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Courbes intégrales : transcendance et géométrie

Tiago Jardim da Fonseca

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Tiago Jardim da Fonseca. Courbes intégrales : transcendance et géométrie. Théorie des nombres [math.NT]. Université Paris Saclay (COmUE), 2017. Français. NNT : 2017SACLS515 . tel-01685449

HAL Id: tel-01685449

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Submitted on 16 Jan 2018

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Courbes intégrales : transcendance et géométrie

Thèse de doctorat de l'Université Paris-Saclay
préparée à l'Université Paris-Sud

École doctorale de mathématiques Hadamard (EDMH, ED 574)
Spécialité de doctorat: Mathématiques fondamentales

Thèse présentée et soutenue à Orsay, le 12 décembre 2017, par

Tiago Jardim da Fonseca

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*“Ah, não há saudades mais dolorosas do que
as das coisas que nunca foram!”*

Fernando Pessoa, *Livro do Desassossego*

Remerciements

Je tiens tout d'abord à remercier Jean-Benoît Bost pour avoir accepté de diriger cette thèse. Je lui suis reconnaissant pour tout ce qu'il m'a appris au long de ces dernières années, pour ses encouragements, pour me faire découvrir un beau sujet de mathématiques et pour avoir partagé un peu de sa vaste culture scientifique.

Cette thèse n'aurait pas existé sans l'influence de Laurent Clozel, qui a su identifier mes goûts mathématiques et m'a proposé d'aller voir M. Bost. Je voudrais le remercier pour son mémorable cours de théorie des nombres en M2 avec Guy Henniart et pour son intérêt dans mes travaux de thèse.

Je remercie Yves André et Carlo Gasbarri pour avoir accepté de rapporter cette thèse dans un court délai. Je remercie également Emmanuel Ullmo, Daniel Bertrand et Hugues Randriambololona pour avoir accepté de composer mon jury de thèse.

J'adresse aussi mes sincères remerciements à tout le personnel du Département de Mathématiques d'Orsay, de la Fondation Mathématique Jacques Hadamard (FMJH), qui a financé mon master et ma thèse, et de l'École Doctorale Jacques Hadamard (EDMH).

Parmi les personnes avec qui j'ai eu des conversations mathématiques, je voudrais remercier particulièrement Mikolaj Fraczyk, Yang Cao, Tuan Huynh, Salim Tayou, Aliaksandr Minets, Raju Krishnamoorthy et Javier Fresán. Je remercie aussi Hossein Movasati pour ses commentaires sur mes articles et Daniel Bertrand pour ses remarques bibliographiques.

J'exprime ma gratitude à tous ceux qui ont contribué de façon indirecte à la conclusion de ce mémoire. Je remercie tous les participants des « séminaires secrets d'Orsay », dont l'anonymat sera préservé, et tous les doctorants du bâtiment 430, spécialement mes anciens compagnons du bureau 14 : Cong, Mikolaj, Lison, Lucile et Yang. Enfin, j'adresse mon *muito obrigado* à mes amis brésiliens à Paris, Danilo, Trang, Davi et Tamy, à mes parents pour leur soutien ultra-marin et à Anita pour rendre ma vie plus heureuse.

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Introduction

“Le seul véritable voyage, le seul bain de Jouvence, ce ne serait pas d’aller vers de nouveaux paysages, mais d’avoir d’autres yeux, de voir l’univers avec les yeux d’un autre, de cent autres, de voir les cent univers que chacun d’eux voit, que chacun d’eux est.”

Marcel Proust, *La Prisonnière*

Ce mémoire de thèse est composé de trois chapitres, correspondant à trois prépublications successives ([arXiv:1612.05081](#), [arXiv:1703.02954](#), [arXiv:1710.00563](#)). Chaque chapitre est pourvu d’une introduction présentant ses résultats les plus importants et d’une courte mise en contexte historique.

Dans cette introduction générale, nous nous proposons, d’une part, de mettre en évidence la cohésion des différentes questions étudiées dans ces trois articles en les ramenant toutes à un même problème racine, le problème de *comprendre la vraie portée des méthodes de Nesterenko sur l’indépendance algébrique de valeurs de formes quasi-modulaires*. D’autre part, nous décrivons quelques développements historiquement importants de la théorie des nombres transcendants, destinés à mettre notre contribution en perspective.

Dans ce mémoire, nous adopterons un point de vue *géométrique*. Le rapport entre la théorie des nombres transcendants et la géométrie sera observé à deux niveaux distincts : l’étude de la transcendance de nombres ayant une nature géométrique — les *périodes* des variétés algébriques — et l’application de techniques géométriques aux méthodes de démonstration de la transcendance de certains nombres.

C’est aussi notre intention, dans les paragraphes qui suivent, de montrer que ce *regard* géométrique, non seulement fournit des outils supplémentaires pour la compréhension des phénomènes classiques de transcendance, mais suscite aussi de nouveaux problèmes tout aussi stimulants.

Indépendance algébrique de valeurs de fonctions analytiques

Le terme « transcendant » en mathématiques est l’antonyme d’« algébrique ». Ainsi, un nombre complexe α est dit *transcendant* s’il n’est pas algébrique, *i.e.*, s’il n’existe pas de polynôme non-nul $P \in \mathbf{Q}[X]$ tel que $P(\alpha) = 0$. Plus généralement, des éléments $\alpha_1, \dots, \alpha_n$ dans un corps K sont dits *algébriquement indépendants* sur un sous-corps k de K s’il n’existe pas de polynôme non-nul $P \in k[X_1, \dots, X_n]$ tel que $P(\alpha_1, \dots, \alpha_n) = 0$; on dit alors que l’ensemble $\{\alpha_1, \dots, \alpha_n\} \subset K$ est algébriquement indépendant sur k .

Un problème de transcendance consiste donc à établir si un certain objet mathématique — un nombre, une fonction, une variété, *etc.* — est ou non algébrique. En termes quantitatifs, il s’agit de calculer, ou tout simplement d’estimer, le *degré de transcendance* d’une extension de corps. Rappelons que, si K est une extension du corps k , un sous-ensemble S de K est dit algébriquement indépendant sur k si tous ses sous-ensembles finis sont algébriquement indépendants sur k ; le degré de transcendance $\text{degtr}_k K$ se définit alors comme la plus grande cardinalité d’un sous-ensemble de K algébriquement indépendant sur k .

Dans le cas arithmétique, où l’on étudie l’indépendance algébrique sur \mathbf{Q} , il est connu depuis le papier fondateur de Liouville [62] qu’un problème de transcendance se ramène souvent à un problème d’*approximation diophantienne*. Ainsi, d’après Liouville, la transcendance ou l’algébricité d’un nombre réel se lit de la façon dont il est approché par des nombres rationnels. Voici l’énoncé qui formalise le célèbre *critère de Liouville* :

THÉORÈME (Liouville; cf. [3] Ch. 1). *Si un nombre réel α est algébrique de degré $d > 1$ sur \mathbf{Q} , alors il existe un réel $\varepsilon > 0$ tel que*

$$\left| \alpha - \frac{p}{q} \right| > \frac{\varepsilon}{q^d}$$

pour tous les nombres rationnels de la forme p/q avec $p, q \in \mathbf{Z}$ copremiers et $q > 0$.

Au fil des années, des critères de transcendance plus généraux se sont développés, tout en restant dans l'esprit de l'idée originale du théorème de Liouville; citons par exemple les sophistiqués critères d'indépendance algébrique de Nesterenko [74] et Philippon [85].

Ce rapport entre la théorie des nombres transcendants et l'approximation diophantienne suggère l'investigation de nombres qui s'obtiennent comme de valeurs de fonctions analytiques; en principe, des propriétés de nature analytique de ces fonctions, comme des conditions de croissance, des équations fonctionnelles ou différentielles, *etc.*, peuvent fournir des outils supplémentaires à l'étude de problèmes d'approximation de ses valeurs.

Historiquement, ces vagues idées se sont matérialisés en des résultats à la fois précis et généraux en deux exemples remarquables: les théories de Siegel-Shidlovsky et de Schneider-Lang.

La théorie de Siegel-Shidlovsky concerne l'indépendance algébrique de valeurs de *E-fonctions de Siegel* en des points algébriques. Rappelons qu'une série

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

définit une *E-fonction* si :

- (1) il existe un corps de nombres $K \subset \mathbf{C}$ tel que $a_n \in K$ pour tout $n \geq 0$;
- (2) pour tout $\varepsilon > 0$, on a $\max_{\sigma} |\sigma(a_n)| = O(n^{\varepsilon n})$, où σ parcourt l'ensemble de tous les plongements de corps de K dans \mathbf{C} ;
- (3) pour tout $\varepsilon > 0$, il existe une suite d'entiers strictement positifs $(q_n)_{n \geq 0}$ telle que $q_n = O(n^{n\varepsilon})$ et $q_n a_k$ est un entier algébrique de K pour tout $0 \leq k \leq n$.

La deuxième condition ci-dessus implique que f est une série entière sur \mathbf{C} . Parmi les exemples remarquables de *E-fonctions*, on rencontre la fonction exponentielle et quelques classes particulières de fonctions hypergéométriques, dont la fonction de Bessel

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{z}{2}\right)^{2n}.$$

THÉORÈME (Siegel-Shidlovsky; cf. [3] Ch. 11). *Soient $n \geq 1$ un entier et f_1, \dots, f_n des fonctions entières sur \mathbf{C} à coefficients de Taylor en l'origine dans un même corps de nombres $K \subset \mathbf{C}$ et supposons qu'ils existent des fonctions rationnelles $g_{ij} \in K(z)$, $1 \leq i, j \leq n$, telles que*

$$\frac{df_i}{dz} = \sum_{j=1}^n g_{ij} f_j$$

pour tout $1 \leq i \leq n$. Si de plus :

- (1) $\text{degtr}_{K(z)} K(z)(f_1, \dots, f_n) = n$, et
- (2) chaque f_i est une *E-fonction de Siegel*,

alors, pour tout nombre algébrique non-nul $\alpha \in \mathbf{C}$ qui n'est pas contenu dans l'ensemble des pôles des g_{ij} , on a

$$\text{degtr}_K K(f_1(\alpha), \dots, f_n(\alpha)) = n.$$

Puisque J_0 satisfait l'équation de Bessel :

$$z^2 \frac{d^2 J_0}{dz^2} + z \frac{dJ_0}{dz} + z^2 J_0 = 0,$$

on déduit du théorème ci-dessus que $J_0(\alpha)$ et $J'_0(\alpha)$ sont algébriquement indépendants pour tout nombre algébrique $\alpha \in \mathbf{C}$.

La théorie de Schneider-Lang, à son tour, considère une autre notion de croissance. Étant donné un nombre réel $\rho > 0$, on dit qu'une fonction f entière sur \mathbf{C} est d'ordre inférieur à ρ s'ils existent des nombres réels $a, b > 0$ tels que

$$|f(z)| \leq ae^{b|z|^\rho}$$

pour tout $z \in \mathbf{C}$. Une fonction méromorphe sur \mathbf{C} est dite d'ordre inférieur à ρ si elle peut s'écrire comme quotient de deux fonctions entières d'ordre inférieur à ρ .

THÉORÈME (Schneider-Lang; cf. [94] Thm. 3.3.1). *Soient $\rho_1, \rho_2 > 0$ des nombres réels, $K \subset \mathbf{C}$ un corps de nombres, $n \geq 2$ un entier et f_1, \dots, f_n des fonctions méromorphes sur \mathbf{C} telles que l'anneau $K[f_1, \dots, f_n]$ est stable par la dérivation $\frac{d}{dz}$. Supposons en plus que :*

- (1) f_1 et f_2 sont algébriquement indépendantes sur K ;
- (2) f_i est d'ordre inférieur à ρ_i , pour $i = 1, 2$.

Alors, si S désigne l'ensemble des $\alpha \in \mathbf{C}$ tels que, pour tout $1 \leq i \leq n$, α n'est pas un pôle de f_i et $f_i(\alpha) \in K$, on a :

$$\text{card}(S) \leq (\rho_1 + \rho_2)[K : \mathbf{Q}].$$

Cet énoncé généralise les théorèmes classiques de Hermite-Lindemann et de Gelfond-Schneider. La transcendance de π et de e , par exemple, se déduit facilement du théorème ci-dessus en prenant $f_1(z) = z$ et $f_2(z) = e^z$.

Même si les conclusions des théorèmes de Siegel-Shidlovsky et de Schneider-Lang sont de nature assez différentes, il est remarquable que les hypothèses de ces deux résultats partagent la même structure. Dans les deux cas, il s'agit de fonctions, holomorphes ou méromorphes, définies sur \mathbf{C} tout entier, reliées par une équation différentielle algébrique à coefficients dans un corps de nombres, et l'on impose additionally : (1) une propriété d'indépendance algébrique fonctionnelle ; (2) des conditions de croissance sur ces fonctions ou sur ses coefficients de Taylor.

Que peut-on dire de fonctions définies sur des domaines de \mathbf{C} plus généraux, comme des disques ? Dans ce cas, Mahler a développé une méthode pour étudier la transcendance de valeurs de certaines fonctions satisfaisant des équations fonctionnelles (cf. [64]).

Considérons, par exemple, l'invariant modulaire j de Klein (cf. [91] VII 3.3). Rappelons que j est une fonction holomorphe sur le demi-plan de Poincaré $\mathbf{H} = \{\tau \in \mathbf{C} \mid \text{Im } \tau > 0\}$, invariante sous l'action de $\text{SL}_2(\mathbf{Z})$ sur \mathbf{H} donnée en $\tau \in \mathbf{H}$ par

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

En particulier, on a $j(\tau + 1) = j(\tau)$ et l'on en déduit que j admet un développement de Fourier : si $D = \{q \in \mathbf{C} \mid |q| < 1\}$ désigne le disque unité centré en l'origine, alors il existe une fonction holomorphe J sur $D \setminus \{0\}$ telle que $J(e^{2\pi i\tau}) = j(\tau)$ pour tout $\tau \in \mathbf{H}$. On peut prouver que J s'étend en une fonction méromorphe sur D et que son développement en série de Laurent est donné par

$$J(q) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)q^n$$

avec $c(n)$ entier (cf. [91] VII 3.3 Remarque 2).

La fonction j classe les courbes elliptiques, ce qui lui accorde un rôle central en théorie de nombres. Ceci motive aussi l'étude de la transcendance de ses valeurs. Dans cette direction, Schneider a prouvé que, pour un nombre algébrique $\tau \in \mathbf{H}$, $j(\tau)$ est algébrique si et seulement si τ est quadratique imaginaire ([90] II.4). La démonstration de Schneider se ramène à une application du théorème de Schneider-Lang à certaines fonctions elliptiques (associées au réseau $\mathbf{Z} + \tau\mathbf{Z}$). Schneider a lui-même posé la question ([90] p. 138), encore ouverte à ce jour, de savoir si son résultat pouvait se déduire d'une étude directe des propriétés de la fonction j .

En 1969, Mahler [64] a posé le problème suivant : sa méthode sur les fonctions satisfaisant des équations fonctionnelles peut-elle être adaptée pour traiter la fonction J ? Cette question l'a amené à conjecturer que $J(z)$ est transcendant pour tout $z \in D \setminus \{0\}$ algébrique.

La question de Mahler a obtenue une réponse positive en 1996, dans un travail en collaboration de Barré-Sirieix, Diaz, Gramain et Philibert.

THÉORÈME (Barré-Sirieix-Diaz-Gramain-Philibert [4]). *Pour tout $z \in D \setminus \{0\}$, on a*

$$\text{degr}_{\mathbf{Q}} \mathbf{Q}(z, J(z)) \geq 1.$$

Peu de temps après, cet énoncé a été généralisé par Nesterenko, qui a réussi à combiner les nouvelles idées introduites dans la démonstration de ce résultat avec ses propres méthodes développées dans ses investigations sur la théorie de Siegel-Shidlovsky et les critères de transcendance.

Considérons les q -expansions des séries d'Eisenstein classiques

$$E_2(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \quad E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \quad E_6(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n},$$

vues comme des fonctions holomorphes sur le disque unitaire complexe D ; ainsi

$$(*) \quad J = 1728 \frac{E_4^3}{E_4^3 - E_6^2}.$$

THÉORÈME (Nesterenko [75]). *Pour tout $z \in D \setminus \{0\}$, on a*

$$\text{trdeg}_{\mathbf{Q}} \mathbf{Q}(z, E_2(z), E_4(z), E_6(z)) \geq 3.$$

À titre d'exemple, ce théorème appliqué à la valeur $z = e^{-2\pi}$ fournit l'indépendance algébrique de $\pi, e^\pi, \Gamma(1/4)$. Il convient de rappeler que l'indépendance algébrique de seuls π et e^π était ouverte avant le théorème de Nesterenko.

Tout aussi frappante que la simplicité et la puissance de ce résultat en est la démonstration par Nesterenko dans [75]. Après réduction de son énoncé à un problème d'approximation diophantienne *via* un critère d'indépendance algébrique dû à Philippon ([85] Théorème 2.11)¹, Nesterenko emploie une *méthode* qui fait intervenir diverses propriétés remarquables des séries d'Eisenstein, comme les *équations de Ramanujan*

$$q \frac{dE_2}{dq} = \frac{E_2^2 - E_4}{12}, \quad q \frac{dE_4}{dq} = \frac{E_2 E_4 - E_6}{3}, \quad q \frac{dE_6}{dq} = \frac{E_2 E_6 - E_4^2}{2}$$

et l'intégralité de ses coefficients de Taylor, en plus de deux conditions techniques :

(N1) [*Condition de croissance*] pour tout $k \in \{1, 2, 3\}$, la suite de coefficients de Taylor en l'origine $(E_{2k}^{(n)}(0)/n!)_{n \geq 0}$ est d'ordre de croissance polynomiale en n , et

(N2) [*Lemme de zéros*] il existe une constante réelle $C > 0$ telle que

$$\text{ord}_{q=0} P(q, E_2(q), E_4(q), E_6(q)) \leq C(\deg P)^4$$

pour tout polynôme non-nul $P \in \mathbf{C}[X_0, X_1, X_2, X_3]$.

Notons que la propriété (N1) est triviale, alors qu'une bonne partie de [75] est dédiée à la preuve d'une version renforcée de (N2).

Remarquons au passage que Philippon a étendu, dans [86], la méthode de Barré-Sirieix, Diaz, Gramain et Philibert à une classe de fonctions, appelées *K-fonctions*, qui contient les séries d'Eisenstein. Ceci lui permet de retrouver en particulier le résultat de Nesterenko.

Le caractère général de la démonstration de Nesterenko suggère que pour n'importe quelle famille de fonctions holomorphes sur le disque unitaire f_1, \dots, f_n à coefficients de Taylor entiers, satisfaisant des équations différentielles algébriques à coefficients rationnels, et vérifiant des conditions semblables à (N1) et (N2) ci-dessus, des arguments analogues à ceux pour les séries d'Eisenstein donneraient un résultat d'indépendance algébrique pour les valeurs de f_1, \dots, f_n .

1. Ceci marque, d'ailleurs, une différence importante avec la démonstration du théorème de Barré-Sirieix, Diaz, Gramain et Philibert qui ne fait pas appel à un critère général de transcendance comme celui de Philippon.

Le problème de l'existence de telles fonctions f_1, \dots, f_n qui ne seraient pas reliées à des formes modulaires classiques (dans un sens assez vague) a été énoncé de façon explicite par Zudilin dans [100], qui a aussi étudié quelques candidats provenant du phénomène de la *symétrie miroir* ([97]); cependant, les cas où Zudilin peut vérifier toutes les propriétés requises par la « méthode de Nesterenko » sont tous de nature modulaire.

L'objectif de cette thèse est d'apporter quelques contributions à ces questions liés au théorème de Nesterenko. Nos investigations ont été axés sur deux lignes de recherche essentiellement indépendantes :

- (a) étude géométrique de la méthode de Nesterenko *per se* ;
- (b) recherche de nouveaux exemples d'application.

À premier abord, on s'attend à que ces nouveaux exemples dans (b) soient trouvés dans des constructions algebro-géométriques naturelles, comme les candidats provenant de la symétrie miroir le suggèrent. L'avantage de reformuler la méthode de Nesterenko en termes d'hypothèses géométriques serait donc de la rendre adaptée aux équations différentielles d'origine géométrique construites en (b).

Dans les prochaines sections, nous exposons nos principales contributions et nous discutons, à la fin de cette introduction, quelques questions ouvertes issues de nos travaux.

Croissance modérée de courbes analytiques

Le point de départ dans l'étude de la partie (a) de notre programme est l'observation de Bost et Randriambololona que la condition de croissance polynomiale dans la méthode de Nesterenko peut être remplacée par une autre condition plus faible de « croissance modérée », formulée de manière purement géométrique.

Fixons (M, h) une variété hermitienne et considérons la $(1, 1)$ -forme réelle positive associée à h

$$\omega = -\operatorname{Im} h ;$$

en coordonnées locales (z_1, \dots, z_n) sur M , si $h = \sum_{1 \leq k, l \leq n} h_{kl} dz_k \otimes d\bar{z}_l$, alors

$$\omega = \frac{i}{2} \sum_{1 \leq k, l \leq n} h_{kl} dz_k \wedge d\bar{z}_l.$$

Soit $R > 0$ un nombre réel, $D_R = \{z \in \mathbf{C} \mid |z| < R\}$ le disque complexe de rayon R centré en l'origine et $\varphi : D_R \rightarrow M$ une application analytique. Pour tout $t \in]0, R[$, l'aire du « disque » $\varphi(D_t) \subset M$ se calcule par

$$A_\varphi(t) = \int_{D_t} \varphi^* \omega.$$

On définit alors la *fonction caractéristique* $T_\varphi :]0, R[\rightarrow \mathbf{R}_{\geq 0}$ de φ en prenant une « intégrale logarithmique »

$$T_\varphi(r) := \int_0^r A_\varphi(t) d \log t.$$

Dans le cas particulier où $M = \mathbf{P}^1(\mathbf{C})$ et h est la métrique de Fubini-Study, on a, sur la carte affine $\mathbf{C} \subset \mathbf{P}^1(\mathbf{C})$,

$$\omega = \frac{i}{2\pi} \partial \bar{\partial} \log(1 + |z|^2) = \frac{i}{2\pi} \frac{1}{(1 + |z|^2)^2} dz \wedge d\bar{z},$$

et l'on constate que la fonction caractéristique T_φ d'une application analytique $\varphi : D_R \rightarrow \mathbf{P}^1(\mathbf{C})$ définie comme ci-dessus n'est autre que la *fonction caractéristique d'Ahlfors-Shimizu* (cf. [93] V) :

$$T_\varphi(r) = \int_{\mathbf{C}} \log^+ \left(\frac{r}{|z|} \right) \varphi^* \omega.$$

DÉFINITION. Soit $R > 0$ un nombre réel et (M, h) une variété hermitienne. On dit qu'une application analytique $\varphi : D_R \rightarrow M$ est à *croissance modérée* s'ils existent des réels $a, b > 0$ tels que

$$T_\varphi(r) \leq a + b \log \frac{1}{1 - \frac{r}{R}}$$

pour tout $r \in]0, R[$.

Remarquons que, si M est une variété complexe *compacte*, alors toutes les métriques hermitiennes sur M sont « comparables » ; il en résulte que la propriété de croissance modérée d’une application analytique $\varphi : D_R \rightarrow M$ ne dépend pas du choix de métrique hermitienne sur M .

Munissons le disque D_R de la métrique de Poincaré :

$$\frac{R}{R^2 - |z|^2} |dz|.$$

Parmi les classes générales d’exemples de courbes analytiques à croissance modérée, on trouve les « courbes à dérivée bornée » ; par définition, ce sont les courbes $\varphi : D_R \rightarrow M$ dont la norme de l’application tangente $D_z\varphi : T_z D_R \rightarrow T_{\varphi(z)} M$ par rapport aux métriques h sur M et de Poincaré sur D_R est uniformément bornée pour $z \in D_R$.

Compte tenu de cette observation, un résultat de Brunella (cf. [34] Théorèmes 15 et 16) entraîne que, pour un feuilletage \mathcal{F} singulier de dimension un générique sur $\mathbf{P}^n(\mathbf{C})$, toute courbe analytique intégrale de \mathcal{F} paramétrée par un disque est à croissance modérée. En ce sens, la croissance modérée est une condition *naturelle* pour les courbes intégrales de feuilletages.

Par ailleurs, dans le cas $R = 1$, une courbe analytique

$$\varphi = (\varphi_1, \dots, \varphi_n) : D \rightarrow \mathbf{C}^n \subset \mathbf{P}^n(\mathbf{C})$$

dont les suites de coefficients de Taylor $(\varphi_k^{(m)}(0)/m!)_{m \geq 0}$ ont une croissance polynomiale en m pour tout $1 \leq k \leq n$, est à croissance modérée (Exemple 3.4.5). La croissance modérée généralise donc la condition de croissance (N1) considérée dans la méthode de Nesterenko.

Dans la recherche de nouveaux exemples d’application de la méthode de Nesterenko, l’une des principales difficultés se doit au fait que la condition de croissance polynomiale sur les coefficients de Taylor est trop restrictive, n’étant pas préservé par des simples manipulations algébriques sur les fonctions. Par exemple, la suite de coefficients de Taylor en l’origine $(c(n))_{n \geq 0}$ de la fonction

$$J = 1728 \frac{E_4^3}{E_4^3 - E_6^2}$$

ne croît pas polynomialement en n ; en fait,

$$c(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$$

lorsque $n \rightarrow +\infty$ (voir [84] ou [87]).

La croissance modérée résout cette difficulté. Si l’on se restreint aux variétés ambiantes M *projectives* — c’est-à-dire, M s’identifie à l’analytifié $X(\mathbf{C})$ d’une variété algébrique projective lisse X sur \mathbf{C} —, alors des arguments standards en théorie de Nevanlinna permettent de démontrer que, sous une hypothèse de non-dégénérescence, la croissance modérée est un *invariant birationnel* :

THÉORÈME 1 (cf. Corollary 3.4.12). *Soit $f : X \rightarrow Y$ un morphisme birationnel de \mathbf{C} -variétés algébriques projectives lisses. Si $R > 0$ est un nombre réel, alors une application analytique $\varphi : D_R \rightarrow X(\mathbf{C})$ dont l’image est Zariski-dense est à croissance modérée si, et seulement si, $f \circ \varphi : D_R \rightarrow Y(\mathbf{C})$ est à croissance modérée.*

Signalons au passage que ce résultat nous permet aussi de définir la notion de croissance modérée d’une courbe analytique Zariski-dense dans une variété *quasi-projective* lisse en considérant des compactifications (cf. Corollaire 3.4.13).

Une généralisation géométrique de la méthode de Nesterenko

Dans le troisième chapitre de cette thèse, nous prouvons un énoncé géométrique qui généralise la méthode de Nesterenko en trois directions : (1) suivant Bost et Randriambololona, la condition de croissance polynomiale est remplacée par la croissance modérée ; (2) l’anneau d’entiers \mathbf{Z} est remplacé par un anneau d’entiers algébriques quelconque ; (3) l’espace affine (variété ambiante) est remplacé par une variété quasi-projective plus générale.

Avant de présenter l'énoncé précis de notre théorème, expliquons comment la condition technique (N2) dans la preuve de Nesterenko se formalise dans notre cadre géométrique général.

Fixons un corps k quelconque.

DÉFINITION. Soit X une variété projective de dimension n sur k , munie d'un fibré en droites ample L . On dit qu'une courbe formelle $\hat{\varphi} : \mathrm{Spf} k[[q]] \rightarrow X$ (i.e., $\hat{\varphi}$ est un morphisme de k -schémas formels) est *ZL-dense* s'il existe une constante réelle $C > 0$ telle que

$$(ZL) \quad \mathrm{ord}_{q=0} \hat{\varphi}^* s \leq Cd^m$$

pour tout entier $d \geq 1$ et toute section non-nulle $s \in \Gamma(X, L^{\otimes d})$.

Il découle de l'amplitude de L que l'image de toute courbe formelle ZL-dense est Zariski-dense. Ainsi, la notion de ZL-densité peut s'interpréter comme une version renforcée de la Zariski-densité.²

Cette définition généralise bien la conclusion du « lemme de zéros » considéré par Nesterenko : la condition en (N2) ci-dessus revient à dire que la courbe formelle

$$\begin{aligned} \hat{\varphi} : \mathrm{Spf} \mathbf{C}[[q]] &\rightarrow \mathbf{A}_{\mathbf{C}}^4 \subset \mathbf{P}_{\mathbf{C}}^4 \\ q &\mapsto (q, E_2(q), E_4(q), E_6(q)) \end{aligned}$$

est ZL-dense dans $\mathbf{P}_{\mathbf{C}}^4$ muni du fibré ample $\mathcal{O}_{\mathbf{P}_{\mathbf{C}}^4}(1)$. La terminologie « ZL-dense » provient de *Zero Lemma*.

La ZL-densité est une notion véritablement géométrique. Tout d'abord, remarquons qu'elle ne dépend pas du choix de L (cf. Proposition 3.2.9). En outre, si X est supposé seulement quasi-projectif, la ZL-densité d'une courbe formelle $\hat{\varphi}$ dans X dont le k -point $\hat{\varphi}(0) \in X(k)$ est régulier ne dépend pas de la compactification projective de X choisie (cf. Corollaire 3.2.16). Finalement, si X est supposée géométriquement intègre, alors une courbe formelle $\hat{\varphi} : \mathrm{Spf} k[[q]] \rightarrow X$ est ZL-dense dans X si, et seulement si, pour toute extension de corps K de k , la courbe $\hat{\varphi}_K : \mathrm{Spf} K[[q]] \rightarrow X_K$, obtenue par changement de corps de base, est ZL-dense dans X_K .

Nous sommes maintenant en mesure d'énoncer notre théorème.

Soit K un corps de nombres et dénotons par \mathcal{O}_K son anneau d'entiers. Rappelons qu'une *variété arithmétique* sur \mathcal{O}_K désigne un schéma intègre \mathcal{X} muni d'un morphisme plat, séparé et de type fini $\mathcal{X} \rightarrow \mathrm{Spec} \mathcal{O}_K$.

THÉORÈME 2 (cf. Théorème 3.1.2). *Soit \mathcal{X} une variété arithmétique quasi-projective sur \mathcal{O}_K de dimension relative $n \geq 2$, avec fibre générique \mathcal{X}_K lisse, et soit $\hat{\varphi} : \mathrm{Spf} \mathcal{O}_K[[q]] \rightarrow \mathcal{X}$ un morphisme de \mathcal{O}_K -schémas formels tel que, pour tout plongement de corps $\sigma : K \hookrightarrow \mathbf{C}$, la courbe formelle $\hat{\varphi}_\sigma : \mathrm{Spf} \mathbf{C}[[q]] \rightarrow \mathcal{X}_\sigma$, obtenue de $\hat{\varphi}$ par changement de base, se relève en une courbe analytique $\varphi_\sigma : D_{R_\sigma} \subset \mathbf{C} \rightarrow \mathcal{X}_\sigma(\mathbf{C})$ définie sur un disque de rayon $R_\sigma > 0$ centré en l'origine.*

Supposons que

$$\prod_{\sigma} R_\sigma = 1$$

et qu'il existe un champ de vecteurs $v \in \Gamma(\mathcal{X}_K, T_{\mathcal{X}_K/K}) \setminus \{0\}$ sur la fibre générique de \mathcal{X} tel que $\hat{\varphi}_K : \mathrm{Spf} K[[q]] \rightarrow \mathcal{X}_K$ satisfait l'équation différentielle

$$q \frac{d\hat{\varphi}_K}{dq} = v \circ \hat{\varphi}_K.$$

Si de plus :

- (1) *la courbe formelle $\hat{\varphi}_K$ est ZL-dense dans \mathcal{X}_K , et*
- (2) *pour chaque plongement de corps $\sigma : K \hookrightarrow \mathbf{C}$, la courbe analytique $\varphi_\sigma : D_{R_\sigma} \rightarrow \mathcal{X}_\sigma(\mathbf{C})$ est à croissance modérée,*

alors, pour tout $\sigma : K \hookrightarrow \mathbf{C}$, et tout $z \in D_{R_\sigma} \setminus \{0\}$, le corps de définition $K(\varphi_\sigma(z))$ du point complexe $\varphi_\sigma(z)$ dans \mathcal{X}_K satisfait

$$\mathrm{degtr}_K K(\varphi_\sigma(z)) \geq n - 1.$$

2. L'exposant $n = \dim X$ dans la borne polynomiale ci-dessus est le plus petit possible (cf. Proposition 3.2.6).

Le théorème de Nesterenko se retrouve comme le cas particulier $K = \mathbf{Q}$, $\mathcal{X} = \mathbf{A}_{\mathbf{Z}}^4$, et $\hat{\varphi} : \text{Spf } \mathbf{Z}[[q]] \rightarrow \mathbf{A}_{\mathbf{Z}}^4$ définie par $\hat{\varphi}(q) = (q, E_2(q), E_4(q), E_6(q))$.

Dans le Théorème 2, la condition (1) est, comme la condition (2), naturelle pour les courbes intégrales de champs de vecteurs algébriques. Ceci est aussi dû à Nesterenko ([75] Théorème 6), qui prouve qu'une solution d'une équation différentielle satisfaisant la *D*-propriété (cf. Définition 3.B.1) satisfait aussi le lemme de zéros (ZL). Ce résultat, et sa démonstration, ont été étendus dans un cadre géométrique par Binyamini [10]. Dans l'annexe 3.B, nous indiquons comment adapter les arguments de Binyamini pour obtenir la généralisation suivante :

THÉORÈME 3 (cf. [10] et Théorème 3.B.2). *Soit X une variété quasi-projective lisse sur un corps k algébriquement clos de caractéristique nulle, $v \in \Gamma(X, T_{X/k}) \setminus \{0\}$ un champ de vecteurs sur X et $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow X$ une courbe formelle lisse satisfaisant l'équation différentielle*

$$q \frac{d\hat{\varphi}}{dq} = v \circ \hat{\varphi}.$$

*Si $\hat{\varphi}$ satisfait la *D*-propriété pour le feuilletage engendré par v , alors $\hat{\varphi}$ est ZL-dense dans X .*

Notre démonstration de la généralisation géométrique du théorème de Nesterenko suit la structure de la preuve de Nesterenko dans [75]. En particulier, nous employons le même critère d'indépendance algébrique de Philippon [85]. Pour cela, nous montrons dans l'annexe 3.A comment ce critère peut être généralisé à des variétés arithmétiques projectives plus générales que \mathbf{P}^n .

Remarquons finalement que les hypothèses du théorème ci-dessus suivent le même schéma général de celles des théorèmes de Siegel-Shidlovsky et de Schneider-Lang. Notre résultat peut s'interpréter donc comme un complément « hyperbolique » à ces méthodes « paraboliques », qui admettent elles aussi des généralisations géométriques (cf. [1], [7], [37], [36], [46]).

Périodes de variétés abéliennes

L'intérêt dans le théorème de Nesterenko provient, dans une large mesure, du fait que les valeurs des séries d'Eisenstein classiques, ou plus généralement des formes quasi-modulaires, sont des « périodes » de courbes elliptiques.

Grosso modo, une période est un nombre complexe qui peut s'exprimer comme la valeur d'une intégrale d'origine algebro-géométrique. Les premiers résultats sur la transcendance de périodes elliptiques remontent à Schneider. Dans les deux dernières décennies, l'étude de ces nombres a connu un regain d'intérêt, dû notamment à son lien étroit avec la *théorie des motifs*. Un survol sur la théorie des périodes en général nous emmènerait très loin ; pour cela, nous renvoyons à [56], [2] et [47]. Nous nous bornerons ici à les *périodes abéliennes*.

Soit $g \geq 1$ un entier, k un sous-corps de \mathbf{C} et X une variété abélienne de dimension g sur k . Alors on dispose, pour tout $i \geq 0$ entier, du k -espace vectoriel de *i*-ème cohomologie de de Rham algébrique :

$$H_{\text{dR}}^i(X/k) := \mathbf{H}^i(\Omega_{X/k}^\bullet),$$

et du \mathbf{Q} -espace vectoriel de *i*-ème cohomologie singulière, ou *cohomologie de Betti*, sur le tore complexe $X(\mathbf{C})$:

$$H^i(X(\mathbf{C}), \mathbf{Q}).$$

Pour n'importe quelle théorie de cohomologie de Weil (de Rham, Betti, ℓ -adique, etc.), le premier groupe de cohomologie $H^1(X)$ est un espace vectoriel de dimension $2g$ et, pour $i \geq 0$ entier, $H^i(X)$ s'identifie canoniquement à la puissance extérieure $\wedge^i H^1(X)$. Pour cette raison, il suffit de considérer les groupes de cohomologie H^1 dans la suite.

Les périodes de X sont des invariants numériques provenant de l'*isomorphisme de comparaison de Grothendieck* :

$$c : H_{\text{dR}}^1(X/k) \otimes_k \mathbf{C} \xrightarrow{\sim} H^1(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}.$$

Si $H_1(X(\mathbf{C}), \mathbf{Q})$ dénote le premier groupe d'homologie singulière de $X(\mathbf{C})$ à coefficients dans \mathbf{Q} , alors l'accouplement induit par l'isomorphisme c

$$H_1(X(\mathbf{C}), \mathbf{Q}) \times H_{\text{DR}}^1(X/k) \longrightarrow \mathbf{C}$$

est précisément l'« accouplement de périodes », défini par l'intégration des classes de cohomologie algébriques à coefficients dans k sur des cycles singuliers, et l'on définit le corps

$$\mathcal{P}(X/k) \subset \mathbf{C}$$

des k -périodes de X comme le sous-corps de \mathbf{C} engendré sur k par les éléments dans l'image de l'accouplement ci-dessus. Autrement dit, $\mathcal{P}(X/k)$ est le corps de rationalité de l'isomorphisme linéaire c .

Comment calculer $\text{degr}_k \mathcal{P}(X/k)$ ou $\text{degr}_{\mathbf{Q}} \mathcal{P}(X/k)$ pour une variété abélienne X donnée ?

Lorsque le corps k est contenu dans $\overline{\mathbf{Q}}$, la clôture algébrique de \mathbf{Q} dans \mathbf{C} , Grothendieck a formulé une conjecture qui donne une réponse géométrique, ou plutôt motivique, à cette question dans un cadre plus large : toutes les relations algébriques parmi les périodes d'une k -variété projective X proviendraient de cycles algébriques sur des puissances $X \times_k \cdots \times_k X$ de X (cf. [40] note de bas de page 10, [60] p. 40-44).

La conjecture de Grothendieck pour une variété abélienne X peut se reformuler en termes du *groupe Mumford-Tate* $\text{MT}(X)$, *i.e.*, le plus petit \mathbf{Q} -sous-groupe algébrique de $\text{GL}_{H^1(X(\mathbf{C}), \mathbf{Q})} \times_{\mathbf{Q}} \mathbf{G}_{m, \mathbf{Q}}$ qui fixe toutes les classes de Hodge dans des puissances tensorielles mixtes de la \mathbf{Q} -structure de Hodge sous-jacente à $H^1(X(\mathbf{C}), \mathbf{Q})$ (cf. [29] I.3). La conjecture suivante est une version de la conjecture de Grothendieck, valable pour des corps $k \subset \mathbf{C}$ arbitraires, proposée par André ([2] 23.4.1) dans un cadre plus général :

CONJECTURE (Grothendieck-André). *Pour toute variété abélienne X sur un corps $k \subset \mathbf{C}$, on a :*

$$\text{degr}_{\mathbf{Q}} \mathcal{P}(X/k) \stackrel{?}{\geq} \dim \text{MT}(X).$$

Deligne [28] (cf. [29] Corollaire I.6.4) a prouvé, comme conséquence de ses travaux sur les cycles de Hodge absolus, que l'on a toujours la borne supérieure :

$$\text{degr}_k \mathcal{P}(X/k) \leq \dim \text{MT}(X).$$

En particulier, dans le cas où $k \subset \overline{\mathbf{Q}}$, la conjecture de Grothendieck devient une égalité :

$$\text{degr}_{\mathbf{Q}} \mathcal{P}(X/k) \stackrel{?}{=} \dim \text{MT}(X).$$

On dispose de très peu d'évidence pour la conjecture de Grothendieck-André. À part un résultat de Wüstholz sur les relations linéaires entre les périodes (cf. [95]), un théorème de Chudnovsky affirme que, dans le cas $k \subset \overline{\mathbf{Q}}$, on a $\text{degr}_{\mathbf{Q}} \mathcal{P}(X/k) \geq 2$ pour n'importe quelle variété abélienne X simple ([24] Ch. 7, Proposition 2.5). Pour les courbes elliptiques — *i.e.*, les variétés abéliennes de dimension $g = 1$ —, le résultat de Chudnovsky prouve la conjecture des périodes dans le cas de « multiplication complexe ». Lorsque X est une courbe elliptique sans multiplication complexe, la conjecture de Grothendieck (pour $k \subset \overline{\mathbf{Q}}$) s'écrit

$$\text{degr}_{\mathbf{Q}} \mathcal{P}(X/k) \stackrel{?}{=} 4$$

et reste encore ouverte.

Le théorème de Nesterenko non seulement renforce le résultat de Chudnovsky pour les courbes elliptiques, mais fournit aussi un autre point de vue sur ce problème, que l'on peut appeler le point de vue *modulaire* sur la conjecture de périodes. Expliquons cela.

Rappelons que le demi-plan de Poincaré \mathbf{H} classe les tores complexes de dimension 1 munis d'une base orientée de leur premier groupe d'homologie singulière à coefficients dans \mathbf{Z} : à $\tau \in \mathbf{H}$, on associe le tore complexe $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ muni de la base $(\gamma_\tau, \delta_\tau)$ de $H_1(\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z}), \mathbf{Z})$ induite par les lacets

$$\begin{aligned} \gamma_\tau : [0, 1] &\longrightarrow \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z}) & \delta_\tau : [0, 1] &\longrightarrow \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z}) \\ t &\longmapsto [t] & t &\longmapsto [t\tau]. \end{aligned}$$

La théorie de Weierstrass implique que, pour chaque $\tau \in \mathbf{H}$, ils existent $g_{2, \tau}, g_{3, \tau} \in \mathbf{Q}(j(\tau))$ tels que $g_{2, \tau}^3 - 27g_{3, \tau}^2 \neq 0$ et que le tore $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ soit isomorphe au complexifié $E_\tau(\mathbf{C})$ de la courbe elliptique E_τ sur $\mathbf{Q}(j(\tau))$ donnée par l'équation $y^2 = 4x^3 - g_{2, \tau}x - g_{3, \tau}$.

En considérant la base $(\frac{dx}{y}, x\frac{dx}{y})$ de $H_{\text{dR}}^1(E_\tau/\mathbf{Q}(j(\tau)))$, on obtient quatre périodes :

$$\omega_{1,\tau} = \int_{\gamma_\tau} \frac{dx}{y}, \quad \omega_{2,\tau} = \int_{\delta_\tau} \frac{dx}{y}, \quad \eta_{1,\tau} = \int_{\gamma_\tau} x \frac{dx}{y}, \quad \eta_{2,\tau} = \int_{\delta_\tau} x \frac{dx}{y},$$

qui engendrent le corps de périodes $\mathcal{P}(E_\tau/\mathbf{Q}(j(\tau)))$:

$$\mathcal{P}(E_\tau/\mathbf{Q}(j(\tau))) = \mathbf{Q}(j(\tau), \omega_{1,\tau}, \omega_{2,\tau}, \eta_{1,\tau}, \eta_{2,\tau}).$$

La théorie classique des formes modulaires entraîne les formules

$$E_2(e^{2\pi i\tau}) = 12 \left(\frac{\omega_{1,\tau}}{2\pi i} \right) \left(\frac{\eta_{1,\tau}}{2\pi i} \right), \quad E_4(e^{2\pi i\tau}) = 12g_{2,\tau} \left(\frac{\omega_{1,\tau}}{2\pi i} \right)^4, \quad E_6(e^{2\pi i\tau}) = -216g_{3,\tau} \left(\frac{\omega_{1,\tau}}{2\pi i} \right)^6.$$

Finalement, compte tenu de l'expression de j en termes de E_4 et E_6 (voir (*)) et de la relation de périodes de Legendre $\omega_{1,\tau}\eta_{2,\tau} - \omega_{2,\tau}\eta_{1,\tau} = 2\pi i$, on conclut que

$$(P) \quad \text{degtr}_{\mathbf{Q}} \mathcal{P}(E_\tau/\mathbf{Q}(j(\tau))) = \text{degtr}_{\mathbf{Q}} \mathbf{Q}(2\pi i, \tau, E_2(e^{2\pi i\tau}), E_4(e^{2\pi i\tau}), E_6(e^{2\pi i\tau})).$$

Le théorème de Nesterenko implique que, pour tout $\tau \in \mathbf{H}$, on a

$$\text{degtr}_{\mathbf{Q}} \mathbf{Q}(E_2(e^{2\pi i\tau}), E_4(e^{2\pi i\tau}), E_6(e^{2\pi i\tau})) \geq 2.$$

Or, si E est une courbe elliptique complexe quelconque, alors il existe $\tau \in \mathbf{H}$ tel que $E(\mathbf{C})$ soit isomorphe au tore complexe $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$, et $\mathbf{Q}(j(\tau))$ est le corps de définition de E . En particulier, on conclut de (P) que le théorème de Chudnovsky pour les courbes elliptiques sur $\overline{\mathbf{Q}}$

$$\text{degtr}_{\mathbf{Q}} \mathcal{P}(E/\overline{\mathbf{Q}}) \geq 2$$

est un corollaire du théorème de Nesterenko.

Il est naturel à ce stade de se demander si cet approche modulaire à la conjecture de périodes se généralise aux variétés abéliennes de dimension quelconque.

Équations de Ramanujan supérieures

Dans les deux premiers chapitres de cette thèse, nous franchissons un premier pas dans l'étude de cette question ci-dessus. En particulier, nous généralisons la formule (P) : pour tout entier $g \geq 1$, il est possible de paramétrer les corps de périodes de variétés abéliennes de dimension g , à extension algébrique près, par une solution de certaines équations différentielles algébriques à coefficients rationnels, les *équations de Ramanujan supérieures*.

Expliquons comment définir ces équations différentielles géométriquement.

Notre construction a été inspiré de la réinterprétation géométrique des équations de Ramanujan « classiques » par Movasati (cf. [69]) et constitue une généralisation de celle-ci en dimensions supérieures (voir aussi [70] pour un autre point de vue). Remarquons qu'une interprétation géométrique des équations de Ramanujan, par le biais de la *dérivée de Serre* sur les formes modulaires, a été considérée précédemment par Deligne (cf. [51] Appendix A).

La cohomologie de de Rham d'une variété abélienne principalement polarisée (X, λ) de dimension g sur un corps k est munie de deux structures supplémentaires : (1) un sous-espace vectoriel $F^1(X/k) \subset H_{\text{dR}}^1(X/k)$ de dimension g , le *sous-espace de Hodge*, canoniquement isomorphe à l'espace de formes holomorphes $H^0(X, \Omega_{X/k}^1)$; (2) une forme k -bilinéaire symplectique (*i.e.*, alternée et non-dégénérée) $\langle \cdot, \cdot \rangle_\lambda$ induite par la polarisation principale λ .

DÉFINITION. Soit (X, λ) une variété abélienne principalement polarisée de dimension g sur un corps k . Une base $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ du k -espace vectoriel $H_{\text{dR}}^1(X/k)$ est dite *base Hodge-symplectique* de (X, λ) si b est une base symplectique par rapport à $\langle \cdot, \cdot \rangle_\lambda$ et si $(\omega_1, \dots, \omega_g)$ est une base de $F^1(X/k)$.

À part la terminologie, cette définition n'est pas nouvelle ; le choix d'une base de $H_{\text{dR}}^1(X/k)$ de la forme ci-dessus est classique dans la théorie des variétés abéliennes.

Nous considérons ensuite, pour un entier $g \geq 1$ donnée, un « champ de modules » sur $\text{Spec } \mathbf{Z}$, que nous dénotons \mathcal{B}_g , classifiant les variétés abéliennes principalement polarisées de dimension g munies d'une base Hodge-symplectique.

THÉORÈME 4 (cf. Théorèmes 1.3.2 et 1.4.1). *Le champ \mathcal{B}_g sur $\text{Spec } \mathbf{Z}$ est un champ de Deligne-Mumford lisse de dimension relative $2g^2 + g$ sur $\text{Spec } \mathbf{Z}$. De plus, le changement de base $\mathcal{B}_g \otimes_{\mathbf{Z}} \mathbf{Z}[1/2]$ est représentable par un schéma quasi-projectif lisse B_g sur $\text{Spec } \mathbf{Z}[1/2]$.*

Le résultat de représentabilité dans l'énoncé ci-dessus repose essentiellement sur un théorème d'Oda [78] sur le module de Dieudonné associé à la cohomologie de de Rham d'une variété abélienne en caractéristique positive.

Nous démontrons ensuite (Théorème 1.5.4) que la théorie de déformations de variétés abéliennes permet de décrire le fibré tangent $T_{\mathcal{B}_g/\mathbf{Z}}$ de \mathcal{B}_g en termes de la cohomologie de de Rham relative du schéma abélien universel sur le champ de modules de variétés abéliennes principalement polarisées de dimension g . Ceci nous permet de définir une famille canonique $(v_{ij})_{1 \leq i < j \leq g}$ de champs de vecteurs sur \mathcal{B}_g que l'on appelle *champs de Ramanujan supérieurs*. Ceux-ci se caractérisent aussi par la propriété suivante.

Soit \mathcal{X}_g le schéma abélien universel sur \mathcal{B}_g et ∇ la connexion de Gauss-Manin sur la cohomologie de de Rham relative $H_{\text{dR}}^1(\mathcal{X}_g/\mathcal{B}_g)$.

THÉORÈME 5 (cf. Proposition 1.5.7 et Corollaire 1.5.10). *Si $(\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ dénote la base Hodge-symplectique universelle sur \mathcal{B}_g , alors $(v_{ij})_{1 \leq i < j \leq g}$ est l'unique famille de champs de vecteurs sur \mathcal{B}_g satisfaisant, pour tout $1 \leq i < j \leq g$:*

- (1) $\nabla_{v_{ij}} \omega_i = \eta_j$, $\nabla_{v_{ij}} \omega_j = \eta_i$, et $\nabla_{v_{ij}} \omega_k = 0$ pour tout $k \notin \{i, j\}$,
- (2) $\nabla_{v_{ij}} \eta_k = 0$ pour tout $1 \leq k \leq g$.

En particulier, les champs v_{ij} commutent entre eux.

Dans le cas $g = 1$, la théorie classique des courbes elliptiques permet d'identifier $B_1 \otimes_{\mathbf{Z}[1/2]} \mathbf{Z}[1/6]$ avec le schéma affine $\text{Spec } \mathbf{Z}[1/6, e_2, e_4, e_6, (e_4^3 - e_6^2)^{-1}]$ et l'on déduit du théorème ci-dessus que

$$v_{11} = \frac{e_2^2 - e_4}{12} \frac{\partial}{\partial e_2} + \frac{e_2 e_4 - e_6}{3} \frac{\partial}{\partial e_4} + \frac{e_2 e_6 - e_4^2}{2} \frac{\partial}{\partial e_6}$$

coïncide avec le champ de vecteurs associé aux équations de Ramanujan classiques.

Nous généralisons la solution (E_2, E_4, E_6) comme suit. Rappelons que, pour $g \geq 1$ entier, le *demi-espace de Siegel*

$$\mathbf{H}_g = \{\tau \in M_{g \times g}(\mathbf{C}) \mid {}^t \tau = \tau \text{ et } \text{Im } \tau > 0\}$$

classifie les tores complexes principalement polarisés de dimension g munis d'une base symplectique du première groupe d'homologie à coefficients entiers : à $\tau \in \mathbf{H}_g$ correspond un tore principalement polarisé $(\mathbf{X}_{g,\tau}, E_{g,\tau})$ muni de la base $\beta_{g,\tau} = (\gamma_{1,\tau}, \dots, \gamma_{g,\tau}, \delta_{1,\tau}, \dots, \delta_{g,\tau})$ de $H_1(\mathbf{X}_{g,\tau}, \mathbf{Z})$. Ici, $\mathbf{X}_{g,\tau} = \mathbf{C}^g / (\mathbf{Z}^g + \tau \mathbf{Z}^g)$, $E_{g,\tau}$ dénote la forme de Riemann principale

$$\begin{aligned} E_{g,\tau} : \mathbf{C}^g \times \mathbf{C}^g &\longrightarrow \mathbf{R} \\ (v, w) &\longmapsto \text{Im}({}^t \bar{v} (\text{Im } \tau)^{-1} w), \end{aligned}$$

et la base $\beta_{g,\tau}$ est donnée par les lacets

$$\begin{aligned} \gamma_{j,\tau} : [0, 1] &\longrightarrow \mathbf{X}_{g,\tau} & \delta_{j,\tau} : [0, 1] &\longrightarrow \mathbf{X}_{g,\tau} \\ t &\longmapsto [te_j] & t &\longmapsto [t\tau_j] \end{aligned}$$

où $e_j \in \mathbf{C}^g$ (resp. $\tau_j \in \mathbf{C}^g$) dénote la j -ème colonne de la matrice identité $\mathbf{1}_g \in M_{g \times g}(\mathbf{C})$ (resp. $\tau \in M_{g \times g}(\mathbf{C})$). Remarquons au passage que tout tore muni d'une polarisation est algébrisable, *i.e.*, s'identifie à l'analytifié d'une variété abélienne complexe polarisée.

L'isomorphisme de comparaison

$$\begin{aligned} \mathcal{H}_{\text{dR}}^1(\mathbf{X}_{g,\tau}) &:= \mathbf{H}^1(\Omega_{\mathbf{X}_{g,\tau}}^\bullet) \xrightarrow{\sim} \text{Hom}_{\mathbf{Z}}(H_1(\mathbf{X}_{g,\tau}, \mathbf{Z}), \mathbf{C}) \\ \alpha &\longmapsto \int_{-} \alpha \end{aligned}$$

nous permet donc de définir une base $\mathbf{b}_{g,\tau} = (\omega_{1,\tau}, \dots, \omega_{g,\tau}, \eta_{1,\tau}, \dots, \eta_{g,\tau})$ de $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_{g,\tau})$ par :

$$\int_{\gamma_{j,\tau}} \omega_{i,\tau} = \delta_{ij}, \quad \int_{\delta_{j,\tau}} \omega_{i,\tau} = \tau_{ij}, \quad \int_{\gamma_{j,\tau}} \eta_{i,\tau} = 0, \quad \int_{\delta_{j,\tau}} \eta_{i,\tau} = \delta_{ij}.$$

Voir Section 2.4.1 pour une définition alternative de $\mathbf{b}_{g,\tau}$.

THÉORÈME 6 (cf. Théorème 2.4.2). *Pour tout $\tau \in \mathbf{H}_g$, la base $\mathbf{b}_{g,\tau}$ de $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_{g,\tau})$ est Hodge-symplectique. De plus, l'application analytique*

$$\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$$

induite par \mathbf{b}_g satisfait les équations différentielles

$$\frac{1}{2\pi i} \frac{\partial \varphi_g}{\partial \tau_{kl}} = v_{kl} \circ \varphi_g, \quad 1 \leq k \leq l \leq g.$$

Remarquons que, sous l'identification $B_1(\mathbf{C}) = \{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_2^3 - z_3^2 \neq 0\}$ ci-dessus, la courbe analytique $\varphi_1 : \mathbf{H} \longrightarrow B_1(\mathbf{C})$ est donnée par

$$\varphi_1(\tau) = (E_2(e^{2\pi i \tau}), E_4(e^{2\pi i \tau}), E_6(e^{2\pi i \tau})).$$

Dans le cas général, nous prouvons quelques résultats de transcendance fonctionnelle.

THÉORÈME 7 (cf. Théorème 2.7.1). *Toute feuille analytique dans $B_g(\mathbf{C})$ du feuilletage engendré par les champs de Ramanujan supérieurs est Zariski-dense dans $B_{g,\mathbf{C}}$.*

Remarquons que des feuilletages algébriques de dimension r (i.e., engendrés par un sous-fibré involutif algébrique de rang r du fibré tangent d'une variété algébrique) dont toutes les feuilles analytiques sont Zariski-denses jouent un rôle important dans les « estimées de multiplicité » en approximation diophantienne, du moins lorsque $r = 1$, où cette propriété implique la D -propriété de Nesterenko.

D'après les Théorèmes 6 et 7, l'image de φ_g est Zariski-dense dans $B_{g,\mathbf{C}}$. Un argument simple permet alors d'en déduire le résultat plus fort que le *graphe* de φ_g :

$$\{(\tau, \varphi_g(\tau)) \in \text{Sym}_g(\mathbf{C}) \times B_g(\mathbf{C}) \mid \tau \in \mathbf{H}_g\}$$

est Zariski-dense dans $\text{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}}$, où Sym_g dénote le \mathbf{Z} -schéma en groupes des matrices symétriques d'ordre g (cf. Corollaire 7.2). Ce dernier résultat est un analogue partiel en dimension supérieure d'un théorème de Mahler [63] sur l'indépendance algébrique des fonctions $\tau, e^{2\pi i \tau}, E_2(\tau), E_4(\tau), E_6(\tau)$.

Remarquons que des généralisations en dimension supérieure des équations de Ramanujan, sous forme de systèmes d'équations aux dérivées partielles satisfaits par certaines *Thetanullwerte*, ainsi que des résultats de transcendance fonctionnelle à la Mahler sur ces dernières, ont été obtenus par Zudilin [97] et Bertrand-Zudilin [8], [9]. Le lien précis entre leur résultats et les nôtres n'est pas complètement clair à ce stade.

Finalement, nous montrons que φ_g paramétrise les corps de périodes de variétés abéliennes principalement polarisées de dimension g . Pour cela, on remarque que toute variété abélienne complexe X admet un plus petit sous-corps algébriquement clos $k \subset \mathbf{C}$ de définition, i.e., pour lequel il existe une variété abélienne X_0 sur k telle que $X_0 \otimes_k \mathbf{C} \cong X$ (cf. Lemme 2.5.1). On définit alors le « corps de périodes absolu » de X par

$$\mathcal{P}(X) := \mathcal{P}(X_0/k);$$

ceci ne dépend pas du choix de X_0 .

Pour $\tau \in \mathbf{H}_g$, on dénote par $\mathbf{Q}(2\pi i, \tau, \varphi_g(\tau))$ le corps de définition du point complexe $(2\pi i, \tau, \varphi_g(\tau))$ de la \mathbf{Q} -variété $\mathbf{G}_{m,\mathbf{Q}} \times_{\mathbf{Q}} \text{Sym}_{g,\mathbf{Q}} \times_{\mathbf{Q}} B_{g,\mathbf{Q}}$.

THÉORÈME 8 (cf. Théorème 2.5.3). *Fixons $g \geq 1$ entier. Pour tout $\tau \in \mathbf{H}_g$, le corps de périodes $\mathcal{P}(\mathbf{X}_{g,\tau})$ est une extension algébrique de $\mathbf{Q}(2\pi i, \tau, \varphi_g(\tau))$.*

Pour tout $\tau \in \mathbf{H}_g$, on a donc

$$\text{degr}_{\mathbf{Q}} \mathcal{P}(\mathbf{X}_{g,\tau}) = \text{degr}_{\mathbf{Q}} \mathbf{Q}(2\pi i, \tau, \varphi_g(\tau)),$$

ce qui généralise (P) ci-dessus. En particulier, la Zariski-densité du graphe de φ_g peut s'interpréter comme une variante fonctionnelle de la conjecture de périodes : outre les relations induites par les données de polarisation, il n'y a pas d'autre relation algébrique simultanément satisfaite par les périodes de toutes les variétés abéliennes principalement polarisées.

Signalons que Bertrand et Zudilin ont aussi obtenu un résultat analogue au Théorème 8 ci-dessus pour le corps différentiel engendré par les formes modulaires de Siegel ([8] Proposition 2).

Quelques questions ouvertes

Comme remarqué par Schneider dans [90] p. 138, il n'y a pas de difficulté à trouver des questions non résolues dans la théorie des nombres transcendants.³ Néanmoins, dans la fin de cette introduction, nous tenons à signaler quelques questions issues des travaux réalisés dans cette thèse et quelques directions d'investigation que nous jugeons intéressantes.

Tout d'abord, le Théorème 2 ci-dessus permet de formuler une version mathématiquement précise du problème concernant l'existence de nouveaux exemples d'application de la méthode de Nesterenko : *existe-t-il un exemple d'application du Théorème 2 dont l'énoncé de transcendance résultant ne soit pas contenu dans le théorème de Nesterenko ?*

Si les candidats provenant de la symétrie miroir semblent prometteurs, à ce stade nous ne pouvons pas exclure la possibilité d'une réponse négative à cette question. Remarquons, cependant, qu'une preuve de ce fait serait *tout aussi remarquable* que la découverte d'un nouveau exemple, puisque ceci impliquerait que les fonctions quasi-modulaires sont les *uniques* fonctions satisfaisant les hypothèses du Théorème 2, lesquelles ne font pas référence explicite à la nature géométrique des fonctions modulaires liées aux courbes elliptiques.

Il est aussi naturel de se demander si le Théorème 2 admet des *généralisations en dimension supérieure*. Nos travaux sur les équations de Ramanujan supérieures indiquent que φ_g , ou une variante de cette construction, seraient des candidats naturels à l'application de telles généralisations.

L'accomplissement de ce programme aurait grand intérêt dans la théorie des nombres transcendants. En effet, d'après le Théorème 8, un résultat d'indépendance algébrique sur les valeurs de φ_g entraînerait des estimés de degrés de transcendance dans la direction de la conjecture de périodes pour les variétés abéliennes.

3. « *Es macht keine Schwierigkeit, ungelöste Fragen aus dem Gebiet der transzendenten Zahlen aufzuwerfen.* »

Higher Ramanujan Equations I : moduli stacks of abelian varieties and higher Ramanujan vector fields

Abstract

We describe a higher dimensional generalization of Ramanujan's differential equations satisfied by the Eisenstein series E_2 , E_4 , and E_6 . This will be obtained geometrically as follows. For every integer $g \geq 1$, we construct a moduli stack \mathcal{B}_g over \mathbf{Z} classifying principally polarized abelian varieties of dimension g equipped with a suitable additional structure : a *symplectic-Hodge basis* of its first algebraic de Rham cohomology. We prove that \mathcal{B}_g is a smooth Deligne-Mumford stack over \mathbf{Z} of relative dimension $2g^2 + g$ and that $\mathcal{B}_g \otimes \mathbf{Z}[1/2]$ is representable by a smooth quasi-projective scheme over $\mathbf{Z}[1/2]$. Our main result is a description of the tangent bundle $T_{\mathcal{B}_g/\mathbf{Z}}$ in terms of the cohomology of the universal abelian scheme over the moduli stack of principally polarized abelian varieties \mathcal{A}_g . We derive from this description a family of $g(g+1)/2$ commuting vector fields $(v_{ij})_{1 \leq i < j \leq g}$ on \mathcal{B}_g ; these are the *higher Ramanujan vector fields*. In the case $g = 1$, we show that v_{11} coincides with the vector field associated to the classical Ramanujan equations.

This geometric framework taking account of integrality issues is mainly motivated by questions in transcendental number theory. In the upcoming second part of this work, we shall relate the values of a particular analytic solution to the differential equations defined by v_{ij} with Grothendieck's periods conjecture on abelian varieties.

1. Introduction

Consider the classical normalized Eisenstein series in $\mathbf{Z}[[q]]$

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \quad E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n}, \quad E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}$$

and let $\theta := q \frac{d}{dq}$. In 1916 [88] Ramanujan proved that these formal series satisfy the system of algebraic differential equations

$$(R) \quad \theta E_2 = \frac{E_2^2 - E_4}{12}, \quad \theta E_4 = \frac{E_2 E_4 - E_6}{3}, \quad \theta E_6 = \frac{E_2 E_6 - E_4^2}{2}.$$

The study of equivalent forms of such differential equations actually predates Ramanujan. To the best of our knowledge, Jacobi was the first to prove in 1848 [49] that his *Thetanullwerte* satisfy a third order algebraic differential equation. In 1881 [45] Halphen found a simpler description of Jacobi's equation by considering logarithmic derivatives. Further, in 1911 [22] Chazy considered a third order differential equation¹ satisfied by the Eisenstein series E_2 :

$$(C) \quad \theta^3 E_2 = E_2 \theta^2 E_2 - \frac{3}{2} (\theta E_2)^2.$$

We refer to [79] for a thorough study of Jacobi's, Halphen's, and Chazy's equations, and the relations between them. We point out that Ramanujan's and Chazy's equations concern level 1 (quasi-)modular forms, whereas the equations of Jacobi and Halphen involve level 2 (quasi-)modular forms.

A higher dimensional generalization of Jacobi's equation concerning *Thetanullwerte* of complex abelian varieties of dimension 2 was first given by Ohyama [80] in 1996, and for any dimension by Zudilin [97] in 2000 (see also [8]).

1. In Chazy's original notation (cf. [22] (4)) the equation he considered is written as $y''' = 2yy'' - 3(y')^2$. If derivatives in this equation are with respect to a variable t , equation (C) is obtained from this one by the change of variables $q = e^{2t}$.

This paper, and its sequel, grew out from our attempt to obtain a more conceptual understanding of the Ramanujan equations and their higher dimensional extensions. This could possibly shed some light on their arithmetical and geometric properties. An important motivation for this program is the central role of the original Ramanujan equations (R) and of the integrality properties of the series E_2 , E_4 , and E_6 , in Nesterenko's celebrated result on the transcendence of their values, when regarded as holomorphic functions on the complex unit disc $D = \{z \in \mathbf{C} \mid |z| < 1\}$:

THEOREM 1.1 (Nesterenko [75] 1996). *For every $q \in D \setminus \{0\}$,*

$$\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(q, E_2(q), E_4(q), E_6(q)) \geq 3.$$

In contrast with the concrete methods of Ohyama and Zudilin based on theta functions, our geometric approach allows us to construct by purely algebraic methods some higher dimensional avatars of the system (R), involving suitable moduli spaces of abelian varieties that enjoy remarkable smoothness properties over \mathbf{Z} . Another important difference between our approach and that of Ohyama and Zudilin is that we work in "level 1", although it should be clear that we can also introduce higher level structures in the picture.

We next explain our main results.

Fix an integer $g \geq 1$. Let k be a field and (X, λ) be a principally polarized abelian variety over k of dimension g (here λ denotes a suitable isomorphism from X onto the dual abelian variety X^t). Then the first algebraic de Rham cohomology $H_{\mathrm{dR}}^1(X/k)$ is a k -vector space of dimension $2g$ endowed with a canonical subspace $F^1(X/k)$ of dimension g (given by the Hodge filtration) and a non-degenerate alternating k -bilinear form

$$\langle \cdot, \cdot \rangle_{\lambda} : H_{\mathrm{dR}}^1(X/k) \times H_{\mathrm{dR}}^1(X/k) \longrightarrow k$$

induced by the principal polarization λ . By a *symplectic-Hodge basis* of (X, λ) , we mean a basis $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ of the k -vector space $H_{\mathrm{dR}}^1(X/k)$, such that

- (1) each ω_i is in $F^1(X/k)$, and
- (2) b is symplectic with respect to $\langle \cdot, \cdot \rangle_{\lambda}$, that is, $\langle \omega_i, \omega_j \rangle_{\lambda} = \langle \eta_i, \eta_j \rangle_{\lambda} = 0$ and $\langle \omega_i, \eta_j \rangle_{\lambda} = \delta_{ij}$ for every $1 \leq i, j \leq g$.

We may consider the moduli stack \mathcal{B}_g classifying principally polarized abelian varieties of dimension g equipped with a symplectic-Hodge basis; we prove that \mathcal{B}_g is a smooth Deligne-Mumford stack over $\mathrm{Spec} \mathbf{Z}$ of relative dimension $2g^2 + g$. This stack is not representable by a scheme (or even an algebraic space). Nevertheless, we prove that $\mathcal{B}_g \otimes \mathbf{Z}[1/2]$ is representable by a smooth quasi-projective scheme B_g over $\mathbf{Z}[1/2]$. This result relies essentially on a theorem of Oda ([78] Corollary 5.11) relating $H_{\mathrm{dR}}^1(X/k)$ to the Dieudonné module associated to the p -torsion subscheme $X[p]$ when k is a perfect field of characteristic p .

