}$ on $\{b\} \times \gamma \times [-1,1]$.

Definition I.1.15. A *pseudo-section* of a pseudo-parallelization of a compact oriented 3-manifold M together with an additional inner parallelization, $\bar{\tau} = (N(\gamma); \tau_e, \tau_d, \tau_g)$, is a 3-chain of UM of the following form :

$$\begin{aligned} \bar{\tau}(M \times \{v\}) &= \tau_e((M \setminus \mathring{N}(\gamma)) \times \{v\}) \\ &\quad + \frac{\tau_d(N(\gamma) \times \{v\}) + \tau_g(N(\gamma) \times \{v\}) + \tau_e(\{b\} \times \gamma \times C_2(v))}{2} \end{aligned}$$

where $v \in \mathbb{S}^2$ and $C_2(v)$ is the 2-chain of $[-1,1] \times \mathbb{S}^1(v)$ of Figure I.1.16 (here $\mathbb{S}^1(w)$ stands for the circle of $\mathbb{S}^2$ that lies on the plane orthogonal to e_1 and passes through $w \in \mathbb{S}^2$). Note that :

$$\begin{aligned} \partial C_2(v) &= \{(u, R_{e_1,\pi+\theta(u)}(v)) \mid u \in [-1,1]\} \\ &\quad + \{(u, R_{e_1,-\pi-\theta(u)}(v)) \mid u \in [-1,1]\} - 2 \cdot [-1,1] \times \{v\}. \end{aligned}$$

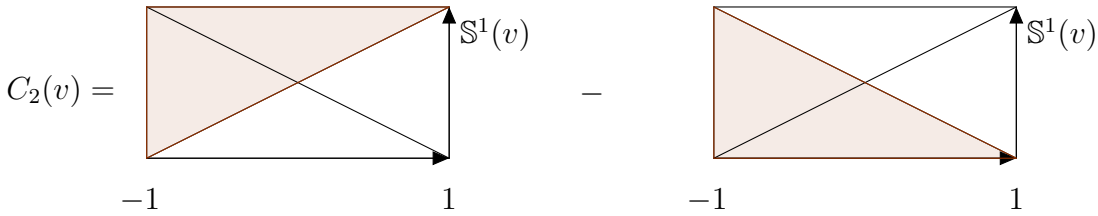


Figure I.1.16 – The 2-chain $C_2(v)$ where we cut the annulus $[-1,1] \times \mathbb{S}^1(v)$ along $[-1,1] \times \{v\}$.

Definition I.1.17. If $\bar{\tau} = (N(\gamma); \tau_e, \tau_d, \tau_g)$ is a pseudo-parallelization of a compact oriented 3-manifold M, let the *Siamese sections* of $\bar{\tau}$ denote the following sections of UM :

$$E_1^d : \begin{cases} m \in M \setminus \mathring{N}(\gamma) \longmapsto \tau_e(m, e_1) \\ m \in N(\gamma) \longmapsto \tau_d(m, e_1) \end{cases} \quad \text{and} \quad E_1^g : \begin{cases} m \in M \setminus \mathring{N}(\gamma) \longmapsto \tau_e(m, e_1) \\ m \in N(\gamma) \longmapsto \tau_g(m, e_1). \end{cases}$$

As already mentioned in the introduction, note that when $\bar{\tau} = (N(\gamma); \tau_e, \tau_d, \tau_g)$ is a pseudo-parallelization of a compact oriented 3-manifold M, its pseudo-section at e_1 reads

$$\bar{\tau}(M \times \{e_1\}) = \frac{E_1^d(M) + E_1^g(M)}{2}$$

where E_1^d and E_1^g are the Siamese sections of $\bar{\tau}$.

2 From parallelizations to pseudo-parallelizations

2.1 Pontrjagin numbers of parallelizations

In this subsection we review the definition of first relative Pontrjagin numbers for parallelizations of compact connected oriented 3-manifolds. For a detailed presentation of these objects we refer to [Les13, Section 5] and [Les15b, Subsection 4.1].

Let C_1 and C_2 be compact connected oriented 3-manifolds with identified boundaries. Recall that a *cobordism from C_1 to C_2* is a compact oriented 4-manifold W whose boundary reads

$$\partial W = -C_1 \bigcup_{\partial C_1 \simeq \{0\} \times C_1} -[0,1] \times \partial C_1 \bigcup_{\partial C_2 \simeq \{1\} \times C_1} C_2.$$

Moreover, we require W to be identified with $[0,1[\times C_1$ or $]0,1]\times C_2$ on collars of ∂W.

Recall that any compact oriented 3-manifold bounds a compact oriented 4-manifold, so that a cobordism from C_1 to C_2 always exists. Also recall that the signature of a 4-manifold is the signature of the intersection form on its second homology group with real coefficients and that any 4-manifold can be turned into a 4-manifold with signature zero by performing connected sums with copies of $\pm\mathbb{C}P^2$. So let us fix a connected cobordism W from C_1 to C_2 with signature zero.

Now consider a parallelization τ_1, resp. τ_2, of C_1, resp. C_2. Define the vector field $\vec{n}$ on a collar of ∂W as follows. Let $\vec{n}$ be the unit tangent vector to $[0,1]\times\{x\}$ where $x\in C_1$ or C_2. Define $\tau(\tau_1,\tau_2)$ as the trivialization of $TW\otimes\mathbb{C}$ over ∂W obtained by stabilizing τ_1 or τ_2 into $\vec{n}\oplus\tau_1$ or $\vec{n}\oplus\tau_2$ and tensoring with $\mathbb{C}$. In general, this trivialization does not extend as a parallelization of W. This leads to a Pontrjagin obstruction class $p_1(W;\tau(\tau_1,\tau_2))$ in $H^4(W,\partial W,\pi_3(SU(4)))$. Since $\pi_3(SU(4))\simeq\mathbb{Z}$, there exists $p_1(\tau_1,\tau_2)\in\mathbb{Z}$ such that $p_1(W;\tau(\tau_1,\tau_2))=p_1(\tau_1,\tau_2)[W,\partial W]$. Let us call $p_1(\tau_1,\tau_2)$ the *first relative Pontrjagin number of* τ_1 *and* τ_2.

Similarly, define the *Pontrjagin number* $p_1(\tau)$ *of a parallelization* τ of a closed connected oriented 3-manifold M, by taking a connected oriented 4-manifold W with boundary M, a collar of ∂W identified with $]0,1]\times M$ and $\vec{n}$ as the outward normal vector field over ∂W.

We have not actually defined the sign of the Pontrjagin numbers. We will not give details here on how to define it, instead we refer to [MS74, §15] or [Les13, p.44]. Let us only mention that p_1 is the opposite of the second Chern class c_2 of the complexified tangent bundle.

2.2 Pontrjagin numbers for pseudo-parallelizations

Definition I.2.1. Let $\bar{\tau}=(N(\gamma);\tau_e,\tau_d,\tau_g)$ be a pseudo-parallelization of a compact oriented 3-manifold M, a *complex trivialization* $\bar{\tau}_{\mathbb{C}}$ associated to $\bar{\tau}$ is a trivialization of $TM\otimes\mathbb{C}$ such that :

- $\bar{\tau}_{\mathbb{C}}$ is *special* (*ie* its determinant is one everywhere) with respect to the trivialization of the determinant bundle induced by the orientation of M,
- on $M\setminus\mathring{N}(\gamma)$, $\bar{\tau}_{\mathbb{C}}=\tau_e\otimes 1_{\mathbb{C}}$,
- for $m=(t,c,u)\in[a,b]\times\gamma\times[-1,1]$, $\bar{\tau}_{\mathbb{C}}(m,.)=(m,\tau_d(t,c,u)(\mathcal{G}(t,u)(.)))$,

where $\mathcal{G}$ is a map so that :

$$\mathcal{G} : \begin{cases} [a,b] \times [-1,1] \longrightarrow SU(3) \\ \qquad\qquad (t,u) \longmapsto \begin{cases} \mathrm{Id}_{SU(3)} & \text{for } |u| > 1-\varepsilon \\ \mathrm{Id}_{SU(3)} & \text{for } t < a+\varepsilon \\ R_{e_1,-\pi-\theta(u)} & \text{for } t > b-\varepsilon. \end{cases} \end{cases}$$

Note that such a smooth map $\mathcal{G}$ on $[a,b] \times [-1,1]$ exists since $\pi_1(SU(3)) = \{1\}$. Moreover, $\mathcal{G}$ is well-defined up to homotopy since $\pi_2(SU(3)) = \{0\}$.

Pseudo-parallelizations, or pseudo-trivializations, have been first used in [Les04a, Section 4.3], but they have been first defined in [Les10, Section 10]. Note that our conventions are slightly different.

Definition I.2.2. Let $\bar{\tau}_1$ and $\bar{\tau}_2$ be pseudo-parallelizations of two compact oriented 3-manifolds M_1 and M_2 with identified boundaries and let W be a cobordism from M_1 to M_2 with signature zero. As in the case of genuine parallelizations, define a trivialization $\tau(\bar{\tau}_1, \bar{\tau}_2)$ of $TW \otimes \mathbb{C}$ over ∂W using the special complex trivializations $\bar{\tau}_{1,\mathbb{C}}$ and $\bar{\tau}_{2,\mathbb{C}}$ associated to $\bar{\tau}_1$ and $\bar{\tau}_2$, respectively. The *first relative Pontrjagin number of* $\bar{\tau}_1$ *and* $\bar{\tau}_2$ is the Pontrjagin obstruction $p_1(\bar{\tau}_1, \bar{\tau}_2)$ to extending the trivialization $\tau(\bar{\tau}_1, \bar{\tau}_2)$ as a trivialization of $TW \otimes \mathbb{C}$ over W.

Finally, if $\bar{\tau}$ is a pseudo-parallelization of a closed oriented 3-manifold M then define the *Pontrjagin number* $p_1(\bar{\tau})$ *of the pseudo-parallelization* $\bar{\tau}$ by using a cobordism from the empty set to M.

2.3 Variation of p_1 as an intersection of three 4-chains

In this subsection, we give a proof of Proposition I.9, which expresses the relative Pontrjagin numbers (resp. the variation of Pontrjagin numbers) of pseudo-parallelizations in compact (resp. closed) oriented 3-manifolds as an algebraic intersection of three 4-chains.

Lemma I.2.3. *If* $\bar{\tau} = (N(\gamma); \tau_e, \tau_d, \tau_g)$ *is a pseudo-parallelization of a compact oriented 3-manifold* M*, if* E_1^d *and* E_1^g *are its Siamese sections and if* E_2^e *denotes the second vector of* τ_e*, then* $P(e_2^M(E_1^{d\perp}, E^e_{2|\partial M})) = -P(e_2^M(E_1^{g\perp}, E^e_{2|\partial M})) = [\gamma]$ *in* $H_1(M;\mathbb{Z})$.

Proof. Since E_1^d and E_1^e coincide on $M \setminus \mathring{N}(\gamma)$, the obstruction to extending E_2^e as a section of $E_1^{d\perp}$ is the obstruction to extending E_2^e as a section of $E^{d\perp}_{1|N(\gamma)}$. However, parallelizing $N(\gamma)$ with τ_d and using that

$$\tau_d = \begin{cases} \tau_e & \text{on } \partial([a,b] \times \gamma \times [-1,1]) \setminus \{b\} \times \gamma \times [-1,1] \\ \tau_e \circ \mathcal{T}_\gamma & \text{on } \{b\} \times \gamma \times [-1,1] \end{cases}$$

we get that E_2^e induces a degree $+1$ map $E^e_{2|\alpha} : \alpha \to \mathbb{S}^1$ on any meridian α of $N(\gamma)$. It follows that

$$P(e_2^M(E_1^{d\perp}, E^e_{2|\partial M})) = +[\gamma].$$

Similarly, parallelizing $N(\gamma)$ with τ_g, E_2^e induces a degree -1 map $E^e_{2|\alpha} : \alpha \to \mathbb{S}^1$ on any meridian α of $N(\gamma)$, so that

$$P(e_2^M(E_1^{g\perp}, E^e_{2|\partial M})) = -[\gamma].$$

□

Recall that for a combing X of a compact oriented 3-manifold M and a pseudo-parallelization $\bar{\tau}$ of M, if X and $\bar{\tau}$ are compatible, then

$$L_{\bar{\tau}=X} = \frac{L_{E_1^d=X} + L_{E_1^g=X}}{2} \quad \text{and} \quad L_{\bar{\tau}=-X} = \frac{L_{E_1^d=-X} + L_{E_1^g=-X}}{2}$$

where E_1^d and E_1^g denote the Siamese sections of $\bar{\tau}$.

Lemma I.2.4. *Let $\bar{\tau} = (N(\gamma); \tau_e, \tau_d, \tau_g)$ be a pseudo-parallelization of a compact oriented 3-manifold M. If X is a torsion combing of M compatible with $\bar{\tau}$, then $L_{\bar{\tau}=X}$ and $L_{\bar{\tau}=-X}$ are rationally null-homologous in M.*

Proof. Let E_1^d and E_1^g be the Siamese sections of $\bar{\tau}$. Using Proposition I.1.10 and the fact that X is a torsion combing, we get, in $H_1(M;\mathbb{Q})$,

$$2 \cdot [L_{X=-E_1^d} + L_{X=-E_1^g}] = [-P(e_2^M(E_1^{d\perp}, E^e_{2|\partial M})) - P(e_2^M(E_1^{g\perp}, E^e_{2|\partial M}))]$$
$$2 \cdot [L_{X=E_1^d} + L_{X=E_1^g}] = [P(e_2^M(E_1^{d\perp}, E^e_{2|\partial M})) + P(e_2^M(E_1^{g\perp}, E^e_{2|\partial M}))]$$

where E_2^e is the second vector of τ_e. Conclude with Lemma I.2.3. □

Definition I.2.5. Let X and Y be ∂-compatible combings of a compact oriented 3-manifold M. For all $t \in [0,1]$, let $\bar{F}_t(X,Y)$ denote the 4-chain of $[0,1] \times UM$:

$$\bar{F}_t(X,Y) = [0,t] \times X(M) + \{t\} \times \bar{F}(X,Y) + [t,1] \times Y(M),$$

where $\bar{F}(X,Y)$ is a 4-chain of UM as in Lemma I.1.7. Note that :

$$\partial \bar{F}_t(X,Y) = \{1\} \times Y(M) - \{0\} \times X(M) - [0,1] \times X(\partial M) + \{t\} \times UM_{L_{X=-Y}}.$$

Lemma I.2.6. *Let $\bar{\tau} = (N(\gamma); \tau_e, \tau_d, \tau_g)$ be a pseudo-parallelization of a compact oriented 3-manifold M. If X is a torsion combing of M compatible with $\bar{\tau}$, then there exist 4-chains of $[0,1] \times UM$, $C_4^{\pm}(\bar{\tau}, X)$ and $C_4^{\pm}(X, \bar{\tau})$, with boundaries :*

$$\partial C_4^{\pm}(\bar{\tau}, X) = \{1\} \times (\pm X)(M) - \{0\} \times \bar{\tau}(M \times \{\pm e_1\}) - [0,1] \times (\pm X)(\partial M)$$
$$\partial C_4^{\pm}(X, \bar{\tau}) = \{1\} \times \bar{\tau}(M \times \{\pm e_1\}) - \{0\} \times (\pm X)(M) - [0,1] \times (\pm X)(\partial M)$$

Proof. Let E_1^d and E_1^g be the Siamese sections of $\bar{\tau}$ and just set

$$C_4^{\pm}(\bar{\tau}, X) = 1/2 \cdot \left(\bar{F}_t(\pm E_1^d, \pm X) + \bar{F}_t(\pm E_1^g, \pm X) - \{t\} \times UM_{|\Sigma(\pm e_1)}\right)$$
$$C_4^{\pm}(X, \bar{\tau}) = 1/2 \cdot \left(\bar{F}_t(\pm X, \pm E_1^d) + \bar{F}_t(\pm X, \pm E_1^g) - \{t\} \times UM_{|-\Sigma(\pm e_1)}\right)$$

where the 4-chains $\bar{F}_t$ are as in Definition I.2.5 and where $\Sigma(\pm e_1)$ are rational 2-chains of M bounded by $\pm(L_{E_1^d=-X} + L_{E_1^g=-X})$ which are rationally null-homologous according to Lemma I.2.4. □

Lemma I.2.7. *Let τ and $\bar{\tau}$ be two pseudo-parallelizations of a compact oriented 3-manifold M that coincide on ∂M and whose links are disjoint. For all $v \in \mathbb{S}^2$, there exists a 4-chain $C_4(M, \tau, \bar{\tau}; v)$ of $[0,1] \times UM$ such that*

$$\partial C_4(M, \tau, \bar{\tau}; v) = \{1\} \times \bar{\tau}(M \times \{v\}) - \{0\} \times \tau(M \times \{v\}) - [0,1] \times \tau(\partial M \times \{v\}).$$

Proof. Let us write $C_4(\tau, \bar{\tau}; v)$ instead of $C_4(M, \tau, \bar{\tau}; v)$ when there is no ambiguity. Since the 3-chains $\partial C_4(\tau, \bar{\tau}; v)$, where $v \in \mathbb{S}^2$, are homologous, it is enough to prove the existence of $C_4(\tau, \bar{\tau}; e_1)$. First, let X be a combing of M such that X is compatible with τ and $\bar{\tau}$. In general, this combing is not a torsion combing. Second, let E_1^d and E_1^g, resp. $\bar{E}_1^d$ and $\bar{E}_1^g$, denote the Siamese section of τ, resp. $\bar{\tau}$, and set

$$\bar{F}(\tau, X) = 1/2 \cdot \left(\bar{F}(E_1^d, X) + \bar{F}(E_1^g, X)\right)$$
$$\bar{F}(X, \bar{\tau}) = 1/2 \cdot \left(\bar{F}(X, \bar{E}_1^d) + \bar{F}(X, \bar{E}_1^g)\right).$$

These chains have boundaries :

$$\partial \bar{F}(\tau, X) = X(M) - \tau(M \times \{e_1\}) + UM_{|L_{\tau=-X}}$$
$$\partial \bar{F}(X, \bar{\tau}) = \bar{\tau}(M \times \{e_1\}) - X(M) + UM_{|-L_{\bar{\tau}=-X}}.$$

Hence, for all $t \in [0,1]$, the 4-chain of $[0,1] \times UM$

$$\bar{F}_t(\tau, \bar{\tau}; e_1) = [0,t] \times \tau(M \times \{e_1\}) + \{t\} \times \left(\bar{F}(\tau, X) + \bar{F}(X, \bar{\tau})\right) + [t,1] \times \bar{\tau}(M \times \{e_1\})$$

has boundary :

$$\begin{aligned}\partial \bar{F}_t(\tau, \bar{\tau}; e_1) = {} & \{1\} \times \bar{\tau}(M \times \{e_1\}) - \{0\} \times \tau(M \times \{e_1\}) \\ & - [0,1] \times \tau(\partial M \times \{e_1\}) + \{t\} \times UM_{|L_{\tau=-X} \cup -L_{\bar{\tau}=-X}}.\end{aligned}$$

Thanks to Proposition I.1.10 and Lemma I.2.3, in $H_1(M; \mathbb{Q})$:

$$\begin{aligned}4 \cdot [L_{\tau=-X}] &= 2 \cdot [L_{E_1^d=-X} + L_{E_1^g=-X}] \\ &= [-2 \cdot P(e_2^M(X^{\perp}, E^e_{2|\partial M})) + P(e_2^M(E_1^{d\perp}, E^e_{2|\partial M})) + P(e_2^M(E_1^{g\perp}, E^e_{2|\partial M}))] \\ &= -2 \cdot [P(e_2^M(X^{\perp}, E^e_{2|\partial M}))]\end{aligned}$$

where E_2^e is the second vector of τ_e. Similarly, $2 \cdot [L_{\bar{\tau}=-X}] = -[P(e_2^M(X^\perp, E^e_{2|\partial M}))]$ in $H_1(M;\mathbb{Q})$. So, the link $L_{\tau=-X} \cup -L_{\bar{\tau}=-X}$ is rationally null-homologous in M, *ie* there exists a rational 2-chain $\Sigma(\tau,\bar{\tau})$ such that $\partial\Sigma(\tau,\bar{\tau}) = L_{\tau=-X} \cup -L_{\bar{\tau}=-X}$. Hence, we get a 4-chain $C_4(\tau,\bar{\tau};e_1)$ as desired by setting

$$C_4(\tau,\bar{\tau};e_1) = \bar{F}_t(\tau,\bar{\tau};e_1) - \{t\} \times UM_{|\Sigma(\tau,\bar{\tau})}.$$

□

Lemma I.2.8. *Let τ and $\bar{\tau}$ be two pseudo-parallelizations of a compact oriented 3-manifold M that coincide on ∂M. If x, y and z are points in $\mathbb{S}^2$ with pairwise different distances to e_1, then there exist pairwise transverse 4-chains $C_4(\tau,\bar{\tau};x)$, $C_4(\tau,\bar{\tau};y)$ and $C_4(\tau,\bar{\tau};z)$ as in Lemma I.2.7 and the algebraic intersection $\langle C_4(\tau,\bar{\tau};x), C_4(\tau,\bar{\tau};y), C_4(\tau,\bar{\tau};z)\rangle_{[0,1]\times UM}$ only depends on τ and $\bar{\tau}$.*

Proof. Pick any x, y and z in $\mathbb{S}^2$ with pairwise different distances to e_1 and consider some 4-chains $C_4(\tau,\bar{\tau};x)$, $C_4(\tau,\bar{\tau};y)$ and $C_4(\tau,\bar{\tau};z)$ such that, for $v \in \{x,y,z\}$,

$$\partial C_4(\tau,\bar{\tau};v) = \{1\} \times \bar{\tau}(M \times \{v\}) - \{0\} \times \tau(M \times \{v\}) - [0,1] \times \tau(\partial M \times \{v\}).$$

The triple intersection of $C_4(\tau,\bar{\tau};x)$, $C_4(\tau,\bar{\tau};y)$ and $C_4(\tau,\bar{\tau};z)$ is in the interior of $[0,1] \times UM$. The algebraic triple intersection of these three 4-chains only depends on the fixed boundaries and on the homology classes of the 4-chains. The space $H_4([0,1] \times UM;\mathbb{Q})$ is generated by the classes of 4-chains $\Sigma \times \mathbb{S}^2$ where Σ is a surface in M. If $\Sigma \times S^2$ is such a 4-chain, then

$$\begin{aligned}
&\langle C_4(\tau,\bar{\tau};x) + \Sigma \times \mathbb{S}^2, C_4(\tau,\bar{\tau};y), C_4(\tau,\bar{\tau};z)\rangle - \langle C_4(\tau,\bar{\tau};x), C_4(\tau,\bar{\tau};y), C_4(\tau,\bar{\tau};z)\rangle \\
&= \langle \Sigma \times \mathbb{S}^2, C_4(\tau,\bar{\tau};y), C_4(\tau,\bar{\tau};z)\rangle \\
&= \begin{cases} \langle \Sigma \times \mathbb{S}^2, [0,1] \times \tau(M \times \{y\}), [0,1] \times \tau(M \times \{z\})\rangle \text{ , pushing } \Sigma \times \mathbb{S}^2 \text{ near } 0, \\ \langle \Sigma \times \mathbb{S}^2, [0,1] \times \bar{\tau}(M \times \{y\}), [0,1] \times \bar{\tau}(M \times \{z\})\rangle \text{ , pushing } \Sigma \times \mathbb{S}^2 \text{ near } 1. \end{cases}
\end{aligned}$$

It follows that $\langle \Sigma \times \mathbb{S}^2, C_4(\tau,\bar{\tau};y), C_4(\tau,\bar{\tau};z)\rangle$ is independent of τ and $\bar{\tau}$. So, use Lemma I.1.13 to extend a trivialization of $TM_{|\Sigma}$ as a pseudo-parallelization τ' that coincides with τ and $\bar{\tau}$ on ∂M. Considering this pseudo-parallelization we get

$$\langle \Sigma \times \mathbb{S}^2, C_4(\tau,\bar{\tau};y), C_4(\tau,\bar{\tau};z)\rangle = \langle \Sigma \times \mathbb{S}^2, [0,1] \times \tau'(M \times \{y\}), [0,1] \times \tau'(M \times \{z\})\rangle = 0,$$

so that the algebraic triple intersection of the three chains $C_4(\tau,\bar{\tau};x)$, $C_4(\tau,\bar{\tau};y)$ and $C_4(\tau,\bar{\tau};z)$ only depends on their fixed boundaries. □

Proof of Proposition I.9. Let τ and $\bar{\tau}$ be two pseudo-parallelizations of a compact oriented 3-manifold M that coincide on ∂M and whose links are disjoint. To

conclude the proof of Proposition I.9, we have to prove that for any x, y and z in $\mathbb{S}^2$ with pairwise different distances to e_1 :

$$p_1(\tau,\bar{\tau}) = 4\cdot\langle C_4(M,\tau,\bar{\tau};x), C_4(M,\tau,\bar{\tau};y), C_4(M,\tau,\bar{\tau};z)\rangle_{[0,1]\times UM}.$$

First, we know from [Les10, Lemma 10.9] that this is true if M is a $\mathbb{Q}$HH of genus 1. Notice that it is also true if M embeds in such a manifold. Indeed, if $\mathbb{H}$ is a $\mathbb{Q}$HH of genus 1 and if M embeds in $\mathbb{H}$ then, using Lemma I.1.13 and using that τ and $\bar{\tau}$ coincide on ∂M, there exists a pseudo-parallelization $\check{\tau}$ of $\mathbb{H}\setminus\mathring{M}$ such that

$$\bar{\tau}_{\mathbb{H}} : \begin{cases} m\in M \mapsto \bar{\tau}(m,.) \\ m\in \mathbb{H}\setminus\mathring{M} \mapsto \check{\tau}(m,.) \end{cases} \quad\text{and}\quad \tau_{\mathbb{H}} : \begin{cases} m\in M \mapsto \tau(m,.) \\ m\in \mathbb{H}\setminus\mathring{M} \mapsto \check{\tau}(m,.) \end{cases}$$

are pseudo-parallelizations of $\mathbb{H}$. Furthermore, for any $v\in\mathbb{S}^2$, let $C_4(\mathbb{H},v)$ be the 4-chain of $[0,1]\times U\mathbb{H}$:

$$C_4(\mathbb{H},\tau_{\mathbb{H}},\bar{\tau}_{\mathbb{H}};v) = C_4(M,\tau,\bar{\tau};v)\cup[0,1]\times\check{\tau}((\mathbb{H}\setminus\mathring{M})\times\{v\})$$

where $C_4(M,\tau,\bar{\tau};v)$ is as in Lemma I.2.7. The boundary of $C_4(\mathbb{H},\tau_{\mathbb{H}},\bar{\tau}_{\mathbb{H}};v)$ is :

$$\partial C_4(\mathbb{H},\tau_{\mathbb{H}},\bar{\tau}_{\mathbb{H}};v) = \{1\}\times\bar{\tau}_{\mathbb{H}}(M\times\{v\})-\{0\}\times\tau_{\mathbb{H}}(M\times\{v\})-[0,1]\times\tau_{\mathbb{H}}(\partial M\times\{v\}).$$

Using the definition of Pontrjagin numbers of pseudo-parallelizations and the hypothesis on $\mathbb{H}$, it follows that if x,y and z are points in $\mathbb{S}^2$ with pairwise different distances to e_1 :

$$p_1(\tau,\bar{\tau}) = p_1(\tau_{\mathbb{H}},\bar{\tau}_{\mathbb{H}}) = 4\cdot\langle C_4(\mathbb{H},\tau_{\mathbb{H}},\bar{\tau}_{\mathbb{H}};x), C_4(\mathbb{H},\tau_{\mathbb{H}},\bar{\tau}_{\mathbb{H}};y), C_4(\mathbb{H},\tau_{\mathbb{H}},\bar{\tau}_{\mathbb{H}};z)\rangle.$$

Now note that :

$$\langle[0,1]\times\check{\tau}(\mathbb{H}\setminus\mathring{M}\times\{x\}),[0,1]\times\check{\tau}(\mathbb{H}\setminus\mathring{M}\times\{y\}),[0,1]\times\check{\tau}(\mathbb{H}\setminus\mathring{M}\times\{z\})\rangle = 0.$$

Indeed, if $\check{\tau} = (N(\check{\gamma});\check{\tau}_e,\check{\tau}_d,\check{\tau}_g)$, for all $v\in\mathbb{S}^2$ the pseudo-section of $\check{\tau}$ reads

$$\begin{aligned}\check{\tau}(M\times\{v\}) &= \check{\tau}_e((M\setminus\mathring{N}(\check{\gamma}))\times\{v\}) \\ &\quad + \frac{\check{\tau}_d(N(\check{\gamma})\times\{v\})+\check{\tau}_g(N(\check{\gamma})\times\{v\})+\check{\tau}_e(\{b\}\times\check{\gamma}\times C_2(v))}{2}.\end{aligned}$$

The 3-chains $\check{\tau}_e((M\setminus\mathring{N}(\check{\gamma}))\times\{v\})$, for $v\in\{x,y,z\}$, are pairwise disjoint since $\check{\tau}_e$ is a genuine parallelization and since x,y and z are pairwise distinct points in $\mathbb{S}^2$. Moreover, the 3-chains $\check{\tau}_e(\{b\}\times\check{\gamma}\times C_2(v))$, for $v\in\{x,y,z\}$, are also pairwise disjoint since they are subsets of the $\check{\tau}_e(\{b\}\times\check{\gamma}\times\mathbb{S}^1(v))$, $v\in\{x,y,z\}$, which are

pairwise disjoint since x, y and z have pairwise different distances to e_1. Finally, we have :

$$\begin{aligned}\langle \check{\tau}_d(N(\check{\gamma}) \times \{x\}) + \check{\tau}_g(N(\check{\gamma}) \times \{x\}), \check{\tau}_d(N(\check{\gamma}) \times \{y\}) + \check{\tau}_g(N(\check{\gamma}) \times \{y\}),\\ \check{\tau}_d(N(\check{\gamma}) \times \{z\}) + \check{\tau}_g(N(\check{\gamma}) \times \{z\})\rangle = 0\end{aligned}$$

since a triple intersection between the 3-chains

$$\{\check{\tau}_d(N(\check{\gamma}) \times \{v\}) + \check{\tau}_g(N(\check{\gamma}) \times \{v\})\}_{v \in \{x,y,z\}}$$

would be contained in an intersection between two of the $\{\check{\tau}_d(N(\check{\gamma}) \times \{v\})\}_{v \in \{x,y,z\}}$ or between two of the $\{\check{\tau}_g(N(\check{\gamma}) \times \{v\})\}_{v \in \{x,y,z\}}$ which must be empty since $\check{\tau}_d$ and $\check{\tau}_g$ are genuine parallelizations. It follows that

$$\begin{aligned}p_1(\tau, \bar{\tau}) &= 4 \cdot \langle C_4(\mathbb{H}, \tau_{\mathbb{H}}, \bar{\tau}_{\mathbb{H}}; x), C_4(\mathbb{H}, \tau_{\mathbb{H}}, \bar{\tau}_{\mathbb{H}}; y), C_4(\mathbb{H}, \tau_{\mathbb{H}}, \bar{\tau}_{\mathbb{H}}; z)\rangle\\ &= 4 \cdot \langle C_4(M, \tau, \bar{\tau}; x), C_4(M, \tau, \bar{\tau}; y), C_4(M, \tau, \bar{\tau}; z)\rangle.\end{aligned}$$

Using the same construction, note also that it is enough to prove the statement when M is a closed oriented 3-manifold since any oriented 3-manifold embeds into a closed one.

