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Maxime Herda

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**Analyse asymptotique et numérique de
quelques modèles pour le transport de
particules chargées**

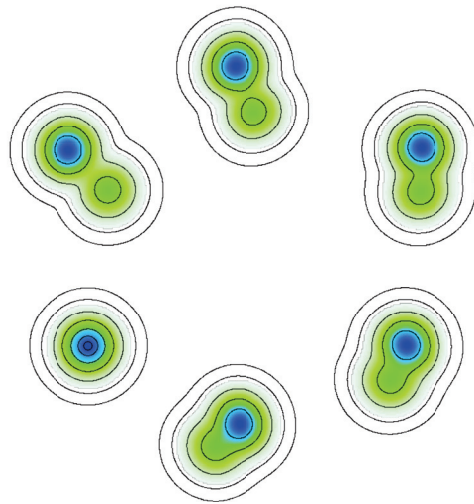
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Analyse asymptotique et numérique de quelques modèles pour le transport de particules chargées



Maxime Herda
Thèse de doctorat

Remerciements

Qu'il est étrange d'écrire ces quelques lignes. C'est l'occasion de faire le bilan, de remercier tous ceux qui m'ont accompagné durant ces trois ans de thèse. Une dernière fois, je cherche l'inspiration dans les nuages qui défilent derrière la vitre du bureau 219.

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1. ©N. C.

2. $n \approx N_{Avogadro}$

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3. Mais qui ?

Avant-propos

Nous présentons dans ce manuscrit les résultats obtenus durant la thèse réalisée au sein de l’Institut Camille Jordan et supervisée par Francis Filbet et Luis Miguel Rodrigues. Ce document est divisé en quatre chapitres rassemblant, dans des versions parfois étendues, des travaux publiés ou soumis pour publication. Une introduction détaillée présente le contexte physique, les modèles considérés et les problématiques centrales de cette thèse et rassemble une présentation des contributions de chaque chapitre. Certaines parties de la thèse pointeront vers l’Annexe B qui contient des développements supplémentaires et vers l’Annexe C composée de résultats techniques.

Les trois premiers chapitres sont consacrés à l’étude théorique de différents régimes asymptotiques de systèmes d’équations aux dérivées partielles cinétiques collisionnelles de type Vlasov-Poisson-Fokker-Planck (VPFP). Ces modèles sont utilisés pour décrire l’évolution d’un ensemble de particules chargées, appelé plasma. Pour de nombreuses raisons, la dynamique de ces systèmes est multi-échelle, en particulier à cause de la présence de différentes espèces de particules aux caractéristiques différentes. En effet, un plasma est constitué de particules «lourdes», les ions de masse m_i et de particules légères, les électrons de masse m_e . Le rapport de masse m_e/m_i est un paramètre essentiel et la majorité de ce manuscrit est lié à l’asymptotique des électrons sans masse, consistant à faire tendre ce rapport vers 0 dans les modèles. Dans les Chapitres 1 et 2, on s’intéresse à cette limite pour des modèles contenant un champ magnétique extérieur. Ensuite, au Chapitre 3, on étudie le comportement en temps long de VPFP sans champ magnétique et l’obtention de modèles asymptotiques. Dans le Chapitre 4, on s’intéresse à la conception d’un schéma numérique pour une classe d’équations paraboliques non-linéaires, parmi lesquelles se trouve l’équation de Fokker-Planck avec champ magnétique. Enfin, dans l’Annexe A on présente quelques résultats numériques pour un système de Vlasov-Poisson à deux espèces dans la limite des électrons sans masse.

Liste des publications :

- Chapitre 1 : *On massless electron limit for a multispecies kinetic system with external magnetic field*, paru [124] dans *Journal of Differential Equations*, 11 :7861–7891, 2016.
- Chapitre 2 : *Anisotropic Boltzmann-Gibbs dynamics of strongly magnetized Vlasov-Fokker-Planck equations*, en collaboration avec L. Miguel Rodrigues, soumis pour publication [126], arXiv preprint arXiv :1610.05138, 2016.
- Chapitre 3 : *Large-time behavior of solutions to Vlasov-Poisson-Fokker-Planck equations : from evanescent collisions to diffusive limit*, en collaboration avec L. Miguel Rodrigues, soumis pour publication [125], arXiv preprint arXiv :1706.05880, 2017.
- Chapitre 4 : *A finite volume scheme for boundary-driven convection-diffusion equations with relative entropy structure*, en collaboration avec Francis Filbet, paru [89] dans *Numerische Mathematik*, pp. 1–43, Apr 2017.
- Annexe A : *Modelling and simulating a multispecies plasma*, en collaboration avec Mehdi Badsì, paru [9] dans *ESAIM : Proceedings and Surveys*, 53 :22–37, 2016.

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Introduction

1 Contexte et motivations physiques

Dans ce travail de thèse, on étudie d'un point de vue mathématique le mouvement de particules chargées en interactions. Plus précisément, on s'intéresse à des problèmes d'analyse asymptotique et numérique de modèles cinétiques et hydrodynamiques provenant de la physique des plasmas. Dans un premier temps, on introduit les phénomènes physiques en jeu dans un gaz de particules chargées. On insiste sur les échelles caractéristiques associées afin de souligner l'importance du rapport de masse. Pour davantage de détails, nous renvoyons le lecteur aux ouvrages d'introduction à la physique des plasmas de Bellan [19], Boyd et Sanderson [32], Goldston et Rutherford [101] et aux livres de Miyamoto [154] et Freidberg [91] axés plus spécifiquement sur les plasmas de fusion.

1.1 Plasmas

Lorsqu'un gaz atteint une température suffisamment élevée, les électrons des atomes qui le composent quittent leur orbite autour des noyaux. En devenant un mélange de particules chargées, la matière entre dans un nouvel état de gaz ionisé. En 1922, le chimiste et physicien américain Irving Langmuir emprunte le mot grec *plasma* pour désigner ce quatrième état de la matière [19], par analogie avec le liquide transparent qui transporte les globules et plaquettes dans le sang. Dans le mélange d'ions et d'électrons, ce sont les interactions électromagnétiques qui transportent les particules et expliquent la dynamique du plasma.

Bien que les conditions thermodynamiques sur Terre rendent impossible sa formation naturelle, plus de 99 % de la matière visible dans l'univers est à l'état plasma. En effet, l'atmosphère des étoiles et les vents soufflant dans l'espace interstellaire sont en grande partie composés d'atomes ionisés et d'électrons. Les applications industrielles utilisant le plasma sont nombreuses, tant dans des technologies actuelles maîtrisées, comme les écrans plasma, les néons et autres lampes à gaz que dans des technologies du futur comme les réacteurs nucléaires à fusion par confinement magnétique, appelés *tokamak*.

Le tokamak est une cavité toroïdale entourée de bobines produisant un champ magnétique à l'intérieur du tore. Lors de son fonctionnement, un plasma obtenu à partir d'atomes légers (hydrogène, hélium, etc...) est confiné grâce au champ magnétique et accéléré. Lorsque les ions ont suffisamment d'énergie cinétique pour contrer la répulsion électrique et collisionner au sein du tokamak, la réaction de fusion nucléaire se produit.

Une partie de l'énergie créée est utilisée pour entretenir la réaction et le reste est récupéré pour produire de l'électricité. La complexité de la dynamique d'un plasma de tokamak empêche encore à l'heure actuelle, plus de 60 ans après la conception du premier prototype par les physiciens Tamm et Sakharov, de comprendre complètement sa physique et de réaliser des simulations numériques complètes et précises en temps raisonnable. C'est pourquoi on utilise un certain nombre d'approximations pour simplifier les modèles, afin des les analyser ou de les simuler. Un des objectifs de ce travail de thèse est la justification mathématique de certains de ces modèles réduits à partir de modèles plus complets.

Nous allons à présent décrire plus précisément la nature des interactions entre particules au sein d'un plasma.

1.2 Interactions électromagnétiques

Un plasma est composé d'ions chargés positivement et d'électrons chargés négativement. Tout au long de cette thèse nous considérerons, pour simplifier la présentation, que les ions sont des protons, si bien que leur charge q_i est égale à la charge élémentaire que nous noterons q et opposée à la charge d'un électron $q_e = -q$. La généralisation des résultats à un mélange de particules contenant différentes espèces d'ions avec des charges différentes pose peu de problèmes mathématiques⁴. Dans la suite, une particule d'espèce $s = i$ (ions) ou e (électrons) sera caractérisée par des quantités physiques indicées par s .

En présence d'un champ électrique E et d'un champ magnétique B , une particule de masse m_s , $s \in \{i, e\}$ voyageant dans l'espace à une vitesse v_s est soumise à la force de Lorentz

$$F_s = q_s (E + v_s \wedge B), \quad (1)$$

où \wedge désigne le produit vectoriel. Les champs électriques et magnétiques sont en toute généralité la somme de champs extérieurs imposés au plasma et de champs auto-induits par l'ensemble des particules. En effet, chaque particule contribue à l'émission d'un champ électromagnétique exerçant une force de Lorentz sur le reste du plasma. La somme des champs auto-induits, notés $E \equiv E(t, x)$ et $B \equiv B(t, x)$ au temps t et à la position x , satisfont les équations de Maxwell dans le vide qui s'écrivent

$$\left\{ \begin{array}{l} \varepsilon_0 \operatorname{div}_x E = \rho, \\ \operatorname{div}_x B = 0, \end{array} \right. \quad (2) \quad (3)$$

$$\operatorname{rot}_x E + \partial_t B = 0, \quad (4)$$

$$c^2 \varepsilon_0 \operatorname{rot}_x B - \varepsilon_0 \partial_t E = j. \quad (5)$$

Les équations portent respectivement les noms de Maxwell-Gauss, Maxwell-Thompson, Maxwell-Faraday et Maxwell-Ampère. La constante ε_0 est la permittivité diélectrique du

4. En effet, les équations pour chaque espèce d'ions sont les mêmes à un paramètre physique prêt (le numéro atomique) qui n'intervient pas dans les différentes limites d'échelle considérée dans cette thèse. Pour un système multi-espèces, il suffit alors de répéter les mêmes techniques d'analyse pour chaque espèce.

vide et c est la vitesse de la lumière dans le vide. Les sources $\rho \equiv \rho(t, x)$ et $j \equiv j(t, x)$ désignent respectivement les densités de charge et de courant. Dans la limite non-relativiste, c'est-à-dire lorsque la vitesse caractéristique des particules est très faible par rapport à la vitesse de la lumière dans le vide, on montre que les équations de Maxwell se réduisent à la loi de Gauss (2) pour un champ électrique irrotationnel [144]. Cela revient, en simplifiant, à prendre $B = 0$ dans les équations ci-dessus. Le champ électrique auto-induit dérive alors d'un potentiel, c'est-à-dire

$$E = -\nabla_x \phi, \quad (6)$$

avec ϕ satisfaisant l'équation de Poisson

$$-\varepsilon_0 \Delta_x \phi = \rho. \quad (7)$$

Sauf mention contraire, nous nous plaçons dorénavant dans ce cadre électrostatique. Lorsque les modèles feront intervenir des champs auto-induits par les particules, nous négligerons le champ magnétique auto-induit et B désignera un champ magnétique extérieur. Notons que cette approximation est classique pour des plasmas de tokamak.

La diversité des effets cinétiques, électriques et magnétiques entrant en jeu dans un plasma est source de nombreuses constantes physiques caractérisant les échelles de temps, d'espace et de vitesse sur lesquelles se déroule la dynamique. Nous introduisons ici ces paramètres et des quantités adimensionnées correspondantes. Dans tous nos modèles, nous supposons toujours la neutralité globale des charges dans le plasma. Le nombre caractéristique d'ions ou d'électrons par unité spatiale sera alors noté \bar{n} . Nous supposons également que les températures caractéristiques des ions et des électrons sont du même ordre θ . Cette dernière hypothèse correspond à considérer un *plasma chaud*, c'est-à-dire un plasma très énergétique, ce qui est classique pour des plasmas de fusion. Par opposition un *plasma froid* est caractérisé par une température ionique très inférieure à la température électronique.

Paramètres cinétiques Les vitesses caractéristiques de déplacement des particules dans un plasma sont les vitesses thermiques. Elles sont reliées à la température caractéristique par la relation suivante

$$V_s = \left(\frac{k_B \theta}{m_s} \right)^{1/2}. \quad (8)$$

où k_B est la constante universelle de Boltzmann.

Paramètres électrostatiques Une propriété fondamentale d'un plasma est l'*écranage électrique*. Dans un plasma globalement neutre et isolé, l'équilibre est atteint si et seulement si le champ électrique est nul en tout point de l'espace. Lorsqu'une perturbation de charge est appliquée localement dans un plasma à l'équilibre, le potentiel électrique est lui-même perturbé sur une longueur caractéristique appelée *longueur de Debye*, et

donnée par l'expression

$$\lambda_D = \left(\frac{\varepsilon_0 k_B \theta}{\bar{n} q^2} \right)^{1/2}. \quad (9)$$

Au delà de cette distance caractéristique (de l'ordre de la dizaine de microns pour un plasma de fusion [154]), la perturbation a un effet négligeable. Lorsqu'un système a une taille caractéristique L très grande devant la longueur de Debye, ce qui est presque toujours le cas, le plasma est proche du régime dit *quasi-neutre*, car les perturbations ne changent presque plus la neutralité locale du plasma. Pour quantifier la quasi-neutralité, on introduit la longueur de Debye adimensionnelle

$$\delta = \frac{\lambda_D}{L}. \quad (10)$$

La longueur de Debye donne une échelle spatiale caractéristique des effets électrostatiques. L'échelle caractéristique temporelle est donnée par l'inverse de la *fréquence plasma* ou *fréquence de Langmuir*, le temps plasma, qui s'écrit pour l'espèce s

$$t_p^{(s)} = \left(\frac{\varepsilon_0 m_s}{q^2 \bar{n}} \right)^{1/2}. \quad (11)$$

Notons que la longueur de Debye, le temps plasma et les vitesses thermiques sont reliées par la relation

$$\lambda_D = V_s t_p^{(s)}.$$

Un autre paramètre sans dimension classique [173, 154] permet de comparer les effets thermiques et électrostatiques. C'est le paramètre de couplage $q \bar{\phi} / k_B \theta$, qui mesure le rapport entre l'énergie potentielle de Coulomb caractéristique et l'énergie cinétique caractéristique. La première dépend du potentiel caractéristique $\bar{\phi}$. Le paramètre de couplage peut s'exprimer [154] comme une puissance positive du *paramètre plasma*, qui quantifie le nombre caractéristique de particules dans une boule de rayon égal à la longueur de Debye. Quitte à renormaliser le potentiel ou le nombre de particules⁵, on inclut ce paramètre dans la quantité $\bar{\phi}$ en posant

$$\bar{\phi} = \frac{k_B \theta}{q}.$$

Paramètres magnétiques Sous l'effet d'un champ magnétique les particules décrivent un mouvement hélicoïdal autour des lignes de champ (voir Figure 1). La fréquence caractéristique de rotation est appelée *fréquence cyclotron*. L'inverse de cette quantité, le

5. Grâce à l'expression physique du potentiel de Coulomb (en dimension 3) on devrait [154, Section 1.2] avoir $\bar{\phi} \approx q/(4\pi\varepsilon_0 d)$ où la distance caractéristique entre deux particules dépend de la densité par la relation $d = \bar{n}^{-1/3}$. En fixant $\bar{\phi}$ on renormalise la densité de particule. Une seconde façon de voir les choses est de désigner par $\bar{\phi}$ le produit entre le paramètre de couplage et le potentiel caractéristique.

temps cyclotron, est donné par la relation

$$t_c^{(s)} = \frac{m_s}{q \bar{B}}, \quad (12)$$

où \bar{B} est l'amplitude caractéristique du champ magnétique. Le rayon de giration autour du champ, définissant l'échelle spatiale caractéristique des effets magnétiques s'appelle *rayon de Larmor* et s'écrit

$$r_L^{(s)} = \frac{m_s V_s}{q \bar{B}}. \quad (13)$$

Comme précédemment, les quantités caractéristiques spatiales et temporelles liées au champ magnétique sont reliées à la vitesse thermique par la relation

$$r_L^{(s)} = V_s t_p^{(s)}.$$

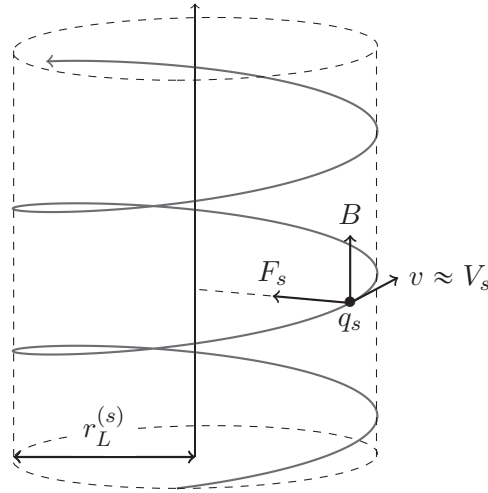


FIGURE 1 – **Interactions électromagnétiques.** Mouvement cyclotron d'une particule autour des lignes du champ magnétique.

Rapport de masse On peut constater dans les paragraphes précédents que la plupart des échelles caractéristiques dépendent de l'espèce s que l'on considère. Cela est dû au fait que ces paramètres sont liés aux propriétés inertielles induites par les champs ou l'agitation thermique. La masse des particules est donc ce qui différencie une même quantité, suivant l'espèce considérée. L'hypothèse de *plasma chaud* est importante dans cette discussion, car elle impose par définition des vitesses thermiques

$$m_i V_i^2 = m_e V_e^2.$$

Or, le rapport entre la masse d'un électron et celle d'un ion est relativement faible (de l'ordre de 10^{-4} pour un plasma de deutérium). Cela signifie que les électrons sont environ cent fois plus rapide que les ions dans le plasma. Ce rapport de masse constitue le paramètre essentiel des deux premiers chapitres de cette thèse. On le note

$$\lambda = \frac{m_e}{m_i}. \quad (14)$$

À partir des expressions (8), (9), (11), (12) et (13), on remarque les liens suivants entre les quantités physiques caractéristiques et le rapport de masse

$$\lambda = \left(\frac{V_i}{V_e}\right)^2 = \left(\frac{t_p^{(e)}}{t_p^{(i)}}\right)^2 = \frac{t_c^{(e)}}{t_c^{(i)}} = \left(\frac{r_L^{(e)}}{r_L^{(i)}}\right)^2. \quad (15)$$

Ces équations illustrent le rôle de la disparité entre masses des particules dans la dynamique multi-échelle du plasma.

1.3 Électrons sans masse

Les échelles caractéristiques diffèrent entre espèces à cause du rapport de masse. Par conséquent, la simulation numérique de modèles multi-espèces complets est extrêmement coûteuse en temps-machine. Afin de capturer la dynamique rapide et localisée des électrons, il faut une discrétisation fine des petites échelles spatio-temporelles. Au contraire, à ces échelles, les ions sont quasiment immobiles et la simulation doit alors être faite dans un domaine spatio-temporel immense! Dans les applications, notamment pour les plasmas de fusion, on s'intéresse surtout au mouvement des ions que l'on souhaite faire fusionner. La description précise de la dynamique des électrons est moins importante et par conséquent, on s'autorise à simplifier les modèles en considérant des *électrons sans masse* ou *électrons sans inertie*. Cette réduction de modèle consiste à

1. Adimensionner les équations décrivant la dynamique des électrons sur l'échelle temporelle caractéristique des ions ;
2. Remplacer le modèle complexe sur les électrons par un modèle réduit consistant avec la limite $\lambda \rightarrow 0$.

Bien entendu, selon le modèle considéré initialement et d'autres considérations d'échelles liées au régime dans lequel se trouve le gaz de particules, cette procédure peut mener à une large gamme de modèles réduits. Un de nos objectifs est l'étude rigoureuse de cette procédure, afin de retrouver certains modèles réduits utilisés par les physiciens.

Avant d'en dire plus, nous introduisons dans la section suivante les modèles cinétiques complets à partir desquels nous allons extraire ces modèles asymptotiques.

2 Modèles cinétiques

Pour commencer, on rappelle les différents niveaux de description d'un système de particules en interaction et la place de la *théorie cinétique* dans cette hiérarchie. Le paramètre d désigne la dimension de l'espace, égale à 3 dans les deux premiers chapitres et à 2 dans le troisième chapitre. On note l'espace physique

$$\Omega = \mathbb{R}^d \text{ ou } \mathbb{T}^d,$$

où $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ est le tore d -dimensionnel correspondant à un domaine périodique. De cette manière on ne se préoccupe pas des conditions de bord. Dans les plasmas de tokamak, des phénomènes de couches limites complexes appelées «gaines de Debye» apparaissent. Ces interactions plasma-paroi constituent un sujet de recherche encore très actifs en physique des plasmas. Pour plus de détails et un point de vue mathématique sur la question, on renvoie à la thèse de Badsì [8] et aux références incluses.

Description microscopique En mécanique classique, la manière la plus complète de décrire mathématiquement un système de particules interagissant entre elles et avec leur environnement est le formalisme des *équations de Newton*. Le mouvement d'une particule est repéré au temps $t \in \mathbb{R}_+$ par un couple position-vitesse vivant dans l'espace des phases $\Omega \times \mathbb{R}^d$. Prenons l'exemple du plasma décrit en Section 1. Il est composé de N ions et N électrons soumis à la force de Lorentz (1). Le principe fondamental de la dynamique pour la k -ième particule d'espèce s s'écrit

$$\begin{cases} \frac{d}{dt} X_k^{(s)} &= V_k^{(s)}, \\ m_s \frac{d}{dt} V_k^{(s)} &= F_s(t, X_k^{(s)}(t), V_k^{(s)}(t)), \end{cases} \quad (16)$$

où $X_k^{(s)} : \mathbb{R}_+ \mapsto \Omega$ désigne la position dans l'espace physique et $V_k^{(s)} : \mathbb{R}_+ \mapsto \mathbb{R}^d$ le vecteur vitesse. Dans un gaz ou un plasma, le nombre de particule est de l'ordre du nombre d'Avogadro ce qui correspond à $N \approx 10^{24}$. Cela correspond à un modèle composé du même nombre d'équations aux dérivées ordinaires (EDO) couplées, puisque le champ de force F_s dépend du champ électrique, lui même relié aux positions de toutes les particules grâce à l'équation de Poisson (7). À cause de cette complexité, l'analyse et la simulation d'un tel modèle est tout simplement impossible. Pour décrire un plasma on utilise un niveau de description différent, soit *macroscopique* soit *mésoscopique*.

Description macroscopique Les modèles macroscopiques résultent d'une modélisation continue de la matière. Au lieu d'appliquer les lois de Newton à des particules discrètes, on résonne par bilan de matière dans des éléments de volumes infinitésimaux dans Ω . Cela aboutit à une ou plusieurs équations aux dérivées partielles (EDP) portant sur des grandeurs macroscopiques fonctions de (t, x) . En voici les principales dans notre contexte :

- La densité macroscopique $n_s : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$;
- La densité de charge $\rho_s : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$, telle que $\rho_s = q_s n_s$;
- La densité de charge totale $\rho = \rho_e + \rho_i$;
- La vitesse moyenne $u_s : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$;
- La densité de courant $j_s : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$, telle que $j_s = \rho_s u_s$;
- La densité de courant totale $j = j_e + j_i$;
- La température $T_s : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$.

Les modèles reliant ces différentes quantités (ainsi que les champs électriques et magnétiques) sont appelés modèles *hydrodynamiques* ou *fluides*. Parmi les modèles hydrodynamiques classiques utilisés en physique des plasmas, on trouve en particulier les équations de la *magnétohydrodynamique* introduites par le physicien suédois Hannes Alfvén dans les années 40. L'approximation des électrons sans masse qui nous intéresse dans cette thèse mène également à des modèles fluides pour la dynamique des électrons. Cependant, la description macroscopique repose sur l'hypothèse que les particules sont à l'équilibre thermodynamique. Cela signifie qu'en un point (t, x) la probabilité qu'une particule ait une vitesse donnée suit une loi normale centrée en la vitesse moyenne et de variance donnée par la température (voir Figure 2). Cette hypothèse est rarement vérifiée pour un plasma de fusion, ce qui nécessite une description supplémentaire de la distribution en vitesse au sein du modèle. C'est le principe de la théorie cinétique.

Théorie cinétique Formalisée par Maxwell et Boltzmann⁶, la théorie cinétique offre une échelle de description intermédiaire entre le microscopique et le macroscopique, appelée *mésoscopique*. Le principe est de décrire l'évolution de la *densité microscopique* ou *fonction de distribution* $f_s(t, x, v)$ de l'espèce s . Celle-ci mesure la densité de particules dans l'espace des phases, si bien que pour des ensembles mesurables $\omega \subset \Omega$ et $\mathcal{V} \subset \mathbb{R}^d$ le nombre de particules d'espèce s contenues dans $\omega \times \mathcal{V}$ à l'instant t est

$$\int_{\omega} \int_{\mathcal{V}} f_s(t, x, v) dx dv .$$

La description cinétique renferme plus d'information que les modèles macroscopiques. Les différentes densités définies au paragraphe précédent correspondent à des moments en vitesses de la fonction de distribution. En particulier

$$n_s(t, x) = \int_{\mathbb{R}^d} f_s(t, x, v) dv , \tag{17}$$

$$j_s(t, x) = q_s n_s(t, x) u_s(t, x) = q_s \int_{\mathbb{R}^d} v f_s(t, x, v) dv , \tag{18}$$

$$T_s(t, x) = \frac{m_s}{d k_B n_s(t, x)} \int_{\mathbb{R}^d} |v - u_s(t, x)|^2 f_s(t, x, v) dv . \tag{19}$$

6. L'équation de Boltzmann fut d'abord écrite [149] par Maxwell en 1867 sous forme faible. Cinq ans plus tard Boltzmann tente d'expliquer pourquoi les solutions de l'équation convergent vers les équilibres calculés par Maxwell, ce qui l'amène à découvrir le fameux théorème H.

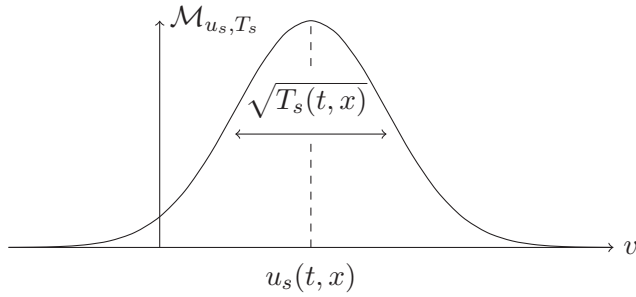


FIGURE 2 – **Modèles cinétiques.** Densité maxwellienne à l'équilibre thermodynamique.

L'équilibre thermodynamique évoqué plus haut comme condition nécessaire de validité de certains modèles macroscopiques correspond alors à une distribution *maxwellienne* des vitesses,

$$\begin{aligned} f_s(t, x, v) &= n_s(t, x) \mathcal{M}_{u_s, T_s}(t, x, v), \\ &= \frac{n_s(t, x)}{(2\pi k_B T_s / m_s)^{d/2}} \exp\left(-\frac{m_s |v - u_s(t, x)|^2}{2k_B T_s(t, x)}\right). \end{aligned} \quad (20)$$

L'équation cinétique la plus simple est l'équation de *transport libre*. Elle s'écrit,

$$\partial_t f_s + v \cdot \nabla_x f_s = 0. \quad (21)$$

Pour une donnée initiale $f_s^{\text{in}} : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ intégrable, il y a une unique solution à cette équation donnée par la formule explicite $f_s(t, x, v) = f_s^{\text{in}}(x - vt, v)$. À v fixé, la densité est transportée dans l'espace physique à vitesse v . L'opérateur $\partial_t + v \cdot \nabla_x$ est fondamental en théorie cinétique et il traduit le fait que la vitesse est la dérivée de la position. Malgré son apparente simplicité, le modèle (21) qui modélise des particules en vol libre sans aucune interaction recèle beaucoup d'information. Notons par exemple que les solutions à l'équilibre thermodynamique, en insérant (20) dans (21), sont telles que (n_s, u_s, T_s) vérifient les équations d'Euler pour un fluide compressible⁷.

En présence de forces électromagnétiques, l'équation cinétique adéquate est l'équation de Vlasov.

2.1 Équation de Vlasov

En présence d'un champ électromagnétique extérieur, l'équation de Vlasov peut-être obtenue simplement à partir des équations de Newton. En effet, si on considère $X_k^{(s)}(t)$, $V_k^{(s)}(t)$, $k = 1 \dots N$, les trajectoires de particules données par les équations de Newton (16) avec force de Lorentz (1) pour un champ électromagnétique extérieur (E, B) régulier.

⁷ Cela peut être lié à la limite hydrodynamique de forte collisions des équations cinétiques (voir Bardos, Golse, Levermore [14, 13]).

Alors la densité empirique

$$f_s(t, x, v) = \frac{1}{N} \sum_{k=1}^N \delta(x - X_k^{(s)}(t)) \otimes \delta(v - V_k^{(s)}(t)),$$

constituée d'une somme de Dirac centrés en les trajectoires, vérifie l'équation de Vlasov

$$\partial_t f_s + v \cdot \nabla_x f_s + \frac{q_s}{m_s} (E + v \wedge B) \cdot \nabla_v f_s = 0. \quad (22)$$

Ce résultat est prouvé dans [30, Proposition 1.4]. Si on considère une infinité de particules la densité limite vérifiera toujours l'équation de Vlasov. En effet la densité empirique est bornée uniformément en N dans l'espace des mesures de Radon finies et positives, si bien que, à l'extraction d'une sous-suite près il existe une densité limite lorsque $N \rightarrow \infty$ et cette densité vérifie l'équation de Vlasov linéaire (22).

Dans le cas d'interactions entre les particules, il est beaucoup plus difficile de justifier le passage du microscopique au cinétique et la justification mathématique de ce processus est souvent associé au *sixième problème de Hilbert* [127]. Dans le cas d'interactions coulombiennes, c'est un problème encore ouvert et on renvoie le lecteur aux articles de Dobrushin et Fritz [71, 72] pour les premiers résultats sur la question et à Hauray et Jabin [118, 119] pour les meilleurs résultats à ce jour.

2.2 Système de Vlasov-Poisson

Le modèle non-linéaire correspondant à des interactions électrostatiques auto-induites est le modèle de Vlasov-Poisson. Dans notre contexte, c'est-à-dire en dimension 3 avec deux espèces et un champ magnétique extérieur, il s'écrit

$$\begin{cases} \partial_t f_e + v \cdot \nabla_x f_e - \frac{q}{m_e} (E + v \wedge B) \cdot \nabla_v f_e = 0, \\ \partial_t f_i + v \cdot \nabla_x f_i + \frac{q}{m_i} (E + v \wedge B) \cdot \nabla_v f_i = 0, \\ E = -\nabla_x \phi, \\ -\varepsilon_0 \Delta_x \phi = \rho = q n_i - q n_e, \end{cases} \quad (23)$$

où l'on rappelle que n_i et n_e sont donnés par (17). Notons que si l'on avait souhaité décrire les effets magnétiques auto-induits, il aurait fallu remplacer (23) par le système de Vlasov-Maxwell obtenu en couplant les équations de Vlasov pour les différentes espèces avec les équations de Maxwell (2)-(5). Le passage à la limite non-relativiste de Vlasov-Maxwell vers Vlasov-Poisson a été étudié mathématiquement par Degond [60], Schaeffer [179], Asano et Ukai [7].

De nombreux travaux se sont concentrés sur le problème d'existence et d'unicité de solutions pour le système de Vlasov-Poisson. Ces travaux concernent pour la plupart une version à une seule espèce et sans champ magnétique extérieur, dans $\Omega = \mathbb{R}^d$, $d \leq 3$. Pour les solutions faibles globales qui nous intéresseront en partie au Chapitre 1 de

cette thèse, les résultats s'adaptent sans difficulté avec ajout d'un champ magnétique extérieur de régularité suffisante, qui ne modifie pas les estimations *a priori*. De même la considération un domaine périodique $\Omega = \mathbb{T}^d$ n'implique que des changements mineurs aux preuves d'existence de solutions faibles globales.

Sous hypothèses «physiques» (énergie totale et masse initiale finies, densité initiale bornée), les premiers théorèmes d'existence globale de solutions faibles pour Vlasov-Poisson sont dus à Arsenev [6] et Illner et Neunzert [131]. Les hypothèses furent ensuite améliorées (contrôle L^p plutôt que L^∞ sur la donnée initiale) par DiPerna et Lions dans [68, 66, 69]. La construction des solutions repose sur la méthode itérative suivante. On montre l'existence et l'unicité de solutions à un problème similaire à (23) où les champs sont remplacés par leur convolution avec une suite régularisante. Cette procédure est effectuée de manière à préserver des estimations *a priori* uniformes en la régularisation. La construction de solutions du problème régularisé est faite itérativement en résolvant à l'étape $n + 1$ (grâce à la méthode des caractéristiques) l'équation de transport avec les champs calculés par la densité de l'étape n . Ensuite, on passe à la limite dans ce processus puis en la régularisation grâce aux estimations *a priori*. La compacité forte nécessaire pour le passage à la limite dans le terme non-linéaire est obtenue sur le champ électrique, grâce à l'effet régularisant de l'équation de Poisson.

Lions et Perthame, ont développé une théorie de la régularité pour Vlasov-Poisson basée sur la propagation des moments en vitesses [141]. En particulier, ils ont montré qu'une donnée initiale aux six premiers moments bornés permet d'obtenir une densité macroscopique bornée uniformément en t et x . Lorsque cette dernière propriété est vérifiée, on a alors unicité des solutions faibles, comme l'a démontré Loeper [142]. L'hypothèse sur ρ a été récemment relaxée par Miot [151] en une croissance au plus linéaire en p de ses normes L^p .

L'existence et l'unicité de solutions classiques globales pour $\Omega = \mathbb{R}^2$ ont été prouvées par Ukai et Okabe dans [187] pour des données initiales régulières à décroissance (polynomiale) rapide en x et v . En dimension 3, l'existence globale a été montrée par Bardos et Degond [11] pour des données initiales suffisamment petites. La preuve repose sur les propriétés dispersives de l'opérateur de transport libre, généralisées ensuite par Castella et Perthame [43] en des estimations de Strichartz. Dans [171], Pfaffelmoser montre, sans hypothèse de petitesse, l'existence et l'unicité de solutions fortes globales en trois dimensions pour des données initiales à support compact. Sa démonstration repose sur une estimation précise de la croissance du support en fonction du temps.

3 Modèles collisionnels

À notre connaissance les modèles réduits pour les électrons sans masse présupposent des électrons qu'ils soient à l'équilibre thermodynamique. Dans le système de Vlasov-Poisson rien ne force les électrons à ce retour vers l'équilibre. Nous verrons en Annexe A grâce à des simulations numériques, que sans collisions, le modèle des électrons boltzmannien n'est pas justifiable.

Électrons boltzmanniens Les *électrons boltzmanniens* ou *électrons adiabatiques* constituent un modèle classique dans la limite de faible rapport de masse. Ils sont caractérisés par leur densité macroscopique qui est donnée par la *densité de boltzmann* ou *densité de Gibbs*⁸

$$n_e = n_0 \exp\left(\frac{q\phi}{k_B T_e}\right), \quad (24)$$

reliant la densité, le potentiel électrique et la température des électrons et une constante de normalisation n_0 . Une autre façon d'écrire cette relation est de relier le gradient du logarithme de n_e multiplié par T_e avec le champ électrique. On parle alors de loi d'Ohm généralisée [181]. Dans la Section B.2 de l'Annexe B on justifie formellement l'obtention de (24) pour des électrons sans masse.

En présence d'un champ magnétique extérieur et dans le contexte de plasmas de tokamak, le modèle (24) est parfois remplacé par une version anisotrope, qui peut également s'écrire comme loi d'Ohm généralisée faisant intervenir le champ magnétique et que l'on peut justifier de la manière heuristique suivante. La justification mathématique fera l'objet du Chapitre 2. Sans champ magnétique, à cause des relations (15) et du fait que l'on observe les électrons sur l'échelle de temps des ions, tout se passe pour les électrons comme si l'on avait accéléré le temps d'un facteur $\lambda^{-1/2}$. Lorsqu'un champ magnétique extérieur est ajouté, l'amplitude relative de la force de Coulomb par rapport à celle de la force de Laplace peut être quantifiée par le paramètre

$$\mu = \frac{\bar{\phi}}{L V_i \bar{B}}. \quad (25)$$

Ce paramètre est petit lorsque le champ magnétique est fort. Dans la définition de μ , nous avons fait le choix d'utiliser la vitesse thermique des ions. C'est naturel en vertu du premier point de la procédure d'obtention de modèles réduits pour les électrons sans masse : la dynamique est observée du point de vue des ions. Si l'on en revient aux électrons, on constate qu'à cause du lien entre rapport de vitesses caractéristiques et rapport de masse dans (15), si l'effet de la force de Laplace est de taille μ^{-1} sur les ions, il est de taille $\mu^{-1}\lambda^{-1/2}$ pour les électrons. Notons également que la force de Laplace est orthogonale au champ magnétique et n'a aucun effet le long des lignes de champ. Ainsi, l'effet accru du champ magnétique sur les électrons va empêcher ces derniers d'atteindre l'équilibre thermodynamique dans la direction orthogonale au champ magnétique. Dans ce cas une approximation raffinée des électrons boltzmanniens est utilisée. À notre connaissance aucune dénomination classique n'existe pour cette approximation que nous appelons *densité*

8. Dans les ouvrages de physique des plasmas, on parlera de densité de boltzmann ou de Maxwell-Boltzmann, la dernière insinuant également une répartition maxwellienne des vitesses. La terminologie densité de Gibbs est plutôt utilisée en chimie, en physique statistique et en probabilité (lorsque l'on se réfère à la mesure stationnaire associée à une équation différentielle stochastique). Dans cette thèse on utilisera indifféremment les terminologies «Maxwell-Boltzmann», «Boltzmann», «Gibbs» et «Boltzmann-Gibbs».

de Boltzmann anisotrope

$$n_e = N_e \exp\left(\frac{q(\phi - \tilde{\phi})}{k_B T_e}\right), \quad (26)$$

où $\tilde{\phi}$ est une version moyennée du potentiel électrique le long des lignes⁹ de champ magnétique et N_e une densité constante le long du champ magnétique. C'est par exemple ce modèle¹⁰ qui est utilisé [110, 94] dans les simulations de modèles gyrocinétiques¹¹ pour le tokamak ITER.

Comme on le verra au Chapitre 2, la dynamique de N_e dans le plan orthogonal au champ magnétique est reliée à celle des centres-guides. En effet, lorsque le champ magnétique est fort, le temps cyclotron et le rayon de Larmor diminuent. À la limite la trajectoire de la particule se confond avec la trajectoire du centre de l'hélice, appelé *centre-guide*. Sur l'échelle de temps adaptée, les effets dominants dans la direction orthogonale au champ magnétique sont les effets de *dérive*. En présence d'un champ électrique, et dans un champ magnétique unidirectionnel et constant, les centres-guides sont transportés à la vitesse

$$\mathbf{v}_{E \times B} = \frac{E \wedge B}{|B|^2}. \quad (27)$$

Cette dérive est appelée *dérive électrique* ou *dérive « E cross B »*¹². On remarque que cette vitesse de dérive est orthogonale au champ magnétique.

Afin d'espérer justifier les modèles (24) ou (26) on doit ajouter un mécanisme de collisions dans le système de Vlasov-Poisson (23).

3.1 Opérateurs de collisions

Contrairement à un gaz neutre où les collisions entre molécules sont de type «boules de billard», dans un plasma, la force électrique de Coulomb interdit de telles interactions. Cependant, à cause d'une densité importante de particules avec de fortes énergies cinétiques, certaines interactions électriques engendrent une déviation de la trajectoire seulement à faible portée (sous la longueur de Debye). On parle alors de *collisions de Coulomb* [154, Section 2.6]. L'angle de déviation est en général faible, mais leur accumulation résulte en un brassage des vitesses des particules et un effet de thermalisation.

Pour tenir compte de ce phénomène dans les modèles cinétiques, et si l'on considère uniquement des collisions entre des particules de la même espèce¹³, l'équation de Vlasov (22) est modifiée de la manière suivante

$$\partial_t f_s + v \cdot \nabla_x f_s + \frac{q_s}{m_s} (E + v \wedge B) \cdot \nabla_v f_s = \nu_s Q_s(f_s), \quad (28)$$

9. ou surfaces, suivant la géométrie du champ magnétique

10. sous forme linéarisée

11. un modèle réduit lorsque $\mu \rightarrow 0$

12. en référence à la notation anglo-saxonne $E \times B$ pour le produit vectoriel

13. Il y a également des collisions de Coulomb entre ions et électrons. Pour davantage de détails nous renvoyons vers le papier de Degond [61]. Ici notre approche sera de considérer un mécanisme de collision plus basique, afin de permettre une étude mathématique rigoureuse.

avec ν_s la fréquence de collision. Pour une étude détaillée des différents opérateurs de collisions, leur signification physique et les problèmes mathématiques associés, on renvoie à l'article de Villani [193].

Fokker-Planck-Landau L'un des opérateurs les plus réalistes pour modéliser des collisions de Coulomb en physique des plasmas est l'opérateur de *Fokker-Planck-Landau* ou simplement *Landau*. Il fut formellement obtenu par Landau comme une approximation de l'opérateur de Boltzmann avec potentiel d'interaction de Debye. Pour plus de détail sur ce processus d'approximation, on renvoie au papier d'Alexandre et Villani [2]. L'opérateur de Landau s'écrit

$$Q_s^L(f_s) = \nabla_v \cdot \left[\int_{\mathbb{R}^3} K_s a(v - v_1) ((f_s)_1 \nabla_v f_s - f_s (\nabla_v f_s)_1) dv_1 \right], \quad (29)$$

où K_s est une constante physique, l'indice 1 est utilisé pour les quantités évaluées en la variable d'intégration v_1 et $a(v)$ est la matrice 3×3 de composantes

$$a_{ij}(v) = \frac{1}{|v|} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right).$$

L'opérateur de Landau a une structure de type Fokker-Planck (ou convection-diffusion) dans la variable de vitesse. En effet,

$$Q_s^L(f_s) = \nabla_v \cdot (A(f_s) \nabla_v f_s + B(f_s) f_s). \quad (30)$$

avec $A(f_s)$ la matrice dont les coefficients sont donnés par la convolution $A_{ij}(f_s) = a_{ij} *_v f_s$ et $B(f_s)$ le vecteur dont les composantes sont $B_j(f_s) = b_j *_v f_s$ avec $b_j(v) = \sum_i \partial_{v_i} a_{ij}(v)$. L'opérateur de Fokker-Planck-Landau jouit des mêmes propriétés d'invariance et d'entropie que l'opérateur de Boltzmann (voir [30, Theorem 2.1]). En effet, il vérifie la conservation de la masse, du moment et de l'énergie cinétique

$$\int_{\mathbb{R}^3} Q_s^L(f_s) dv = 0, \quad \int_{\mathbb{R}^3} Q_s^L(f_s) \begin{pmatrix} v \\ |v|^2 \end{pmatrix} dv = 0,$$

ainsi que la décroissance de l'entropie («théorème H»)

$$\int_{\mathbb{R}^3} Q_s^L(f_s) \ln(f_s) dv \leq 0.$$

Le noyau de Q_s^L est constitué de l'ensemble des densités maxwelliennes (20). Ces deux dernières propriétés caractérisent l'irréversibilité de l'équation cinétique collisionnelle (28)-(29) et le retour vers les équilibres thermiques. Pour l'opérateur de Landau, ces équilibres sont caractérisés par les quantités macroscopiques n_e , u_e et T_e .

Dans plusieurs articles récents de Degond [61], Negulescu, Possaner et DeCecco [160, 44], les auteurs obtiennent formellement des modèles hydrodynamiques réduits pour les électrons sans masse à partir d'opérateurs non-linéaires de Landau ou des versions

simplifiées telles que l'opérateur BGK. Étant donné la forme des équilibres, les modèles obtenus font intervenir une dynamique complexe sur les densités macroscopiques, vitesses moyennes et températures.

Cependant, la littérature mathématique concernant la justification des densités macroscopiques boltzmanniennes (24) ou (26) par un processus de passage à la limite rigoureux est quasi-inexistante. À notre connaissance les seuls travaux en ce sens sont ceux de Bouchut et Dolbeault [29] et le très récent [15] par Bardos, Golse, Nguyen et Sentis. Dans les deux cas, seuls des modèles sans champ magnétique sont considérés. Afin d'étudier le cas magnétisé en s'affranchissant de la dynamique en température et en vitesse moyenne, on remplace dans nos modèles l'opérateur de Landau avec un opérateur de collision à la structure et aux propriétés similaires mais possédant un noyau plus simple, l'opérateur de Fokker-Planck linéaire.

Fokker-Planck linéaire L'opérateur de Fokker-Planck linéaire, introduit par l'astrophysicien et mathématicien Chandrasekhar [51, 49] s'écrit

$$Q_s(f_s) = \nabla_v \cdot (v f_s + \frac{k_B \theta}{m_s} \nabla_s f_s). \quad (31)$$

Ce dernier modélise l'interaction entre des particules et un bain thermique à la température θ . Les collisions sont réduites à la somme de deux phénomènes, une force de friction proportionnelle à la vitesse et un bruit Brownien de température θ . Les caractéristiques de l'équation de Vlasov-Fokker-Planck, c'est-à-dire (28) avec l'opérateur (31) sont données par le système d'équations différentielles stochastiques suivant

$$\begin{cases} dX_t = V_t dt, \\ dV_t = \frac{q_s}{m_s} (E(t, X_t) + v \wedge B(t, X_t)) dt - \nu_s V_t dt - \frac{\nu_s k_B \theta}{m_s} dW_t, \end{cases} \quad (32)$$

où $(W_t)_{t \geq 0}$ est un mouvement Brownien standard. Inversement, l'équation de Vlasov-Fokker-Planck peut être vue comme l'évolution temporelle de la fonction de densité associée à la loi de probabilité du vecteur (X_t, V_t) . L'opérateur de Fokker-Planck linéaire peut être vu comme un opérateur non-borné sur l'espace de Hilbert $L^2(M_{s,\theta}^{-1} dv)$ caractérisé par le produit scalaire

$$\langle f, g \rangle_{L^2(M_{s,\theta} dv)} = \int_{\mathbb{R}^d} f(v) g(v) \frac{dv}{M_{s,\theta}(v)}.$$

où $M_{s,\theta}$ est la maxwellienne centrée suivante

$$M_{s,\theta}(v) = \frac{1}{(2\pi k_B \theta / m_s)^{d/2}} \exp\left(-\frac{m_s |v|^2}{2 k_B \theta}\right).$$

L'opérateur $-Q_s$ est autoadjoint et positif. Son image est constituée des fonctions de $L^2(M_{s,\theta}^{-1} dv)$ à moyenne nulle et son noyau est la droite engendrée par $M_{s,\theta}$ (voir [81]).

Pour des fonctions suffisamment régulières, on a également un théorème H (voir [172])

$$\int_{\mathbb{R}^d} Q_s(f) \ln \left(\frac{f}{M_{s,\theta}} \right) dv \leq 0,$$

avec égalité si et seulement si $f_s = n M_{s,\theta}$ pour une certaine constante (en v) positive n . L'opérateur de Fokker-Planck linéaire dispose d'un unique invariant, la masse

$$\int_{\mathbb{R}^d} Q_s(f) dv = 0,$$

ce qui explique la perte des degrés de liberté sur la température¹⁴ et la vitesse moyenne dans les maxwelliennes.

3.2 Système de Vlasov-Poisson-Fokker-Planck

Le couplage entre les équations de Vlasov collisionnelles (28), les opérateurs de collision de Fokker-Planck (31) et l'équation de Poisson (7) conduit au système de Vlasov-Poisson-Fokker-Planck à deux espèces avec champ magnétique extérieur suivant

$$\begin{cases} \partial_t f_e + v \cdot \nabla_x f_e - \frac{q}{m_e} (E + v \wedge B) \cdot \nabla_v f_e = \nu_e Q_e(f_e), \\ \partial_t f_i + v \cdot \nabla_x f_i + \frac{q}{m_i} (E + v \wedge B) \cdot \nabla_v f_i = \nu_i Q_i(f_i), \\ E = -\nabla_x \phi, \\ -\varepsilon_0 \Delta_x \phi = q(n_i - n_e). \end{cases} \quad (33)$$

La théorie d'existence et d'unicité¹⁵ de solutions pour des systèmes de type Vlasov-Poisson-Fokker-Planck (VPFP) a débuté vers la fin des années 80. Le premier résultat d'existence de solution est celui de Neunzert, Pulvirenti et Triolo [161]. Pour $\Omega = \mathbb{R}^2$, ils montrent par des méthodes probabilistes l'existence de mesures finies solutions de VPFP en se basant sur l'analyse des caractéristiques stochastiques (32). Peu après, Degond prouve dans [59] l'existence et l'unicité de solutions fortes en une et deux dimensions pour VPFP sans terme de frottement. Il justifie également la limite vers le système de Vlasov-Poisson lorsque la fréquence de collision tend vers 0. Victory et O'Dwyer montrent dans [192] l'existence de solutions classiques pour des données initiales arbitraires en deux dimensions et des données petites en dimension supérieure. Leur méthode repose sur le calcul explicite de la fonction de Green de l'opérateur de Fokker-Planck cinétique $\partial_t + v \cdot \nabla_x - \nabla_v \cdot (\nabla_v + v)$. Signalons que nous reproduisons ce calcul dans le cas magnétisé en Section B.3 de l'Annexe B et qu'il existe des résultats similaires pour des opérateurs différentiels plus généraux. On parle alors de formules de Mehler (voir Hörmander [129]). Grâce à une formulation de Duhamel utilisant la solution fondamentale,

14. θ est fixé ici

15. sans champ magnétique extérieur et pour une seule espèce. La généralisation (pour des solutions faibles) ne pose pas plus de problèmes que pour le système de Vlasov-Poisson.

Victory et O’Dwyer reformulent VPF_P en incluant le champ électrique comme un terme source. À partir des estimations sur la fonction de Green, ils parviennent à montrer l’existence de solutions classiques grâce à un argument de point fixe. Cette technique est reprise ensuite par Victory [191] pour les solutions faibles en 3D, par Rein et Weckler [175] pour l’existence globale de solutions classiques en 3D avec donnée initiale petite. Enfin, Bouchut montre dans [28, 26] l’existence et l’unicité de solutions lisses à partir de données initiales intégrables et bornées avec un contrôle de quelques moments en vitesse. Il s’inspire de la méthode, alors récente, de Lions et Perthame sur la propagation de régularité des moments en vitesse [141].

Comme nous le verrons par la suite, les quantités naturellement propagées par les systèmes de type VPF_P sont la masse et l’énergie libre. Les supposer bornées à l’instant initial ne suffit pas à satisfaire les hypothèses des théorèmes précédents et même si l’on admettait l’existence de solutions avec la seule régularité donnée par ces estimations, cela ne suffirait pas à donner un sens à l’équation dans l’espace des distributions ! Cependant, en reformulant l’équation, il est tout de même possible de construire des solutions très faibles, avec ces seules estimations. Ce sont les *solutions renormalisées* de DiPerna et Lions [68, 66, 67, 69], qui dans le cas de VPF_P sont également appelées *solutions d’énergie libre* [29, 74, 81]. Nous expliquerons cette notion en Section 4 et dans le Chapitre 1.

3.3 Modèle adimensionné à la base de notre étude

Après une procédure d’adimensionnement de (33) que l’on détaille en Section B.1 de l’Annexe B, on obtient le modèle suivant

$$\left\{ \begin{array}{l} \lambda^{1/2} \partial_t f_e + v \cdot \nabla_x f_e - E \cdot \nabla_v f_e - \frac{1}{\lambda^{1/2} \mu} v \wedge B \cdot \nabla_v f_e = \gamma_e(\lambda) Q(f_e), \\ \partial_t f_i + v \cdot \nabla_x f_i + E \cdot \nabla_v f_i + \frac{1}{\mu} v \wedge B \cdot \nabla_v f_i = \gamma_i Q(f_i), \\ E = -\nabla_x \phi, \\ -\delta^2 \Delta_x \phi = n_i - n_e. \end{array} \right. \quad (34)$$

avec Q l’opérateur de Fokker-Planck adimensionné

$$Q(f) = \nabla_v \cdot (v f + \nabla_v f) = \nabla_v \cdot \left(M \nabla_v \left(\frac{f}{M} \right) \right) \quad (35)$$

et M est la maxwellienne centrée réduite

$$M(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}. \quad (36)$$

Sous cette forme, on voit apparaître les paramètres sans dimension introduits précédemment dans (10), (14) et (25), et qui sont respectivement la longueur de Debye adimensionnée δ , le rapport de masse λ et le paramètre de champ magnétique fort μ . En plus de ces paramètres, on en trouve deux nouveaux, $\gamma_e(\lambda)$ et γ_i correspondants aux inverses des

libres parcours moyens adimensionnées. On se laisse le choix de prendre un ou deux de ces paramètres égaux à 0, auquel cas on retombera sur un système de Vlasov-Poisson comme dans l'Annexe A ou un système mixte comme au Chapitre 1 (voir la Remarque B.1.1 pour quelques commentaires sur ces paramètres). À cause du scaling sur l'échelle de temps des ions, on remarque que $\gamma_e(\lambda)$ peut dépendre du rapport de masse. C'est exactement sur le choix de cette dépendance que nous allons jouer pour obtenir différents modèles réduits dans les Chapitres 1 et 2.

Dans les quatre sections suivantes nous résumons et commentons nos contributions des Chapitres 1 à 4.

4 Limite de diffusion anisotrope (Chapitre 1)

Dans ce chapitre on s'intéresse à la limite $\varepsilon \rightarrow 0$ du système bi-cinétique suivant

$$\begin{cases} \partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon - \nabla_x \phi^\varepsilon \cdot \nabla_v f_i^\varepsilon + (v \wedge B) \cdot \nabla_v f_i^\varepsilon = 0, \\ \varepsilon \partial_t f_e^\varepsilon + v \cdot \nabla_x f_e^\varepsilon + \nabla_x \phi^\varepsilon \cdot \nabla_v f_e^\varepsilon - \frac{1}{\varepsilon} (v \wedge B) \cdot \nabla_v f_e^\varepsilon = \frac{1}{\varepsilon} Q(f_e^\varepsilon), \\ -\Delta_x \phi^\varepsilon = n_i^\varepsilon - n_e^\varepsilon. \end{cases} \quad (37)$$

Ici on travaille dans l'espace entier¹⁶ $\Omega = \mathbb{R}^3$ et pour un champ magnétique quelconque

$$B(t, x) = (b_1(t, x) \ b_2(t, x) \ b_3(t, x))^\top,$$

que l'on suppose borné uniformément. Ce système correspond à (34) avec $\delta = \mu = 1$, $\varepsilon := \lambda^{1/2}$, $\gamma_i = 0$ et

$$\gamma_e(\lambda) = \frac{1}{\lambda^{1/2}}.$$

Cette mise à l'échelle est également utilisée dans le cadre de faible rapport de masse, pour des opérateurs de collisions différents, dans les travaux de Petit et Darrozes [170], de Degond et Lucquin [64, 65, 61] et plus récemment de Graille, Magin, Massot et Giovangigli [109, 97, 96]. Il est naturel de s'intéresser à ce cette mise à l'échelle car dans l'équation des électrons, le terme correspondant au champ magnétique est alors du même ordre en ε que le terme de Fokker-Planck. Algébriquement, on peut fusionner les deux termes en introduisant l'opérateur de Fokker-Planck anisotrope

$$\mathcal{L}_A(f) = \nabla_v \cdot (A(t, x) v f + \nabla_v f)$$

avec A la matrice telle que $Av = v + v \wedge B$, c'est-à-dire

$$A = \begin{pmatrix} 1 & b_3 & -b_2 \\ -b_3 & 1 & b_1 \\ b_2 & -b_1 & 1 \end{pmatrix}.$$

Puisque $v \wedge B$ est à divergence nulle en vitesse, la seconde équation de (37) se récrit alors

$$\varepsilon \partial_t f_e^\varepsilon + v \cdot \nabla_x f_e^\varepsilon + \nabla_x \phi^\varepsilon \cdot \nabla_v f_e^\varepsilon = \frac{1}{\varepsilon} \mathcal{L}_A(f_e^\varepsilon),$$

et le problème peut s'interpréter comme une limite de diffusion pour l'opérateur de collision anisotrope \mathcal{L}_A . Sans champ magnétique et à une seule espèce, la limite de diffusion

¹⁶. Dans le cas du tore tri-dimensionnel \mathbb{T}^3 , le processus de passage à la limite est essentiellement le même que celui présenté dans ce chapitre. Notons que dans ce cas, on peut s'affranchir du contrôle d'un moment en espace sur la donnée initiale (voir Théorème 1), qui découle de l'intégrabilité des fonctions de distributions en domaine borné. De plus, concernant l'équation de Poisson, on remarquera que la condition de neutralité globale (déjà supposée ici) est capitale pour que l'équation soit bien posée sur le tore.

de VPFPP a été étudiée par plusieurs auteurs. Les premiers résultats majeurs sont dus à Poupaud et Soler [172] qui traitent essentiellement le cas bidimensionnel. Le résultat le plus récent, 3D et global en temps avec des hypothèses de régularité très faibles sur la donnée initiale, est celui de Masmoudi et El Ghani [81]. L'analyse développée dans [81] va nous fournir un bon nombre de clés pour traiter notre propre problème. Néanmoins, le champ magnétique fort et la considération d'un système à deux espèces créent des problèmes nouveaux dans l'analyse. D'abord, décrivons le passage à la limite formel.

4.1 Passage à limite formel et modèle asymptotique

Si une limite (f_i, f_e, ϕ) existe lorsque $\varepsilon \rightarrow 0$ alors on s'attend, par identification des ordres principaux en ε dans les équations, à ce que f_i et ϕ vérifient toujours respectivement les équations de Vlasov et Poisson et que f_e soit dans le noyau de \mathcal{L}_A . On peut montrer que, dans le bon cadre fonctionnel, les seules fonctions de v vérifiant cette propriété sont les maxwelliennes locales. Ainsi on s'attend à obtenir à la limite

$$f_e(t, x, v) = n_e(t, x) M(v).$$

On peut ensuite déterminer l'équation sur n_e par la méthode des moments. Plus précisément, si l'on pose

$$j_e^\varepsilon(t, x) = \frac{1}{\varepsilon} \int_{\mathbb{R}^3} v f_e^\varepsilon(t, x, v) dv, \quad (38)$$

alors en intégrant en vitesse l'équation des électrons, on obtient l'équation de continuité

$$\partial_t n_e^\varepsilon + \nabla_x \cdot j_e^\varepsilon = 0. \quad (39)$$

En multipliant l'équation des électrons par v et en intégrant on a

$$\begin{aligned} \varepsilon^2 \partial_t j_e^\varepsilon + \nabla_x \cdot \int_{\mathbb{R}^3} (v \otimes v - I) f_e^\varepsilon dv + \nabla_x n_e^\varepsilon - \nabla_x \phi^\varepsilon n_e^\varepsilon &= -j_e^\varepsilon \wedge B - j_e^\varepsilon, \\ &= -A(t, x) j_e^\varepsilon, \end{aligned} \quad (40)$$

où I est la matrice identité et $v \otimes v$ la matrice de composantes $v_i v_j$ en i -ème ligne et j -ième colonne. Pour obtenir un modèle limite il faut pouvoir fermer les équations, c'est-à-dire exprimer le moment d'ordre le plus élevé en fonction des autres. C'est ici possible grâce à la convergence vers une maxwellienne locale. En effet pour $i, j = 1, \dots, 3$, on a

$$\int_{\mathbb{R}^3} (v_i v_j - \delta_{ij}) M(v) dv = 0.$$

Par conséquent le système limite attendu, après avoir fait tendre $\varepsilon \rightarrow 0$ est le suivant

$$\left\{ \begin{array}{l} \partial_t f_i + v \cdot \nabla_x f_i - \nabla_x \phi \cdot \nabla_v f_i + (v \wedge B) \cdot \nabla_v f_i = 0, \\ \partial_t n_e + \nabla_x \cdot j_e = 0, \\ j_e + j_e \wedge B = \nabla_x \phi n_e - \nabla_x n_e, \\ -\Delta_x \phi = n_i - n_e, \\ f_i(0, \cdot, \cdot) = f_i^{\text{in}} \quad \text{et} \quad n_e(0, \cdot) = \int f_e^{\text{in}} dv. \end{array} \right. \quad (41)$$

Quelques commentaires sur le modèle limite s'imposent. Il couple l'équation (cinétique) de Vlasov sur les ions avec une équation (fluide) de convection-diffusion sur la densité électronique macroscopique. En effet, en insérant l'expression de j_e dans la seconde équation, on a

$$\partial_t n_e - \nabla_x \cdot (D E n_e + D \nabla_x n_e) = 0, \quad (42)$$

où $E = -\nabla_x \phi$ et la matrice de diffusion $D(t, x) = A^{-1}(t, x)$ est uniformément bornée et uniformément définie positive. Dans cette matrice se cache certains aspects de la dynamique centre-guide. En effet si on pose $E = -\nabla_x \phi$ alors un développement asymptotique formel dans la limite $|B| \rightarrow \infty$ donne

$$D E = \frac{(E \cdot B) B}{|B|^2} - \mathbf{v}_{E \times B} + O\left(\frac{1}{|B|^2}\right)$$

où $\mathbf{v}_{E \times B}$ est la dérive électrique définie en (27). Au premier ordre, on a des effets de la partie du champ électrique parallèle au champ magnétique. À l'ordre suivant, on retrouve la dérive électrique puis les effets d'advection d'ordre supérieur. Notons que les effets de dérive dus au gradient et à la géométrie du champ magnétique se trouvent dans le terme $\nabla_x \cdot D$ (en développant la partie correspondant à la diffusion).

D'autres exemples de modèles hydrodynamiques pour les plasmas avec diffusion asymétrique peuvent être trouvés dans [20, 56, 58, 57, 61]. En particulier, notons que les modèles de départ et limite considérés par Ben Abdallah et El Hajj dans [20] sont très proches du notre. Cependant ils considèrent des situations linéaires, c'est-à-dire sans couplage avec l'équation de Poisson. Sous la forme (41), la troisième équation correspond à une loi d'Ohm. De plus, écrite sous cette forme l'équation sur les électrons coïncide avec le modèle de Van Roosbroeck étudié par Gajewski et Gartner [93] dans le cadre de semi-conducteurs avec champ magnétique.

Remarquons également que les densités de Gibbs isotropes, multiples de e^ϕ , sont des états stationnaires de (41).

4.2 Estimations *a priori* uniformes

Afin de justifier le passage à la limite, nous avons besoin d'estimations. Dans les scaling diffusifs de VPFP, il est difficile d'obtenir des estimations *a priori* uniformes

en ε procurant suffisamment de régularité, au sens large de l'intégrabilité et du gain de compacité, pour passer à la limite, particulièrement dans les termes non-linéaires contenant $\nabla_x \phi^\varepsilon$. Dans la suite on donne et on commente les estimations dont nous disposerons.

Commençons par la propagation des normes L^p . Pour l'équation de Vlasov des ions, grâce à la structure de transport, on a pour tout $p \in [1, +\infty]$ et uniformément en t et ε

$$\|f_i^\varepsilon(t)\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} = \|f_i^{\text{in}}\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)},$$

au moins pour des solutions assez régulières. Dans le cas de l'équation de Vlasov-Fokker-Planck des électrons, le terme de frottement en vitesse $\nabla_v \cdot (v f_e^\varepsilon)$ provoque un phénomène de concentration de la distribution illustré par l'estimation suivante

$$\|f_e^\varepsilon(t)\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} \leq \|f_e^{\text{in}}\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} \exp\left(\frac{3}{\varepsilon^2} \left(1 - \frac{1}{p}\right) t\right)$$

avec $1/p = 0$ si $p = +\infty$. Par conséquent, si l'on considère une donnée initiale f_e^{in} intégrable et bornée, celle-ci le restera jusqu'à n'importe quel temps final $T > 0$ fini. En revanche, on perd toute information dans le régime $\varepsilon \rightarrow 0$ sauf pour $p = 1$ donc supposer davantage d'intégrabilité n'aiderait pas au passage à la limite. On dispose d'une autre estimation naturelle uniforme pour le système, la dissipation d'énergie libre,

$$\mathcal{E}^\varepsilon(t) + \int_0^t D^\varepsilon(\tau) d\tau = \mathcal{E}(0), \quad (43)$$

où l'énergie libre est la somme de l'énergie potentielle électrique, des énergies cinétiques et de l'entropie des électrons

$$\mathcal{E}^\varepsilon = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x \phi^\varepsilon|^2 dx + \frac{1}{2} \iint_{\mathbb{R}^6} |v|^2 (f_i^\varepsilon + f_e^\varepsilon) dv dx + \iint_{\mathbb{R}^6} f_e^\varepsilon \ln(f_e^\varepsilon) dv dx,$$

la dissipation d'énergie libre est la quantité positive suivante

$$D^\varepsilon(t) = \frac{4}{\varepsilon^2} \iint_{\mathbb{R}^6} \left| \nabla_v \sqrt{\frac{f_e^\varepsilon}{M(v)}} \right|^2 M(v) dv dx. \quad (44)$$

L'estimation d'énergie libre fournit un contrôle $L \log L$, meilleur que L^1 , propagé uniformément sur la fonction de distribution des électrons ainsi que deux moment en vitesse. Par un argument de convexité on s'attend alors également à contrôle uniforme $L \log L$ de la densité macroscopique n_e^ε . De plus, le champ électrique est naturellement borné dans $L^\infty(0, T; L^2(\mathbb{R}^3))$. Enfin, grâce au facteur $1/\varepsilon^2$, le contrôle de la dissipation d'énergie libre donne immédiatement la convergence forte vers une maxwellienne locale. En effet, grâce aux inégalités fonctionnelles de Csiszar-Kullback-Pinsker (voir Corollaire C.2.4) et de log-Sobolev (voir Corollaire C.2.2) on sait que la distance entre f_e^ε et $n_e^\varepsilon M$ est d'ordre ε dans $L^2(0, T; L^1(\mathbb{R}^6))$.

L'un des principaux problèmes relatifs aux limites diffusives de VPFPP provient du fait que les seules estimations uniformes ne permettent pas de donner un sens (dans \mathcal{D}') au terme non-linéaire de l'équation des électrons et à $n_e \nabla_x \phi$ dans l'équation (41), même en utilisant les propriétés de régularisation elliptique de l'équation de Poisson. Dans les premiers résultats sur la limite de diffusion de VPFPP de Poupaud et Soler [172] en dimension 2, améliorés ensuite par Goudon dans [106], ce problème est évité par une reformulation du produit $n_e \nabla_x \phi$ utilisant de manière essentielle les symétries de la solution fondamentale du Laplacien et la nature de sa singularité en dimension 2 (voir [172, Section 4]). Cette reformulation permet l'estimation uniforme du produit grâce aux seules quantités contrôlées par les estimations précédentes, et le passage à la limite peut se faire uniquement avec de la convergence faible dans L^1 en espace. Cependant en dimension 3 ce traitement astucieux ne fonctionne pas. Dans [81] et suite au travail de Masmoudi et Tayeb [147] sur la limite de diffusion de Vlasov-Poisson-Boltzmann¹⁷, Masmoudi et El Ghani parviennent à traiter ce cas en utilisant deux autres outils puissants, les solutions renormalisées, que l'on doit principalement à DiPerna et Lions, et la compacité par moyennisation en vitesse découverte par Golse. Les premières permettent de s'affranchir de la discussion du sens à donner au produit non-linéaire au prix d'une formulation plus technique de la notion de solution. Grâce aux lemmes de moyenne, on peut obtenir de la compacité forte sur la densité macroscopique, ce qui règle le problème du passage à la limite. Précisons maintenant comment vont s'appliquer ces outils dans notre cadre.

4.3 Solutions renormalisées et résultat principal

La notion de solution renormalisée pour des équations cinétiques comprenant des termes de type Fokker-Planck est spécifiquement introduite dans le papier de DiPerna et Lions [67]. Ici, on adapte la notion de solution utilisée par Masmoudi et El Ghani pour l'équation des électrons, et la notion de solution faible d'Arsenev [6] pour l'équation de Vlasov des ions. On renvoie à la Définition 1.1.1 pour la définition complète de la notion de solution. Précisons seulement ici que l'on demande $f_e^\varepsilon \in L^\infty(0, T; L^1(\mathbb{R}^6))$, $\nabla_v \sqrt{f_e^\varepsilon} \in L^2(0, T; L^2(\mathbb{R}^6))$, $\nabla_x \phi^\varepsilon \in L^\infty(0, T; L^2(\mathbb{R}^3))$ et que pour toute fonction $\beta \in \mathcal{C}^2(\mathbb{R}_+)$ telle que

$$|\beta(u)| \leq C(\sqrt{u} + 1), \quad |\sqrt{u}\beta'(u)| \leq C, \quad |u\beta''(u)| \leq C,$$

pour un certain $C > 0$, $\beta(f_e^\varepsilon)$ vérifie dans $\mathcal{D}'([0, T] \times \mathbb{R}^6)$

$$\begin{aligned} & \varepsilon \partial_t \beta(f_e^\varepsilon) + v \cdot \nabla_x \beta(f_e^\varepsilon) + \nabla_x \phi^\varepsilon \cdot \nabla_v \beta(f_e^\varepsilon) - \frac{1}{\varepsilon} (v \wedge B) \cdot \nabla_v \beta(f_e^\varepsilon) \\ &= \frac{1}{\varepsilon} \left(3 f_e^\varepsilon \beta'(f_e^\varepsilon) + v \cdot \nabla_v \beta(f_e^\varepsilon) + \Delta_v \beta(f_e^\varepsilon) - 4 f_e^\varepsilon \left| \nabla_v \sqrt{f_e^\varepsilon} \right|^2 \beta''(f_e^\varepsilon) \right). \end{aligned} \quad (45)$$

17. L'opérateur de Boltzmann est linéarisé autour d'une maxwellienne.

avec donnée initiale $\beta(f_e^{\text{in}})$ et pour tout $\lambda > 0$, $\theta_{\varepsilon,\lambda} = \sqrt{f_e^\varepsilon + \lambda M}$ vérifie dans $\mathcal{D}'([0, T] \times \mathbb{R}^6)$

$$\varepsilon \partial_t \theta_{\varepsilon,\lambda} + v \cdot \nabla_x \theta_{\varepsilon,\lambda} + \nabla_x \phi^\varepsilon \cdot \nabla_v \theta_{\varepsilon,\lambda} = \frac{1}{2\varepsilon \theta_{\varepsilon,\lambda}} \mathcal{L}_A(f_e^\varepsilon) - \frac{\lambda M}{2\theta_{\varepsilon,\lambda}} v \cdot \nabla_x \phi^\varepsilon, \quad (46)$$

avec la donnée initiale $\sqrt{f_e^{\text{in}} + \lambda M}$. Dans (45) le membre de droite correspond (si la fonction f_e^ε était régulière) au terme $Q(f_e^\varepsilon)\beta'(f_e^\varepsilon)$. Pour la formulation (46), il faut donner un sens précis à la distribution $\mathcal{L}_A(f_e^\varepsilon)/\theta_{\varepsilon,\lambda}$, et pour éviter de surcharger cette introduction de détails techniques, on renvoie le lecteur à la démonstration de la Proposition 1.5.4 pour davantage de détails sur ce point. Cette équation provient d'une renormalisation de l'équation sur f_e^ε/M par la fonction $s \mapsto \sqrt{s + \lambda}$. Alors que (45) sert simplement à donner un sens alternatif à l'équation de Vlasov-Fokker-Planck des électrons, nous avons besoin de (46) pour passer à la limite. Notons qu'une reformulation similaire à (46) était déjà utilisée par Bouchut et Dolbeault dans [29] pour étudier le comportement en temps long de VPPF.

Nous pouvons maintenant énoncer le résultat principal du Chapitre 1 (Théorème 1.1.6 dans le chapitre).

Théorème 1 (Chapitre 1). *Si les données initiales sont positives presque partout et telle que $f_i^{\text{in}} \in L^1 \cap L^\infty(\mathbb{R}^6)$, $f_e^{\text{in}} \in L^1(\mathbb{R}^6)$ et*

$$\begin{aligned} \sum_{s \in \{i, e\}} \iint_{\mathbb{R}^6} f_s^{\text{in}} (|x| + |v|^2 + |\ln f_s^{\text{in}}|) dv dx + \int_{\mathbb{R}^3} |\nabla_x \phi^{\text{in}}|^2 dx &< +\infty, \\ \iint_{\mathbb{R}^6} f_i^{\text{in}} dv dx &= \iint_{\mathbb{R}^6} f_e^{\text{in}} dv dx. \end{aligned}$$

alors il existe une solution renormalisée $(f_i^\varepsilon, f_e^\varepsilon, \phi^\varepsilon)$ de (1.3). De plus lorsque $\varepsilon \rightarrow 0$ on a, à l'extraction d'une sous-suite près

$$\begin{aligned} f_i^\varepsilon &\longrightarrow f_i && \text{faible-}\star \text{ dans } L^\infty([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3), \\ f_e^\varepsilon &\longrightarrow n_e M && \text{fort dans } L^1(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3)), \\ \phi^\varepsilon &\longrightarrow \phi && \text{fort dans } L^2(0, T; W^{1,p}(\mathbb{R}^3)) \text{ pour tout } 1 \leq p < 2, \end{aligned}$$

et la limite (f_i, n_e, ϕ) est solution faible de (41).

Dans la suite on évoque un point clé de la démonstration relatif à la présence du champ magnétique singulier.

4.4 Compacité forte et contrôles par la dissipation

Contrairement au cas particulier de la dimension 2 où le passage à la limite dans le terme $n_e^\varepsilon \nabla_x \phi^\varepsilon$ de (40) peut être obtenu uniquement par de la convergence faible grâce à la réécriture de Poupaud et Soler, ici il faut de la compacité forte sur l'un des termes du

produit¹⁸. Pour ce faire, nous utilisons le lemme de moyenne suivant énoncé et démontré par Masmoudi et Tayeb dans [147, Lemme 4.2, Appendice 2] et directement adapté des célèbres résultats de Golse, Perthame, Sentis [103], Bardos, Golse, Perthame et Sentis [12], Golse, Lions, Perthame et Sentis [102], DiPerna, Lions et Meyer [70], Bouchut [30] et Golse et Saint-Raymond [104].

Lemme 2 (Masmoudi, Tayeb).

Soit $(h^\varepsilon)_\varepsilon$ une famille bornée dans $L^2(0, T; L^2(\mathbb{R}^3; (L^2_{loc}(\mathbb{R}^3))))$ satisfaisant au sens des distributions

$$\varepsilon \partial_t h^\varepsilon + v \cdot \nabla_x h^\varepsilon = h_0^\varepsilon + \nabla_v \cdot h_1^\varepsilon$$

où $(h_0^\varepsilon)_\varepsilon, (h_1^\varepsilon)_\varepsilon$ sont bornées dans $L^1(0, T; L^1(\mathbb{R}^3; (L^1_{loc}(\mathbb{R}^3))))$. Alors pour tout $\psi \in \mathcal{D}(\mathbb{R}^3)$,

$$\left\| \int_{\mathbb{R}^3} (\tau_y h^\varepsilon - h^\varepsilon) \psi(v) dv \right\|_{L^1(0, T; L^1(\mathbb{R}^3))} \rightarrow 0$$

lorsque $y \rightarrow 0$, uniformément en ε , avec τ_y la translation de vecteur y en la variable x .

Le but est d'appliquer ce lemme à $h^\varepsilon = \beta(f^\varepsilon)$ vérifiant (45). Afin que les hypothèses soient satisfaites, il convient de borner uniformément les termes singuliers de l'équation sur les électrons. Il paraît raisonnable de réussir à traiter les termes du membre de droite grâce au contrôle de la dissipation d'entropie, quantité créée par ces mêmes termes. En revanche il n'est pas *a priori* évident que l'on obtienne un contrôle uniforme du terme singulier de champ magnétique

$$\frac{1}{\varepsilon} (v \wedge B) \cdot \nabla_v \beta(f^\varepsilon).$$

C'est en fait possible grâce au contrôle de la dissipation d'énergie libre et au lemme suivant.

Lemme 3. (Chapitre 1) Soient $f \equiv f(x, v)$ une fonction positive telle que $f(1 + |v|^2) \in L^1(\Omega \times \mathbb{R}^d)$, $\nabla_v(\sqrt{f/M}) \in L^2(M(v) dx dv)$, $B \in L^\infty(\Omega)$ et β une fonction dérivable de dérivée bornée. Alors on a

$$\begin{aligned} & \| (v \wedge B) \cdot \nabla_v \beta(f) \|_{L^1(\Omega \times \mathbb{R}^3)} \\ & \leq \| B \|_{L^\infty(\Omega)} \| \beta'(f) \|_{L^\infty(\Omega)} \left\| \nabla_v \sqrt{f/M} \right\|_{L^2(M dv dx)} \left\| \int_{\mathbb{R}^d} |v|^2 f dv \right\|_{L^1(\Omega)}^{1/2}. \end{aligned} \quad (47)$$

Pour démontrer ce résultat, il suffit d'observer que la force de Laplace est orthogonale à la vitesse des particules, c'est-à-dire $v \cdot (B \wedge v) = 0$. A partir de là on écrit $(v \wedge B) \cdot \nabla_v \beta(f) = \beta'(f) \sqrt{f} (v \wedge B) \cdot (2 \nabla_v \sqrt{f} + v \sqrt{f})$ et on utilise l'inégalité de Cauchy-Schwartz

18. Signalons tout de même que dans notre preuve nous ne passons pas à la limite dans l'équation (40) qui elle non plus n'a pas de sens sans renormalisation. Le passage à la limite se fait directement dans (46)

ainsi que les bornes sur le champ magnétique et la fonction de renormalisation. Cette estimation est très proche de celles de Poupaud et Soler [172, Lemme 2.2, Equation 5.45] estimant les moments d'ordre 1 et 2. Sous les hypothèses du lemme précédent, elles s'écrivent

$$\left\| \int_{\mathbb{R}^d} v f \, dv \right\|_{L^1(\Omega)} \leq \left\| \nabla_v \sqrt{f/M} \right\|_{L^2(M(v) \, dv \, dx)} \|f\|_{L^1(\Omega \times \mathbb{R}^d)}^{1/2}, \quad (48)$$

et

$$\left\| \int_{\mathbb{R}^d} (v \otimes v - I) f \, dv \right\|_{L^1(\Omega)} \leq \left\| \nabla_v \sqrt{f/M} \right\|_{L^2(M(v) \, dv \, dx)} \left\| \int_{\mathbb{R}^d} |v|^2 f \, dv \right\|_{L^1(\Omega)}^{1/2}. \quad (49)$$

Dans les deux cas, des annulations permettent de faire apparaître la quantité $v f + \nabla_v f$ et d'obtenir un contrôle grâce à la dissipation d'énergie libre. Ces estimations sont capitales dans notre preuve. En effet étant donné le facteur $1/\varepsilon^2$ de la dissipation (44), on obtient que la densité de courant des électrons j_e^ε définie dans (38) est d'ordre 1 en ε et que le second terme de (40) tend bien vers 0 lorsque $\varepsilon \rightarrow 0$, ce qui légitime la procédure formelle présentée plus haut.

Pour le terme de champ magnétique, il semble que la forme de la dissipation de l'opérateur de Fokker-Planck soit importante pour établir l'estimation du Lemme 3. Notons que pour l'application du lemme de moyenne on pourrait être tenté choisir d'inclure le terme $\frac{1}{\varepsilon} (v \wedge B) \cdot \nabla_v \beta(f_e^\varepsilon)$ dans h_1^ε et estimer directement $(v \wedge B) f_e^\varepsilon$, par exemple par interpolation avec le moment d'ordre 2. Cependant, à cause de la singularité en ε on n'obtiendrait pas de contrôle uniforme.

5 Densité de Boltzmann anisotrope (Chapitre 2)

On se place ici sur le tore tri-dimensionnel en espace $\Omega = \mathbb{T}^3$ et on considère l'évolution de la distribution $f^\varepsilon : \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}_+$ suivant l'équation de Vlasov-Fokker-Planck

$$\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon - E^\varepsilon \cdot \nabla_v f^\varepsilon + \frac{1}{\varepsilon} v^\perp \cdot \nabla_v f^\varepsilon = \frac{1}{\varepsilon^\alpha} \nabla_v \cdot (v f^\varepsilon + \nabla_v f^\varepsilon), \quad (50)$$

où l'exposant \perp désigne la transformation $v^\perp = (0, 0, 1) \wedge v = (-v_2, v_1, 0)$. Ainsi, le dernier terme du membre de gauche correspond à la contribution de la force de Laplace pour un champ magnétique unidirectionnel et constant $B = (0, 0, 1)^\top$. L'équation (50)

correspond¹⁹ à l'équation des électrons du système (34), pour $\mu = 1$, $\varepsilon = \lambda^{1/2}$ et

$$\gamma_e(\lambda) = \frac{1}{\lambda^{\alpha/2}}. \quad (51)$$

Le scaling des collisions est sous-diffusif, c'est-à-dire

$$\alpha < 1.$$

Dans le Chapitre 1, on a étudié le cas limite $\alpha = 1$. Notre motivation ici est de retrouver une approximation d'électrons boltzmanniens de type (26) comme celle utilisée par les physiciens à la limite $\varepsilon \rightarrow 0$. À cause du choix que l'on a fait dans la modélisation des collisions par un opérateur de Fokker-Planck linéaire, on s'attend à une description triviale de la dynamique de la température. Nous sommes plutôt intéressés par la caractérisation de la moyenne $\tilde{\phi}$ du potentiel le long des lignes de champ ainsi qu'en la dynamique du préfacteur dans le plan orthogonal au champ magnétique. Notons néanmoins que récemment Bardos, Golse, Nguyen et Sentis ont obtenus dans [15] une justification rigoureuse d'une dynamique de type Maxwell-Boltzmann avec une température non constante en temps dans la limite des électrons sans masse à partir d'un système similaire à (34) sans champ magnétique. Cependant, leur résultat n'est valide que sous des hypothèses de régularité dont la propagation par les équations n'est pas prouvée. Dans le cas isotrope (sans champ magnétique) le seul résultat connu à notre connaissance sans hypothèse technique de régularité est celui de Bouchut et Dolbeault qui justifient dans [29] la limite des électrons boltzmanniens à partir d'un système de Vlasov-Poisson-Fokker-Planck. Dans ce papier leur analyse concerne principalement le comportement en temps long de VFPF mais comme nous l'avons évoqué précédemment et le verrons au Chapitre 3, il y a plusieurs similitudes entre l'asymptotique en temps long et la limite des électrons sans masse. Dans le cas magnétisé, on renvoie également le lecteur aux papiers de Negulescu, de Cecco et Possaner [44, 160] où des modèles plus complets que celui obtenu dans ce chapitre sont justifiés formellement. Pour nous, l'important est entre autre d'obtenir une justification rigoureuse d'une approximation similaire à (26).

Le champ électrique E^ε peut être de deux natures différentes. Soit, comme jusqu'à maintenant il dérive d'un potentiel ϕ^ε qui vérifie l'équation de Poisson

$$-\delta^2 \Delta_x \phi^\varepsilon = \underline{n} - n^\varepsilon, \quad n^\varepsilon(t, x) = \int_{\mathbb{R}^3} f^\varepsilon(t, x, v) dv,$$

où $\underline{n} \equiv \underline{n}(t, x)$ est une densité donnée. Dans ce cas le modèle est non-linéaire, de type Vlasov-Poisson-Fokker-Planck et nous ne sommes malheureusement pas en mesure de justifier complètement le passage à la limite à partir de l'équation cinétique. On expliquera les difficultés rencontrées ci-après. Cependant, nous savons montrer que le modèle non-linéaire attendu à la limite est bien posé (Théorèmes 6 et 7). Notre résultat de passage

19. Dans le cas $\alpha = 0$, et quitte à changer le signe des champs de force, la limite $\varepsilon \rightarrow 0$ de l'équation peut également être interprétée comme une limite de champ magnétique fort de l'équation des ions ou des électrons en temps long. On renvoie au Chapitre 2 pour davantage de détails.

à la limite (Théorème 4) concerne donc un modèle linéaire où le champ électrique est extérieur, si bien que lorsque c'est le cas on note $E^\varepsilon = E = -\nabla_x \phi$, pour un potentiel ϕ donné. On commence par présenter l'obtention formelle du modèle limite.

5.1 Passage à limite et modèle asymptotique

On décrit ici formellement comment le modèle limite est obtenu puis on énonce et commente le résultat de convergence. Les collisions sont ici moins fortes que dans le régime diffusif du Chapitre 1. Notamment le champ magnétique est le terme strictement dominant en ε dans l'équation (50). Au premier abord, il n'est donc pas clair que la limite annule l'opérateur de Fokker-Planck. Grâce à des arguments de dissipation d'énergie libre on justifie l'obtention d'une maxwellienne locale à la limite. En effet, l'estimation d'énergie libre, qui est invariante par ajout d'un champ magnétique, est similaire à (43) (on renvoie au Chapitre 2 pour davantage de détails) à condition de remplacer la dissipation d'entropie par

$$D^\varepsilon(t) = \frac{4}{\varepsilon^{1+\alpha}} \iint_{\mathbb{R}^6} \left| \nabla_v \sqrt{\frac{f^\varepsilon}{M(v)}} \right|^2 M(v) dv dx. \quad (52)$$

La restriction additionnelle

$$\alpha > -1$$

sépare un régime asymptotiquement non-collisionnel à celui que l'on considère ici, où lorsque $\varepsilon \rightarrow 0$,

$$f(t, x, v) = n(t, x) M(v).$$

Il reste à caractériser la densité macroscopique limite n . Comme dans la section précédente, l'équation de continuité

$$\partial_t n^\varepsilon + \nabla_x \cdot j^\varepsilon = 0,$$

est vérifiée avec la même définition pour la densité de courant

$$j^\varepsilon(t, x) = \frac{1}{\varepsilon} \int_{\mathbb{R}^3} v f^\varepsilon(t, x, v) dv.$$

Pour l'équation d'évolution de j^ε , on sépare ce qui se passe parallèlement et perpendiculairement au champ magnétique. Au niveau des notations²⁰ cela correspond respectivement à un indice \parallel et \perp . On a

$$\varepsilon^2 \partial_t j_\perp^\varepsilon + \int_{\mathbb{R}^3} v_\perp v \cdot \nabla_x f^\varepsilon dv - \nabla_{x_\perp} \phi^\varepsilon n^\varepsilon = (j_\perp^\varepsilon)^\perp - \varepsilon^{1-\alpha} j_\perp^\varepsilon, \quad (53)$$

$$\varepsilon^2 \partial_t j_\parallel^\varepsilon + \int_{\mathbb{R}^3} v_\parallel v \cdot \nabla_x f^\varepsilon dv - \partial_{x_\parallel} \phi^\varepsilon n^\varepsilon = -\varepsilon^{1-\alpha} j_\parallel^\varepsilon. \quad (54)$$

20. Par exemple $v_\perp = (v_1, v_2)^\top$ et $v_\parallel = v_3$

À partir de là, il s'agit d'estimer la taille de chacun des termes afin de pouvoir identifier la limite. Dans le contexte de solutions d'énergie libre, on a comme on l'a vu au premier chapitre des bornes uniformes sur f^ε , n^ε , $|v|^2 f^\varepsilon$ et $\nabla_x \phi^\varepsilon$ bornés dans les espaces de Lebesgue adéquats. Grâce au Lemme 48, on s'attend à ce que j^ε soit quant à lui, au pire, d'ordre $\varepsilon^{(\alpha-1)/2}$. Par conséquent, à la limite, (54) donne

$$\partial_{x_\parallel} n - \partial_{x_\parallel} \phi n = 0.$$

L'intégration de cette équation nous donne que le rapport entre n et e^ϕ ne dépend pas de x_\parallel . On peut récrire cela sous la forme suivante

$$\begin{aligned} n(t, x) &= N(t, x_\perp) \frac{e^{\phi(t, x)}}{\int_{\mathbb{T}} e^{\phi(t, x_\perp, y_\parallel)} dy_\parallel} \\ &= N(t, x_\perp) e^{\phi(t, x) - \tilde{\phi}(t, x_\perp)}, \end{aligned} \quad (55)$$

avec la densité moyennée

$$N(t, x_\perp) = \int_{\mathbb{T}} n(t, x_\perp, y_\parallel) dy_\parallel, \quad (56)$$

et le potentiel moyenné

$$\tilde{\phi}(t, x_\perp) = \ln \left(\int_{\mathbb{T}} e^{\phi(t, x_\perp, x_\parallel)} dx_\parallel \right). \quad (57)$$

Afin de déterminer l'équation suivie par N on intègre l'équation de continuité et l'équation (53) par rapport à x_\parallel et on obtient respectivement

$$\partial_t N^\varepsilon + \nabla_{x_\perp} \cdot J^\varepsilon = 0$$

et

$$J^\varepsilon = \left(\int_{\mathbb{T}} n^\varepsilon \nabla_{x_\perp} \phi^\varepsilon dx_\parallel \right)^\perp - \left(\iint_{\mathbb{T} \times \mathbb{R}^3} v_\perp v_\perp \cdot \nabla_{x_\perp} f^\varepsilon dv dx_\parallel \right)^\perp - \varepsilon^{1-\alpha} J^{\varepsilon \perp} - \varepsilon^2 \partial_t J^{\varepsilon \perp},$$

où l'on a également appliqué une rotation de $-\pi/2$ et où

$$J^\varepsilon = \int_{\mathbb{T}} j_\perp^\varepsilon dx_\parallel \quad \text{et} \quad N^\varepsilon = \int_{\mathbb{T}} n^\varepsilon dx_\parallel.$$

À partir de là, on substitue l'expression de J^ε dans l'équation de continuité moyennée, on passe à la limite et on utilise l'expression (55) pour finalement obtenir le modèle limite

$$\begin{cases} \partial_t N - \nabla_{x_\perp} \cdot (\tilde{E}^\perp N) = 0, \\ \tilde{E} = -\nabla \tilde{\phi}, \\ \tilde{\phi} = \ln \left(\int_{\mathbb{T}} e^\phi dx_\parallel \right). \end{cases} \quad (58)$$

On énonce maintenant le théorème de passage à la limite du Chapitre 2 (Théorème 2.1.3 dans le chapitre).

Théorème 4 (Chapitre 2). Soit $\phi \in W_{loc}^{1,1}(\mathbb{R}_+; L^\infty(\mathbb{T}^3)) \cap L_{loc}^1(\mathbb{R}_+; W^{1,\infty}(\mathbb{T}^3))$ un potentiel extérieur et f^{in} une donnée initiale positive telle que

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} (1 + |v|^2 + |\ln f^{in}|) f^{in} dx dv < +\infty.$$

Alors il existe une unique solution $f^\varepsilon \in \mathcal{C}(\mathbb{R}_+; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$ à

$$\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \nabla_x \phi \cdot \nabla_v f^\varepsilon + \frac{1}{\varepsilon} v^\perp \cdot \nabla_v f^\varepsilon = \frac{1}{\varepsilon^\alpha} \nabla_v \cdot (v f^\varepsilon + \nabla_v f^\varepsilon)$$

muni de la donnée initiale f^{in} . De plus, si $\alpha \in (-1, 1)$, alors il existe $N \in L^\infty(\mathbb{R}_+; L^1(\mathbb{T}^2))$ et une suite strictement positive $(\varepsilon_n)_n$ et convergeant vers 0 telle que lorsque $n \rightarrow \infty$

$$f^{\varepsilon_n}(t, x, v) \rightarrow N(t, x_\perp) \frac{e^{\phi(t,x)}}{\int_{\mathbb{T}} e^{\phi(t,x_\perp, y_\parallel)} dy_\parallel} M(v) \quad \text{faiblement dans } L_{loc}^1(\mathbb{R}_+; L^1(\mathbb{T}^3 \times \mathbb{R}^3)),$$

et N est solution faible de l'équation de transport linéaire (58) avec la donnée initiale $\iint_{\mathbb{T} \times \mathbb{R}^3} f^{in} dv dx_\parallel$.

Avec la régularité supposée sur le potentiel, il est possible de démontrer ce théorème en suivant l'ébauche de preuve précédente. L'adaptation des techniques du Chapitre 1 ne permet pas d'étendre le résultat au cas non linéaire et de retrouver le modèle complet (55) muni de l'équation de Poisson Boltzmann anisotrope

$$-\delta^2 \Delta_x \phi = \underline{n} - N e^{\phi - \tilde{\phi}}. \quad (59)$$

En particulier, nous ne parvenons pour le moment pas à obtenir de la compacité forte sur la densité macroscopique, nécessaire pour passer à la limite dans le terme non-linéaire. En effet, à cause de la singularité du terme de champ magnétique, il semble difficile d'appliquer un lemme de moyenne comme le Lemme 2. Dans le présent scaling, l'application du Lemme 3 donne

$$\left\| \frac{1}{\varepsilon} v^\perp \cdot \nabla_v f^\varepsilon \right\|_{L^2(0,T; L^1(\mathbb{T}^3 \times \mathbb{R}^3))} = O(\varepsilon^{(\alpha-1)/2}),$$

ce qui ne donne une borne uniforme que lorsque $\alpha \geq 1$.

Malgré cela nous sommes capables de montrer que le modèle limite non-linéaire est bien posé.

5.2 Commentaires sur le modèle limite

Le modèle limite est composé d'une équation de transport (58) sur la densité moyennée $N \equiv N(t, x_\perp)$ dont le champ d'advection dérive d'un potentiel moyenné $\tilde{\phi} \equiv \tilde{\phi}(t, x_\perp)$ qui dépend non-linéairement du potentiel $\phi \equiv \phi(t, x_\perp, x_\parallel)$. Ce dernier est lui-même solution d'une équation elliptique non-linéaire (59) de type Poisson-Boltzmann faisant inter-

venir la densité N . Notons que ce modèle ressemble énormément à l'équation centre-guide (ou Euler 2D en formulation vorticit ) obtenue par Golse, Saint-Raymond [105, 178] et Miot [150]   partir du syst me de Vlasov-Poisson 2D.