The main result in this paper is a description of the tangent bundle $T_{\mathcal{B}_g/\mathbf{Z}}$ in terms of the first relative de Rham cohomology of the universal abelian scheme over the moduli stack \mathcal{A}_g of principally polarized abelian varieties of dimension g (see Theorem 5.4 for a precise statement). From this description, we construct a family $(v_{ij})_{1 \leq i \leq j \leq g}$ of $g(g+1)/2$ commuting vector fields over \mathcal{B}_g ; these are the *higher Ramanujan vector fields*. Concretely, if $(\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ denotes the universal symplectic-Hodge basis over \mathcal{B}_g , and ∇ denotes the Gauss-Manin connection on the first relative de Rham cohomology of the universal abelian scheme over \mathcal{B}_g , then for every $1 \leq i \leq j \leq g$ we have

- (1) $\nabla_{v_{ij}} \omega_i = \eta_j$, $\nabla_{v_{ij}} \omega_j = \eta_i$, and $\nabla_{v_{ij}} \omega_k = 0$ for every $k \notin \{i, j\}$,
- (2) $\nabla_{v_{ij}} \eta_k = 0$ for every $1 \leq k \leq g$,

and these equations completely determine v_{ij} .

When $g = 1$, we shall recall how B_1 may be identified, by means of the classical theory of elliptic curves, with an open subscheme of $\mathbf{A}_{\mathbf{Z}[1/2]}^3 = \mathrm{Spec} \mathbf{Z}[1/2, b_2, b_4, b_6]$. Under this isomorphism, the vector field v_{11} gets identified with

$$2b_4 \frac{\partial}{\partial b_2} + 3b_6 \frac{\partial}{\partial b_4} + (b_2 b_6 - b_4^2) \frac{\partial}{\partial b_6}$$

which is, up to scaling, the vector field associated to Chazy's equation (C)². We also show that $B_1 \otimes \mathbf{Z}[1/6]$ may be identified with the open subscheme $\mathrm{Spec} \mathbf{Z}[1/6, e_2, e_4, e_6, 1/(e_4^3 - e_6^2)]$ of $\mathbf{A}_{\mathbf{Z}[1/6]}^3$, and that, under this

2. An integral curve of this vector field for the derivation θ is given by $q \mapsto (E_2(q), \frac{1}{2}\theta E_2(q), \frac{1}{6}\theta^2 E_2(q))$.

isomorphism, the vector field v_{11} gets identified with the “original” vector field associated to the Ramanujan equations (R) :

$$\frac{e_2^2 - e_4}{12} \frac{\partial}{\partial e_2} + \frac{e_2 e_4 - e_6}{3} \frac{\partial}{\partial e_4} + \frac{e_2 e_6 - e_4^2}{2} \frac{\partial}{\partial e_6}.$$

A geometric description of the above vector field in terms of the universal elliptic curve and the Gauss-Manin connection on its de Rham cohomology has actually been given by Movasati in [67] (see also [69]), and this has been one of the starting points of our construction. Let us remark that this point of view was already implicitly contained in the concept of “Serre derivative” of modular forms ([92] 1.4) and in its geometric interpretation given by Deligne ([51] A1.4).

In the sequel of this paper, *Higher Ramanujan equations II : periods of abelian varieties and transcendence questions*, we shall introduce analytic methods in our construction and we shall tackle some transcendence questions. We shall prove, for instance, that every leaf of the holomorphic foliation on the complex manifold $B_g(\mathbf{C})$ defined by the higher Ramanujan vector fields is Zariski-dense in B_g . We shall also construct a particular solution φ_g to the differential equations defined by the higher Ramanujan vector fields that will constitute a higher dimensional generalization of the solution $q \mapsto (E_2(q), E_4(q), E_6(q))$ when $g = 1$. Finally, we shall give a precise relation between the transcendence degree over \mathbf{Q} of values of φ_g and Grothendieck’s periods conjecture on abelian varieties.

We expect that the results in this paper, and in its sequel, might interest specialists in transcendental number theory. We have tried to keep prerequisites in abelian schemes and algebraic stacks to a minimum by recalling many notions and constructions that are well known to specialists in algebraic geometry, and by citing precise results in the (rather scarce) literature on these subjects.

1.1. Acknowledgments. This work was supported by a public grant as part of the FMJH project, and is part of my PhD thesis under the supervision of Jean-Benoît Bost. I thank him for suggesting me this research theme, and for his careful reading of the manuscript of this paper.

1.2. Terminology and notations.

1.2.1. By a *vector bundle* over a scheme U we mean a locally free sheaf \mathcal{E} over U of finite rank. A *line bundle* is a vector bundle of rank 1. A *subbundle* of \mathcal{E} is a subsheaf \mathcal{F} of \mathcal{E} such that \mathcal{F} and \mathcal{E}/\mathcal{F} are also vector bundles, that is, \mathcal{F} is locally a direct factor of \mathcal{E} . If \mathcal{E} has constant rank r , by a *basis* of \mathcal{E} over U we mean an ordered family of r global sections of \mathcal{E} that generate this sheaf as an \mathcal{O}_U -module. The *dual* of a vector bundle \mathcal{E} is the vector bundle $\mathcal{E}^\vee := \text{Hom}_{\mathcal{O}_U}(\mathcal{E}, \mathcal{O}_U)$.

1.2.2. Let U be a scheme. By an *abelian scheme* over U , we mean a proper and smooth group scheme $p : X \rightarrow U$ over U with geometrically connected fibers. The group law of X over U is commutative (cf. [72] Corollary 6.5) and will be denoted additively. A *morphism of abelian schemes* over U is a morphism of U -group schemes.

When p is projective, the relative Picard functor $\text{Pic}_{X/U}$ is representable by a group scheme over U ([13] Chapter 8). Then, the open group subscheme X^t of $\text{Pic}_{X/U}$, whose geometric points correspond to line bundles some power of which are algebraically equivalent to zero, is a projective abelian scheme over U , called the *dual abelian scheme*; we denote its structural morphism by $p^t : X^t \rightarrow U$. There is a canonical biduality isomorphism $X \xrightarrow{\sim} X^{tt}$ (cf. [13] 8.4 Theorem 5). The formation of both the dual abelian scheme and the biduality isomorphism is compatible with every base change in U . The universal line bundle over $X \times_U X^t$, the so-called *Poincaré line bundle*, will be denoted by $\mathcal{P}_{X/U}$.

A *principal polarization* on a projective abelian scheme X over U is an isomorphism of U -group schemes $\lambda : X \rightarrow X^t$ satisfying the equivalent conditions (cf. [72] 6.2 and [30] 1.4)

- (1) λ is symmetric (i.e. $\lambda = \lambda^t$ under the biduality isomorphism $X \cong X^{tt}$) and $(\text{id}_X, \lambda)^* \mathcal{P}_{X/U}$ is relatively ample over U .
- (2) Étale locally over U , λ is *induced by a line bundle on X* (cf. [72] Definition 6.2) relatively ample over U .

A *principally polarized abelian scheme* over U is a couple (X, λ) , where X is a projective abelian scheme over U and λ is a principal polarization on X .

1.2.3. If $X \rightarrow S$ is a smooth morphism of schemes, the dual \mathcal{O}_X -module of the sheaf of relative differentials $\Omega_{X/S}^1$ (i.e. the sheaf of \mathcal{O}_S -derivations of \mathcal{O}_X) is denoted by $T_{X/S}$. This is a vector bundle over X whose rank is given by the relative dimension of $X \rightarrow S$. If $S = \text{Spec } R$ is affine, we denote $T_{X/S} = T_{X/R}$.

The *Lie bracket* $[\cdot, \cdot] : T_{X/S} \times T_{X/S} \rightarrow T_{X/S}$ is defined on derivations by $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - \theta_2 \circ \theta_1$.

If S is a scheme, and $f : X \rightarrow Y$ is a morphism of smooth S -schemes, then there is a canonical morphism of \mathcal{O}_X -modules $f^* \Omega_{Y/S}^1 \rightarrow \Omega_{X/S}^1$. Further, as $Y \rightarrow S$ is smooth, the canonical morphism of \mathcal{O}_X -modules $f^* T_{Y/S} \rightarrow (f^* \Omega_{Y/S}^1)^\vee$ is an isomorphism. We denote by

$$Df : T_{X/S} \rightarrow f^* T_{Y/S}$$

the dual \mathcal{O}_X -morphism of $f^* \Omega_{Y/S}^1 \rightarrow \Omega_{X/S}^1$ after the identification $(f^* \Omega_{Y/S}^1)^\vee \cong f^* T_{Y/S}$. If f is smooth, we have an exact sequence of vector bundles over X

$$0 \rightarrow T_{X/Y} \rightarrow T_{X/S} \xrightarrow{Df} f^* T_{Y/S} \rightarrow 0.$$

1.2.4. If U is any scheme, the category of U -schemes (resp. U -group schemes) is denoted by Sch/U (resp. GpSch/U). The category of sets is denoted by Set . If \mathbf{C} is any category, its opposite category is denoted by \mathbf{C}^{op} .

1.2.5. We shall use the language of *categories fibered in groupoids* and the elements of the theory of *Deligne-Mumford stacks*. We follow the same conventions and terminology of [82]. In particular, if S is a scheme, whenever we talk about a *stack* over the category of S -schemes Sch/S (cf. [82] Definition 4.6.1), or simply a stack over S (or an S -stack), we shall always assume that Sch/S is endowed with the *Étale topology*.

In view of [82] Corollary 8.3.5, by an *algebraic space* over a scheme S we mean a Deligne-Mumford stack \mathcal{X} over S such that for any S -scheme U the fiber category $\mathcal{X}(U)$ is discrete (i.e. any automorphism is the identity).

The *étale site* of a Deligne-Mumford stack \mathcal{X} is denoted by $\acute{\text{E}}t(\mathcal{X})$ (cf. [82] Paragraph 9.1). We recall that the objects of the underlying category of $\acute{\text{E}}t(\mathcal{X})$ are *étale schemes over \mathcal{X}* , that is, pairs (U, u) where U is an S -scheme and $u : U \rightarrow \mathcal{X}$ is an étale S -morphism; morphisms are given by couples $(f, f^b) : (U', u') \rightarrow (U, u)$, where $f : U' \rightarrow U$ is an S -morphism and $f^b : u' \rightarrow u \circ f$ is an isomorphism of functors $U' \rightarrow \mathcal{X}$. Coverings in $\acute{\text{E}}t(\mathcal{X})$ are given by families of morphisms $\{(f_i, f_i^b) : (U_i, u_i) \rightarrow (U, u)\}_{i \in I}$ such that $\{f_i : U_i \rightarrow U\}_{i \in I}$ is an étale covering of U .

The structural sheaf on $\acute{\text{E}}t(\mathcal{X})$, which to any (U, u) associates the ring $\Gamma(U, \mathcal{O}_U)$, is denoted by $\mathcal{O}_{\mathcal{X}_{\acute{\text{E}}t}}$. We recall that an $\mathcal{O}_{\mathcal{X}_{\acute{\text{E}}t}}$ -module \mathcal{F} is said to be *quasi-coherent* if $u^* \mathcal{F}$ is a quasi-coherent \mathcal{O}_U -module for any object (U, u) of $\acute{\text{E}}t(\mathcal{X})$.

By a *vector bundle* over a Deligne-Mumford stack \mathcal{X} , we mean a locally free $\mathcal{O}_{\mathcal{X}_{\acute{\text{E}}t}}$ -module of finite rank. We define subbundles, bases, and duals as in 1.2.1.

1.2.6. Sheaves of differentials and tangent sheaves can also be defined for Deligne-Mumford stacks. If \mathcal{X} is a Deligne-Mumford stack over S , we define a presheaf of $\mathcal{O}_{\mathcal{X}_{\acute{\text{E}}t}}$ -modules $\Omega_{\mathcal{X}/S}^1$ on $\acute{\text{E}}t(\mathcal{X})$ by

$$\Gamma((U, u), \Omega_{\mathcal{X}/S}^1) := \Gamma(U, \Omega_{U/S}^1)$$

for any étale scheme (U, u) over \mathcal{X} ; restriction maps are defined in the obvious way. Since, for any étale morphism of S -schemes $f : U' \rightarrow U$, the induced morphism $f^* \Omega_{U/S}^1 \rightarrow \Omega_{U'/S}^1$ is an isomorphism of $\mathcal{O}_{U'}$ -modules, and for any S -scheme U the sheaf $\Omega_{U/S}^1$ is a quasi-coherent \mathcal{O}_U -module, we see that $\Omega_{\mathcal{X}/S}^1$ is in fact a quasi-coherent sheaf over \mathcal{X} (cf. [82] Lemma 4.3.3). Note that $u^* \Omega_{\mathcal{X}/S}^1 = \Omega_{U/S}^1$ for any étale scheme (U, u) over \mathcal{X} .

Let $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford stacks over S . If φ is representable by schemes, then there exists a unique morphism of $\mathcal{O}_{\mathcal{Y}}$ -modules $\Omega_{\mathcal{Y}/S}^1 \rightarrow \varphi_* \Omega_{\mathcal{X}/S}^1$ inducing, for any étale scheme (V, v) over \mathcal{Y} , the canonical morphism $\Omega_{V/S}^1 \rightarrow \varphi'_* \Omega_{U/S}^1$, where (U, u) (resp. $\varphi' : U \rightarrow V$) denotes the étale scheme over \mathcal{X} (resp. the morphism of S -schemes) obtained from (V, v) (resp. φ) by base change. If, moreover, φ is quasi-compact and quasi-separated, by adjointness (cf. [82] Proposition 9.3.6), we obtain a morphism of $\mathcal{O}_{\mathcal{X}_{\acute{\text{E}}t}}$ -modules

$$(1.1) \quad \varphi^* \Omega_{\mathcal{Y}/S}^1 \rightarrow \Omega_{\mathcal{X}/S}^1.$$

We then define a quasi-coherent $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ -module

$$\Omega_{\mathcal{X}/\mathcal{Y}}^1 := \text{coker}(\varphi^* \Omega_{\mathcal{Y}/S}^1 \longrightarrow \Omega_{\mathcal{X}/S}^1).$$

If \mathcal{X} is a smooth Deligne-Mumford stack over S , then $\Omega_{\mathcal{X}/S}^1$ is a vector bundle over \mathcal{X} . We define $T_{\mathcal{X}/S}$ as the dual $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ -module of $\Omega_{\mathcal{X}/S}^1$. If $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of smooth Deligne-Mumford stacks over S representable by smooth schemes, then $\Omega_{\mathcal{X}/\mathcal{Y}}^1$ is a vector bundle over \mathcal{X} , and its dual is denoted by $T_{\mathcal{X}/\mathcal{Y}}$. Moreover, in this case, the morphism in (1.1) is injective and induces a surjective morphism of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ -modules $D\varphi : T_{\mathcal{X}/S} \rightarrow \varphi^* T_{\mathcal{Y}/S}$. We thus obtain an exact sequence of quasi-coherent $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ -modules

$$0 \longrightarrow T_{\mathcal{X}/\mathcal{Y}} \longrightarrow T_{\mathcal{X}/S} \xrightarrow{D\varphi} \varphi^* T_{\mathcal{Y}/S} \longrightarrow 0.$$

2. Symplectic-Hodge bases

We start this section by recalling the definition of the de Rham cohomology of an abelian scheme and its main properties. We next explain how to associate to a principal polarization on an abelian scheme a symplectic structure on its first de Rham cohomology. This leads us to the definition of symplectic-Hodge bases.

2.1. De Rham cohomology of abelian schemes. Let $p : X \rightarrow U$ be an abelian scheme of relative dimension g .

We recall that, for any integer $i \geq 0$, the i -th de Rham cohomology sheaf of \mathcal{O}_U -modules associated to p is defined as the i -th left hyperderived functor of p_* applied to the complex of relative differential forms $\Omega_{X/U}^\bullet$:

$$H_{\text{dR}}^i(X/U) := \mathbf{R}^i p_* \Omega_{X/U}^\bullet.$$

If $F : X \rightarrow Y$ is a morphism of abelian schemes over U , we denote by $F^* : H_{\text{dR}}^i(Y/U) \rightarrow H_{\text{dR}}^i(X/U)$ the induced \mathcal{O}_U -morphism on cohomology.

One can prove that there is a canonical isomorphism given by cup product

$$\bigwedge^i H_{\text{dR}}^1(X/U) \xrightarrow{\sim} H_{\text{dR}}^i(X/U),$$

and that $H_{\text{dR}}^1(X/U)$ is a vector bundle over U of rank $2g$. Moreover, the canonical \mathcal{O}_U -morphism $p_* \Omega_{X/U}^1 \rightarrow H_{\text{dR}}^1(X/U)$ induces an isomorphism of $p_* \Omega_{X/U}^1$ with a rank g subbundle of $H_{\text{dR}}^1(X/U)$, its *Hodge subbundle* $F^1(X/U)$. It fits into a canonical exact sequence of \mathcal{O}_U -modules :

$$(2.1) \quad 0 \longrightarrow F^1(X/U) \longrightarrow H_{\text{dR}}^1(X/U) \longrightarrow R^1 p_* \mathcal{O}_X \longrightarrow 0.$$

The formation of $H_{\text{dR}}^1(X/U)$, $F^1(X/U)$, $R^1 p_* \mathcal{O}_X$, and the above exact sequence is compatible with every base change in U .

For a proof of all these facts, the reader may consult [5] 2.5.

2.2. Symplectic form associated to a principal polarization. Let $p : X \rightarrow U$ be a projective abelian scheme of relative dimension g and $\lambda : X \rightarrow X^t$ be a principal polarization. In this paragraph, we recall how to associate to λ a canonical *symplectic* \mathcal{O}_U -bilinear form

$$\langle \cdot, \cdot \rangle_\lambda : H_{\text{dR}}^1(X/U) \times H_{\text{dR}}^1(X/U) \longrightarrow \mathcal{O}_U.$$

We refer to Appendix 1.A for basic definitions and terminology concerning symplectic forms on vector bundles over schemes.

Recall that to any line bundle \mathcal{L} on X we can associate its *first Chern class in de Rham cohomology* $c_{1,\text{dR}}(\mathcal{L})$, namely the global section of $H_{\text{dR}}^2(X/U)$ given by the image of the class of the line bundle \mathcal{L} under the morphism of \mathcal{O}_U -modules

$$R^1 p_* \mathcal{O}_X^\times \longrightarrow \mathbf{R}^1 p_* \Omega_{X/U}^\bullet[1] \cong H_{\text{dR}}^2(X/U)$$

induced by $\mathrm{dlog} : \mathcal{O}_X^\times \rightarrow \Omega_{X/U}^\bullet[1]$.³

We apply the above construction to the Poincaré line bundle $\mathcal{P}_{X/U}$ on the projective abelian scheme $X \times_U X^t$ over U . Let

$$\phi_{X/U} : H_{\mathrm{dR}}^1(X/U)^\vee \rightarrow H_{\mathrm{dR}}^1(X^t/U)$$

be the morphism of \mathcal{O}_U -modules given by the image of $c_{1,\mathrm{dR}}(\mathcal{P}_{X/U})$ in the Künneth component $H_{\mathrm{dR}}^1(X/U) \otimes_{\mathcal{O}_U} H_{\mathrm{dR}}^1(X^t/U)$ of $H_{\mathrm{dR}}^2(X/U)$, and consider the isomorphism of \mathcal{O}_U -modules

$$\lambda^* : H_{\mathrm{dR}}^1(X^t/U) \rightarrow H_{\mathrm{dR}}^1(X/U)$$

induced by the principal polarization $\lambda : X \rightarrow X^t$. For any sections γ and δ of $H_{\mathrm{dR}}^1(X/U)^\vee$, we put

$$Q_\lambda(\gamma, \delta) := \delta \circ \lambda^* \circ \phi_{X/U}(\gamma).$$

It is clear that Q_λ defines an \mathcal{O}_U -bilinear form over $H_{\mathrm{dR}}^1(X/U)^\vee$. By [5] 5.1.3.1, $\phi_{X/U}$ is in fact an isomorphism; in particular, Q_λ is non-degenerate. By duality, we can thus define a non-degenerate bilinear form $\langle \cdot, \cdot \rangle_\lambda$ over $H_{\mathrm{dR}}^1(X/U)$ via

$$\langle Q_\lambda(\gamma, \cdot), Q_\lambda(\delta, \cdot) \rangle_\lambda := Q_\lambda(\gamma, \delta),$$

where we identified $H_{\mathrm{dR}}^1(X/U)^{\vee\vee}$ with $H_{\mathrm{dR}}^1(X/U)$.

LEMMA 2.1. *The non-degenerate bilinear form $\langle \cdot, \cdot \rangle_\lambda$ is alternating, thus symplectic.*

PROOF. It suffices to prove that Q_λ is alternating. Since λ is a polarization, it is étale locally over U induced by a line bundle \mathcal{L} over X relatively ample over U . We consider the first Chern class $c_{1,\mathrm{dR}}(\mathcal{L})$ in $H_{\mathrm{dR}}^2(X/U) \cong \bigwedge^2 H_{\mathrm{dR}}^1(X/U)$. Then, one can verify that Q_λ defined above coincides with the alternating form

$$(\gamma, \delta) \mapsto \gamma \wedge \delta(c_{1,\mathrm{dR}}(\mathcal{L})).$$

We refer to [30], Section 1, for further details. ■

Thus we obtain a symplectic vector bundle $(H_{\mathrm{dR}}^1(X/U), \langle \cdot, \cdot \rangle_\lambda)$ over U in the sense of Definition 1.A.1.

LEMMA 2.2. *$F^1(X/U)$ is a Lagrangian subbundle of $H_{\mathrm{dR}}^1(X/U)$ with respect to the symplectic form $\langle \cdot, \cdot \rangle_\lambda$.*

PROOF. Since the rank of $H_{\mathrm{dR}}^1(X/U)$ is $2g$, and $F^1(X/U)$ is a rank g subbundle of $H_{\mathrm{dR}}^1(X/U)$, it suffices to prove that $F^1(X/U)$ is isotropic with respect to $\langle \cdot, \cdot \rangle_\lambda$ (cf. Corollary 1.A.4). This follows immediately from the compatibility of $\phi_{X/U}$ with the exact sequence (2.1), that is, from the existence of canonical morphisms $\phi_{X/U}^0$ and $\phi_{X/U}^1$ making the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (R^1 p_* \mathcal{O}_X)^\vee & \longrightarrow & H_{\mathrm{dR}}^1(X/U)^\vee & \longrightarrow & F^1(X/U)^\vee \longrightarrow 0 \\ & & \downarrow \phi_{X/U}^0 & & \downarrow \phi_{X/U} & & \downarrow \phi_{X/U}^1 \\ 0 & \longrightarrow & F^1(X^t/U) & \longrightarrow & H_{\mathrm{dR}}^1(X^t/U) & \longrightarrow & R^1 p_*^t \mathcal{O}_{X^t} \longrightarrow 0 \end{array}$$

commute ([5] Lemme 5.1.4; the morphisms $\phi_{X/U}^0$ and $\phi_{X/U}^1$ are uniquely determined by this commutative diagram, and are isomorphisms). ■

REMARK 2.3. It is clear from the above construction that the formation of the symplectic form $\langle \cdot, \cdot \rangle_\lambda$ is compatible with base change. Namely, if $f : U' \rightarrow U$ is a morphism of schemes, and (X', λ') denotes the principally polarized abelian scheme over U' obtained by base change via f , then $f^* \langle \cdot, \cdot \rangle_\lambda$ coincides with $\langle \cdot, \cdot \rangle_{\lambda'}$ under the base change isomorphism $f^* H_{\mathrm{dR}}^1(X/U) \xrightarrow{\sim} H_{\mathrm{dR}}^1(X'/U')$.

³ We adopt the same sign conventions of [5] 0.3 for the differentials of the shifted complex $\Omega_{X/U}^\bullet[1]$ and for the isomorphism $R^1 p_* \Omega_{X/U}^\bullet[1] \cong H_{\mathrm{dR}}^2(X/U)$.

2.3. Symplectic-Hodge bases of $H_{\text{dR}}^1(X/U)$. Let U be a scheme and (X, λ) be a principally polarized abelian scheme over U of relative dimension g .

DEFINITION 2.4. A *symplectic-Hodge basis* of $(X, \lambda)_{/U}$ is a sequence $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ of $2g$ global sections of $H_{\text{dR}}^1(X/U)$ such that :

- (1) $\omega_1, \dots, \omega_g$ are sections of $F^1(X/U)$, and
- (2) b is a symplectic basis of $(H_{\text{dR}}^1(X/U), \langle \cdot, \cdot \rangle_\lambda)$ (Definition 1.A.7).

Let us note that symplectic-Hodge bases may not exist globally, but such bases always exist locally for the Zariski topology over U by Proposition 1.A.8.

3. The moduli stack \mathcal{B}_g

In this section, we define for every integer $g \geq 1$ a category \mathcal{B}_g fibered in groupoids over the category of schemes Sch/\mathbf{Z} classifying principally polarized abelian schemes of relative dimension g endowed with a symplectic-Hodge basis.

We prove that $\mathcal{B}_g \rightarrow \text{Spec } \mathbf{Z}$ is a smooth Deligne-Mumford stack over $\text{Spec } \mathbf{Z}$ of relative dimension $2g^2 + g$. The main point in proving this result will be to remark that for any principally polarized abelian scheme (X, λ) of relative dimension g over an affine scheme $U = \text{Spec } R$, there is a natural free and transitive right action of the Siegel parabolic subgroup $P_g(R)$ of $\text{Sp}_{2g}(R)$, consisting of “upper triangular matrices”, on the set of symplectic-Hodge bases of $(X, \lambda)_{/U}$.

3.1. The moduli stack \mathcal{A}_g . Let $g \geq 1$ be an integer. To fix ideas and notations we recall the definition of the moduli stack of principally polarized abelian schemes of relative dimension g .

For any scheme S , we define a category fibered in groupoids $\mathcal{A}_{g,S} \rightarrow \text{Sch}/_S$ as follows.

- (i) An object of $\mathcal{A}_{g,S}$ is given by an S -scheme U and a principally polarized abelian scheme (X, λ) of relative dimension g over U ; when U is not clear in the context, we shall incorporate it in the notation by writing $(X, \lambda)_{/U}$. A morphism $(X, \lambda)_{/U} \rightarrow (Y, \mu)_{/V}$ in $\mathcal{A}_{g,S}$, denoted $F_{/f}$, is given by a cartesian diagram of S -schemes

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow & \square & \downarrow \\ U & \xrightarrow{f} & V \end{array}$$

preserving the identity sections of the abelian schemes and identifying λ with the pullback of μ by $f : U \rightarrow V$. We shall occasionally denote $F_{/f}$ simply by F when there will be no danger of confusion. We may also denote $(X, \lambda) = (Y, \mu) \times_U V$.

- (ii) The structural functor $\mathcal{A}_{g,S} \rightarrow \text{Sch}/_S$ is given by sending an object $(X, \lambda)_{/U}$ of $\mathcal{A}_{g,S}$ to the S -scheme U , and a morphism $F_{/f}$ to f .

If $S = \text{Spec } R$ is affine (resp. $S = \text{Spec } \mathbf{Z}$), then we denote $\mathcal{A}_{g,S} =: \mathcal{A}_{g,R}$ (resp. $\mathcal{A}_{g,S} =: \mathcal{A}_g$).

Recall that the category of S -schemes can be seen as a subcategory of the 2-category of categories fibered in groupoids over $\text{Sch}/_S$ by sending each S -scheme U to the category $\text{Sch}/_U$ endowed with its natural functor $\text{Sch}/_U \rightarrow \text{Sch}/_S$. In the sequel, we shall adopt the standard convention of denoting $\text{Sch}/_U$ simply by U when working in the context of categories fibered in groupoids. Then $\mathcal{A}_{g,S}$ is canonically equivalent to $\mathcal{A}_g \times_{\mathbf{Z}} S$ as categories fibered in groupoids over S .

We summarize the main properties of $\mathcal{A}_{g,S}$ we are going to use in the form of the next theorem.

THEOREM 3.1. *For any scheme S and any integer $g \geq 1$, $\mathcal{A}_{g,S}$ is a smooth Deligne-Mumford stack over S of relative dimension $g(g+1)/2$.*

A proof that $\mathcal{A}_{g,S}$ is a Deligne-Mumford stack over S is essentially contained in [72] Theorem 7.9 (cf. [81] Theorem 2.1.11). Smoothness and relative dimension are obtained by a theorem of Grothendieck (cf. [83] Proposition 2.4.1).

3.2. Definition of \mathcal{B}_g . Let $F/f : (X, \lambda)_{/U} \rightarrow (Y, \mu)_{/V}$ be a morphism in \mathcal{A}_g . By the compatibility with base change of the symplectic forms induced by principal polarizations (Remark 2.3), the pullback F^*b of every symplectic-Hodge basis b of $(Y, \mu)_{/V}$ is a symplectic-Hodge basis of $(X, \lambda)_{/U}$. We can thus define a functor

$$\underline{\mathcal{B}}_g : \mathcal{A}_g^{\text{op}} \rightarrow \text{Set}$$

that sends every object $(X, \lambda)_{/U}$ of \mathcal{A}_g to the set of symplectic-Hodge bases of $(X, \lambda)_{/U}$, and whose action on morphisms is given by pullbacks as above.

From the functor $\underline{\mathcal{B}}_g$, we form a category fibered in groupoids

$$\mathcal{B}_g \rightarrow \text{Spec } \mathbf{Z}$$

as follows.

- (i) An object of \mathcal{B}_g is a “triple” $(X, \lambda, b)_{/U}$ where $(X, \lambda)_{/U}$ is an object of \mathcal{A}_g and $b \in \underline{\mathcal{B}}_g(X, \lambda)$. An arrow $(X, \lambda, b)_{/U} \rightarrow (Y, \mu, c)_{/V}$ is given by a morphism $F/f : (X, \lambda)_{/U} \rightarrow (Y, \mu)_{/V}$ such that $b = F^*c$. We denote by

$$\pi_g : \mathcal{B}_g \rightarrow \mathcal{A}_g$$

the forgetful functor $(X, \lambda, b)_{/U} \mapsto (X, \lambda)_{/U}$.

- (ii) The structural functor $\mathcal{B}_g \rightarrow \text{Spec } \mathbf{Z}$ is defined as the composition of π_g with the structural functor $\mathcal{A}_g \rightarrow \text{Spec } \mathbf{Z}$.

The rest of this section is devoted to the proof of the next theorem.

THEOREM 3.2. *The category fibered in groupoids $\mathcal{B}_g \rightarrow \text{Spec } \mathbf{Z}$ is a smooth Deligne-Mumford stack over $\text{Spec } \mathbf{Z}$ of relative dimension $2g^2 + g$.*

3.3. Siegel parabolic subgroup and proof of Theorem 3.2. Fix a scheme U and an object (X, λ) of \mathcal{A}_g lying over U . Then we can define a functor

$$\underline{\mathcal{B}}_{(X, \lambda)} : \text{Sch}_{/U}^{\text{op}} \rightarrow \text{Set}$$

that sends a U -scheme U' to the set $\underline{\mathcal{B}}_g((X, \lambda) \times_U U')$. It is clear that this functor defines a sheaf for the Zariski topology over $\text{Sch}_{/U}$.

Let us now consider the *symplectic group* Sp_{2g} , namely the smooth affine group scheme over $\text{Spec } \mathbf{Z}$ of relative dimension $2g^2 + g$ such that for every affine scheme $V = \text{Spec } R$

$$\text{Sp}_{2g}(V) = \left\{ \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \in M_{2g \times 2g}(R) \mid \begin{array}{l} A, B, C, D \in M_{g \times g}(R) \text{ satisfy} \\ AB^\top = BA^\top, CD^\top = DC^\top, \text{ and } AD^\top - BC^\top = \mathbf{1}_g \end{array} \right\}.$$

The *Siegel parabolic subgroup* P_g of Sp_{2g} is defined as the subgroup scheme of Sp_{2g} such that, for every affine scheme $V = \text{Spec } R$,

$$P_g(V) = \left\{ \left(\begin{array}{cc} A & B \\ 0 & (A^\top)^{-1} \end{array} \right) \in M_{2g \times 2g}(R) \mid A \in \text{GL}_g(R) \text{ and } B \in M_{g \times g}(R) \text{ satisfy } AB^\top = BA^\top \right\}.$$

Note that P_g is a smooth affine group scheme over $\text{Spec } \mathbf{Z}$ of relative dimension $g(3g + 1)/2$.

Let (X, λ, b) be an object of \mathcal{B}_g lying over $V = \text{Spec } R$ and consider $b = (\omega \ \eta)$ as a row vector of order $2g$ with coefficients in the R -module $H_{\text{dR}}^1(X/V)$. For any

$$p = \left(\begin{array}{cc} A & B \\ 0 & (A^\top)^{-1} \end{array} \right) \in P_g(V)$$

it is easy to check that

$$b \cdot p := (\omega A \quad \omega B + \eta(A^\top)^{-1})$$

is a symplectic-Hodge basis of $(X, \lambda)_{/V}$. This defines a right action of $P_g(V)$ on $\underline{\mathcal{B}}_g(X, \lambda)$:

$$\underline{\mathcal{B}}_g(X, \lambda) \times P_g(V) \rightarrow \underline{\mathcal{B}}_g(X, \lambda).$$

Moreover, it is clear that if $V' \subset V$ is an affine open subscheme of V , then the natural diagram

$$\begin{array}{ccc} \underline{B}_g(X, \lambda) \times P_g(V) & \longrightarrow & \underline{B}_g(X, \lambda) \\ \downarrow & & \downarrow \\ \underline{B}_g(X', \lambda') \times P_g(V') & \longrightarrow & \underline{B}_g(X', \lambda') \end{array}$$

commutes, where $(X', \lambda') = (X, \lambda) \times_V V'$.

Thus, for any scheme U , and any object (X, λ) of \mathcal{A}_g lying over U , we obtain a right action of the U -group scheme $P_{g,U} = P_g \times_{\mathbf{Z}} U$ on $\underline{B}_{(X,\lambda)}$.

LEMMA 3.3. *The Zariski sheaf $\underline{B}_{(X,\lambda)}$ over Sch/U is a right Zariski $P_{g,U}$ -torsor for the above action.*

PROOF. If V is any affine scheme over U such that $\underline{B}_{(X,\lambda)}(V)$ is non-empty, a routine computation shows that the action of $P_g(V)$ on $\underline{B}_{(X,\lambda)}(V)$ is free and transitive. Moreover, it was already remarked above that symplectic-Hodge bases exist locally for the Zariski topology. ■

Since $P_{g,U}$ is affine, smooth, and of relative dimension $g(3g+1)/2$ over U , Lemma 3.3 immediately implies the following corollary.

COROLLARY 3.4. *For every scheme U , and every object (X, λ) of \mathcal{A}_g lying over U , the functor $\underline{B}_{(X,\lambda)}$ is representable by a smooth affine U -scheme $B(X, \lambda)$ of relative dimension $g(3g+1)/2$.*

REMARK 3.5. Let us keep the notation of the above corollary. Recall that the principally polarized abelian scheme (X, λ) over U corresponds to a morphism $U \rightarrow \mathcal{A}_g$. Then $B(X, \lambda)$ is a scheme representing $\mathcal{B}_g \times_{\mathcal{A}_g} U$.

PROOF OF THEOREM 3.2. Recall that for any scheme U and any abelian scheme X over U , $H_{\text{dR}}^1(X/U)$ is a quasi-coherent sheaf over U , and that any quasi-coherent sheaf over U induces a sheaf over Sch/U endowed with the fppf topology ([82] Lemma 4.3.3). Since the étale topology is coarser than the fppf topology, this shows in particular that $H_{\text{dR}}^1(X/U)$ induces a sheaf over Sch/U endowed with the étale topology; this immediately implies that $\mathcal{B}_g \rightarrow \text{Spec } \mathbf{Z}$ is a stack over $\text{Spec } \mathbf{Z}$.

It follows in particular from Corollary 3.4 that the morphism $\pi_g : \mathcal{B}_g \rightarrow \mathcal{A}_g$ is representable by smooth schemes (Remark 3.8). Hence, as $\mathcal{A}_g \rightarrow \text{Spec } \mathbf{Z}$ is a Deligne-Mumford stack over $\text{Spec } \mathbf{Z}$, the same holds for $\mathcal{B}_g \rightarrow \text{Spec } \mathbf{Z}$ (cf. [82] Proposition 10.2.2). The smoothness of $\mathcal{B}_g \rightarrow \text{Spec } \mathbf{Z}$ follows by composition from that of $\mathcal{A}_g \rightarrow \text{Spec } \mathbf{Z}$ and that of π_g . Finally, we can compute the relative dimension of $\mathcal{B}_g \rightarrow \text{Spec } \mathbf{Z}$ as the sum of that of $\mathcal{A}_g \rightarrow \text{Spec } \mathbf{Z}$ and that of π_g :

$$\frac{g(g+1)}{2} + \frac{g(3g+1)}{2} = 2g^2 + g.$$

■

4. Representability of \mathcal{B}_g by a scheme

It is easy to see that if S is a scheme over \mathbf{F}_2 , then $\mathcal{B}_g \times_{\mathbf{Z}} S \rightarrow S$ is not representable. Indeed, if $(X, \lambda, b)_{/U}$ is an object of \mathcal{B}_g lying over a scheme U over \mathbf{F}_2 , then the involution $[-1] : P \mapsto -P$ on X defines a non-trivial automorphism $[-1]_{/\text{id}_U} : (X, \lambda)_{/U} \rightarrow (X, \lambda)_{/U}$ in $\mathcal{A}_g(U)$ such that

$$[-1]^*b = -b = b,$$

thus a non-trivial automorphism of $(X, \lambda, b)_{/U}$ in $\mathcal{B}_g(U)$.

For any ring R , let us denote $\mathcal{B}_{g,R} := \mathcal{B}_g \otimes_{\mathbf{Z}} R$. In this section we prove the following theorem.

THEOREM 4.1. *The stack $\mathcal{B}_{g,\mathbf{Z}[1/2]} \rightarrow \text{Spec } \mathbf{Z}[1/2]$ is representable by a smooth quasi-projective scheme B_g over $\mathbf{Z}[1/2]$ of relative dimension $2g^2 + g$.*

Let us briefly summarize our proof of Theorem 4.1.

We shall first prove that $\mathcal{B}_{g,\mathbf{Z}[1/2]}$ is an algebraic space over $\mathbf{Z}[1/2]$. This amounts to proving that the functor \underline{B}_g is *rigid* over $\mathbf{Z}[1/2]$ (see Definition 4.2 below). By the classical “rigidity lemma” for abelian schemes (Lemma 4.6), we reduce the proof that \underline{B}_g is rigid over $\mathbf{Z}[1/2]$ to proving that \underline{B}_g is rigid over any

algebraically closed field of characteristic 0 or $p > 2$. In positive characteristic, this will be obtained by a theorem of Oda characterizing the first de Rham cohomology of an abelian variety over a perfect field of characteristic p in terms of its p -torsion subgroup scheme.

Finally, we use the existence of a quasi-projective surjective étale scheme over $\mathcal{A}_{g, \mathbf{Z}[1/2]}$ to conclude, via a simple base-change argument, that $\mathcal{B}_{g, \mathbf{Z}[1/2]}$ is actually representable by a quasi-projective $\mathbf{Z}[1/2]$ -scheme.

4.1. Rigidity over $\mathbf{Z}[1/2]$. Let R be a ring. The following terminology has been borrowed from [52] 4.4.

DEFINITION 4.2. We say that the functor \underline{B}_g (cf. paragraph 3.2) is *rigid* over R if, for every R -scheme U , and every object (X, λ) of \mathcal{A}_g lying over U , the action of $\text{Aut}_U(X, \lambda)$ on $\underline{B}_g(X, \lambda)_{/U}$ is free.

Note that \underline{B}_g is rigid over R if and only if the fiber categories of $\mathcal{B}_{g, R} \rightarrow \text{Spec } R$ are discrete. As \mathcal{B}_g is a Deligne-Mumford stack over $\text{Spec } \mathbf{Z}$, this amounts to saying that $\mathcal{B}_{g, R} \rightarrow \text{Spec } R$ is an algebraic space over $\text{Spec } R$ (see our terminology conventions in 1.2.5).

LEMMA 4.3. *Let k be a field of characteristic 0. Then \underline{B}_g is rigid over k .*

PROOF. Let (X, λ, b) be an object of \mathcal{B}_g lying over k and $\varphi : X \rightarrow X$ be a k -automorphism of (X, λ) such that $\varphi^*b = b$; we must show that $\varphi = \text{id}_X$.

We claim that it is sufficient to treat the case $k = \mathbf{C}$. In fact, as X is of finite type over k , by “elimination of Noetherian hypothesis” (cf. [42] 8.8, 8.9, 8.10, 12.2.1, and [43] 17.7.9), there exists a subfield k_0 of k , of finite type over \mathbf{Q} , and a principally polarized abelian variety (X_0, λ_0) over k_0 endowed with a symplectic-Hodge basis b_0 and a k_0 -automorphism φ_0 of (X_0, λ_0) satisfying $\varphi_0^*b_0 = b_0$, such that (X, λ, b) (resp. φ) is obtained from (X_0, λ_0, b_0) (resp. φ_0) by the base change $\text{Spec } k \rightarrow \text{Spec } k_0$. After fixing an embedding of k_0 in \mathbf{C} , we finally remark that if $\varphi_{0, \mathbf{C}}$ is the identity over $X_0 \otimes_{k_0} \mathbf{C}$, then the same holds for φ_0 , and thus also for φ .

Let then $k = \mathbf{C}$. It is sufficient to prove that the induced automorphism of complex Lie groups $\varphi^{\text{an}} : X^{\text{an}} \rightarrow X^{\text{an}}$ is the identity. As X^{an} is a complex torus, the exponential $\exp : \text{Lie } X \rightarrow X^{\text{an}}$ is a surjective morphism of complex Lie groups. Therefore, it follows from the commutative diagram

$$\begin{array}{ccc} \text{Lie } X & \xrightarrow{\text{Lie } \varphi} & \text{Lie } X \\ \exp \downarrow & & \downarrow \exp \\ X^{\text{an}} & \xrightarrow{\varphi^{\text{an}}} & X^{\text{an}} \end{array}$$

that it is sufficient to prove that $\text{Lie } \varphi = \text{id}_{\text{Lie } X}$. Now, if φ preserves symplectic-Hodge basis of (X, λ) , then in particular the \mathbf{C} -linear map $\varphi^* : H^0(X, \Omega_{X/\mathbf{C}}^1) \rightarrow H^0(X, \Omega_{X/\mathbf{C}}^1)$ is the identity, and thus its dual $\text{Lie } \varphi : \text{Lie } X \rightarrow \text{Lie } X$ is also the identity. ■

We now treat the case of positive characteristic. Let us briefly recall some notions in Dieudonné theory and its relations with abelian varieties.

Let k be a perfect field of characteristic $p > 0$. We denote by W the ring of Witt vectors over k , and by σ the unique ring automorphism of W lifting the absolute Frobenius $x \mapsto x^p$ of k . We can then define a W -algebra D generated by elements F and V subjected to the relations

$$FV = VF = p, \quad Fx = \sigma(x)F, \quad xV = V\sigma(x)$$

for any $x \in W$.

The theory of Dieudonné (cf. [78] Definition 3.12) provides an additive contravariant functor

$$(4.1) \quad G \mapsto M(G)$$

from the category of commutative finite k -group schemes of p -power order to the category of left D -modules. This functor is shown to be faithful and its essential image is given by the category of left D -modules of finite W -length : $M(G)$ is of W -length r if and only if G is of order p^r ([78] Corollary 3.16).

Now, let X be an abelian variety over k and consider the k -vector space $H_{\text{dR}}^1(X/k)$ as a W -module via the canonical map $W \rightarrow k$. Then one can endow $H_{\text{dR}}^1(X/k)$ with the structure of a D -module, the action of F (resp. V) being induced by the relative Frobenius on X (resp. the Cartier operator in degree

1); we refer to [78] Definition 5.3 and Definition 5.6 for further details. This construction is functorial in the sense that for any morphism $\varphi : X \rightarrow Y$ of abelian varieties over k , if we endow $H_{\text{dR}}^1(X/k)$ and $H_{\text{dR}}^1(Y/k)$ with the preceding D -module structure, then the induced morphism on de Rham cohomology $\varphi^* : H_{\text{dR}}^1(Y/k) \rightarrow H_{\text{dR}}^1(X/k)$ is D -linear.

In the next statement, for any abelian variety X over k , we regard $H_{\text{dR}}^1(X/k)$ with the above D -module structure, and we denote its p -torsion subscheme by $X[p]$. Note that $X[p]$ is a commutative finite k -group scheme of order $p^{2 \dim X}$.

THEOREM 4.4 (Oda, [78] Corollary 5.11). *The contravariant functors $X \mapsto M(X[p])$ and $X \mapsto H_{\text{dR}}^1(X/k)$ from the category of abelian varieties over k to the category of (p -torsion) D -modules of finite W -length are naturally equivalent.*

LEMMA 4.5. *Let k be a perfect field of characteristic $p > 2$. Then \underline{B}_g is rigid over k .*

PROOF. Let (X, λ) be a principally polarized abelian variety over k of dimension g and $\varphi : X \rightarrow X$ be a k -automorphism of (X, λ) .

If φ preserves a symplectic-Hodge basis of $(X, \lambda)_{/k}$, then in particular $\varphi^* : H_{\text{dR}}^1(X/k) \rightarrow H_{\text{dR}}^1(X/k)$ is the identity; a fortiori, φ induces the identity on $H_{\text{dR}}^1(X/k)$ regarded as a D -module. Then, by Theorem 4.4, φ induces the identity on the D -module $M(X[p])$. As the functor $G \mapsto M(G)$ in (4.1) is faithful, φ restricts to the identity on the p -torsion subscheme $X[p]$ of X . As φ preserves, in addition, the polarization λ on X , and since $p \geq 3$, then necessarily $\varphi = \text{id}_X$ (cf. [71] IV.21, Theorem 5). ■

Recall the following version of the classical “rigidity lemma” for abelian schemes which follows from the arguments in the proof of Proposition 6.1 in [72].

LEMMA 4.6. *Let A be a local Artinian ring, and X be an abelian scheme over A . If $\varphi : X \rightarrow X$ is an endomorphism of A -group schemes restricting to the identity on the closed fiber of $X \rightarrow \text{Spec } A$, then $\varphi = \text{id}_X$.*

PROPOSITION 4.7. *The functor \underline{B}_g is rigid over $\mathbf{Z}[1/2]$.*

PROOF. Let U be a $\mathbf{Z}[1/2]$ -scheme, (X, λ) be an object of \mathcal{A}_g lying over U , and φ be an automorphism of (X, λ) in the fiber category $\mathcal{A}_g(U)$ preserving an element b of $\underline{B}_g(X, \lambda)$. We must show that $\varphi = \text{id}_X$. This being a local property over U , we can assume that U is affine.

Suppose that U is Noetherian. By Lemmas 4.3 and 4.5, for every geometric point \bar{u} of U , we have $\varphi_{X_{\bar{u}}} = \text{id}_{X_{\bar{u}}}$. Let Z be the closed subscheme of U where $\varphi = \text{id}$. Then Z contains every closed point of U . By Lemma 4.6, and Krull’s intersection theorem, Z is also an open subscheme of U ; hence $Z = U$, which amounts to saying that $\varphi = \text{id}_X$.

In general, by “elimination of Noetherian hypothesis” (cf. [42], 8.8, 8.9, 8.10, 12.2.1, and [43], 17.7.9), there exists an affine Noetherian scheme U_0 under U , and a principally polarized abelian scheme (X_0, λ_0) over U_0 endowed with a symplectic-Hodge basis b_0 , and with an U_0 -automorphism φ_0 , such that $\varphi_0^* b_0 = b_0$, and (X, λ) (resp. b , resp. φ) is deduced from (X_0, λ_0) (resp. b_0 , resp. φ_0) by the base change $U \rightarrow U_0$. The preceding paragraph shows that $\varphi_0 = \text{id}_{X_0}$, hence $\varphi = \text{id}_X$. ■

4.2. Proof of Theorem 4.1. We briefly recollect some facts on quotients of schemes by actions of finite groups.

Let S be a scheme and Γ be a finite constant group scheme over S , that is, an S -group scheme associated to a finite abstract group $|\Gamma|$.

For any S -scheme X , an S -action of Γ on X is equivalent to a morphism of groups $|\Gamma| \rightarrow \text{Aut}_S(X)$. If X is an S -scheme, we say that an action of Γ on X is *free* if the action of $\Gamma(U)$ on $X(U)$ is free for any S -scheme U .

The next lemma easily follows from [44] V and [55] IV.1.

LEMMA 4.8. *Let S be an affine Noetherian scheme and X be a quasi-projective S -scheme equipped with an S -action of a finite constant group scheme Γ over S . Then*

- (1) There exists a quasi-projective S -scheme Y and a Γ -invariant surjective finite morphism $p : X \rightarrow Y$ such that the natural morphism of sheaves of rings over Y

$$\mathcal{O}_Y \rightarrow (p_* \mathcal{O}_X)^{|\Gamma|}$$

is an isomorphism. We denote $Y =: X/\Gamma$.

- (2) If moreover the action of Γ on X is free, then p is étale and

$$\begin{aligned} \Gamma \times_S X &\rightarrow X \times_Y X \\ (\gamma, x) &\mapsto (x, \gamma \cdot x) \end{aligned}$$

is an isomorphism.

REMARK 4.9. Part (2) in the above lemma implies that, when the action of Γ on X is free, then the stacky quotient $[X/\Gamma]$ (cf. [82] Example 8.1.12) is representable by the scheme X/Γ .

PROOF OF THEOREM 4.1. Recall from [72] Theorem 7.9 (cf. [81] proof of Theorem 2.1.11) that there exists a quasi-projective scheme $A_{g,1,4}$ over $\mathbf{Z}[1/2]$ endowed with an action by the constant finite group scheme $\Gamma = \mathrm{GL}_g(\mathbf{Z}/4\mathbf{Z})_{\mathbf{Z}[1/2]}$ over $\mathbf{Z}[1/2]$, and with a surjective étale morphism $A_{g,1,4} \rightarrow \mathcal{A}_{g,\mathbf{Z}[1/2]}$ inducing an isomorphism of the stacky quotient $[A_{g,1,4}/\Gamma]$ with $\mathcal{A}_{g,\mathbf{Z}[1/2]}$; namely, $A_{g,1,4}$ is a scheme representing the functor

$$\begin{aligned} \mathcal{A}_g^{\mathrm{op}} &\rightarrow \mathrm{Set} \\ (X, \lambda)_{/U} &\mapsto \mathrm{Isom}_{\mathrm{GpSch}/U}((\mathbf{Z}/4\mathbf{Z})_U^{2g}, X[4]). \end{aligned}$$

As the morphism of Deligne-Mumford stacks over $\mathrm{Spec} \mathbf{Z}$

$$\pi_g : \mathcal{B}_g \rightarrow \mathcal{A}_g$$

is representable by smooth affine schemes (Remark 3.8), the fiber product

$$\mathcal{F} = A_{g,1,4} \times_{\mathcal{A}_{g,\mathbf{Z}[1/2]}} \mathcal{B}_{g,\mathbf{Z}[1/2]}$$

is representable by a smooth affine scheme B over $A_{g,1,4}$ via the first projection $\mathcal{F} \rightarrow A_{g,1,4}$. In particular, B is affine and of finite type over $A_{g,1,4}$. Since $A_{g,1,4}$ is quasi-projective over $\mathbf{Z}[1/2]$, it follows that B is a quasi-projective $\mathbf{Z}[1/2]$ -scheme.

The action of Γ on $A_{g,1,4}$ naturally induces an action of Γ on \mathcal{F} , thus on B . As $\mathcal{B}_{g,\mathbf{Z}[1/2]}$ is an algebraic space by Proposition 4.7 (cf. remark following Definition 4.2), this action is free. Moreover, by the compatibility of quotients of stacks by group actions with base change (cf. [89] Proposition 2.6), the second projection $\mathcal{F} \rightarrow \mathcal{B}_{g,\mathbf{Z}[1/2]}$ induces an isomorphism of the stacky quotient $[B/\Gamma]$ with $\mathcal{B}_{g,\mathbf{Z}[1/2]}$. Finally, by Lemma 4.8 and Remark 4.9, we conclude that $\mathcal{B}_{g,\mathbf{Z}[1/2]}$ is representable by the quasi-projective $\mathbf{Z}[1/2]$ -scheme B/Γ . ■

5. The vector bundle $T_{\mathcal{B}_g/\mathbf{Z}}$ and the higher Ramanujan vector fields

Fix an integer $g \geq 1$. We define a presheaf \mathcal{H}_g (resp. \mathcal{F}_g) of $\mathcal{O}_{\mathcal{A}_g, \text{ét}}$ -modules on $\text{Ét}(\mathcal{A}_g)$ as follows. Let (U, u) be an étale scheme over \mathcal{A}_g , and (X, λ) be the principally polarized abelian scheme over U corresponding to $u : U \rightarrow \mathcal{A}_g$. We put

$$\Gamma((U, u), \mathcal{H}_g) := \Gamma(U, H_{\mathrm{dR}}^1(X/U)) \quad (\text{resp. } \Gamma((U, u), \mathcal{F}_g) := \Gamma(U, F^1(X/U)))$$

If $(f, f^b) : (U', u') \rightarrow (U, u)$ is a morphism in $\text{Ét}(\mathcal{A}_g)$, the restriction map is given by the base change morphism $f^* H_{\mathrm{dR}}^1(X/U) \rightarrow H_{\mathrm{dR}}^1(X'/U')$ (resp. $f^* F^1(X/U) \rightarrow F^1(X'/U')$), where $(X', \lambda') = (X, \lambda) \times_U U'$. As the base change morphism is actually an isomorphism (i.e. the formation of $H_{\mathrm{dR}}^1(X/U)$ (resp. $F^1(X/U)$) is compatible with base change), and $H_{\mathrm{dR}}^1(X/U)$ (resp. $F^1(X/U)$) is quasi-coherent, \mathcal{H}_g (resp. \mathcal{F}_g) is a quasi-coherent sheaf over \mathcal{A}_g (cf. [82] Lemma 4.3.3). We finally remark that \mathcal{H}_g is actually a vector bundle of rank $2g$ over \mathcal{A}_g and that \mathcal{F}_g is a rank g subbundle of \mathcal{H}_g .

REMARK 5.1. The sheaf \mathcal{H}_g should be thought as the first de Rham cohomology of the universal abelian scheme over \mathcal{A}_g , and \mathcal{F}_g as its Hodge subbundle.

In this section we describe the tangent bundle $T_{\mathcal{B}_g/\mathbf{Z}}$ in terms of \mathcal{H}_g and \mathcal{F}_g ; see Theorem 5.4 for a precise statement. This will be obtained by realizing \mathcal{B}_g as a substack of the stack over \mathcal{A}_g associated to the vector bundle $\mathcal{H}_g^{\oplus g}$.

Further, Theorem 5.4 will allow us to construct a certain family of $g(g+1)/2$ global sections v_{ij} of $T_{\mathcal{B}_g/\mathbf{Z}}$ that we call *higher Ramanujan vector fields*.

5.1. The Gauss-Manin connection and the Kodaira-Spencer morphism on \mathcal{A}_g/\mathbf{Z} .

5.1.1. In order to give a precise statement of Theorem 5.4, we need to recall some basic facts concerning Gauss-Manin connections and Kodaira-Spencer morphisms over abelian schemes.

Fix a base scheme S and let $p : X \rightarrow U$ be a projective abelian scheme, with U a *smooth* S -scheme. Then there is defined an integrable S -connection over the de Rham cohomology sheaves ([53]; see also [50]), the *Gauss-Manin connection*

$$(5.1) \quad \nabla : H_{\mathrm{dR}}^i(X/U) \rightarrow H_{\mathrm{dR}}^i(X/U) \otimes_{\mathcal{O}_U} \Omega_{U/S}^1,$$

whose formation is compatible with every base change $U' \rightarrow U$, where U' is a smooth S -scheme.

The Gauss-Manin connection on $H_{\mathrm{dR}}^1(X/U)$ induces a morphism

$$\begin{aligned} T_{U/S} &\rightarrow \mathcal{H}om_{\mathcal{O}_S}(H_{\mathrm{dR}}^1(X/U), H_{\mathrm{dR}}^1(X/U)) \\ \theta &\mapsto \nabla_{\theta}(\). \end{aligned}$$

Restricting to $F^1(X/U)$ and passing to the quotient (cf. exact sequence (2.1)), we obtain an \mathcal{O}_U -morphism

$$T_{U/S} \rightarrow \mathcal{H}om_{\mathcal{O}_U}(F^1(X/U), R^1p_*\mathcal{O}_X) \cong F^1(X/U)^{\vee} \otimes_{\mathcal{O}_U} R^1p_*\mathcal{O}_X.$$

Applying the inverse of the canonical isomorphism $\phi_{X^t/U}^1 : F^1(X^t/U)^{\vee} \xrightarrow{\sim} R^1p_*\mathcal{O}_X$ (cf. proof of Lemma 2.2, where we identified X with X^{tt} via the canonical biduality isomorphism), we obtain an \mathcal{O}_U -morphism

$$\delta : T_{U/S} \rightarrow F^1(X/U)^{\vee} \otimes_{\mathcal{O}_U} F^1(X^t/U)^{\vee}.$$

This is, possibly up to a sign, the dual of ρ defined in [31] III.9.⁴

5.1.2. Now, with the same notation and hypotheses as above, let $\lambda : X \rightarrow X^t$ be a principal polarization. The Gauss-Manin connection ∇ on $H_{\mathrm{dR}}^1(X/U)$ is compatible with the symplectic form $\langle \ , \ \rangle_{\lambda}$ in the following sense. For every sections θ of $T_{U/S}$, and α and β of $H_{\mathrm{dR}}^1(X/U)$, we have

$$(5.2) \quad \theta \langle \alpha, \beta \rangle_{\lambda} = \langle \nabla_{\theta} \alpha, \beta \rangle_{\lambda} + \langle \alpha, \nabla_{\theta} \beta \rangle_{\lambda}.$$

This can be deduced from the fact that the first Chern class in $H_{\mathrm{dR}}^2(X \times_U X^t/U)$ of the Poincaré line bundle $\mathcal{P}_{X/U}$ is horizontal for the Gauss-Manin connection, since it actually comes from a class in $H_{\mathrm{dR}}^2(X \times_U X^t/S)$.

By composing δ with $((\lambda^*)^{\vee})^{-1} : F^1(X^t/U)^{\vee} \xrightarrow{\sim} F^1(X/U)^{\vee}$, we obtain a morphism

$$(5.3) \quad \kappa : T_{U/S} \rightarrow F^1(X/U)^{\vee} \otimes_{\mathcal{O}_U} F^1(X/U)^{\vee}.$$

This is the *Kodaira-Spencer morphism* associated to $(X, \lambda)_{/U}$ over S . It follows from the compatibility (5.2) that κ factors through the submodule of *symmetric tensors* in $F^1(X/U)^{\vee} \otimes_{\mathcal{O}_U} F^1(X/U)^{\vee}$, denoted $\Gamma^2(F^1(X/U)^{\vee})$.

REMARK 5.2. As $\phi_{X^t/U}^{\vee} = -\phi_{X/U}$ under the canonical biduality isomorphism $X \cong X^{tt}$ (cf. [5] Lemme 5.1.5), one may verify that the composition

$$R^1p_*\mathcal{O}_X \xrightarrow{(\phi_{X^t/U}^1)^{-1}} F^1(X^t/U)^{\vee} \xrightarrow{((\lambda^*)^{\vee})^{-1}} F^1(X/U)^{\vee}$$

considered above is given by the isomorphism of vector bundles $H_{\mathrm{dR}}^1(X/U)/F^1(X/U) \xrightarrow{\sim} F^1(X/U)^{\vee}$ induced by (cf. Lemma 1.A.2)

$$\begin{aligned} H_{\mathrm{dR}}^1(X/U) &\rightarrow H_{\mathrm{dR}}^1(X/U)^{\vee} \\ \alpha &\mapsto \langle \ , \ \alpha \rangle_{\lambda}. \end{aligned}$$

4. With notations as in the proof of Lemma 2.2, there are two natural ways of identifying $R^1p_*\mathcal{O}_X$ with $F^1(X^t/U)^{\vee}$: one by $(\phi_{X/U}^0)^{\vee}$, and another by $\phi_{X^t/U}^1$. These produce the same isomorphisms up to a sign. In [31] this choice is not specified.

Thus, if $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ is a symplectic-Hodge basis of $(X, \lambda)_{/U}$, κ admits the following explicit description in terms of b :

$$\kappa(\theta) = \sum_{i=1}^g \langle \cdot, \eta_i \rangle_\lambda \otimes \langle \cdot, \nabla_\theta \omega_i \rangle_\lambda.$$

Finally, we remark that the Kodaira-Spencer morphism is natural in the following sense. Let U' be a smooth scheme over S and let $F_{/f} : (X', \lambda')_{/U'} \rightarrow (X, \lambda)_{/U'}$ be a morphism in $\mathcal{A}_{g,S}$. Denote by κ (resp. κ') the Kodaira-Spencer morphism associated to $(X, \lambda)_{/U}$ (resp. $(X', \lambda')_{/U'}$) over S . Then the diagram

$$\begin{array}{ccc} T_{U'/S} & \xrightarrow{Df} & f^* T_{U/S} \\ \kappa' \downarrow & & \downarrow f^* \kappa \\ \Gamma^2(F^1(X'/U')^\vee) & \xrightarrow{(f^*)^\vee \otimes (f^*)^\vee} & \Gamma^2(f^* F^1(X/U)^\vee) \end{array}$$

commutes.

5.1.3. Let S be a scheme, and denote by $\mathcal{H}_{g,S}$ (resp. $\mathcal{F}_{g,S}$) the vector bundle over $\mathcal{A}_{g,S}$ obtained from \mathcal{H}_g (resp. \mathcal{F}_g) by the base change $\mathcal{A}_{g,S} \rightarrow \mathcal{A}_g$. As $\mathcal{A}_{g,S} \rightarrow S$ is smooth, the naturality of the Gauss-Manin connection permits us to construct a “universal” Gauss-Manin connection

$$\nabla : \mathcal{H}_{g,S} \rightarrow \mathcal{H}_{g,S} \otimes_{\mathcal{O}_{\mathcal{A}_{g,S}, \text{ét}}} \Omega_{\mathcal{A}_{g,S}/S}^1$$

and the naturality of the Kodaira-Spencer morphism permits us to construct a “universal” Kodaira-Spencer morphism

$$\kappa : T_{\mathcal{A}_{g,S}} \rightarrow \Gamma^2(\mathcal{F}_{g,S}^\vee).$$

These are morphism of sheaves on the étale site of $\mathcal{A}_{g,S}$ given, for any étale scheme (U, u) over $\mathcal{A}_{g,S}$ corresponding to the principally polarized abelian scheme (X, λ) over the S -scheme U , respectively by the Gauss-Manin connection (5.1) and the Kodaira-Spencer morphism of $(X, \lambda)_{/U}$ over S (5.3); note that as $u : U \rightarrow \mathcal{A}_{g,S}$ is étale, then U is smooth over S .

We remark that the universal Kodaira-Spencer morphism $\kappa : T_{\mathcal{A}_{g,S}} \rightarrow \Gamma^2(\mathcal{F}_{g,S}^\vee)$ is actually an *isomorphism* of $\mathcal{O}_{\mathcal{A}_{g,S}, \text{ét}}$ -modules (cf. [31] Theorem 5.7.(3)).

Finally, let \mathcal{U} be a smooth Deligne-Mumford stack over S and $u : \mathcal{U} \rightarrow \mathcal{A}_{g,S}$ be a quasi-compact and quasi-separated morphism of S -stacks representable by schemes. Then, the Gauss-Manin connection over (\mathcal{U}, u) , or simply over \mathcal{U} if u is implicit,

$$\nabla : u^* \mathcal{H}_{g,S} \rightarrow u^* \mathcal{H}_{g,S} \otimes_{\mathcal{O}_{\mathcal{U}, \text{ét}}} \Omega_{\mathcal{U}/S}^1$$

is defined by pulling back the universal Gauss-Manin connection on $\mathcal{A}_{g,S}$. Further, we may define a Kodaira-Spencer morphism over (\mathcal{U}, u) as the composition

$$\kappa_u : T_{\mathcal{U}/S} \xrightarrow{Du} u^* T_{\mathcal{A}_{g,S}/S} \xrightarrow{u^* \kappa} \Gamma^2(u^* \mathcal{F}_{g,S}^\vee).$$

5.2. The embedding $i_g : \mathcal{B}_g \rightarrow \mathcal{V}_g$. We shall employ the following notations in the statement of Theorem 5.4. Consider the morphism of coherent $\mathcal{O}_{\mathcal{B}_g, \text{ét}}$ -modules

$$m_g : \pi_g^* \mathcal{H}_g^{\oplus g} \rightarrow M_{g \times g}(\mathcal{O}_{\mathcal{B}_g, \text{ét}})$$

defined as follows. Let (U, u) be an étale scheme over \mathcal{B}_g , and let $(X, \lambda, b)_{/U}$, $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$, be the corresponding object of the fiber category $\mathcal{B}_g(U)$. Then the morphism m_g sends a section $(\alpha_1, \dots, \alpha_g)$ of $H_{\text{dR}}^1(X/U)^{\oplus g}$ to the section

$$(5.4) \quad (\langle \alpha_i, \eta_j \rangle_\lambda)_{1 \leq i, j \leq g}$$

of $M_{g \times g}(\mathcal{O}_U)$. We can thus define a subbundle \mathcal{S}_g of $\pi_g^* \mathcal{H}_g^{\oplus g}$ as the inverse image of the subbundle of symmetric matrices $\text{Sym}_g(\mathcal{O}_{\mathcal{B}_g, \text{ét}})$ by this morphism. In other words, if (U, u) and $(X, \lambda, b)_{/U}$ are as above, a section $(\alpha_1, \dots, \alpha_g)$ of $H_{\text{dR}}^1(X/U)^{\oplus g}$ is in \mathcal{S}_g if and only if the matrix (5.4) is symmetric.

REMARK 5.3. Note that m_g is surjective : with the above notations, for a given matrix $(a_{ij})_{1 \leq i, j \leq g}$ in $M_{g \times g}(\mathcal{O}_U)$, take $\alpha_i = \sum_{j=1}^g a_{ij} \omega_j$. In particular, \mathcal{S}_g is a subbundle of $\pi_g^* \mathcal{H}_g^{\oplus g}$ of rank $g^2 + g(g+1)/2 = g(3g+1)/2$.

THEOREM 5.4. Consider the morphism of quasi-coherent $\mathcal{O}_{\mathcal{B}_g, \text{ét}}$ -modules

$$c_g : T_{\mathcal{B}_g/\mathbf{Z}} \longrightarrow \Gamma^2(\pi_g^* \mathcal{F}_g^\vee) \oplus \pi_g^* \mathcal{H}_g^{\oplus g}$$

defined by

$$c_g(\theta) = (\kappa_u(\theta), \nabla_\theta \eta_1, \dots, \nabla_\theta \eta_g)$$

for every étale scheme (U, u) over \mathcal{B}_g corresponding to the object $(X, \lambda, b)_{/U}$ of $\mathcal{B}_g(U)$, where $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$, and θ a section of $T_{U/\mathbf{Z}}$. Then c_g induces an isomorphism of $T_{\mathcal{B}_g/\mathbf{Z}}$ onto the subbundle $\Gamma^2(\pi_g^* \mathcal{F}_g^\vee) \oplus \mathcal{S}_g$ of $\Gamma^2(\pi_g^* \mathcal{F}_g^\vee) \oplus \pi_g^* \mathcal{H}_g^{\oplus g}$.

A proof of this result will be given at the end of this paragraph.