Let us finally prove Proposition I.9 when M is a closed oriented 3-manifold. First, consider a Heegaard splitting $M = H_1 \cup_\Sigma H_2$ such that there is a collar $\Sigma \times [0,1] \subset H_2$ of Σ verifying

$$N(\bar{\gamma}) \cap (\Sigma \times [0,1]) = \emptyset \quad \text{and} \quad N(\gamma) \cap (\Sigma \times [0,1]) = \emptyset$$

where γ and $\bar{\gamma}$ are the links of τ and $\bar{\tau}$, respectively, and such that $\Sigma = \Sigma \times \{0\}$. Such a splitting can be obtained by considering a triangulation of M containing γ and $\bar{\gamma}$ in its 1-skeleton, and then defining H_1 as a tubular neighborhood of this 1-skeleton. Second, using Lemma I.1.13, we can construct a pseudo-parallelization τ^c of $\Sigma \times [0,1]$ such that τ^c coincides with $\bar{\tau}$ on $\Sigma \times \{1\}$ and with τ on $\Sigma \times \{0\}$. Then, write $H_1' = H_1 \cup (\Sigma \times [0,1])$ and $H_2' = H_2 \setminus (\Sigma \times [0,1[)$ – see Figure I.2.9 – and set

$$\check{\tau} : \begin{cases} (m,v) \in UH_1 \longmapsto \tau(m,v) \\ (m,v) \in U(\Sigma \times [0,1]) \longmapsto \tau^c(m,v) \\ (m,v) \in UH_2' \longmapsto \bar{\tau}(m,v). \end{cases}$$

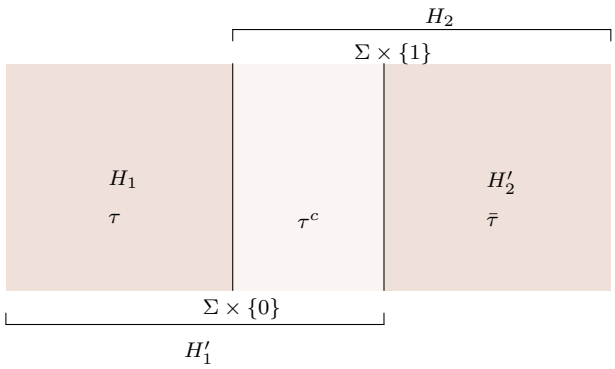


Figure I.2.9

For $v \in \mathbb{S}^2$, consider some 4-chains $C_4(H_1, \tau, \check{\tau}; v)$, $C_4(H_2, \tau, \check{\tau}; v)$, $C_4(H_1', \check{\tau}, \bar{\tau}; v)$ and $C_4(H_2', \check{\tau}, \bar{\tau}; v)$ of $[0,1] \times UH_1$, $[0,1] \times UH_2$, $[0,1] \times UH_1'$ and $[0,1] \times UH_2'$, respectively, such that :

$$\partial C_4(H_1, \tau, \check{\tau}; v) = \{1\} \times \check{\tau}(H_1 \times \{v\}) - \{0\} \times \tau(H_1 \times \{v\}) - [0,1] \times \tau(\partial H_1 \times \{v\})$$
$$\partial C_4(H_2, \tau, \check{\tau}; v) = \{1\} \times \check{\tau}(H_2 \times \{v\}) - \{0\} \times \tau(H_2 \times \{v\}) - [0,1] \times \tau(\partial H_2 \times \{v\})$$

and

$$\partial C_4(H_1', \check{\tau}, \bar{\tau}; v) = \{1\} \times \bar{\tau}(H_1' \times \{v\}) - \{0\} \times \check{\tau}(H_1' \times \{v\}) - [0,1] \times \check{\tau}(\partial H_1' \times \{v\})$$
$$\partial C_4(H_2', \check{\tau}, \bar{\tau}; v) = \{1\} \times \bar{\tau}(H_2' \times \{v\}) - \{0\} \times \check{\tau}(H_2' \times \{v\}) - [0,1] \times \check{\tau}(\partial H_2' \times \{v\})$$

Since H_1 and H_2 embed in rational homology balls, for any x, y and z in $\mathbb{S}^2$ with pairwise different distances to e_1

$$p_1(\tau_{|H_1}, \check{\tau}_{|H_1}) = 4 \cdot \langle C_4(H_1, \tau, \check{\tau}; x), C_4(H_1, \tau, \check{\tau}; y), C_4(H_1, \tau, \check{\tau}; z) \rangle_{[0,1] \times UH_1}$$
$$p_1(\tau_{|H_2}, \check{\tau}_{|H_2}) = 4 \cdot \langle C_4(H_2, \tau, \check{\tau}; x), C_4(H_2, \tau, \check{\tau}; y), C_4(H_2, \tau, \check{\tau}; z) \rangle_{[0,1] \times UH_2}$$

so that, using $C_4(M, \tau, \check{\tau}; v) = C_4(H_1, \tau, \check{\tau}; v) + C_4(H_2, \tau, \check{\tau}; v)$ for $v \in \mathbb{S}^2$,

$$p_1(\tau, \check{\tau}) = 4 \cdot \langle C_4(M, \tau, \check{\tau}; x),\ C_4(M, \tau, \check{\tau}; y),\ C_4(M, \tau, \check{\tau}; z) \rangle_{[0,1] \times UM}.$$

Similarly, since H_1' and H_2' embed in rational homology balls, for any x, y and z in $\mathbb{S}^2$ with pairwise different distances to e_1

$$p_1(\check{\tau}_{|H_1'}, \bar{\tau}_{|H_1'}) = 4 \cdot \langle C_4(H_1', \check{\tau}, \bar{\tau}; x), C_4(H_1', \check{\tau}, \bar{\tau}; y), C_4(H_1', \check{\tau}, \bar{\tau}; z) \rangle_{[0,1] \times UH_1'}$$
$$p_1(\check{\tau}_{|H_2'}, \bar{\tau}_{|H_2'}) = 4 \cdot \langle C_4(H_2', \check{\tau}, \bar{\tau}; x), C_4(H_2', \check{\tau}, \bar{\tau}; y), C_4(H_2', \check{\tau}, \bar{\tau}; z) \rangle_{[0,1] \times UH_2'}$$

so that, using $C_4(M, \check{\tau}, \bar{\tau}; v) = C_4(H_1', \check{\tau}, \bar{\tau}; v) + C_4(H_2', \check{\tau}, \bar{\tau}; v)$ for $v \in \mathbb{S}^2$,

$$p_1(\check{\tau}, \bar{\tau}) = 4 \cdot \langle C_4(M, \check{\tau}, \bar{\tau}; x), C_4(M, \check{\tau}, \bar{\tau}; y), C_4(M, \check{\tau}, \bar{\tau}; z) \rangle_{[0,1] \times UM}.$$

Eventually, reparameterizing and stacking $C_4(M,\tau,\check{\tau};v)$ and $C_4(M,\check{\tau},\bar{\tau};v)$, for all $v \in \mathbb{S}^2$ we get a 4-chain $C_4(M,\tau,\bar{\tau};v)$ of $[0,1]\times UM$ such that

$$\partial C_4(M,\tau,\bar{\tau};v) = \{1\}\times\bar{\tau}(M\times\{v\}) - \{0\}\times\tau(M\times\{v\}) - [0,1]\times\tau(\partial M\times\{v\})$$

and such that for any x, y and z in $\mathbb{S}^2$ with pairwise different distances to e_1

$$p_1(\tau,\bar{\tau}) = 4\cdot\langle C_4(M,\tau,\bar{\tau};x), C_4(M,\tau,\bar{\tau};y), C_4(M,\tau,\bar{\tau};z)\rangle_{[0,1]\times UM}.$$

□

3 From pseudo-parallelizations to torsion combings

3.1 Variation of p_1 as an intersection of two 4-chains

Definition I.3.1. Let M be a compact oriented 3-manifold. A trivialization ρ of $TM_{|\partial M}$ is *admissible* if there exists a section X of UM such that (X,ρ) is a torsion combing of M.

Lemma I.3.2. *Let M be a compact oriented 3-manifold, let ρ be an admissible trivialization of $TM_{|\partial M}$ and let $S_1, S_2, \ldots, S_{\beta_1(M)}$ be surfaces in M comprising a basis of $H_2(M;\mathbb{Q})$. The subspace $H_T^\rho(M)$ of $H_2(UM;\mathbb{Q})$ generated by $\{[X(S_1)],\ldots,[X(S_{\beta_1(M)})]\}$ where X is a section of UM such that (X,ρ) is a torsion combing of M, only depends on ρ.*

Proof. Let (Y,ρ) be another choice of torsion combing of M. Assume, without loss of generality, that (X,ρ) and (Y,ρ) are ∂-compatible, and let $C(X,Y)$ be the 4-chain of UM

$$C(X,Y) = \bar{F}(X,Y) - UM_{|\Sigma_{X=-Y}}$$

constructed using Lemma I.1.7 and Proposition I.1.10, which provide $\bar{F}(X,Y)$ and a 2-chain $\Sigma_{X=-Y}$ of M bounded by $L_{X=-Y}$, respectively. For $i\in\{1,2,\ldots,\beta_1(M)\}$,

$$Y(S_i) - X(S_i) = \partial(C(X,Y)\cap UM_{|S_i}).$$

□

Lemma I.3.3. *Let M be a compact oriented 3-manifold, let ρ be an admissible trivialization of $TM_{|\partial M}$ and let (X,ρ) and (Y,ρ) be ∂-compatible torsion combings of M. There exists a 4-chain $C_4(X,Y)$ of $[0,1]\times UM$ such that*

$$\partial C_4(X,Y) = \{1\}\times Y(M) - \{0\}\times X(M) - [0,1]\times X(\partial M).$$

For any such chain $C_4(X,Y)$, if C is a 2-cycle of $[0,1]\times UM$ then,

$$[C] = \langle C, C_4(X,Y)\rangle_{[0,1]\times UM}[S] \quad \text{in } H_2([0,1]\times UM;\mathbb{Q})/H_T^\rho(M),$$

where $[S]$ is the homology class of the fiber of UM in $H_2([0,1]\times UM;\mathbb{Q})$.

Proof. Observe that $H_2(UM;\mathbb{Q})$ can be generated by $\{[Z(S_1)],\dots,[Z(S_{\beta_1(M)})],[S]\}$ where $S_1,S_2,\dots,S_{\beta_1(M)}$ are surfaces in M comprising a basis of $H_2(M;\mathbb{Q})$ and where Z is a torsion combing of M that coincides with X and Y on ∂M. Let $C_4(X,Y)$ be the 4-chain

$$C_4(X,Y)=\bar{F}_t(X,Y)-\{t\}\times UM_{|\Sigma_{X=-Y}}$$

where $\bar{F}_t(X,Y)$ is a 4-chain as in Definition I.2.5 and $\Sigma_{X=-Y}$ is a 2-chain of M bounded by $L_{X=-Y}$ provided by Proposition I.1.10. The chain $C_4(X,Y)$ has the desired boundary. Note that $\langle[S],C_4(X,Y)\rangle=1$. Moreover, $\langle[Z(\Sigma)],C_4(X,Y)\rangle=0$ for any surface Σ in M. Indeed, notice that

$$\langle[Z(\Sigma)],C_4(X,Y)\rangle=\begin{cases}\langle[Z(\Sigma)],[0,1]\times X(M)\rangle\ \text{, pushing } Z(\Sigma) \text{ before } t,\\ \langle[Z(\Sigma)],[0,1]\times Y(M)\rangle\ \text{, pushing } Z(\Sigma) \text{ after } t.\end{cases}$$

As a consequence, $\langle[Z(\Sigma)],C_4(X,Y)\rangle$ is independent of X and Y. Let us prove that it is possible to construct a torsion combing Z' that coincides with X and Y on ∂M and such that

$$\langle[Z(\Sigma)],[0,1]\times Z'(M)\rangle=0.$$

Using the parallelization $\rho=(E_1^\rho,E_2^\rho,E_3^\rho)$ of ∂M induced by X, define a homotopy $\mathcal{Z}:[0,1]\times\partial M\to[0,1]\times UM_{|\partial M}$ from $-Z_{|\partial M}$ to $Z_{|\partial M}$ along the unique geodesic arc passing through E_3^ρ. Since Σ sits in $\mathring{M}$, we can get a collar $\mathcal{C}\simeq[0,1]\times\partial M$ of ∂M such that $\mathcal{C}\cap\Sigma=\emptyset$ and $\{1\}\times\partial M=\partial M$. Finally set Z' to coincide with $-Z$ on $M\setminus\mathring{\mathcal{C}}$ and with the homotopy $\mathcal{Z}$ on the collar. The combing (Z',ρ) is a torsion combing. Indeed, E_2^ρ can be extended as a nonvanishing section of $Z'^{\perp}_{|\mathcal{C}}$ so that

$$e_2^M(Z'^{\perp},E_2^\rho)=e_2^{M\setminus\mathring{\mathcal{C}}}(Z'^{\perp},E_2^\rho)=e_2^M(-Z^{\perp},E_2^\rho)=-e_2^M(Z^{\perp},E_2^\rho).$$

Finally, using the torsion combing Z', we get $\langle[Z(\Sigma)],[0,1]\times Z'(M)\rangle=0$.

To conclude the proof, assume that $C_4'(X,Y)$ is a 4-chain with same boundary as the chain $C_4(X,Y)$ we constructed, and let C be a 2-cycle of $[0,1]\times UM$. The 2-cycle C is homologous to a 2-cycle in $\{1\}\times UM$. Similarly, $(C_4'(X,Y)-C_4(X,Y))$ is homologous to a 4-cycle in $\{0\}\times UM$. Hence, $\langle C,\ C_4'(X,Y)-C_4(X,Y)\rangle=0$. □

Lemma I.3.4. *Let τ and $\bar{\tau}$ be two pseudo-parallelizations of a compact oriented 3-manifold M that coincide on ∂M. Let $C_4(\tau,\bar{\tau};\pm e_1)$ denote 4-chains of $[0,1]\times UM$ as in Proposition I.9 for $v=\pm e_1$. If the 4-chains $C_4(\tau,\bar{\tau};\pm e_1)$ are transverse to each other, then*

$$\begin{aligned}\partial(C_4(\tau,\bar{\tau};e_1)\cap C_4(\tau,\bar{\tau};-e_1))&={}^1\!/_4\cdot\{1\}\times\Big(\bar{E}_1^d(L_{\bar{E}_1^d=-\bar{E}_1^g})-(-\bar{E}_1^d)(L_{\bar{E}_1^d=-\bar{E}_1^g})\Big)\\&\quad-{}^1\!/_4\cdot\{0\}\times\Big(E_1^d(L_{E_1^d=-E_1^g})-(-E_1^d)(L_{E_1^d=-E_1^g})\Big)\end{aligned}$$

where E_1^d and E_1^g, resp. $\bar{E}_1^d$ and $\bar{E}_1^g$, are the Siamese sections of τ, resp. $\bar{\tau}$.

Proof. Since τ and $\bar{\tau}$ coincide with a trivialization of $TM_{|\partial M}$ on ∂M, we have

$$\begin{aligned}
\partial(C_4(\tau,\bar{\tau};e_1)\cap C_4(\tau,\bar{\tau};-e_1)) &= (\{1\}\times UM)\bigcap(C_4(\tau,\bar{\tau};e_1)\cap C_4(\tau,\bar{\tau};-e_1))\\
&- (\{0\}\times UM)\bigcap(C_4(\tau,\bar{\tau};e_1)\cap C_4(\tau,\bar{\tau};-e_1))\\
&= \{1\}\times(\bar{\tau}(M\times\{e_1\})\cap\bar{\tau}(M\times\{-e_1\}))\\
&- \{0\}\times(\tau(M\times\{e_1\})\cap\tau(M\times\{-e_1\}))\\
&= {}^1\!/_4\cdot\{1\}\times\Big(\bar{E}_1^d(L_{\bar{E}_1^d=-\bar{E}_1^g})+\bar{E}_1^g(L_{\bar{E}_1^g=-\bar{E}_1^d})\Big)\\
&- {}^1\!/_4\cdot\{0\}\times\Big(E_1^d(L_{E_1^d=-E_1^g})+E_1^g(L_{E_1^g=-E_1^d})\Big)\\
&= {}^1\!/_4\cdot\{1\}\times\Big(\bar{E}_1^d(L_{\bar{E}_1^d=-\bar{E}_1^g})-(-\bar{E}_1^d)(L_{\bar{E}_1^d=-\bar{E}_1^g})\Big)\\
&- {}^1\!/_4\cdot\{0\}\times\Big(E_1^d(L_{E_1^d=-E_1^g})-(-E_1^d)(L_{E_1^d=-E_1^g})\Big).
\end{aligned}$$

□

Definition I.3.5. Let $\bar{\tau}=(N(\gamma);\tau_e,\tau_d,\tau_g)$ be a pseudo-parallelization of a compact oriented 3-manifold M, and let E_1^d and E_1^g denote its Siamese sections. Recall from Definition I.1.14 that the map

$$\tau_d^{-1}\circ\tau_g:[a,b]\times\gamma\times[-1,1]\times\mathbb{R}^3\to[a,b]\times\gamma\times[-1,1]\times\mathbb{R}^3$$

is such that

$$\begin{aligned}
&\forall t\in[a,b],\ u\in[-1,1],\ c\in\gamma,\ v\in\mathbb{R}^3:\\
&\tau_d^{-1}\circ\tau_g((t,c,u),v)=\big((t,c,u),P_{\mathbb{R}^3}\circ\mathcal{T}_\gamma^{-1}((t,c,u),\mathcal{F}(t,u)(v))\big),
\end{aligned}$$

where $P_{\mathbb{R}^3}\circ\mathcal{T}_\gamma^{-1}$ does not depend on $c\in\gamma$. Hence, $L_{E_1^d=-E_1^g}$ consists in parallels of γ. Still using the invariance of $\tau_d^{-1}\circ\tau_g$, for all component L of $L_{E_1^d=-E_1^g}$, there exists a point e_2^L in $\mathbb{S}^1(e_2)$ such that $L\times\{e_2^L\}=\tau_d^{-1}\circ\tau_g(L\times\{e_2\})$. Choose a point e_2^Ω in $\mathbb{S}^1(e_2)$ distinct from e_2 and from the points e_2^L. Finally, set

$$\Omega(\bar{\tau})=-\tau_d(L_{E_1^d=-E_1^g}\times[-e_1,e_1]_{e_2^\Omega})$$

where $[-e_1,e_1]_{e_2^\Omega}$ is the geodesic arc from $-e_1$ to e_1 passing through e_2^Ω. The 3-chain $\Omega(\bar{\tau})$ can be seen as the projection of a homotopy from $-E_1^d$ to E_1^d over $L_{E_1^d=-E_1^g}$. The choice of e_2^Ω ensures that $\Omega(\bar{\tau})\cap\bar{\tau}(M\times\{e_2\})=\emptyset$. Note that $\Omega(\tau)=\emptyset$ when τ is a genuine parallelization.

Definition I.3.6. Let M be a compact oriented 3-manifold and let ρ be an admissible trivialization of $TM_{|\partial M}$. Let τ and $\bar{\tau}$ be pseudo-parallelizations of a compact oriented 3-manifold M which coincide with ρ on $TM_{|\partial M}$ and let $C_4(\tau, \bar{\tau}; \pm e_1)$ denote 4-chains of $[0,1] \times UM$ as in Proposition I.9. Set

$$\mathfrak{P}(\tau, \bar{\tau}) = \{0\} \times \Omega(\tau) + 4 \cdot (C_4(\tau, \bar{\tau}; e_1) \cap C_4(\tau, \bar{\tau}; -e_1)) - \{1\} \times \Omega(\bar{\tau}).$$

When (X, ρ) is a torsion combing of M, let $C_4^+(X, \bar{\tau})$ and $C_4^-(X, \bar{\tau})$ be 4-chains of $[0,1] \times UM$ as in Lemma I.2.6 and set

$$\begin{aligned}\mathfrak{P}(X, \bar{\tau}) &= 4 \cdot (C_4^+(X, \bar{\tau}) \cap C_4^-(X, \bar{\tau})) - \{1\} \times \Omega(\bar{\tau}),\\ \mathfrak{P}(\bar{\tau}, X) &= \{0\} \times \Omega(\bar{\tau}) + 4 \cdot (C_4^+(\bar{\tau}, X) \cap C_4^-(\bar{\tau}, X)).\end{aligned}$$

According to Lemma I.3.4 and Definition I.3.5, the 4-chains $\mathfrak{P}(\lambda, \mu)$ of Definition I.3.6 above are cycles. In the remaining of this section, we prove that their classes read $p_1(\lambda, \mu)[S]$, in $H_2([0,1] \times UM; \mathbb{Q})/H_T^\rho(M)$.

Proposition I.3.7. *Let M be a compact oriented 3-manifold, let ρ be an admissible trivialization of $TM_{|\partial M}$ and let τ and $\bar{\tau}$ be two pseudo-parallelizations of M that coincide with ρ on ∂M. Under the assumptions of Definition I.3.6, the class of $\mathfrak{P}(\tau, \bar{\tau})$ in $H_2([0,1] \times UM; \mathbb{Q})/H_T^\rho(M)$ equals $p_1(\tau, \bar{\tau})[S]$ where $[S]$ is the homology class of the fiber of UM in $H_2([0,1] \times UM; \mathbb{Q})$.*

Proof. The class of $\mathfrak{P}(\tau, \bar{\tau})$ in $H_2([0,1] \times UM; \mathbb{Q})/H_T^\rho(M)$ is

$$[\mathfrak{P}(\tau, \bar{\tau})] = \langle \mathfrak{P}(\tau, \bar{\tau}), C_4(X, Y) \rangle \cdot [S]$$

where (X, ρ) and (Y, ρ) are ∂-compatible torsion combings of M and where $C_4(X, Y)$ is any 4-chain of $[0,1] \times UM$ as in Lemma I.3.3. Let us construct a specific $C_4(X, Y)$ as follows. Let $C_4(\tau, \bar{\tau}; e_2)$ be as in Proposition I.9 where $e_2 = (0,1,0)$. Since, $\partial C_4(\tau, \bar{\tau}; e_1)$ and $\partial C_4(\tau, \bar{\tau}; e_2)$ are homologous, it is possible to reparameterize and to stack the 4-chains $C_4^+(X, \tau)$, $C_4(\tau, \bar{\tau}; e_2)$ and $C_4^+(\bar{\tau}, Y)$ where the chains $C_4^+(X, \tau)$ and $C_4^+(\bar{\tau}, Y)$ are as in Lemma I.2.6. It follows that, in $H_2([0,1] \times UM; \mathbb{Q})/H_T^\rho(M)$,

$$\begin{aligned}[\mathfrak{P}(\tau, \bar{\tau})] &= \langle \mathfrak{P}(\tau, \bar{\tau}), C_4(X, Y) \rangle [S]\\ &= \langle \mathfrak{P}(\tau, \bar{\tau}), C_4(\tau, \bar{\tau}; e_2) \rangle [S]\\ &= 4 \cdot \langle C_4(\tau, \bar{\tau}; e_1) \cap C_4(\tau, \bar{\tau}; -e_1), C_4(\tau, \bar{\tau}; e_2) \rangle [S]\\ &\quad + \langle \{0\} \times \Omega(\tau) - \{1\} \times \Omega(\bar{\tau}), C_4(\tau, \bar{\tau}; e_2) \rangle [S].\end{aligned}$$

Now, note that $\langle \{0\} \times \Omega(\tau) - \{1\} \times \Omega(\bar{\tau}), C_4(\tau, \bar{\tau}; e_2) \rangle = 0$ since

$$\Omega(\tau) \cap \tau(M \times \{e_2\}) = \emptyset \quad \text{and} \quad \Omega(\bar{\tau}) \cap \bar{\tau}(M \times \{e_2\}) = \emptyset.$$

Hence,

$$[\mathfrak{P}(\tau, \bar{\tau})] = 4 \cdot \langle C_4(\tau, \bar{\tau}; e_1), C_4(\tau, \bar{\tau}; -e_1), C_4(\tau, \bar{\tau}; e_2) \rangle [S] = p_1(\tau, \bar{\tau})[S].$$

□

3.2 Pontrjagin numbers for combings of compact 3-manifolds Proof of Theorem I.4

Lemma I.3.8. *Let (X,ρ) be a torsion combing of a compact oriented 3-manifold M. Let $\bar\tau$ be a pseudo-parallelization of M compatible with X. Let $C_4^+(\bar\tau,X)$ and $C_4^-(\bar\tau,X)$ be 4-chains of $[0,1]\times UM$ as in Lemma I.2.6. For such 4-chains, the class of the cycle $\mathfrak{P}(\bar\tau,X)$ in $H_2([0,1]\times UM;\mathbb{Q})/H_T^\rho(M)$ only depends on $\bar\tau$ and on the homotopy class of X. It will be denoted $\tilde p_1(\bar\tau,[X])[S]$.*

Proof. Let τ be another pseudo-parallelization of M which is compatible with X. Let $C_4^+(X,\tau)$ and $C_4^-(X,\tau)$ be fixed choices of 4-chains of $[0,1]\times UM$ as in Lemma I.2.6. Using these 4-chains, construct the cycle $\mathfrak{P}(X,\tau)$ as in Definition I.3.6. In $H_2([0,1]\times UM;\mathbb{Q})/H_T^\rho(M)$, we then have

$$\begin{aligned}[\mathfrak{P}(\bar\tau,X)]+[\mathfrak{P}(X,\tau)] &= [\mathfrak{P}(\bar\tau,X)+\mathfrak{P}(X,\tau)]\\ &= [\{0\}\times\Omega(\bar\tau)+4\cdot(C_4^+(\bar\tau,X)\cap C_4^-(\bar\tau,X))\\ &\quad +4\cdot(C_4^+(X,\tau)\cap C_4^-(X,\tau))-\{1\}\times\Omega(\tau)].\end{aligned}$$

By reparameterizing and stacking $C_4^+(\bar\tau,X)$ and $C_4^+(X,\tau)$, resp. $C_4^-(\bar\tau,X)$ and $C_4^-(X,\tau)$, we get a 4-chain $C_4(\bar\tau,\tau,e_1)$, resp. $C_4(\bar\tau,\tau,-e_1)$, as in Lemma I.2.7. It follows that

$$\begin{aligned}[\mathfrak{P}(\bar\tau,X)]+[\mathfrak{P}(X,\tau)] &= [\{0\}\times\Omega(\bar\tau)+4\cdot(C_4(\bar\tau,\tau,e_1)\cap C_4(\bar\tau,\tau,e_1))-\{1\}\times\Omega(\tau)]\\ &= [\mathfrak{P}(\bar\tau,\tau)]\end{aligned}$$

or, equivalently, $[\mathfrak{P}(\bar\tau,X)]=[\mathfrak{P}(\bar\tau,\tau)]-[\mathfrak{P}(X,\tau)]$. This proves the statement since $\mathfrak{P}(X,\tau)$ is independent of the choices made for $C_4^+(\bar\tau,X)$ and $C_4^-(\bar\tau,X)$, and since, according to Proposition I.3.7, the class $[\mathfrak{P}(\bar\tau,\tau)]$ is independent of the choices for $C_4(\bar\tau,\tau,e_1)$ and $C_4(\bar\tau,\tau,-e_1)$. □

Proposition I.3.9. *If $\bar\tau$ is a pseudo-parallelization of a closed oriented 3-manifold M and if X is a torsion combing of M compatible with $\bar\tau$, then*

$$\tilde p_1(\bar\tau,[X])=p_1([X])-p_1(\bar\tau).$$

Proof. According to Lemma I.3.8, $\tilde p_1(\bar\tau,[X])=[\mathfrak{P}(\bar\tau,X)]$. Since $[\mathfrak{P}(\bar\tau,X)]$ is independent of the choices for $C_4^+(\bar\tau,X)$ and $C_4^-(\bar\tau,X)$, let us construct convenient 4-chains $C_4^+(\bar\tau,X)$ and $C_4^-(\bar\tau,X)$. Let τ be a genuine parallelization of M. Thanks to Proposition I.9, there exist two 4-chains of $[0,1]\times UM$, $C_4(\bar\tau,\tau;e_1)$ and $C_4(\bar\tau,\tau;-e_1)$, such that

$$\begin{aligned}\partial C_4(\bar\tau,\tau;e_1) &= \{1\}\times\tau(M\times\{e_1\})-\{0\}\times\bar\tau(M\times\{e_1\}),\\ \partial C_4(\bar\tau,\tau;-e_1) &= \{1\}\times\tau(M\times\{-e_1\})-\{0\}\times\bar\tau(M\times\{-e_1\}).\end{aligned}$$

Using Lemma I.1.7, Definition I.2.5 and $t_1, t_2 \in]0,1[$, there also exist two 4-chains of $[0,1] \times UM$, $\bar{F}_{t_1}(E_1^\tau, X)$ and $\bar{F}_{t_2}(-E_1^\tau, -X)$, which have boundaries :

$$\begin{aligned}\partial \bar{F}_{t_1}(E_1^\tau, X) &= \{1\} \times X(M) - \{0\} \times E_1^\tau(M) + \{t_1\} \times UM_{|L_{E_1^\tau = -X}} \\ \partial \bar{F}_{t_2}(-E_1^\tau, -X) &= \{1\} \times (-X)(M) - \{0\} \times (-E_1^\tau)(M) + \{t_2\} \times UM_{|L_{-E_1^\tau = X}}\end{aligned}$$

where E_1^τ stands for the first vector of the parallelization τ. Now, using Proposition I.1.10, we know there exist two 2-chains of M, $\Sigma_{E_1^\tau = -X}$ and $\Sigma_{-E_1^\tau = X}$, with respective boundaries $L_{E_1^\tau = -X}$ and $L_{-E_1^\tau = X}$ so that we can set

$$\begin{aligned}C_4(E_1^\tau, X) &= \bar{F}_{t_1}(E_1^\tau, X) - \{t_1\} \times UM_{|\Sigma_{E_1^\tau = -X}} \\ C_4(-E_1^\tau, -X) &= \bar{F}_{t_2}(-E_1^\tau, -X) - \{t_2\} \times UM_{|\Sigma_{-E_1^\tau = X}}\end{aligned}$$

and get $C_4^+(\bar{\tau}, X)$, resp. $C_4^-(\bar{\tau}, X)$, by reparameterizing and stacking the 4-chains $C_4(\bar{\tau}, \tau; e_1)$ and $C_4(E_1^\tau, X)$, resp. $C_4(\bar{\tau}, \tau; -e_1)$ and $C_4(-E_1^\tau, -X)$.

Let us finally compute $[\{0\} \times \Omega(\bar{\tau}) + 4 \cdot (C_4^+(\bar{\tau}, X) \cap C_4^-(\bar{\tau}, X))]$. By construction, we have :

$$\begin{aligned}&[\{0\} \times \Omega(\bar{\tau}) + 4 \cdot (C_4^+(\bar{\tau}, X) \cap C_4^-(\bar{\tau}, X))] \\ =&[\{0\} \times \Omega(\bar{\tau}) + 4 \cdot (C_4(\bar{\tau}, \tau; e_1) \cap C_4(\bar{\tau}, \tau; -e_1))] + 4[C_4(E_1^\tau, X) \cap C_4(-E_1^\tau, -X)],\end{aligned}$$

so that, using Proposition I.3.7,

$$\begin{aligned}&[\{0\} \times \Omega(\bar{\tau}) + 4 \cdot (C_4^+(\bar{\tau}, X) \cap C_4^-(\bar{\tau}, X))] \\ &= (p_1(\tau) - p_1(\bar{\tau}))[S] + 4 \cdot [C_4(E_1^\tau, X) \cap C_4(-E_1^\tau, -X)].\end{aligned}$$

Now, using Definition I.2.5,

$$\begin{aligned}C_4(E_1^\tau, X) = &\ [0, t_1] \times E_1^\tau(M) + \{t_1\} \times \bar{F}(E_1^\tau, X) \\ &+ [t_1, 1] \times X(M) - \{t_1\} \times UM_{|\Sigma_{E_1^\tau = -X}} \\ C_4(-E_1^\tau, -X) = &\ [0, t_2] \times (-E_1^\tau)(M) + \{t_2\} \times \bar{F}(-E_1^\tau, -X) \\ &+ [t_2, 1] \times (-X)(M) - \{t_2\} \times UM_{|\Sigma_{-E_1^\tau = X}}\end{aligned}$$

so that, assuming $t_1 < t_2$ without loss of generality,

$$\begin{aligned}C_4(E_1^\tau, X) \cap C_4(-E_1^\tau, -X) = &- \{t_1\} \times (-E_1^\tau) \left(\Sigma_{E_1^\tau = -X}\right) \\ &+ [t_1, t_2] \times (-E_1^\tau) \left(L_{-E_1^\tau = X}\right) \\ &- \{t_2\} \times X(\Sigma_{-E_1^\tau = X}).\end{aligned}$$

It follows that, using Lemma I.3.3 with $C_4(E_1^\tau, E_1^\tau) = [0,1] \times E_1^\tau(M)$,

$$\begin{aligned} 4 \cdot [C_4(E_1^\tau, X) \cap C_4(-E_1^\tau, -X)] &= 4 \cdot \langle C_4(E_1^\tau, X), C_4(-E_1^\tau, -X), [0,1] \times E_1^\tau(M) \rangle [S] \\ &= 4 \cdot lk(L_{E_1^\tau = X}, L_{E_1^\tau = -X})[S] \\ &= (p_1([X]) - p_1(\tau))[S] \end{aligned}$$

in $H_2([0,1] \times UM; \mathbb{Q})/H_T(M)$, and, eventually,

$$\begin{aligned} &[\{0\} \times \Omega(\bar\tau) + 4 \cdot (C_4^+(\bar\tau, X) \cap C_4^-(\bar\tau, X))] \\ &\qquad = (p_1(\tau) - p_1(\bar\tau))[S] + (p_1([X]) - p_1(\tau))[S] \\ &\qquad = (p_1([X]) - p_1(\bar\tau))[S]. \end{aligned}$$

□

Lemma I.3.10. *If (X, ρ) is a torsion combing of a compact oriented 3-manifold M and if $\bar\tau = (N(\gamma); \tau_e, \tau_d, \tau_g)$ is a pseudo-parallelization of M compatible with X, then*

$$\begin{aligned} \tilde p_1(\bar\tau, [X]) = {} & lk_M \left(L_{E_1^d = X} + L_{E_1^g = X} \ , \ L_{E_1^d = -X} + L_{E_1^g = -X} \right) \\ & - lk_{\mathbb{S}^2} \left(e_1 - (-e_1), \ P_{\mathbb{S}^2} \circ \tau_d^{-1} \circ X(L_{E_1^d = -E_1^g}) \right) \end{aligned}$$

where E_1^d and E_1^g denote the Siamese sections of $\bar\tau$.