Th or me 5 (Golse, Saint-Raymond, Miot ; adapt  de [150]). *Soient $\Omega = \mathbb{T}^2$ et $f_{in}^\varepsilon \in L^1 \cap L^\infty(\Omega \times \mathbb{R}^2)$ des fonctions positives   support compact telles que*

$$\sup_{\varepsilon > 0} (\|(1 + |v|^2)f_{in}^\varepsilon\|_{L^1(\Omega \times \mathbb{R}^2)} + \|E_{in}^\varepsilon\|_{L^2(\Omega \times \mathbb{R}^2)}) < +\infty$$

o  $E_{in}^\varepsilon = \nabla_x \Delta_x^{-1} (\int f_{in}^\varepsilon dv - \iint f_{in}^\varepsilon dv dx)$ et

$$\varepsilon \|f_{in}^\varepsilon\|_{L^\infty(\Omega \times \mathbb{R}^2)} \ln(2 + \|f_{in}^\varepsilon\|_{L^\infty(\Omega \times \mathbb{R}^2)}) \longrightarrow 0,$$

lorsque $\varepsilon \rightarrow 0$. Soit $(f^\varepsilon)_{\varepsilon > 0}$ la suite de solutions faibles de

$$\begin{cases} \varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + E^\varepsilon \cdot \nabla_v f^\varepsilon - \frac{1}{\varepsilon} v^\perp \cdot \nabla_v f^\varepsilon = 0, \\ E^\varepsilon = -\nabla_x \phi^\varepsilon, \\ -\Delta_x \phi^\varepsilon = n^\varepsilon - \int_\Omega n^\varepsilon dx \quad n^\varepsilon(t, x) = \int_{\mathbb{R}^2} f^\varepsilon(t, x, v) dv \end{cases} \quad (60)$$

correspondant   la famille de donn es initiales $(f_{in}^\varepsilon)_{\varepsilon > 0}$. Alors, il existe une sous-suite de $(n^\varepsilon)_{\varepsilon > 0}$ convergeant vers n dans $\mathcal{C}(\mathbb{R}_+; \mathcal{M}_+(\Omega) - \text{faible} - \star)$. De plus, la limite est une solution globale de l' quation d'Euler en formulation vorticit 

$$\begin{cases} \partial_t n - \nabla_x \cdot (E^\perp n) = 0 \\ E = -\nabla_x \phi, \\ -\Delta_x \phi = n - \int_\Omega n dx \end{cases} \quad (61)$$

au sens de Delort et Schochet (voir [150, D finition 2.1] et les r f rences cit es).

Outre les non-lin arit s, la diff rence notable entre (58)-(59) et (61) est le couplage entre une  quation de transport en 2D et une  quation elliptique 3D de notre syst me. On remarque d'ailleurs que ce couplage est uniquement d    la d pendance en x_\parallel de la densit  de fond $\underline{n} \equiv \underline{n}(t, x_\perp, x_\parallel)$. En effet si $\partial_{x_\parallel} \underline{n} = 0$ alors en d rivant (59) par rapport   x_\parallel , il est facile de montrer que $\partial_{x_\parallel} \phi$ est  galement nul. Dans ce cas de figure $\tilde{\phi} = \phi$ et on retrouve,   la densit  de fond \underline{n} pr s, le mod le (61).

Notons  galement la consistance entre notre syst me et celui obtenu formellement par Degond et Filbet dans [63]. Dans ce papier, ces derniers s'int ressent   la limite $\varepsilon \rightarrow 0$ de l' quation de Vlasov (60) en version tri-dimensionnelle. Le mod le limite est une densit  radiale dans les deux premi res composantes en vitesse $f(t, x, v) = F(t, x, |v_\perp|, v_\parallel)$

vérifiant

$$\begin{aligned} \partial_t F + \nabla_{x_\perp}^\perp \phi \cdot \nabla_{x_\perp} F + \partial_{x_\parallel} \psi \partial_{v_\parallel} F + v_\parallel \partial_{x_\parallel} P + \partial_{x_\parallel} \phi \partial_{v_\parallel} P &= 0, \\ v_\parallel \partial_{x_\parallel} F + \partial_{x_\parallel} \phi \partial_{v_\parallel} F &= 0, \quad -\Delta_x \phi = \underline{n} - (2\pi) \int_{\mathbb{R}_+ \times \mathbb{R}} F(\cdot, \cdot, p, v_\parallel) p \, dp \, dv_\parallel, \\ -\Delta_x \psi &= \underline{n} - (2\pi) \int_{\mathbb{R}_+ \times \mathbb{R}} P(\cdot, \cdot, p, v_\parallel) p \, dp \, dv_\parallel, \end{aligned}$$

où P est un multiplicateur de Lagrange associé à la contrainte sur F encodée par la première équation de la seconde ligne. Si l'on considère une maxwellienne locale solution de cette équation, alors cette même contrainte impose une densité de Maxwell-Boltzmann anisotrope (55) et en intégrant la première équation en $(x_\parallel, |v_\perp|, v_\parallel)$, (P, ψ) disparaissent et le système devient finalement (58)-(59). Autrement dit les densités macroscopiques des solutions maxwelliennes du système de Degond et Filbet sont solutions de notre système.

Mentionnons enfin que pour ne pas ajouter une complexité superflue aux preuves nous avons considéré un champ magnétique d'amplitude constante.

Dans le cas d'un champ magnétique $B(t, x) = (0, 0, b(t, x_\perp))$ d'amplitude²¹ $b : \mathbb{R}_+ \times \mathbb{T}^2 \rightarrow \mathbb{R}_+$, $(t, x_\perp) \mapsto b(t, x_\perp)$ l'équation limite est la suivante

$$\begin{aligned} \partial_t N + \nabla_{x_\perp} \cdot (N U_\perp) &= 0, \\ U_\perp &= \frac{1}{b} (\nabla_{x_\perp} \tilde{\phi})^\perp + \frac{1}{b^2} (\nabla_{x_\perp} b)^\perp. \end{aligned}$$

Le second terme du champ d'advection est bien connu dans la théorie centre-guide des plasmas magnétisés et s'appelle *dérive de gradient*. Sous des hypothèses adéquates sur l'amplitude (régularité, bornes sur b et son gradient et borne par en dessous uniformément loin de 0), le passage à la limite du Théorème 4 se généralise sans difficulté et l'obtention de la seconde dérive résulte du passage à la limite dans le terme

$$\nabla_{x_\perp} \cdot \left(\frac{1}{b} \iint_{\mathbb{T} \times \mathbb{R}^3} v_\perp v_\perp \cdot \nabla_{x_\perp} f^\varepsilon \, dv \, dx_\parallel \right)^\perp,$$

dans l'expression de $\nabla_{x_\perp} \cdot J^\varepsilon$. Lorsque b est constant en x_\perp ce terme tend vers 0.

Pour des classes plus larges de champs magnétiques, d'autres effets dus à la géométrie du champ sont attendus. Même sans champ électrique et sans collisions, ces derniers sont extrêmement complexes à décrire et on renvoie vers [53, 54] pour davantage de détails à ce sujet.

5.3 Analyse du modèle limite

Expliquons ici les mécanismes de propagation et de gain de régularité en jeu dans le système limite. Une fois ces points compris, la construction des solutions peut se faire

21. L'amplitude b ne dépend pas de x_\parallel afin d'obtenir un champ à divergence nulle et vérifier l'équation de Maxwell-Thompson (3).

par un processus itératif similaire à ceux utilisés pour Vlasov-Poisson [99, 30, 174] ou Euler incompressible en formulation vorticité [145, 52, 143].

Malgré les non-linéarités, le système (58)-(59) hérite d'une riche structure. D'abord, l'équation de conservation (58) peut être écrite sous forme d'équation de transport puisque le champ d'advection est à divergence nulle. Ainsi, en particulier, les normes L^p sont formellement conservées

$$\|N(t)\|_{L^p(\mathbb{T}^2)} = \|N(0)\|_{L^p(\mathbb{T}^2)},$$

quel que soit $p \in [1, +\infty]$. Ensuite, l'équation de Poisson-Boltzmann (59) est l'équation d'Euler-Lagrange associée à la fonctionnelle suivante

$$J[\psi] = \frac{1}{2} \delta^2 \int_{\mathbb{T}^3} |\nabla \psi|^2 + \int_{\mathbb{T}^2} N \ln \left(\int_{\mathbb{T}} e^\psi dx_{\parallel} \right) dx_{\perp} - \int_{\mathbb{T}^3} \underline{n} \psi. \quad (62)$$

Pour des sources $N \in L^{4/3}(\mathbb{T}^2)$ et $\underline{n} \in L^{6/5}(\mathbb{T}^3)$, cette fonctionnelle est coercive, strictement convexe, semi-continue inférieurement et Gâteaux-différentiable dans

$$H_0 = \left\{ h \in \mathcal{D}'(\mathbb{T}^3) \mid \nabla h \in L^2(\mathbb{T}^3) \text{ et } \int_{\mathbb{T}^3} h = 0 \right\}.$$

Elle a donc un unique point critique qui est l'unique solution faible de (59). Concernant le gain de régularité, on parvient à montrer que si de plus $\underline{n} \in L^{3/2,1}(\mathbb{T}^3)$ alors les non-linéarités e^ϕ et $e^{-\tilde{\phi}}$ sont bornées dans L^∞ . À partir de là, (59) se comporte comme une équation elliptique du second ordre linéaire, dans le sens où ϕ est deux dérivées plus régulière que les sources N et \underline{n} (voir Proposition 2.3.3).

Puisque le champ d'advection dépend de $\tilde{\phi}$, il reste à comprendre le lien entre régularité de ϕ et de sa moyenne non-linéaire dans la direction parallèle. Dans toute l'analyse de ce système, il est important de distinguer ce qui se passe parallèlement et perpendiculairement au champ magnétique. Avec quelques ingrédients supplémentaires on obtient le résultat suivant (Théorème 2.1.2 dans le chapitre).

Théorème 6 (Chapitre 2). *Soient $p > 4/3$ et $\delta > 0$.*

Soit $\underline{n} \in \mathcal{C}(\mathbb{R}; L^{6/5}(\mathbb{T}^3)) \cap L_{loc}^\infty(\mathbb{R}; L^p(\mathbb{T}^3) \cap L^{3/2,1}(\mathbb{T}^3))$ une densité de fond positive.

Pour toute donnée initiale $N^{in} \in L^p(\mathbb{T}^2)$ positive telle que $\int_{\mathbb{T}^2} N^{in} = \int_{\mathbb{T}^3} \underline{n}$, il existe une solution faible globale N au système (58), (59) commençant à N^{in} au temps 0.

Commentons simplement le seuil sur p , afin d'illustrer que la régularité est transmise, de manière anisotrope, sans perte dans les estimations. Imaginons que \underline{n} soit constante. Dans ce cas, le système se réduit à (61) et en particulier, l'équation de Poisson devient linéaire et 2D. Par conséquent si la source N est dans L^p le potentiel est dans l'espace de Sobolev $W^{2,p}$. Si $p < 2$, par injection de Sobolev en 2D, le gradient du potentiel est dans $L^{2p/(2-p)}$. Le plus petit p qui assure que le produit d'un élément de L^p avec un élément de $L^{2p/(2-p)}$ soit une distribution est bien $4/3$.

Concernant l'existence et l'unicité de solutions fortes, on démontre le résultat suivant (Théorème 2.1.1 dans le chapitre).

Théorème 7 (Chapitre 2).

Soit $p > 2$, $\delta > 0$ et $\underline{n} \in \mathcal{C}_b(\mathbb{R}_+; W^{1,p}(\mathbb{T}^3))$ une densité de fond positive.

Pour tout $N^{in} \in W^{1,p}(\mathbb{T}^2)$ positive telle que $\int_{\mathbb{T}^2} N^{in} = \int_{\mathbb{T}^3} \underline{n}$, il existe une unique solution maximale (N, I) à l'équation N au système (58), (59) dans $\mathcal{C}(I; L^1(\mathbb{T}^2)) \cap \mathcal{C}(I; W^{1,p}(\mathbb{T}^2) - \text{faible})$ commençant à N^{in} au temps 0. De plus, le temps maximal d'existence est uniformément borné loin de 0 pour des données initiales dans un ensemble borné de $W^{1,p}(\mathbb{T}^2)$.

Ici le seuil correspond à l'obtention de la régularité Lipschitzienne pour le champ d'advection, naturelle pour l'obtention de solutions fortes à une équation de transport. Le seuil $p = 2$ correspond donc à l'exposant critique de l'injection de Sobolev $W^{2,p}(\mathbb{T}^2) \hookrightarrow W^{1,\infty}(\mathbb{T}^2)$. Afin d'obtenir le résultat du Théorème 7 il faut ici étudier la propagation de la régularité $W^{1,p}(\mathbb{T}^2)$ pour la densité N . En dérivant l'équation de transport on obtient

$$\partial_t \partial_{x_i} N + \nabla^\perp \phi \cdot \nabla \partial_{x_i} N = -\partial_{x_i} \nabla^\perp \phi \cdot \nabla N,$$

qui correspond à une structure de transport avec terme source. À cause de la non-linéarité du terme de droite on obtient une borne $W^{1,p}(\mathbb{T}^2)$ jusqu'à un temps fini dépendant de la donnée initiale²². On montre ainsi l'existence de solution fortes sur un intervalle de taille contrôlée par en dessous. Grâce à la régularité Lipschitz du champ d'advection, on a l'unicité et on peut prolonger notre solution en une unique solution maximale sur un intervalle de temps de taille bornée par en dessous. Notons que rien n'assure que cette solution soit globale, sans toutefois réfuter l'existence globale de solutions fortes pour cette équation.

6 Méthodes hypocoercives et taux de convergence (Chapitres 2 et 3)

Dans cette section on présente une méthode permettant d'établir des estimations de régularisation et de retour vers l'équilibre pour des équations de type Vlasov-Fokker-Planck dans les régimes limites de certains paramètres de l'équation. On commence par expliquer la méthode sur un modèle jouet.

6.1 Stratégie illustrée sur un modèle jouet

On considère l'équation de Kolmogorov posée sur le tore $(x, v) \in \mathbb{T}^2$,

$$\partial_t f + \varepsilon^a v \partial_x f = \varepsilon^b \partial_{vv}^2 f, \tag{63}$$

où $\varepsilon > 0$ et $a, b \in \mathbb{R}$. On veut montrer par une estimation globale que f converge vers sa moyenne lorsque $t \rightarrow \infty$, au pire exponentiellement rapidement. De plus on veut déterminer la dépendance du taux exponentiel en ε dans la limite $\varepsilon \rightarrow 0$ et plus généralement la dépendance de normes de la solution en ε .

²². Le temps fini est décroissant par rapport à la taille de la donnée initiale dans $W^{1,p}(\mathbb{T}^2)$

L'équation (63) est un exemple classique d'une équation parabolique dégénérée à la structure hypoelliptique et hypocoercive. On réfère à Hörmander [128] et Villani [194] pour une présentation abstraite et générale de ces deux concepts, traduisant les propriétés de régularisation et de convergence vers l'équilibre. On introduit l'espace de Hilbert $\mathcal{H} = L^2(\mathbb{T}^2)$, que l'on munit de son produit scalaire $\langle \cdot, \cdot \rangle$ canonique et de la norme associée $\| \cdot \|$, et le sous-espace \mathcal{H}_0 des fonctions de \mathcal{H} à moyenne nulle. Si $f \in \mathcal{H}$ est solution de (63) alors $h = f - \iint_{\mathbb{T}^2} f$ est également solution et vérifie $h \in \mathcal{H}_0$. D'autre part, les opérateurs non-bornés $v\partial_x$ et ∂_v sont anti-symétriques sur \mathcal{H} et leur commutateur est $[v\partial_x, \partial_v] = -\partial_x$. Grâce à ces propriétés on montre aisément les égalités suivantes (au moins formellement)

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|h\|^2 + \varepsilon^b \|\partial_v h\|^2 &= 0, & \frac{1}{2} \frac{d}{dt} \|\partial_v h\|^2 + \varepsilon^b \|\partial_{vv}^2 h\|^2 &= -\varepsilon^a \langle \partial_x h, \partial_v h \rangle, \\ \frac{1}{2} \frac{d}{dt} \|\partial_x h\|^2 + \varepsilon^b \|\partial_{vx}^2 h\|^2 &= 0, & \frac{d}{dt} \langle \partial_x h, \partial_v h \rangle + \varepsilon^a \|\partial_x h\|^2 &= -2\varepsilon^b \langle \partial_{xv}^2 h, \partial_{vv}^2 h \rangle. \end{aligned}$$

On voit clairement qu'on ne peut pas obtenir une estimation H^1 classique en sommant les trois premières égalités car le terme de reste $\langle \partial_x h, \partial_v h \rangle$ n'est pas contrôlé par les termes de dissipation dans les membres de gauche des égalités. Cela est dû au défaut de coercivité de $v\partial_x - \partial_{vv}^2$ dans les espaces H^s , $s \in \mathbb{N}$, munis de leur norme canonique. Il est cependant possible de boucler une estimation pour une norme H^1 équivalente en ajoutant le terme croisé. Grâce à des poids en temps il est même possible de rendre compte des effets de régularisation. Cette méthode d'hypoellipticité globale est due à Hérau [121] et Villani et est expliquée pour l'équation de Fokker-Planck cinétique dans [194, Appendice A.21]. On introduit donc

$$\begin{aligned} \|\| h \|^2_{t,\varepsilon,\gamma,\beta} &= \|h\|^2 + \gamma_1 \varepsilon^{\beta_1} \min\left(1, \frac{t}{\varepsilon^b}\right) \|\partial_v h\|^2 \\ &\quad + \gamma_2 \varepsilon^{\beta_2} \min\left(1, \frac{t}{\varepsilon^b}\right)^3 \|\partial_x h\|^2 + 2\gamma_3 \varepsilon^{\beta_3} \min\left(1, \frac{t}{\varepsilon^b}\right)^2 \langle \partial_x h, \partial_v h \rangle, \end{aligned} \quad (64)$$

avec $\gamma \in (0, \infty)^3$ et $\beta \in \mathbb{R}^3$ des paramètres à calibrer. Les puissances du temps peuvent être prédites grâce à la solution fondamentale²³ de (63) (voir [194, Appendice A.21] et Section B.3 en Appendice B). À partir des équations précédentes on s'attend à pouvoir contrôler les termes de reste grâce aux dissipations (par des inégalités de Young) sous un certain nombre de contraintes (sur β) d'inégalités affines

$$M(a, b)\beta \leq V(a, b), \quad (65)$$

où $M(a, b)$ est une matrice à trois colonnes et $V(a, b)$ un vecteur de taille égale au nombre de contraintes. Sous (65) on s'attend alors à ce qu'il existe $\varepsilon_0 > 0$, $\gamma \in (0, \infty)^3$ et $\theta > 0$ tels qu'uniformément en $\varepsilon \in (0, \varepsilon_0)$, on ait l'équivalence entre $\|\| \cdot \|^2_{t,\varepsilon,\gamma,\beta}$ et une norme H^1

23. On peut également introduire d'autres paramètres à calibrer pour les puissances du temps et les optimiser pour obtenir la meilleure fonctionnelle de Lyapunov. Cette méthode a le mérite d'être généralisable à d'autres équations hypoelliptiques dont on ne connaîtrait pas à priori les taux de régularisation temporels.

canonique dès que $t > 0$ et que pour tout $s \in [0, \infty)$, $t \in [0, \infty)$,

$$\|h(t)\|_{t,\varepsilon,\gamma,\beta}^2 + \theta \int_s^t D_{u,\varepsilon,\gamma,\beta}(h(u)) \, du \leq \|h(s)\|_{s,\varepsilon,\gamma,\beta}^2, \quad (66)$$

pour la dissipation

$$\begin{aligned} D_{t,\varepsilon,\gamma,\beta}(h) &= \varepsilon^b \|\partial_v h\|^2 + \gamma_1 \varepsilon^{\beta_1+b} \min\left(1, \frac{t}{\varepsilon^b}\right) \|\partial_{vv}^2 h\|^2 \\ &+ \gamma_2 \varepsilon^{\beta_2+b} \min\left(1, \frac{t}{\varepsilon^b}\right)^3 \|\partial_{xv}^2 h\|^2 + 2\gamma_3 \varepsilon^{\beta_3+a} \min\left(1, \frac{t}{\varepsilon^b}\right)^2 \|\partial_x h\|^2. \end{aligned} \quad (67)$$

À partir de là on obtient des estimations de f et de ses dérivées en fonction de la donnée initiale et ε . On a capturé des effets de régularisation $L^2 \rightarrow H^1$ puisque $\|h(0)\|_{0,\varepsilon,\gamma,\beta} = \|h(0)\|$. Notons que l'on pourrait monter en ordre dans les dérivées pour quantifier la régularisation dans des espaces H^s . De plus, grâce à l'inégalité de Poincaré

$$\|h\|^2 \leq K_P (\|\partial_v h\|^2 + \|\partial_x h\|^2),$$

on a

$$\left\| f(t) - \int_{\mathbb{T}^2} f \right\| \leq \left\| f(0) - \int_{\mathbb{T}^2} f \right\| e^{-\kappa(\varepsilon)t},$$

avec $\kappa(\varepsilon) = C \varepsilon^{\max(b,\beta_3+a,b-\beta_1,\beta_3+a-\beta_2)}$ pour une certaine constante $C > 0$. On peut alors optimiser le taux²⁴ (asymptotiquement dans la limite $\varepsilon \rightarrow 0$) par la résolution du problème de programmation linéaire consistant à minimiser $\max(b, \beta_3 + a, b - \beta_1, \beta_3 + a - \beta_2)$ sous les contraintes (65). De même en minimisant d'autres puissances de ε on optimise les estimations de chaque dérivée.

On adapte cette méthode à un cadre anisotrope dans le Chapitre 2 et non-linéaire dans le Chapitre 3. Notons que cette technique de pondération de la fonctionnelle hypocoercive par des paramètres d'échelles (ici ε) avait été précédemment utilisée pour des fins similaires par Briant [35] pour l'étude quantitative de la limite de faible nombre de Knudsen des solutions de l'équation de Boltzmann linéaire et l'équation de Boltzmann en régime perturbatif.

6.2 Convergence vers les densités de Boltzmann (Chapitre 2)

On adapte la méthode ci-dessus à l'équation de Vlasov-Fokker-Planck linéaire avec champ magnétique extérieur (50). Notre but est de quantifier la convergence vers la densité de Maxwell-Boltzmann. À cause de l'anisotropie du problème, on doit séparer les dérivées parallèles au champ magnétique et les dérivées orthogonales. On règle donc des paramètres dans les deux directions. Cela fournit le résultat suivant.

Théorème 8. (Chapitre 2) Soit un potentiel extérieur stationnaire $\phi \in W^{2,\infty}(\mathbb{T}^3)$.

Alors il existe une constante positive C telle que pour tout $\varepsilon \in (0, 1)$ et toute donnée

²⁴. Notons que le taux obtenu n'est pas optimal, il est le meilleur pour cette technique particulière seulement et ne donne pas la taille du trou spectral en fonction de ε mais simplement une estimation de ce dernier.

initiale $f_0 \in L^2(M^{-1}(v) dx dv)$, la solution f^ε de l'équation de Vlasov-Fokker-Planck (50) commençant en f_0 vérifie

$$\begin{aligned} \|f^\varepsilon - n^\varepsilon M\|_{L^2(\mathbb{R}_+; L^2(M^{-1}))} &\leq C \|f_0\|_{L^2(M^{-1})} \varepsilon^{\frac{\alpha+1}{2}} \\ \|n^\varepsilon - N^\varepsilon e^{\phi-\tilde{\phi}}\|_{L^2(\mathbb{R}_+ \times \mathbb{T}^3)} &\leq C \|f_0\|_{L^2(M^{-1})} \varepsilon^{\frac{1-|\alpha|}{2}} \end{aligned}$$

où

$$n^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon(\cdot, \cdot, v) dv \quad \text{et} \quad N^\varepsilon = \int_{\mathbb{T}} n^\varepsilon(\cdot, \cdot, y_\parallel) dy_\parallel.$$

En particulier, lorsque $\varepsilon \rightarrow 0$ la fonction de distribution f^ε est à une distance d'ordre $\varepsilon^{\frac{1-|\alpha|}{2}}$ de la densité de Maxwell-Boltzmann anisotrope

$$(t, x, v) \mapsto \frac{1}{(2\pi)^{3/2}} N^\varepsilon(t, x_\perp) e^{-\frac{|v|^2}{2} + \phi(x_\perp, x_\parallel) - \tilde{\phi}(x_\perp)}$$

dans $L^2(\mathbb{R}_+; L^2(M^{-1}))$, avec $\tilde{\phi}$ le potentiel moyenné défini par (57).

Quelques commentaires concernant les puissances de ε s'imposent. D'abord, bien que l'anisotropie due au champ magnétique ajoute de la complexité technique dans l'analyse, elle n'influe pas sur les puissances de ε attendues. En effet, on trouverait les mêmes taux en ε sans champ magnétique et pour une densité de Boltzmann isotrope. Ensuite, la convergence vers la maxwellienne locale en $O(\varepsilon^{(1+\alpha)/2})$ pouvait déjà être obtenue (dans un espace fonctionnel différent) à partir de la dissipation d'énergie libre (52). Le résultat intéressant du théorème est la seconde estimation. On voit que le taux n'est pas monotone²⁵ en α et le taux maximal est obtenu lorsque $\alpha = 0$. Cela illustre le fonctionnement des mécanismes hypocoercifs. Lorsque la convergence provient à la fois du transport et de la dissipation, celle-ci est la plus rapide seulement lorsque les deux phénomènes sont de magnitude comparable. En particulier, une dissipation trop forte détériore la convergence, ce qui peut sembler contre-intuitif à première vue. Enfin, le taux obtenu est cohérent avec la restriction $\alpha \in (-1, 1)$.

6.3 Comportement en temps long de VFP (Chapitre 3)

Dans le Chapitre 3 on s'intéresse à l'équation de Vlasov-Poisson-Fokker-Planck suivante

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \frac{1}{\tau} \nabla_v \cdot (v f + \nabla_v f), \\ -\delta^2 \Delta_x \phi = \rho - \rho_h(x), \quad \rho = \int_{\mathbb{R}^2} f dv. \end{cases} \quad (68)$$

²⁵ Ici encore, nous n'avons pas prouvé que le trou spectral est maximal pour $\alpha = 0$. Cependant sa non-monotonie est assurée par le fait qu'en $\alpha = 1$ et $\alpha = -1$, même formellement, les limites attendues pour f^ε n'e sont pas donnée par la densité Maxwell-Boltzmann anisotrope, comme l'atteste l'analyse du Chapitre 1.

Ici, les aspects de modélisation nous intéressent moins que dans les deux premiers chapitres. L'équation décrit simplement l'évolution d'une distribution $f : \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}_+$, $(t, x, v) \mapsto f(t, x, v)$ de particules chargées dans un domaine périodique bi-dimensionnel contenant un fond de particules lourdes de charge opposée caractérisées par la densité macroscopique ρ_h . Le but principal du chapitre est de décrire le comportement en temps long de f dans des régimes de fortes collisions $\tau \rightarrow 0^+$ et de collisions évanescentes $\tau \rightarrow \infty$ et d'étudier la dépendance des estimations en les paramètres τ et δ . À partir de celles-ci, des modèles asymptotiques seront obtenus en lien avec ceux des deux premiers chapitres.

Dans [29], Bouchut et Dolbeault ont montrés que lorsque $t \rightarrow \infty$ la solution converge vers la densité de Maxwell-Boltzmann

$$f^\infty(x, v) = M(v)e^{-\phi_\infty^\delta(x)}, \quad (69)$$

pour un potentiel limite solution de l'équation de Poisson-Boltzmann

$$-\delta^2 \Delta_x \phi_\infty^\delta = e^{-\phi_\infty^\delta} - \rho_h. \quad (70)$$

Le résultat de Bouchut et Dolbeault ne quantifie pas la vitesse de convergence. Récemment Hérau et Thomann ont montrés [123] une convergence exponentielle vers l'équilibre à τ fixé et pour δ suffisamment grand. Dans des travaux précédents, Hérau [121] avait établi des estimations pour le semi-groupe d'évolution de l'équation de Fokker-Planck cinétique (sans champ électrique) par des méthodes hypocoercives et d'hypoellipticité globale similaires à celles exposées en Section 6.1. À partir de ces estimations, Hérau et Thomann montrent l'existence et l'unicité d'une solution à (68) dans un espace capturant régularisation en temps court et décroissance exponentielle en temps long par une méthode de point fixe de Banach. Leur preuve repose sur une formulation de Duhamel de l'équation (68) grâce au semi-groupe précédent et en considérant le terme non-linéaire comme un terme source. Cependant l'utilisation d'estimations «boite-noire» sur le semi-groupe ne permet pas de boucler un tel argument dans les régimes extrêmes $\tau \rightarrow +\infty$ et $\tau \rightarrow 0^+$, c'est-à-dire uniformément pour τ suffisamment petit ou grand. Par la méthode de la Section 6.1 (τ y prend la place de ε) nous montrons par une méthode de point fixe la régularisation et la décroissance exponentielle et exhibons la dépendance des taux en τ et δ . Pour cela nous construisons les estimations et optimisons les taux directement au niveau non-linéaire.

Résultats principaux

Notre premier résultat concerne le régime diffusif.

Théorème 9 (Chapitre 3; $\tau \lesssim 1$). *Quels que soient $\tau_0 > 0$, $R_0 > 1$ et $\rho_h \in W^{1,p}(\mathbb{T}^2)$ avec $p > 2$, tel que $\int_{\mathbb{T}^2} \rho_h = 1$, il existe $\delta_0 > 0$, $\theta_0 > 0$ et $K > 0$ tels que pour tout $\tau \in (0, \tau_0)$, $\delta \in (\delta_0, \infty)$ et f_0 tel que*

$$\|f_0\|_{L^2(M^{-1})} \leq R_0 \quad \text{et} \quad \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 = 1,$$

l'équation (68) possède une unique solution forte f de donnée initiale f_0 , et elle vérifie pour tout $t \geq 0$

$$\|f(t, \cdot, \cdot) - f^\infty(\cdot, \cdot)\|_{L^2(M^{-1})} \leq K \|f_0 - f^\infty\|_{L^2(M^{-1})} e^{-\theta_0 \tau t}.$$

De plus, l'application qui à f_0 associe f est Lipschitzienne uniformément en temps sur la boule de $L^2(M^{-1})$ centrée de rayon R_0 , à valeur dans $\mathcal{C}(\mathbb{R}_+; L^2(M^{-1}))$.

Notre second résultat concerne le régime des collisions évanescences.

Théorème 10 (Chapitre 3; $\tau \gtrsim 1$). *Quels que soient $\varepsilon > 0$, $\tau_0 > 0$, $R_0 > 1$ et $\rho_h \in W^{1,p}(\mathbb{T}^2)$ avec $p > 2$, tel que $\int_{\mathbb{T}^2} \rho_h = 1$, il existe $K_0 > 0$, $\theta_0 > 0$ et $K > 0$ tels que pour tout $\tau \in (\tau_0, \infty)$, $\delta \in (K_0 \tau^{7/15+\varepsilon}, \infty)$ et f_0 tel que*

$$\|f_0\|_{L^2(M^{-1})} \leq R_0 \quad \text{et} \quad \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 = 1,$$

l'équation (68) possède une unique solution forte f de donnée initiale f_0 , et elle vérifie pour tout $t \geq 0$

$$\|f(t, \cdot, \cdot) - f^\infty(\cdot, \cdot)\|_{L^2(M^{-1})} \leq K \|f_0 - f^\infty\|_{L^2(M^{-1})} e^{-\theta_0 \frac{t}{\tau}}.$$

De plus, l'application qui à f_0 associe f est Lipschitzienne uniformément en temps sur la boule de $L^2(M^{-1})$ centrée de rayon R_0 , à valeur dans $\mathcal{C}(\mathbb{R}_+; L^2(M^{-1}))$.

Les théorèmes ci-dessus décrivent le comportement en temps long de la solution. Les effets de régularisation nous permettent d'obtenir la convergence exponentielle pour des données initiales seulement dans $L^2(M^{-1})$. Même si cela n'est pas énoncé ci-dessus, la méthode de la Section 6.1 adaptée ici permet d'obtenir également des informations quantitatives de régularité sur les solutions par des estimations des dérivées.

La même dépendance en τ dans les taux de retour exponentiels est attendue au niveau linéaire, sans le couplage avec l'équation de Poisson. On peut commenter cela à la lumière d'un modèle jouet d'équations différentielles ordinaires $X'(t) = V(t)$ et $V'(t) = -X(t) - \tau^{-1}V(t)$. Même si la comparaison peut sembler simpliste les taux viennent seulement de la structure hypocoercive qu'on retrouve dans ce modèle élémentaire. En effet, le système peut s'écrire $\begin{pmatrix} X \\ V \end{pmatrix}' + L_\tau \begin{pmatrix} X \\ V \end{pmatrix} = 0$ avec L_τ la somme d'une matrice de rotation, antisymétrique pour le produit scalaire canonique de \mathbb{R}^2 , et d'une matrice de projection sur la composante V multipliée par le facteur τ^{-1} qui est symétrique, et coercive dans une seule composante. La première joue le rôle de l'opérateur de transport et la seconde celui de l'opérateur positif dégénéré de Fokker-Planck. Par le calcul des valeurs propres on trouve que le trou spectral de L_τ se comporte en $O(\tau)$ lorsque τ dans le régime diffusif $\tau \rightarrow 0^+$ et en $O(1/\tau)$ dans le régime de collisions évanescences $\tau \rightarrow \infty$.

Nous passons maintenant aux dépendances concernant le paramètre de quasi-neutralité δ et dues à la non-linéarité.

Dans le Théorème 9, en régime diffusif, le paramètre de quasi-neutralité δ doit être supérieur à une constante suffisamment grande dépendant de la taille des données initiales mais pas de τ . En somme, on demande à la non-linéarité d'être suffisamment petite uniformément en τ . Le résultat peut être considéré comme perturbatif mais dans un sens différent de celui du papier de Hwang et Jang [130] où la condition de petitesse est posée sur la donnée initiale, qui doit être suffisamment proche d'une Maxwellienne. Notons que la même était déjà nécessaire à Hérau et Thomann pour boucler des estimations non-linéaires dans [123].

Dans le Théorème 10, le paramètre δ doit être supérieur à $O(\tau^{7/15+\varepsilon})$. Ainsi lorsque τ augmente, nous devons atténuer la non-linéarité pour fermer nos estimations. On peut s'attendre à cette non-uniformité en τ pour plusieurs raisons. D'abord, si nous avons obtenu une condition sur δ uniforme en τ comme au Théorème 9, alors à la limite $\tau \rightarrow 0$, dans le régime de Vlasov-Poisson, nous aurions prouvé la stabilité de la solution pour tout profil initial f_0 dans une boule de $L^2(M^{-1})$. Or, comme cela a été remarqué initialement par Penrose [169] et prouvé par Guo et Strauss [113], certains profils de vitesse tels que la somme de deux Gaussiennes de moyennes opposées non nulles²⁶ sont asymptotiquement instables pour Vlasov-Poisson (voir Appendice A pour une illustration numérique). Concernant la puissance $7/15 + \varepsilon$, nous nous tournons encore une fois vers une analyse heuristique. La non-linéarité de Vlasov-Poisson-(Fokker-Planck) est quadratique en la fonction de distribution. Par conséquent, pour comparer (68) avec un équation différentielle simple ayant une non-linéarité et une relaxation exponentielle similaire avec un taux de $1/\tau$ lorsqu'elle est linéarisé autour de l'équilibre, nous considérons $y' = \frac{1}{\delta^2}y^2 - \frac{1}{\tau}y$. De toute évidence, y tend vers 0 exponentiellement rapidement dès que $0 \leq y(0) < \delta^2/\tau$. Ainsi, δ doit être dans le pire des cas comme $\tau^{1/2}$ afin d'obtenir l'uniformité en les données initiales dans une boule. Nous obtenons ici un meilleur taux, ce qui signifie que la structure de l'équation de Poisson a fournit un gain dans les estimations non-linéaires.

Mentionnons tout de même que pour les plasmas, le paramètre δ est extrêmement petit et qu'aucun des résultats ci-dessus ne permet de décrire la limite quasi-neutre $\delta \rightarrow 0$. Pour plus de détails sur cette dernière on renvoie aux articles de Brenier et Grenier [34, 111, 33] pour les premiers travaux sur le sujet et aux papiers d'Han-Kwan, Hauray et Rousset [115, 117, 116] pour les résultats les plus récents concernant le traitement de cette limite pour des modèles cinétiques.

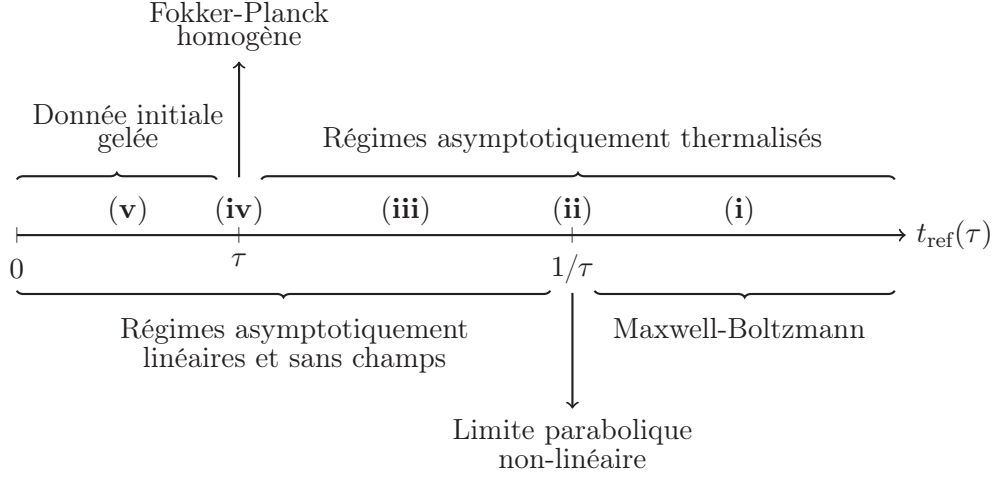
Modèles asymptotiques

Dans le régime diffusif, les contraintes sur δ sont uniformes en τ . Par conséquent, on peut utiliser les estimations établies dans la preuve pour passer à la limite $\tau \rightarrow 0$ sur différentes échelles de temps. On introduit le temps de référence $t_{\text{ref}} \equiv t_{\text{ref}}(\tau)$ et le changement d'échelle

$$f_{\text{ref}}(t, x, v) = f(t_{\text{ref}} t, x, v). \quad (71)$$

Suivant la dépendance de t_{ref} en τ différents modèles asymptotiques peuvent être obtenus. On les décrit dans le diagramme et le théorème suivant.

26. Les profils à deux bosses



On démontre le théorème suivant au Chapitre 3 (Théorème 3.1.4 dans le chapitre).

Théorème 11 (Chapitre 3; $\tau \rightarrow 0$).

Soit f_{ref} la solution du Théorème 9 après le changement d'échelle (71).

- (i). Pour K et $\theta_0 > 0$ dépendant seulement de R_0 , ρ_h et τ_0 , on a pour tout $t \geq 0$ et $\tau \leq \tau_0$

$$\|f_{ref}(t, \cdot, \cdot) - f^\infty(\cdot, \cdot)\|_{L^2(M^{-1})} \leq K \|f_0 - f^\infty\|_{L^2(M^{-1})} e^{-\theta_0 t_{ref} \tau t}$$

si bien que, dans $L^2(M^{-1})$, uniformément pour les temps t dans les compacts de $(0, \infty]$,

$$f_{ref}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} f^\infty$$

à condition que $t_{ref}(\tau) \tau \xrightarrow{\tau \rightarrow 0} \infty$.

- (ii). Lorsque $t_{ref}(\tau) = \tau^{-1}$, il existe $K > 0$ dépendant seulement de R_0 , $\rho_h \in W^{2,p}(\mathbb{T}^2)$ avec $p > 2$ et τ_0 , tel que pour tout $\tau \leq \tau_0$

$$\|f_{ref} - \rho_{ref} M\|_{L^2(\mathbb{R}_+; L^2(M^{-1}))} \leq K \tau \|f_0 - f^\infty\|_{L^2(M^{-1})} \xrightarrow{\tau \rightarrow 0} 0$$

avec $\rho_{ref} = \int_{\mathbb{R}^2} f_{ref}(\cdot, \cdot, v) dv$, et, dans $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^2))$,

$$\rho_{ref}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} \rho_{as}$$

où $\rho_{as} \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^2)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{T}^2))$ est l'unique solution de l'équation de dérive-diffusion

$$\begin{cases} \partial_t \rho_{as} + \nabla_x \cdot (E_{as} \rho_{as} - \nabla_x \rho_{as}) = 0, \\ E_{as} = \delta^{-2} \nabla_x \Delta_x^{-1} (\rho_{as} - \rho_h), \end{cases} \quad (72)$$

de donnée initiale $\rho_0 = \int_{\mathbb{R}^2} f_0(\cdot, v) dv \in L^2(\mathbb{T}^2)$. De plus l'application qui à la

donnée initiale ρ_0 associe ρ_{as} est Lipschitzienne uniformément en temps de la boule de $L^2(\mathbb{T}^2)$ centrée en 0 de rayon R_0 , à valeur dans $\mathcal{C}(\mathbb{R}_+; L^2(\mathbb{T}^2))$ et ρ_{as} converge exponentiellement rapidement en temps long vers la densité de Boltzmann, c'est-à-dire que, pour un certain $\theta_0 > 0$, pour tout $t \geq 0$,

$$\left\| \rho_{as}(t, \cdot) - e^{-\phi_\infty^\delta(\cdot)} \right\|_{L^2(\mathbb{T}^2)} \leq K \|f_0 - f^\infty\|_{L^2(M^{-1})} e^{-\theta_0 t}. \quad (73)$$

(iii). Dans $L^2(M^{-1})$ uniformément pour des temps t dans les compacts de $(0, \infty)$,

$$f_{ref}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} \rho_0 M$$

avec $\rho_0 = \int_{\mathbb{R}^2} f_0(\cdot, v) dv$ si $t_{ref}(\tau) \tau \xrightarrow{\tau \rightarrow 0} 0$ et $t_{ref}(\tau) \tau^{-1} \xrightarrow{\tau \rightarrow 0} \infty$.

(iv). Lorsque $t_{ref}(\tau) = \tau$, dans $L^2(M^{-1})$, uniformément pour des temps t dans les compacts de $[0, \infty)$,

$$f_{ref}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} f_{FP}$$

où f_{FP} est l'unique solution de l'équation de Fokker-Planck homogène

$$\partial_t f_{FP} = \nabla_v \cdot (v f_{FP} + \nabla_v f_{FP}), \quad (74)$$

de donnée initiale f_0 .

(v). Dans $L^2(M^{-1})$, uniformément pour des temps t dans les compacts de $[0, \infty)$,

$$f_{ref}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} f_0$$

à condition que $t_{ref}(\tau) \tau^{-1} \rightarrow 0$ lorsque $\tau \rightarrow 0$.

Dans ce théorème, on montre de la convergence forte sur toutes les échelles de temps. Dans le régime (i), nous démontrons une convergence exponentielle vers l'état stationnaire, la densité de Maxwell-Boltzmann. Cette limite peut être mise en lien avec l'approximation des électrons sans masse. En effet à condition de prendre $\varepsilon = t_{ref}(\tau)^{-1} = \tau^{1/\alpha}$ avec $\alpha \in (0, 1)$, on retrouve le modèle du Chapitre 2 sans champ magnétique et en dimension 2. Il existe un seuil $t_{ref}(\tau) = \tau^{-1}$ entre les régimes asymptotiques qui conservent la trace des effets de champ ((i) et (ii)) et les régimes sans effets des champs de transport, que l'on appelle régimes linéaires ((iii), (iv) et (v)). Pour ces derniers, nous montrons dans la preuve du théorème précédent, que la solution est asymptotiquement proche d'une solution d'une équation homogène de Fokker-Plank. De là, les modèles asymptotiques sont facilement identifiables. Au seuil (ii) apparaît la dynamique non-linéaire la plus intéressante, la limite parabolique ou limite de diffusion. Cela correspond au régime étudié au Chapitre 1 dans le cadre des électrons sans masse. On redémontre donc ici le résultat de Poupaud et Soler [172], Goudon [106] et Masmoudi et el Ghani [81]. On améliore pas ces résultats concernant les hypothèses sur la donnée initiale puisqu'on demande ici plus qu'une masse et énergie libre bornée initialement. Cependant, dans notre méthode de

preuve l'obtention de compacité forte pour passer à la limite est extrêmement simple. Il suffit de passer à la limite pour des données très régulières pour lesquelles on peut obtenir simplement des estimations de propagation de régularité et de la convergence vers la limite avec des taux. Ensuite, grâce à la continuité par rapport à la donnée initiale du Théorème 9, on récupère la convergence²⁷ vers le modèle limite pour une donnée simplement $L^2(M^{-1})$.

7 Un schéma volumes finis pour des problèmes aux limites paraboliques (Chapitre 4)

Dans ce chapitre on s'intéresse à la conception d'un schéma numérique pour la résolution de l'équation de convection-diffusion suivante

$$\begin{cases} \frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{E}(\mathbf{x}) \eta(f) - \nabla \eta(f)) = 0 & \text{pour } \mathbf{x} \in \Omega, t \geq 0, \\ f(t, \mathbf{x}) = f^b(\mathbf{x}) & \text{pour } \mathbf{x} \in \Gamma, t \geq 0, \\ f(0, \mathbf{x}) = f^{\text{in}}(\mathbf{x}) & \text{pour } \mathbf{x} \in \Omega, \end{cases} \quad (75)$$

posée dans un domaine Ω polyédral convexe ouvert de frontière Γ . Le champ d'advection $\mathbf{E} : \Omega \rightarrow \mathbb{R}^d$ et la condition de Dirichlet non-homogène $f^b : \Omega \rightarrow \mathbb{R}_+$ ne dépendent pas du temps et la non-linéarité $\eta : \mathbb{R} \rightarrow \mathbb{R}$ est une fonction régulière strictement croissante et surjective telle que $\eta(0) = 0$. Précisons tout de suite que tous les résultats du chapitre s'adaptent à des conditions de bord mixtes Dirichlet non-homogènes / flux nuls

$$\begin{cases} f = f^b \text{ sur } \Gamma_D, \\ [\mathbf{E} \eta(f) - \nabla \eta(f)] \cdot \mathbf{n}(\mathbf{x}) = 0 \text{ sur } \Gamma_N, \end{cases}$$

où $\Gamma = \Gamma_D \cup \Gamma_N$.

Dans le chapitre on réalisera des simulations numériques sur plusieurs exemples de modèles physiques dont l'équation de Fokker-Planck avec champ magnétique

$$\partial_t f + (v \wedge \mathbf{B}) \cdot \nabla_v f = \nabla_v \cdot (v f + \nabla_v f),$$

avec \mathbf{B} constant, qui correspond à une version homogène (ou intégrée) en espace et sans champ électrique des équations de Vlasov-Fokker-Planck avec champ magnétique étudiées dans les Chapitres 1 et 2. Ce modèle s'intègre dans le cadre général de (75) avec $\eta(s) = s$ et $\mathbf{E}(\mathbf{x}) = -\mathbf{x} + \mathbf{x} \wedge \mathbf{B}$ où la variable \mathbf{x} correspond à la variable v de vitesse des chapitres précédents. Si l'on souhaite réaliser un schéma numérique Eulerien de l'équation précédente, on est confronté à un problème de troncature du domaine \mathbb{R}^3 sur

^{27.} sans taux, car les constantes dépendraient de normes de dérivées que l'on ne contrôle pas initialement.

lequel elle est naturellement posée, afin de se ramener à un domaine borné discrétisable. Après troncature il faut ajouter des conditions de bord pour que le modèle soit bien posé.

7.1 Motivations

Il a été montré dans [24] par Bodineau, Villani, Mouhot et Lebowitz que le modèle (75) admet une grande classe d'entropies relatives à l'état stationnaire

$$\begin{cases} \nabla \cdot (\mathbf{E} \eta(f^\infty) - \nabla \eta(f^\infty)) = 0 & \text{dans } \Omega, \\ f^\infty = f^b & \text{sur } \Gamma. \end{cases} \quad (76)$$

Comme on l'a vu dans la section précédente dans le cadre de l'équation de Vlasov-Poisson-Fokker-Planck, ces fonctionnelles peuvent caractériser la convergence, éventuellement exponentielle, vers l'équilibre. Ici, les entropies relatives sont définies par

$$\mathcal{H}_\phi(t) = \int_\Omega \int_{f^\infty(\mathbf{x})}^{f(t,\mathbf{x})} \phi' \left(\frac{\eta(s)}{\eta(f^\infty(\mathbf{x}))} \right) ds d\mathbf{x},$$

où ϕ est une fonction, dite génératrice, strictement convexe, deux fois continûment dérivable et telle que $\phi(1) = \phi'(1) = 0$. Dans le cas linéaire $\eta(f) = f$, l'entropie relative est réduite à la forme classique

$$\mathcal{H}_\phi(t) = \int_\Omega \phi \left(\frac{f(t,\mathbf{x})}{f^\infty(\mathbf{x})} \right) f^\infty(\mathbf{x}) d\mathbf{x}.$$

D'après [24, Théorème 1.4] on sait que quelle que soit la fonction génératrice ϕ

$$\frac{d\mathcal{H}_\phi}{dt} + \mathcal{D}_\phi \leq 0, \quad (77)$$

avec une dissipation d'entropie donnée par

$$\mathcal{D}_\phi(t) = \int_\Omega \left| \nabla \left(\frac{\eta(f(t,\mathbf{x}))}{\eta(f^\infty(\mathbf{x}))} \right) \right|^2 \phi'' \left(\frac{\eta(f(t,\mathbf{x}))}{\eta(f^\infty(\mathbf{x}))} \right) \eta(f^\infty(\mathbf{x})) d\mathbf{x}.$$

Les exemples typiques d'entropies relatives de la forme \mathcal{H}_ϕ sont l'*entropie physique* de Boltzmann générée par

$$\phi_1(x) = x \ln(x) - (x - 1),$$

ou les *p-entropies* de Tsallis [186] générées par

$$\phi_p(x) = \frac{x^p - px}{p - 1} + 1,$$

pour $p \in (1, 2]$. Les ϕ -entropies caractérisent l'éloignement à l'état stationnaire. En effet \mathcal{H}_ϕ et \mathcal{D}_ϕ sont toujours positives et s'annulent si et seulement si $f = f^\infty$. Dans le cas de la

2-entropie et si $\eta(s) = s$ alors \mathcal{H}_ϕ est même une distance L^2 à poids. Lorsque ces entropies ne sont pas des distances, elles en contrôlent grâce à des inégalités fonctionnelles de type Csiszar-Kullback-Pinsker [138, 55, 189]. D'autre part, si l'on dispose d'une inégalité fonctionnelle de type Poincaré-Sobolev, c'est à dire un contrôle de \mathcal{H}_ϕ par \mathcal{D}_ϕ alors on peut montrer la convergence et obtenir des taux par un argument de type Gronwall. Ainsi, quels que soient les outils à disposition, il est clair que (77) caractérise le comportement en temps long et l'éventuelle convergence vers l'équilibre pour les solutions de (75).

La preuve de cette inégalité différentielle est extrêmement simple si l'on remplace (75) par la reformulation suivante

$$\begin{cases} \frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{U}^\infty h - \eta(f^\infty) \nabla h) = 0, \\ \mathbf{U}^\infty = \mathbf{E} \eta(f^\infty) - \nabla \eta(f^\infty), \\ \nabla \cdot \mathbf{U}^\infty = 0, \end{cases} \quad (78)$$

utilisant à la fois (75) et (76) et une nouvelle inconnue

$$h = \begin{cases} \frac{\eta(f)}{\eta(f^\infty)} & \text{dans } \Omega, \\ 1 & \text{sur } \Gamma. \end{cases} \quad (79)$$

Sous la forme (78), la complexité calculatoire menant à (77) est minimale. On conserve une structure de convection-diffusion, mais le champ d'advection devient incompressible alors que \mathbf{E} est un champ quelconque. Surtout, les conditions de bord sont homogènes sur $h-1$. Dans ce problème la dynamique est complètement entraînée par les conditions de bord. Avec la reformulation (78), ces dernières sont en quelque sorte intégrées à l'équation.

La motivation principale de ce chapitre est la conception d'un schéma numérique reproduisant toutes les inégalités (77) au niveau discret afin d'obtenir un bon comportement en temps long de la solution numérique et une convergence vers l'état stationnaire. L'idée est bien sûr de reproduire les calculs continus menant à (77) et étant donné ce que nous avons présenté ci-dessus, c'est sous la forme (78) que nous allons discrétiser le problème. Cela implique de construire un schéma basé sur l'état stationnaire.

On insiste sur le fait qu'ici le champ d'advection n'est ni irrotationnel, ni incompressible. Lorsqu'il dérive d'un potentiel, l'équation (78) a une structure de flot gradient et de nombreuses méthodes numériques [90, 45, 21, 38, 36, 46] ont été développées. D'autre part, la plupart des travaux concernant la conservation par des schémas numériques de propriétés entropiques en domaine borné concernent des équations munies de conditions de bord conservatives. Dans le cas de conditions de Dirichlet non-homogènes, on rappelle tout de même les travaux de Bessemoulin-Chatard [21] qui propose une extension du schéma de Scharfetter et Gummel [180] pour les modèles de semi-conducteurs à toute une classe d'équations de convection-diffusion. Cependant l'obtention d'inégalités d'en-

tropie discrètes ne concerne que certains cas particuliers.

7.2 Schéma basé sur les flux stationnaires et résultats principaux

On renvoie au Chapitre 4 pour l'introduction complète du schéma numérique. Comme on l'a évoqué précédemment notre schéma est construit grâce à une discrétisation préalable de l'état stationnaire. En effet, on se donne d'abord des flux stationnaire, c'est-à-dire une discrétisation de $\nabla \cdot \mathbf{U}^\infty = 0$ ainsi que des approximations de l'état stationnaire.

Pour un modèle physique, deux situations peuvent se présenter. Dans certains cas, on connaît à l'avance l'état stationnaire du modèle. C'est le cas par exemple lorsque ce dernier correspond à un équilibre thermodynamique comme pour le modèle de Fokker-Planck avec champ magnétique. Dans ce cas, les approximations peuvent être obtenues grâce à la formule explicite, par exemple par une méthode de quadrature des intégrales définissant les flux ou les moyennes sur les cellules.

Dans les autres cas, on ne connaît pas l'état stationnaire à l'avance. On résout alors l'équation (76) par un schéma volumes finis. Notre méthode ne prescrit en rien le schéma précis à utiliser pour l'état stationnaire et différents exemples sont considérés dans le chapitre.

À partir des quantités stationnaires discrètes on discrétise (78). Les flux correspondant à la partie convective et la partie diffusive sont respectivement approchés par un flux décentré amont et un flux à deux points. Les propriétés de monotonie relatives à ces derniers sont importantes pour établir la dissipation des entropies qui nous intéressent. Notons que ces flux sont extrêmement classiques. L'originalité du schéma vient de la reformulation de l'équation et de sa construction à partir d'un autre schéma pour l'état stationnaire. Concernant la discrétisation en temps on réalise dans le chapitre l'analyse et des simulations pour des discrétisations d'Euler explicite et implicite.

On montre un certain nombre de résultats théoriques sur le schéma. Dans le Théorème 4.4.2, on montre à maillage fixé les propriétés suivantes.

- 1) Il existe une unique solution au schéma.
- 2) Les approximations numériques sont positives et bornées uniformément par des constantes dépendant seulement des bornes sur la donnée initiale et l'état stationnaire.
- 3) Le schéma préserve l'état stationnaire.
- 4) Le schéma fait décroître toutes les ϕ -entropies relatives discrètes avec une dissipation consistante.

Dans le cas linéaire, la décroissance des entropies discrètes ainsi qu'une inégalité de Poincaré discrète (voir Proposition 4.4.6), prouvée par Bessemoulin-Chatard, Chainais-Hillairet et Filbet dans [22, Theorem 6] et basée sur des travaux plus anciens cités dans le même article, donne la convergence exponentielle vers l'équilibre en temps long (Théorème 4.4.7).

- 5) Si η est la fonction identité, la solution discrète converge (en norme L^1 et en entropie) au pire exponentiellement rapidement vers l'équilibre discret à un taux qui ne dépend pas de la discrétisation.

On montre finalement la convergence vers une solution faible de (78) (Théorème 4.5.7)

- 6) Si la solution du schéma stationnaire converge (lorsque le paramètre de discrétisation tend vers 0) vers une solution faible de (76) alors la solution de notre schéma en temps converge vers une solution faible de l'équation reformulée (78).

On renvoie le lecteur vers le Chapitre 4 pour des simulations numériques illustrant ces résultats. Concernant l'implémentation des schémas numériques, le code de nos cas test est écrit en C++. Pour le schéma implicite nécessitant la résolution d'un système linéaire à chaque itération temporelle, nous utilisons la librairie SuperLU [140] permettant une factorisation LU efficace²⁸ des matrices creuses non-symétriques. Comme le champ d'advection est indépendant du temps, cette opération n'est effectuée qu'une fois au début de la simulation. Concernant la visualisation des résultats, nous avons utilisé les logiciels libres Gnuplot et VisIt afin de produire les figures du Chapitre 4.

8 Perspectives

8.1 Modèle de Van Roosbroeck avec champ magnétique

Dans le Chapitre 1, on a obtenu un modèle asymptotique de dérive-diffusion anisotrope dont la matrice de diffusion dépend du champ magnétique. Ce modèle est très proche de celui considéré par Gajewski et Gärtner dans [93] comme extension du modèle de Van Roosbroeck [176] en présence d'un champ magnétique extérieur. Ce modèle décrit l'évolution des densités macroscopiques d'électrons $N \equiv N(t, x)$ et de trous $P \equiv P(t, x)$ sous l'effet d'un champ magnétique extérieur $B \equiv B(t, x)$ et des interactions électrostatiques caractérisées par le potentiel ϕ . Le modèle s'écrit

$$\begin{cases} \partial_t N + \nabla \cdot J_N = 0, & J_N \wedge B + J_N = N \nabla \phi - \nabla N, \\ \partial_t P + \nabla \cdot J_P = 0, & J_P \wedge B + J_P = -P \nabla \phi - \nabla P, \\ -\Delta \phi = P - N - C, \end{cases} \quad (80)$$

avec $C \equiv C(t, x)$ le profil de dopage qui caractérise le milieu semi-conducteur. Ce dernier est dans un domaine borné Ω de frontière $\Gamma = \partial\Omega$ et (80) est munie de conditions de bord de type Neumann en présence d'une isolation électrique

$$\nabla N \cdot \nu(x) = 0, \quad \nabla P \cdot \nu(x) = 0, \quad \nabla V \cdot \nu(x) = 0, \quad (81)$$

et de Dirichlet en présence d'un contact ohmique

$$N(t, x) = N^D(x), \quad P(t, x) = P^D(x), \quad V(t, x) = V^D(x). \quad (82)$$

La modélisation des semi-conducteurs connaît toute une hiérarchie de modèles cinétiques et hydrodynamiques (voir les ouvrages de Jüngel [135, 134] et les références incluses). Ici, on pourrait clairement, à partir d'un modèle de type Vlasov-Poisson-Fokker-Planck

²⁸. conservant une structure de matrice creuse

ou Vlasov-Poisson-Boltzmann dans le scaling de l'équation du Chapitre 1 retrouver le modèle (80)-(82). Au niveau de l'analyse, de nouvelles difficultés seraient alors créées à cause des conditions de bord. Cependant quelques travaux dans la littérature existante donnent des pistes de travail. En effet, des conditions compatibles avec (82) au niveau cinétique sont étudiées dans le papier de Masmoudi et Tayeb [147]. En ce qui concerne les conditions de Neumann (81), une analyse basée sur [147] existe dans le papier de Wu, Lin et Liu [195].

8.2 Justification non-linéaire des densités de Boltzmann anisotrope

Dans le Théorème 11 (i), on obtient la convergence exponentielle vers la densité de Boltzmann isotrope à partir de l'équation de Vlasov-Poisson-Fokker-Planck. En présence d'un champ magnétique, on parvient à obtenir un modèle limite dans le scaling diffusif au Chapitre 1. Au Chapitre 2, on réalise le passage à la limite vers la densité de Boltzmann anisotrope pour un potentiel électrique extérieur. Plusieurs pistes sont envisagées pour obtenir le même résultat au niveau non-linéaire. La première piste repose sur l'utilisation d'un argument d'énergie modulée ou entropie relative au modèle asymptotique attendu (voir par exemple Han-Kwan [115] et les références incluses). La construction d'une fonctionnelle de Lyapunov modulant l'énergie libre a été considéré mais nous n'avons pas réussi à fermer les estimations. Un renforcement de cette stratégie avec celle développée en Section 6 fera l'objet de futures recherches.

Si l'on se contente des estimations d'énergie libre, la principale obstruction au passage à la limite non-linéaire est l'obtention de compacité forte. Dans le scaling du Chapitre 2, il ne nous semble pas possible d'utiliser des lemmes de moyenne en vitesse existants (par exemple le Lemme 2) à cause du champ magnétique singulier. Cependant une autre piste consisterait à ré-estimer explicitement le gain de régularité pour des moyennes en vitesse spécifiques (à symétrie radiale par exemple) afin de tirer partie de la structure particulière des termes de champ magnétique et de Fokker-Planck. Plus précisément, si l'on note \hat{f} la transformée de Fourier de $f \equiv f(x, v)$ en x et ξ la variable de Fourier correspondante, il s'agirait d'estimer des quantités comme

$$\mu |\xi|^s \int v^\perp \cdot \nabla_v \hat{f} \frac{\varphi(|v|)}{1 + i v \cdot \xi} dv$$

pour un certain $s > 0$ en fonction de quantités contrôlées uniformément grâce aux estimations a priori et en tirant éventuellement partie des phénomènes de dissipation dus à l'opérateur de Fokker-Planck. Cette piste est en cours d'investigation.

8.3 Autres opérateurs de collisions

Du point de vue de la modélisation d'un plasma, le principal biais dans les travaux présentés dans cette thèse concerne le choix de l'opérateur de collision. Mentionnons d'abord que la majorité de l'analyse des Chapitres 1 et 2 se généralise pour d'autres opérateurs avec un noyau générée par la maxwellienne centrée réduite (par exemple les opérateurs de relaxation ou de Boltzmann linéaire), à quelques exceptions près. Le contrôle

du champ magnétique dans le scaling diffusif du Chapitre 1 et la régularisation hypoelliptique utilisée dans le Chapitre 2 dépendent fortement de la structure de l'opérateur de Fokker-Planck. D'autre part, dans la preuve du Théorème 4, on estime la taille des moments en vitesse grâce à la dissipation d'énergie libre de l'opérateur de Fokker-Planck dans les estimations (48) et (49). Dans le cas d'un opérateur de relaxation²⁹ linéaire $Q(f) = f - nM$ la dissipation d'énergie libre contrôle $\frac{1}{2}\|\sqrt{f} - \sqrt{nM}\|_{L^2}^2$, et en utilisant cette quantité on peut obtenir l'estimation

$$\left\| \int_{\mathbb{R}^d} v f \, dv \right\|_{L^1(\Omega)} \leq \|\sqrt{f} - \sqrt{nM}\|_{L^2}^2 \| |v|^2 f \|_{L^1}^{1/2}.$$

En revanche sans contrôle de moments d'ordre supérieur, il n'est pas clair que l'on obtienne l'équivalent de (49). Nous pensons néanmoins que grâce à des arguments de troncature en vitesse comme dans [178], il est possible de régler le passage à la limite du terme impliquant les seconds moments en vitesse dans (40) sans supposer davantage sur la donnée initiale.

L'extension de nos analyses à des opérateurs plus réalistes comme l'opérateur de Landau est envisagée comme perspective aux travaux réalisés dans les Chapitres 1 et 2. De nombreuses difficultés apparaîtront dans l'étude d'asymptotique de ce type de modèles en présence d'un champ magnétique fort. Ici seule la dynamique spatiale est affectée par le champ magnétique dans les limites considérées. On peut s'attendre à ce qu'avec un opérateur plus complexe, la statistique en vitesse soit également modifiée de manière anisotrope. Comme on l'a déjà mentionné, des résultats formels existent déjà dans la littérature [170, 64, 65, 61, 160, 44, 109, 97, 96] en lien avec ces modèles. Cependant leur analyse rigoureuse est encore, à notre connaissance, complètement ouverte.

8.4 Généralisation des résultats du Chapitre 3 en dimension supérieure

Dans le Chapitre 3, nos résultats de retour exponentiel vers l'équilibre et l'obtention de modèles asymptotiques sont énoncés en dimension 2. La principale difficulté technique relative aux dimensions supérieures est la suivante. Afin de contrôler un certain nombre des termes de restes dans l'estimation de notre fonctionnelle de Lyapunov \mathcal{F} , nous avons besoin d'un contrôle L^∞ du champ électrique. Grâce à l'équation de Poisson et par injection de Sobolev ce contrôle est obtenu dès que la solution est dans H^η avec $\eta > (d - 2)/2$. Le contrôle de la fonctionnelle et de sa dissipation ne fournissent que des estimations H^1 . Par conséquent dès que $d \geq 4$, il faut ajouter des dérivées d'ordre supérieur dans \mathcal{F} afin d'espérer boucler l'argument. En dimension 3 si l'on oublie la régularisation hypoelliptique, c'est à dire si l'on s'autorise un contrôle des dérivées de la donnée initiale et que l'on supprime les poids en temps dans la fonctionnelle, alors on obtient sans difficulté des résultats équivalents. En revanche, la méthode ne semble pas permettre d'obtenir la décroissance exponentielle en dimension 3 avec une donnée initiale dans L^2 . En effet la régularisation pour obtenir un champ L^∞ est trop lente (à cause du seuil sur η) et on est confronté à un problème d'intégrabilité en temps. Dans leur

²⁹. la discussion est la même pour Boltzmann linéaire avec cross-section bornée

papier [123] Hérau et Thomann sont confrontés à un problème similaire et leur analyse suggère qu'il est possible d'établir notre estimation avec une donnée initiale dans H^s avec $s > 1/2$.

8.5 Généralisation des résultats du Chapitre 3 dans \mathbb{R}^d

Dans le Chapitre 3, notre analyse repose comme dans [120, 122, 121, 123] sur l'existence d'une inégalité de Poincaré dans \mathcal{H}_0 (voir Proposition 3.2.1) venant du poids Gaussien en vitesse et de la compacité du domaine spatial. C'est l'absence de contrôle de termes d'ordre 0 dans la dissipation de la fonctionnelle hypocoercive qui pousse à tirer partie du confinement pour obtenir la décroissance exponentielle vers l'équilibre. Nous pensons néanmoins qu'une décroissance exponentielle vers l'équilibre est envisageable également dans l'espace entier, pour une densité de fond quelconque, dans le sens où elle ne prend pas forcément le rôle d'un potentiel confinant comme dans l'article de Hérau et Thomann [123].

Pour argumenter notre propos, repartons de l'analyse menée dans la Section 6.1 de cette introduction. Pour le modèle jouet, on a été capable d'établir (66), sans l'utilisation de l'inégalité de Poincaré. Cette dernière permet néanmoins d'obtenir le résultat de convergence exponentiel final. Si l'on remplaçait \mathbb{T}^2 par \mathbb{R}^2 pour cette équation la convergence vers l'état d'équilibre deviendrait algébrique en temps. Pour le prouver on peut remplacer l'inégalité de Poincaré par l'inégalité de Nash. On renvoie au récent [31] pour davantage de détails et la justification de l'optimalité de cette décroissance. Dans le modèle de Vlasov-Poisson-Fokker-Planck du Chapitre 3, la présence supplémentaire d'interactions de Coulomb modifie la situation. Tout d'abord, de nombreux termes de restes sont estimés grâce à des termes d'ordre 0 (voir Lemme 3.4.2 et Lemme 3.4.3 dans le Chapitre 3). L'inégalité de Poincaré est alors déjà utilisée pour contrôler ces derniers par leur dérivées, présentes dans la dissipation, et établir l'équivalent de (66). Afin de généraliser les résultats du Chapitre 3 dans l'espace entier \mathbb{R}^d , le simple fait d'établir l'inégalité de dissipation d'entropie semble *a priori* incertaine. Néanmoins nous pensons qu'il est possible de tirer partie de la structure de l'équation de Poisson afin de créer une dissipation d'ordre 0 (la norme L^2 de la densité macroscopique, voir la seconde estimation du Lemme B.4.2 en appendice de ce manuscrit, dans la Section B.4) en ajoutant des termes impliquant des quantités macroscopique dans la fonctionnelle à estimer. Cette stratégie, inspirée des estimations micro-macro de Guo a fait ses preuves dans l'article de Hwang et Yang [130] où un résultat perturbatif de retour exponentiel vers l'équilibre est obtenu pour Vlasov-Poisson-Fokker-Planck dans le cas d'une densité de fond constante³⁰.

Dans l'esprit de cette stratégie micro-macro mais indépendamment de la discussion précédente relative à l'influence de l'équation de Poisson, nous pensons qu'il y a également des liens entre la première estimation du Lemme B.4.2 en appendice de ce manuscrit avec l'ajout du dernier terme de l'entropie modifiée de Mouhot, Dolbeault et Schmeiser (voir Section 1.3 dans [75]). Ce terme est constitué de quantités macroscopiques remplaçant la

30. Une densité de fond constante pour des charges opposée correspond implicitement à un potentiel confinant quadratique, comme le fond remarquer Hérau et Thomann [123]. Cependant dans [130] les auteurs ne semblent pas tirer partie de cela.

dérivée croisée $\langle \nabla_x h, \nabla_v h \rangle$ et permettant de boucler les estimations dans un espace L^2 plutôt que H^1 . L'utilisation de ces techniques d'hypocoercivité, différentes des méthodes utilisées dans ce manuscrit s'inspirant essentiellement du traitement de Villani dans le mémoire [194], sont également des pistes pour la généralisation de nos résultats.

8.6 Restriction de non-quasineutralité

Dans le Chapitre 3, on obtient un certain nombre de modèle asymptotique sur différentes échelles de temps lorsque la fréquence de collisions tend vers l'infini. Cependant, il y a une limite diffusive non-triviale de VPFP qui ne peut être obtenu pour le moment avec notre méthode, la «high-field limit». Elle a été étudiée par plusieurs auteurs [163, 3, 25, 107] et donne une équation hyperbolique non-linéaire sur la densité macroscopique. Afin de traiter cette limite, il faudrait pouvoir prendre $\delta^2 = \tau$, ré-échelonner le potentiel en $\tilde{\phi} = \tau \phi$ et prendre la limite $\tau \rightarrow 0$ sur f et $\tilde{\phi}$. Ce n'est pour le moment pas possible car les estimations de la preuve du Théorème 9 ne sont valables que pour δ suffisamment grand, uniformément en τ . Nous pensons néanmoins qu'il est possible de surmonter cette difficulté par l'estimation d'une fonctionnelle différente. En effet la restriction sur δ n'est pas due à la non-linéarité de l'équation et est nécessaire pour arriver aux mêmes conclusions sur un modèle linéarisé autour d'une maxwellienne. Notre idée est de partir d'une fonctionnelle plus proche de l'énergie libre afin de mieux intégrer la structure de l'équation dans les estimations. Pour ce faire, il faut remplacer la mesure statique $e^{-\phi_\infty^\delta(x)} M(v) dv dx$ par une mesure dynamique $e^{-\phi(t,x)} M(v) dv dx$ dans l'espace de Hilbert de travail. De cette façon les termes relatifs au champ électrique s'intégreront mieux dans les estimations au prix de nouveaux restes impliquant des quantités comme $\partial_t \phi$. Cette dernière étant moralement une primitive d'une quantité moyennée en vitesse (à cause de l'équation d'Ampère), il est envisageable de coupler cette approche avec la stratégie micro-macro du paragraphe précédent. Cette piste rejoint également celles évoquées plus haut pour obtenir des estimations globales permettant de traiter la limite non-linéaire du Chapitre 2.

8.7 Extension du schéma du Chapitre 4 à des diffusions anisotropes

Concernant les résultats du Chapitre 4, la première perspective concernerait la généralisation du schéma à des diffusions anisotropes (et / ou à des maillages non-orthogonaux). En effet le résultat théorique de Bodineau, Mouhot, Lebowitz et Villani [24] est également valable pour une matrice de diffusion plus générale. Dans un cadre isotrope sur maillage orthogonal la conception de flux numériques approchés nécessite seulement une discrétisation de gradients orthogonaux au faces ou arêtes du maillage. Une discrétisation du gradient dans toutes les directions est nécessaire lorsqu'une matrice de diffusion est ajoutée. Plusieurs méthodes ont été développées en ce sens par Eymard, Herbin, Gallouët [86, 83, 84, 87] et, Guichard et Cancès [39]. Leurs méthodes sont basées sur des schémas volumes finis hybrides (SUSHI, DDFV, VAG...) qui utilisent en général des inconnues auxiliaires (parfois seulement nécessaires pour écrire le schéma) sur les faces ou les sommets du maillage. Les premières recherches effectuées en ce sens suggèrent

qu'il sera difficile de conserver les propriétés de monotonie des flux permettant d'obtenir la décroissance des entropies. Une option envisagée serait un schéma dépendant d'une fonction génératrice ϕ bien choisie préservant naturellement les bornes sur la solution par la décroissance de l'entropie associée. Le flux de convection pourrait alors être discrétisé grâce au flux centré $F_{K,\sigma}^{M_\phi}$ introduit dans la preuve du Théorème 4.4.2.

8.8 Schémas préservant les états stationnaires quasi-thermiques pour les semi-conducteurs

La généralisation du schéma conçu au Chapitre 4 pour le modèle de Van Roosbroeck (80)-(82) sans champ magnétique ($B = 0$) est en cours d'investigation. L'idée serait de réussir à conserver des équilibres non thermiques pour ce modèle. En effet lorsque les conditions de bord satisfont

$$V^D(x) - \log(P^D(x)) = V^D(x) + \log(N^D(x)) = \alpha, \quad \forall x \in \Gamma_D, \quad (83)$$

pour un α constant on sait qu'il existe un unique équilibre ([146, 188]), appelé équilibre thermique, et tel que la relation (de Boltzmann-Gibbs) (83) est vérifiée à l'intérieur du domaine. Physiquement, cet état stationnaire est atteint lorsqu'aucun potentiel n'est appliqué sur Γ_D . Par conséquent dans un grand nombre de situations physiques, lorsqu'un potentiel de biais est appliqué, il existe des états stationnaires non-thermiques [155]. À notre connaissance, mis à part quelques pistes dans l'ouvrage [155] de Mock, la convergence exponentielle vers ces états n'a jamais été prouvée théoriquement. Notons que c'est une question délicate, car l'unicité des états stationnaires non-thermiques n'est pas toujours assurée [156]. En revanche, dans les cas quasi-thermiques, c'est à dire pour une petite perturbation des conditions d'équilibre thermique nous sommes en mesure de montrer cette convergence exponentielle vers l'unique état stationnaire grâce à une reformulation similaire à celle de l'équation du Chapitre 4 et un argument d'entropie relative. La conception d'un schéma préservant ces états stationnaire est également généralisable pour ce modèle et est l'objet d'un travail en cours.

Chapitre 1

Limite de diffusion anisotrope d'un système bi-cinétique

Ce chapitre est publié en version abrégée dans
Journal of Differential Equations [124].

We consider a three-dimensional kinetic model for a two species plasma consisting of electrons and ions confined by an external nonconstant magnetic field. Then we derive a kinetic-fluid model when the mass ratio m_e/m_i tends to zero.

Each species initially obeys a Vlasov-type equation and the electrostatic coupling follows from a Poisson equation. In our modeling, ions are assumed non-collisional while a Fokker-Planck collision operator is taken into account in the electron equation. As the mass ratio tends to zero we show convergence to a new system where the macroscopic electron density satisfies an anisotropic drift-diffusion equation. To achieve this task, we overcome some specific technical issues of our model such as the strong effect of the magnetic field on electrons and the lack of regularity at the limit. With methods including renormalized solutions, relative entropy dissipation and velocity averages, we establish the rigorous derivation of the limit model.

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1.1 Introduction

In many plasma physics applications, the numerical simulation of full multi-species kinetic systems of equations can be extremely expensive in computer time. Indeed, since the typical time, space and velocity scales of each species differ from several orders of magnitude, it requires a fine discretization to accurately approximate the different scales. We refer to [9] for a discussion on these issues in the two-species Vlasov-Poisson case. Therefore, part of the problem is sometimes overcome by making simplifying assumptions on species with negligible contribution to the whole dynamic. In this chapter, we are interested in reducing a kinetic model by taking the limit when the mass ratio between light and heavy particles, namely electrons and ions, tends to 0.

The charged gas evolves under its self-consistent electrostatic field and an external magnetic field. This configuration is typical of a tokamak plasma [19, 154] where the magnetic field is used to confine particles inside the core of the device. We assume that on the time scale we consider, collisions on ions can be neglected while for electrons, it is entirely modeled with a Fokker-Planck operator. At the formal level, an exhaustive study of asymptotic mass-disparate model with more involved collision operators such as the Boltzmann or Landau operator can be found in the review [61] of Degond. When the mass ratio goes to 0, our model converges to a drift-diffusion equation featuring a magnetic-field dependent diffusion matrix for the electrons coupled with the original kinetic equation for the heavy particles. Similar parabolic equations with non-symmetric diffusion for plasmas can also be found in [20, 56, 58, 57, 61].

Hereafter, we start from the physical equations and propose a detailed scaling with respect to the ions time scale. In the dimensionless system, for the light species equation, the leading order terms with respect to the mass ratio are those related to the magnetic field and the collisions. Therefore, the resulting derivation is in the mean time similar to a strong magnetic field limit as in the papers of Golse and Saint-Raymond [105, 177], and to a diffusive or parabolic limit as in the work of Poupaud, Soler, Masmoudi and El Ghani in [172, 81]. While the latter papers provide us with many tools to handle our own problem, some technical issues in our analysis are closely related to the special features of our model. In a single-species model of charged particles, the other particles density is usually given either as a static regular background or as a function of the electric potential, while our ions are only known to obey a non-trivial kinetic equation. Because of this coupling, it turns out that some extra analysis is needed to recover the regularity required for our limit system to make sense. Besides, we have to control the strong magnetic field. This is done by looking at the interplay between the increasing effect of oscillations and collisions in the asymptotic regime where the mass ratio tends to 0. Indeed, despite the fact that collisions and magnetic forces appear at the same order of magnitude in original and limit equations we consider, we prove, by making the most of special cancellations, that with other respects collisional effects provide a form of control on magnetic contributions. While the rest of our analysis seems rather robust the latter control — that relies on cancellations — does not seem to extend readily to other forms of dissipation operators such as linear Boltzmann collision operators.

1.1.1 The physical model

To avoid unnecessary technicalities, we suppose that there is only one type of ion in the plasma of mass m_i and charge q , while m_e denotes the electron mass and $-q$ their negative charge. The case of a plasma containing several types of ions can be treated in the same way.

The number of particle of type s in the phase space volume $dx dv$ at position $x \in \mathbb{R}^3$, with velocity $v \in \mathbb{R}^3$ between time t and $t+dt$ is $f_s(t, x, v) dx dv dt$. The index s stands for the species of particles and can be either i for ions or e for electrons. The ion distribution function f_i evolves according to a Vlasov equation and the electron distribution function f_e follows a Vlasov-Fokker-Planck equation. The coupling occurs through the Poisson equation that relates the electric field to the densities. The equations of the model, written in physical units, are the following

$$\begin{cases} \partial_t f_i + v \cdot \nabla_x f_i + \frac{q}{m_i} (-\nabla_x \phi + v \wedge B_{\text{ext}}) \cdot \nabla_v f_i = 0, \\ \partial_t f_e + v \cdot \nabla_x f_e - \frac{q}{m_e} (-\nabla_x \phi + v \wedge B_{\text{ext}}) \cdot \nabla_v f_e = \frac{1}{t_{\text{col}}} \nabla_v \cdot \left(v f_e + \frac{k_B \theta}{m_e} \nabla_v f_e \right), \\ -\varepsilon_0 \Delta_x \phi = q(n_i - n_e), \end{cases} \quad (1.1)$$

where t_{col} is the characteristic time between two collisions, k_B is the Boltzmann constant, θ is the average electron temperature and ε_0 is the dielectric constant. The Poisson equation involves the macroscopic densities

$$n_s = \int_{\mathbb{R}^3} f_s dv, \quad \forall s \in \{i, e\}.$$

As mentioned before the typical time scales of ions and electrons largely differ due to the smallness of the mass ratio m_e/m_i . In applications involving magnetic confinement fusion, ions are the particles of interest and approximations are made on the electron distribution function f_e in order to simplify the model. A common reduction supposes that the macroscopic electron density is given by the Maxwell-Boltzmann density

$$n_{\text{MB}}(t, x) = C(t) e^{\frac{q\phi(t, x)}{k_B \theta}},$$

where $C(t)$ is a normalization function. The derivation of the latter from (1.1), with $B_{\text{ext}} = 0$, is discussed in [27] and obtained in [29] for a one species Vlasov-Poisson-Fokker-Planck model. To our knowledge, the case of a magnetized plasma has never been treated before. We stress that the presence of a strong magnetic field modifies even formal computations.

The goal of the present chapter is to derive a fluid equation for electrons, when the mass ratio tends to 0. The evolution of their dynamic shall obey an equation on the macroscopic density. In the next paragraph we write the system in a consistent dimensionless form to receive the fluid model in the asymptotic regime of massless electrons.

1.1.2 Scaling

We denote by L the characteristic length of the system, t_0 the characteristic time and V_s the thermal velocity for the species $s \in \{i, e\}$. For any other physical quantity G , we denote by \bar{G} the characteristic value of G and G' the dimensionless quantity associated to G so that $G = \bar{G}G'$. We assume that the plasma is globally neutral, which means that

$$\bar{n}_i = \bar{n}_e =: N,$$

and that the characteristic temperature (or kinetic energy) of each species are equal. A plasma satisfying the latter hypothesis is called a hot plasma [19] and satisfies with our notation

$$m_i V_i^2 = m_e V_e^2 = k_B \theta.$$

The new unknowns of the system are then defined by the following relations

$$f_s(t, x, v) = \frac{N}{V_s^3} f'_s \left(\frac{t}{t_0}, \frac{x}{L}, \frac{v}{V_s} \right), \quad n_s(t, x) = N n'_s \left(\frac{t}{t_0}, \frac{x}{L} \right),$$

$$\phi(t, x) = \bar{\phi} \phi' \left(\frac{t}{t_0}, \frac{x}{L} \right).$$

The dimensional analysis of (1.1) introduces several important physical constants of the system, namely, for each species $s \in \{i, e\}$

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B \theta}{q^2 N}}, \quad t_p^{(s)} = \frac{\lambda_D}{V_s}, \quad t_c^{(s)} = \frac{m_s}{q \bar{B}_{\text{ext}}}, \quad r_L^{(s)} = V_s t_c^{(s)},$$

which are respectively the Debye length, the plasma time, the cyclotron time and the Larmor radius. More details on these constants can be found in the physics literature (see [19, 154]). We mention that the first two are typical scales of the electrostatic effects while the last two are related to magnetic phenomena. Since the goal is to perform a model reduction for the electron dynamic, we choose a scaling relative to the typical ion time scale. In particular, it means that we choose

$$t_0 = \frac{L}{V_i}.$$

From some of the original physical constants and characteristic quantities arise dimensionless parameters of the system, namely

$$\delta = \frac{\lambda_D}{L}, \quad \eta = \frac{q \bar{\phi}}{k_B \theta}, \quad \mu = \frac{\bar{\phi}}{L V_i \bar{B}_{\text{ext}}}, \quad \varepsilon = \sqrt{\frac{m_e}{m_i}}.$$

In a tokamak plasma these parameters would all be small. Dealing with the asymptotic $\delta \rightarrow 0$ is called the quasineutral limit and has been first investigated for the Vlasov-Poisson system by Brenier and Grenier in [34, 111, 33] and developed recently by Han-Kwan, Hauray and Rousset in [115, 117, 116]. The second parameter η is called the

coupling parameter, for it measures the importance of the electrostatic effects with respect to the thermal agitation. The third parameter μ compares the Coulomb (electric) and the Laplace (magnetic) forces. The limit $\mu \rightarrow 0$ corresponds to the case of a strong magnetic field and has been studied for single-species plasma by Frenod and Sonnendrucker in [92] and Golse and Saint-Raymond in [105, 177]. This asymptotic is called the gyrokinetic or drift-kinetic approximation in the mathematics literature. The last parameter ε , quantifying the mass ratio, is the main concern of this chapter. We are interested in the limit of massless electrons, namely $\varepsilon \rightarrow 0$. The dimensionless equations write

$$\begin{cases} \partial_t f'_i + v \cdot \nabla_x f'_i - \eta \nabla_x \phi' \cdot \nabla_v f'_i + \frac{\eta}{\mu} (v \wedge B_{\text{ext}}) \cdot \nabla_v f'_i = 0, \\ \partial_t f'_e + \frac{1}{\varepsilon} v \cdot \nabla_x f'_e + \left(\frac{\eta}{\varepsilon} \nabla_x \phi' - \frac{\eta}{\mu \varepsilon^2} (v \wedge B_{\text{ext}}) \right) \cdot \nabla_v f'_e = \frac{t_0}{t_{\text{col}}} \nabla_v \cdot (v f'_e + \nabla_v f'_e) \\ - \eta \delta^2 \Delta_x \phi^\varepsilon = n'_i - n'_e. \end{cases} \quad (1.2)$$

The link between the different time scales and dimensionless parameters follows from the previous relations and writes

$$t_p^i = \frac{1}{\varepsilon^2} t_p^e = \delta t_0, \quad t_c^i = \frac{1}{\varepsilon} t_c^e = \frac{\mu}{\eta} t_0.$$

Finally we consider the regime where

$$\frac{t_0}{t_{\text{col}}} = \frac{1}{\varepsilon^2}.$$

which means that the ratio between collision time and the observation time is of the same order than the mass ratio. This can be derived considering that the Knudsen number, namely the ratio between the mean free path of the particles and the observation length L is of order ε . More details on this part of the scaling can be found in the paper of Petit and Darrozes [170] and in the work of Degond and Lucquin [64, 65, 61] and more recently Graille, Magin, Massot and Giovangigli [109, 97, 96]. This makes the collision the leading order term with the magnetic field in the electron equation. In its analysis, the resulting scaling slightly differs from the usual parabolic or hydrodynamic scaling of Vlasov-Poisson-Fokker-Planck since the magnetic field introduces fast oscillations when ε tends to zero and it is *a priori* not clear that the latter can be controlled by the dissipation effect due to collisions.

By taking weaker collisions, say $t_0/t_{\text{col}} = 1/\varepsilon$, one formally derives a Maxwell-Boltzmann density in the direction parallel to the magnetic field and a guiding center model in the perpendicular plane when $\varepsilon \rightarrow 0$. This limit model features interesting and well-known physical phenomena. Its derivation and analysis will be investigated in future work.

1.1.3 The mathematical model

From now on, we keep δ, η, μ fixed and thus, for writing convenience, we assume that $\delta = \eta = \mu = 1$. The rescaled two species (Vlasov)-(Vlasov-Fokker-Planck)-(Poisson)

kinetic system with external magnetic field writes

$$\begin{cases} \partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon - \nabla_x \phi^\varepsilon \cdot \nabla_v f_i^\varepsilon + (v \wedge B) \cdot \nabla_v f_i^\varepsilon = 0, \\ \varepsilon \partial_t f_e^\varepsilon + v \cdot \nabla_x f_e^\varepsilon + \nabla_x \phi^\varepsilon \cdot \nabla_v f_e^\varepsilon - \frac{1}{\varepsilon} (v \wedge B) \cdot \nabla_v f_e^\varepsilon = \frac{1}{\varepsilon} \mathcal{L}_I(f_e^\varepsilon), \\ -\Delta_x \phi^\varepsilon = n_i^\varepsilon - n_e^\varepsilon. \end{cases} \quad (1.3)$$

where we have dropped primes and marked the dependency on ε with an exponent. The Fokker-Planck operator on the right-hand side of the second equation of (1.3) is defined by

$$\mathcal{L}_I : f \mapsto \nabla_v \cdot (vf + \nabla_v f).$$

The external magnetic field is of the form

$$B(t, x) = (b_1 \ b_2 \ b_3)^\top,$$

in the canonical euclidean basis of \mathbb{R}^3 . The magnetic field B is assumed to belong to $L^\infty(0, T; L_x^\infty)$ and to satisfy the Maxwell-Thompson equation $\nabla_x \cdot B = 0$. Its effect on the particle is given by the Lorentz force, appearing in the kinetic equations as $”(v \wedge B) \cdot \nabla_v”$. The Cauchy problem is completed with initial conditions on distribution functions

$$f_s^\varepsilon(t = 0, x, v) = f_s^{\text{in}}(x, v), \quad \forall s \in \{i, e\}.$$

The Poisson equation can be reformulated using its fundamental solution Φ , in dimension 3

$$\Phi(x) = \frac{1}{4\pi|x|}. \quad (1.4)$$

The potential ϕ^ε is then given by

$$\phi^\varepsilon(t, x) = \Phi *_x n_\varepsilon = \int_{\mathbb{R}^3} \Phi(x - y) n_\varepsilon(t, y) dy, \quad (1.5)$$

where n^ε denotes the total charge density and is given by

$$n^\varepsilon(t, x) = n_i^\varepsilon(t, x) - n_e^\varepsilon(t, x).$$

For later use, we also introduce the current density

$$j^\varepsilon(t, x) = j_i^\varepsilon(t, x) - j_e^\varepsilon(t, x), \text{ where } j_i = \int_{\mathbb{R}^3} v f_i^\varepsilon dv \text{ and } j_e = \frac{1}{\varepsilon} \int_{\mathbb{R}^3} v f_e^\varepsilon dv.$$

and the rescaled uniform Maxwellian, in dimension 3

$$M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}. \quad (1.6)$$

The existence of global weak solutions for the Vlasov-Poisson system, namely the first and the last equation of (1.3) with a given electron background n_e^ε , was first established by

Arsenev in [6]. DiPerna and Lions improved conditions on the initial data for this result in [66] and adapted their renormalization techniques for the Vlasov-Poisson system [68]. Concerning classical solutions, global existence in dimension 2 was proved by Ukai and Okabe for well-localized initial data in [187]. The case of dimension 3 was considered by Bardos and Degond in [11].

The theory for the Vlasov-Poisson-Fokker-Planck system, consisting of last two equations of (1.3) with a given ion background n_i^ε , is also well-known. The existence of global weak solutions is proved in [191, 41] and results on the existence and uniqueness of smooth solutions were first obtained by Degond in [59] and generalized by Bouchut in [28]. Here, we shall consider DiPerna-Lions renormalized solutions of the Vlasov-Fokker-Planck equation as introduced in [68]. Our choice is motivated by two observations. First, the presence of the friction part of the Fokker-Planck operator precludes any hope for a uniform (in time and in ε) $L_{x,v}^p$ bound of f_e^ε in the small ε limit when $p > 1$. As Poupaud and Soler proved in [172], a uniform in ε bound can be obtained for the adequate weighted L^p norm but only on a finite time decreasing with p , which is not quite satisfying. On the other hand, knowing merely that f_e^ε lies in $L_{x,v}^1$ and that $\nabla_x \phi^\varepsilon$ belongs to L_x^2 is insufficient to give even a distributional sense to the product $f_e^\varepsilon \nabla_x \phi^\varepsilon$ appearing in the electron equation of (1.3). The concept of renormalized solutions proposes to replace the direct consideration of the troublesome equation for f_e^ε by a family of meaningful equations for $\beta(f_e^\varepsilon)$, β ranging through a sufficiently large class of smooth functions.

For the coupled system (1.3), we consider weak solutions of the ion equation and renormalized solutions of the electron equation, the electric potential being given by (1.5).

Definition 1.1.1. We say that a triplet $(f_i^\varepsilon, f_e^\varepsilon, \phi^\varepsilon)$ is a solution on $[0, T)$ to the Cauchy problem (1.3) with initial data $f_i^{\text{in}}, f_e^{\text{in}}$ if it satisfies

- 1) $f_i^\varepsilon \in L^\infty(0, T; L_{x,v}^1 \cap L_{x,v}^\infty)$; $f_e^\varepsilon \in L^\infty(0, T; L_{x,v}^1)$; $f_s^\varepsilon \geq 0$ almost everywhere for $s \in \{i, e\}$,
- 2) $\phi^\varepsilon \in L^\infty(0, T; \dot{H}_x^1)$; $\nabla_v \sqrt{f_e^\varepsilon} \in L^2(0, T; L_{x,v}^2)$,

and, for every $\varphi \in \mathcal{D}([0, T) \times \mathbb{R}^6)$

- 3) The mapping $t \mapsto \iint_{\mathbb{R}^6} \varphi f_i^\varepsilon dv dx$ is continuous and the first equation of (1.3) holds in the sense of distributions on $[0, T) \times \mathbb{R}^6$ with the initial condition f_i^{in} .
- 4) For every function $\beta \in \mathcal{C}^2(\mathbb{R}_+)$ satisfying

$$|\beta(u)| \leq C(\sqrt{u} + 1), \quad |\sqrt{u}\beta'(u)| \leq C, \quad |u\beta''(u)| \leq C,$$

for some $C > 0$, the mapping $t \mapsto \iint_{\mathbb{R}^6} \varphi \beta(f_e^\varepsilon) dv dx$ is continuous and $\beta(f_e^\varepsilon)$ satisfies in the sense of distributions on $[0, T) \times \mathbb{R}^6$

$$\varepsilon \partial_t \beta(f_e^\varepsilon) + v \cdot \nabla_x \beta(f_e^\varepsilon) + \nabla_x \phi^\varepsilon \cdot \nabla_v \beta(f_e^\varepsilon) - \frac{1}{\varepsilon} (v \wedge B) \cdot \nabla_v \beta(f_e^\varepsilon) = \frac{1}{\varepsilon} \mathcal{L}_I(f_e^\varepsilon) \beta'(f_e^\varepsilon), \quad (1.7)$$

with the initial condition $\beta(f_e^{\text{in}})$.