5.2.1. Consider the associated space of the vector bundle $\mathcal{H}_g^{\oplus g}$ (cf. [82] 10.2)

$$\mathcal{V}_g := \mathbf{V}((\mathcal{H}_g^{\oplus g})^\vee) = \underline{\text{Spec}}_{\mathcal{A}_g}(\text{Sym}(\mathcal{H}_g^{\oplus g})^\vee).$$

This is a Deligne-Mumford stack over $\text{Spec } \mathbf{Z}$ whose objects lying over a scheme U are given by “ $(g+2)$ -uples”

$$(X, \lambda, \alpha_1, \dots, \alpha_g)_{/U},$$

where $(X, \lambda)_{/U}$ is an object of $\mathcal{A}_g(U)$, and α_i is a global section of $H_{\text{dR}}^1(X/U)$ for every $1 \leq i \leq g$. Note that the forgetful functor

$$\Phi_g : \mathcal{V}_g \longrightarrow \mathcal{A}_g$$

defines a morphism of stacks representable by smooth affine schemes.

We define a morphism of stacks

$$i_g : \mathcal{B}_g \longrightarrow \mathcal{V}_g$$

as follows. Let $(X, \lambda, b)_{/U}$ be an object of \mathcal{B}_g and denote $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$. Then i_g sends $(X, \lambda, b)_{/U}$ to the object

$$(X, \lambda, \eta_1, \dots, \eta_g)_{/U}$$

of \mathcal{V}_g . The action of i_g on morphisms is evident. Note that the diagram of morphisms of stacks

$$\begin{array}{ccc} \mathcal{B}_g & \xrightarrow{i_g} & \mathcal{V}_g \\ & \searrow \pi_g & \swarrow \Phi_g \\ & & \mathcal{A}_g \end{array}$$

is (strictly) commutative.

LEMMA 5.5. The morphism $i_g : \mathcal{B}_g \longrightarrow \mathcal{V}_g$ is an immersion of stacks.

PROOF. Let U be a scheme and $U \longrightarrow \mathcal{V}_g$ be a morphism corresponding to the object $(X, \lambda, \alpha_1, \dots, \alpha_g)_{/U}$ of $\mathcal{V}_g(U)$. Then the fiber product $\mathcal{B}_g \times_{\mathcal{V}_g} U$ can be naturally identified with the locally closed subscheme of U defined by the equations

$$\begin{aligned} \bar{\alpha}_1 \wedge \dots \wedge \bar{\alpha}_g &\neq 0 \\ \langle \alpha_i, \alpha_j \rangle_\lambda &= 0, \quad \forall i, j \end{aligned}$$

where $\bar{\alpha}_i$ denotes the image of α_i in $H_{\text{dR}}^1(X/U)/F^1(X/U)$ (cf. Proposition 1.A.8 (2)). ■

5.2.2. The proof Theorem 5.4 relies on Ehresmann's point of view on connections on vector bundles. Let us briefly recall how this goes in our context.

Let S be a scheme, X be a smooth S -scheme and \mathcal{E} be a vector bundle over X . We denote by $E = \mathbf{V}(\mathcal{E}^\vee)$ the associated space and by $p : E \rightarrow X$ the projection morphism. As p is smooth, we have the exact sequence of vector bundles over E

$$0 \rightarrow T_{E/X} \rightarrow T_{E/S} \xrightarrow{Dp} p^*T_{X/S} \rightarrow 0.$$

We claim that every S -connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^1$ induces a canonical splitting of the above exact sequence. In fact, this can be obtained by means of the projection

$$P_\nabla : T_{E/S} \rightarrow T_{E/X}$$

defined as follows. The vector bundle $T_{E/X}$ is canonically isomorphic to $p^*\mathcal{E}$ ([43] Corollaire 16.4.9); it is thus endowed with a universal global section, say s . We put $P_\nabla(\theta) = (p^*\nabla)_\theta s$.

It is not difficult to transpose the above considerations to the case of smooth Deligne-Mumford stacks (cf. 1.2.6).

5.2.3. *Proof of Theorem 5.4.* Let \mathcal{V}_g and $\Phi_g : \mathcal{V}_g \rightarrow \mathcal{A}_g$ be as in 5.2.1. According to the discussion in 5.2.2, the connection on $\mathcal{H}_g^{\oplus g}$ given by the direct sum of the ‘‘universal’’ Gauss-Manin connection $\nabla : \mathcal{H}_g \rightarrow \mathcal{H}_g \otimes_{\mathcal{O}_{\mathcal{A}_g, \text{ét}}} \Omega_{\mathcal{A}_g/\mathbf{Z}}^1$ at each factor induces a splitting of the exact sequence

$$0 \rightarrow T_{\mathcal{V}_g/\mathcal{A}_g} \rightarrow T_{\mathcal{V}_g/\mathbf{Z}} \xrightarrow{D\Phi_g} \Phi_g^*T_{\mathcal{A}_g/\mathbf{Z}} \rightarrow 0.$$

Thus, after identifying $T_{\mathcal{V}_g/\mathcal{A}_g}$ with $\Phi_g^*\mathcal{H}_g^{\oplus g}$, we obtain an isomorphism

$$\bar{c}'_g : T_{\mathcal{V}_g} \xrightarrow{\sim} \Phi_g^*T_{\mathcal{A}_g/\mathbf{Z}} \oplus \Phi_g^*\mathcal{H}_g^{\oplus g}$$

given explicitly by

$$\bar{c}'_g(\theta) = (D\Phi_g(\theta), \nabla_\theta \alpha_1, \dots, \nabla_\theta \alpha_g)$$

for every étale scheme (U, u) over \mathcal{V}_g corresponding to the object $(X, \lambda, \alpha_1, \dots, \alpha_g)_{/U}$ of $\mathcal{V}_g(U)$, and every section θ of $T_{U/\mathbf{Z}}$.

By composing \bar{c}'_g with the Kodaira-Spencer isomorphism $\kappa : T_{\mathcal{A}_g/\mathbf{Z}} \xrightarrow{\sim} \Gamma^2(\mathcal{F}_g^\vee)$ (see 5.1.3), we obtain an isomorphism

$$\bar{c}_g : T_{\mathcal{V}_g/\mathbf{Z}} \xrightarrow{\sim} \Gamma^2(\Phi_g^*\mathcal{F}_g^\vee) \oplus \Phi_g^*\mathcal{H}_g^{\oplus g}$$

given explicitly by

$$\bar{c}_g(\theta) = (\kappa_u(\theta), \nabla_\theta \alpha_1, \dots, \nabla_\theta \alpha_g)$$

with notations as above. Finally, note that the morphism c_g in the statement is defined by restricting \bar{c}_g to \mathcal{B}_g via the immersion $i_g : \mathcal{B}_g \rightarrow \mathcal{V}_g$ (cf. Lemma 5.5). In particular, as \mathcal{B}_g is a smooth substack of \mathcal{V}_g via i_g , then c_g induces an isomorphism of $T_{\mathcal{B}_g/\mathbf{Z}}$ onto a subbundle, say \mathcal{E}_g , of $\Gamma^2(\pi_g^*\mathcal{F}_g^\vee) \oplus \pi_g^*\mathcal{H}_g^{\oplus g}$. To finish the proof, it is sufficient to show that $\mathcal{E}_g = \Gamma^2(\pi_g^*\mathcal{F}_g^\vee) \oplus \mathcal{S}_g$.

Note that the compatibility (5.2) between Gauss-Manin connections and principal polarizations implies that c_g factors by the subbundle $\Gamma^2(\pi_g^*\mathcal{F}_g^\vee) \oplus \mathcal{S}_g$. Indeed, let (U, u) be an étale scheme over \mathcal{B}_g corresponding to the object $(X, \lambda, b)_{/U}$ of $\mathcal{B}_g(U)$, with $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$, and let θ be a section of $T_{U/\mathbf{Z}}$. Then, as $\langle \eta_i, \eta_j \rangle_\lambda = 0$, we obtain

$$0 = \nabla_\theta \langle \eta_i, \eta_j \rangle_\lambda = \langle \nabla_\theta \eta_i, \eta_j \rangle_\lambda + \langle \eta_i, \nabla_\theta \eta_j \rangle_\lambda = \langle \nabla_\theta \eta_i, \eta_j \rangle_\lambda - \langle \nabla_\theta \eta_j, \eta_i \rangle_\lambda.$$

This proves that \mathcal{E}_g is a subbundle of $\Gamma^2(\pi_g^*\mathcal{F}_g^\vee) \oplus \mathcal{S}_g$. To conclude, we simply remark that the ranks of the subbundles \mathcal{E}_g and $\Gamma^2(\pi_g^*\mathcal{F}_g^\vee) \oplus \mathcal{S}_g$ of $\Gamma^2(\pi_g^*\mathcal{F}_g^\vee) \oplus \pi_g^*\mathcal{H}_g^{\oplus g}$ coincide (cf. Remark 5.3). \blacksquare

5.3. The higher Ramanujan vector fields. Let

$$\langle \cdot, \cdot \rangle : \pi_g^* \mathcal{H}_g \times \pi_g^* \mathcal{H}_g \longrightarrow \mathcal{O}_{\mathcal{B}_g, \text{ét}}$$

be the symplectic $\mathcal{O}_{\mathcal{B}_g, \text{ét}}$ -bilinear form given, for each étale scheme (U, u) over \mathcal{B}_g corresponding to the object $(X, \lambda, b)_{/U}$ of $\mathcal{B}_g(U)$, by

$$u^* \langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_\lambda : H_{\text{dR}}^1(X/U) \times H_{\text{dR}}^1(X/U) \longrightarrow \mathcal{O}_U$$

This is well-defined by Remark 2.3.

We denote by

$$b_g = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$$

the “universal” symplectic-Hodge basis over \mathcal{B}_g . Namely, b_g is the basis of the vector bundle $\pi_g^* \mathcal{H}_g$ such that for every étale scheme (U, u) over \mathcal{B}_g corresponding to the object $(X, \lambda, b)_{/U}$ of $\mathcal{B}_g(U)$ we have $u^* b_g = b$.

Note that vector bundle $\pi_g^* \mathcal{F}_g^\vee$ is trivialized over \mathcal{B}_g by the global sections

$$\langle \cdot, \eta_i \rangle : \pi_g^* \mathcal{F}_g \longrightarrow \mathcal{O}_{\mathcal{B}_g, \text{ét}}$$

for $1 \leq i \leq g$ (this is the dual basis of $(\omega_1, \dots, \omega_g)$). Accordingly, $\Gamma^2(\pi_g^* \mathcal{F}_g^\vee)$ is trivialized by the global sections

$$\varphi_{ij} := \begin{cases} \langle \cdot, \eta_i \rangle \otimes \langle \cdot, \eta_i \rangle & i = j \\ \langle \cdot, \eta_i \rangle \otimes \langle \cdot, \eta_j \rangle + \langle \cdot, \eta_j \rangle \otimes \langle \cdot, \eta_i \rangle & i < j \end{cases}$$

for $1 \leq i \leq j \leq g$.

Let $c_g : T_{\mathcal{B}_g/\mathbf{Z}} \xrightarrow{\sim} \Gamma^2(\pi_g^* \mathcal{F}_g^\vee) \oplus \mathcal{S}_g$ be the isomorphism of $\mathcal{O}_{\mathcal{B}_g, \text{ét}}$ -modules defined in Theorem 5.4.

DEFINITION 5.6. For every $1 \leq i \leq j \leq g$, we define the *higher Ramanujan vector field* v_{ij} as being the unique global section of $T_{\mathcal{B}_g/\mathbf{Z}}$ such that $c_g(v_{ij}) = (\varphi_{ij}, 0)$.

Let us denote the “universal” Gauss-Manin connection over \mathcal{B}_g by (cf. 5.1.3)

$$\nabla : \pi_g^* \mathcal{H}_g \longrightarrow \pi_g^* \mathcal{H}_g \otimes_{\mathcal{O}_{\mathcal{B}_g, \text{ét}}} \Omega_{\mathcal{B}_g/\mathbf{Z}}^1.$$

PROPOSITION 5.7. *The higher Ramanujan vector fields are the unique global sections v_{ij} of $T_{\mathcal{B}_g/\mathbf{Z}}$ such that, for every $1 \leq i \leq j \leq g$,*

- (1) $\nabla_{v_{ij}} \omega_i = \eta_j$, $\nabla_{v_{ij}} \omega_j = \eta_i$, and $\nabla_{v_{ij}} \omega_k = 0$ for $k \notin \{i, j\}$.
- (2) $\nabla_{v_{ij}} \eta_k = 0$, for every $1 \leq k \leq g$.

PROOF. The vector fields v_{ij} satisfy (2) by definition of c_g in Theorem 5.4. Moreover, using the explicit expression of the Kodaira-Spencer morphism in Remark 5.2, we see that

$$(5.5) \quad \sum_{k=1}^g \langle \cdot, \eta_k \rangle \otimes \langle \cdot, \nabla_{v_{ij}} \omega_k \rangle = \begin{cases} \langle \cdot, \eta_i \rangle \otimes \langle \cdot, \eta_i \rangle & i = j \\ \langle \cdot, \eta_i \rangle \otimes \langle \cdot, \eta_j \rangle + \langle \cdot, \eta_j \rangle \otimes \langle \cdot, \eta_i \rangle & i < j \end{cases}$$

in $\Gamma^2(\pi_g^* \mathcal{F}_g^\vee)$ for every $1 \leq i \leq j \leq g$. As b_g is symplectic with respect to $\langle \cdot, \cdot \rangle$, by evaluating the second factors at η_l for every $1 \leq l \leq g$ in the above equation, we see that $\nabla_{v_{ij}} \omega_k$ lies in the subbundle of $\pi_g^* \mathcal{H}_g$ generated by η_1, \dots, η_g , for every $1 \leq i \leq j \leq g$ and $1 \leq k \leq g$.

Thus, to prove that the vector fields v_{ij} satisfy (1), it is sufficient to prove that

$$(5.6) \quad \langle \omega_l, \nabla_{v_{ij}} \omega_i \rangle = \delta_{lj}, \langle \omega_l, \nabla_{v_{ij}} \omega_j \rangle = \delta_{li}, \text{ and } \langle \omega_l, \nabla_{v_{ij}} \omega_k \rangle = 0 \text{ for } k \notin \{i, j\}$$

for every $1 \leq l \leq g$. This in turn follows immediately from (5.5) by evaluating the second factors at ω_l .

To prove unicity, let $(w_{ij})_{1 \leq i \leq j \leq g}$ be a family of vector fields on \mathcal{B}_g satisfying (1) and (2). Note that, by the explicit expression of the Kodaira-Spencer morphism in Remark 5.2, equations in (1) imply $\kappa_{\pi_g}(w_{ij}) = \varphi_{ij}$ in the notation preceding Definition 5.6. Thus, by (1) and (2),

$$c_g(w_{ij}) = (\varphi_{ij}, 0) = c_g(v_{ij})$$

for every $1 \leq i \leq j \leq g$, that is, $w_{ij} = v_{ij}$ (cf. Definition 5.6). ■

Let S be a scheme. We denote by $\pi_{g,S} : \mathcal{B}_{g,S} \rightarrow \mathcal{A}_{g,S}$ the base change of $\pi_g : \mathcal{B}_g \rightarrow \mathcal{A}_g$ by $\mathcal{A}_{g,S} \rightarrow \mathcal{A}_g$, and by

$$\nabla : \pi_{g,S}^* \mathcal{H}_{g,S} \rightarrow \pi_{g,S}^* \mathcal{H}_{g,S} \otimes_{\mathcal{O}_{\mathcal{B}_{g,S,\acute{e}t}}} \Omega_{\mathcal{B}_{g,S}/S}^1$$

the ‘‘universal’’ Gauss-Manin S -connection over $\mathcal{B}_{g,S}$.

REMARK 5.8. Let $\mathcal{R}_{g,S}$ be the $\mathcal{O}_{\mathcal{B}_{g,S,\acute{e}t}}$ -submodule of $T_{\mathcal{B}_{g,S}/S}$ generated by all the v_{ij} ; if $S = \text{Spec } \mathbf{Z}$, we denote simply $\mathcal{R}_{g,S} =: \mathcal{R}_g$. It is clear from Theorem 5.4 that $\mathcal{R}_{g,S}$ is the kernel of the surjective $\mathcal{O}_{\mathcal{B}_{g,S,\acute{e}t}}$ -morphism

$$\begin{aligned} T_{\mathcal{B}_{g,S}/S} &\rightarrow \mathcal{S}_g \\ \theta &\mapsto (\nabla_\theta \eta_1, \dots, \nabla_\theta \eta_g) \end{aligned}$$

In particular, $\mathcal{R}_{g,S}$ is a subbundle of $T_{\mathcal{B}_{g,S}/S}$ of rank $g(g+1)/2$.

LEMMA 5.9. *Let θ be a section of $T_{\mathcal{B}_{g,S}/S}$ such that $\nabla_\theta \omega_i = \nabla_\theta \eta_i = 0$ for every $1 \leq i \leq g$. Then $\theta = 0$.*

PROOF. Let θ be as in the statement. By Remark 5.8, θ is in the subbundle $\mathcal{R}_{g,S}$ of $T_{\mathcal{B}_{g,S}/S}$, thus there exist sections $(f_{ij})_{1 \leq i \leq j \leq g}$ of $\mathcal{O}_{\mathcal{B}_{g,S,\acute{e}t}}$ such that

$$\theta = \sum_{1 \leq i \leq j \leq g} f_{ij} v_{ij}.$$

We prove that each $f_{ij} = 0$ by induction on i . For $i = 1$, we have by Proposition 5.7

$$0 = \nabla_\theta \omega_1 = \sum_{1 \leq i \leq j \leq g} f_{ij} \nabla_{v_{ij}} \omega_1 = \sum_{j=1}^g f_{1j} \eta_j,$$

thus $f_{1j} = 0$ for every $1 \leq j \leq g$. If $2 \leq i_0 \leq g$ and $f_{ij} = 0$ for every $i < i_0$ and $i \leq j \leq g$, we have

$$0 = \nabla_\theta \omega_{i_0} = \sum_{i_0 \leq i \leq j \leq g} f_{ij} \nabla_{v_{ij}} \omega_{i_0} = \sum_{j=i_0}^g f_{i_0 j} \eta_j,$$

thus $f_{i_0 j} = 0$ for every $i_0 \leq j \leq g$. ■

Let $[,]$ denote the Lie bracket in $T_{\mathcal{B}_g/\mathbf{Z}}$.

COROLLARY 5.10. *The higher Ramanujan vector fields commute. That is,*

$$[v_{ij}, v_{i'j'}] = 0$$

for any $1 \leq i \leq j \leq g$ and $1 \leq i' \leq j' \leq g$.

PROOF. Let us first remark that, as the Gauss-Manin connection is integrable, for any sections θ and θ' of $T_{\mathcal{B}_g/\mathbf{Z}}$, we have

$$\nabla_{[\theta, \theta']} = \nabla_\theta \nabla_{\theta'} - \nabla_{\theta'} \nabla_\theta.$$

This implies that \mathcal{R}_g is integrable : if θ and θ' are both sections of \mathcal{R}_g , then $[\theta, \theta']$ is a section of \mathcal{R}_g . In particular, $\theta := [v_{ij}, v_{i'j'}]$ is a section of \mathcal{R}_g . By Lemma 5.9, to prove that $\theta = 0$, it is sufficient to prove that $\nabla_\theta \omega_k = 0$ for every $1 \leq k \leq g$.

We have

$$\nabla_\theta \omega_k = \nabla_{v_{ij}} (\nabla_{v_{i'j'}} \omega_k) - \nabla_{v_{i'j'}} (\nabla_{v_{ij}} \omega_k).$$

It follows from Proposition 5.7 that $\nabla_{v_{i'j'}} \omega_k$ (resp. $\nabla_{v_{ij}} \omega_k$) is an element of $\{0, \eta_1, \dots, \eta_g\}$; hence $\nabla_{v_{ij}} (\nabla_{v_{i'j'}} \omega_k) = 0$ (resp. $\nabla_{v_{i'j'}} (\nabla_{v_{ij}} \omega_k) = 0$). ■

5.4. The action of Siegel parabolic subgroup P_g on the higher Ramanujan vector fields. Geometrically, $\pi_g : \mathcal{B}_g \rightarrow \mathcal{A}_g$ may be regarded as a “principal P_g -bundle” over \mathcal{A}_g (cf. Lemma 3.3). It is therefore natural to ask how the integrable subbundle \mathcal{R}_g of $T_{\mathcal{B}_g/\mathbb{Z}}$ (cf. Remark 5.8 and Corollary 5.10) transforms under the action of P_g .

In order to formulate precise statements, fix an affine base scheme $S = \text{Spec } R$ and let $p \in P_g(S)$. Then, p induces an S -automorphism of $\mathcal{B}_{g,S}$ given by

$$\begin{aligned} p : \mathcal{B}_{g,S} &\longrightarrow \mathcal{B}_{g,S} \\ (X, \lambda, b)_{/U} &\longmapsto (X, \lambda, b \cdot p)_{/U} \end{aligned}$$

where we have implicitly identified p with its image by the natural map $P_g(S) \rightarrow P_g(U)$ to compute $b \cdot p$.

PROPOSITION 5.11. *Let us write*

$$p = \begin{pmatrix} A & B \\ 0 & (A^T)^{-1} \end{pmatrix} \in P_g(S),$$

and consider the tangent map

$$Dp : T_{\mathcal{B}_{g,S}/S} \longrightarrow p^*T_{\mathcal{B}_{g,S}/S}.$$

Then $Dp(\mathcal{R}_{g,S}) \subset p^*\mathcal{R}_{g,S}$ if and only if $B = 0$.

Let us introduce some preliminary notation before proving this result. Note that Dp induces an R -automorphism

$$p_* : \Gamma(\mathcal{B}_{g,S}, T_{\mathcal{B}_{g,S}/S}) \longrightarrow \Gamma(\mathcal{B}_{g,S}, T_{\mathcal{B}_{g,S}/S})$$

which is compatible with the “universal” Gauss-Manin S -connection

$$\nabla : \pi_{g,S}^* \mathcal{H}_{g,S} \longrightarrow \pi_{g,S}^* \mathcal{H}_{g,S} \otimes_{\mathcal{O}_{\mathcal{B}_{g,S}, \text{ét}}} \Omega_{\mathcal{B}_{g,S}/S}^1$$

in the following sense. Denote by

$$p^* : \Gamma(\mathcal{B}_{g,S}, \pi_{g,S}^* \mathcal{H}_{g,S}) \longrightarrow \Gamma(\mathcal{B}_{g,S}, \pi_{g,S}^* \mathcal{H}_{g,S})$$

the R -automorphism induced by the isomorphism of vector bundles $p^* \pi_{g,S}^* \mathcal{H}_{g,S} \xrightarrow{\sim} \pi_{g,S}^* \mathcal{H}_{g,S}$ (observe that $\pi_{g,S} \circ p = \pi_{g,S}$). Then, for any $\alpha \in \Gamma(\mathcal{B}_{g,S}, \pi_{g,S}^* \mathcal{H}_{g,S})$, and any $\theta \in \Gamma(\mathcal{B}_{g,S}, T_{\mathcal{B}_{g,S}/S})$, we have

$$(5.7) \quad p^*(\nabla_{p_*\theta} \alpha) = \nabla_{\theta}(p^* \alpha).$$

REMARK 5.12. The automorphism p^* introduced above is characterized by

$$(p^* \omega_1 \quad \cdots \quad p^* \omega_g \quad p^* \eta_1 \quad \cdots \quad p^* \eta_g) = (\omega_1 \quad \cdots \quad \omega_g \quad \eta_1 \quad \cdots \quad \eta_g) \begin{pmatrix} A & B \\ 0 & (A^T)^{-1} \end{pmatrix}$$

where $b_g = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ is the universal symplectic-Hodge basis of $\pi_{g,S}^* \mathcal{H}_{g,S}$.

PROOF OF PROPOSITION 5.11. Since the vector bundle $\mathcal{R}_{g,S}$ is generated by the higher Ramanujan vector fields v_{ij} , we have $Dp(\mathcal{R}_{g,S}) \subset p^*\mathcal{R}_{g,S}$ if and only if

$$p_* v_{ij} \in \Gamma(\mathcal{B}_{g,S}, \mathcal{R}_{g,S})$$

for every $1 \leq i \leq j \leq g$. Further, by Remark 5.8, $p_* v_{ij}$ lies in $\Gamma(\mathcal{B}_{g,S}, \mathcal{R}_{g,S})$ if and only if

$$\nabla_{p_* v_{ij}} \eta_k = 0$$

for every $1 \leq k \leq g$. Finally, by the compatibility (5.7), we conclude that $Dp(\mathcal{R}_{g,S}) \subset p^*\mathcal{R}_{g,S}$ if and only if

$$\nabla_{v_{ij}}(p^* \eta_k) = 0$$

for every $1 \leq i \leq j \leq g$ and $1 \leq k \leq g$.

Now, by Remark 5.12, we have

$$p^* \eta_k = \sum_{l=1}^g \omega_l B_{lk} + \eta_l (A^{-1})_{kl}.$$

Using Proposition 5.7, we obtain

$$\nabla_{v_{ij}}(p^*\eta_k) = \sum_{l=1}^g (\nabla_{v_{ij}}\omega_l)B_{lk} = \begin{cases} \eta_i B_{ik} & i = j \\ \eta_j B_{ik} + \eta_i B_{jk} & i < j \end{cases}$$

The assertion follows. ■

Let L_g be the subgroup scheme of P_g given by

$$L_g(V) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix} \in M_{2g \times 2g}(R_0) \mid A \in \mathrm{GL}_g(R_0) \right\}$$

for any affine scheme $V = \mathrm{Spec} R_0$. The above proposition shows in particular that the action of L_g on \mathcal{B}_g preserves the integrable subbundle \mathcal{R}_g . The next proposition gives a precise transformation law for the higher Ramanujan vector fields v_{ij} under the action of L_g .

PROPOSITION 5.13. *Let $v = (v_{ij})_{1 \leq i, j \leq g}$ be the unique symmetric matrix of global sections of $T_{\mathcal{B}_g, S/S}$ where v_{ij} are the higher Ramanujan vector fields for $1 \leq i \leq j \leq g$. For every*

$$p = \begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix} \in L_g(S)$$

*if we denote $p_*v = (p_*v_{ij})_{1 \leq i, j \leq g}$, then we have the equality of matrices of sections $T_{\mathcal{B}_g, S/S}$ over S*

$$p_*v = AvA^\top,$$

PROOF. For each $1 \leq i \leq j \leq g$, put

$$w_{ij} := \sum_{m, n=1}^g A_{im} v_{mn} A_{jn}.$$

Then we must prove that $p_*v_{ij} = w_{ij}$ for every $1 \leq i \leq j \leq g$, which, by Lemma 5.9, is equivalent to proving that

$$(5.8) \quad \nabla_{p_*v_{ij}}\omega_k = \nabla_{w_{ij}}\omega_k, \quad \nabla_{p_*v_{ij}}\eta_k = \nabla_{w_{ij}}\eta_k$$

for every $1 \leq k \leq g$. By compatibility (5.7), equations (5.8) are equivalent to

$$\nabla_{v_{ij}}(p^*\omega_k) = p^*(\nabla_{w_{ij}}\omega_k), \quad \nabla_{v_{ij}}(p^*\eta_k) = p^*(\nabla_{w_{ij}}\eta_k).$$

As each η_k is horizontal for the Gauss-Manin connection, we have $\nabla_{w_{ij}}\eta_k = 0$. Further, as $p \in L_g(S)$, each $p^*\eta_k$ is an R -linear combination of η_1, \dots, η_g ; thus $p^*\eta_k$ is horizontal for the Gauss-Manin connection.

We are thus reduced to proving that

$$\nabla_{v_{ij}}(p^*\omega_k) = p^*(\nabla_{w_{ij}}\omega_k)$$

for every $1 \leq i \leq j \leq g$ and $1 \leq k \leq g$. On the one hand, we have

$$p^*\omega_k = \sum_{l=1}^g \omega_l A_{lk},$$

so that, by Proposition 5.7,

$$\nabla_{v_{ij}}(p^*\omega_k) = \sum_{l=1}^g (\nabla_{v_{ij}}\omega_l)A_{lk} = \eta_j A_{ik} + \eta_i A_{jk}.$$

On the other hand,

$$\begin{aligned}
\nabla_{w_{ij}}\omega_k &= \sum_{m,n=1}^g A_{im}(\nabla_{v_{mn}}\omega_k)A_{jn} \\
&= \sum_{n=1}^g A_{ik}\eta_n A_{jn} + \sum_{m=1}^g A_{im}\eta_m A_{jk} \\
&= A_{ik} \left(\sum_{n=1}^g \eta_n A_{jn} \right) + \left(\sum_{m=1}^g A_{im}\eta_m \right) A_{jk},
\end{aligned}$$

hence, by Remark 5.12,

$$p^*(\nabla_{w_{ij}}\omega_k) = A_{ik} \left(\sum_{n=1}^g p^*\eta_n (A^\top)_{nj} \right) + \left(\sum_{m=1}^g A_{im}p^*\eta_m \right) A_{jk} = A_{ik}\eta_j + \eta_i A_{jk}.$$

■

6. The case $g = 1$: explicit equations

When $g = 1$, we can compute explicit equations for B_g and for the Ramanujan vector field.

6.1. Explicit equation for the universal elliptic curve X_1 over B_1 and its universal symplectic-Hodge basis. Fix a scheme U . Let us recall that every *elliptic curve* E over U (namely, an abelian scheme of relative dimension 1) has a canonical unique principal polarization $\lambda_E : E \rightarrow E^t$ given, for any U -scheme V and any point $P \in E(V)$, by

$$\lambda_E(P) = \mathcal{O}_E([P] - [O])$$

where $O \in E(V)$ denotes the identity section and $\mathcal{O}_E([P] - [O])$ denotes the class in $E^t(V)$ of the inverse of the ideal sheaf defined by the relative Cartier divisor $[P] - [O]$.

Therefore, the functor

$$E \mapsto (E, \lambda_E)$$

defines an equivalence between the category of elliptic curves over U and that of principally polarized elliptic curves over U . We can thus “forget” the principal polarization : an elliptic curve E will always be assumed to be endowed with its canonical principal polarization λ_E . In particular, an object of \mathcal{B}_1 will be denoted simply by a “couple” $(E, b)_{/U}$.

REMARK 6.1. The symplectic form induced by λ_E coincides with the composition of the cup product in de Rham cohomology $H_{\text{dR}}^1(E/U) \times H_{\text{dR}}^1(E/U) \rightarrow H_{\text{dR}}^2(E/U)$ with the trace map $H_{\text{dR}}^2(E/U) \rightarrow \mathcal{O}_U$.

THEOREM 6.2. *Let*

$$B_1 := \text{Spec } \mathbf{Z}[1/2, b_2, b_4, b_6, \Delta^{-1}]$$

where

$$\Delta := \frac{b_2^2(b_4^2 - b_2b_6)}{4} - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 = 16 \text{ disc} \left(x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4} \right),$$

and let X_1 be the elliptic curve over B_1 given by the equation

$$y^2 = x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4}.$$

Then $b_1 = (\omega_1, \eta_1)$ defined by

$$\omega_1 := \frac{dx}{2y}, \quad \eta_1 := x \frac{dx}{2y}$$

is a symplectic-Hodge basis of X_{1/B_1} and the morphism $B_1 \rightarrow \mathcal{B}_1$ corresponding to $(X_1, b_1)_{/B_1}$ induces an isomorphism of B_1 with the $\mathbf{Z}[1/2]$ -stack $\mathcal{B}_{1, \mathbf{Z}[1/2]}$.

In other words, if $(X_1, b_1)_{/B_1}$ is defined as above, then for any $\mathbf{Z}[1/2]$ -scheme U , and any elliptic curve E over U endowed with a symplectic-Hodge basis b , there exists a unique morphism $F_{/f} : E_{/U} \rightarrow X_{1/B_1}$ in $\mathcal{A}_{1, \mathbf{Z}[1/2]}$ such that $F^*b_1 = b$.

PROOF. It is classical that ω_1 so defined is in $F^1(X_1/B_1)$. To prove that $\langle \omega_1, \eta_1 \rangle_{\lambda_E} = 1$ one can, for instance, use the compatibility with base change to reduce this statement to an analogous statement concerning an elliptic curve over \mathbf{C} , and then apply the classical residue formula (cf. [29] pp. 23-25).

Let U be a $\mathbf{Z}[1/2]$ -scheme and $(E, b)_{/U}$ be an object of $\mathcal{B}_1(U)$, with $b = (\omega, \eta)$. It is sufficient to prove that, locally for the Zariski topology over U , there exists a unique morphism $(E, b)_{/U} \rightarrow (X_1, b_1)_{/B_1}$ in $\mathcal{B}_{1, \mathbf{Z}[1/2]}$.

We follow essentially the same steps in [52] 2.2 to find a Weierstrass equation for an elliptic curve. Let us denote by $O : U \rightarrow E$ the identity section of the elliptic curve E over U and by $p : E \rightarrow U$ its structural morphism. Locally for the Zariski topology on U we can find a formal parameter t in the neighborhood of O such that ω has a formal expansion in t of the form

$$\omega = (1 + O(t))dt,$$

where $O(t)$ stands for a formal power series in t of order ≥ 1 . Up to replacing U by an open subscheme, we can and shall assume from now on that t exists globally over U .

There exist bases $(1, x)$ of $p_*\mathcal{O}_E(2[O])$, and $(1, x, y)$ of $p_*\mathcal{O}_E(3[O])$, such that

$$(6.1) \quad x = \frac{1}{t^2}(1 + O(t)) \quad \text{and} \quad y = \frac{1}{t^3}(1 + O(t)).$$

Then the rational functions x and y necessarily satisfy an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where a_i are uniquely defined global sections of \mathcal{O}_U . Since 2 is invertible in U , the above equation is equivalent to

$$\left(y + \frac{a_1}{2}x + \frac{a_3}{2}\right)^2 = x^3 + \left(\frac{a_1^2 + 4a_2}{4}\right)x^2 + \left(\frac{a_1a_3 + 2a_4}{2}\right)x + \frac{a_3^2 + 4a_6}{4}.$$

Therefore, after the change of coordinates $(x, y) \mapsto (x, y + \frac{a_1}{2}x + \frac{a_3}{2})$, we can assume that x and y satisfy

$$y^2 = x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4},$$

where b_i are global sections of \mathcal{O}_U . Put differently, we obtain a morphism $F_{/f} : E_{/U} \rightarrow X_{1/B_1}$ in $\mathcal{A}_{1, \mathbf{Z}[1/2]}$.

By considering formal expansions in t , we see that $F^*\omega_1 = \omega$. In particular,

$$(\omega, F^*\eta_1) = F^*b_1$$

is a symplectic-Hodge basis of $E_{/U}$, and there exists a section s of \mathcal{O}_U such that $\eta = F^*\eta_1 + s\omega$. Thus, after the change of coordinates $(x, y) \mapsto (x + s, y)$, we have $F^*b_1 = b$. Therefore, we have constructed a morphism $F_{/f} : (E, b)_{/U} \rightarrow (X_1, b_1)_{/B_1}$ in $\mathcal{B}_{1, \mathbf{Z}[1/2]}$.

We now prove that the morphism $F_{/f}$ is unique. Let $F'_{/f'} : (E, b)_{/U} \rightarrow (X_1, b_1)_{/B_1}$ be any morphism in $\mathcal{B}_{1, \mathbf{Z}[1/2]}$. If $f' = (b'_2, b'_4, b'_6)$ are the coordinates of f' , then F' is given by a basis $(1, x', y')$ of $p_*\mathcal{O}_E(3[O])$ satisfying

$$(*) \quad (y')^2 = (x')^3 + \frac{b'_2}{4}(x')^2 + \frac{b'_4}{2}x' + \frac{b'_6}{4}.$$

As both $(1, x, y)$ and $(1, x', y')$ (resp. $(1, x)$ and $(1, x')$) are a basis of $p_*\mathcal{O}_E(3[O])$ (resp. $p_*\mathcal{O}_E(2[O])$), then there exists global sections c_1, c_2, c_3 of \mathcal{O}_U (resp. u, v of \mathcal{O}_U^\times) such that

$$\begin{aligned} x' &= u(x + c_1) \\ y' &= v(y + c_2x + c_3). \end{aligned}$$

Note that equation $(*)$ implies that $u^3 = v^2$.

Now, as $(F')^*\omega_1 = F^*\omega_1$, we obtain

$$\frac{dx'}{2y'} = \frac{dx}{2y} \iff \frac{u}{v} \frac{dx}{2(y + c_2x + c_3)} = \frac{dx}{2y},$$

thus $c_2x + c_3 = 0$ and $u = v$. Since $u^3 = v^2$, we obtain $u = v = 1$ and $(x', y') = (x + c_1, y)$. Finally, as $(F')^*\eta_1 = F^*\eta_1$, we have

$$x' \frac{dx'}{2y'} = \frac{dx}{2y} \iff x \frac{dx}{2y} + c_1 \frac{dx}{2y} = x \frac{dx}{2y},$$

hence $c_1 = 0$. Thus $(x', y') = (x, y)$ and this also implies that $f = f'$. ■

REMARK 6.3. By considering the change of variables

$$\begin{cases} b_2 = e_2 \\ b_4 = (e_2^2 - e_4)/24 \\ b_6 = (4e_2^3 - 12e_2e_4 + 8e_6)/1728 \end{cases} \iff \begin{cases} e_2 = b_2 \\ e_4 = b_2^2 - 24b_4 \\ e_6 = b_2^3 - 36b_2b_4 + 216b_6 \end{cases}$$

we see that $B_1 \otimes_{\mathbf{Z}[1/2]} \mathbf{Z}[1/6]$ is isomorphic to

$$\text{Spec } \mathbf{Z}[1/6, e_2, e_4, e_6, (e_4^3 - e_6^2)^{-1}].$$

Under this identification, the universal elliptic curve X_1 is given by the equation

$$y^2 = 4 \left(x + \frac{e_2}{12} \right)^3 - \frac{e_4}{12} \left(x + \frac{e_2}{12} \right) + \frac{e_6}{216},$$

and the universal symplectic-Hodge basis b_1 by $(dx/y, xdx/y)$.

6.2. Explicit formula for the Ramanujan vector field. It is also possible to give an explicit formula for the Ramanujan vector field v_{11} over B_1 . Indeed, consider the global section of $T_{B_1/\mathbf{Z}[1/2]}$ given by

$$v := 2b_4 \frac{\partial}{\partial b_2} + 3b_6 \frac{\partial}{\partial b_4} + (b_2b_6 - b_4^2) \frac{\partial}{\partial b_6}.$$

One may easily verify using the expression for the Gauss-Manin connection on $H_{\text{dR}}^1(X_1/B_1)$ given in 1.B.3 that

$$\nabla_v \begin{pmatrix} \omega_1 & \eta_1 \end{pmatrix} = \begin{pmatrix} \omega_1 & \eta_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

By Proposition 5.7, v is the Ramanujan vector field v_{11} over B_1 .

REMARK 6.4. Under the isomorphism $B_1 \otimes_{\mathbf{Z}[1/2]} \mathbf{Z}[1/6] \cong \mathbf{Z}[1/6, e_2, e_4, e_6, (e_4^3 - e_6^2)^{-1}]$ of Remark 6.3, v gets identified with the vector field associated to the classical Ramanujan equations :

$$v = \frac{e_2^2 - e_4}{12} \frac{\partial}{\partial e_2} + \frac{e_2e_4 - e_6}{3} \frac{\partial}{\partial e_4} + \frac{e_2e_6 - e_4^2}{2} \frac{\partial}{\partial e_6}.$$

1.A. Symplectic vector bundles

Fix once and for all a scheme U .

1.A.1. Symplectic vector bundles. Let \mathcal{E} a vector bundle over U . An \mathcal{O}_U -bilinear map

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{O}_U$$

is said to be

- (1) *non-degenerate* if the \mathcal{O}_U -morphism $e \mapsto \langle \cdot, e \rangle$ from \mathcal{E} to \mathcal{E}^\vee is an isomorphism,
- (2) *alternating* if $\langle e, e \rangle = 0$ for every section e of \mathcal{E} .

DEFINITION 1.A.1. A *symplectic form* over \mathcal{E} is a non-degenerate and alternating \mathcal{O}_U -bilinear form over \mathcal{E} . A *symplectic vector bundle* over U is a couple $(\mathcal{E}, \langle \cdot, \cdot \rangle)$, where \mathcal{E} is a vector bundle over U and $\langle \cdot, \cdot \rangle$ is a symplectic form over \mathcal{E} .

1.A.2. Lagrangian subbundles. Let $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ be a symplectic vector bundle over U and \mathcal{F} be a subbundle of \mathcal{E} . We denote by $\mathcal{F}^{\langle \cdot, \cdot \rangle}$ the subsheaf of \mathcal{E} consisting of those sections e of \mathcal{E} such that $\langle f, e \rangle = 0$ for every section f of \mathcal{F} .

LEMMA 1.A.2. *We have an exact sequence of \mathcal{O}_U -modules*

$$\begin{aligned} 0 \longrightarrow \mathcal{F}^{\langle \cdot, \cdot \rangle} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}^\vee \longrightarrow 0 \\ e \longmapsto \langle \cdot, e \rangle|_{\mathcal{F}} \end{aligned}$$

In particular, $\mathcal{F}^{\langle \cdot, \cdot \rangle}$ is a subbundle of \mathcal{E} of rank $\text{rank}(\mathcal{E}) - \text{rank}(\mathcal{F})$.

PROOF. The sequence $0 \longrightarrow \mathcal{F}^{\langle \cdot, \cdot \rangle} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}^\vee$ is exact by definition. To see that $\mathcal{E} \longrightarrow \mathcal{F}^\vee$ defined above is surjective, one may work locally and remark that in this case \mathcal{F} is a direct factor of \mathcal{E} , and thus any \mathcal{O}_U -linear functional on \mathcal{F} can be extended to \mathcal{E} ; then one applies the non-degeneracy of the bilinear form $\langle \cdot, \cdot \rangle$. ■

DEFINITION 1.A.3. A subbundle \mathcal{F} of \mathcal{E} is said to be *isotropic* with respect to $\langle \cdot, \cdot \rangle$ if $\mathcal{F} \subset \mathcal{F}^{\langle \cdot, \cdot \rangle}$. An isotropic subbundle of \mathcal{E} such that $\mathcal{F} = \mathcal{F}^{\langle \cdot, \cdot \rangle}$ is said to be a *Lagrangian subbundle*.

The next result easily follows from Lemma 1.A.2.

COROLLARY 1.A.4. *Let \mathcal{F} be an isotropic subbundle of \mathcal{E} . Then $2\text{rank}(\mathcal{F}) \leq \text{rank}(\mathcal{E})$. Moreover, \mathcal{F} is Lagrangian if and only if $2\text{rank}(\mathcal{F}) = \text{rank}(\mathcal{E})$.*

The next lemma shows that Lagrangian subbundles exist locally for the Zariski topology over U . This implies in particular that the rank of every symplectic vector bundle is even.

LEMMA 1.A.5. *Let $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ be a symplectic vector bundle over U and assume that $U = \text{Spec } R$, where R is a local ring. Then there exists a Lagrangian subbundle of \mathcal{E} .*

PROOF. Let S be the set of isotropic subbundles of \mathcal{E} ordered by inclusion. It is sufficient to prove that every maximal element in S is a Lagrangian (maximal elements always exist; consider the rank, for instance).

We proceed by contraposition. Let \mathcal{F} be an element of S that is not a Lagrangian. As R is local and both \mathcal{F} and $\mathcal{F}^{\langle \cdot, \cdot \rangle}$ are subbundles of \mathcal{E} (cf. Lemma 1.A.2), there exists an integer $k \geq 1$ and global sections e_1, \dots, e_k of $\mathcal{F}^{\langle \cdot, \cdot \rangle}$ such that

$$\mathcal{F}^{\langle \cdot, \cdot \rangle} = \mathcal{F} \oplus \mathcal{O}_U e_1 \oplus \dots \oplus \mathcal{O}_U e_k.$$

In particular, $\mathcal{F} \oplus \mathcal{O}_U e_1$ is an element of S strictly containing \mathcal{F} ; thus, \mathcal{F} is not maximal. ■

REMARK 1.A.6. The same proof applies to any ring R such that every projective R -module is free, e.g., R a principal ideal domain, or R a polynomial ring over a field.

1.A.3. Symplectic bases. Let $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ be a symplectic vector bundle of constant rank $2n$ over U .

DEFINITION 1.A.7. A *symplectic basis* of $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ over U is a basis of \mathcal{E} over U of the form $(e_1, \dots, e_n, f_1, \dots, f_n)$ with $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ and $\langle e_i, f_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$.

As Lagrangian subbundles exist locally by Lemma 1.A.5, the next proposition implies in particular that symplectic bases also exist locally.

PROPOSITION 1.A.8. *Let U be an affine scheme, $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ be a symplectic vector bundle over U , and \mathcal{L} be a Lagrangian subbundle of \mathcal{E} . Then*

- (1) *Every basis (e_1, \dots, e_n) of \mathcal{L} over U can be completed to a symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of \mathcal{E} over U .*
- (2) *If \mathcal{F} is a Lagrangian subbundle of \mathcal{E} such that $\mathcal{L} \oplus \mathcal{F} = \mathcal{E}$, and (f_1, \dots, f_n) is a basis of \mathcal{F} over U , then there exists a unique basis (e_1, \dots, e_n) of \mathcal{L} over U such that $(e_1, \dots, e_n, f_1, \dots, f_n)$ is a symplectic basis of \mathcal{E} over U .*

PROOF. Consider the surjective morphism of \mathcal{O}_U -modules (cf. Lemma 1.A.2)

$$\begin{aligned}\mathcal{E} &\longrightarrow \mathcal{L}^\vee \\ e &\longmapsto \langle \cdot, e \rangle|_{\mathcal{L}}.\end{aligned}$$

Since U is affine, there exists a sequence (f'_1, \dots, f'_n) of global sections of \mathcal{E} lifting the dual basis of (e_1, \dots, e_n) in \mathcal{L}^\vee , so that $\langle e_i, f'_j \rangle = \delta_{ij}$ for every $1 \leq i, j \leq n$. As \mathcal{L} is an isotropic subbundle of \mathcal{E} , to prove (1) it is sufficient to show the existence of global sections ℓ_j of \mathcal{L} such that

$$f_j := f'_j + \ell_j$$

satisfy $\langle f_i, f_j \rangle = 0$ for every $1 \leq i, j \leq n$.

Since the bilinear form $\langle \cdot, \cdot \rangle$ is alternating, $A := (\langle f'_i, f'_j \rangle)_{1 \leq i, j \leq n}$ is an antisymmetric matrix in $M_{n \times n}(\mathcal{O}_U(U))$. Thus, there exists a matrix $B = (b_{ij})_{1 \leq i, j \leq n}$ in $M_{n \times n}(\mathcal{O}_U(U))$ such that $A = B - B^\top$. We put

$$\ell_i := - \sum_{j=1}^n b_{ij} e_j,$$

hence

$$\langle f_i, f_j \rangle = \langle f'_i, f'_j \rangle + \langle \ell_i, f'_j \rangle - \langle \ell_j, f'_i \rangle = \langle f'_i, f'_j \rangle + b_{ij} - b_{ji} = 0.$$

We now proceed to the proof of (2). As \mathcal{F} is an isotropic subbundle of \mathcal{E} satisfying $\mathcal{L} \oplus \mathcal{F} = \mathcal{E}$, the morphism of \mathcal{O}_U -modules

$$\begin{aligned}\mathcal{F} &\longrightarrow \mathcal{L}^\vee \\ f &\longmapsto \langle \cdot, f \rangle|_{\mathcal{L}}\end{aligned}$$

is injective by non-degeneracy of $\langle \cdot, \cdot \rangle$, thus an isomorphism since \mathcal{F} and \mathcal{L}^\vee have equal rank. The existence and unicity of (e_1, \dots, e_n) follows from remarking that $(e_1, \dots, e_n, f_1, \dots, f_n)$ is a symplectic basis of \mathcal{E} over U if and only if (e_1, \dots, e_n) is the basis of \mathcal{L} over U dual to the basis $(\langle \cdot, f_1 \rangle|_{\mathcal{L}}, \dots, \langle \cdot, f_n \rangle|_{\mathcal{L}})$ of \mathcal{L}^\vee . \blacksquare

1.B. Gauss-Manin connection on some elliptic curves

1.B.1. The Weierstrass elliptic curve. Let

$$W := \text{Spec } \mathbf{C}[g_2, g_3, \Delta^{-1}]$$

where

$$\Delta := g_2^3 - 27g_3^2.$$

Then we can define an elliptic curve E over W by the classical Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3.$$

Further, we define a symplectic-Hodge basis (ω, η) of E/W by the formulas

$$\omega := \frac{dx}{y}, \quad \eta := x \frac{dx}{y}.$$

LEMMA 1.B.1. *With the above notations, the Gauss-Manin connection ∇ on $H_{\text{dR}}^1(E/W)$ is given by*

$$\nabla \begin{pmatrix} \omega & \eta \end{pmatrix} = \begin{pmatrix} \omega & \eta \end{pmatrix} \otimes \frac{1}{\Delta} \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

where

$$\Omega_{11} = -\frac{1}{4}g_2^2 dg_2 + \frac{9}{2}g_3 dg_3$$

$$\Omega_{12} = \frac{3}{8}g_2g_3 dg_2 - \frac{1}{4}g_2^2 dg_3$$

$$\Omega_{21} = -\frac{9}{2}g_3 dg_2 + 3g_2 dg_3$$

$$\Omega_{22} = -\Omega_{11}.$$

Let us briefly explain how these expressions follow from the description given in [51] A1.3 of the Gauss-Manin connection on the relative first de Rham cohomology of the universal elliptic curve \mathbb{E} over the Poincaré half-plane \mathbb{H} (whose fiber at each $\tau \in \mathbb{H}$ is given by the complex torus $\mathbb{E}_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$).⁵

We first remark that for any $u \in \mathbf{C}^\times$ we can define an automorphism $M_{u/\mu_u} : E/W \rightarrow E/W$ in the category $\mathcal{A}_{1,\mathbf{C}}$ by

$$\mu_u(g_2, g_3) = (u^{-4}g_2, u^{-6}g_3), \quad M_u(x, y) = (u^{-2}x, u^{-3}y).$$

Using that the Gauss-Manin connection commutes with base change and admits regular singularities, we deduce by homogeneity that there exists constants c_1, \dots, c_8 in \mathbf{C} such that

$$\begin{aligned} \Omega_{11} &= c_1 g_2^2 dg_2 + c_2 g_3 dg_3, & \Omega_{12} &= c_3 g_2 g_3 dg_2 + c_4 g_2^2 dg_3, \\ \Omega_{21} &= c_5 g_3 dg_2 + c_6 g_2 dg_3, & \Omega_{22} &= c_7 g_2^2 dg_2 + c_8 g_3 dg_3. \end{aligned}$$

To determine these constants, we consider the cartesian diagram in the category of complex analytic spaces

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\Psi} & E(\mathbf{C}) \\ \downarrow & \square & \downarrow \\ \mathbb{H} & \xrightarrow{\psi} & W(\mathbf{C}) \end{array}$$

given by the classical Weierstrass theory :

$$\psi(\tau) = (g_2(\tau), g_3(\tau)), \quad \Psi_\tau(z) = (\wp_\tau(z), \wp'_\tau(z))$$

Finally, we apply once again that the formation of the Gauss-Manin connection (now in the complex analytic category) commutes with base change, and we use the formulas in [51] A1.3 :

$$\nabla \left(\begin{array}{c} dz \\ \wp_\tau(z) dz \end{array} \right) = \left(\begin{array}{c} dz \\ \wp_\tau(z) dz \end{array} \right) \otimes \frac{1}{2\pi i} \begin{pmatrix} -(2\pi i)^2 E_2(\tau)/12 & -(2\pi i)^4 E_4(\tau)/144 \\ 1 & (2\pi i)^2 E_2(\tau)/12 \end{pmatrix} d\tau$$

1.B.2. The elliptic curve X/B over $\mathbf{Z}[1/6]$. Let

$$B := \text{Spec } \mathbf{Z}[1/6, e_2, e_4, e_6, \Delta^{-1}]$$

where

$$\Delta := e_4^3 - e_6^2.$$

We define an elliptic curve X over B by

$$y^2 = 4 \left(x + \frac{e_2}{12} \right)^3 - \frac{e_4}{12} \left(x + \frac{e_2}{12} \right) + \frac{e_6}{216}.$$

We define a symplectic-Hodge basis (ω, η) of X/B by the formulas

$$\omega := \frac{dx}{y}, \quad \eta := x \frac{dx}{y}.$$

Note that there is a morphism $F/f : (X_{\mathbf{C}})_{/B_{\mathbf{C}}} \rightarrow E/W$ in $\mathcal{A}_{1,\mathbf{C}}$ given by

$$f(e_2, e_4, e_6) = \left(\frac{e_4}{12}, -\frac{e_6}{216} \right), \quad F(x, y) = \left(x + \frac{e_2}{12}, y \right).$$

By pulling back the Gauss-Manin connection on $H_{\text{dR}}^1(E/W)$ described in Lemma 1.B.1 by the morphism F/f , we obtain that the Gauss-Manin connection ∇ on $H_{\text{dR}}^1(X/B)$ over $\mathbf{Z}[1/6]$ is given by

$$\nabla \left(\begin{array}{c} \omega \\ \eta \end{array} \right) = \left(\begin{array}{c} \omega \\ \eta \end{array} \right) \otimes \frac{1}{\Delta} \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

5. A direct algebraic approach is also possible. See for instance [53] 3 and [54] 3.4.

where

$$\begin{aligned}\Omega_{11} &= \left(\frac{e_2 e_6 - e_4^2}{4} \right) de_4 + \left(\frac{e_6 - e_2 e_4}{6} \right) de_6 \\ \Omega_{12} &= -\frac{\Delta}{12} de_2 - \left(\frac{e_4 e_6 - 2e_2 e_4^2 + e_2^2 e_6}{48} \right) de_4 + \left(\frac{e_4^2 - 2e_2 e_6 + e_2^2 e_4}{72} \right) de_6 \\ \Omega_{21} &= 3e_6 de_4 - 2e_4 de_6 \\ \Omega_{22} &= -\Omega_{11}.\end{aligned}$$

1.B.3. The universal elliptic curve X_{1/B_1} over $\mathbf{Z}[1/2]$. Consider the elliptic curve X_1 over B_1 defined in Theorem 6.2 and let $\Phi_{/\varphi} : (X_{1,\mathbf{Z}[1/6]})_{/B_1,\mathbf{Z}[1/6]} \rightarrow X/B$ be the isomorphism in $\mathcal{A}_{1,\mathbf{Z}[1/6]}$ given by

$$\varphi(b_2, b_4, b_6) = (b_2, b_2^2 - 24b_4, b_2^3 - 36b_2 b_4 + 216b_6), \quad \Phi(x, y) = (x, 2y).$$

If (ω_1, η_1) denotes de symplectic-Hodge basis of X_{1/B_1} defined in Theorem 6.2, then by pulling back the Gauss-Manin connection on $H_{\text{dR}}^1(X/B)$ described in 1.B.2 by the isomorphism $\Phi_{/\varphi}$, we obtain that the Gauss-Manin connection ∇ on $H_{\text{dR}}^1(X_1/B_1)$ over $\mathbf{Z}[1/2]$ is given by

$$\nabla \begin{pmatrix} \omega_1 & \eta_1 \end{pmatrix} = \begin{pmatrix} \omega_1 & \eta_1 \end{pmatrix} \otimes \frac{1}{\Delta} \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

where

$$\begin{aligned}\Omega_{11} &= \frac{b_2^2 b_6 - 6b_4 b_6 - b_2 b_4^2}{8} db_2 + \frac{4b_4^2 - 3b_2 b_6}{2} db_6 + \frac{18b_6 - b_2 b_4}{4} db_6 \\ \Omega_{12} &= \frac{2b_4^3 + 9b_6^2 - 2b_2 b_4 b_6}{4} db_2 + \frac{b_2^2 b_6 - b_2 b_4^2 - 6b_4 b_6}{4} db_6 + \frac{4b_4^2 - 3b_2 b_6}{4} db_6 \\ \Omega_{21} &= \frac{3b_2 b_6 - 4b_4^2}{4} db_2 + \frac{b_2 b_4 - 18b_6}{2} db_4 + \frac{24b_4 - b_2^2}{4} db_6 \\ \Omega_{22} &= -\Omega_{11}.\end{aligned}$$

Higher Ramanujan Equations II : periods of abelian varieties and transcendence questions

Abstract

In the first part of this work, we have considered a moduli space B_g classifying principally polarized abelian varieties of dimension g endowed with a symplectic-Hodge basis, and we have constructed the higher Ramanujan vector fields $(v_{kl})_{1 \leq k \leq l \leq g}$ on it. In this second part, we study these objects from a complex analytic viewpoint. We construct a holomorphic map $\varphi_g : \mathbf{H}_g \rightarrow B_g(\mathbf{C})$, where \mathbf{H}_g denotes the Siegel upper half-space of genus g , satisfying the system of differential equations $\frac{1}{2\pi i} \frac{\partial \varphi_g}{\partial \tau_{kl}} = v_{kl} \circ \varphi_g$, $1 \leq k \leq l \leq g$. When $g = 1$, we prove that φ_1 may be identified with the triple of Eisenstein series (E_2, E_4, E_6) , so that the previous differential equations coincide with Ramanujan's classical relations concerning Eisenstein series. We discuss the relation between the values of φ_g and the fields of periods of abelian varieties, and we explain how this relates to Grothendieck's periods conjecture. Finally, we prove that every leaf of the holomorphic foliation on $B_g(\mathbf{C})$ induced by the vector fields v_{kl} is Zariski-dense in $B_g(\mathbf{C})$. This last result implies a "functional version" of Grothendieck's periods conjecture for abelian varieties.

1. Introduction

1.1. In the first part of this work ([32]) we have considered for any integer $g \geq 1$ a smooth moduli stack \mathcal{B}_g over \mathbf{Z} classifying principally polarized abelian varieties of dimension g endowed with a *symplectic-Hodge basis* of its first algebraic de Rham cohomology, and we have constructed a family $(v_{kl})_{1 \leq k \leq l \leq g}$ of $g(g+1)/2$ commuting vector fields on \mathcal{B}_g , the *higher Ramanujan vector fields*. We have also proved that $\mathcal{B}_g \otimes \mathbf{Z}[1/2]$ is representable by a smooth quasi-projective scheme B_g over $\mathbf{Z}[1/2]$. In particular, the set of complex points $B_g(\mathbf{C})$ has a natural structure of a quasi-projective complex manifold.

In this second part, we consider the differential equations defined by the higher Ramanujan vector fields, and we study their complex analytic solutions. Let

$$\mathbf{H}_g := \{\tau \in M_{g \times g}(\mathbf{C}) \mid \tau^T = \tau, \operatorname{Im} \tau > 0\}$$

be the Siegel upper half-space of genus g . This is a complex manifold of dimension $g(g+1)/2$ admitting the holomorphic coordinate system $(\tau_{kl})_{1 \leq k \leq l \leq g}$, where $\tau_{kl} : \mathbf{H}_g \rightarrow \mathbf{C}$ is the holomorphic map associating to $\tau \in \mathbf{H}_g$ its entry in the k th row and l th column. Using the universal property of $B_g(\mathbf{C})$, we shall construct a holomorphic map

$$\varphi_g : \mathbf{H}_g \rightarrow B_g(\mathbf{C})$$

satisfying the system of differential equations

$$(1.1) \quad \frac{1}{2\pi i} \frac{\partial \varphi_g}{\partial \tau_{kl}} = v_{kl} \circ \varphi_g, \quad 1 \leq k \leq l \leq g.$$

When $g = 1$, the Siegel upper half-space \mathbf{H}_1 is the Poincaré upper half-plane $\mathbf{H} = \{\tau \in \mathbf{C} \mid \operatorname{Im} \tau > 0\}$, and the classical theory of elliptic curves provides an isomorphism

$$B_1 \otimes \mathbf{Z}[1/6] \cong \mathbf{Z}[1/6, e_2, e_4, e_6, (e_4^3 - e_6^2)^{-1}],$$

under which the vector field v_{11} becomes the "classical" Ramanujan vector field (cf. [32] Section 6)

$$(1.2) \quad v := \frac{e_2^2 - e_4}{12} \frac{\partial}{\partial e_2} + \frac{e_2 e_4 - e_6}{3} \frac{\partial}{\partial e_4} + \frac{e_2 e_6 - e_4^2}{2} \frac{\partial}{\partial e_6}.$$

In this situation, the holomorphic map φ_1 gets identified with

$$\tau \mapsto (E_2(\tau), E_4(\tau), E_6(\tau)),$$

where $E_2, E_4, E_6 : \mathbf{H} \rightarrow \mathbf{C}$ are the classical (level 1) Eisenstein series normalized by $E_{2k}(+i\infty) = 1$, and we obtain Ramanujan's original relations between Eisenstein series :

$$\frac{1}{2\pi i} \frac{dE_2}{d\tau} = \frac{E_2^2 - E_4}{12}, \quad \frac{1}{2\pi i} \frac{dE_4}{d\tau} = \frac{E_2E_4 - E_6}{3}, \quad \frac{1}{2\pi i} \frac{dE_6}{d\tau} = \frac{E_2E_6 - E_4^2}{2}.$$

1.2. As explained in the introduction of [32], questions in Transcendental Number Theory constitute our main source of motivation for the study of these higher dimensional analogs of Ramanujan's equations. We shall illustrate this point by relating the values of φ_g with Grothendieck's periods conjecture on abelian varieties. In order to fully motivate a precise statement of our result, let us digress into a discussion of periods of abelian varieties and Grothendieck's conjecture on the algebraic relations between them.

Let X be a complex abelian variety and $k \subset \mathbf{C}$ be the smallest algebraically closed subfield of \mathbf{C} over which there exists an abelian variety X_0 such that X is isomorphic to $X_0 \otimes_k \mathbf{C}$ as complex abelian varieties (cf. Lemma 5.1). By a *period* of X , we mean any complex number of the form

$$\int_{\gamma} \alpha$$

where α is an element of the first algebraic de Rham cohomology $H_{\text{dR}}^1(X_0/k)$ and $\gamma \in H_1(X_0(\mathbf{C}), \mathbf{Z})$ is the class of a singular 1-cycle. We define the *field of periods* $\mathcal{P}(X)$ of X as the smallest subfield of \mathbf{C} containing k and all the periods of X .¹ Equivalently, $\mathcal{P}(X)$ may be regarded as the field of rationality of the comparison isomorphism

$$H^1(X_0(\mathbf{C}), \mathbf{C}) = \text{Hom}(H_1(X_0(\mathbf{C}), \mathbf{Z}), \mathbf{C}) \xrightarrow{\sim} H_{\text{dR}}^1(X_0/k) \otimes_k \mathbf{C}.$$

A central problem in the theory of transcendental numbers is to determine, or simply to estimate, the transcendence degree over \mathbf{Q} of the field of periods $\mathcal{P}(X)$.

In a first approach, one might observe that any algebraic cycle in some power X^n of X induces an algebraic relation between its periods (cf. [29] Proposition I.1.6). Broadly speaking, Grothendieck conjectured that *every* algebraic relation between periods of an abelian variety can be "explained" through algebraic cycles on its powers.

A convenient way of giving a precise formulation for Grothendieck's conjecture is by means of Mumford-Tate groups. Let X be a complex abelian variety, and denote by H the \mathbf{Q} -Hodge structure of weight 1 with underlying \mathbf{Q} -vector space given by $H^1(X(\mathbf{C}), \mathbf{Q})$, and Hodge filtration F^1H given by $H^0(X, \Omega_{X/\mathbf{C}}^1) \subset H_{\text{dR}}^1(X/\mathbf{C}) \cong H^1(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$. The decomposition $H_{\mathbf{C}} = F^1H \oplus \overline{F^1H}$ corresponds to the morphism of real algebraic groups

$$h : \mathbf{C}^{\times} \rightarrow \text{GL}(H_{\mathbf{R}}),$$

where $h(z)$ acts on F^1H by a homothety of ratio z^{-1} , and on $\overline{F^1H}$ by a homothety of ratio \bar{z}^{-1} . The *Mumford-Tate group* $\text{MT}(X)$ of X is defined as the smallest \mathbf{Q} -algebraic subgroup of $\text{GL}(H)$ such that h factors through $\text{MT}(X)_{\mathbf{R}}$. It can also be interpreted as the smallest \mathbf{Q} -algebraic subgroup of $\text{GL}(H) \times \mathbf{G}_{m, \mathbf{Q}}$ fixing all Hodge classes in twisted mixed tensor powers of the \mathbf{Q} -Hodge structure H (cf. [29] I.3).

The following formulation of *Grothendieck's periods conjecture* (GPC) for abelian varieties is a specialization of the "generalized Grothendieck's periods conjecture" proposed by André ([2] 23.4.1; see also [60] Historical Note pp. 40-44 and [40] footnote 10).

CONJECTURE (Grothendieck-André). *For any complex abelian variety X , we have*

$$\text{trdeg}_{\mathbf{Q}} \mathcal{P}(X) \stackrel{?}{\geq} \dim \text{MT}(X).$$

1. This definition is not standard. Usually, one starts with an abelian variety X defined over a subfield $K \subset \mathbf{C}$, and one defines K -periods in terms of $H_{\text{dR}}^1(X/K)$. Our "absolute" definition considering a minimal algebraically closed field of definition is convenient for our purposes and will be justified in the sequel.

It follows from Deligne [28] (cf. [29] Corollary I.6.4) that we always have the upper bound

$$\mathrm{trdeg}_{\mathbf{Q}}\mathcal{P}(X) \leq \dim \mathrm{MT}(X) + \mathrm{trdeg}_{\mathbf{Q}}k,$$

where k is the smallest algebraically closed subfield of \mathbf{C} over which X may be defined. In particular, if X is definable over the field of algebraic numbers \mathbf{Q} — the case originally considered by Grothendieck — the above conjectural inequality becomes the conjectural equality

$$\mathrm{trdeg}_{\mathbf{Q}}\mathcal{P}(X) \stackrel{?}{=} \dim \mathrm{MT}(X).$$

In the case $g = 1$, the Mumford-Tate group of a complex elliptic curve E may be easily computed. Its dimension only depends on the existence or not of complex multiplication, and GPC predicts that

$$\mathrm{trdeg}_{\mathbf{Q}}\mathcal{P}(E) \stackrel{?}{\geq} \begin{cases} 2 & \text{if } E \text{ has complex multiplication} \\ 4 & \text{otherwise.} \end{cases}$$

Even in this minimal case, GPC is not yet established in full generality — only the complex multiplication case is understood; see below. Nevertheless, an approach that has been proved fruitful for obtaining non-trivial lower bounds in the direction of GPC relies on a *modular description* of the fields of periods of elliptic curves, which we now recall.

Let E be a complex elliptic curve and let $j \in \mathbf{C}$ be its j -invariant. Then E admits a model

$$E : y^2 = 4x^3 - g_2x - g_3$$

with $g_2, g_3 \in \mathbf{Q}(j)$, and we can consider the algebraic differential forms defined over $\mathbf{Q}(j)$

$$\omega := \frac{dx}{y}, \quad \eta := x \frac{dx}{y}.$$

They form a (symplectic-Hodge) basis of the first algebraic de Rham cohomology $H_{\mathrm{dR}}^1(E/\mathbf{Q}(j))$. If (γ, δ) is any basis of the first singular homology group $H_1(E(\mathbf{C}), \mathbf{Z})$, we may consider the periods

$$\omega_1 = \int_{\gamma} \omega, \quad \omega_2 = \int_{\delta} \omega, \quad \eta_1 = \int_{\gamma} \eta, \quad \eta_2 = \int_{\delta} \eta.$$

We may assume moreover that the basis (γ, δ) is oriented, in the sense that their topological intersection product $\gamma \cap \delta = 1$.