Proof. We just have to evaluate the class of the 4-cycle

$$\mathfrak{P}(\bar\tau, X) = \{0\} \times \Omega(\bar\tau) + 4 \cdot (C_4^+(\bar\tau, X) \cap C_4^-(\bar\tau, X))$$

in $H_2([0,1] \times UM; \mathbb{Q})/H_T^\rho(M)$ for convenient 4-chains $C_4^+(\bar\tau, X)$ and $C_4^-(\bar\tau, X)$ with the prescribed boundaries. Let t_1 and t_2 in $]0,1[$, with $t_1 > t_2$, and set

$$\begin{aligned} C_4^+(\bar\tau, X) &= {}^1\!/_2 \cdot \left(\bar F_{t_1}(E_1^d, X) + \bar F_{t_1}(E_1^g, X) - \{t_1\} \times UM_{|\Sigma(e_1)} \right) \\ C_4^-(\bar\tau, X) &= {}^1\!/_2 \cdot \left(\bar F_{t_2}(-E_1^d, -X) + \bar F_{t_2}(-E_1^g, -X) - \{t_2\} \times UM_{|\Sigma(-e_1)} \right) \end{aligned}$$

where the chains $\bar F_t$ are as in Definition I.2.5 and where, using Lemma I.2.4, $\Sigma(e_1)$ and $\Sigma(-e_1)$ are 2-chains of M so that

$$\partial\Sigma(\pm e_1) = \pm(L_{E_1^d = -X} + L_{E_1^g = -X}).$$

These 4-chains do have the expected boundaries. Let us now describe the intersection $C_4^+(\bar\tau, X) \cap C_4^-(\bar\tau, X)$:

- on $[0, t_2[$: The intersection between $C_4^+(\bar\tau, X)$ and $C_4^-(\bar\tau, X)$ is

$$^1\!/_4 \cdot [0, t_2[\ \times \ E_1^d(L_{E_1^d = -E_1^g}) + {}^1\!/_4 \cdot [0, t_2[\ \times \ E_1^g(L_{E_1^g = -E_1^d}).$$

- on $]t_2, t_1[$: The intersection between $C_4^+(\bar{\tau}, X)$ and $C_4^-(\bar{\tau}, X)$ is

$$1/2 \cdot]t_2, t_1[\times (-X)(L_{E_1^d=-X}) + 1/2 \cdot]t_2, t_1[\times (-X)(L_{E_1^g=-X}).$$

- on $]t_1, 1]$: There is no intersection between $C_4^+(\bar{\tau}, X)$ and $C_4^-(\bar{\tau}, X)$ since they consist in $]t_1, 1] \times X(M)$ and $]t_1, 1] \times (-X)(M)$.
- at t_2 : The intersection between $C_4^+(\bar{\tau}, X)$ and $C_4^-(\bar{\tau}, X)$ is

$$\begin{aligned} &1/2 \cdot \{t_2\} \times E_1^d(M) \cap \{t_2\} \times \bar{F}(-E_1^g, -X) \\ &+1/2 \cdot \{t_2\} \times E_1^g(M) \cap \{t_2\} \times \bar{F}(-E_1^d, -X) \\ &-1/4 \cdot \{t_2\} \times E_1^d(\Sigma(-e_1)) - 1/4 \cdot \{t_2\} \times E_1^g(\Sigma(-e_1)) \end{aligned}$$

- at t_1 : The intersection between $C_4^+(\bar{\tau}, X)$ and $C_4^-(\bar{\tau}, X)$ is

$$\begin{aligned} &1/2 \cdot \{t_1\} \times \bar{F}(E_1^d, X) \cap \{t_1\} \times (-X)(M) \\ &+1/2 \cdot \{t_1\} \times \bar{F}(E_1^g, X) \cap \{t_1\} \times (-X)(M) \\ &-1/2 \cdot \{t_1\} \times (-X)(\Sigma(e_1)). \end{aligned}$$

It follows that :

$$\begin{aligned} &\langle 1/4 \cdot \{0\} \times \Omega(\bar{\tau}) + C_4^+(\bar{\tau}, X) \cap C_4^-(\bar{\tau}, X), [0,1] \times X(M)\rangle_{[0,1]\times UM} \\ &= 1/4 \cdot \langle [0, t_2[\times E_1^d(L_{E_1^d=-E_1^g}) + [0, t_2[\times E_1^g(L_{E_1^g=-E_1^d}), [0,1] \times X(M)\rangle_{[0,1]\times UM} \\ &+ 1/2 \cdot \langle \{t_2\} \times E_1^d(M) \cap \{t_2\} \times \bar{F}(-E_1^g, -X), [0,1] \times X(M)\rangle_{[0,1]\times UM} \\ &+ 1/2 \cdot \langle \{t_2\} \times E_1^g(M) \cap \{t_2\} \times \bar{F}(-E_1^d, -X), [0,1] \times X(M)\rangle_{[0,1]\times UM} \\ &- 1/4 \cdot \langle \{t_2\} \times E_1^d(\Sigma(-e_1)) + \{t_2\} \times E_1^g(\Sigma(-e_1)), [0,1] \times X(M)\rangle_{[0,1]\times UM} \\ &+ 1/4 \cdot \langle \Omega(\bar{\tau}), X(M)\rangle_{UM}. \end{aligned}$$

Since $L_{E_1^g=X} \cap L_{E_1^d=-X}$ and $L_{E_1^g=-X} \cap L_{E_1^d=X}$ are empty :

$$\langle [0, t_2[\times E_1^d(L_{E_1^d=-E_1^g}) + [0, t_2[\times E_1^g(L_{E_1^g=-E_1^d}), [0,1] \times X(M)\rangle_{[0,1]\times UM} = 0.$$

Furthermore, note that if (m, v) is an intersection point of

$$E_1^d(M) \cap \bar{F}(-E_1^g, -X) \cap X(M)$$

then, in particular, $v = E_1^d(m) = X(m)$ so that $-E_1^g(m)$ and $-X(m)$ are not antipodal since $L_{E_1^d=X} \cap L_{E_1^g=-X} = \emptyset$. It follows that $v = E_1^d(m) = X(m)$ should also sit on the shortest geodesic arc from $-E_1^g(m)$ to $-X(m)$. Since such a configuration is impossible, this triple intersection is empty, thus

$$\langle \{t_2\} \times E_1^d(M) \cap \{t_2\} \times \bar{F}(-E_1^g, -X), [0,1] \times X(M)\rangle_{[0,1]\times UM} = 0.$$

Similarly,

$$\langle \{t_2\} \times E_1^g(M) \cap \{t_2\} \times \bar{F}(-E_1^d, -X), [0,1] \times X(M) \rangle_{[0,1]\times UM} = 0.$$

Now, we have

$$\begin{aligned} &\langle \{t_2\} \times E_1^d(\Sigma(-e_1)) + \{t_2\} \times E_1^g(\Sigma(-e_1)), [0,1] \times X(M) \rangle \\ &= -lk_M(L_{E_1^d=X} + L_{E_1^g=X},\ L_{E_1^d=-X} + L_{E_1^g=-X}). \end{aligned}$$

Furthermore,

$$\begin{aligned} \langle \Omega(\bar{\tau}), X(M) \rangle_{UM} &= \langle \tau_d^{-1}(\Omega(\bar{\tau})), \tau_d^{-1} \circ X(L_{E_1^d=-E_1^g}) \rangle_{L_{E_1^d=-E_1^g} \times \mathbb{S}^2} \\ &= -\langle L_{E_1^d=-E_1^g} \times [-e_1, e_1]_{e_2^\Omega}\ ,\ \tau_d^{-1} \circ X(L_{E_1^d=-E_1^g}) \rangle_{L_{E_1^d=-E_1^g} \times \mathbb{S}^2} \end{aligned}$$

where $[-e_1, e_1]_{e_2^\Omega}$ is the geodesic arc of $\mathbb{S}^2$ from $-e_1$ to e_1 passing through e_2^Ω. Now, $L_{E_1^d=-E_1^g} \times \mathbb{S}^2$ is oriented and an intersection

$$(m, v) \in L_{E_1^d=-E_1^g} \times [-e_1, e_1]_{e_2^\Omega} \cap \tau_d^{-1} \circ X(L_{E_1^d=-E_1^g})$$

is positive when

$$T_{(m,v)}(L_{E_1^d=-E_1^g} \times [-e_1, e_1]_{e_2^\Omega}) \oplus T_{(m,v)}(\tau_d^{-1} \circ X(L_{E_1^d=-E_1^g})) = T_{(m,v)}(L_{E_1^d=-E_1^g} \times \mathbb{S}^2)$$

as an oriented sum, which is equivalent to

$$T_v([-e_1, e_1]_{e_2^\Omega}) \oplus T_v(P_{\mathbb{S}^2} \circ \tau_d^{-1} \circ X(L_{E_1^d=-E_1^g})) = T_v(L_{E_1^d=-E_1^g} \times \mathbb{S}^2)$$

as an oriented sum, where $P_{\mathbb{S}^2}$ is the standard projection from $M \times \mathbb{S}^2$ to $\mathbb{S}^2$. See Figure I.3.11.

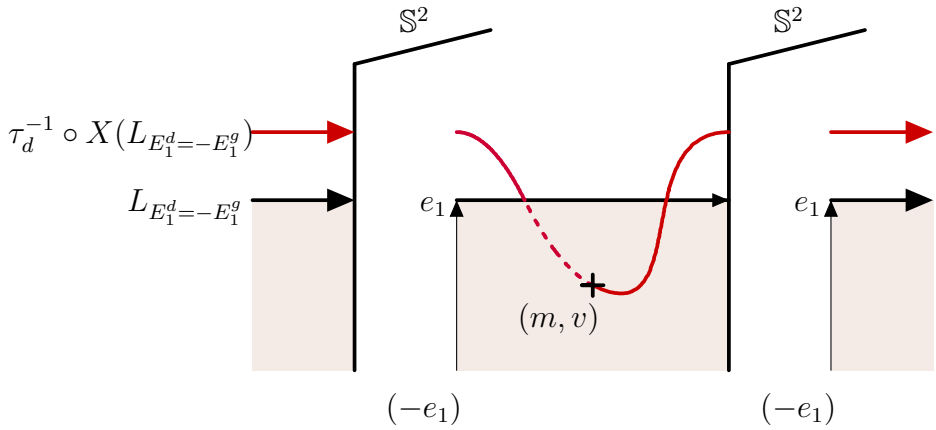


Figure I.3.11 – A positive intersection (m, v) between the 2-chain $L_{E_1^d=-E_1^g} \times [-e_1, e_1]_{e_2^\Omega}$ and $\tau_d^{-1} \circ X(L_{E_1^d=-E_1^g})$ in $L_{E_1^d=-E_1^g} \times \mathbb{S}^2$.

It follows that

$$\begin{aligned}\langle \Omega(\bar\tau), X(M)\rangle_{UM} &= -\langle [-e_1, e_1]_{e_2^\Omega},\ P_{\mathbb{S}^2}\circ \tau_d^{-1}\circ X(L_{E_1^d=-E_1^g})\rangle_{\mathbb{S}^2}\\ &= -lk_{\mathbb{S}^2}\left(e_1-(-e_1),\ P_{\mathbb{S}^2}\circ \tau_d^{-1}\circ X(L_{E_1^d=-E_1^g})\right),\end{aligned}$$

and, finally,

$$\begin{aligned}\langle \mathfrak{P}(\bar\tau, X), [0,1]\times X(M)\rangle_{[0,1]\times UM} &= lk_M\left(L_{E_1^d=X}+L_{E_1^g=X}\ ,\ L_{E_1^d=-X}+L_{E_1^g=-X}\right)\\ &\quad - lk_{\mathbb{S}^2}\left(e_1-(-e_1),\ P_{\mathbb{S}^2}\circ \tau_d^{-1}\circ X(L_{E_1^d=-E_1^g})\right).\end{aligned}$$

□

Proof of Lemma I.3. According to Lemmas I.3.8 and I.3.10, Lemma I.3 is true for $p_1=\tilde p_1$. □

From now on, if X is a torsion combing of a compact oriented 3-manifold M and if $\bar\tau$ is a pseudo-parallelization of M compatible with X, then set

$$p_1([X],\bar\tau)=\tilde p_1([X],\bar\tau)\ \text{ and }\ p_1(\bar\tau,[X])=\tilde p_1(\bar\tau,[X]).$$

As an obvious consequence, we get the following lemma.

Lemma I.3.12. *Under the assumptions of Lemma I.3.8, the class of* $\mathfrak{P}(\bar\tau, X)$ *in* $H_2([0,1]\times UM;\mathbb{Q})/H_T^\rho(M)$ *is* $p_1(\bar\tau, X)[S]$.

Proof of Theorem I.4. Let X_1 and X_2 be torsion combings of two compact oriented 3-manifolds M_1 and M_2 with identified boundaries such that X_1 and X_2 coincide on the boundary. Let also $\bar\tau_1$ and $\bar\tau_1'$ be two pseudo-parallelizations of M_1 that extend the trivialization $\rho(X_1)$ and, similarly, let $\bar\tau_2$ be a pseudo-parallelization of M_2 that extends the trivialization $\rho(X_2)$. In such a context, let

$$\begin{aligned}p_1([X_1],[X_2])(\bar\tau_1,\bar\tau_2) &= p_1([X_1],\bar\tau_1)+p_1(\bar\tau_1,\bar\tau_2)+p_1(\bar\tau_2,[X_2])\\ p_1([X_1],[X_2])(\bar\tau_1',\bar\tau_2) &= p_1([X_1],\bar\tau_1')+p_1(\bar\tau_1',\bar\tau_2)+p_1(\bar\tau_2,[X_2])\end{aligned}$$

and note that

$$p_1([X_1],[X_2])(\bar\tau_1,\bar\tau_2)-p_1([X_1],[X_2])(\bar\tau_1',\bar\tau_2)=p_1(\bar\tau_1,\bar\tau_1')-\langle\mathfrak{P}(\bar\tau_1,\bar\tau_1'),[0,1]\times X_1(M)\rangle.$$

Using Proposition I.3.7, we get $p_1([X_1],[X_2])(\bar\tau_1,\bar\tau_2)-p_1([X_1],[X_2])(\bar\tau_1',\bar\tau_2)=0$, in other words $p_1([X_1],[X_2])(\bar\tau_1,\bar\tau_2)$ is independent of $\bar\tau_1$. Similarly, it is also independent of $\bar\tau_2$ so let us drop the pseudo-parallelizations from the notation. Eventually, using Lemma I.3, we get the formula of the statement.

For the second part of the statement, if M_1 and M_2 are closed, then Proposition I.3.9 ensures that, for $i \in \{1,2\}$, $p_1(\bar{\tau}_i, [X_i]) = p_1([X_i]) - p_1(\bar{\tau}_i)$. Hence,

$$\begin{aligned} p_1([X_1],[X_2]) &= (p_1(\bar{\tau}_2) - p_1(\bar{\tau}_1)) + (p_1([X_2]) - p_1(\bar{\tau}_2)) - (p_1([X_1]) - p_1(\bar{\tau}_1)) \\ &= p_1([X_2]) - p_1([X_1]). \end{aligned}$$

□

Let us now end this section by proving Proposition I.7 and Theorem I.8, starting with the following.

Lemma I.3.13. *Let (X,ρ) and (Y,ρ) be ∂-compatible torsion combings of a compact oriented 3-manifold M. Let $C_4(X,Y)$ and $C_4(-X,-Y)$ be 4-chains of $[0,1]\times UM$ as in Lemma I.3.3. The class of $\mathfrak{P}(X,Y)=4\big(C_4(X,Y)\cap C_4(-X,-Y)\big)$ in $H_2([0,1]\times UM;\mathbb{Q})/H_T^\rho(M)$ reads $p_1([X],[Y])[S]$ where $[S]$ is the homology class of the fiber of UM in $H_2([0,1]\times UM;\mathbb{Q})$.*

Proof. Let $\bar{\tau}$ be a pseudo-parallelization of M compatible with X and Y. By Theorem I.4, in $H_2([0,1]\times UM;\mathbb{Q})/H_T^\rho(M)$:

$$\begin{aligned} p_1([X],[Y])[S] &= \big(p_1([X],\bar{\tau}) + p_1(\bar{\tau},\bar{\tau}) + p_1(\bar{\tau},[Y])\big)[S] \\ &= \big(p_1([X],\bar{\tau}) + p_1(\bar{\tau},[Y])\big)[S] \end{aligned}$$

Then, using Lemma I.3.12,

$$\begin{aligned} p_1([X],[Y])[S] &= [\mathfrak{P}(X,\bar{\tau}) + \mathfrak{P}(\bar{\tau},Y)] \\ &= [4(C_4^+(X,\bar{\tau}) \cap C_4^-(X,\bar{\tau})) + 4(C_4^+(\bar{\tau},Y) \cap C_4^-(\bar{\tau},Y))]. \end{aligned}$$

Hence, reparameterizing and stacking $C_4^+(X,\bar{\tau})$ and $C_4^+(\bar{\tau},Y)$, resp. $C_4^-(X,\bar{\tau})$ and $C_4^-(\bar{\tau},Y)$, we get a 4-chain $C_4(X,Y)$, resp. $C_4(-X,-Y)$, as in Lemma I.3.3 and

$$p_1([X],[Y])[S] = 4\cdot[C_4(X,Y)\cap C_4(-X,-Y)].$$

To conclude the proof, see that if $C_4'(X,Y)$ is 4-chain of $[0,1]\times UM$ such that $\partial C_4'(X,Y) = \partial C_4(X,Y)$, then $C_4'(X,Y) - C_4(X,Y)$ is homologous to a 4-cycle in $\{0\}\times UM$ so that

$$\begin{aligned} &[C_4'(X,Y)\cap C_4(-X,-Y)] - [C_4(X,Y)\cap C_4(-X,-Y)] \\ &= \big[\big(C_4'(X,Y) - C_4(X,Y)\big)\cap C_4(-X,-Y)\big] \end{aligned}$$

sits in $H_T^\rho(M)$. So, the class $[C_4(X,Y)\cap C_4(-X,-Y)]$ in $H_2([0,1]\times UM;\mathbb{Q})/H_T^\rho(M)$ is independent of the choices for $C_4(X,Y)$. Similarly, it is independent of the choices for $C_4(-X,-Y)$. □

Proof of Proposition I.7. According to Lemma I.3.13, it is enough to evaluate the class of $4\big(C_4(X,Y)\cap C_4(-X,-Y)\big)$ in $H_2([0,1]\times UM;\mathbb{Q})/H_T^\rho(M)$ where $\rho=\rho(X)$ and where $C_4(X,Y)$ and $C_4(-X,-Y)$ are 4-chains of $[0,1]\times UM$ as in Lemma I.3.3. Let us consider the 4-chains

$$C_4(X,Y)=\bar{F}_{t_1}(X,Y)-\{t_1\}\times UM_{|\Sigma_{X=-Y}},$$
$$C_4(-X,-Y)=\bar{F}_{t_2}(-X,-Y)-\{t_2\}\times UM_{|\Sigma_{-X=Y}},$$

where $0<t_1<t_2<1$, and where $\bar{F}_{t_1}(X,Y)$, resp. $\bar{F}_{t_2}(-X,-Y)$, is a 4-chain as in Definition I.2.5 and $\Sigma_{X=-Y}$, resp. $\Sigma_{-X=Y}$, is a 2-chain of M bounded by $L_{X=-Y}$, resp. $L_{-X=Y}$, provided by Proposition I.1.10. With these chains,

$$\begin{aligned}&C_4(X,Y)\cap C_4(-X,-Y)\\&=-\{t_1\}\times(-X)(\Sigma_{X=-Y})+[t_1,t_2]\times(-X)(L_{-X=Y})-\{t_2\}\times Y(\Sigma_{-X=Y}).\end{aligned}$$

Hence, using Lemma I.3.3 with $[0,1]\times X(M)$, in $H_2([0,1]\times UM;\mathbb{Q})/H_T^\rho(M)$:

$$\begin{aligned}&[C_4(X,Y)\cap C_4(-X,-Y)]=\langle C_4(X,Y)\cap C_4(-X,-Y),[0,1]\times X(M)\rangle_{[0,1]\times UM}[S]\\&=\langle-\{t_1\}\times(-X)(\Sigma_{X=-Y})+[t_1,t_2]\times(-X)(L_{-X=Y})\\&\qquad\qquad-\{t_2\}\times Y(\Sigma_{-X=Y}),[0,1]\times X(M)\rangle_{[0,1]\times UM}[S]\\&=-lk(L_{X=Y},L_{-X=Y})[S]\\&=lk(L_{X=Y},L_{X=-Y})[S].\end{aligned}$$

□

Proof of Theorem I.8. If X and Y are homotopic relatively to the boundary, then $p_1([X],[Y])=0$. Hence, consider two combings X and Y_0 in the same Spin^c-structure and assume that $p_1([X],[Y_0])=0$. Since Y_0 is in the same Spin^c-structure as X, there exists a homotopy from Y_0 to a combing Y_1 that coincides with X outside a ball $\mathcal{B}$ in $\mathring{M}$.

Let σ be a unit section of $X^\perp_{|\mathcal{B}}$, and let $(X,\sigma,X\wedge\sigma)$ denote the corresponding parallelization over $\mathcal{B}$. Extend the unit section σ as a generic section of $X^\perp$ such that $\sigma_{|\partial M}=\sigma(X)$, and deform Y_1 to Y where

$$Y(m)=\frac{Y_1(m)+\chi(m)\sigma(m)}{\parallel Y_1(m)+\chi(m)\sigma(m)\parallel}$$

for a smooth map χ from M to $[0,\varepsilon]$, such that $\chi^{-1}(0)=\partial M$ and χ maps the complement of a neighborhood of ∂M to ε, where ε is a small positive real number. The link $L_{X=Y}$ is the disjoint union of $L_{X=Y}\cap\mathcal{B}$ and a link L_2 of $M\setminus\mathcal{B}$, the link $L_{X=-Y}$ sits in $\mathcal{B}$, and

$$0=p_1(X,Y_0)=p_1(X,Y)=4lk(L_{X=Y},L_{X=-Y}).$$

In particular, $lk(L_{X=Y}, L_{X=-Y}) = lk(L_{X=Y} \cap \mathcal{B}, L_{X=-Y}) = 0$.

The parallelization $(X, \sigma, X \wedge \sigma)$ turns the restriction $Y_{|\mathcal{B}}$ to a map from $\mathcal{B}$ to $\mathbb{S}^2 = \mathbb{S}(\mathbb{R}X \oplus \mathbb{R}\sigma \oplus \mathbb{R}X \wedge \sigma)$ constant on $\partial\mathcal{B}$, thus to a map from $\mathcal{B}/\partial\mathcal{B} \simeq \mathbb{S}^3$ to $\mathbb{S}^2$, and it suffices to prove that this map is homotopic to the constant map to prove Theorem I.8. For this it suffices to prove that this map represents 0 in $\pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$.

There is a classical isomorphism from $\pi_3(\mathbb{S}^2)$ to $\mathbb{Z}$ that maps the class of a map g from $\mathbb{S}^3$ to $\mathbb{S}^2$ to the linking number of the preimages of two regular points of g under g (see [Pon41, Theorem 2]). It is easy to check that this map is well-defined, depends only on the homotopy class of g, and is a group morphism on $\pi_3(\mathbb{S}^2)$ that maps the class of the Hopf fibration $((z_1, z_2) \in (\mathbb{S}^3 \subset \mathbb{C}^2) \mapsto ({}^{z_1}/_{z_2}) \in (\mathbb{C}P^1 = \mathbb{S}^2))$ to ± 1. Therefore it is an isomorphism from $\pi_3(\mathbb{S}^2)$ to $\mathbb{Z}$. Since Y is in the kernel of this isomorphism, it is homotopically trivial so that Y is homotopic to a constant on B, relatively to the boundary of B, and Y_0 is homotopic to X on M, relatively to the boundary of M. □

4 Variation of Pontrjagin numbers under $\mathrm{LP}_{\mathbb{Q}}$-surgeries

4.1 For pseudo-parallelizations

In this subsection we recall the variation formula and the finite type property of Pontrjagin numbers of pseudo-parallelizations wich are contained in [Les10, Section 11].

Proposition I.4.1. *For M a compact oriented 3-manifold and $({}^B/_A)$ an $LP_{\mathbb{Q}}$-surgery in M, if $\overline{\tau}_M$ and $\overline{\tau}_{M({}^B/_A)}$ are pseudo-parallelizations of M and $M({}^B/_A)$ which coincide on $M \setminus \mathring{A}$ and coincide with a genuine parallelization on ∂A, then*

$$p_1(\overline{\tau}_{M({}^B/_A)}, \overline{\tau}_M) = p_1(\overline{\tau}_{M({}^B/_A)|B}, \overline{\tau}_{M|A}).$$

Proof. Let W^- be a signature zero cobordism from A to B. By definition, the obstruction $p_1(\overline{\tau}_{M({}^B/_A)|B}, \overline{\tau}_{M|A})$ is the Pontrjagin obstruction to extending the complex trivialization $\tau(\overline{\tau}_{M({}^B/_A)|B}, \overline{\tau}_{M|A})$ of $TW^-_{|\partial W^-} \otimes \mathbb{C}$ as a trivialization of $TW^- \otimes \mathbb{C}$. Let $W^+ = [0,1] \times (M \setminus \mathring{A})$ and let $V = -[0,1] \times \partial A$. As shown in [Les13, Proof of Proposition 5.3 item 2], since $({}^B/_A)$ is an $\mathrm{LP}_{\mathbb{Q}}$-surgery in M, the manifold $W = W^+ \cup_V W^-$ has signature zero. Furthermore, since W has signature zero, $p_1(\overline{\tau}_{M({}^B/_A)}, \overline{\tau}_M)$ is the Pontrjagin obstruction to extending the trivialization $\tau(\overline{\tau}_{M({}^B/_A)}, \overline{\tau}_M)$ of $TW_{|\partial W} \otimes \mathbb{C}$

as a trivialization of $TW \otimes \mathbb{C}$. Finally, it is clear that $\tau(\overline{\tau}_{M(B/A)}, \overline{\tau}_M)$ extends as a trivialization of $TW_{|W^+} \otimes \mathbb{C}$ so that

$$p_1(\overline{\tau}_{M(B/A)}, \overline{\tau}_M) = p_1(\overline{\tau}_{M(B/A)|B}, \overline{\tau}_{M|A}).$$

□

Corollary I.4.2. *Let M be a compact oriented 3-manifold and let $\{{}^{B_i}/_{A_i}\}_{i\in\{1,\dots,k\}}$ be a family of disjoint $LP_{\mathbb{Q}}$-surgeries where $k \geqslant 2$. For any family $\{\overline{\tau}_I\}_{I\subset\{1,\dots,k\}}$ of pseudo-parallelizations of the $\{M(\{{}^{B_i}/_{A_i}\}_{i\in I})\}_{I\subset\{1,\dots,k\}}$ whose links sit in $M \setminus (\cup_{i=1}^k \partial A_i)$ and such that, for all subsets $I, J \subset \{1, \dots, k\}$, $\overline{\tau}^I$ and $\overline{\tau}^J$ coincide on $(M \setminus \cup_{I\cup J} A_i) \cup_{I\cap J} B_i$, the following identity holds :*

$$\sum_{I\subset\{2,\dots,k\}} (-1)^{|I|} p_1(\overline{\tau}^I, \overline{\tau}^{I\cup\{1\}}) = 0.$$

4.2 Lemmas for the proof of Theorem I.10

Lemma I.4.3. *If X is a combing of a compact oriented 3-manifold M and if ∂A is the connected boundary of a submanifold of M of dimension 3, then the normal bundle $X^{\perp}_{|\partial A}$ admits a nonvanishing section.*

Proof. Parallelize M so that X induces a map $X_{|\partial A} : \partial A \to \mathbb{S}^2$. This map must have degree 0 since X extends this map to A (so $(X_{|\partial A})_* : H_2(\partial A; \mathbb{Q}) \to H_2(\mathbb{S}^2; \mathbb{Q})$ factors through the inclusion $H_2(\partial A; \mathbb{Q}) \to H_2(A; \mathbb{Q})$ which is zero). It follows that $X_{|\partial A}$ is homotopic to the map $(m \in \partial A \mapsto e_1 \in \mathbb{S}^2)$ whose normal bundle admits a nonvanishing section. □

Lemma I.4.4. *Let (M, X) be a compact oriented 3-manifold equipped with a combing, let $({}^B/_A, X_B)$ be an $LP_{\mathbb{Q}}$-surgery in (M, X) and let σ be a nonvanishing section of $X^{\perp}_{|\partial A}$. Let P stand for Poincaré duality isomorphisms and recall the sequence of isomorphisms induced by the inclusions*

$$H_1(A; \mathbb{Q}) \xleftarrow{i^A_*} \frac{H_1(\partial A; \mathbb{Q})}{\mathfrak{L}_A} = \frac{H_1(\partial B; \mathbb{Q})}{\mathfrak{L}_B} \xrightarrow{i^B_*} H_1(B; \mathbb{Q}).$$

The class $\left(i^A_* \circ (i^B_*)^{-1} \left(P(e^B_2(X^{\perp}_B, \sigma))\right) - P(e^A_2(X^{\perp}_{|A}, \sigma))\right)$ in $H_1(A; \mathbb{Q})$ is independent of the choice of the section σ.

Proof. Let us drop the inclusions i^B_* and i^A_* from the notation. According to Proposition I.1.9, the class $P(e^A_2(X^{\perp}_{|A}, \sigma))$ verifies

$$[X(A) - (-X)(A) + \mathcal{H}^{-X}_{X,\sigma}(\partial A \times [0, 1])] = [P(e^A_2(X^{\perp}_{|A}, \sigma)) \times \mathbb{S}^2] \text{ in } H_1(UA; \mathbb{Q}).$$

It follows that, for another choice σ' of section of $X^{\perp}_{|\partial A}$,

$$\begin{aligned}[P(e_2^B(X_B^{\perp},\sigma))\times\mathbb{S}^2 - P(e_2^B(X_B^{\perp},\sigma'))\times\mathbb{S}^2]\\ &= [\mathcal{H}^{-X}_{X,\sigma}(\partial A\times[0,1]) - \mathcal{H}^{-X}_{X,\sigma'}(\partial A\times[0,1])]\\ &= [P(e_2^A(X^{\perp}_{|A},\sigma))\times\mathbb{S}^2 - P(e_2^A(X^{\perp}_{|A},\sigma'))\times\mathbb{S}^2].\end{aligned}$$

□

Lemma I.4.5. *Let (M,X) be a compact oriented 3-manifold equipped with a combing and let $({}^B/_A, X_B)$ be an $LP_{\mathbb{Q}}$-surgery in (M,X). If (X,σ) is a torsion combing then $(X({}^B/_A),\sigma)$ is a torsion combing if and only if*

$$i_*^A\circ(i_*^B)^{-1}\big(P(e_2^B({X_B}^{\perp},\zeta))\big) - P(e_2^A(X^{\perp}_{|A},\zeta)) = 0 \quad \text{in } H_1(M;\mathbb{Q})$$

for some nonvanishing section ζ of $X^{\perp}_{|\partial A}$.