- 5) For any $\lambda > 0$, $\theta_{\varepsilon,\lambda} = \sqrt{f_e^\varepsilon + \lambda M}$ is such that the mapping $t \mapsto \iint_{\mathbb{R}^6} \varphi \theta_{\varepsilon,\lambda} dv dx$ is continuous and $\theta_{\varepsilon,\lambda}$ satisfies in the sense of distributions on $[0, T) \times \mathbb{R}^6$

$$\varepsilon \partial_t \theta_{\varepsilon,\lambda} + v \cdot \nabla_x \theta_{\varepsilon,\lambda} + \nabla_x \phi^\varepsilon \cdot \nabla_v \theta_{\varepsilon,\lambda} = \frac{1}{2\varepsilon \theta_{\varepsilon,\lambda}} \mathcal{L}_A(f_e^\varepsilon) - \frac{\lambda M}{2\theta_{\varepsilon,\lambda}} v \cdot \nabla_x \phi^\varepsilon, \quad (1.8)$$

where the Fokker-Planck operator and the term related to the magnetic field are gathered within the operator \mathcal{L}_A which may be written in the following form

$$\mathcal{L}_A(f) = \nabla_v \cdot (A(t, x) v f + \nabla_v f) \quad (1.9)$$

where the matrix $A(t, x)$ is given by

$$Av = v + v \wedge B = \begin{pmatrix} 1 & b_3 & -b_2 \\ -b_3 & 1 & b_1 \\ b_2 & -b_1 & 1 \end{pmatrix} v. \quad (1.10)$$

Remark 1.1.2. *One readily checks that weak formulations of 3), 4) and 5) are consistent with estimates in 1) and 2). We stress that we need to use two types of renormalization for the electron Vlasov-Fokker-Planck equation. Actually, points 1) to 4) are sufficient to define a self-consistent notion of solution for which we can prove an existence result. The introduction of the additional equation (1.8) is inspired by [81]. It comes from a renormalization of the equation satisfied by f_e^ε/M with the function $s \mapsto \sqrt{s + \lambda}$. While equation (1.7) provides us with the required alternative meaning for the electron equation in (1.3), we shall pass to the limit $\varepsilon \rightarrow 0$ in (1.8).*

1.1.4 Main result

Our main goal is to prove the convergence of solutions to (1.3)–in the sense of Definition 1.1.1–towards weak solutions of the following coupled kinetic-fluid system

$$\begin{cases} \partial_t f_i + v \cdot \nabla_x f_i - \nabla_x \phi \cdot \nabla_v f_i + (v \wedge B) \cdot \nabla_v f_i = 0, \\ \partial_t n_e + \nabla_x \cdot j_e = 0, \\ j_e = -D(\nabla_x n_e - \nabla_x \phi n_e), \\ -\Delta_x \phi = n_i - n_e, \\ f_i(0, \cdot, \cdot) = f_i^{\text{in}} \text{ and } n_e(0, \cdot, \cdot) = \int f_e^{\text{in}} dv. \end{cases} \quad (1.11)$$

where the diffusion matrix is given by

$$D(t, x) = A^{-1}(t, x) = \frac{1}{1 + |B|^2} \begin{pmatrix} 1 + b_1^2 & -b_3 + b_1 b_2 & b_2 + b_1 b_3 \\ b_3 + b_1 b_2 & 1 + b_2^2 & -b_1 + b_2 b_3 \\ -b_2 + b_1 b_3 & b_1 + b_2 b_3 & 1 + b_3^2 \end{pmatrix}.$$

Since B is essentially bounded, it yields

$$D \in L^\infty(0, T; L_x^\infty).$$

Moreover by denoting I_3 the identity matrix in dimension 3, one sees that

$$\operatorname{Re}(D) = \frac{D + D^\top}{2} \geq \frac{1}{1 + |B|^2} I_3$$

is uniformly positive definite.

Remark 1.1.3. *Let us denote by $E = -\nabla_x \phi$ the electric field. In the strong magnetic field limit, namely $|B| \rightarrow \infty$, the particles follow the guiding center dynamic (see for example [164, 154, 19] in the physics literature). They are mainly advected along the magnetic field and transport in the perpendicular plane, called the electric drift, occurs at the next order in $1/|B|$. The corresponding parallel and drift velocities are respectively*

$$\mathbf{v}_\parallel = -\frac{(E \cdot B)B}{|B|^2}, \quad \mathbf{v}_{\text{drift}} = \frac{E \wedge B}{|B|^2}.$$

The guiding center dynamic can be recovered from the limit electron equation in (1.11). Indeed, note that applying the diffusion matrix to the electric field gives

$$\begin{aligned} DE &= E - \frac{1}{1 + |B|^2} E \wedge B + \frac{1}{1 + |B|^2} (E \wedge B) \wedge B, \\ &= \frac{1}{1 + |B|^2} [(E \cdot B)B - E \wedge B + E]. \end{aligned} \tag{1.12}$$

Therefore the limit equation for the electrons formally rewrites

$$\partial_t n_e + \nabla_x \cdot [(\mathbf{v}_\parallel + \mathbf{v}_{\text{drift}} + O(1/|B|^2))n_e - D\nabla_x n_e] = 0.$$

Definition 1.1.4. A triplet (f_i, n_e, ϕ) is called a weak solution on $[0, T)$ of the Cauchy problem (1.11) if it satisfies

- 1) $f_i \in L^\infty(0, T; L^1_{x,v} \cap L^\infty_{x,v})$; $f_i \geq 0$ almost everywhere,
- 2) $n_e \in L^\infty(0, T; L^1_x)$; $n_e \geq 0$ almost everywhere,
- 3) $\phi = \Phi *_x (\int f_i dv - n_e) \in L^\infty(0, T; \dot{H}^1_x)$ and $n_e \nabla_x \phi \in L^1_{\text{loc}}([0, T) \times \mathbb{R}^6)$,

and for every $\varphi \in \mathcal{D}([0, T) \times \mathbb{R}^6)$, $\psi \in \mathcal{D}([0, T) \times \mathbb{R}^3)$

- 4) The mappings $t \mapsto \iint_{\mathbb{R}^6} \varphi f_i dv dx$ and $t \mapsto \int_{\mathbb{R}^3} \psi n_e dx$ are continuous and (1.11) holds in the sense of distributions.

Let us state the main result of this chapter. We make the following assumptions on the initial data

$$\forall s \in \{i, e\}, f_s^{\text{in}} \geq 0 \text{ almost everywhere,} \tag{A1}$$

$$\sum_{s \in \{i, e\}} \iint_{\mathbb{R}^6} f_s^{\text{in}} (|x| + |v|^2 + |\ln f_s^{\text{in}}|) dv dx + \int_{\mathbb{R}^3} |\nabla_x \phi^{\text{in}}|^2 dx < +\infty, \tag{A2}$$

$$f_i^{\text{in}} \in L^1_{x,v} \cap L^\infty_{x,v}, \quad f_e^{\text{in}} \in L^1_{x,v}, \tag{A3}$$

$$\iint f_i^{\text{in}} dv dx = \iint f_e^{\text{in}} dv dx. \tag{A4}$$

where $\phi^{\text{in}} = \Phi *_x (\int f_i^{\text{in}} dv - \int f_e^{\text{in}} dv)$.

Remark 1.1.5. *The case of initial data depending on ε may be treated in the same way as soon as assumptions (A1) to (A4) are satisfied uniformly in ε . In this case one may pick as limit initial condition for (1.11) any accumulation point of the sequence of initial data.*

Theorem 1.1.6. *Under assumptions (A1) to (A4), there exists a solution $(f_i^\varepsilon, f_e^\varepsilon, \phi^\varepsilon)$ of (1.3) in the sense of Definition 1.1.1. Moreover, for any such solution, one has, when $\varepsilon \rightarrow 0$ and up to the extraction of a subsequence*

$$\begin{aligned} f_i^\varepsilon &\longrightarrow f_i && \text{weakly-}\star \text{ in } L^\infty([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3), \\ f_e^\varepsilon &\longrightarrow n_e M && \text{strongly in } L^1(0, T; L^1_{x,v}), \\ \phi^\varepsilon &\longrightarrow \phi && \text{strongly in } L^2(0, T; \dot{W}_x^{1,p}) \text{ for } 1 \leq p < 2, \end{aligned}$$

and the limit (f_i, n_e, ϕ) is a weak solution of (1.11).

The parabolic limit for the Vlasov-Poisson-Fokker-Planck system with a given ion background has been studied in dimension 2 by Poupaud-Soler in [172] and Goudon in [106]. Masmoudi and El Ghani generalized these results in any dimension in [81], using DiPerna-Lions renormalized solutions and averaging lemmas. The procedure they follow was introduced by Masmoudi and Tayeb to study the diffusion limit of a semi-conductor Boltzmann-Poisson system in [147]. Let us point out, here and later, the importance of the latter paper, which provides efficient tools to derive a global in time result for our own problem. A recent paper [195] of Wu, Lin and Liu treats with this method the case of a multispecies model where several Vlasov-Fokker-Planck equations are coupled by a Poisson equation on a bounded domain. Let us also mention [80], where the author deals with the case of a self-consistent magnetic field in the context of a Vlasov-Maxwell-Fokker-Planck system.

Here, we are considering a multispecies model with an external magnetic field. This brings some new technical difficulties in the analysis. First, the coupling with a non-collisional kinetic equation rather than a fixed background of charged particles makes it harder to recover regularity sufficient to give a distributional sense to the limit problem. Indeed, the regularity given by Vlasov-Poisson in dimension three for the ions impacts that of the electrons (see Lemma 1.6.1 and 1.6.2). Besides, by considering a magnetic field of the same order of magnitude than the collisions, it is not clear at all that one can control the fast Larmor oscillations at the limit especially in the three dimensional setting. As an example of this difficulty in the non-collisional case of Vlasov-Poisson, we refer to the paper of Saint-Raymond [177] where only the two-dimensional case can be treated since the dynamic along the magnetic field lines is too fast to be captured at the limit in this scaling. Here, we are able to control the whole dynamic thanks to the dissipative effect of the collisions and the orthogonality between the Lorentz force and the velocity field (see Lemma 1.3.3 and Proposition 1.4.3). An other proof of the nontrivial effect of the B field can be seen with the diffusion matrix of the limit drift-diffusion equation which is anisotropic, for it contains the magnetic field effects. Let us mention that such a model was derived in a linear setting (*i.e.* with an external electric field) by Ben Abdallah and El Hajj in [20] with a linear Boltzmann collision kernel.

In the rest of this chapter we detail the proof of Theorem 1.1.6. The outline is as follows. In section 1.2, we introduce natural estimates associated with (1.3). These estimates will be crucial to prove our derivation and the existence of solution which is briefly discussed in Section 1.3. After proving the required compactness of the family of solutions in Section 1.4, we will take limits in equations in Section 1.5. Eventually, in Section 1.6, we shall use the algebraic structure of the limit system to gain some regularity which will allow us to recover (1.11) in a distributional—rather than renormalized—sense.

1.2 A priori estimates

The study of the asymptotic $\varepsilon \rightarrow 0$ requires estimates that are uniform with respect to ε . For our coupled system, the only natural identities providing such bounds are mass estimates (see Lemma 1.2.2), free energy and entropy inequalities (see Proposition 1.2.1). Let us introduce kinetic energies associated with each species

$$K^\varepsilon(t) = K_i^\varepsilon(t) + K_e^\varepsilon(t) \text{ where } K_s = \frac{1}{2} \iint_{\mathbb{R}^6} |v|^2 f_s^\varepsilon dv dx \quad \forall s \in \{i, e\}.$$

The characteristic energy due to electrostatic effects is called electric energy and reads

$$U^\varepsilon(t) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x \phi^\varepsilon|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} \phi^\varepsilon n^\varepsilon dx,$$

where the last equality stems from the Poisson equation. Let us also define the entropy of each species

$$S_s^\varepsilon(t) = \iint_{\mathbb{R}^6} f_s^\varepsilon \ln(f_s^\varepsilon) dv dx \quad \forall s \in \{i, e\}.$$

The natural energy associated with Vlasov-Fokker-Planck type equations is called the free energy and writes, for our system

$$\mathcal{E}^\varepsilon(t) = U^\varepsilon(t) + K^\varepsilon(t) + S_e^\varepsilon(t).$$

We also introduce the free energy dissipation given by the following non-negative quantity

$$D^\varepsilon(t) = \frac{1}{\varepsilon^2} \iint_{\mathbb{R}^6} \frac{1}{f_e^\varepsilon} |v f_e^\varepsilon + \nabla_v f_e^\varepsilon|^2 dv dx = \frac{4}{\varepsilon^2} \iint_{\mathbb{R}^6} \left| \nabla_v \sqrt{\frac{f_e^\varepsilon}{M}} \right|^2 M dv dx$$

Proposition 1.2.1 (Free energy and entropy estimates). *Suppose that $(f_i^\varepsilon, f_e^\varepsilon, \phi^\varepsilon)$ is a smooth localized solution of (1.3). One has the following "entropy" estimates, for all $\varepsilon > 0$ and $t \in [0, T)$.*

— *The free energy satisfies*

$$\mathcal{E}^\varepsilon(t) + \int_0^t D^\varepsilon(s) ds = \mathcal{E}(0).$$

— The ion entropy satisfies

$$S_i^\varepsilon(t) = S_i^\varepsilon(0)$$

Proof. Multiplying the first two equations of (1.3) by $|v|^2/2$ and $|v|^2/(2\varepsilon)$ respectively, integrating in v, x and summing the two equations yields, up to an integration by parts,

$$\frac{d}{dt}K^\varepsilon + \int \nabla_x \phi^\varepsilon \cdot j^\varepsilon dx = -\frac{1}{\varepsilon^2} \iint v \cdot (v f_e^\varepsilon + \nabla_v f_e^\varepsilon) dv dx \quad (1.13)$$

The continuity equation is obtained by integrating the first two equations of (1.3) with respect to v, x and summing the resulting equations after dividing the electron equation by ε . It reads

$$\partial_t n^\varepsilon + \nabla_x \cdot j^\varepsilon = 0 \quad (1.14)$$

Then using (1.14) and the Poisson equation, we can rewrite the second term in (1.13)

$$\int \nabla_x \phi^\varepsilon \cdot j^\varepsilon dx = \frac{1}{2} \frac{d}{dt} \int |\nabla_x \phi^\varepsilon|^2 dx \quad (1.15)$$

Hence we get an energy estimate from (1.13) and (1.15), namely

$$\frac{d}{dt}(K^\varepsilon + U^\varepsilon) = -\frac{1}{\varepsilon^2} \iint v \cdot (v f_e^\varepsilon + \nabla_v f_e^\varepsilon) dv dx \quad (1.16)$$

The entropy equations are obtained by multiplying the first two equations of (1.3) by $\ln f_i^\varepsilon + 1$ and $\ln f_e^\varepsilon + 1$ respectively and integrating in x, v . It yields

$$\frac{d}{dt}S_i^\varepsilon = 0, \quad (1.17)$$

$$\varepsilon \frac{d}{dt}S_e^\varepsilon = -\frac{1}{\varepsilon} \iint \frac{\nabla_v f_e^\varepsilon}{f_e^\varepsilon} \cdot (v f_e^\varepsilon + \nabla_v f_e^\varepsilon) dv dx, \quad (1.18)$$

where we performed an integration by part of the right-hand side of the second equation. Equation (1.17) provides the ion entropy estimate. By summing (1.18) divided by ε and (1.16), we obtain the announced estimate up to an integration in time. \square

The other type of natural a priori estimate for Vlasov-type equations is the conservation of L^p norms.

Lemma 1.2.2 (L^p norms). *Suppose that $(f_i^\varepsilon, f_e^\varepsilon, \phi^\varepsilon)$ is a smooth localized solution of (1.3). For all $\varepsilon > 0$ and $t \in [0, T)$,*

— The distribution function of electrons satisfies

$$\|f_e^\varepsilon(t, \cdot, \cdot)\|_{L_{x,v}^1} = \|f_e^{in}\|_{L_{x,v}^1}$$

— The distribution function of ions satisfies

$$\|f_i^\varepsilon(t, \cdot, \cdot)\|_{L_{x,v}^p} = \|f_i^{in}\|_{L_{x,v}^p}, \quad \forall p \in [1, \infty]$$

Proof. Keeping in mind that distribution functions are positive, the integration of the Vlasov-Fokker-Planck equation in (1.3) with respect to x and v provides the first estimate. Let us multiply the ion equation of (1.3) by $(f_i^\varepsilon)^{p-1}/p$ and integrate in x and v to get

$$\frac{d}{dt} \|f_i^\varepsilon\|_{L_{x,v}^p}^p = 0.$$

Hence $\|f_i^\varepsilon\|_{L_{x,v}^p}$ is constant. Letting p go to infinity gives the limit case. \square

As we are working on an unbounded domain in space, we need to control space moments of the distribution functions to ensure that no mass can be "lost" at infinity. It reduces to controlling current densities as shows the following estimate.

Lemma 1.2.3 (First moment in space). *Suppose that $(f_i^\varepsilon, f_e^\varepsilon, \phi^\varepsilon)$ is a smooth localized solution of (1.3). For all $\varepsilon > 0$, $t \in [0, T)$ and $s \in \{i, e\}$*

$$\iint_{\mathbb{R}^6} |x| f_s^\varepsilon dv dx = \int_0^t \int_{\mathbb{R}^3} \frac{x}{|x|} \cdot j_s^\varepsilon dx + \iint_{\mathbb{R}^6} |x| f_s^{in} dv dx$$

Proof. Multiply the first two equations of (1.3) by $|x|$ and integrate in x , v and t to obtain the result. \square

1.3 Existence of solutions and uniform in ε estimates

In this section, we give an existence result for (1.3). The a priori estimates of the previous section are necessary to build these solutions, by a mollification procedure. Let us mention that this result follows from single-species cases. Indeed the coupling between the kinetic equations of (1.3) is weak in the sense that, because of the form of the Poisson equation, it is possible to isolate the contribution of each species in the electric field $\nabla_x \phi^\varepsilon$. The addition of the magnetic field term only cause minor and harmless modifications to usual proofs as it is linear and does not alter a priori estimates. For the Vlasov-Poisson part, the theory of Arsenev may be applied. We refer to [30, Theorem 1.3 and 1.4] for details. The Vlasov-Poisson-Fokker-Planck part of (1.3) may be handled with the DiPerna-Lions theory of renormalized solutions [66, 68]. Some technical details may also be found in [29] and [147, 152, 153] on bounded domains.

Proposition 1.3.1. *Under assumptions (A1) to (A4), the system (1.3) admits a solution in the sense of Definition 1.1.1. In particular, the solution satisfies the continuity equations in the sense of distributions on $[0, T) \times \mathbb{R}^6$*

$$\partial_t n_s^\varepsilon + \nabla_x \cdot j_s^\varepsilon = 0, \quad \forall s \in \{i, e\} \tag{1.19}$$

Moreover, the following estimates hold, uniformly in $t \in [0, T)$, $\varepsilon > 0$ and $s \in \{i, e\}$,

$$\mathcal{E}^\varepsilon(t) + \int_0^t D^\varepsilon(s) ds \leq \mathcal{E}^{in}, \quad (1.20)$$

$$S_i^\varepsilon(t) \leq S_i^{in}, \quad (1.21)$$

$$\|f_e^\varepsilon(t, \cdot, \cdot)\|_{L_{x,v}^1} \leq \|f_e^{in}\|_{L_{x,v}^1}, \quad (1.22)$$

$$\forall p \in [1, \infty], \quad \|f_e^\varepsilon(t, \cdot, \cdot)\|_{L_{x,v}^p} \leq \|f_e^{in}\|_{L_{x,v}^p}, \quad (1.23)$$

$$\iint_{\mathbb{R}^6} |x| f_s^\varepsilon dv dx \leq \int_0^t \int_{\mathbb{R}^3} \frac{x}{|x|} \cdot j_s^\varepsilon dx + \iint_{\mathbb{R}^6} |x| f_s^{in} dv dx. \quad (1.24)$$

and the distribution functions are non-negative almost everywhere.

From estimates obtained in Proposition 1.3.1 we infer uniform in ε estimates that will allow us to take limits in the following sections. Let us first give a name to the particular solutions satisfying these estimates.

Definition 1.3.2. Any triplet $(f_i^\varepsilon, f_e^\varepsilon, \phi^\varepsilon)$ which is a solution of the system (1.3) in the sense of Definition 1.1.1, associated with initial datum satisfying (A1) to (A4), and itself satisfying estimates of Proposition 1.3.1 is called from now on a *physical solution* of (1.3).

Proposition 1.3.3. A physical solution of (1.3) satisfies the following properties

(a) Control of current densities :

$$\|j_i^\varepsilon\|_{L_x^1} \leq C(T) + K_i^\varepsilon, \quad (1.25)$$

$$\|j_e^\varepsilon\|_{L_x^1} \leq C(T) + \frac{1}{2} D^\varepsilon. \quad (1.26)$$

(b) Uniform bounds on the free energy and moments :

$$\sum_{s \in \{i, e\}} \iint_{\mathbb{R}^6} f_s^\varepsilon (|x| + |v|^2 + |\ln f_s^\varepsilon|) dv dx + \int_{\mathbb{R}^3} |\nabla_x \phi^\varepsilon|^2 dx + \int_0^t D^\varepsilon(s) ds \leq C(T) \quad (1.27)$$

(c) Consequences of (b) :

$$\begin{aligned} & \|j_i^\varepsilon\|_{L^1(0, T; L_x^1)} + \|j_e^\varepsilon\|_{L^1(0, T; L_x^1)} + \|\nabla_v \sqrt{f_e^\varepsilon}\|_{L^2(0, T; L_{x,v}^2)} \\ & + \frac{1}{\varepsilon} \|(v \wedge B) \cdot \nabla_v f_e^\varepsilon\|_{L^1(0, T; L_{x,v}^1)} \leq C(T) \end{aligned} \quad (1.28)$$

for some constant $C(T)$ independent of ε and $t \in [0, T)$

Proof. (a) First we can control the ion current density by decomposing the velocity space in the following way,

$$\begin{aligned} \|j_i^\varepsilon\|_{L_x^1} & \leq \iint_{|v| < 2} |v| f_i^\varepsilon dv dx + \iint_{|v| \geq 2} |v| f_i^\varepsilon dv dx \\ & \leq 2 \|f_i^{in}\|_{L^\infty(0, T; L_{x,v}^1)} + \frac{1}{2} \iint_{\mathbb{R}^6} |v|^2 f_i^\varepsilon dv dx. \end{aligned}$$

We can conclude with (A2).

The electron current density can be controlled by the free energy dissipation. Indeed, following the idea in [172, Equation 2.21], it writes

$$j_e = \frac{1}{\varepsilon} \int (v\sqrt{f_e^\varepsilon} + 2\nabla_v\sqrt{f_e^\varepsilon})\sqrt{f_e^\varepsilon}dv$$

and thus

$$\|j_e^\varepsilon\|_{L_x^1} \leq \frac{1}{2}\|f_e^\varepsilon\|_{L_{x,v}^1} + \frac{1}{2\varepsilon^2} \iint \frac{1}{f_e^\varepsilon} |vf_e^\varepsilon + \nabla_v f_e^\varepsilon|^2 dvd x$$

and the desired result follows using (A2).

(b) The key arguments here are the entropy estimates (1.20) and (1.21). Since distribution functions are non-negative, we decompose the free energy in the following form

$$\mathcal{E}^\varepsilon = K^\varepsilon + U^\varepsilon + S_{e,+}^\varepsilon - S_{e,-}^\varepsilon, \quad (1.29)$$

where we define, for $s \in \{i, e\}$,

$$S_{s,+}^\varepsilon = \iint f_s^\varepsilon \ln^+ f_s^\varepsilon dvd x,$$

$$S_{s,-}^\varepsilon = \iint f_s^\varepsilon \ln^- f_s^\varepsilon dvd x,$$

with $\ln^+(s) = \max\{\ln(s), 0\}$ and $\ln^-(s) = \max\{-\ln(s), 0\}$ for $s > 0$. By applying the same arguments as in [172, Lemma 2.3] and estimate (1.24), one can get the following bound on the negative part of the entropy

$$S_{e,-}^\varepsilon \leq C + \frac{1}{2} (K_e^\varepsilon + \|j_e^\varepsilon\|_{L^1(0,t,L_x^1)}) + \frac{1}{2} \iint |x| f_e^{\text{in}} dvd x, \quad (1.30)$$

for some positive constant C . By inequality (1.26) on the electron current density and (1.30)

$$S_{e,-}^\varepsilon \leq \frac{1}{2} \left(K_e^\varepsilon + \int_0^t D^\varepsilon(s) ds \right) + C(T).$$

Now with estimate (1.20) and the decomposition of the free energy (1.29), one can conclude that K^ε , U^ε , $\int_0^t D^\varepsilon(s) ds$, $S_{e,-}^\varepsilon$ and $S_{e,+}^\varepsilon$ are uniformly bounded in ε and t . Replacing the index e by i , inequality (1.30) holds true for the ion related quantities and the bound (1.25) on the ion current density and estimate (1.21) on the ions entropy give the boundedness of $S_{i,-}^\varepsilon$ and $S_{i,+}^\varepsilon$.

(c) The estimates on current densities follow using (a) and (b). Now, as in [81, Corollary 5.3], the following computation

$$\begin{aligned} \|\nabla_v \sqrt{f_e^\varepsilon}\|_{L^2(0,T;L_{x,v}^2)}^2 &= \int_0^T \iint \left(\frac{1}{4} |v\sqrt{f_e^\varepsilon} + 2\nabla_v\sqrt{f_e^\varepsilon}|^2 - \frac{1}{4} |v|^2 f_e^\varepsilon - v\sqrt{f_e^\varepsilon} \cdot \nabla_v \sqrt{f_e^\varepsilon} \right) dvd x dt \\ &\leq \frac{\varepsilon^2}{4} D^\varepsilon + \frac{1}{2} \int_0^T \iint (\nabla_v \cdot v) f_e^\varepsilon dvd x dt \\ &\leq \frac{\varepsilon^2}{4} D^\varepsilon + \frac{3T}{2} \iint f_e^{\text{in}} dvd x, \end{aligned}$$

provides the third bound. Finally we can control the term related to the magnetic field by noticing that, since $(v \wedge B) \cdot v = 0$,

$$\begin{aligned} \frac{1}{\varepsilon} \|(v \wedge B) \cdot \nabla_v f_e^\varepsilon\|_{L^1(0,T;L^1_{x,v})} &= \frac{1}{\varepsilon} \int_0^T \iint \left| (v \wedge B) \cdot (2\nabla_v \sqrt{f_e^\varepsilon} + v\sqrt{f_e^\varepsilon})\sqrt{f_e^\varepsilon} \right| dx dv dt \\ &\leq \|B\|_{L^\infty_{t,x}} \left(\frac{1}{\varepsilon^2} \int_0^T D^\varepsilon dt + T \sup_{[0,T]} K_e^\varepsilon \right). \end{aligned}$$

We conclude with the estimates in (b). \square

Remark 1.3.4. *Actually, the estimate on the ion current density can be largely improved using a classical moment lemma [105, Lemma 3.1]. The latter gives that $(j_i^\varepsilon)_\varepsilon$ is uniformly bounded in $L^\infty(0, T; L_x^{5/4})$.*

1.4 Compactness of the family of solutions

From the estimates of the foregoing section, we can infer some compactness. When ε tends to 0, estimates (1.23) and (1.27) give, up to the extraction of a subsequence,

$$f_i^\varepsilon \rightarrow f_i \text{ weakly-}\star \text{ in } L^\infty(0, T; L^p(\mathbb{R}^6)) \text{ for } p \in (1, \infty], \quad (1.31)$$

and

$$\nabla_x \phi^\varepsilon \rightarrow \nabla_x \phi \text{ weakly-}\star \text{ in } L^\infty(0, T; L^2(\mathbb{R}^3)), \quad (1.32)$$

where the limit is a gradient because it is irrotational in the sense of distributions.

Lemma 1.4.1. *Families of physical solutions of (1.3) satisfy the following properties*

- (a) $(f_i^\varepsilon)_\varepsilon$ and $(f_e^\varepsilon)_\varepsilon$ are weakly relatively compact in $L^1([0, T] \times \mathbb{R}^6)$.
- (b) $(n_i^\varepsilon)_\varepsilon$ and $(n_e^\varepsilon)_\varepsilon$ are weakly relatively compact in $L^1([0, T] \times \mathbb{R}^3)$.
- (c) $f_e^\varepsilon - n_e^\varepsilon M \rightarrow 0$ in $L^1([0, T] \times \mathbb{R}^6)$ when $\varepsilon \rightarrow 0$

Proof. The first two assertions follow from the Dunford-Pettis theorem.

(a, b) Let us define $W : x \mapsto e^{-|x|}/(8\pi)$, $\mathcal{M} : (x, v) \mapsto W(x)M(v)$ and $\Psi : u \geq 0 \mapsto u(\ln^+ u + 1)$. Note that W and \mathcal{M} are of integral 1 and Ψ is a convex, non-negative, increasing function satisfying

$$\lim_{u \rightarrow \infty} \frac{\Psi(u)}{u} = +\infty.$$

Let $s \in \{i, e\}$. One sees that by the Jensen inequality

$$\begin{aligned} \int \Psi\left(\frac{n_s^\varepsilon}{W}\right) W dx &= \int \Psi\left(\int \frac{f_s^\varepsilon}{\mathcal{M}} M dv\right) W dx \\ &\leq \iint \Psi\left(\frac{f_s^\varepsilon}{\mathcal{M}}\right) \mathcal{M} dv dx \\ &= \iint f_s^\varepsilon \ln\left(\frac{f_s^\varepsilon}{\mathcal{M}}\right) dv dx \\ &\leq S_{s,+}^\varepsilon + S_{s,-}^\varepsilon + K_s^\varepsilon + \iint |x| f_s^\varepsilon dv dx + C \iint f_s^\varepsilon dv dx, \end{aligned}$$

for some constant $C > 0$. Because of the uniform estimates (1.23) and (1.27), we hence get by the de la Vallée Poussin lemma the equi-integrability of the bounded families

$$\{f_s^\varepsilon/\mathcal{M}\}_\varepsilon \subset L^1(0, T; L^1(\mathcal{M}dvdx))$$

and

$$\{n_s^\varepsilon/W\}_\varepsilon \subset L^1(0, T; L^1(Wdx)).$$

This yields the announced weak compactness by the Dunford-Pettis theorem.

(c) Using the log-Sobolev inequality (see [112]), we get an upper bound for the relative entropy of the electrons with respect to a local Maxwellian

$$\begin{aligned} \int_0^t \iint f_e^\varepsilon \ln \left(\frac{f_e^\varepsilon}{n^\varepsilon M} \right) dv dx ds &\leq 2 \int_0^t \iint \left| \nabla_v \sqrt{\frac{f_e^\varepsilon}{M}} \right|^2 M dv dx ds \\ &\leq C(T)\varepsilon^2, \end{aligned}$$

where the last inequality comes from estimate (1.27). By the Csiszar-Kullback-Pinsker inequality (see [55, Theorem 3.1 and Section 4]), we have

$$\begin{aligned} \int_0^t \left(\iint |f_e^\varepsilon - n_e^\varepsilon M| dv dx \right)^2 ds &\leq 2 \sup_{[0, T)} \left(\int n_e^\varepsilon dx \right) \int_0^t \iint f_e^\varepsilon \ln \left(\frac{f_e^\varepsilon}{n_e^\varepsilon M} \right) dv dx ds \\ &\leq C(T)\varepsilon^2, \end{aligned}$$

which yields the $L^2(0, T; L_{x,v}^1)$ convergence hence the expected result. \square

The results of Lemma 1.4.1 are not sufficient to take limits in equations. We need to gain strong compactness to deal with non-linear terms. While there is no hope of doing so with the distribution functions, the particular averaging properties of Vlasov type equations allows us to get additional compactness results in space on the macroscopic densities. Indeed, by the dispersion property of the $v \cdot \nabla_x$ transport operator, one gains regularity on velocity averages of the distribution function. Besides, time compactness stems from the continuity equations (1.19) and the uniform bounds on current densities. We shall apply the following averaging lemma that is adapted from the famous DiPerna, Lions and Meyer results in [70].

Lemma 1.4.2. *Let $(h^\varepsilon)_\varepsilon$ be a bounded sequence in $L^2(0, T; L_x^2(L_{loc,v}^2))$ satisfying in the sense of distributions*

$$\varepsilon \partial_t h^\varepsilon + v \cdot \nabla_x h^\varepsilon = h_0^\varepsilon + \nabla_v \cdot h_1^\varepsilon$$

where $(h_0^\varepsilon)_\varepsilon, (h_1^\varepsilon)_\varepsilon$ are bounded sequences in $L^1(0, T; L_x^1(L_{loc,v}^1))$. Then for all $\psi \in \mathcal{D}(\mathbb{R}^3)$,

$$\left\| \int (\tau_y h^\varepsilon - h^\varepsilon) \psi dv \right\|_{L^1(0, T; L_x^1)} \rightarrow 0$$

when $y \rightarrow 0$ uniformly in ε , where τ_y is the translation of vector y in the x variable.

Proof. We refer to [147, Appendix 2] for a proof. \square

Let us highlight that the following result is crucial to the rest of the proof of Theorem 1.1.6 and that in particular, point (b) in Proposition 1.4.3 depends upon the control of the magnetic leading term of the equation by the entropy dissipation.

Proposition 1.4.3. *Families of physical solutions of (1.3) satisfy the following strong compactness properties*

- (a) $(\nabla_x \phi^\varepsilon)_\varepsilon$ is relatively compact in $L^2(0, T; L^p_{loc,x})$ for $1 \leq p < 2$,
- (b) $(n^\varepsilon)_\varepsilon$ is relatively compact in $L^1([0, T] \times \mathbb{R}^3)$.

Proof. The result of (a) stems from applying the Aubin-Lions-Simon lemma [182, Theorem 5, Corollary 4] to the family of electric fields and the proof can be readily adapted from [147, Proposition 3.3 3)].

(b) Let us define

$$\beta_\delta : u \mapsto \frac{u}{1 + \delta u}.$$

There exists $C_\delta > 0$ such that, for all $u \geq 0$

$$\beta_\delta(u) \leq \min\left(\frac{1}{\delta}, u\right), \quad (1 + \sqrt{u})|\beta'_\delta(u)| \leq 1 + \frac{1}{\sqrt{\delta}}, \quad |u\beta''_\delta(u)| \leq \frac{1}{2}.$$

In particular, one checks that β_δ satisfies the requirements of a renormalization function for the Vlasov-Fokker-Planck equation and hence (1.7) holds. Now set

$$\begin{cases} h^\varepsilon = \beta_\delta(f^\varepsilon) \\ h^\varepsilon_0 = -\nabla_v f^\varepsilon \cdot \frac{v f^\varepsilon + \nabla_v f^\varepsilon}{\varepsilon} \beta''_\delta(f^\varepsilon) + \frac{1}{\varepsilon} (v \wedge B) \cdot \nabla_v (\beta_\delta(f^\varepsilon)) \\ h^\varepsilon_1 = -\nabla_x \phi^\varepsilon \beta_\delta(f^\varepsilon) + \frac{v f^\varepsilon + \nabla_v f^\varepsilon}{\varepsilon} \beta'_\delta(f^\varepsilon) \end{cases}$$

Let us check the hypotheses of Lemma 1.4.2. Clearly, we have,

$$(h^\varepsilon)_\varepsilon \text{ is bounded in } L^\infty(0, T; L^1_{x,v} \cap L^\infty_{x,v}) \hookrightarrow L^2(0, T; L^2_{x,v}).$$

Moreover,

$$\left\| \nabla_v f^\varepsilon \cdot \frac{v f^\varepsilon + \nabla_v f^\varepsilon}{\varepsilon} \beta''_\delta(f^\varepsilon) \right\|_{L^1(0, T; L^1_{x,v})} \leq \left\| \nabla_v \sqrt{f^\varepsilon} \right\|_{L^2} \left\| \frac{v f^\varepsilon + \nabla_v f^\varepsilon}{\varepsilon \sqrt{f^\varepsilon}} \right\|_{L^2} \left\| f^\varepsilon \beta''_\delta(f^\varepsilon) \right\|_{L^\infty},$$

and, using that $(v \wedge B) \cdot v = 0$, we have

$$\begin{aligned} & \left\| \frac{1}{\varepsilon} (v \wedge B) \cdot \nabla_v (\beta_\delta(f^\varepsilon)) \right\|_{L^1(0, T; L^1_{x,v})} \\ &= \left\| (v \wedge B) \sqrt{f^\varepsilon} \cdot \frac{1}{\varepsilon} (2\nabla_v \sqrt{f^\varepsilon} + v \sqrt{f^\varepsilon}) \beta'_\delta(f^\varepsilon) \right\|_{L^1(0, T; L^1_{x,v})} \\ &\leq \sqrt{T} \|B\|_{L^\infty(0, T; L^\infty_x)} \| |v|^2 f^\varepsilon \|_{L^\infty(0, T; L^1_{x,v})} \left\| \frac{v f^\varepsilon + \nabla_v f^\varepsilon}{\varepsilon \sqrt{f^\varepsilon}} \right\|_{L^2(0, T; L^2_{x,v})} \left\| \beta'_\delta(f^\varepsilon) \right\|_{L^\infty(0, T; L^\infty_{x,v})}. \end{aligned}$$

Finally, if K is a compact measurable set of \mathbb{R}^3 , then

$$\|\nabla_x \phi^\varepsilon \beta_\delta(f_e^\varepsilon)\|_{L^1(0,T;L_x^1(L^1(K,dv)))} \leq T \sqrt{\text{meas}(K)} \|\nabla_x \phi^\varepsilon\|_{L_t^\infty L_x^2} \|\beta_\delta(f_e^\varepsilon)\|_{L_t^\infty L_{x,v}^2},$$

and,

$$\left\| \frac{v f_e^\varepsilon + \nabla_v f_e^\varepsilon}{\varepsilon} \beta'_\delta(f_e^\varepsilon) \right\|_{L^1(0,T;L_{x,v}^1)} \leq \sqrt{T} \left\| \frac{v f_e^\varepsilon + \nabla_v f_e^\varepsilon}{\varepsilon \sqrt{f_e^\varepsilon}} \right\|_{L^2} \|f_e\|_{L_t^\infty L_{x,v}^1} \|\beta'_\delta(f_e^\varepsilon)\|_{L^\infty}.$$

Thus, the sequences $(h^\varepsilon)_\varepsilon$, $(h_0^\varepsilon)_\varepsilon$ and $(h_1^\varepsilon)_\varepsilon$ satisfy the hypotheses of Lemma 1.4.2. Then for any fixed δ and any $\psi \in \mathcal{D}(\mathbb{R}^3)$

$$\left\| \int (\tau_y \beta_\delta(f_e^\varepsilon) - \beta_\delta(f_e^\varepsilon)) \psi dv \right\|_{L^1(0,T;L_x^1)} \rightarrow 0 \quad (1.33)$$

when $y \rightarrow 0$ uniformly in ε . The $L^\infty(0,T;L^1((1+|v|^2)dvdv))$ uniform bound on f_e^ε from (1.22) and (1.27) allows us to extend this to $\psi(v) \equiv 1$. Now, we can also take the limit $\delta \rightarrow 0$ uniformly in ε , using the equi-integrability of the family $(f_e^\varepsilon)_\varepsilon$. Indeed, since, for any $u > 0$

$$0 \leq \beta_\delta(u) \leq u \quad \text{and} \quad |\beta_\delta(u) - u| \leq \delta u^2,$$

one has

$$\|f_e^\varepsilon - \beta_\delta(f_e^\varepsilon)\|_{L^1([0,T] \times \mathbb{R}^6)} \leq 2 \int_0^T \iint_{f_e^\varepsilon > M} f_e^\varepsilon dv dx dt + \delta \int_0^T \iint_{\{f_e^\varepsilon < M\}} |f_e^\varepsilon|^2 dv dx dt \quad (1.34)$$

for an arbitrary $M > 0$. The first term is $O(1/\ln(|M|))$ because of the uniform bound on the entropy S_e^ε . Using the uniform bound on the mass, the second term is seen to be $O(\delta M)$. Take $M = \delta^{-1/2}$ to conclude. Therefore, using (1.33) and (1.34), one has

$$\|\tau_y n_e^\varepsilon - n_e^\varepsilon\|_{L^1(0,T;L_x^1)} \rightarrow 0$$

when $y \rightarrow 0$ uniformly in ε .

Finally, using the continuity equation (1.19), we get a $L^1(0,T;W_x^{-1,1})$ bound on $\partial_t n_e^\varepsilon$. With the uniform $L^1((1+|x|)dxdt)$ estimate (1.24) on n_e^ε , this gives the relative compactness of the sequence using Lemma C.3.3. \square

Using the results of Lemma 1.4.1 and Proposition 1.4.3, we get the following strong convergence results concerning the macroscopic density of electrons.

Lemma 1.4.4. *The families of physical solutions of (1.3) satisfy the following properties, up to the extraction of a subsequence. There exists $n_e \in L^1([0,T] \times \mathbb{R}^3)$ such that when $\varepsilon \rightarrow 0$*

- (a) $n_e^\varepsilon \rightarrow n_e$ in $L^1(0,T;L_x^1)$,
- (b) $\sqrt{n_e^\varepsilon} \rightarrow \sqrt{n_e}$ in $L^2(0,T;L_x^2)$,

- (c) $f_e^\varepsilon \rightarrow n_e M$ in $L^1(0, T; L^1_{x,v})$,
 (d) $\theta_{\varepsilon, \lambda} = \sqrt{f_e^\varepsilon + \lambda M} \rightarrow \sqrt{n_e + \lambda} \sqrt{M}$ in $L^2(0, T; L^2_{x,v})$,
 (e) $\frac{M}{\theta_{\varepsilon, \lambda}} \rightarrow \frac{\sqrt{M}}{\sqrt{n_e + \lambda}}$ in $L^2(0, T; L^2_{x,v})$,
 (f) $\frac{\sqrt{n_e^\varepsilon} M}{\theta_{\varepsilon, \lambda}} \rightarrow \frac{\sqrt{n_e} M}{\sqrt{n_e + \lambda}}$ in $L^2(0, T; L^2_{x,v})$,

for any $\lambda > 0$.

Proof. Properties (a, b, c, d) are straightforward consequences of Lemma 1.4.1 (c) and Proposition 1.4.3 (b).

(e) Using the definition of $\theta_{\varepsilon, \lambda}$, it yields that,

$$\begin{aligned} \left| \frac{M}{\theta_{\varepsilon, \lambda}} - \frac{\sqrt{M}}{\sqrt{n_e + \lambda}} \right| &\leq \frac{\sqrt{M}}{\theta_{\varepsilon, \lambda} \sqrt{n_e + \lambda}} \left| \sqrt{(n_e + \lambda)M} - \theta_{\varepsilon, \lambda} \right| \\ &\leq \frac{1}{\lambda} \left| \sqrt{(n_e + \lambda)M} - \theta_{\varepsilon, \lambda} \right|. \end{aligned}$$

Then one concludes with (d).

(f) Just notice that,

$$\begin{aligned} &\sqrt{M} \left| \frac{\sqrt{n_e^\varepsilon} M}{\theta_{\varepsilon, \lambda}} - \frac{\sqrt{n_e} M}{\sqrt{n_e + \lambda}} \right| \\ &\leq \frac{\sqrt{M}}{\theta_{\varepsilon, \lambda}} \left| \sqrt{n_e^\varepsilon} M - \sqrt{n_e} M \right| + \frac{\sqrt{n_e} M}{\theta_{\varepsilon, \lambda} \sqrt{n_e + \lambda}} \left| \sqrt{(n_e + \lambda)M} - \theta_{\varepsilon, \lambda} \right| \\ &\leq \frac{1}{\sqrt{\lambda}} \left(\left| \sqrt{n_e^\varepsilon} M - \sqrt{n_e} M \right| + \left| \sqrt{(n_e + \lambda)M} - \theta_{\varepsilon, \lambda} \right| \right), \end{aligned}$$

and conclude with (b) and (d). □

1.5 Taking limits in equation (1.8)

Using the previous compactness results, we can readily take limits in weak formulations of the Vlasov equation for ions in (1.3), the continuity equation for electrons (1.19), and the Poisson equation in (1.3).

Lemma 1.5.1. *The limits $n_e \in L^1([0, T] \times \mathbb{R}^3)$, $f_i \in L^\infty(0, T; L^1_{x,v} \cap L^\infty_{x,v})$ and $\nabla_x \phi \in L^\infty(0, T; L^2_x)$ defined respectively in Lemma 1.4.4, (1.31) and (1.32) satisfy in the sense of distributions,*

$$\begin{cases} \partial_t f_i + v \cdot \nabla_x f_i - \nabla_x \phi \cdot \nabla_v f_i + (v \wedge B) \cdot \nabla_v f_i = 0, \\ \partial_t n_e + \nabla_x \cdot j_e = 0, \\ -\Delta_x \phi = n_i - n_e, \\ n_s = \int f_s dv, \quad \forall s \in \{i, e\}, \end{cases}$$

for any accumulation point j_e of the family $(j_e^\varepsilon)_\varepsilon$.

At this point, we do not know much on limits³¹ of the electron current density j_e^ε . Our goal is now to characterize them. Let us introduce,

$$r_e^\varepsilon = \frac{1}{\varepsilon\sqrt{M}}(\sqrt{f_e^\varepsilon} - \sqrt{n_e^\varepsilon M}), \quad (1.35)$$

and

$$R_e^\varepsilon = \frac{1}{2\varepsilon}(f_e^\varepsilon - n_e^\varepsilon M) \quad (1.36)$$

so that the distribution function of electrons may be written as

$$\begin{aligned} f_e^\varepsilon &= n_e^\varepsilon M + 2\varepsilon r_e^\varepsilon \sqrt{n_e^\varepsilon M} + \varepsilon^2 |r_e^\varepsilon|^2 M \\ &= n_e^\varepsilon M + 2\varepsilon R_e^\varepsilon. \end{aligned} \quad (1.37)$$

With this in hands, one writes the electron current density as

$$j_e^\varepsilon = 2\sqrt{n_e^\varepsilon} \int r_e^\varepsilon M v dv + \varepsilon \int |r_e^\varepsilon|^2 M v dv$$

As in [81, 80], we aim at taking the limit $\varepsilon \rightarrow 0$ on the latter and characterize the limit current. We first gather some useful estimates on r_e^ε .

Proposition 1.5.2. *Let $(f_e^\varepsilon)_\varepsilon$ be the electron distribution functions of a family of physical solutions of (1.3) and define r_e^ε by (1.35). Then, the following uniform estimates hold*

- (a) $(r_e^\varepsilon)_\varepsilon$ is bounded in $L^2(0, T; L^2(M dv dx))$,
- (b) $(\varepsilon |r_e^\varepsilon|^2 |v|^2 M)_\varepsilon$ is bounded in $L^1(0, T; L^1_{x,v})$,
- (c) $(\sqrt{\varepsilon} |r_e^\varepsilon|^2 |v| M)_\varepsilon$ is bounded in $L^1(0, T; L^1_{x,v})$,
- (d) $(\nabla_v r_e^\varepsilon)_\varepsilon$ is bounded in $L^2(0, T; L^2(M dv dx))$.

Proof. ((a, b, c) are adapted from [81, Proposition 5.5])

We begin with some elementary preliminaries. Let us define $h : x \mapsto (1+x) \ln(1+x) - x$ for $x > -1$. The Legendre transform of h , defined by $x^* \mapsto h^*(x^*) = \sup\{xx^* - h(x), x > -1\}$, is given here by $h^*(x^*) = e^{x^*} - x^* - 1$ for $x^* \in \mathbb{R}$. By definition, the Young inequality holds, $xx^* \leq h(x) + h^*(x^*)$.

Moreover it follows from a convexity argument in the variable $\sqrt{1+x}$ that h is bounded from below by $g : x \mapsto (\sqrt{x+1} - 1)^2$. Additionally a direct study shows that for $x \geq -1$, $h(|x|) \leq h(x)$.

Notice finally that one readily obtain from the power expansion of h^* that $h^*(\lambda x^*) \leq \lambda^2 h^*(x^*)$ for $0 \leq \lambda \leq 1$ and $x^* \geq 0$.

(a) One has,

$$|r_e^\varepsilon|^2 M = \frac{n_e^\varepsilon M}{\varepsilon^2} g\left(\frac{f_e^\varepsilon}{n_e^\varepsilon M} - 1\right) \leq \frac{n_e^\varepsilon M}{\varepsilon^2} h\left(\frac{f_e^\varepsilon}{n_e^\varepsilon M} - 1\right).$$

31. From estimate 1.3.3, we only know *a priori* that, up to the extraction of a subsequence, $(j_e^\varepsilon)_\varepsilon$ converges to a finite Radon measure for the weak star topology.

Integrating in v yields,

$$\int |r_e^\varepsilon|^2 M dv \leq \frac{1}{\varepsilon^2} \int n_e^\varepsilon M h \left(\frac{f_e^\varepsilon}{n_e^\varepsilon M} - 1 \right) dv = \frac{1}{\varepsilon^2} \int f_e^\varepsilon \ln \left(\frac{f_e^\varepsilon}{n_e^\varepsilon M} \right) dv.$$

Then after an integration in x, t , we can use the log-Sobolev inequality (see Corollary C.2.2 in Appendix C) and estimate (1.27), to get a uniform bound on $(r_e^\varepsilon)_\varepsilon$ in $L^2(0, T; L^2(M dv dx))$.

(b) Now we use the Young inequality and other properties of h and h^* to obtain, for ε sufficiently small,

$$\begin{aligned} \varepsilon |r_e^\varepsilon|^2 |v|^2 &= \frac{n_e^\varepsilon}{\varepsilon} \left| \sqrt{\frac{f_e^\varepsilon}{n_e^\varepsilon M}} - 1 \right|^2 |v|^2 \\ &\leq \frac{n_e^\varepsilon}{\varepsilon^2} \left| \frac{f_e^\varepsilon}{n_e^\varepsilon M} - 1 \right| \varepsilon |v|^2 \\ &\leq \frac{n_e^\varepsilon}{\varepsilon^2} \left[h \left(\left| \frac{f_e^\varepsilon}{n_e^\varepsilon M} - 1 \right| \right) + h^* (\varepsilon |v|^2) \right] \\ &\leq \frac{n_e^\varepsilon}{\varepsilon^2} \left[h \left(\frac{f_e^\varepsilon}{n_e^\varepsilon M} - 1 \right) + 16\varepsilon^2 h^* \left(\frac{|v|^2}{4} \right) \right]. \end{aligned}$$

After a multiplication by M and an integration in v , one can bound the first term as in (a) and the second one thanks to mass estimate (1.22).

(c) In order to establish the third bound, we apply Young inequality,

$$\sqrt{\varepsilon} |v| |r_e^\varepsilon|^2 \leq \frac{1}{2} \varepsilon |v|^2 |r_e^\varepsilon|^2 + \frac{1}{2} |r_e^\varepsilon|^2,$$

and conclude with (a) and (b).

(d) From the definition of r_e^ε stems

$$\int_0^T \iint |\nabla_v r_e^\varepsilon|^2 M dv dx dt = \frac{1}{\varepsilon^2} \int_0^T \iint \left| \nabla_v \sqrt{\frac{f_e^\varepsilon}{M}} \right|^2 M dv dx dt.$$

The right-hand side is uniformly bounded thanks to estimate (1.27) on the entropy dissipation. □

From now on we consider a subsequence of a family of physical solutions of (1.3) such that all convergence properties following from the previous compactness results hold. Let us denote by r_e the weak $L^2(0, T; L^2(M dv dx))$ limit of $(r_e^\varepsilon)_\varepsilon$.

Lemma 1.5.3. *When $\varepsilon \rightarrow 0$, the family $(j_e^\varepsilon)_\varepsilon$ satisfies*

$$j_e^\varepsilon \rightarrow j_e := 2\sqrt{n_e} \int r_e v M dv \text{ weakly in } L^1(0, T; L_x^1).$$

Proof. Take the limit in the following expression, coming from (1.37),

$$j_e^\varepsilon = 2\sqrt{n_e^\varepsilon} \int vr_e^\varepsilon M dv + \varepsilon \int v|r_e^\varepsilon|^2 M dv. \quad (1.38)$$

The first term in the right-hand side converges weakly in $L^1(0, T; L^1_{x,v})$. Indeed,

$$\sqrt{n_e^\varepsilon} \rightarrow \sqrt{n_e} \text{ strongly in } L^2(0, T; L^2_x),$$

and since $r_e^\varepsilon \rightarrow r_e$ weakly in $L^2(0, T; L^2(Mdvdx))$ and $(t, v) \mapsto v \in L^2(0, T; L^2(Mdv))$

$$\int vr_e^\varepsilon M dv \rightarrow \int vr_e M dv \text{ weakly in } L^2(0, T; L^2_x).$$

The second term in the right-hand side of (1.38) goes to 0 in $L^1(0, T; L^1_x)$ since we know from Proposition 1.5.2 (c) that $(\sqrt{\varepsilon}|r_e^\varepsilon|^2|v|M)_\varepsilon$ is bounded in $L^1(0, T; L^1_{x,v})$. \square

Now we focus on the limit of equation (1.8) from Definition 1.1.1 to derive the last pieces of information we need to characterize the limit of the current density j_e . To do so, we introduce elements of notation associated with the operator \mathcal{L}_A . We denote by L^2_M the Hilbert space $L^2(\mathbb{R}^3, M^{-1}dv)$ endowed with the scalar product

$$\langle f, g \rangle = \int_{\mathbb{R}^3} fgM^{-1}dv.$$

Almost everywhere in t, x ,

$$\mathcal{L}_A : f \mapsto \nabla_v \cdot (A(t, x)vf + \nabla_v f) = \nabla_v \cdot (vf + \nabla_v f) + (v \wedge B(t, x)) \cdot \nabla_v f$$

is an unbounded operator on L^2_M . On

$$\mathcal{L}_A(f) = \nabla_v \cdot \left(M \nabla_v \left(\frac{f}{M} \right) \right) + (v \wedge B) \cdot \nabla_v \left(\frac{f}{\sqrt{M}} \right) \sqrt{M}, \quad (1.39)$$

one sees that the formal adjoint of \mathcal{L}_A in L^2_M is given by

$$\mathcal{L}_A^*(f) = \nabla_v \cdot (vf + \nabla_v f) - (v \wedge B) \cdot \nabla_v f = \nabla_v \cdot (A^\top vf + \nabla_v f) = \mathcal{L}_{A^\top}(f). \quad (1.40)$$

Proposition 1.5.4. *The limit density n_e and electric field $-\nabla_x \phi$ are such that $(\nabla_x \sqrt{n_e} - \frac{1}{2} \nabla_x \phi \sqrt{n_e}) \in L^2(0, T; L^2_x)$. Moreover the limit electron current density j_e satisfies in the sense of distributions*

$$j_e = -2\sqrt{n_e} A^{-1} (\nabla_x \sqrt{n_e} - \frac{1}{2} \nabla_x \phi \sqrt{n_e}),$$

Proof. We know from Lemma 1.5.3 that

$$j_e = 2\sqrt{n_e} \int r_e v M dv.$$

We will now take the limit in the renormalized equation (1.8) in order to get an additional equation characterizing r_e . Let us recall that (1.8) reads

$$\varepsilon \partial_t \theta_{\varepsilon, \lambda} + v \cdot \nabla_x \theta_{\varepsilon, \lambda} + \nabla_x \phi^\varepsilon \cdot \nabla_v \theta_{\varepsilon, \lambda} = \frac{1}{2\varepsilon \theta_{\varepsilon, \lambda}} \mathcal{L}_A(f_e^\varepsilon) - \frac{\lambda M}{2\theta_{\varepsilon, \lambda}} v \cdot \nabla_x \phi^\varepsilon.$$

By the strong convergence of $\theta_{\varepsilon, \lambda}$ from Lemma 1.4.4 (d) and the weak convergence of $\nabla_x \phi^\varepsilon$ from (1.32), the left-hand side of equation (1.8) converges to

$$\begin{aligned} & \nabla_x \cdot (v \sqrt{(n_e + \lambda)M}) + \nabla_v \cdot (\nabla_x \phi \sqrt{(n_e + \lambda)M}) \\ & = v \sqrt{M} \cdot (\nabla_x \sqrt{n_e + \lambda} - \frac{1}{2} \nabla_x \phi \sqrt{n_e + \lambda}). \end{aligned}$$

The first term of the right-hand side of the renormalized equation (1.8) may be written, using (1.36) and the fact that $M \in \text{Ker} \mathcal{L}_A$, as

$$\frac{1}{2\varepsilon \theta_{\varepsilon, \lambda}} \mathcal{L}_A(f_e^\varepsilon) = \frac{1}{\theta_{\varepsilon, \lambda}} \mathcal{L}_A(R_e^\varepsilon). \quad (1.41)$$

Now, for any $\varphi \in \mathcal{D}([0, T] \times \mathbb{R}^6)$, one has

$$I^\varepsilon = \int_0^T \iint \frac{1}{\theta_{\varepsilon, \lambda}} \mathcal{L}_A(R_e^\varepsilon) \varphi dv dx dt := I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon.$$

with

$$\begin{aligned} I_1^\varepsilon &:= \int_0^T \iint \frac{\varphi M}{\theta_{\varepsilon, \lambda}} (v \wedge B) \cdot \nabla_v \left(\frac{R_e^\varepsilon}{M} \right) dv dx dt, \\ I_2^\varepsilon &:= - \int_0^T \iint \frac{\sqrt{M}}{\theta_{\varepsilon, \lambda}} \nabla_v \left(\frac{R_e^\varepsilon}{M} \right) \cdot \nabla_v \left(\frac{\varphi}{\sqrt{M}} \right) M dv dx dt, \\ I_3^\varepsilon &:= - \int_0^T \iint \nabla_v \left(\frac{R_e^\varepsilon}{M} \right) \cdot \nabla_v \left(\frac{\sqrt{M}}{\theta_{\varepsilon, \lambda}} \right) \varphi \sqrt{M} dv dx dt. \end{aligned}$$

where we have performed an integration by part on the Fokker-Planck part of the operator \mathcal{L}_A defined in (1.39) to obtain I_2^ε and I_3^ε . Mark that

$$\nabla_v \left(\frac{\sqrt{M}}{\theta_{\varepsilon, \lambda}} \right) = -\frac{1}{2} \frac{\nabla_v \left(\frac{f_e^\varepsilon}{M} \right)}{\left(\frac{f_e^\varepsilon}{M} + \lambda \right)^{3/2}}$$

so that using the definition of R_e^ε in (1.36), one has, for some constant $C(T)$,

$$|I_3^\varepsilon| \leq \frac{1}{\varepsilon} \int_0^T \iint M \left| \nabla_v \sqrt{\frac{f_e^\varepsilon}{M}} \right|^2 dv dx dt \left\| \frac{f_e^\varepsilon}{\theta_{\varepsilon, \lambda}^2} \right\|_{L^\infty(0, T; L_{x, v}^\infty)} \left\| \frac{\varphi}{\theta_{\varepsilon, \lambda}} \right\|_{L^\infty(0, T; L_{x, v}^\infty)} \leq C(T) \varepsilon,$$

by the control on the free energy dissipation provided by estimate (1.27). Hence I_3^ε goes to 0. To handle I_1^ε and I_2^ε , we decompose R_e^ε according to,

$$\frac{R_e^\varepsilon}{M} = \sqrt{n_e^\varepsilon} r_e^\varepsilon + \frac{\varepsilon}{2} |r_e^\varepsilon|^2$$

Doing so, the contribution of the magnetic field becomes

$$\begin{aligned} I_1^\varepsilon &= \int_0^T \iint \left(\frac{\sqrt{n_e^\varepsilon} M}{\theta_{\varepsilon, \lambda}} - \frac{\sqrt{n_e} M}{\sqrt{n_e + \lambda}} \right) \varphi(v \wedge B) \cdot \nabla_v r_e^\varepsilon dv dx dt \\ &\quad - \int_0^T \iint \frac{\sqrt{n_e}}{\sqrt{n_e + \lambda}} r_e^\varepsilon \sqrt{M} (v \wedge B) \cdot \nabla_v \varphi dv dx dt \\ &\quad + \varepsilon \int_0^T \iint \frac{\varphi}{\theta_{\varepsilon, \lambda}} (v \wedge B) \cdot \nabla_v r_e^\varepsilon \sqrt{M} r_e^\varepsilon \sqrt{M} dv dx dt. \end{aligned}$$

after an integration by parts in the second term. The first and third terms of the right-hand side go to zero by Lemma 1.4.4 (f) and 1.5.2 (a),(d). Therefore

$$\begin{aligned} I_1^\varepsilon &\longrightarrow - \int_0^T \iint \frac{\sqrt{n_e}}{\sqrt{n_e + \lambda}} r_e \sqrt{M} (v \wedge B) \cdot \nabla_v \varphi dv dx dt \\ &= \int_0^T \iint \frac{\sqrt{n_e}}{\sqrt{n_e + \lambda} \sqrt{M}} (v \wedge B) \cdot \nabla_v (r_e M) \varphi dv dx dt \end{aligned}$$

The contribution of the Fokker-Planck part may be written as

$$\begin{aligned} I_2^\varepsilon &= - \int_0^T \iint \left(\frac{\sqrt{n_e^\varepsilon} M}{\theta_{\varepsilon, \lambda}} - \frac{\sqrt{n_e} M}{\sqrt{n_e + \lambda}} \right) \sqrt{M} \nabla_v r_e^\varepsilon \cdot \nabla_v \left(\frac{\varphi}{\sqrt{M}} \right) dv dx dt \\ &\quad + \int_0^T \iint \frac{\sqrt{n_e}}{\sqrt{n_e + \lambda}} r_e^\varepsilon \nabla_v \cdot \left(M \nabla_v \left(\frac{\varphi}{\sqrt{M}} \right) \right) dv dx dt \\ &\quad - \varepsilon \int_0^T \iint \frac{\sqrt{M}}{\theta_{\varepsilon, \lambda}} \sqrt{M} r_e^\varepsilon \sqrt{M} \nabla_v r_e^\varepsilon \cdot \nabla_v \left(\frac{\varphi}{\sqrt{M}} \right) dv dx dt. \end{aligned}$$

As for I_1^ε , the first and third terms go to 0 and therefore

$$I_2^\varepsilon \longrightarrow \int_0^T \iint \frac{\sqrt{n_e}}{\sqrt{n_e + \lambda} \sqrt{M}} \nabla_v \cdot (M \nabla_v (r_e)) \varphi dv dx dt$$

We eventually showed that, in the sense of distributions

$$\frac{1}{2\varepsilon\theta_{\varepsilon, \lambda}} \mathcal{L}_A(f_e^\varepsilon) \longrightarrow \frac{\sqrt{n_e}}{\sqrt{n_e + \lambda} \sqrt{M}} \mathcal{L}_A(r_e M) \quad (1.42)$$

The convergence of the second term of the right-hand side of the renormalized equation (1.8) stems from the strong convergence of $M/\theta_{\varepsilon, \lambda}$ from Lemma 1.4.4 and the weak convergence of $\nabla_x \phi^\varepsilon$ from (1.32). Finally, we receive for any $\lambda > 0$

$$\begin{aligned} &v \sqrt{M} \cdot \left(\nabla_x \sqrt{n_e + \lambda} - \frac{1}{2} \nabla_x \phi \sqrt{n_e + \lambda} \right) \\ &= \frac{\sqrt{n_e}}{\sqrt{(n_e + \lambda) M}} \mathcal{L}_A(r_e M) - \frac{\lambda M}{2\sqrt{(n_e + \lambda) M}} v \cdot \nabla_x \phi. \end{aligned}$$

By dominated convergence, one may take the limit $\lambda \rightarrow 0$ to obtain

$$vM \cdot (\nabla_x \sqrt{n_e} - \frac{1}{2} \nabla_x \phi \sqrt{n_e}) = \mathcal{L}_A(r_e M)$$

in the sense of distributions. Since the left-hand side of the former equality is rapidly decaying in v , one may actually multiply the previous equation by v and integrate in the v variable to derive in the sense of distributions

$$\begin{aligned} (\nabla_x \sqrt{n_e} - \frac{1}{2} \nabla_x \phi \sqrt{n_e}) &= \int v \mathcal{L}_A(r_e M) dv \\ &= \int \mathcal{L}_A^*(vM) r_e dv \\ &= \int [\mathcal{L}_I(vM) - (v \wedge B) \cdot \nabla_v (vM)] r_e dv \\ &= \int -(v + (v \wedge B)) r_e M dv = -A \int r_e v M dv, \end{aligned}$$

Since $A(t, x)$ is invertible one gets the result by combining this identity with the expression of j_e from Lemma 1.5.3. \square

1.6 Regularity of the limit

Let us summarize what we have proved so far. The triplet $(f_i, \nabla_x \phi, n_e) \in L^\infty(0, T; L^1_{x,v} \cap L^\infty_{x,v}) \times L^\infty(0, T; L^2_x) \times L^1(0, T; L^1_x)$ is such that in the sense of distributions

$$\left\{ \begin{array}{l} \partial_t f_i + v \cdot \nabla_x f_i - \nabla_x \phi \cdot \nabla_v f_i + (v \wedge B) \cdot \nabla_v f_i = 0, \\ \partial_t n_e + \nabla_x \cdot j_e = 0, \\ (\nabla_x \sqrt{n_e} - \frac{1}{2} \nabla_x \phi \sqrt{n_e}) \in L^2(0, T, L^2_x) \\ j_e = -2\sqrt{n_e} A^{-1} (\nabla_x \sqrt{n_e} - \frac{1}{2} \nabla_x \phi \sqrt{n_e}) \\ -\Delta_x \phi = n_i - n_e, \end{array} \right.$$

with initial data

$$f_i(0, \cdot, \cdot) = f_i^{\text{in}} \text{ and } n_e(0, \cdot, \cdot) = \int f_e^{\text{in}} dv =: n_e^{\text{in}}.$$

Note that because of uniform bounds on $(\partial_t n_e^\varepsilon)_\varepsilon$ in $L^1(0, T; W_x^{-1,1})$ and on $(\partial_t f_i^\varepsilon)_\varepsilon$ in $L^1(0, T; W_{x,v}^{-1,1})$, we get that, by the Arzela-Ascoli theorem, the limit functions $t \mapsto \int n_e \varphi dx$ and $t \mapsto \int f_i \psi dv dx$ are continuous on $[0, T)$ for any test functions φ and ψ . Thus, we do recover the above initial conditions for the limit system. On the other hand, usual arguments based on the convexity and lower semi-continuity of the energy and entropy functionals and corresponding uniform estimates prove the boundedness of the

following quantities, uniformly in $t \in [0, T)$

$$\begin{aligned} \int n_e |\ln n_e| dx + \int |x| n_e dx + \iint (|v|^2 + |x|) f_i dv dx &\leq C(T) \\ \iint f_i |\ln f_i| dv dx &\leq \|f_i^{\text{in}} \ln f_i^{\text{in}}\|_{L^1_{x,v}} \end{aligned}$$

for some positive constant $C(T)$. Furthermore, the following mass estimates and global neutrality result hold uniformly in $t \in [0, T)$

$$\|n_e\|_{L^1_x} = \|f_i\|_{L^1_{x,v}} = \|f_e^{\text{in}}\|_{L^1_{x,v}} = \|f_i^{\text{in}}\|_{L^1_{x,v}}. \quad (1.43)$$

$$\|f_i\|_{L^p_{x,v}} \leq \|f_i^{\text{in}}\|_{L^p_{x,v}} \text{ for } p \in (1, +\infty] \quad (1.44)$$

Equality is verified for the mass estimate thanks to the tightness of the distribution functions that comes from the control of space and velocity moments.

The classical moment lemma [105, Lemma 3.1] shows that, by the boundedness of f_i in $L^\infty(0, T; L^1((1+|v|^2)dv dx) \cap L^\infty_{x,v})$, the limit macroscopic ion density has the regularity

$$n_i \in L^\infty(0, T; L_x^{5/3}). \quad (1.45)$$

We may gain some additional regularity on n_e using the particular structure of the limit system. The procedure we set up is a generalization of Lemma 7.1 in [147]. Here the situation is trickier because the ion background n_i is not regular enough to reach directly an L^2 regularity in space for n_e in order for the product $n_e \nabla_x \phi$ to make sense. However one may first gain some regularity and conclude by a bootstrap argument in Lemma 1.6.2.

Lemma 1.6.1. *Let $n_e \in L^1([0, T) \times \mathbb{R}^d)$ be a non-negative function that satisfies*

$$\nabla_x \sqrt{n_e} + \frac{1}{2} E \sqrt{n_e} = G, \quad (1.46)$$

$$\nabla_x \cdot E = n_i - n_e \quad (1.47)$$

in the sense of distributions, where $G \in L^2([0, T) \times \mathbb{R}^d)$, $E \in L^2([0, T) \times \mathbb{R}^d)$ and $n_i \in \mathcal{D}'([0, T) \times \mathbb{R}^d)$. Then, for any $p \in [1, 2]$ it holds

$$n_i \in L^p([0, T) \times \mathbb{R}^d) \implies n_e \in L^p([0, T) \times \mathbb{R}^d)$$

Proof. Let $p \in (1, 2]$. The first step of the proof is the renormalization of equation (1.46). We define hereafter the particular renormalization function we use and which is built to recover in the end an L^p bound on n_e . Let us define $\gamma \in C^\infty(\mathbb{R}_+)$ such that $\gamma(s) = s$ on $[0, 1]$, $\gamma(s) = 2$ for $s > 3$ and $0 \leq \gamma' \leq 1$. Now set, for $\delta \in (0, 1]$,

$$\gamma_\delta(s) = \frac{1}{p-1} \left(\frac{1}{\delta} \gamma(\delta s) + 1 \right)^{p-1} \quad \text{and} \quad \gamma'_\delta(s) = \gamma'(\delta s) \left(\frac{1}{\delta} \gamma(\delta s) + 1 \right)^{p-2}.$$

The derivative of the renormalization function satisfies

$$|\gamma'_\delta(s)| \leq 1 \quad \text{and} \quad |s\gamma'_\delta(s)| \leq \frac{3}{\delta}. \quad (1.48)$$

Equation (1.46) implies that $\nabla_x \sqrt{n_e} \in L^1_{\text{loc}}([0, T] \times \mathbb{R}^d)$. Let us renormalize equation (1.46) by multiplying it by $\gamma'_\delta(\sqrt{n_e})$

$$\nabla_x \gamma_\delta(\sqrt{n_e}) + \frac{1}{2} E \sqrt{n_e} \gamma'_\delta(\sqrt{n_e}) = G \gamma'_\delta(\sqrt{n_e}).$$

One can check with (1.48) that every term is square integrable. By taking the L^2 norm of the equation and expanding we obtain

$$\begin{aligned} & \|\nabla_x \gamma_\delta(\sqrt{n_e})\|_{L^2}^2 + \frac{1}{4} \|E \sqrt{n_e} \gamma'_\delta(\sqrt{n_e})\|_{L^2}^2 \\ & + \iint \nabla_x \gamma_\delta(\sqrt{n_e}) \cdot E \sqrt{n_e} \gamma'_\delta(\sqrt{n_e}) dx dt \leq \|G\|_{L^2}^2. \end{aligned} \quad (1.49)$$

We want to rewrite the third term as the scalar product of E with a gradient in order to use (1.47). Let us define

$$\tilde{\gamma}_\delta(s) = \int_0^s (\gamma'_\delta(u))^2 u du. \quad (1.50)$$

Using (1.50) in the third term of (1.49) and dropping the first two non-negative terms yields

$$\iint E \cdot \nabla_x \tilde{\gamma}_\delta(\sqrt{n_e}) dx dt \leq \|G\|_{L^2}^2.$$

By using equation (1.47), one gets after integrating by parts

$$\iint \tilde{\gamma}_\delta(\sqrt{n_e}) (n_e - n_i) \leq \|G\|_{L^2}^2. \quad (1.51)$$

Let us estimate $\tilde{\gamma}_\delta(\sqrt{n_e})$. For $s \in [0, 1/\delta]$, one obtains

$$\tilde{\gamma}_\delta(s) = \int_0^s u(u+1)^{2p-4} du \leq \int_0^s u^{2p-3} du = \frac{s^{2p-2}}{2p-2},$$

and

$$\tilde{\gamma}_\delta(s) = \begin{cases} \frac{(s+1)^{2p-2} - 1}{2p-2} - \frac{(s+1)^{2p-3} - 1}{2p-3} & \text{if } p \neq \frac{3}{2}, \\ s - \ln(s+1) & \text{if } p = \frac{3}{2}. \end{cases}$$

Then, one readily checks that for any $p \in (1, 2]$ there exists $C_1 > 0$ such that for any $\delta \in (0, 1]$ and $s < \frac{1}{\delta}$ it holds

$$\tilde{\gamma}_\delta(s) \geq \frac{1}{4} s^{2p-2} - C_1$$

For $s > 1/\delta$, there exists $C_2 > 0$ depending only on p such that

$$\tilde{\gamma}_\delta(s) \leq \tilde{\gamma}_\delta(3/\delta) \leq \left(\frac{1}{\delta}\right)^{2p-2} \int_0^3 \gamma(u)^{2p-4} u du = C_2 s^{2p-2}$$

This provides the following estimates, for any $s \geq 0$ and $C = \max(C_1, C_2, 1/(2p-2))$

$$\left(\frac{1}{4}s^{2p-2} - C\right) \mathbf{1}_{\{s \leq 1/\delta\}}(s) \leq \tilde{\gamma}_\delta(s) \leq C s^{2p-2}, \quad \forall s \geq 0. \quad (1.52)$$

We now use estimate (1.52) in (1.51). First we get rid of the part involving the ion density n_i . Using Young's inequality, for any $\eta > 0$ there exists some constant $C_\eta > 0$ such that

$$\begin{aligned} \iint \tilde{\gamma}_\delta(\sqrt{n_e}) n_i &\leq C_\eta \|n_i\|_{L^p}^p + \eta \iint (\tilde{\gamma}_\delta(\sqrt{n_e}))^{\frac{p}{p-1}} dx dt. \\ &\leq C_\eta \|n_i\|_{L^p}^p + C^{\frac{1}{p-1}} \eta \iint n_e \tilde{\gamma}_\delta(\sqrt{n_e}) dx dt \end{aligned}$$

where we used estimate (1.52) in the second inequality. On the other hand, we have

$$\iint n_e \tilde{\gamma}_\delta(\sqrt{n_e}) dx dt \geq \frac{1}{4} \iint_{\{n_e \leq 1/\delta^2\}} n_e^p dx dt - C \|n_e\|_{L^1}.$$

As a result, by taking η sufficiently small, one gets

$$0 \leq \iint n_e^p \mathbf{1}_{\{n_e \leq 1/\delta^2\}} dx dt \leq C(\|G\|_{L^2}, \|n_i\|_{L^p}, \|n_e\|_{L^1}),$$

uniformly in δ . Taking the monotone limit $\delta \rightarrow 0$ concludes the proof. \square

The regularity of n_e that one may establish by the previous proof is limited by the regularity of n_i . Nevertheless the available regularity of n_i is sufficient to provide us with a termwise sense for j_e .

Lemma 1.6.2. *Limiting densities of families of physical solutions satisfy $\nabla_x \phi \sqrt{n_e} \in L^1(0, T; L^2_{loc,x})$, $\sqrt{n_e} \in L^1(0, T; H^1_{loc,x})$ and in the sense of distributions*

$$j_e = -D(\nabla_x n_e - \nabla_x \phi n_e),$$

where we recall that $D = A^{-1}$.

Proof. First by an application of Lemma 1.6.1 with $p = 5/3$, we show that the source n in the Poisson equation $-\Delta_x \phi = n_i - n_e =: n$ lies in $L^{5/3}(0, T; L^{5/3}_x) \cap L^\infty(0, T; L^1_x)$ and therefore, with an $L^\infty(0, T; L^2_x)$ electric field, it yields $\nabla_x \phi = \nabla_x \Phi *_x n$. Thus, by the Hardy-Littlewood-Sobolev inequality, $\nabla_x \phi \in L^{5/3}(0, T; L^{15/4}_x)$. Hence, by the Hölder inequality, the product $\nabla_x \phi \sqrt{n_e}$ is in $L^{10/9}(0, T; L^{30/17}_x)$ and since

$$\nabla_x \sqrt{n_e} + \frac{1}{2} E \sqrt{n_e} \in L^2([0, T] \times \mathbb{R}^3),$$

this yields $\sqrt{n_e} \in L^{10/9}(0, T; W_{loc,x}^{1,30/17})$. By Sobolev embedding this gives at least $\sqrt{n_e} \in L^1(0, T; L_{loc,x}^4)$ and since $\nabla_x \phi \in L^\infty(0, T; L_x^2)$, the product $\nabla_x \phi \sqrt{n_e}$ belongs to $L^1(0, T; L_{loc,x}^2)$ which yields the results. \square

This completes the proof of Theorem 1.1.6.

Chapitre 2

Densité de Maxwell-Boltzmann anisotrope

Travail en collaboration avec L. Miguel Rodrigues³².
Ce chapitre est soumis pour publication [126].

We consider various sets of Vlasov-Fokker-Planck equations modeling the dynamics of charged particles in a plasma under the effect of a strong magnetic field. For each of them in a regime where the strength of the magnetic field is effectively stronger than that of collisions we first formally derive asymptotically reduced models. In this regime, strong anisotropic phenomena occur ; while equilibrium along magnetic field lines is asymptotically reached our asymptotic models capture a non trivial dynamics in the perpendicular directions. We do check that in any case the obtained asymptotic model defines a well-posed dynamical system and when self consistent electric fields are neglected we provide a rigorous mathematical justification of the formally derived systems. In this last step we provide a complete control on solutions by developing anisotropic hypocoercive estimates.

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2.1 Introduction

In the present contribution we are interested in understanding on long-term scales interactions between oscillations and dissipative mechanisms when the latter are not the dominant effect and the long-time scale is chosen to match natural times scales of the averaged evolution arising from strong oscillations. We are essentially motivated by the modeling of a strongly magnetized, weakly collisional three-dimensional plasma. Yet our analysis indicates that even with strong but non dominant collisions the presence of a strong magnetic field would essentially prevent the classical Maxwell-Boltzmann/Gibbs regime resulting from entropy dissipation from happening.

The dynamics of a plasma, a charged gas composed of ions and electrons evolving under self-consistent and external electromagnetic interactions, can be accurately modeled by a system of kinetic equations of Vlasov-type. However, in applications, these models are rarely numerically simulated in their full complexity because of computer-time costs. Instead, some reduced models are used in computational experiments. Even at a theoretical level these models are precious to capture coherent dynamics emerging on large-time scales from short-scale oscillations or out of dissipation. Therefore it is crucial to understand how they derive from modeling assumptions in some asymptotic regimes. In tokamak physics [154], a usual reduction consists in approximating the density of electrons by Maxwell distribution in velocity and the so-called Boltzmann-Gibbs macroscopic density which reads

$$n_{\text{BG}}(t, x) = \frac{1}{Z(t)} \exp\left(\frac{q\phi(t, x)}{k_B T(t)}\right), \quad (2.1)$$

where t and x are respectively the time and space variables, q and k_B stand for the elementary charge and the Boltzmann constant, ϕ is the electric potential and T is a temperature. It depends also on a normalizing density Z that could actually be taken to be 1 by a suitable normalization of the electric potential. The approximation by Maxwellian distribution hinges directly on entropy dissipation arguments whereas the derivation of the Boltzmann-Gibbs approximation stems from the interplay between transport and collisions in an appropriate scaling. Depending on the modeling of both phenomena one obtains different asymptotic descriptions reflecting more or less complex dynamics for T . For instance, recently, in [15], Bardos, Golse, Nguyen and Sentis formally derive (2.1) jointly with an evolution rule for T when starting from a magnetic-field-free kinetic description of electrons and prove that the corresponding evolution obtained by coupling with a full kinetic model for ions dynamics possesses global weak solutions. In any case the asymptotic reduction hinges on essentially the same mechanisms driving large-time³³ asymptotics for similar models. On the latter the reader is referred to [29].

In this paper, we are rather interested in a detailed analysis of the normalizing factor $C := 1/Z$ for strongly magnetized plasma. While, as already pointed out, in many applications in tokamak physics the isotropic (2.1) is often used, namely C at (t, x) is $C(t)$, we claim that under the influence of a strong three-dimensional magnetic field the

33. Here in the sense of $t \rightarrow \infty$.

relevant asymptotic form for the slaved density (2.1) features a space dependent density $C(t, x_\perp)$ where x_\perp is the projection of x in the plane orthogonal to magnetic field lines. In other words we claim that in a strongly magnetized regime Boltzmann equilibrium of energy occur in velocity variables and spatially along the magnetic field but not perpendicular to it. Therefore we shall call the resulting approximation *anisotropic Boltzmann-Gibbs approximation*. Note that when C is space-dependent it cannot be eliminated by a suitable convention change in the definition of the electric field. Some similar questions have already been investigated by formal asymptotics both in plasma physics and applied mathematics communities. We point out the articles of Negulescu, de Cecco and Possaner [44, 160] to the attention of the reader as both an instance of a formal analysis similar to that developed here and a source of references to the relevant plasma physics literature. Besides the formal derivation of a suitable anisotropic version of (2.1) and of the corresponding spatio-temporal evolution for C , *our goal is to provide various rigorous validations of our formal analysis and a mathematical analysis of the reduced dynamics for C .*

2.1.1 Our original kinetic model and its approximation

To keep the general picture as simple as possible we restrict our attention to a uniform magnetic field, compact spatial domains without boundaries and collisions with a thermal bath, which will yield a constant temperature T in our anisotropic version of (2.1). Our initial object of study is the evolution of a distribution function $f^\varepsilon : \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}_+$, $(t, x, v) \mapsto f^\varepsilon(t, x, v)$ depending on time t , space x and velocity v (where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a one-dimensional torus) according to a Vlasov–Fokker-Planck equation (VFP)

$$\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \sigma E^\varepsilon \cdot \nabla_v f^\varepsilon - \frac{1}{\varepsilon} \sigma v^\perp \cdot \nabla_v f^\varepsilon = \frac{1}{\varepsilon^\alpha} \operatorname{div}_v (v f^\varepsilon + \nabla_v f^\varepsilon), \quad (2.2)$$

where $v^\perp = (0, 0, 1) \wedge v$ — in coordinates $(v_1, v_2, v_3)^\perp = (-v_2, v_1, 0)$ — stems from the Laplace force induced by the external magnetic field, σ stands for the sign of the particle charge, E^ε is an electric field and $\alpha \in \mathbb{R}$ measures the collisions strength with respect to magnetic effects. Note that as is implicit above the vector $(0, 0, 1)$ provides the direction of the magnetic field. Depending on the situation, we shall consider various forms of prescription for the electric field $E^\varepsilon : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$. In any case it derives from an electric potential $\phi^\varepsilon : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}$, $E^\varepsilon = -\nabla_x \phi^\varepsilon$. The simplest situation corresponds to the case where the electric potential itself is taken as a known applied electric potential. For obvious reasons we shall refer to the latter case as the external field case or the linear case. In this case the electric potential is independent of ε and we sometimes drop the superscript ε on it. Two other situations are considered in the present chapter. Indeed though we have so far mostly discussed motivations from asymptotic reduction in the electronic dynamics we also investigate a similar question for the ions dynamics. This results in two separated prescriptions of the electric field through modeling considerations that are only briefly summarized hereafter but detailed in Section 2.2.1.

In the first nonlinear case, that we shall refer to as the electronic or the light species case, f^ε is the electron distribution function and phenomena are observed on ions characteristic scales. Consistently we should set $\sigma = -1$ here but we shall mostly keep σ undetermined to offer a treatment as unified as possible of all situations. In this first case the scaling parameter ε represents the square root of the ratio m_e/m_i of masses of electrons and ions. In the asymptotics $\varepsilon \rightarrow 0$, particles are massless zero-inertia electrons and the corresponding dynamics is sometimes called adiabatic evolution. Note that here the fact that magnetic effects are *strongly* perceived by electrons arise from the strong separation in time scales induced by a small mass ratio. As we have already stressed this limit plays a particular important role in the understanding of plasma dynamics and accordingly has been investigated in many similar but distinct situations [124, 9, 160, 44, 61, 64, 65, 27, 15]. In a non relativistic regime, the electric field satisfies the classical Poisson equation

$$-\delta^2 \Delta \phi^\varepsilon = \sigma(n^\varepsilon - \underline{n}), \quad (2.3)$$

where $n^\varepsilon : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}_+$ is the macroscopic density of particles under consideration

$$n^\varepsilon(t, x) = \int_{\mathbb{R}^3} f^\varepsilon(t, x, v) \, dv$$

and $\underline{n} : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}$ denotes the background time-dependent density of the other species with opposite charges.

In the second nonlinear model, called the ionic or the heavy species case in the following, f^ε denotes the ion distribution function. Scaling parameter ε stands for a scaled Larmor radius which directly measures the strength of the external magnetic field. In this regime, classical Boltzmann-Gibbs approximation for the macroscopic electron density leads to the so-called Poisson-Boltzmann equation for the electric field

$$-\delta^2 \Delta \phi^\varepsilon = \sigma \left(n^\varepsilon - \frac{1}{\int_{\mathbb{T}^3} e^{\sigma \phi^\varepsilon(y)} \, dy} e^{\sigma \phi^\varepsilon} \right). \quad (2.4)$$

In non-collisional settings the strong magnetic field limit $\varepsilon \rightarrow 0$ has also been thoroughly investigated by many authors [63, 105, 178, 92, 53, 54]. As in those pieces of work time scale is chosen here to observe a non trivial averaged dynamics.

Concerning the strength of collisions, we focus on cases where $|\alpha| < 1$, $\alpha < 0$ corresponding to a regime of evanescent collision frequency and $\alpha > 0$ to a strongly collisional regime. The assumption $\alpha > -1$ ensures that even in weakly collisional regimes collisions are strong enough to lead on the large observation time scales we consider to a global equilibrium if magnetic effects were discarded (see [29, 125] for a related analysis and Remark 2.2.3 for further comments) so that the non-equilibrium features that we shall describe are indeed due to the presence of a strong magnetic field. The condition $\alpha < 1$ enforces that magnetic effects are dominant. We refer the reader to [124] for an extensive list of references concerning the borderline case $\alpha = 1$ that yields a diffusive

limit when $\varepsilon \rightarrow 0$ which provides an asymptotic model retaining some features of the external magnetic field. We point out that in contrast with the case under study here, in the borderline case the specific geometry of the magnetic field plays almost no role in the derivation of an asymptotic model.

In Section 2.2.2 by arguing on formal grounds we provide a heuristic argument and in Section 2.4.2 we offer a rigorous justification in the linear case that both suggest that, for any family $(f^\varepsilon)_{\varepsilon>0}$ of suitable solutions to (2.2) with a given initial datum f_0 , taking the limit $\varepsilon \rightarrow 0$ yields a limiting distribution f and a limiting electric potential ϕ that have reached an adiabatic regime along the magnetic field

$$f(t, x, v) = N(t, x_\perp) \frac{e^{-\sigma\phi(t, x)}}{\int_{\mathbb{T}} e^{-\sigma\phi(t, x_\perp, y_\parallel)} \mathrm{d}y_\parallel} M(v) \quad (2.5)$$

where $x = (x_\perp, x_\parallel) \in \mathbb{T}^2 \times \mathbb{T}$ and M is a Maxwellian distribution

$$M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}.$$

In this asymptotic regime the dynamics is slaved to the evolution of a reduced macroscopic distribution $N : \mathbb{R}_+ \times \mathbb{T}^2 \rightarrow \mathbb{R}_+$ in the perpendicular direction, satisfying

$$\partial_t N + \operatorname{div}_{x_\perp} (N (\nabla_{x_\perp} \tilde{\phi})^\perp) = 0 \quad (2.6)$$

with initial condition

$$N(0, x_\perp) = N_0(x_\perp) = \iint_{\mathbb{T} \times \mathbb{R}^3} f_0(x_\perp, x_\parallel, v) \mathrm{d}x_\parallel \mathrm{d}v,$$

where $\tilde{\phi} : \mathbb{R}_+ \times \mathbb{T}^2 \rightarrow \mathbb{R}$ is an x_\parallel -averaged version of ϕ

$$\tilde{\phi}(t, x_\perp) = -\sigma \ln \left(\int_{\mathbb{T}} e^{-\sigma\phi(t, x_\perp, x_\parallel)} \mathrm{d}x_\parallel \right), \quad (2.7)$$

and ϕ is either the initially prescribed electric field in the external field case, or is obtained by solving (2.3) or (2.4) (dropping the superscript ε) with the anisotropic Boltzmann-Gibbs density

$$n(t, x) = N(t, x_\perp) \frac{e^{-\sigma\phi(t, x)}}{\int_{\mathbb{T}} e^{-\sigma\phi(t, x_\perp, y_\parallel)} \mathrm{d}y_\parallel} = N(t, x_\perp) e^{-\sigma(\phi(t, x) - \tilde{\phi}(t, x_\perp))}. \quad (2.8)$$

To ease comparison with the initial discussion we observe that (2.8) provides an anisotropic version of (2.1) where $C = 1/Z$ would be given by

$$C(t, x_\perp) = N(t, x_\perp) / \left(\int_{\mathbb{T}} e^{-\sigma\phi(t, x_\perp, y_\parallel)} \mathrm{d}y_\parallel \right).$$

The reader may rightfully wonder whether the reduced dynamics described above could in disguise follow a set of purely *two-dimensional* local differential equations. The answer turns out to be positive in the heavy-species case that actually follows

$$\partial_t N + \operatorname{div}_{x_\perp} (N (\nabla_{x_\perp} \tilde{\phi})^\perp) = 0, \quad -\delta^2 \Delta_{x_\perp} \tilde{\phi} = \sigma \left(N - \frac{e^{\sigma \tilde{\phi}}}{\int_{\mathbb{T}^2} e^{\sigma \tilde{\phi}(\cdot, y_\perp)} dy_\perp} \right).$$

In the light-species case the presence of a three-dimensional background \underline{n} prevents any fully two-dimensional reduction from happening. However in the special case where actually \underline{n} does not depend on x_\parallel the light-species case does reduce to a two-dimensional set of equations, the classical guiding center model

$$\partial_t N + \operatorname{div}_{x_\perp} (N \nabla_{x_\perp}^\perp \phi) = 0, \quad -\delta^2 \Delta_{x_\perp} \phi = \sigma (N - \underline{n}),$$

that has been analytically derived by Golse and Saint-Raymond [105, 178] and Miot [150] as the large magnetic field limit of the Vlasov-Poisson system with uniform magnetic fields. Interestingly enough it is also consistent with formal arguments in the recent [63] by Degond and Filbet that in a non-uniform large magnetic field regime obtain from the Vlasov-Poisson system a set of equations for a limiting $f(t, x, v) = F(t, x, |v_\perp|, v_\parallel)$ which when specialized to a constant magnetic field as considered here is

$$\begin{aligned} \partial_t F + \nabla_{x_\perp}^\perp \phi \cdot \nabla_{x_\perp} F - \sigma \partial_{x_\parallel} \psi \partial_{v_\parallel} F + v_\parallel \partial_{x_\parallel} P - \sigma \partial_{x_\parallel} \phi \partial_{v_\parallel} P &= 0 \\ v_\parallel \partial_{x_\parallel} F - \sigma \partial_{x_\parallel} \phi \partial_{v_\parallel} F &= 0, \quad -\Delta_x \phi = \sigma \left((2\pi) \int_{\mathbb{R}_+ \times \mathbb{R}} F(\cdot, \cdot, p, v_\parallel) p dp dv_\parallel - \underline{n} \right) \\ -\Delta_x \psi &= \sigma \left((2\pi) \int_{\mathbb{R}_+ \times \mathbb{R}} P(\cdot, \cdot, p, v_\parallel) p dp dv_\parallel - \underline{n} \right) \end{aligned}$$

where P is thought as a multiplier associated with the constraint on F encoded by the first equation of the second-line. *A priori* for collisionless plasmas nothing seems to force a Maxwellian distribution in velocity but if one inserts a Maxwellian distribution $f(t, x, v) = n(t, x)M(v)$ in the asymptotic model formally derived by Degond and Filbet then the first equation of the second-line of the system becomes (2.8) for some N and, afterwards, integrating in $(x_\parallel, |v_\perp|, v_\parallel)$ the first equation to eliminate (P, ψ) reduces the two first lines of the system to (2.6)-(2.7)-(2.8)-(2.3).

2.1.2 Main analytical results

We now state our two sets of mathematical results, on one hand existence results for nonlinear asymptotic models and on the other hand analytic proofs of validity for the linear asymptotic model.

Roughly speaking, nonlinear asymptotic systems remotely look like transport equations for N by a divergence-free vector-field obtained by applying to N a linear differential operator of order -1 . Yet in both cases the relation between N and $\tilde{\phi}$ is far from being linear and in the light-species case the system is not even truly two-dimensional so that it is not readily apparent that those nonlinear asymptotic systems do enjoy existence

and well-posedness results expected for the foregoing class of equations. In Section 2.3 we prove that it is the case at least in subcritical³⁴ regimes and our strategy does follow the above analogy. Indeed once sufficient control on the construction of $\tilde{\phi}$ has been obtained one may follow standard arguments for similar equations such as Vlasov-Poisson equations [99, 30, 174] or two-dimensional incompressible Euler equations in vorticity formulation [145, 52, 143]. Consequently, the key part of the argument is a detailed analysis of existence, uniqueness, continuous dependence and regularity results for solutions to both elliptic equations (2.4)-(2.8) and (2.3)-(2.8) (as well as estimates on $\tilde{\phi}$ in terms of ϕ) that are for $\tilde{\phi}$ as two-dimensional and linear as possible. From these we obtain the following results for the full asymptotic systems.

Theorem 2.1.1 (Strong solutions).