The field of periods of E is given by

$$\mathcal{P}(E) = \overline{\mathbf{Q}(j)}(\omega_1, \omega_2, \eta_1, \eta_2),$$

where $\overline{\mathbf{Q}(j)}$ denotes the algebraic closure of $\mathbf{Q}(j)$ in \mathbf{C} . Now, observe that $\omega_1 \neq 0$ and let

$$\tau := \frac{\omega_2}{\omega_1}.$$

As the basis (γ, δ) of $H_1(E(\mathbf{C}), \mathbf{Z})$ is oriented, the complex number τ is in \mathbf{H} . By the classical theory of modular forms, we have

$$E_2(\tau) = 12 \left(\frac{\omega_1}{2\pi i} \right) \left(\frac{\eta_1}{2\pi i} \right), \quad E_4(\tau) = 12g_2 \left(\frac{\omega_1}{2\pi i} \right)^4, \quad E_6(\tau) = -216g_3 \left(\frac{\omega_1}{2\pi i} \right)^6.$$

Finally, Legendre's periods relation and the definition of j show that $\mathcal{P}(E)$ is an algebraic extension of the field $\mathbf{Q}(2\pi i, \tau, E_2(\tau), E_4(\tau), E_6(\tau))$, and we obtain

$$(1.3) \quad \mathrm{trdeg}_{\mathbf{Q}}\mathcal{P}(E) = \mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}(2\pi i, \tau, E_2(\tau), E_4(\tau), E_6(\tau)).$$

In this way, the problem of estimating the transcendence degree of fields of periods of elliptic curves translates into the problem of estimating the transcendence degree of values of some analytic functions. Accordingly, the theorem of Nesterenko [75] asserts that, for any $\tau \in \mathbf{H}$,

$$\mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}(e^{2\pi i\tau}, E_2(\tau), E_4(\tau), E_6(\tau)) \geq 3.$$

As an immediate consequence, we obtain

$$\mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}(2\pi i, \tau, E_2(\tau), E_4(\tau), E_6(\tau)) \geq \mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}(E_2(\tau), E_4(\tau), E_6(\tau)) \geq 2$$

for any $\tau \in \mathbf{H}$. Equivalently, by equation (1.3), for any complex elliptic curve E , we obtain the uniform bound

$$\mathrm{trdeg}_{\mathbf{Q}}\mathcal{P}(E) \geq 2,$$

which is sharp when E has complex multiplication. This last result had already been previously established by Chudnovsky (cf. [23]) via elliptic methods.²

1.3. In this paper, we generalize the modular description (1.3). Namely, let $g \geq 1$ be an integer and, for any $\tau \in \mathbf{H}_g$, let X_τ be the complex abelian variety given by the (polarizable) complex torus $\mathbf{C}^g/(\mathbf{Z}^g + \tau\mathbf{Z}^g)$. We shall prove that, for any $\tau \in \mathbf{H}_g$, the field of periods $\mathcal{P}(X_\tau)$ is an algebraic extension of $\mathbf{Q}(2\pi i, \tau, \varphi_g(\tau))$ (the residue field in $\mathbf{A}_{\mathbf{Q}}^1 \times_{\mathbf{Q}} \mathrm{Sym}_{g,\mathbf{Q}} \times_{\mathbf{Q}} B_{g,\mathbf{Q}}$ of the complex point $(2\pi i, \tau, \varphi_g(\tau))$, where Sym_g denotes the group scheme of symmetric matrices of order $g \times g$); in particular, we obtain

$$\mathrm{trdeg}_{\mathbf{Q}}\mathcal{P}(X_\tau) = \mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}(2\pi i, \tau, \varphi_g(\tau)).$$

This generalized modular description raises the question of whether it is possible to adapt Nesterenko's methods to this higher dimensional setting. Guided by this problem, we are naturally lead to study the *higher Ramanujan foliation*, namely, the holomorphic foliation on $B_g(\mathbf{C})$ generated by the higher Ramanujan vector fields.

We shall prove that every leaf of the Ramanujan foliation on $B_g(\mathbf{C})$ is Zariski-dense in $B_{g,\mathbf{C}}$. This property of a foliation plays an important role, at least in the case in which leaves are one dimensional (where it implies Nesterenko's D -property), in the "multiplicity estimates" appearing in applications of differential equations to transcendental number theory (cf. [10], [73], [75]).

The Zariski-density of the image of $\varphi_g : \mathbf{H}_g \rightarrow B_g(\mathbf{C})$ in $B_{g,\mathbf{C}}$ also implies the *a priori* stronger result that its graph

$$\{(\tau, \varphi_g(\tau)) \in \mathrm{Sym}_g(\mathbf{C}) \times B_g(\mathbf{C}) \mid \tau \in \mathbf{H}_g\}$$

is Zariski-dense in $\mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}}$ (cf. [8], Theorem 1, where a similar question is investigated in the context of derivatives of Siegel modular forms). This can be seen as an analog — but not a complete generalization by the lack of the modular parameter q — of Mahler's result [63] on the algebraic independence of the holomorphic functions τ , $e^{2\pi i\tau}$, $E_2(\tau)$, $E_4(\tau)$, and $E_6(\tau)$, of $\tau \in \mathbf{H}$. Under the viewpoint of [8], Bertrand and Zudilin obtained in [9], Theorem 1, a full generalization of Mahler's result.

Geometrically, the Zariski-density of the graph of $\hat{\varphi}$ in $\mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}}$ can also be interpreted as a "functional version" of GPC : loosely speaking, it says that there is no algebraic relation simultaneously satisfied by the periods of every (principally polarized) abelian variety other than the relations given by the polarization data.

Our density results will rely on a characterization of the leaves of the higher Ramanujan foliation in terms of an action by $\mathrm{Sp}_{2g}(\mathbf{C})$. In fact, from the complex analytic viewpoint, the complex manifold $B_g(\mathbf{C})$ and the higher Ramanujan vector fields admit a simple description in terms of algebraic groups.

Namely, we shall explain how to realize $B_g(\mathbf{C})$ as a domain (in the analytic topology) of the quotient manifold $\mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$, and we shall prove that under this identification the higher Ramanujan vector field v_{kl} is induced by the left invariant holomorphic vector field on $\mathrm{Sp}_{2g}(\mathbf{C})$ associated to

$$\frac{1}{2\pi i} \begin{pmatrix} 0 & \mathbf{E}^{kl} \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie} \mathrm{Sp}_{2g}(\mathbf{C}),$$

where \mathbf{E}^{kl} is the symmetric matrix of order $g \times g$ whose entry in the k th row and l th column (resp. l th row and k th column) is 1, and whose all other entries are 0.

Furthermore, the solution $\varphi_g : \mathbf{H}_g \rightarrow B_g(\mathbf{C})$ of the higher Ramanujan equations is identified to

$$\tau \mapsto \left[\begin{pmatrix} \mathbf{1}_g & \tau \\ 0 & \mathbf{1}_g \end{pmatrix} \right] \in \mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C}),$$

2. We should also point out that the modular parameter $e^{2\pi i\tau}$, ignored in our discussion, has a geometric interpretation. Namely, it is a period of a certain 1-motive naturally attached to E . We refer to [6] (cf. [2] 23.4.3) for further discussion on these matters.

where $\mathbf{1}_g$ denotes the identity matrix of order $g \times g$. This enables us to obtain every leaf of the higher Ramanujan foliation as the image of a holomorphic map $\varphi_\delta : U_\delta \rightarrow B_g(\mathbf{C})$ defined on some explicitly defined open subset $U_\delta \subset \mathbf{H}_g$ obtained from φ_g via a “twist” by some element $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$.

In the case $g = 1$, the above twisting procedure may be illustrated as follows. Let

$$\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}),$$

let $U_\delta = \{\tau \in \mathbf{H} \mid c\tau + d \neq 0\}$, and define a holomorphic map $\varphi_\delta : U_\delta \rightarrow B_1(\mathbf{C}) \subset \mathbf{C}^3$ by

$$\varphi_\delta(\tau) = \left((c\tau + d)^2 E_2(\tau) + \frac{12c}{2\pi i} (c\tau + d), (c\tau + d)^4 E_4(\tau), (c\tau + d)^6 E_6(\tau) \right)$$

Then one may easily check that φ_δ satisfy the differential equation

$$\frac{1}{2\pi i} \frac{d\varphi_\delta}{d\tau} = (c\tau + d)^{-2} v \circ \varphi_\delta$$

where v is the classical Ramanujan vector field defined by (1.2).

1.4. As acknowledged in [32], our definition of the moduli stack \mathcal{B}_g was inspired by Movasati’s point of view on the Ramanujan vector field in terms of the Gauss-Manin connection on the de Rham cohomology of the universal elliptic curve (cf. [69] 4.2), which corresponds to the case $g = 1$ of our construction.

After I completed a first version this article, H. Movasati has kindly indicated to me that a number of its results and constructions has some overlap with his article [70]. In this work, he considers complex analytic spaces U classifying lattices in maximal totally real subspaces of some given complex vector space V_0 (i.e. subgroups of V_0 generated by a \mathbf{C} -basis of V_0) satisfying suitable compatibility conditions with a fixed Hodge filtration F_0^\bullet on V_0 , and a fixed polarization ψ_0 ; these spaces come equipped with a natural analytic right action of the complex algebraic group

$$G_0 = \{g \in \mathrm{GL}(V_0) \mid gF_0^i = F_0^i \text{ for every } i, \text{ and } g^*\psi_0 = \psi_0\}.$$

For the particular case where $V_0 = \mathbf{C}^{2g}$,

$$F_0^\bullet = (F_0^0 = V_0 \supset F_0^1 = \mathbf{C}^g \times \{0\} \supset F_0^2 = 0),$$

and ψ_0 is the standard (complex) symplectic form ([70] 5.1), the space U becomes the analytic moduli space $B_g(\mathbf{C})$, investigated in the present article. Of course, the algebraic group G_0 coincides with our $P_g(\mathbf{C})$, and the action of G_0 on U gets identified with the action of $P_g(\mathbf{C})$ on $B_g(\mathbf{C})$ defined in [32] under $U \cong B_g(\mathbf{C})$.

In [70] 3.2, Movasati also describes U as a quotient $\Gamma_{\mathbf{Z}} \backslash P$, where P is the space of “period matrices” and $\Gamma_{\mathbf{Z}}$ is some explicitly defined discrete group. In our particular case, P may be identified with our \mathbf{B}_g (cf. Proposition 6.6) and $\Gamma_{\mathbf{Z}} = \mathrm{Sp}_{2g}(\mathbf{Z})$. Moreover, the map $\mathbf{H}_g \rightarrow P$ defined in [70] p. 584 coincides with our $\varphi_g : \mathbf{H}_g \rightarrow B_g(\mathbf{C})$ constructed via the universal property of $B_g(\mathbf{C})$.

In his article, Movasati explicitly states the problem of algebraizing U — i.e. finding the algebraic variety T over $\overline{\mathbf{Q}}$, in his notations — and the action of G_0 . This is solved “by definition” in our construction, which also shows that T , here called $B_{g, \overline{\mathbf{Q}}}$, is smooth and quasi-projective. Movasati actually conjectures that the complex manifold $B_g(\mathbf{C})$ admit a *unique* structure of complex algebraic variety (in analogy with the Baily-Borel theorem) and that it is actually quasi-affine. On his web page³, Movasati also indicates a construction of what we call “higher Ramanujan vector fields” with slightly different normalizations (cf. [32] Proposition 5.7).

1.5. Acknowledgments. This work was supported by a public grant as part of the FMJH project, and is part of my PhD thesis under the supervision of Jean-Benoît Bost. I thank Mikolaj Fraczyk for sharing his insights on Zariski-density matters and for his interest in this work, Hossein Movasati for his kind remarks on a first version of this article and for making me aware of his work on this circle of questions, and Daniel Bertrand for his comments and bibliographical corrections.

1.6. Terminology and notations. Besides the terminology and notations of [32], we shall consider the following.

3. See “What is a Siegel quasi-modular form?” in <http://w3.impa.br/~hossein/WikiHossein/WikiHossein.html>.

1.6.1. Let M be a complex manifold. Every holomorphic vector bundle $\pi : V \rightarrow M$ may be seen as a (commutative) relative complex Lie group over M . We shall occasionally identify V with its corresponding locally free sheaf of \mathcal{O}_M -modules of holomorphic sections of π .

1.6.2. If R is any ring, we denote the constant sheaf with values in R over some complex manifold M by R_M . A *local system* of R -modules over M is a locally constant sheaf L of R -modules over M . The *dual* of L is denoted by $L^\vee := \mathcal{H}om_R(L, R_M)$.

The *étalé space* of a local system of R -modules L over M will be denoted by $E(L)$; this is a topological covering space over M whose fiber at each $p \in M$ is naturally identified to L_p .

1.6.3. Let $m, n \geq 1$ be integers. The set of matrices of order $m \times n$ over a ring R is denoted by $M_{m \times n}(R)$. We shall frequently adopt a block notation for elements in $M_{2n \times 2n}(R)$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (A \ B ; \ C \ D),$$

where $A, B, C, D \in M_{n \times n}(R)$.

The transpose of a matrix $M \in M_{m \times n}(R)$ is denoted by $M^\top \in M_{n \times m}(R)$. For $1 \leq i \leq n$, $\mathbf{e}_i \in M_{n \times 1}(R)$ denotes for the column vector whose entry in the i th line is 1, and all the others are 0. The identity matrix in $M_{n \times n}(R)$ is denoted by $\mathbf{1}_n$. For every $1 \leq i \leq j \leq n$, we denote by \mathbf{E}^{ij} the unique symmetric matrix $(\mathbf{E}_{kl}^{ij})_{1 \leq k, l \leq n} \in M_{n \times n}(R)$ such that

$$\mathbf{E}_{kl}^{ij} = \begin{cases} 1 & \text{if } (k, l) = (i, j) \text{ or } (k, l) = (j, i) \\ 0 & \text{otherwise.} \end{cases}$$

The *symmetric group* Sym_n is the subgroup scheme of $M_{n \times n}$ consisting of symmetric matrices. The *symplectic group* Sp_{2n} is defined as the subgroup scheme of GL_{2n} such that for every affine scheme $V = \text{Spec } R$

$$\text{Sp}_{2g}(V) = \{M \in \text{GL}_{2n}(R) \mid MJM^\top = J\}$$

where

$$J := \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}.$$

REMARK 1.1. As $J^2 = -\mathbf{1}_{2n}$, the condition $MJM^\top = J$ is equivalent to $M^{-1} = -JM^\top J$; thus $MJM^\top = J$ if and only if $M^\top JM = J$. In particular, if we write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n \times 2n}(R)$$

for some $A, B, C, D \in M_{n \times n}(R)$, then M is in $\text{Sp}_{2n}(R)$ if and only if one of the following two conditions is satisfied

- (1) $AB^\top = BA^\top$, $CD^\top = DC^\top$, and $AD^\top - BC^\top = \mathbf{1}_n$.
- (2) $A^\top C = C^\top A$, $B^\top D = D^\top B$, and $A^\top D - C^\top B = \mathbf{1}_n$.

Finally, the *Siegel parabolic subgroup* P_n of Sp_{2n} consists of matrices $(A \ B ; \ C \ D)$ in Sp_{2n} such that $C = 0$.

1.6.4. Let K be a subfield of \mathbf{C} and X be an algebraic variety over K (i.e. a reduced separated scheme of finite type over K). For any complex point $\bar{x} : \text{Spec } \mathbf{C} \rightarrow X$, if $x \in X$ denotes the point in the image of \bar{x} , and $k(x)$ denotes its residue field, we put

$$K(\bar{x}) := k(x).$$

Let us remark that

$$\text{trdeg}_K K(\bar{x}) = \min\{\dim Y \mid Y \text{ is an integral closed } K\text{-subscheme of } X \text{ such that } \bar{x} \in Y(\mathbf{C})\}.$$

2. Analytic families of complex tori, abelian varieties, and their uniformization

In this section we briefly transpose some of the standard theory of complex tori to a relative situation, that is, we shall consider analytic families of complex tori. To both simplify and shorten our exposition, we shall assume that the parameter space is smooth (i.e. a complex manifold); this largely suffices our needs.

Most of the material included in here, and in the following section, is well known to experts — and may be even considered as “classical” — but we could not find a convenient reference in the literature.

2.1. Relative complex tori. Let M be a complex manifold.

DEFINITION 2.1. A *(relative) complex torus over M* is a relative complex Lie group $\pi : X \rightarrow M$ over M such that π is proper with connected fibers. A morphism of complex tori over M is a morphism of relative complex Lie groups over M .

As any compact connected complex Lie group is a complex torus, every fiber of π in the above definition is a complex torus.

In general, for any relative complex Lie group $\pi : X \rightarrow M$ over M , we may consider its *relative Lie algebra* $\text{Lie}_M X$; this is a holomorphic vector bundle over M whose fiber at each $p \in M$ is the Lie algebra $\text{Lie } X_p$ of the complex torus $X_p := \pi^{-1}(p)$. Moreover, there exists a canonical morphism of relative complex Lie groups over M

$$\exp : \text{Lie}_M X \rightarrow X$$

restricting to the usual exponential map of complex Lie groups at each fiber.

LEMMA 2.2. *Let $\pi : X \rightarrow M$ be a complex torus over M . Then $\exp : \text{Lie}_M X \rightarrow X$ is a surjective submersion, and the sheaf of sections of the relative complex Lie group $\ker(\exp)$ over M is canonically isomorphic to*

$$R_1\pi_*\mathbf{Z}_X := (R^1\pi_*\mathbf{Z}_X)^\vee.$$

This follows from the classical case where M is a point via a fiber-by-fiber consideration (cf. [71] I.1). Note that $R_1\pi_*\mathbf{Z}_X$ is a local system of free abelian groups over M whose fiber at $p \in M$ is given by the first singular homology group $H_1(X_p, \mathbf{Z})$.

DEFINITION 2.3. Let V be a holomorphic vector bundle of rank g over M . By a *lattice* in V , we mean a subsheaf of abelian groups L of $\mathcal{O}_M(V)$ such that

- (1) L is a local system of free abelian groups of rank $2g$,
- (2) for each $p \in M$, the quotient V_p/L_p is compact.

It follows from Lemma 2.2 that, for any complex torus $\pi : X \rightarrow M$ of relative dimension g , $R_1\pi_*\mathbf{Z}_X$ may be canonically identified to a lattice in $\text{Lie}_M X$.

Conversely, if V is a holomorphic vector bundle of rank g over M and L is a lattice in V , then the étalé space $E(L)$ of L is a relative complex Lie subgroup of V over M and $X := V/E(L)$ is a complex torus over M of relative dimension g . Furthermore, the relative Lie algebra $\text{Lie}_M X$ gets canonically identified with V and, under this identification, $E(L)$ is the kernel of the exponential map $\exp : \text{Lie}_M X \rightarrow X$.

REMARK 2.4. The above reasoning actually proves that the category of complex tori over M of relative dimension g is equivalent to the category of couples (V, L) where V is a holomorphic vector bundle of rank g over M and L is a lattice in V ; a morphism $(V, L) \rightarrow (V', L')$ in this category is given by a morphism of holomorphic vector bundles $\varphi : V \rightarrow V'$ such that $\varphi(E(L)) \subset E(L')$.

In what follows, we shall drop the notation $E(L)$ and identify a local system with its étalé space.

2.2. Riemann forms and principally polarized complex tori. Let M be a complex manifold and $\pi : X \rightarrow M$ be a complex torus over M .

DEFINITION 2.5. A *Riemann form* over X is a C^∞ Hermitian metric⁴ H on the vector bundle $\text{Lie}_M X$ over M such that

$$E := \text{Im } H$$

takes integral values on $R_1\pi_*\mathbf{Z}_X$.

Observe that E is an alternating \mathbf{R} -bilinear form. We also remark that the Hermitian metric H is completely determined by E : for any sections v and w of $\text{Lie}_M X$ we have $H(v, w) = E(v, iw) + iE(v, w)$. In particular, by abuse, we may also say that E is Riemann form over X .

DEFINITION 2.6. With the above notations, we say that the Riemann form E is *principal* if the induced morphism of local systems

$$\begin{aligned} R_1\pi_*\mathbf{Z}_X &\longrightarrow (R_1\pi_*\mathbf{Z}_X)^\vee \cong R^1\pi_*\mathbf{Z}_X \\ \gamma &\longmapsto E(\gamma, \) \end{aligned}$$

is an isomorphism.

DEFINITION 2.7. Let M be a complex manifold. A *principally polarized complex torus* over M of relative dimension g is a couple (X, E) , where X is a complex torus over M of relative dimension g and E is a principal Riemann form over X .

EXAMPLE 2.8. Let $g \geq 1$ and consider the Siegel upper half-space

$$\mathbf{H}_g := \{\tau \in M_{g \times g}(\mathbf{C}) \mid \tau = \tau^\top, \text{Im } \tau > 0\}.$$

If $g = 1$, we denote $\mathbf{H} := \mathbf{H}_1$; this is the Poincaré upper half-plane. Let us consider the trivial vector bundle $V := \mathbf{C}^g \times \mathbf{H}_g$ over \mathbf{H}_g and let L be the subsheaf of $\mathcal{O}_{\mathbf{H}_g}(V)$ given by the image of the morphism of sheaves of abelian groups

$$\begin{aligned} (\mathbf{Z}^g \oplus \mathbf{Z}^g)_{\mathbf{H}_g} &\longrightarrow \mathcal{O}_{\mathbf{H}_g}(V) = \mathcal{O}_{\mathbf{H}_g}^{\oplus g} \\ (m, n) &\longmapsto m + \tau n \end{aligned}$$

where m and n are considered as column vectors of order g . Then L is a lattice in V and we denote by

$$\mathbf{p}_g : \mathbf{X}_g \longrightarrow \mathbf{H}_g$$

the corresponding complex torus over \mathbf{H}_g of relative dimension g (cf. Remark 2.4). Let E_g be imaginary part of the Hermitian metric over V given by

$$(v, w) \longmapsto \bar{v}^\top (\text{Im } \tau)^{-1} w.$$

One may easily verify that E_g takes integral values on L and that $\gamma \longmapsto E_g(\gamma, \)$ induces an isomorphism $L \xrightarrow{\sim} L^\vee$. We thus obtain a principally polarized complex torus (\mathbf{X}_g, E_g) over \mathbf{H}_g of relative dimension g .

2.3. The category \mathcal{T}_g of principally polarized complex tori of relative dimension g . Let Man/\mathbf{C} denote the category of complex manifolds. We define a category \mathcal{T}_g fibered in groupoids over Man/\mathbf{C} as follows.

- (1) An object of the category \mathcal{T}_g consists in a complex manifold M and a principally polarized complex torus (X, E) over M of relative dimension g ; we denote such an object by $(X, E)_{/M}$.
- (2) Let $(X, E)_{/M}$ and $(X', E')_{/M'}$ be objects of \mathcal{T}_g . A morphism

$$F_{/f} : (X', E')_{/M'} \longrightarrow (X, E)_{/M}$$

in \mathcal{T}_g is a cartesian diagram of complex manifolds

$$\begin{array}{ccc} X' & \xrightarrow{F} & X \\ \downarrow & \square & \downarrow \\ M' & \xrightarrow{f} & M \end{array}$$

4. Our convention is that Hermitian forms are anti-linear on the first coordinate and linear on the second.

preserving the identity sections of the complex tori and such that $E' = f^*E$ under the isomorphism of holomorphic vector bundles $\text{Lie}_{M'} X' \xrightarrow{\sim} f^* \text{Lie}_M X$ induced by F . We may also denote $(X', E') = (X, E) \times_M M'$.

- (3) The structural functor $\mathcal{T}_g \rightarrow \text{Man}/\mathbf{C}$ sends an object $(X, E)_{/M}$ of \mathcal{T}_g to the complex manifold M , and a morphism $F_{/f}$ as above to f .

EXAMPLE 2.9. We define an action of $\text{Sp}_{2g}(\mathbf{Z})$ on the object $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ of \mathcal{T}_g

$$\begin{aligned} \text{Sp}_{2g}(\mathbf{Z}) &\longrightarrow \text{Aut}_{\mathcal{T}_g}((\mathbf{X}_g, E_g)_{/\mathbf{H}_g}) \\ \gamma &\longmapsto F_{\gamma/f_\gamma} \end{aligned}$$

as follows. Recall that an element $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2g}(\mathbf{R})$ acts on \mathbf{H}_g by

$$\begin{aligned} f_\gamma : \mathbf{H}_g &\longrightarrow \mathbf{H}_g \\ \tau &\longmapsto \gamma \cdot \tau := (A\tau + B)(C\tau + D)^{-1}. \end{aligned}$$

For γ as above, consider the holomorphic map

$$\begin{aligned} \tilde{F}_\gamma : \mathbf{C}^g \times \mathbf{H}_g &\longrightarrow \mathbf{C}^g \times \mathbf{H}_g \\ (z, \tau) &\longmapsto ((j(\gamma, \tau)^\top)^{-1}z, \gamma \cdot \tau) \end{aligned}$$

where

$$j(\gamma, \tau) := C\tau + D \in \text{GL}_g(\mathbf{C}).$$

If $\gamma \in \text{Sp}_{2g}(\mathbf{Z})$, then for every $\tau \in \mathbf{H}_g$ we have

$$\tilde{F}_{\gamma, \tau}(\mathbf{Z}^g + \tau\mathbf{Z}^g) = \mathbf{Z}^g + (\gamma \cdot \tau)\mathbf{Z}^g,$$

so that \tilde{F}_γ induces a holomorphic map $F_\gamma : \mathbf{X}_g \rightarrow \mathbf{X}_g$. One easily verifies that

$$\begin{array}{ccc} \mathbf{X}_g & \xrightarrow{F_\gamma} & \mathbf{X}_g \\ p_g \downarrow & & \downarrow p_g \\ \mathbf{H}_g & \xrightarrow{f_\gamma} & \mathbf{H}_g \end{array}$$

is a cartesian diagram of complex manifolds preserving the identity sections and the Riemann forms E_g , i.e. it defines a morphism $F_{\gamma/f_\gamma} : (\mathbf{X}_g, E_g)_{/\mathbf{H}_g} \rightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ in \mathcal{T}_g . Finally, the formula

$$j(\gamma_1\gamma_2, \tau) = j(\gamma_1, \gamma_2 \cdot \tau)j(\gamma_2, \tau)$$

implies that F_{γ/f_γ} is in fact an automorphism of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ in \mathcal{T}_g and that $\gamma \mapsto F_{\gamma/f_\gamma}$ is a morphism of groups.⁵

2.4. De Rham cohomology of complex tori. Let M be a complex manifold and $\pi : X \rightarrow M$ be a complex torus over M of relative dimension g .

2.4.1. For any integer $i \geq 0$, we define the i th *analytic de Rham cohomology* sheaf of \mathcal{O}_M -modules by

$$\mathcal{H}_{\text{dR}}^i(X/M) := \mathbf{R}^i \pi_* \Omega_{X/M}^\bullet,$$

where $\Omega_{X/M}^\bullet$ is the complex of relative holomorphic differential forms. If M is a point, we denote $\mathcal{H}_{\text{dR}}^i(X) := \mathcal{H}_{\text{dR}}^i(X/M)$.

REMARK 2.10. If M is a point, the analytic de Rham cohomology $\mathcal{H}_{\text{dR}}^i(X)$ is canonically isomorphic to the quotient of the complex vector space of C^∞ closed i -forms over X with values in \mathbf{C} by the subspace of exact i -forms (cf. [29] I.1 p. 16).

⁵ Actually, it follows from Proposition 3.4 below (see also Remark 3.5) that $\gamma \mapsto F_{\gamma/f_\gamma}$ is an *isomorphism* of groups.

The arguments in [5] 2.5 prove, *mutatis mutandis*, that there is a canonical isomorphism of \mathcal{O}_M -modules given by cup product

$$\bigwedge^i \mathcal{H}_{\text{dR}}^1(X/M) \xrightarrow{\sim} \mathcal{H}_{\text{dR}}^i(X/M),$$

and that $\mathcal{H}_{\text{dR}}^1(X/M)$ is (the sheaf of sections of) a holomorphic vector bundle over M of rank $2g$. Moreover, the canonical \mathcal{O}_M -morphism $\pi_*\Omega_{X/M}^1 \rightarrow \mathcal{H}_{\text{dR}}^1(X/M)$ induces an isomorphism of $\pi_*\Omega_{X/M}^1$ onto a rank g subbundle of $\mathcal{H}_{\text{dR}}^1(X/M)$ that we denote by $\mathcal{F}^1(X/M)$.

Analogously, it follows from the arguments of [53] that $\mathcal{H}_{\text{dR}}^1(X/M)$ is equipped with a canonical integrable holomorphic connection

$$\nabla : \mathcal{H}_{\text{dR}}^1(X/M) \rightarrow \mathcal{H}_{\text{dR}}^1(X/M) \otimes_{\mathcal{O}_M} \Omega_M^1,$$

the *Gauss-Manin connection*.

Furthermore, the formation of $\mathcal{H}_{\text{dR}}^1(X/M)$, $\mathcal{F}^1(X/M)$, and ∇ , are compatible with every base change in M .

2.4.2. There is a canonical *comparison isomorphism* of holomorphic vector bundles

$$(2.1) \quad c : \text{Hom}_{\mathbf{Z}}(R_1\pi_*\mathbf{Z}_X, \mathcal{O}_M) \cong R^1\pi_*\mathbf{Z}_X \otimes_{\mathbf{Z}} \mathcal{O}_M \xrightarrow{\sim} \mathcal{H}_{\text{dR}}^1(X/M)$$

identifying the the local system of \mathbf{C} -vector spaces $\text{Hom}_{\mathbf{Z}}(R_1\pi_*\mathbf{Z}_X, \mathbf{C}_M) \cong R^1\pi_*\mathbf{C}_X$ with the subsheaf of $\mathcal{H}_{\text{dR}}^1(X/M)$ consisting of horizontal sections for the Gauss-Manin connection ([26] I Proposition 2.28 and II 7.6-7.7). The induced pairing

$$\begin{aligned} \mathcal{H}_{\text{dR}}^1(X/M) \otimes_{\mathbf{Z}} R_1\pi_*\mathbf{Z}_X &\rightarrow \mathcal{O}_M \\ \alpha \otimes \gamma &\mapsto c^{-1}(\alpha)(\gamma) =: \int_{\gamma} \alpha \end{aligned}$$

is given at each fiber by “integration of differential forms” (cf. Remark 2.10).

REMARK 2.11. In particular, for any section γ of $R_1\pi_*\mathbf{Z}_X$, any C^∞ section α of the vector bundle $\mathcal{H}_{\text{dR}}^1(X/M)$, and any holomorphic vector field θ on M , we have

$$\theta \left(\int_{\gamma} \alpha \right) = \int_{\gamma} \nabla_{\theta} \alpha.$$

Recall that $R_1\pi_*\mathbf{Z}_X$ may be naturally identified with a lattice in the holomorphic vector bundle $\text{Lie}_M X$. Accordingly, the dual bundle $(\text{Lie}_M X)^\vee$ gets naturally identified with a holomorphic subbundle of $\text{Hom}_{\mathbf{Z}}(R_1\pi_*\mathbf{Z}_X, \mathcal{O}_M)$.

LEMMA 2.12. *With notations as above, the comparison isomorphism (2.1) induces an isomorphism of the holomorphic vector bundle $(\text{Lie}_M X)^\vee$ onto $\mathcal{F}^1(X/M)$.*

This also follows from a fiber-by-fiber argument : if M is a point, by identifying $\mathcal{H}_{\text{dR}}^1(X)$ with the C^∞ de Rham cohomology with values in \mathbf{C} (Remark 2.10), the subspace $\mathcal{F}^1(X)$ gets identified with the space of $(1, 0)$ -forms in $\mathcal{H}_{\text{dR}}^1(X)$, and these correspond to $\text{Hom}_{\mathbf{C}}(\text{Lie } X, \mathbf{C})$ under the de Rham isomorphism (cf. [11] Theorem 1.4.1).

2.4.3. If X admits a principal Riemann form E , then, by linearity, we may define a holomorphic symplectic form $\langle \cdot, \cdot \rangle_E$ on the holomorphic vector bundle $\mathcal{H}_{\text{dR}}^1(X/M)$ over M (cf. [32] Appendix A) by

$$\langle E(\gamma, \cdot), E(\delta, \cdot) \rangle_E := \frac{1}{2\pi i} E(\gamma, \delta)$$

for any sections γ and δ of $R_1\pi_*\mathbf{Z}_X$, where $E(\gamma, \cdot)$ and $E(\delta, \cdot)$ are regarded as sections of $\mathcal{H}_{\text{dR}}^1(X/M)$ via the comparison isomorphism (2.1).

Since every section of $R^1\pi_*\mathbf{Z}_X$ is horizontal for the Gauss-Manin connection ∇ on $\mathcal{H}_{\text{dR}}^1(X/M)$ under the comparison isomorphism (2.1), the symplectic form $\langle \cdot, \cdot \rangle_E$ is compatible with ∇ : for every sections α, β of $\mathcal{H}_{\text{dR}}^1(X/M)$, and every holomorphic vector field θ on M , we have

$$(2.2) \quad \theta \langle \alpha, \beta \rangle_E = \langle \nabla_{\theta} \alpha, \beta \rangle_E + \langle \alpha, \nabla_{\theta} \beta \rangle_E.$$

2.5. Relative uniformization of complex abelian schemes. Let U be a smooth separated \mathbf{C} -scheme of finite type and (X, λ) be a principally polarized abelian scheme over U of relative dimension g . Denote by $p : X \rightarrow U$ its structural morphism. Then the associated analytic space U^{an} is a complex manifold, and the analytification $p^{\text{an}} : X^{\text{an}} \rightarrow U^{\text{an}}$ of p is a complex torus over U^{an} of relative dimension g .

Since the analytification of the coherent \mathcal{O}_U -module $H_{\text{dR}}^1(X/U)$ is canonically isomorphic to $\mathcal{H}_{\text{dR}}^1(X^{\text{an}}/U^{\text{an}})$, the symplectic form $\langle \cdot, \cdot \rangle_\lambda$ on $H_{\text{dR}}^1(X/U)$ defined in [32] 2.2 induces a symplectic form $\langle \cdot, \cdot \rangle_\lambda^{\text{an}}$ on the holomorphic vector bundle $\mathcal{H}_{\text{dR}}^1(X^{\text{an}}/U^{\text{an}})$ over U^{an} .

LEMMA 2.13. *Let γ and δ be sections of $R_1 p_*^{\text{an}} \mathbf{Z}_{X^{\text{an}}}$, and let α and β be sections of $\mathcal{H}_{\text{dR}}^1(X^{\text{an}}/U^{\text{an}})$ such that $\gamma = \langle \cdot, \alpha \rangle_\lambda^{\text{an}}$ and $\delta = \langle \cdot, \beta \rangle_\lambda^{\text{an}}$ under (the dual of) the comparison isomorphism (2.1). Then*

(1) *The formula*

$$E_\lambda(\gamma, \delta) := \frac{1}{2\pi i} \langle \alpha, \beta \rangle_\lambda^{\text{an}}$$

defines a Riemann form over X^{an} .

(2) *The holomorphic symplectic forms $\langle \cdot, \cdot \rangle_{E_\lambda}$ and $\langle \cdot, \cdot \rangle_\lambda^{\text{an}}$ over $\mathcal{H}_{\text{dR}}^1(X^{\text{an}}/U^{\text{an}})$ coincide.*

PROOF. We can assume $U = \text{Spec } \mathbf{C}$, so that (X, λ) is a principally polarized complex abelian variety.

Recall from [32] 2.2 that we have constructed an alternating bilinear form Q_λ on $H_{\text{dR}}^1(X/\mathbf{C})^\vee$, and that the bilinear form $\langle \cdot, \cdot \rangle_\lambda$ over $H_{\text{dR}}^1(X/\mathbf{C})$ is obtained from Q_λ by duality. Therefore, to prove (1), it is sufficient to prove that, under the identification of $H_1(X^{\text{an}}, \mathbf{Z})$ with an abelian subgroup of $H_{\text{dR}}^1(X/\mathbf{C})^\vee$ via (the dual of) the comparison isomorphism (2.1), for any elements γ and δ of $H_1(X^{\text{an}}, \mathbf{Z})$,

$$E_\lambda(\gamma, \delta) := \frac{1}{2\pi i} Q_\lambda(\gamma, \delta)$$

is in \mathbf{Z} , and that the induced morphism

$$(*) \quad \begin{aligned} H_1(X^{\text{an}}, \mathbf{Z}) &\longrightarrow \text{Hom}(H_1(X^{\text{an}}, \mathbf{Z}), \mathbf{Z}) \\ \gamma &\longmapsto E_\lambda(\gamma, \cdot) \end{aligned}$$

is an isomorphism of abelian groups.

Note that, with this definition, (2) is automatic, since for any $\gamma, \delta \in H_1(X^{\text{an}}, \mathbf{Z})$ we have

$$\begin{aligned} \langle E_\lambda(\gamma, \cdot), E_\lambda(\delta, \cdot) \rangle_{E_\lambda} &= \frac{1}{2\pi i} E_\lambda(\gamma, \delta) = \frac{1}{(2\pi i)^2} Q_\lambda(\gamma, \delta) = \frac{1}{(2\pi i)^2} \langle Q_\lambda(\gamma, \cdot), Q_\lambda(\delta, \cdot) \rangle_\lambda^{\text{an}} \\ &= \langle \frac{1}{2\pi i} Q_\lambda(\gamma, \cdot), \frac{1}{2\pi i} Q_\lambda(\delta, \cdot) \rangle_\lambda^{\text{an}} = \langle E_\lambda(\gamma, \cdot), E_\lambda(\delta, \cdot) \rangle_\lambda^{\text{an}}. \end{aligned}$$

where we identified the vector space $H_{\text{dR}}^1(X/\mathbf{C})$ with $\mathcal{H}_{\text{dR}}^1(X^{\text{an}})$ via the canonical analytification isomorphism.

Now, the topological Chern class $c_{1, \text{top}} : \text{Pic}(X) \rightarrow H^2(X^{\text{an}}, \mathbf{Z})$, defined via the exponential sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{Z}_{X^{\text{an}}} \longrightarrow \mathcal{O}_{X^{\text{an}}} \longrightarrow \mathcal{O}_{X^{\text{an}}}^\times \longrightarrow 0 \\ f \longmapsto \exp(2\pi i f) \end{aligned}$$

and the de Rham Chern class $c_{1, \text{dR}} : \text{Pic}(X) \rightarrow H_{\text{dR}}^2(X/\mathbf{C})$ (cf. [32] 2.2) are related by the following commutative diagram (cf. [27] 2.2.5.2)

$$\begin{array}{ccc} \text{Pic}(X) & \xrightarrow{c_{1, \text{dR}}} & H_{\text{dR}}^2(X/\mathbf{C}) \\ c_{1, \text{top}} \downarrow & & \downarrow \\ H^2(X^{\text{an}}, \mathbf{Z}) & \xrightarrow{-2\pi i} & H^2(X^{\text{an}}, \mathbf{C}) \end{array}$$

where the arrow $H_{\text{dR}}^2(X/\mathbf{C}) \rightarrow H^2(X^{\text{an}}, \mathbf{C}) \cong \text{Hom}(H_2(X^{\text{an}}, \mathbf{Z}), \mathbf{C})$ is given by the comparison isomorphism.

If \mathcal{L} is an ample line bundle on X inducing λ , then $Q_\lambda = c_{1, \text{dR}}(\mathcal{L})$ under the identification $H_{\text{dR}}^2(X/\mathbf{C})$ with the vector space of alternating bilinear forms on $H_{\text{dR}}^1(X/\mathbf{C})^\vee$ (cf. [32] proof of Lemma 2.1). By the commutativity of the above diagram, we see that $E_\lambda = -c_{1, \text{top}}(\mathcal{L})$ under the identification of $H^2(X^{\text{an}}, \mathbf{Z})$

with the module of alternating (integral) bilinear forms on $H_1(X^{\text{an}}, \mathbf{Z})$. This proves that E_λ takes integral values.

To prove that $(*)$ is an isomorphism, we simply use the fact that λ^{an} is an isomorphism of X^{an} onto its dual torus, hence the determinant of the bilinear form on $H_1(X^{\text{an}}, \mathbf{Z})$ induced by $c_{1,\text{top}}(\mathcal{L})$ is 1 (cf. [11] 2.4.9). \blacksquare

Thus, for any smooth separated \mathbf{C} -scheme of finite type U and any principally polarized abelian scheme (X, λ) over U of relative dimension g , the above construction gives a principally polarized complex torus $(X^{\text{an}}, E_\lambda)$ over U^{an} of relative dimension g .

Recall from [32] 3.1 that we denote by $\mathcal{A}_{g,\mathbf{C}}$ the moduli stack over \mathbf{C} of principally polarized abelian schemes of relative dimension g over \mathbf{C} -schemes. Let SmVar/\mathbf{C} be the full subcategory of Sch/\mathbf{C} consisting of smooth separated \mathbf{C} -schemes of finite type, and $\mathcal{A}_{g,\mathbf{C}}^{\text{sm}}$ be the full subcategory of $\mathcal{A}_{g,\mathbf{C}}$ consisting of objects $(X, \lambda)_{/U}$ of $\mathcal{A}_{g,\mathbf{C}}$ such that U is an object of SmVar/\mathbf{C} .

We can summarize this paragraph by remarking that we have constructed a “relative uniformization functor” $\mathcal{A}_{g,\mathbf{C}}^{\text{sm}} \rightarrow \mathcal{T}_g$ making the diagram

$$\begin{array}{ccc} \mathcal{A}_{g,\mathbf{C}}^{\text{sm}} & \longrightarrow & \mathcal{T}_g \\ \downarrow & & \downarrow \\ \text{SmVar}/\mathbf{C} & \longrightarrow & \text{Man}/\mathbf{C} \end{array}$$

(strictly) commutative, where $\text{SmVar}/\mathbf{C} \rightarrow \text{Man}/\mathbf{C}$ is the classical analytification functor $U \mapsto U^{\text{an}}$.

REMARK 2.14. One can prove that the above diagram is “cartesian” in the sense that it induces an equivalence of categories between $\mathcal{A}_{g,\mathbf{C}}^{\text{sm}}$ and the full subcategory of \mathcal{T}_g formed by the objects lying above the essential image of the analytification functor $\text{SmVar}/\mathbf{C} \rightarrow \text{Man}/\mathbf{C}$ (cf. [27] Rappel 4.4.3 and [12] Theorem 3.10). In particular, for any object U of SmVar/\mathbf{C} and any principally polarized complex torus (X', E) over U^{an} of relative dimension g , there exists up to isomorphism a unique principally polarized abelian scheme (X, λ) over U of relative dimension g such that $(X', E)_{/U^{\text{an}}}$ is isomorphic to $(X^{\text{an}}, E_\lambda)_{/U^{\text{an}}}$ in $\mathcal{T}_g(U^{\text{an}})$. In this paper, we shall only need this algebraization result when $U = \text{Spec } \mathbf{C}$, which is classical (cf. [71] Corollary p. 35).

3. Analytic moduli spaces of complex abelian varieties with a symplectic-Hodge basis

In this section we consider some moduli problems of principally polarized complex tori, regarded as functors

$$\mathcal{T}_g^{\text{op}} \rightarrow \text{Set}$$

where \mathcal{T}_g is the category fibered in groupoids over the category of complex manifolds Man/\mathbf{C} defined in 2.3.

Recall that we denote by B_g the smooth quasi-projective scheme over $\mathbf{Z}[1/2]$ representing the stack $\mathcal{B}_g \otimes_{\mathbf{Z}} \mathbf{Z}[1/2]$ (see [32] Theorem 4.1). We shall prove in particular that the complex manifold $B_g(\mathbf{C}) = B_{g,\mathbf{C}}^{\text{an}}$ is a fine moduli space in the analytic category for principally polarized complex abelian varieties of dimension g endowed with a symplectic-Hodge basis.

3.1. Descent of principally polarized complex tori. Let M be a complex manifold and (X, E) be a principally polarized complex torus over M of relative dimension g .

If M_0 is another complex manifold and $M \rightarrow M_0$ is a holomorphic map, we say that (X, E) *descends* to M_0 if there exists a principally polarized complex torus (X_0, E_0) over M_0 and a morphism $(X, E)_{/M} \rightarrow (X_0, E_0)_{/M_0}$ in \mathcal{T}_g , i.e. a cartesian diagram of complex manifolds

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow & \square & \downarrow \\ M & \longrightarrow & M_0 \end{array}$$

which is compatible with both the identity sections and the principal Riemann forms (cf. Paragraph 2.3).

LEMMA 3.1. *With the above notations, suppose that there exists a proper and free left action of a discrete group Γ on M . If the action of Γ on M lifts to an action of Γ on $(X, E)_{/M}$ in the category \mathcal{T}_g , then $(X, E)_{/M}$ descends to a principally polarized complex torus over the quotient $\Gamma \backslash M$.*

SKETCH OF THE PROOF. Consider X as a pair (V, L) , where V is a holomorphic vector bundle over M of rank g , and L is a lattice in V (cf. Remark 2.4). Then, to every $\gamma \in \Gamma$ there is associated a holomorphic map $\varphi_\gamma : V \rightarrow V$ making the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi_\gamma} & V \\ \downarrow & & \downarrow \\ M & \xrightarrow{\gamma} & M \end{array}$$

commute, and compatible with the vector bundle structures. It follows from the commutativity of this diagram that the action of Γ on V is also proper and free. Thus, there exists a unique holomorphic vector bundle structure on the complex manifold $\Gamma \backslash V$ over $\Gamma \backslash M$ such that the canonical holomorphic map $V \rightarrow \Gamma \backslash V$ induces a vector bundle isomorphism of V onto the pullback to M of the vector bundle $\Gamma \backslash V$ over $\Gamma \backslash M$.

Analogously, one descends the lattice L to a lattice in $\Gamma \backslash V$ (consider the étale space, for instance), and the bilinear form E on V to a bilinear form on $\Gamma \backslash V$, which is seen to be a principal polarization *a posteriori*. \blacksquare

3.2. Integral symplectic bases over complex tori. Let M be a complex manifold and (X, E) be a principally polarized complex torus over M of relative dimension g . We denote by $\pi : X \rightarrow M$ its structural morphism.

DEFINITION 3.2. An *integral symplectic basis* of $(X, E)_{/M}$ is a trivializing $2g$ -uple $(\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$ of global sections of $R_1\pi_*\mathbf{Z}_X$ which is symplectic with respect to the Riemann form E , that is,

$$E(\gamma_i, \gamma_j) = E(\delta_i, \delta_j) = 0 \quad \text{and} \quad E(\gamma_i, \delta_j) = \delta_{ij}$$

for any $1 \leq i, j \leq g$.

EXAMPLE 3.3. Consider the principally polarized complex torus (\mathbf{X}_g, E_g) over \mathbf{H}_g of Example 2.8 and recall that a section of $R_1\mathbf{p}_{g*}\mathbf{Z}_{\mathbf{X}_g}$ is given by a column vector of holomorphic functions on \mathbf{H}_g of the form $\tau \mapsto m + \tau n$, for some sections (m, n) of $(\mathbf{Z}^g \oplus \mathbf{Z}^g)_{\mathbf{H}_g}$. We can thus define an integral symplectic basis

$$\beta_g = (\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$$

of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ by

$$\gamma_i(\tau) := \mathbf{e}_i \quad \text{and} \quad \delta_i(\tau) := \tau \mathbf{e}_i$$

for any $\tau \in \mathbf{H}_g$.

Let $(X', E')_{/M'}$ and $(X, E)_{/M}$ be objects of \mathcal{T}_g with structural morphisms $\pi' : X' \rightarrow M'$ and $\pi : X \rightarrow M$. If $F/f : (X', E')_{/M'} \rightarrow (X, E)_{/M}$ is a morphism in \mathcal{T}_g , then the isomorphism of vector bundles

$$(3.1) \quad \text{Lie}_{M'} X' \xrightarrow{\sim} f^* \text{Lie}_M X$$

induced by F identifies the lattice $R_1\pi'_*\mathbf{Z}_{X'}$ with $f^*R_1\pi_*\mathbf{Z}_X$. If γ is a section of $R_1\pi_*\mathbf{Z}_X$, we denote by $F^*\gamma$ the section of $R_1\pi'_*\mathbf{Z}_{X'}$ mapping to $f^*\gamma$ under (3.1). As the isomorphism (3.1) also preserves the corresponding Riemann forms, for any integral symplectic basis $(\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$ of $(X, E)_{/M}$, the $2g$ -uple of global sections of $R_1\pi'_*\mathbf{Z}_{X'}$ given by

$$F^*\beta := (F^*\gamma_1, \dots, F^*\gamma_g, F^*\delta_1, \dots, F^*\delta_g)$$

is an integral symplectic basis of $(X', E')_{/M'}$.

PROPOSITION 3.4 (cf. [11] Proposition 8.1.2). *The functor $\mathcal{T}_g^{\text{op}} \rightarrow \text{Set}$ sending an object $(X, E)_{/M}$ of \mathcal{T}_g to the set of integral symplectic bases of $(X, E)_{/M}$ is representable by $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$, with universal integral symplectic basis β_g defined in Example 3.3.*

PROOF. Let $(X, E)_{/M}$ be an object of \mathcal{T}_g with structural morphism $\pi : X \rightarrow M$, and let $\beta = (\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$ be an integral symplectic basis of $(X, E)_{/M}$. Let W be the real subbundle of $\text{Lie}_M X$ generated by $\gamma_1, \dots, \gamma_g$. Since E is the imaginary part of a Hermitian metric, for any nontrivial section γ of W , we have $E(\gamma, i\gamma) \neq 0$. As W is isotropic with respect to E , it follows that $\text{Lie}_M X = W \oplus iW$ as a real vector bundle. In particular, $\gamma := (\gamma_1, \dots, \gamma_g)$ trivializes $\text{Lie}_M X$ as a holomorphic vector bundle. Hence, if $\delta := (\delta_1, \dots, \delta_g)$, then there exists a unique holomorphic map $\tau : M \rightarrow \text{GL}_g(\mathbf{C})$ such that $\delta = \gamma\tau$, where γ and δ are regarded as row vectors of global holomorphic sections of $\text{Lie}_M X$.

Let $A := (E(\gamma_k, i\gamma_l))_{1 \leq k, l \leq g} \in M_{g \times g}(\mathbf{C})$. Since

$$\delta = \gamma \text{Re } \tau + i\gamma \text{Im } \tau,$$

the matrix of E in the basis β is given by

$$\begin{pmatrix} 0 & A \text{Im } \tau \\ -(A \text{Im } \tau)^\top & (\text{Re } \tau)^\top A \text{Im } \tau - (\text{Im } \tau)^\top A^\top \text{Re } \tau \end{pmatrix}.$$

Using that β is symplectic with respect to E , and that A is symmetric and positive-definite (recall that E is the imaginary part of a Hermitian metric), we conclude that τ factors through $\mathbf{H}_g \subset \text{GL}_g(\mathbf{C})$.

Finally, writing X as the quotient of $\text{Lie}_M X$ by $R_1\pi_*\mathbf{Z}_X$, we see that τ lifts to a unique morphism in \mathcal{T}_g

$$F/\tau : (X, E)_{/M} \rightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$$

satisfying $F^*\beta_g = \beta$. ■

REMARK 3.5. We may define a *left* action of the group $\text{Sp}_{2g}(\mathbf{Z})$ on the functor $\mathcal{T}_g^{\text{op}} \rightarrow \text{Set}$ of integral symplectic bases, considered in the above proposition, as follows. Let $(X, E)_{/U}$ be an object of \mathcal{T}_g and β be an integral symplectic basis of $(X, E)_{/U}$. Let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2g}(\mathbf{Z})$, and consider $\beta = (\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$ as a row vector of order $2g$; then we define

$$\gamma \cdot \beta := \begin{pmatrix} \gamma_1 & \cdots & \gamma_g & \delta_1 & \cdots & \delta_g \end{pmatrix} \begin{pmatrix} D^\top & B^\top \\ C^\top & A^\top \end{pmatrix}$$

The morphism

$$F_{\gamma/f_\gamma} : (\mathbf{X}_g, E_g)_{/\mathbf{H}_g} \rightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$$

defined in Example 2.9 is the unique morphism in \mathcal{T}_g satisfying

$$F_{\gamma}^*\beta_g = \gamma \cdot \beta_g.$$

3.3. Principal (symplectic) level structures.

3.3.1. Let U be a scheme, and X be an abelian scheme over U . Recall that, for any integer $n \geq 1$, we may define a natural pairing, the so-called *Weil pairing*,

$$X[n] \times X^t[n] \rightarrow \mu_{n,U},$$

where $\mu_{n,U}$ denotes the U -group scheme of n th roots of unity (cf. [71] IV.20).

Fix an integer $n \geq 1$, and let $\zeta_n \in \mathbf{C}$ be the n th root of unity $e^{\frac{2\pi i}{n}}$. For any scheme U over $\mathbf{Z}[1/n, \zeta_n]$, and any principally polarized abelian scheme (X, λ) over U of relative dimension g , by identifying $X^t[n]$ with $X[n]$ via λ , and $\mu_{n,U}$ with $(\mathbf{Z}/n\mathbf{Z})_U$ via ζ_n , we obtain a pairing

$$e_n^\lambda : X[n] \times X[n] \rightarrow (\mathbf{Z}/n\mathbf{Z})_U.$$

The formation of e_n^λ is compatible with every base change in U . Moreover, e_n^λ is skew-symmetric and non-degenerate (cf. [71] IV.23).

Since, for any integer $n \geq 3$, there exists a fine moduli space $A_{g,1,n}$ over $\mathbf{Z}[1/n]$ for principally polarized abelian varieties of dimension g endowed with a full level n -structure (see [72] Theorem 7.9, and the following remark; see also [65] Théorème VII.3.2), there also exists a fine moduli space $A_{g,n}$ over $\mathbf{Z}[1/n, \zeta_n]$ for principally polarized abelian varieties (X, λ) of dimension g endowed with a symplectic basis of $X[n]$ for the pairing e_n^λ (cf. [31] IV.6). The scheme $A_{g,n}$ is quasi-projective and smooth over $\mathbf{Z}[1/n, \zeta_n]$, with connected fibers. In the sequel, we denote the universal principally polarized abelian scheme over $A_{g,n}$ by $(X_{g,n}, \lambda_{g,n})$, and the universal symplectic basis of $X_{g,n}[n]$ by $\alpha_{g,n}$.

3.3.2. Let $(X, E)_{/M}$ be an object of \mathcal{T}_g with structural morphism $\pi : X \rightarrow M$. For any integer $n \geq 1$, by an *integral symplectic basis modulo n* of $(X, E)_{/M}$, we mean a $2g$ -uple of global sections of the local system of $\mathbf{Z}/n\mathbf{Z}$ -modules

$$R_1\pi_*(\mathbf{Z}/n\mathbf{Z})_X = R_1\pi_*\mathbf{Z}_X/nR_1\pi_*\mathbf{Z}_X$$

which is symplectic with respect to the alternating $\mathbf{Z}/n\mathbf{Z}$ -linear form on $R_1\pi_*(\mathbf{Z}/n\mathbf{Z})_X$ induced by E .

REMARK 3.6. Every integral symplectic basis of $(X, E)_{/M}$ induces an integral symplectic basis modulo n of $(X, E)_{/M}$. Conversely, since the natural map $\mathrm{Sp}_{2g}(\mathbf{Z}) \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$ is surjective, locally on M , every integral symplectic basis modulo n of $(X, E)_{/M}$ can be lifted to an integral symplectic basis of $(X, E)_{/M}$.

The notion of integral symplectic bases modulo n is compatible with the notion of principal level n structures of 3.3.1 in the following sense. Let $(X, \lambda)_{/U}$ be an object of $\mathcal{A}_{g, \mathbf{C}}^{\mathrm{sm}}$ (see 2.5) with structural morphism $p : X \rightarrow U$. The étalé space of the local system $R_1p_*^{\mathrm{an}}(\mathbf{Z}/n\mathbf{Z})_{X^{\mathrm{an}}}$ is canonically isomorphic to the n -torsion Lie subgroup $X^{\mathrm{an}}[n]$ of X^{an} . Under this identification, the pairing e_n^λ on $X[n]$ coincides, up to a sign, with the reduction modulo n of the Riemann form E_λ (cf. [71] IV.23 and IV.24), and thus an integral symplectic basis modulo n of $(X^{\mathrm{an}}, E_\lambda)_{/U^{\mathrm{an}}}$ canonically corresponds to a symplectic trivialization of $X^{\mathrm{an}}[n]$ with respect to e_n^λ .

3.3.3. Let $\Gamma(n)$ the kernel of the natural map $\mathrm{Sp}_{2g}(\mathbf{Z}) \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$. Recall that for any $n \geq 3$ the induced action of $\Gamma(n)$ on \mathbf{H}_g is free ([71] IV.21 Theorem 5) and proper.

The following proposition is well known.

PROPOSITION 3.7. *For any integer $n \geq 3$, the complex manifold $A_{g,n}(\mathbf{C}) = A_{g,n,\mathbf{C}}^{\mathrm{an}}$ is canonically biholomorphic to the quotient of \mathbf{H}_g by $\Gamma(n)$, and the functor $\mathcal{T}_g^{\mathrm{op}} \rightarrow \mathbf{Set}$ sending an object $(X, E)_{/M}$ of \mathcal{T}_g to the set of integral symplectic basis modulo n of $(X, E)_{/M}$ is representable by $(X_{g,n,\mathbf{C}}^{\mathrm{an}}, E_{\lambda_{g,n}})_{/A_{g,n,\mathbf{C}}^{\mathrm{an}}}$.*

PROOF. As the action of $\Gamma(n)$ on \mathbf{H}_g is proper and free, the quotient

$$\mathbf{A}_{g,n} := \Gamma(n) \backslash \mathbf{H}_g$$

is a complex manifold, and the canonical holomorphic map $\mathbf{H}_g \rightarrow \mathbf{A}_{g,n}$ is a covering map with Galois group $\Gamma(n)$. Moreover, since the action of $\Gamma(n)$ on \mathbf{H}_g lifts to an action of $\Gamma(n)$ on $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ in the category \mathcal{T}_g , the principally polarized complex torus (\mathbf{X}_g, E_g) over \mathbf{H}_g descends to a principally polarized complex torus $(\mathbf{X}_{g,n}, E_{g,n})$ over $\mathbf{A}_{g,n}$ (Lemma 3.1).

Let $\bar{\beta}_g$ be the integral symplectic basis modulo n of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ obtained from β_g by reduction modulo n . Then $\bar{\beta}_g$ is invariant under the action of $\Gamma(n)$, and thus it descends to an integral symplectic basis modulo n of $(\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$, say $\beta_{g,n}$.

The object $(\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$ of \mathcal{T}_g so constructed represents the functor in the statement with $\beta_{g,n}$ serving as universal symplectic basis modulo n . Indeed, let $(X, E)_{/M}$ be an object of \mathcal{T}_g , and β be an integral symplectic basis modulo n of $(X, E)_{/M}$. By Remark 3.6, there exists an open covering $M = \bigcup_{i \in I} U^i$ and, for each $i \in I$, an integral symplectic basis β^i of $(X, E)_{/U^i}$ lifting β . By Proposition 3.4, we obtain for each $i \in I$ a morphism $F_{/f^i}^i : (X, E)_{/U^i} \rightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ in \mathcal{T}_g satisfying $(F^i)^*\beta_g = \beta^i$. Finally, by construction, for any $i, j \in I$, the compositions of $F_{/f^i}^i$ and $F_{/f^j}^j$ with the projection $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g} \rightarrow (\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$ agree over the intersection $U^i \cap U^j$; hence they glue to a morphism

$$F_{/f} : (X, E)_{/M} \rightarrow (\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$$

satisfying $F^*\beta_{g,n} = \beta$, and uniquely determined by this property.

To finish the proof, it is sufficient to show that $(X_{g,n,\mathbf{C}}^{\mathrm{an}}, E_{\lambda_{g,n}})_{/A_{g,n,\mathbf{C}}^{\mathrm{an}}}$ is isomorphic to $(\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$ in the category \mathcal{T}_g . By the compatibility of principal level n structures with integral symplectic bases modulo n , there exists a unique morphism in \mathcal{T}_g

$$F_{/f} : (X_{g,n,\mathbf{C}}^{\mathrm{an}}, E_{\lambda_{g,n}})_{/A_{g,n,\mathbf{C}}^{\mathrm{an}}} \rightarrow (\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$$

such that $F^*\beta_{g,n}$ is the integral symplectic basis modulo n of $(X_{g,n,\mathbf{C}}^{\mathrm{an}}, E_{\lambda_{g,n}})_{/A_{g,n,\mathbf{C}}^{\mathrm{an}}}$ associated to $\alpha_{g,n}$ (the universal principal level n structure of $(X_{g,n}, \lambda_{g,n})_{/A_{g,n}}$). Since complex tori (over a point) endowed with a

principal Riemann form are algebraizable (cf. Remark 2.14), the holomorphic map

$$f : A_{g,n}(\mathbf{C}) = A_{g,n,\mathbf{C}}^{\text{an}} \longrightarrow \mathbf{A}_{g,n}$$

is bijective. As the complex manifolds $\mathbf{A}_{g,n}$ and $A_{g,n}(\mathbf{C})$ have same dimension, f is necessarily a biholomorphism ([39] p. 19). \blacksquare

3.4. Symplectic-Hodge bases over complex tori.

3.4.1. Let M be a complex manifold and (X, E) be a principally polarized complex torus over M of relative dimension g . As in [32] Definition 2.4, by a *symplectic-Hodge basis* of $(X, E)_{/M}$, we mean a $2g$ -uple $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ of global sections of the holomorphic vector bundle $\mathcal{H}_{\text{dR}}^1(X/M)$ such that $\omega_1, \dots, \omega_g$ are sections of the subbundle $\mathcal{F}^1(X/M)$, and b is symplectic with respect to the holomorphic symplectic form $\langle \cdot, \cdot \rangle_E$.

It follows from Lemma 2.13 that this notion of symplectic-Hodge basis is compatible with its algebraic counterpart ([32] Definition 2.4) via the ‘‘relative uniformization functor’’ in 2.5.

3.4.2. Consider Siegel parabolic subgroup of $\text{Sp}_{2g}(\mathbf{C})$

$$P_g(\mathbf{C}) = \left\{ \begin{pmatrix} A & B \\ 0 & (A^\top)^{-1} \end{pmatrix} \in M_{2g \times 2g}(\mathbf{C}) \mid A \in \text{GL}_g(\mathbf{C}) \text{ and } B \in M_{g \times g}(\mathbf{C}) \text{ satisfy } AB^\top = BA^\top \right\}.$$

Note that $P_g(\mathbf{C})$ is a complex Lie group of dimension $g(3g+1)/2$.

Let (X, E) be a principally polarized complex torus of dimension g . If $b = (\omega \ \eta)$ is a symplectic-Hodge basis of (X, E) , seen as a row vector of order $2g$ with coefficients in $\mathcal{H}_{\text{dR}}^1(X)$, and $p = (A \ B ; 0 \ (A^\top)^{-1}) \in P_g(\mathbf{C})$, then we put

$$b \cdot p := (\ \omega A \ \ \omega B + \eta(A^\top)^{-1} \)$$

It is easy to check that $b \cdot p$ is a symplectic-Hodge basis of (X, E) , and that the above formula defines a free and transitive action of $P_g(\mathbf{C})$ on the set of symplectic-Hodge bases of (X, E) .

3.4.3. For a complex manifold M , let us denote by $\text{Man}_{/M}$ the category of complex manifolds endowed with a holomorphic map to M .

LEMMA 3.8 (cf. [32] Corollary 3.4). *Let M be a complex manifold and (X, E) be a principally polarized complex torus over M of relative dimension g . The functor*

$$\begin{aligned} \text{Man}_{/M}^{\text{op}} &\longrightarrow \text{Set} \\ M' &\longmapsto \{ \text{symplectic-Hodge bases of } (X, E) \times_M M' \} \end{aligned}$$

is representable by a principal $P_g(\mathbf{C})$ -bundle $B(X, E)$ over M .

PROOF. Let us denote by $\pi : V \longrightarrow M$ the holomorphic vector bundle $\mathcal{H}_{\text{dR}}^1(X/M)^{\oplus g}$ over M . For any $p \in M$, the fiber $\pi^{-1}(p) = V_p$ is the vector space of g -uples $(\alpha_1, \dots, \alpha_g)$, with each $\alpha_i \in \mathcal{H}_{\text{dR}}^1(X_p)$. Let B be the locally closed analytic subspace of V consisting of points $v = (\alpha_1, \dots, \alpha_g)$ of V such that

$$L := \mathbf{C}\alpha_1 + \dots + \mathbf{C}\alpha_g$$

is a Lagrangian subspace of $\mathcal{H}_{\text{dR}}^1(X_{\pi(v)})$ with respect to $\langle \cdot, \cdot \rangle_{E_{\pi(v)}}$ satisfying

$$\mathcal{F}^1(X_{\pi(v)}) \oplus L = \mathcal{H}_{\text{dR}}^1(X_{\pi(v)}).$$

By [32] Proposition A.7. (2), a symplectic-Hodge basis $(\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ of a principally polarized complex torus is uniquely determined by (η_1, \dots, η_g) . In particular, for each $p \in M$, the fiber $B_p = B \cap V_p$ may be naturally identified with the set of symplectic-Hodge bases of (X_p, E_p) .

Thus, it follows from 3.4.2 that B is a principal $P_g(\mathbf{C})$ -bundle over M ; in particular, it is a complex manifold. We also conclude from the above paragraph that B represents the functor in the statement. \blacksquare

REMARK 3.9. The above construction is compatible, under analytification, with its algebraic counterpart. Namely, let U be a smooth separated \mathbf{C} -scheme of finite type, and (X, λ) be a principally polarized abelian scheme over U . The complex manifold $B(X^{\text{an}}, E_\lambda)$ over U^{an} constructed in Lemma 3.8 is canonically isomorphic to the analytification of the scheme $B(X, \lambda)$ over U constructed in [32] Corollary 3.4.

Recall that we denote by (X_g, λ_g) the universal principally polarized abelian scheme over B_g , and by b_g the universal symplectic-Hodge basis of $(X_g, \lambda_g)_{/B_g}$.

PROPOSITION 3.10. *The functor $\mathcal{T}_g^{\text{op}} \rightarrow \text{Set}$ sending an object $(X, E)_{/M}$ of \mathcal{T}_g to the set of symplectic-Hodge bases of $(X, E)_{/M}$ is representable by $(X_{g, \mathbf{C}}^{\text{an}}, E_{\lambda_g})_{/B_{g, \mathbf{C}}^{\text{an}}}$, with universal symplectic-Hodge basis b_g .*

PROOF. By Lemma 3.8, there exists a complex manifold $\mathbf{B}_g := B(\mathbf{X}_g, E_g)$ over \mathbf{H}_g representing the functor

$$\begin{aligned} \text{Man}_{/\mathbf{H}_g}^{\text{op}} &\longrightarrow \text{Set} \\ M &\longmapsto \{\text{symplectic-Hodge bases of } (\mathbf{X}_g, E_g) \times_{\mathbf{H}_g} M\} \end{aligned}$$

Let $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g}) = (\mathbf{X}_g, E_g) \times_{\mathbf{H}_g} \mathbf{B}_g$. Note that the principally polarized complex torus $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})$ over \mathbf{B}_g is equipped with a universal symplectic-Hodge basis $b_{\mathbf{B}_g}$, and with an integral symplectic basis $\beta_{\mathbf{B}_g}$ obtained by pullback from β_g via the canonical morphism $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})_{/\mathbf{B}_g} \rightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ in \mathcal{T}_g .

We now remark that $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})_{/\mathbf{B}_g}$ represents the functor $\mathcal{T}_g^{\text{op}} \rightarrow \text{Set}$ sending an object $(X, E)_{/M}$ of \mathcal{T}_g to the cartesian product of the set of symplectic-Hodge bases of $(X, E)_{/M}$ with the set of integral symplectic bases of $(X, E)_{/M}$, with $(b_{\mathbf{B}_g}, \beta_{\mathbf{B}_g})$ serving as a universal object. Thus, for any element $\gamma \in \text{Sp}_{2g}(\mathbf{Z})$, there exists a unique automorphism $\Psi_{\gamma/\psi_\gamma}$ of $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})_{/\mathbf{B}_g}$ in \mathcal{T}_g such that $\Psi_\gamma^* b_{\mathbf{B}_g} = b_{\mathbf{B}_g}$ and $\Psi_\gamma^* \beta_{\mathbf{B}_g} = \gamma \cdot \beta_{\mathbf{B}_g}$ (where the left action of $\text{Sp}_{2g}(\mathbf{Z})$ on integral symplectic bases is defined as in Remark 3.5).

As the functor $\underline{B}_g : \mathcal{A}_g^{\text{op}} \rightarrow \text{Set}$ is rigid over \mathbf{C} ([32] Lemma 4.3), we see that

- (1) $\gamma \mapsto \Psi_{\gamma/\psi_\gamma}$ is in fact an action of $\text{Sp}_{2g}(\mathbf{Z})$ on $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})_{/\mathbf{B}_g}$ in the category \mathcal{T}_g , and
- (2) the action $\gamma \mapsto \psi_\gamma$ of $\text{Sp}_{2g}(\mathbf{Z})$ on the complex manifold \mathbf{B}_g is free; it is also proper since it lifts the action on \mathbf{H}_g .