Proof. By definition, we have

$$\begin{aligned}P(e_2(X^{\perp},\sigma) &= P(e_2^A(X^{\perp}_{|A},\zeta)) + P(e_2^{M\setminus\mathring{A}}(X^{\perp},\sigma\cup\zeta))\\ P(e_2(X({}^B/_A)^{\perp},\sigma) &= P(e_2^B(X_B^{\perp},\zeta)) + P(e_2^{M\setminus\mathring{A}}(X'^{\perp},\sigma\cup\zeta))\end{aligned}$$

where ζ is any nonvanishing section of $X^{\perp}_{|\partial A}$. So, it follows that, using appropriate identifications,

$$P(e_2(X({}^B/_A)^{\perp},\sigma)) - P(e_2(X^{\perp},\sigma)) = P(e_2^B(X_B^{\perp},\zeta)) - P(e_2^A(X^{\perp}_{|A},\zeta)).$$

If X is a torsion combing, then $P(e_2(X^{\perp},\sigma))$ is rationally null-homologous in M. Hence, $X({}^B/_A)$ is a torsion combing if and only if

$$i_*^A\circ(i_*^B)^{-1}\big(P(e_2^B({X_B}^{\perp},\zeta))\big) - P(e_2^A(X^{\perp}_{|A},\zeta)) = 0 \ \text{ in } H_1(M;\mathbb{Q}).$$

□

Lemma I.4.6. *Let (M,X) be a compact oriented 3-manifold equipped with a combing. Let $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in\{1,\dots,k\}}$ be a family of disjoint $LP_{\mathbb{Q}}$-surgeries in (M,X), where $k\in\mathbb{N}\setminus\{0\}$. For all $I\subset\{1,\dots,k\}$, let $M_I = M(\{{}^{B_i}/_{A_i}\}_{i\in I})$ and $X^I = X(\{{}^{B_i}/_{A_i}\}_{i\in I})$. There exists a family of pseudo-parallelizations $\{\bar\tau^I\}_{I\subset\{1,\dots,k\}}$ of the $\{M_I\}_{I\subset\{1,\dots,k\}}$ such that :*

(i) for all $I\subset\{1,\dots,k\}$, $\bar\tau^I$ is compatible with X^I,

(ii) for all $I\subset\{1,\dots,k\}$, if γ^I denotes the link of $\bar\tau_I$, then $N(\gamma^I)\cap\big(\cup_{i=1}^k\partial A_i\big)=\emptyset$,

(iii) the third vector of $\bar\tau$ coincides with X on $\cup_{i=1}^k\partial A_i$,

(iv) for all $I, J \subset \{1, \dots, k\}$, $\bar{\tau}^I$ and $\bar{\tau}^J$ coincide on $(M \setminus \cup_{i \in I \cup J} A_i) \cup_{i \in I \cap J} B_i$,

(v) there exist links $L^{\pm}_{A_i}$ in A_i, $L^{\pm}_{B_i}$ in B_i and $L^{\pm}_{ext}$ in $M \setminus \cup_{i=1}^k \mathring{A}_i$ such that, for $I \subset \{1, \dots, k\}$:

$$2 \cdot L_{\bar{\tau}^I = X^I} = L_{E^{d^I} = X^I} + L_{E^{g^I} = X^I} = L^+_{ext} + \sum_{i \in I} L^+_{B_i} + \sum_{i \in \{1,\dots,k\} \setminus I} L^+_{A_i},$$

$$2 \cdot L_{\bar{\tau}^I = -X^I} = L_{E^{d^I} = -X^I} + L_{E^{g^I} = -X^I} = L^-_{ext} + \sum_{i \in I} L^-_{B_i} + \sum_{i \in \{1,\dots,k\} \setminus I} L^-_{A_i},$$

where E^{d^I} and E^{g^I} are the Siamese sections of $\bar{\tau}^I$.

Proof. Let $\mathcal{C}$ denote a collar of $\cup_{i=1}^k \partial A_i$. Using Lemma I.4.3, construct a trivialization τ_e of $\cup_{i=1}^k TM_{|\mathcal{C}}$ so that its third vector coincides with X on $\mathcal{C}$. Then use Lemma I.1.13 to extend τ_e as pseudo-parallelizations of the $\{A_i\}_{i \in \{1,\dots,k\}}$, of the $\{B_i\}_{i \in \{1,\dots,k\}}$ and of $M \setminus (\cup_{i=1}^k \mathring{A}_i)$. Finally, use these pseudo-parallelizations to construct the pseudo-parallelizations of the 3-manifolds $\{M_I\}_{I \subset \{1,\dots,k\}}$ as in the statement. □

Lemma I.4.7. *In the context of Lemma I.4.6, using the sequence of isomorphisms induced by the inclusions*

$$H_1(A_i; \mathbb{Q}) \xleftarrow{i_*^{A_i}} \frac{H_1(\partial A_i; \mathbb{Q})}{\mathfrak{L}_{A_i}} = \frac{H_1(\partial B_i; \mathbb{Q})}{\mathfrak{L}_{B_i}} \xrightarrow{i_*^{B_i}} H_1(B_i; \mathbb{Q}),$$

for all $i \in \{1, \dots, k\}$, we have that, in $H_1(\partial A_i; \mathbb{Q})/\mathfrak{L}_{A_i}$,

$$[L^{\pm}_{B_i} - L^{\pm}_{A_i}] = \pm\big((i_*^{B_i})^{-1}(P(e_2^{B_i}(X^{\perp}_{B_i}, \sigma_i))) - (i_*^{A_i})^{-1}(P(e_2^{A_i}(X^{\perp}_{|A_i}, \sigma_i)))\big)$$

where σ_i is any nonvanishing section of $X^{\perp}_{|\partial A_i}$.

Proof. For the proof, let us drop the inclusions i_*^B and i_*^A from the notation. Let $i \in \{1, \dots, k\}$. According to Lemma I.4.5, it is enough to prove the statement for a particular non vanishing section σ_i of $X^{\perp}_{|\partial A_i}$. Note that A_i is equipped with a combing $X_{|A_i}$ and a pseudo-parallelization $\bar{\tau}_{|A_i} = (N(\gamma \cap A_i); \tau_{e|A_i}, \tau_{d|A_i}, \tau_{g|A_i})$ such that $X_{|\partial A_i}$ coincides with $E^e_{3|\partial A_i}$ where $\tau_e = (E^e_1, E^e_2, E^e_3)$. Furthermore,

$$L^+_{A_i} = 2 \cdot L_{\bar{\tau}_{|A_i} = X_{|A_i}} \quad \text{and} \quad L^-_{A_i} = 2 \cdot L_{\bar{\tau}_{|A_i} = -X_{|A_i}}.$$

Construct a pseudo-parallelization $\check{\tau} = (N(\check{\gamma}); \check{\tau}_e, \check{\tau}_d, \check{\tau}_g)$ of A_i by modifying $\bar{\tau}$ as follows so that $\check{\tau}$ and X coincide on ∂A_i. Consider a collar $\mathcal{C} = [0,1] \times \partial A_i$ of ∂A_i such that $\{1\} \times \partial A_i = \partial A_i$ and $\mathcal{C} \cap \gamma = \emptyset$. Let $\check{\tau}$ coincide with $\bar{\tau}_{|A_i}$ on $\overline{A_i \setminus \mathcal{C}}$. End the construction of $\check{\tau}$ by requiring

$$\forall (s,b) \in \mathcal{C} = [0,1] \times \partial A_i,\ \forall v \in \mathbb{S}^2 \ :\ \check{\tau}_e((s,b), v) = \tau_e\left((s,b), R_{e_2, \frac{-\pi s}{2}}(v)\right).$$

Without loss of generality, assume that $X_{|A_i}$ coincides with E_3^e on the collar $\mathcal{C}$. Note that $\check{\tau}$ and $X_{|A_i}$ are compatible and that

$$L_{A_i}^+ = 2 \cdot L_{\check{\tau}=X_{|A_i}} \quad \text{and} \quad L_{A_i}^- = 2 \cdot L_{\check{\tau}=-X_{|A_i}}.$$

Using $\check{\tau}$ and Proposition I.1.10, we get

$$\begin{aligned}
[L_{A_i}^+] &= P(e_2^{A_i}(X_{|A_i}^\perp, \check{E}_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{A_i}(\check{E}_1^{d\perp}, \check{E}_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{A_i}(\check{E}_1^{g\perp}, \check{E}_{2|\partial A_i}^e)) \\
[L_{A_i}^-] &= -P(e_2^{A_i}(X_{|A_i}^\perp, \check{E}_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{A_i}(\check{E}_1^{d\perp}, \check{E}_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{A_i}(\check{E}_1^{g\perp}, \check{E}_{2|\partial A_i}^e))
\end{aligned}$$

where $\check{\tau}_e = (\check{E}_1^e, \check{E}_2^e, \check{E}_3^e)$ and where $\check{E}_1^d$ and $\check{E}_1^g$ are the Siamese sections of $\check{\tau}$. By construction, it follows that

$$\begin{aligned}
[L_{A_i}^+] &= P(e_2^{A_i}(X_{|A_i}^\perp, E_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{A_i}(E_{1\,|A_i}^{d\perp}, E_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{A_i}(E_{1\,|A_i}^{g\perp}, E_{2|\partial A_i}^e)) \\
[L_{A_i}^-] &= -P(e_2^{A_i}(X_{|A_i}^\perp, E_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{A_i}(E_{1\,|A_i}^{d\perp}, E_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{A_i}(E_{1\,|A_i}^{g\perp}, E_{2|\partial A_i}^e))
\end{aligned}$$

where E_1^d and E_1^g are the Siamese sections of $\bar{\tau}$. Using the same method, we also get that

$$\begin{aligned}
[L_{B_i}^+] &= P(e_2^{B_i}(X_{B_i}^\perp, E_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{B_i}(E_{1\,|B_i}^{d\perp}, E_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{B_i}(E_{1\,|B_i}^{g\perp}, E_{2|\partial A_i}^e)) \\
[L_{B_i}^-] &= -P(e_2^{B_i}(X_{B_i}^\perp, E_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{B_i}(E_{1\,|B_i}^{d\perp}, E_{2|\partial A_i}^e)) + \frac{1}{2}P(e_2^{B_i}(E_{1\,|B_i}^{g\perp}, E_{2|\partial A_i}^e)).
\end{aligned}$$

Finally, using Lemma I.2.3,

$$[L_{B_i}^\pm - L_{A_i}^\pm] = \pm\big(P(e_2^{B_i}(X_{B_i}^\perp, E_{2|\partial A_i}^e)) - P(e_2^{A_i}(X_{|A_i}^\perp, E_{2|\partial A_i}^e))\big) \text{ in } H_1(\partial A_i;\mathbb{Q})/\mathfrak{L}_{A_i}.$$

□

4.3 Variation formula for torsion combings

Proof of Theorem I.10. Let (M, X) be a compact oriented 3-manifold equipped with a combing. Let $\{({}^{B_i}/_{A_i}, X_{B_i})\}_{i\in\{1,2\}}$ be two disjoint $\mathrm{LP}_{\mathbb{Q}}$-surgeries in (M, X) and assume that $\{X^I = X(\{{}^{B_i}/_{A_i}\}_{i\in I})\}_{I\subset\{1,2\}}$ is a family of torsion combings of the $\{M_I = M(\{{}^{B_i}/_{A_i}\}_{i\in I})\}_{I\subset\{1,2\}}$. Note that, for all $I, J \subset \{1,2\}$, X^I and X^J coincide on $(M \setminus \cup_{i\in I\cup J} A_i) \cup_{i\in I\cap J} B_i$. Finally, let $\{\bar{\tau}^I\}_{I\subset\{1,2\}}$ be a family of pseudo-parallelizations as in Lemma I.4.6, let $E_1^{d^I}$ and $E_1^{g^I}$ denote the Siamese sections of $\bar{\tau}^I$ for all $I \subset \{1,2\}$ and let L_I stand for $L_{E_1^{d^I}=-E_1^{g^I}}$. Using Corollary I.4.2, we have

$$\begin{aligned}
p_1\left([X^{\{2\}}], [X^{\{1,2\}}]\right) - p_1\left([X], [X^{\{1\}}]\right) &= p_1\left([X^{\{2\}}], [X^{\{1,2\}}]\right) - p_1\left(\bar{\tau}^{\{2\}}, \bar{\tau}^{\{1,2\}}\right) \\
&\quad - p_1\left([X], [X^{\{1\}}]\right) + p_1\left(\bar{\tau}, \bar{\tau}^{\{1\}}\right)
\end{aligned}$$

which, using Theorem I.4, reads

$$\begin{aligned}
&4\cdot lk_{M_{\{1,2\}}}(L_{\bar\tau^{\{1,2\}}=X^{\{1,2\}}}\ ,\ L_{\bar\tau^{\{1,2\}}=-X^{\{1,2\}}}) - 4\cdot lk_{M_{\{2\}}}(L_{\bar\tau^{\{2\}}=X^{\{2\}}}\ ,\ L_{\bar\tau^{\{2\}}=-X^{\{2\}}})\\
&- lk_{\mathbb{S}^2}\Big(e_1-(-e_1)\ ,\ P_{\mathbb{S}^2}\circ(\tau_d^{\{1,2\}})^{-1}\circ X^{\{1,2\}}(L_{\{1,2\}}) - P_{\mathbb{S}^2}\circ(\tau_d^{\{2\}})^{-1}\circ X^{\{2\}}(L_{\{2\}})\Big)\\
&- 4\cdot lk_{M_{\{1\}}}(L_{\bar\tau^{\{1\}}=X^{\{1\}}}\ ,\ L_{\bar\tau^{\{1\}}=-X^{\{1\}}}) + 4\cdot lk_M(L_{\bar\tau=X}\ ,\ L_{\bar\tau=-X})\\
&+ lk_{\mathbb{S}^2}\Big(e_1-(-e_1)\ ,\ P_{\mathbb{S}^2}\circ(\tau_d^{\{1\}})^{-1}\circ X^{\{1\}}(L_{\{1\}}) - P_{\mathbb{S}^2}\circ(\tau_d)^{-1}\circ X(L)\Big).
\end{aligned}$$

This can further be reduced to the following by using Lemma I.4.6,

$$\begin{aligned}
&lk_M\left(L^+_{ext}+L^+_{A_1}+L^+_{A_2},\ L^-_{ext}+L^-_{A_1}+L^-_{A_2}\right)\\
&- lk_{M_{\{1\}}}\left(L^+_{ext}+L^+_{B_1}+L^+_{A_2},\ L^-_{ext}+L^-_{B_1}+L^-_{A_2}\right)\\
&- lk_{M_{\{2\}}}\left(L^+_{ext}+L^+_{A_1}+L^+_{B_2},\ L^-_{ext}+L^-_{A_1}+L^-_{B_2}\right)\\
&+ lk_{M_{\{1,2\}}}\left(L^+_{ext}+L^+_{B_1}+L^+_{B_2},\ L^-_{ext}+L^-_{B_1}+L^-_{B_2}\right).
\end{aligned}$$

In order to compute these linking numbers, let us construct specific 2-chains. Let us introduce a more convenient set of notations. For all $i\neq j\in\{1,2\}$, let

$$L^\pm_i = L^\pm_{A_i},\ L^\pm_{ij} = L^\pm_{A_i}+L^\pm_{A_j},\ L^\pm_{eij} = L^\pm_{ext}+L^\pm_{A_i}+L^\pm_{A_j}.$$

Set also similar notations with primed indices where a primed index i', $i\in\{1,2\}$, indicates that $L^\pm_{A_i}$ should be replaced by $L^\pm_{B_i}$. For instance, $L^\pm_{i'} = L^\pm_{B_i}$, $L^\pm_{i,j'} = L^\pm_{A_i}+L^\pm_{B_j}$, *etc.* Using these notations, $p_1\left([X^{\{2\}}],[X^{\{1,2\}}]\right) - p_1\left([X],[X^{\{1\}}]\right)$ reads :

$$lk_M(L^+_{e12},L^-_{e12}) - lk_{M_{\{1\}}}(L^+_{e1'2},L^-_{e1'2}) - lk_{M_{\{2\}}}(L^+_{e12'},L^-_{e12'}) + lk_{M_{\{1,2\}}}(L^+_{e1'2'},L^-_{e1'2'}).$$

Recall from Lemma I.2.4 that there exist rational 2-chains $\Sigma^\pm_{e12}$ of M which are bounded by the links $L^\pm_{e12}$. Similarly, there exist rational two chains $\Sigma^\pm_{e1'2'}$ bounded by $L^\pm_{e1'2'}$ in $M_{\{1,2\}}$. Note that, for all $i\in\{1,2\}$, the 2-chains $\Sigma^\pm_{e12}\cap A_i$ are cobordisms between the $L^\pm_i$ and 1-chains $\ell^\pm_i$ in ∂A_i. Similarly, for all $i\in\{1,2\}$, the 2-chains $\Sigma^\pm_{e1'2'}\cap B_i$ are cobordisms between the $L^\pm_{i'}$ and 1-chains $\ell^\pm_{i'}$ in ∂B_i. Furthermore, according to Lemma I.4.7, for all $i\in\{1,2\}$ and for any nonvanishing section σ_i of $X^\perp_{|\partial A_i}$, in $H_1(\partial A_i;\mathbb{Q})/\mathfrak{L}_{A_i}$:

$$[\ell^\pm_{i'}-\ell^\pm_i] = \pm\big((i^{B_i}_*)^{-1}(P(e_2^{B_i}(X^\perp_{B_i},\sigma_i))) - (i^{A_i}_*)^{-1}(P(e_2^{A_i}(X^\perp_{|A_i},\sigma_i)))\big).$$

So, according to Lemma I.4.5, for all $I\subset\{1,2\}$ containing i, there exists a 2-chain $S^I_{B_i/A_i}$ in M_I which is bounded by $\ell^+_{i'}-\ell^+_i$. Finally, since $({}^{B_1}/_{A_1})$ and $({}^{B_2}/_{A_2})$ are $LP_{\mathbb{Q}}$-surgeries, we can construct these chains so that

$$\begin{aligned}
S^{\{1\}}_{B_1/A_1}\cap(M_{\{1\}}\setminus\mathring{A}_2) &= S^{\{1,2\}}_{B_1/A_1}\cap(M_{\{1,2\}}\setminus\mathring{B}_2),\\
S^{\{2\}}_{B_2/A_2}\cap(M_{\{2\}}\setminus\mathring{A}_1) &= S^{\{1,2\}}_{B_2/A_2}\cap(M_{\{1,2\}})\setminus\mathring{B}_1).
\end{aligned}$$

Let us now return to the computation of $p_1\left([X^{\{2\}}],[X^{\{1,2\}}]\right)-p_1\left([X],[X^{\{1\}}]\right)$. Using the 2-chains we constructed, we have :

$$\begin{aligned}
lk_M(L^+_{e12},L^-_{e12}) &= \langle \Sigma^+_{e12}, L^-_{e12}\rangle\\
lk_{M_{\{1\}}}(L^+_{e1'2},L^-_{e1'2}) &= \langle \Sigma^+_{e12}\cap(M\setminus \mathring{A}_1)+S^{\{1\}}_{B_1/A_1}+\Sigma^+_{e1'2'}\cap B_1, L^-_{e1'2}\rangle\\
lk_{M_{\{2\}}}(L^+_{e12'},L^-_{e12'}) &= \langle \Sigma^+_{e12}\cap(M\setminus \mathring{A}_2)+S^{\{2\}}_{B_2/A_2}+\Sigma^+_{e1'2'}\cap B_2, L^-_{e12'}\rangle\\
lk_{M_{\{1,2\}}}(L^+_{e1'2'},L^-_{e1'2'}) &= \langle \Sigma^+_{e12}\cap(M\setminus (\mathring{A}_1\cup\mathring{A}_2))+S^{\{1,2\}}_{B_1/A_1}\\
&\qquad + S^{\{1,2\}}_{B_2/A_2}+\Sigma^+_{e1'2'}\cap(B_1\cup B_2), L^-_{e1'2'}\rangle.
\end{aligned}$$

So, the contribution of the intersections in $M\setminus(\mathring{A}_1\cup\mathring{A}_2)$ is zero since it reads :

$$\begin{aligned}
&\langle \Sigma^+_{e12}\cap(M\setminus(\mathring{A}_1\cup\mathring{A}_2)), L^-_e\rangle_{M\setminus(\mathring{A}_1\cup\mathring{A}_2)}\\
&-\langle \Sigma^+_{e12}\cap(M\setminus(\mathring{A}_1\cup\mathring{A}_2))+S^{\{1\}}_{B_1/A_1}\cap(M\setminus(\mathring{A}_1\cup\mathring{A}_2)), L^-_e\rangle_{M\setminus(\mathring{A}_1\cup\mathring{A}_2)}\\
&-\langle \Sigma^+_{e12}\cap(M\setminus(\mathring{A}_1\cup\mathring{A}_2))+S^{\{2\}}_{B_2/A_2}\cap(M\setminus(\mathring{A}_1\cup\mathring{A}_2)), L^-_e\rangle_{M\setminus(\mathring{A}_1\cup\mathring{A}_2)}\\
&+\langle \Sigma^+_{e12}\cap(M\setminus(\mathring{A}_1\cup\mathring{A}_2))+S^{\{1,2\}}_{B_1/A_1}\cap(M\setminus(\mathring{A}_1\cup\mathring{A}_2))\\
&\qquad\qquad + S^{\{1,2\}}_{B_2/A_2}\cap(M\setminus(\mathring{A}_1\cup\mathring{A}_2)), L^-_e\rangle_{M\setminus(\mathring{A}_1\cup\mathring{A}_2)}.
\end{aligned}$$

The contribution in A_1 is

$$\langle \Sigma^+_{e12}\cap A_1, L^-_1\rangle_{A_1}-\langle \Sigma^+_{e12}\cap A_1+S^{\{2\}}_{B_2/A_2}\cap A_1, L^-_1\rangle_{A_1}=-\langle S^{\{2\}}_{B_2/A_2}\cap A_1, L^-_1\rangle_{A_1}.$$

In A_2, we get

$$\langle \Sigma^+_{e12}\cap A_2, L^-_2\rangle_{A_2}-\langle \Sigma^+_{e12}\cap A_2+S^{\{1\}}_{B_1/A_1}\cap A_2, L^-_2\rangle_{A_2}=-\langle S^{\{1\}}_{B_1/A_1}\cap A_2, L^-_2\rangle_{A_2}.$$

Similarly, the contribution in B_1 is

$$\begin{aligned}
&-\langle S^{\{1\}}_{B_1/A_1}\cap B_1+\Sigma^+_{e1'2'}\cap B_1, L^-_{1'}\rangle_{B_1}+\langle S^{\{1,2\}}_{B_1/A_1}\cap B_1+S^{\{1,2\}}_{B_2/A_2}\cap B_1+\Sigma^+_{e1'2'}\cap B_1, L^-_{1'}\rangle_{B_1}\\
&=\langle S^{\{1,2\}}_{B_2/A_2}\cap B_1, L^-_{1'}\rangle_{B_1}
\end{aligned}$$

and, in B_2, we have

$$\begin{aligned}
&-\langle S^{\{2\}}_{B_2/A_2}\cap B_2+\Sigma^+_{e1'2'}\cap B_2, L^-_{2'}\rangle_{B_2}+\langle S^{\{1,2\}}_{B_2/A_2}\cap B_2+S^{\{1,2\}}_{B_1/A_1}\cap B_2+\Sigma^+_{e1'2'}\cap B_2, L^-_{2'}\rangle_{B_2}\\
&=\langle S^{\{1,2\}}_{B_1/A_1}\cap B_2, L^-_{2'}\rangle_{B_2}.
\end{aligned}$$

Eventually, for all $i\in\{1,2\}$, set $L_{\{X^I\}}({}^{B_i}/_{A_i})=(i_*^{A_i})^{-1}([\ell^+_{i'}-\ell^+_i])$, recall that

$[\ell_{i'}^{+} - \ell_{i}^{+}] = -[\ell_{i'}^{-} - \ell_{i}^{-}]$ in $H_1(\partial A_i;\mathbb{Q})/\mathfrak{L}_{A_i}$, and complete the computations :

$$\begin{aligned}
&p_1\left([X^{\{2\}}],[X^{\{1,2\}}]\right) - p_1\left([X],[X^{\{1\}}]\right)\\
&\quad= \langle S^{\{1,2\}}_{B_2/A_2} \cap B_1, L^{-}_{1'}\rangle_{B_1} + \langle S^{\{1,2\}}_{B_1/A_1} \cap B_2, L^{-}_{2'}\rangle_{B_2}\\
&\quad- \langle S^{\{2\}}_{B_2/A_2} \cap A_1, L^{-}_{1}\rangle_{A_1} - \langle S^{\{1\}}_{B_1/A_1} \cap A_2, L^{-}_{2}\rangle_{A_2}\\
&\quad= \langle S^{\{1,2\}}_{B_2/A_2}, \ell^{-}_{1'}\rangle_{M_{\{1,2\}}} + \langle S^{\{1,2\}}_{B_1/A_1}, \ell^{-}_{2'}\rangle_{M_{\{1,2\}}}\\
&\quad- \langle S^{\{2\}}_{B_2/A_2}, \ell^{-}_{1}\rangle_{M_{\{2\}}} - \langle S^{\{1\}}_{B_1/A_1}, \ell^{-}_{2}\rangle_{M_{\{1\}}}\\
&\quad= \langle S^{\{1,2\}}_{B_2/A_2}, \ell^{-}_{1'} - \ell^{-}_{1}\rangle_{M_{\{1,2\}}} + \langle S^{\{1,2\}}_{B_1/A_1}, \ell^{-}_{2'} - \ell^{-}_{2}\rangle_{M_{\{1,2\}}}\\
&\quad= lk_{M_{\{1,2\}}}\left(L_{\{X^I\}}({}^{B_2}/_{A_2})\ ,\ -L_{\{X^I\}}({}^{B_1}/_{A_1})\right)\\
&\quad+\ lk_{M_{\{1,2\}}}\left(L_{\{X^I\}}({}^{B_1}/_{A_1})\ ,\ -L_{\{X^I\}}({}^{B_2}/_{A_2})\right)\\
&\quad= -2\cdot lk_{M_{\{1,2\}}}\left(L_{\{X^I\}}({}^{B_1}/_{A_1})\ ,\ L_{\{X^I\}}({}^{B_2}/_{A_2})\right)\\
&\quad= -2\cdot lk_{M}\left(L_{\{X^I\}}({}^{B_1}/_{A_1})\ ,\ L_{\{X^I\}}({}^{B_2}/_{A_2})\right).
\end{aligned}$$

□

Chapitre II

A combinatorial proof of the relation $\Theta = 3 \cdot \lambda_{\mathrm{CW}} + {}^1\!/_4 \cdot p_1$

Introduction

Context

In [Kon94], M. Kontsevich outlined the definition of an invariant of rational homology 3-spheres * Z taking values in a space of Jacobi diagrams by using differential forms on configuration spaces. G. Kuperberg and D. Thurston proved that Z is a universal finite type invariant of integer homology 3-spheres with respect to Torelli surgeries (see [KT99]). Later on, in [Les04a], C. Lescop revisited the construction of Z.

The invariant Z is a graded object, and the above authors also described the lowest degree part of Z which is called the Θ-invariant. More precisely, the Θ-invariant is the simplest invariant of parallelized rational homology 3-spheres, defined using configuration spaces. In [KT99], G. Kuperberg and D. Thurston proved that, for any integer homology 3-sphere M equipped with a parallelization τ, the value $\Theta(M, \tau)$ decomposes as $6 \cdot \lambda(M) + {}^1\!/_4 \cdot p_1(\tau)$ where λ is the Casson invariant and where $p_1(\tau)$ is the Pontrjagin number of τ.

In [Les04b], C. Lescop generalized this formula for rational homology 3-spheres, and in [Les15b], she showed that the Θ-invariant is also a homotopy invariant of combings in rational homology 3-spheres for which this formula extends : if M is a rational homology 3-sphere equipped with a homotopy class of a combing $[X]$,

$*$. Recall that, according to our definition, rational homology spheres are always oriented in this thesis.

then

$$\Theta(M, [X]) = 3 \cdot \lambda_{\mathrm{cw}}(M) + {}^1\!/_4 \cdot p_1([X]) \qquad (\star)$$

where λ_{cw} is Walker's generalization of the Casson invariant which is normalized so that $\lambda_{\mathrm{cw}}(M) = 2\lambda(M)$ for any integer homology 3-spheres M, and where p_1 is the extension of the Pontrjagin numbers for torsion combings of closed oriented 3-manifolds defined at the beginning of Chapter I.

Finally, in [Les15a], C. Lescop gave a combinatorial formula for Θ by using Heegaard diagrams and then she showed the invariance of this formula by using only combinatorics (see [Les16]). Following this work, Chapter II presents a combinatorial proof of the relation $(\star)$.

Setting and statements

In [Wal92], K. Walker developed an extension λ_{cw} of the Casson invariant for all rational homology 3-spheres. This invariant can be defined as follows.

For all real numbers p, q, r, let $\lfloor r \rfloor$ denote the integer part of r and set

$$((r)) = \begin{cases} 0 & \text{if } r \in \mathbb{Z}, \\ r - \lfloor r \rfloor - {}^1\!/_2 & \text{otherwise} \end{cases} \quad \text{and} \quad s(q,p) = \operatorname{sign}(p) \sum_{k=1}^{|p|} \left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{kq}{p}\right)\right).$$

For a knot K in a rational homology 3-sphere M, let $N(K)$ be a tubular neighborhood of K and let Δ_K denote the Alexander polynomial of $M \setminus \mathring{N}(K)$ normalized such that $\Delta_K(1) = 1$ and $\Delta_K(t^{-1}) = \Delta_K(t)$. If ℓ denotes one of the two generators of the Lagrangian with integral coefficients of $M \setminus \mathring{N}(K)$, *ie*

$$\ker(H_1(\partial(M \setminus \mathring{N}(K)); \mathbb{Z}) \longrightarrow H_1(M \setminus \mathring{N}(K); \mathbb{Z})/\mathcal{T})$$

where $\mathcal{T}$ is the torsion subgroup of $H_1(M \setminus \mathring{N}(K); \mathbb{Z})$, if a and b are primitive homology classes in $H_1(\partial(M \setminus \mathring{N}(K)); \mathbb{Z})$ such that $\langle a, \ell \rangle \neq 0$ and $\langle b, \ell \rangle \neq 0$, and if $\{x, y\}$ is a basis of $H_1(\partial(M \setminus \mathring{N}(K)); \mathbb{Z})$ such that $\langle x, y \rangle = 1$ and $\ell = dy$ for some $d \in \mathbb{Z}$, then define

$$\tau(a, b; \ell) = s(\langle x, b \rangle, \langle y, b \rangle) - s(\langle x, a \rangle, \langle y, a \rangle) + \frac{d^2 - 1}{12} \frac{\langle a, b \rangle}{\langle a, \ell \rangle \langle b, \ell \rangle} \qquad \text{([Wal92, p.81])}$$

The number $\tau(a, b; \ell)$ is independent of the choice of basis $\{x, y\}$.

Theorem ([Wal92, Theorem 5.1])**.** *There exists a unique $\mathbb{Q}$-valued invariant of rational homology 3-spheres λ_{cw} such that :*

(i) $\lambda_{\mathrm{cw}}(\mathbb{S}^3) = 0$,

(ii) *if K is a knot in a rational homology 3-sphere M, if ℓ is a generator of the Lagrangian with integral coefficients of $M \setminus \mathring{N}(K)$ and if a and b are primitive homology classes in $H_1(\partial(M \setminus \mathring{N}(K)); \mathbb{Z})$ such that $\langle a, \ell \rangle \neq 0$ and $\langle b, \ell \rangle \neq 0$, then*

$$\lambda_{\mathrm{cw}}(M_b) = \lambda_{\mathrm{cw}}(M_a) + \tau(a, b; \ell) + \frac{\langle a, b \rangle}{\langle a, \ell \rangle \langle b, \ell \rangle} \Delta''_K(1)$$

where M_a and M_b denote the results of Dehn fillings on $M \setminus \mathring{N}(K)$ where a and b bound disks, respectively.

According to [Les15b], the Θ-invariant for rational homology 3-spheres equipped with a homotopy class of a combing can be decomposed using the Casson-Walker invariant and Pontrjagin numbers of torsion combings of closed oriented 3-manifolds.

Theorem II.1. *Let M be a rational homology 3-sphere and $[X]$ be the homotopy class of a combing on M. If λ_{cw} denotes the Casson-Walker invariant and if $p_1([X])$ is the Pontrjagin number of $[X]$, then*

$$\Theta(M, [X]) = 3 \cdot \lambda_{\mathrm{cw}}(M) + {}^1\!/_4 \cdot p_1([X]).$$

In this chapter, we prove this formula starting from the combinatorial definition of the Θ-invariant of [Les16], completing the combinatorial study of Θ, which was started in [Les16]. The key tool of our proof is the following characterization of the Casson-Walker invariant using the finite type invariants theory for rational homology 3-spheres with respect to $\mathrm{LP}_{\mathbb{Q}}$-surgeries (see Section II.2).