Let $p > 2$, $\delta > 0$ and $\underline{n} \in \mathcal{C}_b(\mathbb{R}_+; W^{1,p}(\mathbb{T}^3))$ with mean constant equal to one, $\underline{n} \geq 0$. For any $N^{in} \in W^{1,p}(\mathbb{T}^2)$, $N^{in} \geq 0$, $\int_{\mathbb{T}^2} N^{in} = 1$, there exists a unique maximal solution (N, I) to (2.6)-(2.7)-(2.3)-(2.8) (resp. (2.6)-(2.7)-(2.4)-(2.8)) in $\mathcal{C}(I; L^1(\mathbb{T}^2)) \cap \mathcal{C}_w(I; W^{1,p}(\mathbb{T}^2))$ starting from N^{in} at time 0. Moreover the maximal time of existence is uniformly bounded away from zero on bounded sets of $W^{1,p}(\mathbb{T}^2)$.

In the former Theorem, \mathcal{C}_w denotes the space of functions that are continuous for the weak topology on the space of values and of course in the heavy-species case no \underline{n} is actually needed.

Theorem 2.1.2 (Weak solutions). Let $p > 4/3$, $\delta > 0$.

Let $\underline{n} \in \mathcal{C}(\mathbb{R}; L^{6/5}(\mathbb{T}^3)) \cap L_{loc}^\infty(\mathbb{R}; L^p(\mathbb{T}^3) \cap L^{3/2,1}(\mathbb{T}^3))$ with mean constant equal to one, $\underline{n} \geq 0$.

For any $N^{in} \in L^p(\mathbb{T}^2)$, $N^{in} \geq 0$, $\int_{\mathbb{T}^2} N^{in} = 1$, there exists a global weak solution N to (2.6)-(2.7)-(2.3)-(2.8) (resp. (2.6)-(2.7)-(2.4)-(2.8)) starting from N^{in} at time 0.

The reader unfamiliar with Lorentz spaces as appearing in our last statement may replace $L^p(\mathbb{T}^3) \cap L^{3/2,1}(\mathbb{T}^3)$ with $L^q(\mathbb{T}^3)$ for some $q > \max(\{p, 3/2\})$ to obtain a simpler weaker statement or consult [139, Chapter 2].

Now we elaborate on the analogy mentioned in the paragraph preceding our statements to motivate critical thresholds appearing in our foregoing theorems. For strong solutions of transport equations a natural threshold is obtained by requiring the Lipschitz property for the advection field as it enables to transport regularity. Assuming that $(\nabla_{x_\perp} \tilde{\phi})^\perp$ is indeed one-derivative more regular than N this amounts here to $W^{2,p}(\mathbb{T}^2) \hookrightarrow W^{1,\infty}(\mathbb{T}^2)$, hence $p > 2$. Concerning weak solutions, natural thresholds are obtained by requiring that all terms in the equations should be well-defined as distributions and this is usually an issue only for nonlinear terms, hence here only for $N(\nabla_{x_\perp} \tilde{\phi})^\perp$. Therefore we seek $\nabla_{x_\perp} \tilde{\phi} \in L^{p'}(\mathbb{T}^2)$ since $N \in L^p(\mathbb{T}^2)$ (where $(\cdot)'$ denotes Lebesgue conjugation, hence $1/p + 1/p' = 1$). Assuming again that $(\nabla_{x_\perp} \tilde{\phi})^\perp$ is indeed one-derivative more regular than N this leads to enforce $W^{1,p}(\mathbb{T}^2) \hookrightarrow L^{p'}(\mathbb{T}^2)$ which yields $p = 4/3$ as a critical exponent. Note that in our actual analysis we need to go from N to $\tilde{\phi}$ through ϕ and thus need

34. However we let open more technical issues such as existence of renormalized solutions and well-posedness in critical spaces.

to replace two-dimensional embeddings implicitly used above with suitable anisotropic versions of three-dimensional embeddings to preserve two-dimensional-like scalings.

Though we do not explicitly state it here an inspection of our proofs also provides propagation of regularity for strong solutions, weak-strong uniqueness and global existence of strong solutions for small initial data. Concerning the latter such a form of statement would be slightly inconsistent with modeling considerations since N^{in} can not really be small... However in the same way one may also obtain the following form of global existence in weakly nonlinear regimes : for any $R > 0$ there exists $\delta_0(R)$ such that if $\delta > \delta_0(\mathbb{R})$ and $\|N^{\text{in}}\|_{W^{1,p}(\mathbb{T}^2)} \leq R$, $\|\underline{n}\|_{L^\infty(\mathbb{R};W^{1,p}(\mathbb{T}^3))} \leq R$ then solutions of Theorem 2.1.1 are global. We do not elaborate here on the crucial role of δ that we consider as fixed in the present contribution but we refer the reader to [125] for a detailed discussion and a thorough analysis of a non magnetized version of our original systems.

Second, in the external field case, we prove in Section 2.4 that formal arguments may indeed be replaced with sound rigorous mathematical analysis. We provide such a justification for what is often considered to be the weakest notion of solutions that are physically relevant, that is, for free energy solutions of (2.2) in the sense of Bouchut and Dolbeault [29, 74]. More explicitly we consider all $L \log L$ distributions with finite second moments in velocity or equivalently, distributions with bounded mass, finite kinetic energy and finite entropy.

Theorem 2.1.3. *Assume that $\phi \in W_{loc}^{1,1}(\mathbb{R}_+; L^\infty(\mathbb{T}^3)) \cap L_{loc}^1(\mathbb{R}_+; W^{1,\infty}(\mathbb{T}^3))$ and f_0 is such that*

$$f_0 \geq 0, \quad \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (1 + |v|^2 + \ln_+ f_0) f_0 dx dv < \infty$$

(where $(\cdot)_+$ denotes positive part). Then there exists a unique $f^\varepsilon \in \mathcal{C}(\mathbb{R}_+; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$ solving Equation (2.2) and starting from f_0 at time 0. Moreover there exists $N \in L^\infty(\mathbb{R}_+; L^1(\mathbb{T}^2))$ and a positive sequence (ε_n) converging to zero such that when $n \rightarrow \infty$

$$f^{\varepsilon_n}(t, x, v) \rightarrow N(t, x_\perp) \frac{e^{-\sigma\phi(t,x)}}{\int_{\mathbb{T}} e^{-\sigma\phi(t,x_\perp, y_\parallel)} dy_\parallel} M(v) \quad \text{weakly in } L_{loc}^1(\mathbb{R}_+; L^1(\mathbb{T}^3 \times \mathbb{R}^3)),$$

and N solves the transport equation (2.6)-(2.7).

Note that when ϕ is sufficiently regular to ensure uniqueness for the transport equation one may actually replace convergence along the subsequence (ε_n) with full convergence as $\varepsilon \rightarrow 0$ by the argument relying on compactness and uniqueness of limit points.

Finally, we want to obtain convergence rates in strong norms for asymptotic limits that are expected to be compatible with strong convergence. Indeed one expects three kind of phenomena to co-exist, dissipation by collisions leads to Maxwellian distribution in velocity, interaction between transport and collisions yields anisotropic Boltzmann-Gibbs behavior of macroscopic density for some N^ε , strong oscillations induced by the magnetic field average N^ε to some N solving (2.6). By essence the last part does not lend itself to a strong convergence analysis. For the other ones we develop a form of anisotropic hypocoercive estimates that provide a strong form of control on norms involved

in the limiting process. Note that even for the simplest part of the convergence analysis leading to Maxwellian behavior one needs to assume some strong form of localization of initial data to derive decay rates. Accordingly we assume our initial data to lie in some weighted functional space. The justification of the Boltzmann-Gibbs approximation also requires control on derivatives of the solutions but we assume no regularity on the initial data and instead gain this regularity at later time from hypoelliptic properties of the dynamics. At a technical level our proof proceeds by building a suitable dissipated functional that captures both hypocoercivity observed on relevant commutators as in [194] and hypoellipticity as in [121, 122]. For simplicity we assume ϕ to be time-independent here.

Theorem 2.1.4 (Convergence rates). *Let $\phi \in W^{2,\infty}(\mathbb{T}^3)$.*

Then there exists a positive constant C such that for any $\varepsilon \in (0, 1)$ and any initial data $f_0 \in L^2(M^{-1}(v) dx dv)$, the strong solution f^ε solving (2.2) and starting from f_0 satisfies

$$\|f^\varepsilon - n^\varepsilon M\|_{L^2(\mathbb{R}_+; L^2(M^{-1} dx dv))} \leq C \|f_0\|_{L^2(M^{-1} dx dv)} \varepsilon^{\frac{\alpha+1}{2}}$$

$$\left\| (t, x) \mapsto n^\varepsilon(t, x) - N^\varepsilon(t, x_\perp) e^{-\sigma(\phi(x) - \tilde{\phi}(x_\perp))} \right\|_{L^2(\mathbb{R}_+ \times \mathbb{T}^3)} \leq C \|f_0\|_{L^2(M^{-1} dx dv)} \varepsilon^{\frac{1-|\alpha|}{2}}$$

where

$$n^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon(\cdot, \cdot, v) dv \quad \text{and} \quad N^\varepsilon = \int_{\mathbb{T}} n^\varepsilon(\cdot, \cdot, y_\parallel) dy_\parallel.$$

In particular, as $\varepsilon \rightarrow 0$ the distribution function f^ε gets $\varepsilon^{\frac{1-|\alpha|}{2}}$ -close to the anisotropic Maxwell-Boltzmann density

$$(t, x, v) \mapsto N^\varepsilon(t, x_\perp) e^{-\sigma(\phi(x) - \tilde{\phi}(x_\perp))} \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}}$$

in $L^2(\mathbb{R}_+; L^2(M^{-1}(v) dx dv))$.

A few comments on rates are in order. Rates in our statement are expected to be optimal. Indeed the first rate is naturally associated with the fact that solutions to

$$\varepsilon \partial_t \tilde{f}^\varepsilon = \frac{1}{\varepsilon^\alpha} \operatorname{div}_v(v \tilde{f}^\varepsilon + \nabla_v \tilde{f}^\varepsilon)$$

starting from $\tilde{f}_0 \in L^2(M^{-1}(v) dv)$ decay to a Maxwellian distribution at a rate $e^{-t/\varepsilon^{1+\alpha}}$ in $L^2(M^{-1}(v) dv)$. On the other hand the second one is closely connected to the convergence of solutions to

$$\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \sigma E^\varepsilon \cdot \nabla_v f^\varepsilon = \frac{1}{\varepsilon^\alpha} \operatorname{div}_v(v f^\varepsilon + \nabla_v f^\varepsilon)$$

starting from $f_0 \in L^2(M^{-1}(v) dx dv)$ such that $\nabla_x f_0, \nabla_v f_0, v f_0 \in L^2(M^{-1}(v) dx dv)$ towards a Maxwell-Boltzmann distribution at a rate $e^{-\kappa_0 t/\varepsilon^{1-|\alpha|}}$ in $L^2(M^{-1}(v) dx dv)$ for some $\kappa_0 > 0$. Note that the latter rate is not monotone with respect to α . This reflects that collisions are not the only mechanism involved in this convergence. For a thorough discussion we refer the reader to [125] that analyzes nonlinear magnetic-field-free analogous questions.

Structure of the chapter

The structure of the chapter follows the organization of our introduction. In Section 2.2 we provide motivations for our initial models and a formal derivation of reduced equations. In Section 2.3 we analyze nonlinear reduced asymptotic models to prove Theorems 2.1.1 and 2.1.2. At last in Section 2.4 we prove Theorems 2.1.3 and 2.1.4 hence justify the linear reduced model.

2.2 Heuristic considerations

In the following we give modeling and heuristic justifications for both our starting and limiting models. We explain in Section 2.2.1 how our initial physical models — Equation (2.2) completed with some prescription of the electric field — arise precisely in the regime $\varepsilon \ll 1$ from more complete two-species systems. Then, in Section 2.2.2, starting from (2.2) we argue on formal grounds to identify the asymptotic dynamics obeying (2.6)-(2.7)-(2.8).

2.2.1 Original physical variables

Though our goal is mostly to identify qualitative mechanisms our choice of systems originates in concrete realistic plasma dynamics, from which they are obtained by a combination of arguments — that we expound now — either of asymptotic analysis type or made purely to lower technicalities to their bare minimum.

We begin our discussion by considering the full dynamics of a plasma containing electrons with negative charge $-q$ and mass m_e and ions of mass m_i and charge Zq , where q is the elementary charge and Z is the atomic number of ions. In the following we take into account an external unidirectional magnetic field of constant amplitude \bar{B} , to be thought of as a confinement field for a fusion device, but as is classical in this regime (and rigorously justified in some closely related contexts, see for instance [60]) we neglect self-induced magnetic effects. To reduce technicalities in the analysis we also idealize the mechanism of collisions between particles and describe collisions as if they were occurring with a thermal bath with temperature θ and zero mean velocity. To our opinion this is by far the less realistic of our simplifications and the only one that does not bear principally on modeling considerations but we expect that the phenomena that we identify in our simplified evolution do occur on a much wider range of models. In this direction we recall that the formal asymptotic analysis carried out in Section 2.2.2 does not rely heavily on the specific form of the full collision operator but only on the nature of its kernel, here Maxwellian equilibria with zero mean velocity and fixed temperature.

Denoting respectively by f_e and f_i electron and ion distribution functions, the original

equations, written in physical variables, are

$$\begin{cases} \partial_t f_e + v \cdot \nabla_x f_e - \frac{q}{m_e} (-\nabla_x \phi - \bar{B} v^\perp) \cdot \nabla_v f_e &= \nu_{\text{col}}^{(e)} \nabla_v \cdot \left(v f_e + \frac{k_B \theta}{m_e} \nabla_v f_e \right), \\ \partial_t f_i + v \cdot \nabla_x f_i + \frac{Zq}{m_i} (-\nabla_x \phi - \bar{B} v^\perp) \cdot \nabla_v f_i &= \nu_{\text{col}}^{(i)} \nabla_v \cdot \left(v f_i + \frac{k_B \theta}{m_i} \nabla_v f_i \right), \\ -\varepsilon_0 \Delta_x \phi &= Zq n_i - q n_e, \end{cases} \quad (2.9)$$

where the macroscopic density of the species $s \in \{i, e\}$ is given by $n_s(t, x) = \int_{\mathbb{R}^3} f_s(t, x, v) dv$.

The parameter $\nu_{\text{col}}^{(s)}$ is the characteristic frequency of collisions with the thermal bath for the species $s \in \{i, e\}$, k_B is the Boltzmann constant and ε_0 is the dielectric constant. To ease comparisons of respective sizes of parameters and allow corresponding asymptotic analysis we now aim at turning (2.9) in dimensionless form. To do so, in the following, we denote by L the characteristic length of the system, t_{obs} the characteristic observation time and V_i, V_e the thermal velocity of ions and electrons respectively. For any other physical quantity G , we denote by \bar{G} the characteristic value of G and G' the dimensionless quantity associated to G so that $G = \bar{G} G'$. Accordingly, we introduce

$$\begin{aligned} f_s(t, x, v) &= \frac{\bar{n}_s}{V_s^3} f'_s \left(\frac{t}{t_{\text{obs}}}, \frac{x}{L}, \frac{v}{V_s} \right), & n_s(t, x) &= \bar{n}_s n' \left(\frac{t}{t_{\text{obs}}}, \frac{x}{L} \right), \\ \phi(t, x) &= \bar{\phi} \phi' \left(\frac{t}{t_{\text{obs}}}, \frac{x}{L} \right). \end{aligned}$$

where $s \in \{i, e\}$, and reformulate (2.9) in terms of f'_i, f'_e, n'_i, n'_e and ϕ' .

Physical scales

We assume that the plasma is globally neutral and introduce the characteristic number of electrons N defined by

$$Z\bar{n}_i = \bar{n}_e = N.$$

Moreover we consider a hot plasma [19] meaning that characteristic temperatures (or kinetic energy) are equal to that of the thermal bath. It implies

$$m_i V_i^2 = m_e V_e^2 = k_B \theta.$$

Besides the characteristic potential energy due to electric effects is assumed of same order than that of kinetic effects which yields

$$\bar{\phi} = \frac{k_B \theta}{q}.$$

Now we may introduce several physical time and space scales characterizing each of the electric, magnetic and collision phenomena. Electrostatic constants are the Debye length and plasma time respectively given by

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B \theta}{q^2 N}}, \quad t_p^{(s)} = \frac{\lambda_D}{V_s}.$$

They measure the typical length of influence of an isolated particle and the period of electrostatic waves in the plasma. Magnetic constants are cyclotron time and the Larmor radius defining the period and radius of gyration of particles around magnetic field lines and reading respectively

$$t_c^{(s)} = \frac{m_s}{qB_{\text{ext}}}, \quad r_L^{(s)} = V_s t_c^{(s)}.$$

Finally the mechanism of collisions is characterized by a typical time between two collisions $1/(\nu_{\text{col}}^{(s)})$ and the mean free path of a particle

$$l_s = \frac{V_s}{\nu_{\text{col}}^{(s)}}.$$

We refer to the physics literature [101, 19, 154] for deeper insights and thorough comments on the respective roles of each of the former quantities.

Dimensionless parameters

From now on we use ionic quantities as references and set

$$t_{\text{obs}} = \frac{1}{\nu_{\text{col}}^{(i)}} \tau_{\text{obs}}, \quad L = l_i.$$

This leads to the consideration of quotients

$$\delta = \frac{\lambda_D}{L}, \quad \mu = \frac{r_L^{(i)}}{L}, \quad \gamma = \frac{l_e}{l_i}, \quad \lambda = \frac{m_e}{m_i}.$$

Since we focus on globally-neutral hot plasmas one may expect the ratio between mean free paths γ to depend only on the mass ratio λ . Consistently we set for some real exponent α ,

$$\gamma = \lambda^{\alpha/2}.$$

Incidentally we point to the attention of the reader [61, Section 1.5] as an instance of a detailed modeling analysis of the dependence of collision frequencies with respect to mass ratio λ for more realistic collisional operators and we note that the modeling arguments there suggests that $\alpha = 0$ would be the case of highest practical interest. Going on by dropping primes on dimensionless quantities, we may express the dimensionless form of the equations as

$$\begin{cases} \frac{\lambda^{\frac{1}{2}}}{\tau_{\text{obs}}} \partial_t f_e + v \cdot \nabla_x f_e + \left(\nabla_x \phi + \frac{1}{\lambda^{\frac{1}{2}} \mu} v^\perp \right) \cdot \nabla_v f_e = \frac{1}{\lambda^{\frac{\alpha}{2}}} \nabla_v \cdot (v f_e + \nabla_v f_e), \\ \frac{1}{\tau_{\text{obs}}} \partial_t f_i + v \cdot \nabla_x f_i + Z \left(-\nabla_x \phi - \frac{1}{\mu} v^\perp \right) \cdot \nabla_v f_i = \nabla_v \cdot (v f_i + \nabla_v f_i), \\ -\delta^2 \Delta_x \phi = n_i - n_e, \end{cases} \quad (2.10)$$

We refer to [61, 9] for more details on similar computations. Since the changes required to deal with the general case are of notational nature from now on we assume that the atomic number of ions Z is equal to one. Note that the foregoing assumption, made here purely for expository purposes, is consistent with the presence of light ions in tokamak plasmas [154].

In the situations we have in mind both λ and μ are small. In particular, the mass ratio λ is of order 10^{-4} for a deuterium plasma. As a consequence typical time scales of ions and electrons completely uncouple since, with our previous set of notation,

$$\lambda = \left(\frac{t_p^{(e)}}{t_p^{(i)}} \right)^2 = \frac{t_c^{(e)}}{t_c^{(i)}}.$$

The systems we consider in the core of the present chapter are motivated by the asymptotic analysis of the case where $\lambda \ll \mu \ll 1$. We believe that this distinguished regime is of particular relevance for applications. We stress however that it is far from covering the full range of $(\lambda, \mu) \ll 1$ and we do not intend here to justify a double limit in any case.

Ion dynamics interacting with massless electrons in the strong magnetic field limit.

Neglecting the fact that a strong magnetic field could slow down thermalization effects, one expects, as it is usually assumed in tokamak physics applications, that the electronic zero-inertia regime $\lambda \rightarrow 0$ brings the macroscopic density of electrons to a Boltzmann-Gibbs density provided that the scaled observation time is at least of order one, $\tau_{\text{obs}} \gtrsim 1$. See for instance Remark 2.2.3 for some elements of justification. This yields

$$n_e(t, x) = \frac{e^{\phi(t, x)}}{\int_{\mathbb{T}^3} e^{\phi(t, x)} dx}.$$

This approximation being taken for granted, we are now interested in the limit $\mu \rightarrow 0$, meaning that the Larmor radius becomes infinitely small. We point out that the corresponding asymptotics has already been extensively studied in related contexts, notably for collisionless models in [92, 105, 178].

In the present case, the small parameter of interest is thus

$$\varepsilon = \mu$$

and we recover (2.2)-(2.4), with $\sigma = 1$, $\alpha = 0$ on the observation time scale $\tau_{\text{obs}} = 1/\mu$. As already mentioned the later choice reflects the fact that one needs to consider long-time dynamics to observe a significant averaged evolution.

Electron dynamics in the massless limit

Though the former scenario may seem plausible our intention is precisely to convince the reader that the above-mentioned Boltzmann-Gibbs approximation for n_e fails when

collisions are not the dominant mechanism leading the electron dynamics. To do so, we focus on the kinetic description of electrons in the massless limit $\lambda \rightarrow 0$ when $\alpha < 1$.

In this context the small parameter of interest is

$$\varepsilon = \sqrt{\lambda}.$$

Since the ion evolution equation does not depend explicitly on λ and the coupling of equations is relatively weak we make the further simplification assumption that the macroscopic density of ions n_i is independent of λ and hence can be considered as frozen and given by a background density \underline{n} . This reflects the fact that without this simplification one would expect to recover in the limit $\lambda \rightarrow 0$ the same asymptotic equation derived under the frozen ion density assumption but coupled with the Vlasov-Fokker-Planck equation for the evolution of the "frozen" density \underline{n} . In slightly different related situations several papers dealing with the full coupling [124, 15] have proved that it was indeed a sound expectation.

After the harmless notational simplification consisting in setting $\mu = 1$, we recover the model (2.2)-(2.3), with $\sigma = -1$ when $\tau_{\text{obs}} = 1$.

Remark 2.2.1. *Our analysis not only indicates that the classical isotropic Boltzmann-Gibbs approximation fails to capture the electron dynamics in the massless limit but also provides an anisotropic replacement that may be coupled with the slow dynamics of ions. For completeness' sake we draw now briefly some consequences for the ion dynamics. Our first claim is that the dynamics of ions surrounded by massless electrons could be correctly described by*

$$\left\{ \begin{array}{l} \frac{1}{\tau_{\text{obs}}} \partial_t N_e + \mu \operatorname{div}_{x_\perp} (N_e (\nabla_{x_\perp} \tilde{\phi}_e)^\perp) = 0 \\ \frac{1}{\tau_{\text{obs}}} \partial_t f_i + v \cdot \nabla_x f_i + \left(-\nabla_x \phi - \frac{1}{\mu} v^\perp \right) \cdot \nabla_v f_i = \nabla_v \cdot (v f_i + \nabla_v f_i), \\ -\delta^2 \Delta_x \phi = n_i - N e^{\phi - \tilde{\phi}_e}, \end{array} \right.$$

where $\tilde{\phi}_e$ is obtained from ϕ by the nonlinear averaging (2.7) with $\sigma = -1$. An analysis similar to the one expounded here then shows that on time-scale $\tau_{\text{obs}} = 1/\mu$ the limit $\mu \rightarrow \infty$ leads n_i to an anisotropic Boltzmann-Gibbs regime where corresponding (N_e, N_i) satisfies a system of coupled guiding-center equations

$$\left\{ \begin{array}{l} \partial_t N_i + \operatorname{div}_{x_\perp} (N_i \nabla_{x_\perp}^\perp \phi) = 0, \\ \partial_t N_e + \operatorname{div}_{x_\perp} (N_e \nabla_{x_\perp}^\perp \phi) = 0, \\ -\delta^2 \Delta_{x_\perp} \phi = N_i - N_e. \end{array} \right.$$

Incidentally note that consistently with classical gyrokinetic theory both species are advected by the same electric drifts irrespective of the sign of their charges.

2.2.2 Formal derivation

Now we begin the derivation of (2.6)-(2.7)-(2.8) from (2.2). Here we argue formally and in particular take for granted in this heuristic section that any sequence of functions that is bounded in one norm actually converges in any norm required to take limits and that rates of convergence may indeed be predicted by balancing orders in equations. It goes without telling that conclusions thus derived must then be taken with care. In particular even when formally derived results may be rigorously justified, in general the mathematical validation may not proceed by proving each step of the crude path followed arguing heuristically. Moreover especially in strongly oscillating asymptotics formal arguments may lead to misleading conclusions. Indeed terms that may seem prominent may actually turn to be irrelevant because of averaging effects, that is because for instance they are derivatives of small immaterial terms, or in the converse direction terms that are small may have huge derivatives and thus could yield strong effects. Besides, taking oscillating limits typically do not commute with nonlinear operations. This being stated we begin our formal process leading to an identification of possible asymptotic dynamics but try to use arguments as close as possible from those that we do justify in the linear setting in Section 2.4.2.

To do so we pick a family of solutions (f^ε) and denote (ϕ^ε) corresponding electric potentials (whether they actually depend on ε or not). We may normalize potentials by requiring for any t and ε

$$\int_{\mathbb{T}^3} e^{-\sigma \phi^\varepsilon(t,y)} dy = 1. \quad (2.11)$$

This amounts to replacing the original ϕ^ε with

$$\phi^\varepsilon - \ln \left(\int_{\mathbb{T}^3} e^{-\sigma \phi^\varepsilon(t,y)} dy \right),$$

a process that as expected does not modify any equation.

We stress that aiming at a unified in α derivation precludes any use of an asymptotic expansion in powers of ε . Hence alternatively we rely directly on symmetries of the equations to sort out different terms of the equation. Incidentally note that even in an α -by- α identification several symmetry arguments would be needed to derive that some terms that are expected to sum to zero actually vanish separately.

Free energy identities

To begin with, we introduce the time-dependent measure $d\mu_t^\varepsilon = \mathcal{M}^\varepsilon(t, x, v) dx dv$ where the global Maxwellian is given by

$$\mathcal{M}^\varepsilon(t, x, v) = e^{-\sigma \phi^\varepsilon(t,x)} M(v) \quad (2.12)$$

and $M(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$, so that, roughly speaking, transport terms and collision terms are respectively skew-symmetric and symmetric with respect to $d\mu_t^\varepsilon$. As a consequence, setting

$$h^\varepsilon = f^\varepsilon (\mathcal{M}^\varepsilon)^{-1},$$

we obtain the following family of *a priori* estimates.

Lemma 2.2.2. *Let f^ε be a smooth and localized solution of the magnetized Vlasov-Fokker-Planck equation (2.2) with smooth electric potential ϕ^ε . Then for any smooth function H ,*

$$\begin{aligned} \frac{d}{dt} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} H(h^\varepsilon) d\mu^\varepsilon + \frac{1}{\varepsilon^{\alpha+1}} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} H''(h^\varepsilon) |\nabla_v h^\varepsilon|^2 d\mu^\varepsilon \\ = - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (H'(h^\varepsilon) h^\varepsilon - H(h^\varepsilon)) \sigma \partial_t \phi^\varepsilon d\mu^\varepsilon. \end{aligned} \quad (2.13)$$

In particular,

$$\frac{d}{dt} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f^\varepsilon \left(\ln f^\varepsilon + \frac{|v|^2}{2} + \sigma \phi^\varepsilon \right) + \frac{1}{\varepsilon^{\alpha+1}} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{1}{h^\varepsilon} |\nabla_v h^\varepsilon|^2 d\mu^\varepsilon = - \int_{\mathbb{T}^3} n^\varepsilon \sigma \partial_t \phi^\varepsilon. \quad (2.14)$$

Proof. We omit the detailed computation leading to (2.13) as perfectly standard. Equation (2.14) follows then from the choice $H(h) = h \ln(h)$. \square

Assuming moreover (2.3) or (2.4) one may rewrite (2.14) in a form related to the conservation of some free energy. However since we shall make no use of such considerations we do not dwell on those here.

Now we formally take limits in the free energy identities of the previous lemma. For the limiting functions, we drop the superscript ε (e.g. f , ϕ , n , h , μ ...). Since $\alpha > -1$, we get for all time $t \geq 0$

$$\int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{1}{h(s, \cdot, \cdot)} |\nabla_v h(s, \cdot, \cdot)|^2 d\mu_s \quad ds = 0.$$

As a consequence h only depends on space and time variables which means that the limiting distribution function is a local Maxwellian, namely $f = nM$. By inserting this into equation (2.2) and balancing terms we can surmise³⁵ that

$$\tilde{f}^\varepsilon = \frac{f^\varepsilon - f}{\varepsilon}$$

should possess a limit \tilde{f} that satisfies for any (t, x, v)

$$v \cdot \nabla_x f(t, x, v) - \sigma \nabla_x \phi(t, x) \cdot \nabla_v f(t, x, v) - \sigma v^\perp \cdot \nabla_v \tilde{f}(t, x, v) = 0.$$

Now comes the key observation that the image of $v^\perp \cdot \nabla_v$ consists of functions with zero average with respect to the polar angle θ from cylinder coordinates $(|v_\perp|, \theta, v_\parallel)$ implicitly

³⁵. Unlike the rest of the formal analysis this expectation has no rigorous counterpart in our mathematical analysis. It is highly dubious that it could be fulfilled.

defined by $v = (|v_\perp| \cos(\theta), |v_\perp| \sin(\theta), v_\parallel)$. From this and the radial character of f with respect to velocity variable stems

$$v_\parallel \partial_{x_\parallel} f(t, x, v) - \sigma \partial_{x_\parallel} \phi(t, x) \partial_{v_\parallel} f(t, x, v) = 0.$$

Using once again that f is Maxwellian in velocity we conclude that indeed

$$f(t, x, v) = N(t, x_\perp) \frac{e^{-\sigma \phi(t, x)}}{\int_{\mathbb{T}} e^{-\sigma \phi(t, x_\perp, y_\parallel)} dy_\parallel} \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}}$$

where

$$N(t, x_\perp) = \iint_{\mathbb{T} \times \mathbb{R}^3} f(t, x, v) dx_\parallel dv.$$

Dynamics of velocity moments

The only task left is the identification of an equation for N . To proceed we denote by N^ε the integral of f^ε with respect to (x_\parallel, v) and observe that a corresponding integration of (2.2) leads to a certain form of conservation of charge

$$\partial_t N^\varepsilon + \operatorname{div}_{x_\perp} (J_\perp^\varepsilon) = 0$$

where

$$J^\varepsilon(t, x) = \frac{1}{\varepsilon} \int_{\mathbb{T} \times \mathbb{R}^3} v f^\varepsilon(t, x, v) dx_\parallel dv.$$

This suggests that J_\perp^ε could possess a limit J_\perp . To identify this limit we proceed as above. By multiplying (2.2) by v_\perp and integrating one derives

$$\begin{aligned} \varepsilon^2 \partial_t J_\perp^\varepsilon + \operatorname{div}_{x_\perp} \left(\int_{\mathbb{T} \times \mathbb{R}^3} v_\perp \otimes v_\perp f^\varepsilon(\cdot, \cdot, x_\parallel, v) dx_\parallel dv \right) + \\ \sigma \int_{\mathbb{T}} \nabla_{x_\perp} \phi^\varepsilon(\cdot, \cdot, x_\parallel) n^\varepsilon(\cdot, \cdot, x_\parallel) dx_\parallel = -\sigma (J_\perp^\varepsilon)^\perp - \varepsilon^{1-\alpha} J_\perp^\varepsilon \end{aligned}$$

where $^\perp$ acts on vectors of \mathbb{R}^2 through $(j_1, j_2)^\perp = (-j_2, j_1)$. Taking limits brings

$$-\sigma (J_\perp)^\perp = \operatorname{div}_{x_\perp} \left(\int_{\mathbb{T} \times \mathbb{R}^3} v_\perp \otimes v_\perp f(\cdot, \cdot, x_\parallel, v) dx_\parallel dv \right) + \sigma \int_{\mathbb{T}} \nabla_{x_\perp} \phi(\cdot, \cdot, x_\parallel) n(\cdot, \cdot, x_\parallel) dx_\parallel$$

which, by using the special form of f , may also be written as

$$-\sigma (J_\perp)^\perp = \nabla_{x_\perp} N - N \nabla_{x_\perp} \left(\ln \left(\int_{\mathbb{T}} e^{-\sigma \phi(\cdot, \cdot, x_\parallel)} dx_\parallel \right) \right).$$

Inserting this in the limit of the foregoing equation of conservation of charge achieves the derivation of (2.6) after noticing that $(\nabla_{x_\perp} N)^\perp$ is divergence free.

Remark 2.2.3. *To ease comparisons we briefly sketch how in a collision-dominated scenario a similar formal analysis may be carried out. Therefore in the present remark we insert $\sigma_0 \in \{0, 1\}$ in front of the magnetic field term and allow $\sigma_0 = 0$ (non magnetized case) and $\alpha > 1$ (very strong collisions). The condition $\alpha > -1$ is sufficient to support the expectation that f^ε converges to some $f = nM$. Then one turns to the analysis of moments*

$$n^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon(\cdot, \cdot, v) dv, \quad j^\varepsilon = \frac{1}{\varepsilon} \int_{\mathbb{R}^3} v f^\varepsilon(\cdot, \cdot, v) dv$$

involved in the following version of conservation of charge

$$\partial_t n^\varepsilon + \operatorname{div}_x(j^\varepsilon) = 0$$

and computes that

$$\varepsilon^2 \partial_t j^\varepsilon + \operatorname{div}_x \left(\int_{\mathbb{R}^3} v \otimes v f^\varepsilon dv \right) + \sigma \nabla_x \phi^\varepsilon n^\varepsilon = -\sigma \sigma_0 (j^\varepsilon)^\perp - \varepsilon^{1-\alpha} j^\varepsilon. \quad (2.15)$$

If $\sigma_0 = 0$ and $-1 < \alpha < 1$ taking the limit $\varepsilon \rightarrow 0$ and inserting $f = nM$ yields

$$\nabla_x n + \sigma n \nabla_x \phi = 0$$

hence the classical Maxwell-Boltzmann approximation as for instance in [15]. If $\alpha = 1$ (no matter what σ_0 is) one rather obtains

$$\nabla_x n + \sigma n \nabla_x \phi = -\sigma \sigma_0 j^\perp - j$$

that may be solved for j and inserted in $\partial_t n + \operatorname{div}_x(j) = 0$ to derive a diffusive equation for n as for instance in [124]. At last if $\alpha > 1$ (no matter what σ_0 is) one derives $j = 0$ hence $\partial_t n = 0$ thus the formal analysis suggests convergence to a global equilibrium. In our case we have essentially taken the limit of the third component of (2.15) to obtain Boltzmann-Gibbs behavior along the magnetic field and we have integrated the two first components of (2.15) to derive the equation for N .

2.3 Well-posedness of asymptotic models

2.3.1 Light particles

In the present subsection we analyze the asymptotic system obtained in the electronic case so that, since it is consistent with modeling, for reading convenience we set $\sigma = -1$. Therefore we study the well-posedness of the following system

$$\begin{cases} \partial_t N + \operatorname{div}_{x_\perp} (N \nabla_{x_\perp}^\perp \tilde{\phi}) = 0, \\ -\delta^2 \Delta_x \phi = \underline{n} - N e^{\phi - \tilde{\phi}}, \quad \tilde{\phi}(t, x_\perp) = \ln \left(\int_{\mathbb{T}} e^{\phi(t, x_\perp, x_\parallel)} dx_\parallel \right). \end{cases} \quad (2.16)$$

We are thus trying to solve a conservation law corresponding to advection by a divergence-free velocity field build from a "stream function" $\tilde{\phi}$ obtained by averaging the solution

to a second-order elliptic equation. As already pointed out in the introduction once sufficient control on the construction of $\tilde{\phi}$ from N has been obtained one may follow standard arguments for similar equations such as Vlasov-Poisson equations [99, 30, 174] or two-dimensional incompressible Euler equations in vorticity formulation [145, 52, 143].

Anisotropic Poisson-Boltzmann equation

Accordingly we first focus on the elliptic equation involved in System 2.16. Since time variable is a simple parameter there, we temporarily omit to mention time dependence along the following considerations. Moreover for concision's sake we do not repeat but always assume $N \geq 0$, $\underline{n} \geq 0$, $\int_{\mathbb{T}^3} \underline{n} = \int_{\mathbb{T}^2} N = 1$.

Hence we study the following anisotropic Poisson-Boltzmann equation

$$-\delta^2 \Delta_x \phi(x_\perp, x_\parallel) = \underline{n}(x_\perp, x_\parallel) - N(x_\perp) \frac{e^{\phi(x_\perp, x_\parallel)}}{\int_{\mathbb{T}} e^{\phi(x_\perp, y_\parallel)} dy_\parallel} \quad (2.17)$$

where N and \underline{n} are nonnegative functions with integral equal to one. The key observation is that Equation 2.17 is the Euler-Lagrange equation associated to the energy functional

$$J[\psi] = \frac{1}{2} \delta^2 \int_{\mathbb{T}^3} |\nabla \psi|^2 + \int_{\mathbb{T}^2} N \ln \left(\int_{\mathbb{T}} e^\psi dx_\parallel \right) dx_\perp - \int_{\mathbb{T}^3} \underline{n} \psi. \quad (2.18)$$

Explicitly, at least for smooth pairs (ϕ, ψ) ,

$$\left. \frac{d}{d\tau} \right|_{\tau=0} J[\phi + \tau\psi] = \int_{\mathbb{T}^3} \left(\delta^2 \nabla \phi \cdot \nabla \psi + N \frac{e^\phi}{\int_{\mathbb{T}} e^\phi dx_\parallel} \psi - \underline{n} \psi \right) dx \quad (2.19)$$

We aim now at identifying an appropriate functional setting in which J is strictly convex and coercive. From this shall stem existence and uniqueness for (2.17). Afterwards we focus on corresponding regularity issues.

To do so we introduce

$$H_0 = \left\{ h \in \mathcal{D}'(\mathbb{T}^3) \mid \nabla h \in L^2(\mathbb{T}^3) \text{ and } \int_{\mathbb{T}^3} h = 0 \right\}.$$

The set H_0 is a closed linear subspace of the Sobolev space $H^1(\mathbb{T}^3)$ and by Mazur's theorem it is also weakly closed. Moreover, by Sobolev embedding and Poincaré inequality $H_0 \hookrightarrow L^6(\mathbb{T}^3)$.

Proposition 2.3.1. *Assume $N \in L^{4/3}(\mathbb{T}^2)$ and $\underline{n} \in L^{6/5}(\mathbb{T}^3)$. Then*

- (i) $J : H_0 \rightarrow \mathbb{R}$ is well defined, bounded on bounded sets and coercive;
- (ii) J is strictly convex;
- (iii) J is Gâteaux differentiable;
- (iv) J is weakly lower semi-continuous.

Proof. Since $H_0 \hookrightarrow L^6(\mathbb{T}^3)$, for some C

$$\int_{\mathbb{T}^3} \underline{n} |\psi| dx \leq \|\underline{n}\|_{L^{6/5}(\mathbb{T}^3)} \|\psi\|_{L^6(\mathbb{T}^3)} \leq C \|\underline{n}\|_{L^{6/5}(\mathbb{T}^3)} \|\psi\|_{H^1(\mathbb{T}^3)}.$$

Likewise since³⁶ $H_0 \hookrightarrow L^2(\mathbb{T}^2; H^1(\mathbb{T})) \cap L^6(\mathbb{T}^3) \hookrightarrow L^4(\mathbb{T}^2; L^\infty(\mathbb{T}))$, a pointwise estimate on the exponential function shows that

$$\begin{aligned} \int N \left| \ln \left(\int_{\mathbb{T}} e^{\psi(\cdot, y_{\parallel})} dy_{\parallel} \right) \right| &\leq \|N\|_{L^{4/3}(\mathbb{T}^2)} \left\| \ln \left(\int_{\mathbb{T}} e^{\psi(\cdot, y_{\parallel})} dy_{\parallel} \right) \right\|_{L^4(\mathbb{T}^2)} \\ &\leq C \|N\|_{L^{4/3}(\mathbb{T}^2)} \|\psi\|_{L^4(\mathbb{T}^2; L^\infty(\mathbb{T}))} \\ &\leq C' \|N\|_{L^{4/3}(\mathbb{T}^2)} \|\psi\|_{L^6(\mathbb{T}^3)}^{3/4} \|\psi\|_{L^2(\mathbb{T}^2; H^1(\mathbb{T}))}^{1/4} \\ &\leq C'' \|N\|_{L^{4/3}(\mathbb{T}^2)} \|\psi\|_{H^1(\mathbb{T}^3)} \end{aligned}$$

for some C, C', C'' . Combined with a Poincaré inequality this provides

$$\left| J[\psi] - \frac{1}{2} \delta^2 \|\nabla \psi\|_{L^2(\mathbb{T}^3)}^2 \right| \leq C (\|N\|_{L^{4/3}(\mathbb{T}^2)} + \|\underline{n}\|_{L^{6/5}(\mathbb{T}^3)}) \|\nabla \psi\|_{L^2(\mathbb{T}^3)}, \quad (2.20)$$

for some C , which proves the first claim.

Thanks to the Hölder inequality it holds, for $\theta \in [0, 1]$

$$\int_{\mathbb{T}} e^{\theta \psi_1 + (1-\theta) \psi_2} dx_{\parallel} \leq \left(\int_{\mathbb{T}} e^{\psi_1} dx_{\parallel} \right)^{\theta} \left(\int_{\mathbb{T}} e^{\psi_2} dx_{\parallel} \right)^{1-\theta}.$$

Therefore the second term of the right-hand side of (2.18) is convex. The first is strictly convex and the last is linear. This proves the second claim.

To prove differentiability, since the first and third term of J are respectively quadratic and linear continuous, one may safely focus on the middle term of J . The latter differentiability follows from applying twice the Dominated Convergence Theorem. To do so, we note that when $(\phi, \psi) \in (H_0)^2$ and $(t, s) \in \mathbb{R}^2$, for almost any x_{\perp}

$$\begin{aligned} \left| \ln \left(\int_{\mathbb{T}} e^{(\phi+t\psi)(x_{\perp}, y_{\parallel})} dy_{\parallel} \right) - \ln \left(\int_{\mathbb{T}} e^{(\phi+s\psi)(x_{\perp}, z_{\parallel})} dz_{\parallel} \right) \right| \\ = \ln \left(\int_{\mathbb{T}} e^{(t-s)\psi(x_{\perp}, y_{\parallel})} \frac{e^{(\phi+s\psi)(x_{\perp}, y_{\parallel})} dy_{\parallel}}{\int_{\mathbb{T}} e^{(\phi+s\psi)(x_{\perp}, z_{\parallel})} dz_{\parallel}} \right) \\ \leq \ln \left(\int_{\mathbb{T}} e^{|t-s| \|\psi(x_{\perp}, \cdot)\|_{L^\infty(\mathbb{T})}} \frac{e^{(\phi+s\psi)(x_{\perp}, y_{\parallel})} dy_{\parallel}}{\int_{\mathbb{T}} e^{(\phi+s\psi)(x_{\perp}, z_{\parallel})} dz_{\parallel}} \right) \\ \leq |t-s| \|\psi(x_{\perp}, \cdot)\|_{L^\infty(\mathbb{T})} \end{aligned}$$

36. The last embedding follows from Gagliardo-Nirenberg inequality

$$\|\psi(x_{\perp}, \cdot)\|_{L^\infty(\mathbb{T})} \leq C \|\psi(x_{\perp}, \cdot)\|_{L^6(\mathbb{T})}^{3/4} \|\psi(x_{\perp}, \cdot)\|_{H^1(\mathbb{T})}^{1/4}$$

combined with one Hölder inequality. From now on we shall make use of similar anisotropic Sobolev embeddings without further comment.

where we have broken the symmetry by assuming that

$$\int_{\mathbb{T}} e^{(\phi+s\psi)(x_{\perp},z_{\parallel})} dz_{\parallel} \leq \int_{\mathbb{T}} e^{(\phi+t\psi)(x_{\perp},y_{\parallel})} dy_{\parallel}$$

and used monotonicity of \ln and \exp . From this follows the third claim.

The last claim follows for instance from convexity and local boundedness combined with Mazur's theorem. \square

The foregoing proposition yields by classical arguments the following one.

Proposition 2.3.2 (Existence and uniqueness). *For any $N \in L^{4/3}(\mathbb{T}^2)$ and $\underline{n} \in L^{6/5}(\mathbb{T}^3)$, Equation (2.17) possesses a unique weak solution $\phi \in H_0$. Moreover there exists C such that for any such (N, \underline{n}) the solution ϕ satisfies*

$$\|\phi\|_{H^1(\mathbb{T}^3)} \leq C(\|N\|_{L^{4/3}(\mathbb{T}^2)} + \|\underline{n}\|_{L^{6/5}(\mathbb{T}^3)}).$$

Besides there exists C such that for any pair $(N, \underline{n}), (N', \underline{n}')$ of such couples respective solutions ϕ and ϕ' satisfy

$$\|\phi - \phi'\|_{H^1(\mathbb{T}^3)} \leq C(\|N - N'\|_{L^{4/3}(\mathbb{T}^2)} + \|\underline{n} - \underline{n}'\|_{L^{6/5}(\mathbb{T}^3)})$$

Proof. Existence and uniqueness follows from Proposition 2.3.1 along the classical line of the direct method of calculus of variations. Then the uniform bound stems directly from coercivity estimate (2.20) and the fact that J takes the value 0 at the null function. At last, as we explain now, the Lipschitz estimate follows from quantifying convexity and local boundedness. Temporarily, to make it precise, we mark J with suffixes N, \underline{n} . By using that ϕ and ϕ' are critical points of, respectively, $J_{N, \underline{n}}$ and $J_{N', \underline{n}'}$ and that each of the three terms defining J is convex, one obtains

$$\begin{aligned} \delta^2 \|\nabla(\phi - \phi')\|_{L^2(\mathbb{T}^3)}^2 &\leq J_{N, \underline{n}}(\phi') - J_{N, \underline{n}}(\phi) + J_{N', \underline{n}'}(\phi) - J_{N', \underline{n}'}(\phi') \\ &= \int_{\mathbb{T}^2} (N - N') \ln \left(\frac{\int_{\mathbb{T}} e^{\phi'} dx_{\parallel}}{\int_{\mathbb{T}} e^{\phi} dx_{\parallel}} \right) dx_{\perp} - \int_{\mathbb{T}^3} (\underline{n} - \underline{n}') (\phi' - \phi) dx \\ &\leq C'(\|N - N'\|_{L^{4/3}(\mathbb{T}^2)} + \|\underline{n} - \underline{n}'\|_{L^{6/5}(\mathbb{T}^3)}) \|\phi - \phi'\|_{H^1(\mathbb{T}^3)} \end{aligned}$$

for some C' , where we have used the pointwise bound

$$\ln \left(\frac{\int_{\mathbb{T}} e^{\phi'(x_{\perp}, x_{\parallel})} dx_{\parallel}}{\int_{\mathbb{T}} e^{\phi(x_{\perp}, x_{\parallel})} dx_{\parallel}} \right) \leq \|(\phi - \phi')(x_{\perp}, \cdot)\|_{L^{\infty}(\mathbb{T})} \quad (2.21)$$

and the embedding $H_0 \hookrightarrow L^4(\mathbb{T}^2; L^{\infty}(\mathbb{T}))$. From here a Poincaré inequality yields the result. \square

Now we discuss regularity of solutions to Equation (2.17). In the next proposition we use classical elliptic regularity properties — in Calderón-Zygmund form — and maximum principles — as recalled in Appendix C.4 — to obtain estimates of ϕ in higher order Sobolev spaces. We refer the reader to [183, 184] or [108] on the classical Calderón-Zygmund regularity theory and to [139, Chapter 2] for relevant basic properties of Lorentz spaces appearing in the following.

Proposition 2.3.3 (Regularity). *There exists C such that if $N \in L^{4/3}(\mathbb{T}^2)$ and $\underline{n} \in L^{3/2,1}(\mathbb{T}^3)$ then ϕ the unique solution to (2.17) satisfies*

$$\|e^\phi\|_{L^\infty(\mathbb{T}^3)} + \left\| \frac{1}{\int_{\mathbb{T}} e^{\phi(\cdot, y_{\parallel})} dy_{\parallel}} \right\|_{L^\infty(\mathbb{T}^2)} \leq C e^{C(\|\underline{n}\|_{L^{3/2,1}(\mathbb{T}^3)} + \|N\|_{L^{4/3}(\mathbb{T}^2)})}$$

and if moreover $N \in L^{3/2,1}(\mathbb{T}^2)$

$$\|\phi\|_{L^\infty(\mathbb{T}^3)} \leq C \left(\|\underline{n}\|_{L^{3/2,1}(\mathbb{T}^3)} + \|N\|_{L^{3/2,1}(\mathbb{T}^2)} e^{C(\|\underline{n}\|_{L^{3/2,1}(\mathbb{T}^3)} + \|N\|_{L^{4/3}(\mathbb{T}^2)})} \right).$$

Moreover, for any $1 < p < \infty$, there exists C such that if $N \in L^{4/3}(\mathbb{T}^2) \cap L^p(\mathbb{T}^2)$ and $\underline{n} \in L^{3/2,1}(\mathbb{T}^3) \cap L^p(\mathbb{T}^3)$ then ϕ the unique solution to (2.17) satisfies

$$\|\phi\|_{W^{2,p}(\mathbb{T}^3)} \leq C \left(\|\underline{n}\|_{L^p(\mathbb{T}^3)} + \|N\|_{L^p(\mathbb{T}^2)} e^{C(\|\underline{n}\|_{L^{3/2,1}(\mathbb{T}^3)} + \|N\|_{L^{4/3}(\mathbb{T}^2)})} \right).$$

Besides for any $1 < p < \infty$ and any $1 \leq q \leq \infty$ such that

$$\frac{3}{2p} - 1 < \frac{1}{q} \leq \frac{1}{p}$$

there exists C such that with

$$\frac{1}{r} = \begin{cases} \frac{1}{3} + \frac{1}{p} - \frac{2}{3q} & \text{if } q \geq 2 \\ \frac{1}{2} + \frac{1}{p} - \frac{1}{q} & \text{if } q \leq 2 \end{cases}$$

if $N \in L^q(\mathbb{T}^2) \cap L^r(\mathbb{T}^2) \cap W^{1,p}(\mathbb{T}^2)$ and $\underline{n} \in L^r(\mathbb{T}^3) \cap W^{1,p}(\mathbb{T}^3)$ then ϕ the unique solution to (2.17) satisfies

$$\begin{aligned} \|\phi\|_{W^{3,p}(\mathbb{T}^3)} &\leq C \left(\|\underline{n}\|_{W^{1,p}(\mathbb{T}^3)} + \|N\|_{L^q(\mathbb{T}^2)} \|\underline{n}\|_{L^r(\mathbb{T}^3)} e^{C(\|\underline{n}\|_{L^{3/2,1}(\mathbb{T}^3)} + \|N\|_{L^{4/3}(\mathbb{T}^2)})} \right) \\ &+ C e^{C(\|\underline{n}\|_{L^{3/2,1}(\mathbb{T}^3)} + \|N\|_{L^{4/3}(\mathbb{T}^2)})} (\|N\|_{W^{1,p}(\mathbb{T}^2)} + \|N\|_{L^q(\mathbb{T}^2)} \|\underline{n}\|_{L^r(\mathbb{T}^2)}). \end{aligned}$$

To ease later use of estimates of $\|\phi\|_{W^{3,p}(\mathbb{T}^3)}$ note that for instance the choice

$$\frac{1}{q} = \frac{1}{2} \left(\frac{1}{p} + \frac{1}{\max(\{2, p\})} \right)$$

is always available and that in this case $r \leq q$.

Proof. Bounds in L^∞ follow from repeated use of the maximum principle stated in Lemma C.4.1. Indeed, first, since $L^{3/2,1}(\mathbb{T}^3) \hookrightarrow L^{6/5}(\mathbb{T}^3)$ and $-\delta^2 \Delta \phi \leq \underline{n}$, the bound on e^ϕ follows from $0 \leq e^\phi \leq e^{\text{ess sup } \phi}$. In turn the bound on $(\int_{\mathbb{T}} e^{\phi(x_\perp, y_\parallel)} dy_\parallel)^{-1}$ stems from

$$0 \leq \frac{1}{\int_{\mathbb{T}} e^{\phi(x_\perp, y_\parallel)} dy_\parallel} \leq e^{-\int_{\mathbb{T}} \phi(x_\perp, y_\parallel) dy_\parallel} \leq e^{\text{ess sup}(-\int_{\mathbb{T}} \phi(x_\perp, y_\parallel) dy_\parallel)}$$

and $-\Delta_{x_\perp}(-\int_{\mathbb{T}} \phi dx_\parallel) = N - \int_{\mathbb{T}} \underline{n}(\cdot, y_\parallel) dy_\parallel$ combined with embeddings $W^{2,4/3}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$ and $L^{3/2,1}(\mathbb{T}^3) \hookrightarrow L^{4/3}(\mathbb{T}^2; L^1(\mathbb{T}))$. At last, using the equation, those two estimates yield then estimates on ϕ itself both in $L^\infty(\mathbb{T}^3)$ and $W^{2,p}(\mathbb{T}^3)$ respectively by maximum principle and elliptic regularity.

The bound in $W^{3,p}(\mathbb{T}^3)$ may then be obtained by differentiating once the equation. Indeed differentiation yields

$$-\delta^2 \Delta \nabla \phi = \nabla \underline{n} - \left(\nabla N + N \nabla \phi - N \nabla \tilde{\phi} \right) e^{\phi - \tilde{\phi}}$$

where we have introduced notation $\tilde{\phi}$ defined through nonlinear averaging (2.7) for concision's sake. Moreover, by using Lemma 2.3.5 below to estimate $\tilde{\phi}$ and Hölder inequalities, we obtain for some C independent of N

$$\|N \nabla \phi\|_{L^p(\mathbb{T}^3)} + \|N \nabla \tilde{\phi}\|_{L^p(\mathbb{T}^3)} \leq C e^{C(\|\underline{n}\|_{L^{3/2,1}(\mathbb{T}^3)} + \|N\|_{L^{4/3}(\mathbb{T}^2)})} \|N\|_{L^q(\mathbb{T}^2)} \|\nabla \phi\|_{L^s(\mathbb{T}^2; L^p(\mathbb{T}))}.$$

with $1/s = 1/p - 1/q$. One achieves the proof of the claim through elliptic regularity and bounds on $\|\phi\|_{W^{2,r}(\mathbb{T}^3)}$ combined with embedding $W^{1,r}(\mathbb{T}^3) \hookrightarrow L^s(\mathbb{T}^2; L^p(\mathbb{T}))$. Strictly speaking instead of differentiating the equation and estimating as above terms thus appearing we should have applied finite differences to the equation and estimated those finite differences. But here and elsewhere we skip over those cumbersome details. This is particularly immaterial in our analysis since there is no boundary to deal with. \square

With the obtained regularity one may provide higher-order counter-parts to the Lipschitz bound of Proposition 2.3.2.

Proposition 2.3.4 (Lipschitz dependence). *For any $4/3 \leq p < \infty$, there exists C such that if $N \in L^p(\mathbb{T}^2)$, $N' \in L^p(\mathbb{T}^2)$ and $\underline{n} \in L^{3/2,1}(\mathbb{T}^3)$ then ϕ and ϕ' the respective solutions to (2.17) corresponding to (N, \underline{n}) and (N', \underline{n}) satisfy*

$$\|\phi - \phi'\|_{W^{2,p}(\mathbb{T}^3)} \leq C \|N - N'\|_{L^p(\mathbb{T}^2)} (1 + \|N'\|_{L^p(\mathbb{T}^2)})^2 e^{C(\|\underline{n}\|_{L^{3/2,1}(\mathbb{T}^3)} + \|N\|_{L^{4/3}(\mathbb{T}^2)} + \|N'\|_{L^{4/3}(\mathbb{T}^2)})}.$$

Proof. The power 2 in the above estimate reflects the fact that we provide a three-steps proof to reach the full range of Lebesgue indices p . First observe that $H^1(\mathbb{T}^3) \hookrightarrow L^q(\mathbb{T}^2; L^2(\mathbb{T}))$ for any $1 \leq q < \infty$. Using the equation, we get

$$-\delta^2 \Delta(\phi - \phi') = (N - N')e^{\phi - \tilde{\phi}} + (e^\phi - e^{\phi'})N'e^{-\tilde{\phi}} + (e^{-\tilde{\phi}} - e^{-\tilde{\phi}'})N'e^{\phi'}$$

where again we use notation from (2.7). By using Lemma 2.3.5 this enables us to prove that for any $1 \leq r < 4/3$ there exists C_r such that

$$\|\phi - \phi'\|_{W^{2,r}(\mathbb{T}^3)} \leq C_r \|N - N'\|_{L^{4/3}(\mathbb{T}^2)} e^{C_r (\|n\|_{L^{3/2,1}(\mathbb{T}^3)} + \|N\|_{L^{4/3}(\mathbb{T}^2)} + \|N'\|_{L^{4/3}(\mathbb{T}^2)})}.$$

Using once again the equation and the embedding $W^{2,r_p}(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^2; L^p(\mathbb{T}))$ for a suitable $1 \leq r_p < 4/3$, one deduces the result (with a bound depending linearly on $\|N'\|_{L^p(\mathbb{T}^2)}$) when $p < 4$.

One may then use this intermediate case to conclude the proof by using again the equation and the embedding $W^{2,p_0}(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$ for some $1 \leq p_0 < 4$ when $p \geq 4$. \square

Prior to turning back to Equation (2.16), we provide estimates converting bounds on a potential ϕ into bounds on advection field $\tilde{\phi}$ obtained through nonlinear averaging (2.7). That is the content of the following lemma. It is of course necessary to deal with (2.16) and we have already used it repeatedly in foregoing estimates on ϕ .

Lemma 2.3.5. (i). *For any $\phi \in H_0$, let $\tilde{\phi}$ be defined by (2.7), then for any $1 \leq p \leq \infty$*

$$\|\tilde{\phi}\|_{L^p(\mathbb{T}^2)} \leq \|\phi\|_{L^p(\mathbb{T}^2; L^\infty(\mathbb{T}))}$$

and

$$\|\nabla_{x_\perp} \tilde{\phi}\|_{L^p(\mathbb{T}^2)} \leq \min(\{\|\nabla_{x_\perp} \phi\|_{L^p(\mathbb{T}^2; L^\infty(\mathbb{T}))}, \|e^\phi\|_{L^\infty(\mathbb{T}^3)} \|e^{-\tilde{\phi}}\|_{L^\infty(\mathbb{T}^2)} \|\nabla_{x_\perp} \phi\|_{L^p(\mathbb{T}^2; L^1(\mathbb{T}))}\}).$$

Moreover there exists C such that for any ϕ , p and $\tilde{\phi}$ as above,

$$\|\text{Hess}_{x_\perp} \tilde{\phi}\|_{L^p(\mathbb{T}^2)} \leq C \|e^\phi\|_{L^\infty(\mathbb{T}^3)}^2 \|e^{-\tilde{\phi}}\|_{L^\infty(\mathbb{T}^2)}^2 \left(\|\text{Hess}_{x_\perp} \phi\|_{L^p(\mathbb{T}^2; L^1(\mathbb{T}))} + \|\nabla_{x_\perp} \phi\|_{L^{2p}(\mathbb{T}^2; L^2(\mathbb{T}))} \right).$$

(ii). *For any $\phi, \phi' \in H_0$, let $\tilde{\phi}$ and $\tilde{\phi}'$ be defined by nonlinear averaging (2.7) respectively from ϕ and ϕ' , then for any $1 \leq p \leq \infty$*

$$\begin{aligned} \|\tilde{\phi} - \tilde{\phi}'\|_{L^p(\mathbb{T}^2)} &\leq \min(\{\|\phi - \phi'\|_{L^p(\mathbb{T}^2; L^\infty(\mathbb{T}))}, \\ &(\|e^\phi\|_{L^\infty(\mathbb{T}^3)} + \|e^{\phi'}\|_{L^\infty(\mathbb{T}^3)}) (\|e^{-\tilde{\phi}}\|_{L^\infty(\mathbb{T}^2)} + \|e^{-\tilde{\phi}'}\|_{L^\infty(\mathbb{T}^2)}) \|\phi - \phi'\|_{L^p(\mathbb{T}^2; L^1(\mathbb{T}))}\}) \end{aligned}$$

and

$$\begin{aligned} \|\nabla_{x_\perp}(\tilde{\phi} - \tilde{\phi}')\|_{L^p(\mathbb{T}^2)} &\leq \min(\{\|\nabla_{x_\perp}(\phi - \phi')\|_{L^p(\mathbb{T}^2; L^\infty(\mathbb{T}))}, \|e^\phi\|_{L^\infty(\mathbb{T}^3)} \|e^{-\tilde{\phi}}\|_{L^\infty(\mathbb{T}^2)} \|\nabla_{x_\perp}(\phi - \phi')\|_{L^p(\mathbb{T}^2; L^1(\mathbb{T}))}\}) \\ &+ (\|e^\phi\|_{L^\infty} + \|e^{\phi'}\|_{L^\infty}) (\|e^{-\tilde{\phi}}\|_{L^\infty} + \|e^{-\tilde{\phi}'}\|_{L^\infty}) \|\nabla_{x_\perp} \phi'\|_{L^p(\mathbb{T}^2; L^1(\mathbb{T}))} \|\phi - \phi'\|_{L^p(\mathbb{T}^2; L^\infty(\mathbb{T}))}. \end{aligned}$$

To ease subsequent use of the lemma we stress that

- $W^{2,p}(\mathbb{T}^3) \hookrightarrow W^{1,2p}(\mathbb{T}^2; L^2(\mathbb{T}))$ provided that $p \geq 4/3$;
- $W^{3,q}(\mathbb{T}^3) \hookrightarrow W^{2,p}(\mathbb{T}^2; L^1(\mathbb{T})) \cap W^{1,2p}(\mathbb{T}^2; L^2(\mathbb{T}))$ provided that $1/q \leq 1/2 + 1/p$ when $p < \infty$ and that $1/q < 1/2$ when $p = \infty$;

- $W^{2,q}(\mathbb{T}^3) \hookrightarrow L^p(\mathbb{T}^2; L^\infty(\mathbb{T})) \cap W^{1,p}(\mathbb{T}^2; L^1(\mathbb{T}))$ provided that $1/q \leq 1/2 + 1/p$ when $p < \infty$ and that $1/q < 1/2$ when $p = \infty$.

Proof. Since $H^1(\mathbb{T}^3) \hookrightarrow L^4(\mathbb{T}^2; L^\infty(\mathbb{T}))$, $\tilde{\phi}$ is well-defined. Moreover, the first estimate follows directly from a pointwise bound and the second from the triangle inequality for the $L^p(\mathbb{T}^2)$ -norm. Concerning the latter note that indeed

$$\nabla_{x_\perp} \tilde{\phi}(x_\perp) = e^{-\tilde{\phi}(x_\perp)} \int_{\mathbb{T}} e^{\phi(x_\perp, y_\parallel)} \nabla_{x_\perp} \phi(x_\perp, y_\parallel) dy_\parallel.$$

Likewise

$$\begin{aligned} \text{Hess}_{x_\perp}(\tilde{\phi})(x_\perp) &= e^{-\tilde{\phi}(x_\perp)} \int_{\mathbb{T}} e^{\phi(x_\perp, y_\parallel)} (\text{Hess}_{x_\perp}(\phi) + \nabla_{x_\perp} \phi \otimes \nabla_{x_\perp} \phi)(x_\perp, y_\parallel) dy_\parallel \\ &\quad - e^{-2\tilde{\phi}(x_\perp)} \iint_{\mathbb{T} \times \mathbb{T}} e^{\phi(x_\perp, y_\parallel) + \phi(x_\perp, z_\parallel)} \nabla_{x_\perp} \phi(x_\perp, y_\parallel) \otimes \nabla_{x_\perp} \phi(x_\perp, z_\parallel) dy_\parallel dz_\parallel. \end{aligned}$$

implies the third inequality. This achieves the proof of (i).

Concerning the first inequality of (ii), the $L^p(\mathbb{T}^2; L^\infty(\mathbb{T}))$ estimate follows directly from (2.21). To prove the second part of this first inequality we first observe that, breaking the symmetry as in the proof of (2.21) by assuming $\tilde{\phi}'(x_\perp) \leq \tilde{\phi}(x_\perp)$, one obtains

$$\begin{aligned} |\tilde{\phi}(x_\perp) - \tilde{\phi}'(x_\perp)| &\leq \ln \left(1 + e^{-\tilde{\phi}(x_\perp)} \int_{\mathbb{T}} ((\phi - \phi')e^{\phi'}) (x_\perp, x_\parallel) dx_\parallel \right) \\ &\leq \|e^{\phi'}\|_{L^\infty(\mathbb{T}^3)} \|e^{-\tilde{\phi}}\|_{L^\infty(\mathbb{T}^2)} \|(\phi - \phi')(x_\perp, \cdot)\|_{L^1(\mathbb{T})} \end{aligned}$$

by using both ($\forall z \in \mathbb{R}, e^z \leq 1 + ze^z$) and ($\forall x \in \mathbb{R}_+, \ln(x) \leq x - 1$). Integrating a symmetrized form of the foregoing inequality achieves the proof of the first inequality of (ii). The second one follows almost readily from

$$\begin{aligned} \nabla_{x_\perp} \tilde{\phi}(x_\perp) - \nabla_{x_\perp} \tilde{\phi}'(x_\perp) &= e^{-\tilde{\phi}(x_\perp)} \int_{\mathbb{T}} e^{\phi(x_\perp, y_\parallel)} \nabla_{x_\perp} (\phi - \phi')(x_\perp, y_\parallel) dy_\parallel \\ &\quad + (e^{-\tilde{\phi}(x_\perp)} - e^{-\tilde{\phi}'(x_\perp)}) \int_{\mathbb{T}} e^{\phi(x_\perp, y_\parallel)} \nabla_{x_\perp} \phi'(x_\perp, y_\parallel) dy_\parallel \\ &\quad + e^{-\tilde{\phi}'(x_\perp)} \int_{\mathbb{T}} (e^{\phi(x_\perp, y_\parallel)} - e^{\phi'(x_\perp, y_\parallel)}) \nabla_{x_\perp} \phi'(x_\perp, y_\parallel) dy_\parallel. \end{aligned}$$

□

Existence of solutions

Proof of Theorem 2.1.1 for light particles. The strong framework allows to work with Lipschitz fields $\nabla^\perp \tilde{\phi}$ and propagate $W^{1,p}(\mathbb{T}^2)$ bounds. The other key observation is that it is also sufficient to prove a Lipschitz bound in low-order norms. For instance, with constants depending only on bounds on $\|N\|_{L^{4/3}(\mathbb{T}^2)}$, $\|N'\|_{L^{4/3}(\mathbb{T}^2)}$ and $\|\underline{n}\|_{L^{3/2,1}(\mathbb{T}^3)}$

$$\begin{aligned} \|(\nabla^\perp \tilde{\phi} - \nabla^\perp \tilde{\phi}') \cdot \nabla N'\|_{L^{4/3}(\mathbb{T}^2)} &\leq C \|\nabla(\tilde{\phi} - \tilde{\phi}')\|_{L^4(\mathbb{T}^2)} \|\nabla N'\|_{L^2(\mathbb{T}^2)} \\ &\leq C' \|\phi - \phi'\|_{W^{2,4/3}(\mathbb{T}^3)} \|\nabla N'\|_{L^2(\mathbb{T}^2)} \\ &\leq C'' \|N - N'\|_{L^{4/3}(\mathbb{T}^2)} \|\nabla N'\|_{L^2(\mathbb{T}^2)} \end{aligned}$$

using Proposition 2.3.4 and Lemma 2.3.5 and since $W^{2,4/3}(\mathbb{T}^3) \hookrightarrow W^{1,4}(\mathbb{T}^2; L^1(\mathbb{T})) \cap L^4(\mathbb{T}^2; L^\infty(\mathbb{T}))$. This is sufficient to prove uniqueness (even of weak-strong type) and convergence of existence schemes.

As it is fairly classical we only sketch the main steps of the proof. The uniqueness follows for instance from the fact that if N and N' are two solutions starting with the same initial data then on any compact interval I_0 containing zero on which both N and N' are defined

$$\partial_t(N - N') + \nabla^\perp \tilde{\phi} \cdot \nabla(N - N') = -(\nabla^\perp \tilde{\phi} - \nabla^\perp \tilde{\phi}') \cdot \nabla N'$$

so that for any $t \in I_0$

$$\|(N - N')(t, \cdot)\|_{L^{4/3}(\mathbb{T}^2)} \leq C \left| \int_0^t \|N(s, \cdot) - N'(s, \cdot)\|_{L^{4/3}(\mathbb{T}^2)} ds \right|$$

for some C depending on N and N' . Then $N = N'$ follows from the Gronwall lemma. Existence on an interval of controlled length may be proved by showing convergence of the scheme

$$\partial_t N_{n+1} + \nabla^\perp \tilde{\phi}_n \cdot \nabla N_{n+1} = 0, \quad n \geq 0,$$

where $\tilde{\phi}_{-1} = 0$, with $N_{n+1}(0, \cdot) = N^{\text{in}}$. On any interval of time I containing zero the divergence-free structure of the vector field yields bounds on (N_n) in $L^\infty(I; L^\infty(\mathbb{T}^2))$, independent of I , by $\|N^{\text{in}}\|_{L^\infty(\mathbb{T}^2)}$. Then, on such an interval, by differentiating the equation and using Lemma 2.3.5 and Proposition 2.3.3 one obtains (with obvious modification when $n = 0$) for any $t \in I$

$$\begin{aligned} \|\nabla_{x_\perp} N_{n+1}\|_{L^p(\mathbb{T}^2)} &\leq \|\nabla_{x_\perp} N^{\text{in}}\|_{L^p(\mathbb{T}^2)} + \left| \int_0^t \|\tilde{\phi}_n(s)\|_{W^{2,\infty}(\mathbb{T}^2)} \|\nabla_{x_\perp} N_{n+1}(s)\|_{L^p(\mathbb{T}^2)} ds \right| \\ &\leq \|\nabla_{x_\perp} N^{\text{in}}\|_{L^p(\mathbb{T}^2)} + C \left| \int_0^t \|N_n(s)\|_{W^{1,p}(\mathbb{T}^2)}^2 \|\nabla_{x_\perp} N_{n+1}(s)\|_{L^p(\mathbb{T}^2)} ds \right|. \end{aligned}$$

provided that $2 < p < \infty$ and for some C depending only on p , $\|N^{\text{in}}\|_{L^\infty(\mathbb{T}^2)}$ and $\|\underline{n}\|_{L^\infty(I; W^{1,p}(\mathbb{T}^3))}$. Therefore, on a suitably short interval I one may also obtain uniform bounds of (N_n) in $L^\infty(I; W^{1,p}(\mathbb{T}^2))$ where both the bound on (N_n) and the smallness of I are only constrained by the sizes of $\|N^{\text{in}}\|_{W^{1,p}(\mathbb{T}^2)}$ and of $\|\underline{n}\|_{L^\infty(I; W^{1,p}(\mathbb{T}^3))}$. Now, by already expounded arguments, on the same intervals one may prove convergence of (N_n) in $\mathcal{C}(I; L^{4/3}(\mathbb{T}^2))$ to some N . One then concludes the proof of the claimed existence by upgrading this convergence through interpolation with uniform bounds from Proposition 2.3.3 and taking limits in the equation using Lipschitz bounds on maps $N \mapsto \phi$ and $\phi \mapsto \tilde{\phi}$ from Proposition 2.3.4 and Lemma 2.3.5. At last one achieves the proof by combining the strong uniqueness statement and existence on a controlled interval with classical arguments leading to the existence of a maximal solution. \square

Proof of Theorem 2.1.2 for light particles. While a suitable Lipschitz bound is the key step in proving Theorem 2.1.1, here the main issue is the stability of some approximate solutions (N_ε) and their compactness in compatible norms. With this respect, one key

observation is that in the Lipschitz bound of Proposition 2.3.2 one may replace the $L^{4/3}$ -norm with a $W^{-1/2,2}$ -norm. This follows from

$$\begin{aligned} \left| \ln \left(\frac{\int_{\mathbb{T}} e^{\phi'(x_{\perp}, x_{\parallel})} dx_{\parallel}}{\int_{\mathbb{T}} e^{\phi(x_{\perp}, x_{\parallel})} dx_{\parallel}} \right) - \ln \left(\frac{\int_{\mathbb{T}} e^{\phi'(x'_{\perp}, x_{\parallel})} dx_{\parallel}}{\int_{\mathbb{T}} e^{\phi(x'_{\perp}, x_{\parallel})} dx_{\parallel}} \right) \right| \\ \leq \|(\phi - \phi')(x_{\perp}, \cdot)\|_{L^{\infty}(\mathbb{T})} - \|(\phi - \phi')(x'_{\perp}, \cdot)\|_{L^{\infty}(\mathbb{T})} \end{aligned}$$

and the embedding $H_0 \hookrightarrow W^{1,2}(\mathbb{T}^2; L^2(\mathbb{T})) \cap L^2(\mathbb{T}^2; W^{1,2}(\mathbb{T})) \hookrightarrow W^{1/2,2}(\mathbb{T}^2; L^{\infty}(\mathbb{T}))$ that yield

$$\left\| \ln \left(\frac{\int_{\mathbb{T}} e^{\phi'(\cdot, x_{\parallel})} dx_{\parallel}}{\int_{\mathbb{T}} e^{\phi(\cdot, x_{\parallel})} dx_{\parallel}} \right) \right\|_{W^{1/2,2}(\mathbb{T}^2)} \leq C \|\phi - \phi'\|_{H^1(\mathbb{T}^3)}$$

for some universal C .

To be more specific, let us discuss how to prove existence by a compactness argument on the family of solutions (N_{ε}) to³⁷

$$\partial_t N_{\varepsilon} + \nabla^{\perp} \tilde{\phi}_{\varepsilon} \cdot \nabla N_{\varepsilon} = 0, \quad \varepsilon > 0,$$

starting from N^{in} where $\tilde{\phi}_{\varepsilon}$ is obtained from N_{ε} through

$$-\delta^2 \Delta_x \phi_{\varepsilon} = \underline{n} - N_{\varepsilon} \frac{e^{\phi_{\varepsilon}}}{\int_{\mathbb{T}} e^{\phi_{\varepsilon}(\cdot, y_{\parallel})} dy_{\parallel}}, \quad \tilde{\phi}_{\varepsilon}(t, x_{\perp}) = \zeta_{\varepsilon} * \ln \left(\int_{\mathbb{T}} e^{\phi_{\varepsilon}(t, x_{\perp}, x_{\parallel})} dx_{\parallel} \right)$$

where (ζ_{ε}) is an approximation of unity. The existence of (N_{ε}) may be obtained by an argument similar but much simpler than the one leading to Theorem 2.1.1 with the crucial modification that one does not need to restrict the interval of existence to obtain needed bounds (which are of course ε -dependent).

Now, the divergence-free property of vector fields $(\nabla^{\perp} \tilde{\phi}_{\varepsilon})$ provide control of (N_{ε}) in $L^{\infty}(\mathbb{R}; L^p(\mathbb{T}^2))$ uniformly in ε . Then bounds in $W_{loc}^{1,\infty}(\mathbb{R}; W^{-1,q}(\mathbb{T}^2))$ with $1/q = 2/p - 1/2$ if $p < 2$, any $1 \leq q < 2$ if $p = 2$ and $q = p$ if $p > 2$ may be derived directly from the equation in conservative form that implies for any compact interval I and any $t \in I$

$$\|\partial_t N_{\varepsilon}(t, \cdot)\|_{W^{-1,q}(\mathbb{T}^2)} \leq \|\nabla_{x_{\perp}} \tilde{\phi}_{\varepsilon}(t, \cdot)\|_{L^r(\mathbb{T}^2)} \|N_{\varepsilon}(t, \cdot)\|_{L^p(\mathbb{T}^2)} \leq C.$$

with $1/r = 1/q - 1/p$ for some constant C depending only on $\|N^{\text{in}}\|_{L^p(\mathbb{T}^2)}$, $\|\underline{n}\|_{L^{\infty}(I; L^p(\mathbb{T}^3) \cap L^{3/2,1}(\mathbb{T}^3))}$ (and q if $p = 2$). Therefore, compactness of (N_{ε}) in $\mathcal{C}(\mathbb{R}_+; W^{-1/2,2}(\mathbb{T}^2))$ may be obtained by interpolation between boundedness in $L^p(\mathbb{T}^2)$ and Lipschitz bounds in $W^{-1,q}(\mathbb{T}^2)$ with q as above. The stability argument mentioned above translates this into convergence of $(\tilde{\phi}_{\varepsilon})$ in $\mathcal{C}(\mathbb{R}_+; H^1(\mathbb{T}^2))$. Then interpolation with uniform bounds respectively in $L^p(\mathbb{T}^2)$ and $W^{2,p}(\mathbb{T}^2)$ provides convergence in norms sufficient to take limits in the equations. \square

37. This ε should not be confused with the one involved in our original models.

2.3.2 Heavy particles

Now, consistently with modeling considerations, for reading convenience we analyze the heavy particle case setting $\sigma = 1$. Thus we study the system

$$\begin{cases} \partial_t N + \operatorname{div}_{x_\perp}(N \nabla_{x_\perp}^\perp \tilde{\phi}) = 0, \\ -\delta^2 \Delta_x \phi = \frac{N e^{-\phi}}{\int_{\mathbb{T}} e^{-\phi(\cdot, y_\parallel)} dy_\parallel} - \frac{e^\phi}{\int_{\mathbb{T}^3} e^{\phi(\cdot, y)} dy}, \\ \tilde{\phi}(t, x_\perp) = -\ln \left(\int_{\mathbb{T}} e^{-\phi(t, x_\perp, x_\parallel)} dx_\parallel \right). \end{cases} \quad (2.22)$$

As in the previous subsection we focus on the elliptic equation involved in System 2.22 and while doing so we temporarily omit to mention time dependence and do not always repeat but always assume $N \geq 0$, $\int_{\mathbb{T}^2} N = 1$.

Hence we study the following mixed Poisson-Boltzmann equation

$$-\delta^2 \Delta_x \phi(x_\perp, x_\parallel) = N(x_\perp) \frac{e^{-\phi(x_\perp, x_\parallel)}}{\int_{\mathbb{T}} e^{-\phi(x_\perp, y_\parallel)} dy_\parallel} - \frac{e^{\phi(x_\perp, x_\parallel)}}{\int_{\mathbb{T}^3} e^{\phi(y)} dy} \quad (2.23)$$

where N is a nonnegative function with integral equal to one. Our first observation is that Equation 2.23 is the Euler-Lagrange equation associated to the energy functional

$$J[\psi] = \frac{1}{2} \delta^2 \int_{\mathbb{T}^3} |\nabla \psi|^2 - \int_{\mathbb{T}^2} N \ln \left(\int_{\mathbb{T}} e^{-\psi} dx_\parallel \right) dx_\perp + \ln \left(\int_{\mathbb{T}^3} e^\psi \right). \quad (2.24)$$

Again we consider

$$H_0 = \left\{ h \in \mathcal{D}'(\mathbb{T}^3) \mid \nabla h \in L^2(\mathbb{T}^3) \text{ and } \int_{\mathbb{T}^3} h = 0 \right\}.$$

With minor modifications on the proof of Proposition 2.3.1, one obtains the following.

Proposition 2.3.6. *Assume $N \in L^{4/3}(\mathbb{T}^2)$. Then*

- (i) $J : H_0 \rightarrow \mathbb{R} \cup \{+\infty\}$ is well defined, bounded by below on bounded sets and coercive ;
- (ii) J is strictly convex ;
- (iii) J is Gâteaux differentiable at ϕ in the direction $\psi - \phi$ for any $(\phi, \psi) \in (H_0)^2$ such that $J(\phi) < +\infty$ and $J(\psi) < +\infty$;
- (iv) J is weakly lower semi-continuous.

Proposition 2.3.7 (Existence and uniqueness). *For any $N \in L^{4/3}(\mathbb{T}^2)$, Equation (2.23) possesses a unique weak solution $\phi \in H_0$ such that $\int_{\mathbb{T}^3} e^\phi < +\infty$. Moreover there exists C such that for any such N the solution ϕ satisfies*

$$\|\phi\|_{H^1(\mathbb{T}^3)}^2 + \ln \left(\int_{\mathbb{T}^3} e^\phi \right) \leq C \|N\|_{L^{4/3}(\mathbb{T}^2)}^2.$$

Besides there exists C such that for any pair N, N' of such functions respective solutions ϕ and ϕ' satisfy

$$\|\phi - \phi'\|_{H^1(\mathbb{T}^3)} \leq C \|N - N'\|_{L^{4/3}(\mathbb{T}^2)}$$

To tight more closely both propositions note that

$$J^{-1}(\mathbb{R}) = \left\{ h \in \mathcal{D}'(\mathbb{T}^3) \mid \nabla h \in L^2(\mathbb{T}^3), \int_{\mathbb{T}^3} h = 0 \text{ and } \int_{\mathbb{T}^3} e^\phi < +\infty \right\}.$$

Likewise one may also derive qualitative properties of solutions by paralleling the light particle case. However instead we retain only uniqueness from the foregoing propositions and as already mentioned in the introduction we observe that ϕ is actually independent of its third argument and may be obtained by solving

$$-\delta^2 \Delta_{x_\perp} \phi(x_\perp) = N(x_\perp) - \frac{e^{\phi(x_\perp)}}{\int_{\mathbb{T}^2} e^{\phi(y)} dy}. \quad (2.25)$$

To justify this claim it is sufficient now to prove that (2.25) possesses a solution.

Equation (2.25) is a classical Poisson-Boltzmann equation. The equation itself and various generalizations have been studied thoroughly elsewhere and the reader is referred to [27, 73, 37] as entering gates to the extensive relevant literature. Here we merely apply slight variations on the arguments of the light species case to obtain the following propositions. Generally speaking the modified arguments turn to be actually simpler than the original ones. Implicitly they involve the consideration of

$$J_\perp[\psi] = \frac{1}{2} \delta^2 \int_{\mathbb{T}^2} |\nabla \psi|^2 dx - \int_{\mathbb{T}^2} N \psi + \ln \left(\int_{\mathbb{T}^2} e^\psi \right)$$

on

$$H_\perp = \left\{ h \in \mathcal{D}'(\mathbb{T}^2) \mid \nabla h \in L^2(\mathbb{T}^2) \text{ and } \int_{\mathbb{T}^2} h = 0 \right\}.$$

Proposition 2.3.8 (Existence and uniqueness). *For any³⁸ $N \in H^{-1}(\mathbb{T}^2)$, Equation (2.25) possesses a unique weak solution $\phi \in H_\perp$ such that $\int_{\mathbb{T}^2} e^\phi < +\infty$. Moreover there exists C such that for any such N the solution ϕ satisfies*

$$\|\phi\|_{H^1(\mathbb{T}^2)}^2 + \ln \left(\int_{\mathbb{T}^2} e^\phi \right) \leq C \|N\|_{H^{-1}(\mathbb{T}^2)}^2.$$

Besides there exists C such that for any pair N, N' of such functions respective solutions ϕ and ϕ' satisfy

$$\|\phi - \phi'\|_{H^1(\mathbb{T}^2)} \leq C \|N - N'\|_{H^{-1}(\mathbb{T}^2)}$$

38. At this level of regularity in the definition of J_\perp the term involving N should be interpreted as a duality pairing.

Proposition 2.3.9 (Regularity). *Let $1 < p_0 \leq \infty$. There exists C such that if $N \in L^{p_0}(\mathbb{T}^2)$ then ϕ the unique solution to (2.25) satisfies*

$$\|\phi\|_{L^\infty(\mathbb{T}^2)} \leq C \left(1 + \|N\|_{L^{p_0}(\mathbb{T}^2)}\right).$$

Moreover, for any $1 < p < \infty$, there exists C such that if $N \in L^{p_0}(\mathbb{T}^2) \cap L^p(\mathbb{T}^2)$ then ϕ the unique solution to (2.23) satisfies

$$\|\phi\|_{W^{2,p}(\mathbb{T}^2)} \leq C \left(e^{C\|N\|_{L^{p_0}(\mathbb{T}^2)}} + \|N\|_{L^p(\mathbb{T}^2)} \right).$$

Besides for any $1 < p < \infty$ and any $1 < q < \infty$ such that $\frac{1}{q} \leq \frac{1}{p} + \frac{1}{2}$ there exists C such that with if $N \in L^q(\mathbb{T}^2) \cap L^{p_0}(\mathbb{T}^2) \cap W^{1,p}(\mathbb{T}^2)$ then ϕ the unique solution to (2.23) satisfies

$$\|\phi\|_{W^{3,p}(\mathbb{T}^3)} \leq C \left(\|N\|_{W^{1,p}(\mathbb{T}^2)} + \left(\|N\|_{L^q(\mathbb{T}^2)} + 1 \right) e^{C\|N\|_{L^{p_0}(\mathbb{T}^2)}} \right).$$

Proof. On one hand we first observe that $-\delta^2\Delta\phi \leq N$ so that Lemma C.4.4 and the coercivity estimate of Proposition 2.3.8 provides the needed upper bound on ϕ . On the other hand we also have $-\delta^2\Delta\phi \geq -e^\phi$. Now note that on any bounded Ω' one may chose $c_0 \in \mathbb{R}$ to ensure that $-e^{\|\cdot\|^2+c_0} \geq -\delta^2\Delta(\|\cdot\|^2 + c_0)$. With suitable Ω , Ω' and c_0 one may then apply Lemma C.4.5 to $u = \|\cdot\|^2 + c_0 - \phi$ with $K_0 = 0$. Combined with the coercivity estimate of Proposition 2.3.7 this yields the claimed upper bound on ϕ thus achieves the proof of the estimate on $\|\phi\|_{L^\infty(\mathbb{R})}$.

From here the proof is achieved along the lines of the proof of the light species case. \square

Proposition 2.3.10 (Lipschitz dependence). *For any $1 < p < \infty$ and $1 < p_0 \leq \infty$, there exists C such that if $N \in L^p(\mathbb{T}^2) \cap L^{p_0}(\mathbb{T}^2)$, $N' \in L^p(\mathbb{T}^2) \cap L^{p_0}(\mathbb{T}^2)$ then ϕ and ϕ' the respective solutions to (2.25) corresponding to N and N' satisfy*

$$\|\phi - \phi'\|_{W^{2,p}(\mathbb{T}^3)} \leq C \left(\|N - N'\|_{L^p(\mathbb{T}^2)} + \|N\|_{H^{-1}(\mathbb{T}^2)} \left(e^{C\|N\|_{L^{p_0}(\mathbb{T}^2)}} + e^{C\|N'\|_{L^{p_0}(\mathbb{T}^2)}} \right) \right).$$

From here proofs of Theorems 2.1.1 and 2.1.2 in the ionic case follow arguments completely parallel to those of the electronic case that we omit here.

2.4 Mathematical justification of the limiting process

In this section, we investigate the rigorous derivation of the asymptotic model (2.6)-(2.7) from original equation (2.2) in a linear setting, that is when electric forces derive from an external electric potential.

To do so, we provide in Subsection 2.4.1 an existence result and corresponding *a priori* estimates yielding bounds on solutions uniform with respect to ε . In Subsection 2.4.2 we

prove that those bounds provide enough compactness on the ε -family of solutions to justify formal asymptotic arguments. This leads to Theorem 2.1.3, the first main result of this section.

In the last subsection, we investigate further the asymptotic process yielding convergence of the macroscopic density to a limiting anisotropic Boltzmann-Gibbs density. Specifically we provide decay rates in terms of ε and the exponent α . To achieve this task, we use a different functional setting that is more demanding on the localization of considered initial data but classical for the Fokker-Planck operator. Our task is then achieved by designing an anisotropic hypocoercive strategy based on a suitable global Lyapunov functional. This results in Theorem 2.1.4.

2.4.1 *A priori* estimates

We have already encountered in the formal derivation section, either implicitly or explicitly, the two main *a priori* estimates at our disposal, conservation of mass

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} f^\varepsilon(t, x, v) \, dx dv = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_0(x, v) \, dx dv$$

obtained by integrating (2.2) and free energy dissipation (2.14) obtained by setting $H(h) = h \ln(h)$ in (2.13)

$$\begin{aligned} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f^\varepsilon(t) \ln \left(\frac{f^\varepsilon(t)}{\mathcal{M}(t)} \right) \, dx dv + 4\varepsilon^{-(1+\alpha)} \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left| \nabla_v \sqrt{\frac{f^\varepsilon}{M}} \right|^2 M \, dx dv \, ds \\ = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 \ln \left(\frac{f_0}{\mathcal{M}(0)} \right) \, dx dv - \sigma \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \partial_t \phi f^\varepsilon \, dx dv \, ds \end{aligned}$$

where \mathcal{M} is the global Maxwellian defined in (2.12) that has integral equal to one by our normalization³⁹ (2.11) of ϕ . Note however that, as is customarily expected, our weak solutions will satisfy only a relaxed form of the foregoing free energy dissipation equality.

To unfold controls provided by the above estimates we recall the following well-known lemma.

Lemma 2.4.1. *There exists a constant C such that for any integrable $f : \mathbb{T}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}_+^*$ and any bounded $\phi : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}$*

$$\begin{aligned} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f (|\ln(f)| + |v|^2) \, dx dv \\ \leq C \left((1 + \ln_-(\|f\|_{L^1}) + \|\phi\|_{L^\infty}) \|f\|_{L^1} + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f \ln \left(\frac{f}{e^{-\sigma \phi} M} \right) \, dx dv \right). \end{aligned}$$

³⁹. Note that this differs from normalization in Section 2.3.

Proof. We start by expanding

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} f(x, v) \ln \left(\frac{f(x, v)}{e^{-\sigma \phi(x)} M(v)} \right) dx dv$$

into

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|\ln(f(x, v))| + |v|^2 + \sigma \phi(x) + \frac{3}{2} \ln(2\pi)) f(x, v) dx dv - 2 \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f \ln_- f dx dv$$

where $(\cdot)_-$ denotes negative part. Then we note that for any positive u, u_0 ,

$$u \ln_-(u) \leq \begin{cases} u \ln_-(u_0) & \text{if } u \geq u_0 \\ C \sqrt{u_0} & \text{if } u \leq u_0 \end{cases}$$

where $C = \max_{w \in \mathbb{R}_+^*} \sqrt{w} \ln_-(w)$. Applying the above to $u = f(x, v)$, $u_0 = C_0 e^{-\beta |v|^2}$ for each fixed (x, v) yields for any $\beta > 0$, $C_0 > 0$

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} f \ln_- f dx dv \leq C \sqrt{C_0} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} e^{-\beta |v|^2/2} dx dv + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (\beta |v|^2 + \ln_-(C_0)) f(x, v) dx dv.$$

We conclude by choosing first some $\beta \in (0, 1/2)$ then optimizing in C_0 . \square

Relying on the free energy dissipation one may then obtain the following existence result.

Proposition 2.4.2. *Assume that $\phi \in W_{loc}^{1,1}(\mathbb{R}_+; L^\infty(\mathbb{T}^3))$ and f_0 is such that*

$$f_0 \geq 0, \quad \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (1 + |v|^2 + \ln_+ f_0) f_0 dx dv < \infty.$$

Then there exists a unique $f^\varepsilon \in \mathcal{C}(\mathbb{R}_+, L^1(\mathbb{T}^3 \times \mathbb{R}^3))$ solving Equation (2.2) and starting from f_0 at time 0. Moreover there exists C such that for any such (ϕ, f_0) this solution additionally satisfies for any $t \geq 0$

$$\begin{aligned} & \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (1 + |v|^2 + |\ln f^\varepsilon(t, x, v)|) f^\varepsilon(t, x, v) dx dv + 4\varepsilon^{-(1+\alpha)} \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left| \nabla_v \sqrt{\frac{f^\varepsilon}{M}} \right|^2 M \\ & \leq C \left[(1 + \ln_-(\|f_0\|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)})) + \|\phi\|_{W^{1,1}((0,t); L^\infty(\mathbb{T}^3))} \|f_0\|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)} \right. \\ & \quad \left. + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 + \ln_+ f_0(x, v)) f_0(x, v) dx dv \right]. \quad (2.26) \end{aligned}$$

Bounds on free energy provide enough compactness to reduce the proof of the existence part of the proposition to an existence proof for a dense subset of initial data. Moreover uniqueness may also be obtained by proving an existence result for a dual equation starting from a suitably large set of initial data. Besides one may choose the measure with respect to which identification of L^2 -duals is performed in order to make this dual equation as simple as possible. Here one may enforce that the dual equation has essentially the same structure as the original one; see Section 2.4.3 for details on involved symmetries. At last existence for data in adapted Sobolev spaces — which is sufficient to complete the proof — follows from semigroup arguments [167]. One may for instance use a spectral version of hypocoercive estimates to prove that the equation generates a semigroup when $\phi \equiv 0$ [167, Chapter 4] and recover the full case by a fixed point argument or directly establish frozen-time spectral estimates to prove that the equation generates an evolution system [167, Chapter 5]. We skip these details but refer the reader to [120, Chapter 5] for a detailed analysis that is closely related to the last part of the argument.

2.4.2 Weak compactness arguments

As already announced it turns out that in the linear case the arguments that we have expounded along the formal derivation in Section 2.2.2 have fully justified close counterparts. This is the line we follow to prove Theorem 2.1.3.

Proof of Theorem 2.1.3. Since $(f^\varepsilon M^{-1}(1 + (\ln(f^\varepsilon M^{-1}))_+))$ is uniformly bounded in $L^1_{loc}(\mathbb{R}_+; L^1(\mathbb{T}^3 \times \mathbb{R}^3, M(v) dx dv))$, de la Vallée-Poussin and Dunford-Pettis theorems imply that (f^ε) converges weakly to some f in $L^1_{loc}(\mathbb{R}_+; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$ along some subsequence. Moreover the bound on the dissipation of free energy in (2.26) implies that

$$\int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left| \nabla_v \sqrt{\frac{f(s, x, v)}{M(v)}} \right|^2 M(v) dx dv ds = 0$$

thus f is a local Maxwellian, $f = n M$ with $n = \int_{\mathbb{R}^3} f(\cdot, \cdot, v) dv$. Besides tensorization with the constant function equal to one on \mathbb{R}^3 shows that n^ε does converge weakly to n in $L^1_{loc}(\mathbb{R}_+; L^1(\mathbb{T}^3))$.