Let M be the quotient manifold $\text{Sp}_{2g}(\mathbf{Z}) \backslash \mathbf{B}_g$ and descend $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})$ to a principally polarized complex torus (X, E) over M . Since $b_{\mathbf{B}_g}$ is invariant under the action of $\text{Sp}_{2g}(\mathbf{Z})$, we can descend it to a symplectic-Hodge basis b of $(X, E)_{/M}$. As in the proof of Proposition 3.7, we may check that $(X, E)_{/M}$ represents the functor in the statement, with b serving as universal symplectic-Hodge basis.

To finish the proof, we must prove that $(X, E)_{/M}$ is isomorphic to $(X_{g, \mathbf{C}}^{\text{an}}, E_{\lambda_g})_{/B_{g, \mathbf{C}}^{\text{an}}}$ in \mathcal{T}_g . For this, it is sufficient to remark that, by the universal property of $(X, E)_{/M}$, there exists a unique morphism in \mathcal{T}_g

$$F/f : (X_{g, \mathbf{C}}^{\text{an}}, E_{\lambda_g})_{/B_{g, \mathbf{C}}^{\text{an}}} \rightarrow (X, E)_{/M}$$

satisfying $F^*b = b_g$, and that the holomorphic map

$$f : B_g(\mathbf{C}) = B_{g, \mathbf{C}}^{\text{an}} \rightarrow M$$

is bijective since principally polarized complex tori (over a point) are algebraizable (cf. Remark 2.14); then f is necessarily a biholomorphism ([39] p. 19). ■

4. The higher Ramanujan equations and their analytic solution φ_g

Fix an integer $g \geq 1$. Let us consider the holomorphic coordinate system $(\tau_{kl})_{1 \leq k \leq l \leq g}$ on the complex manifold \mathbf{H}_g , where $\tau_{kl} : \mathbf{H}_g \rightarrow \mathbf{C}$ associates to any $\tau \in \mathbf{H}_g$ its entry in the k th row and l th column. To this system of coordinates is attached a family $(\theta_{kl})_{1 \leq k \leq l \leq g}$ of holomorphic vector fields on \mathbf{H}_g , defined by

$$\theta_{kl} := \frac{1}{2\pi i} \frac{\partial}{\partial \tau_{kl}}.$$

Let $(v_{kl})_{1 \leq k \leq l \leq g}$ be the family of holomorphic vector fields on $B_g(\mathbf{C})$ induced by the higher Ramanujan vector fields on \mathbf{B}_g defined in [32] 5.3.

DEFINITION 4.1. Let U be an open subset of \mathbf{H}_g . We say that a holomorphic map $u : U \rightarrow B_g(\mathbf{C})$ is a solution of the *higher Ramanujan equations* if

$$\theta_{kl}u = v_{kl} \circ u$$

for every $1 \leq k \leq l \leq g$.

In this section, we construct a global holomorphic solution

$$\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$$

of the higher Ramanujan equations. In view of the universal property of the moduli space $B_g(\mathbf{C})$ (Proposition 3.10), the holomorphic map φ_g will be induced by a certain symplectic-Hodge basis of the principally polarized complex torus (\mathbf{X}_g, E_g) over \mathbf{H}_g .

4.1. Definition of φ_g and statement of our main theorem. Recall that the comparison isomorphism (2.1) identifies the holomorphic vector bundle $(\text{Lie}_{\mathbf{H}_g} \mathbf{X}_g)^\vee$ over \mathbf{H}_g with $\mathcal{F}^1(\mathbf{X}_g/\mathbf{H}_g)$ (Lemma 2.12). Moreover, it follows from the construction of \mathbf{X}_g in Example 2.8 that $\text{Lie}_{\mathbf{H}_g} \mathbf{X}_g$ is canonically isomorphic to the trivial vector bundle $\mathbf{C}^g \times \mathbf{H}_g$ over \mathbf{H}_g . Under this isomorphism, we define the holomorphic frame

$$(dz_1, \dots, dz_g)$$

of $\mathcal{F}^1(\mathbf{X}_g/\mathbf{H}_g)$ as the dual of the canonical holomorphic frame of $\mathbf{C}^g \times \mathbf{H}_g$.

THEOREM 4.2. *For each $1 \leq k \leq g$, consider the global sections of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$*

$$\omega_k := 2\pi i dz_k, \quad \eta_k := \nabla_{\theta_{kk}} \omega_k$$

where ∇ denotes the Gauss-Manin connection on $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$. Then,

(1) *The 2g-uple*

$$\mathbf{b}_g := (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$$

of holomorphic global sections of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$ is a symplectic-Hodge basis of the principally polarized complex torus $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$.

(2) *The holomorphic map*

$$\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$$

corresponding to \mathbf{b}_g by the universal property of $B_g(\mathbf{C})$ is a solution of the higher Ramanujan equations (Definition 4.1).

The main idea in our proof is to compute with a C^∞ trivialization of the vector bundle $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$. In the next subsection we develop some preliminary background.

4.2. Preliminary results. Consider the *complex conjugation*, seen as a C^∞ morphism of real vector bundles over \mathbf{H}_g ,

$$\begin{aligned} \mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g) &\longrightarrow \mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g) \\ \alpha &\longmapsto \bar{\alpha} \end{aligned}$$

induced by the comparison isomorphism (2.1), and denote $d\bar{z}_k := \overline{dz_k}$ for every $1 \leq k \leq g$. We may check fiber by fiber that the 2g-uple of C^∞ global sections of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$

$$(dz_1, \dots, dz_g, d\bar{z}_1, \dots, d\bar{z}_g)$$

trivializes $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$ as a C^∞ complex vector bundle over \mathbf{H}_g .

For $1 \leq i \leq j \leq g$ and $1 \leq k \leq g$, let us define

$$\eta_k^{ij} := \nabla_{\theta_{ij}} \omega_k,$$

so that

$$\eta_k = \eta_k^{kk}.$$

PROPOSITION 4.3. *Consider the notations in 1.6.3. For every $1 \leq i \leq j \leq g$ and $1 \leq k \leq g$, we have*

$$\eta_k^{ij} = \sum_{l=1}^g \mathbf{e}_k^\top \mathbf{E}^{ij} (\text{Im } \tau)^{-1} \mathbf{e}_l \text{Im } dz_l$$

as a C^∞ section of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$, where $\text{Im } dz_l := (dz_l - d\bar{z}_l)/2i$.

PROOF. For $1 \leq i \leq j \leq g$ and $1 \leq k, l \leq g$, let λ_{kl}^{ij} and μ_{kl}^{ij} be the C^∞ functions on \mathbf{H}_g with values in \mathbf{C} defined by the equation

$$\boldsymbol{\eta}_k^{ij} = \sum_{l=1}^g (\lambda_{kl}^{ij} dz_l + \mu_{kl}^{ij} d\bar{z}_l).$$

We must prove that $\lambda_{kl}^{ij} + \mu_{kl}^{ij} = 0$ and that $\lambda_{kl}^{ij} = \frac{1}{2i} \mathbf{e}_k^T \mathbf{E}^{ij} (\text{Im } \tau)^{-1} \mathbf{e}_l$.

Let us consider the integral symplectic basis $\beta_g = (\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$ of $R_1 \mathbf{p}_{g*} \mathbf{Z}_{\mathbf{X}_g}$ defined in Example 3.3. For every $1 \leq i \leq j \leq g$ and $1 \leq k, l \leq g$, we have (cf. Remark 2.11)

$$\int_{\gamma_i} \boldsymbol{\eta}_k^{ij} = \int_{\gamma_i} \nabla_{\frac{\partial}{\partial \tau_{ij}}} dz_k = \frac{\partial}{\partial \tau_{ij}} \int_{\gamma_i} dz_k = \frac{\partial}{\partial \tau_{ij}} \delta_{kl} = 0$$

and

$$\int_{\delta_l} \boldsymbol{\eta}_k^{ij} = \int_{\delta_l} \nabla_{\frac{\partial}{\partial \tau_{ij}}} dz_k = \frac{\partial}{\partial \tau_{ij}} \int_{\delta_l} dz_k = \frac{\partial}{\partial \tau_{ij}} \tau_{kl} = \mathbf{E}_{kl}^{ij}.$$

Thus, by definition of λ_{kl}^{ij} and μ_{kl}^{ij} , we obtain

$$0 = \int_{\gamma_i} \boldsymbol{\eta}_k^{ij} = \sum_{m=1}^g \left(\lambda_{km}^{ij} \int_{\gamma_i} dz_m + \mu_{km}^{ij} \int_{\gamma_i} d\bar{z}_m \right) = \lambda_{kl}^{ij} + \mu_{kl}^{ij}$$

and

$$\mathbf{E}_{kl}^{ij} = \int_{\delta_l} \boldsymbol{\eta}_k^{ij} = \sum_{m=1}^g \left(\lambda_{km}^{ij} \int_{\delta_l} dz_m + \mu_{km}^{ij} \int_{\delta_l} d\bar{z}_m \right) = \sum_{m=1}^g \lambda_{km}^{ij} (\tau_{ml} - \bar{\tau}_{ml}) = 2i \sum_{m=1}^g \lambda_{km}^{ij} (\text{Im } \tau)_{ml}.$$

In matricial notation, if we put $\lambda^{ij} := (\lambda_{kl}^{ij})_{1 \leq k, l \leq g} \in M_{g \times g}(\mathbf{C})$, then we have shown that

$$2i \lambda^{ij} \text{Im } \tau = \mathbf{E}^{ij}$$

The assertion follows. ■

Specializing to the case $i = j = k$ in the above proposition, we obtain the following formulas.

COROLLARY 4.4. *For any $1 \leq k \leq g$, we have*

$$\boldsymbol{\eta}_k = \sum_{l=1}^g ((\text{Im } \tau)^{-1})_{kl} \text{Im } dz_l.$$

In particular, $\boldsymbol{\eta}_k$ is the unique global section of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$ satisfying

$$\int_{\gamma_i} \boldsymbol{\eta}_k = 0 \quad \text{and} \quad \int_{\delta_l} \boldsymbol{\eta}_k = \delta_{kl}$$

for every $1 \leq l \leq g$. In other words, $\boldsymbol{\eta}_k$ may be identified with $E_g(\gamma_k, \cdot)$ under the comparison isomorphism (2.1).

Since every section of $R^1 \mathbf{p}_{g*} \mathbf{Z}_{\mathbf{X}_g} = (R_1 \mathbf{p}_{g*} \mathbf{Z}_{\mathbf{X}_g})^\vee$, seen as a section of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$ via the comparison isomorphism (2.1), is horizontal for the Gauss-Manin connection, we obtain the next corollary.

COROLLARY 4.5. *For any $1 \leq k \leq g$, the global section $\boldsymbol{\eta}_k$ of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$ is horizontal for the Gauss-Manin connection :*

$$\nabla \boldsymbol{\eta}_k = 0.$$

Our next goal is to use the duality given by the Riemann form E_g to express dz_l in terms of C^∞ sections of $\text{Lie}_{\mathbf{H}_g} \mathbf{X}_g$.

LEMMA 4.6. *Let $1 \leq k \leq g$, and denote by τ_k the k -th column of $\tau \in \mathbf{H}_g$. Then*

$$dz_k = -E_g(i \text{Im } \tau_k, \cdot) + i E_g(\text{Im } \tau_k, \cdot)$$

as a C^∞ section of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$ under the comparison isomorphism (2.1).

PROOF. Note that $\text{Im } \tau_k = (\text{Im } \tau) \mathbf{e}_k$. Let γ be a section of $R_1 \mathbf{p}_{g*} \mathbf{Z}_{\mathbf{X}_g}$. As $\text{Im } \tau$ is symmetric and $\gamma = \text{Re } \gamma + i \text{Im } \gamma$, we have

$$\begin{aligned} -E_g(i \text{Im } \tau_k, \gamma) + i E_g(\text{Im } \tau_k, \gamma) &= -\text{Im}(i \overline{\text{Im } \tau_k}^\top (\text{Im } \tau)^{-1} \gamma) + i \text{Im}(\overline{\text{Im } \tau_k}^\top (\text{Im } \tau)^{-1} \gamma) \\ &= \text{Im}(i \mathbf{e}_k^\top (\text{Im } \tau) (\text{Im } \tau)^{-1} \gamma) + i \text{Im}(\mathbf{e}_k^\top (\text{Im } \tau) (\text{Im } \tau)^{-1} \gamma) \\ &= \text{Re}(\mathbf{e}_k^\top \gamma) + i \text{Im}(\mathbf{e}_k^\top \gamma) \\ &= \mathbf{e}_k^\top \gamma = dz_k(\gamma). \end{aligned}$$

■

4.3. Proof of Theorem 4.2. We prove parts (1) and (2) separately.

PROOF OF THEOREM 4.2 (1). As each ω_k is by definition a section of $\mathcal{F}^1(\mathbf{X}_g/\mathbf{H}_g)$, to prove that \mathbf{b}_g is a symplectic-Hodge basis of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ it is sufficient to show that it is a symplectic trivialization of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$ with respect to the holomorphic symplectic form $\langle \cdot, \cdot \rangle_{E_g}$. For this, we claim that it is enough to prove that

$$(*) \quad \langle \omega_i, \eta_j \rangle_{E_g} = \delta_{ij}$$

for every $1 \leq i \leq j \leq g$. Indeed, by Corollary 4.5 and by the compatibility (2.2), equation (*) implies that $\langle \eta_i, \eta_j \rangle_{E_g} = 0$ (apply $\nabla_{\theta_{ii}}$). Since we already know that $\mathcal{F}^1(\mathbf{X}_g/\mathbf{H}_g)$ is Lagrangian, this proves indeed that \mathbf{b}_g is a symplectic trivialization of $\mathcal{H}_{\text{dR}}^1(\mathbf{X}_g/\mathbf{H}_g)$.

Fix $1 \leq i \leq j \leq g$. By Corollary 4.4, we have

$$\eta_j = \sum_{l=1}^g ((\text{Im } \tau)^{-1})_{jl} \text{Im } dz_l,$$

thus

$$\langle \omega_i, \eta_j \rangle_{E_g} = 2\pi i \sum_{l=1}^g ((\text{Im } \tau)^{-1})_{jl} \langle dz_i, \text{Im } dz_l \rangle_{E_g}.$$

Now, using Lemma 4.6, we obtain

$$\begin{aligned} \langle dz_i, \text{Im } dz_l \rangle_{E_g} &= \langle -E_g(i \text{Im } \tau_i, \cdot) + i E_g(\text{Im } \tau_i, \cdot), E_g(\text{Im } \tau_l, \cdot) \rangle_{E_g} \\ &= -\langle E_g(i \text{Im } \tau_i, \cdot), E_g(\text{Im } \tau_l, \cdot) \rangle_{E_g} + i \langle E_g(\text{Im } \tau_i, \cdot), E_g(\text{Im } \tau_l, \cdot) \rangle_{E_g} \\ &= \frac{1}{2\pi i} (-E_g(i \text{Im } \tau_i, \text{Im } \tau_l) + i E_g(\text{Im } \tau_i, \text{Im } \tau_l)) \\ &= \frac{1}{2\pi i} \text{Im}(i \text{Im } \tau_i^\top (\text{Im } \tau)^{-1} \text{Im } \tau_l) \\ &= \frac{1}{2\pi i} \mathbf{e}_i^\top (\text{Im } \tau) \mathbf{e}_l = \frac{1}{2\pi i} (\text{Im } \tau)_{il}. \end{aligned}$$

Therefore, since $\text{Im } \tau$ is symmetric,

$$\langle \omega_i, \eta_j \rangle_{E_g} = \sum_{l=1}^g ((\text{Im } \tau)^{-1})_{jl} (\text{Im } \tau)_{li} = \delta_{ij}.$$

■

Let M be a complex manifold and (X, E) be a principally polarized complex torus over M of relative dimension g . Let us denote by ∇ the Gauss-Manin connection on $\mathcal{H}_{\text{dR}}^1(X/M)$. To any symplectic-Hodge basis $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ of $(X, E)_{/M}$ we can associate a morphism of \mathcal{O}_M -modules (cf. [32] Theorem 5.4)

$$\begin{aligned} c : T_M &\longrightarrow \Gamma^2(\mathcal{F}^1(X/M)^\vee) \oplus \mathcal{H}_{\text{dR}}^1(X/M)^{\oplus g} \\ \theta &\longmapsto (\kappa(\theta), \nabla_\theta \eta_1, \dots, \nabla_\theta \eta_g) \end{aligned}$$

where

$$\begin{aligned} \kappa : T_M &\longrightarrow \Gamma^2(\mathcal{F}^1(X/M)^\vee) \\ \theta &\longmapsto \sum_{i=1}^g \langle \cdot, \eta_i \rangle_E \otimes \langle \cdot, \nabla_\theta \omega_i \rangle_E \end{aligned}$$

is the *Kodaira-Spencer morphism* defined as in [32] 5.1.2.⁶

This construction is compatible with base change : if M' is another complex manifold, (X', E') is a principally polarized complex torus over M' of relative dimension g , and b' is a symplectic-Hodge basis of $(X', E')_{/M'}$, then for any morphism $F_{/f} : (X', E')_{/M'} \rightarrow (X, E)_{/M}$ in \mathcal{T}_g such that $F^*b = b'$, the diagram

$$\begin{array}{ccc} T_{M'} & \xrightarrow{Df} & f^*T_M \\ c' \downarrow & & \downarrow f^*c \\ \Gamma^2(\mathcal{F}^1(X'/M')^\vee) \oplus \mathcal{H}_{\text{dR}}^1(X'/M')^{\oplus g} & \xrightarrow{\sim} & f^*(\Gamma^2(\mathcal{F}^1(X/M)^\vee) \oplus \mathcal{H}_{\text{dR}}^1(X/M)^{\oplus g}) \end{array}$$

commutes.

REMARK 4.7. Applying the above construction to the universal symplectic-Hodge basis b_g of $(X_{g, \mathbf{C}}^{\text{an}}, E_{\lambda_g})_{/B_{g, \mathbf{C}}^{\text{an}}}$, we obtain the analytification $c_{g, \mathbf{C}}^{\text{an}}$ of the morphism c_g defined in [32] Theorem 5.4.

Part (2) in Theorem 4.2 will be an easy consequence of the following characterization.

PROPOSITION 4.8. *Let $U \subset \mathbf{H}_g$ be an open subset and $u : U \rightarrow B_g(\mathbf{C})$ be the holomorphic map corresponding to a principally polarized complex torus (X, E) over U endowed with some symplectic-Hodge basis $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$. Then the following are equivalent :*

- (1) *u is a solution of the higher Ramanujan equations.*
- (2) *For every $1 \leq i \leq j \leq g$, we have*

$$c(\theta_{ij}) = u^*c_{g, \mathbf{C}}^{\text{an}}(v_{ij})$$

where $c : T_U \rightarrow \Gamma^2(\mathcal{F}^1(X/U)^\vee) \oplus \mathcal{H}_{\text{dR}}^1(X/U)^{\oplus g}$ is the morphism defined above for the symplectic-Hodge basis b of $(X, E)_{/U}$, and v_{ij} are the higher Ramanujan vector fields on $B_g(\mathbf{C})$.

- (3) *For every $1 \leq i \leq j \leq g$, we have*
 - (i) $\nabla_{\theta_{ij}}\omega_i = \eta_j$, $\nabla_{\theta_{ij}}\omega_j = \eta_i$, and $\nabla_{\theta_{ij}}\omega_k = 0$, for $k \notin \{i, j\}$
 - (ii) $\nabla_{\theta_{ij}}\eta_k = 0$, for $1 \leq k \leq g$.

PROOF. The equivalence between (1) and (2) follows from the commutativity of the diagram

$$\begin{array}{ccc} T_U & \xrightarrow{Du} & u^*T_{B_{g, \mathbf{C}}^{\text{an}}} \\ c \downarrow & & \downarrow u^*c_{g, \mathbf{C}}^{\text{an}} \\ \Gamma^2(\mathcal{F}^1(X/U)^\vee) \oplus \mathcal{H}_{\text{dR}}^1(X/U)^{\oplus g} & \xrightarrow{\sim} & u^*(\Gamma^2(\mathcal{F}^1(X_{g, \mathbf{C}}^{\text{an}}/B_{g, \mathbf{C}}^{\text{an}})^\vee) \oplus \mathcal{H}_{\text{dR}}^1(X_{g, \mathbf{C}}^{\text{an}}/B_{g, \mathbf{C}}^{\text{an}})^{\oplus g}) \end{array}$$

and the injectivity of $u^*c_{g, \mathbf{C}}^{\text{an}}$ (cf. [32] Theorem 5.4).

The same argument in the proof of [32] Proposition 5.7 proves the equivalence between (2) and (3). ■

PROOF OF THEOREM 4.2 (2). By Proposition 4.8, it is sufficient to prove that, for every $1 \leq i \leq j \leq g$, we have

- (i) $\nabla_{\theta_{ij}}\omega_i = \eta_j$, $\nabla_{\theta_{ij}}\omega_j = \eta_i$, and $\nabla_{\theta_{ij}}\omega_k = 0$, for $k \notin \{i, j\}$
- (ii) $\nabla_{\theta_{ij}}\eta_k = 0$, for $1 \leq k \leq g$.

Now, (i) follows directly from Proposition 4.3, and (ii) is the content of Corollary 4.5. ■

6. Recall that $\Gamma^2(\mathcal{F}^1(X/M)^\vee)$ denotes the submodule of symmetric tensors in $\mathcal{F}^1(X/M)^\vee \otimes \mathcal{F}^1(X/M)^\vee$.

4.4. The case $g = 1$. We now explicitly describe the holomorphic map $\varphi_1 : \mathbf{H} \rightarrow B_1(\mathbf{C})$. For every $k \geq 1$, let us denote by $E_{2k} : \mathbf{H} \rightarrow \mathbf{C}$ the classical (level 1) Eisenstein series of weight $2k$ normalized by $E_{2k}(+i\infty) = 1$.

PROPOSITION 4.9. *Let $\varphi_1 : \mathbf{H} \rightarrow B_1(\mathbf{C})$ be the holomorphic map defined in Theorem 4.2 for $g = 1$. Then*

(1) *Under the identification $B_1 \cong \text{Spec } \mathbf{Z}[1/2, b_2, b_4, b_6, \Delta^{-1}]$ of [32] Theorem 6.2, we have*

$$\varphi_1(\tau) = \left(E_2(\tau), \frac{1}{2}\theta E_2(\tau), \frac{1}{6}\theta^2 E_2(\tau) \right),$$

where $\theta = \frac{1}{2\pi i} \frac{d}{d\tau}$.

(2) *Under the identification of [32] Remark 6.3, we have*

$$\varphi_1(\tau) = (E_2(\tau), E_4(\tau), E_6(\tau)).$$

PROOF. By the change-of-coordinates formulas in [32] Remark 6.3, it is sufficient to prove (2). For every $\tau \in \mathbf{H}$, we denote by X_τ the complex elliptic curve defined by the equation

$$y^2 = 4x^3 - \frac{E_4(\tau)}{12}x + \frac{E_6(\tau)}{216}.$$

Recall that there is an isomorphism

$$F_\tau : \mathbf{X}_{1,\tau} \xrightarrow{\sim} X_\tau(\mathbf{C})$$

$$z \mapsto \begin{cases} \left(\left(\frac{1}{2\pi i} \right)^2 \wp_\tau(z) : \left(\frac{1}{2\pi i} \right)^3 \wp'_\tau(z) : 1 \right) & \text{if } z \neq 0 \\ (0 : 1 : 0) & \text{if } z = 0 \end{cases}$$

where \wp_τ denotes the Weierstrass \wp -function associated to the lattice $\mathbf{Z} + \tau\mathbf{Z} \subset \mathbf{C}$. Furthermore, we have (cf. [51] A1.3.16)

$$\eta_1 = \nabla_\theta \omega_1 = \frac{1}{2\pi i} \wp_\tau(z) dz - \frac{E_2(\tau)}{12} \omega_1.$$

Let $\varphi : \mathbf{H} \rightarrow B_1(\mathbf{C})$ be given by $\varphi(\tau) = (E_2(\tau), E_4(\tau), E_6(\tau))$. Then, for every $\tau \in \mathbf{H}$, the isomorphism

$$\Phi_\tau : \mathbf{X}_{1,\tau} \xrightarrow{\sim} X_{1,\varphi(\tau)}(\mathbf{C})$$

$$z \mapsto \begin{cases} \left(\left(\frac{1}{2\pi i} \right)^2 \wp_\tau(z) - \frac{E_2(\tau)}{12} : \left(\frac{1}{2\pi i} \right)^3 \wp'_\tau(z) : 1 \right) & \text{if } z \neq 0 \\ (0 : 1 : 0) & \text{if } z = 0 \end{cases}$$

satisfies

$$\Phi_\tau^* \left(\frac{dx}{y} \right) = \omega_1 \quad \text{and} \quad \Phi_\tau^* \left(x \frac{dx}{y} \right) = \eta_1.$$

By making τ vary in \mathbf{H} , we obtain a morphism $\Phi_\varphi : \mathbf{X}_1 \rightarrow X_{1,\mathbf{C}}^{\text{an}}$ in \mathcal{T}_1 . Since $\Phi^*(dx/y, xdx/y) = \mathbf{b}_1$, we must have $\varphi = \varphi_1$ by definition of φ_1 . \blacksquare

5. Values of φ_g and transcendence degree of fields of periods of abelian varieties

Let X be a complex abelian variety (resp. a complex torus). For any subfield k of \mathbf{C} , we say that X is *definable over k* if there exists an abelian variety X_0 over k such that X is isomorphic to $X_0 \otimes_k \mathbf{C}$ as a complex abelian variety (resp. isomorphic to $X_0(\mathbf{C})$ as a complex torus).

LEMMA 5.1. *For any complex abelian variety X (resp. polarizable complex torus), there exists a smallest algebraically closed subfield k of \mathbf{C} over which X is definable.*

PROOF. Let g be the dimension of X , and let $\lambda : X \rightarrow X^t$ be any polarization on X (not necessarily principal). If λ is of degree d^2 , then the isomorphism class of the couple (X, λ) defines a complex point $\bar{x} \in A_{g,d,1}(\mathbf{C})$, where $A_{g,d,1}$ denotes the coarse moduli space over \mathbf{Q} of abelian varieties of dimension g endowed with a polarization of degree d^2 (cf. [72] Theorem 7.10). Let k be the algebraic closure in \mathbf{C} of the residue field $\mathbf{Q}(\bar{x})$ (see 1.6.4).

It is clear that X is definable over k . To prove that k is the smallest algebraically closed subfield of \mathbf{C} with this property, let k' be any algebraically closed subfield of \mathbf{C} over which there is an abelian variety X' such that X is isomorphic to $X' \otimes_{k'} \mathbf{C}$. As k' is algebraically closed, the polarization λ on X descends to a polarization λ' on X' such that (X, λ) and $(X', \lambda') \otimes_{k'} \mathbf{C}$ are isomorphic as polarized complex abelian varieties⁷. Thus the morphism $\bar{x} : \text{Spec } \mathbf{C} \rightarrow A_{g,d,1}$ factors through $\text{Spec } k'$, which implies that $\mathbf{Q}(\bar{x}) \subset k'$. As k' is algebraically closed, we obtain $k \subset k'$. \blacksquare

DEFINITION 5.2. Let X be a complex abelian variety, k be the smallest algebraically closed subfield of \mathbf{C} over which X is definable, and fix a k -model X_0 of X . The *field of periods* $\mathcal{P}(X)$ of X is defined as the smallest subfield of \mathbf{C} containing k and the image of pairing

$$H_{\text{dR}}^1(X_0/k) \otimes H_1(X_0(\mathbf{C}), \mathbf{Z}) \rightarrow \mathbf{C}$$

$$\alpha \otimes \gamma \mapsto \int_{\gamma} \alpha$$

given by “integration of differential forms” (cf. 2.4.2).

Note that $\mathcal{P}(X)$ does not depend on the choice of X_0 .

This section is devoted to the proof of the following theorem (cf. [8], Proposition 2, for an analog in the context of derivatives of Siegel modular forms).

THEOREM 5.3. *Let $g \geq 1$ be an integer. With notations as in Example 2.8 and Theorem 4.2, for every $\tau \in \mathbf{H}_g$ the field of periods $\mathcal{P}(\mathbf{X}_{g,\tau})$ of the polarizable complex torus $\mathbf{X}_{g,\tau} := \mathbf{C}^g / (\mathbf{Z}^g + \tau \mathbf{Z}^g)$ is an algebraic extension of $\mathbf{Q}(2\pi i, \tau, \varphi_g(\tau))$. In particular,*

$$\text{trdeg}_{\mathbf{Q}} \mathbf{Q}(2\pi i, \tau, \varphi_g(\tau)) = \text{trdeg}_{\mathbf{Q}} \mathcal{P}(\mathbf{X}_{g,\tau}).$$

5.1. Period matrices. Let us consider the *general symplectic group*; namely, the subgroup scheme GSp_{2g} of GL_{2g} over $\text{Spec } \mathbf{Z}$ such that, for every affine scheme $V = \text{Spec } R$, we have

$$\text{GSp}_{2g}(V) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g \times 2g}(R) \mid \begin{array}{l} A, B, C, D \in M_{g \times g}(R) \text{ satisfy} \\ AB^{\top} = BA^{\top}, CD^{\top} = DC^{\top}, \text{ and } AD^{\top} - BC^{\top} \in R^{\times} \mathbf{1}_g \end{array} \right\}.$$

We can define a morphism of group schemes

$$\nu : \text{GSp}_{2g} \rightarrow \mathbf{G}_m$$

as follows : if $s = (A \ B ; C \ D) \in \text{GSp}_{2g}(V)$, then $\nu(s) \in R^{\times}$ satisfies $AD^{\top} - BC^{\top} = \nu(s) \mathbf{1}_g$. Note that Sp_{2g} is the kernel of ν .

We denote by GSp_{2g}^* the open subscheme of GSp_{2g} defined by the condition $A \in \text{GL}_g(R)$ in the above notations.

Let (X, E) be a principally polarized complex torus of dimension g , and $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ (resp. $\beta = (\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$) be a symplectic-Hodge basis (resp. an integral symplectic basis) of (X, E) .

DEFINITION 5.4. The *period matrix* of (X, E) with respect to b and β is defined by

$$P(X, E, b, \beta) := \begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix} \in M_{2g \times 2g}(\mathbf{C}),$$

where

$$(\Omega_1)_{ij} := \int_{\gamma_i} \omega_j \quad (N_1)_{ij} := \int_{\gamma_i} \eta_j$$

$$(\Omega_2)_{ij} := \int_{\delta_i} \omega_j \quad (N_2)_{ij} := \int_{\delta_i} \eta_j.$$

Note that $P(X, E, b, \beta)$ is simply the matrix of the comparison isomorphism (2.1) with respect to the bases b of $\mathcal{H}_{\text{dR}}^1(X)$ and $(E(\cdot, \delta_1), \dots, E(\cdot, \delta_g), E(\gamma_1, \cdot), \dots, E(\gamma_g, \cdot))$ of $\text{Hom}(H_1(X, \mathbf{Z}), \mathbf{C})$.

7. This follows from the fact that, for any abelian varieties X and Y over a field K , the functor $\text{Sch}_{/K}^{\text{op}} \rightarrow \text{Set}$ given by $U \mapsto \text{Hom}_{\text{GpSch}/U}(X \times_K U, Y \times_K U)$ is representable by an étale K -scheme.

REMARK 5.5. In particular, let (X, λ) be a principally polarized complex abelian variety, k be the smallest algebraically closed subfield of \mathbf{C} over which X is definable, and (X_0, λ_0) be a k -model of (X, λ) . Then, if b is any symplectic-Hodge basis of (X_0, λ_0) , and β is any integral symplectic basis of $(X^{\text{an}}, E_\lambda)$, the field of periods $\mathcal{P}(X)$ of X is generated over k by the coefficients of the period matrix $P(X^{\text{an}}, E_\lambda, b, \beta)$.

LEMMA 5.6. *For any (X, E, b, β) as above, we have*

- (1) $P(X, E, b, \beta) \in \text{GSp}_{2g}(\mathbf{C})$ and $\nu(P(X, E, b, \beta)) = 2\pi i$,
- (2) $\Omega^1 \in \text{GL}_g(\mathbf{C})$ (i.e. $P(X, E, b, \beta) \in \text{GSp}_{2g}^*(\mathbf{C})$) and $\Omega_2 \Omega_1^{-1} \in \mathbf{H}_g$.

PROOF. Knowing that $P(X, E, b, \beta)$ is a base change matrix with respect to symplectic bases, (1) is simply a rephrasing of Lemma 2.13 and (2) is a particular case of the classical *Riemann relations* (cf. proof of Proposition 3.4). ■

5.2. Auxiliary lemmas. We shall need the following auxiliary results.

LEMMA 5.7. *The morphism of schemes*

$$\begin{aligned} \text{GSp}_{2g}^* &\longrightarrow \mathbf{G}_m \times_{\mathbf{Z}} \text{Sym}_g \times_{\mathbf{Z}} P_g \\ s &\longmapsto (\nu(s), \tau(s), p(s)) \end{aligned}$$

where

$$\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} := CA^{-1} \quad \text{and} \quad p \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} A^{-1} & -B^\top \\ 0 & A^\top \end{pmatrix}$$

is an isomorphism.

PROOF. We simply remark that

$$\left(\lambda, Z, \begin{pmatrix} X & Y \\ 0 & (X^\top)^{-1} \end{pmatrix} \right) \longmapsto \begin{pmatrix} X^{-1} & -Y^\top \\ ZX^{-1} & (\lambda \mathbf{1}_g - ZX^{-1}Y)X^\top \end{pmatrix}$$

is an inverse to the morphism defined in the statement. ■

LEMMA 5.8. *Let $F : (X, E) \rightarrow (X', E')$ be an isomorphism of principally polarized complex tori of dimension g , $\beta = (\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$ be an integral symplectic basis of (X, E) and b' be a symplectic-Hodge basis of (X', E') . We denote by $F_*\beta$ the integral symplectic basis of (X', E') given by pushforward in singular homology. Then the symplectic-Hodge basis*

$$b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g) := F^*b' \cdot p \left(\frac{1}{2\pi i} P(X', E', b', F_*\beta) \right)$$

of (X, E) satisfies

$$\int_{\gamma_i} \eta_j = 0, \quad \int_{\delta_i} \eta_j = \delta_{ij}$$

for every $1 \leq i, j \leq g$.

The proof this lemma is a straightforward computation.

5.3. Proof of Theorem 5.3. Let A_g be the coarse moduli space associated to the Deligne-Mumford stack $\mathcal{A}_g \rightarrow \text{Spec } \mathbf{Z}$ (which exists as an algebraic space by the Keel-Mori theorem, cf. [82] Theorem 11.1.2). We recall that A_g is a quasi-projective scheme over $\text{Spec } \mathbf{Z}$ (cf. [65] VII Théorème 4.2) endowed with a canonical morphism $\mathcal{A}_g \rightarrow A_g$ inducing, for every algebraically closed field k , a bijection of $A_g(k)$ with the set of isomorphism classes of principally polarized abelian varieties over k .

Since any principally polarized complex torus (X, E) of dimension g is algebraizable, (X, E) defines an isomorphism class in the category $\mathcal{A}_g(\mathbf{C})$ that we shall denote $[(X, E)]$. Let

$$\begin{aligned} j_g : \mathbf{H}_g &\longrightarrow A_g(\mathbf{C}) \\ \tau &\longmapsto [(\mathbf{X}_{g,\tau}, E_{g,\tau})]. \end{aligned}$$

The next result follows immediately from our proof of the Lemma 5.1.

LEMMA 5.9. For any $\tau \in \mathbf{H}_g$, the smallest algebraically closed subfield over which $\mathbf{X}_{g,\tau}$ is definable is given by the algebraic closure in \mathbf{C} of the residue field $\mathbf{Q}(j_g(\tau))$.

PROOF OF THEOREM 5.3. Let $\tau \in \mathbf{H}_g$ and fix any integer $n \geq 3$. The principally polarized complex torus $(\mathbf{X}_{g,\tau}, E_{g,\tau})$ endowed with the integral symplectic basis modulo n induced by $\beta_{g,\tau}$ (cf. Remark 3.6) defines a complex point $u(\tau) \in A_{g,n}(\mathbf{C})$ in the fine moduli space $A_{g,n}$ over $\mathbf{Z}[1/n, \zeta_n]$ of principally polarized abelian varieties of dimension g endowed with a symplectic basis of its n -torsion subscheme (cf. 3.3).

Let $(X_{g,n}, \lambda_{g,n})$ denote the universal principally polarized abelian scheme over $A_{g,n}$. Then, by the remark following [32] Definition 2.4, there exists a Zariski open neighborhood $U \subset A_{g,n, \overline{\mathbf{Q}}}$ of $u(\tau)$ over which $(X_{g,n, \overline{\mathbf{Q}}}, \lambda_{g,n})$ admits a symplectic-Hodge basis b . Let us denote by $(X, \lambda) = (X_{g,n, \overline{\mathbf{Q}}}, \lambda_{g,n}) \times_{A_{g,n, \overline{\mathbf{Q}}}} U$ the restriction of $(X_{g,n, \overline{\mathbf{Q}}}, \lambda_{g,n})$ to U .

In the following, fiber products will be taken with respect to $\overline{\mathbf{Q}}$. The symplectic-Hodge basis b of $(X, \lambda)_{/U}$ induces an isomorphism of principal P_g -bundles over U

$$\begin{aligned} P_{g, \overline{\mathbf{Q}}} \times U &\xrightarrow{\sim} B(X, \lambda) \\ (p, u) &\longmapsto b_u \cdot p, \end{aligned}$$

where $B(X, \lambda)$ is the U -scheme defined in [32] Corollary 3.4. By composing this isomorphism with the isomorphism in Lemma 5.7, we obtain the isomorphism of $\overline{\mathbf{Q}}$ -schemes

$$\begin{aligned} f : \mathrm{GSp}_{2g, \overline{\mathbf{Q}}}^* \times U &\longrightarrow \mathbf{G}_{m, \overline{\mathbf{Q}}} \times \mathrm{Sym}_{g, \overline{\mathbf{Q}}} \times B(X, \lambda) \\ (s, u) &\longmapsto (\nu(s), \tau(s), b_u \cdot p(s)). \end{aligned}$$

Note that the canonical morphism $h : B(X, \lambda) \longrightarrow \mathcal{B}_{g, \overline{\mathbf{Q}}} \cong B_{g, \overline{\mathbf{Q}}}$ is quasi-finite, since it fits into the cartesian diagram of Deligne-Mumford stacks

$$\begin{array}{ccc} B(X, \lambda) & \xrightarrow{h} & \mathcal{B}_{g, \overline{\mathbf{Q}}} \\ \downarrow & \square & \downarrow \\ U & \longrightarrow & \mathcal{A}_{g, \overline{\mathbf{Q}}} \end{array}$$

where the bottom arrow $U \longrightarrow \mathcal{A}_{g, \overline{\mathbf{Q}}}$ is given by the composition of the open immersion $U \subset A_{g,n, \overline{\mathbf{Q}}}$ with the (canonical) finite étale morphism $A_{g,n, \overline{\mathbf{Q}}} \longrightarrow \mathcal{A}_{g, \overline{\mathbf{Q}}}$.

In particular, by composing f with h , we obtain a quasi-finite morphism of $\overline{\mathbf{Q}}$ -schemes

$$q : \mathrm{GSp}_{2g, \overline{\mathbf{Q}}}^* \times U \longrightarrow \mathbf{G}_{m, \overline{\mathbf{Q}}} \times \mathrm{Sym}_{g, \overline{\mathbf{Q}}} \times B_{g, \overline{\mathbf{Q}}}$$

given on geometric points by

$$q(s, u) = (\nu(s), \tau(s), [(X_u, \lambda_u, b_u \cdot p(s))])$$

where $[(X_u, \lambda_u, b_u \cdot p(s))]$ denotes the isomorphism class in $\mathcal{B}_g(k(u))$ of $(X_u, \lambda_u, b_u \cdot p(s))$, and $k(u)$ denotes the residue field of $u \in U$.

Let $F : (\mathbf{X}_{g,\tau}, E_{g,\tau}) \xrightarrow{\sim} (X_{u(\tau)}^{\mathrm{an}}, E_{\lambda_{u(\tau)}})$ be the isomorphism of principally polarized complex tori corresponding, by the universal property of $A_{g,n}$, to the reduction of the integral symplectic basis $\beta_{g,\tau}$ modulo n , and put

$$s(\tau) := \frac{1}{2\pi i} P(X_{u(\tau)}^{\mathrm{an}}, E_{\lambda_{u(\tau)}}, b_{u(\tau)}, F_* \beta_{g,\tau}) \in \mathrm{GSp}_{2g}^*(\mathbf{C}).$$

It follows from Corollary 4.4 and Lemma 5.8 that $F^* b_{u(\tau)} \cdot p(s(\tau)) = \mathbf{b}_{g,\tau}$, so that

$$[(X_{u(\tau)}, \lambda_{u(\tau)}, b_{u(\tau)} \cdot p(s(\tau)))] = [(\mathbf{X}_{g,\tau}, E_{g,\tau}, \mathbf{b}_{g,\tau})] = \varphi_g(\tau).$$

Thus, by Lemma 5.6, we obtain

$$q(s(\tau), u(\tau)) = \left(\frac{1}{2\pi i}, \tau, \varphi_g(\tau) \right) \in \mathbf{G}_m(\mathbf{C}) \times \mathrm{Sym}_g(\mathbf{C}) \times B_g(\mathbf{C}).$$

To finish the proof, it is sufficient to show that $\mathcal{P}(\mathbf{X}_{g,\tau})$ is an algebraic extension of $\overline{\mathbf{Q}}(s(\tau), u(\tau))$. For this, let

$$h' : A_{g,n,\overline{\mathbf{Q}}} \longrightarrow A_{g,\overline{\mathbf{Q}}}$$

be the canonical map; then $h'(u(\tau)) = j_g(\tau)$. As h' is finite (in fact, it identifies $A_{g,n,\overline{\mathbf{Q}}}$ with the quotient $\mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z}) \setminus A_{g,n,\overline{\mathbf{Q}}}$), the residue field $\overline{\mathbf{Q}}(u(\tau))$ is a finite extension of $\overline{\mathbf{Q}}(j_g(\tau))$. Then, it follows from Lemma 5.9 that $\overline{\mathbf{Q}}(u(\tau))$ is contained in the smallest algebraically closed field of \mathbf{C} over which $\mathbf{X}_{g,\tau}$ is definable (namely, the algebraic closure in \mathbf{C} of $\overline{\mathbf{Q}}(j_g(\tau))$). Finally, since $2\pi i \in \mathcal{P}(\mathbf{X}_{g,\tau})$ by Lemma 5.6 (1), the assertion follows from Remark 5.5. \blacksquare

6. Group-theoretic interpretation of $B_g(\mathbf{C})$ and of the higher Ramanujan vector fields

In this section we shall explain how to realize the complex manifold $B_g(\mathbf{C})$ as a domain (in the analytic topology) of the quotient manifold $\mathrm{Sp}_{2g}(\mathbf{Z}) \setminus \mathrm{Sp}_{2g}(\mathbf{C})$ (Corollary 6.7).

We shall also give an explicit expression for the higher Ramanujan vector fields, and for the holomorphic map $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$, under this group-theoretic interpretation. For this, recall that the Lie algebra of $\mathrm{Sp}_{2g}(\mathbf{C})$ is given by

$$\mathrm{Lie} \mathrm{Sp}_{2g}(\mathbf{C}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g \times 2g}(\mathbf{C}) \mid B^\top = B, C^\top = C, D = -A^\top \right\}.$$

For $1 \leq k \leq l \leq g$, let us consider the left invariant holomorphic vector field \tilde{V}_{kl} on $\mathrm{Sp}_{2g}(\mathbf{C})$ corresponding to

$$\frac{1}{2\pi i} \begin{pmatrix} 0 & \mathbf{E}^{kl} \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie} \mathrm{Sp}_{2g}(\mathbf{C});$$

it descends to a holomorphic vector field V_{kl} on the quotient $\mathrm{Sp}_{2g}(\mathbf{Z}) \setminus \mathrm{Sp}_{2g}(\mathbf{C})$.

THEOREM 6.1. *Let $(v_{kl})_{1 \leq k \leq l \leq g}$ be the higher Ramanujan vector fields on $B_g(\mathbf{C})$. Under the identification of $B_g(\mathbf{C})$ with an open submanifold of $\mathrm{Sp}_{2g}(\mathbf{Z}) \setminus \mathrm{Sp}_{2g}(\mathbf{C})$ of Corollary 6.7, we have :*

(1) *For every $1 \leq k \leq l \leq g$,*

$$v_{kl} = V_{kl}|_{B_g(\mathbf{C})}.$$

(2) *The solution of the higher Ramanujan equations $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ is given by*

$$\varphi_g(\tau) = \mathrm{Sp}_{2g}(\mathbf{Z}) \begin{pmatrix} \mathbf{1}_g & \tau \\ 0 & \mathbf{1}_g \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{Z}) \setminus \mathrm{Sp}_{2g}(\mathbf{C}).$$

As an example of application, we shall prove the following easy consequence of the above theorem.

COROLLARY 6.2. *The image of $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ is closed for the analytic topology.*

6.1. Realization of $B_g(\mathbf{C})$ as an open submanifold of $\mathrm{Sp}_{2g}(\mathbf{Z}) \setminus \mathrm{Sp}_{2g}(\mathbf{C})$. Let $\mathbf{B}_g = B(\mathbf{X}_g, E_g)$ be the principal $P_g(\mathbf{C})$ -bundle over \mathbf{H}_g associated to the principally polarized complex torus $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ as defined in Lemma 3.8, so that the fiber of $\mathbf{B}_g \longrightarrow \mathbf{H}_g$ over $\tau \in \mathbf{H}_g$ is given by the set of symplectic-Hodge bases of $(\mathbf{X}_{g,\tau}, E_{g,\tau})$.

We shall first realize \mathbf{B}_g as a ‘‘period domain’’ in $\mathrm{Sp}_{2g}(\mathbf{C})$. For this, let us introduce the following convenient modification of period matrices (Definition 5.4).

DEFINITION 6.3. Let (X, E) be a principally polarized complex torus of dimension g , and b (resp. β) be a symplectic-Hodge basis (resp. an integral symplectic basis) of (X, E) . Let

$$P(X, E, b, \beta) = \begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix} \in \mathrm{GSp}_{2g}(\mathbf{C})$$

be the period matrix of (X, E) with respect to b and β . We define

$$\Pi(X, E, b, \beta) := \begin{pmatrix} N_2 & \frac{1}{2\pi i} \Omega_2 \\ N_1 & \frac{1}{2\pi i} \Omega_1 \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{C})$$

Observe that this matrix is indeed symplectic by Lemma 5.6.

We define a holomorphic map $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ as follows. Let q be a point in \mathbf{B}_g lying above $\tau \in \mathbf{H}_g$, and corresponding to the symplectic-Hodge basis b of $(\mathbf{X}_{g,\tau}, E_{g,\tau})$, then

$$\Pi(q) := \Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, b, \beta_{g,\tau})$$

where β_g is the integral symplectic basis of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ defined in Example 3.3.

REMARK 6.4. Alternatively, recall that \mathbf{H}_g may be regarded as the moduli space for principally polarized complex tori of dimension g endowed with an integral symplectic basis (Proposition 3.4). In particular, as already remarked in the proof of Proposition 3.10, points in \mathbf{B}_g correspond to isomorphism classes $[(X, E, b, \beta)]$ of quadruples (X, E, b, β) , where (X, E) is a principally polarized complex torus of dimension g , and b (resp. β) is a symplectic-Hodge basis (resp. integral symplectic basis) of (X, E) . Under this identification, the map $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ is given by $[(X, E, b, \beta)] \longmapsto \Pi(X, E, b, \beta)$.

Let us consider the moduli-theoretic interpretation of \mathbf{B}_g of the above remark, and recall that \mathbf{B}_g is endowed with a natural left action of the discrete group $\mathrm{Sp}_{2g}(\mathbf{Z})$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot [(X, E, b, \beta)] = \left[\left(X, E, b, \beta \cdot \begin{pmatrix} D^\top & B^\top \\ C^\top & A^\top \end{pmatrix} \right) \right]$$

(cf. Remark 3.5), and a right action of the Siegel parabolic subgroup $P_g(\mathbf{C}) \leq \mathrm{Sp}_{2g}(\mathbf{C})$ given by

$$[(X, E, b, \beta)] \cdot p = [(X, E, b \cdot p, \beta)],$$

where both β and b are regarded as row vectors of order $2g$.

Let us denote by P'_g the subgroup scheme of Sp_{2g} consisting of matrices $(A \ B ; C \ D)$ such that $B = 0$. A simple computation proves the following equivariance properties of $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$.

LEMMA 6.5. *Consider the isomorphism of groups*

$$P_g(\mathbf{C}) \xrightarrow{\sim} P'_g(\mathbf{C})$$

$$p = \begin{pmatrix} A & B \\ 0 & (A^\top)^{-1} \end{pmatrix} \longmapsto p' := \begin{pmatrix} (A^\top)^{-1} & 0 \\ 2\pi i B & A \end{pmatrix}.$$

Then, for any $q \in \mathbf{B}_g$, $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z})$, and $p \in P_g(\mathbf{C})$, we have

$$\Pi(\gamma \cdot q) = \gamma \Pi(q) \quad \text{and} \quad \Pi(q \cdot p) = \Pi(q) p'$$

in $\mathrm{Sp}_{2g}(\mathbf{C})$.

Let us now consider the *Lagrangian Grassmannian*, namely the smooth and quasi-projective \mathbf{C} -scheme of dimension $g(g+1)/2$ obtained as the quotient of complex affine algebraic groups

$$L_g := \mathrm{Sp}_{2g, \mathbf{C}} / P'_{g, \mathbf{C}}.$$

The complex manifold $L_g(\mathbf{C}) = \mathrm{Sp}_{2g}(\mathbf{C}) / P'_g(\mathbf{C})$ may be naturally identified with the quotient of

$$M := \{(Z_1, Z_2) \in M_{g \times g}(\mathbf{C}) \times M_{g \times g}(\mathbf{C}) \mid Z_1^\top Z_2 = Z_2^\top Z_1, \mathrm{rank}(Z_1 \ Z_2) = g\}$$

by the right action of $\mathrm{GL}_g(\mathbf{C})$ defined by matrix multiplication :

$$(Z_1, Z_2) \cdot S := (Z_1 S, Z_2 S).$$

We denote the class in $L_g(\mathbf{C})$ of a point $(Z_1, Z_2) \in M$ by $(Z_1 : Z_2)$. The canonical map

$$\pi : \mathrm{Sp}_{2g, \mathbf{C}} \longrightarrow L_g$$

is then given on complex points by

$$\pi \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (B : D).$$

PROPOSITION 6.6. Let $\iota : \mathbf{H}_g \longrightarrow L_g(\mathbf{C})$ be the open embedding given by $\iota(\tau) = (\tau : \mathbf{1}_g)$. Then the diagram of complex manifolds

$$\begin{array}{ccc} \mathbf{B}_g & \xrightarrow{\Pi} & \mathrm{Sp}_{2g}(\mathbf{C}) \\ \downarrow & & \downarrow \pi \\ \mathbf{H}_g & \xrightarrow{\iota} & L_g(\mathbf{C}) \end{array}$$

is cartesian. That is, $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ induces a biholomorphism of \mathbf{B}_g onto the open submanifold

$$\pi^{-1}(\iota(\mathbf{H}_g)) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{C}) \mid D \in \mathrm{GL}_g(\mathbf{C}), BD^{-1} \in \mathbf{H}_g \right\}$$

of $\mathrm{Sp}_{2g}(\mathbf{C})$, and makes the above diagram commute.

PROOF. The commutativity of the diagram in the statement is easy (cf. proof of Proposition 3.4). In particular, if $q, q' \in \mathbf{B}_g$ satisfy $\Pi(q) = \Pi(q')$, then they lie above the same point $\tau \in \mathbf{H}_g$. Let b (resp. b') be the symplectic-Hodge basis of $(\mathbf{X}_{g,\tau}, E_{g,\tau})$ corresponding to q (resp. q'). Since period matrices are base change matrices for the comparison isomorphism, and

$$\Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, b, \beta_{g,\tau}) = \Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, b', \beta_{g,\tau}),$$

it is clear that $b = b'$. This proves that Π is injective.

Observe that \mathbf{B}_g and $\mathrm{Sp}_{2g}(\mathbf{C})$ are complex manifolds of same dimension. Thus, to finish our proof, it suffices to check that $\Pi(\mathbf{B}_g) = \pi^{-1}(\iota(\mathbf{H}_g))$ ([39] p. 19). Let $s \in \pi^{-1}(\iota(\mathbf{H}_g))$, and let $\tau \in \mathbf{H}_g$ be such that $\iota(\tau) = \pi(s)$. Fix any $q \in \mathbf{B}_g$ lying above $\tau \in \mathbf{H}_g$. Then, there exists a unique $p' \in P'_g(\mathbf{C})$ such that $s = \Pi(q)p'$. Hence, by Lemma 6.5, $s = \Pi(q \cdot p) \in \Pi(\mathbf{B}_g)$. ■

Recall from Proposition 3.10 that the canonical map

$$(6.1) \quad \begin{array}{ccc} \mathbf{B}_g & \longrightarrow & B_g(\mathbf{C}) \\ [(X, E, b, \beta)] & \longmapsto & [(X, E, b)] \end{array}$$

induces a biholomorphism

$$\mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathbf{B}_g \xrightarrow{\sim} B_g(\mathbf{C}).$$

Furthermore, note that Lemma 6.5 implies that the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on $\mathrm{Sp}_{2g}(\mathbf{C})$ by left multiplication preserves the open subset $\Pi(\mathbf{B}_g)$.

COROLLARY 6.7. The map $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ induces a biholomorphism of $B_g(\mathbf{C})$ onto the open submanifold of $\mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$

$$\mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \Pi(\mathbf{B}_g) = \{ \mathrm{Sp}_{2g}(\mathbf{Z})s \in \mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C}) \mid \pi(s) \in \iota(\mathbf{H}_g) \}.$$

6.2. Proof of Theorem 6.1 and of Corollary 6.2. We prove parts (1) and (2) of Theorem 6.1 separately.

PROOF OF THEOREM 6.1 (1). It is sufficient to prove that the solutions of the differential equations defined by v_{kl} and by V_{kl} coincide. More precisely, let U be a simply connected open subset of \mathbf{H}_g , and $u : U \longrightarrow B_g(\mathbf{C})$ be a solution of the higher Ramanujan equations (Definition 4.1); we shall prove that, for any lifting

$$\begin{array}{ccc} & \xrightarrow{\tilde{u}} & \mathbf{B}_g \\ & \searrow & \downarrow \\ U & \xrightarrow{u} & B_g(\mathbf{C}) \end{array}$$

of u , the holomorphic map $h := \Pi \circ \tilde{u} : U \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ is a solution of the differential equations

$$(6.2) \quad \theta_{kl}h = \tilde{V}_{kl} \circ h, \quad 1 \leq k \leq l \leq g.$$

where $\theta_{kl} = \frac{1}{2\pi i} \frac{\partial}{\partial \tau_{kl}}$.

By the universal property of \mathbf{B}_g , the holomorphic map \tilde{u} corresponds to a principally polarized complex torus (X, E) over U , of relative dimension g , endowed with a symplectic-Hodge basis $b = (\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g)$ and an integral symplectic basis $\beta = (\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$. For $\tau \in U$, let us write

$$h(\tau) = \begin{pmatrix} N_2(\tau) & \frac{1}{2\pi i} \Omega_2(\tau) \\ N_1(\tau) & \frac{1}{2\pi i} \Omega_1(\tau) \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{C})$$

where $\Omega_1, \Omega_2, N_1, N_2 : U \rightarrow M_{g \times g}(\mathbf{C})$ are holomorphic.

Now, since u is a solution of the higher Ramanujan equations, it follows from Proposition 4.8 (3) that, for every $1 \leq i \leq j \leq g$,

- (i) $\theta_{ij} \Omega_1 = N_1 \mathbf{E}^{ij}$, $\theta_{ij} \Omega_2 = N_2 \mathbf{E}^{ij}$
- (ii) $\theta_{ij} N_1 = 0$, $\theta_{ij} N_2 = 0$.

As U is connected, (ii) implies that N_1 and N_2 are constant. Thus, (i) implies that $\frac{1}{2\pi i} \Omega_1 - N_1 \tau$ and $\frac{1}{2\pi i} \Omega_2 - N_2 \tau$ are also constant. In other words, there exists a unique element $s \in \mathrm{Sp}_{2g}(\mathbf{C})$ such that

$$h(\tau) = s \begin{pmatrix} \mathbf{1}_g & \tau \\ 0 & \mathbf{1}_g \end{pmatrix}$$

for every $\tau \in U$. Finally, since each \tilde{V}_{kl} is left invariant, it is easy to see that h is a solution of the differential equations (6.2). \blacksquare

LEMMA 6.8. *For any $\tau \in \mathbf{H}_g$, we have*

$$\Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, \mathbf{b}_{g,\tau}, \beta_{g,\tau}) = \begin{pmatrix} \mathbf{1}_g & \tau \\ 0 & \mathbf{1}_g \end{pmatrix}.$$

PROOF. Let us write

$$\Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, \mathbf{b}_{g,\tau}, \beta_{g,\tau}) = \begin{pmatrix} N_2(\tau) & \frac{1}{2\pi i} \Omega_2(\tau) \\ N_1(\tau) & \frac{1}{2\pi i} \Omega_1(\tau) \end{pmatrix}.$$

By definition of β_g and of \mathbf{b}_g , it is clear that $\Omega_1(\tau) = 2\pi i \mathbf{1}_g$ and that $\Omega_2(\tau) = 2\pi i \tau$. That $N_1(\tau) = 0$ and $N_2(\tau) = \mathbf{1}_g$ is a reformulation of Corollary 4.4. \blacksquare

PROOF OF THEOREM 6.1 (2). By definition, φ_g is given by the composition of

$$\begin{aligned} \mathbf{H}_g &\longrightarrow \mathbf{B}_g \\ \tau &\longmapsto [(\mathbf{X}_{g,\tau}, E_{g,\tau}, \mathbf{b}_{g,\tau}, \beta_{g,\tau})] \end{aligned}$$

with the canonical map $\mathbf{B}_g \rightarrow B_g(\mathbf{C})$. The result now follows from Lemma 6.8. \blacksquare

PROOF OF COROLLARY 6.2. Consider the subgroup

$$U_g(\mathbf{C}) := \left\{ \begin{pmatrix} \mathbf{1}_g & Z \\ 0 & \mathbf{1}_g \end{pmatrix} \in M_{2g \times 2g}(\mathbf{C}) \mid Z^\top = Z \right\} \leq \mathrm{Sp}_{2g}(\mathbf{C}).$$

The statement is equivalent to asserting that the image of $U_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$ in the quotient $\mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$ is closed, or, equivalently, that $\mathrm{Sp}_{2g}(\mathbf{Z}) \cdot U_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$ is closed. Let us consider the (holomorphic) map

$$\begin{aligned} f : \mathrm{Sp}_{2g}(\mathbf{C}) &\longrightarrow M_g(\mathbf{C}) \times M_g(\mathbf{C}) \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} &\longmapsto (A, C). \end{aligned}$$

Now, one simply remarks that

$$\mathrm{Sp}_{2g}(\mathbf{Z}) \cdot U_g(\mathbf{C}) = f^{-1}(f(\mathrm{Sp}_{2g}(\mathbf{Z}))).$$

Since $f(\mathrm{Sp}_{2g}(\mathbf{Z})) \subset M_g(\mathbf{Z}) \times M_g(\mathbf{Z})$, and $M_g(\mathbf{Z}) \times M_g(\mathbf{Z})$ is a closed discrete subset of $M_g(\mathbf{C}) \times M_g(\mathbf{C})$ for the analytic topology, we conclude that $\mathrm{Sp}_{2g}(\mathbf{Z}) \cdot U_g(\mathbf{C})$ is closed in $\mathrm{Sp}_{2g}(\mathbf{C})$. \blacksquare

7. Zariski-density of leaves of the higher Ramanujan foliation

Let us denote by \mathcal{R}_g the subbundle of the holomorphic tangent bundle $T_{B_g(\mathbf{C})}$ generated by the higher Ramanujan vector fields v_{ij} , $1 \leq i \leq j \leq g$. Since the vector fields v_{ij} commute ([32] Corollary 5.10), \mathcal{R}_g is an integrable subbundle of $T_{B_g(\mathbf{C})}$. Hence, by holomorphic Frobenius Theorem, \mathcal{R}_g induces a holomorphic foliation on $B_g(\mathbf{C})$; we call it the *higher Ramanujan foliation*.

Using the group-theoretic interpretation of $B_g(\mathbf{C})$ of Section 6, we shall also provide an explicit parametrization of every leaf $L \subset B_g(\mathbf{C})$ ⁸ of the higher Ramanujan foliation (see Proposition 7.7 for a precise statement).

Our main results in this section are the following Zariski-density statements.

THEOREM 7.1. *Every leaf $L \subset B_g(\mathbf{C})$ of the higher Ramanujan foliation is Zariski-dense in $B_{g,\mathbf{C}}$, that is, for every closed subscheme Y of $B_{g,\mathbf{C}}$, if $Y(\mathbf{C})$ contains L , then $Y(\mathbf{C}) = B_g(\mathbf{C})$.*

In particular, we obtain that the image of the solution of the higher Ramanujan equations $\varphi_g : \mathbf{H}_g \rightarrow B_g(\mathbf{C})$ is Zariski-dense in B_g .

Concerning the image of φ_g , we can actually derive the following *a priori* stronger result.

COROLLARY 7.2. *The set $\{(\tau, \varphi_g(\tau)) \in \text{Sym}_g(\mathbf{C}) \times B_g(\mathbf{C}) \mid \tau \in \mathbf{H}_g\}$ is Zariski-dense in $\text{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}}$.*

A similar statement was proven by Bertrand and Zudilin for derivatives of Siegel modular forms (see [8], Theorem 1).

The proof of both Zariski-density results will rely on the following elementary lemma.

LEMMA 7.3 (Fibration method). *Let $p : X \rightarrow S$ be a morphism of separated \mathbf{C} -schemes of finite type and let $E \subset X(\mathbf{C})$ be a subset. If, for every $s \in p(E)$, the set $E \cap X_s$ is Zariski-dense in $X_s := p^{-1}(s)$, and one of the following conditions is satisfied,*

(i) $p(E) = S(\mathbf{C})$,

(ii) p is open (in the Zariski topology) and $p(E)$ is Zariski-dense in S ,

then E is Zariski-dense in X .

PROOF. Let U be a non-empty Zariski open subset of X ; we must show that $E \cap U$ is non-empty. In both cases (i) and (ii) above, there exists a closed point $s \in p(E) \cap p(U)$. Since $E \cap X_s$ is Zariski-dense in X_s and $U \cap X_s$ is a non-empty open subset of X_s , there exists a closed point $x \in E \cap U \cap X_s \subset E \cap U$. ■

7.1. Characterization of the leaves of the higher Ramanujan foliation.

7.1.1. Let U_g be the unipotent subgroup scheme of Sp_{2g} defined by

$$U_g(R) = \left\{ \begin{pmatrix} \mathbf{1}_g & Z \\ 0 & \mathbf{1}_g \end{pmatrix} \in M_{2g \times 2g}(R) \mid Z^T = Z \right\}$$

for any ring R .

The Lie algebra of $U_g(\mathbf{C})$ is given by

$$\text{Lie } U_g(\mathbf{C}) = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \in M_{2g \times 2g}(\mathbf{C}) \mid Z^T = Z \right\},$$

and admit as a basis the vectors

$$\frac{1}{2\pi i} \begin{pmatrix} 0 & \mathbf{E}^{kl} \\ 0 & 0 \end{pmatrix} \in \text{Lie } U_g(\mathbf{C}), \quad 1 \leq k \leq l \leq g,$$

inducing the higher Ramanujan vector fields on the quotient $\text{Sp}_{2g}(\mathbf{Z}) \backslash \text{Sp}_{2g}(\mathbf{C})$ (Section 6). In particular, under the realization of $B_g(\mathbf{C})$ as an open submanifold of $\text{Sp}_{2g}(\mathbf{Z}) \backslash \text{Sp}_{2g}(\mathbf{C})$ of Corollary 6.7, the higher Ramanujan foliation on $B_g(\mathbf{C})$ is induced by the foliation on $\text{Sp}_{2g}(\mathbf{C})$ defined by $U_g(\mathbf{C})$, i.e. the foliation whose leaves are left cosets of $U_g(\mathbf{C})$ in $\text{Sp}_{2g}(\mathbf{C})$.

It follows from the above discussion that, under the identification of \mathbf{B}_g (resp. $B_g(\mathbf{C})$) with an open submanifold of $\text{Sp}_{2g}(\mathbf{C})$ (resp. $\text{Sp}_{2g}(\mathbf{Z}) \backslash \text{Sp}_{2g}(\mathbf{C})$) via Π (cf. Proposition 6.6 and Corollary 6.7), for any leaf L

⁸ By definition, a *leaf* of the higher Ramanujan foliation on $B_g(\mathbf{C})$ is a maximal connected immersed complex submanifold of $B_g(\mathbf{C})$ that is everywhere tangent to \mathcal{R}_g .

of the higher Ramanujan foliation on $B_g(\mathbf{C})$, there exists $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$ such that L is a connected component of the image of $\delta U_g(\mathbf{C}) \cap \mathbf{B}_g$ in $B_g(\mathbf{C})$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$. We shall provide a more precise result in Proposition 7.7.

7.1.2. We may also obtain an explicit *parametrization* of every leaf. For this, let us consider $\mathrm{Sym}_g(\mathbf{C}) = \{Z \in M_{g \times g}(\mathbf{C}) \mid Z^\top = Z\}$ as an open subset of the Lagrangian Grassmannian $L_g(\mathbf{C})$ (cf. discussion preceding Proposition 6.6) via

$$\begin{aligned} \mathrm{Sym}_g(\mathbf{C}) &\longrightarrow L_g(\mathbf{C}) \\ Z &\longmapsto (Z : \mathbf{1}_g), \end{aligned}$$

so that the embedding $\iota : \mathbf{H}_g \rightarrow L_g(\mathbf{C})$ defined in Proposition 6.6 is given by the restriction of $\mathrm{Sym}_g(\mathbf{C}) \rightarrow L_g(\mathbf{C})$ to \mathbf{H}_g . Furthermore, let

$$\begin{aligned} \psi : \mathrm{Sym}_g(\mathbf{C}) &\longrightarrow \mathrm{Sp}_{2g}(\mathbf{C}) \\ Z &\longmapsto \begin{pmatrix} \mathbf{1}_g & Z \\ 0 & \mathbf{1}_g \end{pmatrix}. \end{aligned}$$

REMARK 7.4. Under the obvious identification of $\mathrm{Sym}_g(\mathbf{C})$ with $\mathrm{Lie} U_g(\mathbf{C})$, the map ψ is simply the exponential $\exp : \mathrm{Lie} U_g(\mathbf{C}) \rightarrow U_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$.

Now, the action of $\mathrm{Sp}_{2g}(\mathbf{C})$ on itself by left multiplication descends to a left action of $\mathrm{Sp}_{2g}(\mathbf{C})$ on $L_g(\mathbf{C})$ given explicitly by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot (Z_1 : Z_2) = (AZ_1 + BZ_2 : CZ_1 + DZ_2).$$

For any $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, let us define

$$\begin{aligned} \psi_\delta : \delta^{-1} \cdot \mathrm{Sym}_g(\mathbf{C}) \subset L_g(\mathbf{C}) &\longrightarrow \mathrm{Sp}_{2g}(\mathbf{C}) \\ p &\longmapsto \delta^{-1} \psi(\delta \cdot p). \end{aligned}$$

Then ψ_δ induces a biholomorphism of $\delta^{-1} \cdot \mathrm{Sym}_g(\mathbf{C})$ onto the closed submanifold $\delta^{-1} U_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$.