Theorem II.2. *The Casson-Walker invariant λ_{cw} is the unique invariant of rational homology 3-spheres such that*

(i) *λ_{cw} is additive with respect to the connected sum,*

(ii) *λ_{cw} is a finite type invariant of degree at most 2 with respect to $LP_{\mathbb{Q}}$-surgeries,*

(iii) *$\lambda_{\mathrm{cw}}(M) = -\lambda_{\mathrm{cw}}(-M)$ for any rational homology 3-sphere M where $-M$ denotes the manifold M equipped with the opposite orientation,*

(iv) *$\lambda_{\mathrm{cw}}(\mathcal{P}) = -2$ where $\mathcal{P}$ is the Poincaré homology sphere constructed using a -1 surgery on the left-handed trefoil knot in $\mathbb{S}^3$,*

and where (ii) means that if $\{{}^{B_i}\!/_{A_i}\}_{i \in \{1,2,3\}}$ is a triple of disjoint $LP_{\mathbb{Q}}$-surgeries in a rational homology 3-sphere M, then

$$\sum_{I \subset \{1,2,3\}} (-1)^{|I|} \cdot \lambda_{\mathrm{cw}}(M(\{{}^{B_i}\!/_{A_i}\}_{i \in I})) = 0.$$

The main technical difficulty when applying this characterization in order to combinatorially prove that $\Theta(M, [X]) = 3 \cdot \lambda_{\mathrm{cw}}(M) + {}^1\!/_4 \cdot p_1([X])$, is to see that $\frac{1}{3}(\Theta - \frac{p_1}{4})$ is indeed a finite type invariant of degree at most two. A first step has been done in Chapter I where it has been proved that Pontrjagin numbers of torsion combings of compact oriented 3-manifolds have a finite type behaviour with respect to $\mathrm{LP}_{\mathbb{Q}}$-surgeries. The remaining work is achieved in this chapter by, first, using the work of D. Moussard decomposing $\mathrm{LP}_{\mathbb{Q}}$-surgeries into sequences of three kinds of elementary surgeries : connected sums, T_k-surgeries and Borromean surgeries (see Theorem II.1.19 and [Mou12]), and then, introducing standard Heegaard diagrams with respect to one of these surgeries so that computations using C. Lescop's formula for the Θ-invariant are made possible.

1 Elementary surgeries

1.1 T_k-surgeries

Definition II.1.1. Let $k \in \mathbb{N} \setminus \{0\}$, a *$k$-torus T_k* is a genus one rational homology handlebody such that there exist simple curves, m and ℓ, in ∂T_k verifying

- $H_1(\partial T_k; \mathbb{Z}) = \mathbb{Z}[m] \oplus \mathbb{Z}[\ell]$ and $\langle m, \ell \rangle = 1$,
- $k[m] = 0$ in $H_1(T_k; \mathbb{Z})$,
- there exists a simple curve γ in $\mathring{T}_k$ such that $[\ell] = k[\gamma]$ in $H_1(T_k; \mathbb{Z})$ and such that $H_1(T_k; \mathbb{Z}) = \mathbb{Z}_k[m] \oplus \mathbb{Z}[\gamma]$.

Lemma II.1.2 ([Mou12, Lemma 2.5])**.** *For any $k \in \mathbb{N} \setminus \{0\}$, there exists a k-torus.*

Let us reprove this result by constructing a particular k-torus.

Definition II.1.3. A *standard k-torus of longitude ℓ and meridian m* is a k-torus constructed as follows. Let Σ_2 be the standard closed oriented surface of genus two given with the following curves α_1, α_2, t_3, β_1 and β_2.

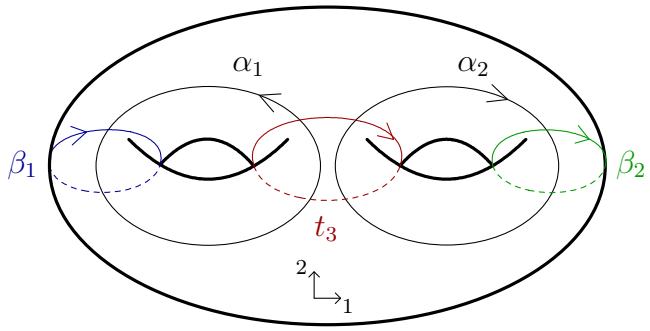


Figure II.1.4 – The oriented surface Σ_2 with the curves α_1, α_2, t_3, β_1 and β_2.

Consider the homeomorphism $h = (\delta_1 \circ \delta_2 \circ \delta_3^{-1})^k$, where δ_1 and δ_2 denote right Dehn twists along β_1 and β_2, and where δ_3 denotes a right Dehn twist along t_3. Define $\beta_{k,1} = h(\alpha_2)$ and $\beta_{k,2} = h(\alpha_1)$. See the case $k = 2$ depicted in Figure II.1.5 below.

Note that in $H_1(\Sigma_2; \mathbb{Z})$, we have $[\beta_{k,1}] = [\alpha_2] + k[\beta_1]$ and $[\beta_{k,2}] = [\alpha_1] + k[\beta_2]$. Indeed, for all $i \in \{1, 2\}$ and for all curve γ in Σ,

$$[\delta_i^k(\gamma)] = [\gamma] + k\langle \beta_i, \gamma \rangle \cdot [\beta_i]$$

and, since $[\beta_1] + [\beta_2] + [t_3] = 0$,

$$\begin{aligned}[\delta_3^{-k}(\gamma)] &= [\gamma] - k\langle t_3, \gamma\rangle \cdot [t_3] \\ &= [\gamma] - k\langle -\beta_1 - \beta_2, \gamma\rangle \cdot (-[\beta_1] - [\beta_2]) \\ &= [\gamma] - k\langle \beta_1 + \beta_2, \gamma\rangle \cdot [\beta_1] - k\langle \beta_1 + \beta_2, \gamma\rangle \cdot [\beta_2].\end{aligned}$$

Hence,

$$\begin{aligned}[\beta_{k,1}] &= [h(\alpha_2)] = [(\delta_1^k \circ \delta_2^k)(\alpha_2 + k\beta_1 + k\beta_2)] = [\delta_1^k(\alpha_2 - k\beta_2 + k\beta_1 + k\beta_2)] \\ &= [\alpha_2] + k[\beta_1], \\ [\beta_{k,2}] &= [h(\alpha_1)] = [(\delta_1^k \circ \delta_2^k)(\alpha_1 + k\beta_1 + k\beta_2)] = [\delta_1^k(\alpha_1 + k\beta_1 + k\beta_2)] \\ &= [\alpha_1 - k\beta_1 + k\beta_1 + k\beta_2] = [\alpha_1] + k[\beta_2].\end{aligned}$$

It follows that considering $\Sigma_2 \times [0,1]$ and gluing thickened disks along $\beta_{k,1} \times \{1\}$, $\beta_{k,2} \times \{1\}$ and $\alpha_1 \times \{0\}$, we get a manifold with two boundary components, ∂_1 homeomorphic to $\mathbb{S}^2$ and ∂_0 homeomorphic to a torus, which can be turned into a k-torus by gluing a standard 3-dimensional ball on ∂_0. Finally, set $m = \beta_2$ and $\ell = -\alpha_2$. With the notations of Definition II.1.1, $\gamma = \beta_1$.

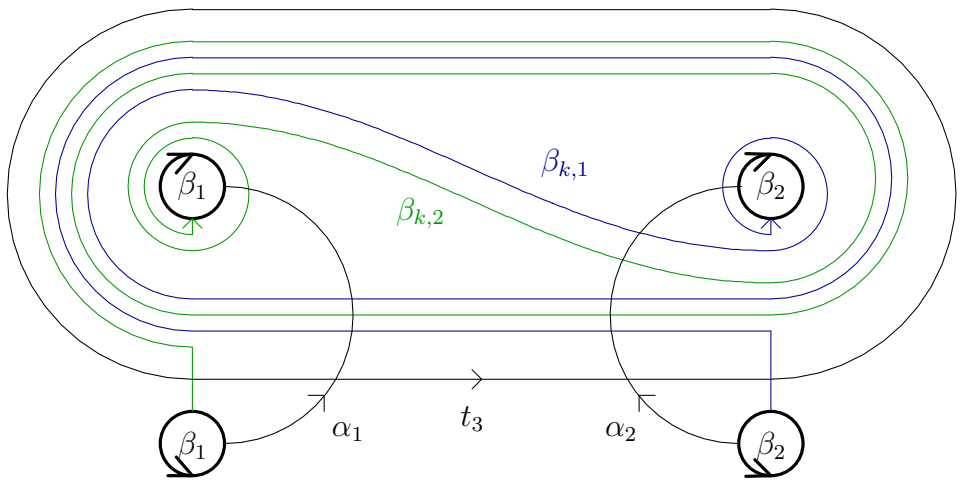


Figure II.1.5 – The curves $\beta_{k,1}$ and $\beta_{k,2}$ for the case $k = 2$. The surface Σ_2 has been cut along β_1 and β_2 in order to be drawn on the plane.

Definition II.1.6. Let M be a compact oriented 3-manifold and let K be a knot in $\mathring{M}$. For $k \in \mathbb{N} \setminus \{0\}$, a *$T_k$-surgery* on M along K consists in removing a tubular neighborhood of K and gluing back a k-torus with an $\mathrm{LP}_{\mathbb{Q}}$-identification of their boundaries.

Definition II.1.7. Let M be a compact oriented 3-manifold and let (K, μ) be a framed knot in $\mathring{M}$. For $k \in \mathbb{N} \setminus \{0\}$, a *standard T_k-surgery* on M along (K, μ) consists in removing a tubular neighborhood of K and gluing back a standard k-torus of longitude ℓ with the $\mathrm{LP}_{\mathbb{Q}}$-identification sending μ to ℓ.

Remark II.1.8. T_k-surgeries and standard T_k-surgeries are $\mathrm{LP}_{\mathbb{Q}}$-surgeries.

1.2 Borromean surgeries

Definition II.1.9. Consider the graph Y_0 below :

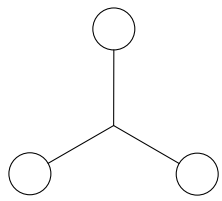


Figure II.1.10 – The graph Y_0.

Let M be a rational homology 3-sphere, let $S(Y_0)$ be a regular neighborhood of Y_0 in the plane and let $\psi : S(Y_0) \longrightarrow M$ be an embedding. The embedded framed graph $Y := \psi(Y_0)$ is called an *embedded Y-graph*. The images of the loops of Y_0 are the *leaves of Y*.

Definition II.1.11. Consider the following six-component framed link L_0 in $S(Y_0) \times [0, 1]$. Here the natural framing of $S(Y_0)$ should coincide with the blackboard framing convention. We will use this latter convention for all framed links.

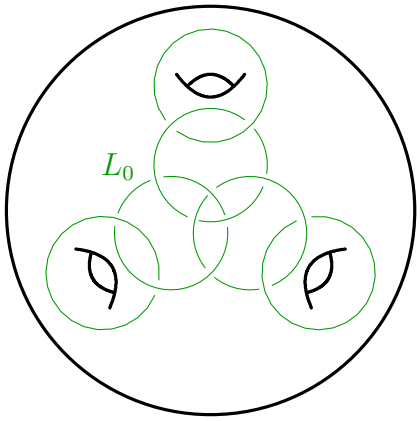


Figure II.1.12 – The link $\mathcal{L}_0$ in $S(Y_0) \times [0, 1]$.

Let M be a compact oriented 3-manifold and $\psi : S(Y_0) \longrightarrow M$ be an embedding and let $Y = \psi(S(Y_0))$. The embedding ψ can be extended to an embedding of

$S(Y_0) \times [0,1]$ onto a tubular neighborhood of Y. *A Borromean surgery along Y in M* is the surgery along the framed link $\psi(L_0)$ in M.

Remark II.1.13. There are two conventions for the link L_0 defining the Borromean surgeries. We use the choice of [Hab00b] and [Mei08]. In [GGP01], [AL05] and [Mou12], the chosen link is the mirror image of L_0. If Y is a Y-graph in a compact oriented 3-manifold A, and if B is the result of the Borromean surgery along Y using our convention, then $-B$ is the result of the Borromean surgery along Y in $-A$ with the convention of [GGP01].

Remark II.1.14. The inverse of a Borromean surgery is a Borromean surgery (see for instance [GGP01, Theorem 3.2]).

Proposition II.1.15. *Borromean surgeries are $LP_{\mathbb{Q}}$-surgeries.*

Proof. In [Mat87], S.V. Matveev proved that a Borromean surgery can be reformulated as removing a genus three handlebody and gluing it back using a non trivial homeomorphism. He even proved that Borromean surgeries are *$LP_{\mathbb{Z}}$-surgeries* which can be defined by replacing $\mathbb{Q}$ by $\mathbb{Z}$ in our definition of $\mathrm{LP}_{\mathbb{Q}}$-surgeries. Here, we give an alternative proof of this reformulation of Borromean surgeries adapted to our setting and prove that the homeomorphism associated to a Borromean surgery is an $\mathrm{LP}_{\mathbb{Q}}$-identification. Actually, we will get that this homeomorphism induces the identity on $H_1(\Sigma;\mathbb{Z})$, where Σ is the boundary of the handlebody.

Let M be a compact oriented 3-manifold, Y be an embedded Y-graph and $\mathcal{L}$ be the framed link defining the Borromean surgery along Y in M. Using standard Kirby calculus, we can reformulate a Borromean surgery as a surgery on the following framed link in a tubular neighborhood of Y denoted by H_Y.

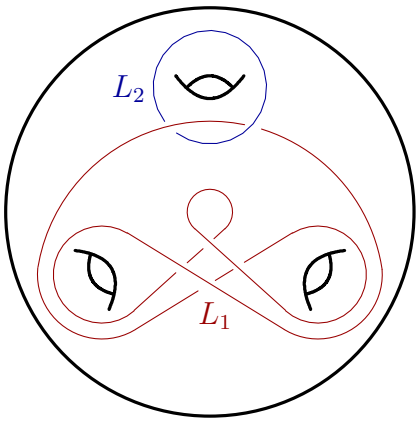


Figure II.1.16 – Alternative link for Borromean surgeries. Present in [Mei08, Lemma 3.2] and already used in [Hab00b] and [GGP01].

Now apply to this link the following generic sequence of Kirby moves and isotopies near two consecutive crossings on a two-component link $L_1 \cup L_2$.

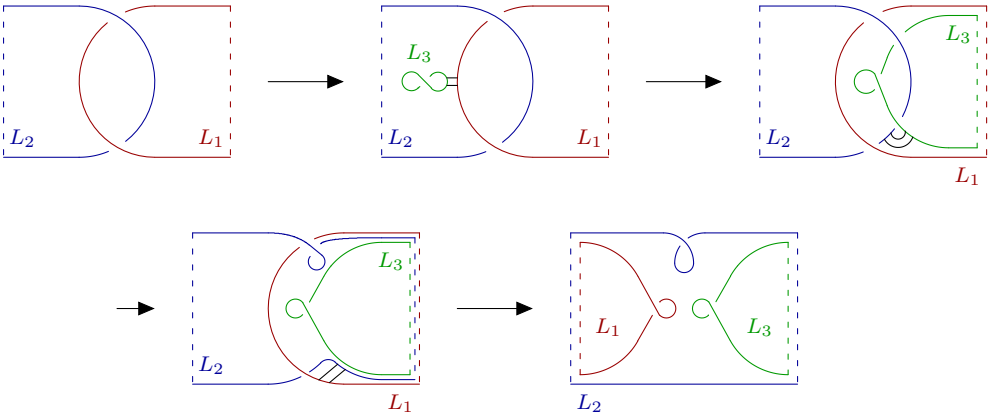


Figure II.1.17 – First add a (-1)-framed trivial component L_3 to the link $L_2 \cup L_1$. Then slide L_3 over L_1. Slide L_2 over L_3. Slide L_1 over L_2.

Finally, notice that the resulting three-component framed link can be pushed as three disjoint simple curves, t_1, t_2 and t_3, on ∂H_Y with some (± 1)-framing corrections (see Figure II.1.18 below). The surgery can now be translated as removing $\mathring{H}_Y$ and gluing it back so that the curves $\{(\delta_1 \circ \delta_2^{-1} \circ \delta_3)(\beta_i)\}_{i\in\{1,2,3\}}$ bound disks, where the $\{\delta_j\}_{j\in\{1,2,3\}}$ are right Dehn twists along the $\{t_j\}_{j\in\{1,2,3\}}$, and where $\{\beta_i\}_{i\in\{1,2,3\}}$ are the curves on ∂H_Y initially bounding disks depicted in Figure II.1.18. Since t_1, t_2 and t_3 cobound a pair of pants on ∂H_Y, the homeomorphism $\delta_1 \circ \delta_2^{-1} \circ \delta_3$ induces the identity on $H_1(\partial H_Y; \mathbb{Z})$ so it is an $\mathrm{LP}_{\mathbb{Q}}$-identification.

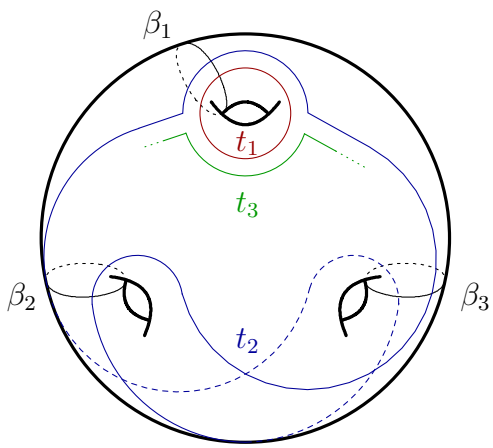


Figure II.1.18 – The curves t_1, t_2 and t_3. The curve t_3 is not entirely drawn : it is the result of sliding t_2 on t_1. □

1.3 Decomposing $\mathrm{LP}_{\mathbb{Q}}$-surgeries

Let us end this section by recalling D. Moussard's decomposition of $\mathrm{LP}_{\mathbb{Q}}$-surgeries and then adapting it to standard T_k-surgeries. Let an *elementary surgery* denote a Borromean surgery, a T_k-surgery or a connected sum with a rational homology 3-sphere. Note that elementary surgeries are $\mathrm{LP}_{\mathbb{Q}}$-surgeries. Furthermore,

Theorem II.1.19 ([Mou12, Theorem 1.15])**.** *If A and B are two rational homology handlebodies and if A and B have $LP_{\mathbb{Q}}$-identified boundaries, then they are related by a finite sequence of elementary surgeries and their inverses.*

This result can be slightly enhanced as follows. See also [Mou12, Remark p.2404 end of Subsection 2.3].

Theorem II.1.20. *If A and B are two rational homology handlebodies and if A and B have $LP_{\mathbb{Q}}$-identified boundaries, then they are related by a finite sequence of standard T_k-surgeries, Borromean surgeries, connected sums with rational homology 3-spheres and their inverses.*

In order to prove Theorem II.1.20, we first prove the next lemma.

Lemma II.1.21. *Let $k \in \mathbb{N} \setminus \{0\}$, let T and T' be two k-tori. There exists an $LP_{\mathbb{Q}}$-identification of ∂T and $\partial T'$ such that T and T' are related by a finite sequence of Borromean surgeries.*

Proof. Since T is a k-torus, there exist simple curves, m and ℓ, in ∂T such that $\langle m, \ell \rangle = 1$, $H_1(\partial T; \mathbb{Z}) = \mathbb{Z}[m] \oplus \mathbb{Z}[\ell]$ and such that $k[m] = 0$ in $H_1(T; \mathbb{Z})$. There also exists a simple curve γ in $\mathring{T}$ such that $[\ell] = k[\gamma]$ in $H_1(T; \mathbb{Z})$ and such that $H_1(T; \mathbb{Z}) = \mathbb{Z}_k[m] \oplus \mathbb{Z}[\gamma]$.

Similarly, since T' is also a k-torus, there exist simple curves, m' and ℓ', in $\partial T'$ such that $\langle m', \ell' \rangle = 1$, $H_1(\partial T'; \mathbb{Z}) = \mathbb{Z}[m'] \oplus \mathbb{Z}[\ell']$ and $k[m'] = 0$ in $H_1(T'; \mathbb{Z})$. Finally, there exists a simple curve γ' in $\mathring{T}'$ such that $[\ell'] = k[\gamma']$ in $H_1(T'; \mathbb{Z})$ and such that $H_1(T'; \mathbb{Z}) = \mathbb{Z}_k[m'] \oplus \mathbb{Z}[\gamma']$.

Let D be a disk in ∂T whose interior contains a simple arc s between two distinct points y and z. Let $\mathbb{D} = [-1, 1]^2$ and let $C = h(\mathbb{D} \times [0, 1])$ be a *tunnel around* γ, *ie* a cylinder in T where h is an embedding such that :

- $h((0, 0), 0) = y$ and $h((0, 0), 1) = z$,
- $h(\mathbb{D} \times \{0\})$ and $h(\mathbb{D} \times \{1\})$ are disjoint disks in $\mathring{D}$,
- $C \cap \partial T = h(\mathbb{D} \times \{0\}) \sqcup h(\mathbb{D} \times \{1\})$,
- $[h((0, 0) \times [0, 1]) \cup s] = [\gamma]$ in $H_1(T; \mathbb{Z})$.

Likewise, let $C' = h'(\mathbb{D} \times [0,1])$ be a tunnel around γ' in T', consider $\bar{T} = T \setminus \mathring{C}$ and $\bar{T}' = T' \setminus \mathring{C}'$, assume that γ and γ' sit on $\partial\bar{T}$ and $\partial\bar{T}'$, and finally, set

$$\delta = \partial h(\mathbb{D} \times \{0\}) \text{ and } \delta' = \partial h'(\mathbb{D} \times \{0\}).$$

According to [Mou12, Lemma 2.9], the manifolds $\bar{T}$ and $\bar{T}'$ are *integer homology handlebodies of genus 2*, *ie* they have the same homology with integral coefficients as the standard genus 2 handlebody. The *Lagrangian with integral coefficients* of an integer homology handlebody H is

$$\mathfrak{L}_H^{\mathbb{Z}} = \ker(i_*^H : H_1(\partial H; \mathbb{Z}) \longrightarrow H_1(H; \mathbb{Z})),$$

where i^H denotes the inclusion of ∂H into H. According to [AL05, Lemma 4.11] (or [Hab00a, Theorem 2.5]), if T and T' have boundaries identified by a homeomorphism ϕ such that $\phi(\mathfrak{L}_T^{\mathbb{Z}}) = \mathfrak{L}_{T'}^{\mathbb{Z}}$, then they are related by a finite sequence of Borromean surgeries. Let us define such an identification.

By construction, the classes $[m]$, $[\ell]$, $[\delta]$ and $[\gamma]$ generate $H_1(\partial\bar{T}; \mathbb{Z})$ and the classes $[m']$, $[\ell']$, $[\delta']$ and $[\gamma']$ generate $H_1(\partial\bar{T}'; \mathbb{Z})$. Furthermore, if S is a surface in T bounded by $k \cdot m$, then

$$\langle S, \gamma \rangle_T = {}^1\!/_k \cdot \langle S, \ell \rangle_T = \langle m, \ell \rangle_{\partial T} = 1.$$

It follows that $k[m] - [\delta] = 0$ in $H_1(\bar{T}; \mathbb{Z})$ and, similarly, $k[m'] - [\delta'] = 0$ in $H_1(\bar{T}'; \mathbb{Z})$. Finally, there exist s and s' in $\mathbb{Z}$ such that $[\ell] - k[\gamma] - s[\delta] = 0$ in $H_1(\bar{T}; \mathbb{Z})$ and $[\ell'] - k[\gamma'] - s'[\delta'] = 0$ in $H_1(\bar{T}'; \mathbb{Z})$. So, a homeomorphism $\phi : \partial\bar{T} \to \partial\bar{T}'$ such that

$$\phi : \begin{cases} m \longmapsto m' \\ \ell \longmapsto \ell' + (s' - s)k \cdot m' \\ \gamma \longmapsto \gamma' \\ \delta \longmapsto \delta' \end{cases}$$

satisfies $\phi(\mathfrak{L}_T^{\mathbb{Z}}) = \mathfrak{L}_{T'}^{\mathbb{Z}}$.

Gluing back C and C' to $\bar{T}$ and $\bar{T}'$ and restricting such a ϕ, we conclude that there is an $\mathrm{LP}_{\mathbb{Q}}$-identification of ∂T and $\partial T'$ such that T and T' are related by a finite sequence of Borromean surgeries. □

Proof of Theorem II.1.20. Let A and B be rational homology handlebodies with $\mathrm{LP}_{\mathbb{Q}}$-identified boundaries. According to Theorem II.1.19, there exists a finite sequence $\{{}^{E_i'}\!/_{E_i}\}_{i \in \{1,\dots,n\}}$ of elementary surgeries or their inverses relating A to B. Let $A_0 = A$, and let $A_i = A_{i-1}({}^{E_i'}\!/_{E_i})$ for all $i \in \{1, \dots, n\}$.

In order to prove Theorem II.1.20, it is enough to show that if $i \in \{1, \dots, n\}$ is such that $({}^{E'_i}\!/_{E_i})$ is a T_k-surgery, then there exists a sequence of standard T_k-surgeries and Borromean surgeries relating A_i and A_{i-1}.

Let i be such an index. By definition, E'_i is a k-torus, for a particular integer $k \in \mathbb{N} \setminus \{0\}$, E_i is the tubular neighborhood of a knot in A_{i-1} and the boundaries of E'_i and E_i are $\mathrm{LP}_{\mathbb{Q}}$-identified by a homeomorphism $h_i : \partial E_i \to \partial E'_i$. Let F be a standard k-torus. According to Lemma II.1.21, there exists an $\mathrm{LP}_{\mathbb{Q}}$-identification $h : \partial E'_i \to \partial F$ of the boundaries of E'_i and F such that F is related to E'_i by a finite sequence of Borromean surgeries $\{{}^{B'_j}\!/_{B_j}\}_{j \in \{1,\dots,q\}}$. The homeomorphism $h \circ h_i$ is an $\mathrm{LP}_{\mathbb{Q}}$-identification of the boundaries of E_i and F, hence $(E_i, F, h \circ h_i)$ is a standard T_k-surgery datum. Furthermore, applying the sequence $\{{}^{B'_j}\!/_{B_j}\}_{j \in \{1,\dots,q\}}$ of Borromean surgeries on $A_{i-1}({}^{F}\!/_{E_i})$ results in A_i, so that A_{i-1} and A_i are related by a sequence of one standard T_k-surgery and Borromean surgeries. □

2 On finite type invariants and λ_{CW}

2.1 Finite type invariants of $\mathbb{Q}$HS w.r.t. $\mathrm{LP}_{\mathbb{Q}}$-surgeries

Let $\mathcal{F}_0$ be the $\mathbb{Q}$-vector space spanned by all rational homology 3-spheres. If M is a $\mathbb{Q}$HS, if $n \in \mathbb{N}$ and if $\{{}^{B_i}\!/_{A_i}\}_{i \in \{1,\dots,n\}}$ is a set of disjoint $\mathrm{LP}_{\mathbb{Q}}$-surgeries, let

$$[M; {}^{B_1}\!/_{A_1}, {}^{B_2}\!/_{A_2}, \dots, {}^{B_n}\!/_{A_n}] := \sum_{I \subset \{1,\dots,n\}} (-1)^{|I|} \cdot M\left(\{{}^{B_i}\!/_{A_i}\}_{i \in I}\right) \in \mathcal{F}_0,$$

and let $\mathcal{F}_n$ denote the subspace of $\mathcal{F}_0$ spanned by the elements $[M; {}^{B_1}\!/_{A_1}, \dots, {}^{B_n}\!/_{A_n}]$. Note that $(\mathcal{F}_n)_{n \in \mathbb{N}}$ is a filtration of $\mathcal{F}_0$. Indeed,

$$\begin{aligned}[M; {}^{B_1}\!/_{A_1}, {}^{B_2}\!/_{A_2}, \dots, {}^{B_{n+1}}\!/_{A_{n+1}}] &= [M; {}^{B_1}\!/_{A_1}, \dots, {}^{B_n}\!/_{A_n}] \\ &\quad - [M\left({}^{B_{n+1}}\!/_{A_{n+1}}\right); {}^{B_1}\!/_{A_1}, \dots, {}^{B_n}\!/_{A_n}]\end{aligned}$$

so that $\mathcal{F}_{n+1} \subset \mathcal{F}_n$. Furthermore, thanks to Theorem II.1.19 :

Lemma II.2.1 ([Mou12, Lemma 5.13])**.** *For $n \in \mathbb{N}$, the space $\mathcal{F}_n / \mathcal{F}_{n+1}$ is generated by the elements $[M; {}^{B_1}\!/_{A_1}, \dots, {}^{B_n}\!/_{A_n}]$ where M is a $\mathbb{Q}$HS and $\{{}^{B_i}\!/_{A_i}\}_{i \in \{1,\dots,n\}}$ is a family of disjoint elementary surgeries.*

Using the same proof as [Mou12, Lemma 5.13] but refering to Theorem II.1.20 instead of Theorem II.1.19, we get the following refinement of Lemma II.2.1.

Lemma II.2.2. *For $n \in \mathbb{N}$, the space $\mathcal{F}_n / \mathcal{F}_{n+1}$ is generated by the elements $[M; {}^{B_1}\!/_{A_1}, \dots, {}^{B_n}\!/_{A_n}]$ where M is a $\mathbb{Q}$HS and $\{{}^{B_i}\!/_{A_i}\}_{i \in \{1,\dots,n\}}$ is a family of disjoint surgeries in M among Borromean surgeries, standard T_k-surgeries, connected sums with a rational homology 3-sphere.*

A $\mathbb{Q}$-linear map $f : \mathcal{F}_0 \to \mathbb{Q}$ is a *finite type invariant of rational homology 3-spheres with respect to $LP_{\mathbb{Q}}$-surgeries* if $f(\mathcal{F}_n) = \{0\}$ for some $n \in \mathbb{N}$. The *degree* of a finite type invariant f is the lowest integer n such that $f(\mathcal{F}_{n+1}) = \{0\}$. For a fixed invariant f of rational homology 3-spheres, looking at $f(\mathcal{F}_n)$ can be thought of as looking at order n derivatives of f with respect to elementary surgeries.

A rational map f defined over the set of rational homology 3-spheres has a unique $\mathbb{Q}$-linear extension on $\mathcal{F}_0$. This extension is still denoted by f.

A natural question about finite type invariants is to find a description of the $\mathbb{Q}$-vector space spanned by finite type invariants of a fixed degree. In the case of finite type invariants of rational homology 3-spheres with respect to $\mathrm{LP}_{\mathbb{Q}}$-surgeries, this has been done by D. Moussard in [Mou12, Theorem 1.7]. Let us only give a partial description needed for the proof of Theorem II.2. For all prime integer p, set

$$\nu_p : M \in \{\text{rational homology 3-spheres}\} \mapsto v_p\left(card(H_1(M;\mathbb{Z}))\right),$$

where, for all $n \in \mathbb{N}$, $v_p(n)$ denotes the highest integer such that $p^{v_p(n)}$ divides n. According to [Mou12, Proposition 1.9 and Corollary 1.10], for any prime integer p, ν_p is a degree one finite type invariant, which is additive with respect to connected sums of 3-manifolds.

Proposition II.2.3. *The space of degree at most two finite type invariants which are additive w.r.t. connected sums of 3-manifolds is freely generated by $\{\nu_p\}_{p\in\mathcal{P}}$, where $\mathcal{P}$ is the set of prime integers, and by a degree two invariant denoted by $\tilde{\lambda}$.*

Proof. See [Mou12, Corollary 1.10 and Proposition 1.11]. □

2.2 Characterization of the Casson-Walker invariant λ_{CW}

This subsection is devoted to the proof of Theorem II.2.

Lemma II.2.4. *The Casson-Walker invariant λ_{CW} verifies (i), (iii) and (iv) of Theorem II.2.*

Proof. For (i) and (iii), let us refer to [Wal92, Proposition 6.1 and Lemma 3.1]. Finally, (iv) can be verified by using that the Poincaré homology sphere is the result of a -1 Dehn surgery on a left-handed trefoil knot in $\mathbb{S}^3$ (see [KS79, p.121]), and using Walker's combinatorial definition of λ_{CW} (see [Wal92, Theorem 5.1] which is recalled in the introduction). □

Let us now check that the Casson-Walker invariant is a degree at most two finite type invariant of rational homology 3-spheres with respect to $\mathrm{LP}_{\mathbb{Q}}$-surgeries.

This result is a direct corollary of the sum formula developed by C. Lescop in [Les98].

If B and A are rational homology handlebodies with $\mathrm{LP}_{\mathbb{Q}}$-identified boundaries, then the Mayer-Vietoris exact sequence associated to $A \cup (-B)$ provides an isomorphism

$$\partial_{B/A} : H_2(A \cup (-B); \mathbb{Q}) \longrightarrow \mathfrak{L}_A$$

such that if S is an embeded surface in $A \cup (-B)$ transverse to ∂A, then $\partial_{B/A}([S])$ is the class of $[S \cap \partial A]$ oriented as $\partial(S \cap A)$. For such manifolds A and B, define an intersection form $\mathcal{I}_{B/A}$ on $\bigwedge^3 \mathfrak{L}_A$ by the expression

$$\mathcal{I}_{B/A}(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) = (\partial_{B/A} \circ P)^{-1}(\alpha_1) \cup (\partial_{B/A} \circ P)^{-1}(\alpha_2) \cup (\partial_{B/A} \circ P)^{-1}(\alpha_3)([A \cup -B])$$

where P denotes the Poincaré duality isomorphism.