Bounds on the dissipation also shows that $(\varepsilon^{-\alpha}(v f^\varepsilon + \nabla_v f^\varepsilon))$ converges strongly to zero in $L^2(\mathbb{R}_+; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$ since

$$v f^\varepsilon + \nabla_v f^\varepsilon = 2\sqrt{f^\varepsilon} \nabla_v \left(\sqrt{\frac{f^\varepsilon}{M}} \right) \sqrt{M}$$

and the Cauchy-Schwarz inequality shows that the relevant norm is $\mathcal{O}(\varepsilon^{(1-\alpha)/2})$ as $\varepsilon \rightarrow 0$. This key observation shows that for any smooth compactly-supported function $\psi : \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ that is independent of the polar angle θ , or equivalently such that $v^\perp \cdot \nabla_v \psi \equiv 0$, taking the limit $\varepsilon \rightarrow 0$ along the relevant subsequence yields

$$\int_{\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3} (v \cdot \nabla_x \psi(t, x, v) - \sigma \nabla_x \phi(t, x) \cdot \nabla_v \psi(t, x, v)) f(t, x, v) dt dx dv = 0$$

which, given that f is a local Maxwellian, is also

$$\int_{\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3} v \cdot (\nabla_x \psi(t, x, v) - \sigma \nabla_x \phi(t, x) \psi(t, x, v)) n(t, x) M(v) dt dx dv = 0$$

thus, by symmetries of ψ and M ,

$$\int_{\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3} v_{\parallel} (\partial_{x_{\parallel}} \psi(t, x, v) - \sigma \partial_{x_{\parallel}} \phi(t, x) \psi(t, x, v)) n(t, x) M(v) dt dx dv = 0.$$

Choosing a non zero smooth radial compactly-supported $\tilde{\psi} : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ and noticing that the foregoing class of ψ functions includes $(t, x, v) \mapsto v_{\parallel} \tilde{\psi}(v) \underline{\psi}(t, x)$ for any smooth compactly-supported $\underline{\psi} : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}_+$, one derives that in distributional sense $\partial_{x_{\parallel}} n = -\sigma n \partial_{x_{\parallel}} \phi$. Hence $n \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{T}^2; W^{1,1}(\mathbb{T}))$ and $\partial_{x_{\parallel}}(e^{\sigma \phi} n) = 0$ thus

$$n(t, x) = N(t, x_{\perp}) \frac{e^{-\sigma \phi(t, x)}}{\int_{\mathbb{T}} e^{-\sigma \phi(t, x_{\perp}, y_{\parallel})} dy_{\parallel}}, \quad N(t, x_{\perp}) = \int_{\mathbb{T}} n(t, x_{\perp}, y_{\parallel}) dy_{\parallel}.$$

Now we prove that N satisfies the claimed equation. To do so we set

$$N^{\varepsilon}(t, x_{\perp}) = \int_{\mathbb{T}} n^{\varepsilon}(t, x_{\perp}, y_{\parallel}) dy_{\parallel}$$

and observe that by using again a tensorization argument one gets that along the relevant subsequence N^{ε} converges weakly to N in $L^1_{loc}(\mathbb{R}_+; L^1(\mathbb{T}^2))$. As in Section 2.2.2 we also introduce

$$J^{\varepsilon}(t, x_{\perp}) = \frac{1}{\varepsilon} \int_{\mathbb{T} \times \mathbb{R}^3} v f^{\varepsilon}(t, x_{\perp}, y_{\parallel}, v) dy_{\parallel} dv$$

and observe that $\partial_t N^{\varepsilon} + \operatorname{div}_{x_{\perp}}(J^{\varepsilon}) = 0$. The same kind of integration shows that

$$\sigma (J^{\varepsilon})^{\perp} + \operatorname{div}_{x_{\perp}} \left(\int_{\mathbb{T} \times \mathbb{R}^3} v_{\perp} \otimes v_{\perp} f^{\varepsilon}(\cdot, \cdot, x_{\parallel}, v) dx_{\parallel} dv \right) + \sigma \int_{\mathbb{T}} \nabla_{x_{\perp}} \phi(\cdot, \cdot, x_{\parallel}) n^{\varepsilon}(\cdot, \cdot, x_{\parallel}) dx_{\parallel}$$

converges weakly to zero. Moreover the last term of the foregoing expression converges weakly to

$$\sigma \int_{\mathbb{T}} \nabla_{x_{\perp}} \phi(\cdot, \cdot, x_{\parallel}) n(\cdot, \cdot, x_{\parallel}) dx_{\parallel} = -N \frac{\nabla_{x_{\perp}} \left(\int_{\mathbb{T}} e^{-\sigma \phi(\cdot, \cdot, x_{\parallel})} dx_{\parallel} \right)}{\int_{\mathbb{T}} e^{-\sigma \phi(\cdot, \cdot, x_{\parallel})} dx_{\parallel}}$$

along the relevant subsequence. Finally we observe that

$$\int_{\mathbb{T} \times \mathbb{R}^3} v_{\perp} \otimes v_{\perp} f^{\varepsilon}(\cdot, \cdot, x_{\parallel}, v) dx_{\parallel} dv = \int_{\mathbb{T} \times \mathbb{R}^3} v_{\perp} \otimes (v_{\perp} + \nabla_{v_{\perp}}) f^{\varepsilon}(\cdot, \cdot, x_{\parallel}, v) dx_{\parallel} dv + N^{\varepsilon} \mathbf{I}$$

and that

$$\begin{aligned} & \int_{\mathbb{T} \times \mathbb{R}^3} v_{\perp} \otimes (v_{\perp} + \nabla_{v_{\perp}}) f^{\varepsilon}(\cdot, \cdot, x_{\parallel}, v) dx_{\parallel} dv \\ &= 2 \int_{\mathbb{T} \times \mathbb{R}^3} v_{\perp} \sqrt{f^{\varepsilon}(\cdot, \cdot, x_{\parallel}, v)} \otimes \nabla_{v_{\perp}} \left(\sqrt{\frac{f^{\varepsilon}}{M}} \right) (\cdot, \cdot, x_{\parallel}, v) \sqrt{M(v)} dx_{\parallel} dv \end{aligned}$$

is seen to converge to zero in $L^2(\mathbb{R}_+; L^1(\mathbb{T}^2))$ by using the Cauchy-Schwarz inequality combined with bounds on second-order moments and dissipation of free energy. Therefore J_\perp^ε converges weakly to

$$\sigma(\nabla_{x_\perp} N)^\perp - \sigma N \left(\nabla_{x_\perp} \ln \left(\int_{\mathbb{T}} e^{-\sigma \phi(\cdot, \cdot, x_\parallel)} dx_\parallel \right) \right)^\perp$$

along the relevant subsequence. Inserting this in $\partial_t N^\varepsilon + \operatorname{div}_{x_\perp}(J^\varepsilon) = 0$ achieves the proof. \square

Remark 2.4.3. *It may be instructive to compare the former proof with the arguments proving [124, Lemma 4.1] as those would apply equally well here. They rely rather on a direct compactness argument on n^ε and log-Sobolev and Csiszár-Kullback-Pinsker inequalities to control $f^\varepsilon - n^\varepsilon M$.*

Remark 2.4.4. *Note that in the last step while we had not enough control on moments in velocities to justify in this way convergence of expression involving terms quadratic in velocity we were able to use cancellations and collisional dissipation to do so. This is a by now classical trick that was crucial in similar studies of the diffusive regime $\alpha = 1$; see [172, Section 5] and [124, Propositions 3.3 & 4.3]. However we could replace it with an argument involving evanescent cut-off functions as in [178].*

2.4.3 Anisotropic hypocoercive estimates

Now we provide convergence rates for some limits involved in the derivation of our asymptotic model. To keep technicalities as low as possible we assume here that ϕ does not depend on time t but only on $x \in \mathbb{T}^3$. While it is likely that one may also deal with time-dependent cases assuming sufficient control on time variations of ϕ this would add unessential complications to already technical estimates.

Part of the asymptotic process involves oscillatory processes and some limits are in essence weak limits that may only be expected to hold in spaces of negative regularity. We only focus here on those that are expected, and proved here, to be strong limits. With this in mind the goal of this section is merely to provide quantitative bounds, explicit with respect to ε and α , on the distance between the distribution function f^ε and an anisotropic Maxwell-Boltzmann density expressed in terms of N^ε . While one could track convergence rates of vanishing terms in the foregoing convergence proof, this would lead to estimates in functional spaces of (highly) negative regularity by lack of control on (spatial) gradients of the solution. Here instead we prove that by assuming that initial data belongs to a stronger space, controlling in particular any number of moments in velocity, we obtain natural convergence rates for

$$f^\varepsilon(t, x, v) - N^\varepsilon(t, x_\perp) \frac{e^{-\sigma \phi(x)}}{\int_{\mathbb{T}} e^{-\sigma \phi(x_\perp, y_\parallel)} dy_\parallel} \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}}$$

in strong norms for some suitable N^ε . Note that while convergence of $f^\varepsilon - n^\varepsilon M$ follows at once from the dissipation of free energy, the next step relies on a subtler interplay between

three kind of terms : electric-and-free transport, magnetic transport and collisions. As we show in Appendix 2.4.4 this interplay may be essentially elucidated by computing commutators of involved terms and then encoded in a global functional of hypocoercive type.

To make the functional setting more precise, let us introduce $h^\varepsilon = f^\varepsilon/\mathcal{M}$ where once again

$$\mathcal{M}(x, v) = \frac{1}{(2\pi)^{3/2}} e^{-\left(\sigma\phi(x) + \frac{|v|^2}{2}\right)},$$

is our global Maxwellian. Associated initial data $h_0 = f_0/\mathcal{M}$ is assumed to belong to $\mathcal{H} = L^2(\mu)$ the Hilbert space \mathcal{H} characterized by its L^2 -norm $\|\cdot\| = (\int |\cdot|^2 d\mu)^{1/2}$ where μ is the probability measure with density \mathcal{M} . In the following we denote by $\langle \cdot; \cdot \rangle$ the corresponding scalar product.

A direct computation shows that the distribution function f^ε solves the linear Vlasov-Fokker-Planck equation (2.2) with external electric potential ϕ if and only if h^ε satisfies

$$\varepsilon \partial_t h^\varepsilon + v \cdot \nabla_x h^\varepsilon - \sigma \nabla_x \phi \cdot \nabla_v h^\varepsilon - \frac{\sigma}{\varepsilon} v^\perp \cdot \nabla_v h^\varepsilon = \frac{1}{\varepsilon^\alpha} (v \cdot \nabla_v h^\varepsilon - \Delta_v h^\varepsilon). \quad (2.27)$$

In the absence of any magnetic field, the original version of the approach that we extend here to serve our purposes would lead to a quantitative proof of convergence as $\varepsilon \rightarrow 0$ to the global Maxwellian \mathcal{M} , and in this case it would really follow from very classical hypocoercive estimates. See [194, 158, 75, 125] for some instances of proofs of hypocoercive convergence to equilibrium through global functionals. The presence of a strong magnetic field prevents return to equilibrium in the perpendicular direction and our goal is to design functionals that capture this anisotropic behavior. We detail this strategy along the proof of the following theorem, which is the main result of this section.

Theorem 2.4.5. *Suppose that $\phi \in W^{2,\infty}(\mathbb{T}^3)$.*

Then for any $h_0 \in \mathcal{H}$ and any $\varepsilon > 0$ there exists a unique global solution h^ε to (2.27) starting from h_0 at time zero such that $h^\varepsilon \in C(\mathbb{R}_+; \mathcal{H})$ and $\nabla_v h^\varepsilon \in L^2(\mathbb{R}_+; \mathcal{H})$.

Moreover there exists a constant $C_\phi > 0$, depending only on the norm of ϕ in $W^{2,\infty}(\mathbb{T}^3)$, such that for any $\varepsilon \in (0, 1)$, $f^\varepsilon = h^\varepsilon \mathcal{M}$ — solving (2.2) with initial data $f_0 = h_0 \mathcal{M}$ and associated to the above solution h^ε — satisfies

$$\|f^\varepsilon - n^\varepsilon M\|_{L^2(\mathbb{R}_+; L^2(M^{-1}(v) dx dv))} \leq C_\phi \|h_0\| \varepsilon^{\frac{\alpha+1}{2}}; \quad (2.28)$$

$$\left\| n^\varepsilon - N^\varepsilon \frac{e^{-\sigma\phi}}{\int_{\mathbb{T}} e^{-\sigma\phi(\cdot, y_\parallel)} dy_\parallel} \right\|_{L^2(\mathbb{R}_+ \times \mathbb{T}^3)} \leq C_\phi \|h_0\| \varepsilon^{\frac{1-|\alpha|}{2}} \quad (2.29)$$

where

$$n^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon(\cdot, \cdot, v) dv \quad \text{and} \quad N^\varepsilon = \int_{\mathbb{T}} n^\varepsilon(\cdot, \cdot, y_\parallel) dy_\parallel.$$

In particular, since $\alpha \in (-1, 1)$, as $\varepsilon \rightarrow 0$ the distribution function f^ε gets $\varepsilon^{\frac{1-|\alpha|}{2}}$ -close to the anisotropic Maxwell-Boltzmann density

$$(t, x, v) \mapsto N^\varepsilon(t, x_\perp) \frac{e^{-\sigma\phi(x)}}{\int_{\mathbb{T}} e^{-\sigma\phi(x_\perp, y_\parallel)} dy_\parallel} \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}}$$

in $L^2(\mathbb{R}_+; L^2(M^{-1}(v) dx dv))$.

Remark 2.4.6. *Again we stress however that the last part of the asymptotic description — the identification of an asymptotic dynamics for limits of N^ε — proceeds from averaging mechanisms and we do not expect to be able to capture it by similar dissipative arguments nor do we expect the corresponding convergence to hold in strong norms.*

Proof. We skip the proof of the existence and uniqueness part but we refer the reader to [120, Chapter 5] for a detailed analysis of a nearly identical problem. The existence proof also provides detailed justification for the rather formal computations that we perform below. The reader may consult [157] for an instance of a similar detailed verification starting directly from the approximation process proving existence.

The starting point is the L^2 -estimate obtained from (2.13) by choosing $H(h) = h^2$,

$$\|h^\varepsilon(t, \cdot, \cdot)\|^2 + 2\varepsilon^{-(\alpha+1)} \int_0^t \|\nabla_v h^\varepsilon(s, \cdot, \cdot)\|^2 ds \leq \|h_0^\varepsilon\|^2.$$

From this, (2.28) follows by Poincaré inequality for the Gaussian measure and boundedness of ϕ , since

$$\begin{aligned} \int_{\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3} |f^\varepsilon - n^\varepsilon M|^2 M^{-1} &\leq \int_{\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3} \left| \frac{f^\varepsilon}{M} - n^\varepsilon \right|^2 M \\ &\leq \int_{\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3} \left| \nabla_v \left(\frac{f^\varepsilon}{M} \right) \right|^2 M \\ &\leq C \int_0^{+\infty} \|\nabla_v h^\varepsilon(t, \cdot, \cdot)\|^2 dt \end{aligned}$$

for some C depending only on $\|\phi\|_{L^\infty(\mathbb{T}^3)}$. In order to prove (2.29), we need to extend this strategy that only yields dissipation of velocity derivatives so that it also provides estimates on an x_{\parallel} -derivative. To do so we exploit the hypocoercive structure and modify the L^2 functional in a H^1 functional whose dissipation contains new terms yielding suitable control on $\partial_{x_{\parallel}}(f^\varepsilon M^{-1})$. To state this in a concise way we define the ε and time dependent norm

$$\begin{aligned} \|h\|_{\varepsilon, t}^2 &= \|h\|^2 + \gamma_{\parallel, 1} \varepsilon^{(|\alpha|-\alpha)/2} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right) \|\partial_{v_{\parallel}} h\|^2 \\ &+ \gamma_{\parallel, 2} \varepsilon^{|\alpha|+(|\alpha|+\alpha)/2} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^3 \|\partial_{x_{\parallel}} h\|^2 \\ &+ 2\gamma_{\parallel, 3} \varepsilon^{|\alpha|} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^2 \langle \partial_{v_{\parallel}} h, \partial_{x_{\parallel}} h \rangle \\ &+ \gamma_{\perp, 1} \varepsilon^{1-\alpha} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right) \|\nabla_{v_{\perp}} h\|^2 + \gamma_{\perp, 2} \varepsilon^2 \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^3 \|\nabla_{x_{\perp}} h\|^2 \\ &+ 2\gamma_{\perp, 3} \varepsilon^{2-\alpha} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^2 \langle \nabla_{v_{\perp}} h, \nabla_{x_{\perp}} h \rangle \end{aligned}$$

and the corresponding ε and time dependent (partial) dissipation

$$\begin{aligned}
 D_{\varepsilon,t}(h) &= \varepsilon^{-(1+\alpha)} \|\nabla_v h\|^2 + \varepsilon^{-1+(|\alpha|-\alpha)/2-\alpha} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right) \|\nabla_v \partial_{v_{\parallel}} h\|^2 \\
 &+ \varepsilon^{-1+|\alpha|+(|\alpha|-\alpha)/2} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^3 \|\nabla_v \partial_{x_{\parallel}} h\|^2 \\
 &+ \varepsilon^{|\alpha|-1} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^2 \|\partial_{x_{\parallel}} h\|^2 \\
 &+ \varepsilon^{-2\alpha} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right) \|\nabla_v \nabla_{v_{\perp}} h\|^2 + \varepsilon^{1-\alpha} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^3 \|\nabla_v \nabla_{x_{\perp}} h\|^2 \\
 &+ \varepsilon^{1-\alpha} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^2 \|\nabla_{x_{\perp}} h\|^2.
 \end{aligned}$$

These quantities are related through the dissipation estimate contained in the following lemma.

Lemma 2.4.7. *Let $\phi \in W^{2,\infty}(\mathbb{T}^3)$. There exist positive constants $\gamma_{\perp,i}$, $\gamma_{\parallel,i}$, $i \in \{1, 2, 3\}$ such that,*

— *uniformly in $(t, \varepsilon) \in \mathbb{R}_+ \times (0, 1)$, the norm $\|\cdot\|_{\varepsilon,t}$ is equivalent to the square root of*

$$\begin{aligned}
 \|\cdot\|^2 &+ \varepsilon^{(|\alpha|-\alpha)/2} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right) \|\partial_{v_{\parallel}}(\cdot)\|^2 \\
 &+ \varepsilon^{|\alpha|+(|\alpha|+\alpha)/2} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^3 \|\partial_{x_{\parallel}}(\cdot)\|^2 \\
 &+ \varepsilon^{1-\alpha} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right) \|\nabla_{v_{\perp}}(\cdot)\|^2 + \varepsilon^2 \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^3 \|\nabla_{x_{\perp}}(\cdot)\|^2.
 \end{aligned}$$

— *there exists $K > 0$ such that for any $h_0 \in \mathcal{H}$ the solution h^ε of (2.27) starting from h_0 satisfies for any $\varepsilon \in (0, 1)$ and any $t \geq 0$,*

$$\|h^\varepsilon(t, \cdot, \cdot)\|_{\varepsilon,t}^2 + K \int_0^t D_{\varepsilon,s}(h^\varepsilon(s, \cdot, \cdot)) ds \leq \|h_0\|^2. \quad (2.30)$$

The proof of this result is given in Appendix 2.4.4. There in particular we explain in details how different choices of power of t and ε arise as necessary constraints to satisfy expected inequalities. Nonetheless, we do provide here some partial insights on the choice of the functional. Standard L^2 -based hypocoercive strategy for the kinetic Fokker-Planck operator suggests to estimate H^1 type functionals with mixed derivative to unravel dissipation of space derivatives that was missing in the original L^2 computation. Here, direct computations performed in Lemma 2.4.9 show that

$$\begin{aligned}
 \frac{d}{dt} \langle \partial_{v_{\parallel}} h^\varepsilon, \partial_{x_{\parallel}} h^\varepsilon \rangle &+ \varepsilon^{-1} \|\partial_{x_{\parallel}} h^\varepsilon\|^2 = \sigma \varepsilon^{-1} \langle (\nabla_v h^\varepsilon \cdot \nabla_x) \partial_{x_{\parallel}} \phi, \partial_{v_{\parallel}} h^\varepsilon \rangle \\
 &- \varepsilon^{-(1+\alpha)} (\langle \partial_{v_{\parallel}} h^\varepsilon, \partial_{x_{\parallel}} h^\varepsilon \rangle + 2 \langle \nabla_v \partial_{v_{\parallel}} h^\varepsilon, \nabla_v \partial_{x_{\parallel}} h^\varepsilon \rangle)
 \end{aligned} \quad (2.31)$$

$$\begin{aligned}
 \frac{d}{dt} \langle \nabla_{v_{\perp}} h^\varepsilon, \nabla_{x_{\perp}} h^\varepsilon \rangle &+ \varepsilon^{-1} \|\nabla_{x_{\perp}} h^\varepsilon\|^2 \\
 &= -\varepsilon^{-(1+\alpha)} \{ \langle \nabla_{v_{\perp}} h^\varepsilon, \nabla_{x_{\perp}} h^\varepsilon \rangle + 2 \langle \nabla_v \nabla_{v_{\perp}} h^\varepsilon, \nabla_v \nabla_{x_{\perp}} h^\varepsilon \rangle \} \\
 &+ \sigma \varepsilon^{-1} \langle (\nabla_v h^\varepsilon \cdot \nabla_x) \nabla_{x_{\perp}} \phi, \nabla_{v_{\perp}} h^\varepsilon \rangle - \sigma \varepsilon^{-2} \langle \nabla_{v_{\perp}}^\perp h^\varepsilon, \nabla_{x_{\perp}} h^\varepsilon \rangle.
 \end{aligned} \quad (2.32)$$

where good useful dissipation terms have been kept on left-hand sides and useless signless terms have been left on the right. When putting everything in one global functional the goal is to balance each bad term with some fraction of good dissipative terms. We stress that in (2.32) the worst term to balance — the one of order ε^{-2} — is precisely the one due to the presence of a strong magnetic field. It stems from non-commutation of ∇_v and $v^\perp \cdot \nabla_v$ and forces the choice of anisotropic weights in our functional. Finally, we note that the presence of time weights in our functionals only reflects our choice to use hypoelliptic regularizing effects to start with L^2 and not H^1 initial data but gain instantaneous H^1 control on the solution. In particular our choice of the powers of time involved in the above definitions follows directly from the hypoelliptic structure of the operator at hand. For the kinetic Fokker-Planck the method essentially originates in work of Hérau [121]; see also [194, Appendix A.21] or [157].

We now use dissipative effects afforded by Lemma 2.4.7 to achieve the proof. The dissipation of the energy functional in (2.30) provides a bound on the size of the perpendicular and parallel space derivatives of h^ε in \mathcal{H} . Indeed, by non-negativity of $\|\cdot\|_{\varepsilon,t}$ we get from the dissipation in (2.30) that

$$\begin{aligned} \|\partial_{x_\parallel} h^\varepsilon\|_{L^2((\varepsilon^{1+\alpha}, +\infty); \mathcal{H})} &\leq C \|h_0\| \varepsilon^{\frac{1-|\alpha|}{2}}, \\ \|\partial_{x_\perp} h^\varepsilon\|_{L^2((\varepsilon^{1+\alpha}, +\infty); \mathcal{H})} &\leq C \|h_0\| \varepsilon^{\frac{\alpha-1}{2}} \end{aligned}$$

for some constant C depending only on $\|\phi\|_{W^{2,\infty}(\mathbb{T}^3)}$. To prove our claim only the first estimate — on the parallel derivative — is needed. Nevertheless, let us stress that the second estimate is essentially useless and in any case does not contradict the non-convergence to a global Maxwellian in the perpendicular direction since $\alpha < 1$. To proceed we first observe that

$$e^{\sigma\phi} n^\varepsilon = \int_{\mathbb{R}^3} h^\varepsilon(\cdot, \cdot, v) M(v) dv$$

hence by the Jensen inequality, for any $t \geq 0$

$$\|\partial_{x_\parallel} (e^{\sigma\phi} n^\varepsilon)(t, \cdot)\|_{L^2(\mathbb{T}^3)} \leq \left(\int_{\mathbb{R}^3} \|\partial_{x_\parallel} h^\varepsilon(t, \cdot, v)\|_{L^2(\mathbb{T}^3)}^2 M(v) dv \right)^{1/2} \leq \|e^{\sigma\phi}\|_{L^\infty}^{\frac{1}{2}} \|\partial_{x_\parallel} h^\varepsilon(t, \cdot, \cdot)\|.$$

Therefore introducing the intermediate quantity

$$\tilde{N}^\varepsilon = \int_{\mathbb{T}} n^\varepsilon(\cdot, \cdot, y_\parallel) e^{\sigma\phi(\cdot, y_\parallel)} dy_\parallel \times \int_{\mathbb{T}} e^{-\sigma\phi(\cdot, y_\parallel)} dy_\parallel$$

the classical Poincaré inequality on \mathbb{T} yields for $E_{\alpha,\varepsilon} = (\varepsilon^{1+\alpha}, +\infty)$

$$\begin{aligned} \left\| n^\varepsilon - \frac{\tilde{N}^\varepsilon e^{-\sigma\phi}}{\int_{\mathbb{T}} e^{-\sigma\phi(\cdot, y_\parallel)} dy_\parallel} \right\|_{L^2(E_{\alpha,\varepsilon} \times \mathbb{T}^3)} &\leq \|e^{-\sigma\phi}\|_{L^\infty} \left\| n^\varepsilon e^{\sigma\phi} - \frac{\tilde{N}^\varepsilon}{\int_{\mathbb{T}} e^{-\sigma\phi(\cdot, y_\parallel)} dy_\parallel} \right\|_{L^2(E_{\alpha,\varepsilon} \times \mathbb{T}^3)} \\ &\leq \|e^{-\sigma\phi}\|_{L^\infty} \|\partial_{x_\parallel} (e^{\sigma\phi} n^\varepsilon)\|_{L^2((\varepsilon^{1+\alpha}, +\infty) \times \mathbb{T}^3)} \\ &\leq C \|e^{-\sigma\phi}\|_{L^\infty(\mathbb{T}^3)} \|e^{\sigma\phi}\|_{L^\infty(\mathbb{T}^3)}^{\frac{1}{2}} \|h_0\| \varepsilon^{\frac{1-|\alpha|}{2}}. \end{aligned}$$

This achieves the proof with \tilde{N}^ε instead of N^ε when combined with

$$\begin{aligned} \left\| n^\varepsilon - \frac{\tilde{N}^\varepsilon e^{-\sigma\phi}}{\int_{\mathbb{T}} e^{-\sigma\phi(\cdot, y_{\parallel})} dy_{\parallel}} \right\|_{L^2((0, \varepsilon^{1+\alpha}) \times \mathbb{T}^3)} &\leq \varepsilon^{\frac{1+\alpha}{2}} \left\| n^\varepsilon - \frac{\tilde{N}^\varepsilon e^{-\sigma\phi}}{\int_{\mathbb{T}} e^{-\sigma\phi(\cdot, y_{\parallel})} dy_{\parallel}} \right\|_{L^\infty((0, \varepsilon^{1+\alpha}); L^2(\mathbb{T}^3))} \\ &\leq C' \varepsilon^{\frac{1+\alpha}{2}} \|h_0\| \leq C' \varepsilon^{\frac{1-|\alpha|}{2}} \|h_0\| \end{aligned}$$

for some C' depending only on $\|\phi\|_{W^{2,\infty}(\mathbb{T}^3)}$, where we have used

$$\|\tilde{N}^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^2))} \leq \|e^{-\sigma\phi}\|_{L^\infty(\mathbb{T}^3)} \|e^{\sigma\phi}\|_{L^\infty(\mathbb{T}^3)} \|n^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3))}$$

and

$$\|n^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^3))} \leq \|e^{\sigma\phi}\|_{L^\infty(\mathbb{T}^3)}^{\frac{1}{2}} \|h^\varepsilon\|_{L^\infty(\mathbb{R}_+; \mathcal{H})} \leq \|e^{\sigma\phi}\|_{L^\infty(\mathbb{T}^3)}^{\frac{1}{2}} \|h_0\|.$$

To conclude the proof we simply observe that

$$N^\varepsilon - \tilde{N}^\varepsilon = \int_{\mathbb{T}} \left(n^\varepsilon(\cdot, \cdot, z_{\parallel}) - \frac{\tilde{N}^\varepsilon e^{-\sigma\phi(\cdot, z_{\parallel})}}{\int_{\mathbb{T}} e^{-\sigma\phi(\cdot, y_{\parallel})} dy_{\parallel}} \right) dz_{\parallel}$$

hence

$$\|N^\varepsilon - \tilde{N}^\varepsilon\|_{L^2(\mathbb{R}_+ \times \mathbb{T}^2)} \leq \left\| n^\varepsilon - \frac{\tilde{N}^\varepsilon e^{-\sigma\phi}}{\int_{\mathbb{T}} e^{-\sigma\phi(\cdot, y_{\parallel})} dy_{\parallel}} \right\|_{L^2(\mathbb{R}_+ \times \mathbb{T}^3)}.$$

□

2.4.4 Hypocoercivity : a technical lemma, Lemma 2.4.7

To motivate the actual choice of powers of ε we prove a slighted extended version of Lemma 2.4.7 for norms involving free powers of ε ,

$$\begin{aligned} \|h\|_{\varepsilon, t}^2 &= \|h\|^2 + \gamma_{\parallel, 1} \varepsilon^{\beta_{\parallel, 1}} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right) \|\partial_{v_{\parallel}} h\|^2 \\ &+ \gamma_{\parallel, 2} \varepsilon^{\beta_{\parallel, 2}} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^3 \|\partial_{x_{\parallel}} h\|^2 \\ &+ 2\gamma_{\parallel, 3} \varepsilon^{\beta_{\parallel, 3}} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^2 \langle \partial_{v_{\parallel}} h, \partial_{x_{\parallel}} h \rangle \\ &+ \gamma_{\perp, 1} \varepsilon^{\beta_{\perp, 1}} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right) \|\nabla_{v_{\perp}} h\|^2 \\ &+ \gamma_{\perp, 2} \varepsilon^{\beta_{\perp, 2}} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^3 \|\nabla_{x_{\perp}} h\|^2 \\ &+ 2\gamma_{\perp, 3} \varepsilon^{\beta_{\perp, 3}} \min\left(\{1, \frac{t}{\varepsilon^{1+\alpha}}\}\right)^2 \langle \nabla_{v_{\perp}} h, \nabla_{x_{\perp}} h \rangle \end{aligned}$$

and the corresponding ε and time dependent (partial) dissipation

$$\begin{aligned}
 D_{\varepsilon,t}(h) &= \varepsilon^{-(1+\alpha)} \|\nabla_v h\|^2 + \varepsilon^{-(1+\alpha)+\beta_{\parallel,1}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right) \|\nabla_v \partial_{v_{\parallel}} h\|^2 \\
 &+ \varepsilon^{-(1+\alpha)+\beta_{\parallel,2}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right)^3 \|\nabla_v \partial_{x_{\parallel}} h\|^2 \\
 &+ \varepsilon^{-1+\beta_{\parallel,3}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right)^2 \|\partial_{x_{\parallel}} h\|^2 \\
 &+ \varepsilon^{-(1+\alpha)+\beta_{\perp,1}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right) \|\nabla_v \nabla_{v_{\perp}} h\|^2 \\
 &+ \varepsilon^{-(1+\alpha)+\beta_{\perp,2}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right)^3 \|\nabla_v \nabla_{x_{\perp}} h\|^2 \\
 &+ \varepsilon^{-1+\beta_{\perp,3}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right)^2 \|\nabla_{x_{\perp}} h\|^2.
 \end{aligned}$$

We prove an extended version of Lemma 2.4.7 under

$$\begin{aligned}
 \max(\{2 - \alpha, (\beta_{\perp,1} + \beta_{\perp,2})/2\}) &\leq \beta_{\perp,3} \leq \min(\{\alpha + 2\beta_{\perp,1}, \alpha + 2\beta_{\perp,2}, \beta_{\perp,2} - \alpha\}), \\
 \max(\{|\alpha|, (\beta_{\parallel,1} + \beta_{\parallel,2})/2\}) &\leq \beta_{\parallel,3} \leq \min(\{\alpha + 2\beta_{\parallel,1}, \alpha + 2\beta_{\parallel,2}, \beta_{\parallel,2} - \alpha\}).
 \end{aligned} \tag{2.33}$$

The actual choice follows from minimization of $\beta_{\parallel,3}$ and $\beta_{\perp,3}$ under this constraint that aims at the best possible control on $\partial_{x_{\parallel}} h^\varepsilon$ and $\nabla_{x_{\perp}} h^\varepsilon$ through the dissipation. Indeed, (2.33) requires $\beta_{\parallel,3} \geq |\alpha|$. And the optimal choice $\beta_{\parallel,3} = |\alpha|$ enforces

$$\beta_{\parallel,1} + \beta_{\parallel,2} \leq 2|\alpha|, \quad \beta_{\parallel,2} - \alpha \geq |\alpha|, \quad 2\beta_{\parallel,1} + \alpha \geq |\alpha|, \quad 2\beta_{\parallel,2} + \alpha \geq |\alpha|$$

which uniquely determine $\beta_{\parallel,1}$ and $\beta_{\parallel,2}$ as $\beta_{\parallel,1} = 0, \beta_{\parallel,2} = 2\alpha$ when $\alpha \geq 0$ and $\beta_{\parallel,1} = -\alpha, \beta_{\parallel,2} = -\alpha$ otherwise. The optimal choice for $\beta_{\perp,3}$ is $\beta_{\perp,3} = 2 - \alpha$ that is available provided that $\beta_{\perp,1} \geq 1 - \alpha, \beta_{\perp,2} \geq 2$ and $\beta_{\perp,1} + \beta_{\perp,2} \leq 4 - 2\alpha$. The latter inequalities do not determine $\beta_{\perp,1}$ and $\beta_{\perp,2}$ uniquely but the best choice is indeed $\beta_{\perp,1} = 1 - \alpha, \beta_{\perp,2} = 2$.

Proposition 2.4.8. *Let $\phi \in W^{2,\infty}(\mathbb{T}^3)$. Under condition (2.33) there exist positive constants $\gamma_{\perp,i}, \gamma_{\parallel,i}, i \in \{1, 2, 3\}$ such that,*

— *uniformly in $(t, \varepsilon) \in \mathbb{R}_+ \times (0, 1)$, the norm $\|\cdot\|_{\varepsilon,t}$ is equivalent to the square root of*

$$\begin{aligned}
 \|\cdot\|^2 &+ \varepsilon^{\beta_{\parallel,1}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right) \|\partial_{v_{\parallel}}(\cdot)\|^2 + \varepsilon^{\beta_{\parallel,2}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right)^3 \|\partial_{x_{\parallel}}(\cdot)\|^2 \\
 &+ \varepsilon^{\beta_{\perp,1}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right) \|\nabla_{v_{\perp}}(\cdot)\|^2 + \varepsilon^{\beta_{\perp,2}} \min\left(\left\{1, \frac{t}{\varepsilon^{1+\alpha}}\right\}\right)^3 \|\nabla_{x_{\perp}}(\cdot)\|^2.
 \end{aligned}$$

— *there exists $K > 0$ such that for any $h_0 \in \mathcal{H}$ the solution h^ε of (2.27) starting from h_0 satisfies (2.30) for any $\varepsilon \in (0, 1)$ and any $t \geq 0$.*

To emphasize the algebraic nature of the proof, as in [194] we introduce abstract notation for operators

$$A = \nabla_v, \quad B = v \cdot \nabla_x - \sigma \nabla_x \phi \cdot \nabla_v,$$

and their coordinate-wise adjoints in \mathcal{H} ,

$$A^* = (v - \nabla_v), \quad B^* = -B.$$

Additionally, we evaluate (and define) the following commutators

$$\begin{aligned} C &:= [A, B] = \nabla_x, & [B, C] &= \sigma \text{Hess}(\phi) \nabla_v, \\ [A_i, A_j^*] &= \delta_{ij}, & [A, v^\perp \cdot \nabla_v] &= -A^\perp \end{aligned}$$

where δ_{ij} is the classical Kronecker symbol. Also, as a preliminary computation we observe, using Einstein's summation convention on repeated indices, that

$$\begin{aligned} \|A^* \cdot Ah\|^2 &= \langle A_i^* A_i h, A_j^* A_j h \rangle \\ &= \langle A_j A_i^* A_i h, A_j h \rangle \\ &= \langle \delta_{ij} A_i h, A_j h \rangle + \langle A_j A_i h, A_i A_j h \rangle \\ &= \|Ah\|^2 + \|A^2 h\|^2, \end{aligned}$$

where A^2 denotes the matrix $A \otimes A$. Finally we set

$$L_\varepsilon = \frac{1}{\varepsilon^{1+\alpha}} A^* \cdot A + \frac{1}{\varepsilon} B.$$

to write (2.27) as

$$\left(\partial_t + L_\varepsilon - \frac{\sigma}{\varepsilon^2} v^\perp \cdot \nabla_v \right) h^\varepsilon = 0.$$

Lemma 2.4.9. *Let h^ε solves (2.27). Then equations (2.31) and (2.32) hold as well as*

$$\frac{1}{2} \frac{d}{dt} \|h^\varepsilon\|^2 + \frac{1}{\varepsilon^{1+\alpha}} \|Ah^\varepsilon\|^2 = 0, \quad (2.34)$$

$$\frac{1}{2} \frac{d}{dt} \|A_\# h^\varepsilon\|^2 + \frac{1}{\varepsilon^{1+\alpha}} (\|A_\# h^\varepsilon\|^2 + \|A_\# Ah^\varepsilon\|^2) = -\frac{1}{\varepsilon} \langle C_\# h^\varepsilon, A_\# h^\varepsilon \rangle, \quad (2.35)$$

$$\frac{1}{2} \frac{d}{dt} \|C_\# h^\varepsilon\|^2 + \frac{1}{\varepsilon^{1+\alpha}} \|C_\# Ah^\varepsilon\|^2 = \frac{\sigma}{\varepsilon} \langle (Ah^\varepsilon \cdot \nabla_x) \nabla_{x_\#} \phi, C_\# h^\varepsilon \rangle, \quad (2.36)$$

where $\#$ stands either for \perp or \parallel .

Proof. As already pointed out, estimate (2.34) follows from (2.13) with $H(h) = h^2$. We prove (2.35) by computing on one hand for any i

$$\langle A_i L_\varepsilon h^\varepsilon, A_i h^\varepsilon \rangle = \frac{1}{\varepsilon^{1+\alpha}} (\|A_i h^\varepsilon\|^2 + \|A_i Ah^\varepsilon\|^2) + \frac{1}{\varepsilon} \langle C_i h^\varepsilon, A_i h^\varepsilon \rangle$$

and on the other hand

$$\langle A_i v^\perp \cdot \nabla_v h^\varepsilon, A_i h^\varepsilon \rangle = -\langle (A^\perp)_i h^\varepsilon, A_i h^\varepsilon \rangle$$

which vanishes when either one sums over $i = 1, 2$ or takes $i = 3$. Equality (2.36) follows from the fact that for any i

$$\langle C_i L_\varepsilon h^\varepsilon, C_i h^\varepsilon \rangle = \frac{1}{\varepsilon^{1+\alpha}} \|C_i A h^\varepsilon\|^2 - \frac{\sigma}{\varepsilon} \langle (A h^\varepsilon \cdot \nabla_x) \partial_{x_i} \phi, C_i h^\varepsilon \rangle$$

and the fact that the $v^\perp \cdot \nabla_v$ commutes with space derivatives hence its contribution vanishes by skew-symmetry. To derive (2.31) and (2.32) we first observe that for any i

$$\begin{aligned} & \langle C_i L_\varepsilon h^\varepsilon, A_i h^\varepsilon \rangle + \langle A_i L_\varepsilon h^\varepsilon, C_i h^\varepsilon \rangle \\ &= \frac{1}{\varepsilon^{1+\alpha}} \langle C_i h^\varepsilon, A_i h^\varepsilon \rangle + \frac{2}{\varepsilon^{1+\alpha}} \langle A C_i h^\varepsilon, A A_i h^\varepsilon \rangle \\ & \quad - \frac{\sigma}{\varepsilon} \langle (A h^\varepsilon \cdot \nabla_x) \partial_{x_i} \phi, A_i h^\varepsilon \rangle + \|C_i h^\varepsilon\|^2 \end{aligned}$$

and conclude with for any i

$$\langle C_i v^\perp \cdot \nabla_v h^\varepsilon, A_i h^\varepsilon \rangle + \langle A_i v^\perp \cdot \nabla_v h^\varepsilon, C_i h^\varepsilon \rangle = - \langle (A^\perp)_i h^\varepsilon, C_i h^\varepsilon \rangle.$$

□

Proof of Proposition 2.4.8. By using estimates (2.34), (2.36), (2.35), (2.32) and (2.31) in each direction we infer that⁴⁰, for any $\varepsilon \in (0, 1)$, $\beta \in \mathbb{R}^6$ and $\gamma \in (\mathbb{R}_+^*)^6$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|h^\varepsilon\|_{\varepsilon, \cdot}^2) \\ & + \sum_{\# \in \{\perp, \parallel\}} \left[\varepsilon^{-1-\alpha} \|A_\# h^\varepsilon\|^2 + \gamma_{1\#} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right) \varepsilon^{\beta_{1\#}-1-\alpha} (\|A_\# h^\varepsilon\|^2 + \|A_\# A h^\varepsilon\|^2) \right. \\ & \quad + \gamma_{2\#} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right)^3 \varepsilon^{\beta_{2\#}-1-\alpha} \|C_\# A h^\varepsilon\|^2 \\ & \quad \left. + \gamma_{3\#} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right)^2 \varepsilon^{\beta_{3\#}-1} \|C_\# h^\varepsilon\|^2 \right] \\ & \leq \sum_{\# \in \{\perp, \parallel\}} \left[\gamma_{1\#} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right) \varepsilon^{\beta_{1\#}-1} \|C_\# h^\varepsilon\| \|A_\# h^\varepsilon\| \right. \\ & \quad + \gamma_{2\#} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right)^3 \|\text{Hess}\phi\|_{L^\infty(\mathbb{T}^3)} \varepsilon^{\beta_{2\#}-1} \|C_\# h^\varepsilon\| \|A h^\varepsilon\| \\ & \quad + \gamma_{3\#} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right)^2 \|\text{Hess}\phi\|_{L^\infty(\mathbb{T}^3)} \varepsilon^{\beta_{3\#}-1} \|A_\# h^\varepsilon\| \|A h^\varepsilon\| \\ & \quad \left. + \gamma_{3\#} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right)^2 \varepsilon^{\beta_{3\#}-1-\alpha} (\|A_\# h^\varepsilon\| \|C_\# h^\varepsilon\| + 2\|A_\# A h^\varepsilon\| \|C_\# A h^\varepsilon\|) \right] \\ & + \gamma_{3\perp} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right)^2 \varepsilon^{\beta_{3\perp}-2} \|A_\perp h^\varepsilon\| \|C_\perp h^\varepsilon\| \\ & + \frac{1}{2} \sum_{\# \in \{\perp, \parallel\}} \left[\gamma_{1\#} \varepsilon^{\beta_{1\#}-1-\alpha} \|A_\# h^\varepsilon\|^2 + 3\gamma_{2\#} \varepsilon^{\beta_{2\#}-1-\alpha} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right)^2 \|C_\# h^\varepsilon\|^2 \right. \\ & \quad \left. + 4\gamma_{3\#} \varepsilon^{\beta_{3\#}-1-\alpha} \min\left(1, \frac{t}{\varepsilon^{1+\alpha}}\right) \|A_\# h^\varepsilon\| \|C_\# h^\varepsilon\| \right] \end{aligned}$$

40. We keep the variable t essentially implicit in the estimate for concision's sake.

where the last sum arises from differentiation of time factors in the definition of $\|h^\varepsilon(t, \cdot, \cdot)\|_{\varepsilon, t}$. In the former inequality, the right-hand side may be controlled by the dissipation on the left-hand side using the following procedure. Young's inequalities implies that for any $(a, b, c) \in \mathbb{R}^3$ and $(K_a, K_b, K_c) \in (\mathbb{R}_+^*)^3$ there exists $K_0 > 0$ such that for any positive N_1, N_2 and any $\varepsilon \in (0, 1)$

$$-K_a \varepsilon^a N_1^2 - K_b \varepsilon^b N_2^2 + 2K_c \varepsilon^c N_1 N_2 \leq -K_0(\varepsilon^a N_1^2 + \varepsilon^b N_2^2),$$

provided that ⁴¹

$$2c \geq a + b \quad \text{and} \quad K_c < K_a K_b.$$

It imposes the following conditions on β

$$\left. \begin{array}{l} \min(\{\alpha + 2\beta_{1\#}, \alpha + 2\beta_{2\#}\}) \geq \beta_{3\#} \\ \max(\{\alpha, -\alpha, (\beta_{1\#} + \beta_{2\#})/2\}) \leq \beta_{3\#} \end{array} \right\} \text{Induced by Fokker-Planck}$$

$$\beta_{\perp, 3} \geq 2 - \alpha \quad \text{Induced by magnetic field}$$

$$\left. \begin{array}{l} \beta_{1\#} \geq 0 \\ \beta_{2\#} - \alpha \geq \beta_{3\#} \\ \alpha \leq \beta_{3\#} \end{array} \right\} \text{Due to regularization in time}$$

By eliminating redundant conditions we obtain (2.33). It remains to find γ s satisfying the remaining constraints " $K_c < K_a K_b$ ". To do so we seek γ s as $\gamma_{i\#} = \eta^{c_i}$ with η positive and sufficiently small. Remaining conditions reduce then to

$$2c_1 > c_3 + 0, \quad 2c_2 > c_3 + 0, \quad c_3 > 0, \quad 2c_3 > c_3 + 0, \quad 2c_3 > c_1 + c_2,$$

$$c_3 > 0, \quad c_1 > 0, \quad c_2 > c_3, \quad c_3 > 0$$

and a smallness condition on η that we do not explicit here. To achieve the proof note that the former condition on c s is satisfied with for instance $c_1 = 1, c_2 = 2$ and $c_3 = 7/4$. At last we observe that those conditions are also sufficient to ensure by similar arguments the claimed equivalence of norms. \square

41. One may relax conditions to

$$2c > a + b \quad \text{or} \quad (2c = a + b \text{ and } K_c < K_a K_b)$$

if one replaces $\varepsilon \in (0, 1)$ with $\varepsilon \in (0, \varepsilon_0)$ for some sufficiently small ε_0 depending on (a, b, c) and (K_a, K_b, K_c) .

Chapitre 3

Comportement asymptotique de l'équation de Vlasov-Poisson-Fokker-Planck

Travail en collaboration avec L. Miguel Rodrigues⁴².
Ce chapitre est soumis pour publication [125].

The present contribution investigates the dynamics generated by the two-dimensional Vlasov-Poisson-Fokker-Planck equation for charged particles in a steady inhomogeneous background of opposite charges. Our goal is to provide global in time estimates that are uniform with respect to initial data taken in a bounded set of a weighted L^2 space, and where dependencies on the mean-free path τ and the Debye length δ are made explicit. In our analysis the mean free path covers the full range of possible values : from the regime of evanescent collisions $\tau \rightarrow \infty$ to the strongly collisional regime $\tau \rightarrow 0$. We pay a special attention to relax as much as possible the τ -dependent constraint on δ ensuring exponential decay with explicit τ -dependent rates towards the stationary solution. In the strongly collisional limit $\tau \rightarrow 0$, we also examine all possible asymptotic regimes selected by a choice of observation time scale. Here also, our emphasis is on strong convergence, uniformity with respect to time and to initial data in bounded sets of a L^2 space. Our proofs rely on a thorough preliminary study of the nonlinear elliptic equation defining stationary solutions and a careful tracking and optimization of parameter dependencies of hypocoercive/hypoelliptic estimates.

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3.1 Introduction

In a periodic box $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we consider a large number of charged particles subject to self-consistent electrostatic forces and interacting with a fixed background of steady heavy particles with opposite charge as well as a thermal bath. The system can be described at a kinetic level by a distribution function $f : \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}_+$, $(t, x, v) \mapsto f(t, x, v)$ obeying the Vlasov-Poisson-Fokker-Planck (VPFP) system which reads in dimensionless form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \frac{1}{\tau} \operatorname{div}_v (vf + \nabla_v f), \\ -\delta^2 \Delta_x \phi = \rho - \rho_h, \quad \rho = \int_{\mathbb{R}^2} f \, dv \end{cases} \quad (3.1)$$

and is completed with the prescription of an initial data, $f(0, \cdot, \cdot) = f_0$.

The number density of background particles is given as $\rho_h : \mathbb{T}^2 \rightarrow \mathbb{R}_+$ whereas $\phi : \mathbb{R}_+ \times \mathbb{T}^2 \rightarrow \mathbb{R}$ is the electrostatic potential satisfying a Poisson equation. Interactions with the thermal bath are modeled by the Fokker-Planck term at the right-hand side of the first equation. The characteristic temperature of the bath is scaled to 1. The respective total charges have also been scaled to 1,

$$\int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 = 1, \quad \int_{\mathbb{T}^2} \rho_h = 1,$$

the constraint being preserved by the time evolution. The parameter τ denotes the scaled mean free path between two "collisions" with the thermal bath. The scaled Debye length δ measures the radius of electrostatic influence of an isolated particle. The asymptotics $\delta \rightarrow 0$ is called the quasineutral limit and the terminology quasineutral parameter is also used for δ . While, as we clarify later on, our strategy has a clear counterpart in any dimension, the precise outcome would be significantly modified by another dimensional choice therefore we choose to restrict ourselves to the two dimensional setting.

The VPFP system, as expounded here or with some variants, including the consideration of gravitational forces instead of electrostatic forces, has a long history. A derivation of the model and references to even earlier derivations may already be found in a seminal piece of work by Chandrasekhar [50]. Concerning the Cauchy problem in two dimensions, first global well-posedness results were obtained by Neunzert, Pulvirenti and Triolo [161] on \mathbb{R}^2 for bounded compactly-supported initial data, by the method of stochastic characteristics; then by Degond [59] for a frictionless version of the system, on \mathbb{R}^2 , for $W^{1,1}$ data with finite moments in velocity of more than second order, by relying mostly on suitable maximum principles. Since then those results have been extended and improved in various ways [192, 41, 165].

Our goal is to provide a description of the dynamics of solutions to Equation (3.1) on every possible time scale, from initial data to exponential convergence towards a stationary state, and in any regime of the collisional parameter τ . We also aim at providing strong convergence results uniform with respect to initial data taken from bounded sets.

Our normalization already contains $\|f_0\|_{L^1(\mathbb{T}^2 \times \mathbb{R}^2)} = 1$. However to benefit from a Hilbert structure we shall use a weighted L^2 space embedded in L^1 instead of L^1 itself. Namely we introduce the norm $\|\cdot\|_{L^2(M^{-1})}$ defined by

$$\|u\|_{L^2(M^{-1})}^2 = \iint_{\mathbb{T}^2 \times \mathbb{R}^2} \frac{|u(x,v)|^2}{M(v)} dx dv$$

where M is the local Maxwellian

$$M(v) = \frac{1}{(2\pi)} e^{-\frac{1}{2}|v|^2}.$$

The main advantage in this particular weight choice stems from the fact that the Fokker-Planck operator is symmetric on $L^2(M^{-1})$. Then we choose $f_0 \geq 0$ such that

$$\|f_0\|_{L^2(M^{-1})} \leq R_0,$$

where $R_0 \geq 1$ is fixed but arbitrary and obtain bounds depending on R_0 but not on f_0 itself.

As a preliminary observation, note that stationary states $(f^\infty, \phi_\infty^\delta)$ are characterized by

$$\begin{cases} v \cdot \nabla_x f^\infty - \nabla_x \phi_\infty^\delta \cdot \nabla_v f^\infty = \frac{1}{\tau} \operatorname{div}_v (v f^\infty + \nabla_v f^\infty), \\ -\delta^2 \Delta_x \phi_\infty^\delta = \rho_\infty - \rho_h, \quad \rho_\infty = \int_{\mathbb{R}^2} f^\infty dv. \end{cases}$$

Since the Fokker-Planck operator is symmetric on $L^2(e^{-\phi_\infty^\delta} M^{-1})$ and the transport part is skew-symmetric on this space, if for instance ϕ_∞^δ is bounded and $f^\infty \in L^2(M^{-1})$ then the two parts must vanish separately when applied to f^∞ . Therefore f^∞ is a global Maxwell-Boltzmann distribution

$$f^\infty(x, v) = M(v) e^{-\phi_\infty^\delta(x)}, \tag{3.2}$$

and the stationary equations reduce to

$$-\delta^2 \Delta_x \phi_\infty^\delta = e^{-\phi_\infty^\delta} - \rho_h. \tag{3.3}$$

Concerning the latter, we observe that

Lemma 3.1.1. *For any $\rho_h \in H^{-1}(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$, for any $\delta > 0$, Equation (3.3) possesses a unique weak solution $\phi_\infty^\delta \in H^1(\mathbb{T}^2)$ and moreover $\int_{\mathbb{T}^2} e^{-\phi_\infty^\delta} = 1$.*

Our analysis involves a more detailed study of Equation (3.3). Though we have not found in the literature directly applicable results, the first steps of our analysis of equilibrium states is however by now essentially standard [100, 76, 73, 98, 74, 37, 79, 78]. It is noteworthy that the existence of nontrivial stationary states holds for most generalizations of (3.1). To prevent it, one essentially needs to look either at frictionless

versions, where the Fokker-Planck operator is replaced with a Laplacian in velocity, or at equations on \mathbb{R}^2 with no background density $\rho_h \equiv 0$ and no confining potential [98, 74]. Correspondingly, in the former case, in various perturbative settings, self-similar decay to zero at algebraic rates has been proven [42, 40, 166, 136].

In [29, 74], Bouchut and Dolbeault prove that in \mathbb{R}^d , $d \geq 3$ in presence of a confining potential, when $\rho_h \equiv 0$, solutions f starting from initial data with finite mass, finite energy, finite entropy and such that $\nabla_x \phi \in L_{loc}^\infty((0, \infty); L^\infty(\mathbb{R}^d))$, converge in $L^1(\mathbb{R}^{2d})$ to a global Maxwell-Boltzmann distribution as time goes to infinity. Their proof is likely to extend to the case under consideration and our focus is not in reproving such behavior but in providing uniform bounds in the same space for the solution and the initial data, and including exponential decay to equilibrium.

Obviously this encompasses asymptotic stability of equilibrium states. For the equation under consideration, Hwang and Jang [130] have already proved some asymptotic stability results for the case where $\rho_h \equiv 1$ so that $\phi_\infty^\delta \equiv 0$, in a topology requiring $(f_0 - f^\infty)/\sqrt{f^\infty}$ to be small in $H^3(\mathbb{T}^2 \times \mathbb{R}^2)$ (instead of L^2). Note that condition $\phi_\infty^\delta \equiv 0$ brings a lot of exceptional cancellations in the analysis and a considerable part of asymptotic stability results for similar equations have been so far focused essentially on this case. Our result contains asymptotic stability in L^2 with explicit dependence on parameters of exponential convergence rates, and arbitrary ρ_h , provided δ is sufficiently large. Though we have not investigated this track, we anticipate that our strategy of proof would also yield exponential asymptotic stability in L^2 for any fixed δ provided $\rho_h - 1$ is sufficiently small, in the spirit of [78]. During the completion of the present contribution, also appeared by Bedrossian [16] an extension of [130], still with $\rho_h \equiv 1$ and high-regularity weighted L^2 spaces (but with algebraic weights instead of Gaussian ones), to a version of the system where collisions are more nonlinear and model self-collisions instead of collisions with a thermal bath. More importantly to us, holding δ fixed, Bedrossian provides a careful study of dependencies on the parameter τ in the limit $\tau \rightarrow \infty$, benefiting both from Landau damping for the limiting Vlasov-Poisson system [159, 17] and from mixing-enhanced dissipation. This extends to the full nonlinear regime the former linearized analysis by Tristani [185].

It is important to note however that our nonlinear parameter shall not be the distance of the initial data to some equilibrium, that is essentially arbitrary here, but the inverse of the quasineutral parameter. In particular even when $\rho_h \equiv 1$ our set of initial data contains data that fail to satisfy Penrose stability criterion, that is known to play a crucial role for the Vlasov-Poisson system both in the large-time limit [169, 113, 159, 17] and in the quasineutral limit [117]. Though the currently available results [59] concerning the approximation of solutions to (3.1) by those of the Vlasov-Poisson system are not precise enough to justify relevant heuristic arguments, this strongly hints at the fact that one cannot benefit from mixing properties at least initially and that one needs to take $\delta \rightarrow \infty$ when $\tau \rightarrow \infty$.

Though they do not state it in this precise way, the recent analysis by Hérau and Thomann [123] proves precisely that for any fixed (τ, R_0) there indeed exists a $\delta_0(\tau, R_0)$ such that when $\delta > \delta_0(\tau, R_0)$ one may obtain global bounds for the solution, for the

version of the system set on \mathbb{R}^2 , with a confining potential but with $\rho_h \equiv 0$. Our main goal here is to provide explicit upper bounds for those $\delta_0(\tau, R_0)$. The analysis of Hérau and Thomann scales badly with respect to τ , due to the anisotropic nature of hypo-coercivity/hypoellipticity. Indeed, their analysis uses directly decay estimates from [121] (see also [122, 120]) for the linearization about a stationary solution. Yet using those would prevent us from keeping track of the anisotropic nature of dissipation that helps in improving estimates of $\delta_0(\tau, R_0)$. Moreover, as we discuss more precisely below, in the limit $\tau \rightarrow 0$, our optimization of involved hypo-coercive Lyapunov functionals differs from what would follow from an optimal treatment of the linearized problem !

Main results

Let us state our main results concerning the latter, that we split in cases including respectively either the strongly collisional regime or the the regime of evanescent collisions.

Our first result concerns the diffusive regime, namely when τ is small.

Theorem 3.1.2 (Diffusive regime; $\tau \lesssim 1$). *For any $\tau_0 > 0$, any $R_0 > 1$ and any $\rho_h \in W^{1,p}(\mathbb{T}^2)$, with $p > 2$, such that $\int_{\mathbb{T}^2} \rho_h = 1$, there exist $\delta_0 > 0$, $\theta_0 > 0$ and $K > 0$ such that for any $\tau \in (0, \tau_0)$, any $\delta \in (\delta_0, \infty)$,*

— *for any f_0 such that*

$$\|f_0\|_{L^2(M^{-1})} \leq R_0 \quad \text{and} \quad \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 = 1,$$

then Equation (3.1) possesses a (unique strong) solution f starting from f_0 , and it satisfies for any $t \geq 0$

$$\|f(t, \cdot, \cdot) - f^\infty\|_{L^2(M^{-1})} \leq K \|f_0 - f^\infty\|_{L^2(M^{-1})} e^{-\theta_0 \tau t}$$

where f^∞ solve (3.2)-(3.3);

— *for any f_0, g_0 such that*

$$\|f_0\|_{L^2(M^{-1})} \leq R_0, \quad \|g_0\|_{L^2(M^{-1})} \leq R_0 \quad \text{and} \quad \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 = \int_{\mathbb{T}^2 \times \mathbb{R}^2} g_0 = 1,$$

then corresponding solutions f and g satisfy for any $t \geq 0$

$$\|f(t, \cdot, \cdot) - g(t, \cdot, \cdot)\|_{L^2(M^{-1})} \leq K \|f_0 - g_0\|_{L^2(M^{-1})} e^{-\theta_0 \tau t}.$$

Actually we may ensure $\delta_0 \leq K(1 + R_0^{1/2})$ for some suitable K depending on ρ_h . The forcing by ρ_h indeed induces inhomogeneity in the bound.

Our second result concerns the regime of evanescent collisions, namely when τ is large.

Theorem 3.1.3 (Evanescent collisions; $\tau \gtrsim 1$). *For any $\varepsilon > 0$, any $\tau_0 > 0$, any $R_0 > 1$ and any $\rho_h \in W^{1,p}(\mathbb{T}^2)$, with $p > 2$, such that $\int_{\mathbb{T}^2} \rho_h = 1$, there exist $K_0 > 0$, $\theta_0 > 0$ and $K > 0$ such that for any $\tau \in (\tau_0, \infty)$, any $\delta \in (K_0 \tau^{7/15+\varepsilon}, \infty)$,*

- for any f_0 such that

$$\|f_0\|_{L^2(M^{-1})} \leq R_0 \quad \text{and} \quad \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 = 1,$$

then Equation (3.1) possesses a (unique strong) solution f starting from f_0 , and it satisfies for any $t \geq 0$

$$\|f(t, \cdot, \cdot) - f^\infty(\cdot, \cdot)\|_{L^2(M^{-1})} \leq K \|f_0 - f^\infty\|_{L^2(M^{-1})} e^{-\theta_0 \frac{t}{\tau}}$$

where f^∞ solve (3.2)-(3.3);

- for any f_0, g_0 such that

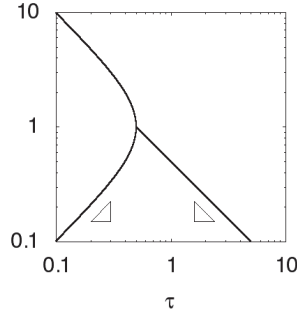
$$\|f_0\|_{L^2(M^{-1})} \leq R_0, \quad \|g_0\|_{L^2(M^{-1})} \leq R_0 \quad \text{and} \quad \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 = \int_{\mathbb{T}^2 \times \mathbb{R}^2} g_0 = 1,$$

then corresponding solutions f and g satisfy for any $t \geq 0$

$$\|f(t, \cdot, \cdot) - g(t, \cdot, \cdot)\|_{L^2(M^{-1})} \leq K \|f_0 - g_0\|_{L^2(M^{-1})} e^{-\theta_0 \frac{t}{\tau}}.$$

We may ensure $K_0 \leq K(1 + R_0^{1/2})$ for some suitable K again depending on ρ_h .

Some comments are in order. Previous theorems prove that the combination of the transport term that mixes space and velocity at typical time scale of size 1 with the Fokker-Planck part that regularizes and dissipates in the velocity variable at time scale τ do lead to both decay and regularity in all variables, regularization being somewhat implicit in our statements, but clearly apparent in our proofs. This type of structure is actually the prototype of systems leading to hypocoercive decay to equilibrium [122, 158, 194, 77, 75, 157] and hypoelliptic regularization [128, 137, 122, 120, 157]. Regularization allows us to obtain exponential convergence starting from initial data in $L^2(M^{-1})$ and to prove Lipschitz dependence on the initial data from the norm topology of $L^2(M^{-1})$ to the norm topology of $L^\infty(\mathbb{R}_+; L^2(M^{-1}))$. The latter is somewhat in contrast with the analysis in [114]. The presence of a constant $K \geq 1$ allowing for some finite transient growth also reflects both the non purely dissipative nature of our nonlinear system and our use of regularizing effects. Indeed the dissipations of functionals involved in our proofs need some time to control enough regularity to prevent the nonlinearity from inducing some norm growth.



Real part of eigenvalues of the toy model vs. τ , in logarithmic scale.

$\tau \rightarrow 0^+$. This is consistent with our results for (3.1). Also note that our decay rates are directly related to the spectral gap of self-adjoint operators

$$-\tau \Delta_x - \frac{1}{\tau} \operatorname{div}_v (v \cdot + \nabla_v \cdot)$$

on $L^2(M^{-1})$, that exhibit the same asymptotic behaviors.

To comment on constraints on δ , let us start with a deliberately oversimplified analogy. Retaining from the foregoing discussion only decay rates and assuming that regularity is not an issue leads to the consideration of scalar differential equations $y'(t) = \frac{1+y(t)}{\delta^2}y(t) - \tau y(t)$ when $\tau \lesssim 1$ and $y'(t) = \frac{1+y(t)}{\delta^2}y(t) - \frac{1}{\tau}y(t)$ when $\tau \gtrsim 1$, the forcing by y/δ^2 modeling in particular the effect of the inhomogeneity of ρ_h . For those equations the ball of center zero and radius R_0 is uniformly attracted to zero respectively when $\delta > \sqrt{1 + R_0} \tau^{-1/2}$ and when $\delta > \sqrt{1 + R_0} \tau^{1/2}$. Though those equations fail to provide relevant predictions for our system, they give an idea of the kind of conclusion that can be drawn when one first derives bounds for the semi-group evolution, without tracking anisotropic dependences in parameters, and then uses them at the nonlinear level. In the regime where $\tau \lesssim 1$ it is obvious that the foregoing scalar equation is way too pessimistic to be relevant since it overlooks the dissipation in v at rate τ^{-1} that helps to bound any term that involves a derivative in v , including all contributions from the electric field, hence all nonlinear terms! It is a bit more surprising that it also fails to predict an accurate threshold in the limit $\tau \rightarrow \infty$.

It may be worth stressing that the necessity to impose $\delta \rightarrow \infty$ when $\tau \rightarrow \infty$ is a short-time/nonlinear constraint and that for the evolution linearized about f^∞ one may obtain uniform estimates assuming only that δ is large uniformly with respect to $\tau \in (0, \infty)$. This follows directly from Proposition 3.7.2 and contrasts with Proposition 3.7.3.

To try a comparison with previous analyses involving an explicit discussion on the parameter τ , let us extrapolate that when $\rho_h \equiv 1$ our constraint could be turned into $\|f_0 - f^\infty\|/\delta^2 \leq K_\varepsilon \min(1, \tau^{-(\frac{14}{15} + \varepsilon)})$ for any $\varepsilon > 0$ and some K , where $\|\cdot\|$ is a suitable norm. As already pointed out, in the limit $\tau \rightarrow \infty$ this compares unfavorably with the

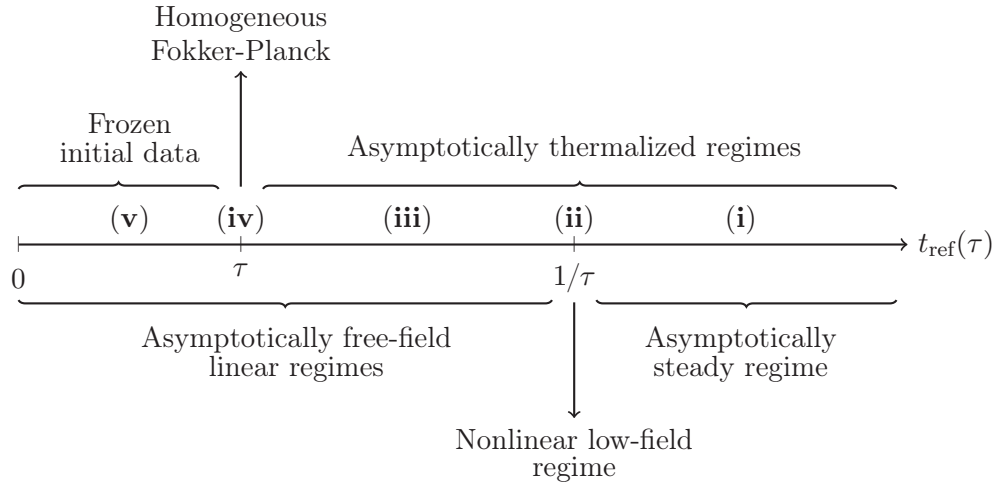
The dependence of decay rates in the collisional parameter τ also stems from the multi-scale anisotropic behavior of the system. As a simple but enlightening toy model consider the system of two ordinary differential equations

$$\begin{aligned} X'(t) &= V(t) \\ V'(t) &= -X(t) - \tau^{-1}V(t) \end{aligned}$$

that mixes X and V at scale 1 and dissipates explicitly V at scale τ , mimicking respectively transport and collisions. For the toy model one readily checks that the rate of exponential decay to zero behaves as $1/(2\tau)$ in the limit $\tau \rightarrow \infty$ and as τ in the limit

recent analysis of Bedrossian [16] that only requires $\|f_0 - f^\infty\| \leq K_\delta \tau^{-1/3}$ (with a norm encompassing strong regularity however) when $\tau \lesssim 1$, and whose cornerstones are Landau damping and mixing-enhanced dissipation — a priori not available in our context — that leads to a dissipation in $\tau^{-1/3}$, instead of τ^{-1} , for inhomogeneities in the x variable. In the limit $\tau \rightarrow 0$ it would compare however extremely favorably with the analysis by Jin and Zhu [132], that also appeared during the finalization of the present contribution and that requires $\|f_0 - f^\infty\|/\delta^2 \leq K \tau$ when $\tau \lesssim 1$ (in a norm also requiring strong regularity of the initial data). In particular their result does not allow to take limits when $\tau \rightarrow 0$ for a fixed nontrivial pair (f_0, δ) .

Now we turn to asymptotic regimes. In the diffusive regime, constraints on δ are uniform with respect to τ . Therefore for any sufficiently large fixed δ one may examine model reductions in the limit $\tau \rightarrow 0$. Those depend on the typical time scale chosen to observe the solution. Therefore we introduce an observation time $t_{\text{ref}} \equiv t_{\text{ref}}(\tau)$ and scale f accordingly to obtain f_{ref} through $f_{\text{ref}}(t, x, v) = f(t_{\text{ref}} t, x, v)$. The following diagramm may help the reader to visualize remarkable time scales appearing in our last main result.



Theorem 3.1.4 (Diffusive limits; $\tau \rightarrow 0$).

Let us denote f_{ref} the rescaled solution provided by Theorem 3.1.2.

(i). With K depending only on R_0, ρ_h and τ_0 , there holds for any $t \geq 0$ and $\tau \leq \tau_0$

$$\|f_{\text{ref}}(t, \cdot, \cdot) - f^\infty(\cdot, \cdot)\|_{L^2(M^{-1})} \leq K \|f_0 - f^\infty\|_{L^2(M^{-1})} e^{-\theta_0 t_{\text{ref}} \tau t}$$

so that, in $L^2(M^{-1})$, uniformly with respect to times t taken in compacts of $(0, \infty]$,

$$f_{\text{ref}}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} f^\infty$$

provided $t_{\text{ref}}(\tau) \tau \xrightarrow{\tau \rightarrow 0} \infty$.

- (ii). When $t_{\text{ref}}(\tau) = \tau^{-1}$, with K depending only on R_0 , any $\rho_h \in W^{2,p}(\mathbb{T}^2)$, with $p > 2$, and τ_0 , there holds for any $\tau \leq \tau_0$

$$\|f_{\text{ref}} - \rho_{\text{ref}} M\|_{L^2(\mathbb{R}_+; L^2(M^{-1}))} \leq K \tau \|f_0 - f^\infty\|_{L^2(M^{-1})} \xrightarrow{\tau \rightarrow 0} 0$$

with $\rho_{\text{ref}} = \int_{\mathbb{R}^2} f_{\text{ref}}(\cdot, \cdot, v) dv$, and, in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^2))$,

$$\rho_{\text{ref}}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} \rho_{as}$$

where $\rho_{as} \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^2)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{T}^2))$ is the unique strong solution to the drift-diffusion equation

$$\begin{cases} \partial_t \rho_{as} + \text{div}_x(E_{as} \rho_{as} - \nabla_x \rho_{as}) = 0, \\ E_{as} = \delta^{-2} \nabla_x \Delta_x^{-1}(\rho_{as} - \rho_h), \end{cases} \quad (3.4)$$

starting from $\rho_0 = \int_{\mathbb{R}^2} f_0(\cdot, v) dv \in L^2(\mathbb{T}^2)$.

- (iii). In $L^2(M^{-1})$ uniformly with respect to times t taken in compacts of $(0, \infty)$,

$$f_{\text{ref}}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} \rho_0 M$$

with $\rho_0 = \int_{\mathbb{R}^2} f_0(\cdot, v) dv$ provided $t_{\text{ref}}(\tau) \tau \xrightarrow{\tau \rightarrow 0} 0$ and $t_{\text{ref}}(\tau) \tau^{-1} \xrightarrow{\tau \rightarrow 0} \infty$.

- (iv). When $t_{\text{ref}}(\tau) = \tau$, in $L^2(M^{-1})$, uniformly with respect to times t taken in compacts of $[0, \infty)$,

$$f_{\text{ref}}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} f_{FP}$$

where f_{FP} is the unique strong solution to the homogeneous Fokker-Planck equation

$$\partial_t f_{FP} = \text{div}_v(v f_{FP} + \nabla_v f_{FP}), \quad (3.5)$$

starting from f_0 .

- (v). In $L^2(M^{-1})$, uniformly with respect to times t taken in compacts of $[0, \infty)$,

$$f_{\text{ref}}(t, \cdot, \cdot) \xrightarrow{\tau \rightarrow 0} f_0$$

provided $t_{\text{ref}}(\tau) \tau^{-1} \rightarrow 0$ when $\tau \rightarrow 0$.

Note that our Lipschitz dependence on initial data also allows us to replace the fixed initial data of the previous theorem with a τ -dependent family converging to f_0 in $L^2(M^{-1})$ as $\tau \rightarrow 0$.

In this theorem we prove strong convergence in each particular regime. In the asymptotically steady regime (i) we prove exponential convergence to the global Maxwellian uniformly with respect to f_0 . Here uniformity in time necessarily excludes a neighborhood of initial time since the asymptotic limit loses any trace of the initial data. Likewise, in regimes (ii) and (iii) part of the initial data is asymptotically lost so that uniformity

near initial time may hold for the asymptotics of ρ_{ref} but not for f_{ref} . There is a threshold $t_{\text{ref}}(\tau) = \tau^{-1}$ between asymptotic regimes where nonlinearity due to field effects play a role ((i) and (ii)) and the asymptotically field-free linear regimes ((iii), (iv) and (v)). For the latter regimes, moreover, the macroscopic density ρ is asymptotically stuck to its initial data, which prevents uniformity in time to hold up to infinite time. Actually those three regimes may be understood via the fact that solutions are asymptotically close to the family of solutions \tilde{f} to homogeneous Fokker-Plank equation

$$\frac{\tau}{t_{\text{ref}}(\tau)} \partial_t \tilde{f} = \text{div}_v(v \tilde{f} + \nabla_v \tilde{f}),$$

starting from f_0 . As for the foregoing linear case, our proof does provide convergence rates for those three asymptotically linear regimes when one assumes more regularity on initial data.

At the threshold (ii) appears the most interesting limit, known as the low-field, parabolic or diffusion limit. Here the asymptotic dynamics for ρ_{ref} is non-linear and capable to connect initial data to large-time equilibrium, resulting in estimates uniform with respect to time in $[0, \infty]$. Note moreover that System (3.4) inherits the same properties of exponential convergence and uniform stability with respect to initial data as (3.1). As for the large-time limit, our main goal here is not to prove that the limit holds but to provide strong convergence in the same L^2 space where the initial data is taken and uniform in time. Indeed the present limit has been extensively investigated over the years, with first results obtained by Poupaud and Soler [172], then improved by Goudon [106] and extended to higher dimensions [81] and multiple-species dynamics [195, 124]. In particular, for various variants of (3.1), the limit is known to hold for initial data with finite mass, finite energy and finite entropy (plus one moment in velocity when the system is set on \mathbb{R}^2). Yet the convergence of ρ proved there is local in time and only weak in the spatial variable, that is, it is proved in $L_{loc}^\infty((0, \infty); L^1 - \text{weak})$. In particular those convergence results cannot be used to transfer the large-time behavior of the limiting (3.4) to the original (3.1). In contrast, assuming here that δ is sufficiently large, uniformly with respect to τ , we prove that the convergence of ρ_{ref} holds in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^2))$ assuming that $f_0 \in L^2(M^{-1})$. Moreover, as for asymptotically linear regimes, assuming more regularity on the initial data, our proof also provides uniform convergence rates.

Strategy

We now provide some more details on our strategy. To begin with, though we do consider initial data not necessarily close to f^∞ , it is convenient for this forthcoming analysis to write solutions in a seemingly perturbative form by introducing the following new unknowns

$$h = \frac{f - f_\infty}{f_\infty}, \quad \psi = \phi - \phi_\infty^\delta.$$

In terms of (h, ψ) , System (3.1) becomes

$$\partial_t h + v \cdot \nabla_x h - \nabla_x \phi_\infty^\delta \cdot \nabla_v h + \frac{1}{\tau} (v - \nabla_v) \cdot \nabla_v h + \nabla_x \psi \cdot v = \nabla_x \psi \cdot (\nabla_v - v) h \quad (3.6)$$

coupled with the Poisson equation

$$-\delta^2 \Delta \psi = n, \quad (3.7)$$

where the source is given by

$$n = \int_{\mathbb{R}^2} h f^\infty dv.$$

Once sufficient bounds have been obtained on ϕ_∞^δ , one may rightfully replace in all our statements canonical norms of $L^2(M^{-1})$ with the equivalent norm arising from its interpretation as $\mathcal{H} := L^2(\mathbb{T}^2 \times \mathbb{R}^2, \mu)$ where μ is the measure with probability density function f^∞ . The space \mathcal{H} is thus endowed with its canonical scalar product

$$\langle \cdot, \cdot \rangle : (f, g) \mapsto \int_{\mathbb{T}^2 \times \mathbb{R}^2} f g d\mu.$$

and we denote by $\| \cdot \|$ the corresponding norm. The main advantage is that now at the linearized level the transport term becomes skew-symmetric for the new structure whereas the Fokker-Planck operator remains symmetric. We stress that $h_0 f^\infty$ is mean-free and so will remain $h(t, \cdot, \cdot) f^\infty$ for later times t . Hence we introduce the following subspace of \mathcal{H}

$$\mathcal{H}_0 = \left\{ h \in \mathcal{H} \text{ such that } \langle \mathbf{1}; h \rangle = \int_{\mathbb{T}^2 \times \mathbb{R}^2} h d\mu = 0 \right\}$$

where $\mathbf{1}$ denotes the constant function with value 1.

As is customary in the field, in particular following the memoir of Villani [194], we shall write estimates proving hypocoercivity using an abstract formulation of the equations. To do so we introduce the following unbounded operators on \mathcal{H}

$$\begin{aligned} A &= \nabla_v, \\ B &= v \cdot \nabla_x - (\nabla_x \phi_\infty^\delta) \cdot \nabla_v. \end{aligned}$$

Let us mention that in order to clarify computations using vectors or higher-order tensors we sometimes use Einstein summation convention on repeated indices. In this way, we also introduce L_τ defined as

$$L_\tau = \frac{1}{\tau} A_i^* A_i + B$$

where we use the superscript $*$ to denote coordinate-wise formal adjoint in \mathcal{H} . On this example our convention explicitly reads

$$A^* = v - \nabla_v, \quad B^* = -B.$$

The perturbative form (3.6)-(3.7) is then equivalently written

$$\begin{cases} \partial_t h + L_\tau h - E \cdot A^*(\mathbf{1}) = E \cdot A^* h, \\ E = \delta^{-2} \nabla_x \Delta_x^{-1} n, \quad n = \int_{\mathbb{R}^2} h f^\infty dv. \end{cases} \quad (3.8)$$

In this abstract form preservation of mass follows from $A(\mathbf{1}) = 0$, $B^*(\mathbf{1}) = 0$, and the non-linear part of the system lies on the right-hand side of the first equation.

Commutators play a crucial role in the analysis so that we define and evaluate

$$\begin{aligned} C &= [A, B] = \nabla_x, \\ [B, C] &= \text{Hess}(\phi_\infty^\delta) \nabla_v, \\ [A_i, A_j^*] &= \delta_{ij}, \end{aligned}$$

where $\text{Hess}(\phi_\infty^\delta)$ is the Hessian matrix of ϕ_∞^δ and δ_{ij} is the Kronecker symbol. At the linearized level, good dissipative terms arise from

$$\langle h; L_\tau h \rangle = \frac{1}{\tau} \|Ah\|^2 \quad \text{and} \quad \langle ABh; Ch \rangle + \langle Ah; CBh \rangle = \|Ch\|^2 - \langle Ah; \text{Hess}(\phi_\infty^\delta) Ah \rangle$$

that are involved in computations of time derivatives of respectively $\|h\|^2$ and $\langle Ah, Ch \rangle$, when h solves (3.8). Incidentally, we point out that, for K, L two vector-valued operators, we shall repeatedly use KL to denote the matrix-valued operator with coefficients $K_i L_j$. For instance the operator yielding the Hessian in the velocity variable is denoted AA or A^2 .

We will prove all parts of Theorems 3.1.2 and 3.1.3 — existence, uniqueness, stability with respect to initial data, regularization and exponential convergence — at once by interpreting (3.8) as the research of a fixed point for a strict contraction on a functional space that encodes regularization and decay and that quantifies precisely dependences on τ . This function space is designed from functionals $\mathcal{E}_{\gamma, \beta, \tau, \delta}$ and $\mathcal{F}_{\gamma, \beta, \tau, \delta}^\theta$ built as follows. First we consider the following weighted Sobolev norm

$$\begin{aligned} \|h\|_{\gamma, \beta, \tau, t}^2 &= \|h\|^2 + \gamma_1 \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right) \|Ah\|^2 \\ &+ \gamma_2 \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|Ch\|^2 + 2\gamma_3 \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \langle Ah, Ch \rangle, \end{aligned}$$

and a corresponding dissipation

$$\begin{aligned} D_{\gamma, \beta, \tau, t}(h) &= \tau^{-1} \|Ah\|^2 + \gamma_1 \tau^{\beta_1 - 1} \min\left(1, \frac{t}{\tau}\right) \|A^* \cdot Ah\|^2 \\ &+ \gamma_2 \tau^{\beta_2 - 1} \min\left(1, \frac{t}{\tau}\right)^3 \|ACH\|^2 + \gamma_3 \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \|Ch\|^2. \end{aligned}$$

The presence of cross terms in $\|h\|_{\gamma, \beta, \tau, t}$ is related to the above mentioned commutator computation. Let us also mention that we have chosen to use weights with pure powers of τ instead of, for instance, some minima of two powers, principally to facilitate reading. However it forces us to split the discussion between regimes $\tau \gtrsim 1$ and $\tau \lesssim 1$.

Note that an estimation of $\|h(t, \cdot, \cdot)\|_{\gamma, \beta, \tau, t}$ in terms of $\|h_0\|_{\gamma, \beta, \tau, 0} = \|h_0\|$ when h solves (3.8) would encode hypoelliptic regularization. Powers of the time variable should be appreciated with this in mind as they are associated with classical gain of regularity afforded by the kinetic Fokker-Planck operator; see for instance [194, Appendix A.21]. Furthermore, if one proves that solutions to (3.8) satisfy for any $t_2 \geq t_1 \geq \tau$

$$\|h(t_2)\|_{\gamma, \beta, \tau, t_2}^2 + \theta \int_{t_1}^{t_2} D_{\gamma, \beta, \tau, s}(h(s))^2 ds \leq \|h(t_1)\|_{\gamma, \beta, \tau, t_1}^2$$

for some $\theta > 0$, then by using Poincaré inequality one deduces exponential decay with rates scaling as $\min(\tau^{-1}, \tau^{\beta_3}, \tau^{\beta_3 - \beta_2})$. Hence closing such form of estimates will prove at the same time hypoelliptic regularization and hypocoercitive decay.

In Section 3.5 we first show what choices of parameters γ and β are available for the reduced toy system $\partial_t h + L_\tau h = 0$ that may be though as a $\delta = \infty$ version of System (3.8). We may then analyze for the original problem what is, among available parameters for the toy system, the optimal choice to relax as much as possible the constraint on δ and still obtain the same entropy/dissipation relations. Since the way in which we prove corresponding nonlinear a priori estimates lends itself to a strict contraction formulation this will lead to Theorems 3.1.3 and 3.1.2. Actually in our study of (3.8) we rather use

$$\mathcal{E}_{\gamma, \beta, \tau, \delta, t}(h) = \|h\|_{\gamma, \beta, \tau, t}^2 + \delta^2(1 + \gamma_1 \tau^{\beta_1} \min(1, \frac{t}{\tau})) \|E\|_{L^2}^2$$

to offer a better account of electric-field contributions. Our goal is then essentially to build solutions such that

$$\mathcal{F}_{\gamma, \beta, \tau, \delta}^\theta(h) \leq K \|h_0\|^2$$

for some constant K , where

$$\mathcal{F}_{\gamma, \beta, \tau, \delta}^\theta(h) = \mathcal{E}_{\gamma, \beta, \tau, \delta}(h) + \theta D_{\gamma, \beta, \tau, \delta}(h) \tag{3.9}$$

with

$$\mathcal{E}_{\gamma, \beta, \tau, \delta}(h) = \sup_{t \geq 0} \mathcal{E}_{\gamma, \beta, \tau, \delta, t}(h(t, \cdot, \cdot)),$$

and

$$D_{\gamma, \beta, \tau, \delta}(h) = \int_0^\infty D_{\gamma, \beta, \tau, t}(h(t, \cdot, \cdot)) dt$$

for a suitable choice of $\theta > 0$, $\beta \in \mathbb{R}^3$ and $\gamma \in (0, +\infty)^3$ under the weakest possible constraint on δ and uniformly with respect to (τ, h_0) taken in relevant spaces. It turns out that our choices are $\beta = (0, 2, 1)$ when $\tau \lesssim 1$ and $\beta = (-8/15, 2/5, -1/15)$ when $\tau \gtrsim 1$, and we amply comment on motivations of these choices along the proof. We stress however here that the latter choice differ from the choice $\beta = (-1, -1, -1)$ optimal for the linearized dynamics when $\tau \gtrsim 1$.

Once Theorem 3.1.2 and 3.1.3 have been proved, corresponding estimates or higher-order versions of those may be used to bound error terms in diffusive asymptotics, leading to Theorem 3.1.4.

Outline

The rest of the chapter is devoted to proofs of our three main theorems. In Section 3.2, we investigate the well-posedness of Equation (3.10) and gather estimates on its solution ϕ_∞^δ , which provides estimates on the steady state f^∞ . Then, in Section 3.3, we glean estimates on E in terms of h when it is obtained from the Poisson equation of (3.8). In Section 3.4, we gather some preliminary pieces of information on solutions to the system obtained by freezing nonlinear terms. In Section 3.5, to support our choice of exponents

β in the functional \mathcal{F} , we close estimates for $\partial_t h + L_\tau h = 0$, a $\delta = \infty$ version of (3.8). Theorems 3.1.2 and 3.1.3 are then proved in Section 3.6 and 3.7. Finally, the last section is devoted to the proof of Theorem 3.1.4.

3.2 The Poisson-Boltzmann equation

In this section we provide well-posedness and regularity results for

$$-\delta^2 \Delta_x \phi_\infty^\delta = e^{-\phi_\infty^\delta} - \rho_h. \quad (3.10)$$

Consistently with our global analysis we insist on uniformity of estimates with respect to δ when δ is bounded away from zero. This turns out to be crucial so as to control all our norms and relative inequalities since they depend on δ through ϕ_∞^δ .

As a key example note the following form of the Poincaré inequality in \mathcal{H}_0 .

Proposition 3.2.1. *There exist a positive constant K such that for any $h \in \mathcal{H}_0$ and any $\delta > 0$, one has*

$$\|h\|^2 \leq K \|e^{\phi_\infty^\delta}\|_{L^\infty(\mathbb{T}^2)} \|e^{-\phi_\infty^\delta}\|_{L^\infty(\mathbb{T}^2)} (\|Ah\|^2 + \|Ch\|^2).$$

The foregoing inequality is actually a straightforward consequence of the tensorization of the classical Poincaré inequality on the torus with the Gaussian Poincaré inequality. The reader is referred to [10, Chapter 4] for a detailed argument.

In the present section our arguments are relatively classical and strongly echo those in [27, 73, 37] and even more those in [126, Section 3]. In particular the existence part follows by identifying (3.10) with an Euler-Lagrange equation. To do so we set $n_h = \rho_h - 1$ so that n_h is mean-free and introduce the functional

$$J(\phi) = \frac{\delta^2}{2} \int_{\mathbb{T}^2} |\nabla_x \phi|^2 + \int_{\mathbb{T}^2} \phi n_h + \ln \left(\int_{\mathbb{T}^2} e^{-\phi} \right)$$

on

$$H_0 = \left\{ \phi \in H^1(\mathbb{T}^2) \mid \int_{\mathbb{T}^2} \phi = 0 \right\}.$$

The functional J is coercive, bounded from below and strictly convex provided that $\rho_h \in H^{-1}(\mathbb{T}^2)$. The main observation leading to strict convexity is that from the Holder inequality stems for any $\theta \in [0, 1]$ and $\phi_1, \phi_2 \in H_0$,

$$\int_{\mathbb{T}^2} e^{-\theta\phi_1 - (1-\theta)\phi_2} dx \leq \left(\int_{\mathbb{T}^2} e^{-\phi_1} dx \right)^\theta \left(\int_{\mathbb{T}^2} e^{-\phi_2} dx \right)^{1-\theta}.$$

In turn, since by Jensen's inequality, for any $\phi \in H_0$

$$\ln \left(\int_{\mathbb{T}^2} e^{-\phi} \right) \geq \ln \left(e^{-\int_{\mathbb{T}^2} \phi} \right) = 0,$$

coercivity and boundedness from below are explicitly derived from

$$J(\phi) \geq \frac{\delta^2}{2} \int_{\mathbb{T}^2} |\nabla_x \phi|^2 - K \|n_h\|_{H^{-1}(\mathbb{T}^2)} \left(\int_{\mathbb{T}^2} |\nabla_x \phi|^2 \right)^{1/2} + \ln \left(\int_{\mathbb{T}^2} e^{-\phi} \right) \quad (3.11)$$

that holds for some constant K and any $\phi \in H_0$. The Euler-Lagrange equation associated with J is actually

$$-\delta^2 \Delta_x \phi_\delta = \frac{e^{-\phi_\delta}}{\int_{\mathbb{T}^2} e^{-\phi_\delta}} - \rho_h.$$

Yet, solutions $\phi_\delta \in H_0$ to the foregoing equation are in one-to-one correspondence with solutions $\phi_\infty^\delta \in H^1(\mathbb{T}^2)$ to Equation (3.10) through

$$\phi_\infty^\delta = \phi_\delta + \ln \left(\int_{\mathbb{T}^2} e^{-\phi_\delta} \right), \quad \phi_\delta = \phi_\infty^\delta - \int_{\mathbb{T}^2} \phi_\infty^\delta, \quad (3.12)$$

(since Equation (3.10) implicitly contains $\int_{\mathbb{T}^2} e^{-\phi_\infty^\delta} = 1$).

Proposition 3.2.2 (Existence, uniqueness and regularity).

1. For any $\rho_h \in H^{-1}(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$, for any $\delta > 0$, Equation (3.10) possesses a unique weak solution $\phi_\infty^\delta \in H^1(\mathbb{T}^2)$ and this solution is such that $\int_{\mathbb{T}^2} e^{-\phi_\infty^\delta} = 1$.
2. Moreover there exists a positive constant K such that for any such ρ_h and any $\delta > 0$, the corresponding solution ϕ_∞^δ satisfies

$$\delta^2 \|\nabla \phi_\infty^\delta\|_{L^2(\mathbb{T}^2)}^2 + \left| \int_{\mathbb{T}^2} \phi_\infty^\delta \right| \leq K \|\rho_h - 1\|_{H^{-1}(\mathbb{T}^2)}^2.$$

3. If additionally, for some $p \in [1, +\infty]$, $\rho_h \in L^p(\mathbb{T}^2)$ then

$$\|e^{-\phi_\infty^\delta}\|_{L^p(\mathbb{T}^2)} \leq \|\rho_h\|_{L^p(\mathbb{T}^2)}.$$

4. In particular, for any $p \in (1, +\infty)$, there exists $K_p > 0$ such that for any $\rho_h \in L^p(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$ and any $\delta > 0$, the unique solution ϕ_∞^δ to Equation (3.10) satisfies

$$\|\nabla_x^2 \phi_\infty^\delta\|_{L^p(\mathbb{T}^2)} \leq \frac{K_p}{\delta^2} \|\rho_h\|_{L^p(\mathbb{T}^2)}$$

and

$$\|\phi_\infty^\delta\|_{L^\infty(\mathbb{T}^2)} \leq \frac{K_p}{\delta^2} \|\rho_h\|_{L^p(\mathbb{T}^2)}.$$

Proof. Existence and uniqueness follow from the properties of J expounded above through a direct minimization of the strictly convex functional J . Then the bound in H^1 stems from (3.11) and $J(0) = 0$ by noticing that

$$\int_{\mathbb{T}^2} \phi_\infty^\delta = \ln \left(\int_{\mathbb{T}^2} e^{-\phi_\delta} \right) \geq 0.$$

Concerning L^p estimate of $e^{-\phi_\infty^\delta}$, the formal argument proceeds by multiplying the equation by $-e^{-(p-1)\phi_\infty^\delta}$ and integrating to derive

$$(p-1)\delta^2 \int_{\mathbb{T}^2} |\nabla_x \phi_\infty^\delta|^2 e^{-(p-1)\phi_\infty^\delta} + \int_{\mathbb{T}^2} e^{-p\phi_\infty^\delta} = \int_{\mathbb{T}^2} e^{-(p-1)\phi_\infty^\delta} \rho_h$$

that implies

$$\|e^{-\phi_\infty^\delta}\|_{L^p(\mathbb{T}^2)}^p \leq \|e^{-\phi_\infty^\delta}\|_{L^p(\mathbb{T}^2)}^{p-1} \|\rho_h\|_{L^p(\mathbb{T}^2)}$$

by Hölder's inequality, from which the bound follows by simple computations. This may be turned into a sound argument by testing instead against $-e^{-(p-1)\max\{\phi_\infty^\delta, \eta\}}$ and letting $\eta \rightarrow -\infty$.