We put

$$U_\delta := \{\tau \in \mathbf{H}_g \mid \delta \cdot (\tau : 1) \in \mathrm{Sym}_g(\mathbf{C}) \subset L_g(\mathbf{C})\} = (\delta^{-1} \cdot \mathrm{Sym}_g(\mathbf{C})) \cap \mathbf{H}_g.$$

Equivalently, if $\delta = (A \ B ; C \ D)$, then

$$U_\delta = \{\tau \in \mathbf{H}_g \mid C\tau + D \in \mathrm{GL}_g(\mathbf{C})\}.$$

DEFINITION 7.5. For any $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, we define a holomorphic map $\varphi_\delta : U_\delta \rightarrow B_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$ by

$$\varphi_\delta(\tau) := \mathrm{Sp}_{2g}(\mathbf{Z}) \psi_\delta(\tau)$$

for any $\tau \in U_\delta$.

Note that $\psi_\delta(U_\delta) = \delta^{-1} U_g(\mathbf{C}) \cap \mathbf{B}_g \subset \mathrm{Sp}_{2g}(\mathbf{C})$ by Lemma 6.5. In particular, the image of φ_δ is indeed in $B_g(\mathbf{C})$. Moreover, if $\delta \in U_g(\mathbf{C})$, then $U_\delta = \mathbf{H}_g$ and $\varphi_\delta = \varphi_g$ (cf. Theorem 6.1 (2)).

LEMMA 7.6. For any $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, U_δ is a dense connected open subset of \mathbf{H}_g .

PROOF. Let $\delta = (A \ B ; C \ D) \in \mathrm{Sp}_{2g}(\mathbf{C})$. By definition, U_δ is the complement in \mathbf{H}_g of the codimension 1 analytic subset $\{\tau \in \mathbf{H}_g \mid \det(C\tau + D) = 0\}$. It is thus a dense open subset of \mathbf{H}_g . Since \mathbf{H}_g is a connected open subset of an affine space, it follows from Riemann's extension theorem (cf. [48] Proposition 1.1.7) that U_δ is connected. \blacksquare

PROPOSITION 7.7. For every $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, the image of the map $\varphi_\delta : U_\delta \rightarrow B_g(\mathbf{C})$ is a leaf of the higher Ramanujan foliation on $B_g(\mathbf{C})$, and coincides with the image of $\delta^{-1} U_g(\mathbf{C}) \cap \mathbf{B}_g$ in $B_g(\mathbf{C})$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$. Moreover, every leaf is of this form.

PROOF. Let $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$. It was already remarked above that $\psi_\delta(U_\delta) = \delta^{-1}U_g(\mathbf{C}) \cap \mathbf{B}_g$; by definition, $\varphi_\delta(U_\delta)$ is the image of $\psi_\delta(U_\delta)$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$. In particular, since the higher Ramanujan foliation on $B_g(\mathbf{C})$ is induced by the foliation on $\mathrm{Sp}_{2g}(\mathbf{C})$ defined by $U_g(\mathbf{C})$ (cf. 7.1.1), to prove that $\varphi_\delta(U_\delta)$ is a leaf of the higher Ramanujan foliation it is sufficient to prove that it is connected. This is an immediate consequence Lemma 7.6.

Conversely, if $L \subset B_g(\mathbf{C})$ is a leaf of the higher Ramanujan foliation, then it follows from 7.1.1 that there exists $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$ such that L is a connected component of the image of $\delta^{-1}U_g(\mathbf{C}) \cap \mathbf{B}_g$ in $B_g(\mathbf{C})$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$. By the last paragraph, $\delta^{-1}U_g(\mathbf{C}) \cap \mathbf{B}_g = \psi_\delta(U_\delta)$ is connected, and we conclude that $L = \varphi_\delta(U_\delta)$. ■

REMARK 7.8. The holomorphic maps $\varphi_\delta : U_\delta \rightarrow B_g(\mathbf{C})$ are immersive but not injective in general. For instance, if $\delta = \mathbf{1}_{2g}$, then one easily verifies that $\varphi_g(\tau) = \varphi_g(\tau')$ if and only if $\tau' \in U_g(\mathbf{Z}) \cdot \tau$. Thus φ_g induces a biholomorphism of the quotient $U_g(\mathbf{Z}) \backslash \mathbf{H}_g$ onto the closed submanifold $\varphi_g(\mathbf{H}_g)$ of $B_g(\mathbf{C})$.

REMARK 7.9. There exist non-closed leaves of the higher Ramanujan foliation on $B_g(\mathbf{C})$. Take for instance

$$\delta = \begin{pmatrix} x\mathbf{1}_g & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$$

where $x \in \mathbf{R} \setminus \mathbf{Q}$. Using the classical fact that the orbit of $(x, 1)$ in \mathbf{R}^2 under the obvious left action of $\mathrm{SL}_2(\mathbf{Z})$ is dense in \mathbf{R}^2 , one may easily deduce that the leaf $L \subset B_g(\mathbf{C})$ given by the image of $\delta U_g(\mathbf{C}) \cap \mathbf{B}_g$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$ has a limit point in $B_g(\mathbf{C}) \setminus L$. In particular, the ‘‘space of leaves’’ of the higher Ramanujan foliation on $B_g(\mathbf{C})$, which may be identified with $\mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C}) / U_g(\mathbf{C})$ by Proposition 7.7, is not a Hausdorff topological space.

The dynamics of the higher Ramanujan foliation in the case $g = 1$ was thoroughly studied by Movasati in [68].

7.1.3. In the sequel, it will be useful to obtain a description of φ_δ purely in terms of the universal property of $B_g(\mathbf{C})$. Let $\delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{C})$ and define a holomorphic map $p_\delta : U_\delta \rightarrow P_g(\mathbf{C})$ by

$$p_\delta(\tau) = p_{\delta,\tau} := \begin{pmatrix} (C\tau + D)^{-1} & -\frac{1}{2\pi i} C^\top \\ 0 & (C\tau + D)^\top \end{pmatrix} \in P_g(\mathbf{C}).$$

The proof of the next lemma is a straightforward computation using the equations defining the symplectic group (cf. Remark 1.1).

LEMMA 7.10. *For every $\tau \in U_\delta \subset \mathbf{H}_g$, we have*

$$\psi_\delta(\tau) = \psi(\tau) p'_{\delta,\tau}$$

in $\mathrm{Sp}_{2g}(\mathbf{C})$, where $p'_{\delta,\tau}$ denotes the image of $p_{\delta,\tau}$ in $P'_g(\mathbf{C})$ under the isomorphism defined in Lemma 6.5.

In particular, by Lemma 6.5 and Lemma 6.8, if \mathbf{B}_g is regarded as the moduli space of principally polarized complex tori of dimension g equipped with a symplectic-Hodge basis and an integral symplectic basis, we have

$$(7.1) \quad \psi_\delta(\tau) = [(\mathbf{X}_{g,\tau}, E_{g,\tau}, \mathbf{b}_{g,\tau} \cdot p_{\delta,\tau}, \beta_{g,\tau})] \in \mathbf{B}_g$$

for every $\tau \in U_\delta$. Composing with the canonical map $\mathbf{B}_g \rightarrow B_g(\mathbf{C})$, we obtain

$$(7.2) \quad \varphi_\delta(\tau) = [(\mathbf{X}_{g,\tau}, E_{g,\tau}, \mathbf{b}_{g,\tau} \cdot p_{\delta,\tau})] \in B_g(\mathbf{C})$$

for every $\tau \in U_\delta$.

7.2. Auxiliary results. Our next objective is to prove that the leaves of the higher Ramanujan foliation on $B_g(\mathbf{C})$ are Zariski-dense in $B_{g,\mathbf{C}}$. We collect in this subsection some auxiliary results. In the last analysis, our proof is a reduction to the fact that $\mathrm{Sp}_{2g}(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sp}_{2g,\mathbf{C}}$ (Lemma 7.13).

Recall that for every $\tau \in \mathbf{H}_g$ and

$$\delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{C})$$

we put

$$j(\delta, \tau) := C\tau + D \in M_{g \times g}(\mathbf{C}),$$

so that $U_\delta = \{\tau \in \mathbf{H}_g \mid j(\delta, \tau) \in \mathrm{GL}_g(\mathbf{C})\}$.

The proof of the next lemma is a simple computation.

LEMMA 7.11. *For $\delta_1, \delta_2 \in \mathrm{Sp}_{2g}(\mathbf{C})$, we have $j(\delta_1 \delta_2, \tau) = j(\delta_1, \delta_2 \cdot \tau) j(\delta_2, \tau)$. In particular, if $\tau \in U_{\delta_2}$ and $\delta_2 \cdot \tau \in U_{\delta_1}$, then $\tau \in U_{\delta_1 \delta_2}$.*

LEMMA 7.12. *Let $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z})$, and $\tau \in U_{\delta\gamma} \subset \mathbf{H}_g$. Then $\gamma \cdot \tau \in U_\delta$ and $\varphi_{\delta\gamma}(\tau) = \varphi_\delta(\gamma \cdot \tau)$.*

PROOF. That $\gamma \cdot \tau \in U_\delta$ is a direct consequence of Lemma 7.11 and the fact that $j(\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})$ (this is true for any $\gamma \in \mathrm{Sp}_{2g}(\mathbf{R})$ and $\tau \in \mathbf{H}_g$). Under the group-theoretic interpretation, we have

$$\begin{aligned} \varphi_{\delta\gamma}(\tau) &= \mathrm{Sp}_{2g}(\mathbf{Z})\psi_{\delta\gamma}(\tau) = \mathrm{Sp}_{2g}(\mathbf{Z})(\delta\gamma)^{-1}\psi((\delta\gamma) \cdot \tau) \\ &= \mathrm{Sp}_{2g}(\mathbf{Z})\delta^{-1}\psi(\delta \cdot (\gamma \cdot \tau)) = \mathrm{Sp}_{2g}(\mathbf{Z})\psi_\delta(\gamma \cdot \tau) = \varphi_\delta(\gamma \cdot \tau). \end{aligned}$$

■

LEMMA 7.13. *The set $\mathrm{Sp}_{2g}(\mathbf{Z}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$ is Zariski-dense in $\mathrm{Sp}_{2g, \mathbf{C}}$.*

PROOF. Let Sp_{2g}^* be the open subscheme of Sp_{2g} defined by $\mathrm{Sp}_{2g}^*(R) = \{(A \ B ; \ C \ D) \in \mathrm{Sp}_{2g}(R) \mid A \in \mathrm{GL}_g(R)\}$ for any ring R . We may define an isomorphism of schemes $\mathrm{Sp}_{2g}^* \xrightarrow{\sim} \mathrm{Sym}_g \times_{\mathbf{Z}} \mathrm{Sym}_g \times_{\mathbf{Z}} \mathrm{GL}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (CA^{-1}, AB^\top, A).$$

Since $\mathrm{Sym}_g \times_{\mathbf{Z}} \mathrm{Sym}_g \times_{\mathbf{Z}} \mathrm{GL}_g$ may be identified to an open subscheme of the affine space $\mathbf{A}_{\mathbf{Z}}^{2g^2+g}$, we see that $\mathrm{Sym}_g(\mathbf{Z}) \times \mathrm{Sym}_g(\mathbf{Z}) \times \mathrm{GL}_g(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sym}_{g, \mathbf{C}} \times_{\mathbf{C}} \mathrm{Sym}_{g, \mathbf{C}} \times_{\mathbf{C}} \mathrm{GL}_{g, \mathbf{C}}$. Thus $\mathrm{Sp}_{2g}^*(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sp}_{2g, \mathbf{C}}^*$. Finally, since $\mathrm{Sp}_{2g, \mathbf{C}}$ is an irreducible scheme, we conclude that $\mathrm{Sp}_{2g}(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sp}_{2g, \mathbf{C}}$.

■

LEMMA 7.14. *Let $\tau \in \mathbf{H}_g$ and $p \in P_g(\mathbf{C})$. Then there exists $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$ such that $\tau \in U_\delta$ and $p = p_{\delta, \tau}$.*

PROOF. Let $A \in \mathrm{GL}_g(\mathbf{C})$ and $B \in M_{g \times g}(\mathbf{C})$ such that

$$p = \begin{pmatrix} A & B \\ 0 & (A^\top)^{-1} \end{pmatrix}.$$

One easily verifies, using the equation $AB^\top = BA^\top$, that

$$\delta := \begin{pmatrix} A^\top & -A^\top \tau \\ -2\pi i B^\top & A^{-1} + 2\pi i B^\top \tau \end{pmatrix} \in M_{2g \times 2g}(\mathbf{C})$$

is in $\mathrm{Sp}_{2g}(\mathbf{C})$ and satisfies the required conditions in the statement.

■

LEMMA 7.15. *For every $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$ and $\tau \in \mathbf{H}_g$, the subset*

$$S_{\delta, \tau} := \{p_{\delta\gamma, \tau} \in P_g(\mathbf{C}) \mid \gamma \in \mathrm{Sp}_{2g}(\mathbf{Z}) \text{ such that } j(\delta\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})\}$$

of $P_g(\mathbf{C})$ is Zariski-dense in $P_{g, \mathbf{C}}$.

PROOF. Let V be the unique open subscheme of $\mathrm{Sp}_{2g, \mathbf{C}}$ such that

$$V(\mathbf{C}) = \{\gamma \in \mathrm{Sp}_{2g}(\mathbf{C}) \mid j(\delta\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})\}$$

and let $h : V \rightarrow P_{g, \mathbf{C}}$ be the morphism of \mathbf{C} -schemes given on complex points by $h(\gamma) = p_{\delta\gamma, \tau}$ (note that V and $P_{g, \mathbf{C}}$ are reduced separated \mathbf{C} -schemes of finite type). It follows from Lemma 7.14 that h is surjective on complex points, thus a dominant morphism of schemes.

Now, we remark that $S_{\delta, \tau} = h(\mathrm{Sp}_{2g}(\mathbf{Z}) \cap V)$. Since $\mathrm{Sp}_{2g, \mathbf{C}}$ is irreducible and $\mathrm{Sp}_{2g}(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sp}_{2g, \mathbf{C}}$ by Lemma 7.13, $\mathrm{Sp}_{2g}(\mathbf{Z}) \cap V$ is also Zariski-dense in $\mathrm{Sp}_{2g, \mathbf{C}}$. Hence, as h is dominant and continuous for the Zariski topology, $S_{\delta, \tau}$ is Zariski-dense in $P_{g, \mathbf{C}}$.

■

7.3. Proof of Theorem 7.1 and Corollary 7.2. Recall from 5.3 that we denote the coarse moduli scheme of \mathcal{A}_g by A_g , and that we have a canonical map $j_g : \mathbf{H}_g \rightarrow A_g(\mathbf{C})$ associating to each $\tau \in \mathbf{H}_g$ the isomorphism class of the principally polarized complex torus $(\mathbf{X}_{g,\tau}, E_{g,\tau})$.

PROOF OF THEOREM 7.1. By Proposition 7.7, we must prove that, for every $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, the image of $\varphi_\delta : U_\delta \rightarrow B_g(\mathbf{C})$ is Zariski-dense in $B_{g,\mathbf{C}}$.

Let

$$\varpi_g : B_{g,\mathbf{C}} \rightarrow A_{g,\mathbf{C}}$$

be the composition of the forgetful functor $\pi_g : B_{g,\mathbf{C}} \cong \mathcal{B}_{g,\mathbf{C}} \rightarrow \mathcal{A}_{g,\mathbf{C}}$ with the canonical morphism $\mathcal{A}_{g,\mathbf{C}} \rightarrow A_{g,\mathbf{C}}$. Note that ϖ_g acts on complex points by sending an isomorphism class in $\mathcal{B}_g(\mathbf{C})$ of a principally polarized complex abelian variety endowed with a symplectic-Hodge basis to the isomorphism class in $\mathcal{A}_g(\mathbf{C})$ of the same principally polarized complex abelian variety.

By Lemma 7.3, we are reduced to proving that, for every $x \in A_g(\mathbf{C})$, the set

$$\varphi_\delta(U_\delta) \cap \varpi_g^{-1}(x)$$

is Zariski-dense in $\varpi_g^{-1}(x) \subset B_{g,\mathbf{C}}$. Indeed, by surjectivity of ϖ_g on the level of complex points, this proves in particular that $\varpi_g(\varphi_\delta(U_\delta)) = A_g(\mathbf{C})$ (cf. condition (i) in Lemma 7.3).

Let (X, λ) be a representative of the isomorphism class x . The set of complex points of the \mathbf{C} -scheme $\varpi_g^{-1}(x)$ can be identified with the set of isomorphism classes of objects of the category $\mathcal{B}_g(\mathbf{C})$ lying over (X, λ) ; we denote these isomorphism classes by $[(X, \lambda, b)]$. Then, we recall that \mathbf{C} -group scheme $P_{g,\mathbf{C}}$ acts transitively on $\varpi_g^{-1}(x)$ by

$$[(X, \lambda, b)] \cdot p := [(X, \lambda, b \cdot p)].$$

Thus, if $\tau \in \mathbf{H}_g$ satisfies $j_g(\tau) = x$, we can define a surjective morphism of \mathbf{C} -schemes⁹

$$\begin{aligned} f_\tau : P_{g,\mathbf{C}} &\rightarrow \varpi_g^{-1}(x) \\ p &\mapsto \varphi_g(\tau) \cdot p. \end{aligned}$$

Now, let $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z})$ be such that $j(\delta\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})$. By Lemma 7.12, we have $\gamma \cdot \tau \in U_\delta$ and $\varphi_{\delta\gamma}(\tau) = \varphi_\delta(\gamma \cdot \tau)$. Thus, by formula (7.2), we obtain

$$f_\tau(p_{\delta\gamma,\tau}) = \varphi_g(\tau) \cdot p_{\delta\gamma,\tau} = \varphi_{\delta\gamma}(\tau) = \varphi_\delta(\gamma \cdot \tau).$$

This proves that

$$S_{\delta,\tau} = \{p_{\delta\gamma,\tau} \in P_{g,\mathbf{C}} \mid \gamma \in \mathrm{Sp}_{2g}(\mathbf{Z}) \text{ such that } j(\delta\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})\} \subset f_\tau^{-1}(\varphi_\delta(U_\delta) \cap \varpi_g^{-1}(x)).$$

By Lemma 7.15, $S_{\delta,\tau}$ is Zariski-dense in $P_{g,\mathbf{C}}$. Hence, as f_τ is surjective and continuous for the Zariski topology, we conclude that $\varphi_\delta(U_\delta) \cap \varpi_g^{-1}(x)$ is Zariski-dense in $\varpi_g^{-1}(x)$. \blacksquare

PROOF OF COROLLARY 7.2. It is clear that $\mathrm{Sym}_g(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sym}_{g,\mathbf{C}}$. Thus, by Theorem 7.1 and Lemma 7.3 (ii) applied to the projection on the second factor

$$\mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}} \rightarrow B_{g,\mathbf{C}},$$

it suffices to prove that for every $N \in \mathrm{Sym}_g(\mathbf{Z})$ and $\tau \in \mathbf{H}_g$ we have $\varphi_g(\tau + N) = \varphi_g(\tau)$. This was already observed in Remark 7.8. \blacksquare

9. Actually, as the automorphism group of a complex principally polarized abelian variety is finite ([71] IV.21 Theorem 5), the stabilizer of $\varphi_g(\tau)$ is a finite subgroup scheme of $P_{g,\mathbf{C}}$. Therefore, f_τ is a finite surjective morphism. We shall not use this fact in our proof.

Algebraic independence for values of integral curves

Abstract

We prove a transcendence theorem concerning values of holomorphic maps from a disk to a quasi-projective variety over $\overline{\mathbf{Q}}$ that are integral curves of some algebraic vector field (defined over $\overline{\mathbf{Q}}$). These maps are required to satisfy some integrality property, besides a growth condition and a strong form of Zariski-density that are natural for integral curves of algebraic vector fields.

This result generalizes a theorem of Nesterenko concerning algebraic independence of values of the Eisenstein series E_2, E_4, E_6 . The main technical improvement in our approach is the replacement of a rather restrictive hypothesis of polynomial growth on Taylor coefficients by a geometric notion of *moderate growth* formulated in terms of Value Distribution Theory.

1. Introduction

1.1. A theorem of Nesterenko. This work was motivated by questions related to the following algebraic independence result.

Let E_2, E_4 , and E_6 be the classical Eisenstein series, seen as holomorphic functions on the complex unit disk $D := \{q \in \mathbf{C} \mid |q| < 1\}$, explicitly defined by

$$E_2(q) = 1 - 24 \sum_{j=1}^{\infty} \sigma_1(j)q^j, \quad E_4(q) = 1 + 240 \sum_{j=1}^{\infty} \sigma_3(j)q^j, \quad E_6(q) = 1 - 504 \sum_{j=1}^{\infty} \sigma_5(j)q^j$$

for every $q \in D$, where $\sigma_k(j) := \sum_{d|j} d^k \in \mathbf{Z}$. Let us also consider the q -expansion of the J -invariant

$$J(q) = 1728 \frac{E_4(q)^3}{E_4(q)^3 - E_6(q)^2} = \frac{1}{q} + 744 + \sum_{j=1}^{\infty} c(j)q^j.$$

THEOREM 1.1 (Nesterenko [75]). *For every $z \in D \setminus \{0\}$, we have*

$$\text{trdeg}_{\mathbf{Q}} \mathbf{Q}(z, E_2(z), E_4(z), E_6(z)) \geq 3.$$

This result is an improvement of Barré-Sirieix's, Diaz's, Gramain's, and Philibert's breakthrough [4] concerning the solution of a conjecture of Mahler : for every algebraic $z \in D \setminus \{0\}$, $J(z)$ is transcendental.

In order to fully motivate our contributions, we next sketch the main steps of Nesterenko's original proof.

In view of an algebraic independence criterion due to Philippon ([85] Théorème 2.11 ; see also [75] Lemma 2.5), it suffices to construct a sequence of polynomials with integral coefficients $Q_n \in \mathbf{Z}[X_0, X_1, X_2, X_3]$, for $n \gg 0$, such that $\deg Q_n = O(n \log n)$, $\log \|Q_n\|_{\infty} = O(n \log^2 n)$ — here, $\|Q_n\|_{\infty}$ denotes the maximum of the absolute values of all the coefficients of Q_n —, and

$$-an^4 \leq \log |Q_n(z, E_2(z), E_4(z), E_6(z))| \leq -bn^4$$

for some real constants $a > b > 0$.

For this, Nesterenko implemented a *method* benefiting from the fact that E_2, E_4 , and E_6 have integral Taylor coefficients in their q -expansion and satisfy the so-called *Ramanujan equations* :

$$q \frac{dE_2}{dq} = \frac{E_2^2 - E_4}{12}, \quad q \frac{dE_4}{dq} = \frac{E_2E_4 - E_6}{3}, \quad q \frac{dE_6}{dq} = \frac{E_2E_6 - E_4^2}{2}.$$

It is also essential in his construction that

- (i) [*Growth condition*] for each $k \in \{1, 2, 3\}$, the sequence of Taylor coefficients $(E_{2k}^{(j)}(0)/j!)_{j \geq 0}$ grows polynomially in j , and

(ii) [*Zero Lemma*]¹ there exists a constant $C > 0$ such that

$$\text{ord}_{q=0} P(q, E_2(q), E_4(q), E_6(q)) \leq C(\deg P)^4$$

for every non-zero polynomial $P \in \mathbf{C}[X_0, X_1, X_2, X_3] \setminus \{0\}$.

The first condition can be easily deduced from the explicit description of the Taylor coefficients of E_{2k} given above. The second, which may be regarded as a strong form of algebraic independence between the functions $q, E_2(q), E_4(q)$, and $E_6(q)$, is a non-trivial consequence of Nesterenko's *D-property* (Definition 3.B.1 below; cf. [75] Paragraph 6), an algebraic property concerning the global behavior of the foliation in \mathbf{C}^4 induced by the vector field

$$(1.1) \quad v := x_0 \frac{\partial}{\partial x_0} + \frac{(x_1^2 - x_2)}{12} \frac{\partial}{\partial x_1} + \frac{(x_1 x_2 - x_3)}{3} \frac{\partial}{\partial x_2} + \frac{(x_1 x_3 - x_2^2)}{2} \frac{\partial}{\partial x_3}.$$

A considerable part of [75] is devoted to a proof of a stronger form of the estimate in (ii).

Nesterenko's method goes as follows.

- (1) Using that the Taylor coefficients of E_2, E_4 , and E_6 are integers of polynomial growth (property (i) above), we may apply Siegel's Lemma ([60] I.1 Lemma 1) to obtain *auxiliary polynomials* with integral coefficients $P_n \in \mathbf{Z}[X_0, X_1, X_2, X_3] \setminus \{0\}$ such that $\deg P_n = n$, $\log \|P_n\|_\infty = O(n \log n)$, and

$$\text{ord}_{q=0} P_n(q, E_2(q), E_4(q), E_6(q)) \geq cn^4$$

for some constant $c > 0$.

- (2) For a fixed $z \in D \setminus \{0\}$, the next step consists in proving the existence of a sequence $j_n = O(n \log n)$ and of constants $\alpha > \beta > 0$ such that the composed function $f_n(q) := P_n(q, E_2(q), E_4(q), E_6(q))$ satisfies

$$-\alpha n^4 \leq \log |f_n^{(j_n)}(z)| \leq -\beta n^4$$

for $n \gg 0$. The main point for obtaining the above lower bound is that, if all the Taylor coefficients of f_n at $q = z$ up to a sufficiently large order are too small, then its first non-zero Taylor coefficient at $q = 0$ will have absolute value < 1 , thereby contradicting its integrality. Here, we also make essential use of property (ii) above. This is the most delicate part of the argument.

- (3) Finally, for $n \gg 0$, if we consider the differential operator

$$v^{[j_n]} := 12^{j_n} v \circ (v - 1) \circ \cdots \circ (v - (j_n - 1)),$$

then the Ramanujan equations imply that $Q_n := v^{[j_n]}(P_n) \in \mathbf{Z}[X_0, X_1, X_2, X_3]$ satisfies

$$(12q)^{j_n} f_n^{(j_n)}(q) = Q_n(q, E_2(q), E_4(q), E_6(q))$$

for every $q \in D$. The required properties for Q_n are now easily deducible from (1) and (2).

1.2. A puzzling remark.

One of the most striking features of the above method is its generality. Indeed, a close inspection of the previous arguments suggests that, if f_1, \dots, f_m are holomorphic functions on the unit disk D with integral Taylor coefficients at $q = 0$, satisfying some algebraic differential equations with rational coefficients, and verifying conditions akin to (i) and (ii) above, then, *mutatis mutandis*, the above method applied to the system (f_1, \dots, f_m) in place of (E_2, E_4, E_6) would produce another transcendence result.

This was certainly known to specialists; see, for instance, [76] Section 3, where the pertinent properties satisfied by E_2, E_4 , and E_6 were axiomatized as above — more generally, see Philippon's notion of *K-functions* introduced in [86]. Clearly, one may produce examples of such f_i *ad libitum* by algebraically manipulating Eisenstein series, but this procedure does not lead to *new* transcendence results. The problem on the existence of functions f_1, \dots, f_m satisfying the above properties, but not “related” to classical modular forms (in some imprecise sense), was explicitly stated by Zudilin in [100].

1. In Diophantine Approximation and Transcendental Number Theory, “Zero Lemma” is an umbrella term covering several auxiliary results involving estimates of number of zeros in a certain region, or zeros multiplicities, of polynomials composed with analytic functions.

Since the publication of [20] and [66], it became apparent that the phenomenon of Mirror Symmetry provides a large class of functions with integral Taylor coefficients with respect to some *canonical coordinate* and which satisfy natural algebraic differential equations — see [61], [99], [57], [58], [25] for integrality issues. In [98], Zudilin studies some candidates within Mirror Symmetry for playing the role of (f_1, \dots, f_m) , but the few cases where he is able to prove all the required properties of Nesterenko’s method — those linked to elliptic curves and $K3$ surfaces — are all of modular nature.

It becomes clear in Zudilin’s work that one of the main obstructions in applying this method for such functions is condition (i) (and, in particular, that the radius of convergence is equal to 1), which is not verified in general. In a more basic level, computing radii of convergence or getting global information on the domain of definition of such functions pertaining to Mirror Symmetry is a current research problem; see [59] for results on certain families of mirror maps.

The following phenomenon provides further evidence that condition (i) is overly restrictive. Let f be the holomorphic function on D given by $f(q) = qJ(q)$ and set $\theta := q \frac{d}{dq}$. Since $\mathbf{Q}(f, \theta f, \theta^2 f) \subset \mathbf{Q}(E_2, E_4, E_6)$, and this field extension is algebraic (see, for instance, the explicit formulas in [75] Paragraph 1), it follows from Theorem 1.1 that

$$\text{trdeg}_{\mathbf{Q}} \mathbf{Q}(z, f(z), \theta f(z), \theta^2 f(z)) \geq 3$$

for any $z \in D \setminus \{0\}$. However, Nesterenko’s method *cannot* be *directly* applied to the system $(f, \theta f, \theta^2 f)$ since the sequence $c(j)$ does not grow polynomially in j .² All the other good properties are nevertheless satisfied: f , θf , and $\theta^2 f$ have integral Taylor coefficients, the Ramanujan equations imply that f satisfies a third order algebraic differential equation with rational coefficients, and a condition similar to (ii) also holds.

This paper grew from an observation of J.-B. Bost and H. Randriambololona that the growth condition (i) in Nesterenko’s method could be replaced by a geometric notion of *moderate growth* formulated in terms of characteristic functions *à la* Nevanlinna Theory. Besides being weaker than the growth condition in (i), which in principle enlarges the domain of application of Nesterenko’s method, this geometric growth condition is preserved under some algebraic manipulations on the input functions, thereby eliminating the odd phenomenon explained in last paragraph.

We next explain our main results. Further directions and open problems are indicated below.

1.3. Our main results : a geometric approach. Our main theorem is a general geometric formulation of Nesterenko’s method valid for arbitrary rings of algebraic integers and more general quasi-projective ambient spaces.

Let us first informally introduce the geometric notions which will replace conditions (i) and (ii) above.

1.3.1. *Moderate growth.* Let X be a smooth projective variety over \mathbf{C} , and h be a C^∞ Hermitian metric on the complex manifold $X(\mathbf{C})$. Let $\omega := -\text{Im } h$ be the positive real (1,1)-form on $X(\mathbf{C})$ associated to h . To fix ideas, the reader may consider the example $X = \mathbf{P}_{\mathbf{C}}^n$ endowed with the Fubini-Study metric, for which ω is given in homogeneous coordinates $\mathbf{z} = (z_0 : \dots : z_n)$ by

$$\omega = \frac{i}{2\pi} \partial \bar{\partial} \log |\mathbf{z}|^2 = \frac{i}{2\pi} \left(\frac{\sum_{j=0}^n dz_j \wedge d\bar{z}_j}{|\mathbf{z}|^2} - \frac{\sum_{j,k=0}^n \bar{z}_j z_k dz_j \wedge d\bar{z}_k}{|\mathbf{z}|^4} \right),$$

where $|\mathbf{z}|^2 = \sum_{j=0}^n |z_j|^2$.

Let $R > 0$ be a real number and denote by $D_R := \{z \in \mathbf{C} \mid |z| < R\}$ the complex disk of radius R centered at the origin. One may measure the growth of an analytic map $\varphi : D_R \rightarrow X(\mathbf{C})$ as follows. For each $t \in (0, R)$, the area of the “disk” $\varphi(D_t)$ in $X(\mathbf{C})$ with respect to the metric h is given by

$$A_\varphi(t) := \int_{D_t} \varphi^* \omega.$$

2. Actually, $c(j) \sim \frac{e^{4\pi\sqrt{j}}}{\sqrt{2j^{3/4}}}$; see [84] or [87].

We may then form the *characteristic function*

$$T_\varphi : (0, R) \longrightarrow \mathbf{R}_{\geq 0}$$

$$r \longmapsto T_\varphi(r) := \int_0^r A_\varphi(t) d \log t,$$

and we say that φ has *moderate growth* in X if

$$\limsup_{r \rightarrow R^-} \frac{T_\varphi(r)}{\log \frac{1}{1-\frac{r}{R}}} < +\infty$$

By the compactness of $X(\mathbf{C})$, moderate growth does not depend on the choice of Hermitian metric.

When $R = 1$, any analytic map $\varphi : D \longrightarrow \mathbf{C}^n \subset \mathbf{P}^n(\mathbf{C})$ whose coordinates have Taylor coefficients of polynomial growth has moderate growth in $\mathbf{P}^n_{\mathbf{C}}$ (see Example 4.5 below). Therefore, moderate growth generalizes the growth condition (i) in Nesterenko's method.

Moderate growth is nonetheless more flexible than polynomial growth on Taylor coefficients. For instance, as long as the image of $\varphi : D_R \longrightarrow X(\mathbf{C})$ is Zariski-dense in X , moderate growth is a birational invariant in the following sense : if $f : X \longrightarrow Y$ is a birational morphism between smooth projective varieties over \mathbf{C} , then φ has moderate growth in X if and only if $f \circ \varphi$ has moderate growth in Y (cf. Theorem 4.11 below). In particular, this allows us to define, via compactifications, an unambiguous notion of moderate growth in smooth *quasi-projective* varieties.

1.3.2. *ZL-density.* Let k be a field, X a projective variety over k of dimension n , and L an ample line bundle on X . Consider a parametrized *formal curve* $\hat{\varphi} : \mathrm{Spf} k[[q]] \longrightarrow X$ in X , i.e., $\hat{\varphi}$ is a morphism of formal k -schemes.

We say that $\hat{\varphi}$ is *ZL-dense* in X if there exists a constant $C > 0$ such that, for every integer $d \geq 1$ and every non-zero global section $s \in \Gamma(X, L^{\otimes d}) \setminus \{0\}$, we have

$$\mathrm{ord}_{q=0} \hat{\varphi}^* s \leq Cd^n.$$

The exponent $n = \dim X$ in the above polynomial bound is the smallest possible (see Proposition 2.6 below). Moreover, since L is ample, a ZL-dense formal curve has Zariski-dense image. Thus, ZL-density may be regarded as a strong form of Zariski-density.

For a formal curve $\hat{\varphi} : \mathrm{Spf} k[[q]] \longrightarrow \mathbf{A}_k^n \subset \mathbf{P}_k^n$ the above notion boils down to a classical Zero Lemma property — here, $L = \mathcal{O}_{\mathbf{P}_k^n}(1)$. For instance, taking $k = \mathbf{C}$, $n = 4$, and $\hat{\varphi}$ defined by the system of formal series $(q, E_2(q), E_4(q), E_6(q))$, the Zero Lemma in condition (ii) above amounts to asserting that $\hat{\varphi}$ is ZL-dense in $\mathbf{P}_{\mathbf{C}}^4$.

We shall prove that ZL-density does not depend on the choice of L (see Proposition 2.9 below). Actually, if X is only quasi-projective, and the closed point $\hat{\varphi}(0)$ in the image of $\hat{\varphi}$ is a regular point of X , then we shall prove that ZL-density does not depend on the choice of a projective compactification of X (see Corollary 2.16 below).

1.3.3. *Statement of our main theorem and proof method.* Let K be a number field and \mathcal{O}_K be its ring of integers. By an *arithmetic variety* over \mathcal{O}_K we mean an integral scheme \mathcal{X} endowed with a separated and flat morphism of finite type $\mathcal{X} \longrightarrow \mathrm{Spec} \mathcal{O}_K$.

The following theorem formalizes and generalizes Nesterenko's method.

THEOREM 1.2. *Let \mathcal{X} be a quasi-projective arithmetic variety over \mathcal{O}_K of relative dimension $n \geq 2$, with smooth generic fiber \mathcal{X}_K , and let $\hat{\varphi} : \mathrm{Spf} \mathcal{O}_K[[q]] \longrightarrow \mathcal{X}$ be a morphism of formal \mathcal{O}_K -schemes such that, for every field embedding $\sigma : K \hookrightarrow \mathbf{C}$, the formal curve $\hat{\varphi}_\sigma : \mathrm{Spf} \mathbf{C}[[q]] \longrightarrow \mathcal{X}_\sigma$, obtained from $\hat{\varphi}$ by base change, lifts to an analytic curve $\varphi_\sigma : D_{R_\sigma} \subset \mathbf{C} \longrightarrow \mathcal{X}_\sigma^{\mathrm{an}}$ defined on a disk of radius $R_\sigma > 0$ centered at the origin.*

Assume that

$$\prod_{\sigma: K \hookrightarrow \mathbf{C}} R_\sigma = 1$$

and that there exists a vector field $v \in \Gamma(\mathcal{X}_K, T_{\mathcal{X}_K/K}) \setminus \{0\}$ on the generic fiber of \mathcal{X} such that $\hat{\varphi}_K : \text{Spf } K[[q]] \rightarrow \mathcal{X}_K$ satisfies the differential equation

$$q \frac{d\hat{\varphi}_K}{dq} = v \circ \hat{\varphi}_K.$$

If, moreover,

(1) the formal curve $\hat{\varphi}_K$ is ZL-dense in \mathcal{X}_K , and

(2) for each field embedding $\sigma : K \hookrightarrow \mathbf{C}$, the analytic curve $\varphi_\sigma : D_{R_\sigma} \rightarrow \mathcal{X}_\sigma^{\text{an}}$ has moderate growth,

then, for every $\sigma : K \hookrightarrow \mathbf{C}$, and every $z \in D_{R_\sigma} \setminus \{0\}$, the field of definition $K(\varphi_\sigma(z))$ of the complex point $\varphi_\sigma(z)$ in \mathcal{X}_K satisfies

$$\text{trdeg}_{\mathbf{Q}} K(\varphi_\sigma(z)) \geq n - 1.$$

Let us remark that the conditions of *ZL-density* and of *moderate growth*, corresponding to conditions (i) and (ii) in Nesterenko's method, are actually very mild hypotheses.

For instance, ZL-density is automatic whenever $\hat{\varphi}$ is a smooth integral curve of some vector field satisfying Nesterenko's *D*-property. When the ambient space is an affine space, this is also a theorem of Nesterenko ([75] Theorem 6), which was recently extended to a geometric framework by Binyamini [10]. In Appendix 3.B we explain how to slightly modify Binyamini's arguments to prove a similar statement for any smooth quasi-projective variety.

Moderate growth, in turn, is satisfied for curves having uniformly bounded derivative on the disk (endowed with the Poincaré metric; see Example 4.3 for a precise statement). In particular, a theorem of Brunella ([34] Theorem 16; see also [34] Theorem 15) implies that, for a generic one dimensional holomorphic foliation (with singularities) \mathcal{F} on $\mathbf{P}^n(\mathbf{C})$, any integral curve to \mathcal{F} parametrized by a disk has moderate growth.

In the broader context of Transcendental Number Theory, our result may be regarded as complementary to the Siegel-Shidlovsky and Schneider-Lang theories, which also deal with algebraic independence of values — here, at algebraic points — of integral curves of algebraic vector fields (see [37], [36], and [46] for general geometric formulations). Indeed, while the Siegel-Shidlovsky and Schneider-Lang criteria handle curves parametrized by *parabolic* Riemann surfaces, our theorem deals with the *hyperbolic* case.

Our proof of Theorem 1.2 bears the same general structure of Nesterenko's method. We also start by reducing it to a diophantine approximation statement : Theorem 7.1 below. This is done via the same algebraic independence criterion of Philippon; we explain in Appendix 3.A how to generalize it to arbitrary quasi-projective varieties. The first step in the method, concerning the construction of “auxiliary polynomials”, is replaced in our geometric framework by a construction of “auxiliary sections” given by Theorem 5.1 below, the proof of which makes essential use of Bost's *method of slopes* in Arakelov Theory ([14]; cf. [15], [17]). The second step also involves estimating some higher order derivative — here, our main tool is a general result comparing, for a section of a Hermitian line bundle on a disk, norms of jets at two distinct points; see Proposition 3.13 below and its corollaries. The third and last step is essentially the same trick using the differential equation as explained above.

1.4. Further directions and open problems. With Theorem 1.2 in hand, we may turn the puzzling remark explained above into a precise mathematical question : *is there any example of application of Theorem 1.2 whose resulting transcendence statement is not contained in Theorem 1.1?* As promising as the potential candidates from the theory of Mirror Symmetry may seem, one must face, given our current state of knowledge, the logical possibility of a negative answer. However, a *proof* of this fact would also be remarkable, since it would imply that modular functions are the only ones satisfying the (quite general) hypotheses of Theorem 1.2, which make no reference to the geometric nature of modular functions in terms of moduli of elliptic curves.

It is also natural to wonder if Theorem 1.2 admits a generalization in several variables; that is, one wishes to replace a disk by a domain in a higher dimensional complex euclidean space. This conjectural higher dimensional statement has actually good candidates of application : for any integer $g \geq 1$, there exists a higher dimensional analog of the Ramanujan equations which lives in some smooth quasi-projective variety of dimension $2g^2 + g$ and admits a solution φ_g sharing many of the relevant properties of $\varphi_1 = (E_2, E_4, E_6)$ (see [32], [33]). In [33] Section 5, we show that the fields of definition of values of φ_g are *fields of periods*

of abelian varieties. Thus, conjecturally, a generalization in several variables of Theorem 1.2 would lead to transcendence degree lower bounds in the direction of Grothendieck’s Period Conjecture for abelian varieties. In this sense, Theorem 1.2 might be seen as a first step in this program.

1.5. Organization of this article. A great effort has been done to isolate all the different techniques intervening in Nesterenko’s method and to place them in their natural generality. This distillation process is aimed not only at improving the readability of our paper, but also at making these techniques suitable for other applications in Diophantine Approximation.

Section 2 contains the definition of ZL-density and some of its basic properties ; here we use elementary Intersection Theory. Section 3 defines characteristic functions and moderate growth for 2-forms on a disk and contains basic versions of the jet estimates we shall need later ; our main result here is Proposition 3.13 (see also Corollary 3.16). Both Sections 2 and 3 are self-contained and are of independent interest.

Section 4 treats the special case of moderate growth for analytic curves (as explained in this introduction) and it depends only on the beginning of Section 3. Its main objective is to prove that, under a non-degeneracy hypothesis, this concept is a birational invariant of the target space (Theorem 4.11) ; this is essentially classical material on Nevanlinna Theory.

Section 5 is devoted to the construction of “auxiliary sections” in a geometric context. Here, we combine the concepts of moderate growth developed in Sections 3 and 4 with Bost’s *slope inequality* to obtain Theorem 5.1. This section contains a review of the prerequisites in Arakelov Theory.

In Section 6 we explain how vector fields induce derivations on global sections of line bundles and we provide some L^∞ estimates. This section is also self-contained and of independent interest.

Section 7 contains a proof of Theorem 1.2. The reader will recognize, in Lemmas 7.2, 7.3, and 7.4, natural generalizations of the three steps of Nesterenko’s method explained above.

Finally, Appendices 3.A and 3.B concern geometric generalizations of results of Philippon and Binyamini originally stated only for affine (or projective) spaces, as explained above. In Appendix 3.B we make use of the basic constructions of Section 6.

1.6. Acknowledgments. This work was supported by a public grant as part of the FMJH project, and is part of my PhD thesis under the supervision of Jean-Benoît Bost. I thank him and Hugues Randriambololona for allowing me to use their preliminary non-published notes on moderate growth as a starting point for this paper. I am also grateful to Dinh Tuan Huynh for a fruitful discussion on Nevanlinna Theory.

1.7. Terminology and notations.

1.7.1. By an (algebraic) *variety* over a field k we mean a separated integral scheme of finite type over k .

1.7.2. Recall that a line bundle L on a scheme X is *semiample* if there exists an integer $m \geq 1$ such that $L^{\otimes m}$ is generated by its global sections. Observe that ample line bundles are semiample, and that semiamplicity is preserved under pullbacks.

1.7.3. A real (1,1)-form ω on a complex manifold M can always be written, in local coordinates (z_1, \dots, z_n) on M , as

$$\omega = \frac{i}{2} \sum_{k,l=1}^n h_{kl} dz_k \wedge d\bar{z}_l$$

where $H := (h_{kl})_{1 \leq k,l \leq n}$ is a Hermitian matrix. We say that ω is *positive* (resp. *semipositive*) if the matrix H is positive-definite (resp. positive-semidefinite). Note that semipositive (1,1)-forms are stable under pullbacks.

1.7.4. By a Hermitian line bundle $\bar{L} = (L, \|\cdot\|)$ on a complex manifold M , we mean a holomorphic line bundle L on M endowed with a C^∞ Hermitian metric $\|\cdot\|$. If Θ denotes the curvature of the Chern connection on L associated to $\|\cdot\|$ (locally, $\Theta = -\partial\bar{\partial} \log \|e\|^2$ where e is some trivialization of L), then we define the *Chern curvature* of \bar{L} by

$$c_1(\bar{L}) = \frac{i}{2\pi} \Theta.$$

This is a closed real C^∞ (1,1)-form on M whose class in $H^2(M, \mathbf{R})$ coincides with the first Chern class $c_1(L)$. We say that \bar{L} is *positive* (resp. *semipositive*) if $c_1(\bar{L})$ is positive (resp. semipositive).

1.7.5. We use the standard notation

$$d^c = \frac{i}{4\pi}(\bar{\partial} - \partial) = \frac{1}{4\pi} \left(r \frac{\partial}{\partial r} \otimes d\theta - \frac{1}{r} \frac{\partial}{\partial \theta} \otimes dr \right),$$

so that $dd^c = \frac{i}{2\pi} \partial\bar{\partial}$.

1.7.6. The continuous function $\log^+ : \mathbf{R} \rightarrow \mathbf{R}$ is defined by

$$\log^+ x = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. ZL-dense formal curves in quasi-projective varieties

In this section we introduce the purely algebraic concept of *ZL-dense* formal curves, and we prove some of its basic properties. This notion refines the property of being Zariski-dense and isolates the content of the Zero Lemma necessary in Nesterenko's method; that is, a formal curve satisfies the Zero Lemma if and only if it is ZL-dense.

2.1. Degree of a divisor with respect to a line bundle. Let k be a field and X be a variety over k .

Recall from [35] 2.5 that a line bundle L on X defines an additive operator

$$\alpha \mapsto c_1(L) \cap \alpha$$

on the abelian group of algebraic cycles in X modulo rational equivalence; if α is the class of a subvariety V of X , then $c_1(L) \cap \alpha$ is by definition the class of the cycle in V associated to any Cartier divisor D of V for which $L|_V \cong \mathcal{O}_V(D)$. The r -fold composition of this operator with itself is denoted by $\alpha \mapsto c_1(L)^r \cap \alpha$.

We say that a cycle class α in X is *semipositive* if there exists an integer $m \geq 1$ such that $m\alpha$ can be represented by a non-negative cycle in X (i.e., a cycle of the form $\sum_i m_i [V_i]$ with each $m_i \geq 0$). For instance, the cycle class of a Cartier divisor D is semipositive if and only if some positive multiple of D is linearly equivalent to an effective divisor.

LEMMA 2.1. *Let L be a semiample line bundle on X . Then, for any semipositive cycle class α in X , $c_1(L) \cap \alpha$ is semipositive.*

PROOF. Let $m \geq 1$ be an integer such that $m\alpha$ is represented by the cycle $\sum_i m_i [V_i]$, with each $m_i \geq 0$.

As L is semiample, there exists an integer $n \geq 1$ such that $L^{\otimes n}$ is generated by global sections. In particular, for any subvariety V of X , the line bundle $L^{\otimes n}|_V$ on V admits a non-zero global section s_V .

For every i , $c_1(L^{\otimes n}) \cap [V_i]$ is the cycle class induced by the effective Cartier divisor $\text{div}(s_{V_i})$ on V_i , so that $nm c_1(L) \cap \alpha = c_1(L^{\otimes n}) \cap m\alpha$ is represented by the non-negative cycle $\sum_i m_i [\text{div}(s_{V_i})]$. ■

Still following the terminology of [35], for any line bundle L on X , and any r -cycle class α in X , the *L -degree* of α is defined by

$$\deg_L \alpha = \deg(c_1(L)^r \cap \alpha),$$

where \deg denotes the degree function on zero-cycle classes. If D is a Cartier divisor in X , then we denote by

$$\deg_L D = \deg(c_1(L)^{\dim X - 1} \cap [D])$$

the L -degree of the cycle class $[D]$ induced by D .

Observe that the degree of a semipositive zero-cycle is non-negative. In the next result, we use the following easy consequence of Lemma 2.1 : if L is semiample and α is semipositive, then $\deg_L \alpha \geq 0$ (cf. [35] Lemma 12.1).

PROPOSITION 2.2. *Let X be a projective variety over a field k and L (resp. M) be an ample (resp. semiample) line bundle on X . Then, there exists an integer $m \geq 1$ such that, for any semipositive r -cycle class α in X , we have*

$$0 \leq \deg_M \alpha \leq m^r \deg_L \alpha.$$

PROOF. Let $m \geq 1$ be an integer such that $N := L^{\otimes m} \otimes M^\vee$ is semiample. For any r -cycle class α in X , we have

$$m^r \deg_L \alpha = \deg_{L^{\otimes m}} \alpha = \deg_{N \otimes M} \alpha = \sum_{s=0}^r \binom{r}{s} \deg_M (c_1(N)^s \cap \alpha).$$

Since N is semiample and α is semipositive, it follows from Lemma 2.1 that each $c_1(N)^s \cap \alpha$ is semipositive. As M is also semiample, we conclude that each term in the right-hand side of the above equation is non-negative, so that $m^r \deg_L \alpha \geq \deg_M \alpha \geq 0$. \blacksquare

REMARK 2.3. By combining the above proposition with a simple induction argument in r , one can actually prove the following stronger statement. Let X be a projective variety over a field k , L be an ample line bundle on X , and M be any line bundle on X . Then, for any integer $0 \leq r \leq \dim X$, there exists a constant $C > 0$ such that $|\deg_M \alpha| \leq C \deg_L \alpha$ for every semipositive r -cycle class α in X .

COROLLARY 2.4. *Let X be a projective variety over a field k . If L and M are ample line bundles on X , then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \deg_M D \leq \deg_L D \leq C_2 \deg_M D$$

for any effective Cartier divisor D in X . \blacksquare

2.2. ZL-dense formal curves in projective varieties. Let X be an algebraic variety over a field k . By a *formal curve* in X we mean a morphism of k -schemes $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$, or, equivalently, a morphism of formal k -schemes $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow X$. The k -point of X obtained by composing the k -point of $\text{Spec } k[[q]]$ given by the ideal $(q) \subset k[[q]]$ with $\hat{\varphi}$ is denoted by $\hat{\varphi}(0)$.

Let $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ be a formal curve in X , and D be an effective Cartier divisor in X . We define the *intersection multiplicity* of D with $\hat{\varphi}$ (at $\hat{\varphi}(0)$) by

$$\text{mult}_{\hat{\varphi}} D := \text{ord}_0 \hat{\varphi}^* f,$$

where $f \in \mathcal{O}_{X, \hat{\varphi}(0)}$ is any local equation for D around $\hat{\varphi}(0)$. This clearly does not depend on the choice of f . The multiplicity function $\text{mult}_{\hat{\varphi}}$ is additive and takes values in $\mathbf{N} \cup \{+\infty\}$.

DEFINITION 2.5. Let X be a projective variety of dimension n over a field k and let L be any ample line bundle on X . We say that a formal curve $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ in X is *ZL-dense* if there exists a constant $C > 0$ such that

$$(2.1) \quad \text{mult}_{\hat{\varphi}} D \leq C (\deg_L D)^n$$

for every effective Cartier divisor D in X .

Observe that the choice of L in the above definition is irrelevant by Corollary 2.4.

Let us remark that the exponent $n = \dim X$ intervening in the polynomial bound (2.1) is the smallest possible one :

PROPOSITION 2.6. *Let X be a projective variety of dimension n over k endowed with an ample line bundle L , and $\hat{\varphi}$ be a formal curve in X . Then there exists $\varepsilon > 0$, and sequence of effective Cartier divisors $(D_i)_{i \geq 1}$ in X , satisfying*

$$\lim_{i \rightarrow +\infty} \deg_L D_i = +\infty \quad \text{and} \quad \text{mult}_{\hat{\varphi}} D_i \geq \varepsilon (\deg_L D_i)^n \quad \text{for every } i \geq 1.$$

In the above statement, we allow the possibility that $\text{mult}_{\hat{\varphi}} D_i = +\infty$ (i.e., the divisor D_i vanishes identically along the formal curve $\hat{\varphi}$) by adopting the standard convention that $+\infty \geq t$ for every $t \in \mathbf{R}$.

PROOF. Assume first that $X = \mathbf{P}_k^n$ and $L = \mathcal{O}(1)$. Consider the natural projection $\mathbf{A}_k^n \setminus \{0\} \rightarrow \mathbf{P}_k^n$ and lift $\hat{\varphi}$ to some $\hat{\psi} : \text{Spec } k[[q]] \rightarrow \mathbf{A}_k^n \setminus \{0\}$. Let $i \geq 1$ be an integer. If $P = \sum_{|I|=i} a_I X^I \in k[X_0, \dots, X_n]$ is a homogeneous polynomial of degree i , seen as a regular function on $\mathbf{A}_k^n \setminus \{0\}$, then we may write

$$\hat{\psi}^* P = \sum_{j=0}^{\infty} \left(\sum_{|I|=i} a_I b_{I,j} \right) q^j \in k[[q]],$$

for some $b_{I,j} \in k$ depending on the coefficients of the $n + 1$ formal series defining $\hat{\psi}$. Since

$$\text{card}\{I \in \mathbf{N}^{n+1} \mid |I| = i\} = \binom{i+n}{n} > \frac{1}{n!} i^n,$$

it follows from elementary linear algebra that there exists a non-zero homogeneous polynomial P_i of degree i such that $\text{ord}_0 \hat{\psi}^* P_i \geq \frac{1}{n!} i^n$. By considering the Cartier divisors D_i of \mathbf{P}_k^n induced by P_i , we see that we may take $\varepsilon = 1/n!$ in this case.

The general case follows from the above one by considering a finite surjective morphism $f : X \rightarrow \mathbf{P}_k^n$ satisfying $f^* \mathcal{O}(1) \cong L^m$ for some $m \geq 1$. \blacksquare

REMARK 2.7. It follows from the above proof that the Cartier divisors D_i can actually be taken in the linear system $|L^{\otimes m i}|$ for some fixed integer $m \geq 1$.

Any ZL-dense formal curve $\hat{\varphi}$ in a projective variety X has a dense image in the Zariski topology. Indeed, since X is projective, any Zariski-closed subset of X is contained in the support of some effective Cartier divisor of X ; then, one simply remarks that ZL-density implies that $\text{mult}_{\hat{\varphi}} D < +\infty$ for any effective Cartier divisor D in X , so that the image of $\hat{\varphi}$ is not contained in the support of D .

The following example shows that the converse is not true in general.

EXAMPLE 2.8 (Lacunary series). Let k be a field and $(n_i)_{i \geq 0}$ be an increasing sequence of natural numbers satisfying $\lim_{i \rightarrow +\infty} \frac{n_{i+1}}{n_i^2} = +\infty$. If $h \in k[[q]]$ is any formal series of the form

$$h(q) = \sum_{i \geq 0} a_i q^{n_i}, \quad a_i \neq 0$$

then the formal curve $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow \mathbf{P}_k^2$, given in homogeneous coordinates by $\hat{\varphi}(q) = (1 : q : h(q))$, is not ZL-dense. Indeed, for any integer $d \geq 1$, we may consider the homogeneous polynomial of degree n_d

$$P_d = X_0^{n_d-1} X_2 - \sum_{i=0}^d a_i X_0^{n_d-n_i} X_1^{n_i}$$

so that

$$\frac{\text{mult}_{\hat{\varphi}} \text{div}(P_d)}{n_d^2} = \frac{n_{d+1}}{n_d^2} \rightarrow +\infty$$

as $d \rightarrow +\infty$.

Observe that the image of $\hat{\varphi}$ is indeed Zariski-dense. By contradiction, if $C \subset \mathbf{P}_k^2$ is an irreducible curve containing the image of $\hat{\varphi}$, then, for any effective Cartier divisor D in \mathbf{P}_k^2 whose support does not contain C , we have $\text{mult}_{\hat{\varphi}} D = i(\hat{\varphi}(0), C \cdot D) \leq \deg C \cdot \deg D$. By construction, this is absurd for $D = \text{div}(P_d)$ and d sufficiently large.

For natural examples of ZL-dense formal curves we refer to Appendix 3.B.

2.3. Reformulation in terms of sections of an ample line bundle. Let X be an algebraic variety over a field k and $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ be a formal curve. If L is any line bundle on X , and s is a section of L on a neighborhood of $\hat{\varphi}(0)$, we may consider the vanishing order of $\hat{\varphi}^* s \in \Gamma(\text{Spec } k[[q]], \hat{\varphi}^* L)$ at $q = 0$, which coincides with the intersection multiplicity of the effective Cartier divisor $\text{div}(s)$ with $\hat{\varphi}$:

$$\text{ord}_0 \hat{\varphi}^* s = \text{mult}_{\hat{\varphi}} \text{div}(s).$$

The next proposition shows that ZL-density is a condition that has to be checked only for Cartier divisors arising from sections of powers of some fixed ample line bundle.

PROPOSITION 2.9. *Let X be a projective variety of dimension n over a field k and L be an ample line bundle on X . A formal curve $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ is ZL-dense if and only if there exists a constant $C > 0$ such that*

$$\text{ord}_0 \hat{\varphi}^* s \leq C d^n$$

for any integer $d \geq 1$, and any $s \in \Gamma(X, L^{\otimes d}) \setminus \{0\}$.

PROOF. The necessity follows from the fact that, for any $s \in \Gamma(X, L^{\otimes d}) \setminus \{0\}$, $\deg_L \operatorname{div}(s) = (\deg_L X)d$.

To prove the sufficiency, fix any finite surjective morphism $f : X \rightarrow \mathbf{P}_k^n$ such that $f^*\mathcal{O}(1)$ is isomorphic to $L^{\otimes m}$ for some $m \geq 1$. If E is an effective Cartier divisor in \mathbf{P}_k^n , then there exists a section $s \in \Gamma(\mathbf{P}_k^n, \mathcal{O}(\deg_{\mathcal{O}(1)} E))$ satisfying $E = \operatorname{div}(s)$, so that

$$(2.2) \quad \operatorname{mult}_{f \circ \hat{\varphi}} E = \operatorname{ord}_0(f \circ \hat{\varphi})^* s = \operatorname{ord}_0 \hat{\varphi}^*(f^* s) \leq C(m \deg_{\mathcal{O}(1)} E)^n = C m^n (\deg_{\mathcal{O}(1)} E)^n.$$

Let D be an effective Cartier divisor in X . Since f is finite and \mathbf{P}_k^n is normal, we may define the pushforward $f_* D$ by taking norms : there is an open affine covering $(U_i)_i$ of \mathbf{P}_k^n such that D admits a local equation h_i on each $f^{-1}(U_i)$, and we define $f_* D = [(\operatorname{Norm}_f(h_i), U_i)_i]$. As $f^* f_* D - D$ is effective, we obtain

$$\operatorname{mult}_{\hat{\varphi}} D \leq \operatorname{mult}_{\hat{\varphi}} f^* f_* D = \operatorname{mult}_{f \circ \hat{\varphi}} f_* D.$$

Note that the Weil divisor associated to $f_* D$ coincides with the pushforward (of cycles) of the Weil divisor associated to D (cf. [35] Proposition 1.4). In particular, the projection formula gives

$$\deg_{\mathcal{O}(1)} f_* D = \deg_{L^{\otimes m}} D = m^{n-1} \deg_L D$$

so that, by (2.2),

$$\operatorname{mult}_{\hat{\varphi}} D \leq C m^n (\deg_{\mathcal{O}(1)} f_* D)^n = C m^{n^2} (\deg_L D)^n. \quad \blacksquare$$

An advantage of considering the above equivalent form of ZL-density stems from the vector space structure of the sets $\Gamma(X, L^{\otimes d})$, $d \geq 1$. In general, a formal curve $\hat{\varphi} : \operatorname{Spec} k[[q]] \rightarrow X$ induces, for every integer $d \geq 1$, a decreasing filtration by linear subspaces $(E_d^i)_{i \geq 0}$ on the k -vector space $E_d := \Gamma(X, L^{\otimes d})$ defined by $E_d^i := \{s \in E_d \mid \operatorname{ord}_0 \hat{\varphi}^* s \geq i\}$.

REMARK 2.10. Since E_d is finite dimensional, there exists $i_d \geq 1$ such that $E_d^{i_d} = \bigcap_{i \geq 0} E_d^i = \{s \in E_d \mid \hat{\varphi}^* s = 0\}$. In other words, for every $s \in E_d$ such that $\hat{\varphi}^* s \neq 0$, we have $\operatorname{ord}_0 \hat{\varphi}^* s < i_d$. In particular, this shows that one may replace in Proposition 2.9 the condition “for any integer $d \geq 1$ ” by the weaker “for any sufficiently large integer d ”.

As a first application of Proposition 2.9, we use the filtration $(E_d^i)_{i \geq 0}$ to show that ZL-density is a geometric property.

PROPOSITION 2.11. *Let X be a geometrically integral projective variety over a field k and $\hat{\varphi} : \operatorname{Spf} k[[q]] \rightarrow X$ be a formal curve. Then, for any field extension K of k , the formal curve $\hat{\varphi}_K : \operatorname{Spf} K[[q]] \rightarrow X_K$, obtained from $\hat{\varphi}$ by base change, is ZL-dense in X_K if and only if $\hat{\varphi}$ is ZL-dense in X .*

PROOF. Let $d \geq 1$ and $i \geq 0$ be integers. Note that $E_d \otimes_k K$ may be canonically identified with $\Gamma(X_K, L_K^{\otimes d})$. Moreover, since E_d^i is the kernel of the k -linear map

$$\begin{aligned} E_d &\longrightarrow \Gamma(\operatorname{Spec} k[[q]], \hat{\varphi}^* L^{\otimes d}) \otimes_k k[[q]] / (q^i) \\ s &\longmapsto \hat{\varphi}^* s \pmod{q^i}, \end{aligned}$$

we conclude that $E_d^i \otimes_k K = \{t \in E_d \otimes_k K \mid \operatorname{ord}_0 \hat{\varphi}_K^* t \geq i\}$.

In particular, for every integer $d \geq 1$, and any real number $\kappa > 0$, $\operatorname{ord}_0 \hat{\varphi}^* s \leq \kappa$ for every $s \in \Gamma(X, L^{\otimes d}) \setminus \{0\}$ if and only if $\operatorname{ord}_0 \hat{\varphi}_K^* t \leq \kappa$ for every $t \in \Gamma(X_K, L_K^{\otimes d}) \setminus \{0\}$. \blacksquare

As another application of Proposition 2.9, we prove the following result which will be used in our proof of Proposition 2.15.

PROPOSITION 2.12. *Let $f : X \rightarrow Y$ be a surjective morphism between projective varieties of dimension n over a field k and let $\hat{\varphi} : \operatorname{Spec} k[[q]] \rightarrow X$ be a formal curve in X . If $\hat{\varphi}$ is ZL-dense in X , then $f \circ \hat{\varphi}$ is ZL-dense in Y .*

PROOF. Let L be an ample line bundle on X admitting a global section $s \in \Gamma(X, L) \setminus \{0\}$, and M be any ample line bundle on Y . Since f^*M is semiample, $N := L \otimes f^*M$ is ample ([41] Proposition 4.5.6 (ii)).

Let $d \geq 1$ be an integer, and $t \in \Gamma(Y, M^{\otimes d}) \setminus \{0\}$. Since f is surjective, $f^*t \neq 0$. Thus $s^{\otimes d} \otimes f^*t$ is a non-zero global section of $N^{\otimes d}$ and, since $\hat{\varphi}$ is ZL-dense in X , there exists a constant $C > 0$ independent of d such that

$$\text{ord}_0 \hat{\varphi}^*(s^{\otimes d} \otimes f^*t) \leq Cd^n.$$

To complete the proof, it is sufficient to remark that

$$\text{ord}_0(f \circ \hat{\varphi})^*(t) = \text{ord}_0 \hat{\varphi}^*(f^*t) \leq \text{ord}_0 \hat{\varphi}^*(s^{\otimes d} \otimes f^*t).$$

■

REMARK 2.13. The above proposition combined with the arguments in the proof of Proposition 2.9 actually show that for any finite surjective morphism $f : X \rightarrow Y$ between projective varieties over a field k , with Y normal, a formal curve $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ is ZL-dense in X if and only if $f \circ \hat{\varphi}$ is ZL-dense in Y .

2.4. ZL-density in quasi-projective varieties. We defined a notion of ZL-density for formal curves in projective varieties. In this paragraph, under a mild technical condition, we extend this notion, via compactification, to formal curves in *quasi-projective* varieties. To assure that we obtain a well defined notion, we must show that this does not depend on the choice of compactification.

Let X be an algebraic variety over a field k , and $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ be a formal curve. If $\hat{\varphi}(0)$ is a regular point of X , then we may define the intersection multiplicity of any *Weil divisor* with $\hat{\varphi}$. Indeed, if U is a regular open neighborhood of $\hat{\varphi}(0)$ and Z is a Weil divisor in X , then $Z \cap U$ is induced by some Cartier divisor D in U , and we define

$$\text{mult}_{\hat{\varphi}} Z = \text{mult}_{\hat{\varphi}} D.$$

We may thus mimic the proof of Proposition 2.9 to obtain the following result.

PROPOSITION 2.14. *Let X be a projective variety of dimension n over a field k , L be an ample line bundle on X , and $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ be a formal curve such that $\hat{\varphi}(0)$ is a regular point of X . Then, $\hat{\varphi}$ is ZL-dense in X if and only if there exists a constant $C > 0$ such that, for every effective Weil divisor Z in X ,*

$$\text{mult}_{\hat{\varphi}} Z \leq C(\deg_L Z)^n$$

We are now in position to prove that a modification away from $\hat{\varphi}$ does not affect ZL-density.

PROPOSITION 2.15. *Let $f : X \rightarrow Y$ be a proper morphism between projective varieties over a field k , and U be an open subset of Y such that f induces an isomorphism $f^{-1}(U) \xrightarrow{\sim} U$. If $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow f^{-1}(U) \subset X$ is a formal curve such that $\hat{\varphi}(0)$ is a regular point of X , then $\hat{\varphi}$ is ZL-dense in X if and only if $f \circ \hat{\varphi}$ is ZL-dense in Y .*

PROOF. Since $f : X \rightarrow Y$ a proper birational morphism, and Y is irreducible, f is surjective. By Proposition 2.12, if $\hat{\varphi}$ is ZL-dense in X , then $f \circ \hat{\varphi}$ is ZL-dense in Y .

Conversely, suppose that $f \circ \hat{\varphi}$ is ZL-dense in Y . Fix an ample line bundle L (resp. M) on X (resp. Y), and let Z be an effective Weil divisor in X . Since f is an isomorphism over U and $\hat{\varphi}$ factors through $f^{-1}(U)$, we have

$$\text{mult}_{\hat{\varphi}} Z = \text{mult}_{f \circ \hat{\varphi}} f_* Z.$$

As $f \circ \hat{\varphi}$ is ZL-dense in Y , there is a constant $C_1 > 0$ (not depending on Z) such that

$$\text{mult}_{f \circ \hat{\varphi}} f_* Z \leq C_1(\deg_M f_* Z)^n,$$

where $n = \dim Y = \dim X$. By the projection formula, $\deg_M f_* Z = \deg_{f^*M} Z$. Since f^*M is semiample, it follows from Proposition 2.2 that there exists a constant $C_2 > 0$ such that $\deg_{f^*M} Z \leq C_2 \deg_L Z$. We conclude that

$$\text{mult}_{\hat{\varphi}} Z \leq C_1 C_2^n (\deg_L Z)^n.$$

■

COROLLARY 2.16. *Let X be a quasi-projective variety over a field k , and $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ be a formal curve such that $\hat{\varphi}(0)$ is a regular point of X . If $j_i : X \hookrightarrow \overline{X}_i$, $i = 1, 2$, are two projective compactifications of X , then $j_1 \circ \hat{\varphi}$ is ZL-dense in \overline{X}_1 if and only if $j_2 \circ \hat{\varphi}$ is ZL-dense in \overline{X}_2 .*

PROOF. Consider the scheme theoretic image \overline{X} of $(j_1, j_2) : X \rightarrow \overline{X}_1 \times_k \overline{X}_2$ and apply Proposition 2.15 to the natural projections $\overline{X} \rightarrow \overline{X}_i$, $i = 1, 2$. \blacksquare

This enables us to define a good notion of ZL-density in a quasi-projective variety.