Theorem II.2.5 ([Les98, Theorem 1.3])**.** *Let A_1, A_2, B_1 and B_2 be rational homology handlebodies of genus $g \in \mathbb{N}$ such that ∂A_1, ∂B_1, $-\partial A_2$ and $-\partial B_2$ are identified, using orientation-preserving homeomorphisms, with the standard closed oriented surface of genus g, Σ_g. Assume that $\mathfrak{L}_{B_1} = \mathfrak{L}_{A_1}$, $\mathfrak{L}_{B_2} = \mathfrak{L}_{A_2}$ and $\mathfrak{L}_{A_1} \cap \mathfrak{L}_{A_2} = \{0\}$. If $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_1, \dots, \beta_g\}$ are basis of $\mathfrak{L}_{A_1}$ and $\mathfrak{L}_{A_2}$, respectively, such that, for all $i, j \in \{1, \dots, g\}$, $\langle \alpha_i, \beta_j \rangle = 1$ when $i = j$ and 0 otherwise, then*

$$\begin{aligned} &\lambda_{\mathrm{CW}}(A_1 \cup A_2) - \lambda_{\mathrm{CW}}(B_1 \cup A_2) - \lambda_{\mathrm{CW}}(A_1 \cup B_2) + \lambda_{\mathrm{CW}}(B_1 \cup B_2) \\ &= -4 \cdot \sum_{\{i,j,k\} \subset \{1,\dots,g\}} \mathcal{I}_{B_1/A_1}(\alpha_i \wedge \alpha_j \wedge \alpha_k)\, \mathcal{I}_{B_2/A_2}(\beta_i \wedge \beta_j \wedge \beta_k). \end{aligned}$$

Lemma II.2.6. *The Casson-Walker invariant λ_{CW} is a degree at most 2 finite type invariant of rational homology 3-spheres with respect to $LP_{\mathbb{Q}}$-surgeries.*

Proof. This is a direct consequence of [Les04b, Theorem 2.4 and Theorem 2.6] or [Mas14]. Here, we give a direct proof using [Les98].

Let M be a rational homology 3-sphere and let $\{{}^{B_1}\!/_{A_1}, {}^{B_2}\!/_{A_2}, {}^{B_3}\!/_{A_3}\}$ be disjoint $\mathrm{LP}_{\mathbb{Q}}$-surgeries in M. For $I \subset \{1, 2, 3\}$, let M_I denote $M(\{{}^{B_i}\!/_{A_i}\}_{i \in I})$ and, if $1 \notin I$, let $M'_I = M_I \setminus \mathring{A}_1$. Recall from the proof of Proposition I.1.2 that the complement of a rational homology handlebody of genus $g \in \mathbb{N}$ in a rational homology 3-sphere is a rational homology handlebody of genus g. Finally, let $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_1, \dots, \beta_g\}$ be basis of $\mathfrak{L}_{A_1}$ and $\mathfrak{L}_{M'_\emptyset}$, respectively, such that, for all $i, j \in \{1, \dots, g\}$, $\langle \alpha_i, \beta_j \rangle = 1$ if $i = j$ and 0 otherwise.

Since $({}^{B_2}/_{A_2})$ is an $\mathrm{LP}_{\mathbb{Q}}$-surgery, the Lagrangians of $M'_{\{2\}}$ and $M'_{\emptyset}$ coincide (see the proof of Proposition I.1.1). So, according to Theorem II.2.5, we have

$$\begin{aligned}\sum_{I\subset\{1,2\}} (-1)^{|I|}\ \lambda_{\mathrm{CW}}(M_I) &= \lambda_{\mathrm{CW}}(A_1\cup M'_{\emptyset}) - \lambda_{\mathrm{CW}}(B_1\cup M'_{\emptyset})\\ &\quad - \lambda_{\mathrm{CW}}(A_1\cup M'_{\{2\}}) + \lambda_{\mathrm{CW}}(B_1\cup M'_{\{2\}})\\ &= -4\cdot\sum_{\{i,j,k\}\subset\{1,\dots,g\}} \mathcal{I}_{B_1/A_1}(\alpha_i\wedge\alpha_j\wedge\alpha_k)\ \mathcal{I}_{M'_{\{2\}}/M'_{\emptyset}}(\beta_i\wedge\beta_j\wedge\beta_k).\end{aligned}$$

Still using that $({}^{B_2}/_{A_2})$ is an $\mathrm{LP}_{\mathbb{Q}}$-surgery, the Lagrangians of $M'_{\{2,3\}}$ and $M'_{\{3\}}$ coincide, and

$$\sum_{\substack{I\subset\{1,2,3\}\\ 3\in I}}(-1)^{|I|}\ \lambda_{\mathrm{CW}}(M_I) = 4\cdot\sum_{\{i,j,k\}\subset\{1,\dots,g\}} \mathcal{I}_{B_1/A_1}(\alpha_i\wedge\alpha_j\wedge\alpha_k)\ \mathcal{I}_{M'_{\{2,3\}}/M'_{\{3\}}}(\beta_i^{\{3\}}\wedge\beta_j^{\{3\}}\wedge\beta_k^{\{3\}}),$$

where $\{\beta_1^{\{3\}},\dots,\beta_g^{\{3\}}\}$ is a basis of $\mathfrak{L}_{M'_{\{3\}}}$ such that, for all $i,j\in\{1,\dots,g\}$, $\langle\alpha_i,\beta_j^{\{3\}}\rangle = 1$ if $i=j$ and 0 otherwise. Since $({}^{B_3}/_{A_3})$ is an $\mathrm{LP}_{\mathbb{Q}}$-surgery the Lagrangians of $M'_{\{3\}}$ and $M'_{\emptyset}$ coincide so that we can choose $\{\beta_1,\dots,\beta_g\}$ as a basis of $\mathfrak{L}_{M'_{\{3\}}}$ and write

$$\begin{aligned}\sum_{I\subset\{1,2,3\}}(-1)^{|I|}\ \lambda_{\mathrm{CW}}(M_I) &= 4\cdot\sum_{\{i,j,k\}\subset\{1,\dots,g\}} \mathcal{I}_{B_1/A_1}(\alpha_i\wedge\alpha_j\wedge\alpha_k)\ \mathcal{I}_{M'_{\{2,3\}}/M'_{\{3\}}}(\beta_i\wedge\beta_j\wedge\beta_k)\\ &\quad - 4\cdot\sum_{\{i,j,k\}\subset\{1,\dots,g\}} \mathcal{I}_{B_1/A_1}(\alpha_i\wedge\alpha_j\wedge\alpha_k)\ \mathcal{I}_{M'_{\{2\}}/M'_{\emptyset}}(\beta_i\wedge\beta_j\wedge\beta_k).\end{aligned}$$

Let us end this proof by showing that, for all pairwise distinct i, j and k in $\{1,\dots,g\}$,

$$\mathcal{I}_{M'_{\{2,3\}}/M'_{\{3\}}}(\beta_i\wedge\beta_j\wedge\beta_k) = \mathcal{I}_{M'_{\{2\}}/M'_{\emptyset}}(\beta_i\wedge\beta_j\wedge\beta_k).$$

Let i, j and k be pairwise distinct elements of $\{1,\dots,g\}$. Since $({}^{B_2}/_{A_2})$ and $({}^{B_3}/_{A_3})$ are $\mathrm{LP}_{\mathbb{Q}}$-surgeries, for all $q\in\{i,j,k\}$, there exists a family $\{S_q^I\}_{I\subset\{2,3\}}$ of surfaces in the $\{M'_I\}_{I\subset\{2,3\}}$ such that $\partial S_q^I = \beta_q$ and such that, for all $I,J\subset\{2,3\}$,

$$S_q^I\cap((M'_{\emptyset}\setminus\cup_{r\in I\cup J}A_r)\cup_{r\in I\cap J}B_r) = S_q^J\cap((M'_{\emptyset}\setminus\cup_{r\in I\cup J}A_r)\cup_{r\in I\cap J}B_r).$$

By definitions, $\mathcal{I}_{M'_{\{2\}}/M'_{\emptyset}}(\beta_i\wedge\beta_j\wedge\beta_k)$ is the triple algebraic intersection

$$\langle S_i^{\emptyset}\cup -S_i^{\{2\}},\ S_j^{\emptyset}\cup -S_j^{\{2\}},\ S_k^{\emptyset}\cup -S_k^{\{2\}}\rangle_{M'_{\emptyset}\cup -M'_{\{2\}}},$$

and $\mathcal{I}_{M'_{\{2,3\}}/M'_{\{3\}}}(\beta_i\wedge\beta_j\wedge\beta_k)$ is the triple algebraic intersection

$$\langle S_i^{\{3\}}\cup -S_i^{\{2,3\}},\ S_j^{\{3\}}\cup -S_j^{\{2,3\}},\ S_k^{\{3\}}\cup -S_k^{\{2,3\}}\rangle_{M'_{\{3\}}\cup -M'_{\{2,3\}}}.$$

Then,

$$\begin{aligned}\mathcal{I}_{M'_{\{2,3\}}/M'_{\{3\}}}(\beta_i \wedge \beta_j \wedge \beta_k) - \mathcal{I}_{M'_{\{2\}}/M'_{\emptyset}}(\beta_i \wedge \beta_j \wedge \beta_k) \\ = \langle S_i^{\{3\}} \cup -S_i^{\{2,3\}},\ S_j^{\{3\}} \cup -S_j^{\{2,3\}},\ S_k^{\{3\}} \cup -S_k^{\{2,3\}} \rangle_{B_3 \cup -B_3} \\ - \langle S_i^{\emptyset} \cup -S_i^{\{2\}},\ S_j^{\emptyset} \cup -S_j^{\{2\}},\ S_k^{\emptyset} \cup -S_k^{\{2\}} \rangle_{A_3 \cup -A_3}.\end{aligned}$$

Finally, see that these two terms are zero. Indeed,

$$\begin{aligned}\langle S_i^{\emptyset} \cup -S_i^{\{2\}},\ S_j^{\emptyset} \cup -S_j^{\{2\}},\ S_k^{\emptyset} \cup -S_k^{\{2\}} \rangle_{A_3 \cup -A_3} \\ = \langle S_i^{\emptyset},\ S_j^{\emptyset},\ S_k^{\emptyset} \rangle_{A_3} + \langle -S_i^{\{2\}},\ -S_j^{\{2\}},\ -S_k^{\{2\}} \rangle_{-A_3}.\end{aligned}$$

However, for all $q \in \{i, j, k\}$, $N(S_q^{\emptyset})_{|A_3} \simeq N(-S_q^{\{2\}})_{|-A_3}$ as oriented vector spaces. It follows that

$$\langle S_i^{\emptyset},\ S_j^{\emptyset},\ S_k^{\emptyset} \rangle_{A_3} = -\langle -S_i^{\{2\}},\ -S_j^{\{2\}},\ -S_k^{\{2\}} \rangle_{-A_3}$$

and similarly that $\langle S_i^{\{3\}} \cup -S_i^{\{2,3\}}, S_j^{\{3\}} \cup -S_j^{\{2,3\}}, S_k^{\{3\}} \cup -S_k^{\{2,3\}} \rangle_{B_3 \cup -B_3} = 0$. □

Lemma II.2.7. *Let f be a degree at most 1 finite type invariant of rational homology 3-spheres w.r.t. $LP_{\mathbb{Q}}$-surgeries and assume that f is additive w.r.t. the connected sum. The invariant f is zero if and only if $f(M) = -f(-M)$ for all rational homology 3-sphere M.*

Proof. Let $\mathcal{P}$ denote the set of prime numbers. Using [Mou12, Corollary 1.10], there exists a set $\{\mu_{1,p}\}_{p \in \mathcal{P}}$ of rational numbers such that

$$f = \sum_{p \in \mathcal{P}} \mu_{1,p} \cdot \nu_p. \tag{†}$$

Let $p \in \mathcal{P}$, let M_p be a rational homology 3-sphere such that $|H_1(M_p; \mathbb{Q})| = p$ (the lens space $L(p, 1)$ is such a manifold),

$$f(M_p) = \mu_{1,p} \cdot \nu_p(M_p) = \mu_{1,p} p \quad \text{and} \quad f(-M_p) = \mu_{1,p} \cdot \nu_p(-M_p) = \mu_{1,p} p.$$

So, if $f(M) = -f(-M)$ for all rational homology 3-sphere M, then

$$\mu_{1,p} p = f(M_p) = -f(-M_p) = -\mu_{1,p} p$$

which gives $\mu_{1,p} = 0$ for all $p \in \mathcal{P}$ and $f = 0$. □

End of the proof of Theorem II.2. According to Lemmas II.2.4 and II.2.6, λ_{CW} is a nontrivial finite type invariant of rational homology 3-spheres w.r.t. $\mathrm{LP}_{\mathbb{Q}}$-surgeries of degree at most 2 such that λ_{CW} is additive w.r.t. the connected sum and

$\lambda_{\mathrm{cw}}(M) = -\lambda_{\mathrm{cw}}(-M)$ for all rational homology 3-sphere M. Hence, according to Lemma II.2.7, λ_{cw} must have degree 2. This last fact could also be proved from Theorem II.2.5.

It follows that Proposition II.2.3 is true with $\tilde{\lambda} = \lambda_{\mathrm{cw}}$. So that, if Φ is a degree at most 2 finite type invariant of rational homology 3-spheres w.r.t $\mathrm{LP}_{\mathbb{Q}}$-surgeries, and if Φ is additive w.r.t. the connected sum (conditions (i) to (ii) of Theorem II.2), then Φ can be decomposed as

$$\Phi = \mu_2 \cdot \lambda_{\mathrm{cw}} + \sum_{p \in \mathcal{E}} \mu_{1,p} \cdot \nu_p$$

where $\mathcal{P}$ is the set of prime numbers and where $\{\mu_2\} \cup \{\mu_{1,p}\}_{p \in \mathcal{P}}$ is a set of rational numbers. If Φ also verifies condition (iii) of Theorem II.2, then $(\Phi - \mu_2 \cdot \lambda_{\mathrm{cw}})$ is a degree at most 1 finite type invariant of rational homology 3-spheres w.r.t. $\mathrm{LP}_{\mathbb{Q}}$-surgeries such that $(\Phi - \mu_2 \cdot \lambda_{\mathrm{cw}})$ is additive w.r.t. the connected sum and $(\Phi - \mu_2 \cdot \lambda_{\mathrm{cw}})(M) = -(\Phi - \mu_2 \cdot \lambda_{\mathrm{cw}})(-M)$ for all rational homology 3-sphere M. Using Lemma II.2.7 again, it follows that $(\Phi - \mu_2 \cdot \lambda_{\mathrm{cw}}) = 0$. Hence, if Φ verifies the conditions (i) to (iv) of Theorem II.2, then $\Phi = \lambda_{\mathrm{cw}}$. □

3 Combinatorial definition of the Θ-invariant

3.1 Heegaard diagrams

Let M be a closed connected oriented 3-manifold. Recall that any splitting of $M : M = H_\alpha \cup_\Sigma H_\beta$ where H_α and H_β are two embedded handlebodies and Σ is their common boundary is called a *Heegaard splitting of* M.

Every 3-manifold M (closed connected and oriented) admits (many) Heegaard splittings. Two standard constructions exist. A first one consists in using a self-indexing Morse function, $f : M \to \mathbb{R}$, and writing

$$M = f^{-1}([0, 3/2]) \bigcup_{f^{-1}(\{3/2\})} f^{-1}([3/2, 3]).$$

Another construction uses a triangulation of M and takes a regular neighborhood of the 1-skeleton of M as a first handlebody and its complement as a second handlebody. For a complete description of Heegaard splittings of 3-manifolds see [Sin33] and for an approach using Morse functions see [Bot88].

Furthermore, if H is a handlebody of genus $g \in \mathbb{N}$, then it can be recovered up to homeomorphism from its boundary and a non-separating family of g disjoint simple curves in ∂H, $\{\gamma_i\}_{i\in\{1,\dots,g\}}$. Proceed as follows : consider $\partial H \times [0,1]$, glue thickened disks along each $\gamma_i \times \{0\}$; the resulting manifold has two boundary components, ∂_0 homeomorphic to $\mathbb{S}^2$ and $\partial_1 = \partial H \times \{1\}$. End the construction by gluing a standard 3-ball to ∂_0. We call such collection $\{\gamma_i\}_i$ of curves a *set of Heegaard curves.*

We call *Heegaard diagram for M* any triple $(\Sigma, \bar{\alpha}, \bar{\beta})$ where Σ is a standard genus g closed connected oriented surface, called *Heegaard surface*, and where $\bar{\alpha}$ and $\bar{\beta}$ are two sets of Heegaard curves ("α-curves" and "β-curves") such that constructing a first handlebody H_α from Σ and $\bar{\alpha}$ and a second one H_β from $-\Sigma$ and $\bar{\beta}$ and finally gluing them gives a 3-manifold $H_\alpha \cup_{\mathrm{id}_{\Sigma\times\{1\}}} H_\beta$ homeomorphic to M.

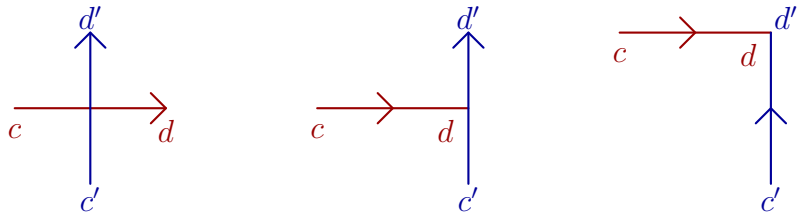
In the case of a Heegaard splitting of a manifold M, constructed from a triangulation, it is easy to see how to choose two sets of Heegaard curves that would give a Heegaard diagram. In the case of a Heegaard splitting using a self-indexing Morse function $f : M \to \mathbb{R}$ with one index 0 and one index 3 critical points, first choose a Morse-Smale metric on M then the ascending manifolds of ∇f out of the index 1 critical points of f intersect the Heegaard surface $f^{-1}(\{{}^3\!/_2\})$ in a set of α-curves and the descending manifolds of ∇f of the index 2 critical points of f intersect $f^{-1}(\{{}^3\!/_2\})$ in a set of β-curves.

Finally, let us fix a *pointed Heegaard diagram for M ie* a quadruple $(\Sigma, \bar{\alpha}, \bar{\beta}, z)$ where $(\Sigma, \bar{\alpha}, \bar{\beta})$ is a Heegaard diagram for M and z is a point in $\Sigma \setminus (\bar{\alpha} \cup \bar{\beta})$. Assume that the curves in $\bar{\alpha}$ are transverse to the curves in $\bar{\beta}$ so that $\alpha \cap \beta$ is finite for every $\alpha \in \bar{\alpha}$ and $\beta \in \bar{\beta}$. Let $g \in \mathbb{N}$ be the genus of Σ. Choose an orientation for each curve in $\bar{\alpha} \cup \bar{\beta}$. Finally let $(\Sigma, \vec{\alpha}, \vec{\beta}, z)$ denote the datum $(\Sigma, \bar{\alpha}, \bar{\beta}, z)$ together with the chosen orientations. Such $(\Sigma, \vec{\alpha}, \vec{\beta}, z)$ is an *oriented pointed Heegaard diagram.*

3.2 Combinatorial formula for the Θ-invariant

Let $(\Sigma, \vec{\alpha}, \vec{\beta}, z)$ be an oriented pointed Heegaard diagram. For $\alpha \in \vec{\alpha}$ and $\beta \in \vec{\beta}$, elements of $\alpha \cap \beta$ are called *crossings*. Let $\mathcal{C}$ be the set of all crossings in Σ. Number the curves in $\vec{\alpha}$ and $\vec{\beta}$. Define $i : \mathcal{C} \to \{1, \dots, g\}$ (resp. $j : \mathcal{C} \to \{1, \dots, g\}$) as the unique map such that for $c \in \mathcal{C}$, $c \in \alpha_{i(c)}$ (resp. $c \in \beta_{j(c)}$) where α_r (resp. β_s) stands for the rth α-curve (resp. sth β-curve). For $c \in \mathcal{C}$, let also $\sigma(c) = \varepsilon_{\alpha_{i(c)} \cap \beta_{j(c)}}(c)$, *ie* if $\vec{v}_{\alpha_{i(c)}}(c)$ and $\vec{v}_{\beta_{j(c)}}(c)$ denote the tangent vectors of $\alpha_{i(c)}$ and $\beta_{j(c)}$ at c, and if $(\vec{v}_{\alpha_{i(c)}}(c), \vec{v}_{\beta_{j(c)}}(c))$ agrees with the orientation of Σ, then $\sigma(c) = 1$, otherwise $\sigma(c) = -1$. Furthermore, for $c, d \in \mathcal{C}$ such that $i(c) = i(d)$, define $[c, d]_\alpha$ as the

segment of $\alpha_{i(c)}$ starting at c and ending at d, following the orientation of $\alpha_{i(c)}$. In the same setting, let also $|c,d|_\alpha$ stand for $[c,d]_\alpha$ where the ends are to be counted with coefficient $1/2$ in the algebraic intersection number formula (see Figure II.3.1 below). Finally, define similar segments of β-curves and set $|c,c|_\alpha = |c,c|_\beta = \emptyset$.



$\langle |c,d|_\alpha, |c',d'|_\beta \rangle = 1$ $\langle |c,d|_\alpha, |c',d'|_\beta \rangle = \frac{1}{2}$ $\langle |c,d|_\alpha, |c',d'|_\beta \rangle = \frac{1}{4}$

Figure II.3.1

Definition II.3.2. Let $\mathfrak{m} = \{m_i\}_{i\in\{1,\dots,g\}} \subset \mathcal{C}$, $\mathfrak{m}$ is called a *matching* if and only if there exists a permutation ϱ of $\{1,\dots,g\}$ such that, for $i \in \{1,\dots,g\}$, $m_i \in \alpha_{\varrho^{-1}(i)} \cap \beta_i$. In other words, a matching $\mathfrak{m}$ is a choice of g crossings such that every Heegaard curve contains exactly one element of $\mathfrak{m}$.

Matchings of Heegaard diagrams are also the generators of the hat Heegaard-Floer chain complex of a Heegaard diagram (see [OS04, Section 4]). Actually, as in Heegaard-Floer theories, we use matchings in a pointed Heegaard diagram as a way to encode combings.

Notation II.3.3. Fix a matching $\mathfrak{m}$ of $(\Sigma, \vec{\alpha}, \vec{\beta}, z)$ and define a combing $X(\mathfrak{m}, z)$ as follows. Assume, without loss of generality, that this diagram is obtained by considering a Morse-Smale metric and a self-indexing Morse function $f : M \to \mathbb{R}$ having a single index 0, resp. index 3, critical point x_0, resp. x_3. Let also $\{x_1^i\}_{i\in\{1,\dots,g\}}$ and $\{x_2^i\}_{i\in\{1,\dots,g\}}$ denote the set of index 1 and 2 critical points of f. Notice that, there exists a unique flow line of ∇f from x_0 to x_3 containing z and a unique g-tuple of flow lines from x_1^i to x_2^i containing m_i, for $i \in \{1,\dots,g\}$. Let N be a tubular neighborhood of these. The vector field $\nabla f|_{M\setminus N}$ can then be extended, in a prescribed way, to a nowhere vanishing vector field $X(\mathfrak{m}, z)$ which is opposite to ∇f on the interior of the flow lines through z and the elements of $\mathfrak{m}$. See [Les16, Subsection 4.2] for details regarding this construction.

For the next three definitions, let M be a $\mathbb{Q}$HS equipped with an oriented pointed Heegaard diagram $\mathcal{D} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$, let $\mathfrak{m} = \{m_i\}_{i\in\{1,\dots,g\}}$ be a matching on $\mathcal{D}$ and number the Heegaard curves of $\mathcal{D}$ so that $m_i \in \alpha_i \cap \beta_i$ for all $i \in \{1,\dots,g\}$.

Definition II.3.4. Let $\tilde{\ell}_{\mathfrak{m}} : \mathcal{C}^2 \to \mathbb{Q}$ be the map defined by :

$$\tilde{\ell}_{\mathfrak{m}}(c,d) = \langle |m_{i(c)}, c|_\alpha, |m_{j(d)}, d|_\beta \rangle - \sum_{r,s\in\{1,\dots,g\}} \mathcal{J}_{sr} \langle |m_{i(c)}, c|_\alpha, \beta_s \rangle \langle \alpha_r, |m_{j(d)}, d|_\beta \rangle$$

where $c, d \in \mathcal{C}$ and $[\mathcal{J}_{ji}]_{i,j\in\{1,\dots,g\}}$ is the inverse of the matrix $[\langle \alpha_i, \beta_j\rangle]_{i,j\in\{1,\dots,g\}}$.

In order to give a combinatorial definition of the Θ-invariant, we need to introduce one last object. First, cut Σ along the curves in $\vec{\alpha}$ and cut a small disk centered at z and avoiding the Heegaard curves. The resulting surface, Σ', can be drawn in the plane as a disk with g pairs of removed inner disks, one for each $\alpha_i \in \vec{\alpha}$, whose boundaries are called α_i' and α_i'', $i \in \{1, \dots, g\}$, and which have opposite orientations. Now we represent the crossings on $\alpha_i \in \vec{\alpha}$, $i \in \{1, \dots, g\}$, as far as possible from the element of $\mathfrak{m}$ on α_i and so that the arcs representing the curves in $\vec{\beta}$ are horizontal near their ends as depicted below

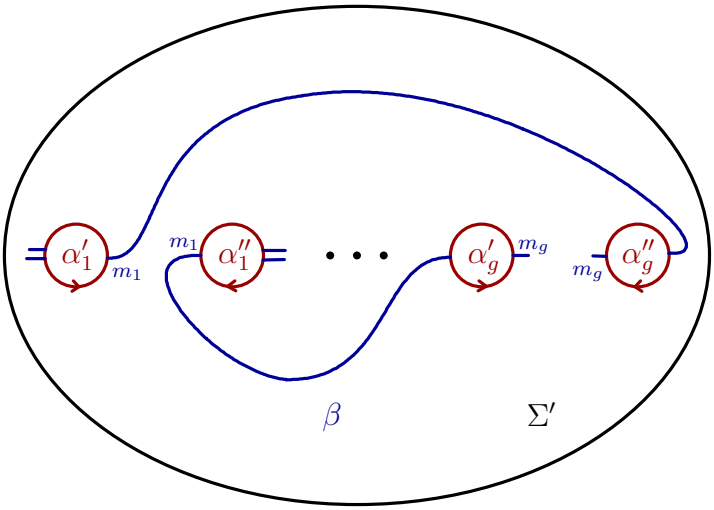


Figure II.3.5 – The surface Σ' in the plane.

Definition II.3.6. In the above setting, define the map d_e from the set of all arcs of β-curves to $\frac{1}{2}\mathbb{Z}$ as follows. For any arc s_β of $\beta \in \vec{\beta}$, the unit tangent vector map of β restricted to s_β induces a map from s_β to $\mathbb{S}^1$ sending ∂s_β to $\{(1,0), (-1,0)\}$. Let $\mathrm{d}_e(s_\beta)$ denote the average of the differential degrees of this map at the points $(0,1)$ and $(0,-1)$ w.r.t. the standard parallelization of the plane.

Definition II.3.7. Finally, set

$$\begin{aligned}\Theta(\mathcal{D}, \mathfrak{m}) = &\sum_{(c,d)\in\mathcal{C}^2} \left(\mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)} + \mathcal{J}_{j(c)i(c)}\mathcal{J}_{j(d)i(d)}\right)\sigma(c)\sigma(d)\tilde{\ell}_{\mathfrak{m}}(c,d)\\ &- \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\tilde{\ell}_{\mathfrak{m}}(c,c)\\ &- \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\left[\mathrm{d}_e(|m_{j(c)}, c|_\beta) - \sum_{r,s\in\{1,\dots,g\}} \mathcal{J}_{sr}\langle \alpha_r, |m_{j(c)}, c|_\beta\rangle \mathrm{d}_e(\beta_s)\right].\end{aligned}$$

Note that the formula of [Les15a, Theorem 3.8] uses a choice of basepoints on the Heegaard curves. Here we have chosen the elements of the matching as basepoints.

Theorem II.3.8 ([Les16, Theorem 6.14])**.** *Let M be a rational homology 3-sphere and X a combing on M. Let $\mathcal{D} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ be an oriented pointed Heegaard diagram for M and $\mathfrak{m}$ a matching on $\mathcal{D}$ such that $X = X(\mathfrak{m}, z)$. Set*

$$\bar{\Theta}(M, [X]) = \Theta(\mathcal{D}, \mathfrak{m}).$$

$\bar{\Theta}$ is an invariant of rational homology 3-spheres equipped with a homotopy class of a combing, called the Θ-invariant.

This theorem has been proved by, first, reformulating the Θ-invariant from the intersection of two propagators (*ie* 4-chains in a configuration space, see [Les15a, Theorem 2.1]) and computing this intersection for particular propagators (see [Les15a, Theorem 3.8]). In [Les16], C. Lescop gave a second independent and combinatorial proof of the invariance of the resulting formula with respect to Heegaard moves on Heegaard diagrams.

4 Diagrams for elementary surgeries

4.1 Adapted diagrams

Let M be a closed oriented 3-manifold and let Y be an embedded Y-graph. A Heegaard splitting of M, $M = H_\alpha \cup H_\beta$, is *adapted to the Y-graph Y* if H_β decomposes as the connected sum along the boundary of a handlebody with a tubular neighborhood of Y. Similarly, we call a Heegaard splitting of M, $M = H_\alpha \cup H_\beta$, *adapted to a knot K* in M if H_β decomposes as the connected sum of a handlebody with a tubular neighborhood of K.

Given $\{G_i\}_{i\in I}$ a family of disjoint Y-graphs and knots in a closed oriented 3-manifold M, there always exists a Heegaard splitting of M adapted to each G_i, $i \in I$. Just consider a triangulation of M and assume that, after sufficiently many subdivisions, the $\{G_i\}_{i\in I}$ are embedded in the 1-skeleton of the triangulation. Then, as in Section 3, consider the splitting $M = H_\alpha \cup H_\beta$ where H_β is a tubular neighborhood of the 1-skeleton and $H_\alpha = M \setminus \mathring{H}_\beta$. Finally, using handleslides on such a splitting, it is possible to ensure that H_β is the result of connected sums

along the boundary of the tubular neighborhoods of the $\{G_i\}_{i\in I}$ with a handlebody $\tilde{H}_\beta$. For a diagram encoding such a splitting, the β-curves can be chosen such that each of them intersects the boundary of at most one of the tubular neighborhoods of the $\{G_i\}_{i\in I}$. An *adapted Heegaard diagram* is a Heegaard diagram encoding such an adapted Heegaard splitting with such a choice of β-curves.

4.2 Standard T_k-surgery on an adapted Heegaard diagram

Let M be a closed oriented 3-manifold and let (K, μ) be a framed knot in M. In this subsection, we describe how to normalize a Heegaard diagram of M adapted to K in order to study the effect of a standard T_k-surgery on M along (K, μ).

First, consider a Heegaard diagram of M adapted to K whose Heegaard surface contains μ. Then perform a stabilization on the genus one part of the diagram containing μ, see Figure II.4.1 below. Let $\mathcal{D}^0 = (\Sigma^0, \bar{\alpha}^0, \bar{\beta}^0)$ be the resulting diagram, let α_1 and β_1 denote the Heegaard curves involved in the stabilization and assume, without loss of generality, that $\mathcal{D}^0$ contains a meridian of K, β_2, in its set of β-curves. Finally, let $\mathbb{T}^0 = (\Sigma_{\mathbb{T}^0}, (\cup_{\alpha\in\bar{\alpha}^0}\alpha) \cap \Sigma_{\mathbb{T}^0}, \{\beta_1, \beta_2\})$ where $\Sigma_{\mathbb{T}^0}$ denotes the genus two surface with one boundary component sitting in Σ^0 and containing the curves μ, α_1, β_1 and β_2.