From here the $W^{2,p}$ bound stems directly from the equation and classical elliptic regularity properties — in Calderón-Zygmund form — for which we refer the reader to [183, 184] or [108]. The L^∞ bound then follows from the bound on $\int_{\mathbb{T}^2} \phi_\infty^\delta$ and a Sobolev embedding applied to $\phi_\infty^\delta - \int_{\mathbb{T}^2} \phi_\infty^\delta$. \square

The foregoing proposition provides an L^∞ bound on $e^{\phi_\infty^\delta}$ that blows up exponentially in δ^{-2} in the quasi-neutral regime $\delta \rightarrow 0$. Though this will be sufficient for our general argument and thus we do not pursue this line of investigation here, let us mention for precision's sake that the bound on $e^{\phi_\infty^\delta}$ may be dramatically improved if ρ_h is bounded away from zero.

Proposition 3.2.3 (Higher regularity). *Let $s \in \mathbb{N}$, $s \geq 2$, $q \in (1, \infty]$ and $p \in (1, \infty)$. There exists a positive constant $K = K_{p,q,s}$ and an integer α_s such that for any $\rho_h \in W^{s-2,p}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$ and any $\delta > 0$, the unique solution ϕ_∞^δ to Equation (3.10) satisfies*

$$\|\nabla_x^s \phi_\infty^\delta\|_{L^p(\mathbb{T}^2)} \leq \frac{K}{\delta^2} \left(1 + \|e^{-\phi_\infty^\delta}\|_{L^\infty}^{\frac{s-2}{2}}\right) \left(1 + \|\phi_\infty^\delta\|_{W^{2,q}}^{\alpha_s}\right) \left(1 + \frac{1}{\delta^{s-2}}\right) \|\rho_h\|_{W^{s-2,p}}. \quad (3.13)$$

Proof. We proceed by induction. The induction estimate

$$\delta^2 \|\nabla^{k+2} \phi_\infty^\delta\|_{L^p(\mathbb{T}^2)} \leq K \|\nabla^k \rho_h\|_{L^p(\mathbb{T}^2)} + K \|e^{-\phi_\infty^\delta}\|_{L^\infty(\mathbb{T}^2)} (1 + \|\phi_\infty^\delta\|_{W^{2,q}(\mathbb{T}^2)}^{k-1}) \|\phi_\infty^\delta\|_{W^{k,p}(\mathbb{T}^2)}$$

is obtained essentially by differentiating the equation and applying suitable Sobolev inequalities in Gagliardo-Nirenberg's form. Namely, the elementary block leading to the foregoing estimates is that if $1 \leq \ell \leq k$ and $\sigma \in (\mathbb{N}^*)^\ell$ is such that $|\sigma| = k$ then

$$\begin{aligned} \left\| \prod_{j=1}^{\ell} \partial^{\sigma_j} \phi_\infty^\delta \right\|_{L^p(\mathbb{T}^2)} &\leq \prod_{j=1}^{\ell} \|\partial^{\sigma_j} \phi_\infty^\delta\|_{L^{p_j}(\mathbb{T}^2)} \\ &\leq K \prod_{j=1}^{\ell} \|\phi_\infty^\delta\|_{W^{2,q}(\mathbb{T}^2)}^{1-\frac{\sigma_j}{k}} \|\nabla^k \phi_\infty^\delta\|_{L^p(\mathbb{T}^2)}^{\frac{\sigma_j}{k}} = \|\phi_\infty^\delta\|_{W^{2,q}(\mathbb{T}^2)}^{\ell-1} \|\nabla^k \phi_\infty^\delta\|_{L^p(\mathbb{T}^2)} \\ &\leq K (1 + \|\phi_\infty^\delta\|_{W^{2,q}(\mathbb{T}^2)}^{k-1}) \|\phi_\infty^\delta\|_{W^{k,p}(\mathbb{T}^2)} \end{aligned}$$

where $p_j = pk/\sigma_j$. Actually some of the derivatives needs first to be replaced with finite differences to justify formal manipulations, but we skip those classical details. \square

From now on we shall always assume but never repeat that ϕ_∞^δ is obtained from ρ_h through Equation (3.10) and we shall keep the dependence on norms of ρ_h implicit. Also for concision's sake we shall use without mention estimates of the present section.

3.3 The Poisson equation

We glean here estimates on E in terms of h when $E = -\nabla\psi$ and

$$-\delta^2\Delta\psi = n, \quad n = \int_{\mathbb{R}^2} h(\cdot, v) f^\infty(\cdot, v) dv$$

whenever $h \in \mathcal{H}_0$. They are naturally obtained from classical estimates on the Poisson equation on one side and estimates on n in terms of h on the other side. We recall that $\|\cdot\|$ denotes the canonical $L^2(d\mu)$ norm for the measure $d\mu = f^\infty dx dv$, that does depend on δ and ρ_h .

Proposition 3.3.1.

1. For any $s \in \mathbb{N}$, $\rho_h \in H^s(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$ and $\delta_0 > 0$ there exists $K_s > 0$ such that for all $\delta \geq \delta_0$, for any h , $n = \int_{\mathbb{R}^2} h(\cdot, v) f^\infty(\cdot, v) dv$ satisfies

$$\|\nabla_x^s n\|_{L^2(\mathbb{T}^2)} \leq K_s \sum_{k \leq s} \|\nabla_x^k h\|$$

and if moreover $h \in \mathcal{H}_0$ and E is the corresponding electric field

$$\|\nabla_x^{s+1} E\|_{L^2(\mathbb{T}^2)} \leq \frac{K_s}{\delta^2} \sum_{k \leq s} \|\nabla_x^k h\|.$$

2. For any $\rho_h \in H^1(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$, $p \in (1, \infty)$ and $\delta_0 > 0$ there exists $K_p > 0$ such that for all $\delta \geq \delta_0$, for any h , $n = \int_{\mathbb{R}^2} h(\cdot, v) f^\infty(\cdot, v) dv$ satisfies

$$\|n\|_{L^p(\mathbb{T}^2)} \leq K_p \|h\|^{2/p} \|\nabla_x h\|^{1-2/p}$$

and if moreover $h \in \mathcal{H}_0$ and E is the corresponding electric field

$$\|\nabla_x E\|_{L^p(\mathbb{T}^2)} \leq \frac{K_p}{\delta^2} \|h\|^{2/p} \|\nabla_x h\|^{1-2/p}.$$

3. For any $\rho_h \in L^2(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$, $p \in (1, \infty)$ and $\delta_0 > 0$ there exists $K_p > 0$ such that for all $\delta \geq \delta_0$ and any $h \in \mathcal{H}_0$, the corresponding electric field E satisfies

$$\|E\|_{L^p(\mathbb{T}^2)} \leq \frac{K_p}{\delta^2} \|h\|.$$

4. For any $\rho_h \in L^2(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$, $\eta \in (0, 1]$ and $\delta_0 > 0$ there exists $K_\eta > 0$ such that for all $\delta \geq \delta_0$ and any $h \in \mathcal{H}_0$, the corresponding electric field E satisfies

$$\|E\|_{L^\infty(\mathbb{T}^2)} \leq \frac{K_\eta}{\delta^2} \|h\|^{1-\eta} \|\nabla_x h\|^\eta.$$

Proof. The Sobolev estimate on the macroscopic density stems from an integration of the point-wise

$$|\nabla^s n(x)|^2 \leq K \sum_{k \leq s} |\nabla_x^{s-k}(e^{-\phi_\infty^\delta})(x)|^2 e^{\phi_\infty^\delta(x)} \int_{\mathbb{R}^2} |\nabla_x^k h(x, v)|^2 f^\infty(x, v) dv$$

that follows from direct differentiation and Jensen's inequality (for the square function). The Lebesgue estimate on the macroscopic density follows from

$$\begin{aligned} \|n\|_{L^p(\mathbb{T}^2)} &\leq \|e^{-\phi_\infty^\delta}\|_{L^\infty(\mathbb{T}^2)} \int_{\mathbb{T}^2} \|h(\cdot, v)\|_{L^p(\mathbb{T}^2)} M(v) dv \\ &\leq K_p \|e^{-\phi_\infty^\delta}\|_{L^\infty(\mathbb{T}^2)} \int_{\mathbb{T}^2} \|h(\cdot, v)\|_{L^2(\mathbb{T}^2)}^{\frac{2}{p}} \|\nabla_x h(\cdot, v)\|_{L^2(\mathbb{T}^2)}^{1-\frac{2}{p}} M(v) dv \\ &\leq K_p \|e^{-\phi_\infty^\delta}\|_{L^\infty(\mathbb{T}^2)} \|e^{\phi_\infty^\delta}\|_{L^\infty(\mathbb{T}^2)} \|h\|_{L^2(\mathbb{T}^2)}^{\frac{2}{p}} \|\nabla_x h\|_{L^2(\mathbb{T}^2)}^{1-\frac{2}{p}} \end{aligned}$$

that is derived by triangle inequality, some Sobolev embeddings and the Hölder inequalities. Remaining estimates are then deduced from classical elliptic regularity. Note in particular that for any $1 \leq p < 2 < q \leq \infty$ there exists $K_{p,q}$ such that

$$\|E\|_{L^\infty(\mathbb{T}^2)} \leq K_{p,q} \|n\|_{L^p(\mathbb{T}^2)}^{\theta_{p,q}} \|n\|_{L^q(\mathbb{T}^2)}^{1-\theta_{p,q}}$$

where $\theta_{p,q} \in (0, 1)$ is defined by $1/2 = \theta_{p,q}/p + (1 - \theta_{p,q})/q$. \square

3.4 Frozen equations

As preliminaries to nonlinear final arguments, in the following section, we gather some pieces of information on solutions to the system obtained by freezing nonlinear terms, namely

$$\partial_t h + L_\tau h - E \cdot v = \tilde{E} \cdot A^* g, \quad (3.14)$$

where

$$E = \delta^{-2} \nabla_x \Delta_x^{-1} n, \quad n = \int_{\mathbb{R}^2} h f^\infty dv,$$

and \tilde{E} and g are given sources, \tilde{E} being derived from some given $\tilde{g} \in \mathcal{H}_0$ through

$$\tilde{E} = \delta^{-2} \nabla_x \Delta_x^{-1} \tilde{n}, \quad \tilde{n} = \int_{\mathbb{R}^2} \tilde{g} f^\infty dv.$$

We first collect algebraic identities describing each elementary piece of the final energy estimate.

Lemma 3.4.1. *Any smooth localized h solving (3.14) satisfies*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|h\|^2 + \delta^2 \|E\|_{L^2}^2) + \frac{1}{\tau} \|Ah\|^2 &= \mathcal{R}_0(h, g, \tilde{g}), \\ \frac{1}{2} \frac{d}{dt} (\|Ah\|^2 + \delta^2 \|E\|_{L^2}^2) + \frac{1}{\tau} (\|Ah\|^2 + \|A^2 h\|^2) &= Q_A(h) + \mathcal{R}_A(h, g, \tilde{g}), \\ \frac{1}{2} \frac{d}{dt} \|Ch\|^2 + \frac{1}{\tau} \|AC h\|^2 &= Q_C(h) + \mathcal{R}_C(h, g, \tilde{g}), \\ \frac{d}{dt} \langle Ah, Ch \rangle + \|Ch\|^2 &= Q_{AC}(h) + \mathcal{R}_{AC}(h, g, \tilde{g}), \end{aligned}$$

with quadratic terms given by

$$\begin{aligned} Q_A(h) &= -\langle Ch, Ah \rangle \\ Q_C(h) &= \langle \text{Hess}(\phi_\infty^\delta) Ah, Ch \rangle + \langle \nabla_x(E \cdot v), Ch \rangle \\ Q_{AC}(h) &= -\frac{1}{\tau} (\langle Ah, Ch \rangle + 2\langle A^2h, ACh \rangle) + \tilde{Q}_{AC}(h) \end{aligned}$$

where

$$\tilde{Q}_{AC}(h) = \langle \text{Hess}(\phi_\infty^\delta) Ah, Ah \rangle + \langle E, Ch \rangle + \langle \nabla_x(E \cdot v), Ah \rangle$$

and trilinear terms

$$\begin{aligned} \mathcal{R}_0(h, g, \tilde{g}) &= \langle Ah, \tilde{E}g \rangle \\ \mathcal{R}_A(h, g, \tilde{g}) &= \langle Ah, \tilde{E}g \rangle + \langle A^2h, \tilde{E}Ag \rangle \\ \mathcal{R}_C(h, g, \tilde{g}) &= \langle ACh, \tilde{E}Cg \rangle + \langle CAh, g\nabla_x\tilde{E} \rangle \\ \mathcal{R}_{AC}(h, g, \tilde{g}) &= \langle A^2h, \tilde{E}Cg \rangle + \langle A^2h, g\nabla_x\tilde{E} \rangle + \langle Ch, \tilde{E}g \rangle + \langle ACh, \tilde{E}Ag \rangle. \end{aligned}$$

For latter use we also observe that if more frozen nonlinear terms of the same form were added to (3.14), this would only result in adding more trilinear terms of the same form in the foregoing identities. Namely, if $\tilde{E} \cdot A^*g$ is replaced with $\tilde{E}_1 \cdot A^*g_1 + \tilde{E}_2 \cdot A^*g_2$, then accordingly trilinear terms $\mathcal{R}_\#(h, g, \tilde{g})$ are turned into $\mathcal{R}_\#(h, g_1, \tilde{g}_1) + \mathcal{R}_\#(h, g_2, \tilde{g}_2)$.

Proof. We evaluate the time derivative of $\|h\|^2$, $\|Ah\|^2$, $\|Ch\|^2$ and $\langle Ah, Ch \rangle$. Using (3.14), we need to compute three kinds of terms, involving respectively L_τ , $E \cdot v$ and the nonlinear product $\tilde{E} \cdot A^*g$. Trilinear remainders \mathcal{R}_0 , \mathcal{R}_A , \mathcal{R}_C and \mathcal{R}_{AC} are exclusively obtained from the latter while dissipation terms and quadratic remainders come from other terms.

To obtain the first energy equality we use the skew-symmetry of B to derive

$$-\langle L_\tau h, h \rangle = -\frac{1}{\tau} \langle A^* Ah, h \rangle = -\frac{1}{\tau} \|Ah\|^2,$$

To proceed we use the continuity equation

$$\partial_t n + \text{div}_x(j) = 0, \quad j := \int_{\mathbb{R}^2} v h f^\infty dv \quad (3.15)$$

obtained by multiplying the first equation of (3.14) by f^∞ and integrating with respect to the velocity variable. By the Poisson equation, written in terms of a potential ψ such that $E = -\nabla\psi$, and (3.15) we obtain

$$\langle E \cdot v, h \rangle = \int_{\mathbb{T}^2} j \cdot E dx = - \int_{\mathbb{T}^2} \psi \partial_t n dx = -\delta^2 \int_{\mathbb{T}^2} \nabla_x \psi \cdot \partial_t \nabla_x \psi dx = -\frac{1}{2} \delta^2 \frac{d}{dt} \|E\|_{L^2}^2.$$

The second equation follows from similar computations using that $C = [A, B]$ and $I = [A, A^*]$, which implies

$$\begin{aligned} \|A^* \cdot Ah\|^2 &= \langle A_i^* A_i h, A_j^* A_j h \rangle \\ &= \langle A_j A_i^* A_i h, A_j h \rangle \\ &= \langle \delta_{ij} A_i h, A_j h \rangle + \langle A_j A_i h, A_i A_j h \rangle \\ &= \|Ah\|^2 + \|A^2 h\|^2. \end{aligned}$$

This leads to

$$\begin{aligned} -\langle AL_\tau h, Ah \rangle &= -\frac{1}{\tau} \|A^* Ah\|^2 - \langle ABh, Ah \rangle \\ &= -\frac{1}{\tau} (\|Ah\|^2 + \|A^2 h\|^2) - \langle Ch, Ah \rangle, \end{aligned}$$

and we also observe that

$$\langle A_i(E_j v_j), A_i h \rangle = \langle E_i, A_i h \rangle = \langle A_i^*(E_i), h \rangle = \langle v \cdot E, h \rangle,$$

which we recognize as a term already computed. The trilinear term \mathcal{R}_A is

$$\langle A_i h, A_i(\tilde{E}_j A_j^* g) \rangle = \langle A_i h, A_i A_j^* \tilde{E}_j g \rangle = \langle A_i h, \tilde{E}_i g \rangle + \langle A_i A_j h, \tilde{E}_j A_i g \rangle.$$

Concerning the third equation we first compute

$$-\langle C_i L_\tau h, C_i h \rangle = -\frac{1}{\tau} \langle A_j C_i h, A_j C_i h \rangle + \langle [B, C_i] h, C_i h \rangle,$$

and use that $[B, C] = \text{Hess}(\phi_\infty^\delta) A$. The remainder term \mathcal{R}_C is

$$\langle C_i h, C_i(\tilde{E}_j A_j^* g) \rangle = \langle C_i h, (\partial_{x_i} \tilde{E}_j) A_j^* g \rangle + \langle C_i h, \tilde{E}_j C_i A_j^* g \rangle.$$

Finally, for the fourth equality, we only explain how a few typical terms are derived, other following by the same kind of arguments. To this purpose note that

$$-\langle ABh, Ch \rangle - \langle Ah, CBh \rangle = -\|Ch\|^2 + \langle Ah, [B, C]h \rangle$$

and

$$-\langle A A^* \cdot Ah, Ch \rangle - \langle Ah, C A^* \cdot Ah \rangle = -\langle Ah, Ch \rangle - 2\langle A^2 h, Ach \rangle.$$

□

In the following lemmas, we indicate how to estimate each right-hand side term of the foregoing lemma. The first lemma is a trivial corollary of the Cauchy-Schwarz inequality and estimates on E and ϕ_∞^δ .

Lemma 3.4.2. For any $\rho_h \in W^{1,p}(\mathbb{T}^2)$, $p > 2$ such that $\int_{\mathbb{T}^2} \rho_h = 1$ and $\delta_0 > 0$ there exists $K > 0$ such that for all $\delta \geq \delta_0$, for any h

$$\begin{aligned} |Q_A(h)| &\leq \|Ch\| \|Ah\| \\ |Q_C(h)| &\leq \frac{K}{\delta^2} (\|Ah\| \|Ch\| + \|h\| \|Ch\|) \\ |Q_{AC}(h) - \tilde{Q}_{AC}(h)| &\leq \frac{1}{\tau} (\|Ah\| \|Ch\| + 2\|A^2h\| \|AC h\|) \\ |\tilde{Q}_{AC}(h)| &\leq \frac{K}{\delta^2} (\|Ah\|^2 + \|h\| \|Ch\| + \|h\| \|Ah\|) . \end{aligned}$$

Now we estimate trilinear terms by norms involving h , g , \tilde{g} and their derivatives.

Lemma 3.4.3. For any $\rho_h \in L^2(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$, $\eta \in (0, 1)^4$ and $\delta_0 > 0$ there exists $K = K_\eta > 0$ such that for all $\delta \geq \delta_0$ and any $(h, g, \tilde{g}) \in (\mathcal{H}_0)^3$,

$$\begin{aligned} |\mathcal{R}_0(h, g, \tilde{g})| &\leq \frac{K}{\delta^2} \|Ah\| \min(\|g\|^{1-\eta_1} \|Cg\|^{\eta_1} \|\tilde{g}\|, \|g\| \|\tilde{g}\|^{1-\eta_2} \|C\tilde{g}\|^{\eta_2}), \\ |\mathcal{R}_A(h, g, \tilde{g})| &\leq \frac{K}{\delta^2} \|Ah\| \|g\|^{1-\eta_1} \|Cg\|^{\eta_1} \|\tilde{g}\| \\ &\quad + \frac{K}{\delta^2} \|A^2h\| \|\tilde{g}\|^{1-\eta_2} \|C\tilde{g}\|^{\eta_2} \|Ag\|, \\ |\mathcal{R}_C(h, g, \tilde{g})| &\leq \frac{K}{\delta^2} \|AC h\| \|Cg\| \|\tilde{g}\|^{1-\eta_1} \|C\tilde{g}\|^{\eta_1} \\ &\quad + \frac{K}{\delta^2} \|AC h\| \|\tilde{g}\|^{1-\eta_2} \|C\tilde{g}\|^{\eta_2} \|g\|^{\eta_2} \|Cg\|^{1-\eta_2}, \\ |\mathcal{R}_{AC}(h, g, \tilde{g})| &\leq \frac{K}{\delta^2} \|A^2h\| \|\tilde{g}\|^{1-\eta_1} \|C\tilde{g}\|^{\eta_1} \|Cg\| \\ &\quad + \frac{K}{\delta^2} \|A^2h\| \|\tilde{g}\|^{1-\eta_2} \|C\tilde{g}\|^{\eta_2} \|g\|^{\eta_2} \|Cg\|^{1-\eta_2} \\ &\quad + \frac{K}{\delta^2} \|Ch\| \|g\|^{1-\eta_3} \|Cg\|^{\eta_3} \|\tilde{g}\| \\ &\quad + \frac{K}{\delta^2} \|AC h\| \|\tilde{g}\|^{1-\eta_4} \|C\tilde{g}\|^{\eta_4} \|Ag\|. \end{aligned}$$

Proof. Half of the estimate on \mathcal{R}_0 stems directly from bounds on $\|\tilde{E}\|_{L^\infty(\mathbb{T}^2)}$. The other half follows, setting $p = 2/\eta \in (2, \infty)$ and defining $q \in (2, \infty)$ by $1/2 = 1/p + 1/q$, from

$$\begin{aligned} \left| \langle Ah, \tilde{E}g \rangle \right| &\leq K \|Ah\| \|\tilde{E}\|_{L^p(\mathbb{T}^2)} \|g\|_{L^2(Mdv, L^q(dx))} \\ &\leq K' \|Ah\| \|\tilde{E}\|_{L^p(\mathbb{T}^2)} \|g\|^{1-\eta} \|Cg\|^\eta \end{aligned}$$

where we have used bounds on ϕ_∞^δ and Hölder and Sobolev inequalities and noticed that $2/q = 1 - \eta$. Then, one concludes thanks to bounds on $\|\tilde{E}\|_{L^p(\mathbb{T}^2)}$.

To estimate \mathcal{R}_A , we simply notice that

$$|\mathcal{R}_A(h, g, \tilde{g})| \leq |\mathcal{R}_0(h, g, \tilde{g})| + \|A^2 h\| \|\tilde{E}\|_{L^\infty(\mathbb{T}^2)} \|Ag\|.$$

As for the third estimate, the first term of \mathcal{R}_C is dealt with similarly using once again the L^∞ bound on \tilde{E} . However the second term requires a more careful distribution of spatial derivatives, essentially as in the proof of the bound on $\|n\|_{L^p(\mathbb{T}^2)}$ of Proposition 3.3.1. Namely, set $p = 2/\eta_2 \in (2, \infty)$ and define $q \in (2, \infty)$ by $1/p + 1/q = 2$, then

$$\left| \langle CAh, g \nabla_x \tilde{E} \rangle \right| \leq K \|AC h\| \|\nabla_x \tilde{E}\|_{L^p(\mathbb{T}^2)} \|g\|_{L^2(Mdv, L^q(\mathbb{T}^2))}$$

with

$$\begin{aligned} \|g\|_{L^2(Mdv, L^q(\mathbb{T}^2))} &\leq K' \left(\int_{\mathbb{R}^2} \|g(\cdot, v)\|_{L^2(\mathbb{T}^2)}^{2\left(1-\frac{2}{p}\right)} \|Cg(\cdot, v)\|_{L^2(\mathbb{T}^2)}^{\frac{4}{p}} M(v) dv \right)^{\frac{1}{2}} \\ &\leq K'' \|g\|^{1-\frac{2}{p}} \|Cg\|^{\frac{2}{p}} \end{aligned}$$

by Sobolev embeddings, Hölder inequalities and L^∞ bounds on ϕ_∞^δ . The estimate is achieved by relying on bounds on $\|\nabla_x \tilde{E}\|_{L^p(\mathbb{T}^2)}$.

We skip the estimate of \mathcal{R}_{AC} as completely similar. \square

3.5 Linear warm-up

For expository purpose and to support our choice of exponents in the functional (3.9) we first develop our strategy on the following equation

$$\partial_t h + L_\tau h = 0, \tag{3.16}$$

supplemented with initial data h_0 . Recall that $-L_\tau$ generates a semi-group of contractions on \mathcal{H} [120, 122]. Moreover, Schwartz functions form a core for L_τ and all following computations are readily justified for Schwartz initial data so that up to a density argument it is immaterial to do as if all computations were performed in the classical sense.

As already expounded in the introduction our goal is to prove that in each regime under suitable conditions on parameters $\beta \in \mathbb{R}^3$ and $\gamma \in (0, +\infty)^3$, the following functional

$$\begin{aligned} \|h\|_{\gamma, \beta, t} &= \|h\|^2 + \gamma_1 \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right) \|Ah\|^2 \\ &\quad + \gamma_2 \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|Ch\|^2 + 2\gamma_3 \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \langle Ah, Ch \rangle, \end{aligned}$$

is decaying in time with dissipation rate at least

$$\begin{aligned} D_{\gamma, \beta, t}(h) &= \tau^{-1} \|Ah\|^2 + \gamma_1 \tau^{\beta_1 - 1} \min\left(1, \frac{t}{\tau}\right) \|A^* \cdot Ah\|^2 \\ &\quad + \gamma_2 \tau^{\beta_2 - 1} \min\left(1, \frac{t}{\tau}\right)^3 \|AC h\|^2 + \gamma_3 \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \|Ch\|^2. \end{aligned}$$

Proposition 3.5.1 (Diffusive regime). *Under the following conditions on $\beta \in \mathbb{R}^3$*

$$\max\left(1, \frac{\beta_1 + \beta_2}{2}\right) \leq \beta_3 \leq \min(2\beta_1 + 1, \beta_2 - 1), \quad (3.17)$$

for any $\tau_0 > 0$ there exist $\gamma \in (0, +\infty)^3$, $c_0 > 0$, $C_0 > 0$ and $\tilde{\theta} > 0$ such that for any $\rho_h \in H^{-1}(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$, any $\delta > 0$ and any $\tau \in (0, \tau_0)$

1. for any h , for any $t \geq 0$,

$$\|h\|_t \geq c_0 \left(\|h\|^2 + \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right) \|Ah\|^2 + \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|Ch\|^2 \right)$$

and

$$\|h\|_t \leq C_0 \left(\|h\|^2 + \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right) \|Ah\|^2 + \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|Ch\|^2 \right);$$

2. for any $h_0 \in \mathcal{H}_0$ the solution to the linear Vlasov-Fokker-Planck equation (3.16) starting from h_0 satisfies for all $t \geq 0$

$$\|h(t, \cdot, \cdot)\|_{\gamma, \beta, t}^2 + \tilde{\theta} \int_0^t D_{\gamma, \beta, s}(h(s, \cdot, \cdot)) ds \leq \|h_0\|^2.$$

Proof. One may adapt Lemma 3.4.1 to (3.16) with resulting modifications being that there is no electric field in time derivatives and in remainder terms no trilinear term, no \tilde{Q}_{AC} and no Q_C . In particular, only Q_A and half of Q_{AC} have non-zero contribution to remainders and those may be bounded without resorting to ϕ_∞^δ bounds. This leads to

$$\frac{1}{2} \|h\|_{\gamma, \beta, t}^2 + \int_0^t D_{\gamma, \beta, s}(h) ds \leq \frac{1}{2} \|h_0\|^2 + \int_0^t \mathcal{R}_{\gamma, \beta, s}(h) ds,$$

with

$$\begin{aligned} \mathcal{R}_{\gamma, \beta, t}(h) &= \gamma_1 \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right) \|Ch\| \|Ah\| \\ &+ \gamma_3 \tau^{\beta_3 - 1} \min\left(1, \frac{t}{\tau}\right)^2 (\|Ah\| \|Ch\| + 2\|A^2 h\| \|AC h\|) \\ &+ \frac{1}{2} \chi_{t < \tau} \left(\gamma_1 \tau^{\beta_1 - 1} \|Ah\|^2 + 3\gamma_2 \tau^{\beta_2 - 1} \left(\frac{t}{\tau}\right)^2 \|Ch\|^2 + 4\gamma_3 \tau^{\beta_3 - 1} \left(\frac{t}{\tau}\right) \|Ah\| \|Ch\| \right) \end{aligned}$$

where we have used notation $\chi_{t < \tau}$ to denote the value at time t of the characteristic function of $[0, \tau)$, namely 0 if $t \geq \tau$, 1 otherwise.

Now we want to ensure that $\mathcal{R}_{\gamma, \beta, t}$ is controlled by an arbitrarily small fraction of $D_{\gamma, \beta, t}$, uniformly in time and in τ . To do so, we seek γ_j under the form ε^{c_j} for some positive c_j and observe that a bound

$$\varepsilon^c \tau^\beta \min\left(1, \frac{t}{\tau}\right)^\alpha K L \stackrel{\varepsilon \rightarrow 0}{\equiv} o(\varepsilon^{c'} \tau^{\beta'} \min\left(1, \frac{t}{\tau}\right)^{\alpha'} K^2 + \varepsilon^{c''} \tau^{\beta''} \min\left(1, \frac{t}{\tau}\right)^{\alpha''} L^2)$$

uniform with respect to K , L , t and $\tau \leq \tau_0$ requires $\alpha \geq \min(\{\alpha', \alpha''\})$. If $\alpha' \neq \alpha''$ and $\alpha \in [\alpha', \alpha'']$ then there exists a unique $\theta \in [0, 1]$ such that $\alpha = \theta \alpha' + (1 - \theta) \alpha''$ and

the estimate also requires $\beta \geq \theta\beta' + (1 - \theta)\beta''$. However once the above conditions are fulfilled the estimate holds provided that $c > \theta c' + (1 - \theta)c''$. With this in hands, one deduces our claimed estimate from

$$\begin{aligned} \beta_1 - 1 &\geq -1, & \beta_2 - 1 &\geq \beta_3, & \beta_3 - 1 &\geq \frac{1}{2}(\beta_3 - 1), \\ \beta_1 &\geq \frac{1}{2}(\beta_3 - 1), & \beta_3 - 1 &\geq \frac{2}{3}(\beta_3 - 1) - \frac{1}{3}, & \beta_3 - 1 &\geq \frac{1}{2}(\beta_1 + \beta_2 - 2), \end{aligned}$$

— which reduces to (3.17) — and

$$\frac{1}{2}(c_1 + c_2) < c_3 < \min(c_2, 2c_1).$$

We conclude by noting that the choice $(c_1, c_2, c_3) = (1, 2, 7/4)$ fulfills the latter constraint.

So far we have omitted the very first constraint on $\|\cdot\|_t$. Yet a sufficient condition is

$$\beta_3 \geq \frac{1}{2}(\beta_1 + \beta_2), \quad \gamma_3 < \sqrt{\gamma_1 \gamma_2},$$

which is redundant with above requirements. \square

The proof of Proposition 3.17 may be readily adapted to cope with the regime of evanescent collisions, yielding the following result.

Proposition 3.5.2 (Evanescent collisions). *Under the following conditions on $\beta \in \mathbb{R}^3$*

$$\min\left(1, \frac{\beta_1 + \beta_2}{2}\right) \geq \beta_3 \geq \max(2\beta_1 + 1, \beta_2 - 1). \quad (3.18)$$

for any $\tau_0 > 0$ there exist $\gamma \in (0, +\infty)^3$, $c_0 > 0$, $C_0 > 0$ and $\tilde{\theta} > 0$ such that for any $\rho_h \in H^{-1}(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_h = 1$, any $\delta > 0$ and any $\tau \in (\tau_0, +\infty)$

1. for any h , for any $t \geq 0$,

$$\|h\|_t \geq c_0 \left(\|h\|^2 + \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right) \|Ah\|^2 + \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|Ch\|^2 \right)$$

and

$$\|h\|_t \leq K_0 \left(\|h\|^2 + \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right) \|Ah\|^2 + \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|Ch\|^2 \right);$$

2. for any $h_0 \in \mathcal{H}_0$ the solution to the linear Vlasov-Fokker-Planck equation (3.16) starting from h_0 satisfies for all $t \geq 0$

$$\|h(t, \cdot, \cdot)\|_{\gamma, \beta, t}^2 + \tilde{\theta} \int_0^t D_{\gamma, \beta, s}(h(s, \cdot, \cdot)) ds \leq \|h_0\|^2.$$

3.6 Strongly collisional regime

In view of Lemmas 3.4.2 and 3.4.3, to analyze terms that have been left over in the foregoing section, we only need to consider

$$\begin{aligned}
 Q_t^\delta(h) &:= \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|Ah\| \|Ch\| + \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|h\| \|Ch\| \\
 &+ \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \|Ah\|^2 + \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \|h\| \|Ch\| \\
 &+ \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \|h\| \|Ah\| \\
 &=: \frac{1}{\delta^2} \sum_{i=1}^5 S_{i,t}(h)
 \end{aligned} \tag{3.19}$$

and choosing some $\eta \in (0, 1)^9$ and $\tilde{\eta}_1 \in (0, 1)$

$$\mathcal{R}_t^\delta(h, g, \tilde{g}) := \frac{1}{\delta^2} \left(T_{1,t}(h, g, \tilde{g}) \chi_{t \geq \tau} + \tilde{T}_{1,t}(h, g, \tilde{g}) \chi_{t \leq \tau} \right) + \frac{1}{\delta^2} \sum_{i=2}^9 T_{i,t}(h, g, \tilde{g}) \tag{3.20}$$

where

$$\begin{aligned}
 T_{1,t}(h, g, \tilde{g}) &= \|Ah\| \|g\|^{1-\eta_1} \|Cg\|^{\eta_1} \|\tilde{g}\|, \\
 \tilde{T}_{1,t}(h, g, \tilde{g}) &= \|Ah\| \|g\| \|\tilde{g}\|^{1-\tilde{\eta}_1} \|C\tilde{g}\|^{\tilde{\eta}_1}, \\
 T_{2,t}(h, g, \tilde{g}) &= \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right) \|Ah\| \|g\|^{1-\eta_2} \|Cg\|^{\eta_2} \|\tilde{g}\|, \\
 T_{3,t}(h, g, \tilde{g}) &= \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right) \|A^2h\| \|\tilde{g}\|^{1-\eta_3} \|C\tilde{g}\|^{\eta_3} \|Ag\|, \\
 T_{4,t}(h, g, \tilde{g}) &= \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|ACH\| \|Cg\| \|\tilde{g}\|^{1-\eta_4} \|C\tilde{g}\|^{\eta_4}, \\
 T_{5,t}(h, g, \tilde{g}) &= \tau^{\beta_2} \min\left(1, \frac{t}{\tau}\right)^3 \|ACH\| \|\tilde{g}\|^{1-\eta_5} \|C\tilde{g}\|^{\eta_5} \|g\|^{\eta_5} \|Cg\|^{1-\eta_5}, \\
 \text{and} \\
 T_{6,t}(h, g, \tilde{g}) &= \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \|A^2h\| \|\tilde{g}\|^{1-\eta_6} \|C\tilde{g}\|^{\eta_6} \|Cg\|, \\
 T_{7,t}(h, g, \tilde{g}) &= \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \|A^2h\| \|\tilde{g}\|^{1-\eta_7} \|C\tilde{g}\|^{\eta_7} \|g\|^{\eta_7} \|Cg\|^{1-\eta_7}, \\
 T_{8,t}(h, g, \tilde{g}) &= \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \|Ch\| \|g\|^{1-\eta_8} \|Cg\|^{\eta_8} \|\tilde{g}\|, \\
 T_{9,t}(h, g, \tilde{g}) &= \tau^{\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 \|ACH\| \|\tilde{g}\|^{1-\eta_9} \|C\tilde{g}\|^{\eta_9} \|Ag\|.
 \end{aligned}$$

We also need to take into account the electric field contributions in time derivatives by augmenting $\|\cdot\|_{\gamma, \beta, \tau, t}^2$ to \mathcal{E}_t defined by

$$\mathcal{E}_t(h) = \|h\|_{\gamma, \beta, \tau, t}^2 + \delta^2 (1 + \gamma_1 \tau^{\beta_1} \min\left(1, \frac{t}{\tau}\right)) \|E\|_{L^2}^2.$$

Prior to carrying on our nonlinear analysis in the diffusive regime, for reading's sake we make a specific choice of β , namely $\beta = (0, 2, 1)$, and fix a corresponding γ accordingly. This choice is motivated by the following remark.

Remark 3.6.1 (Optimality of β). From Proposition 3.5.1, one may derive, through Poincaré's inequality, exponential decay of $\|h(t, \cdot, \cdot)\|_{\gamma, \beta, \tau, t}$ when h solves (3.16), explicitly encoded by a rate $e^{-\theta \tau^{\max(\beta_3, -1, \beta_3 - \beta_2)} t}$ for some uniform $\theta > 0$. To optimize the former decay rate, one must minimize $\max(\beta_3, -1, \beta_3 - \beta_2)$ under constraints (3.17). The optimal choice requires actually $\beta_3 = 1$, that forces $\beta_1 = 0$ and $\beta_2 = 2$. Indeed the existence of a β_3 satisfying constraint (3.17) is equivalent to

$$\beta_1 \geq 0, \quad \beta_2 \geq 2, \quad 3\beta_1 \geq \beta_2 - 2, \quad \beta_1 \leq \beta_2 - 2$$

which is compatible with $(\beta_1 + \beta_2)/2 \leq 1$ only if $\beta_1 = 0$ and $\beta_2 = 2$. In turn, the corresponding choice of β does satisfy (3.17).

Last preliminary results are provided by the following proposition.

Proposition 3.6.2 ($\tau \lesssim 1$; $\beta = (0, 2, 1)$). Set $\beta = (0, 2, 1)$. For any $\tau_0 > 0$ and any $\gamma \in (0, \infty)^3$ satisfying corresponding conditions of Proposition 3.5.1, for any $\rho_h \in W^{1,p}(\mathbb{T}^2)$, $p > 2$, such that $\int_{\mathbb{T}^2} \rho_h = 1$, any $\delta_0 > 0$ and any $\tilde{\eta}_1 \in (0, 1)$, there exist $\eta \in (0, 1)^9$ and $K > 0$ such that for any $t \geq 0$, any $\tau \in (0, \tau_0)$, any $\delta \in (\delta_0, \infty)$ and any $(h, g, \tilde{g}) \in (\mathcal{H}_0)^3$

$$Q_t^\delta(h) \leq \frac{K}{\delta^2} D_t(h),$$

$$\mathcal{R}_t^\delta(h, g, \tilde{g}) \leq \frac{K}{\delta^2} [D_t(h)]^{\frac{1}{2}} \left[(D_t(g))^{\frac{1}{2}} + \tau^{\frac{1}{2} - \tilde{\eta}_1} \left(\frac{t}{\tau}\right)^{-\frac{3\tilde{\eta}_1}{2}} \chi_{t \leq \tau} \|g\| \right] [\mathcal{E}_t(\tilde{g})]^{1/2}.$$

Proof. Besides obvious estimates we point out that Poincaré's inequality implies that for any $h \in \mathcal{H}_0$

$$\tau^{1/2} \min\left(1, \frac{t}{\tau}\right) \|h\| \leq K [D_t(h)]^{1/2}$$

uniformly in $\tau \leq \tau_0$, $\delta \geq \delta_0$, $t \geq 0$. With this in hands one readily deduce for any $h \in \mathcal{H}_0$

$$S_{1,t}(h) + S_{3,t}(h) \leq K \tau^2 \min\left(1, \frac{t}{\tau}\right)^2 D_t(h),$$

$$S_{2,t}(h) + S_{5,t}(h) \leq K \tau \min\left(1, \frac{t}{\tau}\right) D_t(h),$$

$$S_{4,t}(h) \leq K D_t(h),$$

hence proving the first estimate. Likewise one obtains for any $(h, g, \tilde{g}) \in (\mathcal{H}_0)^3$

$$T_{1,t}(h, g, \tilde{g}) \leq K \min\left(1, \frac{t}{\tau}\right)^{-1} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

$$\tilde{T}_{1,t}(h, g, \tilde{g}) \leq K \tau^{1 - \tilde{\eta}_1} D_t(h)^{1/2} \left[\tau^{-1/2} \min\left(1, \frac{t}{\tau}\right)^{-3\tilde{\eta}_1/2} \|g\| \right] \mathcal{E}_t(\tilde{g})^{1/2},$$

$$T_{2,t}(h, g, \tilde{g}) + T_{8,t}(h, g, \tilde{g}) \leq K D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

and for $i \in \{3, 4, 5, 6, 7, 9\}$

$$T_{i,t}(h, g, \tilde{g}) \leq K \tau^{1 - 2\eta_i} \min\left(1, \frac{t}{\tau}\right)^{\frac{1 - 3\eta_i}{2}} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2}.$$

This yields the second estimate by choosing $\eta_i \in (0, 1/3]$ for $i \in \{3, 4, 5, 6, 7, 9\}$. \square

Remark 3.6.3 (Follow-up on the optimality of β). *Estimates of Proposition 3.6.2 will be used to set up a contraction argument with $\beta = (0, 2, 1)$ for large enough δ . As pointed out in Remark 3.6.1, this choice of β is motivated by our will to optimize decay rates. However one may wonder whether with a different choice of β one could improve the foregoing estimates and set up a contraction argument using smallness of τ and not largeness of δ , hence allowing for asymptotically vanishing δ (possibly in a τ -dependent way). Unfortunately, the answer is negative since our estimate of $S_{4,t}$ is actually independent of τ and β .*

Proof of Theorem 3.1.2. To prove Theorem 3.1.2, we introduce

$$X = \{ h \in L^\infty(\mathbb{R}_+; \mathcal{H}) \mid \mathcal{E}(h) < \infty \} \quad \text{and} \quad Y = \{ h \in L^\infty(\mathbb{R}_+; \mathcal{H}) \mid \mathcal{F}(h) < \infty \}$$

endowed with norms $\sqrt{\mathcal{E}}$ and $\sqrt{\mathcal{F}}$. For any $R > 0$ we denote by X_R and Y_R the closed balls of center 0 and radius R of Banach spaces X and Y . We recall that \mathcal{E} and \mathcal{F} are defined by

$$\mathcal{F}(h) = \mathcal{E}(h) + \theta D(h)$$

where

$$\mathcal{E}(h) = \sup_{t \geq 0} \mathcal{E}_t(h(t, \cdot, \cdot)) \quad \text{and} \quad D(h) = \int_0^\infty D_t(h(t, \cdot, \cdot)) dt.$$

We fix $R_0 > 0$ and choose $h_0 \in \mathcal{H}_0$ such that $\|h_0\| \leq R_0$ and for a suitable $R > 0$ we consider the map $\Phi : X_R \rightarrow L_{loc}^\infty(\mathbb{R}_+; \mathcal{H})$, $\tilde{g} \mapsto h$ where h starts from h_0 and solves the linear equation

$$\partial_t h + L_\tau h - E \cdot v = \tilde{E} \cdot A^* h \quad (3.21)$$

where E and \tilde{E} are obtained through the Poisson equation from respectively h and \tilde{g} . Existence and uniqueness in $\mathcal{C}(\mathbb{R}_+; \mathcal{H})$ for (3.21) may be shown for instance using arguments [121, Proposition 5.1] (adapted to our space-periodic setting that does not involve a confining potential) in two steps. First, when considered as given source terms, $E \cdot v$ and $\tilde{E} \cdot A^* h$ satisfy the hypotheses of [121, Proposition 5.1] thanks to estimates

$$\int_0^T |\langle E \cdot v, \varphi \rangle| dt \leq \frac{K}{\delta^2} \|h\|_{L^\infty(0,T;\mathcal{H})} \|A\varphi\|_{L^2(0,T;\mathcal{H})} \sqrt{T}$$

and

$$\int_0^T \left| \langle \tilde{E} \cdot A^* h, \varphi \rangle \right| dt \leq \frac{K R}{\delta^2} \|h\|_{L^\infty(0,T;\mathcal{H})} \|A\varphi\|_{L^2(0,T;\mathcal{H})} \left(\int_0^T \min(1, \frac{t}{\tau})^{-3\eta} \right)^{1/2}$$

for any $\varphi \in L^2(0, T; \mathcal{H})$ such that $A\varphi \in L^2(0, T; \mathcal{H})$ and $\eta > 0$ small enough. Then, by a fixed point argument in $\mathcal{C}(0, t_0; \mathcal{H})$ for a sufficiently small t_0 , one builds a unique solution to (3.21). Since t_0 can be chosen independently of the initial data one may repeat the argument to eventually get a global solution.

Our goal is to show that when δ is large enough one may choose R sufficiently large (independently of δ) such that $\Phi(X_R) \subset Y_R$ and Φ is a strict contraction (with uniform

constant) for norms $\sqrt{\mathcal{E}}$ and $\sqrt{\mathcal{F}}$. Since fixed points of Φ are exactly solutions of (3.8) starting from h_0 , this will prove altogether the existence of a solution in Y_R , its uniqueness in X_R , uniform bounds on the solution and smooth dependence on h_0 . To extend the uniqueness result one shall only need to remark that the above argument may be localized in time and to use a continuity argument based on the fact that any solution belongs to a suitable time-localized version of X_R for sufficiently small time.

Let us be more precise on the order in which parameters are chosen. Positive parameters τ_0 and R_0 are given data and we choose a first $\delta_0 > 0$ arbitrarily, say $\delta_0 = 1$. Then we may set $\beta = (0, 2, 1)$ and a suitable γ is provided by Proposition 3.5.1, constants in corresponding estimates being uniform in the range $\tau \leq \tau_0$, $\delta \geq \delta_0$. The parameter θ could be chosen essentially arbitrarily but it is convenient to set $\theta = \tilde{\theta}/2$ where $\tilde{\theta}$ is provided by Proposition 3.5.1. It turns out that we may also choose $R > R_0$ arbitrarily, say $R = 2R_0$.

Step 1, $\Phi(X_R) \subset Y_R$. Now we show that we may choose $\delta'_0 \geq \delta_0$ such that for any $\delta \geq \delta'_0$, $\tau \leq \tau_0$ and $\|h_0\| \leq R_0$, we do have that $\Phi(X_R) \subset Y_R$. Combining Propositions 3.5.1 and 3.6.2, we obtain indeed that for any (τ, δ, h_0) as above, for any $\tilde{g} \in X_R$, $h = \Phi(\tilde{g})$ satisfies for any $t \geq 0$, for some constants K' and K depending only on $\tilde{\eta}_1 \in (0, 1)$

$$\begin{aligned} \mathcal{E}_t(h(t)) &+ \tilde{\theta} \int_0^t D_s(h(s)) \, ds \\ &\leq \|h_0\|^2 + K' \int_0^t \left[Q_s^\delta(h(s)) + R_s^\delta(h(s), h(s), \tilde{g}(s)) \right] \, ds \\ &\leq \|h_0\|^2 + \frac{K(1+R)}{\delta^2} \left(\int_0^t D_s(h(s)) \, ds + \int_0^{\min(t, \tau)} \tau^{1-2\tilde{\eta}_1} \left(\frac{s}{\tau}\right)^{-3\tilde{\eta}_1} \|h(s)\| \, ds \right) \end{aligned}$$

hence, provided that $(1+R)/(\delta'_0)^2$ is sufficiently small one has

$$\mathcal{E}_t(h(t)) + \theta \int_0^t D_s(h(s)) \, ds \leq \|h_0\|^2 + \frac{K(1+R)}{\delta^2} \int_0^{\min(t, \tau)} \tau^{1-2\tilde{\eta}_1} \left(\frac{s}{\tau}\right)^{-3\tilde{\eta}_1} \|h(s)\| \, ds,$$

and choosing $\tilde{\eta}_1 \in (0, \frac{1}{3})$ yields for any $t \geq 0$ and some constant K depending only on $\tilde{\eta}_1$

$$\|h(t)\| \leq \|h_0\|^2 e^{\frac{K(1+R)\tau^{2(1-\tilde{\eta}_1)}}{\delta^2}}$$

therefore for any $t \geq 0$ and some constant K depending only on $\tilde{\eta}_1$

$$\mathcal{F}(h) \leq \|h_0\|^2 \left(1 + \frac{K(1+R)\tau^{2(1-\tilde{\eta}_1)}}{\delta^2} e^{\frac{K(1+R)\tau^{2(1-\tilde{\eta}_1)}}{\delta^2}} \right)$$

which can be made smaller than R^2 provided that $(1+R)/(\delta'_0)^2$ is small enough. It follows that Φ is well-defined from X_R to Y_R .

Step 2, Contraction. Now we show that Φ is a strict contraction from X_R to Y_R . Provided that $(1+R)/(\delta'_0)$ is sufficiently small, for any (τ, δ, h_0) as above, for any data $(\tilde{g}_1, \tilde{g}_2) \in (X_R)^2$, values $h_1 = \Phi(\tilde{g}_1)$ and $h_2 = \Phi(\tilde{g}_2)$ satisfy

$$(\partial_t + L_\tau)(h_1 - h_2) + (E_1 - E_2) \cdot v = (\tilde{E}_1 - \tilde{E}_2) \cdot A^* h_1 + \tilde{E}_2 \cdot A^*(h_1 - h_2)$$

(with obvious implicit notation for electric fields), thus, for some constant K , for any $t \geq 0$,

$$\mathcal{E}_t((h_1 - h_2)(t)) + \tilde{\theta} \int_0^t D_s((h_1 - h_2)(s)) ds \leq \frac{K(1+R)}{\delta^2} \mathcal{F}(h_1 - h_2)^{\frac{1}{2}} \mathcal{E}(\tilde{g}_1 - \tilde{g}_2)^{\frac{1}{2}}$$

as may be derived using that $h_1 \in Y_R$ and $h_2 \in Y_R$. Factoring out $\mathcal{F}(h_1 - h_2)^{\frac{1}{2}}$, it follows that Φ is Lipschitzian from X_R to Y_R and that its Lipschitz constant may be assumed arbitrarily small provided that $(1+R)/(\delta'_0)^2$ is sufficiently small. This is sufficient to lead to the well-posedness part of Theorem 3.1.2

Step 3, Exponential Decay. The large-time decay may then be deduced from the Poincaré inequality. Indeed the foregoing arguments provide for any $t_2 \geq t_1 \geq \tau$

$$\mathcal{E}_{t_2}(h(t_2)) + \theta \int_{t_1}^{t_2} D_s(h(s)) ds \leq \mathcal{E}_{t_1}(h(t_1)),$$

and for some $K > 0$, our choice of parameters yields uniformly for (τ, δ, h_0) as above that for all $t \geq \tau$

$$K \tau \mathcal{E}_t(h(t)) \leq D_t(h(t)).$$

This leads to

$$\mathcal{E}_t(h(t)) \leq e^{-\theta' \tau t} \mathcal{E}_\tau(h(\tau)) \leq K' e^{-\theta' \tau t} \|h_0\|$$

for any $t \geq \tau$ and some uniform positive K' and θ' . This may be extended to all t using the uniform boundedness of $\mathcal{E}_t(h(t))$. Similar arguments prove the uniform stability with respect to initial data.

3.7 The regime of evanescent collisions

Remark 3.7.1 (Optimality of β). *From Proposition 3.5.2, one may also derive, through Poincaré's inequality, exponential decay in \mathcal{H} of solutions to (3.16), explicitly encoded by a rate $e^{-\theta \tau \min(\beta_3, -1, \beta_3 - \beta_2)t}$ for some uniform $\theta > 0$. In order to optimize the former decay rate under (3.18), first observe that the latter constraint implies $\beta_3 - \beta_2 \geq -1$ hence $\min(\beta_3, -1, \beta_3 - \beta_2) = \min(\beta_3, -1)$. Thus, we only need to ensure that $\beta_3 \geq -1$ and there is a large choice of β that meet this constraint jointly with (3.18). Indeed for any $-2 \leq \beta_1 \leq 0$ one may choose β_2 such that*

$$\max(3\beta_1 + 2, -\beta_1 - 2) \leq \beta_2 \leq \beta_1 + 2$$

and then a suitable β_3 may be chosen according to

$$\min\left(1, \frac{\beta_1 + \beta_2}{2}\right) \geq \beta_3 \geq \max(-1, 2\beta_1 + 1, \beta_2 - 1)$$

and that defines a non empty interval of β_3 s.

A natural guide towards a good choice of β could be the examination of the best analogue of Proposition 3.6.2 in the regime where τ is large. However, another thing that also differs from the strongly collisional regime is that the main obstruction here does not arise from quadratic terms. Indeed one may prove the following estimates.

Proposition 3.7.2 ($\tau \gtrsim 1$; $\beta = (-1, -1, -1)$). *Set $\beta = (-1, -1, -1)$. For any $\tau_0 > 0$ and any $\gamma \in (0, \infty)^3$ satisfying corresponding conditions of Proposition 3.5.2, for any $\rho_h \in W^{1,p}(\mathbb{T}^2)$, $p > 2$, such that $\int_{\mathbb{T}^2} \rho_h = 1$, any $\delta_0 > 0$, there exists $K > 0$ such that for any $t \geq 0$, any $\tau \in (\tau_0, \infty)$, any $\delta \in (\delta_0, \infty)$ and any $h \in \mathcal{H}_0$*

$$Q_t^\delta(h) \leq \frac{K}{\delta^2} D_t(h).$$

Since we shall not make any use of the former proposition we skip its proof. Yet let us point out that the involved uniform estimate enforces $\beta_j \leq -1$, $j = 1, 2, 3$. Indeed constraints (3.18) implies $\beta_1 \leq 0$ and, under this condition, the best possible estimates

$$\begin{aligned} S_{1,t}(h) &\leq K \tau^{\frac{2\beta_2+1-\beta_3}{2}} \min\left(1, \frac{t}{\tau}\right)^2 D_t(h), \\ S_{3,t}(h) &\leq K \tau^{1+\beta_3} \min\left(1, \frac{t}{\tau}\right)^2 D_t(h), \end{aligned}$$

provide uniform bounds only when $2\beta_2 + 1 \leq \beta_3$ and $\beta_3 \leq -1$, which jointly with (3.18) yield the claimed constraint. Note also that if moreover one requires $\beta_3 \geq -1$ then the only possible choice is indeed $\beta = (-1, -1, -1)$.

Unfortunately, in the regime $\tau \gtrsim 1$, trilinear terms leads to a more stringent constraint on δ and the foregoing choice $\beta = (-1, -1, -1)$ does not minimize trilinear constraints. Indeed

$$\begin{aligned} T_{1,t}(h, g, \tilde{g}) &\leq K \tau^{1-\frac{\eta_1}{2}(\beta_3+1)} \min\left(1, \frac{t}{\tau}\right)^{-1} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2}, \\ \tilde{T}_{1,t}(h, g, \tilde{g}) &\leq K \tau^{\frac{1+\max(1, -\beta_3)-\tilde{\eta}_1\beta_2}{2}} D_t(h)^{1/2} \left[\tau^{-1/2} \min\left(1, \frac{t}{\tau}\right)^{-3\tilde{\eta}_1/2} \|g(t)\| \right] \mathcal{E}_t(\tilde{g})^{1/2}, \end{aligned}$$

with $0 < \eta_1 \leq 1$ and $0 < \tilde{\eta}_1 \leq 1$, provides bounds that grow superlinearly in τ unless $\beta_2 \geq 0$ and $\beta_3 \geq -1$. Note in turn that in order not to exceed a linear growth in τ , the bound on $S_{3,t}$ only requires $\beta_3 \leq 0$. Now let us observe that in the final argument $\tilde{\eta}_1$ is constrained by $\tilde{\eta}_1 < \frac{1}{3}$ and that constraints (3.18) yield

$$\frac{3}{2}\beta_3 + \frac{1}{2} \leq \beta_2 \leq \beta_3 + 1.$$

In turn minimizing

$$\max\left(1 - \frac{1}{6}\beta_2, \beta_2 - \frac{1}{2}\beta_3 + \frac{1}{2}\right)$$

under these constraints proves that one cannot do better than a $\tau^{\frac{14}{15}}$ -growth and that one may hope to (almost) realize it only with

$$\beta = \left(-\frac{8}{15}, \frac{2}{5}, -\frac{1}{15}\right).$$

As the following proposition proves this turns out to be indeed possible.

Proposition 3.7.3 ($\tau \gtrsim 1$; $\beta = (-\frac{8}{15}, \frac{2}{5}, -\frac{1}{15})$). Set $\beta = (-\frac{8}{15}, \frac{2}{5}, -\frac{1}{15})$. For any $\varepsilon > 0$, any $\tau_0 > 0$ and any $\gamma \in (0, \infty)^3$ satisfying corresponding conditions of Proposition 3.5.2, for any $\rho_h \in W^{1,p}(\mathbb{T}^2)$, $p > 2$, such that $\int_{\mathbb{T}^2} \rho_h = 1$ and any $\delta_0 > 0$, there exist $\eta \in (0, 1)^9$, $\tilde{\eta}_1 \in (0, 1)$ and $K > 0$ such that for any $t \geq 0$, any $\tau \in (\tau_0, +\infty)$, any $\delta \in (\delta_0, \infty)$ and any $(h, g, \tilde{g}) \in (\mathcal{H}_0)^3$

$$Q_t^\delta(h) \leq \frac{K \tau^{\frac{14}{15}}}{\delta^2} D_t(h),$$

$$\mathcal{R}_t^\delta(h, g, \tilde{g}) \leq \frac{K \tau^{\frac{14}{15} + \varepsilon}}{\delta^2} [D_t(h)]^{\frac{1}{2}} \left[(D_t(g))^{\frac{1}{2}} + \tau^{-\frac{1}{2}} \left(\frac{t}{\tau}\right)^{-\frac{3\tilde{\eta}_1}{2}} \chi_{t \leq \tau} \|g\| \right] [\mathcal{E}_t(\tilde{g})]^{1/2}.$$

Proof. We have already shown how to bound $S_{1,t}$ and $S_{3,t}$. Moreover $S_{2,t}$ may be bounded as $S_{1,t}$, and $S_{4,t}$ and $S_{5,t}$ as $S_{3,t}$. Likewise

$$S_{4,t}(h) \leq K \tau^{\frac{1}{2}(1+\beta_3)} D_t(h) = K \tau^{\frac{7}{15}} D_t(h).$$

We have already explained how to bound $\tilde{T}_{1,t}$, we only need to add that $\tilde{\eta}_1$ is chosen as $\tilde{\eta}_1 = \frac{1}{3} - 5\varepsilon$ if $\varepsilon < \frac{1}{15}$, and arbitrarily otherwise. Besides

$$T_{1,t}(h, g, \tilde{g}) \leq K \tau^{1 - \frac{\eta_1}{2}(\beta_3+1)} \min\left(1, \frac{t}{\tau}\right)^{-1} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

$$T_{2,t}(h, g, \tilde{g}) \leq K \tau^{\beta_3 - \frac{\eta_2}{2}(\beta_3+1)} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

$$T_{8,t}(h, g, \tilde{g}) \leq K \tau^{\frac{1-\eta_8}{2}(\beta_3+1)} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

which are shown to be sufficient by choosing $\eta_1 = \frac{1}{7}$ and any η_2, η_8 . At last

$$T_{3,t}(h, g, \tilde{g}) \leq K \tau^{\frac{1}{2}\beta_1 - \frac{\eta_3}{2}\beta_2 + 1} \min\left(1, \frac{t}{\tau}\right)^{\frac{1-3\eta_3}{2}} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

$$T_{4,t}(h, g, \tilde{g}) \leq K \tau^{\frac{1-\eta_4}{2}\beta_2 - \frac{1}{2}\beta_3 + \frac{1}{2}} \min\left(1, \frac{t}{\tau}\right)^{\frac{1-3\eta_4}{2}} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

$$T_{5,t}(h, g, \tilde{g}) \leq K \tau^{\frac{1}{2}\beta_2 - \frac{1}{2}\beta_3 + \frac{1}{2} + \frac{\eta_5}{2}(1+\beta_3-\beta_2)} \min\left(1, \frac{t}{\tau}\right)^{\frac{1-3\eta_5}{2}} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

$$T_{6,t}(h, g, \tilde{g}) \leq K \tau^{-\frac{1}{2}\beta_1 - \frac{\eta_6}{2}\beta_2 + \frac{1}{2}\beta_3 + \frac{1}{2}} \min\left(1, \frac{t}{\tau}\right)^{\frac{1-3\eta_6}{2}} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

$$T_{7,t}(h, g, \tilde{g}) \leq K \tau^{-\frac{1}{2}\beta_1 + \frac{1}{2}\beta_3 + \frac{1}{2} + \frac{\eta_7}{2}(1+\beta_3-\beta_2)} \min\left(1, \frac{t}{\tau}\right)^{\frac{1-3\eta_7}{2}} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

$$T_{9,t}(h, g, \tilde{g}) \leq K \tau^{-\frac{(1+\eta_9)}{2}\beta_2 + \beta_3 + 1} \min\left(1, \frac{t}{\tau}\right)^{\frac{1-3\eta_9}{2}} D_t(h)^{1/2} D_t(g)^{1/2} \mathcal{E}_t(\tilde{g})^{1/2},$$

which are themselves shown to be sufficient by choosing η_i , $i \in \{3, 4, 5, 6, 7, 9\}$, arbitrarily in $(0, \frac{1}{3}]$. \square

With this in hands the proof of Theorem 3.1.3 is achieved as was proved Theorem 3.1.2.

3.8 Asymptotic models in the diffusive regime

This section is devoted to the proof of Theorem 3.1.4. So far we have aimed at global-in-time estimates and therefore what exactly was the reference time scale was immaterial. Now we turn to asymptotics that are uniform only locally in time thus we explicitly introduce a reference time in the equations. Namely, after choosing an observation time $t_{\text{ref}} = t_{\text{ref}}(\tau)$ in a τ -dependent way, we observe that if f solves the original system then f_{ref} defined by $f_{\text{ref}}(t, x, v) = f(t_{\text{ref}}t, x, v)$ and $h_{\text{ref}} = (f_{\text{ref}} - f^\infty)/f^\infty$ are such that

$$\begin{cases} \frac{1}{t_{\text{ref}}(\tau)} \partial_t h_{\text{ref}} + L_\tau h_{\text{ref}} = E_{\text{ref}} \cdot A^*(\mathbf{1}) + E_{\text{ref}} \cdot A^* h_{\text{ref}}, \\ E_{\text{ref}} = \frac{1}{\delta^2} \nabla_x \Delta_x^{-1} n_{\text{ref}}, \quad n_{\text{ref}} = \int_{\mathbb{R}^3} h_{\text{ref}} f^\infty dv. \end{cases} \quad (3.22)$$

with initial data h_0 .

We already know that for any $\tau_0 > 0$, $R_0 > 0$ there exists δ_0 and uniform positive constants K and θ such that when $\delta > \delta_0$ and $\|h_0\| \leq R_0$, for any $t \geq 0$

$$\begin{aligned} \|h_{\text{ref}}(t)\|^2 &+ \min\left(1, \frac{t_{\text{ref}}}{\tau} t\right) \|Ah_{\text{ref}}(t)\|^2 + \tau^2 \min\left(1, \frac{t_{\text{ref}}}{\tau} t\right)^3 \|Ch_{\text{ref}}(t)\|^2 \\ &+ \frac{t_{\text{ref}}}{\tau} \int_0^t \|Ah_{\text{ref}}(s)\|^2 ds \quad + \quad t_{\text{ref}} \tau \int_0^t \min\left(1, \frac{t_{\text{ref}}}{\tau} s\right)^2 \|Ch_{\text{ref}}(s)\|^2 ds \\ &+ \frac{t_{\text{ref}}}{\tau} \int_0^t \min\left(1, \frac{t_{\text{ref}}}{\tau} s\right) \|A^2 h_{\text{ref}}(s)\|^2 ds + t_{\text{ref}} \tau \int_0^t \min\left(1, \frac{t_{\text{ref}}}{\tau} s\right)^3 \|ACH_{\text{ref}}(s)\|^2 ds \\ &\leq K \|h_0\|^2 \end{aligned}$$

and

$$\|h_{\text{ref}}(t)\| \leq K e^{-\theta t_{\text{ref}} \tau t} \|h_0\|.$$

The latter estimates shows that $\|h_{\text{ref}}(t)\|$ converges to 0 uniformly on compacts of $(0, +\infty]$ provided that $t_{\text{ref}}(\tau)\tau \xrightarrow{\tau \rightarrow 0} \infty$. The foregoing asymptotic regime is stationary. This proves part (i) of Theorem 3.1.4.

3.8.1 Asymptotically linear free-field regimes

In the opposite regime where $t_{\text{ref}}(\tau)\tau \xrightarrow{\tau \rightarrow 0} 0$ we show now that relevant asymptotic models are of evolution type or at least strongly keep track of initial data. However they are linear and one can also drop out convective terms at least in the velocity directions.

Namely, let us denote h_{lin} the solution to

$$\frac{1}{t_{\text{ref}}(\tau)} \partial_t h_{\text{lin}} + v \cdot \nabla_x h_{\text{lin}} - E_\infty^\delta \cdot \nabla_v h_{\text{lin}} - \frac{1}{\tau} A^* A h_{\text{lin}} = 0$$

with initial data h_0 . Observe that for any $t \geq 0$

$$\|h_{\text{lin}}(t)\|^2 + 2 \frac{t_{\text{ref}}}{\tau} \int_0^t \|Ah_{\text{lin}}(s)\|^2 ds \leq \|h_0\|^2.$$

Note that

$$\begin{aligned} \frac{1}{t_{\text{ref}}(\tau)} \partial_t (h_{\text{ref}} - h_{\text{lin}}) &+ v \cdot \nabla_x (h_{\text{ref}} - h_{\text{lin}}) - E_{\infty}^{\delta} \cdot \nabla_v (h_{\text{ref}} - h_{\text{lin}}) - \frac{1}{\tau} A^* A (h_{\text{ref}} - h_{\text{lin}}) \\ &= E_{\text{ref}} \cdot A^* h_{\text{ref}} + E_{\text{ref}} \cdot A^* (\mathbf{1}). \end{aligned}$$

This implies for any $t \geq 0$

$$\begin{aligned} \|(h_{\text{ref}} - h_{\text{lin}})(t)\|^2 &+ \frac{t_{\text{ref}}}{\tau} \int_0^t \|A(h_{\text{ref}} - h_{\text{lin}})(s)\|^2 ds \\ &\leq t_{\text{ref}} \tau \int_0^t (\|E_{\text{ref}}(s) h_{\text{ref}}(s)\|^2 + \|E_{\text{ref}}(s)\|^2) ds \\ &\leq \frac{K_{\eta} (1 + R_0^2)}{\delta^2} \|h_0\|^2 \times \begin{cases} (t_{\text{ref}} \tau t)^{1-\eta} & \text{when } t_{\text{ref}} t \geq \tau \\ \tau^{2(1-\eta)} & \text{when } t_{\text{ref}} t \leq \tau \end{cases}. \end{aligned}$$

for any $0 < \eta < \frac{1}{3}$. The trickiest part of the foregoing bound follows from

$$\begin{aligned} t_{\text{ref}} \tau \int_0^t \|E_{\text{ref}}(s) h_{\text{ref}}(s)\|^2 ds &\leq \frac{K_{\eta}}{\delta^2} t_{\text{ref}} \tau \int_0^t \|C h_{\text{ref}}(s)\|^{2\eta} \|h_{\text{ref}}(s)\|^{2(1-\eta)} ds \\ &\leq \frac{K_{\eta}}{\delta^2} \|h_0\|^{2(1-\eta)} t_{\text{ref}} \tau \int_0^t \frac{\min(1, \frac{t_{\text{ref}}}{\tau} s)^{2\eta} \|C h_{\text{ref}}(s)\|^{2\eta}}{\min(1, \frac{t_{\text{ref}}}{\tau} s)^{2\eta}} ds \\ &\leq \frac{K_{\eta} R_0^2}{\delta^2} \|h_0\|^2 (t_{\text{ref}} \tau)^{1-\eta} \left(\int_0^t \min(1, \frac{t_{\text{ref}}}{\tau} s)^{-\frac{2\eta}{1-\eta}} ds \right)^{1-\eta}. \end{aligned}$$

obtained by the same argument used to bound \mathcal{R}_0 in Lemma 3.4.3 and Hölder estimates. The proof of the claim is then achieved by noticing that

$$\int_0^t \min(1, \frac{t_{\text{ref}}}{\tau} s)^{-\frac{2\eta}{1-\eta}} ds \leq K \max\left(t, \frac{\tau}{t_{\text{ref}}}\right).$$

Hence we are asymptotically close to the linear regime.

Now let us show that contributions of the linear field terms also vanish in these regimes. Let us denote h_{as} the solution to

$$\frac{1}{t_{\text{ref}}(\tau)} \partial_t h_{\text{as}} - \frac{1}{\tau} A^* A h_{\text{as}} = 0,$$

with initial data $h_0 \in \mathcal{H}$, and assume $C h_0 \in \mathcal{H}$. Note that

$$\|h_{\text{as}}(t)\|^2 + 2 \frac{t_{\text{ref}}}{\tau} \int_0^t \|A h_{\text{as}}(s)\|^2 ds \leq \|h_0\|^2.$$

and that, since C and A commute, $C h_{\text{as}}$ solves the same equation, hence satisfies a similar estimate with initial data $C h_0$. Using that $v = A + A^*$, a direct estimate provides

$$\begin{aligned} \|(h_{\text{lin}} - h_{\text{as}})(t)\|^2 &+ \frac{t_{\text{ref}}}{\tau} \int_0^t \|A(h_{\text{lin}} - h_{\text{as}})(s)\|^2 ds \\ &\leq K t_{\text{ref}} \tau \int_0^t \|C h_{\text{as}}(s)\|^2 ds \\ &+ K t_{\text{ref}} \int_0^t [\|A C h_{\text{as}}(s)\| + \|A h_{\text{as}}(s)\|] \|(h_{\text{lin}} - h_{\text{as}})(s)\| ds \\ &\leq K' [t_{\text{ref}} \tau t \|C h_0\|^2 + \sqrt{t_{\text{ref}} \tau t} (\|h_0\|^2 + \|C h_0\|^2)]. \end{aligned}$$

Since such h_0 form a dense set in \mathcal{H} , the corresponding convergence, uniform in t on compact sets of $[0, \infty)$,

$$\|h_{\text{lin}}(t) - h_{\text{as}}(t)\| \xrightarrow{\tau \rightarrow 0} 0 \quad \text{provided} \quad t_{\text{ref}}(\tau) \tau \xrightarrow{\tau \rightarrow 0} 0$$

may be extended to any $h_0 \in \mathcal{H}$. This proves part (iv) of Theorem 3.1.4.

At last observe that, on one hand, for any $t \geq 0$, with $n_0 = \int_{\mathbb{R}^2} h_0 M dv$

$$\|h_{\text{as}}(t) - n_0\| \leq e^{-\theta \frac{t_{\text{ref}}}{\tau} t} \|h_0 - n_0\|$$

for some uniform $\theta > 0$, which proves part (iii) of Theorem 3.1.4. On the other hand, uniformly in t on compact sets of $[0, \infty)$,

$$\|h_{\text{as}}(t) - h_0\| \xrightarrow{\tau \rightarrow 0} 0 \quad \text{provided} \quad \frac{t_{\text{ref}}(\tau)}{\tau} \xrightarrow{\tau \rightarrow 0} 0$$

for any $h_0 \in \mathcal{H}$. The latter follows through a density argument from the explicit estimate

$$\|h_{\text{as}}(t) - h_0\| \leq \frac{t_{\text{ref}}(\tau) t}{\tau} \|A^* A h_0\|$$

that holds when moreover $A^2 h_0 \in \mathcal{H}$. This proves part (v) of Theorem 3.1.4.

3.8.2 Nonlinear diffusive regime

The remaining regime corresponds to the case where $t_{\text{ref}} \tau$ is of order 1 and therefore in this section we set $t_{\text{ref}}(\tau) = \tau^{-1}$. Note that with this choice we already know that

$$\left(\int_0^\infty \|A h_{\text{ref}}(t)\|^2 dt \right)^{\frac{1}{2}} \leq K \tau \|h_0\|$$

and this implies that

$$\left(\int_0^\infty \|f_{\text{ref}}(t) - M \rho_{\text{ref}}(t)\|_{L^2(M^{-1})}^2 dt \right)^{\frac{1}{2}} \leq K' \tau \|h_0\|.$$

Our goal is to also identify some asymptotic limiting behavior for $n_{\text{ref}} = \rho_{\text{ref}} - e^{-\phi_\infty^\delta}$. Note that the proof of Theorem 3.1.2 by a contraction argument also provides us with the fact the map $h_0 \mapsto h_{\text{ref}}$ is Lipschitz from \mathcal{H}_0 to $L^\infty(\mathbb{R}_+; \mathcal{H}_0)$, therefore this is also the case for the map $h_0 \mapsto n_{\text{ref}}$ from \mathcal{H}_0 to $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^2))$. To prove convergence in $L_{loc}^\infty(\mathbb{R}_+; L^2(\mathbb{T}^2))$ (without explicit decay rates) we may therefore restrict to a case where also hold $A h_0 \in \mathcal{H}$, $A C h_0 \in \mathcal{H}$, $C h_0 \in \mathcal{H}$ and $C^2 h_0 \in \mathcal{H}$. The gain we shall use is two-fold. Indeed one may both drop out time weights in our arguments and upgrade it to higher

regularity to obtain

$$\begin{aligned}
 & \|h_{\text{ref}}(t)\|^2 + \|Ah_{\text{ref}}(t)\|^2 + \|Ch_{\text{ref}}(t)\|^2 + \|AC h_{\text{ref}}(t)\|^2 + \tau^2 \|C^2 h_{\text{ref}}(t)\|^2 \\
 & + \frac{1}{\tau^2} \int_0^t \|Ah_{\text{ref}}(s)\|^2 ds + \frac{1}{\tau^2} \int_0^t \|A^2 h_{\text{ref}}(s)\|^2 ds \\
 & + \frac{1}{\tau^2} \int_0^t \|AC h_{\text{ref}}(s)\|^2 ds + \frac{1}{\tau^2} \int_0^t \|A^2 C h_{\text{ref}}(s)\|^2 ds \\
 & + \int_0^t \|AC^2 h_{\text{ref}}(s)\|^2 ds + \int_0^t \|Ch_{\text{ref}}(s)\|^2 ds + \int_0^t \|C^2 h_{\text{ref}}(s)\|^2 ds \\
 & \leq K [\|h_0\|^2 + \|Ah_0\|^2 + \|Ch_0\|^2 + \|AC h_0\|^2 + \tau^2 \|C^2 h_0\|^2].
 \end{aligned}$$

It is very important to note however that to do so we do not need to restrict further δ_0 in a way that would depend on the size of $\|Ah_0\|$, $\|Ch_0\|$, $\|AC h_0\|$ and $\|C^2 h_0\|$. Otherwise this would prevent us from extending the convergence to $h_0 \in \mathcal{H}_0$ by a density argument. In contrast, we will be free to restrict τ_0 in a way depending on above norms. We skip the proof of the foregoing claim as lengthier but otherwise completely similar to estimates that have been proved in detail above.

Now, our starting point is the continuity equation (3.15), that here takes the form

$$\partial_t n_{\text{ref}} + \frac{1}{\tau} \text{div}_x(j_{\text{ref}}) = 0$$

where we recall that

$$n_{\text{ref}} = \int_{\mathbb{R}^2} h_{\text{ref}} f^\infty dv, \quad j_{\text{ref}} := \int_{\mathbb{R}^2} v h_{\text{ref}} f^\infty dv.$$

Similarly, a momentum equation

$$\tau \partial_t j_{\text{ref}} + \frac{1}{\tau} j_{\text{ref}} = n_{\text{ref}} E_\infty^\delta + (\rho_\infty + n_{\text{ref}}) E_{\text{ref}} - \nabla_x n_{\text{ref}} - \text{div}_x(\mathbb{S}_{\text{ref}}) \quad (3.23)$$

may be obtained by multiplying (3.22) by v and integrating in the velocity variable, with

$$\mathbb{S}_{\text{ref}} = \int_{\mathbb{R}^2} (v \otimes v - \mathbf{I}) h_{\text{ref}} f^\infty dv.$$

To give some hints on computations involved in the foregoing derivation, we introduce $\langle \cdot; \cdot \rangle_v$ to denote the spatially dependent scalar product on $L^2(f^\infty dv)$ and notice that

$$n_{\text{ref}} = \langle \mathbf{1}; h_{\text{ref}} \rangle_v, \quad j_{\text{ref}} = \langle A^*(\mathbf{1}); h_{\text{ref}} \rangle_v = \langle \mathbf{1}; Ah_{\text{ref}} \rangle_v, \quad \mathbb{S}_{\text{ref}} = \langle (A^*)^2(\mathbf{1}); h_{\text{ref}} \rangle_v$$

as follows from $v = A + A^*$, $A(\mathbf{1}) = 0$ and commutation properties of A and A^* . Now the key computations leading to the above are

$$\begin{aligned}
 \langle A^*(\mathbf{1}); E_{\text{ref}} \cdot A^*(\mathbf{1} + h_{\text{ref}}) \rangle_v &= (\rho_\infty + n_{\text{ref}}) E_{\text{ref}}, \\
 \langle A^*(\mathbf{1}); E_\infty^\delta \cdot Ah_{\text{ref}} \rangle_v &= \mathbb{S}_{\text{ref}} E_\infty^\delta, \\
 \langle A^*(\mathbf{1}); A^* \cdot Ah_{\text{ref}} \rangle_v &= j_{\text{ref}}, \\
 \langle A^*(\mathbf{1}); \text{div}_x((A + A^*)h_{\text{ref}}) \rangle_v &= \text{div}_x(\langle A^*(\mathbf{1}); (A + A^*)h_{\text{ref}} \rangle_v) \\
 &\quad - \langle A^*(\mathbf{1}); (A + A^*)h_{\text{ref}} \rangle_v E_\infty^\delta \\
 &= \text{div}_x(\mathbb{S}_{\text{ref}} + n_{\text{ref}} \mathbf{I}) - (\mathbb{S}_{\text{ref}} + n_{\text{ref}} \mathbf{I}) E_\infty^\delta.
 \end{aligned}$$

To proceed, with this in hands, the continuity equation may be turned into

$$\partial_t n_{\text{ref}} + \operatorname{div}_x(n_{\text{ref}} E_{\infty}^{\delta} + (\rho_{\infty} + n_{\text{ref}}) E_{\text{ref}} - \nabla_x n_{\text{ref}}) = \operatorname{div}_x(\operatorname{div}_x(\mathbb{S}_{\text{ref}})) + \tau \operatorname{div}_x(\partial_t j_{\text{ref}}).$$

Therefore we set

$$\tilde{n}_{\text{ref}} = n_{\text{ref}} - \tau \operatorname{div}_x(j_{\text{ref}}) \quad \text{and} \quad \tilde{E}_{\text{ref}} = \frac{1}{\delta^2} \nabla_x \Delta_x^{-1} \tilde{n}_{\text{ref}}$$

and introduce n_{as} the solution to

$$\partial_t n_{\text{as}} + \operatorname{div}_x(n_{\text{as}} E_{\infty}^{\delta} + (\rho_{\infty} + n_{\text{as}}) E_{\text{as}} - \nabla_x n_{\text{as}}) = 0$$

starting from n_0 , where $E_{\text{as}} = \delta^{-2} \nabla_x \Delta_x^{-1} n_{\text{as}}$. Note that the well-posedness of the equation for n_{as} may be obtained by a simpler version of the argument proving Theorem 3.1.2. Moreover we may ensure that $\|\tilde{n}_0\| \leq 2R_0$ by requiring $\tau_0 \|\operatorname{div}_x(j_0)\| \leq R_0$ and thus restrict δ_0 in a way that depends only on R_0 in order to deduce for any $t \geq 0$

$$\begin{aligned} \|(\tilde{n}_{\text{ref}} - n_{\text{as}})(t)\|^2 &+ \int_0^t \|\nabla_x(\tilde{n}_{\text{ref}} - n_{\text{as}})(s)\|^2 ds \\ &\leq \tau^2 \|\operatorname{div}_x(j_0)\|^2 \\ &+ K \int_0^t \left[\tau^2 \|\operatorname{div}_x(j_{\text{ref}})(s)\|_{L^4(\mathbb{T}^2)}^2 \|(E_{\infty}^{\delta} + \tilde{E}_{\text{ref}}(s))\|_{L^4(\mathbb{T}^2)}^2 \right. \\ &\quad \left. + \|n_{\text{ref}}(s)\|^2 \|(\tilde{E}_{\text{ref}} - E_{\text{ref}})(s)\|_{L^{\infty}(\mathbb{T}^2)}^2 \right. \\ &\quad \left. + \tau^2 \|\nabla^2(j_{\text{ref}})(s)\|^2 + \|\operatorname{div}_x(\mathbb{S}_{\text{ref}})(s)\|^2 \right] ds \\ &\leq K' \tau^2 [\|h_0\|^2 + \|Ah_0\|^2 + \|Ch_0\|^2 + \|ACH_0\|^2 + \tau^2 \|C^2 h_0\|^2] \end{aligned}$$

since $\|\operatorname{div}_x(\mathbb{S}_{\text{ref}})\| \leq K (\|A^2 Ch_{\text{ref}}\| + \|A^2 h_{\text{ref}}\|)$. This proves part (ii) of Theorem 3.1.4.

Chapitre 4

Un schéma volumes finis pour des problèmes aux limites paraboliques

Travail en collaboration avec Francis Filbet⁴³.
Ce chapitre est publié en version abrégée
dans *Numerische Mathematik* [89].

We propose a finite volume scheme for a class of nonlinear parabolic equations endowed with non-homogeneous Dirichlet boundary conditions and which admit relative entropy functionals. For this kind of models including porous media equations, Fokker-Planck equations for plasma physics or dumbbell models for polymer flows, it has been proved that the transient solution converges to a steady-state when time goes to infinity. The present scheme is built from a discretization of the steady equation and preserves steady-states and natural Lyapunov functionals which provide a satisfying long-time behavior. After proving well-posedness, stability, exponential return to equilibrium and convergence, we present several numerical results which confirm the accuracy and underline the efficiency to preserve large-time asymptotic.

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4.1 Introduction

4.1.1 General Setting

Let Ω be a polyhedral open bounded connected subset of \mathbb{R}^d with boundary $\Gamma = \partial\Omega$. Let us introduce the steady advection field $\mathbf{E} : \Omega \rightarrow \mathbb{R}^d$ and $\eta : \mathbb{R} \rightarrow \mathbb{R}$ a strictly increasing smooth function onto \mathbb{R} satisfying $\eta(0) = 0$. We consider the following nonlinear convection-diffusion equation with non-homogeneous Dirichlet boundary conditions

$$\begin{cases} \frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{E}(\mathbf{x}) \eta(f) - \nabla \eta(f)) = 0 & \text{for } \mathbf{x} \in \Omega, t \geq 0, \\ f(t, \mathbf{x}) = f^b(\mathbf{x}) & \text{for } \mathbf{x} \in \Gamma, t \geq 0, \\ f(0, \mathbf{x}) = f^{\text{in}}(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega. \end{cases} \quad (4.1)$$

In [24], T. Bodineau, C. Villani, C. Mouhot and J. Lebowitz showed that this equation admits a large class of Lyapunov functionals, that we denote, using their denomination, relative ϕ -entropies. Each functional is generated by a convex function ϕ satisfying the following properties.

Definition 4.1.1 (Entropy generating functions). For any non-empty interval J of \mathbb{R} containing 1, we say that $\phi \in \mathcal{C}^2(J, \mathbb{R}_+)$ is an *entropy generating function* or simply *entropy function* if it is strictly convex and satisfies $\phi(1) = 0$ and $\phi'(1) = 0$.

The entropies are defined relatively to a steady state of (4.1). Therefore, we assume that there exists f^∞ which satisfies

$$\begin{cases} \nabla \cdot (\mathbf{E} \eta(f^\infty) - \nabla \eta(f^\infty)) = 0 & \text{in } \Omega, \\ f^\infty = f^b & \text{on } \Gamma. \end{cases} \quad (4.2)$$

Now we can define the relative ϕ -entropies and associated dissipations.

Definition 4.1.2 (Relative ϕ -entropy and dissipation). For any entropy generating function ϕ , we denote by \mathcal{H}_ϕ the *relative ϕ -entropy* defined by

$$\mathcal{H}_\phi(t) = \int_{\Omega} \int_{f^\infty(\mathbf{x})}^{f(t, \mathbf{x})} \phi' \left(\frac{\eta(s)}{\eta(f^\infty(\mathbf{x}))} \right) ds \, d\mathbf{x},$$

and by \mathcal{D}_ϕ the relative ϕ -entropy dissipation defined by

$$\mathcal{D}_\phi(t) = \int_{\Omega} |\nabla h|^2 \phi''(h) \eta(f^\infty) \, d\mathbf{x},$$

where h is the ratio between the transient and stationary nonlinearities

$$h = \begin{cases} \frac{\eta(f)}{\eta(f^\infty)} & \text{in } \Omega, \\ 1 & \text{on } \Gamma. \end{cases} \quad (4.3)$$

Let us note that for the linear problem, namely when η is the identity function, the relative ϕ -entropy rewrites

$$\mathcal{H}_\phi(t) = \int_{\Omega} \phi\left(\frac{f}{f^\infty}\right) f^\infty \, d\mathbf{x}.$$

Typical examples of relative ϕ -entropies are the *physical relative entropy* and *p-entropies* (or *Tsallis relative entropies*) respectively generated, for $p \in (1, 2]$, by

$$\begin{cases} \phi_1(x) &= x \ln(x) - (x - 1), \\ \phi_p(x) &= \frac{x^p - px}{p - 1} + 1. \end{cases} \quad (4.4)$$

One readily sees that, since η and ϕ' are increasing functions satisfying $\eta(0) = 0$ and $\phi'(1) = 0$, the relative ϕ -entropy is a non-negative quantity which cancels if and only if f and f^∞ coincide almost everywhere. The ϕ -entropies are not, in general, distances between the solution and the steady state. However Csiszar-Kullback type inequalities [55, 138, 189] yield a control of the L^1 distance between the solution and the equilibrium. Therefore if a relative ϕ -entropy goes to zero when time goes to infinity, the solution converges to equilibrium in a strong sense.

The following proposition was proved in [24, Theorem 1.4] and yields an entropy-entropy dissipation principle for Equation (4.1). It starts from a reformulation of (4.1) using the new unknown (4.3). It is easily derived using Leibniz product rule in Equation (4.1) together with (4.2) and (4.3) and reads

$$\begin{cases} \frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{U}^\infty h - \eta(f^\infty) \nabla h) = 0, \\ \mathbf{U}^\infty = \mathbf{E} \eta(f^\infty) - \nabla \eta(f^\infty), \\ \nabla \cdot \mathbf{U}^\infty = 0. \end{cases} \quad (4.5)$$

Proposition 4.1.3. *Any L^∞ solution of (4.1) satisfies in the sense of distributions*

$$\frac{d\mathcal{H}_\phi}{dt} = -\mathcal{D}_\phi \leq 0, \quad (4.6)$$

for any entropy generating function ϕ .

The formal computations leading to (4.6) motivate our choices in the elaboration of the discrete scheme. Therefore, we recall the proof yielding the entropy equality.

Proof. First, we integrate (4.5) against $\phi'(h)$, integrate by parts and use the boundary conditions and the fact that $\phi'(1) = 0$ to get

$$\begin{aligned} \frac{d\mathcal{H}_\phi}{dt} &= \int_{\Omega} \phi'(h) \nabla \cdot (-\mathbf{U}^\infty h + \eta(f^\infty) \nabla h) \, d\mathbf{x} \\ &= \int_{\Omega} \mathbf{U}^\infty \cdot \nabla h \phi''(h) h \, d\mathbf{x} - \int_{\Omega} |\nabla h|^2 \phi''(h) \eta(f^\infty) \, d\mathbf{x}. \end{aligned}$$

Let $\varphi : s \mapsto s\phi'(s) - \phi(s)$. It satisfies $\varphi'(s) = \phi''(s)s$ and $\varphi(1) = 0$. Hence, substituting it in the last expression yields

$$\frac{d\mathcal{H}_\phi}{dt} = \int_{\Omega} \mathbf{U}^\infty \cdot \nabla \varphi(h) \, d\mathbf{x} - \mathcal{D}_\phi = -\mathcal{D}_\phi,$$

where we integrated the first term by parts, used the stationary equation (4.2) and the boundary conditions. \square

There are two important facts that justify the use of (4.5) instead of (4.1) to derive the above entropy dissipation inequality. The rewriting transforms the advection field \mathbf{E} on $\eta(f)$ into the incompressible field \mathbf{U}^∞ on h so that the contribution of the convection can vanish when the time derivative of the relative entropy is computed. The underlying cancellations stems from the transformation of $\nabla h \phi''(h)h$ into a gradient thanks to φ and on the fact that f^∞ solves (4.2). The second reason is that considering the equation on h instead of f changes non-homogeneous Dirichlet boundary conditions into homogeneous ones on $h - 1$. Together with properties of ϕ , it enables cancellations of boundary terms. In other words, relative ϕ -entropies are the correct functionals and (4.5) the right form of the equation to capture the boundary-driven dynamics.

The purpose of this work is the design and analysis of a finite volume scheme preserving the whole class of relative ϕ -entropy dissipation inequalities (4.6). This is done by discretizing the reformulated equation (4.5) in a way that enables the counterpart of computations of the proof of Proposition 4.1.3 to hold at the discrete level. Because of the reformulation involving the steady state, the scheme is based on a preliminary discretization of the steady state and flux.

Let us emphasize that \mathbf{E} is a general field and need not to be either incompressible nor irrotational as for parabolic equations with a gradient flow structure [168]. Indeed, assuming some regularity on the advection field, one can apply the Hodge decomposition to get the existence of a potential $V : \Omega \rightarrow \mathbb{R}$ and $\mathbf{F} : \Omega \rightarrow \mathbb{R}^d$ such that

$$\mathbf{E} = -\nabla V + \mathbf{F}, \quad \nabla \cdot \mathbf{F} = 0 \tag{4.7}$$

When $\mathbf{F} = \mathbf{0}$, there are many examples in the literature [90, 45, 21, 38, 36, 46] of finite volume schemes preserving entropy dissipation properties. C. Chainais-Hillairet and F. Filbet studied in [45] a finite volume discretization for nonlinear drift-diffusion system and proved that the numerical solution converges to a steady-state when time goes to infinity. In [36], M. Burger, J. A. Carrillo and M. T. Wolfram proposed a mixed finite element method for nonlinear diffusion equations and proved convergence towards the steady-state in case of a nonlinear Fokker-Planck equation with uniformly convex potential. All these schemes exploit the gradient flow structure of the equation, which gives a natural entropy.

In the non-symmetric case $\mathbf{F} \neq \mathbf{0}$, the gradient structure cannot be exploited anymore, but as Proposition 4.1.3 shows, there is still a relative entropy structure, which may be investigated to prove convergence to a steady state. The relative entropy properties of Fokker-Planck type equations in the whole space are exhaustively studied in the famous

paper [5] of A. Arnold, P. Markowich, G. Toscani, A. Unterreiter and specific properties of the non-symmetric equations have been investigated in [4, 1].

In bounded domains, entropy properties are often used in the context of no-flux boundary conditions or in the whole space, but few results concern Dirichlet boundary conditions. In [21], M. Bessemoulin-Chatard proposed an extension of the Scharfetter-Gummel for finite volume scheme for convection-diffusion equations with nonlinear diffusion and non-homogeneous and unsteady Dirichlet boundary conditions. While in the latter work the author presents a scheme with a satisfying long-time behavior for a similar class of models than those of the present paper, our strategy and objectives differ. Here we aim at preserving a whole class of relative entropies and build our scheme for the transient problem from a discretization of the stationary equation.

Let us precise that we can generalize our approach to the more general boundary conditions

$$\begin{cases} f = f^b \text{ on } \Gamma_D, \\ [\mathbf{E}\eta(f) - \nabla\eta(f)] \cdot \mathbf{n}(\mathbf{x}) = 0 \text{ on } \Gamma_N, \end{cases}$$

with $\Gamma = \Gamma_D \cup \Gamma_N$. Our results hold in this setting with minor modifications but to avoid unnecessary technicalities in the notation and in the analysis we consider non-homogeneous Dirichlet conditions on the whole boundary. However, numerical results will be shown in both cases.

4.1.2 Physical models

Before describing our numerical scheme, let us present some physical models described by equation (4.1) for which the large-time asymptotic has been studied using entropy-entropy dissipation arguments. Some of these models are the homogeneous part of kinetic Fokker-Planck-type equations and this work constitutes a first step towards treating full kinetic models. In future work, we aim at adapting the strategy developed here to ensure the property of convergence to local equilibrium for the solutions of these equations.

The Fokker-Planck equation with magnetic field

A classical model of plasma physics describing the dynamic of charged particles evolving in an external electromagnetic field $(-\nabla_{\mathbf{x}}\phi, \mathbf{B})$ is given by the Vlasov-Fokker-Planck equation reading

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} F - \nabla_{\mathbf{x}} \phi \cdot \nabla_{\mathbf{v}} F + (\mathbf{v} \wedge \mathbf{B}) \cdot \nabla_{\mathbf{v}} F = \nabla_{\mathbf{v}} \cdot (\mathbf{v} F + \nabla_{\mathbf{v}} F). \quad (4.8)$$

For more details on the model we refer to [29, 124]. In [29], Bouchut and Dolbeault proved that the solution of (4.8) in the whole phase space and without magnetic field converges to a global equilibrium. Their proof mainly relies on the decrease of the free energy functional, which corresponds to the physical relative entropy introduced in (4.4). The external magnetic field does not alter the relative entropy inequality. We refer to [124] for the corresponding computations. Here, we consider the phenomena happening in

the velocity space which results in a Fokker-Planck equation with magnetic field, namely equation (4.1) with $\eta(s) = s$ and an advection field given by

$$\mathbf{E}(\mathbf{v}) = -\mathbf{v} + \mathbf{v} \wedge \mathbf{B}, \quad (4.9)$$

with constant magnetic field \mathbf{B} . In applications, the velocity variable \mathbf{v} usually lives in \mathbb{R}^3 . However when performing numerical simulations, one needs to restrict the velocity domain to a bounded set Ω . On the edge of this restricted domain Ω , we shall consider the following non-homogeneous Dirichlet boundary conditions

$$f(t, \mathbf{v}) = f^\infty(\mathbf{v}) \quad \forall \mathbf{v} \in \partial\Omega, \quad (4.10)$$

where f^∞ is the local Maxwellian associated with (4.8) which writes

$$f^\infty(\mathbf{v}) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|\mathbf{v}|^2}{2}} \quad \forall \mathbf{v} \in \Omega, \quad (4.11)$$

and is a stationary state of (4.1) with (4.9)-(4.10). With the boundary conditions (4.10), one recovers the same stationary state as in the whole space while working in a bounded domain. As for the more complicated kinetic model (4.8), free energy (relative ϕ -entropy) decrease holds.