DEFINITION 2.17. Let X be a quasi-projective variety over a field k , and $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ be a formal curve such that $\hat{\varphi}(0)$ is a regular point of X . We say that $\hat{\varphi}$ is ZL-dense in X if there exists a projective compactification $j : X \hookrightarrow \overline{X}$ of X such that $j \circ \hat{\varphi}$ is ZL-dense in \overline{X} .

3. Moderate growth and jet estimates on complex disks

In this section we introduce *characteristic functions* and *moderate growth* of certain 2-forms on a disk; these are purely analytic notions. We then proceed to establishing natural estimates on jets of sections of holomorphic line bundles on disks. In a sense, our exposition is more basic than the usual accounts on Value Distribution Theory, since characteristic functions of analytic curves will be a special case of our construction.

The kind of jet estimates we consider here play a central role in Diophantine Approximation and transcendence proofs. They notably appear in such proofs using the formalism of Arakelov Geometry, to estimate the height of evaluation maps, when applying Bost's method of slopes (see, for instance, [14], [15], [38], [36], [46], [37]).

3.1. Characteristic functions. Let $r > 0$ be a real number, and $p \in D_r := \{z \in \mathbf{C} \mid |z| < r\}$. Recall that the *Green's function* of D_r at p is given by

$$g_{D_r, p}(z) = \log^+ \left| \frac{r^2 - \bar{p}z}{r(z - p)} \right|.$$

This is a continuous function on $\mathbf{C} \setminus \{p\}$, locally integrable over \mathbf{C} , strictly positive on $D_r \setminus \{p\}$, and vanishing identically on $\mathbf{C} \setminus D_r$.

For any locally bounded 2-form α defined on an open neighborhood of D_r , we denote

$$T_{\alpha, p}(r) := \int_{\mathbf{C}} g_{D_r, p} \alpha.$$

REMARK 3.1. An integration by parts with $u(t) = \int_{D_t} \alpha$ and $v(t) = \log t$ shows that

$$T_{\alpha, 0}(r) = \int_0^r \left(\int_{D_t} \alpha \right) \frac{dt}{t}.$$

Since $g_{D_r, p}$ is the composition of $g_{D_r, 0}$ with the automorphism of D_r given by

$$\sigma_{r, p}(z) = \frac{r^2(z - p)}{r^2 - \bar{p}z},$$

we obtain

$$T_{\alpha, p}(r) = \int_0^r \left(\int_{D_t} (\sigma_{r, p}^{-1})^* \alpha \right) \frac{dt}{t}.$$

Let $R > 0$ be a real number, and α be a locally bounded semipositive $(1, 1)$ -form on the disk D_R . The non-decreasing function

$$\begin{aligned} T_\alpha : (0, R) &\longrightarrow \mathbf{R}_{\geq 0} \\ r &\longmapsto T_{\alpha, 0}(r) \end{aligned}$$

is the *characteristic function* of α in D_R .

We shall be particularly interested in the following special case. Let $\overline{L} = (L, \|\cdot\|)$ be a semipositive Hermitian line bundle on D_R . The *characteristic function* of \overline{L} in D_R is defined by

$$T_{\overline{L}} := T_{c_1(\overline{L})}.$$

We also denote $T_{\bar{L},p}(r) = T_{c_1(\bar{L}),p}(r)$ for $p \in D_r \subset D_R$.

REMARK 3.2. Let $d \geq 1$ be an integer. As $c_1(\bar{L}^{\otimes d}) = d \cdot c_1(\bar{L})$, we have $T_{\bar{L}^{\otimes d},p}(r) = d \cdot T_{\bar{L},p}(r)$.

3.2. Forms of moderate growth. Let $R > 0$ be a real number, and α be a locally bounded semipositive (1,1)-form on D_R .

DEFINITION 3.3. We say that α has *moderate growth* if there exist real constants $a, b > 0$ such that

$$T_\alpha(r) \leq a + b \log \frac{1}{1 - \frac{r}{R}}$$

for any $0 < r < R$. When $\alpha = c_1(\bar{L})$ for some semipositive Hermitian line bundle \bar{L} on D_R , we rather say that \bar{L} has *moderate growth*.

The basic example of a form of moderate growth is the following one.

EXAMPLE 3.4 (Poincaré form). Let

$$d\mu_R := \frac{i}{2} \left(\frac{R}{R^2 - |z|^2} \right)^2 dz \wedge d\bar{z}$$

be the 2-form associated to the surface element of the *Poincaré metric* $\frac{R}{R^2 - |z|^2} |dz|$ on D_R . A direct computation shows that, for any $0 < r < R$,

$$T_{d\mu_R}(r) = \frac{\pi}{2} \log \frac{1}{1 + \frac{r}{R}} + \frac{\pi}{2} \log \frac{1}{1 - \frac{r}{R}}.$$

Thus, the 2-form $d\mu_R$ on D_R has moderate growth.

We have defined moderate growth for a form α as a growth condition on $T_{\alpha,0}(r)$ with respect to r . Our next result shows that a similar growth condition for $T_{\alpha,p}(r)$ holds uniformly for p varying in a fixed compact subset.

LEMMA 3.5. *Let $0 < r_0 < r_1 < R_1 < R$ be real numbers. Then there exists a constant $C > 0$ such that, for every $p \in D_{r_0}$, and every $r \in [R_1, R)$, we have*

$$g_{D_r,p} \leq g_{D_{r_1},p} + C g_{D_r,0}.$$

PROOF. Let $r \in [R_1, R)$ and $p \in D_{r_0}$. We set

$$C_{r,p} := \max_{z \in \partial D_{r_1}} \frac{g_{D_r,p}(z)}{g_{D_r,0}(z)}$$

and

$$f_{r,p} := g_{D_r,p} - g_{D_{r_1},p} - C_{r,p} g_{D_r,0}.$$

By definition of $C_{r,p}$, we see that $f_{r,p} \leq 0$ over ∂D_{r_1} . Moreover, $f_{r,p}$ vanishes identically on ∂D_r . Since $f_{r,p}$ is subharmonic over the domains D_{r_1} and $D_r \setminus \bar{D}_{r_1}$, by the Maximum Principle, we conclude that $f_{r,p} \leq 0$ everywhere.

To finish the proof, it is sufficient to remark that $C_{r,p}$ is uniformly bounded for $r \in [R_1, R)$ and $p \in D_{r_0}$. Indeed, for $z \in \partial D_{r_1}$, we have

$$\frac{g_{D_r,p}(z)}{g_{D_r,0}(z)} = \frac{\log \left| \frac{r^2 - \bar{p}z}{r(z-p)} \right|}{\log \left| \frac{r}{z} \right|} \leq \frac{\log \frac{R^2 + r_0 r_1}{R_1(r_1 - r_0)}}{\log \frac{R_1}{r_1}}.$$

■

PROPOSITION 3.6. *Let $R > 0$ be a real number, $K \subset D_R$ be a compact subset, and α be a locally bounded semipositive (1,1)-form on D_R . Fix $0 < R_1 < R$ such that $K \subset D_{R_1}$. If α has moderate growth, then there exist real numbers $a, b > 0$ such that*

$$\sup_{p \in K} T_{\alpha,p}(r) \leq a + b \log \frac{1}{1 - \frac{r}{R}}$$

for every $r \in [R_1, R)$.

PROOF. Let $0 < r_0 < r_1$ be real numbers such that $K \subset D_{r_0}$ and $r_1 < R_1$. By Lemma 3.5, there exists a real constant $C > 0$ such that

$$g_{D_r, p} \leq g_{D_{r_1}, p} + Cg_{D_r, 0}$$

for every $r \in [R_1, R)$ and every $p \in K$, so that

$$T_{\alpha, p}(r) \leq T_{\alpha, p}(r_1) + CT_{\alpha, 0}(r).$$

Since α has moderate growth, to conclude it is sufficient to remark that the function $p \mapsto T_{\alpha, p}(r_1)$ is continuous, thus bounded on the compact K . ■

3.3. Jets and characteristic functions. Let $r > 0$ be a real number and $p \in D_r$. We define a probability measure $\pi_{r, p}$ supported on ∂D_r by

$$\int \psi \pi_{r, p} = \frac{1}{2\pi} \int_0^{2\pi} \psi(\sigma_{r, p}^{-1}(re^{i\theta})) d\theta,$$

where $\sigma_{r, p}$ is the function defined in Remark 3.1. For the next proposition, we shall need the following classical result.

LEMMA 3.7. *As an equality of distributions on \mathbf{C} , we have*

$$-2dd^c g_{D_r, p} = \delta_p - \pi_{r, p}.$$

PROOF. Apply Remark 3.1 and Stokes' Theorem (see also 1.7.5). ■

Let U be an open subset of \mathbf{C} and $\bar{L} = (L, \|\cdot\|)$ be a Hermitian line bundle over U . If $s \in \Gamma(U, L)$ and $z \in U$, the m^{th} jet of s at z is denoted by $j_z^m s$. When s has vanishing order at least m at z , $j_z^m s$ is simply an element of the fiber of $L \otimes (\Omega_V^1)^{\otimes m}$ at z . In this case, if r is a real number strictly greater than $|z|$, we denote by $\|j_z^m s\|_r$ the norm of $j_z^m s$ with respect to the metric $\|\cdot\|$ on L and the norm on $\Omega_{D_r, z}^1$ given by the dual of the Poincaré metric $\frac{r}{r^2 - |z|^2} |dz|$ on D_r .

The following result, relating jets of sections with characteristic functions, is a basic tool in Nevanlinna Theory (see, for instance, [77] Section 2.3); variants of it were used in the context of Diophantine Approximation in [15] Proposition 4.14, [16] Section 3, and [36] Theorem 5.13.

PROPOSITION 3.8. *Let $R > 0$ be a real number, $\bar{L} = (L, \|\cdot\|)$ be a semipositive Hermitian line bundle on D_R , and $p \in D_R$. For every real number $r \in (|p|, R)$ and every global section $s \in \Gamma(D_R, L) \setminus \{0\}$, if $m := \text{ord}_p s$ denotes the vanishing order of s at p , we have*

$$(3.1) \quad \log \|j_p^m s\|_r = T_{\bar{L}, p}(r) + \int \log \|s\| \pi_{r, p} - \int g_{D_r, p} \delta_{\text{div}(s) - m[p]}.$$

We start with a lemma that follows immediately from the explicit formula for the Green's functions on disks (cf. Paragraph 3.1).

LEMMA 3.9. *With the above notation, if $E \subset D_R$ denotes the support of the divisor $\text{div}(s)$, then the distribution $\log \|s\| + mg_{D_r, p}$ on D_R defines a C^∞ function over $(D_r \setminus E) \cup \{p\}$ and a continuous function over $\partial D_r \setminus E$. Moreover,*

$$\lim_{z \rightarrow p} (\log \|s(z)\| + mg_{D_r, p}(z)) = \log \|j_p^m s\|_r.$$

Observe now that, for a fixed s , both sides in formula (3.1) are continuous with respect to r . Since E is discrete, we may thus assume that $\partial D_r \cap E = \emptyset$.

PROOF OF PROPOSITION 3.8. The Poincaré-Lelong formula yields the identity of currents

$$dd^c \log \|s\|^2 = \delta_{\text{div}(s)} - c_1(\bar{L}).$$

Thus, by Lemma 3.7,

$$c_1(\bar{L}) = -2dd^c(\log \|s\| + mg_{D_{r,p}}) + \delta_{\text{div}(s)-m[p]} + m\pi_{r,p}.$$

Since $\pi_{r,p}$ is supported on ∂D_r , and $g_{D_{r,p}}$ vanishes identically on ∂D_r , we obtain

$$T_{\bar{L},p}(r) = \int g_{D_{r,p}} \cdot (-2dd^c(\log \|s\| + mg_{D_{r,p}})) + \int g_{D_{r,p}} \delta_{\text{div}(s)-m[p]}.$$

Note that, by our choice of r and by Lemma 3.9, the above products of distributions are well-defined.

By another application of Lemma 3.7,

$$\begin{aligned} T_{\bar{L},p}(r) &= \int (\log \|s\| + mg_{D_{r,p}})(\delta_p - \pi_{r,p}) + \int g_{D_{r,p}} \delta_{\text{div}(s)-m[p]} \\ &= \int (\log \|s\| + mg_{D_{r,p}})\delta_p - \int \log \|s\| \pi_{r,p} + \int g_{D_{r,p}} \delta_{\text{div}(s)-m[p]} \end{aligned}$$

where in the second equality we used once again that $g_{D_{r,p}}$ vanishes identically on ∂D_r . To conclude, we apply once more Lemma 3.9, which ensures that the function $\log \|s\| + mg_{D_{r,p}}$ tends to $\log \|j_p^m s\|_r$ at p . ■

COROLLARY 3.10. *Let us keep the notations of Proposition 3.8. Then,*

$$(3.2) \quad \log \|j_p^m s\|_r \leq T_{\bar{L},p}(r) + \log \|s\|_{L^\infty(\partial D_r)}.$$

If, moreover, p' is another point of D_r , and m' denotes the vanishing order of s at p' , then

$$(3.3) \quad \log \|j_{p'}^{m'} s\|_r \leq T_{\bar{L},p}(r) + \log \|s\|_{L^\infty(\partial D_r)} - m' g_{D_{r,p}}(p').$$

PROOF. Since $\pi_{r,p}$ is a probability measure over ∂D_r , we have $\int \log \|s\| \pi_{r,p} \leq \log \|s\|_{L^\infty(\partial D_r)}$. Thus, the estimate (3.2) (resp. (3.3)) follows immediately from the non-negativity both of the function $g_{D_{r,p}}$ and of the distribution $\delta_{\text{div}(s)-m[p]}$ (resp. $\delta_{\text{div}(s)-m[p]-m'[p']}$). ■

3.4. A first application of moderate growth. We shall need the following elementary inequality.

LEMMA 3.11. *Let A , B , and R be positive real numbers. Set*

$$r := R \frac{B}{A+B}.$$

Then, if $\log(B/A) \geq 2$, we have

$$A \log \left(\frac{1}{1 - \frac{r}{R}} \right) - B \log r \leq 2A \log \left(\frac{B}{A} \right) - B \log R.$$

PROOF. By homogeneity, we may assume that $A = R = 1$, so that $r = B/(1+B)$ and our statement is equivalent to :

$$\log(1+B) - B \log(B/(1+B)) \leq 2 \log B$$

when $\log B \geq 2$. By subtracting $\log B$ from both sides, we see that this is yet equivalent to :

$$(1+B) \log(1+1/B) \leq \log B$$

when $\log B \geq 2$. Now, this last inequality follows trivially from the fact that $\log(1+1/B) \leq 1/B$. ■

Note that r as above is the minimum of the real function $t \mapsto A \log \left(\frac{1}{1 - \frac{t}{R}} \right) - B \log t$ defined on the open interval $(0, R)$.

PROPOSITION 3.12. *Let $R > 0$ be a real number and $\bar{L} = (L, \|\cdot\|)$ be a semipositive Hermitian line bundle on D_R . If \bar{L} has moderate growth, then there exist constants $\kappa_1, \kappa_2 > 0$ such that for every integer $d \geq 1$ and every bounded global section $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$, if we denote $m := \text{ord}_0 s$, then*

$$\log \|j_0^m s\|_R \leq \kappa_1 d + \kappa_2 d \log^+ m + \log \|s\|_{L^\infty(D_R)}.$$

PROOF. Since \bar{L} has moderate growth, there exist real numbers $a, b > 0$ such that

$$T_{\bar{L}}(r) \leq a + b \log \left(\frac{1}{1 - \frac{r}{R}} \right)$$

for any $0 < r < R$. We may assume that $b > 1$. By the jet estimate (3.2), for any $0 < r < R$, we have

$$\log \|j_0^m s\|_r \leq d \cdot T_{\bar{L}}(r) + \log \|s\|_{L^\infty(\partial D_r)},$$

so that

$$(3.4) \quad \log \|j_0^m s\|_R \leq ad + bd \log \left(\frac{1}{1 - \frac{r}{R}} \right) - m \log r + m \log R + \log \|s\|_{L^\infty(D_R)}.$$

The result is trivial if $m = 0$. We now consider two cases. If $\log(m/bd) < 2$, then we may take $r := R/2$ in (3.4) to obtain

$$\log \|j_0^m s\|_R \leq (a + (1 + e^2)b \log 2)d + \log \|s\|_{L^\infty(D_R)}.$$

If $\log(m/bd) \geq 2$, we apply Lemma 3.11 for $A = bd$ and $B = m$:

$$\log \|j_0^m s\|_R \leq ad + 2bd \log \left(\frac{m}{bd} \right) + \log \|s\|_{L^\infty(D_R)} \leq ad + 2bd \log m + \log \|s\|_{L^\infty(D_R)}.$$

■

3.5. Bounding jets via Taylor coefficients at another point. In this paragraph, we compare Taylor coefficients at different points. We start with a general result, and next we explain how moderate growth improves the estimate.

PROPOSITION 3.13. *Let $R > 0$ be a real number, $\bar{L} = (L, \|\cdot\|)$ be a semipositive Hermitian line bundle on D_R , and $q \in D_R \setminus \{0\}$. Fix a real number R_0 satisfying $|q| < R_0 < R$, and a global holomorphic section $s_0 \in \Gamma(D_R, L)$ such that $s_0(q) \neq 0$. Then there exists a real number $\kappa > 1$ such that, for every integer $J \geq 1$, every integer $d \geq 1$, and every global section $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$, if f denotes the germ of holomorphic function at q such that $s = f s_0^{\otimes d}$ in a neighborhood of q , and if $m := \text{ord}_0 s$, then*

$$\log \|j_0^m s\|_R \leq \log \left(\left(\frac{R_0}{|q|} \right)^{-J} \left(\frac{R}{R_0} \right)^m \|s\|_{L^\infty(\partial D_{R_0})} + \kappa^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) + d \cdot T_{\bar{L}}(R_0)$$

Let us first remark that if such a constant $\kappa > 0$ exists for $s_0 \in \Gamma(D_R, L)$ trivializing L at q , then an analogous constant $\bar{\kappa} > 0$ will exist for *any other* trivialization \bar{s}_0 of L in a neighborhood of q — we *do not* require \bar{s}_0 to be a global section. Indeed, if we write $s = \bar{f} \bar{s}_0^{\otimes d}$ and $\bar{s}_0 = u s_0$ in a neighborhood of q , then $f = \bar{f} u^d$ and

$$\max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \leq \max_{0 \leq j < J} \sum_{k+l=j} \frac{|\bar{f}^{(k)}(q)|}{k!} \frac{|(u^d)^{(l)}(q)|}{l!} \leq \left(J \max_{0 \leq j < J} \frac{|(u^d)^{(j)}(q)|}{j!} \right) \max_{0 \leq j < J} \frac{|\bar{f}^{(j)}(q)|}{j!}.$$

We conclude by the Cauchy inequalities, which ensure that $\max_{0 \leq j < J} \frac{|(u^d)^{(j)}(q)|}{j!}$ grows at most exponentially in $d + J$.

PROOF. By the above remark, up to replacing s_0 by $z^{-\text{ord}_0(s_0)} s_0$, we can assume that s_0 trivializes L both at q and at 0. Let $a_j \in \mathbf{C}$ be defined by the expansion

$$f(z) = z^m \sum_{j=0}^{\infty} a_j (z - q)^j$$

in a neighborhood of q , and set

$$g(z) := z^m \sum_{j=0}^{J-1} a_j (z - q)^j.$$

Note that g extends uniquely to a holomorphic function on D_R . Let $s_1, s_2 \in \Gamma(D_R, L^{\otimes d})$ be given by $s_1 := g s_0^{\otimes d}$ and $s_2 := s - s_1$. Observe that both s_1 and s_2 have vanishing order at least m at 0.

Next, we estimate $\|j_0^m s_i\|_{R_0}$, $i = 1, 2$; for this, we shall first assume that each $j_0^m s_i \neq 0$. By the jet estimate (3.2) for $p = 0$, we have

$$\log \|j_0^m s_1\|_{R_0} \leq d \cdot T_{\bar{L}}(R_0) + \log \|s_1\|_{L^\infty(\partial D_{R_0})}$$

Since $\text{ord}_q s_2 \geq J$, by the jet estimate (3.3) for $p = 0$ and $p' = q$, we have

$$\log \|j_0^m s_2\|_{R_0} \leq d \cdot T_{\bar{L}}(R_0) + \log \|s_2\|_{L^\infty(\partial D_{R_0})} - \log \left(\frac{R_0}{|q|} \right) J.$$

Thus

$$\begin{aligned} \|j_0^m s\|_R &= \|j_0^m s\|_{R_0} \left(\frac{R}{R_0} \right)^m \leq (\|j_0^m s_1\|_{R_0} + \|j_0^m s_2\|_{R_0}) \left(\frac{R}{R_0} \right)^m \\ &\leq \left(\left(\frac{R}{R_0} \right)^m \|s_1\|_{L^\infty(\partial D_{R_0})} + \left(\frac{R_0}{|q|} \right)^{-J} \left(\frac{R}{R_0} \right)^m \|s_2\|_{L^\infty(\partial D_{R_0})} \right) \exp(d \cdot T_{\bar{L}}(R_0)) \end{aligned}$$

Using that $\|s_2\|_{L^\infty(\partial D_{R_0})} \leq \|s_1\|_{L^\infty(\partial D_{R_0})} + \|s\|_{L^\infty(\partial D_{R_0})}$, we get

$$\|j_0^m s\|_R \leq \left(\left(\frac{R_0}{|q|} \right)^{-J} \left(\frac{R}{R_0} \right)^m \|s\|_{L^\infty(\partial D_{R_0})} + \left(1 + \left(\frac{R_0}{|q|} \right)^{-J} \right) \left(\frac{R}{R_0} \right)^m \|s_1\|_{L^\infty(\partial D_{R_0})} \right) \exp(d \cdot T_{\bar{L}}(R_0))$$

It should be clear at this point that the same estimate holds if $j_0^m s_1 = 0$ or $j_0^m s_2 = 0$.

We now estimate $\|s_1\|_{L^\infty(\partial D_{R_0})}$. For any $z \in \partial D_{R_0}$, we have

$$|g(z)| = R_0^m \left| \sum_{j=0}^{J-1} a_j (z - q)^j \right| \leq R_0^m \left(\sum_{j=0}^{J-1} (2R_0)^j \right) \max_{0 \leq j < J} |a_j| \leq J R_0^m \max\{1, (2R_0)^J\} \max_{0 \leq j < J} |a_j|,$$

so that

$$\|s_1\|_{L^\infty(\partial D_{R_0})} = \sup_{z \in \partial D_{R_0}} |g(z)| \|s_0(z)\|^d \leq \|s_0\|_{L^\infty(\partial D_{R_0})}^d J R_0^m \max\{1, (2R_0)^J\} \max_{0 \leq j < J} |a_j|.$$

To finish, we must bound the coefficients a_j . By definition, for any $j \in \mathbf{N}$,

$$a_j = \frac{1}{j!} \frac{d^j}{dz^j} \Big|_{z=q} \left(\frac{f(z)}{z^m} \right) = \sum_{k=0}^j \left(\frac{(-1)^k}{q^{m+k}} \binom{k+m-1}{k} \frac{f^{(j-k)}(q)}{(j-k)!} \right).$$

If $j < J$, then, for any $0 \leq k \leq j$, we have the crude but sufficient estimate

$$\binom{k+m-1}{k} < \binom{J+m-1}{J} < 2^{m+J-1} < 2^{m+J},$$

so that

$$|a_j| \leq \left(\sum_{k=0}^j \frac{1}{|q|^{m+k}} \right) 2^{m+J} \max_{0 \leq k \leq j} \frac{|f^{(k)}(q)|}{k!}.$$

Thus,

$$\max_{0 \leq j < J} |a_j| \leq J(2 \max\{1, |q|^{-1}\})^{m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!}.$$

■

PROPOSITION 3.14. *Let $R > 0$ be a real number, $\bar{L} = (L, \|\cdot\|)$ be a semipositive Hermitian line bundle over D_R , and let $K \subset D_R$ be a compact subset. Fix a real number $R_1 \in (0, R)$ such that $K \subset D_{R_1}$. If \bar{L} has moderate growth, then there exist real numbers $\kappa_0, \kappa_1 > 1$ and an integer $d_0 \geq 1$ such that, for any integer $d \geq d_0$ and any bounded section $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$ for which $m := \text{ord}_0 s$ satisfies $m \geq \kappa_0 d$, we have*

$$\|s(z)\| \leq m^{\kappa_1 d} \left(\frac{|z|}{R} \right)^m \|s\|_{L^\infty(D_R)}$$

for every $z \in K \setminus \{0\}$.

PROOF. By Proposition 3.6, there exist real numbers $a, b > 0$ depending only on (\bar{L}, K, R_1) such that

$$T_{\bar{L},z}(r) \leq a + b \log \frac{1}{1 - \frac{r}{R}}$$

for any $z \in K$ and every $R_1 \leq r < R$.

Let $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$ be a bounded section, and $z \in K \setminus \{0\}$. We may assume that $\text{ord}_z s = 0$. By the jet estimate (3.3) for $p = z$ and $p' = 0$, we have, for every $R_1 \leq r < R$,

$$\begin{aligned} \log \|s(z)\| &\leq d \cdot T_{\bar{L},z}(r) + \log \|s\|_{L^\infty(\partial D_r)} - m \log \frac{r}{|z|} \\ &\leq ad + bd \log \frac{1}{1 - \frac{r}{R}} - m \log r + m \log |z| + \log \|s\|_{L^\infty(D_R)}. \end{aligned}$$

Assume that $m \geq be^2d$. It follows from Lemma 3.11 for $A = bd$ and $B = m$ that, if

$$r := R \frac{m}{m + bd},$$

then

$$bd \log \frac{1}{1 - \frac{r}{R}} - m \log r \leq 2bd \log \frac{m}{bd} - m \log R.$$

If we also require that $m \geq b \frac{R_1}{R - R_1} d$, then $r \geq R_1$, so that

$$\log \|s(z)\| \leq 2bd \log m + (ad - 2bd \log d) + m \log \frac{|z|}{R} + \log \|s\|_{L^\infty(D_R)}.$$

Now, for every integer $d \geq e^{\frac{a}{b}}$, we have $ad - 2bd \log d \leq 0$, and we get

$$\log \|s(z)\| \leq 2bd \log m + m \log \frac{|z|}{R} + \log \|s\|_{L^\infty(M)}.$$

We may thus take $\kappa_0 := b \max \left\{ e^2, \frac{R_1}{R - R_1} \right\}$, $\kappa_1 := 2b$, and $d_0 := \lceil e^{\frac{a}{b}} \rceil$. ■

The following result is a combination of Proposition 3.13 together with the existence of a non-zero global section of L (see the remark following the statement; actually, L is holomorphically trivial on D_R) and Proposition 3.14.

COROLLARY 3.15. *Let $R > 0$ be a real number, $\bar{L} = (L, \|\cdot\|)$ be a semipositive Hermitian line bundle on D_R , $q \in D_R \setminus \{0\}$, R_0 be a real number satisfying $|q| < R_0 < R$, and s_0 be a holomorphic trivialization of L in a neighborhood of q . Assume moreover that \bar{L} has moderate growth. Then there exist real numbers $\kappa_i > 1$, $i = 0, \dots, 4$, and an integer $d_0 \geq 1$, such that, for any integer $J \geq 0$, any integer $d \geq d_0$, and every bounded section $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$ for which $m := \text{ord}_0 s$ satisfies $m \geq \kappa_0 d$, if f denotes the germ of holomorphic function at q such that $s = f s_0^{\otimes d}$ in a neighborhood of q , we have*

$$\log \|j_0^m s\|_R \leq \log \left(\frac{m^{\kappa_1 d}}{\kappa_2^J} \|s\|_{L^\infty(D_R)} + \kappa_3^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) + \kappa_4 d.$$

PROOF. We take $\kappa_2 = R_0/|q|$, $\kappa_3 = \kappa$ given by Proposition 3.13, and $\kappa_4 = T_{\bar{L}}(R_0)$. Since \bar{L} has moderate growth, we may apply Proposition 3.14 to the compact $K = \partial D_{R_0}$ to obtain real numbers $\kappa_0, \kappa_1 > 0$ and an integer $d_0 \geq 1$ such that

$$\|s\|_{L^\infty(\partial D_{R_0})} \leq m^{\kappa_1 d} \left(\frac{R_0}{R} \right)^m \|s\|_{L^\infty(D_R)}$$

for any integer $d \geq d_0$ and any bounded section $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$ such that $m := \text{ord}_0 s \geq \kappa_0 d$. We conclude by combining this bound with the estimate given by Proposition 3.13. ■

In practice, we shall be concerned with the following particular situation.

COROLLARY 3.16. *Let $R > 0$ be a real number, $\bar{L} = (L, \|\cdot\|)$ be a semipositive Hermitian line bundle of moderate growth on D_R , and $q \in D_R \setminus \{0\}$. Fix a holomorphic trivialization s_0 of L in a neighborhood of q , real constants $c_0, c_1, c_2 > 0$, with $c_0 < c_1$, and an integer $n \geq 2$. For any real number $C > 0$, there exist real numbers $\gamma_0, \gamma_1 > 0$ such that, for any sufficiently large integer d , and any bounded section $s \in \Gamma(D_R, L^{\otimes d})$ satisfying*

$$c_0 d^n \leq m := \text{ord}_0 s \leq c_1 d^n, \quad \log \|s\|_{L^\infty(D_R)} \leq c_2 d \log d,$$

and

$$\max_{0 \leq j < \lceil \gamma_0 d \log d \rceil} \log \frac{|f^{(j)}(q)|}{j!} \leq -\gamma_1 d^n,$$

where $s = f s_0^{\otimes d}$ on a neighborhood of q , we have

$$\log \|j_0^m s\|_R \leq -C d \log d.$$

PROOF. Let $\kappa_i > 0$, $i = 0, \dots, 4$, be the constants given by Corollary 3.15. We claim that it suffices to take $\gamma_0 > (\log \kappa_2)^{-1}(n\kappa_1 + c_2 + C)$ and $\gamma_1 > c_1 \log \kappa_3$.

Indeed, let $s \in \Gamma(D_R, L^{\otimes d})$ be as in the statement. Since $n \geq 2$ and $m \geq c_0 d^n$, if d is sufficiently large, we shall have $m \geq \kappa_0 d$, so that the conclusion of Corollary 3.15 for $J := \lceil \gamma_0 d \log d \rceil$ applies :

$$(3.5) \quad \log \|j_0^m s\|_R \leq \log \left(\frac{m^{\kappa_1 d}}{\kappa_2^J} \|s\|_{L^\infty(D_R)} + \kappa_3^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) + \kappa_4 d.$$

Since $m \leq c_1 d^n$, $\log \|s\|_{L^\infty(D_R)} \leq c_2 d \log d$, and $J \geq \gamma_0 d \log d$, we obtain

$$\log \left(\frac{m^{\kappa_1 d}}{\kappa_2^J} \|s\|_{L^\infty(D_R)} \right) \leq (n\kappa_1 + c_2 - (\log \kappa_2)\gamma_0) d \log d + \kappa_1 (\log c_1) d.$$

Thus, by our choice of γ_0 , if d is sufficiently large, we get

$$(3.6) \quad \log \left(\frac{m^{\kappa_1 d}}{\kappa_2^J} \|s\|_{L^\infty(D_R)} \right) \leq -(C + \varepsilon_1) d \log d,$$

for some $\varepsilon_1 > 0$.

Since $m \leq c_1 d^n$, $J \leq \gamma_0 d \log d + 1$, and $\max_{0 \leq j < J} \log \frac{|f^{(j)}(q)|}{j!} \leq -\gamma_1 d^n$, we have

$$\log \left(\kappa_3^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) \leq (c_1 \log \kappa_3 - \gamma_1) d^n + \gamma_0 (\log \kappa_3) d \log d + (\log \kappa_3)(d + 1).$$

Thus, as $n \geq 2$, and by our choice of γ_1 , if d is sufficiently large, we obtain

$$(3.7) \quad \log \left(\kappa_3^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) \leq -\varepsilon_2 d^n$$

for some $\varepsilon_2 > 0$.

We conclude by applying (3.6) and (3.7) in (3.5), and by taking d to be sufficiently large. ■

4. Analytic curves of moderate growth in quasi-projective varieties

This section contains mostly well-known techniques and results in Nevanlinna Theory. These are nevertheless written in the literature in a form not suitable for our purposes. Although our proofs may vary, many of the theory concerning growth of entire analytic maps (“parabolic case”) easily translate into our hyperbolic situation; we refer the reader to the recent monograph [77] for a thorough exposition of the general parabolic theory (in several variables).

4.1. Analytic curves of moderate growth in compact complex manifolds. Let $R > 0$ be a real number, M be a compact complex manifold, and $\varphi : D_R \rightarrow M$ be an analytic map. Fix any Hermitian metric h on M , and let $\omega := -\text{Im } h$ be the positive $(1,1)$ -form associated to h ; in other words, if $h = \sum_{k,l=1}^n h_{kl} dz_k \otimes d\bar{z}_l$ in a local chart (z_1, \dots, z_n) of M , then $\omega = \frac{i}{2} \sum_{k,l=1}^n h_{kl} dz_k \wedge d\bar{z}_l$.

DEFINITION 4.1. We say that $\varphi : D_R \rightarrow M$ has *moderate growth* if the semipositive $(1,1)$ -form $\varphi^*\omega$ on D_R has moderate growth (see Definition 3.3).

This notion does not depend on the choice of the Hermitian metric h . Indeed, since M is compact, any two Hermitian metrics on M are “comparable” : if h_0 is another Hermitian metric on M , then there exist real numbers $\alpha, \beta > 0$ such that $\alpha \| \cdot \|_{h_0} \leq \| \cdot \|_h \leq \beta \| \cdot \|_{h_0}$.

REMARK 4.2. It follows from Remark 3.1 that $T_{\varphi^*\omega}(r) = \int_0^r \left(\int_{D_t} \varphi^*\omega \right) d \log t$ can be thought of as a logarithmic integral of the areas of the disks $\varphi(D_t)$ in M for $0 < t < r$.

We next consider a simple example of curves of moderate growth.

EXAMPLE 4.3 (Bounded derivative). Let $\varphi : D_R \rightarrow M$ be an analytic map, and h be a Hermitian metric on M . Then we can write

$$\varphi^*\omega = \|\varphi'(z)\|_{R,h}^2 d\mu_R,$$

where $d\mu_R$ is the Poincaré form defined in Example 3.4, and $\|\varphi'(z)\|_{R,h}$ denotes the norm of the tangent map $D_z\varphi : T_z D_R \rightarrow T_{\varphi(z)} M$ with respect to the Poincaré metric on D_R , and the Hermitian metric h on M . Since $d\mu_R$ has moderate growth, then the analytic curve φ has moderate growth in M whenever the function $z \mapsto \|\varphi'(z)\|_{R,h}$ is bounded on D_R (e.g., φ extends continuously to $\overline{D_R} \subset \mathbf{C}$).

4.2. Nevanlinna’s characteristic function. Let M be a complex manifold, $\overline{L} = (L, \| \cdot \|)$ be a semi-positive Hermitian line bundle on M , and $s_0 \in \Gamma(M, L) \setminus \{0\}$ be a non-zero global section.

Let $R > 0$ be a real number and $\varphi : D_R \rightarrow M$ be an analytic map whose image is not contained in the support of $\text{div}(s_0)$. We define, for every $0 < r < R$,

$$m_{\varphi, \overline{L}, s_0}(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\|s_0(\varphi(re^{i\theta}))\|} d\theta$$

and

$$N_{\varphi, \overline{L}, s_0}(r) := (\text{ord}_0 \varphi^* s_0) \log r + \sum_{0 < |z| \leq r} (\text{ord}_z \varphi^* s_0) \log \frac{r}{|z|}.$$

Then we can form the *Nevanlinna characteristic function* on the interval $(0, R)$

$$T_{\varphi, \overline{L}, s_0} := m_{\varphi, \overline{L}, s_0} + N_{\varphi, \overline{L}, s_0}.$$

Let us introduce a temporary notation for the next proposition. If s is a global section of φ^*L and $m = \text{ord}_0 s$, we denote by $\ell(s)$ the unique element of the fiber of φ^*L at $0 \in D_R$ such that $j_0^m s = \ell(s) \otimes dz^{\otimes m}$ (the “leading coefficient” of s).

The following classical identity (cf. [77] Theorem 2.3.31) is an immediate corollary of Proposition 3.8 applied to the section $s = \varphi^* s_0$ and the point $p = 0$.

PROPOSITION 4.4 (Nevanlinna’s First Fundamental Theorem). *For every $0 < r < R$, we have*

$$T_{\varphi^* \overline{L}}(r) = T_{\varphi, \overline{L}, s_0}(r) + \log \|\ell(\varphi^* s_0)\|.$$

■

As an application we show that, when $R = 1$, polynomial growth of Taylor coefficients implies moderate growth.

EXAMPLE 4.5. Let $\varphi = (\varphi_1, \dots, \varphi_n) : D \rightarrow \mathbf{C}^n$ be an analytic map with coordinates $\varphi_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j$. Assume that there exist a real number $C > 1$ and an integer $d \geq 1$ such that

$$|a_{ij}| \leq C j^d$$

for every $1 \leq i \leq n$ and $j \geq 0$. Then, when identifying \mathbf{C}^n with the open affine subset $U_0 = \{(p_0 : \cdots : p_n) \in \mathbf{P}^n(\mathbf{C}) \mid p_0 \neq 0\}$ of $\mathbf{P}^n(\mathbf{C})$ via $(z_1, \dots, z_n) \mapsto (1 : z_1 : \cdots : z_n)$, the analytic curve $\varphi : D \rightarrow \mathbf{P}^n(\mathbf{C})$ has moderate growth.

Indeed, let $\overline{\mathcal{O}(1)}$ denote the line bundle $\mathcal{O}(1)$ on $\mathbf{P}^n(\mathbf{C})$ endowed with the Fubini-Study metric; that is,

$$\|X_i(p)\| = \frac{|p_i|}{\sqrt{|p_0|^2 + \cdots + |p_n|^2}}$$

for every $0 \leq i \leq n$, and $p = (p_0 : \cdots : p_n) \in \mathbf{P}^n(\mathbf{C})$. Since $N_{\varphi, \overline{\mathcal{O}(1)}, X_0}$ vanishes identically, by Proposition 4.4, it is sufficient to prove that there exist $a, b > 0$ such that

$$m_{\varphi, \overline{\mathcal{O}(1)}, X_0}(r) \leq a + b \log \frac{1}{1-r}$$

for every $0 < r < 1$.

For any real numbers $t_1, \dots, t_m \geq 0$, we have $\log^+(\sum_{i=1}^m t_i) \leq \sum_{i=1}^m \log^+ t_i + \log m$, so that

$$\log \frac{1}{\|X_0(\varphi(re^{i\theta}))\|} = \log \sqrt{1 + \sum_{i=1}^n |\varphi_i(re^{i\theta})|^2} \leq \sum_{i=1}^n \log^+ |\varphi_i(re^{i\theta})| + \log(\sqrt{1+n}).$$

Since

$$|\varphi_i(re^{i\theta})| \leq C \sum_{j=0}^{\infty} j^d r^j \leq Cd! \left(\frac{1}{1-r}\right)^{d+1},$$

we may take $a = \log(\sqrt{1+n}) + n \log(Cd!)$ and $b = n(d+1)$.

4.3. The field of moderate functions on a disk. In this paragraph we study more closely the case $M = \mathbf{P}^1(\mathbf{C})$. We refer to [93] Chapters V-VII for a survey on the classical work on this subject.

Let $R > 0$ be a real number and f be meromorphic function on D_R , i.e., an analytic map $f : D_R \rightarrow \mathbf{P}^1(\mathbf{C})$ which is not constant equal to $\infty = (0 : 1)$.

DEFINITION 4.6. We say that f is a *moderate function* on D_R if the analytic map $f : D_R \rightarrow \mathbf{P}^1(\mathbf{C})$ has moderated growth.

If $\overline{\mathcal{O}(1)}$ denotes the tautological line bundle $\mathcal{O}(1)$ on $\mathbf{P}^1(\mathbf{C})$ endowed with the Fubini-Study metric (see Example 4.5), we denote

$$T_f := T_{f^* \overline{\mathcal{O}(1)}}.$$

By Proposition 4.4, we have

$$T_f = m_{f, \overline{\mathcal{O}(1)}, X_0} + N_{f, \overline{\mathcal{O}(1)}} + O(1),$$

where $O(1)$ denotes a constant. To lighten the notation, we shall write $m_f = m_{f, \overline{\mathcal{O}(1)}, X_0}$ (resp. $N_f = N_{f, \overline{\mathcal{O}(1)}, X_0}$).

Let \mathcal{K}_{D_R} denote the field of meromorphic functions on D_R . It is classical (and easy to prove) that characteristic functions are compatible with the algebraic structure of \mathcal{K}_{D_R} in the following sense : for $f, g \in \mathcal{K}_{D_R} \setminus \{0\}$ and $n \in \mathbf{Z} \setminus \{0\}$, we have

$$(4.1) \quad T_{f+g} \leq T_f + T_g + O(1), \quad T_{fg} \leq T_f + T_g + O(1), \quad T_{f^n} = |n|T_f + O(1)$$

It follows from the above relations that the subset $\mathcal{K}_{D_R}^m$ of \mathcal{K}_{D_R} consisting of moderate meromorphic functions is a field.

PROPOSITION 4.7 (cf. [77] Lemma 2.5.15). *Let f, f_1, \dots, f_n be meromorphic functions on D_R . If f is algebraic over the field $\mathbf{C}(f_1, \dots, f_n) \subset \mathcal{K}_{D_R}$, then there exist real numbers $a, b > 0$ such that*

$$T_f \leq a + b \sum_{i=1}^n T_{f_i}.$$

PROOF. Let d be the degree of f over $\mathbf{C}(f_1, \dots, f_n)$. If $d = 0$, then the result follows immediately from formulas (4.1). Assume that $d \geq 1$, and let $P = X^d - g_{d-1}X^{d-1} - \dots - g_0 \in \mathbf{C}(f_1, \dots, f_n)[X]$ be the minimal polynomial of f . Since each $g_i \in \mathbf{C}(f_1, \dots, f_n)$, it suffices to prove that $T_f \leq \sum_{i=0}^{d-1} T_{g_i} + O(1)$.

By formulas (4.1), we have

$$T_{f^d} = T_{(g_{d-1}f^{d-2} + \dots + g_1)f + g_0} \leq T_{g_{d-1}f^{d-2} + \dots + g_1} + T_f + T_{g_0} + O(1).$$

By descending induction, we get

$$T_{f^d} \leq (d-1)T_f + \sum_{i=0}^{d-1} T_{g_i} + O(1).$$

As $T_{f^d} = d \cdot T_f + O(1)$, we obtain

$$T_f \leq \sum_{i=0}^{d-1} T_{g_i} + O(1).$$

■

COROLLARY 4.8. *The field of moderate functions $\mathcal{K}_{D_R}^m$ is algebraically closed in \mathcal{K}_{D_R} .* ■

In particular, since the inclusion $D_R \rightarrow \mathbf{P}^1(\mathbf{C})$ is easily seen to be an analytic map of moderate growth (see Example 4.3), the field $\mathcal{K}_{D_R}^m$ contains the field of (univalued) algebraic meromorphic functions on D_R .

4.4. Birational invariance and moderate growth in quasi-projective varieties. In this paragraph, we establish the birational invariance of moderate growth under a non-degeneracy hypothesis. Our arguments follow closely those of [77] 2.5; we claim no originality.

In what follows, if f is a meromorphic function on D_R , we denote the divisor of zeros (resp. poles) of f by $\text{div}_0(f)$ (resp. $\text{div}_\infty(f)$), so that $\text{div}(f) = \text{div}_0(f) - \text{div}_\infty(f)$.

LEMMA 4.9 (cf. [77] Theorem 2.5.7). *Let M be a compact complex manifold endowed with a semipositive Hermitian line bundle $\bar{L} = (L, \|\cdot\|)$. Fix global sections $s_0, s_1 \in \Gamma(M, L)$, with $s_0 \neq 0$. Then, for any analytic map $\varphi : D_R \rightarrow M$ whose image is not contained in the support of $\text{div}(s_0)$, if we denote by f the unique meromorphic function on D_R such that $f\varphi^*s_0 = \varphi^*s_1$, we have*

$$T_f \leq T_{\varphi, \bar{L}, s_0} + O(1).$$

PROOF. Let H be the support of $\text{div}(s_0)$. For $p \in M \setminus H$, we have

$$\log \sqrt{1 + \frac{\|s_1(p)\|^2}{\|s_0(p)\|^2}} = \log \frac{1}{\|s_0(p)\|} + \log \sqrt{\|s_0(p)\|^2 + \|s_1(p)\|^2}.$$

Since M is compact, the functions $\|s_i\|$ on M are bounded, so that

$$\log \sqrt{1 + \frac{\|s_1\|^2}{\|s_0\|^2}} \leq \log \frac{1}{\|s_0\|} + O(1)$$

over $M \setminus H$. In particular, we get

$$m_f \leq m_{\varphi, \bar{L}, s_0} + O(1).$$

Since the $\text{div}_\infty(f) \leq \text{div}(s_0)$, the bound

$$N_f \leq N_{\varphi, \bar{L}, s_0}$$

is trivial. ■

For the next lemma, we endow the line bundle $\mathcal{O}(1)$ over $\mathbf{P}^n(\mathbf{C})$ with the Fubini-Study metric as in Example 4.5. Moreover, if $D = \sum_{z \in D_R} n_z [z]$ is a divisor in D_R , we denote $i(z, D) := n_z$.

LEMMA 4.10. Let $\varphi : D_R \rightarrow \mathbf{P}^n(\mathbf{C})$ be an analytic map whose image is not contained in the support of $\text{div}(X_0)$. For $1 \leq j \leq n$, let us denote by f_j the unique meromorphic function on D_R such that $f_j \varphi^* X_0 = \varphi^* X_j$. Then

$$T_{\varphi, \overline{\mathcal{O}(1)}, X_0} \leq \sum_{j=1}^n T_{f_j} + O(1).$$

PROOF. We first prove that $m_{\varphi, \overline{\mathcal{O}(1)}, X_0} \leq \sum_{j=1}^n m_{f_j}$. For any real numbers $a_1, \dots, a_n \geq 0$, we have

$$\log \left(1 + \sum_{j=1}^n a_j \right) \leq \sum_{j=1}^n \log(1 + a_j).$$

Thus, for any $0 < r < R$,

$$\begin{aligned} m_{\varphi, \overline{\mathcal{O}(1)}, X_0}(r) &= \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + \sum_{j=1}^n |f_j(re^{i\theta})|^2} d\theta \\ &\leq \sum_{j=1}^n \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + |f_j(re^{i\theta})|^2} d\theta = \sum_{j=1}^n m_{f_j}(r) \end{aligned}$$

Next, observe that to prove $N_{\varphi, \overline{\mathcal{O}(1)}, X_0} \leq \sum_{j=1}^n N_{f_j}$ it suffices to show that

$$\text{div}(\varphi^* X_0) \leq \sum_{j=1}^n \text{div}_\infty(f_j).$$

Since each $\text{div}_\infty(f_j)$ is an effective divisor, it is sufficient to prove that, for every $z \in D_R$, there exists $1 \leq j \leq n$ such that $i(z, \text{div}(\varphi^* X_0)) \leq i(z, \text{div}_\infty(f_j))$. Now, for any $1 \leq j \leq n$, since $f_j \varphi^* X_0 = \varphi^* X_j$, we may write

$$\text{div}_\infty(f_j) = \text{div}(\varphi^* X_0) + \text{div}_0(f) - \text{div}(\varphi^* X_j).$$

Finally, we simply remark that for any $z \in D_R$ for which $X_0(\varphi(z)) = 0$ (i.e., $\text{ord}_z \varphi^* X_0 > 0$ or, equivalently, $i(z, \text{div}(\varphi^* X_0)) > 0$), there exists $1 \leq j \leq n$ such that $X_j(\varphi(z)) \neq 0$ (i.e., $\text{ord}_z \varphi^* X_j = 0$), so that $i(z, \text{div}_\infty(f_j)) = i(z, \text{div}(\varphi^* X_0)) + i(z, \text{div}_0(f)) \geq i(z, \text{div}(\varphi^* X_0))$. \blacksquare

Let $f, g : I \rightarrow \mathbf{R}$ be real functions defined on some interval $I \subset \mathbf{R}$. We say that f and g are *comparable* if there exist real numbers $a, b, c, d > 0$ such that

$$af - b \leq g \leq cf + d$$

everywhere on I .

THEOREM 4.11 (cf. [77] Theorem 2.5.18). Let $R > 0$ be a real number, X be a smooth projective variety of dimension n over \mathbf{C} , and $\varphi : D_R \rightarrow X^{\text{an}}$ be an analytic map whose image is Zariski-dense in X . Then, for any positive $(1, 1)$ -form ω on X , and any transcendence basis (f_1, \dots, f_n) of the function field $\mathbf{C}(X)$ of X , the real functions $T_{\varphi^* \omega}$ and $\sum_{j=1}^n T_{f_j \circ \varphi}$ on $(0, R)$ are comparable. In particular, φ has moderate growth in X if and only if $f_j \circ \varphi$ are moderate functions on D_R for every $1 \leq j \leq n$.

Observe that the Zariski-density hypothesis above ensures that, for any rational function f on X , the image of φ is not contained in the indeterminacy locus of f , so that $f \circ \varphi$ is a well-defined meromorphic function on D_R .

PROOF. Let $i : X \rightarrow \mathbf{P}_{\mathbf{C}}^N = \text{Proj} \mathbf{C}[X_0, \dots, X_N]$ be a closed immersion such that $i \circ \varphi(D_R)$ is not contained in the support of $\text{div}(X_0)$, and consider the rational functions $g_j \in \mathbf{C}(X)$, $1 \leq j \leq N$, given by restriction of X_j/X_0 to X .

It follows from the compactness of X^{an} (cf. remark following Definition 3.3), and from Theorem 4.4, that the functions $T_{\varphi^* \omega}$ and $T_{i \circ \varphi, \overline{\mathcal{O}(1)}, X_0}$ are comparable. By Lemmas 4.9 and 4.10, the functions $T_{i \circ \varphi, \overline{\mathcal{O}(1)}, X_0}$ and $\sum_{j=1}^N T_{g_j \circ \varphi}$ are comparable. Furthermore, as $\mathbf{C}(X) = \mathbf{C}(g_1, \dots, g_N)$ is an algebraic extension of $\mathbf{C}(f_1, \dots, f_n)$,

we deduce from formulas (4.1) and from Proposition 4.7 that $\sum_{j=1}^N T_{g_j \circ \varphi}$ and $\sum_{j=1}^n T_{f_j \circ \varphi}$ are comparable. Our statement follows by transitivity of comparability. \blacksquare

In particular, moderate growth in projective varieties is a birational invariant.

COROLLARY 4.12. *Let $f : X \rightarrow Y$ be a birational morphism between smooth projective varieties over \mathbf{C} . If $R > 0$ is a real number, then an analytic map $\varphi : D_R \rightarrow X^{\text{an}}$ with Zariski-dense image has moderated growth if and only if $f \circ \varphi : D_R \rightarrow Y^{\text{an}}$ has moderate growth.* \blacksquare

Combining the standard argument in the proof of Corollary 2.16 with a resolution of singularities yields the following.

COROLLARY 4.13. *Let X be a smooth quasi-projective variety over \mathbf{C} , and let $j_i : X \hookrightarrow \overline{X}_i$, $i = 1, 2$, be smooth projective compactifications of X . If $R > 0$ is a real number and $\varphi : D_R \rightarrow X^{\text{an}}$ is an analytic map with Zariski-dense image, then $j_1 \circ \varphi$ has moderate growth if and only if $j_2 \circ \varphi$ has moderate growth.* \blacksquare

We may thus define an unambiguous notion of moderate growth for Zariski-dense analytic curves in quasi-projective varieties.

DEFINITION 4.14. Let X be a smooth quasi-projective variety, $R > 0$ be a real number, and $\varphi : D_R \rightarrow X^{\text{an}}$ be an analytic map with Zariski-dense image. We say that φ has *moderate growth* if there exists smooth projective compactification $j : X \hookrightarrow \overline{X}$ of X such that $j \circ \varphi : D_R \rightarrow \overline{X}^{\text{an}}$ has moderate growth.

5. Construction of auxiliary sections

We prove in this section Theorem 5.1 below, generalizing the construction of auxiliary polynomials in Nesterenko's method. Our approach, based on Bost's method of slopes, differs from the classical combinatorial one. However, the backbone of the argument remains the same : Minkowski's theorem on minima of lattices (see Proposition 5.7 below).

5.1. Notation and statement. Let K be a number field and \mathcal{O}_K its ring of integers. Recall that, if \mathcal{X} is an arithmetic variety over \mathcal{O}_K (i.e., an integral scheme \mathcal{X} with a separated and flat morphism of finite type $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$) with smooth generic fiber \mathcal{X}_K , a Hermitian line bundle $\overline{L} = (L, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbf{C}})$ over \mathcal{X} is the data of a line bundle L on \mathcal{X} and a family of C^∞ Hermitian metrics $\|\cdot\|_\sigma$ on the holomorphic line bundles L_σ over $\mathcal{X}_\sigma^{\text{an}}$ deduced from L by the field embeddings $\sigma : K \hookrightarrow \mathbf{C}$, that is invariant under complex conjugation.

If $d \geq 1$ is an integer, and $s \in \Gamma(\mathcal{X}, L^{\otimes d})$ is a global section, we denote

$$\|s\|_{\mathcal{X}} := \max_{\sigma} \|s\|_{\sigma, L^\infty(\mathcal{X}_\sigma^{\text{an}})}.$$

This section is devoted to the proof of the following theorem.

THEOREM 5.1. *Let \mathcal{X} be a projective arithmetic variety of relative dimension n over \mathcal{O}_K with smooth generic fiber \mathcal{X}_K , and $\hat{\varphi} : \text{Spf } \mathcal{O}_K[[q]] \rightarrow \mathcal{X}$ be a morphism of formal \mathcal{O}_K -schemes such that, for every field embedding $\sigma : K \hookrightarrow \mathbf{C}$, the formal curve $\hat{\varphi}_\sigma : \text{Spf } \mathbf{C}[[q]] \rightarrow \mathcal{X}_\sigma$ lifts to an analytic curve $\varphi_\sigma : D_{R_\sigma} \rightarrow \mathcal{X}_\sigma^{\text{an}}$ defined on some complex disk of radius $R_\sigma > 0$. Assume that the image of $\hat{\varphi}_K : \text{Spf } K[[q]] \rightarrow \mathcal{X}_K$ is Zariski-dense, that each φ_σ has moderate growth in $\mathcal{X}_\sigma^{\text{an}}$, and that $\prod_{\sigma} R_\sigma = 1$. Fix any Hermitian line bundle $\overline{L} = (L, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbf{C}})$ on \mathcal{X} such that L_K is ample on \mathcal{X}_K . Then, there are constants $C_1, C_2, C_3 > 0$ such that, for every large enough positive integer d , there exists $s \in \Gamma(\mathcal{X}, L^{\otimes d})$ such that*

$$m := \text{ord}_0 \hat{\varphi}^* s > C_1 d^n$$

and

$$\log \|s\|_{\mathcal{X}} \leq C_2 d + C_3 d \log m.$$

REMARK 5.2. By Paragraph 4.4 one could also assume that \mathcal{X} is only quasi-projective over $\text{Spec } \mathcal{O}_K$, and then construct ‘‘auxiliary sections’’ on any projective compactification of \mathcal{X} having smooth generic fiber.

If we require the stronger condition of ZL-density of $\hat{\varphi}_K$ instead of Zariski-density, we obtain the following.

COROLLARY 5.3. *With hypotheses and notation as in Theorem 5.1, if moreover $\hat{\varphi}_K : \mathrm{Spf} K[[q]] \rightarrow \mathcal{X}_K$ is ZL-dense, then there exist constants $c_1, c_2 > 0$ such that, for every large enough positive integer d , there exists $s \in \Gamma(\mathcal{X}, L^{\otimes d}) \setminus \{0\}$ such that*

$$\mathrm{ord}_0 \hat{\varphi}^* s > c_1 d^n$$

and

$$\log \|s\|_{\mathcal{X}} \leq c_2 d \log d.$$

5.2. Recollections on Arakelov theory ; the slope inequality. For the convenience of the reader, we recollect in this paragraph some fundamental notions and results concerning Hermitian vector bundles over rings of algebraic integers. Proofs and further developments can be found in [14] Appendix A, [15] Paragraphs 4.1-4.2, and [19] Paragraphs 3.1-3.3.

Let K be number field, \mathcal{O}_K be its ring of integers, and set $S := \mathrm{Spec} \mathcal{O}_K$. Recall that a *Hermitian vector bundle* over S is a couple $\overline{E} = (E, (\|\cdot\|_{\sigma})_{\sigma:K \hookrightarrow \mathbf{C}})$, where E is a projective \mathcal{O}_K -module of finite type, and $(\|\cdot\|_{\sigma})_{\sigma:K \hookrightarrow \mathbf{C}}$ is a family of Hermitian norms over $E_{\sigma} := E \otimes_{\sigma: \mathcal{O}_K \hookrightarrow \mathbf{C}} \mathbf{C}$, invariant under complex conjugation. If $\mathrm{rk} E = 1$, we say that \overline{E} is a *Hermitian line bundle* over S .

The multilinear constructions in the category of projective modules over \mathcal{O}_K (e.g., tensor products, quotients, Hom) make sense in the category of Hermitian vector bundles over S .

DEFINITION 5.4. Let $\overline{E} = (E, (\|\cdot\|_{\sigma})_{\sigma:K \hookrightarrow \mathbf{C}})$ be a Hermitian vector bundle over S , and fix $s \in \det E \setminus \{0\}$. We define the *Arakelov degree* of \overline{E} by

$$\widehat{\mathrm{deg}}(\overline{E}) := \log |(\det E)/\mathcal{O}_K s| - \sum_{\sigma:K \hookrightarrow \mathbf{C}} \log \|s\|_{\sigma} \in \mathbf{R}.$$

This is easily seen not to depend on the choice of s . We define moreover the *normalized Arakelov degree* of \overline{E} by

$$\widehat{\mathrm{deg}}_n(\overline{E}) := \frac{1}{[K:\mathbf{Q}]} \widehat{\mathrm{deg}}(\overline{E})$$

and the *slope* of \overline{E} by

$$\hat{\mu}(\overline{E}) := \frac{1}{\mathrm{rk} E} \widehat{\mathrm{deg}}_n(\overline{E})$$

when $\mathrm{rk} E > 0$, and $\hat{\mu}(\overline{E}) := -\infty$ when $\mathrm{rk} E = 0$.

PROPOSITION 5.5 (cf. [15] 4.1.1). *The following properties hold :*

(1) *If \overline{L} and \overline{M} are Hermitian line bundles over S , we have*

$$\widehat{\mathrm{deg}}(\overline{L} \otimes \overline{M}) = \widehat{\mathrm{deg}}(\overline{L}) + \widehat{\mathrm{deg}}(\overline{M}).$$

(2) *Let \overline{E} be a Hermitian vector bundle over S and*

$$E = E^0 \supset E^1 \supset \dots \supset E^N \supset \{0\}$$

be a filtration of E by saturated \mathcal{O}_K -submodules. Then

$$\widehat{\mathrm{deg}}(\overline{E}) = \widehat{\mathrm{deg}}(\overline{E^N}) + \sum_{i=0}^{N-1} \widehat{\mathrm{deg}}(\overline{E^i/E^{i+1}}),$$

where $\overline{E^N}$ (resp. $\overline{E^i/E^{i+1}}$) denotes the Hermitian vector bundle with underlying module E^N (resp. E^i/E^{i+1}) and Hermitian structure induced by \overline{E} .

(3) *For every Hermitian vector bundle \overline{E} over S , and every Hermitian line bundle \overline{L} over S , we have*

$$\hat{\mu}(\overline{E} \otimes \overline{L}) = \hat{\mu}(\overline{E}) + \widehat{\mathrm{deg}}_n(\overline{L}).$$

Let \overline{E} and \overline{F} be Hermitian vector bundles over S . For every maximal ideal \mathfrak{p} of \mathcal{O}_K , we denote by $\|\cdot\|_{\mathfrak{p}}$ the non-archimedean norm over $\text{Hom}_{K_{\mathfrak{p}}}(E_{K_{\mathfrak{p}}}, F_{K_{\mathfrak{p}}})$ associated to the $\mathcal{O}_{K,\mathfrak{p}}$ -lattice $\text{Hom}_{\mathcal{O}_{K,\mathfrak{p}}}(E_{\mathcal{O}_{K,\mathfrak{p}}}, F_{\mathcal{O}_{K,\mathfrak{p}}})$; explicitly, if $\varphi \in \text{Hom}_{K_{\mathfrak{p}}}(E_{K_{\mathfrak{p}}}, F_{K_{\mathfrak{p}}}) \setminus \{0\}$, then $\|\varphi\|_{\mathfrak{p}} := |\mathcal{O}_K/\mathfrak{p}|^{-v_{\mathfrak{p}}(\varphi)}$, where $v_{\mathfrak{p}}(\varphi) = \max\{n \in \mathbf{Z} \mid \pi_{\mathfrak{p}}^{-n}\varphi \in \text{Hom}_{\mathcal{O}_{K,\mathfrak{p}}}(E_{\mathcal{O}_{K,\mathfrak{p}}}, F_{\mathcal{O}_{K,\mathfrak{p}}})\}$ and $\pi_{\mathfrak{p}}$ denotes some uniformizer of $\mathcal{O}_{K,\mathfrak{p}}$. For a field embedding $\sigma : K \hookrightarrow \mathbf{C}$, we consider the operator norm on $\text{Hom}_{\mathbf{C}}(E_{\sigma}, F_{\sigma})$:

$$\|\varphi\|_{\sigma} = \max_{v \in E_{\sigma} \setminus \{0\}} \frac{\|\varphi(v)\|_{\sigma}}{\|v\|_{\sigma}}.$$

Then, the *height* of a non-zero K -linear map $\varphi : E_K \rightarrow F_K$ is defined by

$$h_{\overline{E}, \overline{F}}(\varphi) = \frac{1}{[K : \mathbf{Q}]} \left(\sum_{\mathfrak{p}} \log \|\varphi\|_{\mathfrak{p}} + \sum_{\sigma} \log \|\varphi\|_{\sigma} \right).$$

If $\varphi = 0$, our convention is that $h_{\overline{E}, \overline{F}}(\varphi) := -\infty$.

PROPOSITION 5.6 (Slope inequality; [15] Proposition 4.5). *With the above notations, if $\varphi : E_K \rightarrow F_K$ is injective, then*

$$\hat{\mu}(\overline{E}) \leq \hat{\mu}_{\max}(\overline{F}) + h_{\overline{E}, \overline{F}}(\varphi),$$

where $\hat{\mu}_{\max}(\overline{F}) := \sup\{\hat{\mu}(\overline{F}') \mid F' \neq 0 \text{ is an } \mathcal{O}_K\text{-submodule of } F\}$.

Let us point out that $\hat{\mu}_{\max}(\overline{F})$ is attained by a saturated submodule of F (cf. [14] A.3). In particular, if $\text{rk } F = 1$, then $\hat{\mu}_{\max}(\overline{F}) = \hat{\mu}(\overline{F})$.

5.3. Short vectors in filtered Hermitian vector bundles. Let K be a field, \mathcal{O}_K its ring of integers, and $S = \text{Spec } \mathcal{O}_K$. Let $\overline{E} = (E, (\|\cdot\|_{\sigma})_{\sigma:K \hookrightarrow \mathbf{C}})$ be a non-zero Hermitian vector bundle over S ; we denote its *first successive minimum* by

$$\lambda_1(\overline{E}) := \inf \left\{ \max_{\sigma} \|s\|_{\sigma} \mid s \in E \setminus \{0\} \right\}.$$

Since $s \mapsto s \otimes 1$ identifies E with a lattice in the \mathbf{R} -vector space $E \otimes_{\mathbf{Z}} \mathbf{R}$, the first successive minimum is attained by some element $s \in E \setminus \{0\}$.

PROPOSITION 5.7 (Minkowski). *Let \overline{E} be a non-zero Hermitian vector bundle over S . Then*

$$\log \lambda_1(\overline{E}) \leq -\hat{\mu}(\overline{E}) + \frac{1}{2} \log(\text{rk } E) + \frac{\log |\Delta_K|}{2[K : \mathbf{Q}]} + \frac{1}{2} \log[K : \mathbf{Q}],$$

where Δ_K denotes the discriminant of K over \mathbf{Q} .

This statement might be obtained from [19] pp. 1027-1028 by considering the Hermitian vector bundle over $\text{Spec } \mathbf{Z}$ given by the direct image of \overline{E} via $S \rightarrow \text{Spec } \mathbf{Z}$.

Let $(\overline{E}_d)_{d \geq 1}$ be a family of Hermitian vector bundles over S such that

$$r_d := \text{rk } E_d \rightarrow +\infty$$

as $d \rightarrow +\infty$. Assume that, for every $d \geq 1$, we are given a *separated* filtration

$$E_d^0 = E_d \supset E_d^1 \supset E_d^2 \supset \dots$$

by saturated \mathcal{O}_K -submodules. We endow each E_d^m with the Hermitian vector bundle structure induced from \overline{E}_d .

PROPOSITION 5.8. *With the above notation, assume that there exists an integer $k \geq 1$ and a double sequence $(a_{d,m})_{d \geq 1, m \geq 0}$ of positive real numbers, non-decreasing in m for every $d \geq 1$, such that*

$$(5.1) \quad \text{rk}(E_d^m / E_d^{m+1}) \leq k$$

and

$$(5.2) \quad \hat{\mu}(\overline{E}_d^m / \overline{E}_d^{m+1}) \leq a_{d,m}$$

for every $d \geq 1$ and $m \geq 0$. Then, for every $d \geq 1$ such that $r_d \neq 0$, there exists $m \geq \lfloor \frac{r_d}{2k} \rfloor$ and $s \in E_d^m \setminus E_d^{m+1}$ satisfying

$$\max_{\sigma} \log \|s\|_{\sigma} \leq \max\{0, -2\hat{\mu}(\overline{E}_d)\} + \frac{1}{2} \log r_d + a_{d,m} + \frac{\log |\Delta_K|}{2[K : \mathbf{Q}]} + \frac{1}{2} \log [K : \mathbf{Q}].$$

PROOF. Let $d \geq 1$ such that $r_d \neq 0$ and set

$$m' := \lfloor \frac{r_d}{2k} \rfloor,$$

so that, by (5.1),

$$r_d^{m'} := \text{rk } E_d^{m'} \geq r_d - km' \geq \frac{1}{2} r_d > 0.$$

By Proposition 5.5 (2), we have

$$\hat{\mu}(\overline{E}_d) = \frac{r_d^{m'}}{r_d} \hat{\mu}(\overline{E}_d^{m'}) + \frac{1}{r_d} \sum_{0 \leq i < m'} \text{rk}(E_d^i/E_d^{i+1}) \hat{\mu}(\overline{E}_d^i/E_d^{i+1}).$$

(When $E_d^i/E_d^{i+1} = 0$, $\text{rk}(E_d^i/E_d^{i+1}) = 0$ and $\hat{\mu}(\overline{E}_d^i/E_d^{i+1}) = -\infty$, so that $\text{rk}(E_d^i/E_d^{i+1}) \hat{\mu}(\overline{E}_d^i/E_d^{i+1}) = 0$ by convention.) Using hypotheses (5.1) and (5.2), and that $(a_{d,m})$ is non-decreasing in m for every d , we obtain

$$\hat{\mu}(\overline{E}_d) \leq \frac{r_d^{m'}}{r_d} \hat{\mu}(\overline{E}_d^{m'}) + \frac{m' k a_{d,m'}}{r_d},$$

or, equivalently,

$$-\hat{\mu}(\overline{E}_d^{m'}) \leq -\frac{r_d}{r_d^{m'}} \hat{\mu}(\overline{E}_d) + \frac{m' k}{r_d^{m'}} a_{d,m'}.$$

Since $r_d \leq 2r_d^{m'}$ and $m' \leq \frac{r_d}{2k}$, we conclude that

$$-\hat{\mu}(\overline{E}_d^{m'}) \leq \max\{0, -2\hat{\mu}(\overline{E}_d)\} + a_{d,m'}.$$

Let $s \in E_d^{m'}$ be such that $\max_{\sigma} \|s\|_{\sigma} = \lambda_1(\overline{E}_d^{m'})$. Then Proposition 5.7 yields

$$\begin{aligned} \max_{\sigma} \log \|s\|_{\sigma} &\leq -\hat{\mu}(\overline{E}_d^{m'}) + \frac{1}{2} \log r_d^{m'} + \frac{\log |\Delta_K|}{2[K : \mathbf{Q}]} + \frac{1}{2} \log [K : \mathbf{Q}] \\ &\leq \max\{0, -2\hat{\mu}(\overline{E}_d)\} + \frac{1}{2} \log r_d^{m'} + a_{d,m'} + \frac{\log |\Delta_K|}{2[K : \mathbf{Q}]} + \frac{1}{2} \log [K : \mathbf{Q}]. \end{aligned}$$

Thus $m := \max\{i \in \mathbf{N} \mid s \in E_d^i\} \geq m'$ satisfies the conclusion of our statement. \blacksquare

5.4. Proof of Theorem 5.1. Consider the notation and hypotheses of Theorem 5.1. Let us first observe that if Theorem 5.1 holds for some particular choice of Hermitian metric $(\|\cdot\|_{\sigma})_{\sigma:K \hookrightarrow \mathbf{C}}$ on L , then a similar statement holds for any other choice of metric, up to modifying the constant C_2 . We may thus assume that each $(L_{\sigma}, \|\cdot\|_{\sigma})$ is a positive Hermitian line bundle on $\mathcal{X}_{\sigma}^{\text{an}}$.