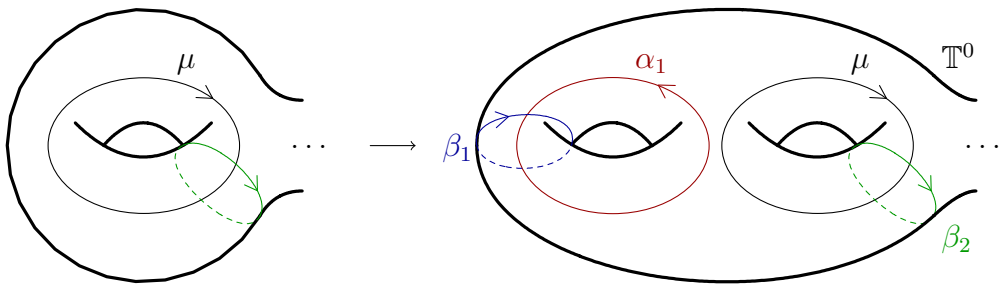


Figure II.4.1 – On the left, the genus one part containing K on the inital adapted diagram. On the right, $\mathbb{T}^0 : \Sigma_{\mathbb{T}^0}$ with the curves μ, α_1, β_1 and β_2.

In general, there are α-curves of $\mathcal{D}^0$ intersecting $\mathbb{T}^0$ other than α_1 but these α-curves do not intersect β_1 since α_1 and β_1 come from a stabilization.

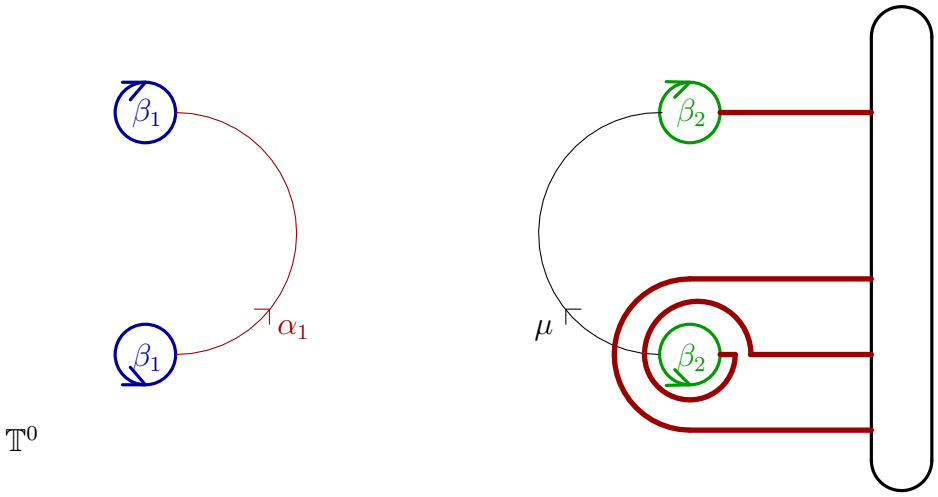


Figure II.4.2 – The surface $\Sigma_{\mathbb{T}^0}$ which has been cut along β_1 and β_2 in order to be drawn in the plane. Thick red segments are drawn where (possibly several) parallel α-curves can appear. In general, the number of twists of the α-curves around β_2 is not known.

According to Definitions II.1.7 and II.1.3, a standard T_k-surgery along (K, μ) in M can be seen on $\mathcal{D}^0$ as the replacement of β_1 and β_2 by some curves $\beta_{k,1}$ and $\beta_{k,2}$ such that $[\beta_{k,1}] = [\mu] + k[\beta_1]$ and $[\beta_{k,2}] = [\alpha_1] + k[\beta_2]$ in $H_1(\Sigma_{\mathbb{T}^0}; \mathbb{Z})$. See Definition II.1.3 for the construction of such curves. Let $\mathbb{T}^0_k$ denote the piece of diagram $\mathbb{T}^0$ equipped with the curves $\beta_{k,1}$ and $\beta_{k,2}$ instead of β_1 and β_2.

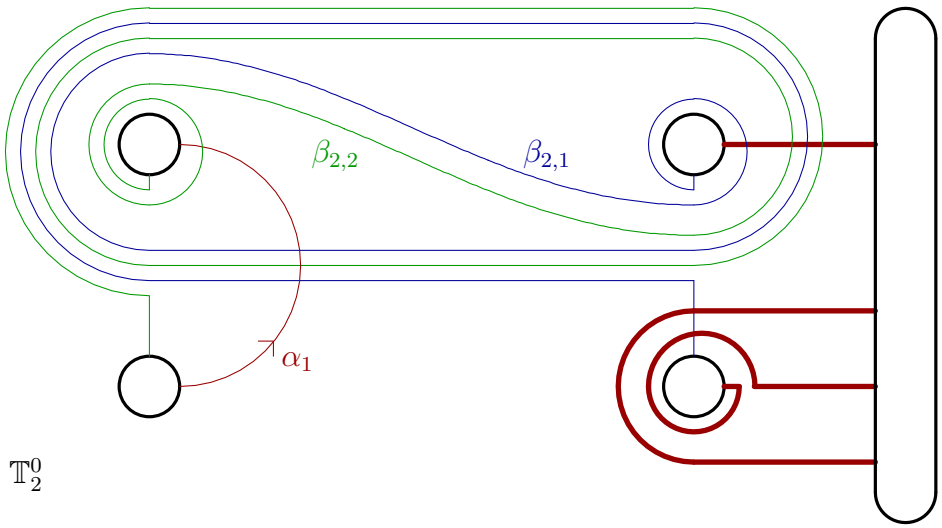


Figure II.4.3 – The piece of diagram $\mathbb{T}^0_2$ replacing $\mathbb{T}^0$ after a standard T_2-surgery.

The diagrams $\mathcal{D}^0$, $\mathbb{T}^0$ and $\mathbb{T}^0_k$ can indeed be used to translate a standard T_k-surgery on Heegaard diagrams, however tracing matchings when performing standard T_k-surgery with these diagrams is not convenient. Instead, consider the diagrams $\mathcal{D}$, $\mathbb{T}$ and $\mathbb{T}_k$ constructed from $\mathcal{D}^0$, $\mathbb{T}^0$ and $\mathbb{T}^0_k$ by performing (yet another) stabilization and some "finger move" isotopies.

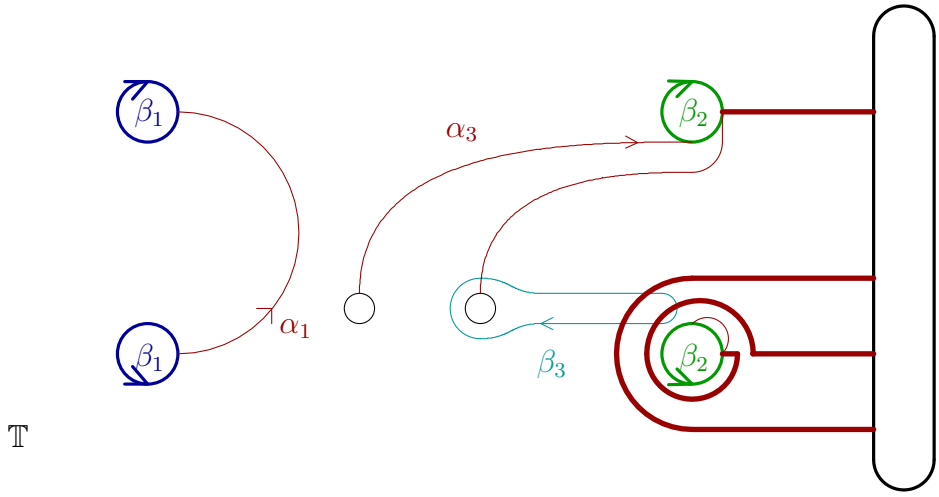


Figure II.4.4 – The surface $\Sigma_{\mathbb{T}}$ and some Heegaard curves of $\mathcal{D}$ forming the diagram $\mathbb{T}$ constructed from $\mathbb{T}^0$ using a stabilization and an isotopy.

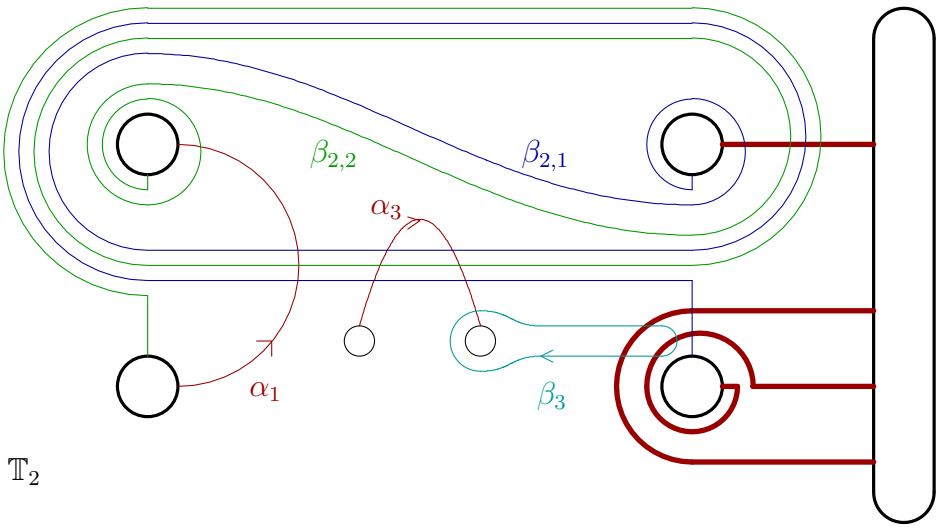


Figure II.4.5 – The surface $\Sigma_{\mathbb{T}_k}$ and some curves forming the diagram $\mathbb{T}_k$ which replaces $\mathbb{T}$ after a standard T_k-surgery in the case $k = 2$. The diagram $\mathbb{T}_k$ is obtained from $\mathbb{T}^0_k$ by performing a stabilization and an isotopy.

4.3 Borromean surgery on an adapted Heegaard diagram

Let M be a closed oriented 3-manifold and let Y be an embedded Y-graph in M. In this subsection, we describe how a Heegaard diagram $\mathcal{D}$ for M adapted to Y can be modified to give a Heegaard diagram $\mathcal{D}^Y$ for $M({}^B/_A)$, where $({}^B/_A)$ is a Borromean surgery along Y. First, let us consider the genus three part of $\mathcal{D}$ in which Y sits.

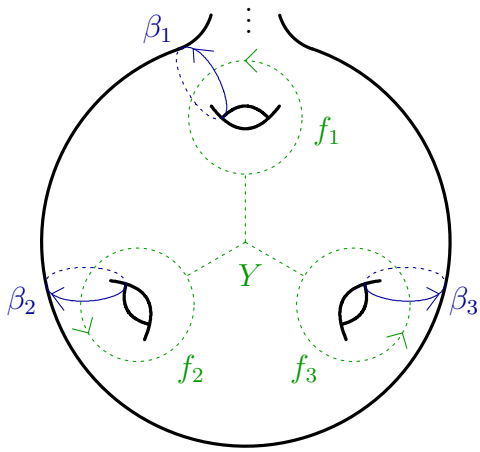


Figure II.4.6 – The graph Y in a genus three part of $\mathcal{D}$. The curves β_1, β_2 and β_3 are Heegaard curves of $\mathcal{D}$.

Following the proof of Lemma II.1.15, performing particular Dehn twists on β_1, β_2 and β_3 along the following curves t_1, t_2 and t_3 gives $\mathcal{D}^Y$. Note that the orientation of the Heegaard surface is the opposite of the orientation of ∂H_Y in Subsection 1.2. In particular, the right Dehn twists $\{\delta_i\}_{i\in\{1,2,3\}}$ along the $\{t_i\}_{i\in\{1,2,3\}}$ on ∂H_Y should be replaced by left Dehn twists on the Heegaard surface.

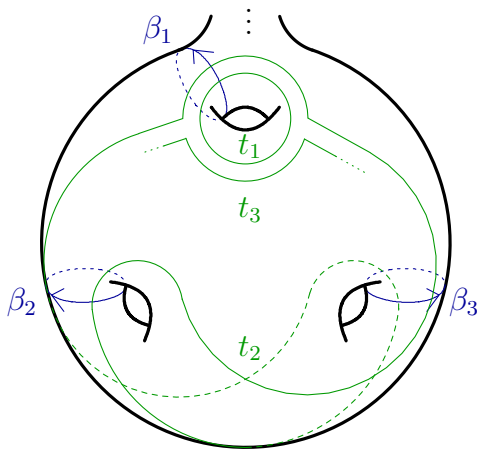


Figure II.4.7 – The curves t_1, t_2 and t_3.

Recall that t_1, t_2 and t_3 cobound a pair of pants $\mathbb{P}$, so that when performing the Borromean surgery along Y, $\mathcal{D}$ is only modified in $\mathbb{P}$. Moreover, see that t_3 is the boundary of a punctured torus S and is homotopic to $[f_2, f_3^{-1}]$ (once we push the leaves of Y on the Heegaard surface). The situation in $\mathbb{P} \cup S$ can be summarized as follows.

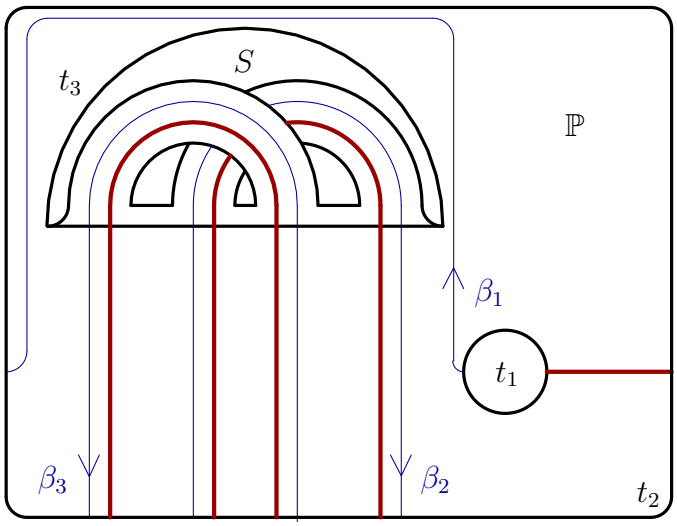


Figure II.4.8 – The pair of pants $\mathbb{P}$ and S before applying the Borromean surgery along Y. Thick red segments are drawn where (possibly several) parallel α-curves can appear.

To see that $\mathbb{P} \cup S$ is indeed crossing-free and that the Heegaard curves cross $\mathbb{P} \cup S$ as depicted, just note that $\mathbb{P} \cup S$ retracts on the following graph G, and that $G \cap \vec{\alpha} \cap \vec{\beta} = \emptyset$ thanks to transversality. Then, $\vec{\alpha}$ can be assumed to cross $\mathbb{P} \cup S$, which is a regular neighborhood of G, as segments parallel to the β-curves.

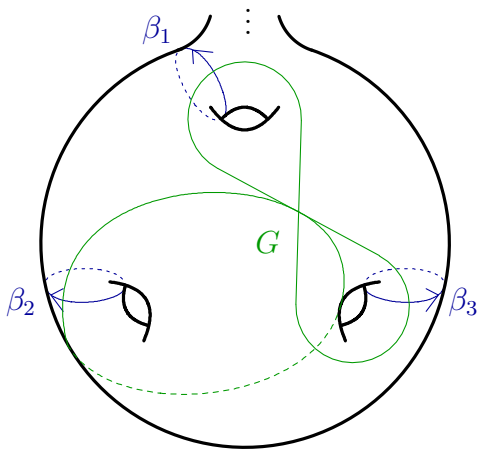


Figure II.4.9 – The graph G and the initial β-curves.

Now, performing the Borromean surgery along Y is just replacing $\mathbb{P}$ with the following pair of pants $\mathbb{P}'$ where we applied the Dehn twists along t_1, t_2 and t_3 according to the reformulation of Borromean surgeries from the proof of Lemma II.1.15.

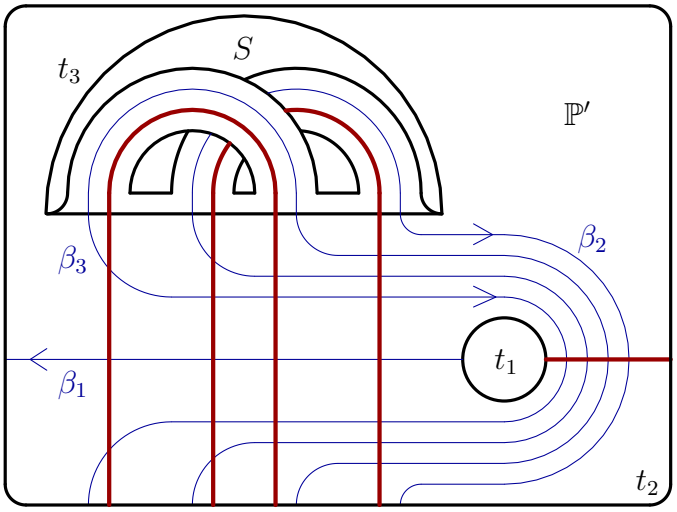


Figure II.4.10 – The pair of pants $\mathbb{P}'$ replacing $\mathbb{P}$ after the Borromean surgery.

5 Variations of Θ

In this section, we compute the variations of Θ when performing Borromean surgeries, standard T_k-surgeries or connected sums with a rational homology 3-sphere. The computation for connected sums is straightforward. The cases of Borromean surgeries and standard T_k-surgeries are more involved and use the normalizations of the previous section.

5.1 Variations w.r.t connected sums

If $\mathcal{D} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ and $\mathcal{D}' = (\Sigma', \vec{\alpha}', \vec{\beta}', z')$ are two oriented pointed Heegaard diagrams for some closed 3-manifolds N and N', then the Heegaard diagram

$$\mathcal{D} \sharp \mathcal{D}' = (\Sigma^{\sharp}, \vec{\alpha} \sqcup \vec{\alpha}', \vec{\beta} \sqcup \vec{\beta}', z)$$

where $\Sigma^{\sharp}$ is the surface constructed by removing a disk near z in Σ, removing a disk near z' in Σ' and gluing the resulting boundaries with an orientation reversing homeomorphism (the disks should avoid the Heegaard curves), is a pointed Heegaard diagram for the usual connected sum of 3-manifolds of N with N' denoted by $N \sharp N'$. Furthermore, using such a construction, it is easy to see that the combinatorial formula for Θ is additive with respect to connected sums.

Lemma II.5.1. *If $\mathcal{D}$ and $\mathcal{D}'$ are oriented pointed Heegaard diagrams and if $\mathfrak{m}$, resp. $\mathfrak{m}'$, is a matching on $\mathcal{D}$, resp. $\mathcal{D}'$, then*

$$\Theta(\mathcal{D} \sharp \mathcal{D}', \mathfrak{m} \sqcup \mathfrak{m}') = \Theta(\mathcal{D}, \mathfrak{m}) + \Theta(\mathcal{D}', \mathfrak{m}').$$

Corollary II.5.2. *Let $\{{}^{B_i}\!/_{A_i}\}_{i\in\{1,2,3\}}$ be three disjoint $LP_{\mathbb{Q}}$-surgeries in a rational homology 3-sphere M where $({}^{B_1}\!/_{A_1})$ is a connected sum. There exists a family $\{X^I\}_{I\subset\{1,2,3\}}$ of combings of the $\{M(\{{}^{B_i}\!/_{A_i}\}_{i\in I})\}_{I\subset\{1,2,3\}}$ such that, for all subsets I, $J \subset \{1,2,3\}$, X^I and X^J coincide on $(M \setminus \cup_{i\in I\cup J} A_i) \cup_{i\in I\cap J} B_i$, and such that*

$$\sum_{I\subset\{1,2,3\}} (-1)^{|I|} \cdot \bar{\Theta}(M(\{{}^{B_i}\!/_{A_i}\}_{i\in I}), [X^I]) = 0.$$

Proof. Since $({}^{B_1}\!/_{A_1})$ is a connected sum, we have $M({}^{B_1}\!/_{A_1}) \simeq M \sharp N$ where N is the $\mathbb{Q}$HS constructed by gluing a standard ball to the rational homology ball B_1. Let $\mathcal{D}$, resp. $\mathcal{D}_N$, be an oriented pointed Heegaard diagram for M, resp. N. Let also $\mathfrak{m}$, resp. $\mathfrak{m}^N$, be a matching in $\mathcal{D}$, resp. $\mathcal{D}_N$. The previous lemma gives

$$\Theta(\mathcal{D}, \mathfrak{m}) - \Theta(\mathcal{D} \sharp \mathcal{D}_N, \mathfrak{m} \sqcup \mathfrak{m}^N) = -\Theta(\mathcal{D}_N, \mathfrak{m}^N).$$

So, consider Heegaard diagrams $\mathcal{D}^I$ equipped with matchings $\mathfrak{m}^I$ for the $\{M(\{{}^{B_i}\!/_{A_i}\}_{i\in I})\}_{I\subset\{2,3\}}$, and corresponding diagrams $\mathcal{D}^{I\cup\{1\}} = D^I \sharp D_N$ equipped with matchings $\mathfrak{m}^{I\cup\{1\}} = m^I \cup m^N$, and set

$$X^J = \begin{cases} X(\mathfrak{m}^J, z) & \text{if } 1 \notin J \\ X(\mathfrak{m}^{J\setminus\{1\}} \sqcup \mathfrak{m}^N, z) & \text{if } 1 \in J, \end{cases}$$

for all $J \subset \{1,2,3\}$ (for the definition of such combings, recall Notation II.3.3 or see [Les16, Subsection 4.2]). The combings $\{X^I\}_{I\subset\{1,2,3\}}$ are such that, for all subsets I, $J \subset \{1,2,3\}$, X^I and X^J coincide on $(M \setminus \cup_{i\in I\cup J} A_i) \cup_{i\in I\cap J} B_i$. Expand the alternated sum as follows :

$$\begin{aligned}
&\sum_{I\subset\{1,2,3\}} (-1)^{|I|} \cdot \bar{\Theta}(M(\{{}^{B_i}\!/_{A_i}\}_{i\in I}), [X^I]) \\
&= \sum_{I\subset\{2,3\}} (-1)^{|I|} \left(\bar{\Theta}(M(\{{}^{B_i}\!/_{A_i}\}_{i\in I}), [X^I]) - \bar{\Theta}(M(\{{}^{B_i}\!/_{A_i}\}_{i\in I\cup\{1\}}), [X^{I\cup\{1\}}])\right) \\
&= \sum_{I\subset\{2,3\}} (-1)^{|I|} \left(\Theta(\mathcal{D}^I, \mathfrak{m}^I) - \Theta(\mathcal{D}^I \sharp \mathcal{D}_N, \mathfrak{m}^I \sqcup \mathfrak{m}^N)\right) \\
&= \sum_{I\subset\{2,3\}} (-1)^{|I|+1} \cdot \Theta(\mathcal{D}_N, \mathfrak{m}^N) = 0.
\end{aligned}$$

□

5.2 Variations w.r.t a Borromean or standard T_k-surgery

Definition II.5.3. Let $\mathcal{D}$ be an oriented pointed Heegaard diagram and let $\mathfrak{m}$ be a matching on $\mathcal{D}$. For c and d crossings in $\mathcal{D}$ and $s \in \{1, \dots, g\}$, set

$$\Theta^{(2)}((\mathcal{D}, \mathfrak{m}); c, d) = \left(\mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)} + \mathcal{J}_{j(c)i(c)}\mathcal{J}_{j(d)i(d)}\right)\sigma(c)\sigma(d)\tilde{\ell}_{\mathfrak{m}}(c, d),$$
$$\Theta^{(1)}((\mathcal{D}, \mathfrak{m}); c) = \mathcal{J}_{j(c)i(c)}\sigma(c)\tilde{\ell}_{\mathfrak{m}}(c, c),$$
$$\Theta^{(1+1)}((\mathcal{D}, \mathfrak{m}); c, s) = \mathcal{J}_{j(c)i(c)}\sigma(c)\left[\frac{\mathrm{d}_e(|m_{j(c)}, c|_\beta)}{g} - \sum_{r\in\{1,\dots,g\}} \mathcal{J}_{sr}\langle\alpha_r, |m_{j(c)}, c|_\beta\rangle \mathrm{d}_e(\beta_s)\right].$$

Furthermore, for all subsets $\mathcal{C}_1$ and $\mathcal{C}_2$ of the set $\mathcal{C}$ of crossings in $\mathcal{D}$ and all subset of indices $\mathcal{S} \subset \{1, \dots, g\}$, set

$$\Theta^{(2)}((\mathcal{D}, \mathfrak{m}); \mathcal{C}_1, \mathcal{C}_2) = \sum_{(c,d)\in\mathcal{C}_1\times\mathcal{C}_2} \Theta^{(2)}((\mathcal{D}, \mathfrak{m}); c, d),$$
$$\Theta^{(1)}((\mathcal{D}, \mathfrak{m}); \mathcal{C}_1) = \sum_{c\in\mathcal{C}_1} \Theta^{(1)}((\mathcal{D}, \mathfrak{m}); c),$$
$$\Theta^{(1+1)}((\mathcal{D}, \mathfrak{m}); \mathcal{C}_1, \mathcal{S}) = \sum_{(c,s)\in\mathcal{C}_1\times\mathcal{S}} \Theta^{(1+1)}((\mathcal{D}, \mathfrak{m}); c, s).$$

With these notations

$$\Theta(\mathcal{D}, \mathfrak{m}) = \Theta^{(2)}((\mathcal{D}, \mathfrak{m}); \mathcal{C}, \mathcal{C}) + \Theta^{(1)}((\mathcal{D}, \mathfrak{m}); \mathcal{C}) + \Theta^{(1+1)}((\mathcal{D}, \mathfrak{m}); \mathcal{C}, \{1, \dots, g\}).$$

Lemma II.5.4. *Recall the notations $\mathbb{P}$ and $\mathbb{P}'$ for Heegaard diagrams adapted to Borromean surgeries from Subsection 4.3. Let $\mathcal{D}$ be an oriented pointed Heegaard diagram containing a copy of $\mathbb{P}$ and let $\mathcal{D}'$ be the diagram obtained from $\mathcal{D}$ by replacing $\mathbb{P}$ by a copy of $\mathbb{P}'$. The set $\mathcal{C}$ of crossings in $\mathcal{D}$ is a subset of the set $\mathcal{C}'$ of crossings in $\mathcal{D}'$. If $\mathfrak{m}$ is a matching in $\mathcal{D}$, then $\mathfrak{m}$ is a matching in $\mathcal{D}'$ and, for all $(c, d) \in \mathcal{C}^2$,*

$$\Theta^{(2)}((\mathcal{D}', \mathfrak{m}); c, d) = \Theta^{(2)}((\mathcal{D}, \mathfrak{m}); c, d) \quad \textit{and} \quad \Theta^{(1)}((\mathcal{D}', \mathfrak{m}); c) = \Theta^{(1)}((\mathcal{D}, \mathfrak{m}); c).$$

Proof. Write $\mathcal{D} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ and number the curves of $\mathcal{D}$ such that $\{\beta_j\}_{j\in\{1,2,3\}}$ are the only β-curves intersecting $\mathbb{P}$ (as in Figure II.4.8) and such that $m_i \in \alpha_i \cap \beta_i$ for all $i \in \{1, \dots, g\}$. Let $\mathcal{D}' = (\Sigma, \vec{\alpha}, \vec{\beta}', z)$ with the numberings of $\vec{\beta}'$ induced by the numberings of $\vec{\beta}$.

For any connected component s_β of $\vec{\beta} \cap \mathbb{P}$, let s'_β be its image in $\mathbb{P}'$ and note that, for all $(c, d) \in \mathcal{C}^2$,

$$\langle [c, d]_\alpha, s_\beta\rangle = \langle [c, d]_\alpha, s'_\beta\rangle.$$

It follows that, for all $(c, d) \in \mathcal{C}^2$ and all $i, j \in \{1, \dots, g\}$,

- $\langle |m_{i(c)}, c|_\alpha, |m_{j(d)}, d|_\beta \rangle_{\mathcal{D}} = \langle |m_{i(c)}, c|_\alpha, |m_{j(d)}, d|_\beta \rangle_{\mathcal{D}'}$,
- $\langle |m_{i(c)}, c|_\alpha, \beta_j \rangle = \langle |m_{i(c)}, c|_\alpha, \beta'_j \rangle$,
- $\langle \alpha_i, |m_{j(d)}, d|_\beta \rangle_{\mathcal{D}'} = \langle \alpha_i, |m_{j(d)}, d|_\beta \rangle_{\mathcal{D}}$,

and also $[\langle \alpha_i, \beta_j \rangle]_{(i,j)\in\{1,\dots,g\}^2} = [\langle \alpha_i, \beta'_j \rangle]_{(i,j)\in\{1,\dots,g\}^2}$ so that

$$\mathcal{J} := [\langle \alpha_i, \beta_j \rangle]^{-1}_{(i,j)\in\{1,\dots,g\}^2} = [\langle \alpha_i, \beta'_j \rangle]^{-1}_{(i,j)\in\{1,\dots,g\}^2} =: \mathcal{J}'.$$

Hence, for all $(c,d) \in \mathcal{C}^2$, $\tilde{\ell}_{\mathfrak{m}}(c,d)_{\mathcal{D}'} = \tilde{\ell}_{\mathfrak{m}}(c,d)_{\mathcal{D}}$,

$$\Theta^{(1)}((\mathcal{D}', \mathfrak{m}); c) = \Theta^{(1)}((\mathcal{D}, \mathfrak{m}); c) \quad \text{and} \quad \Theta^{(2)}((\mathcal{D}', \mathfrak{m}); c, d) = \Theta^{(2)}((\mathcal{D}, \mathfrak{m}); c, d).$$

□

Definition II.5.5. Let $\mathcal{D} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ be an oriented pointed Heegaard diagram containing a copy of $\mathbb{T}$ and let $\mathcal{D}' = (\Sigma, \vec{\alpha}, \vec{\beta}', z)$ be the diagram obtained from $\mathcal{D}$ by replacing $\mathbb{T}$ by a copy of $\mathbb{T}_k$ (for some $k \in \mathbb{N} \setminus \{0\}$). Number the curves of $\mathcal{D}$ so that $\{\alpha_i\}_{i\in\{1,3\}} \cup \{\beta_j\}_{j\in\{1,2,3\}}$ are the curves as in Figure II.4.4. Set $\beta'_1 = \beta_{k,1}$, $\beta'_2 = \beta_{k,2}$ and number the curves in $\vec{\beta} \setminus \{\beta'_1, \beta'_2\}$ with the numbering induced by the numbering of $\vec{\beta}$.

Define *admissible pairs of matchings of $\mathcal{D}$ and $\mathcal{D}'$* as pairs $(\mathfrak{m}, \mathfrak{m}')$ where

- $\mathfrak{m}$, resp. $\mathfrak{m}'$, is a matching in $\mathcal{D}$, resp. in $\mathcal{D}'$,
- $\mathfrak{m} \cap (\mathcal{D} \setminus \mathbb{T}) = \mathfrak{m}' \cap (\mathcal{D}' \setminus \mathbb{T}_k)$,
- $\mathfrak{m} \cap \mathbb{T}$ and $\mathfrak{m}' \cap \mathbb{T}_k$ consist in crossings in $\alpha_1 \cap \beta_1$ and $\alpha_1 \cap \beta'_1$, respectively, crossings in $\alpha_3 \cap \beta_2$ and $\alpha_3 \cap \beta'_2$, respectively, and a shared crossing in $\vec{\alpha} \cap \beta_3$ which is outside the pair of pants cobounded by the curves β_1, β_2 and t_3 as in Figure II.1.5.

It is left to the reader to check that such matchings exist, using Figures II.4.4 and II.4.5, where the only choice is for the crossing on β_3.