Our approach is particularly promising for this kind of problem when the solution develops some micro-instabilities around a steady state. In this situation, it is important that numerical artefacts do not generate spurious oscillations.

The dumbbell model for the density of polymers in a dilute solution

The following kinetic equation describes the evolution of the density $F \equiv F(t, \mathbf{x}, \mathbf{k})$ of polymers at time t and position \mathbf{x} diluted in a fluid flow of velocity $\mathbf{u}(\mathbf{x})$ from a mesoscopic point of view

$$\frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} F = -\nabla_{\mathbf{k}} \cdot \left[\left(\nabla_{\mathbf{x}} \mathbf{u} \mathbf{k} - \frac{1}{2} \nabla_{\mathbf{k}} \Pi(\mathbf{k}) \right) F - \frac{1}{2} \nabla_{\mathbf{k}} F \right]. \quad (4.12)$$

The polymers are pictured as two beads linked by a spring and the variable \mathbf{k} stands for the vector indicating the length and orientation of the molecules. The potential Π is given by $\Pi(\mathbf{k}) = |\mathbf{k}|^2/2$ in the case of Hookean dumbbells or by $\Pi(\mathbf{k}) = -\ln(1 - |\mathbf{k}|^2)/2$ in the case of Finite Extensible Nonlinear Elastic dumbbells. In the complete model, the velocity of the fluid \mathbf{u} follows an incompressible Navier-Stokes equation featuring an additional force term modeling for the contribution of the polymers on the dynamic of the fluid which results in a nonlinear kinetic-fluid coupling. Here, we consider the simpler case where $\mathbf{u}(\mathbf{x})$ is a given incompressible field. For more details on the modeling, we refer to [133] and references therein. We also refer to the paper [148] of Masmoudi that treats the well-posedness and provides additional information on the model.

Once again we aim at approximating numerically the ‘‘velocity’’ part of the kinetic equation (4.12) which rewrites as (4.1) with $\eta(s) = s$ and an advection field given by

$$\mathbf{E}(\mathbf{k}) = \mathbf{A} \mathbf{k} - \frac{1}{2} \nabla_{\mathbf{k}} \Pi(\mathbf{k}), \quad (4.13)$$

with constant matrix \mathbf{A} satisfying $\text{tr}(\mathbf{A}) = 0$. This matrix is the gradient of an incompressible velocity field at some space location. Natural boundary conditions for this model are given by null outward flux. In [133], the long-time behavior of (4.12) and of the latter reduced model are investigated using relative ϕ -entropies with ϕ given by (4.4).

A nonlinear model, the porous medium equation

The porous medium equation is a nonlinear PDE writing

$$\frac{\partial f}{\partial t} = \Delta f^m, \quad (4.14)$$

with $m > 1$. It can model many physical applications and generally describes processes involving fluid flow, heat transfer or diffusion. The typical example is the description of the flow of an isentropic gas through a porous medium. There is a huge literature on this equation and we refer to the book of Vásquez [190] for the detailed mathematical theory.

Here, equation (4.14) is set in a bounded domain Ω with non-homogeneous Dirichlet boundary conditions

$$f(t, \mathbf{x}) = f^b(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \partial\Omega,$$

such that it might be recast like (4.1) with a null advection field and with $\eta(s) = s^m$. Using their relative ϕ -entropy method, Bodineau, Mouhot, Villani and Lebowitz show exponential convergence to equilibrium for this nonlinear equation.

4.1.3 Outline and main results

The plan of the paper is as follows. In Section 4.2, we present the finite volume scheme and the discrete version of the relative ϕ -entropies. Then, in Section 4.3, we prove the main properties of our discrete flux, namely the preservation of steady states in Lemma 4.3.1 and the non-negativity of discrete ϕ entropy dissipations in Proposition 4.3.2. In Section 4.4, we analyze our fully-discrete schemes. In Theorem 4.4.2, we prove well-posedness, stability and decay of discrete ϕ -entropies for the time implicit version of our scheme. For the explicit version, we prove the same results in Theorem 4.4.4, under a parabolic Courant-Friedrichs-Lewy (CFL) condition. Besides, the long-time behavior of the discrete solution, for both schemes, is investigated in this section and we prove in Theorem 4.4.7 that discrete solutions return to equilibrium exponentially fast, with a rate that does not depend on the size of the discretization. In Section 4.5, we show convergence of the discrete solution to a solution of (4.5), when the size of the discretization goes to zero. Finally in Section 4.6, we end by providing numerical illustrations of the properties of our schemes on the models presented above.

4.2 Presentation of the numerical schemes

In this section, we introduce our finite volume schemes. In the following, T is a positive real number and Ω_T denotes the cylinder $[0, T) \times \Omega$. We start with some notations associated with the discretization of Ω_T .

4.2.1 Mesh and time discretization

An admissible mesh of Ω is defined by the triplet $(\mathcal{T}, \mathcal{E}, \mathcal{P})$. The set \mathcal{T} is a finite family of nonempty connected open disjoint subsets $K \subset \Omega$ called control volumes or cells. The closure of the union of all control volumes is equal to $\bar{\Omega}$. The set \mathcal{E} is a finite family of nonempty subsets $\bar{\Omega}$ called edges. Each edge is a subset of an affine hyperplane in \mathbb{R}^{d-1} . Moreover, for any control volume $K \in \mathcal{T}$ there exists a subset \mathcal{E}_K of \mathcal{E} such that the closure of the union of all the edges in \mathcal{E}_K is equal to $\partial K = \bar{K} \setminus K$. We also define several subsets of \mathcal{E} . The family of interior edges \mathcal{E}_{int} is given by $\{\sigma \in \mathcal{E}, \sigma \not\subseteq \Gamma\}$ and the family of exterior edges by $\mathcal{E}_{\text{ext}} = \mathcal{E} \setminus \mathcal{E}_{\text{int}}$. Similarly, for any control volume $K \in \mathcal{T}$, we define $\mathcal{E}_{\text{int},K} = \mathcal{E}_{\text{int}} \cap \mathcal{E}_K$ and $\mathcal{E}_{\text{ext},K} = \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$. We assume that for any edge σ , the number of control volumes sharing the edge σ is exactly 2 for interior edges and 1 for exterior edges. With these assumptions, every interior edge is shared by two control volumes, say K and L , so that we may use the notation $\sigma = K|L$ whenever $\sigma \in \mathcal{E}_{\text{int}}$. The set $\mathcal{P} = \{\mathbf{x}_K\}_{K \in \mathcal{T}}$ is a finite family of points satisfying that for any control volume $K \in \mathcal{T}$, $\mathbf{x}_K \in K$. We introduce the transmissibility of the edge σ , given by

$$\tau_\sigma = \frac{m(\sigma)}{d_\sigma},$$

where

$$d_\sigma = \begin{cases} d(\mathbf{x}_K, \mathbf{x}_L), & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, \\ d(\mathbf{x}_K, \sigma), & \text{if } \sigma \in \mathcal{E}_{\text{ext},K}, \end{cases}$$

with $d(\cdot, \cdot)$ the euclidean distance in \mathbb{R}^d . The size of the mesh is defined by

$$\Delta x = \max_{K \in \mathcal{T}} \sup_{\mathbf{x}, \mathbf{y} \in K} d(\mathbf{x}, \mathbf{y}).$$

The Dirichlet condition on the boundary is given by $f^b \in L^\infty(\Gamma)$. Endowed with these boundary conditions, a discrete solution of the scheme at some fixed time is an element of the set

$$X_{f^b} = \left\{ f \in \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\mathcal{E}_{\text{ext}}} : f_\sigma = \frac{1}{m(\sigma)} \int_\sigma f^b dm, \forall \sigma \in \mathcal{E}_{\text{ext}} \right\}.$$

Remark 4.2.1. *The particular formula providing f_σ on the exterior edges is cosmetic here since by the reformulation of the equation, we will only use the constant boundary values of a reformulated unknown defined in (4.15).*

Let us mention that with a slight abuse we keep the same notation for discrete and continuous unknowns. Concerning the initial condition we assume that

$$f^{\text{in}} \geq 0 \quad \text{and} \quad K_{\text{in}} := \|f^{\text{in}}\|_{L^\infty(\Omega)} < +\infty, \tag{H1}$$

and the discrete initial condition is given by

$$f_K^{\text{in}} = \frac{1}{m(K)} \int_K f^{\text{in}}(\mathbf{x}) \, d\mathbf{x},$$

for all $K \in \mathcal{T}$. For any function $\psi : \mathbb{R} \rightarrow \mathbb{R}$, and $f \in X_{fb}$ we shall define the component-wise composition with the intuitive notation $\psi(f) = ((\psi(f_K))_{K \in \mathcal{T}}, (\psi(f_\sigma))_{\sigma \in \mathcal{E}_{\text{ext}}})$.

We denote the time step by Δt and set $t^n = n \Delta t$. From this, a time discretization $[0, T)$ is given by the integer $N_T = \lfloor T/\Delta t \rfloor$ and the sequence $(t^n)_{0 \leq n \leq N_T}$. A spacetime discretization \mathcal{D} of Ω_T is composed of an admissible mesh of Ω and the time discretization parameters Δt and N_T . Finally, the size of the discretization \mathcal{D} is given by

$$\delta = \max(\Delta x, \Delta t).$$

4.2.2 Discretization of the steady equation

In order to build our numerical scheme for the reformulated equation (4.5), we need a discrete version of the steady state as well as corresponding discrete steady flux at interfaces. Hence we introduce an approximation of $\eta(f^\infty)$ on cells $(\eta(f^\infty)_K)_{K \in \mathcal{T}}$ and on edges $(\eta(f^\infty)_\sigma)_{\sigma \in \mathcal{E}}$. We suppose that there exists positive constants m_∞ and M_∞ that do not depend on the discretization \mathcal{D} and such that for all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$

$$0 < m_\infty \leq \eta(f^\infty)_K, \quad \eta(f^\infty)_\sigma \leq M_\infty. \quad (\text{H2})$$

Remark 4.2.2. *Of course this hypothesis requires some kind of maximum principle holding at the continuous level. It is the case at least for divergence free fields \mathbf{E} , if f^b is bounded and bounded from below by a positive constant then by [82, Section 6.4, Theorem 1].*

Moreover for each cell $K \in \mathcal{T}$ and edge $\sigma \in \mathcal{E}_K$, we introduce a discrete flux $F_{K,\sigma}^\infty$ approximating $\int_\sigma \mathbf{U}^\infty(\mathbf{x}) \cdot \mathbf{n}_{K,\sigma}(\mathbf{x}) \, d\mathbf{x}$ along with the divergence free hypothesis

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^\infty = 0, \quad (\text{H3})$$

and the interior continuity condition

$$\forall \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad F_{K,\sigma}^\infty = -F_{L,\sigma}^\infty. \quad (\text{H4})$$

Let us emphasize that this last condition is required and satisfied for all the discrete flux we define in the following. From the flux we also introduce the discrete velocities

$$U_{K,\sigma}^\infty = \frac{1}{m(\sigma)} F_{K,\sigma}^\infty.$$

On one hand, for some models, the global equilibrium f^∞ may be known analytically. In this case, we may build a discrete approximation in X_{fb} by a standard projection on the mesh and the numerical flux $F_{K,\sigma}^\infty$ may be computed exactly or approximated with a quadrature formula. In any case, hypotheses (H2)-(H4) must be satisfied. In Section 4.6.2 and Section 4.6.3, this strategy is used for computing f_K^∞ and $F_{K,\sigma}^\infty$ on our test cases.

On the other hand, when the steady state is not known we apply a finite volume scheme to compute a numerical approximation. Our method does not impose any scheme for solving the stationary equation as long as assumptions (H2)-(H4) are satisfied. However to fix ideas let us provide an example here.

Example 4.2.3. One can determine the discrete steady state by solving (H3) on each cell $K \in \mathcal{T}$ with the flux

$$F_{K,\sigma}^\infty = \begin{cases} m(\sigma) \left[E_{K,\sigma}^+ \eta(f^\infty)_K - E_{K,\sigma}^- \eta(f^\infty)_L - \frac{\eta(f^\infty)_L - \eta(f^\infty)_K}{d_\sigma} \right], & \text{if } \sigma = K|L, \\ m(\sigma) \left[E_{K,\sigma}^+ \eta(f^\infty)_K - E_{K,\sigma}^- \eta(f^\infty)_\sigma - \frac{\eta(f^\infty)_\sigma - \eta(f^\infty)_K}{d_\sigma} \right], & \text{otherwise,} \end{cases}$$

where for any real number u , we denote by $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ the positive and negative parts of u . The quantity $E_{K,\sigma}$ is a consistent approximation of $\mathbf{E} \cdot \mathbf{n}_{K,\sigma}$ on the edge σ such as $\mathbf{E}(\mathbf{x}_\sigma) \cdot \mathbf{n}_{K,\sigma}$, where \mathbf{x}_σ is the center of mass of the edge σ . Then, for $\sigma = K|L$, the approximation of $\eta(f^\infty)$ on the edge σ can be given by $\eta(f^\infty)_\sigma = (\eta(f^\infty)_K + \eta(f^\infty)_L)/2$ and for each cell $K \in \mathcal{T}$, $f_K^\infty = \eta^{-1}(\eta(f^\infty)_K)$.

In Section 4.6.4, we provide another example of resolution of the steady equation by a finite volume discretization.

4.2.3 Discretization of the evolution equation

Now we treat the time evolution problem and use the discrete steady state to build a numerical approximation of the reformulated equation (4.5). We start by introducing our finite volume scheme in a general implicit and explicit form. Then we define the flux.

Fully discrete schemes

In the continuous setting, we defined a new unknown to reformulate the convection-diffusion equation. Its discrete equivalent is still denoted h and belongs to the space $X_1^{N_T+1}$. It is defined for any $K \in \mathcal{T}$ and $n \in \{0, \dots, N_T\}$ by

$$h_K^n = \frac{\eta(f_K^n)}{\eta(f^\infty)_K}. \quad (4.15)$$

Let us insist on the fact that boundary conditions are contained in the definition of the approximation space so that here, $h_\sigma = 1$ for $\sigma \in \mathcal{E}_{\text{ext}}$. Solving the fully discrete implicit scheme consists in finding $f = (f_K^n)_{n \in \{0, \dots, N_T\}} \in X_{fb}^{N_T+1}$ such that for all $K \in \mathcal{T}$

$$\begin{cases} m(K) \frac{f_K^{n+1} - f_K^n}{\Delta t} + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(h^{n+1}) = 0, & \forall n \in \{0, \dots, N_T - 1\}, \\ f_K^0 = f_K^{\text{in}}. \end{cases} \quad (4.16)$$

In explicit form, it amounts to building f sequentially by

$$\forall K \in \mathcal{T}, \begin{cases} m(K) \frac{f_K^{n+1} - f_K^n}{\Delta t} + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(h^n) = 0, & \forall n \in \{0, \dots, N_T - 1\} \\ f_K^0 = f_K^{\text{in}}, \end{cases} \quad (4.17)$$

In order to lighten the notation we sometimes write $F_{K,\sigma}(h^n) = F_{K,\sigma}^n$ in the following. For any discrete element $u \in X_{u^b}^{N_T+1}$, its reconstruction is given almost everywhere on Ω_T by

$$u_\delta(t, x) = \begin{cases} u_K^{n+1} & \text{if } x \in K \text{ and } t \in [t^n, t^{n+1}), \quad (\text{Implicit case}) \\ u_K^n & \text{if } x \in K \text{ and } t \in [t^n, t^{n+1}). \quad (\text{Explicit case}) \end{cases}$$

Definition of the flux

The flux is divided in two parts,

$$F_{K,\sigma} = F_{K,\sigma}^{\text{conv}} + F_{K,\sigma}^{\text{diss}}, \quad (4.18)$$

corresponding respectively to the discretization of the convective term $\int_\sigma h \mathbf{U}^\infty \cdot \mathbf{n}_{K,\sigma}(\mathbf{x}) \, dm$ and of the dissipative term $-\int_\sigma \eta(f^\infty) \nabla h \cdot \mathbf{n}_{K,\sigma}(\mathbf{x}) \, dm$. Accordingly, we call $F_{K,\sigma}^{\text{conv}}$ the convective flux and $F_{K,\sigma}^{\text{diss}}$ the dissipative flux. The former is discretized thanks to an monotone upstream discretization reading

$$F_{K,\sigma}^{\text{conv}}(h) = \begin{cases} m(\sigma) \left[U_{K,\sigma}^{\infty,+} g(h_K, h_L) - U_{K,\sigma}^{\infty,-} g(h_L, h_K) \right], & \text{if } \sigma = K|L, \\ m(\sigma) \left[U_{K,\sigma}^{\infty,+} g(h_K, h_\sigma) - U_{K,\sigma}^{\infty,-} g(h_\sigma, h_K) \right], & \text{otherwise,} \end{cases} \quad (4.19)$$

where $U_{K,\sigma}^{\infty,+} = \max(U_{K,\sigma}^\infty, 0)$ and $U_{K,\sigma}^{\infty,-} = \max(-U_{K,\sigma}^\infty, 0)$ denote the positive and negative parts of $U_{K,\sigma}^\infty$ respectively. The classical upwind flux, that we use for our numerical simulations is obtained by choosing $g(s, t) = s$. With this more general version, we aim at highlighting the importance of the monotony property of this flux concerning stability of the scheme and decay of discrete relative ϕ -entropy. Accordingly, $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ ought to satisfy the following assumptions.

$$\begin{cases} g \text{ is locally Lipschitz-continuous,} \\ g \text{ is non-decreasing in the first variable and non-increasing in the second variable,} \\ g(s, s) = s, \text{ for all } s \in \mathbb{R}. \end{cases} \quad (\text{H5})$$

The first and last conditions ensure consistency of the approximation.

The dissipative flux is built on a two-point approximation of the derivative along the outward normal vector of each edge, namely

$$F_{K,\sigma}^{\text{diss}}(h) = -\tau_\sigma \eta(f^\infty)_\sigma D_{K,\sigma} h, \quad (4.20)$$

where the difference operator $D_{K,\sigma}$ is defined for any $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$ and $u \in X_{u^b}$ by

$$D_{K,\sigma} u = \begin{cases} u_L - u_K & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, \\ u_\sigma - u_K & \text{if } \sigma \in \mathcal{E}_{\text{ext},K}. \end{cases}$$

For consistency of discrete gradients, we require an orthogonality condition for the mesh, namely

$$\forall \mathbf{x}, \mathbf{y} \in \sigma = K|L, \quad (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x}_K - \mathbf{x}_L) = 0. \quad (\text{H6})$$

Remark 4.2.4. Hypothesis (H6) is necessary for establishing the convergence of the scheme in Section 4.5 and even if it is standard [85], it remains a restrictive condition on the shape of the mesh. The necessity for this hypothesis stems from the fact that we choose a two-point flux (4.20) for the diffusion. This choice is motivated by the monotony properties of this flux which enable the decay of ϕ -entropies at the discrete level.

Remark 4.2.5. Even if the flux (4.19) and (4.20) seem fairly classical, one must not forget that they act on the unknown h while the discrete time derivative in (4.16) or (4.17) is on f . As in the continuous setting, this is the most important part of the strategy.

4.2.4 Discrete relative ϕ -entropies and dissipations

For $f \in X_{fb}$, the discrete equivalent of the relative ϕ -entropy in Definition 4.1.2 is given by

$$H_\phi(f) = \sum_{K \in \mathcal{T}} m(K) e_{\phi,K}(f), \quad (4.21)$$

where $e_\phi = (e_{\phi,K})_{K \in \mathcal{T}}$ is the local discrete relative ϕ -entropy writing, for $K \in \mathcal{T}$

$$e_{\phi,K}(f) = \int_{f_K^\infty}^{f_K} \phi' \left(\frac{\eta(s)}{\eta(f_K^\infty)} \right) ds.$$

Contributions of the convective and diffusive part of the equation to the relative entropy variation are defined, for $h \in X_1$ by

$$C_\phi(h) = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \phi'(h_K) F_{K,\sigma}^{\text{conv}}(h), \quad D_\phi(h) = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \phi'(h_K) F_{K,\sigma}^{\text{diss}}(h). \quad (4.22)$$

We write H_ϕ^n , C_ϕ^n and D_ϕ^n to denote respectively $H_\phi(f^n)$, $C_\phi(h^n)$ and $D_\phi(h^n)$. The precise relation between these quantities is derived in the proof of Theorem 4.4.2 and 4.4.4. For the moment, $C_\phi^n + D_\phi^n$ should be thought as the “time derivative” of H_ϕ^n . In Proposition 4.3.2 we show that D_ϕ is as expected consistent with its continuous analogue \mathcal{D}_ϕ and non-negative. Moreover, thanks to the monotonicity properties of the convective flux (4.19), we also prove that $C_\phi(f)$ remains non-negative, creating an additional numerical dissipation consistent with 0 as $\delta \rightarrow 0$.

4.3 Properties of the discrete flux

We present here, independently of time discretizations (4.16) or (4.17), the properties of the flux introduced in the previous section. In this section, f is an element of X_{fb} and h the corresponding element of X_1 using relation (4.15).

4.3.1 Preservation of the steady state

The next lemma show that the discrete steady state is a steady state of our schemes.

Lemma 4.3.1. *Under hypotheses (H2)-(H3), if for all cells $K \in \mathcal{T}$, $\eta(f_K) = \eta(f^\infty)_K$, then*

$$\forall K \in \mathcal{T}, \quad \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(h) = 0.$$

Proof. Just observe that every component of h equals 1. Hence, for any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, $F_{K,\sigma}^{\text{diss}}(h) = 0$ and

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{\text{conv}}(h) = \sum_{\sigma \in \mathcal{E}_K} m(\sigma) \left(U_{K,\sigma}^{\infty,+} - U_{K,\sigma}^{\infty,-} \right) = \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^\infty = 0,$$

using (H3). □

4.3.2 Non-negativity of dissipations

Proposition 4.3.2. *Let ϕ be any entropy generating function and $h \in X_1$. Under hypotheses (H2)-(H5), the following results hold.*

(i) *The numerical dissipation is non-negative, namely,*

$$C_\phi(h) \geq 0.$$

(ii) *The physical dissipation rewrites*

$$\begin{aligned} D_\phi(h) &= \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma D_{K,\sigma} h D_{K,\sigma} \phi'(h) \eta(f^\infty)_\sigma \\ &+ \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{ext},K}} \tau_\sigma D_{K,\sigma} h D_{K,\sigma} \phi'(h) \eta(f^\infty)_\sigma \geq 0. \end{aligned} \tag{4.23}$$

In order to prove the non-negativity of the numerical dissipation we compare it to $C_\phi^{M_\phi}$ which is the numerical dissipation of a centered convective flux that we define hereafter as well as some complementary notation.

Definition 4.3.3. A function $M : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is called a *mean function* if it satisfies for all $s, t \in \mathbb{R}_+$,

1. $M(s, t) = M(t, s)$,
2. $M(s, s) = s$,
3. If $s \leq t$, then $s \leq M(s, t) \leq t$.

We also define $M_\sigma : X_{u^b} \rightarrow \mathbb{R}$ by $M_\sigma(u) = M(u_K, u_L)$ if $\sigma = K|L$ and $M_\sigma(u) = M(u_K, u_\sigma)$ otherwise. This is well defined thanks to the symmetry of M . For any such function M , we define the centered convective flux associated to M by

$$F_{K,\sigma}^M(h) = m(\sigma) U_{K,\sigma}^\infty M_\sigma(h)$$

and

$$C_\phi^M = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \phi'(h_K) F_{K,\sigma}^M(h).$$

Finally for any entropy generating function ϕ , it is elementary to show that

$$M^\phi(s, t) = \frac{\varphi(s) - \varphi(t)}{\phi'(s) - \phi'(t)},$$

where $\varphi(s) = s\phi'(s) - \phi(s)$, defines a continuous mean function. We call it the ϕ -mean.

Remark 4.3.4. *Let us note that for the 2-entropy generating function $\phi_2(s) = (s-1)^2$, the corresponding ϕ_2 -mean is the arithmetic average and therefore $F_{K,\sigma}^{M_{\phi_2}}$ is a centered approximation for the convective flux, namely for $\sigma \in \mathcal{E}_{\text{int}}$*

$$F_{K,\sigma}^{M_{\phi_2}}(h) = m(\sigma) U_{K,\sigma}^\infty \frac{h_K + h_L}{2}.$$

When choosing the generator of the physical entropy $\phi_1(s) = s \log(s) - s + 1$, the corresponding mean function is the logarithmic average reading $M_{\phi_1}(s, t) = (s-t)/(\log(s) - \log(t))$.

We are now ready to prove Proposition 4.3.2.

Proof of Proposition 4.3.2. In order to prove (4.23), we use (4.22) and (4.20) to get

$$D_\phi(f) = - \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma D_{K,\sigma} h \phi'(h_K) \eta(f_\sigma^\infty)$$

and the result stems from a discrete integration by parts.

Now, let us prove the non-negativity of C_ϕ . For M a mean function (see Definition 4.3.3), we perform a discrete integration by parts of $C_\phi - C_\phi^M$ which yields

$$C_\phi - C_\phi^M = - \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} (F_{K,\sigma}^{\text{conv}} - F_{K,\sigma}^M) D_{K,\sigma}(\phi'(h)) - \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{ext},K}} (F_{K,\sigma}^{\text{conv}} - F_{K,\sigma}^M) D_{K,\sigma}(\phi'(h)).$$

Now let us just remark that for any $\sigma \in \mathcal{E}_{\text{int}}$ with $\sigma = K|L$,

$$\begin{aligned} - (F_{K,\sigma}^{\text{conv}} - F_{K,\sigma}^M) D_{K,\sigma}(\phi'(h)) &= m(\sigma) U_{K,\sigma}^{\infty,+} (g(h_K, h_K) - g(h_K, h_L)) (\phi'(h_L) - \phi'(h_K)) \\ &+ m(\sigma) U_{K,\sigma}^{\infty,+} (M_\sigma(h) - h_K) (\phi'(h_L) - \phi'(h_K)) \\ &+ m(\sigma) U_{K,\sigma}^{\infty,-} (h_L - M_\sigma(h)) (\phi'(h_L) - \phi'(h_K)) \\ &+ m(\sigma) U_{K,\sigma}^{\infty,-} (g(h_L, h_K) - g(h_L, h_L)) (\phi'(h_L) - \phi'(h_K)), \end{aligned}$$

where we used that $g(s, s) = s$. If $\sigma \in \mathcal{E}_{\text{ext}}$, the same equation holds replacing h_L with h_σ . Therefore, since ϕ' and $g(s, \cdot)$ are monotonically non-decreasing functions and $M_\sigma(h)$ is always between h_K and h_L (resp. h_σ), the above quantity is non-negative. Hence, it proves that $C_\phi \geq C_\phi^M$.

Finally, a simple computation using two integrations by parts yields

$$\begin{aligned}
 C_\phi^{M_\phi} &= - \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} F_{K,\sigma}^{M_\phi} D_{K,\sigma}(\phi'(h)) - \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{ext},K}} F_{K,\sigma}^{M_\phi} D_{K,\sigma}(\phi'(h)), \\
 &= - \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} F_{K,\sigma}^\infty D_{K,\sigma}(\varphi(h)) - \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{ext},K}} F_{K,\sigma}^\infty D_{K,\sigma}(\varphi(h)), \\
 &= \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^\infty \varphi(h_K) = 0,
 \end{aligned}$$

where we used (H3) in the last equality. Thus, $C_\phi \geq C_\phi^{M_\phi} = 0$. \square

Remark 4.3.5. *Let us note that if we had used the fluxes $F_{K,\sigma}^{M_\phi}$ instead of $F_{K,\sigma}^{\text{conv}}$ in our scheme, then the contribution of the convection would be null in the discrete relative ϕ -entropy variation. However, the scheme would have been ϕ -dependent. With the upstream flux, we get the whole class of relative entropy inequalities at the cost of an additional numerical dissipation.*

4.4 Analysis of the schemes

4.4.1 Implicit Euler

Before stating our main result for the implicit scheme (4.16)-(4.20). Let us show that the control of a large class of relative ϕ -entropies yields L^∞ stability of the discrete solution. To establish this result with bounds that are independent of the discretization \mathcal{D} , L^∞ bound on the initial data, given by (H1) is mandatory.

Lemma 4.4.1. *Assume that for any entropy function ϕ , and $n \in \{0, \dots, N_T\}$*

$$H_\phi(h^n) \leq H_\phi(h^0).$$

Then, under hypotheses (H1) and (H2), there exists a positive constant K_∞ depending only on K_{in} , η , m_∞ and M_∞ such that for all $(t, x) \in [0, T) \times \mathbb{R}^d$,

$$0 \leq f_\delta(t, x), \quad h_\delta(t, x) \leq K_\infty.$$

Proof. Let us define $\phi_{u_0}(\cdot) = (\cdot - u_0)^+$ if $u_0 > 1$ and $\phi_{u_0}(\cdot) = (\cdot - u_0)^-$ if $u_0 < 1$. It is differentiable on $\mathbb{R} \setminus \{u_0\}$ with derivative $\phi'_{u_0} = \mathbb{1}_{[u_0, +\infty)}$ if $u_0 > 1$ and $\phi'_{u_0} = -\mathbb{1}_{(-\infty, u_0]}$ if $u_0 < 1$, where $\mathbb{1}_A$ denotes the indicator function of the set A . The function ϕ_{u_0} is not an entropy function. However, if for any $n \in \mathbb{N}$ one has $H_{\phi_{u_0}}(h^n) = 0$ for $u_0 = 0$ and $u_0 = \eta(K_{\text{in}})/m_\infty$, then $e_{\phi_{u_0},K}(h^n) = 0$ and $h_K^n \in [0, \eta(K_{\text{in}})/m_\infty]$ for all cell K . Hence let us show that $H_{\phi_{u_0}}(h^n) = 0$ by an approximation argument.

Let $B : x \mapsto x/(\exp(x) - 1)$ be the Bernoulli function, which is a strictly convex \mathcal{C}^2 function. Then, for any $\varepsilon > 0$ and $u_0 \in \mathbb{R}$, one readily checks that

$$\phi_{\varepsilon, u_0} : u \mapsto \phi_{\varepsilon, u_0}(u) = \varepsilon \left[B\left(\frac{u - u_0}{\varepsilon}\right) - B\left(\frac{1 - u_0}{\varepsilon}\right) \right] + B'\left(\frac{1 - u_0}{\varepsilon}\right) (1 - u),$$

are entropy generating functions. When ε tends to 0, ϕ'_{ε, u_0} converges pointwise to ϕ'_{u_0} on $\mathbb{R} \setminus \{u_0\}$. Therefore, by dominated convergence, one has for any $n \in \{0, \dots, N_T\}$

$$0 \leq H_{\phi_{u_0}}^n \leq H_{\phi_{u_0}}^0.$$

Since, by (H2) and (H1), $H_{u_0}^0 = 0$ for $u_0 = \eta(K_{\text{in}})/m_\infty$ and $u_0 = 0$ then $H_{u_0}^n = 0$ for all $n \in \{0, \dots, N_T\}$. Hence we infer the uniform bounds on h_δ , and consequently on f_δ . \square

Theorem 4.4.2 (Implicit Euler). *Under hypotheses (H1)-(H5) the scheme (4.16) together with (4.18)-(4.20) satisfies the following properties.*

- (i) *There exists a unique discrete solution $f \in X_{f^b}^{N_T+1}$;*
- (ii) *there is a positive constant K_∞ depending only on K_{in} , η , m_∞ and M_∞ such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$,*

$$0 \leq f_\delta(t, x), h_\delta(t, x) \leq K_\infty;$$

- (iii) *the scheme preserves the steady state f^∞ and for any entropy function ϕ and $n \in \{0, \dots, N_T - 1\}$,*

$$\frac{H_\phi^{n+1} - H_\phi^n}{\Delta t} + D_\phi^{n+1} \leq 0 \quad \text{and} \quad D_\phi^{n+1} \geq 0. \quad (4.24)$$

Proof. (i) : The existence of a unique solution to the implicit scheme can be shown with a fixed point strategy close to that in [85, Remark 4.9] and we do not detail this part.

(iii) : Let us derive the entropy inequality. The Taylor-Young theorem provides the existence of $\theta_K^{n, n+1} \in (\min(f_K^n, f_K^{n+1}), \max(f_K^n, f_K^{n+1}))$ such that

$$\begin{aligned} e_{\phi, K}^{n+1} - e_{\phi, K}^n &= \int_{f_K^n}^{f_K^{n+1}} \phi' \left(\frac{\eta(s)}{\eta(f_K^\infty)} \right) ds \\ &= \phi'(h_K^{n+1})(f_K^{n+1} - f_K^n) - \frac{1}{2} \psi_K(\theta_K^{n, n+1})(f_K^{n+1} - f_K^n)^2 \\ &= -\frac{\Delta t}{m(K)} \phi'(h_K^{n+1}) \sum_{\sigma \in \mathcal{E}_K} F_{K, \sigma}^{n+1} - \frac{1}{2} \psi_K(\theta_K^{n, n+1})(f_K^{n+1} - f_K^n)^2, \end{aligned}$$

where ψ_K is given by

$$\psi_K : x \mapsto \frac{\eta'(x)}{\eta(f_K^\infty)} \phi'' \left(\frac{\eta(x)}{\eta(f_K^\infty)} \right). \quad (4.25)$$

Note that ψ_K is a positive function thanks to the positive monotony of η and ϕ' . From the definition of the dissipation in (4.22), it yields

$$\frac{H_\phi^{n+1} - H_\phi^n}{\Delta t} + D_\phi^{n+1} \leq -C_\phi^{n+1} - \frac{1}{2\Delta t} \sum_{K \in \mathcal{T}} \psi_K(\theta_K^{n,n+1})(f_K^{n+1} - f_K^n)^2 m(K) \leq 0,$$

where we applied Proposition 4.3.2 to control the first term of the right-hand side.

(ii) : By summing (4.24) over n and using the positivity of D_ϕ^n , we obtain the boundedness of H_ϕ^n and we can use Lemma 4.4.1 to conclude. \square

4.4.2 Explicit Euler

Before stating the main result on this scheme, let us introduce

$$a_{K,\sigma}^n = \begin{cases} d_\sigma \left(U_{K,\sigma}^{\infty,+} \frac{h_K^n - g(h_K^n, h_L^n)}{D_{K,\sigma} h^n} + U_{K,\sigma}^{\infty,-} \frac{g(h_L^n, h_K^n) - h_K^n}{D_{K,\sigma} h^n} \right), & \text{if } \sigma = K|L, \\ d_\sigma \left(U_{K,\sigma}^{\infty,+} \frac{h_K^n - g(h_K^n, h_\sigma)}{D_{K,\sigma} h^n} + U_{K,\sigma}^{\infty,-} \frac{g(h_\sigma, h_K^n) - h_K^n}{D_{K,\sigma} h^n} \right), & \text{otherwise,} \end{cases} \quad (4.26)$$

with the convention $a_{K,\sigma}^n = 0$ if $D_{K,\sigma} h^n = 0$. Then observe that we can use this $a_{K,\sigma}$ to reformulate the convective part of the scheme as a “diffusive term”, thanks to the incompressibility of $U_{K,\sigma}^\infty$, which is a consequence of (H3). Indeed, for all $K \in \mathcal{T}$, we have

$$- \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma a_{K,\sigma} D_{K,\sigma} h = \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{\text{conv}} - h_K \sum_{\sigma \in \mathcal{E}_K} m(\sigma) U_{K,\sigma}^\infty = \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{\text{conv}},$$

where we used (H3) in the last equality. Now we suppose that there is a positive constant V_∞ that does not depend on $\delta > 0$ and such that

$$\max_{K \in \mathcal{T}} \max_{\sigma \in \mathcal{E}_K} |U_{K,\sigma}^\infty| \leq V_\infty. \quad (\text{H7})$$

Thanks to the monotonicity and regularity properties of g from (H5), one has

$$0 \leq a_{K,\sigma} \leq C_g V_\infty \text{diam}(\Omega),$$

where C_g is the Lipschitz constant of g on $[0, \eta(K_{\text{in}})/m_\infty]^2$ and $\text{diam}(\Omega)$ is the diameter of Ω .

Remark 4.4.3. Hypothesis (H7) is the most restrictive of our assumptions since it implicitly demand uniform $W^{1,\infty}$ bound on the discrete steady state $\eta(f^\infty)$. A more natural and similar hypothesis is a uniform L^2 control on the velocity (instead of L^∞). It corresponds to discrete H^1 control on $\eta(f^\infty)$, uniformly in $\delta > 0$, which can be easily obtained for a finite volume discretization of an elliptic equation like (4.2). For the convergence analysis in Section 4.5, this hypothesis follows from (H10). Unfortunately, we need the stronger (H7) to obtain L^∞ stability for the explicit scheme.

Theorem 4.4.4 (Explicit Euler). *Let $f \in X_{fb}^{N_T+1}$ be defined by the scheme (4.17)-(4.20). Then, under hypotheses (H1)-(H5), (H7) the following properties hold.*

(i) *There exists a positive constant C_{L^∞} depending only on K_{in} , M_∞ , V_∞ , g , η and Ω such that under the CFL condition*

$$\max_{K \in \mathcal{T}} \frac{\Delta t}{m(K)} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma \leq C_{L^\infty},$$

there is a positive constant K_∞ depending only on K_{in} , η , m_∞ and M_∞ such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$0 \leq f_\delta(t, x), \quad h_\delta(t, x) \leq K_\infty.$$

(ii) *If Φ is a family of entropy functions with second derivate bounded between m_Φ and M_Φ , then there exists a positive constant C_{ent} depending only on K_{in} , m_∞ , M_∞ , V_∞ , g , η and Ω such that for every $\varepsilon \in (0, 1)$, under the CFL condition*

$$\max_{K \in \mathcal{T}} \frac{\Delta t}{m(K)} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma \leq \min \left(C_{L^\infty}, C_{ent} \frac{m_\Phi}{M_\Phi} \varepsilon \right) \quad (4.27)$$

for any $\phi \in \Phi$ and $n \in \{0, \dots, N_T - 1\}$,

$$\frac{H_\phi^{n+1} - H_\phi^n}{\Delta t} + (1 - \varepsilon) D_\phi^n \leq 0 \quad \text{and} \quad D_\phi^n \geq 0. \quad (4.28)$$

Moreover the scheme preserves the stationary state f^∞ .

Proof. (i) : For explicit time discretization it is rather classical (see [85]) to use the convexity property of the scheme to show L^∞ stability. However due to the fact that flux are expressed in terms of h instead of f adds up some technicalities here. We have to proceed in two steps to show the heredity of the induction hypothesis $h^n \in J = [0, \eta(K_{in})/m_\infty]$.

First note that for any $\delta \geq 0$, under the CFL condition

$$\max_{K \in \mathcal{T}} \frac{\Delta t}{m(K)} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma \leq C_1(\delta) := \frac{\delta}{K_{in} (C_g V_\infty \text{diam}(\Omega) + M_\infty)},$$

one has $f_K^{n+1} \in I_\delta = [-\delta, K_{in} + \delta]$ for all control volume $K \in \mathcal{T}$ since the scheme rewrites

$$f_K^{n+1} = f_K^n + \frac{\Delta t}{m(K)} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma (a_{K,\sigma} + \eta(f_K^\infty)) D_{K,\sigma} h^n.$$

Let us define $M_\eta^\delta = \sup_{s \in I_\delta} \eta'(s)$. Now we show that under a possibly more restrictive CFL condition, h_K^n belongs to J . By the mean value theorem, there exists $g_K^n \in I_\delta$ such that

$$h_K^{n+1} - h_K^n = \frac{\eta'(g_K^n)}{\eta(f_K^\infty)} (f_K^{n+1} - f_K^n).$$

The scheme can then be rewritten as

$$\begin{aligned} h_K^{n+1} &= \left(1 - \frac{\eta'(g_K^n) \Delta t}{\eta(f_K^\infty) m(K)} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma (a_{K,\sigma} + \eta(f_K^\infty)) \right) h_K^n \\ &+ \frac{\eta'(g_K^n) \Delta t}{\eta(f_K^\infty) m(K)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int},K} \\ \sigma = K|L}} \tau_\sigma (a_{K,\sigma} + \eta(f_K^\infty)) h_L^n \\ &+ \frac{\eta'(g_K^n) \Delta t}{\eta(f_K^\infty) m(K)} \sum_{\sigma \in \mathcal{E}_{\text{ext},K}} \tau_\sigma (a_{K,\sigma} + \eta(f_K^\infty)) h_\sigma^n. \end{aligned}$$

Under the CFL condition

$$\max_{K \in \mathcal{T}} \frac{\Delta t}{m(K)} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma \leq C_2(\delta) := \frac{m_\infty}{M_\eta^\delta (C_g V_\infty \text{diam}(\Omega) + M_\infty)},$$

it provides h_K^{n+1} as a convex combination of elements of J and hence $\{h_K^{n+1}\}_{K \in \mathcal{T}} \subseteq J$. The CFL constant C_{L^∞} can then be taken as the supremum of $\min(C_1(\delta), C_2(\delta))$ when $\delta > 0$.

(ii) : We proceed exactly as in the proof of Theorem 4.4.2 to get the existence of $\theta_K^{n,n+1}$ such that

$$\frac{H_\phi^{n+1} - H_\phi^n}{\Delta t} + D_\phi^n \leq -C_\phi^n + \frac{1}{2\Delta t} \sum_{K \in \mathcal{T}} \psi_K(\theta_K^{n,n+1})(f_K^{n+1} - f_K^n)^2 m(K), \quad (4.29)$$

for ψ_K defined by (4.25). Note that the sign of the last term has changed compared to the implicit scheme. Using the scheme, the last term in (4.29) can be estimated with the Cauchy-Schwartz inequality as

$$\begin{aligned} \frac{\Delta t}{2} \sum_{K \in \mathcal{T}} \frac{1}{m(K)} \psi_K(\theta_K^{n,n+1}) \left(\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^n \right)^2 &\leq \\ \frac{\Delta t M_\Phi M_\eta^0 (C_g V_\infty \text{diam}(\Omega) + M_\infty)^2}{m_\infty} \sum_{K \in \mathcal{T}} \frac{1}{m(K)} \left(\sum_{\sigma \in \mathcal{E}_K} \tau_\sigma \right) \left(\sum_{\sigma \in \mathcal{E}_K} \tau_\sigma (D_{K,\sigma} h^n)^2 \right), \end{aligned}$$

Then, using that

$$D_\phi^n \geq \frac{m_\Phi m_\infty}{2} \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma (D_{K,\sigma} h^n)^2,$$

it yields (4.28) provided that the CFL condition is satisfied with constant

$$C_{\text{ent}} = \frac{m_\infty^2}{2 M_\eta^0 (C_g V_\infty \text{diam}(\Omega) + M_\infty)^2}.$$

□

4.4.3 Long time behavior

In this section, we study the long-time behavior of the discrete solution in the linear case $\eta(s) = s$. We prove exponential decay to equilibrium with the following strategy. From a classical discrete Poincaré inequality, we establish a discrete Poincaré-Sobolev inequality for controlling H_{ϕ_2} by D_{ϕ_2} , where we recall that $\phi_2(s) = (s - 1)^2$. Then, properties (4.24) and (4.28) provide exponential decay to equilibrium by a discrete Gronwall-type argument for H_{ϕ_2} and all relative ϕ -entropies controlled by the latter.

Remark 4.4.5. *The question of the existence of a general ϕ -Poincaré-Sobolev functional inequality for any entropy generating function ϕ , even in the continuous setting, goes way beyond the scope of this paper and we refer to [24, 5] for discussions on this matter.*

The 2-entropy and its dissipation are closely related to, respectively, the L^2 norm

$$\|u\|_{0,2} = \left(\sum_{K \in \mathcal{T}} m(K) |u_K|^2 \right)^{1/2},$$

and the discrete H^1 semi-norm

$$|u|_{1,2,\mathcal{T}} = \left(\sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma |D_{K,\sigma} u|^2 + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{ext},K}} \tau_\sigma |D_{K,\sigma} u|^2 \right)^{1/2}, \quad (4.30)$$

for which M. Bessemoulin-Chatard, C. Chainais-Hillairet and F. Filbet proved the following discrete Poincaré inequality in [22, Theorem 6], based on older work that may be found in references therein. Before stating it, we need to introduce the following regularity constraint on the mesh. There is a positive constant ξ_1 that does not depend on \mathcal{D} such that

$$\forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K, \quad d_{K,\sigma} \geq \xi_1 d_\sigma \quad (\text{H8})$$

Proposition 4.4.6 ([22]). *Under hypothesis (H8), there exists a constant C_P only depending on Ω such that for all $u \in X_0$, it holds*

$$\|u\|_{0,2} \leq \frac{C_P}{\xi_1^{1/2}} |u|_{1,2,\mathcal{T}}.$$

As a consequence we have the following chain of inequalities. For any $h \in X_1$

$$H_{\phi_2}(h) \leq M_\infty \|h - 1\|_{0,2}^2 \leq \frac{C_P^2 M_\infty}{\xi_1} |h|_{1,2,\mathcal{T}}^2 \leq \frac{C_P^2 M_\infty}{\xi_1 m_\infty} D_{\phi_2}(h). \quad (4.31)$$

Theorem 4.4.7. *(Exponential return to equilibrium)*

We suppose that η is the identity function, and that the assumptions of Theorem 4.4.2 (respectively those of Theorem 4.4.4 as well as the CFL condition (4.27) for $\Phi = \{\phi_2\}$)

and (H8) are satisfied. In the implicit case, we also assume that Δt is smaller than a given constant k . Then, the solution f of the scheme (4.16),(4.18)-(4.20) (respectively (4.18)-(4.20)) is such that there are two positive constants, κ depending only on Ω , m_∞ , M_∞ , ξ_1 , and C_{in} depending additionally on $H_{\phi_2}^0$ (and k in the implicit case) such that for all $n \in \{0, \dots, N_T - 1\}$,

$$H_{\phi_2}^n \leq C_{in} e^{-\kappa t^n}.$$

As a consequence, for all $n \in \{0, \dots, N_T - 1\}$, it holds

$$\|f_\delta(t^n, \cdot) - f_\delta^\infty(\cdot)\|_{L^1(\Omega)}^2 \leq C_{in} e^{-\kappa t^n}.$$

Proof. By combining (4.31) and (4.24) (respectively (4.28)), we obtain $H_{\phi_2}^{n+1} \leq (1 + \kappa \Delta t)^{-1} H_{\phi_2}^n$ in the implicit case with $\kappa = \xi_1 m_\infty / (C_P^2 M_\infty)$ and $H_{\phi_2}^{n+1} \leq (1 - \kappa \Delta t) H_{\phi_2}^n$ in the explicit case with $\kappa = (1 - \varepsilon) \xi_1 m_\infty / (C_P^2 M_\infty)$ where ε is in the CFL condition (4.27). Hence we get exponential decay of the 2-entropy, with $C_{in} = H_{\phi_2}^0$ in the explicit case and $C_{in} = H_{\phi_2}^0 \exp(\kappa^2 k^2 / 2)$ in the implicit case. Finally, the estimate in L^1 is obtained from a Cauchy-Schwartz inequality

$$\|f_\delta(t^n) - f_\delta^\infty\|_{L^1(\Omega)}^2 \leq \|f_\delta^\infty\|_{L^1(\Omega)} \|f_\delta(t^n) / \sqrt{f_\delta^\infty} - \sqrt{f_\delta^\infty}\|_{L^2(\Omega)}^2 = \|f_\delta^\infty\|_{L^1(\Omega)} H_{\phi_2}^n,$$

and the decay of H_{ϕ_2} . □

Remark 4.4.8. The exponential decay holds for many other ϕ -entropies. For instance, let us restrict our class of entropy generating functions to that introduced by Arnold, Markowich, Toscani and Unterreiter in [5], that is those satisfying $\phi \in C^4(\mathbb{R}_+)$ and

$$\left(\phi'''\right)^2 \leq \frac{1}{2} \phi'' \phi^{IV}.$$

Mark that the physical and p -entropies are generated by these entropy functions. As a consequence of [5, Lemma 2.6], one has that any such ϕ is bounded from above by a quadratic entropy function, namely

$$\frac{\phi(s)}{\phi''(1)} \leq \phi_2(s) = (s - 1)^2.$$

Therefore, since the same inequality holds for the corresponding relative entropies, namely $H_\phi \leq \phi''(1) H_{\phi_2}$, the exponential decay holds for any relative ϕ -entropy in this class.

4.5 Convergence

In this section, we prove that when $\delta \rightarrow 0$, the sequence $(h_\delta)_{\delta > 0}$ converges to a solution of (4.5) in the following sense.

Definition 4.5.1. We say that h is a solution of (4.5) starting at initial data f^{in} if

- $h - 1 \in L^2(0, T; H_0^1(\Omega))$
- $f = \eta^{-1}(\eta(f^\infty)h) \in L^1(\Omega_T)$
- For any test function $\psi \in \mathcal{C}_c^\infty(\Omega_T)$, it holds

$$-\iint_{\Omega_T} [f \partial_t \psi + (\mathbf{U}^\infty h - \eta(f^\infty) \nabla h) \cdot \nabla \psi] - \int_{\Omega} f^{\text{in}} \psi(0, \cdot) = 0.$$

Thanks to the reformulation of Equation (4.1) into (4.5), the only nonlinearity lies in the first term. In order to prove convergence of the scheme we need strong compactness to recover pointwise convergence and identify the limit of $(f_\delta)_\delta$ as $\eta^{-1}(\eta(f^\infty)h)$ with h the limit of $(h_\delta)_\delta$.

Except for this originality, the strategy is fairly standard. We first derive uniform-in- δ estimates on h_δ and its (discrete) gradient in \mathbf{x} in order to get compactness in the space variable. Then thanks to the structure of the equation compactness in time is obtained from previous estimates. Using the consistency we shall then take limits in our scheme and recover a global weak solution in the sense of Definition 4.5.1.

4.5.1 Hypotheses

We discuss specific hypotheses needed for the convergence result to hold. While, it was not crucial until now, the orthogonality condition (H6) is very important in this part, for consistency of discrete gradients. As in [47, 48, 21], the latter are introduced on a dual mesh. For $\sigma = K|L$, we define T_σ as the cell with vertices $\mathbf{x}_K, \mathbf{x}_L$ and those of σ . If $\sigma \in \mathcal{E}_{\text{ext}}$ then T_σ is the cell with vertices \mathbf{x}_K and those of σ , where K is the only cell having $\sigma \in \mathcal{E}_K$. We refer to [48, Fig. 1] for a visualization. On this dual mesh we set

$$\nabla^\delta h_\delta(t, \mathbf{x}) = \begin{cases} \frac{m(\sigma)}{m(T_\sigma)} D_{K,\sigma} h^{n+1} \mathbf{n}_{K,\sigma}, & \text{if } \mathbf{x} \in T_\sigma \text{ and } t \in [t^n, t^{n+1}) \text{ (Implicit case)} \\ \frac{m(\sigma)}{m(T_\sigma)} D_{K,\sigma} h^n \mathbf{n}_{K,\sigma}, & \text{if } \mathbf{x} \in T_\sigma \text{ and } t \in [t^n, t^{n+1}) \text{ (Explicit case)} \end{cases}$$

Remark 4.5.2. *The discrete gradients are well-defined since for $u \in X_{u_b}$ and $\sigma = K|L$, the product $D_{K,\sigma} u \mathbf{n}_{K,\sigma}$ is independent of the cell K . For $\sigma \in \mathcal{E}_{\text{ext}}$, there is no ambiguity since only one cell is sharing the edge.*

In order to obtain convergence of these gradients we need the following regularity hypothesis on the mesh (see [48] and references therein for related comments). There exist a constant $\xi_2 > 1$ that does not depend on the discretization such that

$$\forall \sigma \in \mathcal{E}, \quad m(T_\sigma) \leq m(\sigma) d_\sigma \leq \xi_2 m(T_\sigma). \quad (\text{H9})$$

Since our scheme is based on a discretization of the stationary equation (4.2), one needs the convergence of the latter to obtain that of the former. Therefore we assume that

both the discrete steady state and the discrete velocity converge to their continuous counterparts. First, let us define them on the dual mesh, by

$$\eta(f^\infty)_\delta(\mathbf{x}) = \eta(f^\infty)_\sigma \quad \text{for } \mathbf{x} \in T_\sigma,$$

and

$$\mathbf{U}_\delta^\infty(\mathbf{x}) = \frac{m(\sigma)d_\sigma}{m(T_\sigma)} U_{K,\sigma}^\infty \mathbf{n}_{K,\sigma} \quad \text{for } \mathbf{x} \in T_\sigma.$$

Then, when $\delta \rightarrow 0$, we assume that

$$\begin{cases} \eta(f^\infty)_\delta & \longrightarrow \eta(f^\infty) & \text{strongly in } L^2(\Omega) \\ \mathbf{U}_\delta^\infty & \longrightarrow \mathbf{U}^\infty & \text{weakly in } L^2(\Omega) \end{cases} \quad (\text{H10})$$

Remark 4.5.3. *Hypothesis (H10) is very natural. Indeed, since $\eta(f^\infty)$ satisfies a linear non-degenerate second order elliptic equation, one expects to get uniform H^1 control on discrete solutions of any reasonable finite volume scheme.*

In the rest of Section 4.5, all results are stated for the implicit scheme. However, everything holds also for the explicit scheme with minor modifications provided that one adds hypothesis (H7) and the CFL condition (4.27) with $\Phi = \{\phi_2\}$ to each proposition.

4.5.2 Compactness

In the following proposition, we provide the uniform L^∞ and $L^2(0, T; H^1)$ estimates we need for compactness. The discrete $L^2(0, T; H^1)$ norm is defined by

$$\|h_\delta\|_{1,2,\mathcal{D}} = \left(\sum_{n=0}^{N_T-1} \Delta t |h^{n+1}|_{1,2,\mathcal{T}}^2 \right)^{1/2}$$

for the implicit scheme.

Proposition 4.5.4. *(Uniform estimates) Under hypotheses (H1)-(H6) and (H9)-(H10), there are positive constants K_∞ and $K_{1,2}$ depending only on K_{in} , η , m_∞ and M_∞ such that the solution of the scheme (4.16) together with (4.18)-(4.20) satisfies*

$$0 \leq f_\delta(t, \mathbf{x}), h_\delta(t, \mathbf{x}) \leq K_\infty,$$

and

$$\|h_\delta\|_{1,2,\mathcal{D}} \leq K_{1,2}.$$

Proof. The first estimate is proved in Theorem 4.4.2 and 4.4.4. The second estimate is a consequence of the boundedness of the dissipation of 2-entropy and of the lower bound on the steady state since

$$m_\infty |h^n|_{1,2,\mathcal{T}}^2 \leq D_{\phi_2}^n.$$

By summing (4.24) over n and inserting the above inequality one obtains the result. \square

From the first result of Proposition 4.5.4 one obtains weak- \star compactness in L^∞ . Now, we need to gain weak compactness on the sequence of discrete gradients as well as strong compactness on the sequence of discrete solutions. The strong compactness is obtained by the Riesz-Fréchet-Kolmogorov criterion on spacetime translates. In order to recover right boundary conditions at the limit the criterion is stated on the whole space \mathbb{R}^{d+1} for \tilde{h}_δ defined by

$$\tilde{h}_\delta(t, x) = \begin{cases} h_\delta(t, \mathbf{x}) - 1 & \text{if } (t, x) \in \Omega_T, \\ 0 & \text{if } (t, x) \in \mathbb{R}^{d+1} \setminus \overline{\Omega_T}. \end{cases}$$

The proof of the following lemma can be readily adapted from [85, Lemma 4.3 and 4.7].

Lemma 4.5.5. *Under hypotheses (H1)-(H6) and (H9)-(H10), there is a positive constant K_2 depending only on $K_\infty, K_{1,2}, \Omega, T$ and η such that the solution of the scheme (4.16) together with (4.18)-(4.20) satisfies for all $\mathbf{y} \in \mathbb{R}^d$*

$$\|\tilde{h}_\delta(\cdot, \cdot + \mathbf{y}) - \tilde{h}_\delta(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})}^2 \leq K_2 |\mathbf{y}| (|\mathbf{y}| + \delta),$$

and for all $\tau \in (0, T)$

$$\|\tilde{h}_\delta(\cdot + \tau, \cdot) - \tilde{h}_\delta(\cdot, \cdot)\|_{L^2((0, T-\tau) \times \mathbb{R}^d)}^2 \leq K_2 |\tau|.$$

From the previous results we obtain the following convergences.

Proposition 4.5.6. *There is a function $h \in L^\infty(0, T; H^1(\Omega))$ such that as $\delta \rightarrow 0$ and up to the extraction of a subsequence,*

$$\begin{aligned} h_\delta &\longrightarrow h && \text{strongly in } L^2(\Omega_T), \\ \nabla^\delta h_\delta &\longrightarrow \nabla h && \text{weakly in } (L^2(\Omega_T))^d, \\ f_\delta &\longrightarrow f && \text{strongly in } L^1(\Omega_T), \end{aligned}$$

with $f(t, \mathbf{x}) = \eta^{-1}(\eta(f^\infty(t, \mathbf{x})) h(t, \mathbf{x}))$, (t, \mathbf{x}) -almost everywhere in Ω_T . Moreover $h - 1 \in L^\infty(0, T; H_0^1(\Omega))$.

Proof. The first result is a consequence of the Riesz-Fréchet-Kolmogorov L^p compactness criterion. It yields the convergence of $(\tilde{h}_\delta)_{\delta>0}$ to a function \tilde{h} strongly in $L^2([0, T] \times \mathbb{R}^d)$. By taking limits in the first space translates estimate, we know that $\tilde{h} \in L^2(0, T; H^1(\mathbb{R}^d))$. Since by definition $\tilde{h}_\delta(t, \cdot) \equiv 0$ on $\mathbb{R}^d \setminus \overline{\Omega}$, we get that $\tilde{h} \in L^2(0, T; H_0^1(\Omega))$. Hence $(h_\delta)_\delta$ converges strongly in $L^2(\Omega_T)$ to $h = 1 + \tilde{h}$ and up to the extraction of a sparser subsequence, it also converges almost everywhere. Since h_δ and $\eta(f^\infty)_\delta$ are uniformly bounded, η^{-1} is continuous and the cylinder Ω_T is bounded, one can apply the dominated convergence theorem to obtain the convergence of f_δ towards f . For the weak convergence of discrete gradients we refer to [47, Lemma 4.4] with minor modifications. \square

4.5.3 Convergence of the scheme

Theorem 4.5.7. *Under hypotheses (H1)-(H6) and (H9)-(H10), the function h defined in Proposition 4.5.6 is a solution of Equation (4.5) in the sense of Definition 4.5.1.*

Let us mention that we follow hereafter the methods of proof from [47, 48].

Proof. Let us consider a test function $\psi \in \mathcal{C}_c^\infty([0, T] \times \Omega)$ and set $\psi_K^n = \psi(t^n, \mathbf{x}_K)$ for all $K \in \mathcal{T}$ and $n = 0, \dots, N_T$. We suppose that the mesh size δ is sufficiently small for the inclusion $\text{supp}(\psi) \subset [0, (N_T - 1)\Delta t] \times \{\mathbf{x} \in \Omega, d(\mathbf{x}, \Gamma) > \delta\}$ to hold. With this assumption sums on exterior edges disappear as one may remark in various terms hereafter. Let us define

$$\begin{cases} T_{10}(\delta) = - \iint_{\Omega_T} f_\delta(t, \mathbf{x}) \partial_t \psi(t, \mathbf{x}) \, d\mathbf{x} \, dt - \int_{\Omega} f_\delta^{\text{in}}(\mathbf{x}) \psi(0, \mathbf{x}) \, d\mathbf{x}, \\ T_{20}(\delta) = \iint_{\Omega_T} \eta(f^\infty)_\delta \nabla^\delta h_\delta \cdot \nabla \psi \, d\mathbf{x} \, dt, \\ T_{30}(\delta) = - \iint_{\Omega_T} h_\delta \mathbf{U}_\delta^\infty \cdot \nabla \psi \, d\mathbf{x} \, dt. \end{cases}$$

From the results of Proposition 4.5.6 and assumption (H10), it is clear that

$$T_{10}(\delta) + T_{20}(\delta) + T_{30}(\delta) \longrightarrow - \iint_{[0, T] \times \Omega} [f \partial_t \psi + (h \mathbf{U}^\infty - \eta(f^\infty) \nabla h) \cdot \nabla \psi] - \int_{\Omega} f^{\text{in}} \psi(0, \cdot)$$

as $\delta \rightarrow 0$. Now let us show that it also converges to 0. By multiplying the scheme by $\Delta t \psi_K^n$ and summing over n and K we obtain

$$T_1(\delta) + T_2(\delta) + T_3(\delta) = 0,$$

with

$$\begin{cases} T_1(\delta) = \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}} m(K) (f_K^{n+1} - f_K^n) \psi_K^n, \\ T_2(\delta) = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \eta(f^\infty)_\sigma (h_K^{n+1} - h_L^{n+1}) \psi_K^n, \\ T_3(\delta) = \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) \left[U_{K, \sigma}^{\infty, +} g(h_K^{n+1}, h_L^{n+1}) - U_{K, \sigma}^{\infty, -} g(h_L^{n+1}, h_K^{n+1}) \right] \psi_K^n. \end{cases}$$

Let us show that each $T_i(\delta)$ gets asymptotically close to $T_{i0}(\delta)$, for $i = 1, 2$ or 3 , as δ

goes to 0. After a discrete integration by parts, one gets

$$\begin{aligned} T_1(\delta) &= \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}} m(K) f_K^{n+1} (\psi_K^n - \psi_K^{n+1}) - \sum_{K \in \mathcal{T}} m(K) f_K^0 \psi_K^0, \\ &= - \sum_{n=0}^{N_T} \sum_{K \in \mathcal{T}} \int_{t^n}^{t^{n+1}} \int_K f_K^{n+1} \partial_t \psi(t, \mathbf{x}_K) \, d\mathbf{x} \, dt - \sum_{K \in \mathcal{T}} \int_K f_K^0 \psi(0, \mathbf{x}_K) \, d\mathbf{x}, \end{aligned}$$

which yields

$$|T_1(\delta) - T_{10}(\delta)| \leq \delta (T + 1) m(\Omega) \|f_\delta\|_{L^\infty(\Omega_T)} \|\psi\|_{\mathcal{C}^2(\overline{\Omega_T})} \rightarrow 0,$$

as $\delta \rightarrow 0$ by Proposition 4.5.4. Concerning the diffusion term one has, integrating by parts

$$T_2(\delta) = \sum_{n=0}^{N_T} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \eta(f_\sigma^\infty) (h_K^{n+1} - h_L^{n+1}) (\psi_K^n - \psi_L^n)$$

and

$$T_{20}(\delta) = \sum_{n=0}^{N_T} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) \eta(f_\sigma^\infty) (h_L^{n+1} - h_K^{n+1}) \int_{t^n}^{t^{n+1}} \frac{1}{m(T_\sigma)} \int_{T_\sigma} \nabla \psi \cdot \mathbf{n}_{K,\sigma} \, d\mathbf{x} \, dt.$$

Now just note that by the orthogonality hypothesis (H6) on the mesh and the regularity of ψ one has,

$$\left| \psi_L^n - \psi_K^n - \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \frac{1}{m(T_\sigma)} \int_{T_\sigma} \nabla \psi \cdot d_\sigma \mathbf{n}_{K,\sigma} \, d\mathbf{x} \, dt \right| \leq \|\psi\|_{\mathcal{C}^2(\overline{\Omega_T})} \delta. \quad (4.32)$$

Therefore, with a Cauchy-Schwartz inequality and thanks to the regularity of the mesh one obtains

$$|T_2(\delta) - T_{20}(\delta)| \leq \delta \sqrt{\xi_2 T} \|\eta(f^\infty)_\delta\|_{L^2(\Omega)} \|h_\delta\|_{1,2,\mathcal{D}} \|\psi\|_{\mathcal{C}^2(\overline{\Omega_T})} \rightarrow 0,$$

as $\delta \rightarrow 0$, by Proposition 4.5.4 and assumption (H10). Finally let us deal with the convection term. As it is classical we transform the upwind form of $T_3(\delta)$ into the sum of a numerical diffusion and a centered flux yielding $T_3(\delta) = T_{31}(\delta) + T_{32}(\delta)$, with

$$\begin{aligned} T_{31}(\delta) &= \frac{1}{2} \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) |U_{K,\sigma}^\infty| (g(h_K^{n+1}, h_L^{n+1}) - g(h_L^{n+1}, h_K^{n+1})) \psi_K^n, \\ &= \frac{1}{2} \sum_{n=0}^{N_T} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) |U_{K,\sigma}^\infty| (g(h_K^{n+1}, h_L^{n+1}) - g(h_L^{n+1}, h_K^{n+1})) (\psi_K^n - \psi_L^n), \end{aligned}$$

whereas $T_{32}(\delta)$ is

$$\begin{aligned} T_{32}(\delta) &= \frac{1}{2} \sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) U_{K,\sigma}^\infty (g(h_K^{n+1}, h_L^{n+1}) + g(h_L^{n+1}, h_K^{n+1})) \psi_K^n, \\ &= \sum_{n=0}^{N_T} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) U_{K,\sigma}^\infty g(h_K^{n+1}, h_L^{n+1}) (\psi_K^n - \psi_L^n), \\ &= T_{321}(\delta) + T_{322}(\delta), \end{aligned}$$

with $T_{321}(\delta)$ and $T_{322}(\delta)$ given by

$$\begin{aligned} T_{321}(\delta) &= \sum_{n=0}^{N_T} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) U_{K,\sigma}^\infty (g(h_K^{n+1}, h_L^{n+1}) - h_K^{n+1}) (\psi_K^n - \psi_L^n), \\ T_{322}(\delta) &= \sum_{n=0}^{N_T} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) U_{K,\sigma}^\infty h_K^{n+1} (\psi_K^n - \psi_L^n). \end{aligned}$$

Similarly we introduce the decomposition $T_{30}(\delta) = T_{310}(\delta) + T_{320}(\delta)$ with

$$\begin{aligned} T_{310}(\delta) &= - \sum_{n=0}^{N_T} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \int_{t^n}^{t^{n+1}} \int_{T_\sigma \cap L} (h_L^{n+1} - h_K^{n+1}) \mathbf{U}_\delta^\infty \cdot \nabla \psi \, \mathbf{d}\mathbf{x} \, dt, \\ T_{320}(\delta) &= - \sum_{n=0}^{N_T} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \int_{t^n}^{t^{n+1}} \int_{T_\sigma} h_K^{n+1} \mathbf{U}_\delta^\infty \cdot \nabla \psi \, \mathbf{d}\mathbf{x} \, dt. \end{aligned}$$

Let us prove that $T_{31}(\delta)$, $T_{310}(\delta)$, $T_{321}(\delta)$ and $T_{322}(\delta) - T_{320}(\delta)$ converge to zero when δ goes to zero. First, notice that using the Lipschitz continuity of g , the definition of \mathbf{U}_δ^∞ and eventually the Cauchy-Schwartz inequality yields

$$|T_{321}(\delta)| + |T_{31}(\delta)| \leq 2\delta \|g\|_{W^{1,\infty}([0, \|h_\delta\|_{L^\infty}]^2)} \|\psi\|_{C^1(\overline{\Omega_T})} \|h_\delta\|_{1,2,\mathcal{D}} \|\mathbf{U}_\delta^\infty\|_{L^2(\Omega_T)}.$$

Then, similarly one obtains

$$|T_{310}(\delta)| \leq \delta \|\psi\|_{C^1(\overline{\Omega_T})} \|h_\delta\|_{1,2,\mathcal{D}} \|\mathbf{U}_\delta^\infty\|_{L^2(\Omega_T)},$$

and by Proposition 4.5.4 and hypotheses (H5) (H10) both right-hand sides go to 0 as δ goes to 0. Finally

$$\begin{aligned} T_{322}(\delta) - T_{320}(\delta) &= \\ &= \sum_{n=0}^{N_T} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) U_{K,\sigma}^\infty h_K^{n+1} \int_{t^n}^{t^{n+1}} \left(\psi_K^n - \psi_L^n - \frac{1}{m(T_\sigma)} \int_{T_\sigma} \nabla \psi \cdot d_\sigma \mathbf{n}_{K,\sigma} \, \mathbf{d}\mathbf{x} \right) dt, \end{aligned}$$

and by using (4.32) we obtain,

$$|T_{322}(\delta) - T_{320}(\delta)| \leq \delta \sqrt{T m(\Omega)} \|\psi\|_{C^2(\overline{\Omega_T})} \|h_\delta\|_{L^\infty(\Omega_T)} \|\mathbf{U}_\delta^\infty\|_{L^2(\Omega_T)}.$$

Hence, $|T_i(\delta) - T_{i0}(\delta)| \rightarrow 0$ as $\delta \rightarrow 0$, for any $i \in \{1, 2, 3\}$. \square

4.6 Numerical simulations

4.6.1 Implementation

Before presenting our numerical results let us state an important remark concerning the implementation of our scheme. By Theorem 4.4.7, we expect the solution h_δ of the scheme (4.16),(4.18)-(4.20) or (4.17)-(4.20) to converge to 1 when time goes to infinity. Due to floating point numbers repartition there can be non-negligible numerical errors in the computation of difference operators leading to saturation of ϕ -entropies. To avoid this issue, the scheme should be implemented in the following way. We introduce new unknowns defined by,

$$\tilde{f}_K^n = f_K^n - f_K^\infty, \quad \tilde{h}_K^n = h_K^n - 1,$$

for $n \in \{0, \dots, N_T\}$ and $K \in \mathcal{T}$. One can readily check that in the upwind case $g(s, t) = s$, thanks to (H3), the schemes remain unchanged by replacing f by \tilde{f} and h by \tilde{h} . Even for other g , this modified scheme is the discretization of

$$\frac{\partial}{\partial t}(f - f^\infty) + \nabla \cdot (\mathbf{U}^\infty (h - 1) - \eta(f^\infty) \nabla (h - 1)) = 0,$$

which is the same as (4.5) since f^∞ is steady and \mathbf{U}^∞ is incompressible.

The new unknowns converge to 0 when time goes to infinity and thus differences are computed with a better precision. Hence one should solve the scheme (and compute ϕ -entropies, dissipations, *etc...*) on the new unknowns. Moreover boundary conditions on \tilde{h} become homogeneous, which actually makes the implementation easier.

In the following test cases, the implicit scheme (4.16),(4.18)-(4.20) is used only for linear models in Section 4.6.2 - 4.6.4. The nonlinear model in Section 4.6.5 is solved with the explicit scheme (4.17)-(4.20). For the first test cases we use the linear solver SuperLU [140], which provides efficient results for large sparse and non-symmetric systems by performing a sparsity-preserving LU factorization. Since the advection field is steady in Equation (4.5), the resolution matrix can be factorized only once at the beginning of the simulation.

4.6.2 Proof of concept

In this part, we provide a numerical experiment showing the spatial accuracy of our scheme, especially in the long-time dynamics. It is performed on the following one dimensional toy model. The test case is the linear ($\eta(s) = s$) drift-diffusion equation (4.1)

endowed with the (scalar) advection $E(x) = 1$ and set on the domain $\Omega = (0, 1)$. With the boundary conditions $f(t, 0) = 2$ and $f(t, 1) = 1 + \exp(1)$ the function

$$f(t, x) = 1 + \exp(x) + \exp\left(\frac{x}{2} - \left(\pi^2 + \frac{1}{4}\right)t\right) \sin(\pi x)$$

is the exact solution of (4.1) and converges to the stationary state

$$f^\infty(x) = 1 + \exp(x),$$

as time goes to infinity.

In order to illustrate the advantage of our approach compared to the one consisting in a direct approximation of (4.1), we perform numerical simulations using our scheme (4.16), (4.18)-(4.20) and a classical finite volume discretization of (4.1) with upwind flux for the convective term and two-points approximation of the gradient for the diffusive term. Both schemes are implicit in time and we shall call the former “ ϕ -entropic” and the latter “upwind”.

The domain is discretized by the regular Cartesian mesh $\mathcal{T} = \{K_i := (x_i - \Delta x/2, x_i + \Delta x/2), i = 0, \dots, N - 1\}$ where $x_i = \Delta x/2 + i\Delta x$ and $\Delta x = 1/N$. Concerning the time discretization, the final time is $T = 5$ and we choose the small time step $\Delta t = 10^{-6}$ in order to minimize the error due to time discretization. For the implementation of our scheme we explicitly compute both the discrete steady solution and the steady flux.

We define the L^p error at time t between the reconstruction of the approximate solution f_N (corresponding to f_δ with previous notation) and the projection of the analytic solution on the mesh by

$$\epsilon_N^p(t) = \|\Pi_N f(t, \cdot) - f_N(t, \cdot)\|_{L^p(\Omega)}$$

where $\Pi_N f(t, x) = f_i^n$ if $(t, x) \in [t^n, t^{n+1}) \times K_i$ with f_i^n a numerical approximation of the average of $f(t^{n+1}, \cdot)$ on the cell K_i , computed with the trapezoidal rule. In Table 4.1, we measure, for both schemes and for different number of points N , the global error and experimental order of accuracy, respectively given by

$$e_N^p = \sup_{t \in [0, T)} \epsilon_N^p(t), \quad k_{2N}^p = |\log(e_{2N}) - \log(e_N)| / \log(2).$$

N	Error e_N^1 ϕ -entropic	Order	Error e_N^1 Upwind	Order	Error e_N^∞ ϕ -entropic	Order	Error e_N^∞ Upwind	Order
20	$2.07 \cdot 10^{-3}$		$4.28 \cdot 10^{-3}$		$3.33 \cdot 10^{-3}$		$7.38 \cdot 10^{-3}$	
40	$1.21 \cdot 10^{-3}$	0.77	$2.36 \cdot 10^{-3}$	0.86	$1.93 \cdot 10^{-3}$	0.79	$4.03 \cdot 10^{-3}$	0.87
80	$6.45 \cdot 10^{-4}$	0.91	$1.24 \cdot 10^{-3}$	0.93	$1.02 \cdot 10^{-3}$	0.91	$2.11 \cdot 10^{-3}$	0.94
160	$3.30 \cdot 10^{-4}$	0.97	$6.30 \cdot 10^{-4}$	0.97	$5.22 \cdot 10^{-4}$	0.97	$1.07 \cdot 10^{-3}$	0.98
320	$1.64 \cdot 10^{-4}$	1.01	$3.15 \cdot 10^{-4}$	1.00	$2.59 \cdot 10^{-4}$	1.01	$5.31 \cdot 10^{-4}$	1.00
640	$7.87 \cdot 10^{-5}$	1.06	$1.55 \cdot 10^{-4}$	1.03	$1.26 \cdot 10^{-4}$	1.06	$2.61 \cdot 10^{-4}$	1.02
1280	$3.57 \cdot 10^{-5}$	1.14	$7.38 \cdot 10^{-5}$	1.07	$5.65 \cdot 10^{-5}$	1.14	$1.25 \cdot 10^{-4}$	1.06

TABLE 4.1 – **Proof of concept.** Experimental spatial order of convergence in L^1 and L^∞ .

Both schemes are first order accurate, but we observe that our ϕ -entropic scheme (4.16), (4.18)-(4.20) is performing better since the numerical error is smaller than the classical upwind scheme. Furthermore in Figure 4.1, we observe that for large time the numerical error corresponding to the entropy preserving scheme (4.16),(4.18)-(4.20) decays to zero and the error becomes negligible compared to that of the upwind scheme.

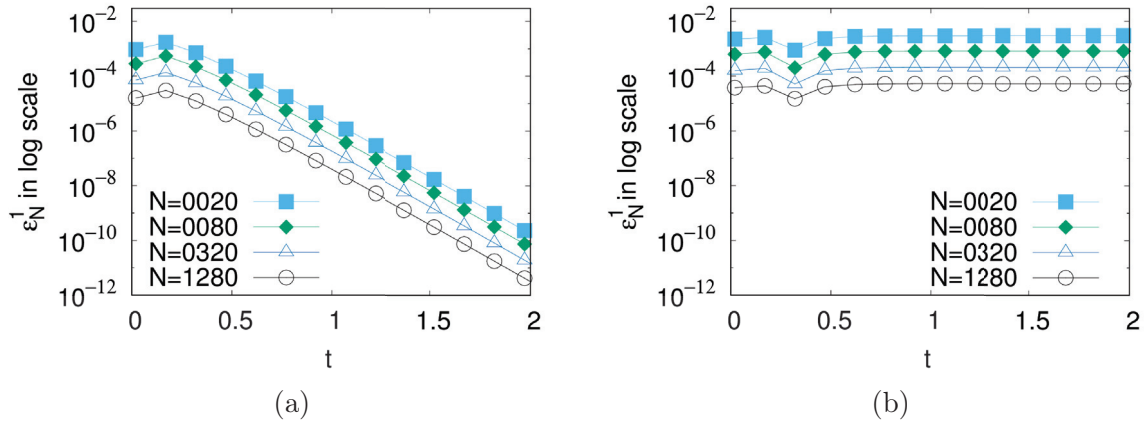


FIGURE 4.1 – **Proof of concept.** Time evolution of the ϵ_N^1 error for (a) the ϕ -entropic scheme and (b) the classical upwind scheme.

The accurate long-time behavior is confirmed by Figure 4.2 where the time variation of the distance to the discrete solution is represented. While the upwind scheme saturates quickly, our scheme reproduces perfectly the exponential decay to 0 of the solution, even if $N = 40$, illustrating the result of Theorem 4.4.7.

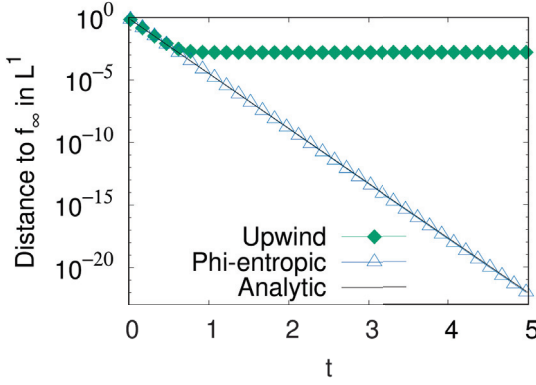


FIGURE 4.2 – **Proof of concept.** Time evolution of the $L^1(\Omega)$ distance to the discrete steady state for $N = 40$.

4.6.3 Fokker-Planck with magnetic field

We now consider the two-dimensional version of homogeneous Fokker-Planck equation with an external magnetic field

$$\begin{cases} \frac{\partial f}{\partial t} + b \mathbf{v}^\perp \cdot \nabla_{\mathbf{v}} f = \nabla_{\mathbf{v}} \cdot (\mathbf{v} f + \nabla_{\mathbf{v}} f) & \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \\ f(t=0) = f_0 & \text{in } \mathbb{R}^2. \end{cases}$$

The external magnetic field is along a third direction that is orthogonal to the plane under consideration and has amplitude $b = 4$. Compared to the 3D case, the vector $\mathbf{v}^\perp = (v_y, -v_x)$ replaces the cross product between \mathbf{v} and the direction of the magnetic field. The initial datum f_0 is given by the sum of two Gaussian distributions

$$f_0(\mathbf{v}) = \frac{1}{2\pi} \left[\alpha \exp\left(-\frac{|\mathbf{v} - \mathbf{v}_1|^2}{2}\right) + (1 - \alpha) \exp\left(-\frac{|\mathbf{v} - \mathbf{v}_2|^2}{2}\right) \right],$$

with $\alpha = 3/4$, $\mathbf{v}_1 = (-1, 2)$ and $\mathbf{v}_2 = (2, -1)$.

This equation is solved numerically in a bounded domain $\Omega = (-8, 8)^2$ on various regular Cartesian meshes from $N = 40^2$ to $N = 640^2$ points. We use our implicit scheme (4.16),(4.18)-(4.20) with a time step $\Delta t = 0.001$ until the final time $T = 10$. We choose non homogeneous Dirichlet boundary conditions $f^b = f^\infty$, where f^∞ is the steady state, that is, the Maxwellian distribution

$$f^\infty(\mathbf{v}) = \frac{1}{2\pi} \exp\left(-\frac{|\mathbf{v}|^2}{2}\right).$$

Here the knowledge of the steady state f^∞ allows us to compute the steady flux $F_{K,\sigma}^\infty$ analytically. Indeed, since $\mathbf{v} f^\infty + \nabla_{\mathbf{v}} f^\infty = 0$, one only needs to evaluate

$$F_{K,\sigma}^\infty = b \int_{\sigma} \mathbf{n}_{K,\sigma} \cdot \mathbf{v}^\perp f^\infty = b \int_{\sigma} \mathbf{n}_{K,\sigma}^\perp \cdot \nabla_{\mathbf{v}} f^\infty,$$

which is, up to a multiplicative constant, the difference between f^∞ evaluated at each endpoint of σ .

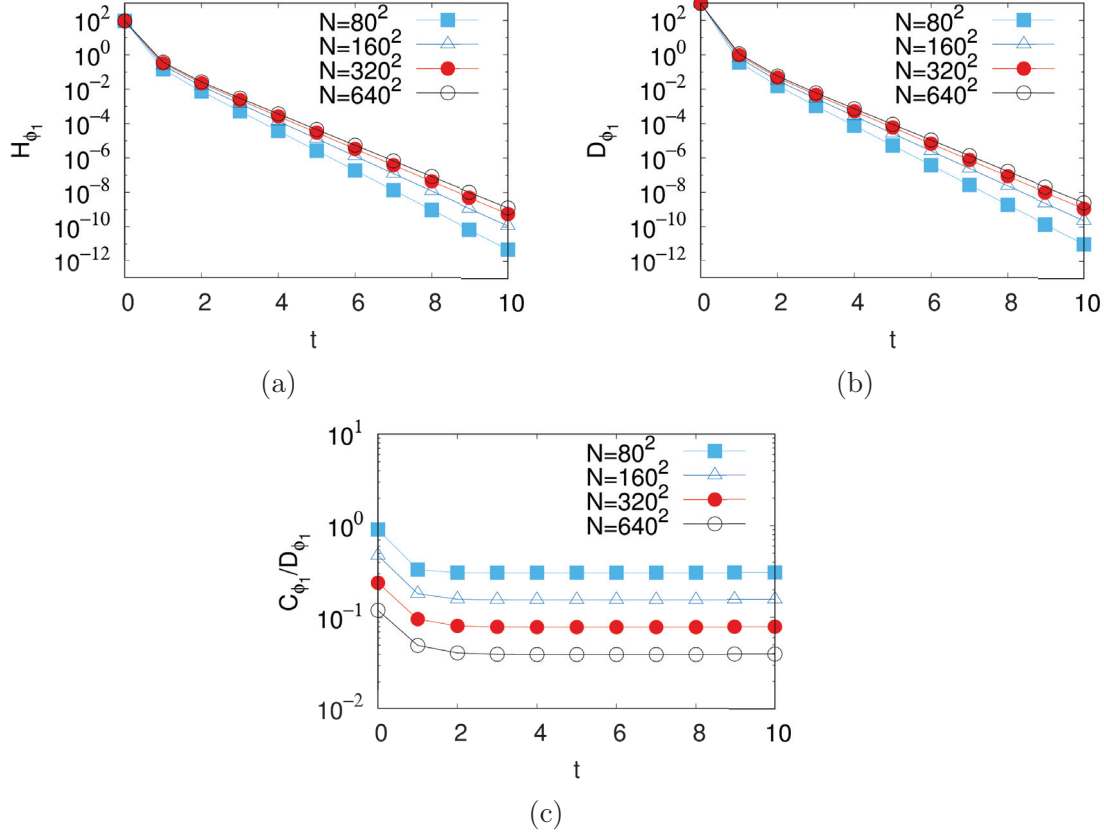


FIGURE 4.3 – **Fokker-Planck equation with magnetic field.** Time evolution of (a) the entropy H_{ϕ_1} (b) the physical dissipation D_{ϕ_1} and (c) the (normalized) numerical dissipation C_{ϕ_1}/H_{ϕ_1} .

We recall that the physical entropy generating function is $\phi_1 : s \mapsto s \log(s) - s + 1$. In Figure 4.3, we represent the time evolution of the entropy H_{ϕ_1} , the physical dissipation D_{ϕ_1} and the ratio between the numerical and physical dissipation C_{ϕ_1}/D_{ϕ_1} in log scale. On one hand, H_{ϕ_1} and D_{ϕ_1} decay exponentially and are well approximated when N is larger than 160^2 . Figure 4.3 (a) and (b) also illustrate the convergence to equilibrium at exponential rate, when time goes to infinity. On the other hand, the numerical dissipation C_{ϕ_1} converges to zero when the space step goes to zero, but since the scheme is only first order accurate, it is relatively slow. Besides, the numerical dissipation C_{ϕ_1} also converges to zero at the same exponential rate than D_{ϕ_1} , hence it does not affect the accuracy on the decay rate for large time. Also note that for the chosen meshes the numerical dissipation is smaller than the physical dissipation. From these numerical experiments,

we get some numerical evidence of the uniform accuracy of the scheme with respect to time.

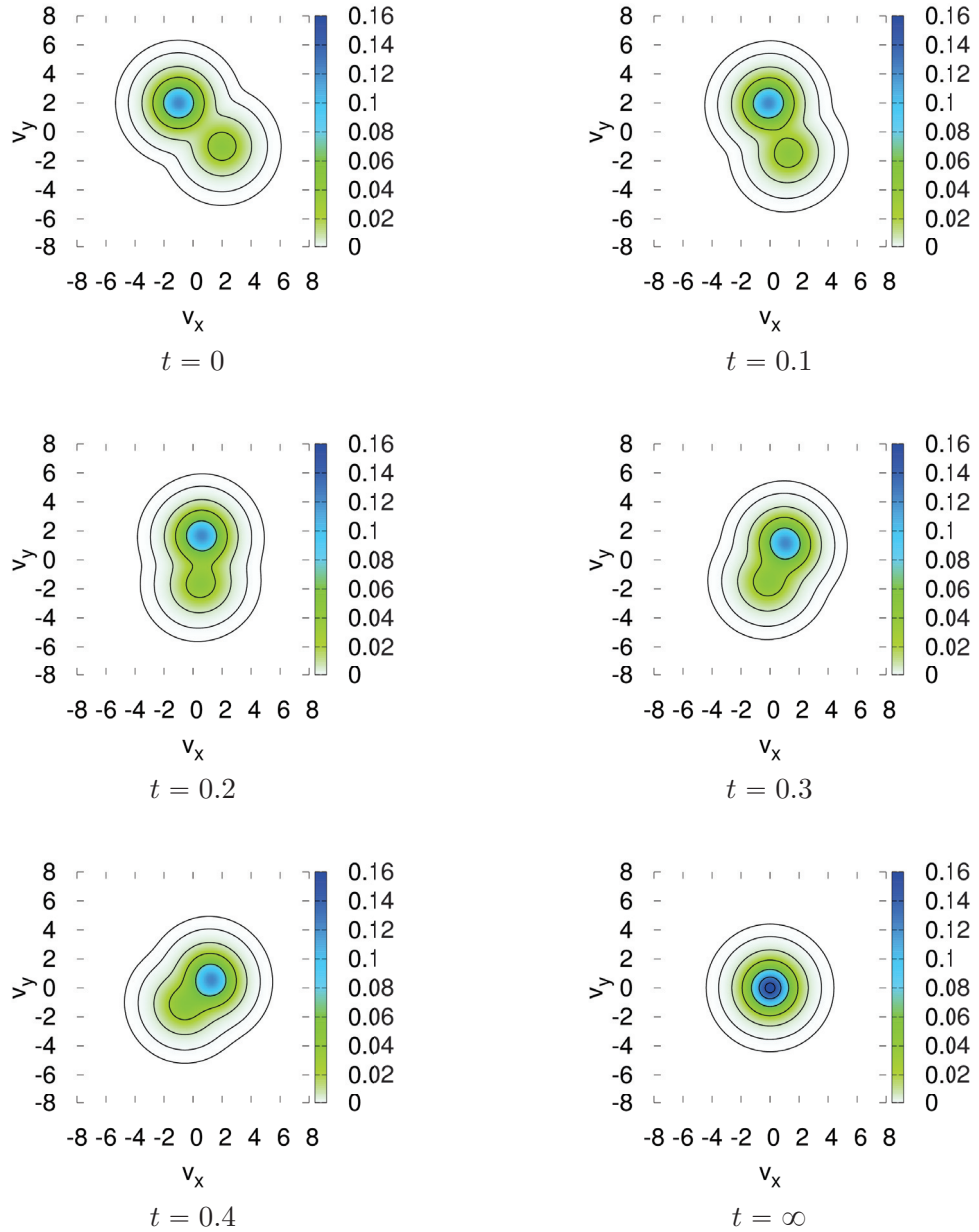


FIGURE 4.4 – Fokker-Planck equation with magnetic field. Time evolution of the distribution on the fine mesh $N = 160^2$.

Remark 4.6.1. For the same mesh if we take a larger magnetic field, the numerical

dissipation can become larger than the physical one. Even if both of them seem to converge to zero with the same decay rate, the dissipation is amplified.

Finally, in Figures 4.4, we propose the time evolution of the distribution function at different times. The black lines represent the isovalues $f(t, \mathbf{v}) \equiv 1.10^{-5}, 3.10^{-4}, 5.10^{-3}, 2.5.10^{-3}, 7.10^{-2}, 1.2.10^{-1}, 1.5.10^{-1}$ of the distribution function. We observe the effect of the magnetic field by the rotation of the two bumps and under the effect of the Fokker-Planck operator, the solution converges to a Maxwellian distribution.

4.6.4 Polymer flow in a dilute solution

We investigate the numerical approximation of the Fokker-Planck part of the kinetic Fokker-Planck equation for polymers [148]

$$\begin{cases} \frac{\partial F}{\partial t} = -\nabla_{\mathbf{k}} \cdot \left[\left(\mathbf{A} \mathbf{k} - \frac{1}{2} \nabla_{\mathbf{k}} \Pi(\mathbf{k}) \right) F - \frac{1}{2} \nabla_{\mathbf{k}} F \right], \\ F(t=0) = F_0 \quad \text{in } \Omega \subset \mathbb{R}^3, \end{cases}$$

where the matrix \mathbf{A} represents the gradient of an external velocity field and is given by

$$\mathbf{A} = \begin{pmatrix} 1/4 & -1/2 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The domain is $\Omega = (-4, 4)^3$ and we choose the Hookean model $\Pi(\mathbf{k}) = |\mathbf{k}|^2/2$. The initial datum F_0 is given by the sum of two Gaussian distributions

$$F_0(\mathbf{k}) = \frac{1}{2(2\pi)^{3/2}} \left[\exp\left(-\frac{|\mathbf{k} - \mathbf{k}_1|^2}{2}\right) + \exp\left(-\frac{|\mathbf{k} - \mathbf{k}_2|^2}{2}\right) \right],$$

with $\mathbf{k}_1 = (-3/2, 1, 0)$ and $\mathbf{k}_2 = (1, -3/2, 0)$. This equation is supplemented with homogeneous Neumann boundary conditions such that global mass is conserved. For numerical simulations we choose various meshes from $N = 24^3$ to 64^3 points with $\Delta t = 0.01$ using a time implicit scheme until $T = 5$. In this case, the steady state is not known, hence the steady equation is first solved numerically to compute a consistent approximation of the equilibrium $(f_K^\infty)_{K \in \mathcal{T}}$ and the stationary flux $F_{K,\sigma}^\infty$. If the matrix \mathbf{A} were equal to zero we would expect the steady state to be close to a Gaussian $G(\mathbf{k}) = \exp(-\Pi(\mathbf{k}))$, at least far from edges since G do not satisfy the boundary conditions. Thus we can expect f^∞/G to be close to some constant in the domain, which seems easier to approximate numerically. Hence for all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_{\text{int}}$, we define the quantities G_K and G_σ as the evaluation of G at the center of the respective cell or edge, as well as

$$h_K^\infty = \frac{f_K^\infty}{G_K}.$$

The scheme is solved on this new unknown h^∞ . More precisely, it is given by (H3) with the flux

$$F_{K,\sigma}^\infty = m(\sigma) G_\sigma \left[A_{K,\sigma}^+ h_K^\infty - A_{K,\sigma}^- h_L^\infty - \frac{1}{2d_\sigma} (h_L^\infty - h_K^\infty) \right],$$

if $\sigma = K|L$ and $F_{K,\sigma}^\infty = 0$ if $\sigma \in \mathcal{E}_{\text{ext}}$. The quantity $A_{K,\sigma}$ is the evaluation of $(\mathbf{A}\mathbf{k}) \cdot \mathbf{n}_{K,\sigma}$ at the center of the edge σ . Besides, because of the conservative boundary conditions, we need to specify the mass of the steady state to be that of the initial data in order to get a unique solution to the scheme. Then we define the steady state on interior edges by $f_\sigma^\infty = (f_K^\infty + f_L^\infty)/2$ if $\sigma = K|L$. Observe that this scheme is consistent with the following equation

$$\nabla \cdot \left[G(\mathbf{k}) \left((\mathbf{A}\mathbf{k}) h^\infty - \frac{1}{2} \nabla h^\infty \right) \right] = 0,$$

which is only a reformulation of the original steady equation.

In Figure 4.5, we represent the time evolution of the entropy H_{ϕ_2} , its dissipation D_{ϕ_2} and the numerical dissipation due to the convective term C_{ϕ_2} in log scale. First when $N \geq 32^3$ points, the entropy and the physical dissipation are well approximated compared to the solution computed with a fine mesh $N \geq 64^3$. Once again both of them are decreasing function of time and converge to zero with an exponential decay rate. The numerical dissipation of the convective term C_{ϕ_2} is much smaller than the physical one and also converges to zero when times goes to infinity exponentially fast, hence it does not affect the accuracy on the decay rate for large time.