For every integer $d \geq 1$,

$$E_d := \Gamma(\mathcal{X}, L^{\otimes d})$$

is a projective \mathcal{O}_K -module of finite type. For each field embedding $\sigma : K \hookrightarrow \mathbf{C}$, we may consider the uniform norm $\|\cdot\|_{L^{\infty}(\mathcal{X}_{\sigma}^{\text{an}})}$ on $E_{d,\sigma}$ induced by the Hermitian metric $\|\cdot\|_{\sigma}$ on L_{σ} .

Note that the norm $\|\cdot\|_{L^{\infty}(\mathcal{X}_{\sigma}^{\text{an}})}$ is not Hermitian in general. We denote by $\|\cdot\|_{d,\sigma}$ the *John norm* on $E_{d,\sigma}$ attached to $\|\cdot\|_{L^{\infty}(\mathcal{X}_{\sigma}^{\text{an}})}$ (cf. [18] Appendix F); this is a Hermitian norm on $E_{d,\sigma}$ satisfying

$$(5.3) \quad \|\cdot\|_{L^{\infty}(\mathcal{X}_{\sigma}^{\text{an}})} \leq \|\cdot\|_{d,\sigma} \leq (2 \text{rk } E_d)^{1/2} \|\cdot\|_{L^{\infty}(\mathcal{X}_{\sigma}^{\text{an}})}.$$

We may thus consider the Hermitian vector bundle over $\text{Spec } \mathcal{O}_K$

$$\overline{E}_d := (E_d, (\|\cdot\|_{d,\sigma})_{\sigma:K \hookrightarrow \mathbf{C}}).$$

We define a decreasing filtration $(E_d^m)_{m \geq 0}$ by saturated submodules on E_d via

$$E_d^m := \{s \in E_d \mid \text{ord}_0 \hat{\varphi}^* s \geq m\}.$$

Since the image of $\hat{\varphi}_K : \text{Spf } K[[q]] \rightarrow \mathcal{X}_K$ is Zariski-dense, $(E_d^m)_{m \geq 0}$ is a separated filtration. The subquotients E_d^m/E_d^{m+1} bear Hermitian vector bundle structures $\overline{E_d^m/E_d^{m+1}}$ induced by $\overline{E_d}$.

Let us denote by Ω the fiber of coherent sheaf $\Omega_{\text{Spf } \mathcal{O}_K[[q]]/\mathcal{O}_K}^1$ at the point of $\text{Spf } \mathcal{O}_K[[q]]$ given by the ideal $(q) \subset \mathcal{O}_K[[q]]$. This is a trivial \mathcal{O}_K -module generated by dq . In what follows, we endow Ω with a structure of Hermitian line bundle $\overline{\Omega}$, defined by

$$\|\alpha dq\|_\sigma = |\sigma(\alpha)|$$

for any $\alpha \in \mathcal{O}_K$ and any embedding $\sigma : K \hookrightarrow \mathbf{C}$. Observe that $\overline{\Omega}$ is isomorphic to the trivial Hermitian line bundle, and therefore $\text{deg } \overline{\Omega} = 0$.

Let $\hat{\varphi}(0) : \text{Spec } \mathcal{O}_K \rightarrow \mathcal{X}$ denote the reduction of $\hat{\varphi}$ modulo q , i.e., the composition of $\hat{\varphi}$ with the closed immersion $\text{Spec } \mathcal{O}_K \rightarrow \text{Spf } \mathcal{O}_K[[q]]$ associated to the ideal (q) . The Hermitian structure on L endows $\hat{\varphi}(0)^*L$ with the structure of a Hermitian line bundle over $\text{Spec } \mathcal{O}_K$.

For every integers $d \geq 1$ and $m \geq 0$, we have an injective \mathcal{O}_K -linear map

$$\gamma_d^m : E_d^m/E_d^{m+1} \rightarrow \hat{\varphi}(0)^*L^{\otimes d} \otimes_{\mathcal{O}_K} \Omega^{\otimes m}$$

defined by mapping the class $[s] \in E_d^m/E_d^{m+1}$ of $s \in E_d^m$ to $j_0^m \hat{\varphi}^* s$ (the jet of order m at $q = 0$ of $\hat{\varphi}^* s$).

LEMMA 5.9. *There exist constants $\kappa_1, \kappa_2 > 0$ such that for every integers $d \geq 1$ and $m \geq 0$ we have*

$$h(\gamma_{d,K}^m) \leq \kappa_1 d + \kappa_2 d \log^+ m,$$

where h denotes the height of $\gamma_{d,K}^m$ with respect to the Hermitian vector bundles $\overline{E_d^m/E_d^{m+1}}$ and $\hat{\varphi}(0)^*\overline{L}^{\otimes d} \otimes_{\mathcal{O}_K} \overline{\Omega}^{\otimes m}$.

PROOF. Let $\sigma : K \hookrightarrow \mathbf{C}$ be a field embedding. Since $\varphi_\sigma : D_{R_\sigma} \rightarrow \mathcal{X}_\sigma^{\text{an}}$ has moderate growth, it follows from Proposition 3.12 that there exist constants $\kappa_{1,\sigma}, \kappa_{2,\sigma} > 0$ such that, for any $d \geq 1$ and $m \geq 0$ for which $E_d^m/E_d^{m+1} \neq 0$, and any $s \in E_{d,\sigma}^m \setminus E_{d,\sigma}^{m+1}$, we have

$$(5.4) \quad \log \|j_0^m \varphi_\sigma^* s\|_{R_\sigma} - \log \|s\|_{L^\infty(\mathcal{X}_\sigma^{\text{an}})} \leq \kappa_{1,\sigma} d + \kappa_{2,\sigma} d \log^+ m.$$

Here, the norm $\|\cdot\|_{R_\sigma}$ is the norm $\|\cdot\|_r$, introduced before Proposition 3.8, in the special case where $r = R_\sigma$ and \overline{L} is $\varphi_\sigma^* L_\sigma$ equipped with the pullback of $\|\cdot\|_\sigma$.

Note that

$$\|j_0^m \hat{\varphi}^* s\|_\sigma = R_\sigma^{-m} \|j_0^m \varphi_\sigma^* s\|_{R_\sigma},$$

where $\|j_0^m \hat{\varphi}^* s\|_\sigma$ denotes the norm of $j_0^m \hat{\varphi}^* s = \gamma_d^m([s])$ with respect to the Hermitian structure of $\hat{\varphi}(0)^*\overline{L}^{\otimes d} \otimes_{\mathcal{O}_K} \overline{\Omega}^{\otimes m}$. The estimate (5.4), together with (5.3), shows that

$$\log \|\gamma_d^m\|_\sigma + m \log R_\sigma \leq \kappa_{1,\sigma} d + \kappa_{2,\sigma} d \log^+ m.$$

Since $\prod_{\sigma: K \hookrightarrow \mathbf{C}} R_\sigma = 1$, we obtain

$$\sum_\sigma \log \|\gamma_d^m\|_\sigma \leq \left(\sum_\sigma \kappa_{1,\sigma} \right) d + \left(\sum_\sigma \kappa_{2,\sigma} \right) d \log^+ m.$$

Since γ_d^m is defined over \mathcal{O}_K , we have $\|\gamma_{d,K}^m\|_{\mathfrak{p}} \leq 1$ for every maximal ideal \mathfrak{p} of \mathcal{O}_K , so that

$$h(\gamma_{d,K}^m) \leq \frac{1}{[K:\mathbf{Q}]} \sum_\sigma \log \|\gamma_d^m\|_\sigma \leq \frac{1}{[K:\mathbf{Q}]} \left(\sum_\sigma \kappa_{1,\sigma} \right) d + \frac{1}{[K:\mathbf{Q}]} \left(\sum_\sigma \kappa_{2,\sigma} \right) d \log^+ m.$$

■

END OF PROOF OF THEOREM 5.1. Let us first remark that, as L_K is ample, we have

$$r_d = \text{rk } E_d = \dim \Gamma(\mathcal{X}_K, L_K^{\otimes d}) \sim_{d \rightarrow +\infty} \frac{\deg_{L_K} \mathcal{X}_K}{n!} d^n.$$

In particular, $\log r_d = O(d)$ as $d \rightarrow +\infty$. We shall apply Proposition 5.8 for $(E_d^m)_{d \geq 1, m \geq 0}$ defined as above. This suffices by the estimates (5.3).

Note that condition (5.1) is trivially verified for $k = 1$. Moreover, by the same argument of [15] Proposition 4.4 (cf. [15] Lemma 4.1) and by the estimates (5.3), there exists a constant $c > 0$ such that $-\hat{\mu}(\overline{E_d}) \leq cd$ for every $d \geq 1$. Thus, to finish our proof, it is sufficient to find constants $a, b > 0$ such that

$$\hat{\mu}(\overline{E_d^m/E_d^{m+1}}) \leq a_{d,m} := ad + bd \log^+ m$$

for every $d \geq 1$ and $m \geq 0$ (condition (5.2)).

By Lemma 5.9, there exist constants $\kappa_1, \kappa_2 > 0$ such that, for every $d \geq 1$ and $m \geq 0$ such that $E_d^m/E_d^{m+1} \neq 0$, we have

$$h(\gamma_{d,K}^m) \leq \kappa_1 d + \kappa_2 d \log^+ m.$$

Thus, since $\gamma_{d,K}^m$ is injective, we may apply the Slope Inequality (Proposition 5.6) to obtain

$$\hat{\mu}(\overline{E_d^m/E_d^{m+1}}) \leq \hat{\mu}(\hat{\varphi}(0)^* \overline{L}^{\otimes d} \otimes_{\mathcal{O}_K} \overline{\Omega}^{\otimes m}) + h(\gamma_{d,K}^m) \leq (\kappa_1 + \hat{\mu}(\hat{\varphi}(0)^* \overline{L}))d + \kappa_2 d \log^+ m.$$

■

6. Derivatives of sections of line bundles along vector fields

A crucial step in Nesterenko's method involves applying a certain differential operator (deduced from the Ramanujan equations) to auxiliary polynomials. It is also important to understand how this differential operator affects the degree and the norm $\|\cdot\|_\infty$ of a polynomial.

Our generalization of Nesterenko's proof replaces polynomials of degree d by global sections of the d th tensor power of some ample line bundle. In this section we explain how to derive global sections of tensor powers of a line bundle L along a vector field v . Under a projectivity hypothesis, we also explain how L^∞ norms with respect to some Hermitian metric on L are affected by a differential operator deduced from v .

6.1. The basic definition. Let M be a compact connected complex manifold, and L be a line bundle over M endowed with a global holomorphic section $s_0 \in \Gamma(M, L) \setminus \{0\}$. To L is associated the graded ring $R = \bigoplus_{d \geq 0} R_d$, where $R_d := \Gamma(M, L^{\otimes d})$.

Let v be a meromorphic vector field on M , and assume that v is holomorphic on the open subset $M_{s_0} := \{p \in M \mid s_0(p) \neq 0\}$. Then there is a smallest integer $k \geq 0$, the "order of pole of v at $\text{div}(s_0)$ ", such that $v \otimes s_0^{\otimes k}$ defines a global holomorphic section of $TM \otimes L^{\otimes k}$.

The vector field v induces a \mathbf{C} -derivation of degree $k + 1$ of the graded ring R

$$\partial_v : R \longrightarrow R$$

given as follows. By definition, ∂_v is the zero map on $R_0 \cong \mathbf{C}$. Let $d \geq 1$ be an integer, $s \in R_d$, and $f : M_{s_0} \rightarrow \mathbf{C}$ be the holomorphic function for which $s = f s_0^{\otimes d}$ over M_{s_0} . Then $\partial_v s \in R_{d+k+1}$ is defined as the unique global section of $L^{\otimes d+k+1}$ such that $\partial_v s = v(f) s_0^{\otimes d+k+1}$ over M_{s_0} . The next lemma guarantees that this is well defined.

LEMMA 6.1. *With the above notations, $v(f) s_0^{\otimes d+k+1}$ extends to a global holomorphic section of $L^{\otimes d+k+1}$.*

PROOF. The couple (L, s_0) corresponds canonically to an effective analytic Cartier divisor E on M ; let $(g_i, U_i)_{i \in I}$ be a family of local equations of E for some open covering $M = \bigcup_{i \in I} U_i$. By hypothesis, for every $i \in I$, $g_i^k v$ extends to a holomorphic vector field on U_i .

An element $s \in R_d$ might be identified with a meromorphic function $f = s/s_0^{\otimes d}$ on M having pole of order at most d on E , i.e., such that $g_i^d f$ defines a holomorphic function on U_i for every $i \in I$. Under this identification, our statement is equivalent to the assertion that $g_i^{d+k+1} v(f)$ defines a holomorphic function on U_i for every $i \in I$.

Now, for $i \in I$, we have

$$g_i^{k+1}v(g_i^d f) = g_i^{k+1}(d \cdot g_i^{d-1}v(g_i)f + g_i^d v(f)) = d \cdot g_i^k v(g_i)g_i^d f + g_i^{d+k+1}v(f),$$

so that $g_i^{d+k+1}v(f)$ defines a holomorphic function on U_i . ■

Finally, it is easy to see that the \mathbf{C} -linear map $\partial_v : R \rightarrow R$ satisfies Leibniz's rule : if $s \in R_d$ and $t \in R_e$, then

$$\partial_v(s \otimes t) = \partial_v s \otimes t + s \otimes \partial_v t$$

in $R_{d+e+k+1}$.

6.2. Estimates of uniform norms. Let us keep the notation of the last paragraph and fix once and for all some $t \in R_{k+1}$ (recall that k denotes the "order of pole of v at $\text{div}(s_0)$ ").

For any integer $j \geq 1$ and $d \geq 1$, we define a differential operator of degree $j(k+1)$

$$\partial_v^{[j]} : R \rightarrow R$$

as the composition

$$\partial_v^{[j]} = \partial_v \circ (\partial_v - t) \circ \cdots \circ (\partial_v - (j-1)t).$$

PROPOSITION 6.2. *With the above notation, assume moreover that $L = i^* \mathcal{O}(1)$ for some projectively normal embedding $i : M \hookrightarrow \mathbf{P}^n(\mathbf{C})$. Let $\| \cdot \|$ be a Hermitian metric on L . Then there exists a constant $C > 0$ such that, for any integer $j \geq 1$, any sufficiently large positive integer d , and any $s \in R_d$, we have*

$$\|\partial_v^{[j]} s\|_{L^\infty(M)} \leq C^{j+d} (j+d)^j \|s\|_{L^\infty(M)},$$

where $\|\partial_v^{[j]}(s)\|_{L^\infty(M)}$ (resp. $\|s\|_{L^\infty(M)}$) denotes the uniform norm on M with respect to the Hermitian metric on $L^{\otimes d+j(k+1)}$ (resp. $L^{\otimes d}$) induced by $\| \cdot \|$.

Our proof is a reduction to the case $M = \mathbf{P}^n(\mathbf{C})$. Let $\| \cdot \|$ denote the Fubini-Study metric on the line bundle $\mathcal{O}(1)$ over $\mathbf{P}^n_{\mathbf{C}} = \text{Proj } \mathbf{C}[X_0, \dots, X_n]$ (cf. Example 4.5), and let us identify $\Gamma(\mathbf{P}^n_{\mathbf{C}}, \mathcal{O}(d))$ with the \mathbf{C} -vector space $\mathbf{C}[X_0, \dots, X_n]_d$ of homogeneous polynomials of degree d . If $P = \sum_{|I|=d} a_I X^I$, we consider the norms

$$\|P\|_\infty := \max_{|I|=d} |a_I| \quad \text{and} \quad \|P\|_1 := \sum_{|I|=d} |a_I|.$$

The uniform norm of P , seen as an element of $\Gamma(\mathbf{P}^n(\mathbf{C}), \mathcal{O}(d))$, with $\mathcal{O}(1)$ equipped with the Fubini-Study metric, is given by

$$\|P\|_{L^\infty(\mathbf{P}^n(\mathbf{C}))} = \sup_{z \in \mathbf{C}^{n+1} \setminus \{0\}} \frac{|P(z)|}{\left(\sum_{i=0}^n |z_i|^2\right)^{\frac{d}{2}}}.$$

LEMMA 6.3. *For any $P \in \Gamma(\mathbf{P}^n_{\mathbf{C}}, \mathcal{O}(d))$, we have*

$$(n+1)^{-\frac{d}{2}} \|P\|_\infty \leq \|P\|_{L^\infty(\mathbf{P}^n(\mathbf{C}))} \leq \|P\|_1 \leq \binom{d+n}{n} \|P\|_\infty.$$

PROOF. If we write $P = \sum_{|I|=d} a_I X^I$, then Cauchy's integral formula gives, for any multi-index I ,

$$a_I = \frac{1}{(2\pi i)^{n+1}} \int_{(\partial D)^{n+1}} \frac{P(z)}{z^{I+\mathbf{1}}} dz_0 \cdots dz_n,$$

where D denotes the unit disk in \mathbf{C} and $\mathbf{1}$ the multi-index of order $n+1$ having 1 at each coordinate. Thus, if $[z]$ denotes the image in $\mathbf{P}^n(\mathbf{C})$ of a point $z \in \mathbf{C}^{n+1} \setminus \{0\}$,

$$|a_I| \leq \sup_{z \in (\partial D)^{n+1}} |P(z)| = (n+1)^{\frac{d}{2}} \sup_{z \in (\partial D)^{n+1}} \|P([z])\|.$$

This proves that $(n+1)^{-\frac{d}{2}} \|P\|_\infty \leq \|P\|_{L^\infty(\mathbf{P}^n(\mathbf{C}))}$.

For any $z \in \mathbf{C}^{n+1} \setminus \{0\}$, we have

$$\|P([z])\| = \frac{|P(z)|}{\left(\sum_{i=0}^n |z_i|^2\right)^{\frac{d}{2}}} \leq \frac{\sum_{|I|=d} |a_I| |z^I|}{\left(\sum_{i=0}^n |z_i|^2\right)^{\frac{d}{2}}} \leq \frac{\max_{|I|=d} |z^I|}{\left(\sum_{i=0}^n |z_i|^2\right)^{\frac{d}{2}}} \|P\|_1.$$

Now, if $I = (i_0, \dots, i_n)$ is a multi-index satisfying $|I| = d$, then it is clear that

$$|z^I|^2 = (|z_0|^2)^{i_0} \cdots (|z_n|^2)^{i_n} \leq \left(\sum_{i=0}^n |z_i|^2\right)^d.$$

We thus obtain $\|P\|_{L^\infty(\mathbf{P}^n(\mathbf{C}))} \leq \|P\|_1$.

The inequality $\|P\|_1 \leq \binom{n+d}{n} \|P\|_\infty$ is an immediate consequence of $\dim \Gamma(\mathbf{P}^n(\mathbf{C}), \mathcal{O}(d)) = \binom{n+d}{n}$. \blacksquare

PROOF OF PROPOSITION 6.2. Since M is compact, if the conclusion of the statement holds for some Hermitian metric $\|\cdot\|$, then, up to replacing the constant C , it also holds for any other Hermitian metric on L . We may thus assume that $\|\cdot\|$ is induced by the Fubini-Study metric on $\mathcal{O}(1)$ via the embedding i .

Let (X_0, \dots, X_n) denote the projective coordinates of $\mathbf{P}^n(\mathbf{C})$, seen as global sections of $\mathcal{O}(1)$, and let $t_j \in R_1$ be the restriction of X_j to M for every $0 \leq j \leq n$. Since $i : M \hookrightarrow \mathbf{P}^n(\mathbf{C})$ is projectively normal, for any integer $d \geq 1$, R_d is generated as a \mathbf{C} -vector space by the monomials of degree d in t_0, \dots, t_n .

We lift v to $\mathbf{P}^n(\mathbf{C})$ as follows. For every $0 \leq j \leq n$, let $P_j \in \Gamma(\mathbf{P}^n(\mathbf{C}), \mathcal{O}(k+2)) = \mathbf{C}[X_0, \dots, X_n]_{k+2}$ be a lifting of $\partial_v t_j \in R_{k+2}$. Then there exists a unique \mathbf{C} -derivation ∂ of $\bigoplus_{d \geq 0} \Gamma(\mathbf{P}^n(\mathbf{C}), \mathcal{O}(d)) = \mathbf{C}[X_0, \dots, X_n]$, of degree $k+1$, such that $\partial X_j = P_j$ for every $0 \leq j \leq n$. It is easy to see that, for every integer $d \geq 0$, the diagram

$$\begin{array}{ccc} \mathbf{C}[X_0, \dots, X_n]_d & \xrightarrow{\partial} & \mathbf{C}[X_0, \dots, X_n]_{d+k+1} \\ i^* \downarrow & & \downarrow i^* \\ R_d & \xrightarrow{\partial_v} & R_{d+k+1} \end{array}$$

commutes. Moreover, if $Q \in \mathbf{C}[X_0, \dots, X_n]_{k+1}$ is any lifting of $t \in R_{k+1}$, then it is clear that

$$\partial^{[j]} := \partial \circ (\partial - Q) \circ \cdots \circ (\partial - (j-1)Q)$$

make the diagrams

$$\begin{array}{ccc} \mathbf{C}[X_0, \dots, X_n]_d & \xrightarrow{\partial^{[j]}} & \mathbf{C}[X_0, \dots, X_n]_{d+j(k+1)} \\ i^* \downarrow & & \downarrow i^* \\ R_d & \xrightarrow{\partial_v^{[j]}} & R_{d+j(k+1)} \end{array}$$

commute for any $j \geq 1$.

For every multi-index $I \in \mathbf{N}^{n+1}$, we obtain, by a straightforward computation, the upper bound

$$\|\partial X^I\|_\infty \leq |I| \max_{0 \leq i \leq n} \|P_i\|_\infty.$$

This implies that, for any $d \geq 1$ and any homogeneous polynomial $P \in \mathbf{C}[X_0, \dots, X_n]_d$,

$$\|\partial P\|_\infty \leq d \left(\max_{0 \leq i \leq n} \|P_i\|_\infty \right) \|P\|_\infty.$$

Thus, if $\kappa := \|Q\|_\infty + (k+1) \max_{0 \leq i \leq n} \|P_i\|_\infty$ and $S \in \mathbf{C}[X_0, \dots, X_n]$ is a homogeneous polynomial of degree $d + m(k+1)$ for some $0 \leq m \leq j-1$, we have

$$\|(\partial - mQ)S\|_\infty \leq \left((d + m(k+1)) \left(\max_{0 \leq i \leq n} \|P_i\|_\infty \right) + m\|Q\|_\infty \right) \|S\|_\infty \leq \kappa(d+j) \|S\|_\infty.$$

By induction, we conclude that, for any $d \geq 1$ and any $P \in \mathbf{C}[X_0, \dots, X_n]_d$, we have

$$(6.1) \quad \|\partial^{[j]} P\|_\infty \leq \kappa^j (d+j)^j \|P\|_\infty.$$

To complete our proof, we apply a lifting argument. By [16] Proposition 3.5, there exists a constant $C_0 > 0$ such that, for every sufficiently large integer d and every $s \in \Gamma(M, L^{\otimes d})$, there exists a lifting $P \in \Gamma(\mathbf{P}_{\mathbf{C}}^n, \mathcal{O}(d))$ of s such that

$$(6.2) \quad \|P\|_{L^\infty(\mathbf{P}^n(\mathbf{C}))} \leq C_0^d \|s\|_{L^\infty(M)}.$$

Thus, for any $j \geq 1$,

$$\begin{aligned} \|\partial_v^{[j]} s\|_{L^\infty(M)} &\leq \|\partial^{[j]} P\|_{L^\infty(\mathbf{P}^n(\mathbf{C}))} \leq \|\partial^{[j]} P\|_1 \leq \binom{d+j(k+1)}{j(k+1)} \|\partial^{[j]} P\|_\infty && \text{by Lemma 6.3} \\ &\leq 2^{d+j(k+1)} \|\partial^{[j]} P\|_\infty \\ &\leq 2^{d+j(k+1)} \kappa^j (d+j)^j \|P\|_\infty && \text{by (6.1)} \\ &\leq 2^{d+j(k+1)} \kappa^j (d+j)^j (n+1)^{\frac{d}{2}} \|P\|_{L^\infty(\mathbf{P}^n(\mathbf{C}))} && \text{by Lemma 6.3} \\ &\leq 2^{d+j(k+1)} \kappa^j (d+j)^j (n+1)^{\frac{d}{2}} C_0^d \|s\|_{L^\infty(M)} && \text{by (6.2)}. \end{aligned}$$

■

6.3. The arithmetic case. We shall actually need an arithmetic variant of the above constructions.

Consider the notation and terminology of Paragraph 5.1. Let K be a number field, \mathcal{X} be a projective arithmetic variety over $S = \text{Spec } \mathcal{O}_K$ with smooth generic fiber, and L be a line bundle over \mathcal{X} endowed with a global section $s_0 \in \Gamma(\mathcal{X}, L) \setminus \{0\}$. Arguing as above, we see that a section $w \in \Gamma(\mathcal{X}_{s_0}, \text{Der}_{\mathcal{O}_S}(\mathcal{O}_{\mathcal{X}}))$ induces an \mathcal{O}_K -derivation ∂_w of the ring $\bigoplus_{d \geq 0} \Gamma(\mathcal{X}, L^{\otimes d})$.

Let us fix $t \in \Gamma(\mathcal{X}, L^{k+1})$, where $k \geq 0$ is the ‘‘order of pole of w at $\text{div}(s_0)$ ’’, and consider the differential operators $\partial_w^{[j]} = \partial_w \circ (\partial_w - t) \circ \cdots \circ (\partial_w - (j-1)t)$, for $j \geq 0$, as above.

By applying Proposition 6.2 for each projective embedding $\sigma : K \hookrightarrow \mathbf{C}$, we obtain the following corollary.

COROLLARY 6.4. *With the above notations, assume moreover that $L = i^* \mathcal{O}(1)$ for some closed immersion $i : \mathcal{X} \hookrightarrow \mathbf{P}_{\mathcal{O}_K}^n$ over S such that $i_K : \mathcal{X}_K \hookrightarrow \mathbf{P}_K^n$ is projectively normal. Let $(\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbf{C}}$ be a Hermitian structure on L . Then, there exists a constant $C > 0$ such that, for any integer $j \geq 1$, any sufficiently large positive integer d , and any $s \in \Gamma(\mathcal{X}, L^{\otimes d})$, we have*

$$\|\partial_w^{[j]} s\|_{\mathcal{X}} \leq C^{j+d} (j+d)^j \|s\|_{\mathcal{X}}.$$

7. Proof of Theorem 1.2

Recall the notation and hypotheses of Theorem 1.2 : \mathcal{X} is a quasi-projective arithmetic variety over \mathcal{O}_K of relative dimension $n \geq 2$ with smooth generic fiber, and $\hat{\varphi} : \text{Spf } \mathcal{O}_K[[q]] \rightarrow \mathcal{X}$ is a morphism of formal \mathcal{O}_K -schemes such that

- (i) the formal curve $\hat{\varphi}_K : \text{Spf } K[[q]] \rightarrow \mathcal{X}_K$ is ZL-dense in \mathcal{X}_K and satisfies the differential equation

$$q \frac{d\hat{\varphi}_K}{dq} = v \circ \hat{\varphi}_K;$$

- (ii) for any field embedding $\sigma : K \hookrightarrow \mathbf{C}$, the formal curve $\hat{\varphi}_\sigma : \text{Spf } \mathbf{C}[[q]] \rightarrow \mathcal{X}_\sigma$ lifts to an analytic curve $\varphi_\sigma : D_{R_\sigma} \subset \mathbf{C} \rightarrow \mathcal{X}_\sigma^{\text{an}}$ of moderate growth. We also assume that $\prod_{\sigma:K \hookrightarrow \mathbf{C}} R_\sigma = 1$.

Let $\bar{\mathcal{X}}$ be some projective compactification with smooth generic fiber of the arithmetic variety \mathcal{X} over \mathcal{O}_K . Fix a Hermitian line bundle $\bar{L} = (L, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbf{C}})$ over $\bar{\mathcal{X}}$ such that L_K is ample and $(L_\sigma, \|\cdot\|_\sigma)$ over $\mathcal{X}_\sigma^{\text{an}}$ is positive for every $\sigma : K \hookrightarrow \mathbf{C}$.

In view of Philippon’s algebraic independence criterion (Theorem 3.A.1), Theorem 1.2 will be a direct consequence of the following.

THEOREM 7.1. *With the above notation, for any field embedding $\sigma : K \hookrightarrow \mathbf{C}$, and any $z \in D_{R_\sigma} \setminus \{0\}$, there exist real constants $c_0, c_1, c_2, c_3 > 0$ such that, for every sufficiently large positive integer d , there exists a positive integer $d' \leq c_0 d \log d$, and $t \in \Gamma(\bar{\mathcal{X}}, L^{\otimes d'})$ satisfying*

$$\log \|t\|_{\bar{\mathcal{X}}} \leq c_1 d \log^2 d$$

and

$$-c_2 d^n \leq \log \|t(\varphi_\sigma(z))\|_\sigma \leq -c_3 d^n.$$

We shall prove this theorem in three steps corresponding to the next three lemmas.

LEMMA 7.2 (Auxiliary sections). *There exist constants $a, b, c > 0$ such that, for every sufficiently large positive integer d , there is a global section $s \in \Gamma(\overline{\mathcal{X}}, L^{\otimes d})$ such that*

$$(7.1) \quad ad^n < \text{ord}_0 \hat{\varphi}^* s \leq bd^n$$

and

$$(7.2) \quad \log \|s\|_{\overline{\mathcal{X}}} \leq cd \log d.$$

PROOF. Since $\hat{\varphi}_K$ is ZL-dense in \mathcal{X}_K , and φ_τ has moderate growth in $\mathcal{X}_\tau^{\text{an}}$ for every embedding $\tau : K \hookrightarrow \mathbf{C}$, our statement follows immediately from Corollary 5.3. \blacksquare

Fix a field embedding $\sigma : K \hookrightarrow \mathbf{C}$ and $z \in D_{R_\sigma} \setminus \{0\}$. By the projective Prime Avoidance Lemma, there is an integer $k \geq 1$, and a global section $s_0 \in \Gamma(\overline{\mathcal{X}}, L^{\otimes k})$ such that $\overline{\mathcal{X}}_{s_0} := (s_0 \neq 0) \subset \overline{\mathcal{X}}$ and $\varphi_\sigma(z) \in \overline{\mathcal{X}}_{s_0, \sigma}^{\text{an}}$. Up to replacing L by $L^{\otimes k}$, we may assume that $k = 1$ (cf. Remark 3.A.2).

LEMMA 7.3. *There exist constants $\gamma_0, \gamma_1, \gamma_2 > 0$ such that, for every sufficiently large positive integer d , and every $s \in \Gamma(\overline{\mathcal{X}}, L^{\otimes d})$ as in Lemma 7.2, there exists $j \leq \gamma_0 d \log d$ such that, if we write $s = f s_0^{\otimes d}$ over $\overline{\mathcal{X}}_{s_0}$, then*

$$-\gamma_1 d^m \leq \log |(\varphi_\sigma^* f)^{(j)}(z)| \leq -\gamma_2 d^m.$$

PROOF. Let d be a sufficiently large positive integer and $s \in \Gamma(\overline{\mathcal{X}}, L^{\otimes d})$ be as in Lemma 7.2. Set $m := \text{ord}_0 \hat{\varphi}^* s$.

According to Proposition 3.12 and to the bounds (7.1) and (7.2), for every embedding $\tau \neq \sigma$, there is a constant κ_τ (not depending on d or s) such that

$$(7.3) \quad \log \|j_0^m \varphi_\tau^* s\|_{R_\tau} \leq \kappa_\tau d \log d.$$

Here, the norm $\|\cdot\|_{R_\tau}$ is the norm $\|\cdot\|_r$, introduced before Proposition 3.8, in the special case where $r = R_\tau$ and \overline{L} is $\varphi_\tau^* L_\tau$ equipped with the pullback of $\|\cdot\|_\tau$.

Fix any constant $C > \sum_{\tau \neq \sigma} \kappa_\tau$. Then Corollary 3.16 shows that there exist real numbers $\gamma_0, \gamma_1 > 0$ such that, for sufficiently large d , if

$$(7.4) \quad \max_{0 \leq j \leq \lfloor \gamma_0 d \log d \rfloor} \log |(\varphi_\sigma^* f)^{(j)}(z)| < -\gamma_1 d^m$$

then

$$(7.5) \quad \log \|j_0^m \varphi_\sigma^* s\|_{R_\sigma} \leq -Cd \log d.$$

By contradiction, assume that (7.4) holds. Observe that $j_0^m \hat{\varphi}^* s$ is an element of $\hat{\varphi}(0)^* L^{\otimes d} \otimes \Omega^{\otimes m}$. The Hermitian structure on $\hat{\varphi}(0)^* \overline{L}^{\otimes d} \otimes \overline{\Omega}^{\otimes m}$ allows us to consider its norms $(\|j_0^m \hat{\varphi}^* s\|_\tau)_{\tau:K \hookrightarrow \mathbf{C}}$. For every field embedding $\tau : K \hookrightarrow \mathbf{C}$, we have

$$\|j_0^m \hat{\varphi}^* s\|_\tau = R_\tau^{-m} \|j_0^m \varphi_\tau^* s\|_{R_\tau}.$$

Thus, since $\prod_{\tau:K \hookrightarrow \mathbf{C}} R_\tau = 1$, we obtain from (7.3) and (7.5)

$$\sum_{\tau:K \hookrightarrow \mathbf{C}} \log \|j_0^m \hat{\varphi}^* s\|_\tau \leq - \left(C - \sum_{\tau \neq \sigma} \kappa_\tau \right) d \log d.$$

On the other hand, by definition of the Arakelov degree, we have

$$\sum_{\tau:K \hookrightarrow \mathbf{C}} \log \|j_0^m \hat{\varphi}^* s\|_\tau \geq -\widehat{\deg}(\hat{\varphi}(0)^* \overline{L}^{\otimes d} \otimes \overline{\Omega}^{\otimes m}) = -\widehat{\deg}(\hat{\varphi}(0)^* \overline{L})d.$$

This contradicts our choice of C for $d \gg 0$. We conclude that, for sufficiently large d , (7.4) cannot hold, so that there exists an integer $j \leq \gamma_0 d \log d$ for which

$$(7.6) \quad \log |(\varphi_\sigma^* f)^{(j)}(z)| \geq -\gamma_1 d^n.$$

Next, we bound $\log |(\varphi_\sigma^* f)^{(j)}(z)|$ from above. Let Δ be a disk centered in z , of radius $\varepsilon > 0$ small enough so that $\overline{\Delta} \subset \varphi_\sigma^{-1}(\mathcal{X}_{s_0, \sigma}^{\text{an}})$. It follows from Proposition 3.14, and bounds (7.1) and (7.2), that there is a constant $c' > 0$ such that

$$\log \max_{\zeta \in \partial \Delta} |\varphi_\sigma^* f(\zeta)| \leq -c' d^n.$$

By the Cauchy inequalities, we have

$$\frac{|(\varphi_\sigma^* f)^{(j)}(z)|}{j!} \leq \frac{\max_{\zeta \in \partial \Delta} |\varphi_\sigma^* f(\zeta)|}{\varepsilon^j},$$

so that

$$\log |(\varphi_\sigma^* f)^{(j)}(z)| \leq -c' d^n + \log j! - j \log \varepsilon.$$

Since $j = O(d \log d)$, we have $\log j! = O(d \log^2 d)$, and we conclude that there is a constant $\gamma_2 > 0$ (not depending on d or s) such that

$$(7.7) \quad \log |(\varphi_\sigma^* f)^{(j)}(z)| \leq -\gamma_2 d^n$$

for every sufficiently large d . ■

Again by Remark 3.A.2, up to replacing L by a sufficiently large tensor power of itself, we may assume that there exists a closed immersion $i : \overline{\mathcal{X}} \hookrightarrow \mathbf{P}_{\mathcal{O}_K}^n$ over \mathcal{O}_K such that $i_K : \overline{\mathcal{X}}_K \hookrightarrow \mathbf{P}_K^n$ is projectively normal and $L = i^* \mathcal{O}(1)$.

Fix any $\alpha \in \mathcal{O}_K \setminus \{0\}$ that “clears the denominators of v ”, i.e., such that $w := \alpha v$ defines a non-zero global section of $\mathcal{D}er_{\mathcal{O}_K}(\mathcal{O}_{\overline{\mathcal{X}}})$. Let $k \geq 0$ be the smallest integer for which $w \otimes s_0^{\otimes k}$ defines a global section of $\mathcal{D}er_{\mathcal{O}_K}(\mathcal{O}_{\overline{\mathcal{X}}}) \otimes L^{\otimes k}$, and let ∂_w be the \mathcal{O}_K -derivation of degree $k+1$ of the ring $\bigoplus_{d \geq 0} \Gamma(\overline{\mathcal{X}}, L^{\otimes d})$ defined in Section 6. For any integer $j \geq 1$, set

$$\partial_w^{[j]} = \partial_w \circ (\partial_w - \alpha s_0^{\otimes k+1}) \circ \cdots \circ (\partial_w - (j-1)\alpha s_0^{\otimes k+1}).$$

LEMMA 7.4. *There exist constants $c_1, c_2, c_3 > 0$ such that, for every sufficiently large positive integer d , and every $s \in \Gamma(\overline{\mathcal{X}}, L^{\otimes d})$ as in Lemma 7.2, if j denotes the integer constructed in Lemma 7.3, then the section $t := \partial_w^{[j]}(s) \in \Gamma(\overline{\mathcal{X}}, L^{\otimes d+j(k+1)})$ satisfies*

$$\log \|t\|_{\overline{\mathcal{X}}} \leq c_1 d \log^2 d$$

and

$$-c_2 d^n \leq \log \|t(\varphi_\sigma(z))\|_\sigma \leq -c_3 d^n.$$

PROOF. Since j grows at the order of $d \log d$, by Corollary 6.4 and bound (7.2), there exists $c_1 > 0$ such that

$$\log \|t\|_{\overline{\mathcal{X}}} \leq c_1 d \log^2 d.$$

In order to bound $\|t(\varphi_\sigma(z))\|_\sigma$, we first remark that the formal identity of differential operators

$$q^j \frac{d^j}{dq^j} = q \frac{d}{dq} \left(q \frac{d}{dq} - 1 \right) \cdots \left(q \frac{d}{dq} - (j-1) \right)$$

and the differential equation

$$D\hat{\varphi}_K \left(q \frac{d}{dq} \right) = \hat{\varphi}_K^* v$$

yield :

$$\begin{aligned}\hat{\varphi}^* t &= \hat{\varphi}^* (\partial_w (\partial_w - \alpha s_0^{\otimes k+1}) \cdots (\partial_w - (j-1)\alpha s_0^{\otimes k+1})(s)) \\ &= \alpha^j \left[q \frac{d}{dq} \left(q \frac{d}{dq} - 1 \right) \cdots \left(q \frac{d}{dq} - (j-1) \right) \hat{\varphi}^*(f) \right] \hat{\varphi}^*(s_0)^{\otimes d+j(k+1)} = (\alpha q)^j \frac{d^j \hat{\varphi}^*(f)}{dq^j} \hat{\varphi}^*(s_0)^{\otimes d+j(k+1)}.\end{aligned}$$

A similar formula holds for φ_σ . Thus

$$\log \|t(\varphi_\sigma(z))\|_\sigma = \log |(\varphi_\sigma^* f)^{(j)}(z)| + j \log |\alpha z| + (d + j(k+1)) \log \|\varphi_\sigma^* s_0(z)\|_\sigma.$$

Since j grows at the order of $d \log d$, we conclude from (7.6) and (7.7) that there exist real constants $c_2 > c_3 > 0$ such that

$$-c_2 d^m \leq \log \|t(\varphi_\sigma(z))\|_\sigma \leq -c_3 d^m$$

for sufficiently large d . ■

To finish the proof, one simply remarks that, if $c_0 := (k+1)\gamma_0$, then the degree $d' := d + j(k+1)$ of t constructed above satisfies $d' \leq d + (k+1)\gamma_0 d \log d \leq c_0 d \log d$, for d sufficiently large.

3.A. Philippon's algebraic independence criterion for projective varieties

Let K be a number field, \mathcal{X} be a projective arithmetic variety over \mathcal{O}_K of relative dimension $n \geq 2$, and $\bar{L} = (L, (\|\cdot\|_\sigma)_{\sigma: K \hookrightarrow \mathbf{C}})$ be a Hermitian line bundle over \mathcal{X} with L relatively ample over $\text{Spec } \mathcal{O}_K$. Recall that, if $s \in \Gamma(\mathcal{X}, L^{\otimes d})$ for some integer $d \geq 1$, then we denote $\|s\|_{\mathcal{X}} = \max_\sigma \|s\|_{\sigma, L^\infty(\mathcal{X}_\sigma^{\text{an}})}$, where σ runs through the set of field embeddings of K in \mathbf{C} .

The proof of the main theorem of this article relies on the following generalized version of an algebraic independence criterion of Philippon (cf. [85] Théorème 2.11 and [75] Lemma 2.5).

THEOREM 3.A.1. *Let $\sigma : K \hookrightarrow \mathbf{C}$ be a field embedding and $p \in \mathcal{X}_\sigma(\mathbf{C})$. Suppose that there exist an integer m such that $2 \leq m \leq n$, a non-decreasing sequence of positive real numbers $(\ell_d)_{d \geq 1}$ satisfying $\ell_d^{m-1} = o(d)$ as $d \rightarrow +\infty$, and real constants $a > b > 0$ such that, for every sufficiently large positive integer d , there exists an integer $d' \leq d\ell_d$ and a section $s \in \Gamma(\mathcal{X}, L^{\otimes d'})$ satisfying*

$$\log \|s\|_{\mathcal{X}} \leq d\ell_d$$

and

$$-ad^m \leq \log \|s_\sigma(p)\|_\sigma \leq -bd^m.$$

Then the field of definition $K(p)$ of the complex point p in \mathcal{X}_K satisfies

$$\text{trdeg}_{\mathbf{Q}} K(p) \geq m - 1.$$

REMARK 3.A.2. For any integer $k \geq 1$, the conditions in the above statement are verified for the Hermitian line bundle \bar{L} if and only if similar conditions hold for the tensor power $\bar{L}^{\otimes k}$ of \bar{L} (up to multiplying ℓ_d , a , and b by suitable constants).

Moreover, since \mathcal{X} is proper over $\text{Spec } \mathcal{O}_K$, it is easy to see that if the above statement is true for a particular choice of Hermitian structure on L , then it also holds for any other Hermitian structure on L .

In what follows, we explain how to deduce the above statement from Philippon's original result concerning $\mathcal{X} = \mathbf{P}_{\mathcal{O}_K}^n$. The main technical tool is the following "integral lifting lemma".

LEMMA 3.A.3. *Let \mathcal{X} and \mathcal{Y} be projective arithmetic varieties over \mathcal{O}_K , \bar{L} be a Hermitian line bundle over \mathcal{X} , with L relatively ample over $\text{Spec } \mathcal{O}_K$, and $\mathcal{Y} \rightarrow \mathcal{X}$ be a closed immersion over $\text{Spec } \mathcal{O}_K$. Endow $L|_{\mathcal{Y}}$ with the induced Hermitian structure. Then, there exists a real number $C > 0$ such that, for every sufficiently large positive integer d , any section $s \in \Gamma(\mathcal{Y}, L|_{\mathcal{Y}}^{\otimes d})$ can be lifted to a section $\tilde{s} \in \Gamma(\mathcal{X}, L^{\otimes d})$ satisfying*

$$\|\tilde{s}\|_{\mathcal{X}} \leq C^d \|s\|_{\mathcal{Y}}.$$

This type of result is well known in Arakelov Geometry and goes back to Zhang's work on arithmetic ampleness [96]. For lack of reference, we include a proof.

PROOF. Let I be the ideal of \mathcal{Y} in $\mathcal{O}_{\mathcal{X}}$. For sufficiently large d , we have an exact sequence of \mathcal{O}_K -modules

$$\begin{aligned} 0 \longrightarrow \Gamma(\mathcal{X}, I \otimes L^{\otimes d}) \longrightarrow \Gamma(\mathcal{X}, L^{\otimes d}) \longrightarrow \Gamma(\mathcal{Y}, L|_{\mathcal{Y}}^{\otimes d}) \longrightarrow 0 \\ t \longmapsto t|_{\mathcal{Y}} \end{aligned}$$

and thus also an exact sequence of \mathbf{R} -vector spaces

$$0 \longrightarrow \Gamma(\mathcal{X}, I \otimes L^{\otimes d}) \otimes_{\mathbf{Z}} \mathbf{R} \longrightarrow \Gamma(\mathcal{X}, L^{\otimes d}) \otimes_{\mathbf{Z}} \mathbf{R} \longrightarrow \Gamma(\mathcal{Y}, L|_{\mathcal{Y}}^{\otimes d}) \otimes_{\mathbf{Z}} \mathbf{R} \longrightarrow 0.$$

Let Ω_d be a fundamental domain in $\Gamma(\mathcal{X}, I \otimes L^{\otimes d}) \otimes_{\mathbf{Z}} \mathbf{R}$ of the lattice $\Gamma(\mathcal{X}, I \otimes L^{\otimes d})$.

By [16] Proposition 3.5, there exists a constant $C_1 > 0$ such that, for every sufficiently large positive integer d , every field embedding $\sigma : K \hookrightarrow \mathbf{C}$, and every $s \in \Gamma(\mathcal{Y}, L|_{\mathcal{Y}}^{\otimes d})$, there exists $t_\sigma \in \Gamma(\mathcal{X}_\sigma, L_\sigma^{\otimes d})$ such that $t_\sigma|_{\mathcal{Y}_\sigma} = s_\sigma$ and

$$\|t_\sigma\|_{\sigma, L^\infty(\mathcal{X}_\sigma^{\text{an}})} \leq C_1^d \|s_\sigma\|_{\sigma, L^\infty(\mathcal{Y}_\sigma^{\text{an}})}.$$

We thus obtain an element $(t_\sigma)_\sigma \in \bigoplus_\sigma \Gamma(\mathcal{X}_\sigma, L_\sigma^{\otimes d}) \cong \Gamma(\mathcal{X}, L^{\otimes d}) \otimes_{\mathbf{Z}} \mathbf{C}$ and we can define

$$t := \frac{1}{2} \left((t_\sigma)_\sigma + \overline{(t_\sigma)_\sigma} \right) \in \Gamma(\mathcal{X}, L^{\otimes d}) \otimes_{\mathbf{Z}} \mathbf{R}.$$

Note that $t|_{\mathcal{Y}} = s \in \Gamma(\mathcal{Y}, L|_{\mathcal{Y}}^{\otimes d}) \otimes_{\mathbf{Z}} \mathbf{R}$ and

$$\|t\|_{\mathcal{X}} \leq C_1^d \|s\|_{\mathcal{Y}}.$$

Let \tilde{s}_0 be any element of $\Gamma(\mathcal{X}, L^{\otimes d})$ lifting s , so that $t - \tilde{s}_0 \in \Gamma(\mathcal{X}, I \otimes L^{\otimes d}) \otimes_{\mathbf{Z}} \mathbf{R}$. Since Ω_d is a fundamental domain, there exists $\tilde{s}_1 \in \Gamma(\mathcal{X}, I \otimes L^{\otimes d})$ such that $t - \tilde{s}_0 - \tilde{s}_1 \in \Omega_d$. We define

$$\tilde{s} := \tilde{s}_0 + \tilde{s}_1 \in \Gamma(\mathcal{X}, L^{\otimes d}).$$

Then $\tilde{s}|_{\mathcal{Y}} = s$ and

$$\|\tilde{s}\|_{\mathcal{X}} \leq \|\tilde{s} - t\|_{\mathcal{X}} + \|t\|_{\mathcal{X}} \leq \text{diam}(\Omega_d) + C_1^d \|s\|_{\mathcal{Y}},$$

where $\text{diam}(\Omega_d)$ denotes the diameter of $\Omega_d \subset \Gamma(\mathcal{X}, I \otimes L^{\otimes d}) \otimes_{\mathbf{Z}} \mathbf{R}$ with respect to the norm $\|\cdot\|_{\mathcal{X}}$. Thus, to finish our proof, it is sufficient to show that there exists a constant $C_0 > 0$ such that, for every sufficiently large integer d ,

$$\text{diam}(\Omega_d) \leq C_0^d.$$

We mimic the argument in the proof of [21] Proposition 2.5. Since L is ample, there exists an integer $n \geq 1$ such that, for any sufficiently large integer r , and any positive integer q , the morphism

$$\Gamma(\mathcal{X}, I \otimes L^{\otimes r}) \otimes_{\mathbf{Z}} \Gamma(\mathcal{X}, L^{\otimes n})^{\otimes q} \longrightarrow \Gamma(\mathcal{X}, I \otimes L^{\otimes nq+r})$$

is surjective. Choose sufficiently large integers r_1, \dots, r_n forming a complete residue system modulo n . Fixing bases of the finite free \mathbf{Z} -modules $\Gamma(\mathcal{X}, L^{\otimes n}), \Gamma(\mathcal{X}, I \otimes L^{\otimes r_1}), \dots, \Gamma(\mathcal{X}, I \otimes L^{\otimes r_n})$, we see that there exists a constant $B > 1$ such that any $\Gamma(\mathcal{X}, I \otimes L^{\otimes nq+r_i})$ admits a full rank submodule having a basis whose elements have norm bounded by B^q . By [96] Lemma 1.7, the \mathbf{Z} -module $\Gamma(\mathcal{X}, I \otimes L^{\otimes nq+r_i})$ admits a basis whose elements have norm bounded by rB^q , where r denotes the rank of $\Gamma(\mathcal{X}, I \otimes L^{\otimes nq+r_i})$. Since r grows polynomially in q , and r_1, \dots, r_n form a complete residue system modulo n , we conclude that there exists a constant $C_0 > 0$ such that, for any sufficiently large integer d , the \mathbf{Z} -module $\Gamma(\mathcal{X}, I \otimes L^{\otimes d})$ admits a basis consisting of elements with norm bounded by C_0^d , i.e., $\text{diam}(\Omega_d) \leq C_0^d$. \blacksquare

To handle the case $\mathcal{X} = \mathbf{P}_{\mathcal{O}_K}^n$, we compare the *height* $\bar{h}(P)$ of a homogeneous polynomial $P \in \mathcal{O}_K[X_0, \dots, X_n]$ of degree d used in [85] with the Fubini-Study norm $\|s\|_{\mathbf{P}_{\mathcal{O}_K}^n}$ of the corresponding section $s \in \Gamma(\mathbf{P}_{\mathcal{O}_K}^n, \mathcal{O}(d))$. By definition,

$$\bar{h}(P) = \frac{1}{[K : \mathbf{Q}]} \sum_{\sigma: K \hookrightarrow \mathbf{C}} \log^+ M_\sigma(P),$$

where, for any field embedding $\sigma : K \hookrightarrow \mathbf{C}$, we set

$$M_\sigma(P) := \exp \left(\frac{1}{(2\pi)^{n+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \log |P^\sigma(e^{i\theta_0}, \dots, e^{i\theta_n})| d\theta_0 \cdots d\theta_n \right).$$

LEMMA 3.A.4. Let $(\|\cdot\|_\sigma)_{\sigma:K\hookrightarrow\mathbf{C}}$ denote the Fubini-Study Hermitian structure on the line bundle $\mathcal{O}(1)$ over the arithmetic variety $\mathbf{P}_{\mathcal{O}_K}^n$. For any integer $d \geq 1$, and any section $s \in \Gamma(\mathbf{P}_{\mathcal{O}_K}^n, \mathcal{O}(d))$, if $P \in \mathcal{O}_K[X_0, \dots, X_n]$ denotes the homogeneous polynomial of degree d corresponding to s , then

$$\bar{h}(P) \leq \log^+ \|s\|_{\mathbf{P}_{\mathcal{O}_K}^n} + \frac{(n+1)}{2}d.$$

PROOF. For any field embedding $\sigma : K \hookrightarrow \mathbf{C}$, and any $(\theta_0, \dots, \theta_n) \in [0, 2\pi]^{n+1}$, we have

$$\|s_\sigma(e^{i\theta_0} : \dots : e^{i\theta_n})\|_\sigma = \frac{|P^\sigma(e^{i\theta_0}, \dots, e^{i\theta_n})|}{(n+1)^{\frac{d}{2}}},$$

so that

$$\log M_\sigma(P) \leq \log \|s\|_{\sigma, L^\infty(\mathbf{P}^n(\mathbf{C}))} + \frac{(n+1)}{2}d \leq \log \|s\|_{\mathbf{P}_{\mathcal{O}_K}^n} + \frac{(n+1)}{2}d.$$

Clearly, a similar inequality holds with \log^+ in place of \log . The result follows by taking the arithmetic mean over all $\sigma : K \hookrightarrow \mathbf{C}$. \blacksquare

PROOF OF THEOREM 3.A.1. The case where $\mathcal{X} = \mathbf{P}_{\mathcal{O}_K}^n$ and \bar{L} is given by $\mathcal{O}(1)$ endowed with the Fubini-Study metric follows from Lemma 3.A.4 and [85] Théorème 2.11 (cf. [75] Lemma 2.5).

The general case follows from this one by considering a closed immersion $i : \mathcal{X} \rightarrow \mathbf{P}_{\mathcal{O}_K}^n$ over \mathcal{O}_K satisfying $i^*\mathcal{O}(1) = L^{\otimes k}$ for some $k \geq 1$, and by applying Lemma 3.A.3 and Remark 3.A.2. \blacksquare

3.B. D -property and ZL-density in quasi-projective varieties

Let k be a field, X be a smooth quasi-projective variety over k , and \mathcal{F} be a \mathcal{O}_X -submodule of rank one of the tangent bundle $T_{X/k}$ such that the quotient $T_{X/k}/\mathcal{F}$ is torsion-free, i.e., a one dimensional (possibly singular) *foliation* on X .

Let $p \in X(k)$ be a k -point of X . We say that a formal curve $\hat{\varphi} : \mathrm{Spf} k[[q]] \rightarrow X$ is an *integral curve* of \mathcal{F} at p if $\hat{\varphi}(0) = p$ and if the image of the tangent map

$$D\hat{\varphi} : T_{\mathrm{Spf} k[[q]]/k} \rightarrow \hat{\varphi}^*T_{X/k}$$

factors through the subbundle $\hat{\varphi}^*\mathcal{F}$ of $\hat{\varphi}^*T_{X/k}$. Moreover, if $\mathcal{F}(p) := \Gamma(\mathrm{Spec} k, p^*\mathcal{F})$ denotes the fiber of \mathcal{F} at p , we say that $\hat{\varphi}$ is *smooth* if $\hat{\varphi}'(0) := D_0\hat{\varphi}(\frac{d}{dq}) \in \mathcal{F}(p)$ is non-zero.

From now on, we assume that k has *characteristic 0*. By a formal version of the Frobenius Theorem, for every $p \in X(k)$ such that $\mathcal{F}(p) \neq 0$, there exists a unique smooth integral curve $\hat{\varphi}$ of \mathcal{F} at p , up to composition by an automorphism of $\mathrm{Spf} k[[q]]$.

We say that a closed subscheme Y of X is \mathcal{F} -invariant if the ideal of Y in \mathcal{O}_X is stable under the derivations of $\mathcal{F} \subset T_{X/k} = \mathrm{Der}_k(\mathcal{O}_X)$.

DEFINITION 3.B.1. Let X be a smooth quasi-projective variety over the field k , let \mathcal{F} be a one dimensional foliation on X , and let $\hat{\varphi} : \mathrm{Spf} k[[q]] \rightarrow X$ be a formal integral curve of \mathcal{F} . We say that $\hat{\varphi}$ satisfies the *D -property* for \mathcal{F} if there exists a constant $C > 0$ such that, for every \mathcal{F} -invariant closed subvariety Y of X , there exists a Cartier divisor D whose support contains Y satisfying

$$\mathrm{mult}_{\hat{\varphi}} D \leq C.$$

Observe that, if $\hat{\varphi}$ satisfies the D -property, then its image is Zariski-dense in X . Indeed, the Zariski-closure of the image of an integral curve of \mathcal{F} is \mathcal{F} -invariant.

THEOREM 3.B.2 (Nesterenko-Binyamini). *Let X be a smooth quasi-projective variety over an algebraically closed field k of characteristic 0, $v \in \Gamma(X, T_{X/k}) \setminus \{0\}$ be a vector field on X , and $\hat{\varphi} : \mathrm{Spf} k[[q]] \rightarrow X$ be a smooth formal curve satisfying the differential equation*

$$q \frac{d\hat{\varphi}}{dq} = v \circ \hat{\varphi}.$$

If $\hat{\varphi}$ satisfies the D -property for the foliation generated by v , then $\hat{\varphi}$ is ZL-dense in X .

Note that $\hat{\varphi}(0)$ is a singular point of v . In the non-singular case, i.e., $\hat{\varphi}$ satisfies the differential equation $\frac{d\hat{\varphi}}{dq} = v \circ \hat{\varphi}$, stronger statements are true (cf. [10] Theorem 2), but an analogous of the above result may be also obtained by virtually the same proof.

Binyamini's original result ([10] Corollary 3) builds on ideas of Nesterenko and concerns the case of an analytic integral curve of a polynomial vector field on some affine space over \mathbf{C} . In what follows, we briefly indicate how a slight modification of the geometric methods of Binyamini may be used to prove the Theorem 3.B.2 above.

We start by recasting the D -property into a more workable form.

PROPOSITION 3.B.3. *Let \overline{X} be any projective compactification of X and L be an ample line bundle on \overline{X} . Then, a formal curve $\hat{\varphi} : \mathrm{Spf} k[[q]] \rightarrow X$ satisfies the D -property for a one dimensional foliation \mathcal{F} on X if and only if there exists a constant $C > 0$ such that, for every \mathcal{F} -invariant closed subvariety Y of X , there exists an integer $d \geq 1$, and a global section $s \in \Gamma(\overline{X}, L^{\otimes d})$ vanishing identically on Y such that $\mathrm{ord}_0 \hat{\varphi}^* s \leq C$.*

PROOF. The sufficiency is clear : consider the divisors $\mathrm{div}(s)$.

Conversely, suppose that $\hat{\varphi}$ satisfies the D -property for \mathcal{F} with constant $C > 0$ and let Y be a \mathcal{F} -invariant closed subvariety of X . Since L is ample, we may assume that Y contains $\hat{\varphi}(0)$; otherwise there exists an integer $d \geq 1$ and a section $s \in \Gamma(\overline{X}, L^{\otimes d})$ vanishing on Y such that $s(\hat{\varphi}(0)) \neq 0$, so that $\mathrm{ord}_0 \hat{\varphi}^* s = 0 \leq C$.

Let D be a divisor whose support contains Y such that $\mathrm{mult}_{\hat{\varphi}} D \leq C$, and let f be a local equation for D on some open neighborhood U of $\hat{\varphi}(0)$. Since L is ample, there exists an integer $m \geq 1$ and a section $s_0 \in \Gamma(\overline{X}, L^{\otimes m})$ such that $\hat{\varphi}(0) \in \overline{X}_{s_0}$ and $\overline{X}_{s_0} \subset U$. Now, there exists an integer $n \geq 1$, and a global section $s \in \Gamma(\overline{X}, L^{\otimes mn})$ such that $s = f s_0^{\otimes mn}$ over \overline{X}_{s_0} . It is clear that s vanishes identically on Y and satisfies $\mathrm{ord}_0 \hat{\varphi}^* s = \mathrm{ord}_0 \hat{\varphi}^* f = \mathrm{mult}_{\hat{\varphi}} D \leq C$. \blacksquare

Consider the hypotheses and notation of Theorem 3.B.2. Fix a projective compactification \overline{X} of X , and an ample line bundle L on \overline{X} endowed with a global section $s_0 \in \Gamma(\overline{X}, L)$ satisfying $\overline{X}_{s_0} \subset X$. Recall from Section 6 that v defines a k -derivation ∂_v on the ring $\bigoplus_{d \geq 0} \Gamma(\overline{X}, L^{\otimes d})$.

Let $p = \hat{\varphi}(0)$. By a *formal cycle* of X at p , we mean a cycle in the scheme $\mathrm{Spec} \hat{\mathcal{O}}_{X,p}$, where $\hat{\mathcal{O}}_{X,p}$ denotes the completion of the local ring $\mathcal{O}_{X,p}$ with respect to its maximal ideal. Note that every (global) cycle of X induces, by localization and formal completion, a formal cycle of X at p .

Let Y be a prime formal cycle of X at p corresponding to the prime ideal \mathfrak{p} of $\hat{\mathcal{O}}_{X,p}$ and denote by $I_{\hat{\varphi}}$ the ideal of $\mathrm{im} \hat{\varphi}$ in $\hat{\mathcal{O}}_{X,p}$. Assume that $I_{\hat{\varphi}}$ does not contain \mathfrak{p} (i.e., Y does not contain the image of $\hat{\varphi}$). Since $\hat{\varphi} : \mathrm{Spf} k[[q]] \rightarrow X$ is smooth, the image of $I_{\hat{\varphi}}$ in the local ring $\hat{\mathcal{O}}_{X,p}/\mathfrak{p}$ contains some power of the maximal ideal. We may thus consider the *Samuel multiplicity*

$$\mathrm{mult}_{\hat{\varphi}} Y := e_{I_{\hat{\varphi}}/I_{\hat{\varphi}} \cap \mathfrak{p}}(\hat{\mathcal{O}}_{X,p}/\mathfrak{p}).$$

By additivity, we may extend this definition to every formal cycle of X at p whose components do not contain the image of $\hat{\varphi}$. By abuse of notation, if Z is a (global) cycle of X , we denote by $\mathrm{mult}_{\hat{\varphi}} Z$ the multiplicity $\mathrm{mult}_{\hat{\varphi}} \hat{Z}$ of its completion at p .

PROPOSITION 3.B.4. *The multiplicity function constructed above satisfies the following properties :*

- (1) *If $Z = \mathrm{div}(f)$, for some $f \in \hat{\mathcal{O}}_{X,p}$, then $\mathrm{mult}_{\hat{\varphi}}(Z) = \mathrm{ord}_0 \hat{\varphi}^* f$.*
- (2) *If $Z = p$, then $\mathrm{mult}_{\hat{\varphi}}(Z) = 1$.*
- (3) *For any closed subvariety Y of X , any integer $d \geq 1$, and any $s \in \Gamma(\overline{X}, L^{\otimes d}) \setminus \{0\}$ vanishing identically on Y , we have $\mathrm{mult}_{\hat{\varphi}}(Y) \leq \mathrm{ord}_0 \hat{\varphi}^* s \cdot \mathrm{mult}_p(Y)$.*
- (4) *For any closed subvariety Y of X , any integer $d \geq 1$, and any $s \in \Gamma(\overline{X}, L^{\otimes d}) \setminus \{0\}$ vanishing identically on Y for which $\partial_v s$ does not vanish identically on Y , we have $\mathrm{mult}_{\hat{\varphi}}(Y) \leq \mathrm{mult}_{\hat{\varphi}}(Y \cdot \mathrm{div}(\partial_v s))$.*
- (5) *There is an integer $n_0 \geq 0$ such that, for every closed subvariety Y of X not contained in a v -invariant subvariety of X , if $d \geq 1$ is the smallest integer for which there is $s \in \Gamma(\overline{X}, L^{\otimes d}) \setminus \{0\}$ vanishing identically on Y , then $\min\{n \mid \partial_v^n s \text{ does not vanish identically on } Y\} \leq n_0$.*

Properties (1) and (2) are easy. For properties (3) and (4), see [10] Lemma 8 and Proposition 9. Finally, property (5) follows by an adaptation of the arguments in [10] Section 3.

Once this is established, the proof Theorem 3.B.2 becomes completely analogous to the proof of [10] Theorem 3.

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Titre : Courbes intégrales : transcendance et géométrie

Mots clés : courbes intégrales, indépendance algébrique, croissance modérée, variétés abéliennes, espaces de modules, périodes

Résumé : Cette thèse est consacrée à l'étude de quelques questions soulevées par le théorème de Nesterenko sur l'indépendance algébrique de valeurs des séries d'Eisenstein E_2 , E_4 , E_6 . Elle est divisée en deux parties.

Dans la première partie, constituée des deux premiers chapitres, on généralise les équations différentielles algébriques satisfaites par les séries d'Eisenstein qui se trouvent dans le cœur de la méthode de Nesterenko, les *équations de Ramanujan*. Ces généralisations, appelées *équations de Ramanujan supérieures*, sont obtenues géométriquement à partir de champs de vecteurs définis, de manière naturelle, sur certains espaces de modules de variétés abéliennes. Afin de justifier l'intérêt des équations de Ramanujan supérieures en théorie

de transcendance, on montre aussi que les valeurs de'une solution particulière remarquable de ces équations sont liées aux périodes de variétés abéliennes.

Dans la deuxième partie (troisième chapitre), on étudie la méthode de Nesterenko *per se*. On établit un énoncé géométrique, contenant le théorème de Nesterenko, sur la transcendance de valeurs d'applications holomorphes d'un disque vers une variété quasi-projective sur $\overline{\mathbf{Q}}$ définies comme des courbes intégrales d'un champ de vecteurs. Ces applications doivent aussi satisfaire une propriété d'intégralité, ainsi qu'une condition de croissance et une forme renforcée de la densité de Zariski, conditions qui sont naturelles pour des courbes intégrales de champs de vecteurs.

Title : Integral curves : transcendence and geometry

Keywords : integral curves, algebraic independence, moderate growth, abelian varieties, moduli spaces, periods

Abstract : This thesis is devoted to the study of some questions motivated by Nesterenko's theorem on the algebraic independence of values of the Eisenstein series E_2 , E_4 , E_6 . It is divided in two parts.

In the first part, comprising the first two chapters, we generalize the algebraic differential equations satisfied by Eisenstein series that lie in the heart of Nesterenko's method, the *Ramanujan equations*. These generalizations, called *higher Ramanujan equations*, are obtained geometrically from vector fields naturally defined on certain moduli spaces of abelian varieties. In order to justify the interest of higher Ramanujan

equations in Transcendence Theory, we also show that values of a remarkable solution of these equations are related to *periods* of abelian varieties.

In the second part (third chapter), we study Nesterenko's method *per se*. We establish a geometric statement, containing the theorem of Nesterenko, on the transcendence of values of holomorphic maps from a disk to a quasi-projective variety over $\overline{\mathbf{Q}}$ defined as integral curves of some vector field. These maps are required to satisfy some integrality property, besides a growth condition and a strong form of Zariski-density that are natural for integral curves of algebraic vector fields.