Lemma II.5.6. *Let $k \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{D}$ be an oriented pointed Heegaard diagram containing a copy of $\mathbb{T}$ and let $\mathcal{D}'$ be the diagram obtained from $\mathcal{D}$ by replacing $\mathbb{T}$ by a copy of $\mathbb{T}_k$. Let $(\mathfrak{m}, \mathfrak{m}')$ be an admissible pair of matchings in $\mathcal{D}$ and $\mathcal{D}'$. Let $\mathcal{C}$ denote the set of crossings in $\mathcal{D}$. The set $\mathcal{C}_0 = \{c \in \mathcal{C} \mid j(c) \notin \{1,2,3\}\}$ is a subset of the set $\mathcal{C}'$ of crossings in $\mathcal{D}'$ and, for all $(c,d) \in \mathcal{C}_0^2$,*

$$\Theta^{(2)}((\mathcal{D}', \mathfrak{m}'); c, d) = \Theta^{(2)}((\mathcal{D}, \mathfrak{m}); c, d) \quad \textit{and} \quad \Theta^{(1)}((\mathcal{D}', \mathfrak{m}'); c) = \Theta^{(1)}((\mathcal{D}, \mathfrak{m}); c).$$

Proof. Write $\mathcal{D} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ and $\mathcal{D}' = (\Sigma, \vec{\alpha}, \vec{\beta}', z)$. Number the curves of $\mathcal{D}$ so that $\{\alpha_i\}_{i\in\{1,3\}} \cup \{\beta_j\}_{j\in\{1,2,3\}}$ are the curves as in Figure II.4.4. Set $\beta'_1 = \beta_{k,1}$,

$\beta'_2 = \beta_{k,2}$ and number the curves in $\vec{\beta} \setminus \{\beta'_1, \beta'_2\}$ with the numbering induced by the numbering of $\vec{\beta}$. Recall that, in $H_1(\Sigma_{\mathbb{T}_k}; \mathbb{Z})$,

$$\begin{cases} [\beta'_1] = [\mu] + k[\beta_1] \\ [\beta'_2] = [\alpha_1] + k[\beta_2]. \end{cases}$$

It follows that, for all $j \in \{1, \dots, g\}$, $\langle \alpha_1, \beta'_j \rangle = k \cdot \langle \alpha_1, \beta_j \rangle$ and

$$\left[\langle \alpha'_i, \beta'_j \rangle\right]_{i,j} = \begin{pmatrix} k & 0 & 0 & 0 & \cdots & 0 \\ \langle \alpha_2, \mu \rangle & k \cdot \langle \alpha_2, \beta_2 \rangle & \langle \alpha_2, \beta_3 \rangle & \langle \alpha_2, \beta_4 \rangle & \cdots & \langle \alpha_2, \beta_g \rangle \\ \langle \alpha_3, \mu \rangle & k \cdot \langle \alpha_3, \beta_2 \rangle & \langle \alpha_3, \beta_3 \rangle & \langle \alpha_3, \beta_4 \rangle & \cdots & \langle \alpha_3, \beta_g \rangle \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \langle \alpha_g, \mu \rangle & k \cdot \langle \alpha_g, \beta_2 \rangle & \langle \alpha_g, \beta_3 \rangle & \langle \alpha_g, \beta_4 \rangle & \cdots & \langle \alpha_g, \beta_g \rangle \end{pmatrix}$$

Hence, $\det(\left[\langle \alpha'_i, \beta'_j \rangle\right]_{i,j}) = k^2 \cdot \det([\langle \alpha_i, \beta_j \rangle]_{i,j})$ and

$$\mathcal{J}' = \begin{pmatrix} 1/k & 0 & 0 & 0 & \cdots & 0 \\ * & 1/k \cdot \mathcal{J}_{2,2} & 1/k \cdot \mathcal{J}_{2,3} & 1/k \cdot \mathcal{J}_{2,4} & \cdots & 1/k \cdot \mathcal{J}_{2,g} \\ * & \mathcal{J}_{3,2} & \mathcal{J}_{3,3} & \mathcal{J}_{3,4} & \cdots & \mathcal{J}_{3,g} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ * & \mathcal{J}_{g,2} & \mathcal{J}_{g,3} & \mathcal{J}_{g,4} & \cdots & \mathcal{J}_{g,g} \end{pmatrix}$$

where

$$\mathcal{J} := [\langle \alpha_i, \beta_j \rangle]^{-1}_{(i,j) \in \{1,\dots,g\}^2} \quad \text{and} \quad \mathcal{J}' = [\langle \alpha'_i, \beta'_j \rangle]^{-1}_{(i,j) \in \{1,\dots,g\}^2}.$$

For all $(c, d) \in \mathcal{C}_0^2$ and all indices $(i, j) \in \{1, \dots, g\}^2$ where $j \geqslant 3$, we also have

- $m'_{i(c)} = m_{i(c)}$ and $m'_{j(d)} = m_{j(d)}$ since $(\mathfrak{m}, \mathfrak{m}')$ is an admissible pair of matchings,
- $\langle \alpha_i, |m_{j(d)}, d|_\beta \rangle_{\mathcal{D}'} = \langle \alpha_i, |m_{j(d)}, d|_\beta \rangle_{\mathcal{D}}$, since $|m_{j(d)}, d|_\beta$ sits in $\mathcal{D} \setminus \mathbb{T}$,
- $\langle |m_{i(c)}, c|_\alpha, |m_{j(d)}, d|_\beta \rangle_{\mathcal{D}'} = \langle |m_{i(c)}, c|_\alpha, |m_{j(d)}, d|_\beta \rangle_{\mathcal{D}}$, for the same reason,
- $\langle |m_{i(c)}, c|_\alpha, \beta'_j \rangle = \langle |m_{i(c)}, c|_\alpha, \beta_j \rangle$, since $\beta'_j = \beta_j$ for all $j \in \{3, \dots, g\}$.

Furthermore, for any crossing $c \in \mathcal{C}_0$ and $i \in \{2, \dots, g\}$, since $m_{i(c)}$ is not in the pair of pant of $\Sigma_{\mathbb{T}}$ where Dehn twists are applied in order to construct $\mathbb{T}_k$,

- $\mathcal{J}'_{2i} \langle |m_{i(c)}, c|_\alpha, \beta'_2 \rangle = 1/k \cdot \mathcal{J}_{2i} \cdot k \cdot \langle |m_{i(c)}, c|_\alpha, \beta_2 \rangle = \mathcal{J}_{2i} \langle |m_{i(c)}, c|_\alpha, \beta_2 \rangle$,
- $\mathcal{J}'_{1i} = 0 = \mathcal{J}_{1i}$, hence $\mathcal{J}'_{1i} \langle |m_{i(c)}, c|_\alpha, \beta'_1 \rangle = 0 = \mathcal{J}_{1i} \langle |m_{i(c)}, c|_\alpha, \beta_1 \rangle$,
- $\langle \alpha_1, |m_{j(d)}, d|_\beta \rangle = 0$, for all $d \in \mathcal{C}_0$, so that for all $j \in \{1, \dots, g\}$

$$\mathcal{J}'_{j1} \langle |m_{i(c)}, c|_\alpha, \beta'_j \rangle \langle \alpha_1, |m_{j(d)}, d|_\beta \rangle = 0 = \mathcal{J}_{j1} \langle |m_{i(c)}, c|_\alpha, \beta_j \rangle \langle \alpha_1, |m_{j(d)}, d|_\beta \rangle.$$

Eventually, for all $(c,d) \in \mathcal{C}_0^2$, $\tilde{\ell}_{\mathfrak{m}'}(c,d)_{\mathcal{D}'} = \tilde{\ell}_{\mathfrak{m}}(c,d)_{\mathcal{D}}$,

$$\Theta^{(1)}((\mathcal{D}',\mathfrak{m}');c) = \Theta^{(1)}((\mathcal{D},\mathfrak{m});c) \quad \text{and} \quad \Theta^{(2)}((\mathcal{D}',\mathfrak{m}');c,d) = \Theta^{(2)}((\mathcal{D},\mathfrak{m});c,d).$$

□

5.3 Variations w.r.t three elementary surgeries

In Subsection 5.1, we proved that the iterated variations of Θ with respect to a family of three disjoint $\mathrm{LP}_{\mathbb{Q}}$-surgeries containing a connected sum with a rational homology 3-sphere is zero (Corollary II.5.2). Following Lemma II.2.2 in order to show that Θ has a finite type behaviour with respect to $\mathrm{LP}_{\mathbb{Q}}$-surgeries, this subsection contains a proof of the following.

Proposition II.5.7. *Let M be a rational homology 3-sphere and let $\mathcal{T} \sqcup \mathcal{B}$ be a partition of $\{1,2,3\}$. Let $\{{}^{B_i}\!/_{A_i}\}_{i\in\mathcal{T}}$ be a family of standard T_k-surgeries and let $\{{}^{B_i}\!/_{A_i}\}_{i\in\mathcal{B}}$ be a family of Borromean surgeries such that $\{{}^{B_i}\!/_{A_i}\}_{i\in\mathcal{T}\cup\mathcal{B}}$ is a family of disjoint surgeries in M. There exists a family $\{X^I\}_{I\subset\{1,2,3\}}$ of combings of the $\{M(\{{}^{B_i}\!/_{A_i}\}_{i\in I})\}_{I\subset\{1,2,3\}}$ such that, for all subsets I, $J \subset \{1,2,3\}$, X^I and X^J coincide on $(M \setminus \cup_{i\in I\cup J} A_i) \cup_{i\in I\cap J} B_i$, and such that*

$$\sum_{I\subset\{1,2,3\}} (-1)^{|I|} \cdot \bar{\Theta}(M(\{{}^{B_i}\!/_{A_i}\}_{i\in I}), [X^I]) = 0.$$

Consider the following setting for this whole subsection. Set $M_I = M(\{{}^{B_i}\!/_{A_i}\}_{i\in I})$ for all $I \subset \{1,2,3\}$. Let $\mathcal{D}$ be an oriented pointed Heegaard diagram of M adapted to the surgeries $\{{}^{B_i}\!/_{A_i}\}_{i\in\{1,2,3\}}$ and stabilized near the genus one parts where the knots of the T_k-surgeries $\{{}^{B_i}\!/_{A_i}\}_{i\in\mathcal{T}}$ sit (see Figure II.4.4). For all $i \in \mathcal{T}$, resp. $i \in \mathcal{B}$, the diagram $\mathcal{D}$ contains a copy $\mathbb{X}_i$ of the piece of diagram $\mathbb{T}$, resp. $\mathbb{P}$. Recall that we have chosen the β-curves of $\mathcal{D}$ so that each of them crosses at most one of the $\{\mathbb{X}_i\}_{i\in\mathcal{T}\cup\mathcal{B}}$. For all $I \subset \{1,2,3\}$, an oriented pointed Heegaard diagram $\mathcal{D}^I$ for M_I is obtained by replacing the pieces of diagram $\{\mathbb{X}_i\}_{i\in\mathcal{T}\cap I}$ by copies $\{\mathbb{X}'_i\}_{i\in\mathcal{T}\cap I}$ of $\mathbb{T}_{k_i}$ (for some $k_i \in \mathbb{N}\setminus\{0\}$) and replacing the $\{\mathbb{X}_i\}_{i\in\mathcal{B}\cap I}$ by copies $\{\mathbb{X}'_i\}_{i\in\mathcal{B}\cap I}$ of $\mathbb{P}'$. For all $I \subset \{1,2,3\}$, let $\mathcal{C}^I$ be the set of crossings in $\mathcal{D}^I$. The set $\mathcal{C}^I$ splits as

$$\mathcal{C}^I = \mathcal{C}_0^I \sqcup \mathcal{C}_1^I \sqcup \mathcal{C}_2^I \sqcup \mathcal{C}_3^I$$

where $\mathcal{C}_0^I = \mathcal{C}_0 = \{c \in \mathcal{C}^\emptyset \mid \beta_{j(c)} \cap (\cup_{i\in\mathcal{T}\cup\mathcal{B}} \mathbb{X}_i) = \emptyset\}$, and where, for all $i \in \{1,2,3\}$,

$$\mathcal{C}_i^I = \begin{cases} \{c \in \mathcal{C}^I \mid \beta_{j(c)} \cap \mathbb{X}_i \neq \emptyset\} & \text{if } i \notin I, \\ \{c \in \mathcal{C}^I \mid \beta_{j(c)} \cap \mathbb{X}'_i \neq \emptyset\} & \text{if } i \in I. \end{cases}$$

Finally, let $\{\mathfrak{m}^I\}_{I\subset\{1,2,3\}}$ be a family of matchings of the $\{\mathcal{D}^I\}_{I\subset\{1,2,3\}}$ such that for all $j \in \{1,2,3\}$ and all $I \subset \{1,2,3\}\setminus\{j\}$, if $j \in \mathcal{B}$ then $\mathfrak{m}^I = \mathfrak{m}^{I\cup\{j\}}$, and if $j \in \mathcal{T}$ then $(\mathfrak{m}^I, \mathfrak{m}^{I\cup\{j\}})$ is an admissible pair of matchings of $\mathcal{D}^I$ and $\mathcal{D}^{I\cup\{j\}}$.

Lemma II.5.8. *In the setting of this subsection,*

$$\sum_{I\subset\{1,2\}} (-1)^{|I|}\cdot\Theta^{(1)}((\mathcal{D}^I,\mathfrak{m}^I);\mathcal{C}^I) = 0 \quad \textit{and} \quad \sum_{I\subset\{1,2,3\}} (-1)^{|I|}\cdot\Theta^{(2)}((\mathcal{D}^I,\mathfrak{m}^I);\mathcal{C}^I,\mathcal{C}^I) = 0.$$

Proof. Let $i \in \{0,1,2,3\}$. Let $j \in \{1,2\}$ such that $j \neq i$. If $I \subset \{1,2\} \setminus \{j\}$, then $\mathcal{C}_i^I = \mathcal{C}_i^{I\cup\{j\}}$ and, according to Lemmas II.5.4 and II.5.6, for all $c \in \mathcal{C}_i^I$,

$$\Theta^{(1)}((\mathcal{D}^I,\mathfrak{m}^I);c) = \Theta^{(1)}((\mathcal{D}^{I\cup\{j\}},\mathfrak{m}^{I\cup\{j\}});c).$$

It follows that,

$$\begin{aligned}
&\sum_{J\subset\{1,2\}} (-1)^{|J|}\cdot\Theta^{(1)}((\mathcal{D}^J,\mathfrak{m}^J);\mathcal{C}_i^J)\\
&= \sum_{I\subset\{1,2\}\setminus\{j\}} (-1)^{|I|}\left(\Theta^{(1)}((\mathcal{D}^I,\mathfrak{m}^I);\mathcal{C}_i^I) - \Theta^{(1)}((\mathcal{D}^{I\cup\{j\}},\mathfrak{m}^{I\cup\{j\}});\mathcal{C}_i^{I\cup\{j\}})\right) = 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{I\subset\{1,2\}} (-1)^{|I|}\cdot\Theta^{(1)}((\mathcal{D}^I,\mathfrak{m}^I);\mathcal{C}^I) &= \sum_{I\subset\{1,2\}} (-1)^{|I|}\cdot\Theta^{(1)}((\mathcal{D}^I,\mathfrak{m}^I);\mathcal{C}_0\sqcup\mathcal{C}_1^I\sqcup\mathcal{C}_2^I\sqcup\mathcal{C}_3^I)\\
&= \sum_{I\subset\{1,2\}} (-1)^{|I|}\sum_{i=0}^{3}\Theta^{(1)}((\mathcal{D}^I,\mathfrak{m}^I);\mathcal{C}_i^I)\\
&= \sum_{i=0}^{3}\sum_{I\subset\{1,2\}} (-1)^{|I|}\cdot\Theta^{(1)}((\mathcal{D}^I,\mathfrak{m}^I);\mathcal{C}_i^I)\\
&= 0.
\end{aligned}$$

Similarly, let $(i,j) \in \{0,1,2,3\}^2$. Let $k \in \{1,2,3\}$ such that $k \neq i$ and $k \neq j$. If $I \subset \{1,2,3\} \setminus \{k\}$, then $\mathcal{C}_i^I = \mathcal{C}_i^{I\cup\{k\}}$, $\mathcal{C}_j^I = \mathcal{C}_j^{I\cup\{k\}}$, and, for all $(c,d) \in \mathcal{C}_i^I \times \mathcal{C}_j^I$,

$$\Theta^{(2)}((\mathcal{D}^I,\mathfrak{m}^I);c,d) = \Theta^{(2)}((\mathcal{D}^{I\cup\{k\}},\mathfrak{m}^{I\cup\{k\}});c,d).$$

Therefore,

$$\begin{aligned}
&\sum_{J\subset\{1,2,3\}} (-1)^{|J|}\cdot\Theta^{(2)}((\mathcal{D}^J,\mathfrak{m}^J);\mathcal{C}_i^J,\mathcal{C}_j^J)\\
&= \sum_{I\subset\{1,2,3\}\setminus\{k\}} (-1)^{|I|}\left(\Theta^{(2)}((\mathcal{D}^I,\mathfrak{m}^I);\mathcal{C}_i^I,\mathcal{C}_j^I) - \Theta^{(2)}((\mathcal{D}^{I\cup\{k\}},\mathfrak{m}^{I\cup\{k\}});\mathcal{C}_i^{I\cup\{k\}},\mathcal{C}_j^{I\cup\{k\}})\right)\\
&= 0,
\end{aligned}$$

and, finally,

$$\begin{aligned}
&\sum_{I\subset\{1,2,3\}} (-1)^{|I|} \cdot \Theta^{(2)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}^I, \mathcal{C}^I) \\
&= \sum_{I\subset\{1,2,3\}} (-1)^{|I|} \cdot \Theta^{(2)}((\mathcal{D}^I, \mathfrak{m}^I); (\mathcal{C}_0 \sqcup \mathcal{C}_1^I \sqcup \mathcal{C}_2^I \sqcup \mathcal{C}_3^I),\ (\mathcal{C}_0 \sqcup \mathcal{C}_1^I \sqcup \mathcal{C}_2^I \sqcup \mathcal{C}_3^I)) \\
&= \sum_{I\subset\{1,2,3\}} (-1)^{|I|} \sum_{(i,j)\in\{0,1,2,3\}^2} \Theta^{(2)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}_i^I, \mathcal{C}_j^I) \\
&= \sum_{(i,j)\in\{0,1,2,3\}^2} \sum_{I\subset\{1,2,3\}} (-1)^{|I|} \cdot \Theta^{(2)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}_i^I, \mathcal{C}_j^I) \\
&= 0.
\end{aligned}$$

□

Lemma II.5.9. *In the setting of this subsection,*

$$\sum_{I\subset\{1,2,3\}} (-1)^{|I|} \cdot \Theta^{(1+1)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}^I, \{1,\dots,g\}) = 0.$$

Proof. Let $(i,j) \in \{1,2,3\}^2$, let $\mathcal{S}_i = \{s \in \{1,\dots,g\} \mid \beta_s \cap \mathbb{X}_i \neq \emptyset\}$ and set $\mathcal{S}_0 = \{1,\dots,g\} \setminus (\cup_{i\in\{1,2,3\}}\mathcal{S}_i)$. Let $k \in \{1,2,3\}$ such that $k \neq i$ and $k \neq j$. If $I \subset \{1,2,3\} \setminus \{k\}$, then $\mathcal{C}_i^I = \mathcal{C}_i^{I\cup\{k\}}$, and, for all $(c,s) \in \mathcal{C}_i^I \times \mathcal{S}_j$,

$$\Theta^{(1+1)}((\mathcal{D}^I, \mathfrak{m}^I); c, s) = \Theta^{(1+1)}((\mathcal{D}^{I\cup\{k\}}, \mathfrak{m}^{I\cup\{k\}}); c, s).$$

Indeed, for such I, i, j, k, c and s,

- $\mathrm{d}_e(\beta_s^I) = \mathrm{d}_e(\beta_s^{I\cup\{k\}})$, since $\beta_s^I = \beta_s^{I\cup\{k\}}$,
- $m_{j(c)}^{I\cup\{k\}} = m_{j(c)}^I$, since $\beta_{j(c)} \cap \mathbb{X}_k = \emptyset$ ($c \in \mathcal{C}_i^I$ which is disjoint from $\mathcal{C}_k^I$),
- $\langle \alpha_r, |m_{j(c)}^{I\cup\{k\}}, c|_\beta \rangle_{\mathcal{D}^{I\cup\{k\}}} = \langle \alpha_r, |m_{j(c)}^I, c|_\beta \rangle_{\mathcal{D}^I}$, since $|m_{j(c)}^I, c|_\beta$ sits in $\mathcal{D}^I \setminus \mathbb{X}_k$,
- as we showed for Lemmas II.5.4 and II.5.6, $\mathcal{J}_{j(c)i(c)}^I \sigma^I(c) = \mathcal{J}_{j(c)i(c)}^{I\cup\{k\}} \sigma^{I\cup\{k\}}(c)$ and $\mathcal{J}_{sr}^{I\cup\{k\}} = \mathcal{J}_{sr}^I$ for all $r \in \{1,\dots,g\}$ such that α_r is not an "α_1-type curve" in a copy of a $\mathbb{T}_k$. Otherwise,

$$\begin{aligned}
\mathcal{J}_{sr}^I \langle \alpha_r, |m_{j(c)}^I, c|_\beta \rangle_{\mathcal{D}^I} &= \mathcal{J}_{sr}^I \cdot 0 = 0, \\
\mathcal{J}_{sr}^{I\cup\{k\}} \langle \alpha_r, |m_{j(c)}^{I\cup\{k\}}, c|_\beta \rangle_{\mathcal{D}^{I\cup\{k\}}} &= \mathcal{J}_{sr}^{I\cup\{k\}} \cdot 0 = 0.
\end{aligned}$$

It follows that, for all $(i,j) \in \{0,1,2,3\}^2$ and $k \in \{1,2,3\}$ such that $k \neq i$ and $k \neq j$,

$$\begin{aligned}
&\sum_{J \subset \{1,2,3\}} (-1)^{|J|} \cdot \Theta^{(1+1)}((\mathcal{D}^J, \mathfrak{m}^J); \mathcal{C}_i^J, \mathcal{S}_j) \\
&= \sum_{I \subset \{1,2,3\} \setminus \{k\}} (-1)^{|I|} \left(\Theta^{(1+1)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}_i^I, \mathcal{S}_j) - \Theta^{(1+1)}((\mathcal{D}^{I \cup \{k\}}, \mathfrak{m}^{I \cup \{k\}}); \mathcal{C}_i^{I \cup \{k\}}, \mathcal{S}_j) \right) \\
&= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{I \subset \{1,2,3\}} (-1)^{|I|} \cdot \Theta^{(1+1)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}^I, \{1, \ldots, g\}) \\
&= \sum_{I \subset \{1,2,3\}} (-1)^{|I|} \sum_{(i,j) \in \{0,1,2,3\}^2} \Theta^{(1+1)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}_i^I, \mathcal{S}_j) \\
&= \sum_{(i,j) \in \{0,1,2,3\}^2} \sum_{I \subset \{1,2,3\}} (-1)^{|I|} \cdot \Theta^{(1+1)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}_i^I, \mathcal{S}_j) \\
&= 0.
\end{aligned}$$

□

End of the proof of Proposition II.5.7. Still using the setting of this subsection, for all $I \subset \{1,2,3\}$, let X^I denote the combing $X(\mathfrak{m}^I, z)$ of M_I constructed from $(\mathfrak{m}^I, z)$ and a Morse fonction defining $\mathcal{D}^I$ (recall Notation II.3.3 or see [Les16, Subsection 4.2]). The family $\{X^I\}_{I \subset \{1,2,3\}}$ of combings of the $\{M(\{{}^{B_i}/_{A_i}\}_{i \in I})\}_{I \subset \{1,2,3\}}$ is such that, for all I, $J \subset \{1,2,3\}$, X^I and X^J coincide on $(M \setminus \cup_{i \in I \cup J} A_i) \cup_{i \in I \cap J} B_i$. Furthermore,

$$\begin{aligned}
&\sum_{I \subset \{1,2,3\}} (-1)^{|I|} \cdot \bar{\Theta}(M(\{{}^{B_i}/_{A_i}\}_{i \in I}), [X^I]) \\
&= \sum_{I \subset \{1,2,3\}} (-1)^{|I|} \cdot \Theta(\mathcal{D}^I, \mathfrak{m}^I) \\
&= \sum_{I \subset \{1,2,3\}} (-1)^{|I|} \big(\Theta^{(2)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}^I, \mathcal{C}^I) + \Theta^{(1)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}^I) \\
&\qquad\qquad + \Theta^{(1+1)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}^I, \{1, \ldots, g\}) \big) \\
&= \sum_{I \subset \{1,2,3\}} (-1)^{|I|} \cdot \Theta^{(2)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}^I, \mathcal{C}^I) + \sum_{I \subset \{1,2,3\}} (-1)^{|I|} \cdot \Theta^{(1)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}^I) \\
&+ \sum_{I \subset \{1,2,3\}} (-1)^{|I|} \cdot \Theta^{(1+1)}((\mathcal{D}^I, \mathfrak{m}^I); \mathcal{C}^I, \{1, \ldots, g\})
\end{aligned}$$

which are zero according to Lemmas II.5.8 and II.5.9. □

6 Conclusion

This section is devoted to the end of the proof of Theorem II.1.

Lemma II.6.1 (see [Les16, Proposition 5.4 and §6]). *The map :*

$$\bar{\lambda} : \begin{cases} \{\textit{oriented pointed Heegaard diagrams}\} \longrightarrow \mathbb{Q} \\ \qquad\qquad \mathcal{D} = (\Sigma, \vec{\alpha}, \vec{\beta}, z) \longmapsto \dfrac{1}{3}\left(\Theta(\mathcal{D}, \mathfrak{m}) - \dfrac{1}{4} p_1([X(\mathfrak{m}, z)])\right), \end{cases}$$

where $\mathfrak{m}$ is any matching of $\mathcal{D}$, is a well-defined map and induces a $\mathbb{Q}$-valued invariant of rational homology 3-spheres.

If $\mathcal{D} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ is an oriented pointed Heegaard diagram of a rational homology 3-sphere M and if $\mathfrak{m}$ denotes a matching of $\mathcal{D}$, then [Les16, Proposition 5.4 and Corollary 5.3] show that $\Theta(\mathcal{D}, \mathfrak{m}) - {}^1\!/_4 \cdot p_1([X(\mathfrak{m}, z)])$ is independent of the choices of $\mathfrak{m}$ and z. C. Lescop proved this property by studying the variations of the combinatorial formula of $\Theta(\mathcal{D}, \mathfrak{m})$ when changing $\mathfrak{m}$ and z, and then relating them to the variations of $p_1([X(\mathfrak{m}, z)])$, which could be computed using the definition of the Pontrjagin numbers of torsion combings of closed oriented 3-manifolds ([Les15b, Theorem 1.2]). Then, in order to get the invariant $\bar{\lambda}$, C. Lescop showed in [Les16, §6] that the expression $(\Theta(\mathcal{D}, \mathfrak{m}) - {}^1\!/_4 \cdot p_1([X(\mathfrak{m}, z)]))$ is invariant with respect to Heegaard moves on $\mathcal{D}$.

End of the proof of Theorem II.1. Using Lemma II.5.1 and the definition of Pontrjagin numbers of torsion combings of closed oriented 3-manifolds, it is easy to see that $\bar{\lambda}$ is additive with respect to connected sums. In other words $\bar{\lambda}$ verifies the condition (i) of Theorem II.2.

Moreover, if M is a rational homology 3-sphere and if $\mathcal{D} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ is an oriented pointed Heegaard diagram for M, then reversing the orientation of Σ gives a diagram $\mathcal{D}' = (-\Sigma, \vec{\beta}, \vec{\alpha}, z)$ representing $(-M)$. Note that reversing the orientation of Σ also acts as a multiplication by (-1) on the coefficients of $\mathcal{J}$, on the signs of the crossings $\sigma(.)$, on $\tilde{\ell}_{\mathfrak{m}}$ and on d_e. So, according to the combinatorial definition of Θ (Definition II.3.7), if $\mathfrak{m}$ is a matching in $\mathcal{D}$ and if $\mathfrak{m}'$ denotes its canonical image in $\mathcal{D}'$, then

$$\Theta(\mathcal{D}, \mathfrak{m}) = -\Theta(\mathcal{D}', \mathfrak{m}'),$$

and the combings $X(\mathcal{D}', \mathfrak{m}', z)$ is the opposite of $X(\mathcal{D}, \mathfrak{m}, z) = X(\mathfrak{m}, z)$ so that $p_1(X(\mathcal{D}', \mathfrak{m}, z) \subset (-M)) = -p_1(X(\mathcal{D}, \mathfrak{m}, z) \subset M)$. Hence, $\bar{\lambda}$ verifies the condition (iii) of Theorem II.2.

According to [Les16, Theorem 7.1], $\frac{1}{2}\bar{\lambda}$ verifies the same surgery formula than the Casson invariant for integer homology 3-spheres. It follows that $\bar{\lambda}$ verifies the condition (iv) of Theorem II.2.

Finally, using the finite type behaviour of Pontrjagin numbers of torsion combings of closed oriented 3-manifolds – Corollary I.11 – and the finite type behaviour of the combinatorial definition of Θ – Corollary II.5.2 and Propositions II.5.7 – we get that $\bar{\lambda}$ is a finite type invariant of rational homology 3-spheres with respect to $\mathrm{LP}_{\mathbb{Q}}$-surgeries of degree at most two. In other words, $\bar{\lambda}$ verifies the condition (ii) of Theorem II.2.

According to Theorem II.2, $\bar{\lambda}$ must be the Casson-Walker invariant so that for any rational homology 3-sphere M and for any combing X on M

$$\Theta(M,[X]) = 3 \cdot \lambda_{\mathrm{cw}}(M) + {}^{1}\!/_{4} \cdot p_1([X]).$$

$\square$

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Résumé

En 1998, R. Gompf a défini un invariant homotopique des champs de plans orientés des 3-variétés fermées orientées, noté θ_G. Cet invariant est défini pour les champs de plans orientés ξ de toute 3-variété fermée orientée M dont la première classe de Chern $c_1(\xi)$ est un élément de torsion de $H_2(M;\mathbb{Z})$. Dans le premier chapitre de la thèse, nous définissons une extension de l'invariant de Gompf pour toutes les 3-variétés compactes orientées à bord et nous étudions ses variations lors de chirurgies lagrangiennes. Il en résulte que l'invariant de Gompf étendu peut être vu comme un invariant de type fini de degré 2.

L'invariant Θ est un invariant de variétés de dimension 3 parallélisées qui provient de la partie de degré 1 du développement perturbatif de la théorie de Chern-Simons. G. Kuperberg et D. Thurston ont identifié l'invariant $\Theta(M,\tau)$ d'une sphère d'homologie entière M munie d'une parallélisation τ à $3\cdot\lambda_{\mathrm{cw}}(M)+{}^1\!/_4\cdot p_1(\tau)$ où λ_{cw} désigne la généralisation de Walker de l'invariant de Casson et p_1 est un invariant de la parallélisation définie à partir d'une première classe de Pontrjagin. C. Lescop a étendu l'invariant Θ aux sphères d'homologie rationnelle munies d'une classe d'homotopie de combings et elle a montré que pour toute sphère d'homologie rationnelle M munie d'un combing X, la formule $\Theta(M,[X]) = 3\cdot\lambda_{\mathrm{cw}}(M)+{}^1\!/_4\cdot p_1([X])$ était encore valable pour une extension *ad hoc* des nombres de Pontrjagin aux combings. Elle a aussi donné une formule combinatoire pour l'invariant Θ d'une sphère d'homologie rationnelle présentée par un diagramme de Heegaard et munie d'un combing associé au diagramme, et elle a démontré combinatoirement que cette formule définit un invariant homotopique des couples $(M,[X])$. Dans le prolongement de ce travail, le deuxième chapitre de la thèse présente une preuve combinatoire de la décomposition de cet invariant combinatoire comme $3\cdot\lambda_{\mathrm{cw}}(M)+{}^1\!/_4\cdot p_1([X])$. Cette preuve repose sur la théorie des invariants de type fini des sphères d'homologie rationnelle relativement aux chirurgies lagrangiennes établie par D. Moussard en 2012.

Abstract

In 1998, R. Gompf defined a homotopy invariant θ_G of oriented 2-plane fields in 3-manifolds. This invariant is defined for oriented 2-plane fields ξ in a closed oriented 3-manifold M when the first Chern class $c_1(\xi)$ is a torsion element of $H^2(M;\mathbb{Z})$. In Chapter I, we define an extension of the Gompf invariant for all compact oriented 3-manifolds with boundary and we study its iterated variations under Lagrangian-preserving surgeries. It follows that the extended Gompf invariant has degree two for a suitable finite type invariant theory.

The Θ-invariant is an invariant of parallelized 3-manifolds constructed from the degree one part of the perturbative expansion of Chern–Simons theory. G. Kuperberg and D. Thurston identified the invariant $\Theta(M,\tau)$ of a rational homology 3-sphere M equipped with a parallelization τ with $3\cdot\lambda_{\mathrm{cw}}(M)+{}^1\!/_4\cdot p_1(\tau)$ where λ_{cw} denotes Walker's generalization of the Casson invariant and where p_1 is an invariant of parallelizations defined using a first Pontrjagin class. C. Lescop extended the Θ-invariant to rational homology 3-spheres equipped with a homotopy class of combings and she showed that for all rational homology 3-sphere M equipped with a combing X, the relation $\Theta(M,[X]) = 3\cdot\lambda_{\mathrm{cw}}(M)+{}^1\!/_4\cdot p_1([X])$ still holds using an *ad hoc* extension of the Pontrjagin numbers for combings. She also gave a combinatorial formula for the Θ-invariant of a rational homology 3-sphere represented by a Heeagaard diagram and equipped with a combing associated to the diagram, and she proved that this formula defines a homotopy invariant of the pair $(M,[X])$, in a combinatorial way. Following this work, Chapter II presents a combinatorial proof of the decomposition of this combinatorial invariant as $3\cdot\lambda_{\mathrm{cw}}(M)+{}^1\!/_4\cdot p_1([X])$. This proof relies on the finite type invariant theory for rational homology 3-spheres with respect to Lagrangian-preserving surgeries established by D. Moussard in 2012.