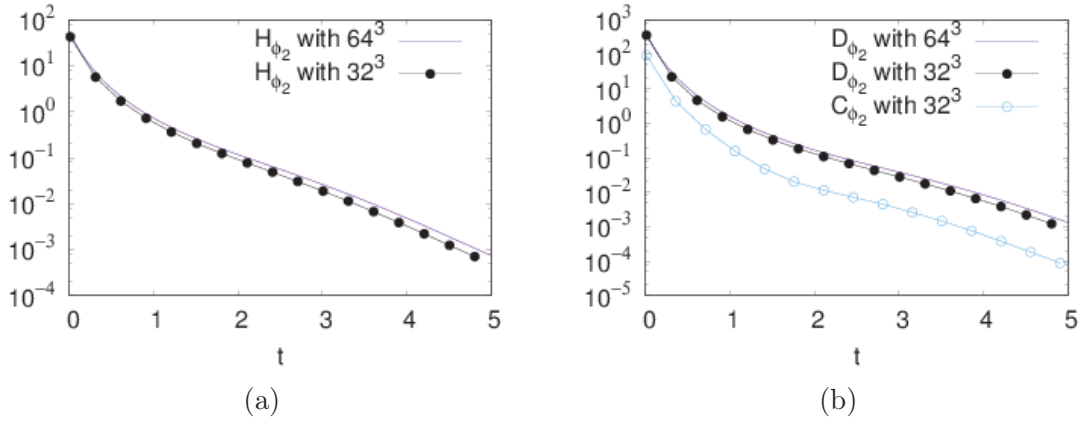


FIGURE 4.5 – **Polymer flow in a dilute solution.** Time evolution of the 2-entropy H_{ϕ_2} and the corresponding physical dissipation and numerical dissipation (D_{ϕ_2}, C_{ϕ_2}) with $N = 32^3$ mesh points.

Finally, in Figure 4.6, we set forth the time evolution of the distribution function at different time. The first column represents an isovalue $f(t, \mathbf{k}) \equiv 0.02$ of the distribution function whereas the second column is a two dimensional projection in the plane $k_x -$

k_y , the solution converges to the discrete steady state, which is consistent with the equilibrium.

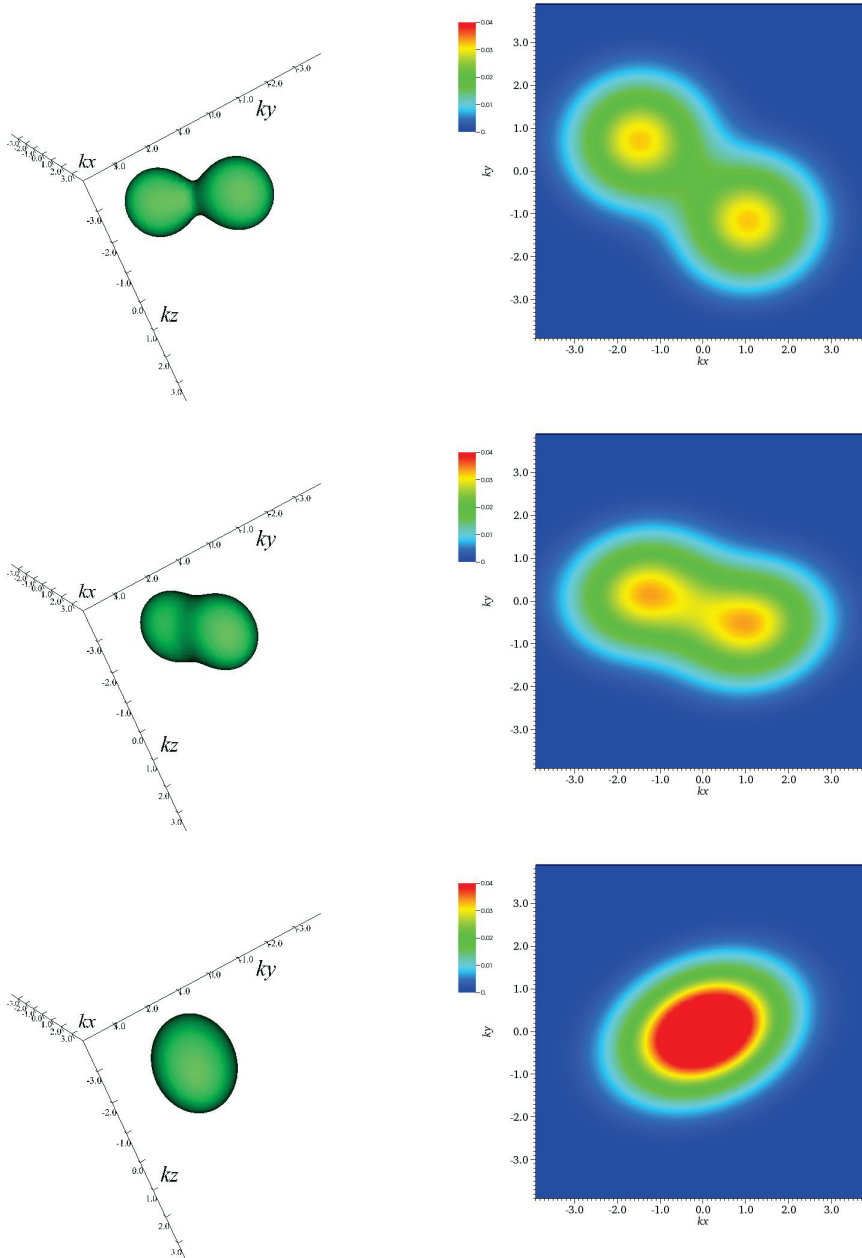


FIGURE 4.6 – Polymer flow in a dilute solution. (a) one isovalue $F(t, \mathbf{k}) = 0.02$ (b) $k_x - k_y$ projection of the distribution in the \mathbf{k} space at time $t = 0.2$, $t = 0.7$ and $t = 5$.

4.6.5 Porous medium equation

We finally study the numerical approximation of the porous medium equation

$$\begin{cases} \frac{\partial f}{\partial t} = \Delta f^m, \\ f(t=0) = f_0 \quad \text{in } \Omega = (0, 1) \times (-1, 1)^2, \end{cases}$$

with $m = 2$ and $f_0 \equiv 0$ together with the non-homogeneous Dirichlet boundary conditions

$$f^b = \begin{cases} 2.5, & \text{if } x = 1 \text{ and } y^2 + z^2 \leq 1/8, \\ 1, & \text{else.} \end{cases}$$

This model is nonlinear, with $\eta(s) = s^m$, and without convective terms. Moreover, since $\eta'(0) = 0$ the equation is degenerate. As a consequence, with our choice of initial data and boundary conditions, we expect the solution to be equal to zero on some subset of $\omega \subset \Omega$, having non-zero measure, for some positive time.

Since the steady state is not known, we first compute a numerical approximation $(f_K^\infty)_{K \in \mathcal{T}}$ and the corresponding stationary flux $F_{K,\sigma}^\infty$ using the scheme given in Example 4.2.3, which amounts to solving (H3) with

$$F_{K,\sigma}^\infty = -\tau_\sigma D_{K,\sigma} \eta(f^\infty).$$

Then from the knowledge of $\eta(f^\infty) \in X_{\eta(f^b)}^{N_T+1}$, we set $f_K^\infty = \eta^{-1}(\eta(f^\infty)_K)$ on each cell $K \in \mathcal{T}$ and $\eta(f^\infty)_\sigma = (\eta(f^\infty)_K + \eta(f^\infty)_L)/2$ for interior edges $\sigma = K|L$.

For the numerical simulations we choose two Cartesian meshes with $N = 30^3$ and 60^3 points using the explicit scheme (4.17)-(4.20). Hence the time step now satisfies a CFL condition $\Delta t = O(\Delta x^2)$ (precisely defined in Theorem 4.4.4) which is satisfied for the two meshes with $\Delta t = 10^{-5}$. In Figure 4.7, we represent the time evolution of the relative entropy and the physical and numerical dissipation until final time $T = 0.5$. These results are in good agreement with those obtained using a finer mesh and the numerical dissipation is several orders of magnitude smaller than the physical dissipation.

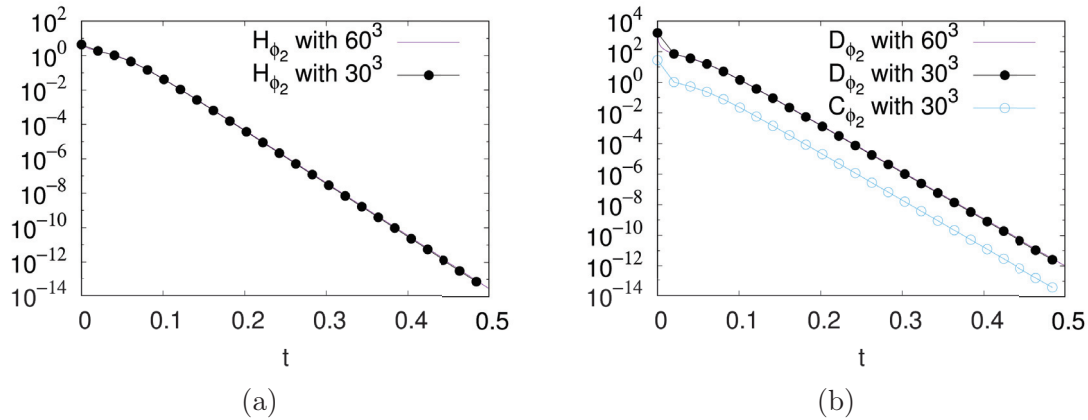


FIGURE 4.7 – **Porous medium equation.** Time evolution of the 2-entropy H_{ϕ_2} and the corresponding physical dissipation and numerical dissipation (D_{ϕ_2}, C_{ϕ_2}) with $N = 30^3$ mesh points.

Finally in Figure 4.8 we represent the intersection of the graph of the discrete solution with the plane $z = 0$ at different time. The black line represents the isovalue 10^{-16} that surrounds the zone where the diffusion has not yet happened, namely where the distribution is null. This illustrates the good behavior of our scheme with respect to the degeneracy of the porous medium equation.

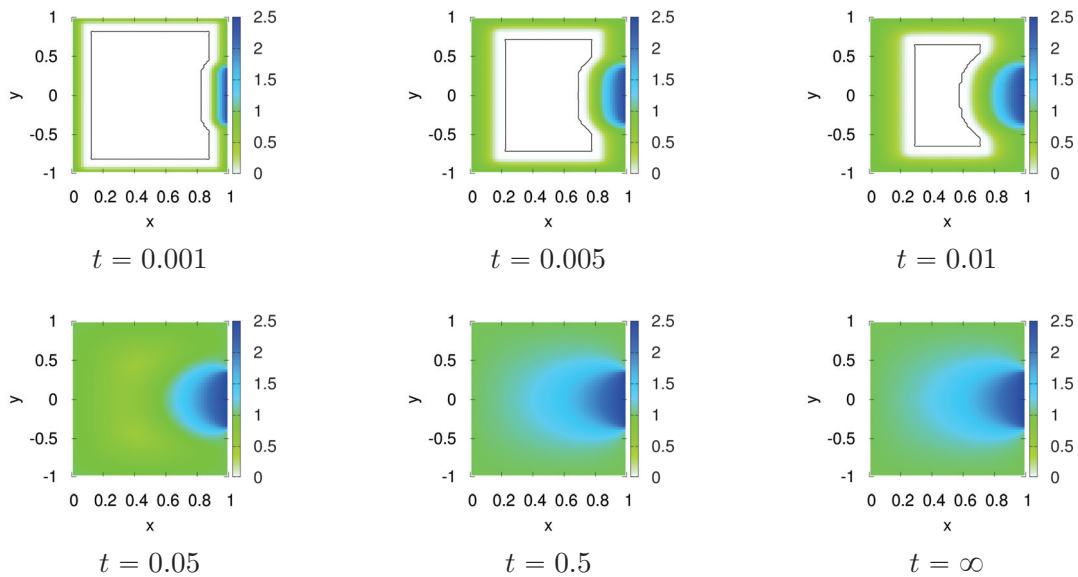


FIGURE 4.8 – **Porous medium equation.** Time evolution of the distribution at $z = 0$ for $N = 60^3$. The black line is the isovalue $f(t, x, y, 0) \equiv 10^{-16}$.

Annexe A

Simulations numériques d'un système bi-cinétique non-collisionnel

Travail en collaboration avec Mehdi Badsì⁴⁴.
Ce chapitre est publié en version étendue dans
ESAIM Proceedings and Surveys [9].

In this chapter, we present part of the work done for the *Multisplash* project (Multispecies plasma simulations) during the CEMRACS 2014. In the framework of a non-magnetized collisionless two-species one dimensional Vlasov-Poisson model we focus on two-asymptotics. On the first hand we adapt an asymptotic preserving numerical scheme dealing with the quasineutral approximation originally proposed by Belaouar, Crouseilles, Degond and Sonnendrücker [18] to our model. On the second hand we focus on the massless electron asymptotics. We make the mass ratio tends to zero and compare the density of electrons obtained for two species simulations with the isotropic Boltzmann density approximating the latter in most one species simulations. We conclude on the invalidity of the Boltzmann approximation in unstable cases.

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We consider the following one dimensional Vlasov-Poisson system corresponding to the model (34) in the introduction with $B = \gamma_i = \gamma_e = 0$ and $\varepsilon := \lambda^{1/2}$. It writes,

$$\begin{cases} \partial_t f_i + v \partial_x f_i + E \partial_v f_i = 0, \\ \partial_t f_e + \frac{1}{\varepsilon} v \partial_x f_e - \frac{1}{\varepsilon} E \partial_v f_e = 0, \\ \delta^2 \partial_x E = n_i - n_e, \end{cases} \quad (\text{A.1})$$

where,

$$E = -\partial_x \phi,$$

is the electric field associated with the potential ϕ . The velocity v varies on the whole real line \mathbb{R} while the space domain is periodic of length L , *i.e.* $x \in \mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$. Let us recall some important features of this model. Under compatible assumptions on the initial data the following properties hold for classical solutions of (A.1) (see [30] for more details), for $\alpha \in \{i, e\}$,

Positivity :

$$f_\alpha \geq 0.$$

L^p norm conservation :

$$\frac{d}{dt} \iint_{\mathbb{R} \times \mathbb{T}_L} f_\alpha^p dv dx = 0, \quad \forall p \in [1, +\infty].$$

Energy conservation :

$$\frac{d}{dt} \mathcal{E} = \frac{d}{dt} \left[\frac{1}{2} \iint_{\mathbb{R} \times \mathbb{T}_L} |v|^2 (f_i + f_e) dv dx + \frac{1}{2} \int_{\mathbb{T}_L} |E|^2 dx \right] = 0.$$

Entropy conservation :

$$\frac{d}{dt} \mathcal{S}_\alpha = \frac{d}{dt} \iint_{\mathbb{R} \times \mathbb{T}_L} f_\alpha \ln f_\alpha dv dx = 0.$$

Moreover, for initial data $f_i^{\text{in}}, f_e^{\text{in}}$, a smooth solution writes, for $\alpha \in \{i, e\}$,

$$f_\alpha(t, x, v) = f_\alpha^{\text{in}}(X_\alpha(0; t, x), V_\alpha(0; t, v)), \quad (\text{A.2})$$

where X_α and V_α solve the characteristics associated with the system (A.1),

$$\begin{cases} \frac{dX_i}{ds}(s; t, x) = V_i(s; t, v), \\ \frac{dV_i}{ds}(s; t, v) = E(t, X_i(s; t, x)), \end{cases} \quad \begin{cases} \frac{dX_e}{ds}(s; t, x) = \frac{1}{\varepsilon} V_e(s; t, v), \\ \frac{dV_e}{ds}(s; t, v) = -\frac{1}{\varepsilon} E(t, X_e(s; t, x)), \end{cases} \quad (\text{A.3})$$

and, for $\alpha \in \{i, e\}$,

$$X_\alpha(t; t, x) = x \quad V_\alpha(t; t, v) = v.$$

A.1 Numerical methods

In this section, we present the numerical methods implemented to solve (A.1). After a brief state of the art, we describe the backward semi-Lagrangian method used for the resolution of the Vlasov equations and adapt an asymptotic preserving scheme for the Poisson equation in the quasineutral regime $\delta \ll 1$. The latter was introduced in [18] and we shall detail its implementation in our model. However, we left its numerical validation with respect to both the massless electron limit and quasineutral regime for future work.

A.1.1 State of the art

The accurate simulation of the Vlasov equation is a delicate problem. Indeed the system has some inherent properties that must be preserved in order to describe the physics underneath the equations. Namely, the invariants such as the mass, the energy or the L^p norms must be preserved by the scheme. Consequently, either the latter is designed to naturally conserve these macroscopic quantities, or they are used as control variates for the simulation. In this context of multispecies simulations, the stiffness towards the parameters ε and δ makes it even harder to fulfill these objectives.

There are two major categories of numerical methods for solving the Vlasov equation. The first and oldest, is the category of particle methods such as the *Particle-In-Cell* method. Its principle is based on the statistical description of the origin of the Vlasov equation. It is a Monte-Carlo type method. One approaches the initial densities with a large number of particles which trajectories are followed in time with the characteristic ODEs. The literature on the subject is vast and includes a more physical description of the method in Birdsall and Langdon [23] and mathematical proofs of convergence in Neunzert and Wick [162]. PIC methods still serve as benchmark in terms of full 6D simulations of Vlasov equations because by construction, the computational cost depends weakly on the dimension. Only the complexity of fields reconstruction increases drastically with the dimension of the physical space.

The second category consists of methods known as *Eulerian*, including the semi-Lagrangian, finite volume, Galerkin or finite difference methods. Eulerian schemes aim at solving the equations on a fixed discretization of the phase space. A comparison of some of these methods for the one species Vlasov-Poisson system can be found in the survey of Filbet and Sonnendrücker [88]. We adapt a backward semi-Lagrangian code to our two species model.

A.1.2 The Backward Semi-Lagrangian method

The semi-Lagrangian method is based on the fact, coming from equation (A.2), that the densities f_i and f_e are constant along their characteristics. Assuming that the densities are known at time $t^n = n\Delta t$ then we get their value at time t^{n+1} by the following relations, for $\alpha \in \{i, e\}$,

$$f_\alpha(t^{n+1}, x, v) = f_\alpha(t^n, X_\alpha(t^n; t^{n+1}, x), V_\alpha(t^n; t^{n+1}, v)).$$

One goes one step back in time and evaluates the densities at the feet of the characteristics. To solve the characteristics, we use here a classical Strang splitting technique that allows us, in this case, to approximate the characteristics with an explicit formula of order 2 in time.

Remark A.1.1. *For our one dimensional unmagnetized model, the explicit formula for the characteristics allows us to handle small values of ε without having to numerically solve a stiff ODE. However, to ensure a uniform approximation for the study of the $\varepsilon \rightarrow 0$ limit, one should take a time step such that $\Delta t = o(\sqrt{\varepsilon})$.*

A sketch of the principle can be found on Figure A.1 for a Cartesian mesh in the phase space $(x_i, v_j)_{i \in I, j \in J}$. In general, the characteristics feet don't coincide with the

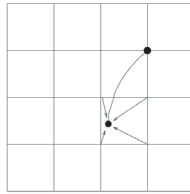


FIGURE A.1 – Reconstruction phase for the backward semi-Lagrangian method

mesh points. Therefore, a reconstruction of the densities by interpolation is required. A wide literature exists on this important phase. Let us just mention that in our case we used a high order Hermite weighted essentially non oscillatory (HWENO) method designed by Filbet and Yang [196]. This method gives an excellent accuracy because of the order of interpolation while avoiding the drawback of numerical oscillations.

The introduction of the mesh brings the question of the boundary conditions. In space, system (A.1) is set on the torus of length L . Consequently, we consider periodic boundary conditions for the $(x_i)_{i \in I}$. The velocity variable is defined on the whole real line for the continuous problem. However, we restrain it to a segment $[-V, V]$ with V large enough to keep the main part of the mass. This is possible since the typical densities quickly decrease in $|v|$ at infinity. In other words most of the particles don't go too fast. Therefore we associate homogeneous Dirichlet conditions for the boundary velocities.

A.1.3 An asymptotic preserving scheme for the Poisson equation in the quasineutral limit

In dimension 1, it is trivial to solve numerically the Poisson equation,

$$\delta^2 \partial_x E = n_i - n_e,$$

from the knowledge of the macroscopic densities, by a direct numerical integration. However, with this method the error blows up when the quasineutrality parameter δ tends to 0. This comes directly from the continuous problem. Indeed the formal limit of the Poisson equation is reduced to the quasineutrality constraint $n_i = n_e$. Namely, we lose the

ability to compute the potential or the electric field. In order to propose an asymptotic preserving scheme to manage small values of δ , the first step is to derive a reformulated Poisson equation adapted to the quasineutral limit.

By multiplying the Vlasov equations of (A.1) by 1 and v and integrating in the v variable, one gets the following moment equations,

$$\partial_t n_i + \partial_x j_i = 0, \quad (\text{A.4})$$

$$\partial_t n_e + \frac{1}{\varepsilon} \partial_x j_e = 0, \quad (\text{A.5})$$

$$\partial_t j_i + \partial_x S_i = -n_i \partial_x \phi, \quad (\text{A.6})$$

$$\partial_t j_e + \frac{1}{\varepsilon} \partial_x S_e = \frac{n_e}{\varepsilon} \partial_x \phi. \quad (\text{A.7})$$

where the first order moments write, for $\alpha \in \{i, e\}$,

$$j_\alpha = \int v f_\alpha dv \quad S_\alpha = \int v^2 f_\alpha dv.$$

Subtracting (A.4) with (A.5) and (A.6) with (A.7) divided by ε , one gets respectively,

$$\partial_t (n_i - n_e) + \partial_x (j_i - \frac{j_e}{\varepsilon}) = 0, \quad (\text{A.8})$$

$$\partial_t (j_i - \frac{j_e}{\varepsilon}) + \partial_x (S_i - \frac{S_e}{\varepsilon^2}) = - \left(n_i + \frac{n_e}{\varepsilon^2} \right) \partial_x \phi. \quad (\text{A.9})$$

By taking the partial derivative in time of (A.8) and subtracting it with the partial derivative of (A.9), we get, using the Poisson equation,

$$\partial_x \left(\left[n_i + \frac{n_e}{\varepsilon^2} + \delta^2 \partial_t^2 \right] E \right) = \partial_x^2 (S_i - \frac{S_e}{\varepsilon^2}). \quad (\text{A.10})$$

Providing some additional conditions on the initial conditions, one can show that we can equivalently use the Poisson equation or (A.10).

The scheme presented hereafter was introduced by Belaouar, Crouseilles, Degond and Sonnendrücker [18] for the resolution of the one species Vlasov Poisson system with a semi-Lagrangian method. The two species case was treated in [62] with a PIC method. Let us discretize equation (A.10) semi-implicitly in time,

$$\partial_x \left(\left[n_i^{(m)} + \frac{n_e^{(m)}}{\varepsilon^2} \right] E^{(m+1)} + \frac{\delta^2}{(\Delta t)^2} (E^{(m+1)} - 2E^{(m)} + E^{(m-1)}) \right) = \partial_x^2 (S_i^{(m)} - \frac{S_e^{(m)}}{\varepsilon^2}).$$

Now, in the last two terms of the left-hand side, we can use the Poisson equation at time $m\Delta t$ and $(m-1)\Delta t$ to get,

$$\partial_x \left(\left[n_i^{(m)} + \frac{n_e^{(m)}}{\varepsilon^2} + \frac{\delta^2}{(\Delta t)^2} \right] E^{(m+1)} \right) = \partial_x^2 (S_i^{(m)} - \frac{S_e^{(m)}}{\varepsilon^2}) + \frac{2(n_i^{(m)} - n_e^{(m)}) - (n_i^{(m-1)} - n_e^{(m-1)})}{(\Delta t)^2}.$$

In order to rewrite the right-hand side we can approach $\partial_t(n_i - n_e)|_{t=m\Delta t}$ with

$$\frac{(n_i^{(m)} - n_e^{(m)}) - (n_i^{(m-1)} - n_e^{(m-1)})}{\Delta t}.$$

Then, using equation (A.8), we get,

$$\partial_x \left(\left[n_i^{(m)} + \frac{n_e^{(m)}}{\varepsilon^2} + \frac{\delta^2}{(\Delta t)^2} \right] E^{(m+1)} \right) = \partial_x^2 \left(S_i^{(m)} - \frac{S_e^{(m)}}{\varepsilon^2} \right) + \frac{n_i^{(m)} - n_e^{(m)}}{(\Delta t)^2} - \frac{\partial_x \left(j_i^{(m)} - \frac{j_e^{(m)}}{\varepsilon} \right)}{\Delta t}.$$

Now, for clarity, let us define,

$$a(x) = (\Delta t)^2 \left(n_i^{(m)} + \frac{n_e^{(m)}}{\varepsilon^2} \right) + \delta^2,$$

$$b(x) = (\Delta t)^2 \left(S_i^{(m)} - \frac{S_e^{(m)}}{\varepsilon^2} \right),$$

$$c(x) = n_i^{(m)} - n_e^{(m)},$$

$$d(x) = -\Delta t \left(j_i^{(m)} - \frac{j_e^{(m)}}{\varepsilon} \right).$$

At this point we need to solve the following equation,

$$\partial_x (aE^{(m+1)}) = \partial_x^2 b + c + \partial_x d. \quad (\text{A.11})$$

We integrate numerically equation (A.11) on the uniform mesh $\{x_0, \dots, x_{n-2}\}$ (with $x_{n-1} = x_0$) with the space step Δx . Between x_0 et x_k , it gives,

$$a_k E_k - a_0 E_0 = \partial_x b|_{x=x_k} - \partial_x b|_{x=x_0} + \sum_{j=0}^{k-1} c_j \Delta x + d_k - d_0,$$

which is approached by,

$$a_k E_k - a_0 E_0 = \frac{b_{k+1} - b_k}{\Delta x} - \frac{b_1 - b_0}{\Delta x} + \sum_{j=0}^{k-1} c_j \Delta x + d_k - d_0$$

and then,

$$E_k = \frac{1}{a_k} \left(a_0 E_0 + \frac{b_{k+1} - b_k}{\Delta x} - \frac{b_1 - b_0}{\Delta x} + \sum_{j=0}^{k-1} c_j \Delta x + d_k - d_0 \right).$$

We need an additional equation to close the system. We know that the electric field E has zero mean value since it derives from a potential. Therefore, the following equation holds,

$$\sum_{k=0}^{n-2} E_k = 0,$$

which gives E_0 . Indeed,

$$-a_0 E_0 \sum_{k=0}^{n-2} \frac{1}{a_k} = \sum_{k=0}^{n-2} \left(E_k - \frac{a_0 E_0}{a_k} \right) = \sum_{k=0}^{n-2} \frac{1}{a_k} \left(\frac{b_{k+1} - b_k}{\Delta x} - \frac{b_1 - b_0}{\Delta x} + \sum_{j=0}^{k-1} c_j \Delta x + d_k - d_0 \right).$$

and so,

$$E_0 = (a_0 \Sigma_1)^{-1} \times \left(-\frac{\Sigma_2}{\Delta x} + \frac{b_1 - b_0}{\Delta x} \Sigma_1 - \Sigma_3 \Delta x - \Sigma_4 + d_0 \Sigma_1 \right),$$

with,

$$\Sigma_1 = \sum_{k=0}^{n-2} \frac{1}{a_k}, \quad \Sigma_2 = \sum_{k=0}^{n-2} \frac{b_{k+1} - b_k}{a_k}, \quad \Sigma_3 = \sum_{k=0}^{n-2} \sum_{j=0}^{k-1} \frac{c_j}{a_k}, \quad \Sigma_4 = \sum_{k=0}^{n-2} \frac{d_k}{a_k}.$$

The asymptotic preserving Poisson solver is adapted to large scale regimes and we refer to [18] for relevant test cases. In the next section, we shall focus on micro kinetic phenomena in the massless electron asymptotic. For this purpose, we will use a classical Poisson solver, setting $\delta = 1$.

A.2 Numerical results

We want to investigate the massless electron approximation for our modeling. In our initial modeling we neglected the collisions between particles. However, if we suppose that the electron density is at thermodynamical equilibrium, namely,

$$f_e(t, x, v) = n_e(t, x) M(v),$$

where M is the rescaled Maxwellian,

$$M(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}.$$

Injecting this expression in the electron equation of (A.1) yields,

$$\varepsilon \partial_t n_e + v (\partial_x n_e + E n_e) = 0.$$

Letting ε goes to zero gives formally,

$$\partial_x n_e + E n_e = 0.$$

since the equality before the last hold for any v . Now, we can use the fact that $E = -\partial_x \phi$ to get,

$$n_e(t, x) = C(t)e^{\phi(t, x)},$$

with

$$C(t) = \frac{\int n_i dx}{\int e^{\phi} dx},$$

because of the global neutrality. The goal of the following numerical test cases is to determine whether the electron density gets close to the latter rescaled Maxwell-Boltzmann density as ε gets small, when following the two-species non-collisional equation (A.1). We will suppose that the initial electron density is at global equilibrium, namely

$$f_e(0, x, v) = M(v),$$

whereas the ions will be such that they impose their dynamic on the coupled system. Actually their initial density will be of the following form,

$$f_i(0, x, v) = \left(1 + A \cos\left(\frac{2\pi}{L}x\right) \right) \nu(v),$$

where $A \ll 1$ is the amplitude of the perturbation. We will try two velocity profiles ν , one being stable and the other being unstable in the sense of Penrose (see Definition A.2.1).

Definition A.2.1 ([116]). We say that an homogeneous even profile $\nu(v)$, such that $\int \nu dv = 1$, satisfies the Penrose instability criterion if μ has a local minimum point in 0 and,

$$\int_{\mathbb{R}} \frac{\nu(v) - \nu(0)}{v^2} dv > \frac{4\pi^2}{L^2}.$$

Under some additional smoothness properties for the profile ν , Guo and Strauss proved the equivalence between the Penrose condition and the nonlinear instability of ν for the one species Vlasov Poisson system [113]. The kinetic instabilities created by these profiles are often referred to as *two-stream instabilities*, because they appear for ν with two or more maxima.

In order to measure the distance between n_e associated with a solution of (A.1) and the Maxwell-Boltzmann density, we introduce the following quantity,

$$d(t) = \frac{1}{L} \|n_e(t, \cdot) - C(t)e^{\phi(t, \cdot)}\|_{L^2}^2.$$

A.2.1 Test case 1 : stability

In this first test case, we consider a grid of 512 points in the $x \in [0, 21]$ direction and of 512 points in the $v \in [-7, 7]$ direction. The time step is $\Delta t = 0.01$ and we stop the

simulation at $t_{\max} = 15$. The amplitude of the perturbation is $A = 0.001$ and the ion velocity profile is the following Maxwellian distribution,

$$\nu(v) = M(v).$$

We perform the simulation of system (A.1) for the following mass ratio,

$$\varepsilon \in \{1, 0.5, 0.1, 0.05, 0.01, 0.005\}$$

The error on the conservation of mass, total energy and entropy can be found in Table A.1.

$\varepsilon =$	1	0.5	0.1	0.05	0.01	0.005
Mass (ions) : $\iint f_i dv dx$	10^{-12}	10^{-12}	10^{-12}	10^{-12}	10^{-12}	10^{-12}
Mass (electrons) : $\iint f_e dv dx$	10^{-12}	10^{-12}	10^{-12}	10^{-12}	10^{-10}	10^{-9}
Total energy : $\mathcal{E}(t)$	10^{-8}	10^{-9}	10^{-8}	10^{-9}	10^{-8}	10^{-8}
Entropy : $\mathcal{S}_i(t) + \mathcal{S}_e(t)$	10^{-12}	10^{-12}	10^{-12}	10^{-12}	10^{-11}	10^{-10}

TABLE A.1 – **Test case 1 : stability.** Relative error on mass, total energy and entropy at time t_{\max} for different values of ε .

As we can observe on Figure A.2, in this stable case, the macroscopic electron density is exponentially damped in time towards the Maxwell-Boltzmann density with a rate that increases as the mass ratio ε diminishes. The accuracy of this simulation is confirmed by the weak relative error on the control variates of Table A.1.

A.2.2 Test case 2 : instability

In this second test case, we consider a grid of 512 points in the $x \in [0, 21]$ direction and of 2048 points in the $v \in [-5, 5]$ direction. The time step is $\Delta t = 0.02$ and we stop the simulation at $t_{\max} = 100$. The amplitude of the perturbation is $A = 0.01$ and the ion velocity profile is the following two-stream density,

$$\nu(v) = 8v^2 M(2v),$$

which satisfies the Penrose instability criterion of Definition A.2.1. We perform the simulation of system (A.1) for the following mass ratio,

$$\varepsilon \in \{1, 0.5, 0.1, 0.05, 0.01, 0.005\}$$

The error on the conservation of mass, total energy and entropy can be found in Table A.2.

We observe on Figure A.3 that in this case, there does not seem to be any convergence to the Maxwell-Boltzmann density neither in time nor in ε . Moreover, as witnesses Figures A.4 and A.5, the two-stream instability arises at similar time and with comparable electric energy amplitude, whatever the value of ε . Therefore, the smallness of the mass ratio does

$\varepsilon =$	1	0.5	0.1	0.05	0.01	0.005
Mass (ions)	$4.1.10^{-3}$	$4.2.10^{-3}$	$3.2.10^{-3}$	$2.9.10^{-3}$	$2.2.10^{-3}$	$3.2.10^{-3}$
Mass (electrons)	$5.2.10^{-4}$	$5.3.10^{-4}$	$5.3.10^{-5}$	$7.1.10^{-5}$	$1.0.10^{-5}$	$1.2.10^{-4}$
Total energy	$7.0.10^{-4}$	$6.9.10^{-4}$	$6.0.10^{-4}$	$5.4.10^{-4}$	$4.2.10^{-4}$	$3.9.10^{-4}$
Entropy	$3.2.10^{-3}$	$4.7.10^{-3}$	$3.3.10^{-3}$	$3.0.10^{-3}$	$2.7.10^{-3}$	$3.2.10^{-3}$

TABLE A.2 – **Test case 2 : instability.** Relative error on mass, total energy and entropy at time t_{\max} for different values of ε .

not seem to prevent from the creation of micro kinetic instabilities such as two-stream instabilities. Compared to the first test case, the relative errors on the control variates in Table A.2 are larger but uniform in ε . We can infer that this is due to the turbulent dynamic and the fact that the simulation is longer.

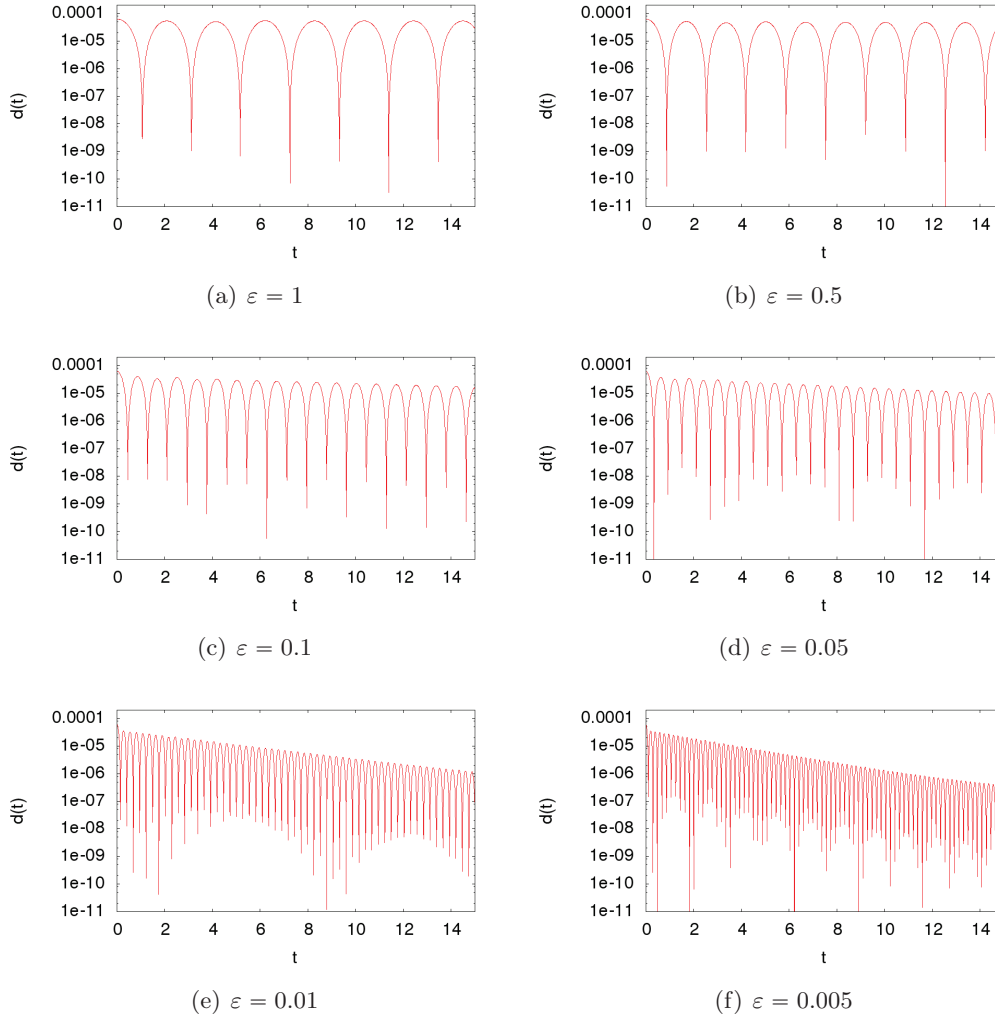


FIGURE A.2 – **Test case 1 : stability.** Distance to the Maxwell Boltzmann density $d(t)$ over time in logscale for different values of mass ratio ε .

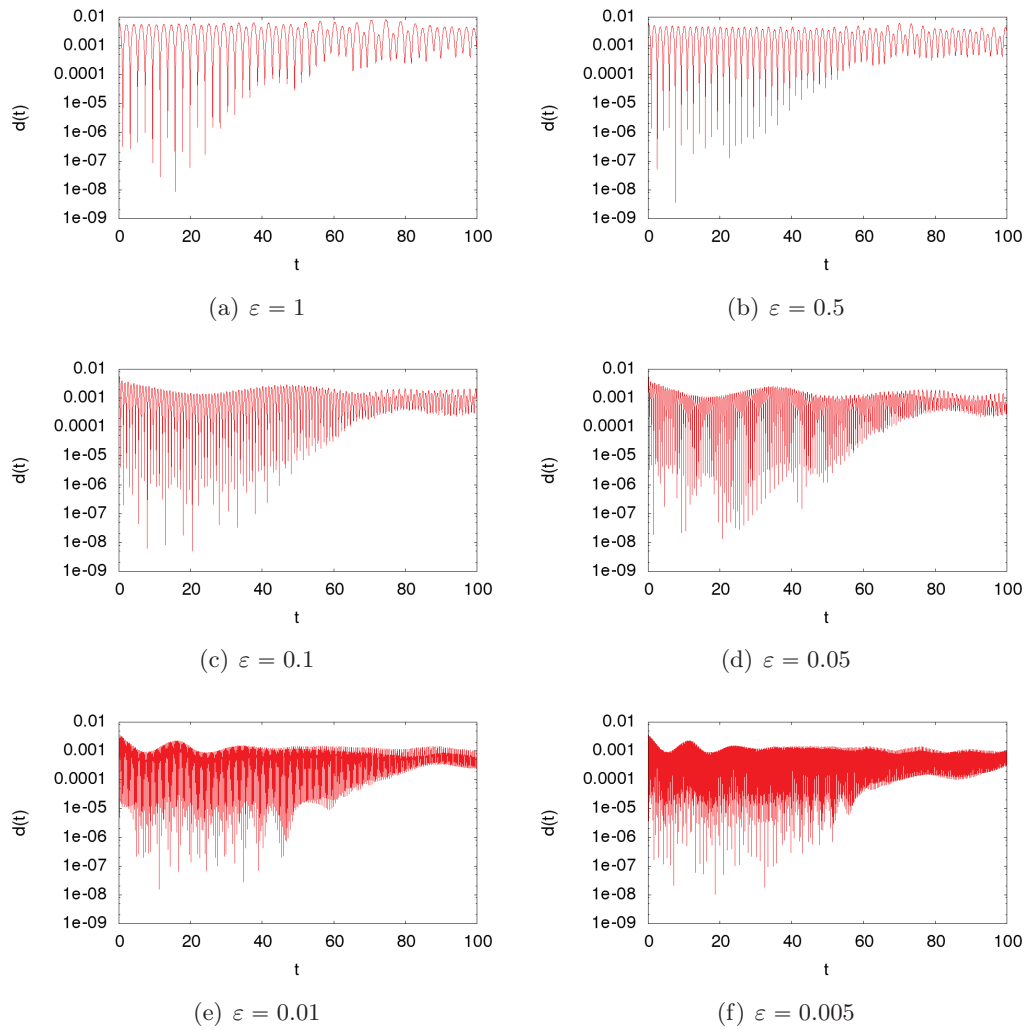


FIGURE A.3 – **Test case 2 : instability.** Distance to the Maxwell Boltzmann density $d(t)$ over time in logscale for different values of ε .

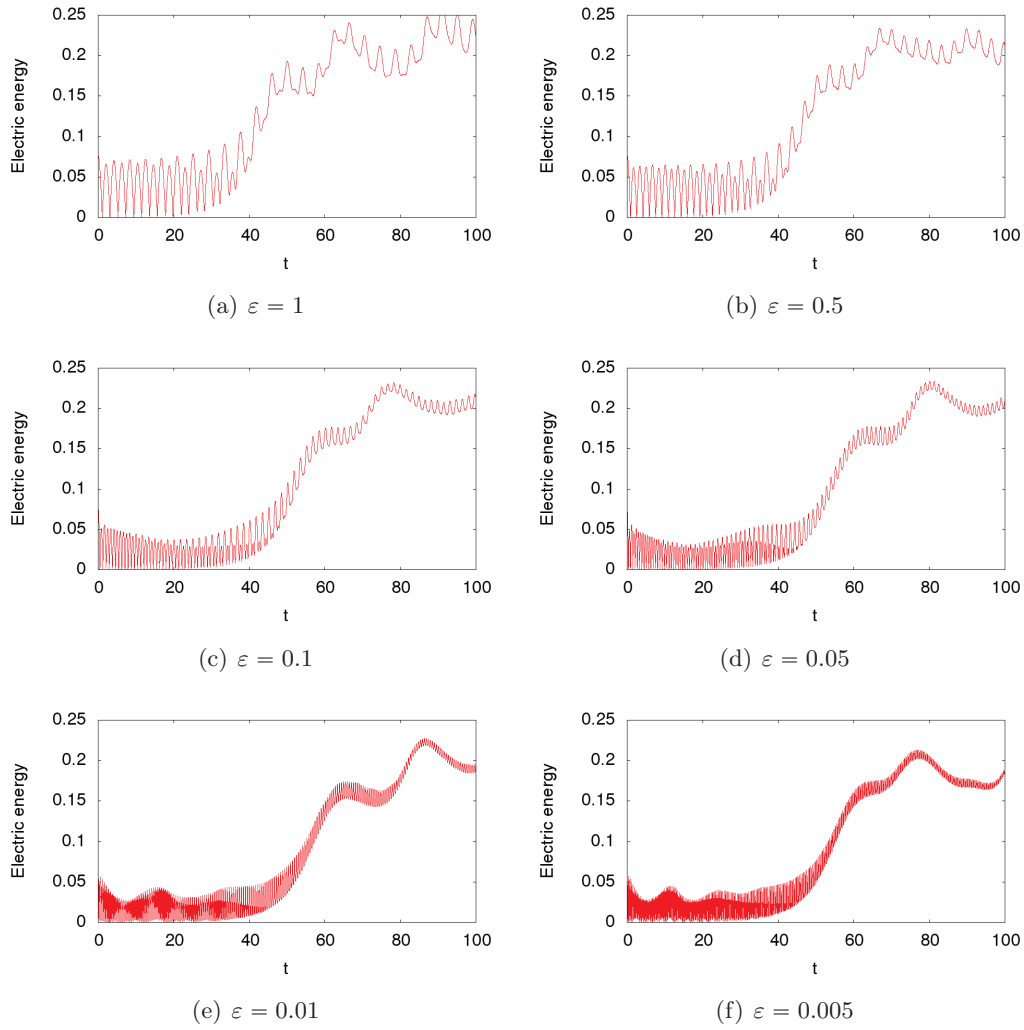


FIGURE A.4 – **Test case 2 : instability.** Time evolution of the electric energy $\frac{1}{2} \int |\partial_x \phi|^2 dx$ for different values of ε .

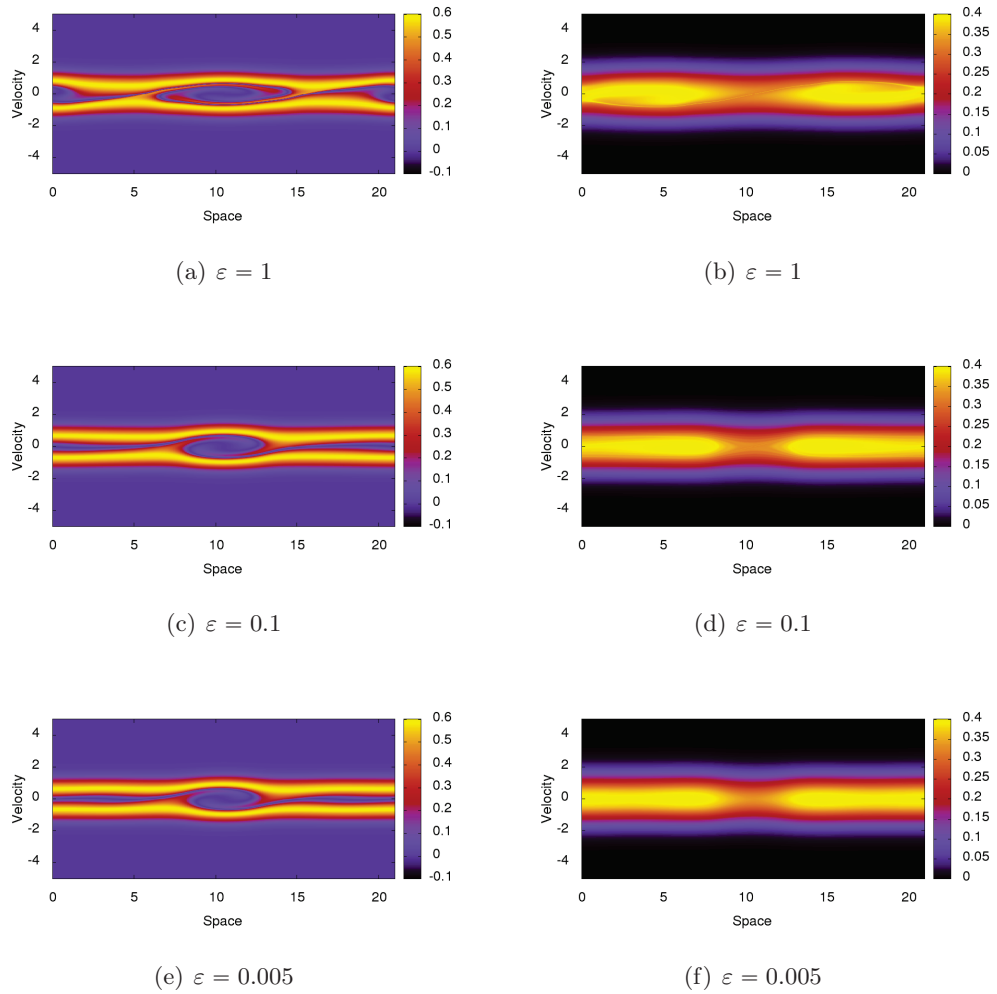


FIGURE A.5 – **Test case 2 : instability.** Densities during the creation of the instability ($t = 60$; Left : Ions ; Right : Electrons) for different values of ε .

Annexe B

Résultats et remarques complémentaires

Sommaire

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B.1 Scaling of the bi-kinetic system

In physical units the two-species VFPF system with magnetic field reads

$$\begin{cases} \partial_t f_i + v \cdot \nabla_x f_i + \frac{q}{m_i} (-\nabla_x \phi + v \wedge B) \cdot \nabla_v f_i &= \nu_i \nabla_v \cdot (v f_i + \frac{k_B \theta}{m_i} \nabla_v f_i), \\ \partial_t f_e + v \cdot \nabla_x f_e - \frac{q}{m_e} (-\nabla_x \phi + v \wedge B) \cdot \nabla_v f_e &= \nu_e \nabla_v \cdot (v f_e + \frac{k_B \theta}{m_e} \nabla_v f_e), \\ -\varepsilon_0 \Delta_x \phi &= q n_i - q n_e. \end{cases}$$

We refer to Table B.1 for a list of characteristic scales. Let us recall appropriate expressions of the various dimensionless parameters defined in the introduction

$$\delta = \frac{\lambda_D}{L} = \sqrt{\frac{\varepsilon_0 k_B \theta}{q^2 \bar{n} L^2}}, \quad \mu = \frac{\bar{\phi}}{LV_i \bar{B}}, \quad \lambda = \frac{m_e}{m_i}.$$

We additionally introduce here the inverses of the scaled mean free path of each species

$$\gamma_e(\lambda) = \frac{L}{l_e}, \quad \gamma_i = \frac{L}{l_i}.$$

Remark B.1.1. *The dependence of γ_e on the mass ratio λ implicitly means that the ratio between the mean free paths of each species depends on the mass ratio. Undoubtedly, a more convincing way of introducing these parameters would be a successive definition of γ_i as above (since ions are the reference species), then l_i/l_e and finally $\gamma_e = \gamma_i l_i/l_e$. However it would prevent us from putting the ions in a collisionless setting as in Chapter 1 without forcing the electrons to be also in a collisionless setting. The modeling argument that justifies of approximation of γ_i by 0 in (1.3) is that γ_i is extremely small in applications and it does not depend on the mass ratio, unlike γ_e which goes to infinity when λ goes to 0. From a mathematical point of view, we also justify this choice by the will of analyzing a mixed collisional / collisionless model. Finally let us mention that the results of Chapter 1 hold replacing the Vlasov equation for the ions by a Vlasov-Fokker-Planck equation with constant γ_i under minor changes.*

We use a «prime» exponent to denote scaled variables and unknowns. Namely, we introduce for $s \in \{i, e\}$

$$\begin{aligned} f_s(t, x, v) &= \frac{\bar{n}}{V_s^3} f'_s \left(\frac{t}{t_0}, \frac{x}{L}, \frac{v}{V_s} \right), & n_s(t, x) &= \bar{n} n'_s \left(\frac{t}{t_0}, \frac{x}{L} \right), \\ \phi(t, x) &= \bar{\phi} \phi' \left(\frac{t}{t_0}, \frac{x}{L} \right), & B(t, x) &= \bar{B} B' \left(\frac{t}{t_0}, \frac{x}{L} \right). \end{aligned}$$

Injecting these expressions in the system yields

$$\left\{ \begin{array}{l} \partial_{t'} f'_i + \frac{V_i t_0}{L} v' \cdot \nabla_{x'} f'_i + \frac{q \bar{\phi} t_0}{m_i V_i L} \left(-\nabla_{x'} \phi' + \frac{\bar{B} V_i L}{\phi} v' \wedge B' \right) \cdot \nabla_{v'} f'_i \\ = \frac{V_i t_0}{L} \frac{L}{l_i} \nabla_{v'} \cdot (v' f'_i + \frac{k_B \theta}{m_i V_i^2} \nabla_{v'} f'_i), \\ \partial_{t'} f'_e + \frac{V_e V_i t_0}{V_i L} v' \cdot \nabla_{x'} f'_e - \frac{m_i V_i}{m_e V_e} \frac{q \bar{\phi} t_0}{m_i V_i L} \left(-\nabla_{x'} \phi' + \frac{V_e \bar{B} V_i L}{V_i \phi} v' \wedge B' \right) \cdot \nabla_{v'} f'_e \\ = \frac{V_i t_0}{L} \frac{V_e L}{V_i l_e} \nabla_{v'} \cdot (v' f'_e + \frac{k_B \theta}{m_e V_e^2} \nabla_{v'} f'_e), \\ - \frac{\varepsilon_0 \bar{\phi}}{q \bar{n} L^2} \Delta_{x'} \phi' = n'_i - n'_e. \end{array} \right.$$

By using the relations of Table B.1 and the expressions of the dimensionless parameters, we get in particular that that

$$\frac{V_i t_0}{L} = 1, \quad \frac{q \bar{\phi} t_0}{m_i V_i L} = 1, \quad \frac{\varepsilon_0 \bar{\phi}}{q \bar{n} L^2} = \delta^2, \quad \frac{V_i}{V_e} = \lambda^{\frac{1}{2}}$$

and recover system (34) of the introduction by dropping the primes.

B.2 Derivation of the Boltzmann approximation with temperature

For a thorough treatment of the following type of arguments concerning the derivation of the isotropic Boltzmann density, we refer to the recent [15].

Lemma B.2.1 (Isotropic Boltzmann density). *Assume that $f_e, f_i : \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R} \mapsto \mathbb{R}_+$ and $\phi : \mathbb{R}_+ \times \mathbb{T} \mapsto \mathbb{R}_+$ are smooth functions and f_e is given by the centered Maxwellian distribution*

$$f_e(t, x, v) = \frac{n_e(t, x)}{(2\pi T_e(t, x))^{1/2}} \exp\left(-\frac{v^2}{2T_e(t, x)}\right),$$

for smooth macroscopic density n_e and temperature T_e . If moreover f_i, f_e, ϕ satisfy massless electron Vlasov-Poisson equation

$$\left\{ \begin{array}{l} \partial_t f_i + v \partial_x f_i - \partial_x \phi \partial_v f_i = 0, \\ v \partial_x f_e + \partial_x \phi \partial_v f_e = 0, \\ -\delta^2 \partial_{xx} \phi = n_i(t, x) - n_e(t, x) \end{array} \right.$$

and the conservation of energy

$$\frac{1}{2} \iint v^2 (f_i(t) + f_e(t)) dv dx + \frac{\delta^2}{2} \int |\partial_x \phi(t)|^2 dx = \mathcal{E}_0,$$

for some given total energy $\mathcal{E}_0 > 0$, then $\partial_x T_e = 0$ and n_e is given by the Boltzmann distribution

$$n_e(t, x) = C(t) e^{\frac{\phi(t, x)}{T(t)}},$$

with the normalization constant

$$C(t) = \left(\iint f_i dx dv \right) / \left(\int e^{\phi/T_e} dx \right)$$

Moreover the electron temperature can be determined from the conservation of energy by the relation

$$T_e(t) = \frac{1}{\iint f_i dx dv} \left(2\mathcal{E}_0 - \iint |v|^2 f_i(t) dv dx - \delta^2 \int |\partial_x \phi(t)|^2 dx \right).$$

Proof. Just note that by plugging the ansatz in the equation, one has for all $v \in \mathbb{R}$,

$$-v^3 \partial_x (T_e^{-1}) n_e + v (\partial_x n_e + \partial_x (T_e^{-1/2}) \sqrt{T_e} n_e - \partial_x \phi n_e T_e^{-1}) = 0.$$

It yields the Boltzmann density and the constant temperature in space by identifying each coefficient of the v polynomial. Then plug the Maxwellian in the energy conservation relation to obtain the last formula. \square

B.3 Green function for magnetized Vlasov-Fokker-Planck

In this section we apply the method of Chandrasekhar [50] to compute the fundamental solution of

$$\partial_t f + v \cdot \nabla_x f + \mu v^\perp \cdot \nabla_v f = \frac{1}{\tau} \nabla_v \cdot (v f + \theta \nabla_v f), \quad (\text{B.1})$$

with $v^\perp = (-v_2, v_1, 0)^\top$, that is the function $\Gamma \equiv \Gamma(t, s, x, y, v, w)$ such that $f \equiv f(t, x, v) = \Gamma(t, s, x, y, v, w)$ satisfy (B.1) for all s, y, w and the Dirac initial condition

$$\Gamma(s, s, x, y, v, w) = \delta(x - y, v - w).$$

First, we introduce a change of variable corresponding to first integrals of the characteristic curves of the transport operator $\partial_t + v \cdot \nabla_x + \mu v^\perp \cdot \nabla_v - \frac{1}{\tau} v \cdot \nabla_v$. It reads

$$\begin{cases} T(t) &= t, \\ X(x, v) &= x - \tau \mu x^\perp + \tau v, \\ V(t, v) &= R_{-\mu t} e^{t/\tau} v, \end{cases}$$

where R_θ is the rotation matrix of angle θ with axis parallel to the magnetic field. Note that

$$\begin{pmatrix} X \\ V \end{pmatrix} = A(t) \begin{pmatrix} x \\ v \end{pmatrix},$$

with $A(t)$ the matrix

$$A(t) = \left(\begin{array}{ccc|c} 1 & -\tau \mu & 0 & \tau I_3 \\ \tau \mu & 1 & 0 & \\ 0 & 0 & 1 & \\ \hline & 0_3 & & R_{-\mu t} e^{t/\tau} \end{array} \right),$$

with I_3 and 0_3 the identity and null matrices in dimension 3 respectively. It yields

$$\begin{cases} \partial_t &= \partial_T + \frac{1}{\tau} [R_{-\mu t} e^{t/\tau} (v - \mu \tau v^\perp)] \cdot \nabla_V \\ v \cdot \nabla_x &= (v - \mu \tau v^\perp) \cdot \nabla_X \\ \nabla_v &= \tau \nabla_X + e^{t/\tau} R_{\mu t} \nabla_V \\ \Delta_v &= \tau^2 \Delta_X + 2\tau e^{t/\tau} \left(\cos(\mu t) \nabla_{X_\perp} \cdot \nabla_{V_\perp} + \partial_{X_\parallel} \partial_{V_\parallel} \right) + e^{2t/\tau} \Delta_V \end{cases}$$

Hence by setting

$$f(t, x, v) = g(T(t), X(x, v), V(t, v)),$$

one easily gets that g satisfies the following PDE

$$\partial_T g = \frac{3}{\tau} g + \theta \left(\tau \Delta_X g + 2 e^{t/\tau} \left(\cos(\mu t) \nabla_{X_\perp} \cdot \nabla_{V_\perp} g + \partial_{X_\parallel} \partial_{V_\parallel} g \right) + \frac{1}{\tau} e^{2t/\tau} \Delta_V g \right)$$

Then we absorb the 0-th order term by setting $h(T, X, V) = e^{-3T/\tau} g(T, X, V)$ and switching to Fourier variables (η, ξ) in (X, V) we get

$$\partial_T \hat{h} = -\theta \left(\tau |\xi|^2 \hat{h} + 2 e^{t/\tau} \left(\cos(\mu t) \xi_\perp \cdot \eta_\perp \hat{h} + \xi_\parallel \eta_\parallel \hat{h} \right) + \frac{1}{\tau} e^{2t/\tau} |\eta|^2 \hat{h} \right).$$

The latter equation is solved by

$$\hat{h}(t, \xi, \eta) = \hat{h}(s, \xi, \eta) \exp \left(-\theta \begin{pmatrix} \xi \\ \eta \end{pmatrix}^\top \Sigma(t-s) \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right)$$

with

$$\Sigma(t) = \left(\begin{array}{ccc|ccc} & & & \tau \chi_{\tau, \mu}(t) & 0 & 0 \\ & \tau t I_3 & & 0 & \tau \chi_{\tau, \mu}(t) & 0 \\ & & & 0 & 0 & \tau \chi_{\tau, 0}(t) \\ \hline \tau \chi_{\tau, \mu}(t) & 0 & 0 & & & \\ 0 & \tau \chi_{\tau, \mu}(t) & 0 & & & \\ 0 & 0 & \tau \chi_{\tau, 0}(t) & & \frac{1}{2} \chi_{\tau, 0}(2t) I_3 & \end{array} \right),$$

where

$$\chi_{\tau, \mu}(t) = \frac{e^{t/\tau} (\cos(\mu t) + \tau \mu \sin(\mu t)) - 1}{1 + (\tau \mu)^2}.$$

Hence, back to real variables we get

$$h(t, X, V) = (2\pi)^{-3/2} |\det \Sigma(t-s)^{-1}| h(s, X, V) *_{X, V} \exp \left(-\theta \begin{pmatrix} X \\ V \end{pmatrix}^\top \Sigma(t-s)^{-1} \begin{pmatrix} X \\ V \end{pmatrix} \right).$$

It yields the following proposition

Proposition B.3.1. *The fundamental solution of (B.1) with initial condition at $t = s$ given by a Dirac mass centered in $(y, w) \in \mathbb{R}^3 \times \mathbb{R}^3$*

$$\Gamma(t, s, x, y, v, w) = \frac{e^{3(t-s)/\tau}}{(2\pi)^{3/2} |\det \Sigma(t-s)|} \exp \left(-\theta \begin{pmatrix} X(x, v) - X(y, w) \\ V(t-s, v) - w \end{pmatrix}^\top \Sigma(t-s)^{-1} \begin{pmatrix} X(x, v) - X(y, w) \\ V(t-s, v) - w \end{pmatrix} \right)$$

From this formula we can for example recover regularization time powers by noticing that for $\mu = 0$, after taking the Fourier transform of $\Gamma(t, 0, x, 0, v, 0)$ (to simplify) in x, v the quadratic form in the Gaussian has matrix given by

$$A^{-1}(t)\Sigma(t)A^{-\top}(t) = \frac{1}{\tau} \begin{pmatrix} \frac{t^3}{3}I_3 + o(t^3) & -\frac{t^2}{2}I_3 + o(t^2) \\ -\frac{t^2}{2}I_3 + o(t^2) & tI_3 + o(t) \end{pmatrix},$$

when $t \rightarrow 0$. This is consistent with our choice of time weights in Chapter 2 and 3.

B.4 Micro-macro estimates

In this section we use the notations of Chapter 3. We consider the same Vlasov-Poisson-Fokker-Planck equation in perturbative abstract form

$$\begin{cases} \partial_t h + L_\tau h - E \cdot v = E \cdot A^* h. \\ E = \nabla_x \Delta_x^{-1} n \end{cases} \quad (\text{B.2})$$

For any $h \in \mathcal{H}$ we define

$$n(t, x) = \int_{\mathbb{R}^d} h(t, x, v) f^\infty(x, v) dv, \quad j(t, x) = \int_{\mathbb{R}^d} v h(t, x, v) f^\infty(x, v) dv.$$

Lemma B.4.1. *Let $h \in \mathcal{H}$ be a smooth solution of (B.2). Then (n, j) satisfies*

$$\partial_t n + \nabla_x \cdot j = 0 \quad (\text{B.3})$$

$$\partial_t j + \nabla_x n + \frac{1}{\tau} j - E \rho_\infty - (E_\infty^\delta + E) n = \nabla_x \cdot \int_{\mathbb{R}^d} (I - v \otimes v) h f^\infty dv \quad (\text{B.4})$$

Proof. Let us integrate

$$\partial_t h + L_\tau h - E \cdot v = E \cdot A^* h,$$

against f^∞ in the velocity variable. First,

$$\int_{\mathbb{R}^d} A^* \cdot A h f^\infty dv = \int_{\mathbb{R}^d} A h \cdot A 1 f^\infty dv = 0 \quad \text{and} \quad E \cdot \int_{\mathbb{R}^d} v f^\infty dv = 0.$$

Then,

$$\begin{aligned} \int_{\mathbb{R}^d} v \cdot \nabla_x h f^\infty dv &= \nabla_x \cdot \int_{\mathbb{R}^d} v h f^\infty dv - E_\infty^\delta \cdot \int_{\mathbb{R}^d} v h f^\infty dv, \\ &= \nabla_x \cdot j - E_\infty^\delta \cdot j. \end{aligned}$$

Finally

$$E_\infty^\delta \cdot \int_{\mathbb{R}^d} \nabla_v h f^\infty dv = E_\infty^\delta \cdot \int_{\mathbb{R}^d} h A^*(1) f^\infty dv = E_\infty^\delta \cdot j.$$

To derive the equation on j , we integrate the kinetic equation against $v f^\infty$ in the velocity variable and use that

$$\int_{\mathbb{R}^d} v E \cdot v f^\infty dv = E \rho_\infty.$$

Then mark that $AA^*(1) = I$ to get

$$\int_{\mathbb{R}^d} vA^* \cdot Ahf^\infty dv = \int_{\mathbb{R}^d} A^*(1)hf^\infty dv = j.$$

The transport in real space yields

$$\int_{\mathbb{R}^d} vv \cdot \nabla_x hf^\infty dv = \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v hf^\infty dv - E_\infty^\delta \cdot \int_{\mathbb{R}^d} v \otimes v hf^\infty dv.$$

For the transport in velocity space part, it holds

$$\begin{aligned} \int_{\mathbb{R}^d} vE_\infty^\delta \cdot \nabla_v hf^\infty dv &= \int_{\mathbb{R}^d} vA \cdot (E_\infty^\delta h) f^\infty dv \\ &= E_\infty^\delta \cdot \int_{\mathbb{R}^d} (v \otimes v - I) hf^\infty dv, \end{aligned}$$

where we used $A^*(v) = v \otimes v - I$. The last equality stems from similar computations and using that $A(v) = I$ and yields

$$\int_{\mathbb{R}^d} vE \cdot A^* hf^\infty dv = nE.$$

□

Lemma B.4.2. *Let h be a smooth solution of (B.2). Then, the following macroscopic estimates hold.*

$$\frac{d}{dt} \left(\int_{\mathbb{T}^d} j \cdot \nabla_x n dx + \frac{1}{2\tau} \|n\|_{L^2}^2 \right) + \frac{1}{\delta^2} \|\sqrt{\rho_\infty} n\|_{L^2}^2 + \|\nabla_x n\|_{L^2}^2 \quad (\text{B.5})$$

$$= \|\nabla_x \cdot j\|_{L^2}^2 + \int_{\mathbb{T}^d} nE_\infty^\delta \cdot (\nabla_x n - \rho_\infty E) dx - \frac{1}{2\delta^2} \int_{\mathbb{T}^d} n^3 dx + \left\langle \nabla_x n, \nabla_x \cdot \int_{\mathbb{R}^d} (I - v \otimes v) h f^\infty dv \right\rangle_{L^2(\mathbb{T}^d)}$$

$$\frac{d}{dt} \left(- \int_{\mathbb{T}^d} j \cdot E dx + \frac{\delta^2}{2\tau} \|E\|_{L^2}^2 \right) + \|\sqrt{\rho_\infty} E\|_{L^2}^2 + \frac{1}{\delta^2} \|n\|_{L^2}^2 \quad (\text{B.6})$$

$$= \delta^2 \|\partial_t \nabla_x \psi\|_{L^2}^2 - \int_{\mathbb{T}^d} nE_\infty^\delta \cdot E dx - \int_{\mathbb{T}^d} n|E|^2 dx - \left\langle E, \nabla_x \cdot \int_{\mathbb{R}^d} (I - v \otimes v) h f^\infty dv \right\rangle_{L^2(\mathbb{T}^d)}$$

Proof. For the first macroscopic estimates, multiply (B.4),

$$\partial_t j + \nabla_x n + \frac{1}{\tau} j - E\rho_\infty - E_\infty^\delta n = \nabla_x \cdot \int_{\mathbb{R}^d} (I - v \otimes v) h f^\infty dv$$

by $\nabla_x n$ and integrate in the space variable to get

$$\begin{aligned} &\int_{\mathbb{T}^d} \partial_t j \cdot \nabla_x n dx + \|\nabla_x n\|_{L^2}^2 + \frac{1}{\tau} \int_{\mathbb{T}^d} j \cdot \nabla_x n dx - \int_{\mathbb{T}^d} (\rho_\infty + n) E \cdot \nabla_x n dx \\ &= \int_{\mathbb{T}^d} nE_\infty^\delta \cdot \nabla_x n dx + \left\langle \nabla_x n, \nabla_x \cdot \int_{\mathbb{R}^d} (I - v \otimes v) h f^\infty dv \right\rangle_{L^2(\mathbb{T}^d)}. \end{aligned}$$

The first term of the left-hand side rewrites

$$\int_{\mathbb{T}^d} \partial_t j \cdot \nabla_x n dx = \frac{d}{dt} \int_{\mathbb{T}^d} j \cdot \nabla_x n dx - \int_{\mathbb{T}^d} j \cdot \nabla_x \partial_t n dx,$$

and an integration by parts and Equation (B.3) yields

$$\int_{\mathbb{T}^d} j \cdot \nabla_x \partial_t n dx = \|\nabla_x \cdot j\|_{L^2}^2.$$

The third term of the left-hand side rewrites

$$\frac{1}{\tau} \int_{\mathbb{T}^d} j \cdot \nabla_x n dx = \frac{1}{2\tau} \frac{d}{dt} \|n\|_{L^2}^2$$

and the fourth term gives

$$- \int_{\mathbb{T}^d} \rho_\infty E \cdot \nabla_x n dx = \frac{1}{\delta^2} \|\sqrt{\rho_\infty} n\|_{L^2}^2 + \int_{\mathbb{T}^d} n \rho_\infty E_\infty^\delta \cdot E dx.$$

Finally, the nonlinear term yields,

$$- \int_{\mathbb{T}^d} n E \cdot \nabla_x n dx = \frac{1}{2\delta^2} \int_{\mathbb{T}^d} n^3 dx$$

For the second macroscopic estimates, multiply (B.4), by $-E$ and integrate in the space variable to get

$$\begin{aligned} & - \int_{\mathbb{T}^d} \partial_t j \cdot E dx - \int_{\mathbb{T}^d} \nabla_x n \cdot E dx - \frac{1}{\tau} \int_{\mathbb{T}^d} j \cdot E dx + \int_{\mathbb{T}^d} (\rho_\infty + n) |E|^2 dx \\ & = - \int_{\mathbb{T}^d} n E_\infty^\delta \cdot E dx - \left\langle E, \nabla_x \cdot \int_{\mathbb{R}^d} (I - v \otimes v) h f^\infty dv \right\rangle_{L^2(\mathbb{T}^d)}. \end{aligned}$$

The first term of the left-hand side rewrites

$$\int_{\mathbb{T}^d} \partial_t j \cdot \nabla_x \psi dx = \frac{d}{dt} \int_{\mathbb{T}^d} j \cdot \nabla_x \psi dx - \int_{\mathbb{T}^d} j \cdot \nabla_x \partial_t \psi dx,$$

and an integration by parts and Equation (B.3) yields

$$\int_{\mathbb{T}^d} j \cdot \nabla_x \partial_t \psi dx = \int_{\mathbb{T}^d} \partial_t n \partial_t \psi dx = \delta^2 \|\partial_t \nabla_x \psi\|_{L^2}^2$$

The second term of the left-hand side yields

$$- \int_{\mathbb{T}^d} \nabla_x n \cdot E dx = \frac{1}{\delta^2} \|n\|_{L^2}^2$$

The third term of the left-hand side rewrites

$$-\frac{1}{\tau} \int_{\mathbb{T}^d} j \cdot E dx = \frac{1}{\tau} \int_{\mathbb{T}^d} \partial_t n \psi dx = \frac{\delta^2}{2\tau} \frac{d}{dt} \|E\|_{L^2}^2$$

□

Physical quantities	Characteristic quantities
Number densities	\bar{n}
Temperatures	θ
Ions thermal velocity	$V_i = \sqrt{\frac{k_B \theta}{m_i}}$
Electrons thermal velocity	$V_e = \sqrt{\frac{k_B \theta}{m_e}}$
Ions distribution function	$\bar{f}_i = \frac{\bar{n}}{V_i^3}$
Electrons distribution function	$\bar{f}_e = \frac{\bar{n}}{V_e^3}$
Ions collision frequency	ν_i
Electrons collision frequency	ν_e
Ions mean free path	$l_i = \frac{V_i}{\nu_i}$
Electrons mean free path	$l_e = \frac{V_e}{\nu_e}$
Reference length	L
Ion reference time	$t_0 = \frac{L}{V_i}$
Electric potential	$\bar{\phi} = \frac{k_B \theta}{q}$
Magnetic field amplitude	\bar{B}

TABLE B.1 – **Scaling of the bi-kinetic system.** Physical characteristic quantities

Annexe C

Lemmes techniques

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C.1 Interpolation results

Lemma C.1.1. *Let $f \equiv f(x, v) \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be an almost everywhere non-negative measurable function such that $(x, v) \mapsto |v|^m f(x, v) \in L^1_{x,v}$ for some $m \geq 0$. Then for any $0 \leq k \leq m$,*

$$\left\| \int f|v|^k dv \right\|_{L_x^{\frac{m+d}{d+k}}} \leq C_{d,k} \|f\|_{L_{x,v}^{\frac{m-k}{d+m}}} \left(\iint_{\mathbb{R}^{2d}} f|v|^m dv dx \right)^{\frac{k+d}{d+m}} \quad (\text{C.1})$$

where $C_{d,k}$ is a positive constant depending only on d and k .

See [105, Lemma 3.1] for a proof.

C.2 Functional inequalities

Theorem C.2.1 (Gross's Log-Sobolev inequality [112]). *Let ν denote a Gaussian measure on \mathbb{R}^d and let $u : \mathbb{R}^d \mapsto \mathbb{C}$. If $u \in H^1(\mathbb{R}^d, d\nu)$, then u is in the Orlicz space $L^2 \ln^+ L$ and,*

$$\int_{\mathbb{R}^d} |u|^2 \ln |u| d\nu \leq \int_{\mathbb{R}^d} |\nabla u|^2 d\nu + \|u\|_2^2 \ln \|u\|_2,$$

where $\|\cdot\|_2$ is the usual $L^2(\mathbb{R}^d, d\nu)$ norm.

See [112] for a proof.

Corollary C.2.2. *Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}_+$ be an integrable function such that $\|\nabla_v \sqrt{\frac{f}{M}}\|_{L^2(Mdvdx)}$ is finite. Then,*

$$\iint_{\mathbb{R}^6} f \ln \left(\frac{f}{nM} \right) dvdx \leq 2 \iint_{\mathbb{R}^6} \left| \nabla_v \sqrt{\frac{f}{M}} \right|^2 M dvdx,$$

where $n = \int_{\mathbb{R}^3} f dv$ and M is defined in (1.6).

Démonstration. Take $d\nu = Mdv$ and $u = \sqrt{\frac{f}{M}}$ in Theorem C.2.1. □

Lemma C.2.3 (Csiszar-Kullback-Pinsker inequality [55]). *Let μ denote a probability measure and $u \in L^1(\mathbb{R}^d, d\mu)$ a non negative function such that $\int_{\mathbb{R}^d} u d\mu = 1$. Then it holds,*

$$\left(\int_{\mathbb{R}^d} |u - 1| d\mu \right)^2 \leq 2 \int_{\mathbb{R}^d} u \ln u d\mu.$$

Démonstration. Define $h : x \geq 0 \mapsto x \ln x - x + 1$. It is easy to see that since $x - 1 \geq \ln x$, $1/x - 1 \geq -\ln x$ and $h(x) \geq 0$. Now let

$$g : x \mapsto (x - 1)^2 - \left(\frac{4}{3} + \frac{2}{3}x \right) h(x).$$

One has $g(1) = g'(1) = 0$ and $g''(x) = -\frac{4h(x)}{3x} \leq 0$. Therefore by the Taylor-Lagrange theorem, for every $x > 0$, there exists $\xi > 0$ such that $|\xi - 1| < |x - 1|$ and,

$$g(x) = \frac{g''(\xi)}{2} (x - 1)^2 = -\frac{2h(\xi)}{3\xi} (x - 1)^2 \leq 0.$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^d} |u - 1| d\mu &\leq \int_{\mathbb{R}^d} \sqrt{\left(\frac{4}{3} + \frac{2}{3}u \right) h(u)} d\mu \\ &\leq \sqrt{\int_{\mathbb{R}^d} \left(\frac{4}{3} + \frac{2}{3}u \right) d\mu} \sqrt{\int_{\mathbb{R}^d} h(u) d\mu} \\ &= \sqrt{2 \int_{\mathbb{R}^d} u \ln u d\mu}. \end{aligned}$$

where we used the Cauchy-Schwartz inequality. □

Corollary C.2.4. *Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}_+$ be an integrable function. Then,*

$$\left(\iint_{\mathbb{R}^6} |f - nM| dvdx \right)^2 \leq 2 \left(\int_{\mathbb{R}^3} ndx \right) \iint_{\mathbb{R}^6} f \ln \left(\frac{f}{nM} \right) dvdx$$

where $n = \int_{\mathbb{R}^3} f dv$ and M is defined in (1.6).

Démonstration. Take $d\mu = \frac{n}{\int_{\mathbb{R}^3} ndx} Mdvdx$ and $u = \frac{f}{nM}$ in Lemma C.2.4. □

C.3 Compactness results

Theorem C.3.1 (Aubin-Lions-Simon [182]). *Let $X \hookrightarrow B \hookrightarrow Y$ be three Banach spaces such that the first embedding $X \hookrightarrow B$ is compact. Assume that for some $1 \leq p \leq +\infty$,*

- F is bounded in $L^p(0, T; X)$,
- $\|\tau_s f - f\|_{L^p(0, T-s; Y)} \rightarrow 0$ as $s \rightarrow 0$, uniformly for $f \in F$,

Then F is relatively compact in $L^p(0, T; B)$ (and $C(0, T; B)$ if $p = +\infty$).

See [182, Theorem 5, Corollary 4] for a proof.

Theorem C.3.2. *Let $n \in L^p(\mathbb{R}^3)$, $1 < p < \infty$, and let $\phi = \Phi * n$, where Φ is defined in (1.4). Then $-\Delta\phi = n$ almost everywhere and, for any i, j ,*

- *Hardy-Littlewood-Sobolev inequality : If $p < 3$, then, for $q = 3p/(3 - p)$,*

$$\|\partial_{x_i} \phi\|_{L^q(\mathbb{R}^3)} \leq C_{p,q} \|n\|_{L^p(\mathbb{R}^3)}$$

where $C_{p,q}$ is a constant that depends only on p .

- *Calderon-Zygmund inequality :*

$$\|\partial_{x_i} \partial_{x_j} \phi\|_{L^p(\mathbb{R}^3)} \leq C_p \|n\|_{L^p(\mathbb{R}^3)}$$

where C_p is a constant that depends only on p .

Both inequalities are proved in Stein [183]. For the Calderon-Zygmund inequality, let us also mention that the case of a source n with compact support is treated in [95, Theorem 9.9] and extended to any source in dimension 3 or higher, in the remark following the proof.

Lemma C.3.3. *Let $(n^\varepsilon : (t, x) \in [0, T] \times \mathbb{R}^d \mapsto \bar{\mathbb{R}})_{\varepsilon > 0}$ be a family of measurable functions. Suppose that,*

- $(n^\varepsilon)_{\varepsilon > 0}$ is a bounded sequence of $L^1(0, T; L^1((1 + |x|)dx))$,
- $\lim_{y \rightarrow 0} \sup_\varepsilon \|\tau_y n^\varepsilon - n^\varepsilon\|_{L^1([0, T] \times \mathbb{R}^d)} = 0$,
- $(\partial_t n^\varepsilon)_{\varepsilon > 0}$ is bounded in $L^1(0, T; W^{-1,1}(\mathbb{R}^d))$,

where τ_y is the translation of y in the space variable and $W^{-1,1}(\mathbb{R}^d)$ denotes the topological dual of the Sobolev space $W^{1,\infty}(\mathbb{R}^d)$. Then,

$$(n^\varepsilon)_{\varepsilon > 0} \text{ is relatively compact in } L^1([0, T] \times \mathbb{R}^d)$$

Démonstration. The goal is to use the Riesz-Fréchet-Kolmogorov theorem but we miss the uniform smallness of the translation in time. Through a mollifying process, we can use the bound on the time derivative and the space compactness to recover this property. Let $(\rho_\delta)_{\delta > 0}$ be the following sequence of standard mollifiers,

$$\rho_\delta(x) = \frac{1}{\delta^d \int \rho dx} \rho(x/\delta),$$

where ρ is given by,

$$\rho(x) = \begin{cases} \frac{1}{e^{|x|^2} - 1} & \text{if } x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

One sees that there exists a universal constant $C > 0$ such that,

$$\|\rho_\delta\|_{W^{1,\infty}(\mathbb{R}^d)} \leq \frac{C}{\delta^{d+1}}.$$

Let us denote by τ_s is the translation of s in the time variable. Using this estimate we have, almost everywhere in t, x ,

$$\begin{aligned} |(\tau_s - 1)n^\varepsilon *_x \rho_\delta| &= \left| \int_{\mathbb{R}^d} (\tau_s - 1)n^\varepsilon(\cdot, y)\rho_\delta(x - y)dy \right| \\ &= \left| \int_{\mathbb{R}^d} \left(\int_t^{t+s} \partial_t n^\varepsilon(\tau, y)d\tau \right) \rho_\delta(x - y)dy \right| \\ &\leq \int_t^{t+s} \left| \langle \partial_t n^\varepsilon(\tau, \cdot), \rho_\delta(x - \cdot) \rangle_{W^{-1,1}, W^{1,\infty}} \right| d\tau \\ &\leq \frac{C}{\delta^{d+1}} \int_t^{t+s} \|\partial_t n^\varepsilon(\tau, \cdot)\|_{W^{-1,1}} d\tau. \end{aligned}$$

Therefore if we integrate on $[0, T) \times B_R$ where B_R is the ball of radius R in \mathbb{R}^d , it yields,

$$\|(\tau_s - 1)n^\varepsilon *_x \rho_\delta\|_{L^1([0,T) \times B_R)} = O\left(\frac{R^d |s|}{\delta^{d+1}}\right)$$

and thanks to the $L^1(0, T; L^1((1 + |x|)dx))$ bound,

$$\sup_{\varepsilon > 0} \|(\tau_s - 1)n^\varepsilon *_x \rho_\delta\|_{L^1([0,T) \times \mathbb{R}^d)} = O\left(\frac{R^d |s|}{\delta^{d+1}} + \frac{1}{R}\right)$$

and optimizing in R yields,

$$\sup_{\varepsilon > 0} \|(\tau_s - 1)n^\varepsilon *_x \rho_\delta\|_{L^1([0,T) \times \mathbb{R}^d)} = O\left(\frac{|s|^{\frac{1}{d+1}}}{\delta}\right) \quad (\text{C.2})$$

Now as in the proof of the Riesz-Fr chet-Kolmogorov theorem, we shall remark that,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} |n^\varepsilon *_x \rho_\delta - n^\varepsilon| dx dt &\leq \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |n^\varepsilon(t, x - y) - n^\varepsilon(t, x)| \rho_\delta(y) dy dx dt \\ &\leq \int_{B_\delta} \rho_\delta(y) dy \sup_{y \in B_\delta} \|\tau_y n^\varepsilon - n^\varepsilon\|_{L^1([0,T) \times \mathbb{R}^d)} \end{aligned}$$

and hence,

$$\limsup_{\delta \rightarrow 0} \sup_{\varepsilon > 0} \|n^\varepsilon *_x \rho_\delta - n^\varepsilon\|_{L^1([0,T) \times \mathbb{R}^d)} = 0.$$

Finally, by taking, say, $\delta = |s|^{\frac{1}{2(d+1)}}$ in (C.2), and using that,

$$\|(\tau_s - 1)n^\varepsilon\|_{L^1([0,T] \times \mathbb{R}^d)} \leq 2\|n^\varepsilon *_x \rho_\delta - n^\varepsilon\|_{L^1([0,T] \times \mathbb{R}^d)} + \|(\tau_s - 1)n^\varepsilon *_x \rho_\delta\|_{L^1([0,T] \times \mathbb{R}^d)}$$

we showed that,

$$\limsup_{s \rightarrow 0} \sup_{\varepsilon > 0} \|\tau_s n^\varepsilon - n^\varepsilon\|_{L^1([0,T] \times \mathbb{R}^d)} = 0.$$

Relative compactness follows from the Riesz-Fréchet-Kolmogorov theorem. \square

C.4 Maximum principle on the torus

For convenience we state and prove here maximum principles adapted to our purposes.

Lemma C.4.1 (Maximum principle on the torus, three-dimensional case). *There exists a constant C such that if $u \in W^{1,1}(\mathbb{T}^3)$ and $-\Delta u \leq K_0$ with $K_0 \in L^{3/2,1}(\mathbb{T}^3)$ then*

$$\operatorname{ess\,sup}_{\mathbb{T}^3} u \leq C (\|u\|_{W^{1,1}(\mathbb{T}^3)} + \|K_0\|_{L^{3/2,1}(\mathbb{T}^3)}).$$

We refer for instance to [139, Chapter 2] for relevant basic properties of Lorentz spaces appearing in the statement of the former lemma.

The foregoing lemma is readily derived from the following one through the canonical identification of functions on \mathbb{T}^3 and periodic functions on \mathbb{R}^3 by choosing Ω containing a fundamental domain and Ω' a suitably larger bounded domain.

Lemma C.4.2 (Maximum principle, three-dimensional case). *Let Ω' be an open subset of \mathbb{R}^3 and Ω be a bounded open subset of Ω' such that $\bar{\Omega} \subset \Omega'$. There exists a constant C such that if $u \in W^{1,1}(\Omega')$ and $-\Delta u \leq K_0$ with $K_0 \in L^{3/2,1}(\Omega')$ then*

$$\operatorname{ess\,sup}_{\Omega} u \leq C (\|u\|_{W^{1,1}(\Omega')} + \|K_0\|_{L^{3/2,1}(\Omega')}).$$

Démonstration. It is sufficient to prove the estimate when u and K_0 are smooth. From there a classical approximation argument yields the full result by providing upper bounds at any Lebesgue point of u .

For any $0 \leq \varepsilon < \delta$ and $x_0 \in \mathbb{R}^3$, we set

$$\Omega_{\varepsilon,\delta}(x_0) = \{ y \in \mathbb{R}^3 \mid \varepsilon < \|y - x_0\| < \delta \}$$

and for any $\varepsilon > 0$ and $x_0 \in \mathbb{R}^3$, we denote

$$S_\varepsilon(x_0) = \{ y \in \mathbb{R}^3 \mid \|y - x_0\| = \varepsilon \}.$$

We also consider $G : \mathbb{R}^3 \rightarrow \mathbb{R}$, $y \mapsto \|y\|^{-1}$, which is a multiple of the Green function of the Laplacian on \mathbb{R}^3 .

First we choose $R > 0$ such that for any $x_0 \in \Omega$, $\overline{\Omega_{0,\delta}(x_0)} \subset \Omega'$. Then, for any $x_0 \in \Omega$, repeated use of the Green formula provides for any $0 < \varepsilon < R$

$$\begin{aligned}
 \frac{1}{\varepsilon^2} \int_{S_\varepsilon(x_0)} u &= \frac{1}{R^2} \int_{S_R(x_0)} u + \left(\frac{1}{\varepsilon} - \frac{1}{R} \right) \int_{\Omega_{0,\varepsilon}(x_0)} (-\Delta u) \\
 &\quad + \int_{\Omega_{\varepsilon,R}(x_0)} \left(G(\cdot - x_0) - \frac{1}{R} \right) (-\Delta u) \\
 &\leq \frac{1}{R^2} \int_{S_R(x_0)} u + \left(\frac{1}{\varepsilon} - \frac{1}{R} \right) \int_{\Omega_{0,\varepsilon}(x_0)} K_0 \\
 &\quad + \int_{\Omega_{\varepsilon,R}(x_0)} \left(G(\cdot - x_0) - \frac{1}{R} \right) K_0 \\
 &\leq \frac{1}{R^2} \|u\|_{L^1(S_R(x_0))} + \frac{1}{\varepsilon} \|K_0\|_{L^1(\Omega_{0,\varepsilon}(x_0))} \\
 &\quad + \left\| \left(G(\cdot - x_0) - \frac{1}{R} \right) K_0 \right\|_{L^1(\Omega_{\varepsilon,R}(x_0))} \\
 &\leq C' \|u\|_{W^{1,1}(\Omega')} + C' \|K_0\|_{L^{3/2}(\Omega_{0,\varepsilon}(x_0))} + C' \|K_0\|_{L^{3/2,1}(\Omega')}
 \end{aligned}$$

for some constant C' depending only on R (for instance through $\|G - R^{-1}\|_{L^{3,\infty}(\Omega_{0,R}(0))}$). Taking the limit $\varepsilon \rightarrow 0$ provides the estimate at x_0 . \square

We shall also use the foregoing slight generalization.

Lemma C.4.3 (Maximum principle, three-dimensional case, second version). *Let Ω' be an open subset of \mathbb{R}^3 and Ω be a bounded open subset of Ω' such that $\overline{\Omega} \subset \Omega'$. There exists a constant C such that if $u \in W^{1,1}(\Omega')$ and, for some $K_0 \in L^{3/2,1}(\Omega')$, stands $\chi_{u \geq 0}(-\Delta u) \leq K_0$ on Ω' then*

$$\operatorname{ess\,sup}_\Omega u \leq C (\|u\|_{W^{1,1}(\Omega')} + \|K_0\|_{L^{3/2,1}(\Omega')}).$$

Proof. Since $u \leq u_+$ (where $(\cdot)_+$ denotes positive part) the proof is achieved by applying Lemma C.4.2 to u_+ since $\nabla u_+ = \chi_{u > 0} \nabla u$ and

$$-\Delta u_+ \leq \chi_{u > 0}(-\Delta u) + \frac{1}{2} \chi_{u=0}(-\Delta u)$$

as is classical and derived by inspecting the limit $\varepsilon \rightarrow 0$ of

$$\frac{1}{2} (\sqrt{\varepsilon + u^2} + u).$$

\square

At last we observe that a slight variation on the proof of Lemma C.4.2 provides the following two-dimensional versions of Lemmas C.4.1 and C.4.3.

Lemma C.4.4 (Maximum principle on the torus, two-dimensional case). *For any $1 < p \leq \infty$ there exists a constant C such that if $u \in W^{1,1}(\mathbb{T}^2)$ and $-\Delta u \leq K_0$ with $K_0 \in L^p(\mathbb{T}^2)$ then*

$$\operatorname{ess\,sup}_{\mathbb{T}^2} u \leq C (\|u\|_{W^{1,1}(\mathbb{T}^2)} + \|K_0\|_{L^p(\mathbb{T}^2)}).$$

Lemma C.4.5 (Maximum principle, two-dimensional case, second version). *Let $1 < p < \infty$, Ω' be an open subset of \mathbb{R}^2 and Ω be a bounded open subset of Ω' such that $\bar{\Omega} \subset \Omega'$. There exists a constant C such that if $u \in W^{1,1}(\Omega')$ and $\chi_{u \geq 0}(-\Delta u) \leq K_0$ on with $K_0 \in L^p(\Omega')$ then*

$$\operatorname{ess\,sup}_{\Omega} u \leq C (\|u\|_{W^{1,1}(\Omega')} + \|K_0\|_{L^p(\Omega')}).$$

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Analyse asymptotique et numérique de quelques modèles pour le transport de particules chargées

Résumé : Cette thèse est consacrée à l'étude mathématique de quelques modèles d'équations aux dérivées partielles issues de la physique des plasmas. On s'intéresse principalement à l'analyse théorique de différents régimes asymptotiques de systèmes d'équations cinétiques de type Vlasov-Poisson-Fokker-Planck. Dans un premier temps, en présence d'un champ magnétique extérieur on se concentre sur l'approximation des électrons sans masse fournissant des modèles réduits lorsque le rapport m_e/m_i entre la masse m_e d'un électron et la masse m_i d'un ion tend vers 0 dans les modèles. Suivant le régime considéré, on montre qu'à la limite les solutions vérifient des modèles hydrodynamiques de type convection-diffusion ou sont données par des densités de type Maxwell-Boltzmann-Gibbs, suivant l'intensité des collisions dans la mise à l'échelle. En utilisant les propriétés hypocoercives et hypoelliptiques des équations, on est capable d'obtenir des taux de convergence en fonction du rapport de masse. Dans un second temps, par des méthodes similaires, on montre la convergence exponentielle en temps long vers l'équilibre des solutions du système de Vlasov-Poisson-Fokker-Planck sans champ magnétique avec des taux explicites en les paramètres du modèles. Enfin, on conçoit un nouveau type de schéma volumes finis pour des équations de convection-diffusion non-linéaires assurant le bon comportement en temps long des solutions discrètes. Ces propriétés sont vérifiées numériquement sur plusieurs modèles dont l'équation de Fokker-Planck avec champ magnétique.

Mots clés : équations cinétiques ; plasma ; électrons sans masse ; Vlasov-Poisson ; Fokker-Planck ; entropie relative ; hypocoercivité ; limite de diffusion ; Maxwell-Boltzmann ; schémas volumes finis ; convection-diffusion.

Asymptotic and numerical analysis of kinetic and fluid models for the transport of charged particles

Abstract : This thesis is devoted to the mathematical study of some models of partial differential equations from plasma physics. We are mainly interested in the theoretical study of various asymptotic regimes of Vlasov-Poisson-Fokker-Planck systems. First, in the presence of an external magnetic field, we focus on the approximation of massless electrons providing reduced models when the ratio m_e/m_i between the mass m_e of an electron and the mass m_i of an ion tends to 0 in the equations. Depending on the scaling, it is shown that, at the limit, solutions satisfy hydrodynamic models of convection-diffusion type or are given by Maxwell-Boltzmann-Gibbs densities depending on the intensity of collisions. Using hypocoercive and hypoelliptic properties of the equations, we are able to obtain convergence rates as a function of the mass ratio. In a second step, by similar methods, we show exponential convergence of solutions of the Vlasov-Poisson-Fokker-Planck system without magnetic field towards the steady state, with explicit rates depending on the parameters of the model. Finally, we design a new type of finite volume scheme for a class of nonlinear convection-diffusion equations ensuring the satisfying long-time behavior of discrete solutions. These properties are verified numerically on several models including the Fokker-Planck equation with magnetic field.

Keywords : Kinetic equations ; plasma ; massless electrons ; Vlasov-Poisson ; Fokker-Planck ; relative entropy ; hypocoercivity ; diffusion limit ; Maxwell-Boltzmann ; finite volume schemes ; convection-diffusion.

Image en couverture : Évolution temporelle de la distribution en vitesse de particules soumises à des collisions et à un champ magnétique orthogonal à la page. Dans le sens négatif, de l'état initial en haut à gauche à l'état stationnaire en bas à gauche.

